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Existence, multiplicity and stability results for Stieltjes differential equations and applications

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Dedication

To my beloved family,
to my parents,
to my sister, and my brother.
Thank you for always being here.

Lamiae MAIA, May 30th 2024.

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Abstract

This manuscript investigates existence and multiplicity of solutions for Stieltjes differential equations using the lower and upper solutions method and fixed point index theory. It also examines the prolongation of solutions over the positive real half-line to introduce Lyapunov-like stability results within the context of these equations. Practical examples are provided to demonstrate the real-world implications of our results and showcase the applicability of Stieltjes differential equations in modeling various phenomena. These equations employ a novel concept known as the *Stieltjes derivative*, which involves differentiation with respect to a left-continuous, non-decreasing function referred to as the *derivoator*. Differentiating with respect to this function, simplifies modeling phenomena with discontinuities and stationary periods, reducing the right-hand side term, and integrates some properties of the derivoator such as jumps and stationary periods into the solution. After introducing the preliminaries and the fundamental tools from Stieltjes calculus, we delve into the motivation behind introducing the Stieltjes derivative. We then establish existence and multiplicity results for first-order Stieltjes equations with periodic or initial value conditions, using the method of lower and upper solutions and fixed point index theory. We then generalize these results to systems of Stieltjes differential equations, establishing existence results for functional systems as well. Our main results do not involve monotonicity conditions imposed on the second member. Finally, in the last chapter concerning Stieltjes dynamical systems, we focus on the extension of solutions on the positive real half-line and the notion of maximal solutions to introduce stability results in the Lyapunov sense. Practical applications are provided at the end of each of the last three chapters, illustrating the theoretical findings.

Keywords: Existence; multiplicity; Stieltjes differential equations; Lebesgue-Stieltjes integral; Lebesgue-Stieltjes measure; periodic boundary condition; initial value problems; lower and upper solution; strict lower and upper solution; systems of Stieltjes differential equations; functional Stieltjes differential equations; predator-prey model; prolongation of solutions; maximal solution; Lyapunov stability; Lyapunov's function; applications of Stieltjes differential equations.

Résumé

Ce manuscrit étudie l'existence et la multiplicité de solutions pour les équations différentielles de Stieltjes en utilisant la méthode des sous et sur-solutions ainsi que la théorie de l'indice de point fixe. Il examine également la prolongation des solutions sur la demi-droite réelle positive pour introduire des résultats de stabilité de type Lyapunov dans le contexte de ces équations. Des exemples pratiques sont fournis pour démontrer les implications concrètes de nos résultats et mettre en évidence l'applicabilité des équations différentielles de Stieltjes dans la modélisation de divers phénomènes. Ces équations utilisent la *dérivée de Stieltjes*, qui implique une différentiation par rapport à une fonction continue à gauche et croissante appelée le *dérivateur*. En dérivant par rapport à cette fonction, on simplifie la modélisation des phénomènes comportant des discontinuités et des périodes stationnaires, ce qui réduit le terme du second membre, et intègre des propriétés du dérivateur telles que les sauts et les périodes stationnaires dans la solution. Après avoir introduit les préliminaires et les outils fondamentaux du calcul de Stieltjes, nous nous penchons sur la motivation derrière l'introduction de la dérivée de Stieltjes. Ensuite, nous établissons des résultats d'existence et de multiplicité pour les équations de Stieltjes du premier ordre avec des conditions aux limites périodiques ou initiales, en utilisant la méthode des sous et sur-solutions ainsi que la théorie de l'indice de point fixe. Nous généralisons ensuite ces résultats aux systèmes d'équations différentielles de Stieltjes, établissant également des résultats d'existence pour les systèmes fonctionnels. Nos principaux résultats n'impliquent pas de conditions de monotonie imposées au second membre. Enfin, dans le dernier chapitre concernant les systèmes dynamiques de Stieltjes, nous nous concentrons sur l'extension des solutions sur la demi-droite réelle positive et la notion de solutions maximales pour introduire des résultats de stabilité au sens de Lyapunov. Des applications concrètes sont fournies à la fin de chacun des trois derniers chapitres, illustrant les découvertes théoriques.

Mots-clés : Existence; multiplicité; équations différentielles de Stieltjes; intégrale de Lebesgue-Stieltjes; mesure de Lebesgue-Stieltjes; condition périodique; Problèmes à valeur initiale; sous et sur-solutions; sous et sur-solutions strictes; solutions multiples; systèmes d'équations différentielles de Stieltjes; équations différentielles fonctionnelles de Stieltjes; modèle proie-prédateur; prolongation des solutions; Solution maximale; stabilité au sens de Lyapunov; fonction de Lyapunov; applications des équations différentielles de Stieltjes.

Résumé étendu

Dans cette thèse, l'autrice présente le travail de recherche réalisé lors de sa phase prédoctorale. Comme l'indique le titre "Existence, multiplicité et résultats de stabilité pour les équations différentielles de Stieltjes et applications", l'objet central de cette investigation réside dans le domaine des équations différentielles de Stieltjes. Ce cadre implique une notion modifiée de la différentiation connue sous le nom de *dérivée de Stieltjes*. Généralisant la dérivée classique, la dérivée de Stieltjes implique de prendre la variation par rapport à une fonction croissante et continue à gauche.

Par la suite, nous présentons un aperçu condensé des cinq chapitres qui composent ce manuscrit. Dans le Chapitre 1, nous présentons en premier lieu les fondements de la théorie de la mesure, en se concentrant particulièrement sur les intégrales et mesures de Lebesgue-Stieltjes, explorant la relation entre les fonctions continues à gauche et croissantes et la mesure de Lebesgue-Stieltjes. La deuxième partie concerne les notions de topologie et de continuité générées par une fonction continue à gauche, croissante $g : \mathbb{R} \rightarrow \mathbb{R}$, introduisant le concept de g -continuité et g -continuité absolue, et présentant la dérivée de Stieltjes et ses propriétés. Le chapitre se conclut par des préliminaires de la théorie des points fixes, posant les bases pour les résultats principaux des chapitres suivants.

Dans le Chapitre 2, nous nous intéressons à la motivation derrière l'introduction des équations différentielles de Stieltjes. Celles-ci formeraient un cadre pour aborder divers problèmes d'un même point de vue. Nous établissons en particulier le lien entre les équations différentielles de Stieltjes et plusieurs problèmes classiques, y compris les équations différentielles ordinaires, les équations différentielles impulsives, les équations de différence et les équations sur des échelles de temps.

Dans le Chapitre 3, nous introduisons les résultats d'existence et de multiplicité que nous avons établis dans [60], liés aux équations différentielles de Stieltjes par la méthode des sous et sur-solutions en utilisant les propriétés de l'indice du point fixe. Nous nous intéressons aux problèmes de la forme

$$\begin{aligned} x'_g(t) &= f(t, x(t)) \quad g\text{-presque pour tout } t \in [0, T], \\ x &\in \mathfrak{B}; \end{aligned}$$

où x'_g désigne la dérivée de Stieltjes de x par rapport à $g : \mathbb{R} \rightarrow \mathbb{R}$ une fonction continue à gauche et croissante, $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, et $\mathfrak{B}$ désigne une condition périodique ou initiale:

$$\begin{aligned} x(0) &= x(T); \\ x(0) &= x_0. \end{aligned}$$

Le chapitre est divisé en deux sections. Dans la première section, nous étudions l'existence de solutions des problèmes périodiques et à valeurs initiales. Ce faisant, nous n'imposons pas de conditions de monotonie, généralisant ainsi les résultats existants dans la littérature. Dans la deuxième section, nous introduisons des résultats

de multiplicité pour les deux problèmes, introduisant les concepts de sous et sur-solutions strictes. Le chapitre se termine par une application du résultat d'existence pour modéliser la persistance d'une population après un événement extrême à l'aide d'une équation différentielle de Stieltjes.

Dans le Chapitre 4, nous présentons nos résultats à partir de l'article [61] en nous appuyant sur les résultats d'existence établis pour les équations différentielles de Stieltjes avec un dérivateur $g : \mathbb{R} \rightarrow \mathbb{R}$, afin d'étendre notre analyse aux systèmes d'équations différentielles de Stieltjes impliquant un dérivateur à valeurs vectorielles g :

$$\begin{aligned} \mathbf{x}'_g(t) &= \mathbf{f}(t, \mathbf{x}(t)) \quad g\text{-presque pour tout } t \in [0, T], \\ \mathbf{x} &\in \mathfrak{B}; \end{aligned}$$

où $\mathbf{f} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, et ceux avec une dépendence fonctionnelle:

$$\begin{aligned} \mathbf{x}'_g(t) &= \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}) \quad g\text{-presque pour tout } t \in [0, T], \\ \mathbf{x} &\in \mathfrak{B}; \end{aligned}$$

où $\mathbf{f} : [0, T] \times \mathbb{R}^n \times \mathcal{C}$, $\mathcal{C}$ un sous-ensemble de l'espace fonctionnel $\mathcal{BC}_g([0, T], \mathbb{R}^n)$ qui sera défini ultérieurement, et $g = (g_1, \dots, g_n) : \mathbb{R} \rightarrow \mathbb{R}^n$ est un dérivateur à valeurs vectorielles tel que g_i est une fonction continue à gauche et croissante pour $i = 1, \dots, n$. La condition $\mathbf{x} \in \mathfrak{B}$ désigne une condition initiale ou périodique:

$$\begin{aligned} \mathbf{x}(0) &= \mathbf{x}_0 = (\mathbf{x}_{0,1}, \dots, \mathbf{x}_{0,n}) \in \mathbb{R}^n; \\ \mathbf{x}(0) &= \mathbf{x}(T). \end{aligned}$$

Nous introduisons ensuite les concepts de fonctions g -Carathéodory et de sous et sur-solutions de valeurs vectorielles des systèmes considérés. De plus, nous éliminons également les conditions de type monotonie courantes dans la littérature liée à la fonction $\mathbf{f}$. La dernière section du chapitre illustre une application de ces résultats à la dynamique d'une population de poissons soumise à la prédation et à la pêche saisonnière.

Enfin, le Chapitre 5 est divisé en trois parties. Dans la première partie, nous nous concentrons sur l'étude des prolongations des solutions sur la demi-droite réelle positive lorsqu'il s'agit de systèmes dynamiques d'équations différentielles de Stieltjes de la forme

$$\begin{aligned} \mathbf{x}'_g(t) &= \mathbf{f}(t, \mathbf{x}(t)) \quad g\text{-presque pour tout } t \geq t_0 \geq 0, \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \in \mathbb{R}^n; \end{aligned}$$

où $\mathbf{f} = (f_1, \dots, f_n) : [t_0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, et $g : \mathbb{R} \rightarrow \mathbb{R}$ continue à gauche et croissante. Ce faisant, nous utilisons des ensembles compacts pour caractériser la solution maximale, et nous établissons des hypothèses garantissant l'existence globale des solutions. Dans la deuxième partie, nous appliquons la *Deuxième Méthode de Lyapunov* pour réaliser une étude qualitative du comportement du système au voisinage d'un point d'équilibre. Cette approche, bien connue dans le cas classique de la dérivation, repose sur la recherche d'une *fonction de Lyapunov*, pour évaluer différents types de stabilité d'un équilibre: stabilité, stabilité asymptotique, et leurs propriétés uniformes. Dans la dernière partie de ce chapitre, nous suggérons des applications à

la dynamique des populations: la première est un modèle de dynamique de population avec effet Allee, impactée négativement par les vibrations d'un train qui passe, et la deuxième représente la dynamique d'une population de *Cyanobactéries* dans un environnement cultivé observant les niveaux d'ammoniac dans le processus.

Introduction

Real-world phenomena frequently exhibit abrupt changes. Epidemic outbreaks surge, financial markets undergo sudden crashes, and ecological populations respond dynamically to various factors. These factors include natural processes, environmental shocks such as earthquakes, sudden migrations..., as well as external events resulting from human interventions, forest implantation, fires.... Rigorously modeling such phenomena necessitates the incorporation of discontinuities into our models. Using classical derivatives, there is a possible way to solve this by means of introducing impulses into these models to include the sudden changes. However, certain species exhibit dormant phases during which reproductive activity significantly diminishes or ceases altogether. For instance, hibernation, migration, adverse conditions impact population dynamics. Since the dynamics barely change during such stationary periods, some of these periods can be regarded as 'negligible' during the whole cycle, or even may necessitate specific parameter choices for accurate modeling. In such cases, equations on time scales can be very useful, since they provide irregular time steps. Whereas, through the Stieltjes derivative, we will be permitted to model these dormant phases in a more descriptive manner within the framework of Stieltjes differential equations.

At the intersection of mathematical disciplines, specifically analysis, and integration, Stieltjes differential equations emerge as a captivating field of study. These equations introduce a novel concept of derivation: a derivative with respect to a function. The idea of deriving with respect to a function has roots in early works of Daniell [17, 18] and Young [87]. In addition, similar concepts were investigated in [6, 12, 23, 39, 52, 74]. Recently, López Pouso and Rodríguez focused their attention on the definition and meaning of Stieltjes derivatives in [58]. They derived a practical definition of the Stieltjes derivative with respect to a nondecreasing function $g : \mathbb{R} \rightarrow \mathbb{R}$ that is continuous from the left, which we will present rigorously later in this work. Initially, the Stieltjes derivative of a real-valued function f at a point t_0 such that the derivator g is not constant on a neighborhood of t_0 , can be formulated as

$$f'_g(t_0) = \begin{cases} \lim_{s \rightarrow t_0} \frac{f(s) - f(t_0)}{g(s) - g(t_0)}, & \text{if } g \text{ is continuous at } t_0, \\ \lim_{s \rightarrow t_0^+} \frac{f(s) - f(t_0)}{g(s) - g(t_0)}, & \text{if } g \text{ is discontinuous at } t_0. \end{cases}$$

The Stieltjes derivative is notable for its ability to be defined even at points where a function is discontinuous. This allows for the tracking of the function's behavior at those points, unlike classical differentiation, which requires the function to be continuous at the point of differentiation. This particular feature make the Stieltjes derivative both relevant and intriguing, motivating further exploration of its properties in light of the classical derivative properties and limitations. At the first glance, one would notice that the Stieltjes derivative is not defined for points where g is constant on a neighborhood of these points, and this conforms an additional feature of

this derivative, since g is constant in such neighborhoods, g does not ‘separate’ those moments. Considering this latter fact, modeling phenomena with stationary periods through Stieltjes differential equations with a derivator having the same constancy periods proves to be both concise and descriptive. Specifically, as we will explore, those stationary periods shall be negligible through derivation with respect to g , allowing us to focus solely on the remaining dynamics. Moreover, when considering phenomena presenting impulses, we gain a concise reformulation of impulsive differential equations, via Stieltjes differential equations with a derivator presenting the same discontinuities at these jumps. Through these settings, Stieltjes differential equations encode prior information about the process through the derivator g , which makes the solutions inherit the same discontinuities and the stationary periods, as we will discover while defining the regularity of solutions.

In modeling a process using Stieltjes differential equations, a fundamental question arises regarding the existence of solutions. This question was one of the primary focuses of the author’s research during her predoctoral stage, as explored in works [60, 61]. In [60], the focus was on the Stieltjes differential equation

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, T], \quad x \in \mathfrak{B}, \quad (1)$$

where x'_g is the Stieltjes derivative of x with respect to $g : \mathbb{R} \rightarrow \mathbb{R}$ a left-continuous and nondecreasing function, $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $\mathfrak{B}$ refers to a periodic boundary condition or an initial value condition. In [60] tackling both the periodic and the initial value problems, via the method of the lower and the upper solutions, we were able to prove existence under weaker conditions on a g -Carathéodory function f . Moreover, we established multiplicity results using the notions of strict lower and upper solutions for both problems. At this point, it is interesting to mention that this method not only guarantees existence but also an a priori bound of solutions is ensured. This can be regarded as a viability result ensuring existence of a solution which lies in the region between the lower and upper solution. The method of the lower and upper solutions has a long history in the classical case where $g(t) = t$. This method has been applied to periodic boundary value problems when f is continuous function by Mawhin in [68], by Nieto in [71], by Nieto and Rodríguez-López in [72], and also by Nkashama in [73] when f is a Carathéodory function. Several authors have extended the concepts of upper and lower solutions, for instance in [53] the authors considered functions which are not necessarily absolutely continuous and may exhibit discontinuities. In the literature of Stieltjes differentiation, it is worthwhile to mention that lower and upper solutions were used in [54] imposing a monotonicity condition on the function f at every discontinuity point $t \in [0, T]$:

$$\text{the function } x \mapsto x + f(t, x)(g(t^+) - g(t)) \quad \text{is nondecreasing on } [\alpha(t), \beta(t)], \quad (2)$$

where α and β are lower and upper solutions respectively.

The notions of lower and upper solutions have been generalized in [69] to study problems with an initial value boundary condition, specifically, to investigate solutions of measure differential equations of the form

$$x(t) = x_0 + \int_0^t f(s, x(s)) \, dg(s),$$

where the integral on the right side is understood in the Kurzweil–Stieltjes sense [70] which coincides with the Lebesgue–Stieltjes integral in our context. In the literature related to Stieltjes differential equations, regarding the periodic boundary value, Satco and Smyrlis [78] were the pioneers in achieving an existence result for the problem:

$$x'_g(t) + b(t)x(t) = f(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, T]; \quad x(0) = x(T). \quad (3)$$

In their work [78], they examined the case where the function f is g -integrally bounded. Furthermore, they extended their findings in [67, 79] to encompass Stieltjes differential inclusions. Our contribution in the first part of [60], was to drop the monotonicity condition imposed on the function f as in (2), and the function f need not be μ_g -integrally bounded, where μ_g is the Lebesgue–Stieltjes measure generated by g . Moreover, as an application, we suggest a model describing the dynamics of population subject to an extreme event, aiming to find a solution that ensures the population's persistence. In the second part of [60], we established results regarding the existence of multiple solutions of (1), which, to the best of our knowledge, constitute the first in the existing literature. These results were based on the notions of strict lower and upper solutions, which ensure that the solutions will lie strictly in the region between the strict lower and upper solutions without reaching the boundary of this region. In a recent paper [49], via topological methods, new existence and multiplicity results have been established by Krajšáková and Tojo for the problem (3) with linear boundary conditions, using fixed point index methods on cones, assuming that $f \geq 0$ and $b > 0$. Even in the case of the classical derivation, which corresponds to $g \equiv \text{id}_{\mathbb{R}}$, there are few results in the literature regarding the existence of several solutions of first-order differential equations. Ambrosetti–Prodi type results have been obtained by Mawhin in [68] and by Nkashama in [73] for periodic boundary value problems. In addition, multiplicity results have been established in [32], for systems of first-order differential inclusions, and also for equations with nonlinear differential operators in [20].

In [61], we generalized the existence results established in [60], to systems of Stieltjes differential equations using vector-valued lower and upper solutions. The study of systems of Stieltjes differential equations involving several Stieltjes derivators traces its origins to the work of López Pouso and Márquez Albés in [56]. In [56], they established Peano-type existence results were established for initial value problems, extending the first existence results in the literature due to Frigon and López Pouso [31], that dealt with systems using the same derivator for each component. Recently, Márquez Albés and Tojo in [66] continued the study initiated in [56], focusing on existence and uniqueness results involving Osgood and Montel–Tonelli conditions, generalizing those in . In the work, we will be considering systems of the form:

$$\mathbf{x}'_g(t) = \mathbf{f}(t, \mathbf{x}(t)) \quad \text{for } g\text{-almost every } t \in [0, T], \quad \mathbf{x} \in \mathfrak{B}, \quad (4)$$

where $\mathbf{f} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and those with a functional dependence:

$$\mathbf{x}'_g(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}) \quad \text{for } g\text{-almost every } t \in [0, T], \quad \mathbf{x} \in \mathfrak{B}, \quad (5)$$

where $\mathbf{f} : [0, T] \times \mathbb{R}^n \times \mathcal{C}$, $\mathcal{C}$ a subset of $\mathcal{BC}_g([0, T], \mathbb{R}^n)$ which is a functional space that we define in the sequel, and $g = (g_1, \dots, g_n) : \mathbb{R} \rightarrow \mathbb{R}^n$ is a vector-valued derivator

such that g_i is a left-continuous nondecreasing function; for $i = 1, \dots, n$. Here, $\mathbf{x} \in \mathfrak{B}$ refers to an initial value condition or a periodic boundary condition:

$$\mathbf{x}(0) = \mathbf{x}_0 = (\mathbf{x}_{0,1}, \dots, \mathbf{x}_{0,n}) \in \mathbb{R}^n; \quad (6)$$

$$\mathbf{x}(0) = \mathbf{x}(T). \quad (7)$$

Existence results were established via the method of the lower and upper solutions in [57] for initial value problems only, imposing monotonicity conditions on the function $\mathbf{f}$. These monotonicity conditions are dropped in our work, ensuring existence and a priori bound of solutions for additional initial value problems, and interestingly for periodic problems as well. The functional dependency in the problem (5) covers more general problems including delayed differential equations [4, 21] and integro-differential equations [2], where the history or the memory of the solution may influence the dynamics. As an application of our results, we address the dynamics of a predator-prey model in an exploited fishery subject to a seasonal fishing strategy. Our results rely on the extension of the notions of the lower and the upper solutions to systems of Stieltjes differential equations. The method of the lower and upper solutions, to our knowledge, was first introduced by Knobloch in [47] for systems of differential equations with periodic boundary conditions. In [57], the method of lower and upper solutions has been extended to initial value problems for systems involving Kurzweil-Stieltjes integrals with a vector-valued measure, (for more details on Kurzweil-Stieltjes integrals, see [70]) to obtain existence results for systems (4), (6) and (5), (6). These extensions led to existence results for the considered systems, relying on monotonicity conditions for right-hand side member $\mathbf{f}$, and other monotonicity conditions involving $f_i(t, \cdot)$ for $t \in D_{g_i}$ for $i = 1, \dots, n$.

In contrast, few results can be found in the literature on periodic problems for systems of Stieltjes differential equations. In particular, in the case of a single-valued derivator where the right-hand side member is bounded by a function of the form $h(t)(1 + \|\mathbf{x}\|)$ for some g -integrable function h , existence results have been obtained for the problem (4), (7) in [78]. These results have been extended in [67, 79] to Stieltjes differential inclusions on $\mathbb{R}^n$ where $g_1 = \dots = g_n$. To the best of our knowledge, our work represents the first exploration of existence results for systems of Stieltjes differential equations (4), (7) with vector-valued derivators, and for the functional systems of Stieltjes differential equations (5), (7) with single and vector-valued derivators. As an application of our result, we model the dynamics of a predator-prey model in a fishery subject to a seasonal fishing strategy.

In the last part of this work, we turn out attention to Stieltjes differential equations on the positive real half-line rather than limiting ourselves to a finite interval. This is motivated by the fact that many natural processes evolve over time without any inherent time limit, while others can exhibit finite-time blow-up. Essentially, we will be considering Stieltjes dynamical systems of the form:

$$\mathbf{x}'_g(t) = \mathbf{f}(t, \mathbf{x}(t)) \quad \text{for } g\text{-almost every } t \geq t_0 \geq 0, \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n;$$

where $\mathbf{f} = (f_1, \dots, f_n) : [t_0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $g : \mathbb{R} \rightarrow \mathbb{R}$ a left-continuous and non-decreasing. Leveraging the existence of the maximal solution, and the generalized Grönwall lemma for the Stieltjes derivatives established by Larivière in [50], we add

results characterizing the maximal solution through compact sets, and others yielding global existence. Along this study, inspired from the Lyapunov stability results in [22, 37] involving measure differential equations and dynamic equations on time scales, we apply *Lyapunov's Second Method* [59], to guarantee different types of stability of an *equilibrium*: stability, asymptotic stability, and their uniform properties. Our results extend some classical results from [44, 48, 84], in particular, the condition on the upper bound of the g -derivative of the Lyapunov function along maximal solutions, is improved. In the last part of this chapter, we suggest applications to the dynamics of population: in the first application, we model the dynamics of a population with Allee's effect negatively impacted by train vibrations, and another application related to the dynamics of a population of *Cyanobacteria* in a cultured environment observing the levels of ammonia in the process.

Measure theory, continuity notions, and fixed point theory preliminaries

In the opening chapter of this thesis, we delve into the essential groundwork required for our study. In the first section, we commence by introducing concepts from measure and integration theory [5, 16, 76, 77]. This will pave the way for a comprehensive understanding of our subsequent study relying on the integral in the sense of the Lebesgue-Stieltjes. The integration in this sense incorporates a left-continuous and nondecreasing function $g : \mathbb{R} \rightarrow \mathbb{R}$ acting like a measure, expanding on the standard Lebesgue integral. Furthermore, we present some recent notions related to the theory of the Stieltjes differentiation, appealing to new notions analogous to continuity and absolute continuity from classical analysis. The last section is devoted to a brief preliminary from the fixed point theory [40], enhancing the fixed point theorems and the notion of the fixed point index used to establish the existence results in Chapter 3 and Chapter 4.

1.1 Introduction to the Lebesgue-Stieltjes Integral

This section is dedicated to preliminaries from the measure theory to introduce the Lebesgue-Stieltjes measure and its associated integral. In doing so, we appeal to necessary definitions and results related to notions of measurable and measure spaces, measurable maps, Lebesgue's integral, and fundamental theorems of integration following [5, 16, 76, 77].

In what follows, given a set X , we denote by $\mathcal{P}(X)$ the set of all subsets of X , and for any subset $A \subset X$, we denote by $A^c := X \setminus A$ its complement.

Definition 1.1.1 (Measurable Space). Let X be a set, and a collection $\mathcal{M} \subset \mathcal{P}(X)$. $\mathcal{M}$ is said to be a σ -algebra if the following assumptions hold:

1. $X \in \mathcal{M}$;
2. If $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$;
3. Every countable union of elements of $\mathcal{M}$ is an element of $\mathcal{M}$, i.e. if $A_i \in \mathcal{M}$ for every $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$.

The pair $(X, \mathcal{M})$ is a *measurable space* if $\mathcal{M} \subset \mathcal{P}(X)$ is a σ -algebra. In this case, the elements of the σ -algebra $\mathcal{M}$ are called *measurable sets*.

Remark 1.1.2. Every σ -algebra $\mathcal{M} \subset \mathcal{P}(X)$ satisfies in particular the following statements:

1. $\emptyset \in \mathcal{M}$;
2. If $A_i \in \mathcal{M}$ for $i \in \{1, \dots, n\}$; $n \in \mathbb{N}$, then $\bigcup_{i=1}^n A_i \in \mathcal{M}$;
3. Every countable intersection of elements of $\mathcal{M}$ is an element of $\mathcal{M}$;
4. If $A, B \in \mathcal{M}$ then $A \setminus B \in \mathcal{M}$.

Remark 1.1.3. The notion of σ -algebra is quite similar to the notion of topology. However, one may observe that topological spaces are closed under finite intersections and arbitrary unions while σ -algebras are closed under complementation and countable union only.

The following notion connects the concepts of σ -algebra and topology. Specifically, it guarantees the construction of a σ -algebra from open sets within a topological space, thereby nurturing a rich variety of sets beyond the open and closed sets in a topological space.

Definition 1.1.4. Let (X, τ) be a topological space, and let $\mathcal{B} \subset \mathcal{P}(X)$ be the smallest σ -algebra containing τ . Then $\mathcal{B}$ is called the *Borel σ -algebra* of (X, τ) . For sake of precision, it is usually denoted by $\mathcal{B}(\tau)$, and its elements are called *Borel (measurable) sets*.

1.1.1 Measurable Functions

Analogous to continuous maps between topological spaces, measurable maps between measurable spaces can be defined as maps where the pre-images of measurable sets are also measurable. Rudin [76] takes a slightly different approach, defining measurable maps as those where pre-images of open sets are measurable. Both definitions align when the target space is endowed with its Borel σ -algebra.

Definition 1.1.5 (Measurable Function). Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be measurable spaces. A map $f : X \rightarrow Y$ is called *measurable* if:

$$B \in \mathcal{M}_Y \implies f^{-1}(B) \in \mathcal{M}_X.$$

It is worthwhile to mention that the measurability of a function depends on the σ -algebras associated with the spaces (X, τ_X) and (Y, τ_Y) . The next definition is a special case of the previous definition taking $\mathcal{M}_X \subset \mathcal{P}(X)$ and $\mathcal{M}_Y \subset \mathcal{P}(Y)$ the Borel σ -algebras.

Definition 1.1.6. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \rightarrow Y$ is called *Borel measurable* if the pre-image of every Borel measurable subset of Y under f is a Borel measurable subset of X .

In particular, if $(X, \mathcal{M})$ be a measurable space, consider the Borel σ -algebra on $\mathbb{R}$, and let $A \subset X$ be any subset. Then the characteristic function χ_A of the set A is measurable if and only if A is a measurable set.

Theorem 1.1.7 (Measurable and Continuous Maps). *Let $(X, \mathcal{M}_X)$, $(Y, \mathcal{M}_Y)$ be measurable spaces. Let $\tau_Y \subset \mathcal{P}(Y)$ be a topology on Y such that $\mathcal{M}_Y$ is the Borel σ -algebra of (Y, τ_Y) .*

1. *A map $f : X \rightarrow Y$ is measurable if and only if*

$$B \in \tau_Y \implies f^{-1}(B) \in \mathcal{M}_X.$$

2. *If $\tau_X \subset \mathcal{P}(X)$ is a topology on X such that $\mathcal{M}_X$ is the Borel σ -algebra of (X, τ_X) . Then, every continuous map $f : X \rightarrow Y$ is Borel measurable.*

We conclude this subsection with a concise exploration of a specific class of measurable functions: *step functions*, also known as *simple functions*. These functions will be pivotal in approximating measurable functions and, consequently, in defining the Lebesgue integral.

Definition 1.1.8 (Step Function). Let $(X, \mathcal{M})$ be a measurable set. A function $S : X \rightarrow \mathbb{R}$ is called a *step function* (or *simple function*) if the image $S(X)$ is a finite subset of $\mathbb{R}$.

Remark 1.1.9. Given a step function $S : X \rightarrow \mathbb{R}$, we can write $S(X) = \{s_1, \dots, s_m\}$ with $s_i \neq s_j$ for $i \neq j$, and define $A_i := S^{-1}(\{s_i\}) = \{x \in X : S(x) = s_i\}$ for $i = 1, \dots, m$. Based on this setting, the family $\{A_i\}_{i=1}^m$ is a partition of X , i.e.

$$X = \bigcup_{i=1}^m A_i, \quad \text{and } A_i \cap A_j = \emptyset \quad \text{for } i \neq j.$$

Moreover,

$$S = \sum_{i=1}^m s_i \chi_{A_i},$$

where $\chi_{A_i} : X \rightarrow \mathbb{R}$ is the characteristic function of the set A_i , for $i = 1, \dots, m$. In this case, S is measurable if and only if the set $A_i \subset X$ is measurable for $i = 1, \dots, m$.

These functions are advantageous, as shows the next theorem [76, Theorem 1.17], a measurable can be approximated by a sequence of measurable step functions.

Theorem 1.1.10 (Approximation Theorem). *Let $(X, \mathcal{M})$ be a measurable space and let $f : X \rightarrow [0, \infty]$ be a function. f is measurable if and only if there exists a sequence of measurable step functions $S_n : X \rightarrow [0, \infty)$; $n \in \mathbb{N}$, such that*

$$0 \leq S_1(x) \leq S_2(x) \leq \dots \leq f(x), \quad \text{and} \quad f(x) = \lim_{n \rightarrow \infty} S_n(x), \quad \text{for every } x \in X.$$

As a consequence of the previous theorem, step functions provide a characterization of regulated functions in the following sense: a function is regulated if and only if it is a uniform limit of a sequence of step functions. The reader is referred to [77, Theorem 1.26].

1.1.2 Integration of Nonnegative Functions

Our goal in this subsection is to define the integral of a measurable step function, followed by the integral of a nonnegative measurable function through approximation. We start by defining positive measures.

Definition 1.1.11 (Measure). Let $(X, \mathcal{M})$ be a measurable space. A *positive measure* on $(X, \mathcal{M})$ is a function

$$\mu : \mathcal{M} \rightarrow [0, \infty]$$

satisfying the following properties:

1. σ -additivity: i.e. if $\{A_i\}_{i \in \mathbb{N}} \subset \mathcal{M}$ is a sequence of pairwise disjoint measurable sets, then

$$\mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i).$$

2. There exists a measurable set $A \in \mathcal{M}$ such that $\mu(A) < \infty$.

Given a set X , a triple $(X, \mathcal{M}, \mu)$ is said to be a *measure space* if $\mathcal{M} \subset \mathcal{P}(X)$ is a σ -algebra on X , and $\mu : \mathcal{M} \rightarrow [0, \infty]$ is a measure.

Definition 1.1.12. Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. A null-measure set is a measurable set $N \in \mathcal{M}$ such that $\mu(N) = 0$.
2. Let $E \subset X$ be fixed. A property $\mathcal{P}$ is said to be satisfied μ -almost everywhere in E (μ -a.e. in E) or that it holds for μ -almost all $x \in E$ (μ -a.a. $x \in E$) if there exists a null-measure set $N \in \mathcal{M}$ such that $\mathcal{P}$ holds in $E \setminus N$.

We can concisely summarize the basic properties of measures in the theorem below.

Theorem 1.1.13 (Properties of Measures). *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then the following assumptions hold:*

1. $\mu(\emptyset) = 0$;
2. If $A, B \in \mathcal{M}$ such that $A \subset B$ then $\mu(A) \leq \mu(B)$;
3. If $\{A_i\}_{i=1}^{\infty} \subset \mathcal{M}$ is a sequence such that $A_i \subset A_{i+1}$ for all $i \in \mathbb{N}$. Then

$$\mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \lim_{i \rightarrow \infty} \mu(A_i);$$

4. Let $A_i \in \mathcal{M}$ be a sequence such that $A_{i+1} \subset A_i$ for all $i \in \mathbb{N}$. Then

$$\mu(A_1) < \infty \implies \mu \left(\bigcap_{i=1}^{\infty} A_i \right) = \lim_{i \rightarrow \infty} \mu(A_i).$$

Having stated these elementary notions, we are now able to present the Lebesgue integral of a nonnegative measurable function.

Definition 1.1.14 (Lebesgue Integral). Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $E \in \mathcal{M}$ be a measurable set.

1. Let $S : X \rightarrow [0, \infty)$ be a measurable step function defined by

$$S = \sum_{i=1}^m s_i \chi_{A_i},$$

where $s_i \in [0, \infty)$ and $A_i \in \mathcal{M}$ for $i = 1, \dots, m$. The *Lebesgue integral* of S over E is defined as:

$$\int_E S \, d\mu := \sum_{i=1}^m s_i \mu(E \cap A_i) \in [0, \infty].$$

2. If $f : X \rightarrow [0, \infty]$ is a measurable function, then the Lebesgue integral of f over E is defined as

$$\int_E f \, d\mu := \sup \left\{ \int_E S \, d\mu : S \in \mathcal{S} : 0 \leq S(x) \leq f(x) \text{ for every } x \in X \right\} \in [0, \infty],$$

where $\mathcal{S}$ denotes the space of measurable step functions $S : X \rightarrow [0, \infty)$.

Theorem 1.1.15. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. Let $a_1, a_2 \geq 0$, and let $f, f_1, f_2 : X \rightarrow [0, \infty]$ be measurable function. Then the following properties hold:*

1. *the map $a_1 f_1 + a_2 f_2$ is measurable, and we have that*

$$\int_E (a_1 f_1 + a_2 f_2) \, d\mu = a_1 \int_E f_1 \, d\mu + a_2 \int_E f_2 \, d\mu;$$

2. *If $f_1 \leq f_2$ on E then $\int_E f_1 \, d\mu \leq \int_E f_2 \, d\mu$;*

3. *$\int_E f \, d\mu = \int_X f \chi_E \, d\mu$;*

4. *If $f(x) = 0$ for all $x \in E$ then $\int_E f \, d\mu = 0$;*

5. *If $\mu(E) = 0$ then $\int_E f \, d\mu = 0$;*

6. *If $A \in \mathcal{M}$ and $E \subset A$ then $\int_E f \, d\mu \leq \int_A f \, d\mu$;*

7. *If $\mu_1, \mu_2 : \mathcal{M} \rightarrow [0, \infty]$ are measures, then*

$$\int_E f \, d(a_1 \mu_1 + a_2 \mu_2) = a_1 \int_E f \, d\mu_1 + a_2 \int_E f \, d\mu_2.$$

The reader is referred to [76, Section 1.24] for the proof. The next result [76, Theorem 1.29] provides a way to build a measure given a nonnegative measurable function.

Proposition 1.1.16. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \rightarrow [0, \infty]$ a measurable function. The set function defined by*

$$\mu_f(E) = \int_E f \, d\mu, \quad \text{for all } E \in \mathcal{M},$$

is a measure. Moreover, for every measurable function $g : X \rightarrow [0, \infty]$ on X , we have that

$$\int_X g \, d\mu_f = \int_X g f \, d\mu.$$

The converse of Proposition 1.1.16 is a significant tool within the realm of measure theory, serving in deriving a new measure from an existing one, both defined on the same measurable space. This process involves assigning a density function, known as the *Radon-Nikodým derivative*, to each point in the space and integrate over the relevant measurable subset. To state this theorem, we need to define the notion of absolute continuity related to measures and the property of σ -finiteness.

Definition 1.1.17. Let $(X, \mathcal{M})$ be a measurable space, and $\mu, \nu : \mathcal{M} \rightarrow [0, +\infty]$ be two measures. The measure ν is said to be *absolutely continuous* with respect to μ , and we denote $\nu \ll \mu$, if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $E \in \mathcal{M}$:

$$\mu(E) < \delta \Rightarrow \nu(E) < \epsilon.$$

Now, we define the notion of σ -finiteness .

Definition 1.1.18. Let $(X, \mathcal{M}, \mu)$ be a measurable space. μ is said to be σ -finite if there exists a sequence $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ satisfying the following:

1. $\mu(A_n) < \infty$, for all $n \in \mathbb{N}$;
2. $X = \bigcup_{n \in \mathbb{N}} A_n$.

Theorem 1.1.19 (Radon-Nikodým Theorem). *Let $(X, \mathcal{M})$ be a measurable space, and $\mu, \nu : \mathcal{M} \rightarrow [0, +\infty]$ two σ -finite measures satisfying $\nu \ll \mu$, then there is a measurable function $f : X \rightarrow [0, +\infty)$ such that*

$$\nu(E) = \int_E f \, d\mu, \quad \text{for every } E \in \mathcal{M}.$$

Moreover, the function f is unique up to a μ -almost everywhere equality, and it is called the Radon-Nikodým derivative.

In the context of the Lebesgue-Stieltjes measure, the Radon-Nikodým Theorem [16, Theorem 4.2.2] plays a crucial role in deriving the Fundamental Theorem of Calculus for the Lebesgue-Stieltjes integrals, which we will state further later. In the statement of the Radon-Nikodým Theorem, notice that the σ -finiteness is indispensable, the reader is referred to [5, Remark 4.1.1] for a counter-example.

We conclude this subsection by stating the Lebesgue Monotone Convergence Theorem [76, Theorem 1.26].

Theorem 1.1.20 (Lebesgue Monotone Convergence Theorem). *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f_n : X \rightarrow [0, \infty]$ be a sequence of measurable functions such that*

$$f_n(x) \leq f_{n+1}(x) \quad \text{for every } x \in X \text{ and } n \in \mathbb{N}.$$

Let us define $f : X \rightarrow [0, \infty]$ by

$$f(x) := \lim_{n \rightarrow \infty} f_n(x) \quad \text{for } x \in X.$$

Then, f is measurable and

$$\lim_{n \rightarrow \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

1.1.3 Integration of Real Valued Functions

The Lebesgue integral of a real-valued measurable function is derived by subtracting the integrals of its positive and negative part. This definition is applicable when at least one of these integrals is not infinite leading to the following definition of the Lebesgue integral.

Definition 1.1.21 (Lebesgue Integrable Functions). Let $(X, \mathcal{M}, \mu)$ be a measure space and $A \in \mathcal{M}$. A function $f : X \rightarrow \mathbb{R}$ is said to be *Lebesgue integrable* or μ -*integrable* over A if f is μ -measurable and $\int_A |f| \, d\mu < \infty$. We denote by $\mathcal{L}_\mu^1(A, \mathbb{R})$ the space of μ -integrable functions over A equipped with the norm $\|\cdot\|_{\mathcal{L}_\mu^1(A, \mathbb{R})}$ defined by

$$\|f\|_{\mathcal{L}_\mu^1(A, \mathbb{R})} := \int_A |f| \, d\mu, \quad \text{for all } f \in \mathcal{L}_\mu^1(A, \mathbb{R}).$$

When there is no ambiguity, we use the notation $\|\cdot\|_{\mathcal{L}_\mu^1}$ to refer to $\|\cdot\|_{\mathcal{L}_\mu^1(A, \mathbb{R})}$. The Lebesgue integral of a function $f \in \mathcal{L}_\mu^1(A, \mathbb{R})$ over the set A is defined as the real number

$$\int_A f \, d\mu := \int_A f^+ \, d\mu - \int_A f^- \, d\mu,$$

where the functions $f^\pm : X \rightarrow [0, \infty)$ are defined for every $x \in X$ by

$$f^+(x) := \max\{f(x), 0\}, \quad f^-(x) := \max\{-f(x), 0\}.$$

Remark 1.1.22. The functions $f^\pm$ are measurable satisfying $0 \leq f^\pm \leq |f|$. Hence, their integrals over A are finite by Condition (1) of Theorem 1.1.15.

Remark 1.1.23. In subsequent sections of this work, when considering a measure space $(X, \mathcal{M}, \mu)$, $A \in \mathcal{M}$ and a function $f : X \rightarrow Y$, where $Y \subset \mathbb{R}$ and $f \in \mathcal{L}_\mu^1(A, \mathbb{R})$, we will adopt the notation $f \in \mathcal{L}_\mu^1(A, Y)$ instead of $f \in \mathcal{L}_\mu^1(A, \mathbb{R})$ to explicitly indicate the codomain Y .

Now that we have defined the Lebesgue integral, we provide some properties resulting from Theorem 1.1.15, the reader is referred to [76, Theorems 1.32 and 1.33] for the proof and further results.

Theorem 1.1.24 (Properties of the Lebesgue Integral). *Let $(X, \mathcal{M}, \mu)$ be a measure space and $A \in \mathcal{M}$. Then the following statements hold:*

1. *The function $h : \mathcal{L}_\mu^1(A, \mathbb{R}) \rightarrow \mathbb{R} : f \mapsto \int_A f \, d\mu$ is linear;*
2. *For all $f, g \in \mathcal{L}_\mu^1(A, \mathbb{R})$, we have:*

$$f \leq g \quad \text{on } A \quad \implies \quad \int_A f \, d\mu \leq \int_A g \, d\mu;$$

3. *If $f \in \mathcal{L}_\mu^1(A, \mathbb{R})$, then $|f| \in \mathcal{L}_\mu^1(A, \mathbb{R}^+)$ and,*

$$\left| \int_A f \, d\mu \right| \leq \int_A |f| \, d\mu;$$

4. If $f \in \mathcal{L}_\mu^1(A, \mathbb{R})$ and $\{A_i\}_{i \in \mathbb{N}}$ is a sequence of pairwise disjoint measurable sets such that $A := \bigcup_{i \in \mathbb{N}} A_i$, then

$$\int_A f \, d\mu = \sum_{i=1}^{\infty} \int_{A_i} f \, d\mu;$$

5. For $f \in \mathcal{L}_\mu^1(A, \mathbb{R})$, we have that

$$\int_A f \, d\mu = \int_X f \chi_A \, d\mu;$$

6. For $f \in \mathcal{L}_\mu^1(A, \mathbb{R})$, if $\mu(A) = 0$ or $f|_A = 0$, then $\int_A f \, d\mu = 0$.

Theorem 1.1.25 (Lebesgue's Dominated Convergence Theorem). *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $h : X \rightarrow [0, \infty)$ be a μ -integrable function, and $\{f_n\}$ be a sequence of μ -integrable functions $f_n : X \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, satisfying*

$$|f_n(x)| \leq h(x) \quad \text{for all } x \in X, \text{ and } n \in \mathbb{N},$$

and converging pointwise to $f : X \rightarrow \mathbb{R}$, i.e.

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \text{for every } x \in X.$$

Then f is μ -integrable and, for every $A \in \mathcal{M}$,

$$\int_A f \, d\mu = \lim_{n \rightarrow \infty} \int_A f_n \, d\mu.$$

Now, we define *Carathéodory functions* with respect to a measure μ .

Definition 1.1.26. Let $I, X \subset \mathbb{R}$ such that $(I, \mathcal{M}, \mu)$ be a measure space. A function $f : I \times X \rightarrow \mathbb{R}$ is said to be μ -Carathéodory if it satisfies the following assumptions:

- (i) for every $x \in X$, $t \mapsto f(t, x)$ is μ -measurable;
- (ii) for μ -almost every $t \in I$, $x \mapsto f(t, x)$ is continuous on $\mathbb{R}$;
- (iii) $\forall r > 0$, $\exists h_r \in \mathcal{L}_\mu^1(I, [0, \infty))$ such that

$$|f(t, x)| \leq h_r(t), \quad \text{for } \mu\text{-almost all } t \in I, \text{ for all } x \in X, \text{ such that } |x| \leq r.$$

1.1.4 The Lebesgue-Stieltjes integral

The aim of this subsection is to introduce the Lebesgue-Stieltjes integral over the real line. Later on, this integral would be viewed as a generalization of the Lebesgue measure.

We start by the defining the Lebesgue-Stieltjes measure. Here and afterwards, we take $X \subset \mathbb{R}$ and we denote τ_u the usual topology on X .

Definition 1.1.27 (The Lebesgue-Stieltjes measure). Given a measure space $(X, \mathcal{M}, \mu)$. μ is the *Lebesgue-Stieltjes measure* if the it fulfills the following assumptions:

1. $\mathcal{B}(\tau_u) \subset \mathcal{M}$;
2. For any bounded set $B \in \mathcal{M}$, $\mu(B) < \infty$.

Remark 1.1.28. It is noteworthy that the Lebesgue-Stieltjes measure is a generalization of the Lebesgue measure m , observe that the σ -algebra $\mathcal{L}$ of Lebesgue measurable sets is a Borel σ -algebra. Moreover, for any bounded set $B \in \mathcal{L}$, there exist $K_B \geq 0$ such that $B \subset [-K_B, K_B]$. Thus, $0 \leq m(B) \leq m([-K_B, K_B]) = 2K_B < \infty$.

As we progress through this work, the Lebesgue-Stieltjes measure, which is introduced in Definition 1.1.27, will be defined as a measure generated by a left-continuous nondecreasing function known as the *integrator*. This measure will be essentially determined by its associated *outer measure* using a *semialgebra*.

Definition 1.1.29 (Outer measure). Let X be a set. A set map $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ is said to be an *outer measure* if it satisfies the following conditions:

1. $\mu^*(\emptyset) = 0$;
2. for $A \subset B$, we have $\mu^*(A) \leq \mu^*(B)$;
3. for any sequence $\{A_i\}_{i \in \mathbb{N}} \subset \mathcal{P}(X)$, we have that

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

Definition 1.1.30 (Semialgebra). Let X be a set. A class $\mathcal{C} \subset \mathcal{P}(X)$ is said to be a *semialgebra* if it satisfies the following assumptions:

1. For $A, B \in \mathcal{C}$, we have $A \cap B \in \mathcal{C}$;
2. For any $A \in \mathcal{C}$, there exists a collection of pairwise disjoint sets $\{A_i\}_{i=1}^m \subset \mathcal{C}$ for some $m \in \mathbb{N}$, and $A^c = \bigcup_{i=1}^m A_i$.

Definition 1.1.31. Given a measure μ on a semialgebra $\mathcal{C}$, the outer measure induced by μ is the set function $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ defined by

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) : \{A_i\}_{i \in \mathbb{N}} \subset \mathcal{C}, A \subset \bigcup_{i \in \mathbb{N}} A_i \right\}.$$

Using Carathéodory's Extension Theorem [5, Theorem 1.3.3], we can construct a *complete* measure space from a given outer measure μ^* using a semialgebra. Recall that a *complete* measure space has the property that any subset of a null-measure set is measurable, i.e. given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$ such that $\mu(E) = 0$, if $A \subset E$ then $A \in \mathcal{M}$.

Theorem 1.1.32 (Carathéodory's Extension Theorem). *Given a set X , a semialgebra $\mathcal{C} \subset \mathcal{P}(X)$, and a map $\phi : \mathcal{C} \rightarrow [0, \infty]$ satisfying:*

1. $\phi(\emptyset) = 0$;

2. for every sequence of pairwise disjoint sets $\{A_i\}_{i=1}^{\infty} \subset \mathcal{C}$, such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$, we have that

$$\phi\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \phi(A_i);$$

Then, the map $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ defined by

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \phi(A_i) : \{A_i\}_{i=1}^{\infty} \subset \mathcal{C}, A \subset \bigcup_{i=1}^{\infty} A_i \right\}, \quad (1.1)$$

is an outer measure, and $(X, \mathcal{M}_{\mu^*}, \mu)$ is a measure space where

$$\mathcal{M}_{\mu^*} = \{A \in \mathcal{P}(X) : \mu^*(E) = \mu^*(E \cap A^c) \text{ for all } E \in \mathcal{P}(X)\} \quad (1.2)$$

is a σ -algebra on X , and $\mu = \mu^*|_{\mathcal{M}_{\mu^*}}$ is a measure on $\mathcal{M}_{\mu^*}$.

Moreover, if $\{B_i\}_{i \in \mathbb{N}} \subset \mathcal{C}$ such that

$$X = \bigcup_{i=1}^{\infty} B_i, \quad \phi(B_i) < \infty, \text{ for all } i \in \mathbb{N}.$$

Then, if $\hat{\mu}$ is a measure defined on the smallest σ -algebra containing $\mathcal{C}$, $\sigma(\mathcal{C})$, such that $\hat{\mu} = \phi$ on $\mathcal{C}$, then $\hat{\mu} = \mu^*$ on $\sigma(\mathcal{C})$. In other terms, the measure μ is unique on $\sigma(\mathcal{C})$.

Remark 1.1.33. In Carathéodory's Extension Theorem, note that since $\mathcal{C} \subset \mathcal{M}_{\mu^*}$, then the σ -algebra generated by $\mathcal{C}$, $\sigma(\mathcal{C})$, is contained in $\mathcal{M}_{\mu^*}$ as $\mathcal{M}_{\mu^*}$ is a σ -algebra. This fact motivates calling μ^* an *extension* of μ . In addition, $(X, \sigma(\mathcal{C}), \mu^*)$ is also a measure space, however, it may fail to be complete, see [5, Section 1.4]. Furthermore, observe that the completeness of the resulting measure space $(X, \mathcal{M}_{\mu^*}, \mu)$ results in essentially from Condition (2) of Definition 1.1.29. Indeed, given a μ^* -measurable set $E \subset \mathcal{M}_{\mu^*}$ such that $\mu(E) = 0$, and $A \subset E$, then

$$0 \leq \mu^*(A) \leq \mu^*(E) = \mu(E) = 0.$$

Consequently, $\mu^*(A) = 0$ and therefore, by [5, Theorem 1.3.2], $\mathcal{P}(A) \subset \mathcal{M}_{\mu^*}$, and hence $A \in \mathcal{M}_{\mu^*}$.

Now, given a left-continuous, nondecreasing function $g : \mathbb{R} \rightarrow \mathbb{R}$, it is possible to define a Lebesgue-Stieltjes measure space using the measure associated to g [5, Definition 1.3.7]. This definition relies on the notion of the outer measure previously introduced using Carathéodory's Extension Theorem. In [62, Example 1.46], Márquez Albés provided a constructive proof for this alternative definition. The primary idea was through considering the semialgebra $\mathcal{C}$ of subsets of $\mathbb{R}$:

$$\mathcal{C} = \{[a, b) := \infty < a \leq b \leq +\infty\} \cup \{(-\infty, a) : -\infty < a \leq +\infty\}, \quad (1.3)$$

and the map $\phi : \mathcal{C} \rightarrow [0, \infty]$ defined by

$$\phi(I) = \begin{cases} 0, & \text{if } I = \emptyset, \\ g(b) - g(a), & \text{if } I = [a, b), a, b \in \mathbb{R}, a < b, \\ \lim_{t \rightarrow +\infty} g(t) - g(a), & \text{if } I = [a, +\infty), a \in \mathbb{R}, \\ g(a) - \lim_{t \rightarrow -\infty} g(t), & \text{if } I = (-\infty, a), a \in \mathbb{R}, \\ \lim_{t \rightarrow +\infty} g(t) - \lim_{t \rightarrow -\infty} g(t), & \text{if } I = \mathbb{R}. \end{cases} \quad (1.4)$$

Consequently, it has been shown that this map fulfills the conditions of Theorem 1.1.32. Precisely, the map μ^* in (1.1) is an outer measure, the set $\mathcal{M}_{\mu^*}$ is a σ -algebra, and $\mu^*|_{\mathcal{M}_{\mu^*}}$ is a measure. Furthermore μ is a Lebesgue-Stieltjes measure in the sense of Definition 1.1.27. In the sequel, we denote this measure by μ_g and the σ -algebra $\mathcal{M}_{\mu^*}$ by $\mathcal{M}_g$ relatively to the function g . In addition, observe that for a given $x \in \mathbb{R}$, we have that $\mathbb{R} = (-\infty, x) \cup [x, +\infty)$, also

$$[x, +\infty) = \bigcup_{i=1}^{\infty} [x + i - 1, x + i),$$

and

$$(-\infty, x) = \bigcup_{i=1}^{\infty} [x - i, x - i + 1).$$

This implies that every element of the semialgebra $\mathcal{C}$ defined in (1.3) can be written as a countable pairwise disjoint semi closed intervals $[a, b) \subset \mathbb{R}$, yielding that the outer measure μ^* in (1.1) can be given as

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (g(b_i) - g(a_i)) : A \subset \bigcup_{i=1}^{\infty} [a_i, b_i), \{[a_i, b_i)\}_{i \in \mathbb{N}} \subset \mathcal{I} \text{ pairwise disjoint} \right\},$$

where $\mathcal{I} = \{[a, b) : a, b \in \mathbb{R} \text{ with } a < b\}$, the reader is referred to [62, Theorem 1.50] for the proof.

This expression does not only permit to compute the outer measure μ_g^* , but it guarantees also an explicit expression of the sum of Lebesgue-Stieltjes measures generated by a finite family of left-continuous and nondecreasing functions, as shows the following proposition [66, Proposition 2.4].

Proposition 1.1.34. *Let $\{g_i\}_{i=1}^n$ be a family of left-continuous and nondecreasing functions $g_i : \mathbb{R} \rightarrow \mathbb{R}$, for $i = 1, \dots, n$. Let us define the left-continuous and nondecreasing function $\hat{g} : \mathbb{R} \rightarrow \mathbb{R}$ as*

$$\hat{g}(t) = g_1(t) + \dots + g_n(t), \quad \text{for all } t \in \mathbb{R}. \quad (1.5)$$

Then,

$$\mu_{\hat{g}}^*(A) = \mu_{g_1}^*(A) + \dots + \mu_{g_n}^*(A), \quad \text{for all } A \in \mathcal{P}(\mathbb{R}).$$

In particular, if A is μ_{g_i} -measurable, then A is $\mu_{\hat{g}}$ -measurable, and

$$\mu_{\hat{g}}(A) = \mu_{g_1}(A) + \dots + \mu_{g_n}(A), \quad \text{for all } A \in \mathcal{M}_g = \bigcap_{i=1}^n \mathcal{M}_{g_i}.$$

Remark 1.1.35. Observe that Proposition 1.1.34 provides a way to convey the concept of measurability through functions as well.

Through Carathéodory's Extension Theorem, a left-continuity and nondecreasing function $g : \mathbb{R} \rightarrow \mathbb{R}$ generates a Lebesgue-Stieltjes measure in the sense of Definition 1.1.27. Now, one may ask the question about the converse, whether for each Lebesgue-Stieltjes measure on $\mathbb{R}$ we can find a left-continuous and nondecreasing function that defines the measure over the class of Borel sets. According to [5, Chapter 1, Section 2, Subsection 3], the answer is affirmative. Given a Radon measure μ , i.e.

a measure defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which is finite on bounded intervals, we can construct a function $g : \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$g(t) = \begin{cases} \mu([a, t)) & \text{if } t > a, \\ 0 & \text{if } t = a, \\ -\mu((t, a]) & \text{if } t < a. \end{cases} \quad (1.6)$$

The function g generates a Lebesgue-Stieltjes measure μ_g , such that $\mu_g = \mu$ on the semialgebra $\mathcal{C}$ of subsets of $\mathbb{R}$ defined in (1.3). In particular, from Carathéodory's extension theorem, Theorem 1.1.32, it follows that $\mu_g = \mu$ on $\sigma(\mathcal{C}) \supset \mathcal{B}(\mathbb{R})$, and we obtain the following proposition.

Proposition 1.1.36. *Let $(\mathbb{R}, \mathcal{M}, \mu)$ a measure space, such that μ is a Lebesgue-Stieltjes measure. Then, there exists a left-continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfying*

$$\mu([a, b)) = g(b) - g(a), \text{ for all } a, b \in \mathbb{R}, \text{ such that } a < b. \quad (1.7)$$

Remark 1.1.37. In the previous proposition, observe that the function g satisfying (1.7) is not unique. Instead, there is an infinite family of functions $\{g_r\}_{r \in \mathbb{R}}$ such that $g_r(t) = g(t) + r$, for all $r \in \mathbb{R}$, where g as given in (1.6).

Now, we are able to define the *Lebesgue-Stieltjes integral*.

Definition 1.1.38. The *Lebesgue-Stieltjes integral* is the integral with respect to the Lebesgue-Stieltjes measure. We denote the Lebesgue-Stieltjes integral of a μ_g -measurable function f over $A \in \mathcal{M}_g$ by

$$\int_A f \, d\mu_g, \quad \text{for } A \in \mathcal{M}_g.$$

For $A \in \mathcal{M}_g$, we denote by $\mathcal{L}_g^1(A, \mathbb{R})$ the space of integrable functions on A with respect to the Lebesgue-Stieltjes measure μ_g , and we endow it with the norm $\|\cdot\|_{\mathcal{L}_g^1}$ as in Definition 1.1.21. Recall that for $A = [a, b) \in \mathcal{I}$ and $f \equiv \chi_{[a, b)}$, we recover $\mu_g([a, b)) = g(b) - g(a)$ as given in (1.4). In the sequel, we adopt the following terminology:

- g -measurability: to refer to the measurability with respect to the σ -algebra $\mathcal{M}_g$;
- Null g -measure sets: to refer to null-measure sets with respect to the measure μ_g ;
- g -Carathéodory functions: to refer to μ_g -Carathéodory functions.

Additionally, we say that a property holds for g -almost everywhere (g -a.e.) in some set $E \subset X$ or that it holds for g -almost all $x \in E$ (g -a.a. $x \in E$), if it holds except on a null g -measure subset of E .

Based on Proposition 1.1.34 regarding the function $\hat{g} : \mathbb{R} \rightarrow \mathbb{R}$ given in (1.5), the following proposition [66] provides an explicit expression to compute the integral of $\hat{g}$ -integrable function with respect to the measure $\mu_{\hat{g}}$.

Proposition 1.1.39. *Let $\hat{g}$ be the function given in (1.5), if $X \subset \mathbb{R}$ and $f \in \mathcal{L}_{g_i}^1(X, \mathbb{R})$ for every $i \in \{1, \dots, n\}$, then $h \in \mathcal{L}_{\hat{g}}^1(X, \mathbb{R})$ and*

$$\int_X h \, d\mu_{\hat{g}} = \int_X f \, d\mu_{g_1} + \dots + \int_X f \, d\mu_{g_n}.$$

In this part of this section, we focus on the sets associated to $g : \mathbb{R} \rightarrow \mathbb{R}$ being left-continuous and nondecreasing, to derive further properties about the measure μ_g associated to g . First, throughout this work, for a given regulated function $u : [a, b] \rightarrow \mathbb{R}$, we employ the notation $u(t^+)$ to denote

$$u(t^+) := \lim_{\epsilon \rightarrow 0^+} u(t + \epsilon),$$

for all $t \in [a, b)$.

Let us define the set C_g by

$$C_g = \{t \in \mathbb{R} : g \text{ is constant on } (t - \epsilon, t + \epsilon) \text{ for some } \epsilon > 0\}, \quad (1.8)$$

which is the union of all connected components where g remains constant, and the set of the discontinuity point D_g :

$$D_g = \{t \in \mathbb{R} : g(t^+) - g(t) > 0\}. \quad (1.9)$$

The set C_g is open in the usual topology τ_u of $\mathbb{R}$, and since τ_u is second countable [46], the set C_g can be written as a union of pairwise disjoint open intervals:

$$C_g = \bigcup_{n \in \Lambda} (a_n, b_n),$$

where $\Lambda \subset \mathbb{N}$. Based on this notation, we define the set:

$$N_g = N_g^- \cup N_g^+, \quad (1.10)$$

where

$$N_g^- := \{a_n\} \setminus D_g, \text{ and } N_g^+ := \{b_n\} \setminus D_g. \quad (1.11)$$

In particular, we have the following properties, the reader is referred to [58, Proposition 2.5] for the proof of Item (1), while Item (2) is commonly referred to as Froda's theorem [36].

Proposition 1.1.40. *The measure μ_g satisfies the following statements:*

1. $\mu_g(C_g) = \mu_g(N_g) = 0$.
2. The set D_g is at most countable.

