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Problème de moment et inégalités opérationnelles

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Dédicace

À mes parents et mon frère,

Pour leur amour inconditionnel, leur soutien indéfectible, et leurs sacrifices sans limite. Vous êtes ma source de force et d'inspiration.

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Abstract

In the first part of this thesis, we study birth and death processes with rates $\lambda_n = q^n(1 - aq^{n+1}),\ \mu_n = aq^n(1 - q^n),\ n \geq 0$ and $0 < a, q < 1$. We show that the corresponding orthogonal polynomials generalize little q-Laguerre polynomials. We also give the minimal solution of the three-term recurrence relation, and we obtain some formulas for the convergent of continued fractions associated with the little q-Laguerre orthogonal polynomials.
Furthermore, we examine the matrix-valued truncated complex moment problem. We show that finite-dimensional completion of a truncated data provides a necessary and sufficient condition, and hence a solution, for the matrix-valued truncated complex moment problem. As a consequence, we obtain a matrix generalization of Curto-Fialkow's result on flat positive extensions of moment matrices.

In the second part of this dissertation, we explore the class of $n-$tuples of doubly commuting bounded operators $\mathbf{T} = (T_1, \cdots, T_n)$ and its applications. we generalize and refine several inequalities involving the joint numerical radius and the joint operator norm of $\mathbf{T}$. Moreover, we investigate the link between nontrivial joint invariant subspaces of the generalized spherical Aluthge transform and the original of $n-$tuples of doubly commuting operators. Additionally we show that $\mathbf{T}$ satisfies wandering subspace property, Beurling-type theorem or admitting a Wold-type decomposition if and only if it the coordinates does, we provide an explicit Wold-type decomposition for the doubly commuting tuples of left invertible operators.

Keywords: Moment problem, birth and death process, operator $n-$tuples, Wold decomposition, Aluthge operator.

Résumé

Dans la première partie de cette thèse, nous étudions des processus de naissance et de mort avec des taux $\lambda_n = q^n(1 - aq^{n+1})$, $\mu_n = aq^n(1 - q^n)$, où $n \geq 0$ et $0 < a, q < 1$. Nous montrons que les polynômes orthogonaux correspondants généralisent les petits polynômes de $q-$Laguerre . De plus, nous donnons la solution minimale de la relation de récurrence à trois termes et obtenons quelques formules pour la convergence de fractions continues associées aux petits polynômes de $q-$Laguerre .

En outre, nous abordons le problème des moments complexes tronqués à valeurs matricielles. Nous montrons que l'achèvement de dimension finie des données tronquées fournit une condition nécessaire et suffisante, et donc une solution, pour le problème des moments complexes tronqués à valeurs matricielles. En conséquence, nous obtenons une généralisation matricielle du résultat de Curto-Fialkow sur les extensions positives plates des matrices de moments.

Dans la seconde partie de cette dissertation, nous étudions la classe des n-uplets d'opérateurs bornés qui commutent deux à deux $\mathbf{T} = (T_1, \cdots, T_n)$ et ses applications. Nous généralisons et affinons plusieurs inégalités impliquant le rayon numérique conjoint et la norme d'opérateur conjointe de $\mathbf{T}$. De plus, nous examinons le lien entre les sous-espaces invariants conjoints non triviaux de la transformée d'Aluthge sphérique généralisée et les n-uplets originaux d'opérateurs qui commutent deux à deux. De plus, nous démontrons que $\mathbf{T}$ satisfait la propriété d'espace errant, le théorème de Beurling ou admet une décomposition de type Wold si et seulement si ses coordonnées le font. Nous présentons une décomposition explicite de type Wold pour les n-uplets qui commutent deux à deux d'opérateurs inversibles à gauche.

Mots-clefs: Problème des moments, processus de naissance et de mort, opérateur n-uplets, décomposition de Wold, opérateut d'Aluthge.

Résumé étendu

Dans la première partie de cette dissertation, nous étudions les processus de naissance et de mort avec des taux $\lambda_n = q^n(1 - aq^{n+1})$ et $\mu_n = aq^n(1 - q^n),\ n \geq 0,\ \ 0 < a, q < 1$. En utilisant la méthode de la fonction génératrice, nous montrons que les polynômes orthogonaux correspondants généralisent les petits polynômes de q-Laguerre du premier genre, notés $(Q_n(x))_{n\geq 1}$, donnés par l'expression

$$Q_n(x) = \sum_{k=0}^{n} \frac{(qx)^k (q^{-n}; q)_k}{(aq; q)_k (q; q)_k}.$$

De plus, nous appliquons la représentation intégrale d'Akheizer pour dériver une formule explicite pour les petits polynômes de q-Laguerre du second genre, notés $(Q_n^*(x))_{n\geq 1}$, donnés par

$$Q_n^*(x) = \sum_{k=0}^{n-1} \Big[\sum_{l=k+1}^{n} \frac{(q^{-n}; q)_l q^l}{(aq; q)_l (q; q)_l (aq^{l-k}; q)_\infty} \Big] x^k.$$

Nous utilisons le Théorème de Pincherle [76] pour déterminer la solution minimale de la relation de récurrence à trois termes et obtenons quelques formules pour les convergents des fractions continues associées aux petits polynômes de q-Laguerre .

En outre, nous abordons le problème des moments complexes tronqués à valeurs matricielles, nous allons démontrer que si une séquence tronquée donnée de matrices hermitiennes $p \times p$ $H^{(n)} \equiv \{H_{ij}\}_{i,j\in\mathbb{Z}_+; i+j\leq 2n}$ admet une mesure représentative, alors

$$\sum_{i+j=1}^{n} \sum_{h+k=1}^{n} \mathbf{c}_{hk}^* H_{i+k,j+h} \mathbf{c}_{ij} \geq 0 \qquad \text{pour tout } \{\mathbf{c}_{ij}\}_{i+j\leq n} \subset \mathbb{C}^p.$$

Cette dernière condition (nécessaire) nous permet de définir, en utilisant $H^{(n)}$, une famille d'espaces de produits scalaires $\{V_i\}_{i\leq n}$, voir (3.9). Nous montrons que l'existence d'une extension infinie à valeurs matricielles positives $\tilde{H} \equiv \{H_{ij}\}_{i,j\in\mathbb{Z}_+}$ de $H^{(n)}$, telle que les espaces de produits scalaires associés $\{V_i\}_{i\in\mathbb{Z}_+}$ vérifient

$$\dim V_0 < \dim V_1 < \ldots < \dim V_r = \dim V_{r+1} = \ldots = \dim V_\infty \quad \text{pour un certain} r \in \mathbb{Z}_+,$$

est également une condition nécessaire pour le problème des moments matriciels tronqué, que nous appellerons complétion finie-dimensionnelle. Cela nous a motivés à examiner une telle complétion des séquences tronquées à valeurs matricielles.

En conséquence, nous prouvons que la complétion finie-dimensionnelle est aussi une condition suffisante et fournit donc une solution au problème des moments matriciels tronqué complexe. Comme conséquence immédiate, nous montrons que si une séquence tronquée à valeurs

matricielles $H^{(n)}$ est positive (c'est-à-dire, vérifie (1.4)) et les espaces de produits scalaires associés vérifient $\dim V_{s-1} = \dim V_s = \ldots = \dim V_n$ pour un entier positif $s \leq n$, alors $H^{(n)}$ admet une unique complétion finie-dimensionnelle, et donc $H^{(n)}$ a une mesure représentative unique qui est $(\dim V_s)$-atomique (voir Théorème 3.4). Ce dernier résultat fournit une généralisation du célèbre théorème de l'extension plate de Curto-Fialkow au cadre matriciel.

Dans la seconde partie de cette dissertation, nous étudions la classe des n-uplets d'opérateurs bornés qui commutent deux à deux $\mathbf{T} = (T_1, \cdots, T_n)$ et ses applications. Nous prouvons des inégalités impliquant la norme conjointe des opérateurs et le rayon numérique conjoint dans le cas de la transformation sphérique généralisée d'Aluthge. Principalement, nous étendons deux résultats récents de K. Feki et T. Yamazaki [35, Théorème 2.1 et Théorème 2.2] en montrant que pour un d-uplet arbitraire $\mathbf{T}$ et $0 \leq s \leq 1$, nous avons

$$\|\widehat{\mathbf{T}}^s\| \leq \|\mathbf{T}\| \quad \text{and} \quad w(\widehat{\mathbf{T}}^s + \widehat{\mathbf{T}}^{1-s}) \leq w(\mathbf{T}) + w(\mathbf{T}^D) \leq 2w(\mathbf{T}).$$

De plus, nous fournissons une preuve simple du résultat récent [3, Théorème 2.6] qui améliore [35, Théorème 3.1]. Plus précisément, si $\mathbf{T} = (T_1, ..., T_d) \in \mathcal{B}(\mathcal{H})^d$ et $0 < s < 1$, alors

$$w(\mathbf{T}) \leq \inf_{0 \leq s \leq 1} (\frac{1}{4}(\|\mathbf{T}\|^{2s} + \|\mathbf{T}\|^{2(1-s)}) + \frac{1}{2} w(\widehat{\mathbf{T}}^s)).$$

Nous utilisons cette extension pour obtenir d'autres résultats qui étendent ceux connus, comme pour le Corollaire 4.1, le Corollaire 4.2 et la Proposition 4.5 ci-dessous. En application de nos calculs, nous retrouvons certains résultats récents de [35, 107] en utilisant des preuves alternatives. De plus nous étudions le lien entre l'existence de sous-espaces invariants conjoints non triviaux pour la transformation d'Aluthge généralisée et pour l'original d-uplet d'opérateurs bornés.

La dernière partie de cette dissertation examine le lien entre la propriété des sous-espaces errants, le Théorème de type Beurling et la décomposition de type Wold pour un n-uplet doublement commutant et ses coordonnées opérateurs. Nous visons à unifier divers résultats connus sur ce sujet concernant la propriété des sous-espaces errants, le Théorème de type Beurling et la décomposition de type Wold d'un n-uplet doublement commutant. En particulier, nous avons pour objectif d'étendre le travail de Sarkar sur la décomposition de type Wold pour les n-uplets d'isométries doublement commutantes à une classe plus générale d'n-uplets d'opérateurs doublement commutants. Nous récupérerons également de manière simple et étendrons plusieurs résultats obtenus dans [20, 50] et dans [74] au cas général des n-uplets doublement commutants d'opérateurs inversibles à gauche .

Contents

Chapter 1

Introduction

The theory of orthogonal polynomials sequences (OPS) originated with the contributions of Chebyshev and Stieltjes in the 19th century, in their work on continued fractions and the moment problem. It represents a rich and diverse branch of mathematics that investigates the properties and characteristics of sequences of polynomials. Classical families of these polynomials, such as Legendre, Chebyshev, Hermite, Jacobi, and Laguerre have given rise to a profound theory in numerous branches of science and mathematics, including approximation theory, numerical analysis, and quantum mechanics.

A system of polynomials $(p_n(x))_{n\geq 1}$ where $p_n(x)$ is of degree n for all $n \in \{0, 1, 2, ...\}$, is called orthogonal on an interval $[a, b]$ with respect to a non negative measure $d\mu(x)$ if

$$\int_a^b p_m(x)p_n(x)d\mu(x) = 0, \quad m \neq n, \quad m, n \in \{0, 1, 2, ...\}$$

and

$$\int_a^b p_n^2(x)d\mu(x) = \alpha_n \neq 0, \quad n \in \{0, 1, 2, ...\}.$$

It is clear that $(p_k)_{k\leq n}$ is a basis in $\mathbb{R}_n[X]$ the space of polynomials of degree less or equal to n. It follows that p_{n+1} is orthogonal to $\mathbb{R}_n[X]$, for every $n \geq 0$. Since $xp_n(x) \in \mathbb{R}_{n+1}[X]$, we have

$$xp_n(x) = \sum_{k=0}^{n+1} a_{n,k}p_k(x), \quad a_{n,k} = \frac{1}{\alpha_k}\int_a^b xp_n(x)p_k(x)d\mu(x).$$

And from $xp_k(x) \in \mathbb{R}_{k+1}[X]$, we derive that

$$a_{n,k} = 0, \text{ for } 0 \leq k < n-1.$$

Thus orthogonal polynomials satisfy a three-term recurrence relation of the form

$$xp_n(x) = a_{n,n+1}p_{n+1}(x) + a_{n,n}p_n(x) + a_{n,n-1}p_{n-1}(x), \text{ for } n \geq 0, \tag{1.1}$$

where $p_{-1}(x) = 0,\ p_0(x) = 1,\ a_{0,-1} = 0$.

Conversely, in 1935, J. Favard established that equation (1.1) characterises the family of orthogonal polynomials with respect to a measure μ (not necessarily unique) on the real line if

$$\frac{a_{n,n-1}}{a_{n-1,n}} > 0, \text{ for } n > 0.$$

Orthogonal polynomials have found applications in the field of probability theory and stochastic processes, elegantly modeling phenomena such as particle diffusion and transitions between different states. These transitions can be interpreted as events in a birth and death process BDP, where particles are born and die according to specific probabilities. S. Karlin and J. L. McGregor in[52] expressed their transition probabilities in terms of a sequence of orthogonal polynomials and a spectral measure.

In Chapter 2, We study birth and death processes with rates $\lambda_n = q^n(1-aq^{n+1})$, $\mu_n = aq^n(1-q^n)$, $n \geq 0$ and $0 < a, q < 1$. Using the generating function method, we show that corresponding orthogonal polynomials generalize little q-Laguerre polynomials of the first kind, denoted by $(Q_n(x))_{n\geq 1}$, given by the expression

$$Q_n(x) = \sum_{k=0}^{n} \frac{q^k(q^{-n};q)_k}{(aq;q)_k(q;q)_k} x^k.$$

Additionally, we apply Akheizer's integral representation to derive an explicit formula for the little q-Laguerre polynomials of second kind, denoted as $(Q_n^*(x))_{n\geq 1}$, given by

$$Q_n^*(x) = \sum_{k=0}^{n-1} \Big[\sum_{l=k+1}^{n} \frac{(q^{-n};q)_l q^l}{(aq;q)_l(q;q)_l(aq^{l-k};q)_\infty} \Big] x^k.$$

We involve Pincherle's Theorem [76], to determine the minimal solution of the three-term recurrence relation, and we obtain some formulas for the convergent of continued fractions associated with the little q-Laguerre orthogonal polynomials.

A fundamental question in functional analysis, related to orthogonal polynomial sequences , is the moment problem, which was first introduced by T.J. Stieltjes in his famous paper "Recherches sur les fractions continues" [98], aiming to establish an integral representation for certain continued fractions. This problem can be stated as follows: Given a real sequence $\{\gamma_n\}_{n\geq 0}$, is there a positive Borel measure μ on a closed set K of $\mathbb{R}$, such that

$$\gamma_n = \int_K x^n d\mu(x) \quad \text{for all } n \in \mathbb{N} \ ? \tag{1.2}$$

This problem has been solved when μ is defined on various sets, such as, $K = \mathbb{R}$ (Hamburger), $K = \mathbb{R}_+$ (Stieltjes), $K = [0,1]$ (Hausdorff), and $K = \mathbb{T}$ (Toeplitz). In the case where $n = +\infty$ it is known as the full K-moment problem represented by equation (1.2), and for $n < +\infty$, it is termed the truncated K-moment problem.

When a measure exists, and if it is unique, the moment problem is determinate; otherwise, it is referred to as an indeterminate moment problem. In 1923, M.Riesz proved that the full moment problem is equivalent to characterizing positive polynomials on certain subsets of the real line.

Precisely, for a real sequence $\{\gamma_n\}_{n\geq 0}$, let Λ be the linear functional on the algebra of polynomials $\mathbb{R}[X]$, defined as

$$\Lambda : p(x) \equiv \sum_i a_i x^i \mapsto \Lambda(p) = \sum_i a_i \gamma_i.$$

The following statements are equivalent:

(i) $\Lambda(p) \geq 0$ for any polynomial $p \in \mathbb{R}[X]$ such that $p \geq 0$ on K.