Remark 1.1.41. It is worthwhile to emphasize that, the measure μ_g may have some *atoms* which are sets of positive measures containing no subset with a smaller positive measure. This holds particularly in the case when $D_g \neq \emptyset$. This primary fact differentiates the Lebesgue-Stieltjes measure from the Lebesgue measure, since this latter is non-atomic on the real line. Indeed, for $t \in D_g$, and using the property in Statement 5 from Theorem 1.1.13, observe that

$$\begin{aligned} \mu_g(\{t\}) &= \mu_g\left(\bigcap_{n=1}^{\infty} \left[t, t + \frac{1}{n}\right)\right) = \lim_{n \rightarrow +\infty} \mu_g\left(\left[t, t + \frac{1}{n}\right)\right) \\ &= \lim_{n \rightarrow +\infty} g\left(t + \frac{1}{n}\right) - g(t) = g(t^+) - g(t) > 0. \end{aligned}$$

It is noteworthy that an analogous approach can be applied for right-continuous and nondecreasing integrators, in fact, such integrators form the usual case in the theory of probabilities involving right-continuous cumulative distribution functions [5, 38].

1.2 The g -topology in the real line

Given a left and nondecreasing function $g : \mathbb{R} \rightarrow \mathbb{R}$, g defines a pseudometric $\rho_g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$ given by

$$\rho_g(s, t) = |g(t) - g(s)|, \quad \text{for every } s, t \in \mathbb{R}.$$

The pseudometric ρ_g induces a topology on $\mathbb{R}$, which we refer to as the g -topology (or τ_g -topology) and denote by τ_g . The reader is referred to [62, Section 2.1] for further details. The basic neighborhoods of a point $t_0 \in \mathbb{R}$ in the g -topology are given by the open balls:

$$\mathcal{B}(t_0, \epsilon) = \{t \in \mathbb{R} : \rho_g(t, t_0) < \epsilon\}, \quad \text{for } \epsilon > 0.$$

In the case where $g = \text{id}_{\mathbb{R}}$, the topology τ_g coincides with the usual topology τ_u of $\mathbb{R}$.

Remark 1.2.1. The topology τ_g is not necessarily Hausdorff as mentioned in [31, Section 2]. Indeed, let us consider:

$$g(t) = \begin{cases} t & \text{for } t < 1, \\ 1 & \text{for } 1 \leq t \leq 2, \\ t - 1 & \text{for } t > 2. \end{cases}$$

$\rho_g(1, 2) = |g(1) - g(2)| = 0$, while $1 \neq 2$. Thus, this pseudometric fails to be a genuine metric.

Nevertheless, in [31, Proposition 2.1], the authors established that $(\mathbb{R}, \tau_g)$ is second countable.

Proposition 1.2.2. *Let us consider $\mathfrak{F}_1$ and $\mathfrak{F}_2$ the collections of intervals defined as*

$$\mathfrak{F}_1 = \{(a, b) \subset \mathbb{R} : a, b \in \mathbb{Q}\} \text{ and } \mathfrak{F}_2 = \{(c, d] \subset \mathbb{R} : c \in \mathbb{Q}, d \in D_g\}.$$

$\mathfrak{B} := \mathfrak{F}_1 \cup \mathfrak{F}_2$ is a countable base of τ_g . Moreover, τ_g is a second countable topology.

Definition 1.2.3. A set $U \subset \mathbb{R}$ is g -open if, for all $t_0 \in U$, there exists $\delta > 0$ such that

$$\mathcal{B}(t_0, \delta) \subset U.$$

In particular, if $t_0 \in D_g$, then $(t_0 - \delta, t_0]$ is a g -open set for some $\delta > 0$ sufficiently small.

In [66, Theorem 3.12], the authors provided a classification of the topologies of derivators based on their sets of constancy and discontinuity points.

Theorem 1.2.4 (Classification of derivator topologies). *Let G be the family of all derivators that are both left-continuous and nondecreasing. Let us define the equivalence relation $\sim$ on G by:*

$$g_1 \sim g_2 :\iff \tau_{g_1} = \tau_{g_2},$$

the class $G := G|_{\sim}$ and $H = \{(C, D) \in \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) : C \text{ open, } D \text{ at most countable, } C \cap D = \emptyset\}$. Then, the map:

$$\begin{aligned} \Phi : G &\rightarrow H \\ g &\mapsto \Phi([g]) = (C_g, D_g) \end{aligned}$$

is a bijection.

Corollary 1.2.5. *Given $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$ two left-continuous and nondecreasing function, then*

$$g_1 \sim g_2 :\iff \tau_{g_1} = \tau_{g_2} \iff D_{g_1} = D_{g_2} \text{ and } C_{g_1} = C_{g_2}.$$

This classification of derivators was primarily developed based on the notion of continuity with respect to a function, in the sense that $g_1 \sim g_2$ if and only if g_1 is continuous with respect to g_2 and vice-versa [66, Corollary 3.11]. The concept of *g-continuity* will be discussed in the next section.

1.3 *g*-Continuity

Throughout this section, we consider $g : \mathbb{R} \rightarrow \mathbb{R}$ a left-continuous and nondecreasing function. Let us denote by $\|\cdot\|$ the maximum norm in $\mathbb{R}^n$, i.e. $\|\mathbf{x}\| = \max\{|x_1|, \dots, |x_n|\}$ for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Definition 1.3.1. Let $\mathbf{u} : A \subset \mathbb{R} \rightarrow \mathbb{R}^n$. The function $\mathbf{u}$ is said to be *g-continuous* at $t \in A$ if, for each $\epsilon > 0$, there is some $\delta > 0$ such that

$$s \in A, \rho_g(s, t) < \delta \implies \|\mathbf{u}(s) - \mathbf{u}(t)\| < \epsilon.$$

The space of *g*-continuous functions $\mathbf{u} : [a, b] \rightarrow \mathbb{R}^n$ is denoted by $\mathcal{C}_g([a, b], \mathbb{R}^n)$. In [31, Proposition 3.2], the authors reveal nice properties of *g*-continuous functions.

Proposition 1.3.2. *If $\mathbf{u} : [a, b] \rightarrow \mathbb{R}^n$ is *g*-continuous function on $[a, b]$, then:*

- (1) *$\mathbf{u}$ is left-continuous at every $t_0 \in (a, b]$.*
- (2) *If g is continuous at $t_0 \in [a, b)$, then so is $\mathbf{u}$.*
- (3) *If g is constant on some $[c, d] \subset [a, b]$, then so is $\mathbf{u}$.*

Example 1.3.3. Constant functions are clearly *g*-continuous. Furthermore, the functions λg for $\lambda \in \mathbb{R}$ are nontrivial examples of *g*-continuous functions.

A *g*-continuous function has also interesting measurability properties, as shown in [31, Corollary 3.5].

Lemma 1.3.4. *Given a Borel set $A \subset \mathbb{R}$ and a function $\mathbf{u} : A \rightarrow \mathbb{R}^n$, the following statements hold:*

- (1) *If $\mathbf{u}$ is *g*-continuous, then $\mathbf{u}$ is Borel-measurable.*
- (2) *If $\mathbf{u}$ is Borel measurable, then $\mathbf{u}$ is *g*-measurable.*

Although *g*-continuous functions have advantageous properties regarding continuity and measurability, they can fail to be bounded or even regulated on bounded intervals unlike in the classical case where $g = \text{id}_{\mathbb{R}}$, as shown in the counter-example in [31, Example 3.3].

Example 1.3.5. Let consider the functions $g, f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$g(t) = \begin{cases} t & \text{if } t \leq 0, \\ t + 1 & \text{if } t > 0, \end{cases} \quad \text{and} \quad f(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \frac{\sin(1/t)}{t} & \text{if } t > 0. \end{cases}$$

The function f is g -continuous on $\mathbb{R}$ but not bounded. Figure 1.3.1 shows the behaviour of g and f around $t = 0$.

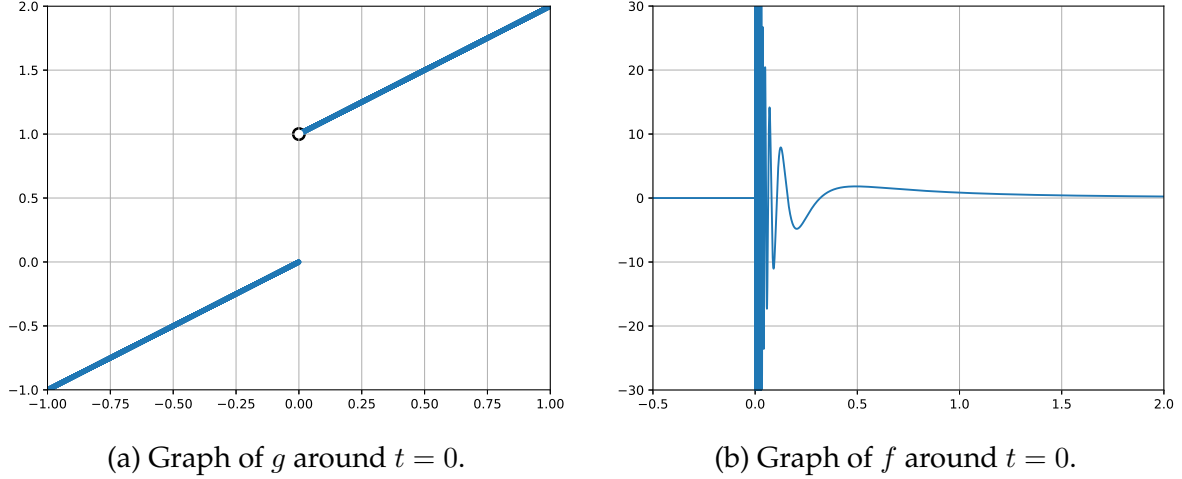


Figure 1.3.1 – Graphs of f and g showing their behavior around $t = 0$.

In the sequel, we denote by $\mathcal{BC}_g([a, b], \mathbb{R}^n)$ the space of bounded g -continuous functions $u : [a, b] \rightarrow \mathbb{R}^n$. The space $\mathcal{BC}_g([a, b], \mathbb{R}^n)$ equipped with the supremum norm is a Banach space, see [31, Theorem 3.4].

1.4 The Stieltjes derivative

In this section, we shed light on the notion of the Stieltjes derivative. As mentioned in the general introduction, the concept of taking derivatives with respect to a function can be traced back to the early works of Daniell [17, 18] and Young [87], this concept involves deriving with respect to a left-continuous nondecreasing function $g : \mathbb{R} \rightarrow \mathbb{R}$, referred to as the *derivator*. Similar ideas were also explored in other works, see for instance [6, 12, 23, 39, 52, 74]. Recently in [58], López Pouso and Rodríguez delved into the significance of the Stieltjes derivative, coming up with a practical definition of the Stieltjes derivative with respect to a left-continuous and nondecreasing function. Thanks to their definition, a generalized framework unifying discrete and continuous calculus has been established, providing various connections between classical problems, as we will explore in Chapter 2. In the sequel, we fix $g : \mathbb{R} \rightarrow \mathbb{R}$ a left-continuous and nondecreasing function.

Definition 1.4.1. Let $u : A \subset \mathbb{R} \rightarrow \mathbb{R}$ be a function, and let $t \in A \setminus C_g$. The *Stieltjes derivative* of u with respect to g , also referred to as the g -*derivative*, at the point t is

defined by

$$u'_g(t) = \begin{cases} \lim_{s \rightarrow t} \frac{u(s) - u(t)}{g(s) - g(t)}, & \text{if } t \notin D_g, \\ \frac{u(t^+) - u(t)}{g(t^+) - g(t)}, & \text{if } t \in D_g, \end{cases} \quad (1.12)$$

provided that the limit exists, where C_g and D_g are defined as in (1.8) and (1.9). In this case, u is said to be g -differentiable at t .

In [58, Proposition 2.2] and [62, Proposition 3.13], the author explored classical results regarding the properties of the derivative within the context of the Stieltjes derivative.

Proposition 1.4.2. *Let $t \in \mathbb{R}$ and $f_1, f_2 : A \subset \mathbb{R} \rightarrow \mathbb{R}$ functions defined on a neighborhood of $t \in A \setminus C_g$. If f_1 and f_2 are g -differentiable at t , then:*

(i) *The function $\lambda_1 f_1 + \lambda_2 f_2$ is g -differentiable at t for any $\lambda_1, \lambda_2 \in \mathbb{R}$, and*

$$(\lambda_1 f_1 + \lambda_2 f_2)'_g(t) = \lambda_1 (f_1)'_g(t) + \lambda_2 (f_2)'_g(t).$$

(ii) *The product $f_1 f_2$ is g -differentiable at t and*

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t) + f_1(t) (f_2)'_g(t) + (f_1)'_g(t) (f_2)'_g(t) \mu_g(\{t\}).$$

In particular, if $t \in D_g$, then

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t^+) + f_1(t) (f_2)'_g(t).$$

(iii) *If $f_2(t) \left(f_2(t) + (f_2)'_g(t) \mu_g(\{t\}) \right) \neq 0$, then the quotient f_1/f_2 is g -differentiable at t and*

$$\left(\frac{f_1}{f_2} \right)'_g(t) = \frac{(f_1)'_g(t) f_2(t) - f_1(t) (f_2)'_g(t)}{f_2(t) \left(f_2(t) + (f_2)'_g(t) \mu_g(\{t\}) \right)}.$$

Remark 1.4.3. Observe that the product formula in Statement (ii) in Proposition 1.4.2 for $t \in D_g$:

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t^+) + f_1(t) (f_2)'_g(t) \quad (1.13)$$

is derived from

$$f_2(t^+) = f_2(t) + (f_2)'_g(t) \mu_g(\{t\}).$$

However, (1.13) does not hold in general at any point in $\mathbb{R} \setminus C_g$. For instance, for $g : \mathbb{R} \rightarrow \mathbb{R}$ defined for $t \leq 1$ by $g(t) = t$, and $g(t) = 1$ for $t > 1$. Let us consider $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that $f_1 \equiv \text{id}_{\mathbb{R}}$ and f_2 is defined by

$$f_2(t) = \begin{cases} t - 1 & \text{if } t \leq 1, \\ 1 & \text{if } t > 1. \end{cases}$$

We have that

$$(f_1 f_2)(t) = \begin{cases} t(t - 1) & \text{if } t \leq 1, \\ t & \text{if } t > 1. \end{cases}$$

$$(f_1 f_2)'_g(1) = \lim_{t \rightarrow 1^-} \frac{(f_1 f_2)(t) - (f_1 f_2)(1)}{g(t) - g(1)} = \lim_{t \rightarrow 1^-} \frac{t(t-1)}{t-1} = 1,$$

while

$$(f_1)'_g(1)f_2(1^+) + f_1(1)(f_2)'_g(1) = 1.1 + 1.1 = 2 \neq 1 = (f_1 f_2)'_g(1).$$

In [24], the authors extended the product rule for a finite number of functions. In doing so, they introduce the following notation:

For $n, k \in \mathbb{N}$ and $(x_1, \dots, x_n) \in \{0, 1\}^n$, let us set

$$|(x_1, \dots, x_n)| := \text{card} \{k \in \{1, \dots, n\} : x_k = 1\}, \quad F_{n,k} := \{\sigma \in \{0, 1\}^n : |\sigma| = k\},$$

and denote $f_g^{(0)} \equiv f, f_g^{(1)} \equiv f'_g$. The following proposition is an adaptation of [24, Proposition 2.7].

Proposition 1.4.4 (Product rule). *Let $n \in \mathbb{N}$, such that $n \geq 2$, and $f_i : \mathbb{R} \rightarrow \mathbb{R}$; for $i = 1, 2, \dots, n$, be g -differentiable at $t \in \mathbb{R}$. Then, the product $\prod_{i=1}^n f_i$ is also g -differentiable at t and*

$$\left(\prod_{i=1}^n f_i \right)'_g(t) = \sum_{k=0}^{n-1} (\mu_g(\{t\}))^k \left(\sum_{\sigma \in F_{n,k+1}} \prod_{i=1}^n (f_i)^{(\sigma_i)}_g(t) \right). \quad (1.14)$$

Remark 1.4.5. When $k = 0$, one may notice that the term $(\mu_g(\{t\}))^k$ might not be properly defined. This is due to (1.14) being just a concise way to write

$$\left(\prod_{i=1}^n f_i \right)'_g(t) = \sum_{\sigma \in F_{n,1}} \prod_{i=1}^n (f_i)^{(\sigma_i)}_g(t) + \sum_{k=1}^{n-1} (\mu_g(\{t\}))^k \left(\sum_{\sigma \in F_{n,k+1}} \prod_{i=1}^n (f_i)^{(\sigma_i)}_g(t) \right).$$

In other terms, in (1.14), $(\mu_g(\{t\}))^0$ is treated as 1.

In [62, Proposition 3.15], a formula for the g -derivative of the composition of two functions was established. Another version of this formula can be found in [25, Proposition 4.1].

Proposition 1.4.6. *Let $t \in \mathbb{R} \setminus C_g$, $f : \mathbb{R} \rightarrow \mathbb{R}$ and h a real function defined on a neighborhood of $f(t)$. We assume that there exist $h'(f(t))$, $f'_g(t)$ and that the function h is continuous at $f(t^+)$. Then, the composition $h \circ f$ is g -differentiable in t_0 , and we have the formulae*

$$(h \circ f)'_g(t) = \begin{cases} h'(f(t))f'_g(t) & \text{if } t \notin D_g, \\ \frac{h(f(t^+)) - h(f(t))}{f(t^+) - f(t)} f'_g(t) & \text{if } t \in D_g. \end{cases}$$

A more general chain rule formula was established in [35, Proposition 5.4] with respect to two derivators that are not necessary monotonic.

Remark 1.4.7. In the context of classical differentiation, differentiability of a function implies its continuity at each point. However, in the context of the Stieltjes derivative, this implication does not generally hold. This remark was also mentioned [25, Remark 3.3] providing an example of g -continuity loss at a point of the set N_g defined

in (1.10). In the sequel, we provide a counter-example on the set D_g . Let us consider the derivator $g : \mathbb{R} \rightarrow \mathbb{R}$ and the function $f : [0, 2] \rightarrow \mathbb{R}$ defined by:

$$g(t) = \begin{cases} t & \text{if } t \leq 1, \\ t + 1 & \text{if } t > 1, \end{cases} \quad \text{and} \quad f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1, \\ 1 & \text{if } t = 1, \\ 2 & \text{if } 1 < t \leq 2. \end{cases}$$

The function f is g -differentiable at 1 and $f'_g(1) = 0$. However, by Proposition 1.3.2, f is not g -continuous at 1 on since it is not left-continuous at 1.

In the particular case where the derivator $g : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and continuous on an interval $[a, b] \subset \mathbb{R}$, we can easily derive a generalized version of the Mean Value Theorem in the context of Stieltjes differentiation.

Theorem 1.4.8. *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a left-continuous and nondecreasing function, continuous and increasing on an interval $[a, b] \subset \mathbb{R}$. Let $f : [a, b] \rightarrow \mathbb{R}$ be g -continuous on $[a, b]$ and g -differentiable on (a, b) . Then, there exists $c \in (a, b)$ such that $f'_g(c) = \frac{f(b) - f(a)}{g(b) - g(a)}$.*

Proof. Let us consider the function $F : [a, b] \rightarrow \mathbb{R}$ defined by

$$F(t) = f(t) - \left(\frac{f(b) - f(a)}{g(b) - g(a)} (g(t) - g(a)) + f(a) \right), \quad \text{for all } t \in [a, b].$$

Clearly F is g -continuous on $[a, b]$, g -differentiable on (a, b) , and

$$F'_g(t) = f'_g(t) - \frac{f(b) - f(a)}{g(b) - g(a)} \quad \text{for all } t \in (a, b).$$

It satisfies also $F(a) = F(b)$.

Proposition 1.3.2 implies that F is continuous on $[a, b]$. We set

$$m = \min_{t \in [a, b]} F(t) \quad \text{and} \quad M = \max_{t \in [a, b]} F(t).$$

If $m = M$, then F is constant on $[a, b]$ and $F'_g(t) = 0$ for every $t \in (a, b)$. Otherwise if $m < M$, it follows that $F(a) \neq m$ or $F(a) \neq M$. As $F(a) = F(b)$, without loss of generality, we assume that $F(a) = F(b) \neq M$. Thus, there exists $c \in (a, b)$ such that $F(c) = M$. Therefore, there is some $\delta > 0$ such that $(c - \delta, c + \delta) \subset (a, b)$ and $F(s) \leq F(c) = M$ for all $s \in (c - \delta, c + \delta)$. Since g is increasing and continuous on (a, b) ,

$$0 \leq F'_g(c) = \lim_{s \rightarrow c^+} \frac{F(s) - F(c)}{g(s) - g(c)} = \lim_{s \rightarrow c^-} \frac{F(s) - F(c)}{g(s) - g(c)} \leq 0.$$

We deduce that $F'_g(c) = 0$. Hence, there exists $c \in (a, b)$ such that $f'_g(c) = \frac{f(b) - f(a)}{g(b) - g(a)}$. $\square$

1.5 g -Absolute continuity

In the classical case of derivation, certain functions can be reconstructed as the integral of their derivatives, which is only possible when the function is absolutely continuous. In the setting of the Stieltjes derivative, an analogous definition was introduced through the notion of absolute continuity with respect to a left-continuous and nondecreasing function $g : \mathbb{R} \rightarrow \mathbb{R}$ by [58, Definition 5.1].

Definition 1.5.1. A map $F : [a, b] \rightarrow \mathbb{R}$ is said to be g -absolutely continuous, if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for any family $\{(a_i, b_i)\}_{i=1}^m$ of pairwise disjoint open subintervals of $[a, b]$,

$$\sum_{i=1}^m g(b_i) - g(a_i) < \delta \Rightarrow \sum_{i=1}^m |F(b_i) - F(a_i)| < \varepsilon.$$

We denote by $\mathcal{AC}_g([a, b], \mathbb{R})$ the vector space of g -absolutely continuous functions $F : [a, b] \rightarrow \mathbb{R}$ on the interval $[a, b]$. Furthermore, the product of two g -absolutely continuous functions on $[a, b]$ is g -absolutely continuous on $[a, b]$, the reader is referred to [31, Proposition 5.4] for the proof.

Analogously to the case of classical derivation, we state the Fundamental Theorem of Calculus for the Lebesgue-Stieltjes integral. In [58, Theorem 5.4], the authors used modern elementary techniques in the proof based on the definition of g -absolute continuity stated in Definition 1.5.1.

Theorem 1.5.2 (Fundamental Theorem of Calculus for the Lebesgue-Stieltjes integral). *Let $F : [a, b] \rightarrow \mathbb{R}$. The next statements are equivalent:*

- (1) F is g -absolutely continuous;
- (2) F satisfies the following properties:
 - (i) $F'_g(t)$ exists g -almost everywhere in $[a, b]$,
 - (ii) $F'_g \in \mathcal{L}_g^1([a, b], \mathbb{R})$,
 - (iii) Barrow's rule: for every $t \in [a, b]$,

$$F(t) = F(a) + \int_{[a, t)} F'_g(s) \, d\mu_g(s).$$

Remark 1.5.3. If $F : [a, b] \rightarrow \mathbb{R}$ is a g -absolutely continuous and $t \in D_g \cap [a, b)$, it follows that $\mu_g(\{t\}) = g(t^+) - g(t) > 0$ and

$$F(t^+) - F(t) = \int_{\{t\}} F'_g(s) \, d\mu_g(s) = F'_g(t) (g(t^+) - g(t)) = F'_g(t) \mu_g(\{t\}).$$

This implies, in particular, that functions that are g -absolutely continuous may exhibit discontinuities at points where the function g is discontinuous.

Nevertheless, despite the fact that g -absolutely continuous functions can exhibit discontinuities, the following result [58, Proposition 5.3] ensures that they have to be g -continuous.

Proposition 1.5.4. *Given a g -absolutely continuous function $F : [a, b] \rightarrow \mathbb{R}$, then F is g -continuous and has bounded variation.*

The next theorem combines the results in [58, Theorem 2.4 and Proposition 5.2] to enhance the g -differentiability of Lebesgue-Stieltjes integrals g -almost everywhere, moreover, it will play a crucial role to deduce the g -absolute continuity of functions defined by Lebesgue-Stieltjes integrals.

Theorem 1.5.5. *Let $f \in \mathcal{L}_g^1([a, b], \mathbb{R})$ and consider its Lebesgue-Stieltjes primitive defined by*

$$F(t) = \int_{[a, t)} f(s) \, d\mu_g(s) \quad \text{for all } t \in [a, b].$$

Then, there exists a null g -measure set $N \subset [a, b]$ such that

$$F'_g(t) = f(t) \quad \text{for all } t \in [a, b] \setminus N.$$

Furthermore, F is g -absolutely continuous in $[a, b]$.

The proposition below [31, Proposition 5.6] provides sufficient conditions to guarantee the relative compactness in $\mathcal{BC}_g([a, b], \mathbb{R})$ of a set in $\mathcal{AC}_g([a, b], \mathbb{R})$. The proof of this result relies on [31, Theorem 4.2] where the authors established further relative compactness results for a family of g -continuous functions. This result is based on Fraňková's Theorem [27, Proposition 2.3] characterizing relative compactness in spaces of regulated functions. Given that g -continuous functions may not be regulated at the discontinuity points of g , as shown earlier in Example 1.3.5, additional assumptions were necessary to control the behavior of these functions at such points.

Proposition 1.5.6. *Let $\mathcal{D} \subset \mathcal{AC}_g([a, b], \mathbb{R})$ be such that $\{F(a) : F \in \mathcal{D}\}$ is bounded. If there exists a function $h \in \mathcal{L}_g^1([a, b], [0, +\infty))$ such that*

$$|F'_g(t)| \leq h(t) \quad \text{for } g\text{-almost all } t \in [a, b], \text{ and all } F \in \mathcal{D},$$

then, $\mathcal{D}$ is relatively compact in $\mathcal{BC}_g([a, b], \mathbb{R})$.

The previous proposition will play a fundamental role in ensuring complete continuity of operators that we use in Chapter 3.

The following principle that we established in [60, Lemma 2.13] serves to compare two g -absolutely continuous functions through comparing their g -derivatives g -almost everywhere. In the classical case of derivation, for $g \equiv \text{id}_{\mathbb{R}}$, this result was established in [20, Lemma 2.5], based on the comparison of the derivatives composed with an increasing homeomorphism. Whereas, when we shift our focus to the general case of Stieltjes differentiation, where the derivator g may exhibit discontinuities, we encounter the necessity of an additional assumption. To the best of our knowledge, a result of this type has not yet been established within the context of Stieltjes derivatives.

Lemma 1.5.7 (Comparison principle). *Let $u, v \in \mathcal{AC}_g([a, b], \mathbb{R})$ be such that*

- (i) $u'_g(t) \leq v'_g(t)$ g -almost everywhere in $\{t \in [a, b] : v(t^+) < u(t^+)\}$;

(ii) $u(a) - v(a) \leq u(b) - v(b)$ or $u(a) \leq v(b)$.

Then, one of the following alternatives holds:

(A1) there exists $c > 0$ such that $u(t) = v(t) + c$ for all $t \in [a, b]$,

(A2) $u(t) \leq v(t)$ for every $t \in [a, b]$.

Proof. Assume that $\{t \in [a, b] : v(t) < u(t)\} \neq \emptyset$.

Observe that if $t \in D_g$ and $v(t^+) < u(t^+)$, then by Remark 1.5.3 and Condition (i), we obtain

$$\begin{aligned} u(t) - v(t) &= u(t^+) - u'_g(t)\mu_g(\{t\}) - v(t^+) + v'_g(t)\mu_g(\{t\}) \\ &\geq u(t^+) - v(t^+) > 0. \end{aligned} \quad (1.15)$$

Let

$$c = \sup_{t \in [a, b]} (u(t) - v(t)) > 0.$$

By (1.15), and since $u - v$ is a left-continuous function, it follows that

$$A = \{\tau \in [a, b] : u(\tau) - v(\tau) = c\} \neq \emptyset.$$

Case 1: If $A = [a, b]$, then

$$u(t) - v(t) = c \quad \text{for all } t \in [a, b]. \quad (1.16)$$

Observe that this case cannot hold if $v(a) \geq u(a)$.

Case 2: If $A \neq [a, b]$, we choose $t_1 \in A \cap (a, b]$ as follows:

$$t_1 \in A \text{ if } a \notin A, \quad \text{and} \quad t_1 = b \text{ if } a \in A. \quad (1.17)$$

Indeed, in this last case, $u(a) - v(a) \leq u(b) - v(b)$ is satisfied in Condition (ii), and $\{a, b\} \subset A$.

Since $u - v$ is left-continuous, then there is some $\delta > 0$ such that

$$u(t) - v(t) \geq \frac{c}{2} \quad \text{for every } t \in [t_1 - \delta, t_1]. \quad (1.18)$$

Let us define

$$t_0 = \inf \left\{ s \in [0, t_1] : u - v \geq \frac{c}{2} \text{ on } [s, t_1] \right\},$$

which is such that $0 \leq t_0 < t_1$ by (1.18). Moreover, one has that

$$u(t^+) - v(t^+) \geq \frac{c}{2} > 0 \quad \text{for each } t \in [t_0, t_1).$$

Combining the Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral, Theorem 1.5.2, and Condition (i), we obtain

$$u(t_1) - u(t) = \int_{[t, t_1)} u'_g(s) \, d\mu_g(s) \leq \int_{[t, t_1)} v'_g(s) \, d\mu_g(s) = v(t_1) - v(t) \quad \text{for every } t \in [t_0, t_1).$$

Therefore,

$$c = u(t_1) - v(t_1) \leq u(t) - v(t) \quad \text{for every } t \in [t_0, t_1).$$

Hence, $[t_0, t_1] \subset A$. If $t_0 > a$, this contradicts t_0 being the infimum, as $u - v$ is left continuous and given (1.15). Otherwise, $t_0 = a$, and by (1.17), $t_1 = b$. This contradicts that $A \neq [a, b]$.

From both cases, we deduce that $u(t) \leq v(t)$ for every $t \in [a, b]$, or there exists some $c > 0$ such that $v(t) = u(t) + c$ for all $t \in [a, b]$. $\square$

Condition (i) of Lemma 1.5.7 cannot be omitted. In the following example, this condition fails for $t = 1$. Consequently, neither alternative holds.

Example 1.5.8. Let us consider the derivator $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(t) = t$ for $t \leq 1$ and $g(t) = t + 1$ for $t > 1$. Let us take the functions: $u, v : [0, 2] \rightarrow \mathbb{R}$ defined by

$$u(t) = \begin{cases} 0 & t \in [0, 1], \\ 2 & t \in (1, 2], \end{cases} \quad \text{and} \quad v(t) = \begin{cases} t^3 & t \in [0, 1], \\ 1 & t \in (1, 2]. \end{cases}$$

For $t \in [0, 1]$ $u(t) \leq v(t)$, and for $t \in (1, 2]$ $u(t) > v(t)$ on $(1, 2]$. Observe that $u'_g(1) > v'_g(1)$ while $u(1^+) > v(1^+)$.

As we proceed, the comparison principle 1.5.7 will play a technical role in establishing the existence and a priori bound of solutions of the Stieltjes differential equations under study.

1.6 An exponential

The aim of this section is to explore the notion of exponentials within the context of linear Stieltjes differential equations, as introduced in [31, Definition 6.1], while considering improvements made in [63].

Definition 1.6.1. Let $h \in \mathcal{L}_g^1([a, b], \mathbb{R})$ be such that

$$1 + h(t)(g(t^+) - g(t)) > 0 \text{ for each } t \in [a, b] \cap D_g. \quad (1.19)$$

Let us define the function $e_h(\cdot, a) : [a, b] \rightarrow (0, \infty)$ for each $t \in [a, b]$ by

$$e_h(t, a) = e^{\int_{[a, t)} \tilde{h}(s) \, d\mu_g(s)},$$

where

$$\tilde{h}(t) = \begin{cases} h(t) & \text{if } t \in [a, b] \setminus D_g, \\ \frac{\log(1 + h(t)(g(t^+) - g(t)))}{g(t^+) - g(t)} & \text{if } t \in [a, b] \cap D_g. \end{cases} \quad (1.20)$$

The next lemma [31, Lemma 6.2] enhances the g -integrability of the function $\tilde{h}$, which will guarantee some nice properties of the exponential function with respect to the Stieltjes derivative.

Lemma 1.6.2. Let $h \in \mathcal{L}_g^1([a, b], \mathbb{R})$ be a function satisfying Condition (1.19). Then, $\tilde{h} \in \mathcal{L}_g^1([a, b], \mathbb{R})$.

Remark 1.6.3. It follows from Lemma 1.6.2, that the map $e_h(\cdot, a)$ is well-defined. Moreover, for every $t \in [a, b) \cap D_g$, it follows that

$$\begin{aligned}
 e_h(t^+, a) &= \lim_{s \rightarrow t^+} e^{\int_{[a,s]} \tilde{h}(s) \, d\mu_g(s)} \\
 &= \lim_{s \rightarrow t^+} e^{\int_{[a,t]} \tilde{h}(s) \, d\mu_g(s) + \int_{[t,s]} \tilde{h}(s) \, d\mu_g(s)} \\
 &= \lim_{s \rightarrow t^+} e^{\int_{[a,t]} \tilde{h}(s) \, d\mu_g(s)} e^{\int_{[t,s]} \tilde{h}(s) \, d\mu_g(s)} \\
 &= e_h(t, a) e^{\int_{\{t\}} \tilde{h}(s) \, d\mu_g(s)} \\
 &= e_h(t, a) e^{\tilde{h}(t) \mu_g(\{t\})} \\
 &= e_h(t, a) e^{\log(1+h(t)\mu_g(\{t\}))} \\
 &= e_h(t, a) (1 + h(t)\mu_g(\{t\})) = e_h(t, a) (1 + h(t)(g(t^+) - g(t))),
 \end{aligned} \tag{1.21}$$

which implies that

$$\left(e_h(\cdot, a) \right)'_g(t) = \frac{e_h(t^+, a) - e_h(t, a)}{g(t^+) - g(t)} = h(t) e_h(t, a).$$

In light of this remark at the discontinuity points, the following theorem ensures that the exponential function can be viewed as a solution of a linear Stieltjes differential equation g -almost everywhere, the reader is referred to [31, 63] for the proof.

Theorem 1.6.4. Let $h \in \mathcal{L}_g^1([a, b], \mathbb{R})$ be a function satisfying Condition (1.19). Then, the function defined by

$$x(\cdot) = x_0 e_h(\cdot, a), \tag{1.22}$$

is g -absolutely continuous, moreover, it solves the initial value problem

$$x'_g(t) = h(t)x(t) \quad \text{for } g\text{-almost every } t \in [a, b), \quad x(a) = x_0. \tag{1.23}$$

Equivalently,

$$x(t) = x_0 e_h(t, a) = x_0 + \int_{[a,t]} h(s) x_0 e_h(s, a) \, d\mu_g(s) \quad \text{for every } t \in [a, b].$$

In the classical case of derivation, we know that the multiplicative inverse of the exponential map solves the linear equation $x'(t) = -h(t)x(t)$. Now, given the definition in the context of the Stieltjes derivative, a natural question arises: Does the multiplicative inverse of the exponential map (as defined in Definition 1.6.1) solve a different linear equation? Indeed, [62, Proposition 4.15] provides an answer to this question.

Proposition 1.6.5. Let $h \in \mathcal{L}_g^1([a, b], \mathbb{R})$ satisfy Condition (1.19) in Definition 1.6.1. Then the map $y : [a, b] \rightarrow \mathbb{R}$ defined by

$$y(t) = (e_h(t, a))^{-1}, \quad t \in [a, b],$$

is well-defined, $y \in \mathcal{AC}_g([a, b], \mathbb{R})$ and

$$y'_g(t) = \frac{-h(t)}{e_h(t, a)(1 + h(t)(g(t^+) - g(t)))}, \quad \text{for } g\text{-almost all } t \in [a, b).$$

Remark 1.6.6. It is worthwhile to mention that Proposition 1.6.5 shows that if $h \in \mathcal{L}_g^1([a, b], \mathbb{R})$ satisfies Condition (1.19), then $(e_h(t, a))^{-1}$ solves the Stieltjes differential equation

$$x'_g(t) = -h(t)x(t),$$

for g -almost every $t \in [a, b] \setminus D_g$.

In order to recover an equality at g -almost everywhere in $[a, b]$, we need to modify the map h analogously to (1.20) so that $-h$ satisfies Condition (1.19), which is the framework of Theorem 1.6.4.

In [31], the authors defined another exponential map that solves a similar problem, this required a condition weaker than Condition (1.19) in Definition 1.6.1. In the proposition below, we combine [31, Lemmata 6.4 and 6.7], taking into account improvements from [62, Theorem 3.4].

Proposition 1.6.7. *Let $h \in \mathcal{L}_g^1([a, b], \mathbb{R})$ satisfy*

$$1 + h(t)(g(t^+) - g(t)) \neq 0 \quad \text{for every } t \in [a, b] \cap D_g.$$

Then the set $T_h^- := \{t \in [a, b] \cap D_g : 1 + h(t)(g(t^+) - g(t)) < 0\}$ has finite cardinality. Moreover, if $T_h^- = \{t_1, \dots, t_k\}$ for some $a \leq t_1 < t_2 < \dots < t_k < t_{k+1} = b$, then it follows that:

1. *the map $\hat{h} : [a, b] \rightarrow \mathbb{R}$ defined by*

$$\hat{h}(t) = \begin{cases} h(t) & \text{if } t \in [a, b] \setminus D_g, \\ \frac{\log(|1 + h(t)(g(t^+) - g(t))|)}{g(t^+) - g(t)} & \text{if } t \in [a, b] \cap D_g, \end{cases}$$

belongs to $\mathcal{L}_g^1([a, b], \mathbb{R})$;

2. *the map $\hat{e}_h(\cdot, a) : [a, b] \rightarrow \mathbb{R}^*$ given by*

$$\hat{e}_h(t, a) = \begin{cases} e^{\int_{[a, t)} \hat{h}(s) \, d\mu_g(s)} & \text{if } a \leq t \leq t_1, \\ (-1)^j e^{\int_{[a, t)} \hat{h}(s) \, d\mu_g(s)} & \text{if } t_j < t \leq t_{j+1}, j = 1, \dots, k, \end{cases}$$

is g -absolutely continuous on $[a, b]$;

3. *the map $x : [a, b] \rightarrow \mathbb{R}$ defined by $x(t) = x_0 \hat{e}_h(t, a)$, for all $t \in [a, b]$, solves the initial value problem*

$$x'_g(t) = h(t)x(t), \quad \text{for } g\text{-almost every } t \in [a, b], \quad x(a) = x_0.$$

The reader is referred to [31, 63] and [50, Sections 2.1.2 and 2.2.2] for more properties of the exponential function, also explicit solutions of nonhomogeneous Stieltjes differential equations have been explored in this context.

1.7 Vector-valued derivators $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^n$

In what follows, let us consider a vector-valued function $\mathbf{g} = (g_1, \dots, g_n) : \mathbb{R} \rightarrow \mathbb{R}^n$ such that $g_i : \mathbb{R} \rightarrow \mathbb{R}$ is a left-continuous and nondecreasing function, for $i = 1, \dots, n$. In this subsection, we explore the notion of continuity with respect to the vector-valued function $\mathbf{g}$.

In [56], the authors introduced the notion of $\vec{\mathbf{g}}$ -continuity.

Definition 1.7.1. Let $\mathbf{u} : A \subset \mathbb{R} \rightarrow \mathbb{R}^n$. $\mathbf{u}$ is said to be $\vec{\mathbf{g}}$ -continuous at $t \in A$ if, for each $\epsilon > 0$, there is some $\delta > 0$ such that

$$s \in A, \|\mathbf{g}(s) - \mathbf{g}(t)\| < \delta \implies \|\mathbf{u}(s) - \mathbf{u}(t)\| < \epsilon.$$

We can readily observe that $\mathbf{u} : A \subset \mathbb{R} \rightarrow \mathbb{R}^n$ is $\vec{\mathbf{g}}$ -continuous if and only if it is $\hat{\mathbf{g}}$ -continuous in the sense of Definition 1.3.1.

Remark 1.7.2. Given $\mathbf{u} = (u_1, \dots, u_n) : A \subset \mathbb{R} \rightarrow \mathbb{R}^n$ such that u_i is g_i -continuous for every $i = 1, \dots, n$, this implies that $\mathbf{u}$ is $\vec{\mathbf{g}}$ -continuous. The converse does not hold in general. Take for instance the derivator $\mathbf{g} = (g_1, g_2) : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by

$$g_1(t) = \begin{cases} t & \text{if } t \leq 0, \\ 0 & \text{if } t \in (0, 1], \\ t - 1 & \text{if } t > 1; \end{cases}$$

$$g_2(t) = \begin{cases} t & \text{if } t \leq 1, \\ 1 & \text{if } t \in (1, 2], \\ t - 1 & \text{if } t > 2. \end{cases}$$

The function $\mathbf{u} = (u_1, u_2) : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $u_1(t) = u_2(t) = t$ is $\vec{\mathbf{g}}$ -continuous, however, u_i is not g_i -continuous for $i = 1, 2$. Indeed, for $t = 1$ and $\epsilon = 1/2$, there is no $\delta > 0$ such that if $\rho_{g_i}(s, t) = |g_i(s) - g_i(t)| < \delta$, one has $|u_i(s) - u_i(t)| < \epsilon$. Take $s_1 = 0$ and $s_2 = 2$. One has

$$\rho_{g_i}(s_i, 1) = |g_i(s_i) - g_i(1)| = 0 \quad \text{and} \quad |u_i(s_i) - u_i(1)| = 1 > \epsilon \quad \text{for } i = 1, 2.$$

We denote

$$\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n) = \prod_{i=1}^n \mathcal{BC}_{g_i}([a, b], \mathbb{R}),$$

equipped with the supremum norm $\|\cdot\|_0$:

$$\|\mathbf{x}\|_0 = \sup_{t \in [a, b]} \|\mathbf{x}(t)\|, \quad \text{for all } \mathbf{x} \in \mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n).$$

Observe that, according to Remark 1.7.2, we have in general:

$$\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n) \subsetneq \mathcal{BC}_{\vec{\mathbf{g}}}([a, b], \mathbb{R}^n) := \left\{ \mathbf{u} : [a, b] \rightarrow \mathbb{R}^n : \right. \\ \left. \mathbf{u} \text{ is } \vec{\mathbf{g}}\text{-continuous and bounded} \right\}.$$

Similarly, we will denote

$$\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n) = \prod_{i=1}^n \mathcal{AC}_{g_i}([a, b], \mathbb{R}).$$

On $\mathbb{R}^n$ and $\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)$, we consider the following partial orders:

$$\begin{aligned} \mathbf{a} = (a_1, \dots, a_n) \leq \mathbf{b} = (b_1, \dots, b_n) &\iff a_i \leq b_i \quad \forall i = 1, \dots, n; \\ \mathbf{x} \leq \mathbf{y} &\iff \mathbf{x}(t) \leq \mathbf{y}(t) \quad \forall t \in [a, b]. \end{aligned}$$

For $\mathbf{a} \leq \mathbf{b}$ in $\mathbb{R}^n$ and $\mathbf{u} \leq \mathbf{v}$ in $\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)$, we denote

$$\begin{aligned} [\mathbf{a}, \mathbf{b}] &= \{\mathbf{c} \in \mathbb{R}^n : \mathbf{a} \leq \mathbf{c} \leq \mathbf{b}\}, \\ [\mathbf{u}, \mathbf{v}]_{\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)} &= \{\mathbf{x} \in \mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n) : \mathbf{u} \leq \mathbf{x} \leq \mathbf{v}\}, \\ [\mathbf{u}, \mathbf{v}]_{\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n)} &= \{\mathbf{x} \in \mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n) : \mathbf{u} \leq \mathbf{x} \leq \mathbf{v}\}. \end{aligned}$$

Arguing as in the proof of [31, Proposition 5.3], we establish the following lemma for g -absolutely continuous vector-valued functions.

Lemma 1.7.3. *Let $\mathbf{u} = (u_1, \dots, u_n) : [a, b] \rightarrow B \subset \mathbb{R}^n$ be a $\mathbf{g}$ -absolutely continuous function, i.e. $u_i \in \mathcal{AC}_{g_i}([a, b], \mathbb{R})$ for $i = 1, \dots, n$, and let $v : B \rightarrow \mathbb{R}$ be a Lipschitz continuous function on B . Then, the composition $v \circ \mathbf{u} \in \mathcal{AC}_{g_i}([a, b], \mathbb{R})$ for $i = 1, \dots, n$.*

Proof. Let $L > 0$ denote the Lipschitz constant of the function v , and let us fix $\epsilon > 0$ and $i \in \{1, \dots, n\}$. Since $u_i \in \mathcal{AC}_{g_i}([a, b], \mathbb{R})$, then for $\epsilon_i = \frac{\epsilon}{Ln} > 0$, there is some $\delta_{\epsilon_i} > 0$ such that, for any family $\{(a_j, b_j)\}_{j=1}^m$ of pairwise disjoint open subintervals of $[a, b]$,

$$\sum_{j=1}^m (g_i(b_j) - g_i(a_j)) < \delta_{\epsilon_i} \implies \sum_{j=1}^m |u_i(b_j) - u_i(a_j)| < \epsilon_i.$$

Now, for a fixed family $\{(a_j, b_j)\}_{j=1}^m$ of pairwise disjoint open subintervals of $[a, b]$, such that

$$\sum_{j=1}^m (g_i(b_j) - g_i(a_j)) < \delta_{\epsilon_i},$$

we have that

$$\begin{aligned} \sum_{j=1}^m |v \circ \mathbf{u}(b_j) - v \circ \mathbf{u}(a_j)| &\leq \sum_{j=1}^m L \|\mathbf{u}(b_j) - \mathbf{u}(a_j)\| \\ &\leq \sum_{j=1}^m L \sum_{i=1}^n |u_i(b_j) - u_i(a_j)| \\ &= L \sum_{i=1}^n \sum_{j=1}^m |u_i(b_j) - u_i(a_j)| \\ &< L \sum_{i=1}^n \epsilon_i = \epsilon. \end{aligned}$$

Hence, the composition $v \circ \mathbf{u}$ is g_i -absolutely continuous on $[a, b]$. □

Remark 1.7.4. In the case where $\mathbf{g} = (g_1, \dots, g_n) = (g, \dots, g)$ for some left-continuous and nondecreasing function $g : \mathbb{R} \rightarrow \mathbb{R}$, we adopt the notations $\mathcal{BC}_g([a, b], \mathbb{R}^n)$ and $\mathcal{AC}_g([a, b], \mathbb{R}^n)$ instead of $\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)$ and $\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n)$ respectively.

In the same context, we appeal to the generalization of the Grönwall lemma to the context of the Stieltjes derivative introduced by Larivière in [50, Proposition 4.1.3]. We concisely present this result in the next lemma, providing a simpler proof. Let us define the space of vector-valued g -integrable functions as

$$\mathcal{L}_g^1([a, b], \mathbb{R}^n) := \{\mathbf{f} = (f_1, \dots, f_n) : f_i \in \mathcal{L}_g^1([a, b], \mathbb{R}) \text{ for } i = 1, \dots, n\}.$$

Lemma 1.7.5. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a left-continuous and nondecreasing function, and $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{AC}_g([a, b], \mathbb{R}^n)$. Assume that there exist $\mathbf{k} = (k_1, \dots, k_n) \in \mathcal{L}_g^1([a, b], \mathbb{R}^n)$, and $p \in \mathcal{L}_g^1([a, b], [0, \infty))$ such that

$$\mathbf{x}'_g(t) \leq \mathbf{k}(t) + p(t)\mathbf{x}(t) \quad \text{for } g\text{-almost every } t \in [a, b].$$

Then,

$$\mathbf{x}(t) \leq e_p(t, a) \left(\int_{[a, t)} \frac{e_p^{-1}(s, a) \mathbf{k}(s)}{1 + p(s)\mu_g(\{s\})} \, d\mu_g(s) + \mathbf{x}(a) \right),$$

for all $i = 1, \dots, n$ and $t \in [a, b]$.

Proof. Let $i \in \{1, \dots, n\}$ be fixed. For g -almost every $t \in [a, b]$, since $(x_i)'_g(t) - p(t)x_i(t) \leq k_i(t)$, and $e_p(t, a)(1 + p(t)\mu_g(\{t\})) > 0$, it follows that

$$\frac{(x_i)'_g(t) - p(t)x_i(t)}{e_p(t, a)(1 + p(t)\mu_g(\{t\}))} \leq \frac{k_i(t)}{e_p(t, a)(1 + p(t)\mu_g(\{t\}))}.$$

Using the derivation formula of the quotient in Proposition 1.4.2, we obtain for g -almost all $t \in [a, b]$,

$$\left(\frac{x_i(\cdot)}{e_p(\cdot, a)} \right)'_g(t) = \frac{e_p(t, a)(x_i)'_g(t) - p(t)e_p(t, a)x_i(t)}{e_p(t, a)(e_p(t, a) + p(t)e_p(t, a)\mu_g(\{t\}))} = \frac{(x_i)'_g(t) - p(t)x_i(t)}{e_p(t, a)(1 + p(t)\mu_g(\{t\}))}.$$

It follows from the Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral, Theorem 1.5.2, that

$$\left(\frac{x_i(\cdot)}{e_p(\cdot, a)} \right)'_g(t) \leq \frac{e_p^{-1}(t, a)k_i(t)}{1 + p(t)\mu_g(\{t\})}. \quad (1.24)$$

Integrating (1.24) from a to t yields

$$\frac{x_i(t)}{e_p(t, a)} \leq \int_{[a, t)} \frac{e_p^{-1}(s, a)k_i(s)}{1 + p(s)\mu_g(\{s\})} \, d\mu_g(s) + x_i(a).$$

Therefore,

$$x_i(t) \leq e_p(t, a) \left(\int_{[a, t)} \frac{e_p^{-1}(s, a)k_i(s)}{1 + p(s)\mu_g(\{s\})} \, d\mu_g(s) + x_i(a) \right).$$

□

Using this lemma, we deduce a concise expression of the upper bound in [50, Proposition 4.1.4] deriving an integral version of the Grönwall lemma.

Proposition 1.7.6. *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a left-continuous and nondecreasing function, and $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{AC}_g([a, b], \mathbb{R}^n)$. Assume that there exist $\mathbf{q} = (q_1, \dots, q_n) \in \mathcal{L}_g^1([a, b], \mathbb{R}^n)$ with $q_i \geq 0$ for $i = 1, \dots, n$, and $p \in \mathcal{L}_g^1([a, b], [0, \infty))$ such that for all $t \in [a, b]$:*

$$\mathbf{x}(t) \leq \mathbf{q}(t) + \int_{[a, t)} p(s) \mathbf{x}(s) \, d\mu_g(s). \quad (1.25)$$

Then,

$$\mathbf{x}(t) \leq \mathbf{q}(t) + e_p(t, a) \int_{[a, t)} \frac{e_p^{-1}(s, a) p(s) \mathbf{q}(s)}{1 + p(s) \mu_g(\{s\})} \, d\mu_g(s) \quad \text{for all } t \in [a, b].$$

Proof. For $t \in [a, b]$ and $i \in \{1, \dots, n\}$, let us consider the function

$$y_i(t) := \int_{[a, t)} p(s) x_i(s) \, d\mu_g(s).$$

Observe that $(y_i)'_g(t) = p(t)x_i(t) \leq p(t)(q_i(t) + y_i(t)) = p(t)q_i(t) + p(t)y_i(t)$ for g -almost all $t \in [a, b]$, by (1.25). Thus, using the previous lemma, for $k_i \equiv p \cdot q_i$, we obtain

$$\begin{aligned} y_i(t) &\leq e_p(t, a) \left(\int_{[a, t)} \frac{e_p^{-1}(s, a) p(s) q_i(s)}{1 + p(s) \mu_g(\{s\})} \, d\mu_g(s) + y_i(a) \right) \\ &= e_p(t, a) \int_{[a, t)} \frac{e_p^{-1}(s, a) p(s) q_i(s)}{1 + p(s) \mu_g(\{s\})} \, d\mu_g(s). \end{aligned}$$

Hence,

$$x_i(t) \leq q_i(t) + y_i(t) \leq q_i(t) + e_p(t, a) \int_{[a, t)} \frac{e_p^{-1}(s, a) p(s) q_i(s)}{1 + p(s) \mu_g(\{s\})} \, d\mu_g(s).$$

□

1.8 Preliminaries on fixed point theory

In this section, we compile relevant results from the literature on fixed point theory, primarily focusing on the results in [1, 40]. As we delve into Chapters 3 and 4, we demonstrate that the problems under investigation can be reduced to the search for fixed points of certain constructed operators within a specified bounded "solution region". This motivates our exploration of various properties of the fixed point index throughout this section.

1.8.1 Fixed point theorems

We start this subsection by some definitions.

Definition 1.8.1. Let X, Y be topological spaces.

- (i) A map $f : X \rightarrow Y$ is said to be *compact* if $f(X)$ is relatively compact.
- (ii) If X is a metric space, a function $f : X \rightarrow Y$ is said to be *completely continuous* if $f(B)$ is relatively compact for every B a bounded set of X .

We state Schauder's fixed point theorem, which generalizes the Brouwer fixed point theorem [40], from the unit ball of $\mathbb{R}^n$ to normed linear spaces.

Theorem 1.8.2 (Schauder fixed point theorem). *Let C be a nonempty closed convex of a normed linear space E and let $f : C \rightarrow C$ be a continuous and compact map, then f has at least one fixed point.*

The proof of Schauder's theorem [1, 40] relies on approximations of the compact map by maps with finite rank, and on the concept of the ϵ -fixed point. Further versions of the Schauder theorem are discussed in [88].

The alternative of Leray-Schauder theorem [40, Theorem 5.4] pertains to completely continuous operators.

Theorem 1.8.3 (Leray-Schauder alternative). *Let C be a nonempty closed convex of a Banach space E such that $0 \in C$. Let $f : C \rightarrow C$ be a continuous and completely continuous operator. Consider the set*

$$\mathcal{E}(f) = \{x \in C : x = \lambda f(x) \text{ for some } \lambda \in (0, 1)\}.$$

Then, one of the following alternatives holds:

- (A1) $\mathcal{E}(f)$ is unbounded;
- (A2) f has a fixed point.

1.8.2 Fixed point index on normed spaces

In this subsection, we define the notion of the *fixed point index* on normed spaces. Given a normed space X , and an open $U \subset X$, roughly speaking, the index of a function $f : U \rightarrow X$ corresponds to an algebraic count of the fixed points of f within the set U . If f is compact, then the fixed point index of f is naturally related to the Leray-Schauder topological degree [40].

Given a normed space X , and an open set $U \subset X$. We denote by $\mathcal{K}(\overline{U}, X)$ the set of compact, continuous maps $f : \overline{U} \rightarrow X$. Furthermore, we denote by $\mathcal{K}_{\partial U}(\overline{U}, X)$ the set of the functions from $\mathcal{K}(\overline{U}, X)$ that are fixed point free on ∂U .

The following theorem defines and provides essential properties of the fixed point index.

Theorem 1.8.4. *Let X be a normed space, and an open set $U \subset X$. Then there exists an integer-valued fixed point index function $\text{index}(\cdot, U) : \mathcal{K}_{\partial U}(\overline{U}, X) \rightarrow \mathbb{Z}$ satisfying the following properties:*

1. **Existence:** $\text{index}(f, U) \neq 0$, then $\text{Fix}(f) \neq \emptyset$.

2. Normalization: If $f \in \mathcal{K}_{\partial U}(\overline{U}, X)$ is a constant map $x \mapsto x_0$, then

$$\text{index}(f, U) = \begin{cases} 1, & \text{if } x_0 \in U, \\ 0, & \text{otherwise.} \end{cases}$$

3. Homotopy: If $H : [0, 1] \times U \rightarrow X$ is a continuous and compact homotopy such that $H(\lambda, \cdot) \in \mathcal{K}_{\partial U}(\overline{U}, X)$ for all $\lambda \in [0, 1]$. Then

$$\text{index}(H(\lambda, \cdot), U) = \text{index}(H(0, \cdot), U) \text{ for all } \lambda \in [0, 1].$$

4. Excision: If $f \in \mathcal{K}_{\partial U}(\overline{U}, X)$ and V is an open in U such that $\text{Fix}(f) \subset V$, then $f|_V \in \mathcal{K}_{\partial U}(\overline{V}, X)$, and we have that

$$\text{index}(f, U) = \text{index}(f|_V, V).$$

Theorem 1.8.5. Let X be a normed space, and an open set $U \subset X$. Let U_1 and U_2 be two open subsets of U satisfying $U_1 \cap U_2 \neq \emptyset$, and $f \in \mathcal{K}_{\partial U}(\overline{U}, X)$ such that $\text{Fix}(f) \subset U_1 \cup U_2$. Then, we have the following properties:

5. Additivity:

$$\text{index}(f, U) = \text{index}(f, U_1) + \text{index}(f, U_2).$$

6. Localization: If $\text{index}(f, U) \neq 0$ and $\text{index}(f, U_1) = 0$, then $\text{Fix}(f|_{U_2}) \neq \emptyset$.

7. Multiplicity: If $\text{index}(f, U) = 0$ and $\text{index}(f, U_1) \neq 0$, then

$$\text{Fix}(f|_{U_1}) \neq \emptyset \text{ and } \text{Fix}(f|_{U_2}) \neq \emptyset.$$

It is worth mentioning that the localization and multiplicity properties hold as consequences of the additivity and the normalization properties.

The following theorem provides additional properties of the fixed point index.

Theorem 1.8.6. Let X_1, X_2 be normed spaces, and let U_1 and U_2 be open subsets of X_1 and X_2 respectively. Then, we have the following properties:

8. Commutativity : Let $f_1 : \overline{U_1} \rightarrow X_2$ and $f_2 : \overline{U_2} \rightarrow X_1$ be continuous functions such that f_1 or f_2 is compact. Let us define $V_1 = U_1 \cap f_1^{-1}(U_2)$ and $V_2 = U_2 \cap f_2^{-1}(U_1)$. Then

- (i) $f_2 \circ f_1 : \overline{V_1} \rightarrow X_1$ and $f_1 \circ f_2 : \overline{V_2} \rightarrow X_2$ compact maps;
- (ii) If $\text{Fix}(f_2 \circ f_1) \subset V_1$ and $\text{Fix}(f_1 \circ f_2) \subset V_2$, then

$$\text{index}(g \circ f, V_1) = \text{index}(f \circ g, V_2).$$

9. Multiplicativity: If $f_1 \in \mathcal{K}_{\partial U_1}(\overline{U_1}, X_1)$ and $f_2 \in \mathcal{K}_{\partial U_2}(\overline{U_2}, X_2)$, then

$$f_1 \times f_2 \in \mathcal{K}_{\partial(U_1 \times U_2)}(\overline{U_1} \times \overline{U_2}, X_1 \times X_2),$$

and

$$\text{index}(f_1 \times f_2, U_1 \times U_2) = \text{index}(f_1, U_1) \text{index}(f_2, U_2).$$

The following property follows from the commutativity property of the index.

Theorem 1.8.7. *Let (X, D) be a pair of normed spaces such that D is a closed subset of X , U is an open in X , and let $f \in \mathcal{K}_{\partial U}(\overline{U}, X)$.*

10. Contraction *If $f(U) \subset D$, let us define the map*

$$\begin{aligned} f|_{U \cap D} : U \cap D &\rightarrow D \\ x &\mapsto f|_{U \cap D}(x) = f(x). \end{aligned}$$

Then

$$\text{index}(f, U) = \text{index}(f|_{U \cap D}, U \cap D).$$

Now, we introduce the notion of the local index, which provides insights into the behavior of isolated fixed points. Our objective is to derive fundamental index formulae to compute the fixed point index for particular cases of functions. This will be one of the tools to establish existence of solutions in Chapters 3 and 4.

Definition 1.8.8. Let $x_0 \in U$ be an isolated fixed point of a function $f \in \mathcal{K}_{\partial U}(\overline{U}, X)$. The local index of the isolated point x_0 for f is defined by

$$\text{index}_{loc}(f, x_0) := \text{index}(f, B(x_0, \epsilon)),$$

where $B(x_0, \epsilon) \subset U$ is an open ball of radius $\epsilon > 0$ centered at $x_0 \in U$ such that

$$\text{Fix}(f|_{B(x_0, \epsilon)}) = \{x_0\}.$$

Remark 1.8.9. Since $B(x_0, \epsilon) \subset U$ is an open ball such that $\text{Fix}(f|_{B(x_0, \epsilon)}) = \{x_0\}$. Then, from the excision property of the fixed point index, Theorem 1.8.4, it follows that

$$\text{index}(f, B(x_0, \epsilon)) = \text{index}(f|_{B(x_0, \delta)}, B(x_0, \delta)),$$

for any choice of $\delta > 0$ such that $B(x_0, \delta) \subset U$. Hence, $\text{index}_{loc}(f, x_0)$ does not depend on the choice of ϵ .

The additivity property of the fixed point index yields the following theorem [40, Chapter IV, Section 12, Theorem 8.2].

Theorem 1.8.10. *Let $f \in \mathcal{K}_{\partial U}(\overline{U}, X)$ be a map such that $\text{Fix}(f) = \{x_1, \dots, x_\ell\}$, then*

$$\text{index}(f, U) = \sum_{i=1}^{\ell} \text{index}_{loc}(f, x_i).$$

We conclude this part by some results providing formulae to compute the local point index for some particular maps. The reader is referred to [40, Chapter IV, Section 12, Subsection 8, Examples 1 and 2].