(ii) Λ is a moment functional, meaning there exists a measure μ on K such that $\Lambda(p) = \int_K p d\mu$ for all $p \in \mathbb{R}[X]$.

To apply this result, positive polynomials need to be expressed as sums of squares. When such a characterization is achieved, it leads to conditions on the sequence, and it can be transformed as follows:

- $(\gamma) \equiv \{\gamma_n\}_{n\geq 0}$ is a Hamburger moment sequence $\iff$ the matrice $(\gamma_{i+j})_{i+j\leq n}$ is positive semi-definite.
- $(\gamma) \equiv \{\gamma_n\}_{n\geq 0}$ is a Stieltjes moment sequence $\iff$ the matrices $(\gamma_{i+j})_{i+j\leq n}$ and $(\gamma_{i+j+1})_{i+j\leq n}$ are positive semi-definite.
- $(\gamma) \equiv \{\gamma_n\}_{n\geq 0}$ is a Hausdorff moment sequence $\iff$ the matrices $(\gamma_{i+j})_{i+j\leq n}$ and $(\gamma_{i+j+1} - \gamma_{i+j+2})_{i+j\leq n}$ are positive semi-definite.

In 1935, E.K. Haviland employed Riesz methods to establish a result applicable to any subset K of $\mathbb{R}^d$. This result implies that a solution exists for the $K-$moment problem on $K \subset \mathbb{R}^d$ if and only if the Riesz functional exhibits positivity on the cone of non-negative polynomials defined on K. However in the multidimensional case, non-negative polynomials on a given set $K \subset \mathbb{R}^d$ cannot be characterized in a simple manner. Indeed, Hilbert demonstrated that there exist positive polynomials in d variables that cannot be expressed as sums of squares. Consequently, the problem of moments in multiple dimensions remains unsolved for general set $K \subset \mathbb{R}^d$.

In this dissertation, we consider the following moment problem. Given a bi-indexed sequence of Hermitian $p \times p$-matrices $H^{(n)} \equiv \{H_{ij}\}_{i,j\in\mathbb{Z}_+; i+j\leq 2n}$, where $n \in \mathbb{Z}_+$, the truncated matrix-valued complex moment problem asks when does there exist a positive matrix-valued Borel measure μ, supported in $\mathbb{C}$, such that

$$H_{ij} = \int_{\mathbb{C}} \overline{z}^i z^j d\mu \qquad \text{for all } i, j \in \mathbb{Z}_+ \text{ with } i + j \leq 2n. \tag{1.3}$$

When (1.3) owns a solution μ, then μ is said to be a representing measure of the moment sequence $H^{(n)}$. In view of its fundamental importance in various field of mathematics and applied science, the matrix-moment problems are the subject of a growing number of ongoing research, see [17, 59, 62, 94] and their references. In the particular case when $p = 1$, that is, $H^{(n)} \equiv \gamma^{(n)}$ is a finite sequence of complex numbers, the problem (1.3) is known as the truncated complex moment problem, see [6, 24, 43, 44]. This last has been intensively studied by R. Curto and L. Fialkow who build a complete approach based on positivity and flat extensions of the moment matrix, see [23–25]. The key results characterize the existence of representing measure μ for which card supp(μ) = rank $M(n)$, where $M(n) \equiv M(n)(\gamma) := (\gamma_{i+k,j+h})_{i+j,h+k=0}^{n}$ is the moment matrix associated with the truncated complex sequence $\gamma^{(n)} \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+;i+j\leq 2n}$. Explicitly, if $M(n) \geq 0$ and $M(n)$ is flat, i.e., rank $M(n)$ = rank $M(n-1)$, then γ has a unique representing measure, which is rank $M(n)$-atomic. F.H. Vasilescu [105, 106] provided an equivalent approach to the concept of flatness, based on the dimensional stability of some Hilbert spaces. In addition to its use in moment problems, the theory of flatness has also been applied successfully in operator theory [26, 67] and in polynomial optimisation [63–65]. Then, it is natural to wonder whether the classical flat extension theory (or equivalently, dimensional stability) extends to an appropriate matrix-valued settings.

In Chapter 3, we will see that if a given truncated sequence of Hermitian $p \times p$-matrices $H^{(n)} \equiv \{H_{ij}\}_{i,j\in\mathbb{Z}_+;i+j\leq 2n}$ admits a representing measure, then

$$\sum_{i+j=1}^{n} \sum_{h+k=1}^{n} \mathbf{c}_{hk}^{*} H_{i+k,j+h} \mathbf{c}_{ij} \geq 0 \qquad \text{for every } \{\mathbf{c}_{ij}\}_{i+j\leq n} \subset \mathbb{C}^{p}. \tag{1.4}$$

This last (necessary condition) allow us to define, using $H^{(n)}$, a family of inner product spaces $\{V_i\}_{i\leq n}$, see (3.9). We show that the existence of an infinite positive matrix-valued extension $\tilde{H} \equiv \{H_{ij}\}_{i,j\in\mathbb{Z}_+}$, of $H^{(n)}$, so that the associate inner product spaces $\{V_i\}_{i\in\mathbb{Z}_+}$ obey

$$\dim V_0 < \dim V_1 < \ldots < \dim V_r = \dim V_{r+1} = \ldots = \dim V_\infty \qquad \text{for some } r \in \mathbb{Z}_+, \tag{1.5}$$

is also a necessary condition for the truncated matrix-valued moment problem, which we will call finite-dimensional completion. This motivated us to investigate such completion of truncated matrix-valued sequences. As a result, we prove that the finite-dimensional completion is also a sufficient condition and hence it provides a solution to the truncated complex matrix-valued moment problem. As an immediate consequence, we show that if a truncated matrix-valued sequence $H^{(n)}$ is positive (that is, obey (1.4)) and the associated inner product spaces verify $\dim V_{s-1} = \dim V_s = \ldots = \dim V_n$ for some positive integer $s \leq n$, then $H^{(n)}$ admits a unique finite-dimensional completion, and thus $H^{(n)}$ has a unique representing measure which is $(\dim V_s)$-atomic (see Theorem 3.4). This last result furnishes a generalization of the celebrate

Curto-Fialkow flat extension theorem to the matrix-valued setting.

There is a substantial connection between the theory of operators and the moment problem, extensively used to expand results or simplify concepts and proofs in both directions. The operator-theoretic approach to the moment problem, introduced by Stone in [99], was later generalized and developed by several authors see for example [86]. For instance, the interplay of Humburger moment problems and Jacobi operators is examined in [90]. The link between the truncated moment problem and subnormal operators has also attracted growing interest from various authors, see [102], this approach has been also employed to address the challenges associated with the multidimensional moment problem.

In the second part of this dissertation, our focus will be directed towards studying several classes of operators and its applications. Let $\mathcal{B}(\mathcal{H})$ be the Banach space of all bounded linear operators. The Aluthge transform for a $T \in \mathcal{B}(\mathcal{H})$ was introduced by A. Aluthge in [4], defined by the expression $\hat{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $T = U|T|$ is the canonical polar decomposition of T. Okubo further introduced in [73] the generalized Aluthge transform for T defined as $\hat{T}^s = |T|^s U|T|^{1-s}$, for $0 \le s \le 1$. Over the last two decades, the Aluthge transform and the generalized Aluthge transform attracted considerable attention because of wide range of possible applications. The guiding principle in work related to these transformations is to identify the links between an operator and its Aluthge transform. In particular it has been shown that $\hat{T}$ and $\hat{T}^s$ share several spectral properties with T. See for instance, [36, 57, 66, 107] for further details and additional information. Due to the importance multi-variable operator theory, the interest in studying extensions of the Aluthge transform for $n-$tuples of operators has grown considerably in the recent few years. The spherical generalized Aluthge transform of $\mathbf{T} = (T_1, ..., T_n)$, denoted as $\widehat{\mathbf{T}}^s$, is defined below, has been considered recently as a field of investigations on $d-$tuples by several authors. We refer to [3, 8–10, 12, 27, 28, 38, 69, 109] and the references therein.

in Chapter 4, we generalize and refine several operator inequalities involving the joint numerical radius and the joint operator norm of spherical Aluthge transform to generalized spherical Aluthge transforms,precisely, we show that for an arbitrary $n-$tuple $\mathbf{T}$ and $0 \le s \le 1$, we have

$$\|\widehat{\mathbf{T}}^s\| \le \|\mathbf{T}\| \quad \text{and} \quad w(\widehat{\mathbf{T}}^s + \widehat{\mathbf{T}}^{1-s}) \le w(\mathbf{T}) + w(\mathbf{T}^D) \le 2w(\mathbf{T}).$$

In addition, we provide that

$$w(\mathbf{T}) \le \inf_{0 \le s \le 1} (\frac{1}{4}(\|\mathbf{T}\|^{2s} + \|\mathbf{T}\|^{2(1-s)}) + \frac{1}{2}w(\widehat{\mathbf{T}}^s)).$$

Moreover, we investigate the link between nontrivial joint invariant subspaces of the generalized spherical Aluthge transform and the original commuting $n-$tuples of bounded operators. Indeed, we prove that $\mathbf{T}$ has a nontrivial joint invariant subspace if and only if $\widehat{\mathbf{T}}^s$ does, see Theorem (4.7).

The final chapter of this dissertation focuses on the classical Wold decomposition Theorem, which asserts that any isometry on a Hilbert space can be decomposed as a direct sum of a unitary operator and a unilateral shift. More precisely, Let V be an isometry on a Hilbert space $\mathcal{H}$, then there is a decomposition of $\mathcal{H}$ as a direct sum of two orthogonale subspaces $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s$, reducing for V, such that $V_{|\mathcal{H}_u}$ is unitary and $V_{|\mathcal{H}_s}$ is unitarily equivalent to a unilateral shift, the canonical subspaces are defined by $\mathcal{H}_u := \bigcap_{n=1}^{\infty} V^n\mathcal{H}$ and $\mathcal{H}_s := \bigoplus_{n=0}^{\infty} V^n\mathcal{W}$. The subspace $\mathcal{W}$ is called a wandering subspace for V, characterized by the property $V^m(\mathcal{W}) \perp V^n(\mathcal{W})$, for every $m \neq n$.

The main purpose of investigations into Wold-type decomposition is to relax the condition that T is isometric to weaker assumptions. Some profound results are obtained for operators close to isometries, where the operator T is assumed to satisfy operator inequalities as in Proposition (5.1).

For an abstract operator T, it is said to possess the wandering subspace property if it has a generating wandering subspace and satisfies the Beurling-type theorem if, when restricted to any closed T-invariant subspace, it exhibits the wandering subspace property.

A natural issue is the extension of Wold-type decomposition from a single isometry to commuting $n-$tuples $V = (V_1, ..., V_1) \in \mathcal{B}(\mathcal{H}, \mathcal{H}^{\mathbf{n}})$ of isometries, where $\mathcal{H}^n = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ stands for the direct sum of n copies of $\mathcal{H}$. Several interesting results on Wold-type decomposition of commuting $n-$tuples have been obtained in many directions. First, M. Słociński extended in [96] the definition of Wold-type decomposition to commuting pairs of isometries. More precisely, if V_1 and V_2 are commuting isometries on $\mathcal{H}$, the pair (V_1, V_2) is said to have Wold-type decomposition if, $\mathcal{H}$ can be decomposed in 4 reducing spaces of V_1 and V_2,

$$\mathcal{H} = \mathcal{H}_{uu} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{ss},$$

Where $V_i|\mathcal{H}_{uu}$ are unitary, $V_i|\mathcal{H}_{ss}$ are cnu, for $i = 1, 2$, and $V_i|\mathcal{H}_{us}$ is unitary for $i = 1$ and cnu, for $i = 2$, while $V_i|\mathcal{H}_{su}$ is unitary for $i = 2$ and cnu, for $i = 1$. Słociński obtained several necessary condition on commuting pairs of isometries to admit a Wold-type decomposition and proved in particular [96, Theorem 3] that, every pair of doubly commuting isomet ries has a Wold-type decomposition. D. Popovici [78], introduced the concept of a weak bi-shift, to set a Wold-type decomposition for arbitrary pairs of commuting isometries. This allowed A. Chattopadhyay *et al.* to extend Shimorin's work to $n-$tuples of doubly commuting operators in [20]. Also, J. Sarkar extended in [87] Słociński's decomposition for $n-$tuples of doubly commuting isometries ($n \geq 2$). He also provided a characterization of the closed subspaces in the orthogonal decomposition. The existence of a Wold-type decomposition for general $n-$tuples of commuting isometries was shown by Z. Burdak, M. Kosiek and M. Słociński in [19], under the condition that the isometric operators V_i, with $i = 1, ..., n$, have finite dimensional wandering subspaces. In [16], the existence of a Wold-Słociński decomposition is investigated for arbitrary commuting isometric $n-$tuples. More specifically, for arbitrary commuting isometric triples (V_1, V_2, V_3), it is shown that such decomposition exists if and only if the pairs $(V_1V_2, V_3), (V_2V_3, V_1)$ and (V_1V_3, V_2) admit Wold–Słociński decomposition. More recently Wold-type decomposition for $\mathcal{U}_n-$twisted isometries and for $\mathcal{U}_n-$twisted contractions has been obtained in [79, 89].

In Chapter 5, we investigate the link between each of the wandering subspace property, Beurling-type theorem and Wold-type decomposition of a doubly commuting $n-$tuple and of its operator coordinates. Precisely It is shown that a doubly commuting $n-$tuples $\mathbf{T} = (T_1, \cdots, T_n)$ satisfies wandering subspace property, Beurling-type theorem or Wold-type decomposition if and only if T_i does for every i. Applications are given in the case of Hilbert spaces of analytic functions and various recent results are extended.The chapter also presents several results concerning Beurling-type theorem for doubly commuting $n-$tuples. Finally, in the case where T_i admits a Wold-type decomposition for every i, we exhibit a Wold-type decomposition for the doubly commuting tuples $\mathbf{T}$. In particular, we aim to extend Sarkar's work on Wold-type decomposition for $n-$tuples of doubly commuting isometries to the more general class of $n-$tuples of doubly commuting operators. We will also recover in a simple way and extend several results obtained in [20, 50, 87] and in [74] to the case general setting of doubly commuting tuples of left invertible operators.

Chapter 2

Birth and death processes associated with little $q-$Laguerre orthogonal polynomials

A birth-death model is a continuous-time Markov process (denoted in the sequel BDP) that is often used to study how the number of individuals in a population changes through time. More precisely, a BDP is a special case of continuous time Markov process, in the non-negative integers $\{X_t, t \in [0; \infty[\}$ on the state space $\{0; 1; 2; ...\}$, in which only jumps to adjacent states are permitted and such that

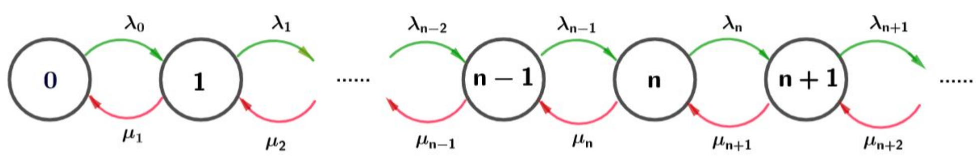
$$P_{m,n}(t) = Pr\{X_{t+s} = m/X_s = n\} = Pr\{X_t = n/X_0 = m\}.$$

Here $P_{m,n}(t)$ is the transition probability that a particle moves from state m to state n during the period t.