Theorem 1.8.11. *Let $X = \mathbb{R}$, $U = (a, b) \subset \mathbb{R}$, and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f(a) \neq a$ and $f(b) \neq b$. Thus,*

$$\text{index}(f, U) = \frac{1}{2} \text{sgn}(b - f(b)) - \frac{1}{2} \text{sgn}(a - f(a)).$$

In addition, if f is differentiable at $x_0 = f(x_0)$ and $f'(x_0) \neq 1$, then

$$\text{index}_{loc}(f, U) = \text{sgn} \frac{d}{dx}(x - f(x)) \Big|_{x=x_0}.$$

Theorem 1.8.12. *Let $X = \mathbb{R}^n$, and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear operator such that $1 \notin \sigma(T)$, where $\sigma(T)$ stands for the spectrum of T . Then*

$$\text{index}_{loc}(T, \mathbf{0}) = \text{sgn det}(I - T) = (-1)^\xi,$$

where I is the identity matrix of $\mathcal{M}(n \times n)$ and $\xi := \text{card}(\{\lambda \in \sigma(T) : \lambda > 1\}) \in \{1, \dots, n\}$.

Taking $T = 2\text{id}_{\mathbb{R}^n}$ in the previous theorem, we have that $\sigma(T) = \{2\}$ and $\xi = n$. Thus, we derive the following corollary.

Corollary 1.8.13. *Let $U = B_{\mathbb{R}^n}(\mathbf{0}, \epsilon) \subset \mathbb{R}^n$ for some $\epsilon > 0$. Then,*

$$\text{index}(2\text{id}_{\mathbb{R}^n}, U) = \text{index}_{loc}(2\text{id}_{\mathbb{R}^n}, \mathbf{0}) = (-1)^n.$$

Motivation behind the notion of the Stieltjes derivative

As seen in the previous chapter, the Stieltjes derivative allows differentiating a function f at the discontinuity points as well, unlike the classical derivative which requires f to be continuous at the derivation point. Moreover, the Stieltjes derivative does not have meaning when the function g is constant on an neighborhood of the derivation point, since such neighborhoods are negligible sets with respect to μ_g . Leveraging these features, provides a descriptive framework to model phenomena presenting discontinuities or stationary periods using Stieltjes differential equations. This requires a judicious choice the derivator g , controlling the following parameters:

- The discontinuities, D_g : should account for the sudden jumps that occur in the process, for instance in dynamics of population: moments of hatching of eggs, implantations... Frequent environmental shocks and extreme events such as earthquakes, fire forests, epidemics.
- The constancy set, C_g : should align will periods where the dynamics are stationary, for instance closed seasons of fishing, hibernation, or periods when it is not possible to keep track of the population...
- The slopes-variations of g : have an important role in enhancing the "weight" of every time in the process, g taking greater slopes, implies that the process goes through important dynamics during the corresponding time intervals.

As we defined the notion of g -absolutely continuous functions in Chapter 1, Section 5, we can anticipate in this chapter that we will be looking for solutions that are g -absolutely continuous. This regularity is motivated by practical applications and proves to be descriptive. Indeed, with a suitable choice of g , this setting will encode prior information about the process, yielding solutions that effectively account for the same discontinuities and stationary periods.

In the following section, we emphasize another motivation behind introducing the Stieltjes derivative. By means of a particular choice of the derivator g , we recover a particular type of differential equations. This highlights that the Stieltjes derivative offers a unification of discrete and continuous calculus, providing that when a result is proven for a Stieltjes differential equation, it will also apply to various types of differential equations.

2.1 Stieltjes differential equations and their relation with other problems

The aim of this section is to establish the connection between Stieltjes differential equations and various classical differential equations, contingent upon the choice of

the function g . It is evident that when $g \equiv \text{id}_{\mathbb{R}}$, we recover the conventional ordinary differential equations. In the subsequent subsections, we will investigate the relationships between Stieltjes differential equations and classical problems, including difference equations, impulsive differential equations, and equations defined on time scales.

2.1.1 Difference equations

In this subsection, we show that *Difference Equations* are Stieltjes differential equations with a particular choice of derivator $g : \mathbb{R} \rightarrow \mathbb{R}$. First of all, let us consider a difference equation of the form

$$\Delta x_n = f(n, x_n), \quad n \in \mathbb{N}, \quad x_0 = r,$$

where Δ is the difference operator defined by

$$\Delta x_n := x_{n+1} - x_n.$$

Based on this setting, the solution of this difference equation changes as every natural $n \in \mathbb{N}$. Thus, our derivator g should have discontinuities such that $D_g = \mathbb{N}$, and for all $t \in D_g$:

$$\mu_g(\{t\}) = 1.$$

A natural choice of function g is therefore:

$$g(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \lceil t \rceil & \text{if } t > 0, \end{cases}$$

where $\lceil \cdot \rceil$ is the ceiling function, and $\lceil t \rceil$ the smallest integer greater than or equal to t . In this case, the difference equation can be viewed as the Stieltjes differential equation

$$x'_g(t) = f(t, x(t)), \quad \text{for } g\text{-almost every } t \geq 0, \quad x(0) = x_0.$$

where the solution x is a piecewise constant function that coincides with the solution $\{x_n\}_{n \in \mathbb{N}}$ of the difference equation on $D_g = \mathbb{N}$.

Observe that the subsets of $C_g = \mathbb{R} \setminus \mathbb{N}$ are the stationary periods of $\tilde{x}$ since the solution of the difference equation changes only on $\mathbb{N}$.

Hence, based on the definition of the Stieltjes derivative (1.12), the g -derivative of the solution $x : \mathbb{R}^+ \rightarrow \mathbb{R}$ exists for $t = n \in \mathbb{N}$.

Since the function g is constant and left-continuous on $(n, n+1]$, then x is constant on $(n, n+1)$, and $x(s) = x(n+1)$, for $s \in (n, n+1]$. In particular, if $t = n \in D_g$, then $x(t^+) = x(n+1)$.

Therefore, for $t = n \in \mathbb{N}$:

$$x'_g(t) = \frac{x(n^+) - x(n)}{g(n^+) - g(n)} = \frac{x(n+1) - x(n)}{g(n+1) - g(n)} = x_{n+1} - x_n.$$

Hence,

$$x'_g(n) = \Delta x_n.$$

2.1.2 Impulsive differential equations

In this section, we emphasize the connection between Stieltjes differential equations and impulsive differential equations, based on the ideas in [31, 58]. Let $J = \{t_k : k \in \mathbb{N}\}$ be a countable subset of an interval $[a, b)$ such that the set of its accumulation points J' is a null-measure set. Let us consider the impulsive ordinary differential equation

$$x'(t) = f(t, x(t)), \quad \text{for almost all } t \in [a, b) \setminus J, \quad (2.1)$$

$$\Delta x(t) := x(t^+) - x(t) = I_t(x(t)), \quad \text{for } t \in J, \quad (2.2)$$

where $f : [a, b) \times \mathbb{R} \rightarrow \mathbb{R}$, and $I_t : \mathbb{R} \rightarrow \mathbb{R}$ are some given functions for $t \in J$.

Let us consider the left-continuous and nondecreasing function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined for $t \in \mathbb{R}$ by

$$g(t) = t + \sum_{\{n \in \mathbb{N} : t_n < t\}} 2^{-n},$$

where the sum is assumed to take the value zero if $\{n \in \mathbb{N} : t_n < t\} = \emptyset$.

g is well-defined, since the series $\sum_{n=1}^{\infty} 2^{-n}$ converges to 1. Moreover, g is an increasing derivator. Indeed, for $s, t \in \mathbb{R}$ with $s < t$, observe that

$$g(t) = t + \sum_{\{n \in \mathbb{N} : t_n < t\}} 2^{-n} = t - s + g(s) + \sum_{\{n \in \mathbb{N} : s \leq t_n < t\}} 2^{-n} > g(s),$$

which yields also

$$\lim_{s \rightarrow t^-} g(s) - g(t) = \lim_{s \rightarrow t^-} \left(t - s + \sum_{\{n \in \mathbb{N} : s \leq t_n < t\}} 2^{-n} \right) = 0.$$

Therefore, g is left-continuous on $\mathbb{R}$. Similarly, we can prove that g is right-continuous on $\mathbb{R} \setminus J$. Indeed, for $t \in \mathbb{R} \setminus J$:

$$\lim_{s \rightarrow t^+} g(s) - g(t) = \lim_{s \rightarrow t^+} \left(s - t + \sum_{\{n \in \mathbb{N} : t \leq t_n < s\}} 2^{-n} \right) = \lim_{s \rightarrow t^+} \left(s - t + \sum_{\{n \in \mathbb{N} : t < t_n < s\}} 2^{-n} \right) = 0$$

as $t \neq t_n$ for all $n \in \mathbb{N}$, which yields that g is continuous on $\mathbb{R} \setminus J$. Moreover, for every $t_n \in J$, we have that

$$g(t_n^+) - g(t_n) = 2^{-n} > 0.$$

Remark 2.1.1. In particular, we observe that g is continuous on the points of J' , which implies that J' is a null g -measure set.

Now considering the function $\tilde{f} : [a, b) \times \mathbb{R} \rightarrow \mathbb{R}$ defined for $(t, x) \in [a, b) \times \mathbb{R}$ by

$$\tilde{f}(t, x) = \begin{cases} f(t, x), & \text{if } t \notin J, \\ \frac{I_t(x)}{\mu_g(\{t\})}, & \text{otherwise,} \end{cases}$$

we can show that the impulsive equation (2.1), (2.2) can be reduced to the Stieltjes differential equation

$$x'_g(t) = \tilde{f}(t, x(t)), \text{ for } g\text{-almost every } t \in [a, b]. \quad (2.3)$$

For $t \notin (J \cup J'), t \notin D_g$, it follows that

$$\lim_{s \rightarrow t} \frac{x(s) - x(t)}{g(s) - g(t)} = \lim_{s \rightarrow t} \frac{x(s) - x(t)}{s - t}.$$

This means that $x'_g(t)$ exists if and only if $x'(t)$ does and $x'_g(t) = x'(t)$. Moreover, since $\mu_g(J') = 0$, then x solves (2.1), (2.2) if and only if x solves (2.3) on $[a, b] \setminus J$. What remains to show is that x fulfills (2.2) if and only if x solves (2.3) on J . If x fulfills (2.2), then for $t \in J$, we have that $x(t^+) = x(t) + I_t(x(t))$ exists and so does $x'_g(t)$. Thus,

$$x'_g(t) = \frac{x(t^+) - x(t)}{g(t^+) - g(t)} = \frac{x(t) + I_t(x(t)) - x(t)}{g(t^+) - g(t)} = \tilde{f}(t, x(t)).$$

Now conversely, if the function x solves (2.3) on J , then $x(t^+)$ exists and we have

$$x(t^+) = x(t) + x'_g(t)\mu_g(\{t\}) = x(t) + \frac{I_t(x(t))}{\mu_g(\{t\})}\mu_g(\{t\}).$$

Hence, $\Delta x(t) = x(t^+) - x(t) = I_t(x(t))$.

2.1.3 Equations on time scales

In this subsection, we will explore a connection between equations defined on time scales and Stieltjes differential equations using Stieltjes derivatives. The theory of *time scales* goes back to Stefan Hilger developed in his PhD thesis [43], studying calculus on time scales as a means to unify discrete and continuous analysis. This equivalence between problems allows us to use differential equations with Stieltjes derivatives to study both discrete and continuous problems. To emphasize this connection, we follow the ideas in [31, 58, 65]. For this purpose, we need to delve into essential concepts related to time scales from [10], in order to define equations on time scales.

A *time scale* refers to any nonempty closed subset of $\mathbb{R}$. For instance, $\mathbb{N}$, $\mathbb{Z}$, $[0, 1] \cup [2, \sqrt{13}]$, the Cantor set, are time scales, while $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$, $\mathbb{C}$ are not time scales.

Definition 2.1.2. Given a time scale $\mathbb{T}$. We define the *forward jump* operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ for all $t \in \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : t < s\}, \quad (2.4)$$

and the *backward jump* operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ for every $t \in \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : t > s\}, \quad (2.5)$$

with $\inf \emptyset = \sup \mathbb{T}$, i.e. $\sigma(t) = t$ if t is the maximum of $\mathbb{T}$, and $\sup \emptyset = \inf \mathbb{T}$, i.e. $\rho(t) = t$ if t is the minimum of $\mathbb{T}$. Moreover, a point t is said to be:

- (i) *right-scattered* if $\sigma(t) > t$;

- (ii) *left-scattered* if $\rho(t) < t$;
- (iii) *isolated* if t is right and left-scattered;
- (iv) *right-dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$;
- (v) *left-dense* if $\inf \mathbb{T} < t$ and $\rho(t) = t$;
- (vi) *dense* if t is right and left-dense.

In the sequel, we will define the Δ -derivative, also known as the Hilger derivative given a time scale $\mathbb{T}$. This requires introducing the set $\mathbb{T}^\kappa$ associated to $\mathbb{T}$ defined as:

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Definition 2.1.3. Let $\mathbb{T}$ be time scale, and $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function. Let $t \in \mathbb{T}^\kappa$. The Δ -derivative of f at t , denoted as $f^\Delta(t)$ (provided that it exists), is defined as the real number such that for each $\epsilon > 0$, there is a neighborhood of t of the form $V = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$, such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \epsilon |\sigma(t) - s|, \text{ for every } s \in V.$$

In the case, where $f^\Delta(t)$ exists, f is said to be Δ -differentiable at t (or Hilger differentiable). Moreover, if f is Δ -differentiable at every $t \in \mathbb{T}^\kappa$, then f is said Δ -differentiable on $\mathbb{T}^\kappa$.

The next proposition enhances useful properties of the Δ -derivative, the reader is referred to [10, Theorem 1.16] for the proof and more interesting properties of the Δ -derivative.

Proposition 2.1.4. Given a time scale $\mathbb{T}$, and a function $f : \mathbb{T} \rightarrow \mathbb{R}$. If $t \in \mathbb{T}^\kappa$, then the following statement hold:

- (i) If f is Δ -differentiable at t , then f is continuous at t ;
- (ii) If f is continuous at t and t is right-scattered, then f is Δ -differentiable at t and

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t};$$

- (iii) If t is right-dense, then f is Δ -differentiable if and only if the limit

$$\lim_{s \rightarrow t} \frac{f(s) - f(t)}{s - t}$$

exists. In this case,

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(s) - f(t)}{s - t};$$

In the sequel, let us fix a time scale $\mathbb{T}$ and $a, b \in \mathbb{T}$ such that $a < b$. The aim in what follows is to show the connection between solutions of the equation on the time scale of the form:

$$x^\Delta(t) = f(t, x(t)), \quad \text{for every } t \in \mathbb{T} \cap [a, b) \quad (2.6)$$

and the Stieltjes differential equation

$$\tilde{x}'_g(t) = f(t, \tilde{x}(t)) \quad \text{for } g\text{-almost every } t \in [a, b), \quad (2.7)$$

given a convenient choice of g . In doing so, let us consider the function

$$g(t) = \begin{cases} a, & \text{if } t \leq a, \\ \inf\{s \in \mathbb{T} : t \leq s\}, & \text{if } a < t \leq b, \\ b, & \text{if } t > b. \end{cases} \quad (2.8)$$

By definition, g is left-continuous and nondecreasing such that $C_g = \mathbb{R} \setminus \mathbb{T}$, and $g(t) = t$ for every $t \in \mathbb{T}$.

Now, given $f : \mathbb{T} \rightarrow \mathbb{R}$, let us consider its *Slavík* extension $\bar{f} : \mathbb{R} \rightarrow \mathbb{R}$ defined for every $t \in \mathbb{R}$ by

$$\bar{f}(t) = f(g(t)). \quad (2.9)$$

The next lemma is consequence of [81, Lemma 4].

Lemma 2.1.5. *Let us consider a function $f : \mathbb{T} \rightarrow \mathbb{R}$. If f is left-continuous on $\mathbb{T}$, then $\bar{f}$ is left-continuous on $\mathbb{T}^\kappa$. If f is right-continuous on $\mathbb{T}$, then $\bar{f}$ is right-continuous at right-dense points of $\mathbb{T}$.*

The following theorem [62, Theorem 3.49] outlines the conditions under which we can convert a solution of a Stieltjes differential equation into a solution for an equation on a time scale.

Theorem 2.1.6. *Let $x : \mathbb{T} \rightarrow \mathbb{R}$ be a left-continuous function at each right-scattered point of $\mathbb{T}$. Let us consider the function g from (2.8) and $\bar{x}$ the Slavík extension of x given in (2.9). If x solves (2.6), then $\bar{x}$ solves (2.7).*

This theorem is an adaptation of [58, Theorem 3.1], we include the proof as in [62, Theorem 3.49] for sake of completeness.

Proof. Let $t \in [a, b) \cap \mathbb{T}$ be fixed. Since $b \in \mathbb{T}$, then $t \in \mathbb{T}^\kappa$. We distinguish two cases: either t is right-scattered or t is right-dense.

Case 1: If t is right-scattered: In this case, x is left-continuous at t . By Statement (ii) from Proposition 2.1.4, it follows that $x^\Delta(t)$ exists and

$$x^\Delta(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Now, for every $s \in (t, \sigma(t))$, we have $g(s) = \sigma(t)$. Thus, since $g(t) = t$, we obtain

$$x^\Delta(t) = \frac{x(g(s)) - x(g(t))}{g(s) - g(t)}, \quad s \in (t, \sigma(t)). \quad (2.10)$$

On the other hand, t being right-scattered, implies that g is discontinuous at t . Indeed, observe that since $t \in \mathbb{T}^\kappa$, we must have that

$$\sigma(t) = \inf\{s \in \mathbb{T} : t < s\},$$

as the other definition of σ in (2.4) can only happen when $t = \sup \mathbb{T}^\kappa$, which would contradict $t \in \mathbb{T}^\kappa = \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] = \mathbb{T} \setminus (\rho(t), t]$. Thus, since t is right-scattered, this implies that $\sigma(t) > t$, and

$$\lim_{s \rightarrow t^+} g(s) = \lim_{s \rightarrow t^+} \inf\{r \in \mathbb{T} : s \leq r\} = \inf\{r \in \mathbb{T} : t < r\} = \sigma(t) > t = g(t). \quad (2.11)$$

Consequently $t \in D_g$. Hence, it follows from (2.10) that $\bar{x}'_g(t)$ exists and

$$\bar{x}'_g(t) = \lim_{s \rightarrow t^+} \frac{\bar{x}(s) - \bar{x}(t)}{g(s) - g(t)} = \lim_{s \rightarrow t^+} \frac{x(g(s)) - x(g(t))}{g(s) - g(t)} = x^\Delta(t),$$

which concludes **Case 1**.

Case 2: If t is right-dense, it follows from Proposition 2.1.4 that x is Δ -differentiable at t if and only if the limit

$$\lim_{\substack{s \rightarrow t \\ s \in \mathbb{T}}} \frac{x(s) - x(t)}{s - t} \quad (2.12)$$

exists. In this case, $x^\Delta(t)$ takes the value of that limit. Moreover, since g is continuous at t by Lemma 2.1.5, we know that $\bar{x}$ is g -differentiable at t if and only if the following limit exists

$$\lim_{s \rightarrow t} \frac{\bar{x}(s) - \bar{x}(t)}{g(s) - g(t)} \quad (2.13)$$

and in that case, $\bar{x}'_g(t)$ takes that value. Hence, it is enough to show that (2.12) exists if and only if (2.13) exists, and that in that case, both of them take the same value.

First, assume that (2.12) exists. Since t is right-dense, there exist sequences $\{t_n\}_{n \in \mathbb{N}}$ such that $\{t_n\}_{n \in \mathbb{N}}$ converges to t and $g(t_n) \neq t$, $n \in \mathbb{N}$. Indeed, we have that

$$t = g(t) = \inf\{r \in \mathbb{T} : t \leq r\} < \inf\{r \in \mathbb{T} : s \leq r\} = g(s), \quad s > t,$$

Thus, we can at least construct one of those sequences. Let us take an arbitrary sequence $\{t_n\}_{n \in \mathbb{N}}$ such that $t_n \rightarrow t$ and $g(t_n) \neq t$, for every $n \in \mathbb{N}$. Since g is continuous at t , it follows that $g(t_n) \rightarrow g(t) = t$. Therefore, $g(t_n) \in \mathbb{T} \setminus \{t\}$, and

$$\lim_{n \rightarrow \infty} \frac{\bar{x}(t_n) - \bar{x}(t)}{g(t_n) - g(t)} = \lim_{n \rightarrow \infty} \frac{x(g(t_n)) - x(g(t))}{g(t_n) - g(t)} = \lim_{\substack{s \rightarrow t \\ s \in \mathbb{T}}} \frac{x(s) - x(t)}{s - t}.$$

Since the sequence was arbitrarily chosen, it follows that (2.13) exists and coincides with (2.12).

Conversely, assume that (2.13) exists. In that case, consider a sequence $\{t_n\}_{n \in \mathbb{N}} \subset (\mathbb{T} \setminus \{t\})$ such that $t_n \rightarrow t$. It follows that (2.12) exists since

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(t)}{t_n - t} = \lim_{n \rightarrow \infty} \frac{x(g(t_n)) - x(g(t))}{g(t_n) - g(t)} = \lim_{n \rightarrow \infty} \frac{\bar{x}(t_n) - \bar{x}(t)}{g(t_n) - g(t)} = \lim_{s \rightarrow t} \frac{\bar{x}(s) - \bar{x}(t)}{g(s) - g(t)}.$$

From **Case 1** and **Case 2**, it follows that $\bar{x}'_g(t)$ exists for all $t \in \mathbb{T} \cap [a, b)$ and

$$\bar{x}'_g(t) = x^\Delta(t) = f(t, x(t)) = f(t, x(g(t))) = f(t, \bar{x}(t)).$$

Since $[a, b) \setminus \mathbb{T} = [a, b) \cap C_g$, then, it is a null g -measure set, which completes the proof. $\square$

The converse of this theorem does not yield an equivalence between the problems (2.7) and (2.6). Indeed, observe that given a solution of the problem (2.7), this equation holds true for g -almost every point in the interval $[a, b)$, which may not include all points of $\mathbb{T}$. Nevertheless, we can still construct a solution for the problem (2.6), given a solution of the Stieltjes differential equation (2.7) provided that the latter holds everywhere in $[a, b) \cap \mathbb{T}$, and we have the following theorem [62, Theorem 3.51], again, we include its proof for sake of completeness.

Theorem 2.1.7. *Let g be the function from (2.8). If $\tilde{x} : \mathbb{R} \rightarrow \mathbb{R}$ is a g -continuous solving the problem*

$$\tilde{x}'_g(t) = f(t, \tilde{x}(t)) \quad \text{for every } t \in [a, b) \cap \mathbb{T},$$

then, $x := \tilde{x}|_{[a, b) \cap \mathbb{T}}$ is a solution of (2.6).

Proof. Let $t \in [a, b) \cap \mathbb{T}$. Recall that this implies that $t \in \mathbb{T}^*$. Again, We distinguish two cases: either t is right-scattered or t is right-dense.

Case 1: If t is right-scattered: In this case, $t \in D_g$, and we have

$$f(t, \tilde{x}(t)) = \tilde{x}'_g(t) = \frac{\tilde{x}(t^+) - \tilde{x}(t)}{g(t^+) - g(t)}.$$

In addition, since t is right-scattered, then g is constant on $(t, \sigma(t))$. Thus, $\tilde{x}$ being g -continuous, yields that $\tilde{x}$ is constant on $(t, \sigma(t))$ as well. Therefore, since $\tilde{x}$ is left-continuous, then $\tilde{x}(t^+) = \tilde{x}(\sigma(t))$ and as mentioned in (2.11), we have $g(t^+) = \sigma(t)$. Moreover $g(t) = t$, then

$$f(t, \tilde{x}(t)) = \frac{\tilde{x}(t^+) - \tilde{x}(t)}{g(t^+) - g(t)} = \frac{\tilde{x}(\sigma(t)) - \tilde{x}(t)}{\sigma(t) - t}.$$

Now, since $t, \sigma(t) \in \mathbb{T}$ and using Proposition 2.1.4, we obtain that

$$f(t, x(t)) = f(t, \tilde{x}(t)) = \frac{\tilde{x}(\sigma(t)) - \tilde{x}(t)}{\sigma(t) - t} = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t} = x^\Delta(t).$$

Case 2: If t is right-dense, in this case we need to prove that $x^\Delta(t)$ exists and equals $f(t, x(t))$. Equivalently, it follows from Proposition 2.1.4 that we can prove that the following limit exists,

$$\lim_{\substack{s \rightarrow t \\ s \in \mathbb{T}}} \frac{x(s) - x(t)}{s - t}, \tag{2.14}$$

and that this limit equals to $f(t, x(t))$.

Since t is a right-dense point, it follows that $g(s) > g(t)$ for all $s \in [a, b] \cap \mathbb{T}$, with $s > t$. Consequently, for each $s \in (t, b] \cap \mathbb{T}$, we obtain

$$\frac{x(s) - x(t)}{s - t} = \frac{\tilde{x}(s) - \tilde{x}(t)}{g(s) - g(t)},$$

this implies that the following limit exists and

$$\lim_{\substack{s \rightarrow t^+ \\ s \in \mathbb{T}}} \frac{x(s) - x(t)}{s - t} = \lim_{\substack{s \rightarrow t^+ \\ s \in \mathbb{T}}} \frac{\tilde{x}(s) - \tilde{x}(t)}{g(s) - g(t)} = \tilde{x}'_g(t) = f(t, \tilde{x}(t)) = f(t, x(t)).$$

If t is left-scattered, then this is equivalent to (2.14) and the result follows immediately. Otherwise, if t is left-dense, then g is continuous at t , and it follows that $g(s) < g(t)$ for $s \in [a, b] \cap \mathbb{T}$, $s < t$. Therefore, for each $s \in [a, t) \cap \mathbb{T}$, one has

$$\frac{x(s) - x(t)}{s - t} = \frac{\tilde{x}(s) - \tilde{x}(t)}{g(s) - g(t)}.$$

Hence, the following limit exists and

$$\lim_{\substack{s \rightarrow t^- \\ s \in \mathbb{T}}} \frac{x(s) - x(t)}{s - t} = \lim_{\substack{s \rightarrow t^- \\ s \in \mathbb{T}}} \frac{\tilde{x}(s) - \tilde{x}(t)}{g(s) - g(t)} = \tilde{x}'_g(t) = f(t, \tilde{x}(t)) = f(t, x(t)),$$

and the proof is complete. □

Existence and multiplicity results for first-order Stieltjes differential equations

In this chapter, we investigate existence and multiplicity of solutions for Stieltjes differential equations. Our approach incorporates the method of lower and upper solutions, alongside the properties of the fixed point index revisited in Chapter 1. We investigate solutions for Stieltjes differential equations of the form

$$\begin{aligned} x'_g(t) &= f(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, T], \\ x &\in \mathfrak{B}; \end{aligned} \tag{3.1}$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is left-continuous and nondecreasing, $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ a g -Carathéodory function, and $\mathfrak{B}$ stands for the boundary conditions: the periodic boundary condition

$$x(0) = x(T); \tag{3.2}$$

or the initial value condition:

$$x(0) = x_0. \tag{3.3}$$

In order to establish the existence of a solution that is a priori bounded by a pair of lower and upper solution, the next monotonicity condition was required in previous studies [54, 57, 67]:

$$\text{The map } x \mapsto x + f(t, x)\mu_g(\{t\}) \text{ is nondecreasing on } [\alpha(t), \beta(t)] \tag{3.4}$$

for every $t \in [0, T] \cap D_g$, where α and β are respectively lower and upper solutions of the problem (3.1). In this chapter, we present existence and multiplicity results for the solutions for the problem (3.1) without imposing the monotonicity condition (3.4).

3.1 Existence results for first-order Stieltjes differential equations

3.1.1 g -Carathéodory function

Let us fix $a, b \in \mathbb{R}$ such that $a < b$, and let us recall the definition of a g -Carathéodory function according to in Definition 1.1.26 with respect to the measure μ_g .

Definition 3.1.1 (g -Carathéodory function). A function $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be g -Carathéodory, if it fulfills the following conditions:

- (i) for each $x \in \mathbb{R}$, $t \mapsto f(t, x)$ is g -measurable;

- (ii) for g -almost every $t \in [a, b]$, $x \mapsto f(t, x)$ is continuous on $\mathbb{R}$;
- (iii) for every $r > 0$, there is a function $h_r \in \mathcal{L}_g^1([a, b], [0, +\infty))$ such that

$$|f(t, x)| \leq h_r(t), \quad \text{for } g\text{-almost all } t \in [a, b], \text{ for all } x \in \mathbb{R}, \text{ such that } |x| \leq r.$$

In the classical case of derivation, it is well-known that given a Carathéodory function f and a continuous function x , the composition $f(\cdot, x(\cdot))$ is a measurable function. By Condition (iii) of Definition 1.1.26, if x is a bounded function, then the composition is integrable. The following lemma [31, Lemma 7.2] asserts that a similar result can be deduced for g -Carathéodory functions.

Lemma 3.1.2. *Let $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a g -Carathéodory function. Then, the map $f(\cdot, x(\cdot)) \in \mathcal{L}_g^1([a, b], \mathbb{R})$ for every $x \in \mathcal{BC}_g([a, b], \mathbb{R})$.*

Given a g -Carathéodory function f , the following lemma states that f is associated to a completely continuous operator.

Lemma 3.1.3. *Let $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a g -Carathéodory function. Then the operator $N_f : \mathcal{BC}_g([a, b], \mathbb{R}) \rightarrow \mathcal{BC}_g([a, b], \mathbb{R})$ defined by*

$$N_f(x)(t) = \int_{[a, t)} f(s, x(s)) \, d\mu_g(s),$$

is continuous and completely continuous.

Proof. N_f is well defined, since f is a g -Carathéodory function. Now, we prove that $N_f : \mathcal{BC}_g([a, b], \mathbb{R}) \rightarrow \mathcal{BC}_g([a, b], \mathbb{R})$ is continuous.

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence converging to $x \in \mathcal{BC}_g([a, b], \mathbb{R})$, then $\{x_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{BC}_g([a, b], \mathbb{R})$ by some $r > 0$. Since $f(t, \cdot)$ is a continuous function for g -almost every $t \in [a, b]$, it follows that

$$f(t, x_n(t)) \rightarrow f(t, x(t)) \quad \text{for } g\text{-almost every } t \in [a, b].$$

Using Definition 3.1.1 (iii), there is a function $h_r \in \mathcal{L}_g^1([a, b], [0, +\infty))$ such that

$$|f(t, x_n(t))| \leq h_r(t) \quad \text{for } g\text{-almost every } t \in [a, b].$$

By means of Lebesgue's Dominated Convergence Theorem, Theorem 1.1.25, we obtain

$$\int_{[a, t)} f(s, x_n(s)) \, d\mu_g(s) \rightarrow \int_{[a, t)} f(s, x(s)) \, d\mu_g(s) \quad \text{for every } t \in [a, b].$$

Therefore,

$$\|N_f(x_n) - N_f(x)\|_0 \leq \int_{[a, b)} |f(s, x_n(s)) - f(s, x(s))| \, d\mu_g(s) \rightarrow 0.$$

Hence, the operator N_f is continuous.

Now, let $M > 0$ be fixed. Thus, there is a function $h_M \in \mathcal{L}_g^1([a, b], [0, +\infty))$ such that, for all $x \in \overline{B(0, M)} \subset \mathcal{BC}_g([a, b], \mathbb{R})$,

$$|f(t, x(t))| \leq h_M(t) \quad \text{for } g\text{-almost every } t \in [a, b].$$

Combining Lemma 3.1.2, Theorem 1.5.5 and Proposition 1.5.6 yields that $N_f(\overline{B(0, M)})$ is relatively compact. Hence, N_f is completely continuous. $\square$

3.1.2 Notions of lower and upper solutions

We start this subsection by defining a solution of (3.1) assuming that $T \notin D_g$, and some remarks regarding the solvability of the problem (3.1) even in the case when $T \in D_g$.

Definition 3.1.4. A function $x : [0, T] \rightarrow \mathbb{R}$ is said to be a *solution of the problem (3.1)*, if x is g -absolutely continuous on $[0, T]$, $x \in \mathfrak{B}$ and

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost all } t \in [0, T].$$

The concept of lower and upper solutions has been extended to Stieltjes differential equations. In the following definition, we introduce these tools which will serve as an a priori bounds of our solutions.

Definition 3.1.5. A function $\alpha \in \mathcal{AC}_g([0, T], \mathbb{R})$ is said to be a *lower solution of (3.1)*, if

- (i) $\alpha'_g(t) \leq f(t, \alpha(t))$ for g -almost every $t \in [0, T]$;
- (ii) $\alpha(0) \leq \alpha(T)$, if $\mathfrak{B}$ refers to the periodic boundary condition (3.2), or $\alpha(0) \leq x_0$, if $\mathfrak{B}$ refers to the initial value condition (3.3).

A function $\beta \in \mathcal{AC}_g([0, T], \mathbb{R})$ is said to be an *upper solution of (3.1)*, if it satisfies (i) and (ii) with the reversed inequalities.

3.1.3 Existence results

In this subsection, we present results ensuring the existence of solutions for Stieltjes differential equations with a periodic boundary condition (3.1), (3.2), as well as for those with an initial value condition (3.1), (3.3). In doing so, we will make use of the following assumptions:

- (H_f) $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a g -Carathéodory function.
- (L- $\mathfrak{U}_{\mathfrak{B}}$) There exists $\alpha, \beta \in \mathcal{AC}_g([a, b], \mathbb{R})$ respectively lower and upper solutions of (3.1) such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, T]$.

Remark 3.1.6. For the sake of accuracy in notations, we will replace the assumption (L- $\mathfrak{U}_{\mathfrak{B}}$) with (L- $\mathfrak{U}_{\mathcal{P}}$) if $\mathfrak{B}$ corresponds to the periodic boundary condition (3.2), and with (L- $\mathfrak{U}_{\mathcal{I}}$) when $\mathfrak{B}$ refers to the initial value condition (3.3).

Problems with a periodic boundary value condition

In this subsection, we consider the periodic problem (3.1), (3.2):

$$\begin{aligned} x'_g(t) &= f(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, T], \\ x(0) &= x(T). \end{aligned}$$

In the classical problem with $g(t) = \text{id}_{\mathbb{R}}$ and f being a Carathéodory function, the existence of lower and upper solutions is sufficient to ensure the existence of a solution (see, for instance [73]). However, when the problem exhibits discontinuities, this requires additional conditions at these discontinuity points in addition to (L- $\mathfrak{U}_{\mathfrak{B}}$). To this aim, we assume the additional condition:

$(\mathcal{D}_{\alpha,\beta}) \quad \alpha(t^+) \leq x + f(t, x)\mu_g(\{t\}) \leq \beta(t^+)$ for every $t \in D_g \cap [0, T)$ and $x \in [\alpha(t), \beta(t)]$.

The following theorem is the main result of this subsection. It demonstrates the existence of a solution for the problem (3.1), (3.2).

Theorem 3.1.7. *Let us assume that (H_f) , $(L-U_P)$ and $(\mathcal{D}_{\alpha,\beta})$ are satisfied. Then, the problem (3.1), (3.2) admits at least one solution $x \in \mathcal{AC}_g([0, T], \mathbb{R})$ such that $\alpha(t) \leq x(t) \leq \beta(t)$ for every $t \in [0, T]$.*

In order to demonstrate the previous theorem, let us first introduce some notations. Consider the family of problems defined for $\lambda \in [0, 1]$:

$$\begin{aligned} x'_g(t) &= \lambda \tilde{f}(t, x(t)) + \frac{k - \lambda}{g(T) - g(0)} \int_{[0, T)} \tilde{f}(s, x(s)) \, d\mu_g(s), \text{ for } g\text{-a.a. } t \in [0, T), \\ x(0) &= x(T), \end{aligned} \quad (3.5_\lambda)$$

where $k > 0$ is chosen such that

$$k + \lambda g(0) \neq 0 \quad \forall \lambda \in [0, 1], \quad (3.6)$$

and $\tilde{f} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\tilde{f}(t, x) = \begin{cases} f(t, \beta(t)) - w_\beta(t)(x - \beta(t)) & \text{if } x > \beta(t), \\ f(t, x) & \text{if } \alpha(t) \leq x \leq \beta(t), \\ f(t, \alpha(t)) + w_\alpha(t)(x - \alpha(t)) & \text{if } x < \alpha(t); \end{cases} \quad (3.7)$$

with $w_\alpha, w_\beta \in \mathcal{L}_g^1([0, T], \mathbb{R})$ functions chosen such that

$$w_\alpha(t) < \min\{0, f(t, \alpha(t))\} \quad \text{and} \quad w_\beta(t) > \max\{0, f(t, \beta(t))\}, \quad \text{if } t \in [0, T) \setminus D_g, \quad (3.8)$$

and, if $t \in [0, T) \cap D_g$,

$$\begin{aligned} w_\beta(t) &:= \max \left\{ 0, \min \left\{ f(t, \beta(t)), \frac{1}{\mu_g(\{t\})} \right\} \right\}, \\ w_\alpha(t) &:= \min \left\{ 0, \max \left\{ f(t, \alpha(t)), \frac{-1}{\mu_g(\{t\})} \right\} \right\}. \end{aligned} \quad (3.9)$$

Moreover, let us denote the sets

$$\begin{aligned} \mathcal{T}_\beta &= \left\{ t \in [0, T) \cap D_g : w_\beta(t) = \frac{1}{\mu_g(\{t\})} \right\}, \\ \mathcal{T}_\alpha &= \left\{ t \in [0, T) \cap D_g : w_\alpha(t) = \frac{-1}{\mu_g(\{t\})} \right\}. \end{aligned} \quad (3.10)$$

Remark 3.1.8. Observe that the sets $\mathcal{T}_\alpha, \mathcal{T}_\beta \subset D_g$ are finite. Indeed,

$$\text{card}(\mathcal{T}_\beta) = \int_{\mathcal{T}_\beta} \frac{1}{\mu_g(\{t\})} \, d\mu_g(t) \leq \int_{\mathcal{T}_\beta} f(t, \beta(t)) \, d\mu_g(t) \leq \int_{[0, T)} |f(t, \beta(t))| \, d\mu_g(t) < \infty.$$

Similarly, we deduce that $\text{card}(\mathcal{T}_\alpha) < \infty$.

Now, let us consider the operator $\mathcal{H} : [0, 1] \times \mathcal{BC}_g([0, T], \mathbb{R}) \rightarrow \mathcal{BC}_g([0, T], \mathbb{R})$ defined by

$$\mathcal{H}(\lambda, x)(t) = x(0) - \frac{k + \lambda g(t)}{g(T) - g(0)} N_{\tilde{f}}(x)(T) + \lambda N_{\tilde{f}}(x)(t), \quad \forall t \in [0, T]; \quad (3.11)$$

where $N_{\tilde{f}}$ stands for the operator defined in Lemma 3.1.3; associated to $\tilde{f}$.

First of all, let us establish the existence of a solution to (3.5 $_{\lambda}$).

Proposition 3.1.9. *Let us assume that the assumptions (H_f) , $(L-U_P)$ and $(\mathcal{D}_{\alpha, \beta})$ hold. Consider the operator $\mathcal{H}$ as in (3.11). Then, there exists $R > \max\{\|\alpha\|_0, \|\beta\|_0\}$ such that*

$$\text{index}(\mathcal{H}(\lambda, \cdot), \mathcal{U}) = -1 \quad \text{for all } \lambda \in [0, 1],$$

where $\mathcal{U} = \{x \in \mathcal{BC}_g([0, T], \mathbb{R}) : \|x\|_0 < R\}$.

In particular, the problem (3.5 $_{\lambda}$) has at least one solution for every $\lambda \in [0, 1]$.

Proof. Using Lemma 3.1.3, the operator $\mathcal{H}$ is continuous and completely continuous.

Now, let us fix $\lambda \in [0, 1]$. We claim that the fixed points of $\mathcal{H}(\lambda, \cdot)$ are solutions of the problem (3.5 $_{\lambda}$). Indeed, if $x \in \mathcal{BC}_g([0, T], \mathbb{R})$ such that $x = \mathcal{H}(\lambda, x)$, then, for all $t \in [0, T]$,

$$x(t) = x(0) + \lambda \int_{[0, t)} \tilde{f}(s, x(s)) \, d\mu_g(s) - \frac{k + \lambda g(t)}{g(T) - g(0)} \int_{[0, T)} \tilde{f}(s, x(s)) \, d\mu_g(s). \quad (3.12)$$

Taking the g -derivatives of both sides of (3.12) yields

$$x'_g(t) = \lambda \tilde{f}(t, x(t)) - \frac{\lambda}{g(T) - g(0)} N_{\tilde{f}}(x)(T). \quad (3.13)$$

Moreover, for $t = 0$ in (3.12), we obtain

$$x(0) = x(0) - \frac{k + \lambda g(0)}{g(T) - g(0)} N_{\tilde{f}}(x)(T).$$

By (3.6), $k + \lambda g(0) \neq 0$, which yields

$$N_{\tilde{f}}(x)(T) = 0. \quad (3.14)$$

On the other hand, taking $t = T$ in (3.12), we have

$$x(T) = x(0) + \lambda N_{\tilde{f}}(x)(T) - \frac{k + \lambda g(T)}{g(T) - g(0)} N_{\tilde{f}}(x)(T).$$

This equality combined with (3.14) yields

$$x(0) = x(T). \quad (3.15)$$

Also for all $t \in [0, T]$, it follows from (3.13) and (3.14) that

$$x'_g(t) = \lambda \tilde{f}(t, x(t)) = \lambda \tilde{f}(t, x(t)) + \frac{k - \lambda}{g(T) - g(0)} N_{\tilde{f}}(x)(T). \quad (3.16)$$

Hence, x is a solution of the family of problems (3.5 _{λ}).

Let $\tilde{R} > 0$ be such that

$$\tilde{R} \geq 1 + \sup_{t \in [0, T]} \{\max\{\beta(t), -\alpha(t)\}\}. \quad (3.17)$$

Now, by means of the comparison principle, Lemma 1.5.7, we aim to establish a priori bound of solutions of the family of problems (3.5 _{λ}). Initially, let us show that

$$\|x\|_0 \leq \tilde{R}, \text{ for any solution of (3.5}_{\lambda}). \quad (3.18)$$

For this purpose, we prove that $x'_g(t) \leq 0$ for g -almost all $t \in [0, T]$ such that $x(t^+) > \tilde{R}$, by considering the following two cases.

Case 1: For g -almost every $t \in [0, T] \setminus D_g$ such that $x(t^+) > \tilde{R}$, it follows from (L-U _{$\mathcal{P}$}), (3.7), (3.8) and (3.17), that $x(t) = x(t^+) > \tilde{R}$ and

$$\begin{aligned} x'_g(t) &= \lambda \left(f(t, \beta(t)) - w_\beta(t)(x(t) - \beta(t)) \right) \\ &\leq \lambda \left(f(t, \beta(t)) - w_\beta(t) \right) \\ &\leq 0. \end{aligned} \quad (3.19)$$

Case 2: For each $t \in [0, T] \cap D_g$ such that $x(t^+) > \tilde{R}$, we distinguish three subcases.

Subcase 1: If $x(t) \in [\alpha(t), \beta(t)]$, then by ($\mathcal{D}_{\alpha, \beta}$) and since β is an upper solution of (3.1), (3.2), we have

$$x'_g(t) = \lambda f(t, x(t)) \leq \lambda \left(\frac{\beta(t^+) - x(t)}{\mu_g(\{t\})} \right). \quad (3.20)$$

Subcase 2: If $x(t) > \beta(t)$, it follows from (L-U _{$\mathcal{P}$}), (3.7) and (3.9) that

$$\begin{aligned} x'_g(t) &= \lambda \left(f(t, \beta(t)) - w_\beta(t)(x(t) - \beta(t)) \right) \\ &= \begin{cases} \lambda f(t, \beta(t)) & \text{if } f(t, \beta(t)) \leq 0, \\ \lambda \left(f(t, \beta(t)) - f(t, \beta(t))(x(t) - \beta(t)) \right) & \text{if } f(t, \beta(t)) \in \left(0, \frac{1}{\mu_g(\{t\})} \right], \\ \lambda \left(f(t, \beta(t)) - \frac{1}{\mu_g(\{t\})}(x(t) - \beta(t)) \right) & \text{if } f(t, \beta(t)) > \frac{1}{\mu_g(\{t\})}, \end{cases} \\ &\leq \begin{cases} 0 & \text{if } f(t, \beta(t)) \leq 0, \\ \lambda \left(f(t, \beta(t))(1 + \beta(t) - x(t)) \right) & \text{if } f(t, \beta(t)) \in \left(0, \frac{1}{\mu_g(\{t\})} \right] \\ & \text{and } x(t) \geq 1 + \beta(t), \\ \lambda \left(\frac{\beta(t^+) - \beta(t)}{\mu_g(\{t\})} - f(t, \beta(t))(x(t) - \beta(t)) \right) & \text{if } f(t, \beta(t)) \in \left(0, \frac{1}{\mu_g(\{t\})} \right] \\ & \text{and } x(t) \in (\beta(t), 1 + \beta(t)), \\ \lambda \left(\frac{\beta(t^+) - x(t)}{\mu_g(\{t\})} \right) & \text{if } f(t, \beta(t)) > \frac{1}{\mu_g(\{t\})}, \end{cases} \end{aligned}$$

$$\leq \begin{cases} 0 & \text{if } f(t, \beta(t)) \leq 0, \\ 0 & \text{if } f(t, \beta(t)) \in \left(0, \frac{1}{\mu_g(\{t\})}\right] \\ & \text{and } x(t) \geq 1 + \beta(t), \\ \lambda \left(\frac{\beta(t^+) + 1 - x(t)}{\mu_g(\{t\})} \right) & \text{if } f(t, \beta(t)) \in \left(0, \frac{1}{\mu_g(\{t\})}\right] \\ & \text{and } x(t) \in (\beta(t), 1 + \beta(t)), \\ \lambda \left(\frac{\beta(t^+) - x(t)}{\mu_g(\{t\})} \right) & \text{if } f(t, \beta(t)) > \frac{1}{\mu_g(\{t\})}. \end{cases}$$

Therefore,

$$x'_g(t) \leq \max \left\{ 0, \frac{\lambda(\beta(t^+) + 1 - x(t))}{\mu_g(\{t\})} \right\}. \quad (3.21)$$

Subcase 3: If $x(t) < \alpha(t)$, then given (L-U_P), $(\mathcal{D}_{\alpha, \beta})$ and (3.7), we obtain

$$\begin{aligned} x'_g(t) &= \lambda \left(f(t, \alpha(t)) + w_\alpha(t)(x(t) - \alpha(t)) \right) \\ &= \begin{cases} \lambda f(t, \alpha(t)) & \text{if } f(t, \alpha(t)) \geq 0, \\ \lambda (f(t, \alpha(t)) + f(t, \alpha(t))(x(t) - \alpha(t))) & \text{if } f(t, \alpha(t)) \in \left[\frac{-1}{\mu_g(\{t\})}, 0 \right), \\ \lambda \left(f(t, \alpha(t)) - \frac{1}{\mu_g(\{t\})}(x(t) - \alpha(t)) \right) & \text{if } f(t, \alpha(t)) < \frac{-1}{\mu_g(\{t\})}, \end{cases} \\ &\leq \begin{cases} \lambda \left(\frac{\beta(t^+) - \alpha(t)}{\mu_g(\{t\})} \right) & \text{if } f(t, \alpha(t)) \geq 0, \\ \lambda \left(\frac{\beta(t^+) - \alpha(t)}{\mu_g(\{t\})} + f(t, \alpha(t))(x(t) - \alpha(t)) \right) & \text{if } f(t, \alpha(t)) \in \left[\frac{-1}{\mu_g(\{t\})}, 0 \right), \\ \lambda \left(\frac{\beta(t^+) - x(t)}{\mu_g(\{t\})} \right) & \text{if } f(t, \alpha(t)) < \frac{-1}{\mu_g(\{t\})}, \end{cases} \\ &\leq \lambda \left(\frac{\beta(t^+) - x(t)}{\mu_g(\{t\})} \right) \\ &\quad + \begin{cases} 0 & \text{if } f(t, \alpha(t)) \geq 0, \\ \lambda \left(\frac{1}{\mu_g(\{t\})} + f(t, \alpha(t)) \right) (x(t) - \alpha(t)) & \text{if } f(t, \alpha(t)) \in \left[\frac{-1}{\mu_g(\{t\})}, 0 \right), \\ 0 & \text{if } f(t, \alpha(t)) < \frac{-1}{\mu_g(\{t\})}. \end{cases} \end{aligned}$$

Consequently,

$$x'_g(t) \leq \lambda \left(\frac{\beta(t^+) - x(t)}{\mu_g(\{t\})} \right). \quad (3.22)$$

To sum up **Case 2**, it follows from (3.20), (3.21) and (3.22) that

$$x'_g(t) \leq \max \left\{ 0, \frac{\lambda(\beta(t^+) + 1 - x(t))}{\mu_g(\{t\})} \right\}.$$

Since $x(t^+) > \tilde{R}$, (3.17) implies that

$$x'_g(t) = \frac{x(t^+) - x(t)}{\mu_g(\{t\})} > \frac{\beta(t^+) + 1 - x(t)}{\mu_g(\{t\})},$$

From the two previous inequalities, we obtain

$$x'_g(t) \leq 0 \quad \text{for } g\text{-almost every } t \in [0, T) \cap D_g \text{ such that } x(t^+) > \tilde{R}. \quad (3.23)$$

Hence, combining (3.19) and (3.23) permit to deduce that

$$x'_g(t) \leq 0 \quad \text{for } g\text{-almost all } t \in [0, T) \text{ such that } x(t^+) > \tilde{R}. \quad (3.24)$$

It follows from the comparison principle, Lemma 1.5.7, that $x(t) \leq \tilde{R}$ for all $t \in [0, T]$ or there is $c > 0$ such that $x(t) = \tilde{R} + c$ for all $t \in [0, T]$.

Let us assume that last case holds, then for all $t \in [0, T]$, $x(t^+) = x(t) = \tilde{R} + c$. By (3.7), (3.8), and (3.9), we have

$$\begin{aligned} w_\beta(t) &> \beta'_g(t) \quad \text{for } g\text{-almost every } t \in [0, T) \setminus D_g, \\ w_\beta(t) &\geq \beta'_g(t) \quad \text{for every } t \in [0, T) \cap (D_g \setminus \mathcal{T}_\beta), \\ w_\beta(t) &= \frac{1}{\mu_g(\{t\})} \quad \text{for every } t \in \mathcal{T}_\beta. \end{aligned} \quad (3.25)$$

This inequality combined with (3.14), (3.7), (3.16), and (3.25) yields

$$\begin{aligned} 0 &= N_{\tilde{f}}(x)(T) = \int_{[0, T)} \tilde{f}(t, x(t)) \, d\mu_g(t) \\ &= \int_{[0, T) \setminus D_g} \tilde{f}(t, x(t)) \, d\mu_g(t) + \int_{\mathcal{T}_\beta} \tilde{f}(t, x(t)) \, d\mu_g(t) + \int_{[0, T) \cap (D_g \setminus \mathcal{T}_\beta)} \tilde{f}(t, x(t)) \, d\mu_g(t) \\ &= \int_{[0, T) \setminus D_g} f(t, \beta(t)) - w_\beta(t)(\tilde{R} + c - \beta(t)) \, d\mu_g(t) \\ &\quad + \int_{\mathcal{T}_\beta} f(t, \beta(t)) - w_\beta(t)(\tilde{R} + c - \beta(t)) \, d\mu_g(t) \\ &\quad + \int_{[0, T) \cap (D_g \setminus \mathcal{T}_\beta)} f(t, \beta(t)) - w_\beta(t)(\tilde{R} + c - \beta(t)) \, d\mu_g(t) \\ &\leq \int_{[0, T) \setminus D_g} f(t, \beta(t)) - w_\beta(t)(\tilde{R} + c - \beta(t)) \, d\mu_g(t) \\ &\quad + \int_{[0, T) \cap (D_g \setminus \mathcal{T}_\beta)} f(t, \beta(t)) - w_\beta(t)(\tilde{R} + c - \beta(t)) \, d\mu_g(t) \\ &\quad + \int_{\mathcal{T}_\beta} f(t, \beta(t)) - w_\beta(t)(\tilde{R} + c - \beta(t)) \, d\mu_g(t) \\ &< \int_{\mathcal{T}_\beta} f(t, \beta(t)) - w_\beta(t)(\tilde{R} + c - \beta(t)) \, d\mu_g(t) \\ &\leq \int_{\mathcal{T}_\beta} \beta'_g(t) - \frac{1}{\mu_g(\{t\})}(\tilde{R} + c - \beta(t)) \, d\mu_g(t) \\ &= \sum_{t \in \mathcal{T}_\beta} \left(\beta'_g(t) - \frac{\tilde{R} + c - \beta(t)}{\mu_g(\{t\})} \right) \mu_g(\{t\}) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{t \in \mathcal{T}_\beta} \frac{\beta(t^+) - \tilde{R} - c}{\mu_g(\{t\})} \mu_g(\{t\}) \\
 &\leq 0.
 \end{aligned}$$

This is a contradiction. Therefore, $x(t) \leq \tilde{R}$ for every $t \in [0, T]$.

Using an analogous argument, we can show that $x(t) \geq -\tilde{R}$ for every $t \in [0, T]$. Hence, we conclude that

$$\|x\|_0 \leq \tilde{R}, \text{ for any solution of (3.5}_\lambda\text{).}$$

Now, let us fix $R > \tilde{R}$ such that

$$\begin{aligned}
 \text{card}(\mathcal{T}_\beta)R &\geq \text{card}(\mathcal{T}_\beta)\beta(t) + \sum_{t \in \mathcal{T}_\beta} f(t, \beta(t))\mu_g(\{t\}) \\
 \text{card}(\mathcal{T}_\alpha)R &\geq -\text{card}(\mathcal{T}_\alpha)\alpha(t) - \sum_{t \in \mathcal{T}_\alpha} f(t, \alpha(t))\mu_g(\{t\}),
 \end{aligned} \tag{3.26}$$

where $\mathcal{T}_\beta$ and $\mathcal{T}_\alpha$ are the finite subsets of D_g , defined in (3.10).

We denote the open convex set of the Banach space $\mathcal{BC}_g([0, T], \mathbb{R})$:

$$\mathcal{U} = \{x \in \mathcal{BC}_g([0, T], \mathbb{R}) : \|x\|_0 < R\}. \tag{3.27}$$

It follows from (3.18) that $\mathcal{H}(\lambda, \cdot)$ has no fixed point on $\partial\mathcal{U}$ for every $\lambda \in [0, 1]$. Thus, using the homotopy property of the fixed point index in Theorem 1.8.4, we obtain

$$\text{index}(\mathcal{H}(\lambda, \cdot), \mathcal{U}) = \text{index}(\mathcal{H}(0, \cdot), \mathcal{U}), \quad \text{for every } \lambda \in [0, 1]. \tag{3.28}$$

Observe that, for every $x \in \mathcal{BC}_g([0, T], \mathbb{R})$,

$$\mathcal{H}(0, x)(t) = x(0) - \frac{k}{g(T) - g(0)} N_{\tilde{f}}(x)(T) \quad \text{for every } t \in [0, T].$$

Let us consider $X_{\mathbb{R}}$ the one dimensional subspace of $\mathcal{BC}_g([0, T], \mathbb{R})$ given by

$$X_{\mathbb{R}} = \{u \in \mathcal{BC}_g([0, T], \mathbb{R}) : u(t) = u(0) \text{ for every } t \in [0, T]\}.$$

Notice that $\mathcal{U} \cap X_{\mathbb{R}} = \{u \in \mathcal{BC}_g([0, T], \mathbb{R}) : u(t) = \text{cte with cte} \in [-R, R]\}$. Thus, $\partial(\mathcal{U} \cap X_{\mathbb{R}})$, the boundary of $\mathcal{U} \cap X_{\mathbb{R}}$ in $X_{\mathbb{R}}$ is $\{-R, R\}$, where $\pm R$ are the constant maps such that $\pm R(t) = \pm R$ for all $t \in [0, T]$. Therefore, one has

$$\begin{aligned}
 \mathcal{H}(0, R) &= R - \frac{k}{g(T) - g(0)} \int_{[0, T]} \tilde{f}(t, R) \, d\mu_g(t) \\
 &= R - \frac{k}{g(T) - g(0)} \int_{[0, T]} (f(t, \beta(t)) - w_\beta(t)(R - \beta(t))) \, d\mu_g(t) \\
 &= R - \frac{k}{g(T) - g(0)} \int_{[0, T] \setminus \mathcal{T}_\beta} (f(t, \beta(t)) - w_\beta(t)(R - \beta(t))) \, d\mu_g(t) \\
 &\quad - \frac{k}{g(T) - g(0)} \int_{\mathcal{T}_\beta} \left(f(t, \beta(t)) - \frac{1}{\mu_g(\{t\})} (R - \beta(t)) \right) \, d\mu_g(t),
 \end{aligned}$$

since $w_\beta(t) = 1/\mu_g(\{t\})$ for $t \in \mathcal{T}_\beta$ by (3.10). Moreover, (3.8) and (3.9) ensure that

$$\begin{aligned} w_\beta(t) &= \max \{0, f(t, \beta(t))\} \quad \text{on } [0, T) \cap (D_g \setminus \mathcal{T}_\beta), \\ w_\beta(t) &> \max \{0, f(t, \beta(t))\} \quad g\text{-almost everywhere in } [0, T) \setminus D_g. \end{aligned}$$

So, since $R > \tilde{R}$, by (3.17) and (3.26), one has

$$\begin{aligned} \mathcal{H}(0, R) &= R - \frac{k}{g(T) - g(0)} \int_{[0, T) \setminus \mathcal{T}_\beta} (f(t, \beta(t)) - w_\beta(t)(R - \beta(t))) \, d\mu_g(t) \\ &\quad - \frac{k}{g(T) - g(0)} \int_{\mathcal{T}_\beta} \left(f(t, \beta(t)) - \frac{1}{\mu_g(\{t\})} (R - \beta(t)) \right) \, d\mu_g(t) \\ &> R - \frac{k}{g(T) - g(0)} \int_{[0, T) \setminus \mathcal{T}_\beta} (f(t, \beta(t)) - w_\beta(t)) \, d\mu_g(t) \\ &\quad + \frac{k}{g(T) - g(0)} \left(\text{card}(\mathcal{T}_\beta) R - \int_{\mathcal{T}_\beta} \left(f(t, \beta(t)) + \frac{\beta(t)}{\mu_g(\{t\})} \right) \, d\mu_g(t) \right) \\ &= R - \frac{k}{g(T) - g(0)} \int_{[0, T) \setminus \mathcal{T}_\beta} (f(t, \beta(t)) - w_\beta(t)) \, d\mu_g(t) \\ &\quad + \frac{k}{g(T) - g(0)} \left(\text{card}(\mathcal{T}_\beta) R - \sum_{t \in \mathcal{T}_\beta} \left(f(t, \beta(t)) + \frac{\beta(t)}{\mu_g(\{t\})} \right) \mu_g(\{t\}) \right) \\ &= R - \frac{k}{g(T) - g(0)} \int_{[0, T) \setminus \mathcal{T}_\beta} (f(t, \beta(t)) - w_\beta(t)) \, d\mu_g \\ &\quad + \frac{k}{g(T) - g(0)} \left(\text{card}(\mathcal{T}_\beta) R - \beta(t) \text{card}(\mathcal{T}_\beta) - \sum_{t \in \mathcal{T}_\beta} f(t, \beta(t)) \mu_g(\{t\}) \right) \\ &> R. \end{aligned}$$

Similarly, we obtain that $\mathcal{H}(0, -R) < -R$.

Using the contraction property of the fixed point index in Theorem 1.8.7, and Theorem 1.8.11, we obtain

$$\begin{aligned} \text{index}(\mathcal{H}(0, \cdot), \mathcal{U}) &= \text{index}(\mathcal{H}(0, \cdot)|_{X_{\mathbb{R}}}, (\mathcal{U} \cap X_{\mathbb{R}})) \\ &= \frac{1}{2} \text{sgn}(R - \mathcal{H}(0, R)) - \frac{1}{2} \text{sgn}(-R - \mathcal{H}(0, -R)) \\ &= -1. \end{aligned}$$

Combining this with (3.28), we deduce that

$$\text{index}(\mathcal{H}(\lambda, \cdot), \mathcal{U}) = -1, \quad \text{for every } \lambda \in [0, 1].$$

Therefore, for every $\lambda \in [0, 1]$, the operator $\mathcal{H}(\lambda, \cdot)$ has a fixed point. Hence (3.5 _{λ}) has a solution. $\square$

Now, let us demonstrate the main result of this subsection.

Proof of Theorem 3.1.7. Proposition 3.1.9 ensures that the family of problems (3.5_λ) has a solution for each $\lambda \in [0, 1]$. Let x be a solution of (3.5_λ) for $\lambda = 1$. Thus, x satisfies

$$\begin{aligned} x'_g(t) &= \tilde{f}(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, T], \\ x(0) &= x(T). \end{aligned}$$

Given (3.7), it remains to prove that $\alpha(t) \leq x(t) \leq \beta(t)$ for every $t \in [0, T]$. In doing so, we apply again the comparison principle, Lemma 1.5.7.