These processes have been used as models in populations growth, in queuing systems and in many other fields of both theoretical and applied interest [18, 82]. To construct a general BDP, we need to define the rules according to which the number of particles evolves. To this aim, we specify the behaviour of the process for a very short time dt, when there are n particles in the system. If dt is very small, the probability of a birth or a death during the interval $(t, t + dt)$ occurs with rate r is approximately rdt. Therefore, the transition probability $P_{m,n}(t)$, for small t, of the process $\{X_t\}$ so that the system moves from state m to state n will be defined by

$$P_{m,n}(t) = \begin{cases} \lambda_m t + o(t) & \text{if } n = m+1 \\ 1 - (\lambda_m + \mu_m)t + o(t) & \text{if } n = m \\ \mu_m t + o(t) & \text{if } n = m-1 \\ o(t) & \text{if } |n-m| > 1 \end{cases}, \tag{2.1}$$

where λ_n and μ_n are the birth and death rates at the state n respectively.
It is always assumed that $\lambda_n > 0,\ \mu_{n+1} > 0$ for $n \geq 0$ and $\mu_0 = 0$.



The solution of (2.1) always satisfies the further conditions

$$0 < \sum_{n\geq 0} P_{m,n}(t) \leq 1, \text{ and } P_{m,n}(s+t) = \sum_{k\geq 0} P_{m,k}(s)P_{k,n}(t), \tag{2.2}$$

known as Chapman–Kolmogorov equation and

$$P_{m,n}(0) = \delta_{m,n},$$

where $\delta_{m,n}$ is the Kronecker delta function.

In the sequel, we only consider $\sum_{n\geq 0} P_{m,n}(t) = 1$. The inequality $\sum_{n\geq 0} P_{m,n}(t) < 1$ expresses a model where $\mu_0 > 0$ (no honest), the particle may disappear in the initial state either by going to infinity, or by absorption at state -1.

It is a classical fact that $P_{m,n}(t)$ satisfies the following system of differential equations, called forward Chapman–Kolmogorov equation

$$\frac{d}{dt} P_{m,n}(t) = \mu_n P_{m,n-1}(t) - (\lambda_n + \mu_n)P_{m,n}(t) + \lambda_n P_{m,n+1}(t). \tag{2.3}$$

The correspondence between BDP, continued fractions and orthogonal polynomial systems have been investigated early in the literature by S. Karlin and J. L. McGregor in [55]. Karlin and McGregor developed a formal theory of general BDP that expresses their transition probabilities in terms of a sequence of orthogonal polynomials and a spectral measure. More precisely, they show in [55] that the solution of (2.1) can be derived from the family of polynomials $(Q_n(x))_{n\geq 1}$, solution of the next recursion propositionerty,

$$-xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x), \tag{2.4}$$

with the initial conditions

$$Q_0(x) = 1, \quad Q_1(x) = \frac{\lambda_0 + \mu_0 - x}{\lambda_0}. \tag{2.5}$$

The transition probability is given by the next expression

$$P_{m,n}(t) = \pi_m \int_0^{\infty} Q_m(x)Q_n(x)e^{-tx}\, \mathrm{d}\mu(x), \tag{2.6}$$

where $\pi_0 = 1$, $\pi_n = \dfrac{\lambda_0\lambda_1 \ldots \lambda_{n-1}}{\mu_1\mu_2 \ldots \mu_n}$, for $n > 0$ and $d\mu$ is non negative mass distribution with respect to which the polynomials $(Q_n(x))_{n\geq 1}$, are orthogonal.

We mention that G. E. H. Reuter showed in [81] that if

$$\sum_{n\geq 0} (\pi_n + \frac{1}{\lambda_n \pi_n}) = \infty, \tag{2.7}$$

then μ is a unique (the problem is then called determinate).

For the relation with continued fractions, we start with the initial state $m = 0$ and write $f(s) =$

$\mathcal{L}(P_{0,0})(s) = \int_0^\infty P_{0,0}(t)e^{-st}dt$ for the Laplace transform of $P_{0,0}$. Then, the expansion of $f(s)$ as a continued fraction is given by,

$$f(s) = \cfrac{1}{s+\lambda_0 - \cfrac{\lambda_0\mu_1}{s+\lambda_1+\mu_1-\cfrac{\lambda_2\mu_1}{s+\lambda_2+\mu_2+\cdots}}} \tag{2.8}$$

To simplify the previous notation, let us denote $\frac{a}{b+}* = \frac{a}{b+*}$ and we pose $a_1 = 1$, $a_n = -\lambda_{n-2}\mu_{n-1}$, $b_1 = s+\lambda_0$ and $b_n = s+\lambda_{n-1}+\mu_{n-1}$ for $n \geq 2$. Then (2.8) becomes

$$f(s) = \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots$$

We denote the kth convergent of the Laplace transform $f(s)$ by

$$f^{(k)}(s) := \frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots\frac{a_k}{b_k},$$

which is, thanks to the definition of a_i and b_i $(1 \leq i \leq k)$, a rational function with the unknown s. In this context, we introduce by Q^* and Q the numerator and the denominator respectively so that $f^{(k)}(s) = \frac{Q_k^*(s)}{Q_k(s)}$.

In [22] F. W. Crawford and M. A. Suchard obtained expressions for the Laplace transforms of these transition probabilities and make explicit an important derivation connecting transition probabilities and continued fractions. The computational method in [22] is based on a work of J. Murphy and M. O'donohoe [71], for numerically computing the transition probabilities for a general BDP with arbitrary birth and death rates. Murphy and O'Donohoe have shown that it is possible to find exact expressions for Laplace transforms of the transition probabilities of a general BDP using continued fractions. However this approach do not lead in general to the exact expression of the transition probabilities. Efficient approximation of these transition probabilities is obtained by the inverse Laplace transform of truncation of the associated continued fraction.

Despite the simplicity of the representation (2.6) for the transitions probabilities, it is usually difficult to find the polynomials$(Q_n(x))_{n\geq 1}$ even for simple models. Indeed, the expression analytic for transition probabilities of BDP are known only is some simple cases of rates arising from specific applicable phenomena. Among these models, we have

- The general linear case $\lambda_n = an+b,\ \mu_n = cn+d$ when $b = d = 0$ was treated by Kendall [56] and exhibits Charlier polynomials. The case $d = 0$ was solved in [52], it leads to Meixner polynomials for $a = c$ and to Laguerre polynomials for $a \neq c$. The case of $d \neq 0$ is analysed in [5, 47].

- In the quadratic case, the coefficients $\lambda_n = (N-n)(n+a),\ \mu_n = n(n+b)$ and $\lambda_n = n^2+an+b,\ \mu_n = n^2+cn+d$ first appeared in applications concerned with genetic models [48, 53, 84] and are linked to continuous dual Han polynomials.
- The quartic case $\lambda_n = (4n+1)(4n+2)^2(4n+3),\ \mu_n = (4n-1)(4n)^2(4n+1)$ and cubic case $\lambda_n = (3n+3c+1)^2(3n+3c+2),\ \mu_n = (3n+3c)^2(3n+3c+1)$ were treated by G. Valent in [40, 104].

Because of the difficulty in identifying the orthogonal polynomials associated to a BDP, numerous areas where birth-death process are applicable are not covered by the previous situations (even by any possible polynomial model). For example, in queuing theory, a birth and death processes where the states of the system represent the number of customers in a queue with a single server and an infinite waiting room. A number of authors have studied the birth and death process for a queuing model [29, 72], E. H. Ismail in [46] has explicitly found the orthogonal polynomials (Al Salam-Carlitz q-polynomials) corresponding to arrival rate $\lambda_n = \lambda q^n$ and departure rate $\mu_n = \mu q(1-q^n),\ n = 0, 1..., \ 0 < q < 1$.

The purpose of this Chapter is to give the spectral representation of the transition probabilities of the birth-death process with the next geometric parameters

$$\lambda_n = q^n(1-aq^{n+1}) \text{ and } \mu_n = aq^n(1-q^n),\ n \geq 0,\ \ 0 < a, q < 1. \tag{2.9}$$

Section 2.1 Contains some basic relations between birth and death processes and orthogonal polynomial theory and the preliminary material needed in the remainder of the Chapter. In section 2.2, we study the orthogonal polynomials associated with the birth and death process (2.9). We compute the generating functions and give the expression of the polynomials of the first kind $(Q_n(x))_{n\geq 1}$. We outline an approach previously used by several authors [7, 68].

We derive an explicit formula for the numerator polynomials $(Q_n^*(x))_{n\geq 1}$ in the continued fractions whose denominators are $(Q_n(x))_{n\geq 1}$.Moreover, we study the convergence for continued fractions (2.8). Finally, we use in section 2.3 a theorem of Pincherle [76] concerning minimal solutions of three-term recurrence relations to evaluate explicitly the continued fractions (2.8) for the little q-Laguerre polynomials.

2.1 basic tools

2.1.1 Askey-scheme and classical orthogonal polynomials associated with BDP

The expression of classical orthogonal polynomials is usually presented as a generalized hypergeometric series by using classical expansion methods. The identification of such is however difficult to reach in general, see [61, page 5] for example.

A hypergeometric series is defined as

$$ {}_rF_s(a_1, ..., a_r; b_1, ..., b_s; x) = \sum_{n\geq 0} \frac{(a_1, ..., a_r)_n}{(b_1, ..., b_s; x)_n} \frac{x^n}{n!}, $$

where $(a)_k$ stands for the Pochhammer symbol defined by
$(a)_0 = 1$, $(a)_k = \prod_{i=1}^{k}(a+i-1)$, and $(a_1, ..., a_r)_k = (a_1)_k...(a_r)_k$, for $k = 1, 2, 3, ...$
In which the q-shifted factorials are given by

$$ (a;q)_0 = 1, \quad (a;q)_n = \prod_{i=1}^{n}(1-aq^{i-1}), \quad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^{i}). $$

And the multiple q-shifted factorials are

$$ (a_1, a_2, ..., a_k; q)_n = \prod_{j=1}^{k}(a_j; q)_n. $$

Askey-scheme of hypergeometric lists several interesting finite systems of orthogonal polynomials. Next are some of these polynomials known in the literature.

- Laguerre polynomials: $a_n = -(n+1),\ b_n = -(a_n + c_n)$ and $c_n = -(n+\alpha)$,

  $$ L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_1F_1(-n; \alpha+1; x), \quad \alpha > -1. $$

- Dual Hahn polynomials: $a_n = (n+\alpha+1)(n-N)$, $b_n = -(a_n+c_n)$ and $c_n = n(n-\beta-N-1)$ for $-1 < \alpha,\ \beta < -N$ and $n \leq N$,

  $$ R_n(\lambda(x); \alpha, \beta, N) = {}_3F_2(-n, -x, x+\alpha+\beta+1; \alpha+1, -N; 1). $$

  Where

  $$ \lambda(x) = x(x+\alpha+\beta+1). $$

- Meixner polynomials : $a_n = \gamma(\beta+n)(1-\gamma)^{-1}$, $b_n = -(a_n+c_n)$ and $c_n = n(1-\gamma)^{-1}$ for $\beta > 0$ and $0 < \gamma < 1$,

  $$ M_n(x; \beta, \gamma) = {}_2F_1(-n, -x; \beta; 1-1/\gamma). $$

- Charlier polynomials : $a_n = -a,\ b_n = -(a_n + c_n)$ and $c_n = n$,

$$C_n(x; a) = {}_0F_1(-n, -x; -1/a) \text{ for } a > 0.$$

- Hermite polynomials: $a_n = \frac{1}{2},\ b_n = 0$ and $c_n = n$,

$$H_n(x) = (2x)^n\ {}_2F_0(-n/2, -(n-1)/2; -1/x^2).$$

- Al Salam–Carlitz q-polynomials: $a_n = 1,\ b_n = (a+1)q^n$ and $c_n = -aq^{n-1}(1-q^n)$,

$$U_n^{(a)}(x; q) = (-a)^n q^{\binom{n}{2}}\ {}_2\phi_1(q^{-n}, 1/x; 0; q, qx/a),\ 0 < q < 1.$$

2.1.2 Continued Fractions

Let $\frac{Q_n^*(s)}{Q_n(s)}$ be the nth convergent of the continued fraction

$$\frac{1}{x+\lambda_0 -}\frac{\lambda_0\mu_0}{x+\lambda_1+\mu_1 -}\cdots\frac{\lambda_{n-1}\mu_n}{x+\lambda_{n-1}+\mu_{n-1}}\cdots,$$

with $\lambda_n\mu_{n+1} \neq 0$, for $n \geq 0$. Then the polynomials $Q_n^*(x)$ and $Q_n(x)$ are solutions of the recurrence relation

$$y_{n+1}(x) = [x - (\lambda_n + \mu_n)]y_n(x) - \lambda_{n-1}\mu_n y_{n-1}(x), n > 0, \tag{2.10}$$

with the initial values $Q_0^*(x) = 0,\ Q_1^*(x) = 1,\ Q_0(x) = 1,\ Q_1(x) = x + 1 - aq$.
A solution $(Q_n^{**}(x))_{n\geq 1}$ of (2.10) is called a minimal solution if $\lim\limits_{n\to+\infty}\dfrac{Q_n^{**}(x)}{y_n(x)} = 0$ for any other solution $y_n(x)$, of orthogonal polynomials.
It is shown in [39] that the existence of a minimal solution of a three term recurrence relation is closely related to the determinancy of the associated moment problem. In addition, S. Pincherle in [76, Capitulo 3] showed that the continued fraction (2.8) converges if and only if the recurrence relation (2.10) has a minimal solution.

2.2 Little q-Laguerre birth and death process.

2.2.1 Little q-Laguerre polynomials

We introduce the q-Laguerre rates as follows

$$\lambda_n = q^n(1 - aq^{n+1}), \quad \mu_n = aq^n(1 - q^n), \quad n = 0, 1..., \tag{2.11}$$

where a, q are such that $0 < a, q < 1$. From direct computations, we obtain $\pi_n = \dfrac{(aq;q)_n}{(aq)^n(q;q)_n}$, from what it follows that $\sum\limits_{n\geq 0}\dfrac{1}{\lambda_n\pi_n} = \infty$. We derive that the associated series (2.7) is divergent, and then the transition rates (2.11) induces a unique process. We have

Theorem 2.1

The orthogonal polynomials $(Q_n(x))_{n\geq 1}$ associated with the BDP (2.9) are precisely the little q-Laguerre polynomials of first kind given by the expression

$$Q_n(x) = \sum_{k=0}^{n} \frac{q^k(q^{-n};q)_k}{(aq;q)_k(q;q)_k} x^k.$$

Proof. Let $(Q_n(x))_{n\geq 1}$ be the orthogonal polynomials associated with the birth and death process (2.11). The polynomials $(Q_n(x))_{n\geq 1}$ are generated recursively by the next three-term recurrence relation

$$-xQ_n(x) = q^n(1-aq^{n+1})Q_{n+1}(x) - q^n(1-a(q^{n+1}+q^n-1))Q_n(x) + aq^n(1-q^n)Q_{n-1}(x),$$

with $Q_0(x) = 1$ and $Q_1(x) = \frac{1-aq-x}{(1-aq)}$.

The famlily of polynomials $P_n(x) = \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} Q_n(x)$ satisfy the next modified three term recurrence relation

$$xP_n(x) = (1-(a+1)q^{n+1}+aq^{2n+2})P_{n+1}(x)+(q^n(1+a)-aq^{2n}(1+q))P_n(x)+aq^{2n-1}P_{n-1}(x), \tag{2.12}$$

with

$$P_0(x) = 1, \qquad P_1(x) = \frac{1-aq-x}{(1-aq)(1-q)}. \tag{2.13}$$

We start by computing the generating function associated with polynomials $(P_n(x))_{n\geq 1}$ given by $P(x,t) := \sum_{n\geq 0} P_n(x)t^n$. We will use the generating function in order to obtain the explicit representation of the polynomial $P_n(x)$, by following the technique of generating functions explained in [68]. Notice also that since $\sum_{n\geq 0} P_n(x)$ is convergent, we have $P(x,t)$ is an analytic function in t for $|t| < 1$.