Observe that

$$\begin{aligned} x'_g(t) &= f(t, \beta(t)) - w_\beta(t)(x(t) - \beta(t)) \leq f(t, \beta(t)) \leq \beta'_g(t) \\ g\text{-a.e. in } \{t \in [0, T] \setminus D_g : \beta(t) < x(t)\} &= \{t \in [0, T] \setminus D_g : \beta(t^+) < x(t^+)\}. \end{aligned}$$

On the other hand, for each $t \in [0, T) \cap D_g$ such that $x(t^+) > \beta(t^+)$, one has

$$x'_g(t) = f(t, \beta(t)) - w_\beta(t)(x(t) - \beta(t)) < f(t, \beta(t)) \leq \beta'_g(t) \quad \text{if } x(t) > \beta(t);$$

while if $x(t) \leq \beta(t)$, we obtain using $(\mathcal{D}_{\alpha, \beta})$

$$\begin{aligned} x'_g(t) &= \begin{cases} f(t, x(t)) & \text{if } \alpha(t) \leq x(t), \\ f(t, \alpha(t)) + w_\alpha(t)(x(t) - \alpha(t)) & \text{if } x(t) < \alpha(t), \end{cases} \\ &= \begin{cases} f(t, x(t)) & \text{if } \alpha(t) \leq x(t), \\ f(t, \alpha(t)) & \text{if } x(t) < \alpha(t) \text{ and } f(t, \alpha(t)) \geq 0, \\ f(t, \alpha(t)) + f(t, \alpha(t))(x(t) - \alpha(t)) & \text{if } x(t) < \alpha(t) \\ & \text{and } f(t, \alpha(t)) \in \left[\frac{-1}{\mu_g(\{t\})}, 0 \right), \\ f(t, \alpha(t)) - \frac{1}{\mu_g(\{t\})}(x(t) - \alpha(t)) & \text{if } x(t) < \alpha(t) \\ & \text{and } f(t, \alpha(t)) < \frac{-1}{\mu_g(\{t\})}, \end{cases} \\ &\leq \frac{\beta(t^+) - x(t)}{\mu_g(\{t\})} \\ &\quad + \begin{cases} 0 & \text{if } \alpha(t) \leq x(t), \\ \frac{x(t) - \alpha(t)}{\mu_g(\{t\})} & \text{if } x(t) < \alpha(t) \text{ and } f(t, \alpha(t)) \geq 0, \\ \left(\frac{1}{\mu_g(\{t\})} + f(t, \alpha(t)) \right)(x(t) - \alpha(t)) & \text{if } x(t) < \alpha(t) \\ & \text{and } f(t, \alpha(t)) \in \left[\frac{-1}{\mu_g(\{t\})}, 0 \right), \\ 0 & \text{if } x(t) < \alpha(t) \\ & \text{and } f(t, \alpha(t)) < \frac{-1}{\mu_g(\{t\})}, \end{cases} \\ &\leq \frac{\beta(t^+) - x(t)}{\mu_g(\{t\})}. \end{aligned}$$

This is a contradiction, since,

$$x'_g(t) = \frac{x(t^+) - x(t)}{\mu_g(\{t\})} > \frac{\beta(t^+) - x(t)}{\mu_g(\{t\})}.$$

Thus,

$$x'_g(t) \leq \beta'_g(t) \quad g\text{-almost everywhere in } \{t \in [0, T) : x(t^+) > \beta(t^+)\}.$$

By Lemma 1.5.7, it follows that either $x(t) \leq \beta(t)$ for every $t \in [0, T]$ or there is a constant $c > 0$ such that $x(t) = \beta(t) + c$ for all $t \in [0, T]$. If we assume that the last case holds, then by (3.7), (3.8), (3.9), and (L- U_P), we obtain

$$\begin{aligned} \int_{[0, T)} \beta'_g(t) \, d\mu_g(t) &= \int_{[0, T)} x'_g(t) \, d\mu_g(t) = \int_{[0, T)} \tilde{f}(t, \mathbf{x}(t)) \, d\mu_g(t) \\ &= \int_{[0, T)} f(t, \beta(t)) - w_\beta(t)(x(t) - \beta(t)) \, d\mu_g(t) \\ &\leq \int_{[0, T)} \beta'_g(t) - cw_\beta(t) \, d\mu_g(t) \\ &< \int_{[0, T)} \beta'_g(t) \, d\mu_g(t), \end{aligned}$$

which is a contraction.

Hence, we deduce that $x(t) \leq \beta(t)$ for all $t \in [0, T]$. Similarly, we obtain that $\alpha(t) \leq x(t)$ for all $t \in [0, T]$.

Finally, from the two previous inequalities, (3.7) and (3.14), and since x is a solution of (3.5 $_\lambda$) for $\lambda = 1$, one has that $x(0) = x(T)$ and

$$x'_g(t) = \lambda \tilde{f}(t, x(t)) + \frac{k-1}{g(T) - g(0)} N_{\tilde{f}}(x)(T) = f(t, x(t)) \text{ for } g\text{-almost every } t \in [0, T).$$

Thus, x is a solution of (3.1), (3.2). □

Example 3.1.10. Consider the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(t) = t$ for $t \leq 1$, and

$$g(t) = t + n \quad \text{for all } t \in (n, n+1], \text{ and } n \in \mathbb{N} = \{1, 2, \dots\}.$$

The function g is left-continuous and nondecreasing. In addition, $D_g = \mathbb{N}$ and $\mu_g(\{t\}) = 1$ for each $t \in D_g$.

Let $T > 1$ be fixed, and consider the periodic problem

$$\begin{aligned} x'_g(t) &= f(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, T), \\ x(0) &= x(T), \end{aligned} \tag{3.29}$$

where $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(t, x) = \begin{cases} \frac{1}{2}x^2 - 2x + \frac{1}{2} & \text{if } t \in [0, T) \cap D_g, \\ -tx^3 - x + 1 & \text{otherwise.} \end{cases} \tag{3.30}$$

f is a g -Carathéodory function. In addition, the constant maps $\alpha = 0$ and $\beta = 1$ are respectively lower and upper solutions of (3.29). Indeed,

$$\begin{aligned}\alpha'_g(t) = 0 &\leq f(t, \alpha(t)) = \begin{cases} \frac{1}{2} & \text{if } t \in [0, T) \cap D_g, \\ 1 & \text{otherwise,} \end{cases} \\ \beta'_g(t) = 0 &\geq f(t, \beta(t)) = \begin{cases} -1 & \text{if } t \in [0, T) \cap D_g, \\ -t & \text{otherwise.} \end{cases}\end{aligned}$$

In addition, the condition $(\mathcal{D}_{\alpha, \beta})$ is satisfied since

$$\alpha(t^+) = 0 \leq h_t(x) \leq 1 = \beta(t^+) \text{ for each } t \in D_g \cap [0, T) \text{ and } x \in [\alpha(t), \beta(t)],$$

where

$$h_t(x) = x + f(t, x)\mu_g(\{t\}) = \frac{1}{2}x^2 - x + \frac{1}{2}.$$

Indeed, h_t is decreasing on $[0, 1]$, thus $h_t([0, 1]) = [0, 1/2]$.

Finally, it follows from Theorem 3.1.7 that the problem (3.29) has at least one solution $x : [0, T] \rightarrow \mathbb{R}$ such that $0 \leq x(t) \leq 1$ for every $t \in [0, T]$. Figure 3.1.1 illustrates a periodic solution of the problem (3.29) on the interval $[0, T] = [0, 4]$.

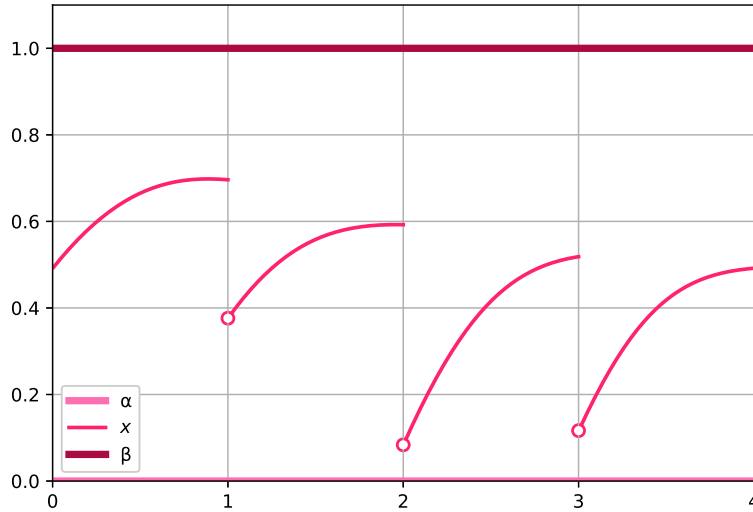


Figure 3.1.1 – A periodic solution of the problem (3.29) for $T = 4$.

Problems with an initial value condition

In this subsection, we provide existence of a solution to (3.1), (3.3), establishing an existence result analogous to the one obtained for the periodic problem in the previous subsection, assuming (H_f) , $(L-U_I)$ and $(\mathcal{D}_{\alpha, \beta})$. To this aim, we consider the family of problems defined for $\lambda \in [0, 1]$:

$$\begin{aligned}x'_g(t) &= \lambda \tilde{f}(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, T), \\ x(0) &= x_0 \in \mathbb{R},\end{aligned}\tag{3.31}_\lambda$$

where $\tilde{f}$ is the function defined in (3.7). Let $\mathcal{F} : [0, 1] \times \mathcal{BC}_g([0, T], \mathbb{R}) \rightarrow \mathcal{BC}_g([0, T], \mathbb{R})$ be the operator defined by

$$\mathcal{F}(\lambda, x)(t) = x_0 + \lambda N_{\tilde{f}}(x)(t), \quad \text{for every } t \in [0, T], \quad (3.32)$$

where $N_{\tilde{f}}$ is the operator defined in Lemma 3.1.3 and associated to $\tilde{f}$.

Proposition 3.1.11. *Let us assume that (H_f) , $(L-U_{\mathcal{I}})$ and $(\mathcal{D}_{\alpha, \beta})$ hold. Then, there exists $R > \max\{\|\alpha\|_0, \|\beta\|_0\}$ such that*

$$\text{index}(\mathcal{F}(\lambda, \cdot), \mathcal{U}) = 1 \quad \text{for every } \lambda \in [0, 1],$$

where $\mathcal{U} = \{x \in \mathcal{BC}_g([0, T], \mathbb{R}) : \|x\|_0 < R\}$. In particular, the problem (3.31) $_{\lambda}$ has at least one solution for every $\lambda \in [0, 1]$.

Proof. Arguing as in the proof of Proposition 3.1.9, and using Lemma 3.1.3 yields that the operator $\mathcal{F}$ is continuous and completely continuous. Moreover, for every $\lambda \in [0, 1]$, the fixed points of $\mathcal{F}(\lambda, \cdot)$ are solutions of (3.31) $_{\lambda}$.

We fix $R > 0$ such that

$$R > 1 + \sup_{t \in [0, T]} \{\max\{\beta(t), -\alpha(t)\}\}, \quad (3.33)$$

and we denote

$$\mathcal{U} = \{x \in \mathcal{BC}_g([0, T], \mathbb{R}) : \|x\|_0 < R\}. \quad (3.34)$$

Lemma 1.5.7 and an argument analogous to the proof of Proposition 3.1.9 imply that $\mathcal{F}(\lambda, \cdot)$ has no fixed point on $\partial\mathcal{U}$ for all $\lambda \in [0, 1]$.

Observe that $\mathcal{F}(0, x) = x_0 \in \mathcal{U}$ for all $x \in \mathcal{U}$. Thus, the normalization property of the fixed point index from Theorem 1.8.4 yields

$$\text{index}(\mathcal{F}(0, \cdot), \mathcal{U}) = 1. \quad (3.35)$$

The homotopy property of the fixed point index from Theorem 1.8.4 ensures that

$$\text{index}(\mathcal{F}(\lambda, \cdot), \mathcal{U}) = \text{index}(\mathcal{F}(0, \cdot), \mathcal{U}), \quad \text{for every } \lambda \in [0, 1]. \quad (3.36)$$

This combined with (3.35), we obtain that

$$\text{index}(\mathcal{F}(\lambda, \cdot), \mathcal{U}) = 1, \quad \text{for every } \lambda \in [0, 1].$$

Therefore, for every $\lambda \in [0, 1]$ the operator $\mathcal{F}(\lambda, \cdot)$ has a fixed point in $\mathcal{U}$. Hence (3.31) $_{\lambda}$ has a solution. $\square$

Arguing similarly to the proof of Theorem 3.1.7, we derive an analogous existence result for the initial value problem (3.1), (3.3).

Theorem 3.1.12. *Assume that (H_f) , $(L-U_{\mathcal{I}})$ and $(\mathcal{D}_{\alpha, \beta})$ are satisfied. Then, the initial value problem (3.1), (3.3) has at least one solution $x \in \mathcal{AC}_g([0, T], \mathbb{R})$ such that $\alpha(t) \leq x(t) \leq \beta(t)$ for every $t \in [0, T]$.*

Remark 3.1.13. Let us consider the problem in Example 3.1.10, with an initial value instead:

$$\begin{aligned} x'_g(t) &= f(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, T], \\ x(0) &= 0, \end{aligned} \tag{3.37}$$

where $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is the function defined in (3.30). It is immediate to show that the functions $\alpha = 0$ and $\beta = 1$ are respectively lower and upper solutions of the problem (3.37). Also, $(\mathcal{D}_{\alpha, \beta})$ is satisfied. Therefore, using Theorem 3.1.12, we deduce that the problem (3.37) admits a solution x such that $x(t) \in [0, 1]$ for every $t \in [0, T]$. It is worthwhile to mention that Theorem 3.1.12 ensured the existence of a solution for (3.37) even though monotonicity condition introduced in [69],

(C_f) for every $t \in [0, T] \cap D_g$, the map

$$x \in [\alpha(t), \beta(t)] \mapsto x + f(t, x)\mu_g(\{t\}) \text{ is nondecreasing,}$$

does not hold, as mentioned in Example 3.1.10. This is an example of a case where [54, Theorem 2.5] is inapplicable.

3.2 Multiplicity results

This section is devoted to multiplicity results for the problem (3.1). First, let us introduce notions of Strict lower and upper solutions which are our main tool.

3.2.1 Notions of strict lower and upper solutions

Definition 3.2.1. (1) Let $\alpha \in \mathcal{AC}_g([0, T], \mathbb{R})$. The function α is said to be a *strict lower solution* of (3.1), if,

- (i) for any $t_0 \in (0, T] \setminus C_g$, there exist $\epsilon > 0$ and $\mathcal{N}_{t_0}$ a g -neighborhood of t_0 such that

$$\alpha'_g(t) \leq f(t, x) \quad \text{for } g\text{-almost every } t \in \mathcal{N}_{t_0}, \forall x \in [\alpha(t), \alpha(t) + \epsilon];$$

- (ii) $\alpha(0) < \alpha(T)$, if $\mathfrak{B}$ corresponds to the periodic boundary condition (3.2), or $\alpha(0) < x_0$, if $\mathfrak{B}$ corresponds to the initial value condition (3.3).

(2) Let $\beta \in \mathcal{AC}_g([0, T], \mathbb{R})$. The function β is said to be a *strict upper solution* of (3.1), if,

- (i) for any $t_0 \in (0, T] \setminus C_g$, there exist $\epsilon > 0$ and $\mathcal{N}_{t_0}$ a g -neighborhood of t_0 such that

$$\beta'_g(t) \geq f(t, x) \quad \text{for } g\text{-almost every } t \in \mathcal{N}_{t_0}, \forall x \in [\beta(t) - \epsilon, \beta(t)];$$

- (ii) $\beta(0) > \beta(T)$, if $\mathfrak{B}$ refers to the periodic boundary condition (3.2), or $\beta(0) > x_0$, if $\mathfrak{B}$ refers to the initial value condition (3.3).

3.2.2 Multiplicity of solutions to the periodic boundary problem

In this subsection, we will establish a multiplicity result for (3.1), (3.2) based on the existence of strict lower and upper solutions. Specifically, we will search for solutions strictly lying within some regions defined by pairs of well-ordered strict lower and upper solutions (α, β) . With this intention, the assumption $(\mathcal{D}_{\alpha, \beta})$ should be replaced by a more stringent condition:

$$(\mathcal{D}_{\alpha, \beta}^s) \quad \alpha(t^+) < x + f(t, x)\mu_g(\{t\}) < \beta(t^+) \quad \text{for every } t \in [0, T) \cap D_g \text{ and } x \in [\alpha(t), \beta(t)].$$

The following lemma provides conditions that guarantee that the solution of (3.1) lies strictly between α and β .

Lemma 3.2.2. *Let us assume that (H_f) is satisfied. Let $\alpha, \beta \in \mathcal{AC}_g([0, T], \mathbb{R})$ be respectively strict lower and upper solutions of (3.1) satisfying $\alpha(t) < \beta(t)$ for all $t \in [0, T]$. Let us assume also that $(\mathcal{D}_{\alpha, \beta}^s)$ is fulfilled, and $x \in \mathcal{AC}_g([0, T], \mathbb{R})$ is a solution of (3.1) such that $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in [0, T]$. Then there is $\epsilon > 0$ such that*

$$\alpha(t) + \epsilon \leq x(t) \leq \beta(t) - \epsilon \quad \text{for every } t \in [0, T].$$

Proof. First, observe that if $x(t^+) = \alpha(t^+)$ for some $t \in [0, T) \cap D_g$, then, by using $(\mathcal{D}_{\alpha, \beta}^s)$, we obtain

$$\alpha(t^+) = x(t^+) = x(t) + x'_g(t)\mu_g(\{t\}) = x(t) + f(t, x(t))\mu_g(\{t\}) > \alpha(t^+).$$

This is a contradiction. Consequently,

$$x(t^+) > \alpha(t^+) \quad \text{for every } t \in [0, T) \cap D_g. \quad (3.38)$$

Assume that

$$A = \{t \in [0, T] : \alpha(t) = x(t)\} \neq \emptyset.$$

We claim that

$$0 \notin A. \quad (3.39)$$

Indeed, by Definition 3.2.1, if $\mathfrak{B}$ denotes (3.3), then (3.39) is immediate. Otherwise, if $\mathfrak{B}$ denotes (3.2), then

$$x(0) - \alpha(0) > x(T) - \alpha(T) \geq 0. \quad (3.40)$$

Let $t_1 = \inf A \in [0, T]$. It follows from (3.38) and (3.40) that $t_1 \in A \cap (0, T]$. Thus,

$$x(t_1) = \alpha(t_1) \text{ and } x(t) > \alpha(t) \quad \text{for every } t \in [0, t_1). \quad (3.41)$$

By the definition of strict lower solution, there exist $\epsilon > 0$ and $\mathcal{N}_{t_1}$ a g -neighborhood of t_1 such that

$$\alpha'_g(t) \leq f(t, y) \quad \text{for } g\text{-almost all } t \in \mathcal{N}_{t_1}, \text{ and all } y \in [\alpha(t), \alpha(t) + \epsilon].$$

Since x is left-continuous, there is some $t_0 \in (0, t_1)$ such that $x(t) \in (\alpha(t), \alpha(t) + \epsilon]$ for every $t \in [t_0, t_1)$. Hence,

$$\alpha(t_1) - \alpha(t_0) > x(t_1) - x(t_0) = \int_{[t_0, t_1)} x'_g(t) \, d\mu_g(t)$$

$$\begin{aligned}
 &= \int_{[t_0, t_1)} f(t, x(t)) \, d\mu_g(t) \\
 &\geq \int_{[t_0, t_1)} \alpha'_g(t) \, d\mu_g(t) = \alpha(t_1) - \alpha(t_0),
 \end{aligned}$$

which is a contradiction. Therefore, $A = \emptyset$ and hence $\alpha(t) < x(t)$ for every $t \in [0, T]$.

The last inequality combined with (3.38) permit to deduce that there is $\epsilon > 0$ such that $x(t) \geq \alpha(t) + \epsilon$ for every $t \in [0, T]$.

Similarly, we deduce that $x(t) \leq \beta(t) - \epsilon$ for every $t \in [0, T]$. $\square$

Theorem 3.2.3. *Assume that (H_f) holds. Assume also there exist α_1, α_2 and $\beta_1, \beta_2 \in \mathcal{AC}_g([0, T], \mathbb{R})$ respectively strict lower and upper solutions of (3.1), (3.2) such that*

- (i) $\alpha_1(t) < \beta_1(t) \leq \beta_2(t)$ and $\alpha_1(t) \leq \alpha_2(t) < \beta_2(t)$ for all $t \in [0, T]$;
- (ii) the set $\{t \in [0, T] : \beta_1(t) < \alpha_2(t)\} \neq \emptyset$;
- (iii) $(\mathcal{D}_{\alpha_1, \beta_2}^s)$, $(\mathcal{D}_{\alpha_1, \beta_1}^s)$ and $(\mathcal{D}_{\alpha_2, \beta_2}^s)$ are satisfied.

Then the periodic boundary value problem (3.1), (3.2) has at least three solutions $x_0, x_1, x_2 \in \mathcal{AC}_g([0, T], \mathbb{R})$ such that

$$\text{for } i = 1, 2, \quad \alpha_i(t) < x_i(t) < \beta_i(t) \text{ for every } t \in [0, T],$$

and

$$\alpha_1(t) < x_0(t) < \beta_2(t) \text{ for every } t \in [0, T] \text{ and } \{t \in [0, T] : \beta_1(t) \leq x_0(t) \leq \alpha_2(t)\} \neq \emptyset.$$

Proof. Let $\mathcal{H}_0, \mathcal{H}_1$ and $\mathcal{H}_2$ be the operators defined in (3.11) and associated to the pairs of lower and upper solutions (α_1, β_2) , (α_1, β_1) and (α_2, β_2) respectively. We consider the open sets of $\mathcal{BC}_g([0, T], \mathbb{R})$, $\mathcal{U}_0, \mathcal{U}_1$ and $\mathcal{U}_2$, defined as in (3.27) and associated to $\mathcal{H}_0, \mathcal{H}_1$ and $\mathcal{H}_2$ respectively. It follows from Proposition 3.1.9 that for $\lambda = 1$, we have

$$\text{index}(\mathcal{H}_i(1, \cdot), \mathcal{U}_i) = -1, \quad \text{for } i = 0, 1, 2.$$

Arguing as in the proof of Theorem 3.1.7, every fixed point x of $\mathcal{H}_i(1, \cdot)$ is a solution of the problem (3.1), (3.2) such that, for all $t \in [0, T]$,

$$\alpha_1(t) \leq x(t) \leq \beta_2(t) \text{ if } i = 0, \text{ and } \alpha_i(t) \leq x(t) \leq \beta_i(t) \text{ if } i = 1, 2. \quad (3.42)$$

Given the assumptions, and since α_1, α_2 and β_1, β_2 are respectively strict lower and upper solutions of the problem (3.1), (3.2), Lemma 3.2.2 and (3.42) imply that, for every $x = \mathcal{H}_i(1, x)$,

$$\begin{aligned}
 \exists \epsilon > 0 \quad \text{such that} \quad &x(t) \in [\alpha_1(t) + \epsilon, \beta_2(t) - \epsilon] \text{ if } i = 0, \\
 \text{and} \quad &x(t) \in [\alpha_i(t) + \epsilon, \beta_i(t) - \epsilon] \text{ if } i = 1, 2.
 \end{aligned}$$

Thus, for $i = 0, 1, 2$, $\mathcal{H}_i(1, \cdot)$ has no fixed points in $\mathcal{U}_i \setminus \overline{\mathcal{V}_i}$, where $\mathcal{V}_i$ are the open sets given by

$$\mathcal{V}_0 = \left\{ x \in \mathcal{BC}_g([0, T], \mathbb{R}) : \exists \epsilon > 0 \text{ such that } \alpha_1(t) + \epsilon \leq x(t) \leq \beta_2(t) - \epsilon \quad \forall t \in [0, T] \right\},$$

and, for $i = 1, 2$,

$$\mathcal{V}_i = \left\{ x \in \mathcal{BC}_g([0, T], \mathbb{R}) : \exists \epsilon > 0 \text{ such that } \alpha_i(t) + \epsilon \leq x(t) \leq \beta_i(t) - \epsilon \quad \forall t \in [0, T] \right\}.$$

By means of the excision property of the fixed point index from Theorem 1.8.4, we obtain

$$\text{index}(\mathcal{H}_i(1, \cdot), \mathcal{V}_i) = -1, \quad \text{for } i = 0, 1, 2. \quad (3.43)$$

Using Conditions (i) and (ii), we deduce that

$$\mathcal{V}_1 \cup \mathcal{V}_2 \subset \mathcal{V}_0 \text{ and } \mathcal{V}_0 \setminus (\overline{\mathcal{V}_1 \cup \mathcal{V}_2}) \neq \emptyset.$$

Therefore,

$$\mathcal{H}_i(1, x) = \mathcal{H}_0(1, x), \quad \forall x \in \mathcal{V}_i, i = 1, 2.$$

Combining this with (3.43), and using the additivity property of the fixed point index from Theorem 1.8.4, we obtain

$$\begin{aligned} & \text{index} \left(\mathcal{H}_0(1, \cdot), \mathcal{V}_0 \setminus (\overline{\mathcal{V}_1 \cup \mathcal{V}_2}) \right) \\ &= \text{index}(\mathcal{H}_0(1, \cdot), \mathcal{V}_0) - \text{index}(\mathcal{H}_0(1, \cdot), \mathcal{V}_1) - \text{index}(\mathcal{H}_0(1, \cdot), \mathcal{V}_2) \\ &= 1. \end{aligned}$$

Hence, the problem (3.1), (3.2) has at least three solutions x_0, x_1, x_2 such that

$$x_0 \in \mathcal{V}_0 \setminus (\overline{\mathcal{V}_1 \cup \mathcal{V}_2}), \quad x_1 \in \mathcal{V}_1, \quad x_2 \in \mathcal{V}_2.$$

□

In Figure 3.2.1, we illustrate the positioning of the multiple periodic solutions ensured by Theorem 3.2.3.

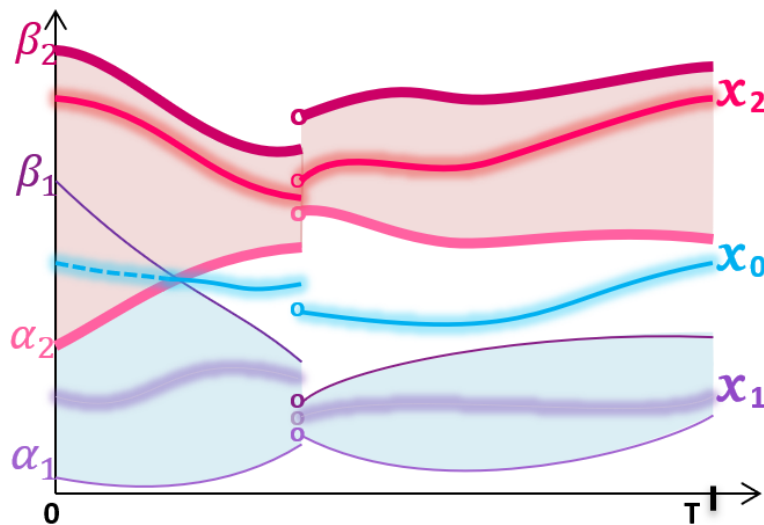


Figure 3.2.1 – Illustration of the positioning of multiple periodic solutions.

Example 3.2.4. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined for $t \leq 1$ by $g(t) = t$, and $g(t) = t + 1$ for all $t > 1$. Let $T > 1$ be fixed and consider the periodic problem:

$$\begin{aligned} x'_g(t) &= f(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, T), \\ x(0) &= x(T), \end{aligned} \tag{3.44}$$

where the function $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(t, x) = \begin{cases} -\frac{1}{2}x^3 - x^2 + \frac{1}{4} & \text{if } t = 1, \\ \frac{1}{3} \cos(3x + 7) & \text{otherwise.} \end{cases}$$

We will show that the problem (3.44) has at least three solutions.

Observe that f is g -Carathéodory. Now, let us define the functions $\alpha_1, \alpha_2, \beta_1, \beta_2 : [0, T] \rightarrow \mathbb{R}$ by

$$\alpha_1(t) = \begin{cases} -2 & \text{if } t \in [0, 1], \\ -2 + \frac{1}{10} & \text{otherwise,} \end{cases} \quad \beta_1(t) = \begin{cases} -1 & \text{if } t \in [0, 1], \\ -1 - \frac{1}{10} & \text{otherwise,} \end{cases}$$

and

$$\alpha_2(t) = \begin{cases} 0 & \text{if } t \in [0, 1], \\ \frac{1}{10} & \text{otherwise,} \end{cases} \quad \beta_2(t) = \begin{cases} \frac{4}{5} & \text{if } t \in [0, 1], \\ \frac{3}{5} & \text{otherwise.} \end{cases}$$

Let us show that α_1, α_2 , and β_1, β_2 are respectively strict lower and upper solutions of (3.44) with $\epsilon = 1/100$.

For each $t \in [0, T) \cap D_g = \{1\}$ and $x \in \mathbb{R}$,

$$\frac{\partial f}{\partial x}(1, x) = -\frac{3}{2}x^2 - 2x = -x \left(\frac{3}{2}x + 2 \right).$$

Thus, $f(1, \cdot)$ is increasing on $[-\frac{4}{3}, 0]$ and decreasing on $(-\infty, -\frac{4}{3}] \cup [0, +\infty)$.

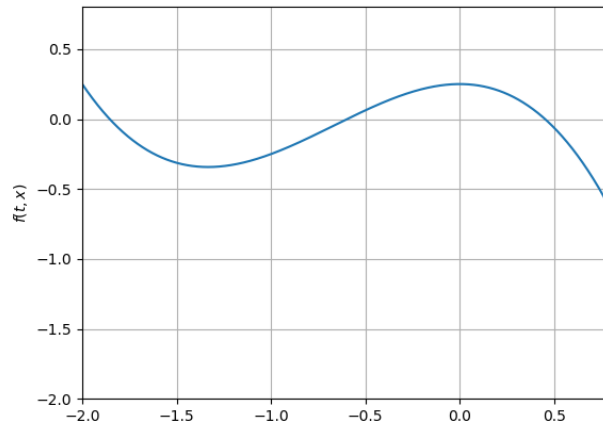


Figure 3.2.2 – $f(1, \cdot)$ on $[-2, 4/5]$.

Therefore,

$$\begin{aligned} f(1, x) &\geq f(1, \alpha_i(1) + \epsilon) & \forall x \in [\alpha_i(1), \alpha_i(1) + \epsilon], & \text{ for } i = 1, 2, \\ f(1, x) &\leq f(1, \beta_1(1)) & \forall x \in [\beta_1(1) - \epsilon, \beta_1(1)], \\ f(1, x) &\leq f(1, \beta_2(1) - \epsilon) & \forall x \in [\beta_2(1) - \epsilon, \beta_2(1)]. \end{aligned}$$

By simple computations, one can obtain

$$\begin{aligned} f(1, \alpha_i(1) + \epsilon) &\geq \alpha'_{ig}(1) = \frac{1}{10}, \quad \text{for } i = 1, 2, \\ f(1, \beta_1(1)) &\leq \beta'_{1g}(1) = -\frac{1}{10}, \\ f(1, \beta_2(1) - \epsilon) &\leq \beta'_{2g}(1) = -\frac{1}{5}. \end{aligned}$$

For every $t \in [0, T] \setminus D_g$ and $x \in \mathbb{R}$, we have

$$\frac{\partial f}{\partial x}(t, x) = -\sin(3x + 7).$$

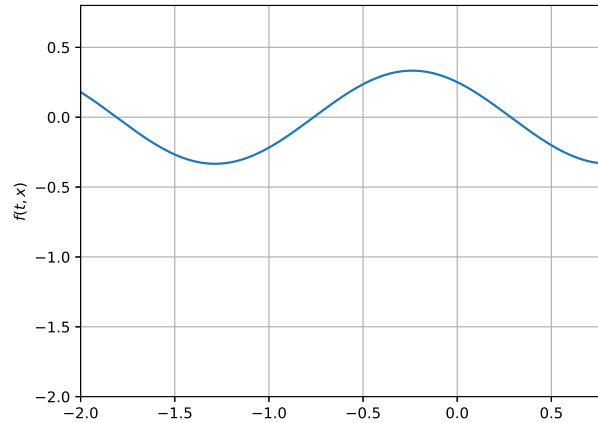


Figure 3.2.3 – $f(t, \cdot)$ on $\left[-2, \frac{4}{5}\right]$ for $t \in [0, T] \setminus D_g$.

Since,

$$3x + 7 \in \begin{cases} (0, \pi) & \text{for } x \in [\alpha_1(t), \alpha_1(t) + \epsilon], \\ (2\pi, 3\pi) & \text{for } x \in [\alpha_2(t), \alpha_2(t) + \epsilon] \cup [\beta_2(t) - \epsilon, \beta_2(t)], \\ (\pi, 2\pi) & \text{for } x \in [\beta_1(t) - \epsilon, \beta_1(t)], \end{cases}$$

$f(t, \cdot)$ is decreasing on $[\alpha_1(t), \alpha_1(t) + \epsilon] \cup [\alpha_2(t), \alpha_2(t) + \epsilon] \cup [\beta_2(t) - \epsilon, \beta_2(t)]$, and increasing on $[\beta_1(t) - \epsilon, \beta_1(t)]$. Thus,

$$f(t, x) \geq f(t, \alpha_i(t) + \epsilon) \quad \forall x \in [\alpha_i(t), \alpha_i(t) + \epsilon], \quad \text{for } i = 1, 2,$$

$$\begin{aligned} f(t, x) &\leq f(t, \beta_1(t)) & \forall x \in [\beta_1(t) - \epsilon, \beta_1(t)], \\ f(t, x) &\leq f(t, \beta_2(t) - \epsilon) & \forall x \in [\beta_2(t) - \epsilon, \beta_2(t)]. \end{aligned}$$

By computations, one can show also that

$$\begin{aligned} f(t, \alpha_i(t) + \epsilon) &\geq (\alpha_i)'_g(t) = 0, \quad \text{for } i = 1, 2, \\ f(t, \beta_1(t)) &\leq (\beta_1)'_g(t) = 0, \\ f(t, \beta_2(t) - \epsilon) &\leq (\beta_2)'_g(t) = 0. \end{aligned}$$

We conclude that α_1, α_2 , and β_1, β_2 are respectively strict lower and upper solutions of (3.44) with $\epsilon = 1/100$.

Observe that

$$\alpha_1(t) < \beta_1(t) < \alpha_2(t) < \beta_2(t) \quad \text{for every } t \in [0, T].$$

Now, it remains to prove that $(\mathcal{D}_{\alpha_1, \beta_1}^s)$, $(\mathcal{D}_{\alpha_2, \beta_2}^s)$ and $(\mathcal{D}_{\alpha_1, \beta_2}^s)$ are satisfied. For every $t \in \{1\} = [0, T) \cap D_g$, we consider

$$h_1 : x \mapsto x + f(1, x)\mu_g(\{1\}) = x - \frac{1}{2}x^3 - x^2 + \frac{1}{4}.$$

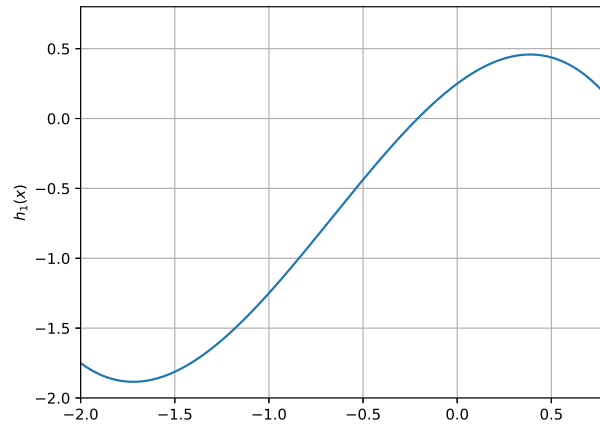


Figure 3.2.4 – h_1 on $\left[-2, \frac{4}{5}\right]$.

Observe that $h_1'(x) = -\frac{3}{2}x^2 - 2x + 1$ and h_1 is increasing on $[z_1, z_2]$, and decreasing on $[-2, z_1] \cup \left[z_2, \frac{4}{5}\right]$, with

$$z_1 = -\frac{2 + \sqrt{10}}{3}, \quad z_2 = -\frac{2 - \sqrt{10}}{3}.$$

Also, it is easy to verify that

$$h_1(z_1) < h_1(-2) < h_1(-1) < h_1\left(\frac{4}{5}\right) < h_1(0) < h_1(z_2).$$

Thus, by simple computations, we obtain

$$\begin{aligned} \alpha_1(1^+) &= -2 + \frac{1}{10} < h_1(z_1) \leq h_1(x) & \text{for all } x \in \left[-2, \frac{4}{5}\right] = [\alpha_1(1), \beta_2(1)], \\ \alpha_2(1^+) &= 0 < h_1\left(\frac{4}{5}\right) \leq h_1(x) & \text{for all } x \in \left[0, \frac{4}{5}\right] = [\alpha_2(1), \beta_2(1)], \end{aligned}$$

and

$$\begin{aligned} \beta_1(1^+) &= -1 - \frac{1}{10} > h_1(-1) \geq h_1(x) & \text{for all } x \in [-2, -1] = [\alpha_1(1), \beta_1(1)], \\ \beta_2(1^+) &= \frac{3}{5} > h_1(z_2) \geq h_1(x) & \text{for all } x \in \left[-2, \frac{4}{5}\right] = [\alpha_1(1), \beta_2(1)]. \end{aligned}$$

Hence, $(\mathcal{D}_{\alpha_1, \beta_1}^s)$, $(\mathcal{D}_{\alpha_2, \beta_2}^s)$ and $(\mathcal{D}_{\alpha_1, \beta_2}^s)$ hold. It follows from Theorem 3.2.3 that the problem (3.44) has at least three solutions. Figure 3.2.5 illustrates a three periodic solutions of the problem (3.44) on the interval $[0, T] = [0, 4]$.

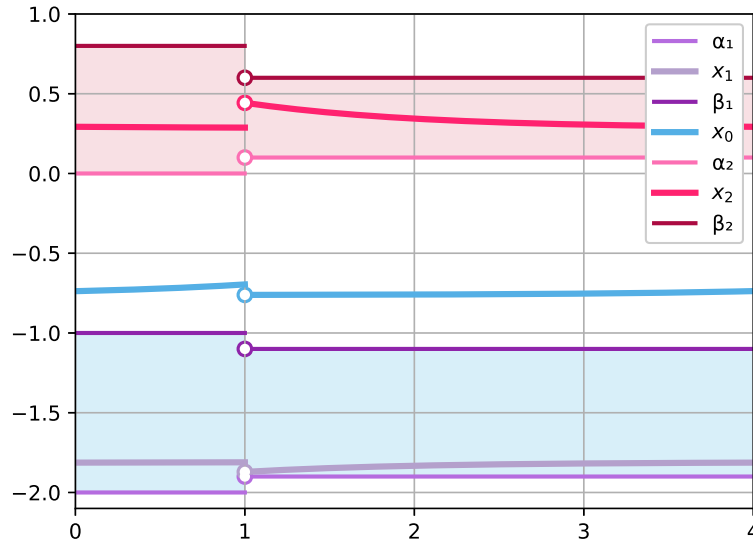


Figure 3.2.5 – Three periodic solutions for the problem (3.44) for $T = 4$.

3.2.3 Multiplicity of solutions to the initial value problem

Using an approach analogous to the one used in the previous Subsection, given two pairs of strict lower and upper solutions, we deduce that the initial value problem (3.1), (3.3) has at least three solutions.

Theorem 3.2.5. Assume that (H_f) holds, and that there exist α_1, α_2 and $\beta_1, \beta_2 \in \mathcal{AC}_g([0, T], \mathbb{R})$ respectively strict lower and upper solutions of (3.1), (3.3) such that

- (i) $\alpha_1(t) < \beta_1(t) \leq \beta_2(t)$ and $\alpha_1(t) \leq \alpha_2(t) < \beta_2(t)$ for every $t \in [0, T]$;
- (ii) $\{t \in [0, T] : \beta_1(t) < \alpha_2(t)\} \neq \emptyset$;
- (iii) $(\mathcal{D}_{\alpha_1, \beta_1}^s)$, $(\mathcal{D}_{\alpha_2, \beta_2}^s)$ and $(\mathcal{D}_{\alpha_1, \beta_2}^s)$ are satisfied.

Then, the initial value problem (3.1), (3.3) has at least three solutions $x_1, x_2, x_3 \in \mathcal{AC}_g([0, T], \mathbb{R})$ such that

$$\text{for } i = 1, 2, \quad \alpha_i(t) < x_i(t) < \beta_i(t) \quad \text{for every } t \in [0, T],$$

and

$$\alpha_1(t) < x_3(t) < \beta_2(t) \text{ for every } t \in [0, T] \text{ and } \{t \in [0, T] : \beta_1(t) \leq x_3(t) \leq \alpha_2(t)\} \neq \emptyset.$$

In Figure 3.2.6, we illustrate the positioning of the multiple solutions ensured by Theorem 3.2.5.

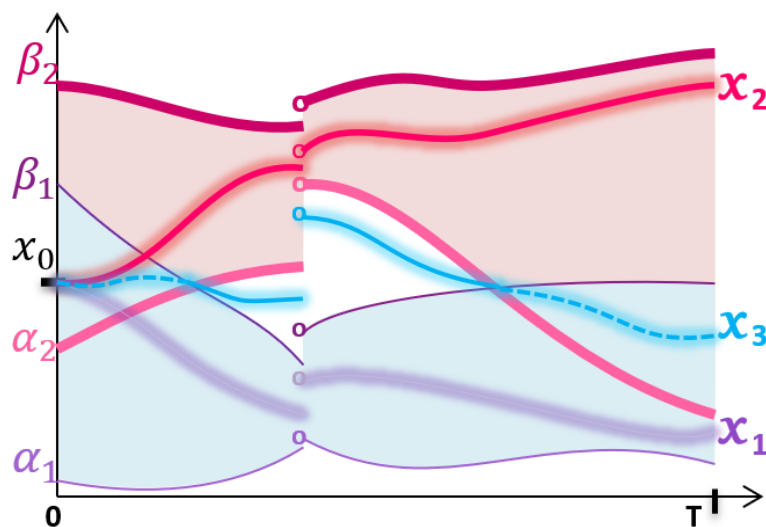


Figure 3.2.6 – Illustration of the positioning of multiple solutions for the initial value problem.

3.3 Application to a population model with an extreme event

The persistence of a species is a critical concern in dynamics of populations. In order to determine the likelihood of a population persisting, several models with periodic boundary conditions have been studied, as referenced in [9, 51, 89].

As climate change continues to worsen and pollution levels rise [28], there is a noticeable increase in the frequency of extreme events compared to previous years. Therefore, it is realistic to incorporate models that account for the occurrence of at least one extreme event annually. Hence, modeling such events involves the presence of discontinuities.

In addition, in some models, it is appropriate to consider that the population size does not change during specific periods of time. These period may be followed by a sudden variation. This might be the case, for instance, when there are periods of dormancy or when population changes are not observable during specific times like winter or migration between habitats.

Impulsive differential equations and the theory of dynamic equations on time scales have been applied to address problems involving discontinuities, see [9, 51, 89].

However, Stieltjes differential equations prove to be particularly advantageous in modeling various situations occurring in the dynamics of a population. Indeed, the derivator g can also account for fluctuations corresponding to particular periods of the year. Specifically, the periods when the variations of the population size are likely more important correspond to higher slopes of g .

In our example, we consider a population (such as fish, trees, perennials...) as it progresses through its life cycle. We assume that the population goes into a form of predictive dormancy during the final month of autumn, as a way to prepare for the harsh winter conditions. Once winter sets in, the species shifts into a strict dormancy phase to protect itself from the unfavorable weather conditions. Towards the last month of winter; at time $t = 2$, a number of dead individuals are observed, and the species begins to resume growth at a slower pace. In the early stages of spring, the conditions become more favorable, leading to increased growth. One month later; for $t = 4$, new individuals are introduced through fish seedlings, new plantations... During the summer season, the natural growth of the species diminishes in while the harvest activity begins. One month before the end of summer; for $t = 8$, we assume that an extreme event causes a decline in the population, which subsequently impacts the overall rate of natural growth. In the early autumn months, natural growth stops but harvesting activities persist until dormancy is triggered one month prior to winter. The dynamics can be incorporated in the diagram given in Figure 3.3.1.

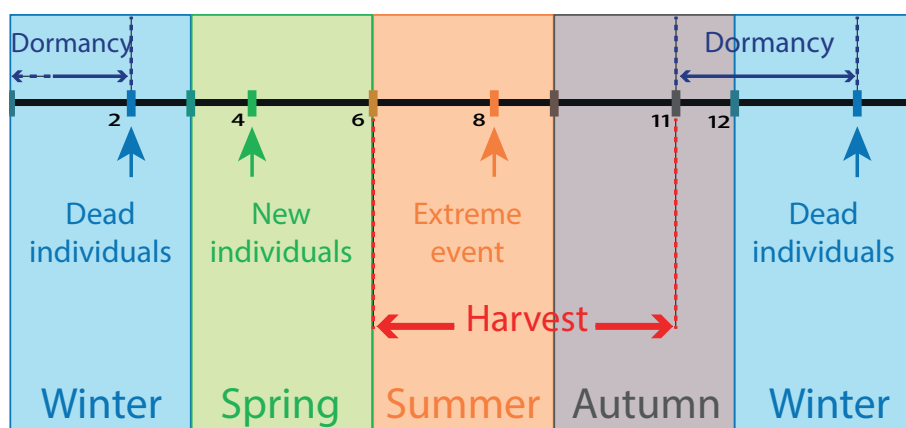


Figure 3.3.1 – Diagram of dynamics.

Let us denote the size of the population at time t by $x(t)$. Then, the dynamics of this population can be modeled using the Stieltjes differential equation:

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, 12), \quad (3.45)$$

where $f : [0, 12] \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(t, x) = \begin{cases} \frac{-x}{3} & \text{if } t \in [0, 2] \cup (11, 12], \\ \frac{x}{4} & \text{if } t \in (2, 6] \setminus \{4\}, \\ 1 & \text{if } t = 4, \\ \frac{x}{3} - \frac{2}{3}x^2 & \text{if } t \in (6, 8), \\ \frac{-x}{2} & \text{if } t = 8, \\ -\frac{2}{3}x^2 & \text{if } t \in (8, 11]. \end{cases} \quad (3.46)$$

In order to ensure the persistence of the population, we seek a solution satisfying the periodic condition:

$$x(0) = x(12). \quad (3.47)$$

Additionally, our model should take into account that the variation is smaller during autumn, late winter, and late spring. Therefore, we introduce the following derivator $g : \mathbb{R} \rightarrow \mathbb{R}$ which incorporates the aforementioned discontinuities and the stationary periods:

$$g(t) = \begin{cases} 0 & \text{if } t \in [0, 2], \\ 1 + 2\left(\frac{t}{2} - 1\right)^2 & \text{if } t \in (2, 4], \\ t & \text{if } t \in (4, 8], \\ t + 1 & \text{if } t \in (8, 9], \\ 11 - \left(\frac{t - 11}{2}\right)^2 & \text{if } t \in (9, 11], \\ 11 & \text{if } t \in (11, 12], \\ 11n + g(t - 12n) & \text{if } t - 12n \in [0, 12), n \in \mathbb{Z} \setminus \{0\}. \end{cases} \quad (3.48)$$

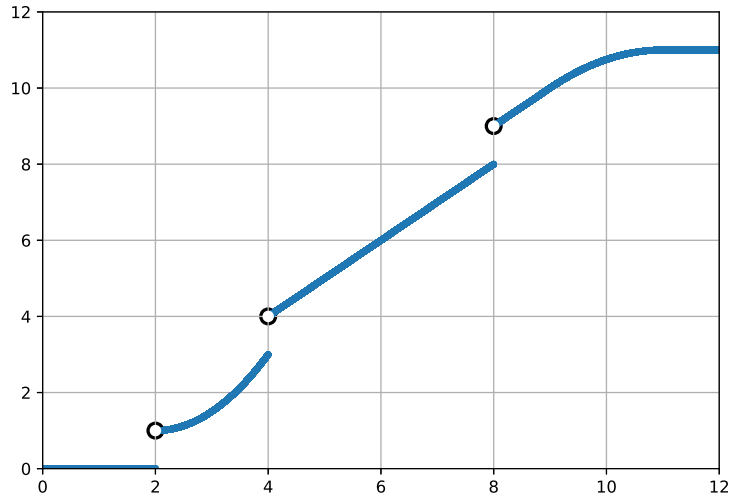


Figure 3.3.2 – Graph of the derivator g .

We notice that

$$D_g = \bigcup_{n \in \mathbb{Z}} \{2 + 12n, 4 + 12n, 8 + 12n\},$$

$$C_g = \bigcup_{n \in \mathbb{Z}} (12n, 2 + 12n) \cup (11 + 12n, 12 + 12n).$$

In what follows, we show that the problem (3.45), (3.47) has a solution. Clearly f defined in (3.46) is a g -Carathéodory function. Now, let us consider $\alpha, \beta : [0, 12] \rightarrow \mathbb{R}$ defined by

$$\alpha(t) = 0 \quad \text{for every } t \in [0, 12],$$

and

$$\beta(t) = \begin{cases} 1 & \text{if } t \in [0, 2], \\ (t-1)^2 & \text{if } t \in (2, 4], \\ 2+t^2 & \text{if } t \in (4, 6], \\ 6+16(8-t) & \text{if } t \in (6, 8], \\ \frac{3(10-t)}{2} & \text{if } t \in (8, 9], \\ 1 + \frac{(t-11)^2}{8} & \text{if } t \in (9, 11], \\ 1 & \text{if } t \in (11, 12]. \end{cases}$$

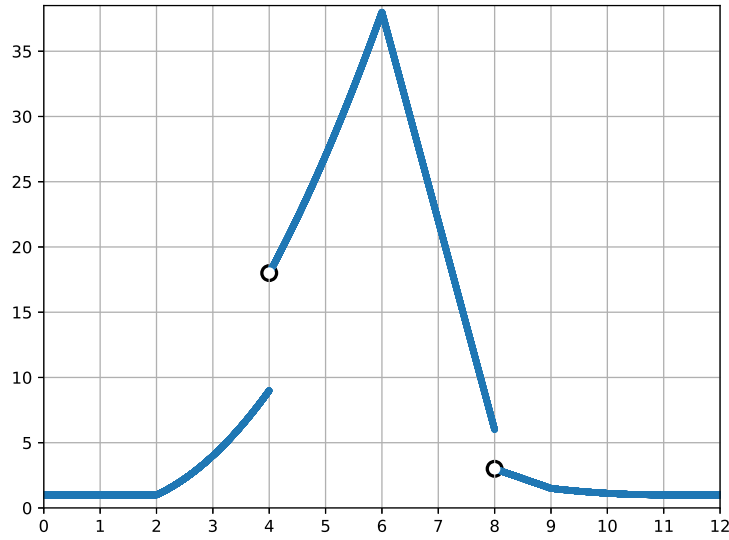


Figure 3.3.3 – Graph of β .

Observe that $\alpha, \beta \in \mathcal{AC}_g([0, 12], \mathbb{R})$, and $\alpha(t) \leq \beta(t)$ for every $t \in [0, 12]$. In addition, $\alpha'_g(t) = 0$ for every $t \in [0, 12] \setminus C_g$, and

$$\beta'_g(t) = \begin{cases} 0 & \text{if } t = 2, \\ \frac{2t-2}{t-2} & \text{if } t \in (2, 4), \\ 9 & \text{if } t = 4, \\ 2t & \text{if } t \in (4, 6), \\ -16 & \text{if } t \in (6, 8), \\ -3 & \text{if } t = 8, \\ \frac{-3}{2} & \text{if } t \in (8, 9), \\ \frac{-1}{2} & \text{if } t \in (9, 11). \end{cases}$$

Also, we have

$$\alpha(0) = 0 = \alpha(12) \quad \text{and} \quad \alpha'_g(t) = 0 \leq f(t, 0) = \begin{cases} 0 & \text{if } t \in [2, 4) \cup (4, 11), \\ 1 & \text{if } t = 4. \end{cases}$$

The function β satisfies

$$\beta(0) = 1 = \beta(12),$$

and

$$\begin{aligned}
 \text{for } t = 2, \quad \beta'_g(2) = 0 &\geq f(2, \beta(2)) = \frac{-\beta(2)}{3} = \frac{-1}{3}, \\
 \text{for } t \in (2, 4), \quad \beta'_g(t) = \frac{2t-2}{t-2} &\geq f(t, \beta(t)) = \frac{\beta(t)}{4} = \frac{(t-1)^2}{4}, \\
 \text{for } t = 4, \quad \beta'_g(4) = 9 &\geq f(4, \beta(4)) = 1, \\
 \text{for } t \in (4, 6), \quad \beta'_g(t) = 2t &\geq f(t, \beta(t)) = \frac{t^2+2}{4}, \\
 \text{for } t \in (6, 8), \quad \beta'_g(t) = -16 &\geq \frac{6-2(6)^2}{3} \\
 &\geq f(t, \beta(t)) = \frac{(6+16(8-t)) - 2(6+16(8-t))^2}{3}, \\
 \text{for } t = 8, \quad \beta'_g(8) = -3 &\geq f(8, \beta(8)) = \frac{-\beta(8)}{2} = -3, \\
 \text{for } t \in (8, 9), \quad \beta'_g(t) = \frac{-3}{2} &\geq f(t, \beta(t)) = \frac{-2\beta(t)^2}{3} = \frac{-3(10-t)^2}{2}, \\
 \text{for } t \in (9, 11), \quad \beta'_g(t) = \frac{-1}{2} &\geq f(t, \beta(t)) = \frac{-2\beta(t)^2}{3} = \frac{-2}{3} \left(1 + \frac{(t-11)^2}{8}\right)^2.
 \end{aligned}$$

Therefore, the functions α and β are respectively lower and upper solutions of the problem (3.45), (3.47).

We claim that Condition $(\mathcal{D}_{\alpha,\beta})$ is satisfied. Indeed, for $t \in D_g \cap [0, 12] = \{2, 4, 8\}$ and $x \in [\alpha(t), \beta(t)]$,

$$\alpha(t^+) = 0 \leq x + f(t, x)\mu_g(\{t\}) = x + f(t, x) = \begin{cases} \frac{2x}{3} & \text{if } t = 2, \\ 1 + x & \text{if } t = 4, \\ \frac{x}{2} & \text{if } t = 8. \end{cases}$$

Also,

$$\begin{aligned}
 \text{for } t = 2 \quad \beta(2^+) = 1 &\geq \frac{2x}{3}, & \forall x \in [\alpha(2), \beta(2)] = [0, 1], \\
 \text{for } t = 4 \quad \beta(4^+) = 18 &\geq 1 + x, & \forall x \in [\alpha(4), \beta(4)] = [0, 9], \\
 \text{for } t = 8 \quad \beta(8^+) = 3 &\geq \frac{x}{2}, & \forall x \in [\alpha(8), \beta(8)] = [0, 6].
 \end{aligned}$$

Hence, applying Theorem 3.1.7, it follows that there exists $x \in \mathcal{AC}_g([0, 12], \mathbb{R})$ a solution of the problem (3.45), (3.47) such that $\alpha(t) \leq x(t) \leq \beta(t)$ for every $t \in [0, 12]$.

Comments and discussion

We have demonstrated that there exists at least one positive periodic solution x defined on $[0, 12]$ for the problem (3.45), (3.47). To ensure the persistence of the species, it is essential to confirm that this solution is positive for all t . The following study will enable us to do so and to determine the trajectory shape. Applying the resolution

approach outlined in [55], we obtain that solutions of the problem (3.45), (3.47) have the form

$$x(t) = \begin{cases} r & \text{if } t \in [0, 2], \\ \frac{2}{3}r e^{\frac{1}{4}(g(t)-g(2^+))} & \text{if } t \in (2, 4], \\ (x(4) + 1)e^{\frac{1}{4}(g(t)-g(4^+))} & \text{if } t \in (4, 6], \\ \frac{1}{2 + (\frac{1}{x(6)} - 2)e^{\frac{1}{3}(g(6)-g(t))}} & \text{if } t \in (6, 8], \\ \frac{1}{\frac{2}{x(8)} + \frac{2}{3}(g(t) - g(8^+))} & \text{if } t \in (8, 11], \\ r & \text{if } t \in (11, 12], \end{cases} \quad (3.49)$$

for some $r \in \mathbb{R}$. In order to find the value $r \in \mathbb{R}$, one must first observe that since $11 \notin D_g$, it follows that $x(11) = r$, i.e.

$$\frac{1}{\frac{2}{x(8)} + \frac{2}{3}(g(11) - g(8^+))} = r. \quad (3.50)$$

Using (3.49), $x(8)$ can be expressed in terms of r . Consequently, (3.50) yields

$$\frac{8}{3}e^{\frac{1}{2}}\left(\frac{4}{3} - e^{-\frac{2}{3}}\right)r^2 + 2\left(\frac{8}{3} + e^{-\frac{7}{3}} - 2e^{-\frac{2}{3}} - \frac{e^{\frac{1}{2}}}{3}\right)r - 1 = 0. \quad (3.51)$$

The equation (3.51) has only one positive root r_+ . So, x given in (3.49) with $r = r_+$ is the unique positive periodic solution of (3.45), (3.47), and it lies between α and β .

Figure 3.3.4 shows the solution x . We observe that the dynamics of this population over the course of a year can be represented by this model as outlined at the beginning of this section. Notice that this solution takes into account the dormancy periods in autumn and winter, as well as the abrupt changes that occur for $t = 2$, $t = 4$ and $t = 8$. Additionally, the model accounts for the change in both the growth rate and the harvesting activity, which are represented by different slopes in the function g . Ultimately, the persistence of this species has been concluded.

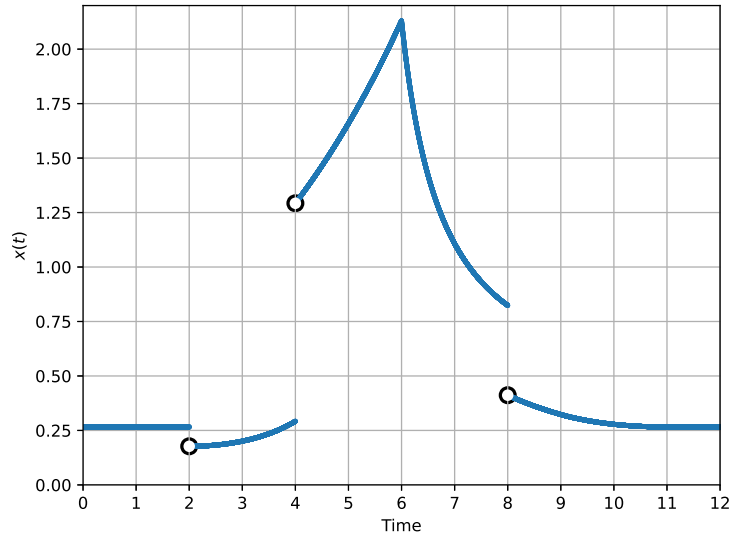


Figure 3.3.4 – Graph of x .

Remark 3.3.1. The numbers $-\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}$, and $-\frac{1}{2}$ in (3.46), can be respectively replaced by the following parameters:

$0 < d < 1$: constant referring to the proportion of the population that could not survive the adverse conditions of winter.

$b > 0$: the growth rate during spring.

$a \geq 0$: the number of individuals added due to fish seedings or new plantations...

$b' > 0$: the growth rate during summer before the occurrence of the extreme event.

$h > 0$: A harvest rate implying that the harvest activity plan reduces the population proportionally to the competitive term x^2 .

$0 < u < 1$: Constant indicating the effect of the extreme event on the population.

In this case of some given parameters, one may check that $\alpha(t) = 0$ for all $t \in [0, 12]$ is a lower solution for the problem. Additionally, once the linear problem

$$\beta'_g(t) = F(t, \beta(t)) \quad \text{for } g\text{-almost every } t \in [0, 12), \quad \beta(0) = \beta(12), \quad (3.52)$$

with $F : [0, 12] \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$F(t, x) = \begin{cases} 0 & \text{if } t \in [0, 2] \cup (8, 12], \\ bx & \text{if } t \in (2, 6] \setminus \{4\}, \\ a & \text{if } t = 4, \\ b'x & \text{if } t \in (6, 8], \end{cases} \quad (3.53)$$

has a positive solution β , then β is an upper solution. Indeed, it suffices to notice that f will satisfy $f(t, \beta(t)) \leq F(t, \beta(t)) = \beta'_g(t)$ for g -almost every $t \in [0, 12)$. Thus, a positive solution is always guaranteed between the lower and the upper solutions.

Systems of first-order Stieltjes differential equations with several derivators

This chapter is devoted to existence results for systems of Stieltjes differential equations involving several derivators. While the theoretical interest is to provide a more general setting, another motivation lies in the applied aspect of this setting. When analyzing a system with multiple interacting components (variables), it becomes crucial to understand how each component evolves over time. For this purpose, by deriving with respect to a vector-valued derivator, we can effectively capture these changes even when some variables experience abrupt shifts or remain constant while others do not. This approach allows us to describe the dynamic behavior of a multidimensional system, where each variable exhibits its unique patterns of discontinuity and stationary periods.

The study of systems of Stieltjes differential equations involving several Stieltjes derivators traces its origins to the work of López Pouso and Márquez Albés in [56]. In their work [56], they established results related to existence, uniqueness, and Peano-type results were established for the initial value boundary problem (4.1), (4.3), extending the first existence results in the literature due to Frigon and López Pouso [31], that dealt with systems using the same derivator for each component. Recently, Márquez Albés and Tojo in [66] continued the study initiated in [56], focusing on existence and uniqueness results involving Osgood and Montel-Tonelli conditions.