Multiplying both sides of the equation (2.12) by t^{n+1} gives

$$\begin{aligned} xtP_n(x)t^n = \;& P_{n+1}(x)t^{n+1} - (a+1)P_{n+1}(x)(tq)^{n+1} + aP_{n+1}(x)(tq^2)^{n+1} \\ & + t(a+1)P_n(x)(tq)^n - ta(q+1)P_n(x)(tq^2)^n + at^2qP_{n-1}(x)(tq^2)^{n-1}. \end{aligned}$$

Summing the resulting equations for $n = 1, 2, ...,$ and using (2.13), we derive that $P(x,t)$ satisfies the next functional equation

$$(1-xt)P(x,t) - (a+1)(1-t)P(x,qt) + a(1-t)(1-tq)P(x,tq^2) = 0. \tag{2.14}$$

Now, by setting the change of function

$$H(x,t) = \frac{(xt;q)_\infty}{(t;q)_\infty} P(x,t), \tag{2.15}$$

and the formula

$$(t;q)_n = \frac{(t;q)_\infty}{(tq^n;q)_\infty}, \tag{2.16}$$

in equation (2.14), we get a more convenient functional equation

$$H(x,t) - (a+1)H(x,qt) + a(1-qxt)H(x,q^2t) = 0. \tag{2.17}$$

If we write

$$H(x,t) = \sum_{n\geq 0} h_n(x)t^n, \tag{2.18}$$

and we replace (2.18) in (2.17), we obtain the following recursive relation satisfied by the sequence $(h_n(x))_{n\geq 0}$.

$$(1-aq^n)(1-q^n)h_n(x) - q^{2n-1}axh_{n-1}(x) = 0.$$

From an easy induction, we derive the expression of h_n as follows

$$h_n(x) = \frac{q^{n^2}a^nx^n}{(aq;q)_n(q;q)_n}, \quad n>0, \quad h_0(x) = 1. \tag{2.19}$$

To find the expression of $P(x,t)$, it suffices to insert (2.19) into (2.15). It comes

$$P(x,t) = \frac{(t;q)_\infty}{(xt;q)_\infty}\sum_{n\geq 0}\frac{q^{n^2}a^nx^n}{(aq;q)_n(q;q)_n}t^n.$$

To identify the obtained expression of $P(x,t)$, we apply Heine transformation, see [95, page 91] for example. We obtain

$$\frac{1}{(t;q)_\infty}\,{}_0\phi_1(0,c;q,cz) = {}_2\phi_1(0,0,c;q,z).$$

Where the q-hypergeometric series are defined as in [61, page 15] by

$${}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}z^n.$$

Thus

$$P(x,t) = (t;q)_\infty\,{}_2\phi_1(0,0,aq;q,xt).$$

From $\binom{n}{2} = \dfrac{n(n+1)}{2}$ for $n \geq 0$ and by applying Euler's formula [95, page 93], we have

$$(t;q)_\infty = \sum_{n\geq 0}\frac{(-1)^nq^{\binom{n}{2}}}{(q;q)_n}t^n.$$

We deduce that

$$\begin{aligned} P(x,t) &= \sum_{n\geq 0}\frac{(-1)^n}{(q;q)_n}q^{\binom{n}{2}}t^n\sum_{k\geq 0}\frac{(xt)^k}{(aq;q)_k(q;q)_k} \\ &= \sum_{n\geq 0}\sum_{k=0}^{n}\frac{(-1)^{n-k}x^k}{(q;q)_{n-k}(q;q)_k(aq;q)_k}q^{\binom{n-k}{2}}t^n, \end{aligned}$$

and substituting by Salter formula from [95, p.241]

$$(q;q)_{n-k} = \frac{(q;q)_n}{(q^{-n};q)_k(-q)^k q^{nk}} q^{\binom{k+1}{2}},$$

gives

$$P(x,t) = \sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} \sum_{k=0}^{n} \frac{(qx)^k (q^{-n};q)_k}{(aq;q)_k(q;q)_k} t^n.$$

Finally, using the expression

$$P(x,t) = \sum_{n\geq 0} \frac{(-1)^n}{(q;q)_n} q^{\binom{n}{2}} Q_n(x) t^n.$$

we obtain

$$Q_n(x) = \sum_{k=0}^{n} \frac{(qx)^k (q^{-n};q)_k}{(aq;q)_k(q;q)_k},$$

that are precisely the Little q-Laguerre polynomials of first kind.

2.2.2 Denominator Polynomials associated with q-Laguerre BDP

We introduce next a second solution of the recurrence relation (2.10) given by the family of the polynomials $(Q_n^*(x))_{n\geq 1}$ of degree $(n-1)$ associated with the initial conditions $Q_0^*(x) = 0$ and $Q_1^*(x) = 1$.

It is not difficult to see that the polynomials $(Q_n^*(x))_{n\geq 1}$ coincide with the denominator of the associated continued fraction (2.8). Moreover, from Markov's Theorem [101, page 57], we have

$$\lim_{n\to+\infty} \frac{Q_n^*(x)}{Q_n(x)} = \int_0^{+\infty} \frac{d\mu(y)}{x-y}, \text{ with } x \notin \{q^k \ /k \in \mathbb{N}\}.$$

Also, it is well known that the polynomials $(Q_n^*(x))_{n\geq 1}$ have the next integral representation

$$Q_n^*(x) = \int_0^{+\infty} \frac{Q_n(x) - Q_n(y)}{x-y} d\mu(y), \quad n \geq 0.$$

See N .I .Akheizer [1, page 8] for further details.

Proposition 2.1

The little q-Laguerre polynomials of second kind $(Q_n^*(x))_{n\geq 1}$ associated with the associated with the BDP (2.9) are given by the expression

$$Q_n^*(x) = \sum_{k=0}^{n-1} \left[\sum_{l=k+1}^{n} a_l^{(n)} s_{l-k-1} \right] x^k.$$

Proof. Writing $Q_n(x) = \sum_{k=0}^{n} a_k^{(n)} x^k$, we obtain

$$a_k^{(n)} = \frac{(q^{-n};q)_k q^k}{(aq;q)_k (q;q)_k}.$$

For any fixed $y \in]0, +\infty[$ and $n \in \mathbb{N}^*$, we get

$$\frac{Q_n(x) - Q_n(y)}{x-y} = \sum_{k=1}^{n} \sum_{l=0}^{k-1} a_k^{(n)} y^{k-l-1} x^l = \sum_{k=0}^{n-1} \Big[\sum_{l=k+1}^{n} a_l^{(n)} y^{l-k-1} \Big] x^k,$$

and by integrating the above expression with respect to $d\mu$, we deduce that

$$Q_n^*(x) = \sum_{k=0}^{n-1} \Big[\sum_{l=k+1}^{n} a_l^{(n)} s_{l-k-1} \Big] x^k. \tag{2.20}$$

Since moreover, the discrete measure of orthogonality of little q-Laguerre polynomials is given by $\mu = \sum_{k \geq 0} \frac{(aq)^k}{(q;q)_k} \delta_{(q^k)}$ see [61, page 519] and the sequences $(s_n)_{n \in \mathbb{N}}$ of moments is given by

$$s_n = \int_0^{+\infty} y^n d\mu(y) = \sum_{k=0}^{+\infty} \frac{(aq^{n+1})^k}{(q;q)_k} = \frac{1}{(aq^{n+1};q)_\infty},$$

it follows that

$$Q_n^*(x) = \sum_{k=0}^{n-1} \Big[\sum_{l=k+1}^{n} \frac{(q^{-n};q)_l q^l}{(aq;q)_l (q;q)_l (aq^{l-k};q)_\infty} \Big] x^k.$$

2.2.3 The convergence of the associated continuous fraction

We have

Proposition 2.2

The continued fraction associated with the BDP (2.9) is convergent.