In this chapter, we present the results that we established in [61], generalizing the existence results presented in the previous chapter to systems of Stieltjes differential equations of the form

$$\begin{aligned} \mathbf{x}'_{\mathbf{g}}(t) &= \mathbf{f}(t, \mathbf{x}(t)) \quad \text{for } \mathbf{g}\text{-almost every } t \in [0, T], \\ \mathbf{x} &\in \mathfrak{B}; \end{aligned} \tag{4.1}$$

where $\mathbf{f} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and those with a functional dependence

$$\begin{aligned} \mathbf{x}'_{\mathbf{g}}(t) &= \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}) \quad \text{for } \mathbf{g}\text{-almost every } t \in [0, T], \\ \mathbf{x} &\in \mathfrak{B}; \end{aligned} \tag{4.2}$$

where $\mathbf{f} : [0, T] \times \mathbb{R}^n \times \mathcal{C}$, $\mathcal{C}$ is a subset of $\mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$, and $\mathbf{g} = (g_1, \dots, g_n) : \mathbb{R} \rightarrow \mathbb{R}^n$ is a vector-valued derivator such that g_i is a left-continuous nondecreasing function for $i = 1, \dots, n$. Here, $\mathbf{x} \in \mathfrak{B}$ means that it satisfies an initial value condition or a periodic boundary condition:

$$\mathbf{x}(0) = \mathbf{x}_0 = (\mathbf{x}_{0,1}, \dots, \mathbf{x}_{0,n}) \in \mathbb{R}^n; \tag{4.3}$$

$$\mathbf{x}(0) = \mathbf{x}(T). \quad (4.4)$$

In particular, the problem (4.1) (resp. the functional problem (4.2)) to be understood, component-by-component, for $i = 1, \dots, n$, as

$$(x_i)'_{g_i}(t) = f_i(t, \mathbf{x}(t)) \text{ (resp. } (x_i)'_{g_i}(t) = f_i(t, \mathbf{x}(t), \mathbf{x}) \text{) for } g_i\text{-almost all } t \in [0, T),$$

subject to $x_i(0) = x_{0,i}$ or $x_i(0) = x_i(T)$ depending on $\mathfrak{B}$.

To establish existence results for the systems of Stieltjes differential equations (4.1), and (4.2), we introduce the concept of ordered pairs of lower and upper solutions for these problems. Again, the right-hand side member $\mathbf{f}$ need not satisfy monotonicity conditions or be bounded by a specific map. Subsequently, we utilize these results to prove the existence of solutions for systems of functional Stieltjes differential equations, involving either (4.3) or (4.4), without assuming monotonicity conditions. We conclude this chapter with an example of application that enhances the applicability of our existence results. Specifically, we describe the dynamics in a fishery with predator-prey interactions subject to seasonal fishing. In this context, the dynamics involve a function $\mathbf{f}$ not satisfying one of the monotonicity conditions required in the literature.

4.1 g -Carathéodory function

Throughout this section, let $\mathbf{g} = (g_1, \dots, g_n) : \mathbb{R} \rightarrow \mathbb{R}^n$ be a vector-valued derivator, such that for $i = 1, \dots, n$, $g_i : \mathbb{R} \rightarrow \mathbb{R}$ is a left-continuous and nondecreasing function. Based on Definition 3.1.1 in the case of a single-valued derivator, let us define the notion of g -Carathéodory function in the context of a vector-valued derivator $\mathbf{g}$.

Definition 4.1.1. A function $\mathbf{f} = (f_1, \dots, f_n) : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be g -Carathéodory if, for $i = 1, \dots, n$, f_i is g_i -Carathéodory, i.e. it fulfills the following conditions:

- (i) for every $\mathbf{x} \in \mathbb{R}^n$, $t \mapsto f_i(t, \mathbf{x})$ is g_i -measurable;
- (ii) for g_i -almost every $t \in [a, b]$, $\mathbf{x} \mapsto f_i(t, \mathbf{x})$ is continuous on $\mathbb{R}^n$;
- (iii) for all $r > 0$, there is a function $h_{r,i} \in \mathcal{L}_{g_i}^1([a, b], [0, +\infty))$ such that $|f_i(t, \mathbf{x})| \leq h_{r,i}(t)$ for g_i -almost all $t \in [a, b]$ and all $\mathbf{x} \in \mathbb{R}^n$ such that $\|\mathbf{x}\| \leq r$.

By means of Lemma 1.3.4, notice that a map $\mathbf{x} \in \mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)$ is bounded and Borel measurable. Thus, arguing as in the proof of [31, Lemma 7.2], we obtain the next lemma.

Lemma 4.1.2. Let us consider a g -Carathéodory function $\mathbf{f} = (f_1, \dots, f_n) : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then, for every $\mathbf{x} \in \mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)$, we have that

$$\mathbf{f}(\cdot, \mathbf{x}(\cdot)) \in \prod_{i=1}^n \mathcal{L}_{g_i}^1([a, b], [0, +\infty)).$$

Again, using arguments analogous to those in the proof of Lemma 4.1.3, we deduce that an operator associated to a g -Carathéodory function has nice properties.

Lemma 4.1.3. Let $\mathbf{f} = (f_1, \dots, f_n) : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a $\mathbf{g}$ -Carathéodory function and let $\mathbf{N}_{\mathbf{f}} : \mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n) \rightarrow \mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)$ be defined by

$$\mathbf{N}_{\mathbf{f}}(\mathbf{x}) = (N_{f_1}(\mathbf{x}), \dots, N_{f_n}(\mathbf{x})), \quad \text{for all } \mathbf{x} \in \mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n), \quad (4.5)$$

where, for $i = 1, \dots, n$,

$$N_{f_i}(\mathbf{x})(t) = \int_{[a, t]} f_i(s, \mathbf{x}(s)) \, d\mu_{g_i}(s) \quad \text{for every } t \in [a, b].$$

Then, $\mathbf{N}_{\mathbf{f}}$ is continuous and completely continuous.

4.2 Ordered pair of vector-valued lower and upper solutions

In this section, we introduce the concept of an ordered pair of vector-valued lower and upper solutions for the system of Stieltjes differential equations (4.1). This notion will play an essential role in the subsequent existence results. Along this section, let $\mathbf{g} = (g_1, \dots, g_n) : \mathbb{R} \rightarrow \mathbb{R}^n$ be a vector-valued derivator, such that for $i = 1, \dots, n$, $g_i : \mathbb{R} \rightarrow \mathbb{R}$ is a left-continuous and nondecreasing function, and let $\mathbf{e}_i$ denotes a vector of the canonical basis of $\mathbb{R}^n$ such that $\mathbf{e}_i$ has 1 at its i^{th} coordinate and zero at the others.

Let us recall the definition of solutions of the problems (4.1) and (4.2).

Definition 4.2.1. A function $\mathbf{x} \in \mathcal{AC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ is a *solution* of (4.1) (resp. (4.2)) if satisfies $\mathbf{x} \in \mathfrak{B}$ (where $\mathfrak{B}$ denotes (4.3) or (4.4)), and

$$\mathbf{x}'_{\mathbf{g}}(t) = \mathbf{f}(t, \mathbf{x}(t)) \quad (\text{resp. } \mathbf{x}'_{\mathbf{g}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x})) \quad \text{for } \mathbf{g}\text{-almost every } t \in [0, T],$$

i.e. for $i = 1, \dots, n$,

$$(x_i)'_{g_i}(t) = f_i(t, \mathbf{x}(t)) \quad (\text{resp. } \mathbf{x}'_{g_i}(t) = f_i(t, \mathbf{x}(t), \mathbf{x}))$$

for g_i -almost every $t \in [0, T]$.

Definition 4.2.2. Let $\alpha, \beta \in \mathcal{AC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$. A pair (α, β) is said to be an *ordered pair of vector-valued lower and upper solutions* of (4.1) if $\alpha \leq \beta$, and, for $i = 1, \dots, n$,

- (i) $(\alpha_i)'_{g_i}(t) \leq f_i(t, (\alpha_i(t) - c_i)\mathbf{e}_i + \mathbf{c})$ for g_i -almost all $t \in [0, T]$ and for every $\mathbf{c} \in [\alpha(t), \beta(t)]$;
- (ii) $(\beta_i)'_{g_i}(t) \geq f_i(t, (\beta_i(t) - c_i)\mathbf{e}_i + \mathbf{c})$ for g_i -a.a. $t \in [0, T]$ and for every $\mathbf{c} \in [\alpha(t), \beta(t)]$;
- (iii) for every $t \in [0, T] \cap D_{g_i}$ and every $\mathbf{c} \in [\alpha(t), \beta(t)]$,

$$\alpha_i(t^+) \leq c_i + f_i(t, \mathbf{c})\mu_{g_i}(\{t\}) \leq \beta_i(t^+);$$

- (iv) - if $\mathfrak{B}$ denotes (4.3), $\alpha_i(0) \leq \mathbf{x}_{0,i} \leq \beta_i(0)$;
- if $\mathfrak{B}$ denotes (4.4), $\alpha_i(0) \leq \alpha_i(T)$ and $\beta_i(0) \geq \beta_i(T)$.

Remark 4.2.3. Conditions (i)–(iii) of the previous definition can appear simpler if we impose monotonicity conditions on the function $\mathbf{f}$. Let $\alpha, \beta \in \mathcal{AC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ be such that $\alpha \leq \beta$ and let $\mathbf{f} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Suppose that the function $\mathbf{f}$ is *quasimonotone nondecreasing* within the set E , defined as $E = \{(t, \mathbf{c}) \in [0, T] \times \mathbb{R}^n : \alpha(t) \leq \mathbf{c} \leq \beta(t), \forall t \in [0, T]\}$ (i.e. $\mathbf{a} \leq \mathbf{b}$ implies that $f_i(t, \mathbf{a}) \leq f_i(t, \mathbf{b})$ for each $i \in \{j \in \{1, \dots, n\} : a_j = b_j\}$).

(1) If $(\alpha_i)'_{g_i}(t) \leq f_i(t, \alpha(t))$ for g_i -almost every $t \in [0, T]$ and every $i \in \{1, \dots, n\}$, then Condition (i) of the previous definition is satisfied.

Similarly, if $(\beta_i)'_{g_i}(t) \geq f_i(t, \beta(t))$ for g_i -almost every $t \in [0, T]$ and every $i \in \{1, \dots, n\}$, then Condition (ii) of the previous definition holds.

(2) In addition, assume that, for each $t \in [0, T] \cap D_{g_i}$ and every $\mathbf{c} \in [\alpha(t), \beta(t)]$,

$$a_i \in [\alpha_i(t), \beta_i(t)] \mapsto a_i + f_i\left(t, (a_i - c_i)\mathbf{e}_i + \mathbf{c}\right)\mu_{g_i}(\{t\}) \text{ is nondecreasing.}$$

If

$$(\alpha_i)'_{g_i}(t) \leq f_i(t, \alpha(t)) \quad \text{and} \quad (\beta_i)'_{g_i}(t) \geq f_i(t, \beta(t)),$$

then Condition (iii) of the previous definition is satisfied.

It is worthwhile to mention that, in [57], the assumption was made that the function $\mathbf{f}$ is quasimonotone nondecreasing and it satisfies the monotonicity condition in (2). However, it is important to note that no such monotonicity conditions will be imposed in the theorems that establish the existence of solutions, as we will see in the subsequent sections.

Remark 4.2.4. In the case of the classical derivation where $\mathbf{g}$ is such that $g_1(t) = \dots = g_n(t) = t$ for all $t \in \mathbb{R}$. Observe that, $D_{g_i} = \emptyset$ for every $i = 1, \dots, n$, and Condition (iii) of the previous definition becomes trivially satisfied. In this specific case, Definition 4.2.2 aligns with the concept of component-wise lower and upper solutions introduced by Knobloch [47] in the context of periodic boundary value problems for systems of differential equations with a continuous right-hand side member.

4.3 Existence of solutions

4.3.1 Systems with an initial value condition

The aim of this subsection is to build upon our previous result, Theorem 3.1.12, leveraging the concept of upper and lower solutions for systems of Stieltjes differential equations, without imposing a monotonicity condition on the right-hand side member $\mathbf{f}$. Let $\mathbf{g} = (g_1, \dots, g_n) : \mathbb{R} \rightarrow \mathbb{R}^n$ be a vector-valued derivator such that $g_i : \mathbb{R} \rightarrow \mathbb{R}$ is a left-continuous and nondecreasing function.

We state the next existence result for the nonfunctional problem with initial value condition (4.1), (4.3).

Theorem 4.3.1. *Let $\mathbf{f} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a $\mathbf{g}$ -Carathéodory function. Let us assume that $(H_{\alpha, \beta})$ there exist $\alpha, \beta \in \mathcal{AC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ such that (α, β) is an ordered pair of vector-valued lower and upper solutions of (4.1), (4.3).*

Then, the problem (4.1), (4.3) has at least one solution $\mathbf{x} \in \mathcal{AC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ such that $\alpha \leq \mathbf{x} \leq \beta$.

In order to prove this result, let us consider the following family of problems defined for $\lambda \in [0, 1]$:

$$\begin{aligned} \mathbf{x}'_{\mathbf{g}}(t) &= \lambda \tilde{\mathbf{f}}(t, \mathbf{x}(t)), \\ \mathbf{x}(0) &= \mathbf{x}_0 = (\mathbf{x}_{0,1}, \dots, \mathbf{x}_{0,n}), \end{aligned} \quad (4.6_{\lambda})$$

where the function $\tilde{\mathbf{f}} = (\tilde{f}_1, \dots, \tilde{f}_n) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined as follows. For $i = 1, \dots, n$, and $\mathbf{x} = (x_1, \dots, x_n)$,

$$\tilde{f}_i(t, \mathbf{x}) = \begin{cases} f_i(t, \bar{\mathbf{x}}) - w_{\beta,i}(t)(x_i - \beta_i(t)) & \text{if } x_i > \beta_i(t), \\ f_i(t, \bar{\mathbf{x}}) & \text{if } \alpha_i(t) \leq x_i \leq \beta_i(t), \\ f_i(t, \bar{\mathbf{x}}) + w_{\alpha,i}(t)(x_i - \alpha_i(t)) & \text{if } x_i < \alpha_i(t); \end{cases} \quad (4.7)$$

with $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n)$ given by

$$\bar{x}_i = \begin{cases} \beta_i(t) & \text{if } x_i > \beta_i(t), \\ x_i & \text{if } x_i \in [\alpha_i(t), \beta_i(t)], \\ \alpha_i(t) & \text{if } x_i < \alpha_i(t); \end{cases}$$

and $\mathbf{w}_{\alpha} = (w_{\alpha,1}, \dots, w_{\alpha,n})$, $\mathbf{w}_{\beta} = (w_{\beta,1}, \dots, w_{\beta,n})$, with $w_{\alpha,i}, w_{\beta,i} \in \mathcal{L}_{g_i}^1([0, T], \mathbb{R})$ chosen such that

$$w_{\alpha,i}(t) < \min \{0, (\alpha_i)'_{g_i}(t)\} \text{ and } w_{\beta,i}(t) > \max \{0, (\beta_i)'_{g_i}(t)\}, \quad \text{if } t \notin D_{g_i}, \quad (4.8)$$

and, if $t \in D_{g_i}$,

$$\begin{aligned} w_{\beta,i}(t) &= \max \left\{ 0, \min \left\{ (\beta_i)'_{g_i}(t), \frac{1}{\mu_{g_i}(\{t\})} \right\} \right\}, \\ w_{\alpha,i}(t) &= \min \left\{ 0, \max \left\{ (\alpha_i)'_{g_i}(t), \frac{-1}{\mu_{g_i}(\{t\})} \right\} \right\}. \end{aligned} \quad (4.9)$$

We denote

$$\begin{aligned} \mathcal{T}_{\beta,i} &= \left\{ t \in D_{g_i} : w_{\beta,i}(t) = \frac{1}{\mu_{g_i}(\{t\})} \right\}, \\ \mathcal{T}_{\alpha,i} &= \left\{ t \in D_{g_i} : w_{\alpha,i}(t) = \frac{-1}{\mu_{g_i}(\{t\})} \right\}. \end{aligned} \quad (4.10)$$

Let $\mathcal{F} : [0, 1] \times \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n) \rightarrow \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ be the operator defined by

$$\mathcal{F}(\lambda, \mathbf{x})(t) = \mathbf{x}_0 + \lambda \mathbf{N}_{\tilde{\mathbf{f}}}(\mathbf{x})(t), \quad \text{for every } t \in [0, T], \quad (4.11)$$

where $\mathbf{N}_{\tilde{\mathbf{f}}}$ is given in (4.5) and associated to $\tilde{\mathbf{f}}$. It follows from the definition of $\tilde{\mathbf{f}}$ and from Lemma 4.1.3 that $\mathcal{F}$ is continuous and completely continuous.

The proposition below is crucial for establishing the existence of solutions using the fixed point index.

Proposition 4.3.2. *Under the hypotheses of the previous theorem, there exists a constant $R > \max\{\|\alpha\|_0, \|\beta\|_0\}$ such that*

$$\text{index}(\mathcal{F}(\lambda, \cdot), \mathcal{U}) = 1 \quad \text{for every } \lambda \in [0, 1],$$

where $\mathcal{U} = \{\mathbf{x} \in \mathcal{BC}_g([0, T], \mathbb{R}^n) : \|\mathbf{x}\|_0 < R\}$. In particular, the problem (4.6 _{λ}) has at least one solution for every $\lambda \in [0, 1]$.

Proof. First of all, it is easy to see that the fixed points of $\mathcal{F}(\lambda, \cdot)$ correspond to solutions of (4.6 _{λ}).

Let $\tilde{R} > 0$ be such that

$$\tilde{R} \geq 1 + \max_{i \in \{1, \dots, n\}} \sup_{t \in [0, T]} \left\{ \max\{\beta_i(t), -\alpha_i(t)\} \right\}. \quad (4.12)$$

We claim that

$$\|\mathbf{x}\|_0 \leq \tilde{R}, \text{ for any solution of (4.6}_\lambda\text{)}. \quad (4.13)$$

Indeed, for a fixed $i \in \{1, \dots, n\}$, let us first show that Condition (ii) of Lemma 1.5.7 is satisfied with $u = x_i$ and $v = \tilde{R}$, i.e. let us verify that $(x_i)'_{g_i}(t) \leq 0$ for g_i -almost all $t \in [0, T)$ such that $x_i(t^+) > \tilde{R}$ by considering the following two cases.

Case 1: For g_i -almost all $t \in [0, T) \setminus D_{g_i}$ such that $x_i(t^+) > \tilde{R}$, it follows from $(H_{\alpha, \beta})$, (4.7), (4.8), and (4.12), that $x_i(t) = x_i(t^+) > \tilde{R}$ and

$$\begin{aligned} (x_i)'_{g_i}(t) &= \lambda \left(f_i(t, \overline{\mathbf{x}(t)}) - w_{\beta, i}(t)(x_i(t) - \beta_i(t)) \right) \\ &\leq \lambda \left((\beta_i)'_{g_i}(t) - w_{\beta, i}(t) \right) \\ &\leq 0. \end{aligned} \quad (4.14)$$

Case 2: For g_i -almost every $t \in [0, T) \cap D_{g_i}$ such that $x_i(t^+) > \tilde{R}$, we distinguish three subcases.

Subcase 1: If $x_i(t) \in [\alpha_i(t), \beta_i(t)]$, then, by $(H_{\alpha, \beta})$ and (iii) of Definition 4.2.2, one has

$$(x_i)'_{g_i}(t) = \lambda f_i(t, \overline{\mathbf{x}(t)}) \leq \lambda \left(\frac{\beta_i(t^+) - x_i(t)}{\mu_{g_i}(\{t\})} \right). \quad (4.15)$$

Subcase 2: If $x_i(t) > \beta_i(t)$, it follows from $(H_{\alpha, \beta})$, (4.7) and (4.9) that

$$\begin{aligned} (x_i)'_{g_i}(t) &= \lambda \left(f_i(t, \overline{\mathbf{x}(t)}) - w_{\beta, i}(t)(x_i(t) - \beta_i(t)) \right) \\ &\leq \lambda \left((\beta_i)'_{g_i}(t) - w_{\beta, i}(t)(x_i(t) - \beta_i(t)) \right) \\ &\leq \begin{cases} \lambda(\beta_i)'_{g_i}(t) & \text{if } (\beta_i)'_{g_i}(t) \leq 0, \\ \lambda \left((\beta_i)'_{g_i}(t) - (\beta_i)'_{g_i}(t)(x_i(t) - \beta_i(t)) \right) & \text{if } (\beta_i)'_{g_i}(t) \in \left(0, \frac{1}{\mu_{g_i}(\{t\})} \right], \\ \lambda \left((\beta_i)'_{g_i}(t) - \frac{1}{\mu_{g_i}(\{t\})}(x_i(t) - \beta_i(t)) \right) & \text{if } (\beta_i)'_{g_i}(t) > \frac{1}{\mu_{g_i}(\{t\})}, \end{cases} \end{aligned}$$

$$\begin{aligned}
 &\leq \begin{cases} 0 & \text{if } (\beta_i)'_{g_i}(t) \leq 0, \\ \lambda \left((\beta_i)'_{g_i}(t)(1 + \beta_i(t) - x_i(t)) \right) & \text{if } (\beta_i)'_{g_i}(t) \in \left(0, \frac{1}{\mu_{g_i}(\{t\})} \right], \\ \lambda \left(\frac{\beta_i(t^+) - x_i(t)}{\mu_{g_i}(\{t\})} \right) & \text{if } (\beta_i)'_{g_i}(t) > \frac{1}{\mu_{g_i}(\{t\})}, \end{cases} \\
 &\leq \begin{cases} 0 & \text{if } (\beta_i)'_{g_i}(t) \leq 0, \\ 0 & \text{if } (\beta_i)'_{g_i}(t) \in \left(0, \frac{1}{\mu_{g_i}(\{t\})} \right] \\ & \text{and } x_i(t) \geq 1 + \beta_i(t), \\ \lambda \left(\frac{\beta_i(t^+) + 1 - x_i(t)}{\mu_{g_i}(\{t\})} \right) & \text{if } (\beta_i)'_{g_i}(t) \in \left(0, \frac{1}{\mu_{g_i}(\{t\})} \right] \\ & \text{and } x_i(t) \in (\beta_i(t), 1 + \beta_i(t)), \\ \lambda \left(\frac{\beta_i(t^+) - x_i(t)}{\mu_{g_i}(\{t\})} \right) & \text{if } (\beta_i)'_{g_i}(t) > \frac{1}{\mu_{g_i}(\{t\})}. \end{cases}
 \end{aligned}$$

Consequently,

$$(x_i)'_{g_i}(t) \leq \max \left\{ 0, \frac{\lambda(\beta_i(t^+) + 1 - x_i(t))}{\mu_{g_i}(\{t\})} \right\}. \quad (4.16)$$

Subcase 3: If $x_i(t) < \alpha_i(t)$, then, by $(H_{\alpha,\beta})$, (4.7) and (4.9), it yields

$$\begin{aligned}
 (x_i)'_{g_i}(t) &= \lambda \left(f_i(t, \bar{\mathbf{x}}(t)) + w_{\alpha,i}(t)(x_i(t) - \alpha_i(t)) \right) \\
 &\leq \begin{cases} \lambda \left(\frac{\beta_i(t^+) - \alpha_i(t)}{\mu_{g_i}(\{t\})} \right) & \text{if } (\alpha_i)'_{g_i}(t) \geq 0, \\ \lambda \left(\frac{\beta_i(t^+) - \alpha_i(t)}{\mu_{g_i}(\{t\})} + (\alpha_i)'_{g_i}(t)(x_i(t) - \alpha_i(t)) \right) & \text{if } (\alpha_i)'_{g_i}(t) \in \left[\frac{-1}{\mu_{g_i}(\{t\})}, 0 \right), \\ \lambda \left(\frac{\beta_i(t^+) - \alpha_i(t)}{\mu_{g_i}(\{t\})} - \frac{1}{\mu_{g_i}(\{t\})}(x_i(t) - \alpha_i(t)) \right) & \text{if } (\alpha_i)'_{g_i}(t) < \frac{-1}{\mu_{g_i}(\{t\})}, \end{cases} \\
 &\leq \begin{cases} \lambda \left(\frac{\beta_i(t^+) - \alpha_i(t)}{\mu_{g_i}(\{t\})} \right) & \text{if } (\alpha_i)'_{g_i}(t) \geq 0, \\ \lambda \left(\frac{\beta_i(t^+) - \alpha_i(t)}{\mu_{g_i}(\{t\})} - \frac{1}{\mu_{g_i}(\{t\})}(x_i(t) - \alpha_i(t)) \right) & \text{if } (\alpha_i)'_{g_i}(t) < 0, \end{cases} \\
 &\leq \lambda \left(\frac{\beta_i(t^+) - x_i(t)}{\mu_{g_i}(\{t\})} \right).
 \end{aligned}$$

Consequently,

$$(x_i)'_{g_i}(t) \leq \lambda \left(\frac{\beta_i(t^+) - x_i(t)}{\mu_{g_i}(\{t\})} \right). \quad (4.17)$$

To conclude Case 2, it follows from (4.15), (4.16) and (4.17) that,

$$(x_i)'_{g_i}(t) \leq \max \left\{ 0, \frac{\lambda(\beta_i(t^+) + 1 - x_i(t))}{\mu_{g_i}(\{t\})} \right\}$$

for $t \in [0, T) \cap D_{g_i}$ such that $x_i(t^+) > \tilde{R}$.

By (4.12),

$$(x_i)_{g_i}'(t) = \frac{x_i(t^+) - x_i(t)}{\mu_{g_i}(\{t\})} > \frac{\beta_i(t^+) + 1 - x_i(t)}{\mu_{g_i}(\{t\})}.$$

Since $\lambda \in [0, 1]$, from the two previous inequalities, we deduce that

$$(x_i)_{g_i}'(t) \leq 0 \quad \text{for } g_i\text{-almost all } t \in [0, T) \cap D_{g_i} \text{ such that } x_i(t^+) > \tilde{R}. \quad (4.18)$$

Hence, in both cases (4.14) and (4.18), we have

$$(x_i)_{g_i}'(t) \leq 0 \quad g_i\text{-almost everywhere in } \{t \in [0, T) : x_i(t^+) > \tilde{R}\}.$$

Consequently, Condition (ii) of Lemma 1.5.7 is satisfied with $u = x_i$ and $v = \tilde{R}$.

Condition (i) of this lemma is satisfied as well by $(H_{\alpha, \beta})$, (4.12) and the fact that $x_i(0) = \mathbf{x}_{0,i}$.

By Lemma 1.5.7, it follows that $x_i(t) \leq \tilde{R}$ for every $t \in [0, T]$. Using an analogous argument, we can show that $x_i(t) \geq -\tilde{R}$ for every $t \in [0, T]$. Since $i \in \{1, \dots, n\}$ is arbitrary, (4.13) is satisfied.

Now, we fix $R > \tilde{R}$ and we denote

$$\mathcal{U} = \{\mathbf{x} \in \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n) : \|\mathbf{x}\|_0 < R\}.$$

It follows from (4.13) that $\mathcal{F}(\lambda, \cdot)$ is fixed point free on $\partial\mathcal{U}$ for all $\lambda \in [0, 1]$. Since $\mathcal{F}(0, \mathbf{x}) = \mathbf{x}_0 \in \mathcal{U}$ for all $\mathbf{x} \in \mathcal{U}$, the normalization property of the fixed point index from Theorem 1.8.4 yields

$$\text{index}(\mathcal{F}(\lambda, \cdot), \mathcal{U}) = \text{index}(\mathcal{F}(0, \cdot), \mathcal{U}) = \text{index}(\mathbf{x}_0, \mathcal{U}) = 1 \quad \forall \lambda \in [0, 1].$$

Therefore, for all $\lambda \in [0, 1]$, $\mathcal{F}(\lambda, \cdot)$ has a fixed point in $\mathcal{U}$. Hence (4.6 _{λ}) has a solution. $\square$

We are now ready to demonstrate Theorem 4.3.1.

Proof of Theorem 4.3.1. Using Proposition 4.3.2, we know that the family of problems as at least a solution of each $\lambda \in [0, 1]$. Let us consider $\mathbf{x}$ a solution of (4.6 _{λ}) for $\lambda = 1$. Since $\mathbf{f}(t, \mathbf{c}) = \tilde{\mathbf{f}}(t, \mathbf{c})$ for $\alpha(t) \leq \mathbf{c} \leq \beta(t)$, it remains to show that $\alpha(t) \leq \mathbf{x}(t) \leq \beta(t)$ for every $t \in [0, T]$.

Assume that there exists $i \in \{1, \dots, n\}$ such that

$$A = \{t \in [0, T] : x_i(t) > \beta_i(t)\} \neq \emptyset.$$

We claim that

$$(x_i)_{g_i}'(t) \leq (\beta_i)_{g_i}'(t) \quad \text{for } g_i\text{-almost every } t \in [0, T) \text{ such that } x_i(t^+) > \beta_i(t^+). \quad (4.19)$$

Observe that for g_i -almost every $t \in A \setminus D_{g_i}$,

$$x_i(t^+) = x_i(t) > \beta_i(t) = \beta_i(t^+).$$

By $(H_{\alpha,\beta})$, (4.7) and (4.8), one has

$$\begin{aligned} (x_i)'_{g_i}(t) &= f_i(t, \overline{\mathbf{x}(t)}) - w_{\beta,i}(t)(x_i(t) - \beta_i(t)) \\ &\leq (\beta_i)'_{g_i}(t) - w_{\beta,i}(t)(x_i(t) - \beta_i(t)) \leq (\beta_i)'_{g_i}(t). \end{aligned} \quad (4.20)$$

On the other hand, let $t \in D_{g_i}$ be such that $x_i(t^+) > \beta_i(t^+)$. If $x_i(t) > \beta_i(t)$, then, by $(H_{\alpha,\beta})$, (4.7), and (4.9), we obtain that

$$\begin{aligned} (x_i)'_{g_i}(t) &= f_i(t, \overline{\mathbf{x}(t)}) - w_{\beta,i}(t)(x_i(t) - \beta_i(t)) \\ &\leq (\beta_i)'_{g_i}(t) - w_{\beta,i}(t)(x_i(t) - \beta_i(t)) \leq (\beta_i)'_{g_i}(t). \end{aligned} \quad (4.21)$$

If $x_i(t) \leq \beta_i(t)$, $(H_{\alpha,\beta})$, (4.7), (4.9) and (4.10) imply that

$$\begin{aligned} (x_i)'_{g_i}(t) &= \begin{cases} f_i(t, \overline{\mathbf{x}(t)}) & \text{if } \alpha_i(t) \leq x_i(t), \\ f_i(t, \overline{\mathbf{x}(t)}) + w_{\alpha,i}(t)(x_i(t) - \alpha_i(t)) & \text{if } x_i(t) < \alpha_i(t), \end{cases} \\ &\leq \begin{cases} \frac{\beta_i(t^+) - x_i(t)}{\mu_{g_i}(\{t\})} & \text{if } \alpha_i(t) \leq x_i(t), \\ f_i(t, \overline{\mathbf{x}(t)}) + w_{\alpha,i}(t)(x_i(t) - \alpha_i(t)) & \text{if } x_i(t) < \alpha_i(t), \end{cases} \\ &\leq \begin{cases} \frac{\beta_i(t^+) - x_i(t)}{\mu_{g_i}(\{t\})} & \text{if } \alpha_i(t) \leq x_i(t), \\ \frac{\beta_i(t^+) - \alpha_i(t)}{\mu_{g_i}(\{t\})} + w_{\alpha,i}(t)(x_i(t) - \alpha_i(t)) & \text{if } x_i(t) < \alpha_i(t), \end{cases} \\ &\leq \begin{cases} \frac{\beta_i(t^+) - x_i(t)}{\mu_{g_i}(\{t\})} & \text{if } \alpha_i(t) \leq x_i(t), \\ \frac{\beta_i(t^+) - \alpha_i(t)}{\mu_{g_i}(\{t\})} & \text{if } x_i(t) < \alpha_i(t) \\ & \text{and } (\alpha_i)'_{g_i}(t) \geq 0, \\ \frac{\beta_i(t^+) - \alpha_i(t)}{\mu_{g_i}(\{t\})} + (\alpha_i)'_{g_i}(t)(x_i(t) - \alpha_i(t)) & \text{if } x_i(t) < \alpha_i(t) \\ & \text{and } (\alpha_i)'_{g_i}(t) \in \left[\frac{-1}{\mu_{g_i}(\{t\})}, 0 \right), \\ \frac{\beta_i(t^+) - \alpha_i(t)}{\mu_{g_i}(\{t\})} - \frac{1}{\mu_{g_i}(\{t\})}(x_i(t) - \alpha_i(t)) & \text{if } x_i(t) < \alpha_i(t) \\ & \text{and } (\alpha_i)'_{g_i}(t) < \frac{-1}{\mu_{g_i}(\{t\})}, \end{cases} \end{aligned}$$

$$\begin{aligned}
 & \leq \begin{cases} \frac{\beta_i(t^+) - x_i(t)}{\mu_{g_i}(\{t\})} & \text{if } \alpha_i(t) \leq x_i(t), \\ \frac{\beta_i(t^+) - \alpha_i(t)}{\mu_{g_i}(\{t\})} & \text{if } x_i(t) < \alpha_i(t) \\ & \text{and } (\alpha_i)'_{g_i}(t) \geq 0, \\ \frac{\beta_i(t^+) - \alpha_i(t)}{\mu_{g_i}(\{t\})} - \frac{1}{\mu_{g_i}(\{t\})}(x_i(t) - \alpha_i(t)) & \text{if } x_i(t) < \alpha_i(t) \\ & \text{and } (\alpha_i)'_{g_i}(t) < 0, \end{cases} \\
 & \leq \frac{\beta_i(t^+) - x_i(t)}{\mu_{g_i}(\{t\})}.
 \end{aligned}$$

This is a contradiction, since

$$(\alpha_i)'_{g_i}(t) = \frac{x_i(t^+) - x_i(t)}{\mu_{g_i}(\{t\})} > \frac{\beta_i(t^+) - x_i(t)}{\mu_{g_i}(\{t\})}.$$

Thus,

$$\{t \in [0, T) \cap D_{g_i} : x_i(t^+) > \beta_i(t^+)\} \subset \{t \in [0, T) \cap D_{g_i} : x_i(t) > \beta_i(t)\}. \quad (4.22)$$

We deduce that (4.19) is satisfied from (4.20), (4.21) and (4.22). Also, by $(H_{\alpha, \beta})$ and the fact that $\mathbf{x}(0) = \mathbf{x}_0$, one has that $x_i(0) \leq \beta_i(0)$.

It follows from Lemma 1.5.7 that $x_i(t) \leq \beta_i(t)$ for every $t \in [0, T]$. Hence, $\mathbf{x}(t) \leq \beta(t)$ for all $t \in [0, T]$ given that $i \in \{1, \dots, n\}$ is arbitrary.

Similarly, we can show that $\alpha(t) \leq \mathbf{x}(t)$ for every $t \in [0, T]$, and hence, we conclude that $\mathbf{x} \in [\alpha, \beta]$ is a solution of (4.1), (4.3). $\square$

Remark 4.3.3. Notice that in Theorem 4.3.1, $\mathbf{f}$ is assumed to be $\mathbf{g}$ -Carathéodory on $[0, T] \times \mathbb{R}^n$. It could be $\mathbf{g}$ -Carathéodory on a smaller set containing $[\alpha, \beta]$. Indeed, it suffices to have $(t, \mathbf{c}) \mapsto \mathbf{f}(t, \bar{\mathbf{c}})$ $\mathbf{g}$ -Carathéodory.

4.3.2 Systems with a periodic boundary value condition

In this subsection, we establish an existence result analogous to Theorem 4.3.1 with the initial value condition replaced by the periodic boundary condition (4.4). In particular, we extend our result in Theorem 3.1.7 obtained in the case of one Stieltjes differential equation (i.e. $n = 1$). Again, we do not impose any monotonicity condition on the right-hand side member $\mathbf{f}$. Let $\mathbf{g} = (g_1, \dots, g_n) : \mathbb{R} \rightarrow \mathbb{R}^n$ be a vector-valued derivator such that $g_i : \mathbb{R} \rightarrow \mathbb{R}$ is a left-continuous and nondecreasing function.

Theorem 4.3.4. *Let $\mathbf{f} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a $\mathbf{g}$ -Carathéodory function, and assume that $(H_{\alpha, \beta}^p)$ there exist $\alpha, \beta \in \mathcal{AC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ such that (α, β) is an ordered pair of vector-valued lower and upper solutions of (4.1), (4.4).*

Then, the problem (4.1), (4.4) has at least one solution $\mathbf{x} \in \mathcal{AC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ such that $\alpha(t) \leq \mathbf{x}(t) \leq \beta(t)$ for every $t \in [0, T]$.

In order to demonstrate this theorem, let us consider the following family of problems defined for $\lambda \in [0, 1]$ by:

$$\begin{aligned} \mathbf{x}'_{\mathbf{g}}(t) &= \lambda \tilde{\mathbf{f}}(t, \mathbf{x}(t)) + \mathbf{K}(\lambda, \cdot) \circ \mathbf{N}_{\tilde{\mathbf{f}}}(\mathbf{x})(T), \\ \mathbf{x}(0) &= \mathbf{x}(T), \end{aligned} \quad (4.23_{\lambda})$$

where $\tilde{\mathbf{f}}$, $\mathbf{N}_{\tilde{\mathbf{f}}}$ are defined in (4.7) and (4.5) respectively, and $\mathbf{K} : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given for every $\lambda \in [0, 1]$ and $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$ by

$$\mathbf{K}(\lambda, \mathbf{c}) = \left(\frac{(k_1 - \lambda)c_1}{g_1(T) - g_1(0)}, \dots, \frac{(k_n - \lambda)c_n}{g_n(T) - g_n(0)} \right), \quad (4.24)$$

with, for $i = 1, \dots, n$, $k_i > 0$ chosen such that

$$k_i + \lambda g_i(0) \neq 0 \quad \forall \lambda \in [0, 1]. \quad (4.25)$$

Let us consider the operator $\mathcal{H} : [0, 1] \times \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n) \rightarrow \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ defined for every $(\lambda, \mathbf{x}) \in [0, 1] \times \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ by

$$\mathcal{H}(\lambda, \mathbf{x})(t) = \mathbf{x}(0) + \lambda \mathbf{N}_{\tilde{\mathbf{f}}}(\mathbf{x})(t) - \tilde{\mathbf{K}}(\lambda, t, \cdot) \circ \mathbf{N}_{\tilde{\mathbf{f}}}(\mathbf{x})(T) \quad \forall t \in [0, T], \quad (4.26)$$

where $\tilde{\mathbf{K}} : [0, 1] \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by

$$\tilde{\mathbf{K}}(\lambda, t, \mathbf{c}) = \left(\frac{(k_1 + \lambda g_1(t))c_1}{g_1(T) - g_1(0)}, \dots, \frac{(k_n + \lambda g_n(t))c_n}{g_n(T) - g_n(0)} \right), \quad (4.27)$$

for all $(\lambda, t) \in [0, 1] \times [0, T]$ and $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$.

Similarly to Proposition 4.3.2, the following result ensures the existence of fixed points of $\mathcal{H}$.

Proposition 4.3.5. *Under the hypotheses of Theorem 4.3.4, there exists a constant $R > \max\{\|\alpha\|_0, \|\beta\|_0\}$ such that*

$$\text{index}(\mathcal{H}(\lambda, \cdot), \mathcal{U}) = (-1)^n \quad \text{for every } \lambda \in [0, 1],$$

where $\mathcal{U} = \{\mathbf{x} \in \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n) : \|\mathbf{x}\|_0 < R\}$. In particular, the problem (4.23 $_{\lambda}$) has at least one solution for every $\lambda \in [0, 1]$.

Proof. By Lemma 4.1.3, the operator $\mathcal{H}$ is continuous and completely continuous.

Let us fix $\lambda \in [0, 1]$. We claim that the fixed points of $\mathcal{H}(\lambda, \cdot)$ are solutions of (4.23 $_{\lambda}$). Indeed, if $\mathbf{x} \in \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ is such that $\mathbf{x} = \mathcal{H}(\lambda, \mathbf{x})$, then, for all $t \in [0, T]$,

$$\mathbf{x}(t) = \mathbf{x}(0) + \lambda \mathbf{N}_{\tilde{\mathbf{f}}}(\mathbf{x})(t) - \tilde{\mathbf{K}}(\lambda, t, \cdot) \circ \mathbf{N}_{\tilde{\mathbf{f}}}(\mathbf{x})(T). \quad (4.28)$$

Observe that for $t = 0$, this equation becomes

$$\mathbf{x}(0) = \mathbf{x}(0) - \tilde{\mathbf{K}}(\lambda, 0, \cdot) \circ \mathbf{N}_{\tilde{\mathbf{f}}}(\mathbf{x})(T).$$

Thus, using (4.25), we deduce that

$$\mathbf{N}_{\tilde{\mathbf{f}}}(\mathbf{x})(T) = (0, \dots, 0). \quad (4.29)$$

Now, taking $t = T$ in (4.28) gives

$$\mathbf{x}(T) = \mathbf{x}(0) + \lambda \mathbf{N}_{\tilde{\mathbf{f}}}(\mathbf{x})(T) - \widetilde{\mathbf{K}}(\lambda, T, \cdot) \circ \mathbf{N}_{\tilde{\mathbf{f}}}(\mathbf{x})(T).$$

This equality combined with (4.29) ensures that

$$\mathbf{x}(0) = \mathbf{x}(T). \quad (4.30)$$

Moreover, for $i = 1, \dots, n$, deriving both sides of the i^{th} equation in (4.28) with respect to g_i gives

$$\begin{aligned} (x_i)_{g_i}'(t) &= \lambda \tilde{f}_i(t, \mathbf{x}(t)) - \frac{\lambda}{g_i(T) - g_i(0)} N_{\tilde{f}_i}(\mathbf{x})(T) \\ &= \lambda \tilde{f}_i(t, \mathbf{x}(t)) \\ &= \lambda \tilde{f}_i(t, \mathbf{x}(t)) + \frac{k_i - \lambda}{g_i(T) - g_i(0)} N_{\tilde{f}_i}(\mathbf{x})(T). \end{aligned} \quad (4.31)$$

Hence, $\mathbf{x}$ is a solution of (4.23 _{λ}).

Let $\tilde{R} > 0$ be as in (4.12). Arguing as in the proof of Proposition 4.3.2, it can be shown that, for every fixed point $\mathbf{x}$ of $\mathcal{H}(\lambda, \cdot)$ and all $i \in \{1, \dots, n\}$,

$$\begin{aligned} (-1)^j (x_i)_{g_i}'(t) &\leq 0 \\ &\text{for } g_i\text{-almost every } t \in \{t \in [0, T) : (-1)^j x_i(t^+) > \tilde{R}\}, \text{ for } j = 0, 1. \end{aligned}$$

Lemma 1.5.7 implies that

$$|x_i(t)| \leq \tilde{R} \quad \text{for every } t \in [0, T], \quad (4.32)$$

or

$$\exists c > 0 \text{ such that } x_i(t) = \tilde{R} + c \text{ (or } x_i(t) = -\tilde{R} - c) \quad \forall t \in [0, T]. \quad (4.33)$$

If Alternative (4.33) holds, without loss of generality, $x_i(t) = \tilde{R} + c > \tilde{R}$. Observe that, by (4.8), (4.9) and (4.10), we have

$$\begin{aligned} w_{\beta,i}(t) &> (\beta_i)_{g_i}'(t) \quad \text{for } g_i\text{-almost every } t \in [0, T) \setminus D_{g_i}, \\ w_{\beta,i}(t) &\geq (\beta_i)_{g_i}'(t) \quad \text{for all } t \in D_{g_i} \setminus \mathcal{T}_{\beta,i}, \\ w_{\beta,i}(t) &= \frac{1}{\mu_{g_i}(\{t\})} \quad \text{for all } t \in \mathcal{T}_{\beta,i}. \end{aligned} \quad (4.34)$$

Thus, by (4.7), (4.12), (4.29), (4.34), and $(H_{\alpha,\beta}^p)$, one has that

$$\begin{aligned} 0 &= N_{\tilde{f}_i}(\mathbf{x})(T) = \int_{[0,T)} \tilde{f}_i(t, \mathbf{x}(t)) \, d\mu_{g_i}(t) \\ &= \int_{[0,T)} f_i(t, \overline{\mathbf{x}(t)}) - w_{\beta,i}(t)(x_i(t) - \beta_i(t)) \, d\mu_{g_i}(t) \\ &= \int_{[0,T)} f_i(t, \overline{\mathbf{x}(t)}) - w_{\beta,i}(t)(\tilde{R} + c - \beta_i(t)) \, d\mu_{g_i}(t) \\ &\leq \int_{[0,T) \setminus D_{g_i}} (\beta_i)_{g_i}'(t) - w_{\beta,i}(t)(\tilde{R} + c - \beta_i(t)) \, d\mu_{g_i}(t) \end{aligned}$$

$$\begin{aligned}
 & + \int_{([0,T] \cap D_{g_i}) \setminus \mathcal{T}_{\beta,i}} (\beta_i)'_{g_i}(t) - w_{\beta,i}(t)(\tilde{R} + c - \beta_i(t)) \, d\mu_{g_i}(t) \\
 & + \int_{\mathcal{T}_{\beta,i}} (\beta_i)'_{g_i}(t) - w_{\beta,i}(t)(\tilde{R} + c - \beta_i(t)) \, d\mu_{g_i}(t) \\
 & < \int_{\mathcal{T}_{\beta,i}} (\beta_i)'_{g_i}(t) - w_{\beta,i}(t)(\tilde{R} + c - \beta_i(t)) \, d\mu_{g_i}(t) \\
 & = \int_{\mathcal{T}_{\beta,i}} \frac{1}{\mu_{g_i}(\{t\})} (\beta_i(t^+) - \tilde{R} - c) \, d\mu_{g_i}(t) \\
 & \leq 0,
 \end{aligned}$$

which is a contradiction. Thus (4.32) holds.

Therefore, since $i \in \{1, \dots, n\}$ is arbitrary, we deduce that

$$\|\mathbf{x}\|_0 \leq \tilde{R} \quad \text{for any fixed point } \mathbf{x} \text{ of } \mathcal{H}(\lambda, \cdot), \text{ for } \lambda \in [0, 1]. \quad (4.35)$$

Now, let us fix $R > \tilde{R}$ be such that, for all $i \in \{1, \dots, n\}$,

$$\text{card}(\mathcal{T}_{\beta,i})R \geq \sum_{t \in \mathcal{T}_{\beta,i}} \beta_i(t^+), \quad \text{and} \quad -\text{card}(\mathcal{T}_{\alpha,i})R \leq \sum_{t \in \mathcal{T}_{\alpha,i}} \alpha_i(t^+). \quad (4.36)$$

Observe that

$$\text{card}(\mathcal{T}_{\beta,i}) = \int_{\mathcal{T}_{\beta,i}} \frac{1}{\mu_{g_i}(\{t\})} \, d\mu_{g_i}(t) \leq \int_{\mathcal{T}_{\beta,i}} (\beta_i)'_{g_i}(t) \, d\mu_{g_i}(t) \leq \int_{[a,b]} |(\beta_i)'_{g_i}(t)| \, d\mu_{g_i}(t) < \infty.$$

Similarly,

$$\text{card}(\mathcal{T}_{\alpha,i}) < \infty \quad \text{and} \quad w_{\alpha,i}(t) \geq (\alpha_i)'_{g_i}(t) \quad \text{for all } t \in D_{g_i} \setminus \mathcal{T}_{\alpha,i}.$$

We set

$$\mathcal{U} = \{\mathbf{x} \in \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n) : \|\mathbf{x}\|_0 < R\}. \quad (4.37)$$

It follows from (4.35) that $\mathcal{H}(\lambda, \cdot)$ is fixed point free on $\partial\mathcal{U}$ for all $\lambda \in [0, 1]$. We want to show that $\text{index}(\mathcal{H}(\lambda, \cdot), \mathcal{U})$ is different from zero. Using the homotopy property of the fixed point index from Theorem 1.8.4, we obtain

$$\text{index}(\mathcal{H}(\lambda, \cdot), \mathcal{U}) = \text{index}(\mathcal{H}(0, \cdot), \mathcal{U}) \quad \forall \lambda \in [0, 1]. \quad (4.38)$$

Observe that, for each $\mathbf{x} \in \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$, the map $\mathcal{H}(0, \mathbf{x})$ is constant since

$$\mathcal{H}(0, \mathbf{x})(t) = \mathbf{x}(0) - \widetilde{\mathbf{K}}(0, t, \cdot) \circ \mathbf{N}_{\widetilde{\mathbf{f}}}(\mathbf{x})(T) = \mathbf{x}(0) - \mathbf{K}(0, \cdot) \circ \mathbf{N}_{\widetilde{\mathbf{f}}}(\mathbf{x})(T) \quad \forall t \in [0, T].$$

Also, let us consider the n -dimensional subspace of $\mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ defined by:

$$X_{\mathbb{R}^n} = \{\mathbf{u} \in \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n) : \mathbf{u}(t) = \mathbf{u}(0), \quad \forall t \in [0, T]\}.$$

Thus, $\mathcal{U} \cap X_{\mathbb{R}^n} = B_{\mathbb{R}^n}(\mathbf{0}, R)$ the open ball of radius R centered at $\mathbf{0} = (0, \dots, 0)$ in $\mathbb{R}^n$. The contraction property of the fixed point index from Theorem 1.8.7 yields

$$\text{index}(\mathcal{H}(0, \cdot), \mathcal{U}) = \text{index}(\mathcal{H}(0, \cdot)|_{X_{\mathbb{R}^n}}, B_{\mathbb{R}^n}(\mathbf{0}, R)). \quad (4.39)$$

Let $\mathcal{H}^0 : [0, 1] \times \overline{B_{\mathbb{R}^n}(\mathbf{0}, R)} \rightarrow \mathbb{R}^n$ be defined by

$$\mathcal{H}^0(\lambda, \mathbf{x}) = 2(1 - \lambda)\mathbf{x} + \lambda\mathcal{H}(0, \mathbf{x}).$$

Notice that

$$\text{index} \left(\mathcal{H}(0, \cdot)|_{X_{\mathbb{R}^n}}, B_{\mathbb{R}^n}(\mathbf{0}, R) \right) = \text{index} \left(\mathcal{H}^0(1, \cdot), B_{\mathbb{R}^n}(\mathbf{0}, R) \right). \quad (4.40)$$

We claim that $\mathcal{H}^0(\lambda, \cdot)$ has no fixed point on $\partial(B_{\mathbb{R}^n}(\mathbf{0}, R))$ for all $\lambda \in [0, 1]$. Indeed, assume that there exists $(\lambda, \mathbf{x}) \in [0, 1] \times \partial(B_{\mathbb{R}^n}(\mathbf{0}, R))$ such that $\mathbf{x} = \mathcal{H}^0(\lambda, \mathbf{x})$. Without loss of generality, let $i \in \{1, \dots, n\}$ be such that $x_i = R$. Then, denoting $\mathcal{H}_i^0(\lambda, \mathbf{x})$ the i^{th} coordinate of $\mathcal{H}^0(\lambda, \mathbf{x})$, we obtain

$$\begin{aligned} R &= \mathcal{H}_i^0(\lambda, \mathbf{x}) = 2(1 - \lambda)R + \lambda \left(R - \frac{k_i}{g_i(T) - g_i(0)} N_{\tilde{f}_i}(\mathbf{x})(T) \right) \\ &= (2 - \lambda)R - \frac{\lambda k_i}{g_i(T) - g_i(0)} \int_{[0, T)} \tilde{f}_i(t, \mathbf{x}(t)) \, d\mu_{g_i}(t) \\ &= (2 - \lambda)R - \frac{\lambda k_i}{g_i(T) - g_i(0)} \int_{[0, T)} \left(f_i(t, \overline{\mathbf{x}(t)}) - w_{\beta, i}(t)(R - \beta_i(t)) \right) \, d\mu_{g_i}(t) \\ &\geq (2 - \lambda)R - \frac{\lambda k_i}{g_i(T) - g_i(0)} \int_{[0, T)} \left((\beta_i)'_{g_i}(t) - w_{\beta, i}(t)(R - \beta_i(t)) \right) \, d\mu_{g_i}(t) \\ &\geq (2 - \lambda)R - \frac{\lambda k_i}{g_i(T) - g_i(0)} \int_{[0, T) \setminus \mathcal{T}_{\beta, i}} \left((\beta_i)'_{g_i}(t) - w_{\beta, i}(t)(R - \beta_i(t)) \right) \, d\mu_{g_i}(t) \\ &\quad - \frac{\lambda k_i}{g_i(T) - g_i(0)} \int_{\mathcal{T}_{\beta, i}} \left((\beta_i)'_{g_i}(t) - \frac{1}{\mu_{g_i}(\{t\})}(R - \beta_i(t)) \right) \, d\mu_{g_i}(t) \\ &> R - \frac{\lambda k_i}{g_i(T) - g_i(0)} \int_{[0, T) \setminus \mathcal{T}_{\beta, i}} \left((\beta_i)'_{g_i}(t) - w_{\beta, i}(t) \right) \, d\mu_{g_i}(t) \\ &\quad - \frac{\lambda k_i}{g_i(T) - g_i(0)} \int_{\mathcal{T}_{\beta, i}} \left(\frac{\beta_i(t^+) - R}{\mu_{g_i}(\{t\})} \right) \, d\mu_{g_i}(t) \\ &= R - \frac{\lambda k_i}{g_i(T) - g_i(0)} \int_{[0, T) \setminus \mathcal{T}_{\beta, i}} \left((\beta_i)'_{g_i}(t) - w_{\beta, i}(t) \right) \, d\mu_{g_i}(t) \\ &\quad + \frac{\lambda k_i}{g_i(T) - g_i(0)} \left(\text{card}(\mathcal{T}_{\beta, i})R - \sum_{t \in \mathcal{T}_{\beta, i}} \beta_i(t^+) \right) \\ &\geq R. \end{aligned}$$

The last inequality follows from (4.36), which yields a contradiction.

Using the homotopy property of the fixed point index from Theorem 1.8.4, and by Corollary 1.8.13, we obtain

$$\begin{aligned} \text{index} \left(\mathcal{H}^0(1, \cdot), B_{\mathbb{R}^n}(\mathbf{0}, R) \right) &= \text{index} \left(\mathcal{H}^0(0, \cdot), B_{\mathbb{R}^n}(\mathbf{0}, R) \right) \\ &= \text{index} \left(2 \text{id}_{\mathbb{R}^n}, B_{\mathbb{R}^n}(\mathbf{0}, R) \right) \\ &= (-1)^n. \end{aligned}$$

Combining this with (4.38), (4.39) and (4.40), we obtain

$$\text{index} \left(\mathcal{H}(\lambda, \cdot), \mathcal{U} \right) = (-1)^n \quad \forall \lambda \in [0, 1].$$

Therefore, for all $\lambda \in [0, 1]$, $\mathcal{H}(\lambda, \cdot)$ has a fixed point in $\mathcal{U}$. Hence, the problem (4.23 $_{\lambda}$) has a solution. $\square$

Now, we are ready to prove Theorem 4.3.4.

Proof of Theorem 4.3.4. Let $\mathbf{x}$ be a solution of (4.23 $_{\lambda}$) for $\lambda = 1$ guaranteed by Proposition 4.3.5. Since $\mathbf{f}(t, \mathbf{c}) = \tilde{\mathbf{f}}(t, \mathbf{c})$ for $\alpha(t) \leq \mathbf{c} \leq \beta(t)$, and $\mathbf{N}_{\tilde{\mathbf{f}}}(\mathbf{x})(T) = \mathbf{0} = (0, \dots, 0)$ by (4.29), it remains to show that $\alpha(t) \leq \mathbf{x}(t) \leq \beta(t)$ for every $t \in [0, T]$.

Arguing as in the proof of Theorem 4.3.1, we obtain that

$$(x_i)_{g_i}'(t) \leq (\beta_i)_{g_i}'(t) \quad g_i\text{-almost everywhere in } \{t \in [0, T] : x_i(t^+) > \beta_i(t^+)\}.$$

Thus, it follows from Lemma 1.5.7 that $x_i(t) \leq \beta_i(t)$ for every $t \in [0, T]$ or there exists $c > 0$ such that $x_i(t) = \beta_i(t) + c$ for every $t \in [0, T]$.

If we assume that the last case holds, then by (4.7), (4.8), (4.9), and $(H_{\alpha, \beta}^p)$, we obtain

$$\begin{aligned} \int_{[0, T)} (\beta_i)_{g_i}'(t) \, d\mu_{g_i}(t) &= \int_{[0, T)} (x_i)_{g_i}'(t) \, d\mu_{g_i}(t) = \int_{[0, T)} \tilde{f}_i(t, \mathbf{x}(t)) \, d\mu_{g_i}(t) \\ &= \int_{[0, T)} f_i(t, \overline{\mathbf{x}(t)}) - w_{\beta, i}(t) (x_i(t) - \beta_i(t)) \, d\mu_{g_i}(t) \\ &\leq \int_{[0, T)} (\beta_i)_{g_i}'(t) - cw_{\beta, i}(t) \, d\mu_{g_i}(t) \\ &< \int_{[0, T)} (\beta_i)_{g_i}'(t) \, d\mu_{g_i}(t), \end{aligned}$$

which is a contraction.

Hence, we deduce that $x_i(t) \leq \beta_i(t)$ for every $t \in [0, T]$ and $i \in \{1, \dots, n\}$.

Arguing similarly, we can show that $\alpha(t) \leq \mathbf{x}(t)$ for every $t \in [0, T]$. Consequently, we conclude that $\mathbf{x} \in [\alpha, \beta]_{\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R})}$ is a solution of (4.1), (4.4). $\square$

4.4 Existence of solutions for functional problems

In this section, we prove the existence of a solution for the system of functional Stieltjes differential equations (4.2), (4.3) and (4.2), (4.4), where $\mathbf{g} = (g_1, \dots, g_n) : \mathbb{R} \rightarrow \mathbb{R}^n$ is a vector-valued derivator such that $g_i : \mathbb{R} \rightarrow \mathbb{R}$ is a left-continuous and nondecreasing function. For sake of simplicity, we assume that $\mathbf{g}$ is continuous at 0, which is not a restrictive condition as shown in [31, Page 21].

4.4.1 Functional systems with an initial value condition

In this subsection, we establish an existence result for the initial value problem for the system of functional Stieltjes differential equations (4.2), (4.3). To achieve this, we rely on the following lemma that ensures the uniqueness in case of existence of a solution. Its proof follows similar arguments as those used in the proof of [56, Theorem 4.3].

Lemma 4.4.1. *Let $X \subset [0, T] \times \mathbb{R}^n$ and $\mathbf{f} : X \rightarrow \mathbb{R}^n$ be a $\mathbf{g}$ -Carathéodory function satisfying*

(H_L) *there exists $L : [0, T) \rightarrow [0, \infty)$ such that, for all $i = 1, \dots, n$, $L \in \mathcal{L}_{g_i}^1([0, T), [0, \infty))$ and*

$$|f_i(t, \mathbf{x}) - f_i(t, \mathbf{y})| \leq L(t) \|\mathbf{x} - \mathbf{y}\|,$$

for g_i -almost every $t \in [0, T)$ and for all $\mathbf{x}, \mathbf{y}$ such that $(t, \mathbf{x}), (t, \mathbf{y}) \in X$.

Then, for $(0, \mathbf{x}_0) \in X$, the problem (4.1), (4.3) has at most one solution $\mathbf{x}$ such that $(t, \mathbf{x}(t)) \in X$ for all $t \in [0, T]$.

Now, for $\mathcal{C} \subset \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$, $\mathbf{f} : [0, T] \times \mathbb{R}^n \times \mathcal{C} \rightarrow \mathbb{R}^n$ and $\gamma \in \mathcal{C}$, let us define the function $\mathbf{f}_\gamma : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\mathbf{f}_\gamma(\cdot, \cdot) := \mathbf{f}(\cdot, \cdot, \gamma).$$

To establish the existence of solutions to the functional problem (4.2), we introduce a family of problems. Indeed, for $\gamma \in \mathcal{C}$, we consider:

$$\begin{aligned} \mathbf{x}'_{\mathbf{g}}(t) &= \mathbf{f}_\gamma(t, \mathbf{x}(t)), \\ \mathbf{x}(0) &= \mathbf{x}_0 = (\mathbf{x}_{0,1}, \dots, \mathbf{x}_{0,n}). \end{aligned} \tag{4.41}_\gamma$$

Now, we can state our main existence result for the functional problem with an initial value condition.

Theorem 4.4.2. *Let $\mathcal{C}$ be a bounded, closed, convex subset of $\mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$, and let a function $\mathbf{f} : [0, T] \times \mathbb{R}^n \times \mathcal{C} \rightarrow \mathbb{R}^n$ satisfy the following assumptions:*

- (H1) *for each $\gamma \in \mathcal{C}$, $\mathbf{f}_\gamma$ is a $\mathbf{g}$ -Carathéodory function;*
- (H2) *for each $\gamma \in \mathcal{C}$, there exist $\alpha_\gamma, \beta_\gamma \in \mathcal{AC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ such that $(\alpha_\gamma, \beta_\gamma)$ is an ordered pair of vector-valued lower and upper solutions of (4.41) _{γ} , and $[\alpha_\gamma, \beta_\gamma]_{\mathcal{AC}_{\mathbf{g}}([a,b], \mathbb{R}^n)} \subset \mathcal{C}$;*
- (H3) *for every $r > 0$, there exist $\phi_r : [0, \infty) \rightarrow [0, \infty)$ continuous at 0, bounded on bounded sets satisfying $\phi_r(0) = 0$, and $L_r, \omega_r \in \mathcal{L}_{g_i}^1([0, T), [0, \infty))$ for all $i = 1, \dots, n$, such that*

$$|f_i(t, \mathbf{x}, \gamma) - f_i(t, \mathbf{y}, \hat{\gamma})| \leq L_r(t) \|\mathbf{x} - \mathbf{y}\| + \omega_r(t) \phi_r(\|\gamma - \hat{\gamma}\|_0),$$

for g_i -almost every $t \in [0, T)$, all $\gamma, \hat{\gamma} \in \mathcal{C}$ and for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\|\mathbf{x}\| \leq r$, $\|\mathbf{y}\| \leq r$.

Then, the problem (4.2), (4.3) has a solution $\mathbf{x} \in [\alpha_{\mathbf{x}}, \beta_{\mathbf{x}}]_{\mathcal{AC}_{\mathbf{g}}([a,b], \mathbb{R}^n)} \subset \mathcal{C}$.

Proof. For $\gamma \in \mathcal{C}$, Conditions (H1) and (H2) imply that $\mathbf{f}_\gamma$ and the pair $(\alpha_\gamma, \beta_\gamma)$ satisfy the assumptions of Theorem 4.3.1. Thus, the problem (4.41) _{γ} has a solution $\mathbf{x}_\gamma \in [\alpha_\gamma, \beta_\gamma]_{\mathcal{AC}_{\mathbf{g}}([a,b], \mathbb{R}^n)} \subset \mathcal{C}$.

By assumption, $\mathcal{C}$ is bounded. Let $R > 0$ be such that $\|\gamma\|_0 \leq R$ for all $\gamma \in \mathcal{C}$. By means of Condition (H3) and Lemma 4.4.1, we deduce that $\mathbf{x}_\gamma$ is the unique solution of (4.41) _{γ} in $\mathcal{C}$.

Let us define the operator $\mathcal{S} : \mathcal{C} \rightarrow \mathcal{C}$ by

$$\mathcal{S}(\gamma) = \mathbf{x}_\gamma.$$

To conclude that (4.2), (4.3) has a solution in $\mathcal{C}$, it suffices to prove that $\mathcal{S}$ has a fixed point. To this aim, we prove that $\mathcal{S}$ satisfies the hypotheses of the Schauder fixed point theorem, Theorem 1.8.2.

First of all, we show that the operator $\mathcal{S}$ is continuous. Indeed, on $\mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$, let us consider the norm $\|\cdot\|_*$ defined by

$$\|\mathbf{x}\|_* = \sup_{t \in [0, T]} \frac{\|\mathbf{x}(t)\|}{e_{L_R}(t, 0)}, \quad \text{for all } \mathbf{x} \in \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n),$$

where $e_{L_R}(\cdot, 0)$ is given in Definition 1.6.1. Observe that $\|\cdot\|_*$ is equivalent to $\|\cdot\|_0$ since

$$e_{L_R}^{-1}(T, 0)\|\mathbf{x}\|_0 \leq \|\mathbf{x}\|_* \leq \|\mathbf{x}\|_0.$$

It follows from Lemma 1.6.4 and (H3) that $e_{L_R}(\cdot, 0) \in \mathcal{AC}_{\hat{g}}([0, T], \mathbb{R})$, where $\hat{g}$ is defined in (1.5). Let $\{\gamma_k\}$ be a sequence converging to $\gamma \in \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n) \cap \mathcal{C}$. Using (4.41_γ),

$$\|\mathcal{S}(\gamma_k) - \mathcal{S}(\gamma)\|_* = \sup_{t \in [0, T]} e_{L_R}^{-1}(t, 0) \left\| \int_{[0, t)} \mathbf{f}(s, \mathbf{x}_{\gamma_k}(s), \gamma_k) - \mathbf{f}(s, \mathbf{x}_{\gamma}(s), \gamma) \, d\mu_{\mathbf{g}}(s) \right\|.$$

Thus, there exists $j \in \{1, \dots, n\}$ such that

$$\begin{aligned} \|\mathcal{S}(\gamma_k) - \mathcal{S}(\gamma)\|_* &= \sup_{t \in [0, T]} e_{L_R}^{-1}(t, 0) \left| \int_{[0, t)} f_j(s, \mathbf{x}_{\gamma_k}(s), \gamma_k) - f_j(s, \mathbf{x}_{\gamma}(s), \gamma) \, d\mu_{g_j}(s) \right| \\ &\leq \sup_{t \in [0, T]} e_{L_R}^{-1}(t, 0) \int_{[0, t)} |f_j(s, \mathbf{x}_{\gamma_k}(s), \gamma_k) - f_j(s, \mathbf{x}_{\gamma}(s), \gamma)| \, d\mu_{g_j}(s). \end{aligned}$$

Using (H3) and (1.5), we obtain

$$\begin{aligned} \|\mathcal{S}(\gamma_k) - \mathcal{S}(\gamma)\|_* &\leq \sup_{t \in [0, T]} \left(e_{L_R}^{-1}(t, 0) \int_{[0, t)} L_R(s) \|\mathbf{x}_{\gamma_k}(s) - \mathbf{x}_{\gamma}(s)\| \right. \\ &\quad \left. + \omega_R(s) \phi_R(\|\gamma_k - \gamma\|_0) \, d\mu_{g_j}(s) \right) \\ &\leq \sup_{t \in [0, T]} \left(e_{L_R}^{-1}(t, 0) \int_{[0, t)} L_R(s) \|\mathbf{x}_{\gamma_k}(s) - \mathbf{x}_{\gamma}(s)\| \right. \\ &\quad \left. + \omega_R(s) \phi_R(\|\gamma_k - \gamma\|_0) \, d\mu_{\hat{g}}(s) \right) \\ &\leq \|\mathbf{x}_{\gamma_k} - \mathbf{x}_{\gamma}\|_* \sup_{t \in [0, T]} \left(e_{L_R}^{-1}(t, 0) \int_{[0, t)} L_R(s) e_{L_R}(s, 0) \, d\mu_{\hat{g}}(s) \right) \\ &\quad + \|\omega_R\|_{\mathcal{L}_{\hat{g}}^1} \phi_R(\|\gamma_k - \gamma\|_0) \end{aligned}$$

By Lemma 1.6.4, we have that

$$\begin{aligned} \|\mathcal{S}(\gamma_k) - \mathcal{S}(\gamma)\|_* &\leq \|\mathbf{x}_{\gamma_k} - \mathbf{x}_{\gamma}\|_* \sup_{t \in [0, T]} \left(e_{L_R}^{-1}(t, 0) (e_{L_R}(t, 0) - 1) \right) \\ &\quad + \|\omega_R\|_{\mathcal{L}_{\hat{g}}^1} \phi_R(\|\gamma_k - \gamma\|_0) \end{aligned}$$

$$\leq \|\mathcal{S}(\gamma_k) - \mathcal{S}(\gamma)\|_* \left(1 - e_{L_R}^{-1}(T, 0)\right) + \|\omega_R\|_{\mathcal{L}_g^1} \phi_R(\|\gamma_k - \gamma\|_0).$$

Consequently,

$$\|\mathcal{S}(\gamma_k) - \mathcal{S}(\gamma)\|_* \leq e_{L_R}(T, 0) \phi_R(\|\gamma_k - \gamma\|_0) \|\omega_R\|_{\mathcal{L}_g^1}. \quad (4.42)$$

Therefore, again by (H3) and since $\|\gamma_k - \gamma\|_0 \rightarrow 0$, we deduce that

$$\phi_R(\|\gamma_k - \gamma\|_0) \rightarrow 0 \quad \text{and} \quad \|\mathcal{S}(\gamma_k) - \mathcal{S}(\gamma)\|_* \rightarrow 0.$$

Hence, $\mathcal{S}$ is continuous.

Now, it remains to show that $\mathcal{S}(\mathcal{C})$ is relatively compact in $\mathcal{BC}_g([0, T], \mathbb{R}^n)$. Since $\mathcal{S}$ has its values in $\mathcal{C}$, then

$$\|\mathcal{S}(\gamma)\|_0 \leq R \quad \text{for all } \gamma \in \mathcal{C}. \quad (4.43)$$

Moreover, let us fix $\hat{\gamma} \in \mathcal{C}$. By (H1), for $i = 1, \dots, n$, there is a function $l_i \in \mathcal{L}_{g_i}^1([0, T], [0, +\infty))$ such that

$$|f_i(t, \mathbf{x}_{\hat{\gamma}}(t), \hat{\gamma})| \leq l_i(t) \quad \text{for } g_i\text{-almost every } t \in [0, T].$$

For $i = 1, \dots, n$ and $\gamma \in \mathcal{C}$, we deduce from (H3) that for g -almost every $t \in [0, T]$

$$\begin{aligned} \left| \left((\mathcal{S}(\gamma))_i \right)'_{g_i}(t) \right| &= |f_i(t, \mathbf{x}_{\gamma}(t), \gamma)| \\ &\leq |f_i(t, \mathbf{x}_{\hat{\gamma}}(t), \hat{\gamma})| + |f_i(t, \mathbf{x}_{\gamma}(t), \gamma) - f_i(t, \mathbf{x}_{\hat{\gamma}}(t), \hat{\gamma})| \\ &\leq l_i(t) + L_R(t) \|\mathbf{x}_{\gamma}(t) - \mathbf{x}_{\hat{\gamma}}(t)\| + \omega_R(t) \phi_R(\|\mathbf{x}_{\gamma} - \mathbf{x}_{\hat{\gamma}}\|_0) \\ &\leq l_i(t) + 2RL_R(t) + \hat{R}\omega_R(t), \end{aligned} \quad (4.44)$$

where $\hat{R} = \sup\{\phi_R(c) : c \in [0, 2R]\}$. Thus,

$$\begin{aligned} \forall i \in \{1, \dots, n\}, \exists h_i \in \mathcal{L}_{g_i}^1([0, T], [0, +\infty)) \quad \text{such that} \quad \left| \left((\mathcal{S}(\gamma))_i \right)'_{g_i}(t) \right| &\leq h_i(t) \\ \text{for } g_i\text{-almost every } t \in [0, T] \text{ and every } \gamma \in \mathcal{C}. \end{aligned} \quad (4.45)$$

It follows from Proposition 1.5.6, and the inequalities (4.43) and (4.45) that $\mathcal{S}(\mathcal{C})$ is relatively compact in $\mathcal{BC}_g([0, T], \mathbb{R}^n)$.

The Schauder fixed point theorem, Theorem 1.8.2, implies that $\mathcal{S}$ has at least one fixed point in $\mathcal{C}$. Hence, the problem (4.2), (4.3) has at least one solution

$$\mathbf{x} \in [\alpha_{\mathbf{x}}, \beta_{\mathbf{x}}]_{\mathcal{AC}_g([a, b], \mathbb{R}^n)} \subset \mathcal{C}.$$

□

In the case where the ordered pair of lower and upper solutions is assumed to be the same for all γ in Theorem 4.4.2, we can derive the next corollary.

Corollary 4.4.3. *Let $\alpha, \beta \in \mathcal{AC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ and let $\mathbf{f} : [0, T] \times \mathbb{R}^n \times [\alpha, \beta]_{\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)} \rightarrow \mathbb{R}^n$ satisfy the following assumptions:*

- (H1') *for each $\gamma \in [\alpha, \beta]_{\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)}$, $\mathbf{f}_{\gamma}$ is a $\mathbf{g}$ -Carathéodory function;*
- (H2') *(α, β) is an ordered pair of vector-valued lower and upper solutions of (4.41 _{γ}) for every $\gamma \in [\alpha, \beta]_{\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)}$;*
- (H3') *for every $r > 0$, there exist $\phi_r : [0, \infty) \rightarrow [0, \infty)$ continuous at 0, bounded on bounded sets satisfying $\phi_r(0) = 0$, and $L_r, \omega_r \in \mathcal{L}_{g_i}^1([0, T], [0, \infty))$ for all $i = 1, \dots, n$, such that*

$$|f_i(t, \mathbf{x}, \gamma) - f_i(t, \mathbf{y}, \hat{\gamma})| \leq L_r(t) \|\mathbf{x} - \mathbf{y}\| + \omega_r(t) \phi_r(\|\gamma - \hat{\gamma}\|_0),$$

for g_i -almost every $t \in [0, T]$, all $\gamma, \hat{\gamma} \in [\alpha, \beta]_{\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)}$, and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\|\mathbf{x}\| \leq r, \|\mathbf{y}\| \leq r$.

Then, the problem (4.2), (4.3) has a solution $\mathbf{x} \in [\alpha, \beta]_{\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n)}$.

Remark 4.4.4. In the previous theorem (resp. corollary), one can observe that the domain of $\mathbf{f}$ can be restricted to the set $X \times \mathcal{C} \subset [0, T] \times \mathbb{R}^n \times \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ (resp. $X \times [\alpha, \beta]_{\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)}$) for some X such that $\text{graph}(\gamma) \subset X$ for every $\gamma \in \mathcal{C}$ (resp. $\gamma \in [\alpha, \beta]_{\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)}$).

Remark 4.4.5. In the case where $\phi_r \equiv \text{id}_{[0, \infty)}$, observe that the function $\mathbf{f}$ would be assumed to satisfy a Lipschitz condition with respect to the functional argument. However, intriguingly, this condition does not guarantee the uniqueness of the solution. Thus, without additional assumptions on the functions ω_r and ϕ_r , uniqueness cannot be guaranteed. Take for instance the following counter-example. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined for $t \leq 1$ by $g(t) = t$, and $g(t) = t + 1$ for $t \geq 1$. Let us consider the following functional problem

$$\begin{aligned} x'_g(t) &= f(t, x(t), x) \quad \text{for } g\text{-almost every } t \in [0, 2], \\ x(0) &= 0, \end{aligned} \tag{4.46}$$

where $f : [0, 2] \times \mathbb{R} \times \mathcal{BC}_g([0, 2], \mathbb{R}) \rightarrow \mathbb{R}$ is defined for every $(t, x, \gamma) \in [0, 2] \times \mathbb{R} \times \mathcal{BC}_g([0, 2], \mathbb{R})$ by

$$f(t, x, \gamma) = \begin{cases} 0 & \text{if } t \neq 1, \\ \gamma(3/2) & \text{if } t = 1. \end{cases} \tag{4.47}$$

Let us consider the g -absolutely continuous functions $\alpha, \beta : [0, 2] \rightarrow \mathbb{R}$, defined by $\alpha(t) = 0$ for all $t \in [0, 2]$, and $\beta(t) = 0$ for $t \in [1, 2]$ and $\beta(t) = 1$ for $t \in (0, 1]$. The conditions of Corollary 4.4.3 are satisfied. Indeed, observe that (H1') holds since $f_{\gamma} := f(\cdot, \cdot, \gamma)$ is a g -Carathéodory function for each $\gamma \in [\alpha, \beta]_{\mathcal{BC}_g([0, 2], \mathbb{R})}$.

Now, we show that (H2') is satisfied, i.e. for all $\gamma \in [\alpha, \beta]_{\mathcal{BC}_g([0, 2], \mathbb{R})}$, (α, β) is an ordered pair of vector-valued lower and upper solutions of the problem:

$$x'_g(t) = f_{\gamma}(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, 2], \quad x(0) = 0. \tag{4.48}$$

First, observe that $\alpha(0) = 0 = \beta(0)$. In addition, for g -almost all $t \in [0, 2]$, we have that

$$\alpha'_g(t) = 0 \leq f_{\gamma}(t, \alpha(t)) = \begin{cases} 0 & \text{if } t \neq 1, \\ \gamma(3/2) & \text{if } t = 1. \end{cases}$$

For g -almost all $t \in [0, 2) \setminus D_g$, we have that

$$f_\gamma(t, \beta(t)) = 0 \leq \beta'_g(t) = 0,$$

and for $t \in [0, 2) \cap D_g = \{1\}$, we have that

$$f_\gamma(1, \beta(1)) = \gamma(3/2) \leq 1 = \beta'_g(1).$$

Moreover, for all $t \in [0, 2) \cap D_g = \{1\}$, and $y \in [\alpha(1), \beta(1)]$, we have that

$$\alpha(1^+) = 0 \leq y + \mu_g(\{1\})f_\gamma(1, y) = y + \mu_g(\{1\})\gamma(3/2) = \gamma(3/2) \leq 1 = \beta(1^+).$$

Furthermore, (H3') holds with $L_r \equiv 0$, $\omega_r \equiv 1$, and $\phi_r \equiv \text{id}_{\mathbb{R}^+}$. Thus, the conditions of Corollary 4.4.3 are satisfied, and the problem (4.46) admits at least one solution $x \in [\alpha, \beta]_{\mathcal{BC}_g([0, 2], \mathbb{R})}$. Now, observe that α and β are solutions of the problem (4.46). Indeed, $\alpha(0) = 0 = \beta(0)$. Moreover, for g -almost all $t \in [0, 2)$,

$$\alpha'_g(t) = 0 = f(t, \alpha(t), \alpha).$$

For g -almost every $t \in [0, 2) \setminus \{1\}$, we have that

$$\beta'_g(t) = 0 = f(t, \beta(t), \beta),$$

and for $t = 1$, we have

$$\beta'_g(1) = 1 = \beta(3/2) = f(t, \beta(t), \beta).$$

Hence, uniqueness does not hold.

In the previous corollary, the assumptions (H2') and (H3') can be weakened by incorporating a monotonicity condition.

Corollary 4.4.6. *Let $\alpha, \beta \in \mathcal{AC}_g([0, T], \mathbb{R}^n)$ and let $\mathbf{f} : [0, T] \times \mathbb{R}^n \times [\alpha, \beta]_{\mathcal{BC}_g([a, b], \mathbb{R}^n)} \rightarrow \mathbb{R}^n$ satisfy (H1'). In addition, assume that*

- (H2'') *(α, β) is an ordered pair of vector-valued lower and upper solutions of (4.41 _{γ}) for $\gamma \in \{\alpha, \beta\}$;*
- (H3'') *for every $r > 0$, there exist $\phi_r : [0, \infty) \rightarrow [0, \infty)$ continuous at 0 satisfying $\phi_r(0) = 0$, and $L_r, \omega_r \in \mathcal{L}_{g_i}^1([0, T], [0, \infty))$ for all $i = 1, \dots, n$, such that*

$$|f_i(t, \mathbf{x}, \gamma) - f_i(t, \mathbf{y}, \hat{\gamma})| \leq L_r(t)\|\mathbf{x} - \mathbf{y}\| + \omega_r(t)\phi_r(\|\gamma - \hat{\gamma}\|_0),$$

for g_i -almost every $t \in [0, T]$, every $\gamma, \hat{\gamma} \in [\alpha, \beta]_{\mathcal{BC}_g([a, b], \mathbb{R}^n)}$ and for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\|\mathbf{x}\| \leq r, \|\mathbf{y}\| \leq r$.

- (H4) *for $i = 1, \dots, n$, the function $f_i(t, \mathbf{x}, \cdot)$ is nondecreasing on $[\alpha, \beta]_{\mathcal{BC}_g([a, b], \mathbb{R}^n)}$ for g_i -almost all $t \in [0, T]$ and for every $\mathbf{x} \in [\alpha(t), \beta(t)]$.*

Then, the problem (4.2), (4.3) has a solution $\mathbf{x} \in [\alpha, \beta]_{\mathcal{AC}_g([a, b], \mathbb{R}^n)}$.

Proof. The conclusion can be deduced from the previous corollary. Indeed, it is easy to verify that (H2'') and (H4) imply (H2').

Let $r > 0$. We define $\hat{\phi}_r : [0, \infty) \rightarrow [0, \infty)$ by

$$\hat{\phi}_r(s) = \begin{cases} \phi_r(s) & \text{if } \phi_r \text{ is bounded on bounded sets,} \\ \min \left\{ \phi_r(s), \phi_r(\|\beta - \alpha\|_0) \right\} & \text{otherwise.} \end{cases}$$

Using (H3''), the function $\hat{\phi}_r$ is continuous at 0 and bounded on bounded sets. Obviously, for $\gamma, \hat{\gamma} \in [\alpha, \beta]_{\mathcal{BC}_{\mathbf{g}}([a,b], \mathbb{R}^n)}$ such that $\hat{\phi}_r(\|\gamma - \hat{\gamma}\|_0) = \phi_r(\|\gamma - \hat{\gamma}\|_0)$, one has, for $i = 1, \dots, n$,

$$\begin{aligned} |f_i(t, \mathbf{x}, \gamma) - f_i(t, \mathbf{y}, \hat{\gamma})| &\leq L_r(t) \|\mathbf{x} - \mathbf{y}\| + \omega_r(t) \phi_r(\|\gamma - \hat{\gamma}\|_0) \\ &= L_r(t) \|\mathbf{x} - \mathbf{y}\| + \omega_r(t) \hat{\phi}_r(\|\gamma - \hat{\gamma}\|_0), \end{aligned}$$

for g_i -almost every $t \in [0, T)$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\|\mathbf{x}\| \leq r, \|\mathbf{y}\| \leq r$.

On the other hand, let $\gamma, \hat{\gamma} \in [\alpha, \beta]_{\mathcal{BC}_{\mathbf{g}}([a,b], \mathbb{R}^n)}$ be such that

$$\hat{\phi}_r(\|\gamma - \hat{\gamma}\|_0) < \phi_r(\|\gamma - \hat{\gamma}\|_0).$$

For $i = 1, \dots, n$ and for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\|\mathbf{x}\| \leq r, \|\mathbf{y}\| \leq r$, without loss of generality, one has that

$$f_i(t, \mathbf{x}, \gamma) \leq f_i(t, \mathbf{y}, \hat{\gamma}).$$

Thus, using (H4),

$$f_i(t, \mathbf{x}, \alpha) \leq f_i(t, \mathbf{x}, \gamma) \quad \text{and} \quad f_i(t, \mathbf{y}, \hat{\gamma}) \leq f_i(t, \mathbf{y}, \beta).$$

Therefore, (H3'') implies that

$$\begin{aligned} |f_i(t, \mathbf{x}, \gamma) - f_i(t, \mathbf{y}, \hat{\gamma})| &\leq |f_i(t, \mathbf{x}, \alpha) - f_i(t, \mathbf{y}, \beta)| \\ &\leq L_r(t) \|\mathbf{x} - \mathbf{y}\| + \omega_r(t) \phi_r(\|\alpha - \beta\|_0) \\ &= L_r(t) \|\mathbf{x} - \mathbf{y}\| + \omega_r(t) \hat{\phi}_r(\|\gamma - \hat{\gamma}\|_0). \end{aligned}$$

We conclude that (H3') is satisfied with $\hat{\phi}_r$. □

4.4.2 Functional systems with a periodic boundary value condition

In this subsection, we prove the existence of a periodic solution for the system of functional Stieltjes differential equations (4.2), (4.4). As previously mentioned, we consider $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}^n$ a vector-valued derivator assuming its continuity at 0 for the sake of simplicity.

Our existence results for the initial value problem for the system of functional Stieltjes differential equations (4.2), (4.3) (see Theorem 4.4.2) rely on the uniqueness result, Lemma 4.4.1. However, it is important to note that an analogous result does

not hold if we replace the initial value condition (4.3) with the periodic boundary condition (4.4). Particularly, Condition (H_L) does not guarantee the uniqueness of solutions in this context. Nevertheless, we can still establish the existence of solutions for functional systems of Stieltjes differential equations with a periodic boundary condition.

Theorem 4.4.7. *Let $\mathcal{C}$ be a bounded, closed, convex subset of $\mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$. Let us consider a function $\mathbf{f} : [0, T] \times \mathbb{R}^n \times \mathcal{C} \rightarrow \mathbb{R}^n$ satisfying Conditions (H1) and (H3) of Theorem 4.4.2. In addition, assume that*

$$(H_C) \quad \mathcal{C}_T = \{\gamma(T) \in \mathbb{R}^n : \gamma \in \mathcal{C}\} \subset \{\gamma(0) \in \mathbb{R}^n : \gamma \in \mathcal{C}\};$$

(H2^p) *for each $(\mathbf{x}_0, \gamma) \in \mathcal{C}_T \times \mathcal{C}$, there exist functions $\alpha_{(\mathbf{x}_0, \gamma)}, \beta_{(\mathbf{x}_0, \gamma)} \in \mathcal{AC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ such that $(\alpha_{(\mathbf{x}_0, \gamma)}, \beta_{(\mathbf{x}_0, \gamma)})$ is an ordered pair of vectored-valued lower and upper solutions of (4.41_γ), and $[\alpha_{(\mathbf{x}_0, \gamma)}, \beta_{(\mathbf{x}_0, \gamma)}]_{\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n)} \subset \mathcal{C}$.*

Then, the problem (4.2), (4.4) has a solution $\mathbf{x} \in [\alpha_{(\mathbf{x}(0), \mathbf{x})}, \beta_{(\mathbf{x}(0), \mathbf{x})}]_{\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n)} \subset \mathcal{C}$.

Proof. For $(\mathbf{x}_0, \gamma) \in \mathcal{C}_T \times \mathcal{C}$, Conditions (H1), (H2^p) and (H3) imply that $\mathbf{f}_\gamma$ and the pair $(\alpha_{(\mathbf{x}_0, \gamma)}, \beta_{(\mathbf{x}_0, \gamma)})$ satisfy the assumptions of Theorem 4.3.1 and Lemma 4.4.1. Thus, the problem (4.41_γ) has a unique solution $\mathbf{x}_{(\mathbf{x}_0, \gamma)} \in [\alpha_{(\mathbf{x}_0, \gamma)}, \beta_{(\mathbf{x}_0, \gamma)}]_{\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n)} \subset \mathcal{C}$.

This combined with (H_C) ensures that

$$\mathbf{x}_{(\mathbf{x}_0, \gamma)}(T) \in \mathcal{C}_T.$$

Let us define the operator $\mathcal{S}^p : \mathcal{C}_T \times \mathcal{C} \rightarrow \mathcal{C}_T \times \mathcal{C}$ by

$$\mathcal{S}^p(\mathbf{x}_0, \gamma) = (\mathbf{x}_{(\mathbf{x}_0, \gamma)}(T), \mathbf{x}_{(\mathbf{x}_0, \gamma)}).$$

To conclude that the problem (4.2), (4.4) has a periodic solution in $\mathcal{C}$, it suffices to show that $\mathcal{S}^p$ has a fixed point.

By arguments analogous to those used in the proof of Theorem 4.4.2, we show that $\mathcal{S}^p$ is continuous and compact using the norm

$$\|(\mathbf{x}, \gamma)\|^* = (1 - e_{L_R}^{-1}(T, 0))\|\mathbf{x}\| + \|\gamma\|_* \quad \forall (\mathbf{x}, \gamma) \in \mathbb{R}^n \times \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n).$$

Indeed, let us show that $\mathcal{S}^p$ is continuous. Let $\{(\mathbf{x}_{0,k}, \gamma_k)\}$ be a sequence converging to $(\mathbf{x}_0, \gamma) \in (\mathbb{R}^n \times \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)) \cap (\mathcal{C}_T \times \mathcal{C})$. Thus

$$\begin{aligned} \|\mathcal{S}^p(\mathbf{x}_{0,k}, \gamma_k) - \mathcal{S}^p(\mathbf{x}_0, \gamma)\|^* &= \left\| (\mathbf{x}_{(\mathbf{x}_{0,k}, \gamma_k)}(T) - \mathbf{x}_{(\mathbf{x}_0, \gamma)}(T), \mathbf{x}_{(\mathbf{x}_{0,k}, \gamma_k)} - \mathbf{x}_{(\mathbf{x}_0, \gamma)}) \right\|^* \\ &= (1 - e_{L_R}^{-1}(T, 0))\|\mathbf{x}_{(\mathbf{x}_{0,k}, \gamma_k)}(T) - \mathbf{x}_{(\mathbf{x}_0, \gamma)}(T)\| \\ &\quad + \|\mathbf{x}_{(\mathbf{x}_{0,k}, \gamma_k)} - \mathbf{x}_{(\mathbf{x}_0, \gamma)}\|_* \\ &= (1 - e_{L_R}^{-1}(T, 0))\|\mathbf{x}_{(\mathbf{x}_{0,k}, \gamma_k)}(T) - \mathbf{x}_{(\mathbf{x}_0, \gamma)}(T)\| \\ &\quad + \sup_{t \in [0, T]} e_{L_R}^{-1}(t, 0)\|\mathbf{x}_{(\mathbf{x}_{0,k}, \gamma_k)}(t) - \mathbf{x}_{(\mathbf{x}_0, \gamma)}(t)\| \\ &= (1 - e_{L_R}^{-1}(T, 0))\|\mathbf{x}_{(\mathbf{x}_{0,k}, \gamma_k)}(T) - \mathbf{x}_{(\mathbf{x}_0, \gamma)}(T)\| \end{aligned}$$

$$\begin{aligned}
 & + \sup_{t \in [0, T]} e_{L_R}^{-1}(t, 0) \left\| \mathbf{x}_{0,k} - \mathbf{x}_0 \right\| \\
 & + \int_{[0, t)} \mathbf{f}(s, \mathbf{x}_{(\mathbf{x}_{0,k}, \gamma_k)}(s), \gamma_k) - \mathbf{f}(s, \mathbf{x}_{(\mathbf{x}_0, \gamma)}(s), \gamma) \, d\mu_{\mathbf{g}}(s) \Big\|.
 \end{aligned}$$

Arguing as in the proof of Theorem 4.4.2, we obtain

$$\begin{aligned}
 \|\mathcal{S}^p(\mathbf{x}_{0,k}, \gamma_k) - \mathcal{S}^p(\mathbf{x}_0, \gamma)\|^* & \leq (1 - e_{L_R}^{-1}(T, 0)) \|\mathbf{x}_{(\mathbf{x}_{0,k}, \gamma_k)}(T) - \mathbf{x}_{(\mathbf{x}_0, \gamma)}(T)\| \\
 & + (1 - e_{L_R}^{-1}(T, 0)) \|\mathbf{x}_{(\mathbf{x}_{0,k}, \gamma_k)} - \mathbf{x}_{(\mathbf{x}_0, \gamma)}\|_* \\
 & + \|\omega_R\|_{\mathcal{L}_{\hat{g}}^1} \phi_R(\|\gamma_k - \gamma\|_0) \\
 & = (1 - e_{L_R}^{-1}(T, 0)) \|\mathcal{S}^p(\mathbf{x}_{0,k}, \gamma_k) - \mathcal{S}^p(\mathbf{x}_0, \gamma)\|^* \\
 & + \|\omega_R\|_{\mathcal{L}_{\hat{g}}^1} \phi_R(\|\gamma_k - \gamma\|_0).
 \end{aligned}$$

Consequently,

$$\|\mathcal{S}^p(\mathbf{x}_{0,k}, \gamma_k) - \mathcal{S}^p(\mathbf{x}_0, \gamma)\|^* \leq e_{L_R}(T, 0) \phi_R(\|\gamma_k - \gamma\|_0) \|\omega_R\|_{\mathcal{L}_{\hat{g}}^1}. \quad (4.49)$$

Therefore, again using (H3) and since $\|\gamma_k - \gamma\|_0 \rightarrow 0$, we deduce that

$$\|\mathcal{S}^p(\mathbf{x}_{0,k}, \gamma_k) - \mathcal{S}^p(\mathbf{x}_0, \gamma)\|^* \rightarrow 0.$$

Hence, $\mathcal{S}^p$ is continuous.

Now, we show that $\mathcal{S}^p(\mathcal{C}_T \times \mathcal{C})$ is relatively compact in $\mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$. Since $\mathcal{S}$ has its values in $\mathcal{C}$, then

$$\|\mathcal{S}^p(\mathbf{x}_0, \gamma)\|_0 \leq R \quad \text{for all } (\mathbf{x}_0, \gamma) \in \mathcal{C}_T \times \mathcal{C}. \quad (4.50)$$

Moreover, let us fix $\hat{\gamma} \in \mathcal{C}$. By (H1), for $i = 1, \dots, n$, there exists $l_i \in \mathcal{L}_{g_i}^1([0, T], [0, +\infty))$ such that for g -almost every $t \in [0, T]$

$$|f_i(t, \mathbf{x}_{\hat{\gamma}}(t), \hat{\gamma})| \leq l_i(t) \quad \text{for } g_i\text{-almost all } t \in [0, T].$$

For $i = 1, \dots, n$, $\mathbf{x}_0 \in \mathcal{C}_T$, and $\gamma \in \mathcal{C}$, we deduce from (H3) that

$$\begin{aligned}
 \left\| \left((\mathcal{S}^p(\mathbf{x}_0, \gamma))_i \right)'_{g_i}(t) \right\|_{\mathbb{R}^2} & = \left\| \left(0, f_i(t, \mathbf{x}_{(\mathbf{x}_0, \gamma)}(t), \gamma) \right) \right\|_{\mathbb{R}^2} \\
 & \leq |f_i(t, \mathbf{x}_{(\mathbf{x}_0, \hat{\gamma})}(t), \hat{\gamma})| + |f_i(t, \mathbf{x}_{(\mathbf{x}_0, \gamma)}(t), \gamma) - f_i(t, \mathbf{x}_{(\mathbf{x}_0, \hat{\gamma})}(t), \hat{\gamma})| \\
 & \leq l_i(t) + L_R(t) \|\mathbf{x}_{(\mathbf{x}_0, \gamma)}(t) - \mathbf{x}_{(\mathbf{x}_0, \hat{\gamma})}(t)\| \\
 & \quad + \omega_R(t) \phi_R(\|\mathbf{x}_{(\mathbf{x}_0, \gamma)} - \mathbf{x}_{(\mathbf{x}_0, \hat{\gamma})}\|_0) \\
 & \leq l_i(t) + 2RL_R(t) + \hat{R}\omega_R(t),
 \end{aligned} \quad (4.51)$$

where $\hat{R} = \sup\{\phi_R(c) : c \in [0, 2R]\}$. Thus,

$$\forall i \in \{1, \dots, n\}, \exists h_i \in \mathcal{L}_{g_i}^1([0, T], [0, +\infty)) \text{ such that } \left\| \left(\left(\mathcal{S}^p(\mathbf{x}_0, \gamma) \right)_i \right)'_{g_i}(t) \right\|_{\mathbb{R}^2} \leq h_i(t) \\ \text{for } g_i\text{-almost every } t \in [0, T) \text{ and every } \mathbf{x}_0 \in \mathcal{C}_T \text{ and } \gamma \in \mathcal{C}. \quad (4.52)$$

It follows from Proposition 1.5.6, and the inequalities (4.50) and (4.52) that $\mathcal{S}^p(\mathcal{C}_T \times \mathcal{C})$ is relatively compact in $\mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$.

The Schauder fixed point theorem implies that $\mathcal{S}^p$ has at least one fixed point $(\mathbf{x}_0, \mathbf{x}) \in \mathcal{C}_T \times \mathcal{C}$. Hence, $\mathbf{x}$ is a solution of the problem (4.2), (4.4) such that

$$\mathbf{x} \in [\alpha_{(\mathbf{x}(0), \mathbf{x})}, \beta_{(\mathbf{x}(0), \mathbf{x})}]_{\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n)} \subset \mathcal{C}.$$

□

In the case where the ordered pair of lower and upper solutions is the same for every γ , we obtain the following corollary.

Corollary 4.4.8. *Let $\alpha, \beta \in \mathcal{AC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ and let $\mathbf{f} : [0, T] \times \mathbb{R}^n \times [\alpha, \beta]_{\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)} \rightarrow \mathbb{R}^n$ satisfy Conditions (H1') and (H3') of Corollary 4.4.3. In addition, assume that*

(H5) $\alpha(0) \leq \alpha(T)$ and $\beta(0) \geq \beta(T)$;

(H6) (α, β) is an ordered pair of vector-valued lower and upper solutions of (4.41 _{γ}) for every $(\mathbf{x}_0, \gamma) \in [\alpha(T), \beta(T)] \times [\alpha, \beta]_{\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)}$.

Then, the problem (4.2), (4.4) has a solution $\mathbf{x} \in [\alpha, \beta]_{\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n)}$.

As in the initial value problem in the previous subsection, the hypothesis (H6) can be weakened if the monotonicity condition (H4) is added.

Corollary 4.4.9. *Let $\alpha, \beta \in \mathcal{AC}_{\mathbf{g}}([0, T], \mathbb{R}^n)$ and let $\mathbf{f} : [0, T] \times \mathbb{R}^n \times [\alpha, \beta]_{\mathcal{BC}_{\mathbf{g}}([a, b], \mathbb{R}^n)} \rightarrow \mathbb{R}^n$ satisfy (H1'), (H3''), (H4) and (H5). In addition, assume that*

(H6') (α, β) is an ordered pair of vector-valued lower and upper solutions of (4.41 _{γ}) for every $(\mathbf{x}_0, \gamma) \in \{(\alpha(T), \alpha), (\beta(T), \beta)\}$.

Then, the problem (4.2), (4.4) has a solution $\mathbf{x} \in [\alpha, \beta]_{\mathcal{AC}_{\mathbf{g}}([a, b], \mathbb{R}^n)}$.

4.5 Application to the dynamics of an exploited fishery

In this section, we model the dynamics of a fish population affected by both predation and fishing using a system of Stieltjes differential equations. In practice, regulations such as seasonal fishing, quotas, licenses, and other controls play a crucial role in determining the intensity of fishing activity [14]. The intent of these measures is to protect the fishing industry and ensure a sustainable harvesting of resources.

Let $p(t)$ denote the number of individuals in a fish population at time t subjected to a predator population denoted by $q(t)$. The prey population p is also subject to fishing by fishing efforts denoted by $E(t)$. We assume that that new individuals are added to the prey population at times $t = 1 + 8n, n \in \mathbb{Z}$, due to hatching with an average incubation period of $0 < \delta < 1$. To protect the reproductive cycle and newborn fish, the fishing activity is only allowed during certain periods of the year, known as the

open seasons. Since hatching occurs at every $t = 1 + 8n$, the intervals $(2, 7] + 8n; n \in \mathbb{Z}$, can be considered as reasonable estimates for open season intervals.

In addition, we denote by $h(t)$ the quantity of individuals caught by a fishing effort $E(t)$ at time t , known as the catch. It is a common concept in fisheries literature that the catch $h(t)$ is directly related to the size of the prey population and the level of fishing effort $E(t)$, with a catchability coefficient $C > 0$ indicating the efficiency of catching fish per unit of effort. This relationship can be expressed as:

$$h(t) = CE(t)p(t).$$

Classical models typically focus on the population reduction caused by the catch only (see for instance [14, 19, 75] and their references). However, by incorporating additional factors in our parameters, such as the ecological cost of the fishing methods, we can better understand the overall impact on fisheries. In fact, fishing methods can significantly impact marine resources. For instance, ghost gear, which are abandoned tools, can contribute to lower water quality and harm the health of the ocean. This can have a negative impact on the ecosystem and cause dispersion or emigration of many species.

In the following analysis, we aim to model the dynamics of both populations p and q while monitoring fishing effort E . In doing so, Stieltjes differential equations are convenient to anticipate a jump in the prey population size at the hatching times $t = 1 + 8n$, E has a discontinuity at the beginning and at the end of the fishing season and it has stationary periods $(8n, 2 + 8n) \cup (7 + 8n, 10 + 8n); n \in \mathbb{Z}$, when the fishing season is closed. This can be achieved by selecting appropriate derivators g_1, g_2 and g_3 of the functions p, E and q respectively. Specifically, the derivators g_1 and g_2 are sought to satisfy the following requirements:

1. g_1 should have discontinuities at the hatching times $t = 1 + 8n, n \in \mathbb{Z}$;
2. g_1 should have greater slopes immediately following each hatching time as a result of the high mortality rate of the newborns fish;
3. g_2 should remain constant during closed seasons, i.e. on $(8n, 2 + 8n) \cup (7 + 8n, 10 + 8n), n \in \mathbb{Z}$;
4. g_2 must present discontinuities at $t \in \{m + 8n : m \in \{2, 7\}, n \in \mathbb{Z}\}$, when fishing resumes and stops;
5. g_1 and g_2 must have greater slopes when fishing resumes reflecting the fishing intensity at the beginning of open seasons due to the abundance of resources. This also describes the regulation strategy of deploying fishing vessels during these periods.

A possible option for derivators could be:

$$g_1(t) = \begin{cases} t & \text{if } t \in [0, 1], \\ 2 + t - (t - 2)^2 & \text{if } t \in (1, 2], \\ 4 + t - \frac{(t - 4)^2}{2} & \text{if } t \in (2, 4], \\ 4 + t & \text{if } t \in (4, 8]; \end{cases}$$

$$g_2(t) = \begin{cases} 0 & \text{if } t \in [0, 2], \\ 6 - \frac{(t - 7)^2}{5} & \text{if } t \in (2, 7], \\ 8 & \text{if } t \in (7, 8]; \end{cases}$$

$$g_3(t) = t;$$

with $g_1(t) = 12n + g_1(t - 8n)$ and $g_i(t) = 8n + g_i(t - 8n)$ for $i = 2, 3$ if $t - 8n \in [0, 8)$, $n \in \mathbb{Z} \setminus \{0\}$. Observe that

$$D_{g_1} = \{1 + 8n : n \in \mathbb{Z}\}, \quad D_{g_2} = \{m + 8n : m \in \{2, 7\}, n \in \mathbb{Z}\}, \quad C_{g_2} = \bigcup_{n \in \mathbb{Z}} (-1, 2) + 8n,$$

and $D_{g_3} = C_{g_3} = C_{g_1} = \emptyset$. Moreover, for $n \in \mathbb{Z}$,

$$1 = \mu_{g_1}(\{1 + 8n\}) = \mu_{g_2}(\{2 + 8n\}) \quad \text{and} \quad \mu_{g_2}(\{7 + 8n\}) = 2.$$

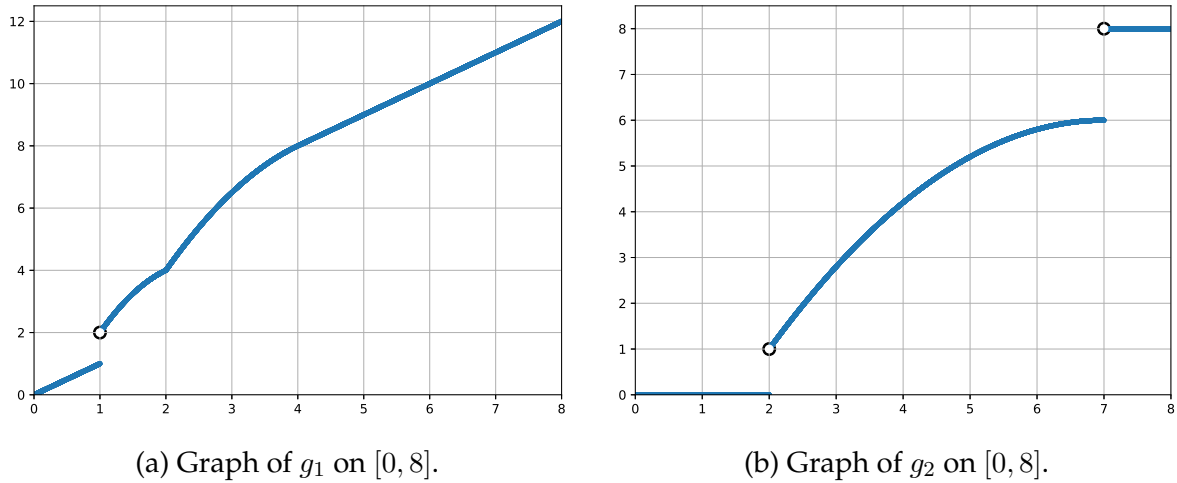


Figure 4.5.1 – Graphs of the derivators g_1 and g_2 .

Furthermore, our model must take into consideration the following dynamics:

1. p should increase at the hatching times $t = 1 + 8n$, $n \in \{0, 1, 2, \dots\}$, i.e. $p'_{g_1}(1 + 8n) > 0$, $n \in \{0, 1, 2, \dots\}$;
2. E should increase at the beginning of open seasons and at the end, it should decrease, meaning that $E'_{g_2}(2 + 8n) > 0$ initially and $E'_{g_2}(7 + 8n) < 0$ later, with $n \in \{0, 1, 2, \dots\}$;

3. Due to regulation, we assume that we start open seasons by catching a proportion of the prey population's stock level p , i.e. $h(2+8n^+) = \rho p(2+8n)$, $0 < \rho < 1$, $n \in \{0, 1, 2, \dots\}$. Consequently, a fishing effort $E(2+8n^+) \geq 0$ needs to be deployed based on the value of $p(2+8n)$, $n \in \{0, 1, 2, \dots\}$;
4. During open seasons, E is assumed to be decreasing during open seasons due to regulation.

Thus, we can model the whole dynamics by means of the following system of Stieltjes differential equations, for sake of simplicity, this model is given on the interval $[0, 8]$

$$\begin{aligned} \mathbf{x}'_{\mathbf{g}}(t) &= \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}), \text{ for } \mathbf{g}\text{-almost every } t \in [0, 8), \\ \mathbf{x}(0) &= \mathbf{x}_0 = (p_0, E_0, q_0) = (p_0, 0, q_0), \end{aligned} \quad (4.53)$$

where $\mathbf{g} = (g_1, g_2, g_3)$, $\mathbf{x} = (p, E, q)$ and

$$\mathbf{f} = (f_1, f_2, f_3) : [0, 8] \times \mathbb{R}^3 \times \mathcal{BC}_{\mathbf{g}}([0, 8], \mathbb{R}^3) \rightarrow \mathbb{R}^3$$

is defined by

$$\begin{aligned} f_1(t, p, E, q, \gamma_1, \gamma_2, \gamma_3) &= \begin{cases} -(a_1 E + b_1 q)p & \text{if } t \in [0, 8] \setminus \{1\}, \\ d_1 \int_{1-\delta}^1 \gamma_1(s) \, d s & \text{if } t = 1; \end{cases} \\ f_2(t, p, E, q, \gamma_1, \gamma_2, \gamma_3) &= \begin{cases} -(a_2 p + b_2 q)E & \text{if } t \in [0, 8] \setminus \{2, 7\}, \\ d_2 \rho p - E & \text{if } t = 2, \\ -\frac{E}{2} & \text{if } t = 7; \end{cases} \\ f_3(t, p, E, q, \gamma_1, \gamma_2, \gamma_3) &= (a_3 p - b_3 E - d_3)q. \end{aligned} \quad (4.54)$$

The parameters of the model could be interpreted as follows:

$a_1 > 0$: Constant representing the negative impact of fishing effort on prey numbers.

$b_1 > 0$: Prey predation rate in proportion to the number of predators.

$d_1 \in \left(0, \frac{1}{\delta}\right]$: Constant corresponding to the number of eggs laid by prey during a laying season. We assume that the number of eggs is proportional to the average number of prey during the incubation period δ , which provides an estimate of the number of eggs that will hatch successfully.

$a_2 \in (0, 1)$: Decay rate of the fishing effort E , proportionally to the prey population p .

$b_2 > 0$: Decay rate of the fishing effort E due to the population of predators. The more abundant the predators, the fewer become the prey individuals, which results in a rapid decline in fishing activity.

d_2 : Constant representing the initial fishing effort deployed at the beginning of open seasons to catch a proportion $0 < \rho < 1$ of the prey population.

$a_3 > 0$: Constant related to the growth rate of the predator population proportionally to the number of the prey population.

$b_3 > 0$: The rate at which the number of predators declines due to the impact of the fishing activity.

$d_3 \in (0, 1)$: Rate of extinction of predators as a result of natural death or migration.

It is worthwhile to mention that, thanks to the use of Stieltjes differential equations and the choice of derivators that are not all $g(t) = t$, the changes in variations of g_1 and g_2 allow incorporating the dynamical effects of predation and fishing, despite considering $a_1, b_2, \dots, d_3$ as fixed constants in the model. Modeling such dynamics using through classical derivation would have necessitated a more complex non-autonomous system.

Now, in order to prove the existence of a solution for the model (4.53), let us consider the g -absolutely continuous functions $\alpha = (\alpha_1, \alpha_2, \alpha_3), \beta = (\beta_1, \beta_2, \beta_3) : [0, 8] \rightarrow \mathbb{R}^3$ defined by

$$\alpha_1(t) = \alpha_2(t) = \alpha_3(t) = 0,$$

and

$$\begin{aligned} \beta_1(t) &= \begin{cases} p_0 & \text{if } t \in [0, 1], \\ 2p_0 & \text{if } t \in (1, 8]; \end{cases} \\ \beta_2(t) &= k; \\ \beta_3(t) &= q_0 e^{lt}; \end{aligned}$$

where $k \geq 2d_2p_0$ and $l \geq \max\{0, 2a_3p_0 - d_3\}$.

We claim that conditions of Corollary 4.4.3 are satisfied. Indeed, observe that (H1') holds since $\mathbf{f}_\gamma := \mathbf{f}(\cdot, \cdot, \gamma)$ is a g -Carathéodory function for every $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in [\alpha, \beta]_{\mathcal{BC}_g([0,8], \mathbb{R}^3)}$.

Now, we show that (H2') is satisfied, i.e. for every $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in [\alpha, \beta]_{\mathcal{BC}_g([0,8], \mathbb{R}^3)}$, (α, β) is an ordered pair of vector-valued lower and upper solutions of the system:

$$\mathbf{x}'_g(t) = \mathbf{f}_\gamma(t, \mathbf{x}(t)) \quad \text{for } g\text{-almost all } t \in [0, 8), \quad \mathbf{x}(0) = \mathbf{x}_0. \quad (4.55)$$

First of all, observe that

$$\alpha_1(0) \leq p_0 \leq \beta_1(0), \quad \alpha_2(0) \leq E_0 = 0 \leq \beta_2(0) \quad \text{and} \quad \alpha_3(0) \leq q_0 \leq \beta_3(0).$$

For g_1 -almost every $t \in [0, 8)$ and $\mathbf{c} = (c_1, c_2, c_3) \in [\alpha(t), \beta(t)]$, one has

$$\begin{aligned} (\alpha_1)'_{g_1}(t) &= 0 \\ &\leq \begin{cases} 0 & \text{if } t \in [0, 8) \setminus \{1\}, \\ d_1 \int_{1-\delta}^1 \gamma_1(s) \, d s & \text{if } t = 1, \end{cases} \\ &= f_{1,\gamma}(t, 0, c_2, c_3) \\ &= f_{1,\gamma}(t, \alpha_1(t), c_2, c_3); \\ (\beta_1)'_{g_1}(t) &= \begin{cases} 0 & \text{if } t \in [0, 8) \setminus \{1\}, \\ p_0 & \text{if } t = 1; \end{cases} \\ &\geq \begin{cases} 0 & \text{if } t \in [0, 8) \setminus \{1\}, \\ d_1 \delta p_0 & \text{if } t = 1; \end{cases} \end{aligned}$$

$$\begin{aligned}
 &\geq \begin{cases} -(a_1c_2 + b_1c_3)\beta_1(t) & \text{if } t \in [0, 8) \setminus \{1\}, \\ d_1 \int_{1-\delta}^1 \gamma_1(s) \, d s & \text{if } t = 1; \end{cases} \\
 &= f_{1,\gamma}(t, \beta_1(t), c_2, c_3).
 \end{aligned}$$

For g_2 -almost every $t \in [0, 8) \setminus C_{g_2}$ and $\mathbf{c} = (c_1, c_2, c_3) \in [\alpha(t), \beta(t)]$, one has

$$\begin{aligned}
 (\alpha_2)'_{g_2}(t) = 0 &\leq \begin{cases} 0 & \text{if } t \in (2, 7), \\ d_2\rho c_1 & \text{if } t = 2, \\ 0 & \text{if } t = 7, \end{cases} \\
 &= f_{2,\gamma}(t, c_1, 0, c_3) \\
 &= f_{2,\gamma}(t, c_1, \alpha_2(t), c_3); \\
 (\beta_2)'_{g_2}(t) = 0 &\geq \begin{cases} 0 & \text{if } t \in (2, 7), \\ 2d_2p_0 - \beta_2(2) & \text{if } t = 2, \\ 0 & \text{if } t = 7; \end{cases} \\
 &\geq \begin{cases} 0 & \text{if } t \in (2, 7), \\ d_2\rho 2p_0 - \beta_2(2) & \text{if } t = 2, \\ 0 & \text{if } t = 7; \end{cases} \\
 &\geq \begin{cases} -(a_2c_1 + b_2c_3)\beta_2(t) & \text{if } t \in (2, 7), \\ d_2\rho c_1 - \beta_2(2) & \text{if } t = 2, \\ -\frac{\beta_2(7)}{2} & \text{if } t = 7; \end{cases} \\
 &= f_{2,\gamma}(t, c_1, \beta_2(t), c_3).
 \end{aligned}$$

For g_3 -almost every $t \in [0, 8)$ and $\mathbf{c} = (c_1, c_2, c_3) \in [\alpha(t), \beta(t)]$, one has

$$\begin{aligned}
 (\alpha_3)'_{g_3}(t) = 0 &= f_{3,\gamma}(t, c_1, c_2, 0) \leq f_{3,\gamma}(t, c_1, c_2, \alpha_3(t)); \\
 (\beta_3)'_{g_3}(t) &= lq_0e^{lt} = l\beta_3(t) \geq (2a_3p_0 - d_3)\beta_3(t) \geq (a_3c_1 - b_3c_2 - d_3)\beta_3(t) \\
 &\geq f_{3,\gamma}(t, c_1, c_2, \beta_3(t)).
 \end{aligned}$$

Now, we verify Condition (iii) of Definition 4.2.2 at points $t \in D_{g_1} \cup D_{g_2} = \{1, 2, 7\}$.

For $\mathbf{c} = (c_1, c_2, c_3) \in [\alpha(t), \beta(t)]$, one has

$$\begin{aligned}
 \alpha_1(1^+) = 0 &\leq c_1 + f_{1,\gamma}(1, \mathbf{c})\mu_{g_1}(\{1\}) = c_1 + d_1 \int_{1-\delta}^1 \gamma_1(s) \, d s \leq p_0 + d_1\delta p_0 \leq 2p_0 \\
 &\leq \beta_1(1^+) \quad \text{if } t = 1; \\
 \alpha_2(2^+) = 0 &\leq c_2 + f_{2,\gamma}(2, \mathbf{c})\mu_{g_2}(\{2\}) = c_2 + (d_2\rho c_1 - c_2) = d_2\rho c_1 \leq d_2\rho 2p_0 \leq 2d_2p_0 \\
 &\leq \beta_2(2^+) \quad \text{if } t = 2; \\
 \alpha_2(7^+) = 0 &\leq c_2 + f_{2,\gamma}(7, \mathbf{c})\mu_{g_2}(\{7\}) = c_2 - \frac{2c_2}{2} \\
 &\leq \beta_2(7^+) \quad \text{if } t = 7.
 \end{aligned}$$

Therefore, (H2') is satisfied since (α, β) is an ordered pair of vector-valued lower and upper solutions of system (4.55) for $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in [\alpha, \beta]_{\mathcal{BC}_{\mathbf{g}}([0,8], \mathbb{R}^3)}$.

Additionally, $f_{i,\gamma}(t, \cdot)$ is locally Lipschitz for all $t \in [0, 8)$. More precisely, for every $r > 0$, there exist $L_r, K_r \geq 0$ such that

$$|f_i(t, \mathbf{x}, \gamma) - f_i(t, \mathbf{y}, \hat{\gamma})| \leq L_r \|\mathbf{x} - \mathbf{y}\| + K_r \|\gamma - \hat{\gamma}\|_0,$$

for all $t \in [0, 8)$, all $\gamma, \hat{\gamma} \in [\alpha, \beta]_{\mathcal{BC}_{\mathbf{g}}([0,8], \mathbb{R}^3)}$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ such that $\|\mathbf{x}\| \leq r$ and $\|\mathbf{y}\| \leq r$. Thus, (H3') is satisfied.

Hence, Corollary 4.4.3 ensures the existence of at least one solution $\mathbf{x}$ such that

$$\alpha(t) \leq \mathbf{x}(t) \leq \beta(t) \quad \text{for every } t \in [0, 8].$$

The graph of a numerical simulation of the solution is given in Figure 4.5.2, (the reader is referred to [26] for more information on the method to obtain such a simulation).

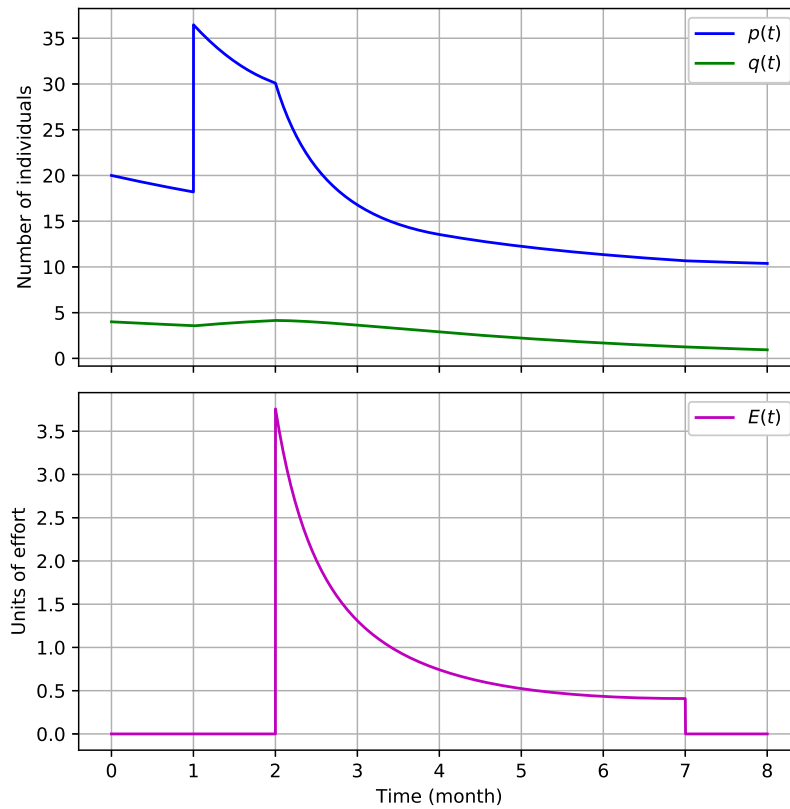


Figure 4.5.2 – Numerical approximation of the solution $\mathbf{x} = (p, E, q)$ with $p_0 = 20$, $E_0 = 0$, $q_0 = 4$, $a_1 = 0.06$, $b_1 = 0.025$, $d_1 = 10$, $\delta = 1/10$, $a_2 = 0.025$, $b_2 = 0.01$, $d_2 = 0.02$, $\rho = 0.25$, $a_3 = 0.02$, $b_3 = 0.03$ and $d_3 = 0.5$, and obtained with a time-discretization step-size of 10^{-3} .

Comments

As assumed earlier in this section, the prey exhibits an abrupt increase at each hatching time, and the fishing effort vanishes during closed seasons, and jumps at the

beginning and at the end of open seasons, which illustrates the regulation strategy. In addition, the prey decreases rapidly during open seasons due to being subject to both predation and fishing. In contrast, this competition for the prey between natural predators and human fishing activities also impacts the number of predators present during open seasons.

Prolongation of solutions and Lyapunov stability

In the realm of dynamical systems theory, Lyapunov's Second Method stands as a fundamental approach for assessing the stability properties of a system near an *equilibrium* point. This point represents a state where the system remains unchanged over time. This stability analysis provides insights into whether small perturbations around an equilibrium lead to convergence or divergence of solutions. The core of this method lies in the concept of the *Lyapunov function*. This function was the subject of numerous works in the classical literature starting from the first works [41, 42, 45, 48, 59, 86] and their references. In this chapter, we provide an adequate theoretical framework to extend classical results from the literature to the Stieltjes case by means of the Stieltjes derivative. In doing so, we start first with the prolongation of solutions considering the Stieltjes dynamical system:

$$\begin{aligned} \mathbf{x}'_g(t) &= \mathbf{f}(t, \mathbf{x}(t)) \quad \text{for } g\text{-almost all } t \geq t_0 \geq 0, \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \in \mathbb{R}^n; \end{aligned} \tag{5.1}$$

where $\mathbf{f} = (f_1, \dots, f_n) : [0, +\infty) \times B \rightarrow \mathbb{R}^n$, $B \in \mathbb{R}^n$. Existence of a maximal solution of the problem (5.1) was established first by Larivière [50]. In this work, we obtain corollary results characterizing the maximal solution. Then, based on a generalized version of the Grönwall lemma for the Stieltjes derivative, we establish the existence of global solutions over the whole positive real half-line. The prolongation of solution will be essentially used later on to establish Lyapunov-like stability results for the system (5.1), inspired by results from the classical theory and works such as [44, 48, 84, 85] dealing with dynamic equations on time scales and impulsive differential equations. As far as the author knows, this is the first work introducing Lyapunov's method adapted to Stieltjes differential equations.

This chapter is organized as follows: In Section 5.1, we focus on prolongation of solutions and the characterization of the maximal solution. Section 5.2 is devoted to Lyapunov-like stability result. We begin by defining notions of stability (resp. uniform stability) and asymptotic stability (resp. asymptotic uniform stability). Afterward, we present stability results for each type of stability inspired from the works [22, 37], to extend some classical results from [44, 48]. In the last section of this chapter, we present two applications to dynamics of population to study the asymptotic stability of some critical equilibria. In the first application, we model the dynamics of a population with Allee's effect [7, 82] negatively impacted by train vibrations. The second application concerns the dynamics of a population of *Cyanobacteria* in a cultured environment, keeping track of ammonia levels in the process.

5.1 Prolongation of solutions and maximal interval of existence

Let O be a nonempty open set of $\mathbb{R}^n$, and I an open interval in the topology τ_g containing $t_0 \geq 0$ with $\sup I > t_0$. We set $\Omega := I \times O$. Ω is an open set with respect to the topology $\hat{\tau}_g$ on $\mathbb{R} \times \mathbb{R}^n$, generated by the open sets $U \times V$ such that $U \in \tau_g$ and $V \in \tau_{\mathbb{R}^n}$. Let us consider $(t_0, \mathbf{x}_0) \in \Omega$ such that $t_0 < \sup I$, and the Stieltjes dynamical system:

$$\begin{aligned} \mathbf{x}'_g(t) &= \mathbf{f}(t, \mathbf{x}(t)) \quad \text{for } g\text{-almost all } t \geq t_0, t \in I, \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \in O, \end{aligned} \tag{5.2}$$

where $\mathbf{f} = (f_1, \dots, f_n) : \Omega \cap ([t_0, \infty) \times \mathbb{R}^n) \rightarrow \mathbb{R}^n$ and Ω defined above satisfying the following assumptions:

(H $_{\Omega}$) For every $(t, \mathbf{x}) \in \Omega$ with $t \geq t_0$,

(a) one of the following conditions hold:

(a1) there exists $\delta > 0$ such that $(t - \delta, t + \delta) \times B_{\mathbb{R}^n}(\mathbf{x}, \delta) \subset \Omega$;

(a2) if for every $\delta > 0$, $(t - \delta, t + \delta) \times B_{\mathbb{R}^n}(\mathbf{x}, \delta) \not\subset \Omega$, then $t \in D_g$ and there exists $\epsilon > 0$ such that $(t - \epsilon, t] \times B_{\mathbb{R}^n}(\mathbf{x}, \delta) \subset \Omega$;

(b) $(t_0, \mathbf{x}_{\mathbf{f}, t_0}^+) \in \Omega$, and if $(t, \mathbf{x}) \in \Omega \cap (D_g \times \mathbb{R}^n)$ such that $(t, \mathbf{x}_{\mathbf{f}, t}^+) \in \Omega$, then $(t, \mathbf{x}_{\mathbf{f}, t}^+)$ satisfies Condition (H $_{\Omega}$)(a)(1). Here and afterward, the notation $\mathbf{x}_{\mathbf{f}, t}^+$ refers to

$$\mathbf{x}_{\mathbf{f}, t}^+ := \mathbf{x} + \mu_g(\{t\})\mathbf{f}(t, \mathbf{x});$$

(H $_{\mathbf{f}}$) (i) for all $\mathbf{x} \in O$, $\mathbf{f}(\cdot, \mathbf{x})$ is g -measurable;

(ii) $\mathbf{f}(\cdot, \mathbf{x}_0) \in \mathcal{L}_{g, \text{loc}}^1([t_0, \infty), \mathbb{R}^n)$;

(iii) $\mathbf{f}$ is g -integrally locally Lipschitz continuous, i.e. for every $r > 0$, there exists a function $L_r \in \mathcal{L}_{g, \text{loc}}^1([t_0, \infty), [0, \infty))$ such that

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})\| \leq L_r(t)\|\mathbf{x} - \mathbf{y}\|,$$

for g -almost all $t \in I \cap [t_0, \infty)$ and all $\mathbf{x}, \mathbf{y} \in \overline{B_{\mathbb{R}^n}(\mathbf{x}_0, r)} \cap O$.

We recall the local existence result [31, Theorem 7.4].

Theorem 5.1.1. *Assume that the conditions in (H $_{\mathbf{f}}$) hold. Then there exists $\delta > 0$ such that the system (5.1) has a unique solution $\mathbf{x} \in \mathcal{AC}_g([t_0, t_0 + \delta], \mathbb{R}^n)$.*

In the sequel, as shown in [50, Section 4.2], the solution given by Theorem 5.1.1 can be extended to intervals larger than $[t_0, t_0 + \tau]$ up to a maximal interval as long as the graph of the solution does not leave Ω and the right-hand side limit of the solution at $t_0 + \tau$ belongs to O . It should be noted that the derivator g may not necessarily be continuous at t_0 . For this purpose, Condition (H $_{\Omega}$) is necessary to seek the maximal interval of existence of the solution. Indeed, notice that Condition (H $_{\Omega}$) guarantees existence of a local solution on an interval $[t_0, t_0 + \delta]$ even in the case where $t_0 \in D_g$, and ensures extension of the solution $\mathbf{x}$ on some larger interval $[t_0, t_0 + \epsilon]$, $\epsilon > \delta$ if

$$t_0 + \delta \in D_g \text{ and } \mathbf{x}(t_0 + \delta^+) \in O.$$

Let us define the following set

$$\mathcal{S}(t_0, \mathbf{x}_0) := \{\mathbf{x} : I_{\mathbf{x}} = J_{\mathbf{x}} \cap [t_0, \infty) \rightarrow \mathbb{R}^n : \mathbf{x} \text{ is a solution of (5.2),} \\ J_{\mathbf{x}} \text{ is an open interval of } \tau_g \text{ such that } \sup J_{\mathbf{x}} > t_0 \in J_{\mathbf{x}}\}.$$

For $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$, it is worth mentioning that $\mathbf{x} \in \mathcal{AC}_g([a, b], \mathbb{R}^n)$ for every interval $[a, b] \subset I_{\mathbf{x}}$. In the sequel, for $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$, we adopt the notation $t_{sup} := \sup I_{\mathbf{x}}$.

Definition 5.1.2. Let $\mathbf{x}, \mathbf{y} \in \mathcal{S}(t_0, \mathbf{x}_0)$.

(1) We say that $\mathbf{x}$ is *smaller* than $\mathbf{y}$ (and we denote $\mathbf{x} \prec \mathbf{y}$), if and only if

- (i) $I_{\mathbf{x}} \subset I_{\mathbf{y}}$;
- (ii) $\sup I_{\mathbf{y}} > \sup I_{\mathbf{x}}$;
- (iii) $\mathbf{y}|_{I_{\mathbf{x}}} = \mathbf{x}$.

In this case, we say that $\mathbf{x}$ is *extendible to the right* and $\mathbf{y}$ is a *prolongation to the right* of $\mathbf{x}$.

(2) We write that $\mathbf{x} \preceq \mathbf{y} \iff \mathbf{x} \prec \mathbf{y} \text{ or } \mathbf{x} = \mathbf{y}$.

Remark 5.1.3. It is worth mentioning that given a solution $\mathbf{x} : [t_0, t_1) \rightarrow \mathbb{R}^n$ such that $\mathbf{x}(t_1^-) := \lim_{t \rightarrow t_1^-} \mathbf{x}(t) \in O$ and $t_1 \in D_g$, to not increase the notation, we will replace $\mathbf{x}$ by the function $\mathbf{x} : [t_0, t_1] \rightarrow \mathbb{R}^n$ defined by

$$\mathbf{x} = \begin{cases} \mathbf{x}(t) & \text{if } t \in [t_0, t_1), \\ \mathbf{x}(t_1^-), & \text{if } t = t_1. \end{cases}$$

The next result asserts that two prolongations to the right of a solution are equal on the common interval of existence, the reader is referred to [50, Theorem 4.2.3] for the proof.

Theorem 5.1.4. Assume that (H_f) holds. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{S}(t_0, \mathbf{x}_0)$ be such that $\mathbf{y}$ and $\mathbf{z}$ are two prolongations of $\mathbf{x}$ then $\mathbf{y} = \mathbf{z}$ on $I_{\mathbf{y}} \cap I_{\mathbf{z}}$.

In [50, Theorem 4.2.4], extendible solutions to the right were characterized as follows.

Theorem 5.1.5. Assume that (H_f) holds. Let $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$. The following assumptions are equivalent:

- (1) $\mathbf{x}$ is extendible to the right;
- (2) (i) $\text{Graph}(\mathbf{x}) := \{(t, \mathbf{x}(t)) : t \in I_{\mathbf{x}}\}$ is bounded;
- (ii) $A \cup A^+ \subset \Omega$ where

$$A = \{(t_{sup}, \mathbf{s}) \in [t_0, \infty) \times \mathbb{R}^n : \exists \{t_n\}_n \subset I_{\mathbf{x}}, t_n \nearrow t_{sup} \text{ and } \mathbf{s} = \lim_{n \rightarrow \infty} \mathbf{x}(t_n)\},$$

$$A^+ = \{(t_{sup}, \mathbf{s}_{f, t_{sup}}^+) : (t_{sup}, \mathbf{s}) \in A\}.$$

Definition 5.1.6. Let $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$. We say that $\mathbf{x}$ is a *maximal solution* of (5.2) defined on an interval $I_{\mathbf{x}}$ if, for every $\mathbf{y} \in \mathcal{S}(t_0, \mathbf{x}_0)$ satisfying $\mathbf{x} \preceq \mathbf{y}$, we have $\mathbf{x} = \mathbf{y}$. $I_{\mathbf{x}}$ is referred to as the *maximal interval of existence*.

As shown in [50, Theorem 4.2.5], the existence of the maximal solution holds as a consequence of using Zorn's lemma [90]. For this purpose, let us consider $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$, and define the set $\mathcal{X}$:

$$\mathcal{X} = \{\mathbf{y} \in \mathcal{S}(t_0, \mathbf{x}_0) : \mathbf{x} \preceq \mathbf{y}\}.$$

$\mathcal{X}$ is a nonempty partially ordered set since $\mathbf{x} \in \mathcal{X}$. Now, following the same argument as in the proof of [50, Theorem 4.2.5], by using the partial order $\preceq$ defined in Definition 5.1.2 instead, we deduce that every chain of $\mathcal{X}$ has the largest element. This yields the existence of a maximal solution defined on $\bigcup_{\mathbf{y} \in \mathcal{X}} I_{\mathbf{y}}$, constructed by taking the union of all the prolongations to the right of $\mathbf{x}$. In addition, Theorem 5.1.4 guarantees the uniqueness of the maximal solution, and we obtain the next theorem.

Theorem 5.1.7. *Assume that (H_{Ω}) and (H_f) hold. Then, there exists a unique maximal solution $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$ such that $\omega(t_0, \mathbf{x}_0) := \sup I_{\mathbf{x}} \leq \infty$.*

The next theorem highlights three alternative cases that occur, the reader is referred to [50, Theorem 4.2.6] for the proof.

Theorem 5.1.8. *Assume that (H_{Ω}) and (H_f) hold. Let $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$ be the maximal solution of (5.2), then one of the alternatives holds:*

- (A1) $\omega(t_0, \mathbf{x}_0) = \infty$;
- (A2) $\omega(t_0, \mathbf{x}_0) < \infty$, and for every $\{t_n\}_n \subset I_{\mathbf{x}}$ such that $t_n \nearrow \omega(t_0, \mathbf{x}_0)$, $\{\mathbf{x}(t_n)\}_n$ is not bounded;
- (A3) $\omega(t_0, \mathbf{x}_0) < \infty$, and there exists $\{t_n\}_n \subset I_{\mathbf{x}}$ satisfying $t_n \nearrow \omega(t_0, \mathbf{x}_0)$ and $\{\mathbf{x}(t_n)\}_n$ is a bounded sequence, such that for each subsequence $\{t_{n_k}\}_k$ verifying $\mathbf{x}(t_{n_k}) \rightarrow \mathbf{m}$, we have that

$$\{(\omega(t_0, \mathbf{x}_0), \mathbf{m}), (\omega(t_0, \mathbf{x}_0), \mathbf{m}_{f, \omega(t_0, \mathbf{x}_0)}^+)\} \not\subset \Omega.$$

Remark 5.1.9. Notice that if Alternative (A1) holds then $I_{\mathbf{x}} = [t_0, \infty)$, if Alternative (A2) or (A3) holds, then $I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0))$ or $I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0)]$.

Thanks to Theorem 5.1.8 established by Larivière, we obtain the following corollary, which ensures the global existence of the maximal solution over the whole interval $[t_0, \infty)$ in the case where $[t_0, \infty) \subset I$.

Corollary 5.1.10. *Let I be an interval containing $[t_0, \infty)$. Assume that (H_{Ω}) and (H_f) hold. Let $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$ be the maximal solution of (5.2). If*

$$\Omega^+ := \{(t, \mathbf{u}) \in \Omega : (t, \mathbf{u}_{f,t}^+) \in \Omega\} = \Omega,$$

and there exists a compact set $D \subset O$ such that $\mathbf{x}(t) \in D$ for every $t \in I_{\mathbf{x}}$, then $\omega(t_0, \mathbf{x}_0) = \infty$.

Proof. Let us assume that $\omega(t_0, \mathbf{x}_0) < \infty$. Since D is compact, for all $\{t_n\}_n \subset I_{\mathbf{x}}$ such that $t_n \nearrow \omega(t_0, \mathbf{x}_0)$, $\{\mathbf{x}(t_n)\}_n$ is bounded. Thus, it results from Theorem 5.1.8 that Alternative (A3) holds. Therefore, there exists $\{t_n\}_n \subset I_{\mathbf{x}}$ with $t_n \nearrow \omega(t_0, \mathbf{x}_0)$

and $\{(t_n, \mathbf{x}(t_n))\}_n$ is bounded and such that, for a fixed subsequence $\{t_{n_k}\}_k$ verifying $\mathbf{x}(t_{n_k}) \rightarrow \mathbf{m}$, we have that

$$\left\{(\omega(t_0, \mathbf{x}_0), \mathbf{m}), (\omega(t_0, \mathbf{x}_0), \mathbf{m}_{\mathbf{f}, \omega(t_0, \mathbf{x}_0)}^+)\right\} \not\subset \Omega. \quad (5.3)$$

On the other hand, D being a compact set yields that

$$(t_{n_k}, \mathbf{x}(t_{n_k})) \rightarrow (\omega(t_0, \mathbf{x}_0), \mathbf{m}) \in [t_0, \omega(t_0, \mathbf{x}_0)] \times D \subset \Omega.$$

Now, since $\Omega^+ = \Omega$, we deduce that $(\omega(t_0, \mathbf{x}_0), \mathbf{m}_{\mathbf{f}, \omega(t_0, \mathbf{x}_0)}^+) \in \Omega$, which contradicts (5.3). Hence, $\omega(t_0, \mathbf{x}_0) = \infty$. $\square$

Now based on Theorems 5.1.5 and 5.1.8, we provide a characterization of the maximal solution of the problem (5.2).

Theorem 5.1.11. *Assume that (H_Ω) and (H_f) hold. Let $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$. The following assumptions are equivalent:*

- (1) $\mathbf{x}$ is maximal;
- (2) for every compact set $K \subset \Omega$, there exists $t_K \in I_{\mathbf{x}}$ such that

$$\left\{(t, \mathbf{x}(t)), (t, (\mathbf{x}(t))_{\mathbf{f}, t}^+)\right\} \not\subset K,$$

for all $t \geq t_K, t \in I_{\mathbf{x}}$.

Proof. Assume that $\mathbf{x}$ is maximal. By Theorem 5.1.8, two cases may occur:

Case 1: if $\omega(t_0, \mathbf{x}_0) = \sup I_{\mathbf{x}} = \infty$, by contradiction assume that there exist a compact set $K \subset \Omega$ and a sequence $\{t_n\}_n \subset I_{\mathbf{x}}$ such that

$$t_n \nearrow \omega(t_0, \mathbf{x}_0) \quad \text{and} \quad \left\{(t_n, \mathbf{x}(t_n)), (t_n, (\mathbf{x}(t_n))_{\mathbf{f}, t_n}^+)\right\} \subset K \text{ for all } n \in \mathbb{N}.$$

This implies that $\{(t_n, \mathbf{x}(t_n))\}_n$ is bounded. Therefore, there exists a convergent subsequence $\{(t_{n_k}, \mathbf{x}(t_{n_k}))\}_k$ such that

$$(t_{n_k}, \mathbf{x}(t_{n_k})) \rightarrow (\tau, \mathbf{u}) \in K \subset [t_0, \infty) \times O.$$

This contradicts that $t_n \nearrow \omega(t_0, \mathbf{x}_0) = \infty$.

Case 2: if $\omega(t_0, \mathbf{x}_0) < \infty$, then by Theorem 5.1.8, there are two subcases:

Subcase 1: if Alternative (A2) holds, then $I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0))$ and $\|\mathbf{x}(t)\| \rightarrow \infty$ as $t \nearrow \omega(t_0, \mathbf{x}_0)$. Thus, for every $M > 0$ there exists $t^* \in I_{\mathbf{x}}$ such that $\|\mathbf{x}(t)\| \geq M$ for all $t \geq t^*$ with $t \in I_{\mathbf{x}}$. Hence, for every compact $K \subset \Omega$ there exists $t_K \in I_{\mathbf{x}}$ such that for all $t \geq t_K$ with $t \in I_{\mathbf{x}}$, we have $(t, \mathbf{x}(t)) \notin K$. In particular,

$$\left\{(t, \mathbf{x}(t)), (t, (\mathbf{x}(t))_{\mathbf{f}, t}^+)\right\} \not\subset K.$$

Subcase 2: if Alternative (A3) holds, then we distinguish two cases. If $I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0))$, then $\text{Graph}(\mathbf{x})$ approaches the boundary of Ω and

$$(\omega(t_0, \mathbf{x}_0), \mathbf{x}(\omega(t_0, \mathbf{x}_0)^-)) \notin \Omega.$$

Thus, for every compact $K \subset \Omega$, there exists $t_K \in [t_0, \omega(t_0, \mathbf{x}_0))$ such that, for every $t \geq t_K$ with $t \in I_{\mathbf{x}}$, $(t, \mathbf{x}(t)) \notin K$, which yields

$$\{(t, \mathbf{x}(t)), (t, (\mathbf{x}(t))_{\mathbf{f},t}^+)\} \not\subset K.$$

Now, if $I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0)]$, then $\omega(t_0, \mathbf{x}_0) \in D_g$, $(\omega(t_0, \mathbf{x}_0), \mathbf{x}(\omega(t_0, \mathbf{x}_0))) \in \Omega$ and

$$(\omega(t_0, \mathbf{x}_0), (\mathbf{x}(\omega(t_0, \mathbf{x}_0)))_{\mathbf{f},\omega(t_0,\mathbf{x}_0)}^+) \notin \Omega.$$

Thus, for every compact $K \subset \Omega$, and for $t_K = \omega(t_0, \mathbf{x}_0)$,

$$(t_K, (\mathbf{x}(t_K))_{\mathbf{f},t_K}^+) \notin K, \quad \text{hence} \quad \{(t_K, \mathbf{x}(t_K)), (t_K, (\mathbf{x}(t_K))_{\mathbf{f},t_K}^+)\} \not\subset K.$$

Conversely, by contradiction, let us assume that $\mathbf{x} : I_{\mathbf{x}} \rightarrow \mathbb{R}^n$ is not maximal. Thus, $t_{sup} = \sup I_{\mathbf{x}} < \infty$ and $\mathbf{x}$ is extendible to the right. From Theorem 5.1.5, it follows that $\text{Graph}(\mathbf{x})$ is bounded and $A \cup A^+ \subset \Omega$. Thus, $[t_0, t_{sup}] \times \overline{\text{Graph}(\mathbf{x})}$ is compact,

$$\mathbf{x}(t_{sup}^-) = \mathbf{x}(t_{sup}) = \mathbf{s} \in O \text{ and } \mathbf{s}_{\mathbf{f},t_{sup}}^+ \in O.$$

Therefore, for the compact $K = \overline{\text{Graph}(\mathbf{x})} \cup \{(t_{sup}, \mathbf{s}_{\mathbf{f},t_{sup}}^+)\} \subset \Omega$, we have that

$$\{(t, \mathbf{x}(t)), (t, (\mathbf{x}(t))_{\mathbf{f},t}^+)\} \subset K \quad \text{for all } t \in I_{\mathbf{x}},$$

which yields a contradiction. Hence, $\mathbf{x}$ is maximal. $\square$

The negation of Theorem 5.1.11 provides also an interesting characterization of extendible solutions.

Corollary 5.1.12. *Assume that (H_{Ω}) and $(H_{\mathbf{f}})$ hold. Let $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$. The following statements are equivalent:*

- (1) $\mathbf{x}$ extendible to the right;
- (2) there exist a compact set $K \subset \Omega$ and a sequence $\{t_n\} \subset I_{\mathbf{x}}$ with $t_n \nearrow \omega(t_0, \mathbf{x}_0)$ such that

$$\{(t_n, \mathbf{x}(t_n)), (t_n, (\mathbf{x}(t_n))_{\mathbf{f},t_n}^+)\} \subset K \quad \text{for all } n \in \mathbb{N}.$$

Given the generalization of the Grönwall lemma, Lemma 1.7.5, we can state the next theorem which provides the global existence of the solution over $[t_0, \infty)$ and a priori bound of the solution when $[t_0, \infty) \subset I$ and $O = \mathbb{R}^n$.

Theorem 5.1.13. *Assume that (H_{Ω}) and $(H_{\mathbf{f}})$ hold. Let $[t_0, \infty) \subset I$ and $O = \mathbb{R}^n$ and assume that the conditions in $(H_{\mathbf{f}})$ hold for $\mathbf{f} : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying the linear growth condition:*

(H_{LG}) there exist functions $k \in \mathcal{L}_{g,loc}^1([t_0, \infty), \mathbb{R})$, and $p \in \mathcal{L}_{g,loc}^1([t_0, \infty), [0, \infty))$ such that

$$\|\mathbf{f}(t, \mathbf{u})\| \leq k(t) + p(t)\|\mathbf{u}\| \quad \text{for } g\text{-almost all } t \in [t_0, \infty) \text{ and all } \mathbf{u} \in \mathbb{R}^n.$$

If $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$ is the maximal solution of (5.2), then

$$\|\mathbf{x}(t)\| \leq e_p(t, t_0) \left(\int_{[t_0, t)} \frac{e_p^{-1}(s, t_0) k(s)}{1 + p(s) \mu_g(\{s\})} d\mu_g(s) + \|\mathbf{x}_0\| \right), \quad (5.4)$$

for all $t \in [t_0, T]$, with $T \in I_{\mathbf{x}}$. Moreover, we have $I_{\mathbf{x}} = [t_0, \infty)$.

Proof. Let $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$ be the maximal solution of (5.2) defined on the maximal interval $I_{\mathbf{x}}$ with $\omega(t_0, \mathbf{x}_0) = \sup I_{\mathbf{x}}$. By (H_{LG}) , we have for $T > t_0$ with $T \in I_{\mathbf{x}}$ that

$$(\|\mathbf{x}\|)'_g(t) \leq \|\mathbf{x}'_g(t)\| = \|\mathbf{f}(t, \mathbf{x}(t))\| \leq k(t) + p(t)\|\mathbf{x}(t)\| \quad \text{for } g\text{-almost all } t \in [t_0, T].$$

Observe that $\|\cdot\|$ is Lipschitz continuous on $\mathbb{R}^n$. Thus, by Lemma 1.7.3, $\|\mathbf{x}(\cdot)\| \in \mathcal{AC}_g([t_0, T], \mathbb{R})$. Using the generalized version of Grönwall's Lemma, Lemma 1.7.5, we obtain

$$\|\mathbf{x}(t)\| \leq e_p(t, t_0) \left(\int_{[t_0, t)} \frac{e_p^{-1}(s, t_0) k(s)}{1 + p(s) \mu_g(\{s\})} d\mu_g(s) + \|\mathbf{x}_0\| \right) \quad \text{for all } t \in [t_0, T].$$

Assume by contradiction that $\omega(t_0, \mathbf{x}_0) < \infty$. Since $p \in \mathcal{L}_{g,loc}^1([t_0, \infty), [0, \infty))$, then $e_p(t, t_0)(1 + p(t)\mu_g(\{t\})) \geq 1$ for all $t \in [t_0, \infty)$. Consequently,

$$\begin{aligned} \|\mathbf{x}(t)\| &\leq \sup_{t \in [t_0, T]} e_p(t, t_0) \left(\int_{[t_0, t)} |k(s)| d\mu_g(s) + \|\mathbf{x}_0\| \right) \\ &\leq e_p(\omega(t_0, \mathbf{x}_0), t_0) \left(\|k\|_{\mathcal{L}_g^1([t_0, \omega(t_0, \mathbf{x}_0)), \mathbb{R})} + \|\mathbf{x}_0\| \right) \\ &= M_1. \end{aligned}$$

As $T \nearrow \omega(t_0, \mathbf{x}_0) < \infty$, $\mathbf{x}(t) \in \overline{B_{\mathbb{R}^n}(\mathbf{0}, M_1)}$ for all $t \in I_{\mathbf{x}}$. As $\mathbf{x}(\omega(t_0, \mathbf{x}_0)^-) = \mathbf{u} \in \mathbb{R}^n$, then

$$\mathbf{u}_{\mathbf{f}, \omega(t_0, \mathbf{x}_0)}^+ \in \mathbb{R}^n.$$

Therefore, for the compact set

$$K = [t_0, \omega(t_0, \mathbf{x}_0)] \times \overline{B_{\mathbb{R}^n}(\mathbf{0}, M)} \quad \text{with} \quad M = \max \left\{ M_1, \|\mathbf{u}_{\mathbf{f}, \omega(t_0, \mathbf{x}_0)}^+\| \right\},$$

we have that

$$\left\{ (t, \mathbf{x}(t)), (t, (\mathbf{x}(t))_{\mathbf{f}, t}^+) \right\} \subset K \quad \text{for all } t \in I_{\mathbf{x}}.$$

By Corollary 5.1.12, we obtain that $\mathbf{x}$ is extendible to the right which is a contradiction. Hence, $\omega(t_0, \mathbf{x}_0) = \infty$. $\square$

5.2 Lyapunov-like stability results

In the study of dynamical systems, the stability of equilibria holds significant importance. Here, the term equilibrium refers to a state that does not change dynamically, in the sense that if a system starts at an equilibrium point, it will stay in that state indefinitely. The interaction between species within many ecosystems often relies on

feedback loops. This makes stability crucial for their functioning, and resilience. Although stability is typically desirable, there are some scenarios where stable equilibrium at zero can be critical and raise concerns for several reasons. For instance, in ecological systems, this concern arises from the vulnerability to perturbations of certain species, which may increase the risk of their extinction.

In this section, we introduce Lyapunov-type results in the context of Stieltjes differential systems, based on the classical Lyapunov's second method, considering the Stieltjes dynamical system of the form:

$$\mathbf{x}'_g(t) = \mathbf{f}(t, \mathbf{x}(t)) \quad \text{for } g\text{-almost all } t \geq t_0 \geq 0, t \in I, \quad (5.5)$$

where $\mathbf{f} = (f_1, \dots, f_n) : \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r_0) \rightarrow \mathbb{R}^n$ satisfying $\mathbf{f}(t, \mathbf{0}) = \mathbf{0}$ for all $t \geq 0$. This study permits to draw conclusions about the behavior of solutions of the dynamical system (5.5) around the equilibrium $\mathbf{x} = \mathbf{0}$, which is also called the *trivial solution*. Before stating the assumptions on $\mathbf{f}$ to guarantee the existence and the uniqueness of solution of the system (5.5) starting at $t_0 \geq 0$ from $\mathbf{x}(t_0) = \mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, r_0)$, notice that if $t_0 \in D_g$ is such that $\mathbf{x}_0 + \mu_g(\{t_0\})\mathbf{f}(t_0, \mathbf{x}_0) \notin B_{\mathbb{R}^n}(\mathbf{0}, r_0)$, then existence of a solution $\mathbf{x}$ satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$ cannot be guaranteed. Thus, we assume

(H_r) there exists $r \in (0, r_0]$ such that for all $(t, \mathbf{u}) \in (\mathbb{R}^+ \cap D_g) \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, we have

$$\mathbf{u} + \mu_g(\{t\})\mathbf{f}(t, \mathbf{u}) \in B_{\mathbb{R}^n}(\mathbf{0}, r_0),$$

in other terms, for all $t \in (\mathbb{R}^+ \cap D_g)$,

$$B_{\mathbb{R}^n}(\mathbf{0}, r) \subset \{\mathbf{u} \in B_{\mathbb{R}^n}(\mathbf{0}, r_0) : \mathbf{u} + \mu_g(\{t\})\mathbf{f}(t, \mathbf{u}) \in B_{\mathbb{R}^n}(\mathbf{0}, r_0)\}.$$

Now, let us assume that $\mathbf{f}$ fulfills (H_f), where Ω is an open set of the form $\Omega = I \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ satisfying (H_Ω) as in Section 5.1 with I a g -open set containing the whole $\mathbb{R}^+$. Therefore, under hypotheses (H_Ω), (H_f), and (H_r), it follows from Theorem 5.1.7 that, for every $(t_0, \mathbf{x}_0) \in \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, there exists a unique maximal solution $\mathbf{x} = \mathbf{x}(\cdot, t_0, \mathbf{x}_0) \in \mathcal{S}(t_0, \mathbf{x}_0) \cap \mathcal{AC}_{g,loc}(I_{t_0, \mathbf{x}_0}, \mathbb{R}^n)$ of (5.5) defined on a maximal interval of existence that we denote $I_{t_0, \mathbf{x}_0}$ since it depends on the initial data $(t_0, \mathbf{x}_0)$. As before, we denote $\omega(t_0, \mathbf{x}_0) = \sup I_{t_0, \mathbf{x}_0} \leq \infty$.

5.2.1 Lyapunov stability notions

In this subsection, we present stability concepts within the framework of Stieltjes' differentiation. Through illustrative examples, we highlight the influence of the sets C_g and D_g on the change of stability properties.

Definition 5.2.1. The trivial solution $\mathbf{x} = \mathbf{0}$ of the system (5.5) is said to be

- *stable* if, for all $\epsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists $\delta = \delta(\epsilon, t_0) \in (0, r)$ such that

$$\|\mathbf{x}_0\| < \delta \quad \text{implies that} \quad \|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \epsilon \quad \text{for all } t \in I_{t_0, \mathbf{x}_0};$$

- *uniformly stable* if, for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) \in (0, r)$ such that for every $t_0 \in \mathbb{R}^+$,

$$\|\mathbf{x}_0\| < \delta \quad \text{implies that} \quad \|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \epsilon \quad \text{for all } t \in I_{t_0, \mathbf{x}_0}.$$

Remark 5.2.2. In the case where the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (5.5) is stable, for every $t_0 \geq 0$ fixed, observe that for $\epsilon = r$, there exists $\delta = \delta(\epsilon, t_0) \in (0, r)$ such that for all $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta)$,

$$\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \epsilon \text{ for all } t \in I_{t_0, \mathbf{x}_0}.$$

Using Theorem 5.1.8 and (H_Ω) and (H_r) , we deduce that $I_{t_0, \mathbf{x}_0} = [t_0, \infty)$. Furthermore, observe that if the stability of the trivial solution $\mathbf{x} = \mathbf{0}$ is uniform, then δ do not depend on t_0 .

The following definition is a notion of *asymptotic stability*. This concerns the behavior of solutions as $t \rightarrow \infty$.

Definition 5.2.3. The trivial solution $\mathbf{x} = \mathbf{0}$ of the system (5.5) is said to be

- *asymptotically stable* if it is *stable* and, for every $t_0 \in \mathbb{R}^+$, there is $\delta = \delta(t_0) \in (0, r)$ such that, for all $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta)$ and $\epsilon > 0$, there exists $\sigma = \sigma(t_0, \mathbf{x}_0, \epsilon) > 0$ such that

$$\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \epsilon \text{ for all } t \in [t_0 + \sigma, \infty) \cap I_{t_0, \mathbf{x}_0};$$

- *uniformly asymptotically stable* if it is *uniformly stable* and there is $\delta \in (0, r)$ such that, for every $\epsilon > 0$, there exists $\sigma = \sigma(\epsilon) > 0$ such that, for all $t_0 \in \mathbb{R}^+$ and $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta)$,

$$\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \epsilon \text{ for all } t \in [t_0 + \sigma, \infty) \cap I_{t_0, \mathbf{x}_0}.$$

In the following example, we compare the stability properties of the trivial solution of a linear Stieltjes dynamical system to the ones in the classical case, observing the change of the stability properties depending on the sets C_g and D_g . The resolution of linear Stieltjes differential equations has been studied in the literature, see for instance [25, 31, 50, 62, 63].

Example 5.2.4. Let us consider the linear Stieltjes dynamical system

$$x'_g(t) = cx(t) \quad \text{for } g\text{-almost every } t \geq t_0 \geq 0, \quad (5.6)$$

for $c \in \mathbb{R}$. In the classical case of derivation where $g \equiv \text{id}_{\mathbb{R}}$, for $c > 0$, the equilibrium $x = 0$ is not stable given that the solutions satisfying $x(t_0) = x_0 \in \mathbb{R}$ have the form $x_0 e^{c(t-t_0)}$ and they are not bounded on $[t_0, \infty)$ for $x_0 \neq 0$. Thus, for every $\epsilon > 0$ and $t_0 \geq 0$, there is no $\delta > 0$ such that

$$|x_0| < \delta \quad \text{implies that} \quad |x_0 e^{c(t-t_0)}| < \epsilon \text{ for all } t \geq t_0.$$

However, for $c < 0$, the equilibrium $x = 0$ is asymptotically stable since stability holds and given that the solutions satisfying $x(t_0) = x_0$ have the form $x_0 e^{c(t-t_0)} \xrightarrow{t \rightarrow \infty} 0$ for all $x_0 \in \mathbb{R}$. In the case where $c = 0$, every constant $z \in \mathbb{R}$ is a uniformly stable equilibrium.

Now, let us reconsider the dynamical system (5.6), where $g : \mathbb{R} \rightarrow \mathbb{R}$ is defined for $t \leq 1$ by $g(t) = t$, and $g(t) = 1$ for $t \geq 1$. Thus, the solutions of the problem (5.6) have the form $x_0 e^{c(g(t)-g(t_0))}$, $t \geq t_0$ for every $x_0 \in \mathbb{R}$ and $t_0 \geq 0$. Now, we show that the stability properties of the trivial solution $x = 0$ of the system (5.6) differ from the

classical case, for $c > 0$ and $c < 0$. For $c > 0$, we deduce the stability of the trivial solution $x = 0$. Indeed, for all $\epsilon > 0$ and $t_0 \geq 0$, there is $\delta = \frac{\epsilon}{e^{c(g(1)-g(t_0))}}$ such that

$$|x_0| < \delta \quad \text{implies that} \quad |x_0 e^{c(g(t)-g(t_0))}| < \epsilon \quad \text{for every } t \geq t_0.$$

Whereas, for $c < 0$ the equilibrium $x = 0$ is merely uniformly stable compared to the classical case, since for all $\epsilon > 0$, there exists $\delta = \epsilon > 0$ such that

$$|x_0| < \delta \quad \text{implies that} \quad |x_0 e^{c(g(t)-g(t_0))}| < \epsilon \quad \text{for every } t \geq t_0.$$

Asymptotic stability does not hold for any $c \in \mathbb{R}$, since $x_0 e^{c(g(t)-g(t_0))} \rightarrow x_0 e^{c(g(1)-g(t_0))} \not\rightarrow 0$ as $t \rightarrow \infty$ for all $t_0 \geq 0$ and $x_0 \in \mathbb{R}^*$.

Next, we change slightly the dynamical system (5.6), to incorporate jumps. Let us consider the derivator $g_1(t) = t + \sum_{n \in \mathbb{N}} \chi_{[n, \infty)}(t)$ for every $t \in \mathbb{R}$, and reconsider the dynamical system with g_1 instead:

$$x'_{g_1}(t) = \begin{cases} cx(t) & \text{if } t \geq t_0 \geq 0, \text{ with } t \notin D_{g_1} = \mathbb{N}, \\ \nu x(t), & \text{if } t \geq t_0 \geq 0, \text{ with } t \in D_{g_1}, \end{cases} \quad (5.7)$$

where $c, \nu \in \mathbb{R}$ with $\nu \in (-1, \infty) \setminus \{0\}$. Again the solution of (5.7) satisfying $x(t_0) = x_0$ has the form

$$x(t, t_0, x_0) = x_0 e_{c_*}(t, t_0) = x_0 e^{\int_{[t_0, t)} c_*(s) \, d\mu_{g_1}(s)},$$

where

$$c_*(t) = \begin{cases} c & \text{if } t \in [t_0, \infty) \setminus D_{g_1}, \\ \log(1 + \nu) & \text{if } t \in [t_0, \infty) \cap D_{g_1}. \end{cases}$$

In Figure 5.2.1, we can observe different patterns depending on the values of c and ν . This implies that the presence of discontinuities can destabilize or restore the stability properties of a dynamical system.

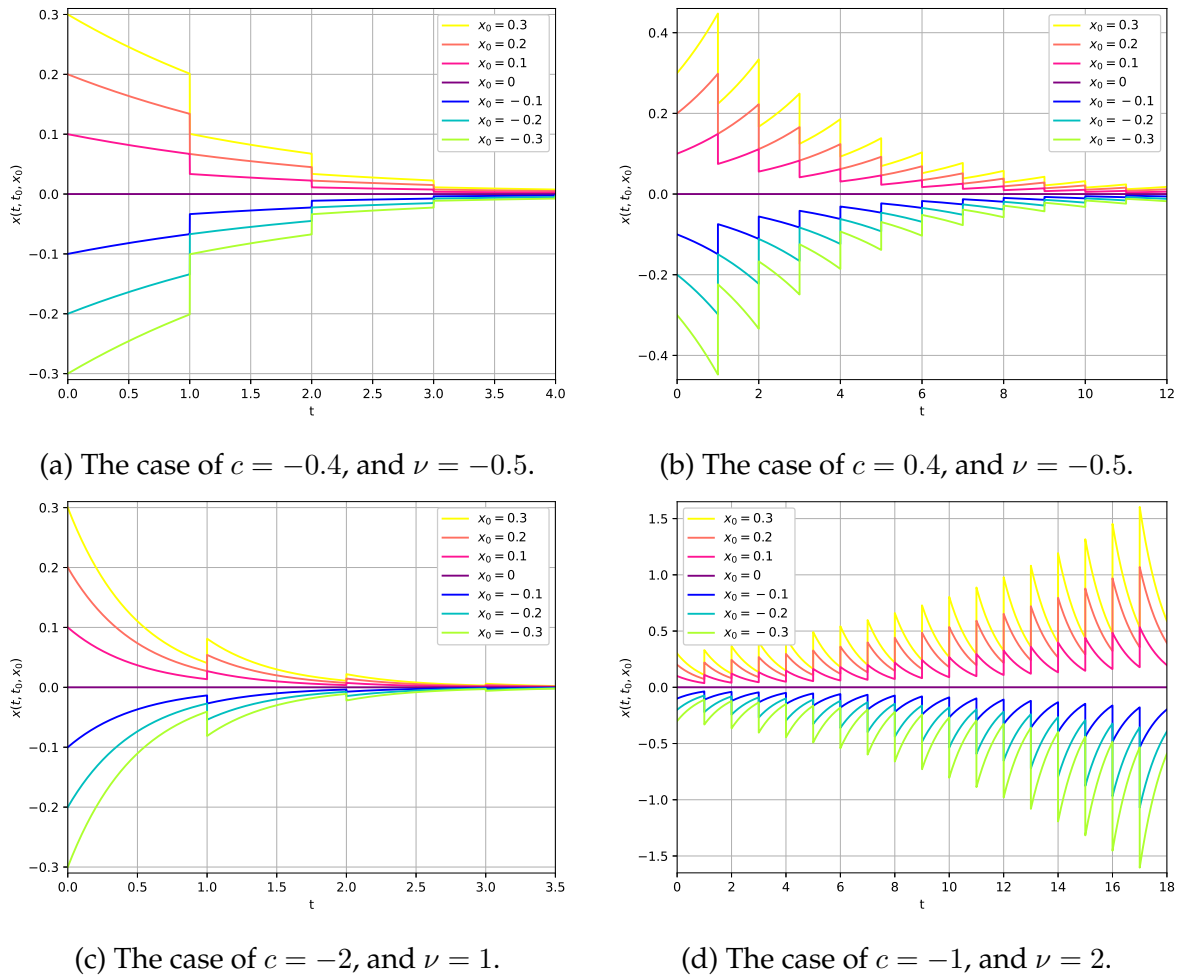


Figure 5.2.1 – Behaviour of solutions of (5.7) satisfying $x(t_0) = x_0$, in a neighborhood of $x = 0$.

5.2.2 Stability results based on Lyapunov's function

In the classical case where $g \equiv \text{id}_{\mathbb{R}}$, Lyapunov's method is used to study the behavior of a trajectory of the system in a neighborhood of the trivial solution $\mathbf{x} = 0$, by means of a function V depending on time and state; known as a *Lyapunov function*. This function can be understood as an energy representation of the system (5.5), since in numerous applications, the function considered is the total energy of the system (5.5) through time, see for instance [8] for an example of an energy-based Lyapunov function for physical systems. In our context, the derivator g takes into account the relevance of each moment during the process by means of the changes of the slopes of g accordingly. Put differently, g amplifies an alternative measurement for time, which may differ from the linear timeline typically used in the classical case where $g \equiv \text{id}_{\mathbb{R}}$, see for instance the works [54–57, 60, 61] where g represents the life cycle of some populations, also we refer to [3, 31] for more applications. Nevertheless, in the context of Stieltjes differentiation, we still can rely on Lyapunov's function, particularly, based on the g -derivative of its composition with maximal solutions of the

system (5.5) under consideration. This g -derivative of the composition will then permit a better understanding of how the energy of the system (5.5) changes, but with respect to this new observed time described by g . More precisely, it will describe how the energy of the system varies in response to the variation of this new "curved" scale of time.

Based on the definition of the Stieltjes derivative in Definition 1.4.1, we defined the partial Stieltjes derivative as follows.

Definition 5.2.5. Given a function $V : \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r_0) \rightarrow \mathbb{R}$ and $t, x_1, \dots, x_n$ its arguments. The partial g -derivative of V with respect to the first argument at a point $(t, \mathbf{x}) \in (\mathbb{R}^+ \setminus C_g) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ is defined as:

$$\frac{\partial V}{\partial_g t}(t, \mathbf{x}) = \begin{cases} \lim_{s \rightarrow t} \frac{V(s, \mathbf{x}) - V(t, \mathbf{x})}{g(s) - g(t)}, & \text{if } t \in \mathbb{R}^+ \setminus D_g, \\ \frac{V(t^+, \mathbf{x}) - V(t, \mathbf{x})}{g(t^+) - g(t)}, & \text{if } t \in \mathbb{R}^+ \cap D_g, \end{cases}$$

provided that the limit exists.

Combining Proposition 1.4.6 and Definition 5.2.5, we obtain the following technical proposition. The proof involves a formula related to the g -derivative of the composition involving a function with two variables. Formulae of this fashion were stated without proof in [78, Lemma 11] for $t \notin D_g$. In the next proposition, we derive formulae in the case where D_g and N_g are discrete.

Proposition 5.2.6. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a left-continuous and nondecreasing function such that D_g and N_g are discrete. Given a function $V : \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r_0) \rightarrow \mathbb{R}$ satisfying the following assumptions:

- (1) $V(\cdot, \mathbf{u})$ is g -differentiable on $\mathbb{R}^+ \setminus (D_g \cup C_g)$ for all $\mathbf{u} \in B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;
- (2) $V(t, \cdot) \in C^1(B_{\mathbb{R}^n}(\mathbf{0}, r_0), \mathbb{R})$ for all $t \geq 0$;
- (3) $\frac{\partial V}{\partial_g t}(\cdot, \mathbf{u})$ is continuous on $\mathbb{R}^+ \setminus (D_g \cup C_g)$ for all $\mathbf{u} \in B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;
- (4) for $(t_0, \mathbf{x}_0) \in \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, $V(\cdot, \mathbf{x}(\cdot)) \in \mathcal{AC}_{g, \text{loc}}(I_{\mathbf{x}}, \mathbb{R})$ for every solution $\mathbf{x} : I_{\mathbf{x}} \rightarrow B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ of the system (5.5).

Then, for all $[a, b] \subset I_{\mathbf{x}}$, we obtain for g -almost every $t \in [a, b] \setminus (D_g \cup C_g)$ that

$$V'_g(t, \mathbf{x}(t)) = \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)) + \sum_{i=1}^n \frac{\partial V}{\partial x_i}(t, \mathbf{x}(t)) f_i(t, \mathbf{x}(t)). \quad (5.8)$$

Moreover, if $t \in D_g$, then

$$V'_g(t, \mathbf{x}(t)) = \frac{V(t^+, \mathbf{x}(t) + \mu_g(\{t\})\mathbf{f}(t, \mathbf{x}(t))) - V(t, \mathbf{x}(t))}{g(t^+) - g(t)}. \quad (5.9)$$

Proof. For $(t_0, \mathbf{x}_0) \in \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, let us consider a solution $\mathbf{x} : I_{\mathbf{x}} \rightarrow B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ of the system (5.5). Thus, the composition $V(\cdot, \mathbf{x}(\cdot)) \in \mathcal{AC}_g(I_{\mathbf{x}}, \mathbb{R})$. Let $[a, b] \subset I_{\mathbf{x}}$. For g -almost every $t \in [a, b] \setminus (D_g \cup C_g)$:

$$V'_g(t, \mathbf{x}(t)) = \lim_{s \rightarrow t} \frac{V(s, \mathbf{x}(s)) - V(t, \mathbf{x}(t))}{g(s) - g(t)}$$

$$\begin{aligned}
 &= \lim_{s \rightarrow t} \frac{V(s, \mathbf{x}(s)) - V(t, \mathbf{x}(s))}{g(s) - g(t)} + \frac{V(t, \mathbf{x}(s)) - V(t, \mathbf{x}(t))}{g(s) - g(t)} \\
 &= \lim_{s \rightarrow t} \frac{V(s, \mathbf{x}(s)) - V(t, \mathbf{x}(s))}{g(s) - g(t)} \\
 &\quad + \sum_{i=1}^n \left(\frac{V(t, (x_1(t), \dots, x_{i-1}(t), x_i(s), \dots, x_n(s)))}{g(s) - g(t)} \right. \\
 &\quad \left. - \frac{V(t, (x_1(t), \dots, x_i(t), x_{i+1}(s), \dots, x_n(s)))}{g(s) - g(t)} \right).
 \end{aligned}$$

For s sufficiently close to t , and since D_g and N_g are discrete, g is continuous and increasing on the interval with endpoint points s and t . By applying Theorem 1.4.8 to the function $V(\cdot, \mathbf{x}(s))$, we obtain that there exists c between s and t such that

$$\frac{V(s, \mathbf{x}(s)) - V(t, \mathbf{x}(s))}{g(s) - g(t)} = \frac{\partial V}{\partial_g t}(c, \mathbf{x}(s)).$$

As $s \rightarrow t$, $c \rightarrow t$, and using Condition (3) we obtain

$$\frac{V(s, \mathbf{x}(s)) - V(t, \mathbf{x}(s))}{g(s) - g(t)} = \frac{\partial V}{\partial_g t}(c, \mathbf{x}(s)) \rightarrow \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)).$$

Therefore,

$$\begin{aligned}
 V'_g(t, \mathbf{x}(t)) &= \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)) + \sum_{i=1}^n \frac{\partial V}{\partial x_i}(t, \mathbf{x}(t))(x_i)'_g(t) \\
 &= \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)) + \sum_{i=1}^n \frac{\partial V}{\partial x_i}(t, \mathbf{x}(t))f_i(t, \mathbf{x}(t)).
 \end{aligned}$$

For $t \in [a, b) \cap D_g$, we obtain immediately that

$$\begin{aligned}
 V'_g(t, \mathbf{x}(t)) &= \frac{V(t^+, \mathbf{x}(t^+)) - V(t, \mathbf{x}(t))}{g(t^+) - g(t)} \\
 &= \frac{V(t^+, \mathbf{x}(t) + \mu_g(\{t\})\mathbf{f}(t, \mathbf{x}(t))) - V(t, \mathbf{x}(t))}{g(t^+) - g(t)}.
 \end{aligned}$$

□

In order to establish sufficient conditions for different types of stability of the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (5.5) as defined in Subsection 5.2.1, we introduce specific sets of functions.

Definition 5.2.7. A function $V : \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r_0) \rightarrow \mathbb{R}$ is said to belong to class $\mathcal{V}_1^g$ if it satisfies the following conditions:

1. $V(t, \cdot)$ is continuous for all $t \geq 0$;
2. $V(\cdot, \mathbf{x}(\cdot)) \in \mathcal{AC}_{g,loc}(I_{t_0, \mathbf{x}_0}, \mathbb{R})$ for all $\mathbf{x} : I_{t_0, \mathbf{x}_0} \rightarrow B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ of $\mathcal{AC}_{g,loc}(I_{t_0, \mathbf{x}_0}, \mathbb{R}^n)$ maximal solution of the system (5.5);
3. $V(t, \mathbf{0}) = 0$ for all $t \geq 0$.

Definition 5.2.8. A function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ belongs to the class $\mathcal{K}$ if it fulfills the following assumptions:

1. φ is continuous;
2. $\varphi(0) = 0$;
3. φ is increasing.

Now, we state the first stability result.

Theorem 5.2.9. Assume that Conditions (H_Ω) , (H_f) and (H_r) hold. If there exist functions $V \in \mathcal{V}_1^g$ and $a \in \mathcal{K}$ such that

- (a) $a(\|\mathbf{u}\|) \leq V(t, \mathbf{u})$, for all $(t, \mathbf{u}) \in \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;
- (b) for every $(t_0, \mathbf{x}_0) \in \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, if $\mathbf{x} : I_{t_0, \mathbf{x}_0} \rightarrow B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ is a maximal solution of the system (5.5), then $V'_g(t, \mathbf{x}(t)) \leq 0$, for g -almost all $t \in I_{t_0, \mathbf{x}_0}$.

Then, the trivial solution of the system (5.5) is

- (i) stable.
- (ii) uniformly stable if there exists $b \in \mathcal{K}$ such that

$$V(t, \mathbf{u}) \leq b(\|\mathbf{u}\|), \quad \text{for every } (t, \mathbf{u}) \in \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r_0). \quad (5.10)$$

Proof. (i) Since $V \in \mathcal{V}_1^g$, then, for all $\epsilon > 0$ and $t_0 \in \mathbb{R}^+$, there is some $\delta = \delta(\epsilon, t_0) \in (0, r)$ such that

$$\sup_{\|\mathbf{u}\| < \delta} V(t_0, \mathbf{u}) < a(\epsilon).$$

For $x_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta)$, let $\mathbf{x} : I_{t_0, \mathbf{x}_0} \rightarrow B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ be a maximal solution of the system (5.5) and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$. It follows from Conditions (a) and (b) that

$$a(\|\mathbf{x}(t)\|) \leq V(t, \mathbf{x}(t)) \leq V(t_0, \mathbf{x}_0) < a(\epsilon).$$

Thus, for every $t \in I_{t_0, \mathbf{x}_0}$, we have that

$$\|\mathbf{x}(t)\| = \|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \epsilon.$$

Therefore, the trivial solution $\mathbf{x} = \mathbf{0}$ is stable.

(ii) Arguing as in (i), we can choose a $\delta = \delta(\epsilon) \in (0, r)$ independent of t_0 such that

$$b(\delta) < a(\epsilon).$$

Thus, using (5.10), we obtain for all $t \in I_{t_0, \mathbf{x}_0}$,

$$a(\|\mathbf{x}(t)\|) \leq V(t, \mathbf{x}(t)) \leq V(t_0, \mathbf{x}_0) \leq b(\|\mathbf{x}_0\|) < b(\delta) < a(\epsilon).$$

This yields that the trivial solution $\mathbf{x} = \mathbf{0}$ is uniformly stable. $\square$

In the following result, we impose additional assumptions which will permit to ensure the asymptotical stability of the trivial solution $\mathbf{x} = \mathbf{0}$ to the system (5.5).

Theorem 5.2.10. Assume that Conditions (H_Ω) , (H_f) and (H_r) hold. Let $V \in \mathcal{V}_1^g$, $a, b \in \mathcal{K}$, $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ continuous, and a g -measurable function $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be such that

- (a) $a(\|\mathbf{u}\|) \leq V(t, \mathbf{u})$ for every $(t, \mathbf{u}) \in \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;
- (b) $\phi(s) = 0$ if and only if $s = 0$;
- (c) for every $(t_0, \mathbf{x}_0) \in \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, the maximal solution $\mathbf{x} : I_{t_0, \mathbf{x}_0} \rightarrow B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ of the system (5.5) satisfies

$$V'_g(t, \mathbf{x}(t)) \leq -w(t)\phi(\|\mathbf{x}(t)\|) \quad \text{for } g\text{-almost every } t \in I_{t_0, \mathbf{x}_0};$$

- (d) $\inf_{t_0 \in \mathbb{R}^+} \lim_{t \rightarrow +\infty} \int_{[t_0, t_0+t)} w(s) \, d\mu_g(s) = +\infty$.

If the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (5.5) is uniformly stable, then $\mathbf{x} = \mathbf{0}$ is asymptotically stable.

Proof. (i) The stability of the trivial solution $\mathbf{x} = \mathbf{0}$ holds from uniform stability. Let us choose $\delta_0 \in (0, r)$ associated to an $\epsilon_0 \leq r$ given by the uniform stability. Now, for a fixed $t_0 \geq 0$, let $\epsilon > 0$. Again, by the uniform stability, there exists $\delta \in (0, \delta_0)$ such that, for all $\hat{t} \in \mathbb{R}^+$ and every $\hat{\mathbf{x}}_0$ such that $\|\hat{\mathbf{x}}_0\| < \delta$, one has

$$\|\hat{\mathbf{x}}(t, \hat{t}, \hat{\mathbf{x}}_0)\| < \epsilon \text{ for every } t \in [\hat{t}, \infty) \cap I_{\hat{t}, \hat{\mathbf{x}}_0}.$$

We denote

$$M = \inf_{s \in [\delta, r_0)} |\phi(s)|. \quad (5.11)$$

By Condition (b), observe that $M > 0$.

Let $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta_0)$. Since

$$\lim_{t \rightarrow +\infty} \int_{[t_0, t_0+t)} w(s) \, d\mu_g(s) = +\infty,$$

we can choose $\sigma > 0$ such that

$$\int_{[t_0, t_0+\sigma)} w(s) \, d\mu_g(s) > \frac{V(t_0, \mathbf{x}_0)}{M}.$$

Let $\mathbf{x} : I_{t_0, \mathbf{x}_0} \rightarrow B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ be the maximal solution of (5.5) and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$. Using Remark 5.2.2, notice that $I_{t_0, \mathbf{x}_0} = [t_0, \infty)$. Now, if there is some $\hat{t} \in [t_0, t_0 + \sigma]$ such that $\|\mathbf{x}(\hat{t})\| < \delta$, then, by the uniform stability

$$\|\hat{\mathbf{x}}(t)\| < \epsilon \quad \text{for every } t \in [\hat{t}, \infty) \cap I_{\hat{t}, \mathbf{x}(\hat{t})},$$

where $\hat{\mathbf{x}} : I_{\hat{t}, \mathbf{x}(\hat{t})} \rightarrow B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ is the maximal solution of (5.5) satisfying the initial condition $\hat{\mathbf{x}}(\hat{t}) = \mathbf{x}(\hat{t})$. By the uniqueness of the maximal solution, one has

$$\omega(t_0, \mathbf{x}_0) = \omega(\hat{t}, \mathbf{x}(\hat{t})) = \infty \quad \text{and} \quad \mathbf{x}(t) = \hat{\mathbf{x}}(t) \text{ for all } t \in [\hat{t}, \infty).$$

Hence,

$$\|\mathbf{x}(t)\| < \epsilon \quad \text{for all } t \in [t_0 + \sigma, \infty) \cap I_{t_0, \mathbf{x}_0}.$$

On the other hand, if $\|\mathbf{x}(t)\| \geq \delta$ for every $t \in [t_0, t_0 + \sigma]$, then using Conditions (a), (a), Theorem 1.5.2, and (5.11), we obtain

$$\begin{aligned} a(\|\mathbf{x}(t_0 + \sigma)\|) &\leq V(t_0 + \sigma, \mathbf{x}(t_0 + \sigma)) \\ &= V(t_0, \mathbf{x}(t_0, t_0, \mathbf{x}_0)) + \int_{[t_0, t_0 + \sigma)} V'_g(s, \mathbf{x}(s)) \, d\mu_g(s) \\ &\leq V(t_0, \mathbf{x}_0) - \int_{[t_0, t_0 + \sigma)} w(s) \phi(\|\mathbf{x}(s)\|) \, d\mu_g(s) \\ &\leq V(t_0, \mathbf{x}_0) - M \int_{[t_0, t_0 + \sigma)} w(s) \, d\mu_g(s) \\ &< 0. \end{aligned}$$

This is a contradiction. Therefore, $\mathbf{x} = \mathbf{0}$ is asymptotically stable. $\square$

In the example below, we present an application of Theorem 5.2.10.

Example 5.2.11. Let us consider the dynamical system

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost all } t \geq t_0 \geq 0, \quad (5.12)$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(t) = t$ for $t \leq 1$, and $g(t) = t + 1$ for $t > 1$, and $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a function defined by

$$f(t, x) = \begin{cases} -\frac{xt}{1+t^2} & \text{if } t \in \mathbb{R}^+ \setminus D_g, \\ \nu x & \text{if } t \in \mathbb{R}^+ \cap D_g, \end{cases}$$

for some $\nu \in \mathbb{R} \setminus \{-1, 0\}$. The function f satisfies conditions of Theorem 5.1.13, thus, for every $(t_0, x_0) \in [0, \infty) \times \mathbb{R}$, the problem (5.12) has a maximal solution $x : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfying $x(t_0) = x_0$. Observe that $x = 0$ is an equilibrium of the dynamical system (5.12).

Let us define the function $V : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ for every $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ by

$$V(t, x) = \begin{cases} x^2 & \text{if } t \in [0, 1], \\ \frac{x^2}{(1+\nu)^2} & \text{if } t > 1. \end{cases}$$

Clearly $V \in \mathcal{V}_g^1$, and $a(|u|) \leq V(t, u) \leq b(|u|)$ for every $(t, u) \in \mathbb{R}^+ \times \mathbb{R}$, where $a, b \in \mathcal{K}$ are given by

$$a(s) = \min \left\{ s^2, \frac{s^2}{(1+\nu)^2} \right\} \quad \text{and} \quad b(s) = \max \left\{ s^2, \frac{s^2}{(1+\nu)^2} \right\} \quad \text{for all } s \in \mathbb{R}^+.$$

In addition, for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$:

$$\begin{aligned} \frac{\partial V}{\partial_g t}(t, x) &= \begin{cases} 0 & \text{if } t \in [0, 1) \cup (1, \infty), \\ \frac{\frac{x^2}{(1+\nu)^2} - x^2}{g(1^+) - g(1)} & \text{if } t = 1, \end{cases} \\ &= \begin{cases} 0 & \text{if } t \in [0, 1) \cup (1, \infty), \\ \frac{1 - (1+\nu)^2}{(1+\nu)^2} x^2 & \text{if } t = 1. \end{cases} \end{aligned}$$

Thus, by means of Proposition 5.2.6, for $t \in [t_0, \infty) \setminus D_g$, we obtain

$$\begin{aligned} V'_g(t, x(t)) &= \frac{\partial V}{\partial g t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t))f(t, x(t)) \\ &= \begin{cases} -\frac{2t}{1+t^2}x(t)^2 & \text{if } t \in [t_0, \infty) \cap [0, 1), \\ -\frac{2t}{(1+t^2)(1+\nu)^2}x(t)^2 & \text{if } t \in [t_0, \infty) \cap (1, \infty). \end{cases} \end{aligned}$$

For $t \in [t_0, \infty) \cap D_g$, if $t_0 \leq 1$, then $t = 1$ and we have that

$$\begin{aligned} V'_g(1, x(1)) &= \frac{V(1^+, x(1^+)) - V(1, x(1))}{g(1^+) - g(1)} \\ &= \frac{V(1^+, x(1) + \mu_g(\{1\})f(1, x(1))) - V(1, x(1))}{g(1^+) - g(1)} \\ &= \frac{\frac{(1+\nu)^2 x(1)^2}{(1+\nu)^2} - x(1)^2}{g(1^+) - g(1)} \\ &= 0. \end{aligned}$$

This implies that

$$V'_g(t, x(t)) \leq -\omega(t)\phi(|x(t)|), \text{ for } g\text{-almost all } t \geq t_0,$$

with $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, defined by

$$w(t) = \begin{cases} \frac{2t}{1+t^2} & \text{if } t \in [0, 1), \\ 0 & \text{if } t = 1, \\ \frac{2t}{(1+t^2)(1+\nu)^2} & \text{if } t \in (1, \infty), \end{cases}$$

and $\phi \in \mathcal{K}$, defined by $\phi(y) = 2y^2$ for all $y \in \mathbb{R}^+$. Moreover, for every $t_0 \in \mathbb{R}^+$ and $t > 1$, we have

$$\begin{aligned} \int_{[t_0, t_0+t)} w(s) \, d\mu_g(s) &= \int_{[t_0, t_0+t) \cap ([0,1) \cup \{1\} \cup (1,\infty))} w(s) \, d\mu_g(s) \\ &\geq \frac{1}{(1+\nu)^2} \int_{[t_0, t_0+t) \cap (1,\infty)} \frac{2s}{1+s^2} \, ds \\ &= \frac{1}{(1+\nu)^2} \left(\log(1 + (t_0 + t)^2) - \sup\{\log(2), \log(1 + t_0^2)\} \right). \end{aligned}$$

As, $\inf_{t_0 \in \mathbb{R}^+} \lim_{t \rightarrow +\infty} \log(1 + (t_0 + t)^2) - \sup\{\log(2), \log(1 + t_0^2)\} = +\infty$, we obtain

$$\inf_{t_0 \in \mathbb{R}^+} \lim_{t \rightarrow +\infty} \int_{[t_0, t_0+t)} w(s) \, d\mu_g(s) = +\infty.$$

Thus, Condition (d) holds. Therefore, by means of Theorem 5.2.10, we deduce that $x = 0$ is an asymptotically stable equilibrium. Figure 5.2.2 illustrates the asymptotic behavior of solutions of the dynamical system (5.12).

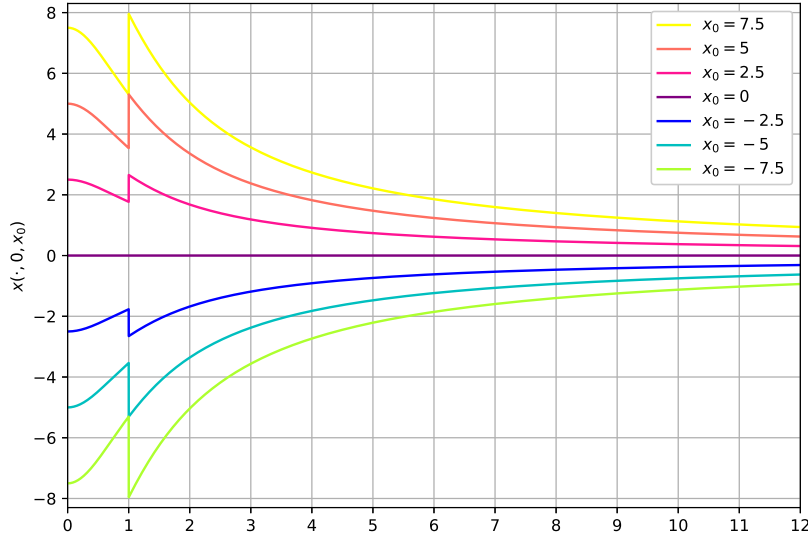


Figure 5.2.2 – Asymptotic behavior of solutions of the dynamical system (5.12) obtained with a time-discretization step-size of 10^{-3} , with $\nu = 0.5$.

The following result gives conditions to ensure the uniform asymptotic stability of the trivial solution of the system (5.5)

Theorem 5.2.12. Assume that Conditions (H_Ω) , (H_f) and (H_r) hold. If there exist $V \in \mathcal{V}_1^g$, $a, b \in \mathcal{K}$, $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ continuous, and a g -measurable function $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

- (a) $a(\|\mathbf{u}\|) \leq V(t, \mathbf{u}) \leq b(\|\mathbf{u}\|)$ for every $(t, \mathbf{u}) \in \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;
- (b) $\phi(s) = 0$ if and only if $s = 0$;
- (c) for every $(t_0, \mathbf{x}_0) \in \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, the maximal solution $\mathbf{x} : I_{t_0, \mathbf{x}_0} \rightarrow B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ of the system (5.5) satisfies

$$V'_g(t, \mathbf{x}(t)) \leq -w(t)\phi(\|\mathbf{x}(t)\|) \quad \text{for } g\text{-almost all } t \in I_{t_0, \mathbf{x}_0}.$$

- (d) $\lim_{t \rightarrow +\infty} \inf_{t_0 \in \mathbb{R}^+} \int_{[t_0, t_0+t)} w(s) \, d\mu_g(s) = +\infty$.

Then the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (5.5) is uniformly asymptotically stable.

Proof. By (ii) of Theorem 5.2.9, the trivial solution $\mathbf{x} = \mathbf{0}$ is uniformly stable. Thus, let us choose $\delta_0 \in (0, r)$ associated to an $\epsilon_0 \leq r$. Let $\epsilon > 0$. Again, by uniform stability, there is $\delta \in (0, r)$ such that, for all $\hat{t} \in \mathbb{R}^+$ and every $\hat{\mathbf{x}}_0$ such that $\|\hat{\mathbf{x}}_0\| < \delta$, one has

$$\|\hat{\mathbf{x}}(t, \hat{t}, \hat{\mathbf{x}}_0)\| < \epsilon \quad \text{for every } t \in [\hat{t}, \infty) \cap I_{\hat{t}, \hat{\mathbf{x}}_0}.$$

Let M be as defined in (5.11). Since

$$\lim_{t \rightarrow +\infty} \inf_{t_0 \in \mathbb{R}^+} \int_{[t_0, t_0+t)} w(s) \, d\mu_g(s) = +\infty,$$

we can choose $\sigma > 0$ such that

$$\int_{[t_0, t_0+\sigma)} w(s) \, d\mu_g(s) > \frac{b(\delta_0)}{M} \quad \text{for all } t_0 \in \mathbb{R}^+.$$

Let $(t_0, \mathbf{x}_0) \in \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, \delta_0)$ and $\mathbf{x} : I_{t_0, \mathbf{x}_0} \rightarrow B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ a maximal solution of (5.5) satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$.

If there exists $\hat{t} \in [t_0, t_0 + \sigma] \cap I_{t_0, \mathbf{x}_0}$ such that $\|\mathbf{x}(\hat{t})\| < \delta$, then

$$\|\hat{\mathbf{x}}(t)\| < \epsilon \quad \text{for every } t \in [\hat{t}, \infty) \cap I_{\hat{t}, \mathbf{x}(\hat{t})},$$

where $\hat{\mathbf{x}} : I_{\hat{t}, \mathbf{x}(\hat{t})} \rightarrow B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ is the maximal solution of (5.5) satisfying the initial condition $\hat{\mathbf{x}}(\hat{t}) = \mathbf{x}(\hat{t})$. By the uniqueness of the maximal solution, one has

$$\omega(t_0, \mathbf{x}_0) = \omega(\hat{t}, \mathbf{x}(\hat{t})) \quad \text{and} \quad \mathbf{x}(t) = \hat{\mathbf{x}}(t) \quad \text{for every } t \in [\hat{t}, \infty) \cap I_{t_0, \mathbf{x}_0}.$$

Hence,

$$\|\mathbf{x}(t)\| < \epsilon \quad \text{for all } t \in [t_0 + \sigma, \infty) \cap I_{t_0, \mathbf{x}_0}.$$

On the other hand, if $\|\mathbf{x}(t)\| \geq \delta$ for every $t \in [t_0, t_0 + \sigma]$, then using Conditions (a), (c), Theorem 1.5.2, and (5.11), we obtain

$$\begin{aligned} a(\|\mathbf{x}(t_0 + \sigma)\|) &\leq V(t_0 + \sigma, \mathbf{x}(t_0 + \sigma)) \\ &= V(t_0, \mathbf{x}(t_0, t_0, \mathbf{x}_0)) + \int_{[t_0, t_0 + \sigma)} V'_g(s, \mathbf{x}(s)) \, d\mu_g(s) \\ &\leq V(t_0, \mathbf{x}_0) - \int_{[t_0, t_0 + \sigma)} w(s) \phi(\|\mathbf{x}(s)\|) \, d\mu_g(s) \\ &\leq V(t_0, \mathbf{x}_0) - M \int_{[t_0, t_0 + \sigma)} w(s) \, d\mu_g(s) \\ &< b(\|\mathbf{x}_0\|) - b(\|\mathbf{x}_0\|) \\ &= 0. \end{aligned}$$

This is a contradiction. Hence, we conclude that $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable. $\square$

In the next example, we provide an application of Theorem 5.2.12 for a system subject to a derivator having an infinite number of discontinuities and for which the trivial solution $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable.

Example 5.2.13. Let us consider the dynamical system

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost every } t \geq t_0 \geq 0, \quad x(t_0) = x_0 \in \mathbb{R}, \quad (5.13)$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is the derivator defined by

$$g(t) = t + \sum_{k \in \mathbb{N}} \chi_{[t_k, +\infty)}(t) \quad \text{for every } t \in \mathbb{R},$$

where $\{t_k\}$ is an unbounded, increasing sequence in $(0, \infty)$, and $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is the function defined by

$$f(t, x) = \begin{cases} -t \arctan(x) & \text{if } t \in \mathbb{R}^+ \setminus D_g, \\ \nu_k x & \text{if } t = t_k, k \in \mathbb{N} \end{cases}$$

where $\{\nu_k\}_k \subset \mathbb{R}_+^*$ is a sequence satisfying

$$\lim_{k \rightarrow \infty} \frac{1}{\prod_{i=1}^k (1 + \nu_i)^2} = a_0 > 0.$$

The map f satisfies conditions of Theorem 5.1.13, thus the problem (5.13) has a maximal solution $x : [t_0, +\infty) \rightarrow \mathbb{R}$ for every $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}$. Observe that $x = 0$ is an equilibrium of the dynamical system (5.13).

Let us define the function $V : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ for every $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ by

$$V(t, x) = \begin{cases} x^2 & \text{if } t \in [0, t_1], \\ \frac{x^2}{\prod_{i=1}^k (1 + \nu_i)^2} & \text{if } t \in (t_k, t_{k+1}], k \in \mathbb{N}. \end{cases}$$

Clearly $V \in \mathcal{V}_g^1$, and $a(|x|) \leq V(t, x) \leq b(|x|)$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, where $a, b \in \mathcal{K}$ are functions defined by $a(s) = a_0 s^2$ and $b(s) = s^2$ for all $s \in \mathbb{R}^+$. In addition, for every $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$:

$$\begin{aligned} \frac{\partial V}{\partial_g t}(t, x) &= \begin{cases} 0 & \text{if } t \in [0, t_1] \cup (t_k, t_{k+1}), k \in \mathbb{N}, \\ \frac{\frac{x^2}{\prod_{i=1}^k (1 + \nu_i)^2} - \frac{x^2}{\prod_{i=1}^{k-1} (1 + \nu_i)^2}}{g(t_k^+) - g(t_k)} & \text{if } t = t_k, k \in \mathbb{N}, \end{cases} \\ &= \begin{cases} 0 & \text{if } t \in [0, t_1] \cup (t_k, t_{k+1}), k \in \mathbb{N}, \\ \frac{1 - (1 + \nu_k)^2}{\prod_{i=1}^k (1 + \nu_i)^2} x^2 & \text{if } t = t_k, k \in \mathbb{N}. \end{cases} \end{aligned}$$

For g -almost every $t \in [t_0, \infty) \setminus D_g$, we have that $t \in [0, t_1)$ or there is some $k \in \mathbb{N}$ such that $t \in (t_k, t_{k+1})$. Thus, by means of Proposition 5.2.6, we obtain if $t \in [0, t_1)$:

$$\begin{aligned} V'_g(t, x(t)) &= \frac{\partial V}{\partial_g t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t))f(t, x(t)) \\ &= -2x(t)t \arctan(x(t)) \\ &\leq -2t \frac{x(t)^2}{1 + x(t)^2}, \end{aligned}$$

where the last inequality follows from the Mean Value Theorem. While if there is some $k \in \mathbb{N}$ such that $t \in (t_k, t_{k+1})$, then

$$\begin{aligned} V'_g(t, x(t)) &= \frac{\partial V}{\partial_g t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t))f(t, x(t)) \\ &= -\frac{2x(t)}{\prod_{i=1}^k (1 + \nu_i)^2} t \arctan(x(t)) \\ &\leq -\frac{2t}{\prod_{i=1}^k (1 + \nu_i)^2} \frac{x(t)^2}{1 + x(t)^2}. \end{aligned}$$

For $t_k \in [t_0, \infty) \cap D_g$, we have that

$$\begin{aligned}
 V'_g(t_k, x(t_k)) &= \frac{V(t_k^+, x(t_k^+)) - V(t_k, x(t_k))}{g(t_k^+) - g(t_k)} \\
 &= \frac{V(t_k^+, x(t_k) + \mu_g(\{t_k\})f(t_k, x(t_k))) - V(t_k, x(t_k))}{g(t_k^+) - g(t_k)} \\
 &= \frac{\frac{(1+\nu_k)^2 x(t_k)^2}{\prod_{i=1}^k (1+\nu_i)^2} - \frac{x(t_k)^2}{\prod_{i=1}^{k-1} (1+\nu_i)^2}}{g(t_k^+) - g(t_k)} \\
 &= 0.
 \end{aligned}$$

Therefore, we conclude that

$$V'_g(t, x(t)) \leq -\omega(t)\phi(|x(t)|), \text{ for } g\text{-almost all } t \geq t_0,$$

where $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is the function defined for every $t \in \mathbb{R}^+$ by

$$w(t) = \begin{cases} 2t & \text{if } t \in [0, t_1), \\ 0 & \text{if } t = t_k, k \in \mathbb{N}, \\ \frac{2t}{\prod_{i=1}^k (1 + \nu_i)^2} & \text{if } t \in (t_k, t_{k+1}), k \in \mathbb{N}, \end{cases}$$

and $\phi \in \mathcal{K}$ the function given by $\phi(y) = \frac{y^2}{1 + y^2}$ for all $y \in \mathbb{R}^+$.

Observe that the function w satisfies

$$\begin{aligned}
 \lim_{t \rightarrow +\infty} \inf_{t_0 \in \mathbb{R}^+} \int_{[t_0, t_0+t)} w(s) \, d\mu_g(s) &\geq \lim_{t \rightarrow +\infty} \inf_{t_0 \in \mathbb{R}^+} \int_{t_0}^{t_0+t} a_0 2s \, ds \\
 &= a_0 \lim_{t \rightarrow +\infty} \inf_{t_0 \in \mathbb{R}^+} 2tt_0 + t^2 \\
 &= +\infty.
 \end{aligned}$$

By means of Theorem 5.2.12, we deduce that $x = 0$ is uniformly asymptotically stable equilibrium. Figure 5.2.3 illustrates the asymptotic behavior of solutions of the dynamical system (5.13) with $D_g = \mathbb{N}$.

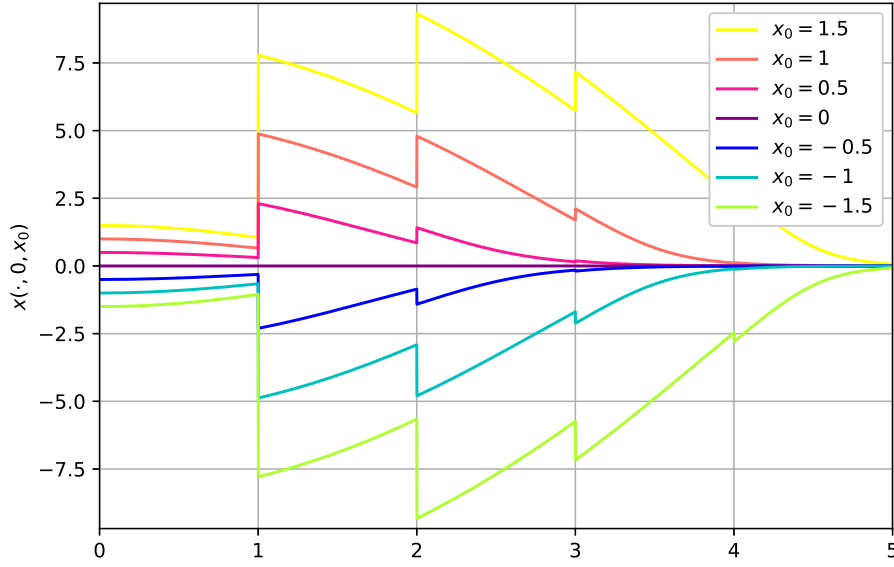


Figure 5.2.3 – Asymptotic behavior of solutions of the dynamical system (5.13) with $D_g = \mathbb{N}$, obtained with a time-discretization step-size of 10^{-3} , with $\nu_k = e^{\frac{2}{k^2}} - 1$ for all $k \in \mathbb{N}$.

Remark 5.2.14. Observe in Example 5.2.11 that the function w satisfies

$$\lim_{t \rightarrow +\infty} \inf_{t_0 \in \mathbb{R}^+} \int_{[t_0, t_0+t)} w(s) \, d\mu_g(s) = 0 \neq +\infty.$$

Thus, Condition (d) of Theorem 5.2.12 does not hold. Consequently, uniform asymptotic stability cannot be deduced.

In the classical case where $g \equiv \text{id}_{\mathbb{R}}$, corollary results [42, Theorem 4.2] and [59] are well-known when Conditions (b) of Theorem 5.2.10 is replaced by $V'_g(t, \mathbf{x}(t))$ being negative definite along each maximal solution $\mathbf{x}$ for every $t \geq t_0$. However, to present an analogous statement, we require an additional assumption to avoid the case when $\lim_{t \rightarrow +\infty} g(t) = l < +\infty$, and in particular, when there exists $T \geq 0$ such that $(T, +\infty) \subset C_g$, Example 5.2.4 provides an interesting illustration of attractivity lack for asymptotic stability.

Corollary 5.2.15. Assume that g is not bounded from above and Conditions (H_Ω) , (H_f) and (H_r) hold. If there exist $V : \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r_0) \rightarrow \mathbb{R}$; $V \in \mathcal{V}_1^g$ and $a, b, \phi \in \mathcal{K}$ such that

- (a) $a(\|\mathbf{u}\|) \leq V(t, \mathbf{u}) \leq b(\|\mathbf{u}\|)$, for all $(t, \mathbf{u}) \in \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;
- (b) for every $(t_0, \mathbf{x}_0) \in \mathbb{R}^+ \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, the maximal solution $\mathbf{x} : I_{t_0, \mathbf{x}_0} \rightarrow B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ of the system (5.5) satisfies

$$V'_g(t, \mathbf{x}(t)) \leq -\phi(\|\mathbf{x}(t)\|), \text{ for } g\text{-almost all } t \geq t_0.$$

Then, the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (5.5) is asymptotically stable. Furthermore, if

$$\lim_{t \rightarrow +\infty} \inf_{t_0 \in \mathbb{R}^+} g(t + t_0) - g(t_0) = +\infty,$$

then, the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (5.5) is uniformly asymptotically stable.

Proof. Observe that Conditions of Theorem 5.2.10 hold for $\omega \equiv 1$ as

$$\inf_{t_0 \in \mathbb{R}^+} \lim_{t \rightarrow +\infty} \int_{[t_0, t+t_0)} \omega(s) \, d\mu_g(s) = \inf_{t_0 \in \mathbb{R}^+} \lim_{t \rightarrow +\infty} g(t+t_0) - g(t_0) = +\infty.$$

Hence, the trivial solution is $\mathbf{x} = \mathbf{0}$ of the system (5.5) is asymptotically stable. Moreover, if

$$\lim_{t \rightarrow +\infty} \inf_{t_0 \in \mathbb{R}^+} g(t+t_0) - g(t_0) = +\infty,$$

then, Theorem 5.2.12 ensures that $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable. $\square$

5.3 Applications to dynamics of population

5.3.1 Stable equilibrium of a population subject to train vibrations

In this subsection, by means of a system of Stieltjes differential equations, we study the long-term impact of high-speed train vibrations and noise pollution on a population of animals living near railways. Depending on the species and their sensitivity to vibrations, various implications can be observed, we mention for instance:

- Hearing damage resulting in from the significant noise and vibrations that can potentially harm animals with sensitive hearing such as certain small mammals, birds, and bats which rely heavily on their hearing for communication, navigation, detection of predators, and finding food, thus, prolonged exposure to train vibrations may lead to hearing impairment or damage, disrupting their normal behaviors and increasing their death rate.
- Increased stress levels since some animals may be startled by the vibrations. This can affect their feeding patterns, reduce their reproduction rate, or lead to emigration resulting in a loss of suitable habitat and altering the composition and diversity of the local ecosystem
- Ecological interactions disruption which can affect pollination for instance if the vibrations deter insects that are important pollinators, disturb ground-dwelling organisms (insects, reptiles, and small mammals...) which can implicitly impact other species that rely on them as a food source.

In particular, an Allee effect can be observed in this regard, especially when the survival of the population depends on a minimum threshold size $M > 0$. In the follows, we denote by $x(t)$ as the number of individuals of a population living in a region near a railway with a carrying capacity $K > M$.

Let us assume that a certain number $m > 0$ of trains pass through the area every day. We refer to $\{\delta_i\}_{i=1}^{i=m}$ as the moments when trains pass in a single day. Once a train pass by the area, its impact is significant for a proportion of individuals living near the railway. They may experience vibrations or direct injuries. In the following analysis, we use a Stieltjes differential equation to model the dynamics of this population affected by train vibrations, and we study the asymptotic behavior of its solutions. In doing so, we require a derivator $g : \mathbb{R} \rightarrow \mathbb{R}$ presenting discontinuities for $t \in \{\delta_i\}_{i=1}^{i=m} + 24\mathbb{N}$ such that $\mu_g(\{t\})$ quantifies the rate at which the risk of damage

varies. Depending on the specific δ_i , this rate can either increase or decrease, reflecting the varying impact of vibrations during daylight and nighttime hours. For simplicity, we can take for instance:

$$g(t) = t + \sum_{k \in \mathbb{N}} \sum_{i=1}^m \chi_{[\delta_i + 24k, \infty)}(t) \quad \text{for all } t \in \mathbb{R},$$

with $\mu_g(\{\cdot\}) \equiv 1$ on $D_g = \{\delta_i : i = 1, \dots, m\} + 24\mathbb{N}$.

In the sequel, we suggest to analyse the asymptotic behaviour of the dynamics of this population, through the study of asymptotic stability of the zero equilibrium of the Stieltjes differential equation:

$$x'_g(t) = f(t, x(t)), \text{ for } g\text{-almost every } t \geq t_0 \geq 0, \quad (5.14)$$

where $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(t, x) = \begin{cases} \rho x \left(1 - \frac{x}{K}\right) \left(\frac{x}{M} - 1\right), & \text{if } t \notin \{\delta_i + 24k\}_{i=1}^m, \quad k = 0, 1, 2, \dots \\ -dx, & \text{if } t \in \{\delta_i + 24k\}_{i=1}^m, \quad k = 0, 1, 2, \dots \end{cases} \quad (5.15)$$

The parameters of the model can be understood as:

$K > M$: the carrying capacity of the environment.

$\rho > 0$: intrinsic rate of reproduction of the population.

$d \in (0, 1)$: Constant related to the impact induced by trains: either a migration rate immediately following the passage of trains or a mortality rate for certain populations that live in close proximity to the railway.

To simplify the analysis, we make the assumption that a train passes every hour over a 24-hour period. Thus, we consider

$$g(t) = t + \sum_{n \in \mathbb{N}} \chi_{[n, \infty)}(t) \quad \text{for every } t \in \mathbb{R},$$

which is such that $D_g = \mathbb{N}$ and $\mu_g(\{t\}) = 1$ for all $t \in D_g$.

Now, we will focus on the zero equilibrium to study its local asymptotic stability within a region $(-r_0, r_0)$ for some $r_0 > 0$ that will be determined to enhance the impact of environmental factors threatening this population.

Since the trivial solution $x = 0$ is an equilibrium of the dynamical system (5.14), let us consider $r_0 \leq M$. For all $(t_0, x_0) \in (\mathbb{R}^+ \cap D_g) \times (-r_0, r_0)$, if x is a solution of (5.14) satisfying $x(t_0) = x_0$, then

$$x(t_0^+) = x(t_0) + \mu_g(\{t_0\})f(t_0, x(t_0)) = (1 - d)x(t_0) \in (-r_0, r_0).$$

Thus, f satisfies (H_r) for $r = r_0$. One can also verify that f satisfies (H_Ω) and (H_f) . Hence, Theorem 5.1.7 yields the existence of a maximal solution $x : I_{t_0, x_0} \rightarrow \mathbb{R}$ for every $(t_0, x_0) \in \mathbb{R}^+ \times (-r_0, r_0)$. Let us show that $\omega(t_0, x_0) = \infty$. First of all, let $\tau \in (t_0, \omega(t_0, x_0))$ be such that $x : [t_0, \tau] \rightarrow (-r_0, r_0)$ is a solution of (5.14). Observe that if $x(\tau) = 0$, by uniqueness of the solution, we deduce that $x \equiv 0$ which lies in $(-r_0, r_0)$. Thus, two other cases occur when $x(\tau) \neq 0$:

Case 1: if $\tau \in D_g$, then $x(\tau^+) = x(\tau) + \mu_g(\{\tau\})f(\tau, x(\tau)) = (1 - d)x(\tau) \in (-r_0, r_0)$.

Case 2: if $\tau \notin D_g$, then we distinguish two subcases:

Subcase 1: if $x(\tau) \in (0, r_0)$ then $x'_g(\tau) = f(\tau, x(\tau)) < 0$. Thus, by g -continuity of x at τ there exists $\tau_1 \in (\tau, \omega(t_0, x_0))$ such that $x : [t_0, \tau_1] \rightarrow (0, r_0)$.

Subcase 2: if $x(\tau) \in (-r_0, 0)$ then $x'_g(\tau) = f(\tau, x(\tau)) > 0$. Thus, there exists $\tau_2 > 0$ such that $x : [\tau, \tau_2] \rightarrow (-r_0, 0)$.

Repeating the same argument for each subinterval of I_{t_0, x_0} , we deduce that

$$x(t) \in [-\lambda_{x_0}, \lambda_{x_0}] \subset (-r_0, r_0) \text{ for all } t \in I_{t_0, x_0},$$

where

$$\lambda_{x_0} = \sup_{t \in [t_0, \tau]} |x(t, t_0, x_0)|.$$

By Corollary 5.1.10, we deduce that $\omega(t_0, x_0) = \infty$. Hence, $x : [t_0, \infty) \rightarrow (-r_0, r_0)$ is a maximal solution of (5.14).

Now, we define the function $V : \mathbb{R}^+ \times (-r_0, r_0) \rightarrow \mathbb{R}$ by $V(t, x) = x^2$ for every $(t, x) \in \mathbb{R}^+ \times (-r_0, r_0)$. Clearly $V \in \mathcal{V}_g^1$. For all $(t_0, x_0) \in \mathbb{R}^+ \times (-r_0, r_0)$, let $x : [t_0, \infty) \rightarrow (-r_0, r_0)$ be the maximal solution of (5.14). Thus, using Proposition 5.2.6, we obtain for $t_0 \in \mathbb{R}^+$ and for g -almost every $t \in [t_0, \infty) \setminus D_g$:

$$\begin{aligned} V'_g(t, x(t)) &= \frac{\partial V}{\partial_g t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t))x'_g(t) \\ &= \frac{\partial V}{\partial_g t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t))f(t, x(t)) \\ &= 2\rho x(t)^2 \left(1 - \frac{x(t)}{K}\right) \left(\frac{x(t)}{M} - 1\right). \end{aligned}$$

While for $t \in [t_0, \infty) \cap D_g$, we obtain

$$\begin{aligned} V'_g(t, x(t)) &= \frac{V(t^+, x(t^+)) - V(t, x(t))}{g(t^+) - g(t)} \\ &= \frac{V(t^+, x(t) + \mu_g(\{t\})f(t, x(t))) - V(t, x(t))}{g(t^+) - g(t)} \\ &= (1 - d)^2 x(t)^2 - x(t)^2 \\ &= (-2d + d^2)x(t)^2. \end{aligned}$$

As $(-2d + d^2) < 0$, it follows that $V'_g(t, x(t))$ is negative definite. Since

$$a(|x|) \leq V(t, x) \leq b(|x|) \quad \text{for all } (t, x) \in \mathbb{R}^+ \times (-r_0, r_0)$$

with $a, b \in \mathcal{K}$ defined by $a(s) = b(s) = s^2$ for all $y \in \mathbb{R}^+$, Corollary 5.2.15 ensures that the trivial solution $x = 0$ is uniformly asymptotically stable. The asymptotic behavior of solutions is illustrated in Figure 5.3.1.

Comments. Figure 5.3.1 illustrates the long-term effect of the high-speed trains vibrations. As shown in the figure, if the vibrations lead to an Allee effect, the population will decline significantly over time. This outcome raises alarm about the overall stability and the persistence of the whole ecosystem in this region. Specifically, it indicates a high likelihood of population extinction when there is no estimation showing that the initial population exceeds M .

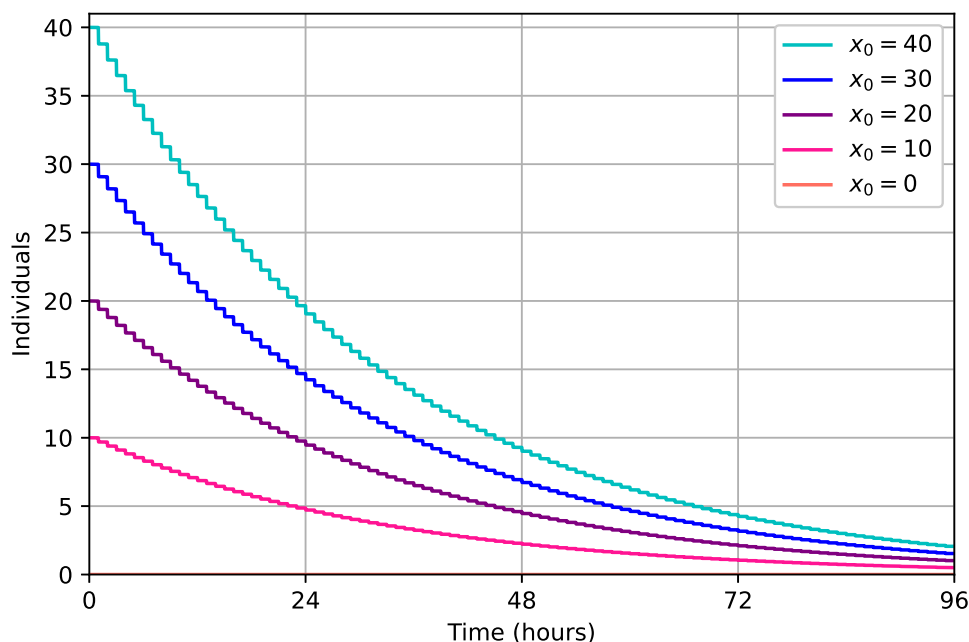


Figure 5.3.1 – Asymptotic behavior of solutions of the dynamical system (5.14) obtained with a time-discretization step-size of 10^{-3} , with $\rho = 0.001$, $K = 1000$, $M = 50$, and $d = 0.03$.

5.3.2 Stable equilibrium of Bacteria-Ammonia dynamics

Cyanobacteria, similar to plants, participate in oxygenic photosynthesis as they are photosynthetic bacteria. *Photosynthesis* is the biochemical process through which organisms convert light energy into chemical energy in the form of glucose or other organic compounds resulting oxygen release. The energy captured is then used to fuel the synthesis of organic molecules such as glucose, serving as an energy source for the bacteria.

Cyanobacteria are found in diverse habitats, including freshwater, marine environments, and terrestrial ecosystems. In this subsection, we consider a species of cyanobacteria that has also the ability to fix nitrogen such as *Anabaena cyanobacteria*. Through nitrogen fixation, atmospheric nitrogen gas (N_2) is converted into a form that can be used by plants and other organisms by means the enzyme *nitrogenase*. This enzyme catalyzes the conversion of nitrogen gas (N_2) into ammonium ions (NH_4^+) based on a considerable amount of energy obtained from photosynthesis, these ammonium ions are assimilated into amino acids and proteins, which are essential for the growth and survival of Cyanobacteria. As a by product of nitrogen fixation, ammonia gas

(NH_3) can be released. Some of the assimilated ammonium ions (NH_4^+) are released back into the environment providing neighboring vegetative cells with a source of nitrogen.

To avoid losing valuable nitrogen nutrients and optimizing nitrogen utilization efficiency, this population has mechanisms allowing ammonium ions (NH_4^+) reabsorption, and ammonia gas (NH_3) assimilation through converting ammonia gas (NH_3) into ammonium ions (NH_4^+). In what follows, the term "ammonia" refers to both the protonated and unprotonated forms, which are denoted as (NH_3) and (NH_4^+) respectively [11]. It is worth mentioning that ammonia is commonly used in cleaning products, fertilizers. Beyond that, ammonia's cooling properties make it an essential refrigerant in air conditioning systems and refrigerators.

To optimize resource utilization and adapt to the varying environmental conditions, the population undergoes a day-night cycling of nitrogen fixation and carbon consumption [83]. This is due to the sensitivity of the nitrogenase enzyme responsible for nitrogen fixation to oxygen. Thus, nitrogen fixation is carried out during daylight hours when the photosynthesis can provide the necessary energy and oxygen levels are relatively low. During the nights, since oxygen levels within the cells would be higher, this population of cyanobacteria reduces their metabolic activity and growth and relies on stored carbon compounds to fulfill their energy needs. Our objective in the sequel is to observe the dynamics of this population which thrives in the presence of ammonia in a culture room without exposure to artificial light, tracking the levels of the ammonia during this process. Here, we assume that the carbon dioxide (CO_2), the nitrogen gas (N_2) and nutrients supply are maintained steady as well as the PH level. Since the population undergoes dormancy phases during the nights, we identify the days with intervals of the form $[2k, 2k + 1]$, $k = 0, 1, 2, \dots$, and the night with intervals $[2k + 1, 2k + 2]$, $k = 0, 1, 2, \dots$. In our example, we differentiate with respect to a derivator g whose slopes describe the intensity of light, which is necessary for the photosynthesis process. We require that g presents smaller slopes at the beginning and at the end of the daylight hours, with a maximal slope of 1 representing the peak light intensity at middays where $t = 2k + 1/2$, and remains constant during the dormancy phases in the night $[2k + 1, 2k + 2]$, $k = 0, 1, 2, \dots$. For instance, we consider $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$g(t) = \begin{cases} \frac{\sin(\pi(t - 1/2)) + 1}{\pi} & \text{if } t \in [0, 1] \\ 2/\pi & \text{if } t \in (1, 2], \end{cases}$$

and $g(t) = 2/\pi + g(t - 2)$ for $t \geq 2$.

We denote by $N(t)$ the biomass of cyanobacteria (grams per liter), and $A(t)$ the ammonia concentration in the environment at time $t \geq 0$. Since the population thrives in the presence of ammonia, we can assume that the growth rate is proportional to the level of ammonia. We denote ρ the maximal intrinsic coefficient of reproduction that the population can reach in the presence of one unit of ammonia with maximal sunlight intensity. Thus, the dynamics can be modeled using the autonomous system of Stieltjes differential equations:

$$\mathbf{u}'_g(t) = \mathbf{F}(\mathbf{u}(t)), \text{ for } g\text{-almost every } t \geq t_0 \geq 0, \quad (5.16)$$

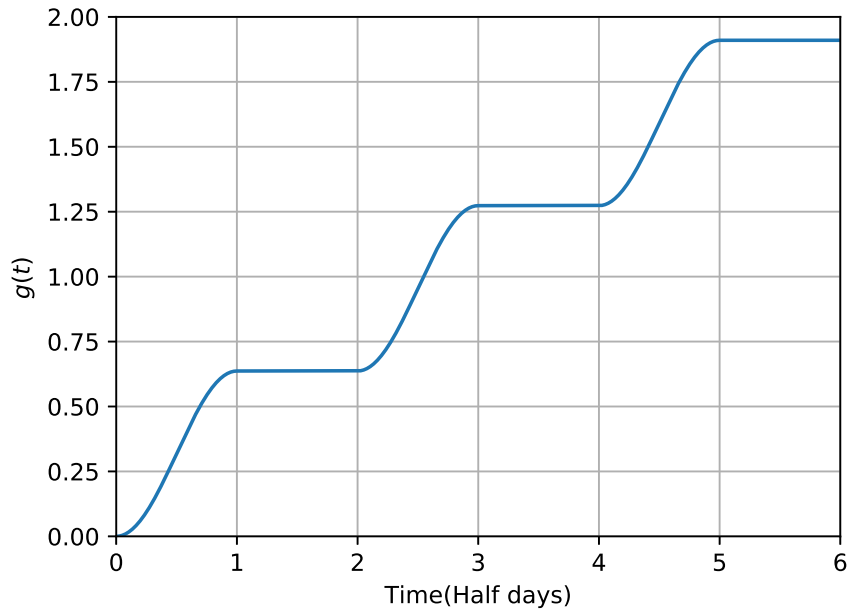


Figure 5.3.2 – Graph of the derivator g .

where $\mathbf{u} = (N, A)$ and $\mathbf{F} = (F_1, F_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by

$$\begin{aligned} F_1(N, A) &= \rho AN \left(1 - \frac{N}{K}\right) \\ F_2(N, A) &= \alpha N - \beta AN, \end{aligned} \tag{5.17}$$

where the parameters of the model can be understood as:

$K > 0$: the carrying capacity of the culture room, which forms a spacial constraint for growth.

$\alpha > 0$: a constant related to the production of ammonia through nitrogen fixation.

$\beta > 0$: a constant related to the proportion of the reabsorption of ammonia by the population depending on the level of ammonia in the environment.

Observe that $\mathbf{u}^* = (K, \alpha/\beta)$ is an equilibrium of the system (5.16) among other equilibria. Its asymptotic stability would guarantee the persistence of the population with nonzero ammonia production. Therefore, we study local asymptotic stability in a neighborhood $B_{\mathbb{R}^2}(\mathbf{u}^*, r_0)$ for some $r_0 > 0$ of this equilibrium. To this aim, we translate our study in a neighborhood of $\mathbf{x} = \mathbf{0} = (0, 0) \in \mathbb{R}^2$ with the variable change $\mathbf{x} = (x_1, x_2) := \mathbf{u} - \mathbf{u}^*$, we obtain the system

$$\mathbf{x}'_g(t) = \mathbf{f}(\mathbf{x}(t)), \text{ for } g\text{-almost every } t \geq t_0 \geq 0, \tag{5.18}$$

where $\mathbf{f} = (f_1, f_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by

$$\begin{aligned} f_1(x_1, x_2) &= -\rho \frac{x_1}{K} (x_2 + \alpha/\beta) (x_1 + K) \\ f_2(x_1, x_2) &= -\beta x_2 (x_1 + K). \end{aligned} \tag{5.19}$$

So, $\mathbf{x} = \mathbf{0} = (0, 0) \in \mathbb{R}^2$ is an equilibrium of the dynamical system (5.18). Next, we prove that $\mathbf{x} = \mathbf{0}$ is asymptotically stable, and hence $\mathbf{u}^* = (K, \alpha/\beta)$ is an asymptotically stable equilibrium of the system (5.16).

Let us consider $r_0 = \min\{K, \alpha/\beta\}$, and let $(t_0, \mathbf{x}_0) \in \mathbb{R}^+ \times B_{\mathbb{R}^2}(\mathbf{0}, r_0)$. Arguing as in the previous subsection, since the function $\mathbf{f}$ satisfies conditions of Theorem 5.1.7, there exists $\mathbf{x} = (x_1, x_2) : [t_0, \infty) \rightarrow B_{\mathbb{R}^2}(\mathbf{0}, r_0)$ the maximal solution of the system (5.18) satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$. Let $\tau \in (0, \omega(t_0, \mathbf{x}_0)) \setminus C_g$ be such that $\mathbf{x} = (x_1, x_2) : [t_0, \tau] \rightarrow B_{\mathbb{R}^2}(\mathbf{0}, r_0)$. In what follows, we analyze the possible cases:

Case 1: if $x_1(\tau), x_2(\tau) \in (0, r_0)$ (resp. $x_1(\tau), x_2(\tau) \in (-r_0, 0)$), then, for $i = 1, 2$, we have $(x_i)'_g(\tau) = f_i(\mathbf{x}(\tau)) < 0$ (resp. $(x_i)'_g(\tau) = f_i(\mathbf{x}(\tau)) > 0$). Therefore, by g -continuity of $\mathbf{x}$ at τ , there exists $\tau_1 \in (\tau, \omega(t_0, \mathbf{x}_0))$ (τ_1 can be chosen such that $\tau_1 \notin C_g$) such that

$$x_i : [t_0, \tau_1] \rightarrow (0, r_0) \quad (\text{resp. } x_i : [t_0, \tau_1] \rightarrow (-r_0, 0)).$$

Case 2: if $x_1(\tau) \in (0, r_0)$ and $x_2(\tau) \in (-r_0, 0)$, then $(x_1)'_g(\tau) = f_1(\mathbf{x}(\tau)) < 0$ and $(x_2)'_g(\tau) = f_2(\mathbf{x}(\tau)) > 0$. Therefore, there exists $\tau_2 \in (\tau, \omega(t_0, \mathbf{x}_0)) \setminus C_g$ such that

$$x_1 : [\tau, \tau_2] \rightarrow (0, r_0), \quad \text{and} \quad x_2 : [\tau, \tau_2] \rightarrow (-r_0, 0).$$

Case 3: if $x_1(\tau) \in (-r_0, 0)$ and $x_2(\tau) \in (0, r_0)$, then similarly to **Case 2**, we deduce that there exists $\tau_3 \in (\tau, \omega(t_0, \mathbf{x}_0))$ such that

$$x_1 : [\tau, \tau_3] \rightarrow (-r_0, 0), \quad \text{and} \quad x_2 : [\tau, \tau_3] \rightarrow (0, r_0).$$

Repeating the same argument for each subinterval of $I_{t_0, \mathbf{x}_0}$, we deduce that

$$\mathbf{x}(t) \in [-\lambda_{\mathbf{x}_{0,1}}, \lambda_{\mathbf{x}_{0,1}}] \times [-\lambda_{\mathbf{x}_{0,2}}, \lambda_{\mathbf{x}_{0,2}}] \subset B_{\mathbb{R}^2}(\mathbf{0}, r_0) \text{ for all } t \in I_{t_0, \mathbf{x}_0},$$

where

$$\lambda_{\mathbf{x}_{0,i}} = \sup_{t \in [t_0, \tau]} |x_i(t, t_0, \mathbf{x}_0)|, \quad \text{for } i = 1, 2 \text{ and } \mathbf{x}_0 = (\mathbf{x}_{0,1}, \mathbf{x}_{0,2}).$$

Using Corollary 5.1.10, we deduce that $\omega(t_0, \mathbf{x}_0) = \infty$.

Now, let us consider the function $V : \mathbb{R}^+ \times B_{\mathbb{R}^2}(\mathbf{0}, r_0) \rightarrow \mathbb{R}$ defined by $V(t, \mathbf{x}) = x_1^2 + x_2^2$ for all $t \geq 0$ and $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. It is clear that $V \in \mathcal{V}_1^g$. Let $(t_0, \mathbf{x}_0) \in \mathbb{R}^+ \times B_{\mathbb{R}^2}(\mathbf{0}, r_0)$, and $\mathbf{x} : [t_0, \infty) \rightarrow B_{\mathbb{R}^2}(\mathbf{0}, r_0)$ be the maximal solution of (5.18) satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$. By means of Proposition 5.2.6, for g -almost all $t \in [t_0, \infty)$, we obtain

$$\begin{aligned} V'_g(t, \mathbf{x}(t)) &= \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)) + \sum_{i=1}^2 \frac{\partial V}{\partial x_i}(t, \mathbf{x}(t))(x_i)'_g(t) \\ &= \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)) + \sum_{i=1}^2 \frac{\partial V}{\partial x_i}(t, \mathbf{x}(t))f_i(t, \mathbf{x}(t)) \\ &= -2\rho \frac{x_1(t)^2}{K}(x_2(t) + \alpha/\beta)(x_1(t) + K) - 2\beta x_2(t)^2(x_1(t) + K). \end{aligned} \tag{5.20}$$

Thus, $V'_g(t, \mathbf{x}(t))$ is negative definite. Since

$$a(\|\mathbf{z}\|) \leq V(t, \mathbf{z}) \leq b(\|\mathbf{z}\|) \quad \text{for all } (t, \mathbf{z}) \in \mathbb{R}^+ \times B_{\mathbb{R}^2}(\mathbf{0}, r_0),$$

where $a, b \in \mathcal{K}$ are defined by $a(y) = y^2$ and $b(y) = 2y^2$ for all $y \in \mathbb{R}^+$. It follows from Corollary 5.2.15 that $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable. Hence, $\mathbf{u}^* = (K, \alpha/\beta)$ is a uniformly asymptotically stable equilibrium of the system (5.16). The graph of asymptotic behavior of solutions near the equilibrium $\mathbf{u}^* = (K, \alpha/\beta)$ is given in Figure 5.3.3.

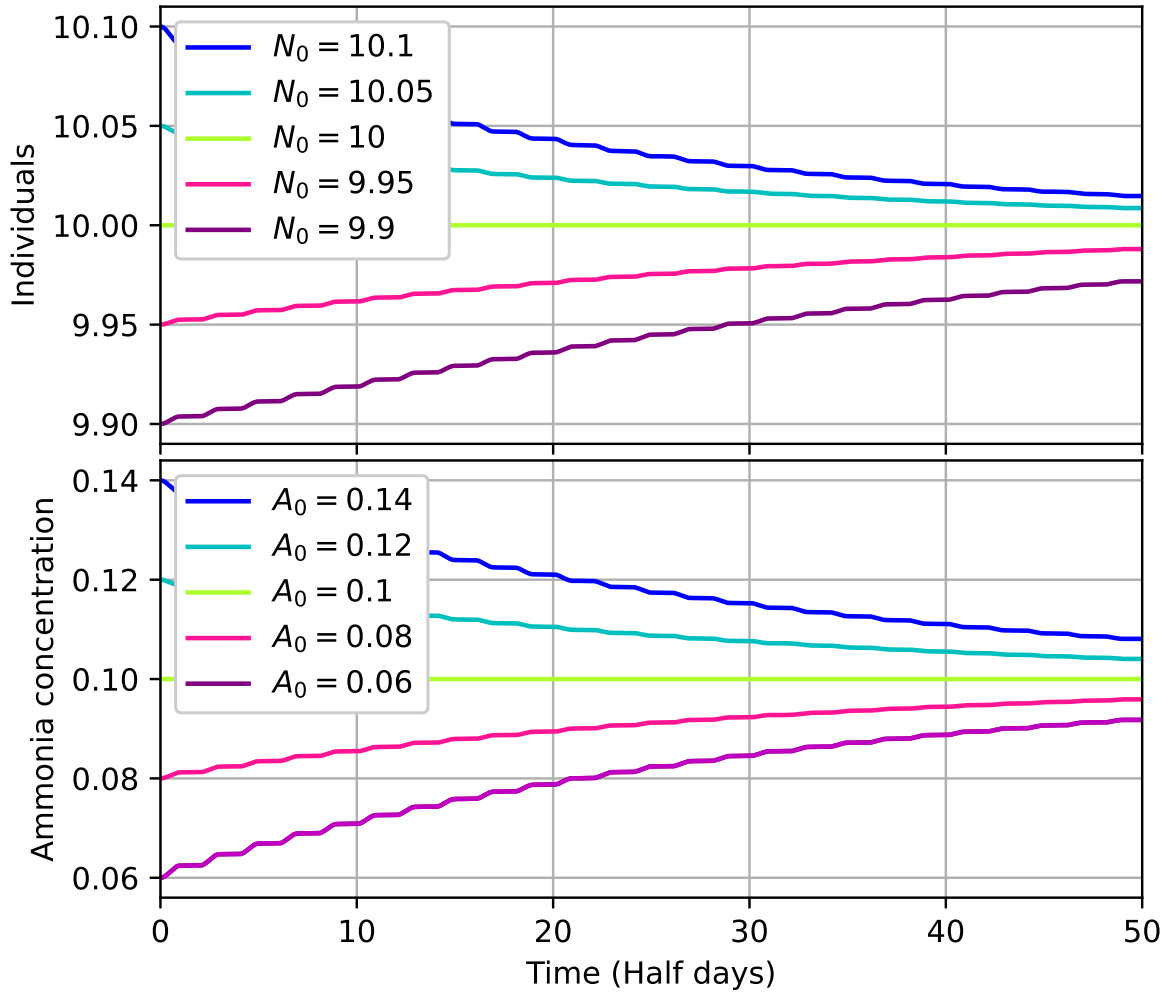


Figure 5.3.3 – Asymptotic behavior of solutions of the dynamical system (5.16) obtained with a time-discretization step-size of 10^{-3} , with different initial densities N_0 and ammonia concentrations A_0 (g/L), where $\rho = 1$, $K = 10$, $\alpha = 0.001$, and $\beta = 0.01$.

Conclusion and perspectives

To conclude this thesis, we provide a summary of the findings carried out in this manuscript. Afterwards, we propose personal research projects to be pursued within both theoretical and applied framework of Stieltjes differential equations.

Research findings

Along this thesis, we have delved into different mathematical objects in depth. The first chapter of this manuscript consists, for the most part, of a bibliographical revision of integration theory, new notions of continuity, and brief elements from fixed point theory related to the Fixed point index on normed spaces. It is in this chapter where we define the notion of the Stieltjes derivative with respect to a left-continuous and nondecreasing derivator $g : \mathbb{R} \rightarrow \mathbb{R}$, also we explore the properties of this Stieltjes derivative, introducing the notion of g -absolute continuity which is the regularity for solutions of the problems under study. At this point, we established the interesting Comparison Principle, Lemma 1.5.7, which is one of the key results to establish a priori bound of solutions, since it provides a straightforward generalization of the classical comparison principle to the context of Stieltjes derivatives where the compared g -absolutely continuous functions may presents discontinuities (interestingly not necessarily separated). In Chapter 2, we relate the problems with Stieltjes derivatives to difference equations, impulsive differential equations, and equations on time scales. The third Chapter in the one of cores of our investigation. In this chapter, we used the method of the lower and upper solutions to establish existence, a priori bounds and multiplicity results for the Stieltjes differential equations of the form

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost every } t \in [0, T), \quad x \in \mathfrak{B}; \quad (1)$$

where $\mathfrak{B}$ refers to the periodic or the initial value condition. In particular, we have been able to generalize the results in the literature dropping a monotonicity condition involving f , a g -Carathéodory function, in Theorems 3.1.7 and 3.1.12. In doing so, we proved that a continuous and completely continuous operator is associated to the g -Carathéodory function f in Lemma 4.1.3, which allowed us to leverage the fixed point theory in the search of solutions. In the second part, we introduced the first multiplicity results in the literature based on notions of strict lower and upper solutions for the periodic boundary problem in Theorem 3.2.3, and for the initial value problem in Theorem 3.2.5. At the end of this chapter, we provided an illustrative example in dynamics of population to ensure the persistence of a population subject to an extreme event, through a model involving a periodic Stieltjes differential equation.

Chapter 4 conforms an extension of Chapter 3 in the existential part mainly. In this chapter, we extended our existence results to systems of Stieltjes differential equations with several derivators:

$$\mathbf{x}'_g(t) = \mathbf{f}(t, \mathbf{x}(t)) \quad \text{for } g\text{-almost every } t \in [0, T), \quad \mathbf{x} \in \mathfrak{B}; \quad (4)$$

where $\mathbf{g} = (g_1, \dots, g_n) : \mathbb{R} \rightarrow \mathbb{R}^n$ is such that g_i is left-continuous nondecreasing for $i = 1, \dots, n$, $\mathbf{f} = (f_1, \dots, f_n) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $\mathbf{x} \in \mathfrak{B}$ refers to an initial value condition or a periodic boundary condition:

$$\mathbf{x}(0) = \mathbf{x}_0 = (\mathbf{x}_{0,1}, \dots, \mathbf{x}_{0,n}) \in \mathbb{R}^n; \quad (6)$$

$$\mathbf{x}(0) = \mathbf{x}(T). \quad (7)$$

Through our method, we have been able to establish new existence results dropping monotonicity conditions existing in the literature for initial value problems in Theorem 4.3.1, and approaching for the first time problems with periodic boundary condition in the literature involving the Stieltjes derivative in Theorem 4.3.4. Furthermore, we have been interested in systems of Stieltjes differential equations with a functional dependence:

$$\mathbf{x}'_{\mathbf{g}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}) \quad \text{for } \mathbf{g}\text{-almost every } t \in [0, T], \quad \mathbf{x} \in \mathfrak{B}; \quad (5)$$

where $\mathbf{f} : [0, T] \times \mathbb{R}^n \times \mathcal{BC}_{\mathbf{g}}([0, T], \mathbb{R}^n) \rightarrow \mathbb{R}^n$. Again in this part, we impose no monotonicity conditions on the function $\mathbf{f}$, dropping the quasimonotonicity nondecreasing of $\mathbf{f}$ with respect to the second argument, and the monotonicity of $\mathbf{f}$ with respect to the functional argument, that were imposed in previous works related to systems of functional Stieltjes differential equations with an initial value condition. This permitted us to establish Theorem 4.4.2 and its corollary results, and inspired us to introduce, for the first time, lower and upper solutions based results for systems of periodic Stieltjes differential equations with functional dependence, presented in Theorem 4.4.7 and its corollary results. In the last part of this chapter, we present an example of application describing the dynamics of a fish population subject to predation and seasonal fishing, through a model involving a system of Stieltjes differential equations. Interestingly, this model was our first motivation to study systems with functional dependence, since the theoretical framework could not yield existence of solutions when one of the monotonicity conditions involving $\mathbf{f}$ fails.

In the last chapter of this thesis, we discuss prolongation of solutions of Stieltjes differential equations on the positive real half-line, providing a new characterization of the maximal solution and global existence result. This provides a framework to define stability notions in the Lyapunov sense, to describe the behavior of solutions of a system around an equilibrium though the Lyapunov function involving the Stieltjes derivative. At this point, Theorems 5.2.9, 5.2.10, and 5.2.12 form the main results of this part providing sufficient conditions for stability, uniform stability, asymptotic stability and uniform asymptotic stability of an equilibrium. Finally, we conclude this chapter with two applications of our results in the dynamics of population ensuring uniform asymptotic stability of their equilibria.

Research Perspectives

While this study has shed light mainly on the method of lower and upper solution applied to systems of Stieltjes differential equations, several promising directions remain unexplored. Given that the region limited by the lower and upper solutions is

a rectangular surface of $\mathbb{R}^n$ at each point t , future research could consider generalizing these results using other regions of solutions, that appear in the case of classical derivation. For instance, through:

- the method of the *Solution-Tubes*: introduced in [33, 34], and for differential inclusions in [29, 32], based on a pair of solution-tubes (v, ρ) satisfying particular assumptions, where $v \in W^{1,1}([0, T], \mathbb{R}^n)$, and $\rho \in W^{1,1}([0, T], \mathbb{R}^+)$. The notion of the solution-tube coincides with the notion of lower and upper solution in $\mathbb{R}$, while, in $\mathbb{R}^n$, it permits to seek solutions a priori bounded in a tube-like region centered on the function v , with a variable radius $\rho(t) \geq 0$, for $t \in [0, T]$.
- the method of the *Solution-Regions*: first introduced by Frigon in [30], generalizes the method of lower and the upper solutions and the notion of Solution-tubes as well. These regions can take various shapes, allowing for a rich variety of solution behaviors.

In the sequel, we discuss some questions that arise from the applied aspect of Stieltjes differential equations. Assume that we want to model the dynamics of a process using Stieltjes differential equations, given some a priori information about its discontinuities and its stationary periods. According to the classification of derivators in Theorem 1.2.4, the possible derivators capturing such a priori information would be equivalent, generating the same topology, even if they have different slopes. This arises a pertinent question about the sensitivity of the model with respect to the choice of slopes of g . Furthermore, this refers us to two questions: how does slightly perturbing the slopes of g affect the model's prediction? and is there any connection between two solutions resulting from two problems with equivalent derivators? The first question comes with an early recommendation, consisting of relating the slopes of the derivator g with their meaning in the process as an additional parameter characterizing the process. While the second question may lead to explore a new classification of solutions depending on their regularity with respect to the derivator.

Possible generalizations can also be incorporated in the Stieltjes derivative. In [35], Frigon and Tojo explored an interesting path, weakening the monotonicity condition of the derivator g , considering derivators with *controlled variation* among the class of functions with a locally bounded variation. These functions exhibit monotonic behavior within each connected component, which is determined by excluding a negligible set. The study of such derivatives beyond the monotonicity conditions framework presents an intriguing and worthwhile task and might provide a bridge to study Stieltjes differential equations and measure differential equations from a common standpoint.

Studying such general cases might lay the ground for further generalizations motivated by real-life phenomena. In fact, natural processes exhibit inherent variability, influenced by factors beyond our control. These factors might include unpredictable environmental fluctuations, human behavior, or even quantum effects. With this in view, when modeling such phenomena we face a challenge: How do we account for this inherent randomness? To account for this, there is the extension of classical differential equations to stochastic differential equations by adding a stochastic term introducing randomness or noise into our system:

$$dX(t, \omega) = f(t, X(t, \omega))dt + S(t, X(t, \omega))dW(t, \omega),$$

where

- $f(t, X(t, \omega))dt$ represents the deterministic part of the change, where $f(t, X(t, \omega))$ represents the drift coefficient indicating how the expected change in X depends on the current value of X and time t .
- $S(t, X(t, \omega))dW(t, \omega)$ represents the stochastic part due to random fluctuations.

In this framework, an intriguing question arises: whether it is possible to incorporate the randomness of our system into the derivator g , to reduce our Stochastic differential equation to concise Stieltjes differential equations of the form

$$X'_g(t) = F(t, X(t)) \quad \text{or} \quad X'_g(t) = F(t, X(t), g),$$

with a well-behaving right-hand side member F , where the possible solutions X would be functions generated by a random derivator.

Future research could also focus on investigating controlled systems of Stieltjes differential equations. In fact, the investigation of controlled systems involving Stieltjes differential equations represents an interesting yet underexplored framework in control theory [15]. Despite its significance, the existing body of work in this field remains limited, since there are only two papers [13, 80] that have addressed control concepts in relation to Stieltjes differential equations. The unification offered by Stieltjes differential equations could enhance our comprehension of complex dynamical systems and help expanding practical techniques to various differential equations from a unified perspective.

In closing, we investigated new cases within the literature of Stieltjes differential equations in this thesis. Our contribution not only bridged some gaps in existing research but also underscored their practical applicability in modeling real-world phenomena across diverse differential problems. The unification achieved through the choice of the function g as a derivator, and the concepts it introduces, can question many connections existing between their classical counterparts. Ultimately, at the intersection of curiosity and rigor, lies an invitation to explore this framework further as we describe reality through the language of equations.

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Résumé

Ce manuscrit étudie l'existence et la multiplicité de solutions pour les équations différentielles de Stieltjes en utilisant la méthode des sous et sur-solutions ainsi que la théorie de l'indice de point fixe. Il examine également la prolongation des solutions sur la demi-droite réelle positive pour introduire des résultats de stabilité de type Lyapunov dans le contexte de ces équations. Des exemples pratiques sont fournis pour démontrer les implications concrètes de nos résultats et mettre en évidence l'applicabilité des équations différentielles de Stieltjes dans la modélisation de divers phénomènes. Ces équations utilisent la dérivée de Stieltjes, qui implique une différentiation par rapport à une fonction continue à gauche et croissante appelée le dérivateur. En dérivant par rapport à cette fonction, on simplifie la modélisation des phénomènes comportant des discontinuités et des périodes stationnaires, ce qui réduit le terme du second membre, et intègre des propriétés du dérivateur telles que les sauts et les périodes stationnaires dans la solution.

Après avoir introduit les préliminaires et les outils fondamentaux du calcul de Stieltjes, nous nous penchons sur la motivation derrière l'introduction de la dérivée de Stieltjes. Ensuite, nous établissons des résultats d'existence et de multiplicité pour les équations de Stieltjes du premier ordre avec des conditions aux limites périodiques ou initiales, en utilisant la méthode des sous et sur-solutions ainsi que la théorie de l'indice de point fixe. Nous généralisons ensuite ces résultats aux systèmes d'équations différentielles de Stieltjes, établissant également des résultats d'existence pour les systèmes fonctionnels. Nos principaux résultats n'impliquent pas de conditions de monotonie imposées au second membre. Enfin, dans le dernier chapitre concernant les systèmes dynamiques de Stieltjes, nous nous concentrons sur l'extension des solutions sur la demi-droite réelle positive et la notion de solutions maximales pour introduire des résultats de stabilité au sens de Lyapunov. Des applications concrètes sont fournies à la fin de chacun des trois derniers chapitres, illustrant les découvertes théoriques.

Mots-clefs : Équations différentielles de Stieltjes et applications; existence et multiplicité; sous et sur-solutions; prolongation de solutions; stabilité au sens de Lyapunov.

Abstract

This manuscript investigates existence and multiplicity of solutions for Stieltjes differential equations using the lower and upper solutions method and fixed point index theory. It also examines the prolongation of solutions over the positive real half-line to introduce Lyapunov-like stability results within the context of these equations. Practical examples are provided to demonstrate the real-world implications of our results and showcase the applicability of Stieltjes differential equations in modeling various phenomena. These equations employ a novel concept known as the Stieltjes derivative, which involves differentiation with respect to a left-continuous, non-decreasing function referred to as the derivator. Differentiating with respect to this function, simplifies modeling phenomena with discontinuities and stationary periods, reducing the right-hand side term, and integrates some properties of the derivator such as jumps and stationary periods into the solution.

After introducing the preliminaries and the fundamental tools from Stieltjes calculus, we delve into the motivation behind introducing the Stieltjes derivative. We then establish existence and multiplicity results for first-order Stieltjes equations with periodic or initial value conditions, using the method of lower and upper solutions and fixed point index theory. We then generalize these results to systems of Stieltjes differential equations, establishing existence results for functional systems as well. Our main results do not involve monotonicity conditions imposed on the second member. Finally, in the last chapter concerning Stieltjes dynamical systems, we focus on the extension of solutions on the positive real half-line and the notion of maximal solutions to introduce stability results in the Lyapunov sense. Practical applications are provided at the end of each of the last three chapters, illustrating the theoretical findings.

Keywords : Stieltjes differential equations and applications; existence and multiplicity; lower and upper solutions; prolongation of solutions; Lyapunov stability.