Proof. According to (2.8) for any $x \notin \operatorname{supp} \mu = \{ q^k \ /k \in \mathbb{N} \}$ we formally have

$$f(x) = \lim_{n \to +\infty} \frac{Q_n^*(x)}{Q_n(x)} = \int_0^{+\infty} \frac{1}{x-y} d\mu(y)$$

and because of the expression of μ, we obtain

$$\int_0^{+\infty} \frac{1}{x-y} d\mu(y) = \sum_{n=0}^{+\infty} \frac{(aq)^n}{(x-q^n)(q;q)_n} = \sum_{n=0}^{+\infty} a_n,$$

where

$$a_n(x) = \frac{(aq)^n}{|x-q^n|(q;q)_n}.$$

Now, since

$$\lim_{n \to +\infty} \frac{a_{n+1}(x)}{a_n(x)} = aq < 1,$$

we deduce that $f(x)$ is convergent for every $x \in \operatorname{supp} \mu$.

2.3 Minimal solution

In this section we shall derive a closed form of the minimal solution of the recurrence relation (2.10), by using the Rodrigues formula from [45].

Proposition 2.3

The minimal solution $(Q_n^{**}(x))_{n\geq 1}$ of the BDP (2.9) exists and has the next expansion formula

$$Q_n^{**}(x) = (-1)^n q^{n\alpha+\binom{n+1}{2}}(1-q)\frac{(\frac{q^{n+1}}{x},q;q)_\infty (q;q)_n}{(q^{\alpha+1};q)_n(\frac{1}{x},qx;q)_\infty x^{n+1+\alpha}}\, {}_2\phi_1(\frac{1}{x},0;\frac{q^{n+1}}{x};q,q^{\alpha+n+1}). \tag{2.21}$$

and the associated continued fraction has the form

$$f(x) = \frac{1}{(x-q)\sum_{k=0}^{\alpha} q^{k-\alpha-1}} \frac{{}_2\phi_1(\frac{1}{x},0,\frac{q^2}{x};q,q^{\alpha+2})}{{}_2\phi_1(\frac{1}{x},0;\frac{q}{x};q,q^{\alpha+1})}, \text{ for } \; x \notin \operatorname{supp}\mu.$$

Proof. According to Pincherle result [76, Capitulo 3], the minimal solution $(Q_n^{**}(x))_{n\geq 1}$ of (2.10) exists since the continued fraction (2.8) converges as shown in the previous section. Moreover, $(Q_n^{**}(x))_{n\geq 1}$ is given by the formula

$$w(x)Q_n^{**}(x) = Q_n(x)w(x)Q_0^{**}(x) - Q_n^*(x),$$

where $d\mu(y) = w(y)dx,\ (w(y) \geq 0)$ is absolutely continuous with respect to Lebesque measure.

By using [90, proposition 5.21], it will follow that $Q_n^{**}(x)$ has the next integral representation,

$$Q_n^{**}(x) = \frac{1}{w(x)}\int_0^\infty \frac{Q_n(y)}{x-y}d\mu(y), \quad \text{for } \; n>0, \; \text{ and } \; x\notin \operatorname{supp} w\ (=\operatorname{supp}\mu). \tag{2.22}$$

On the other hand, the little since q-Laguerre polynomials have the basic hypergeometric representation given by, see [61, page 518]

$$Q_n(x) = {}_2\phi_1(q^{-n},0;aq;q,qx),$$

we derive the next the orthogonality relation as in [61, page 519]

$$\sum_{k=0}^\infty \frac{(aq)^k}{(q;q)_k}Q_m(q^k)Q_n(q^k) = \frac{1}{\pi_n(aq;q)_\infty}\delta_{mn}, \quad 0<a,q<1.$$

The associated Rodrigues-type formula stated in [61, page 519] becomes

$$w(x;\alpha;q)Q_n(x) = \frac{q^{n\alpha+\binom{n}{2}}(1-q)^n}{(q^{\alpha+1};q)_n} D^n_{q^{-1}}[w(x;\alpha+n;q)], \quad \alpha > -1, \tag{2.23}$$

where $w(x;\alpha;q) = (qx;q)_\infty x^\alpha$ and where the q-difference operator D_q is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{x - qx}.$$

By replacing (2.23) in (2.22), we get that

$$w(x;\alpha;q)Q^{**}_n(x) = \frac{q^{n\alpha+\binom{n}{2}}(1-q)^n}{(q^{\alpha+1};q)_n} \int_0^\infty \frac{1}{x-y} D^n_{q^{-1}}[w(y;\alpha+n;q)] d_q y. \tag{2.24}$$

The last factor in the equation above corresponds to the Jackson q-integral defined for an arbitrary function f over $[0;\infty[$ by

$$\int_0^\infty f(x) d_q(x) = (1-q) \sum_{-\infty}^{+\infty} q^n f(q^n). \tag{2.25}$$

Using the product rule for D_q is

$$D_q(f(x)g(x)) = f(x)D_q g(x) + g(qx)D_q f(x) = f(qx)D_q g(x) + g(x)D_q f(x),$$

and since, $D_q(f(x)g(x))$ vanishes at the boundary, the q-integration by parts becomes

$$\int_a^b f(t) D_q g(t) d_q t = -\frac{1}{q} \int_{qa}^{qb} g(t) D_{q^{-1}} f(t) d_q t.$$

We obtain

$$w(x;\alpha;q)Q^{**}_n(x) = \frac{(-1)^n q^{n\alpha+\binom{n+1}{2}}(1-q)^n}{(q^{\alpha+1};q)_n} \int_0^\infty w(y;\alpha+n;q) D^n_{q^{-1}} \frac{1}{x-y} d_q y.$$

Now, remarking that

$$D^n_{q^{-1},y} \frac{1}{x-y} = \frac{(q;q)_n}{(1-q)^n} \frac{1}{x^{n+1}(y/x;q)_{n+1}} = \frac{(q;q)_n}{(1-q)^n} \frac{(yq^{n+1}/x;q)_\infty}{x^{n+1}(y/x;q)_\infty},$$

and by using the expression $(t;q)_n = \dfrac{(t;q)_\infty}{(tq^n;q)_\infty}$ in (2.16), we can rewrite the formula (2.3) in the following form

$$w(x;\alpha,q)Q^{**}_n(x) = \frac{(-1)^n q^{n\alpha+\binom{n+1}{2}}(q;q)_n}{(q^{\alpha+1};q)_n} \int_0^\infty \frac{(yq^{n+1}/x;q)_\infty}{x^{n+1}(y/x;q)_\infty} (yq;q)_\infty y^{\alpha+n} d_q y.$$

This proves that

$$w(x;\alpha;q)Q^{**}_n(x) = \frac{(-1)^n q^{n\alpha+\binom{n+1}{2}}(q;q)_n}{(q^{\alpha+1};q)_n x^{n+1}} \sum_{k\geq 0} \frac{(-1)^k q^{\binom{k+1}{2}}}{(q;q)_k} \int_0^\infty \frac{(yq^{n+1}/x;q)_\infty}{(y/x;q)_\infty} y^{\alpha+n+k} d_q y. \tag{2.26}$$

Let now

$$I = \int_0^\infty \frac{(yq^{n+1}/x;q)_\infty}{(y/x;q)_\infty} y^{\alpha+n+k} d_q y. \tag{2.27}$$

Now applying (2.25), we obtain

$$I=(1-q)\frac{(\frac{q^{n+1}}{x};q)_\infty(q;q)_\infty}{(\frac{1}{x};q)_\infty}\sum_{p\geq 0}\frac{(\frac{1}{x};q)_p}{(\frac{q^{n+1}}{x};q)_p(q;q)_p}(q^{\alpha+n+1})^p, \tag{2.28}$$

and substituting with (2.28) in (2.26) yields

$$Q_n^{**}(x)=(-1)^n q^{n\alpha+\binom{n+1}{2}}(1-q)\frac{(\frac{q^{n+1}}{x},q;q)_\infty(q;q)_n}{(q^{\alpha+1};q)_n(\frac{1}{x},qx;q)_\infty x^{n+1+\alpha}}\,{}_2\phi_1(\frac{1}{x},0;\frac{q^{n+1}}{x};q,q^{\alpha+n+1}).$$

For the second assertion, we use [76, Capitulo 3], where it is shown that, when a minimal solution $(Q_n^{**}(x))_{n\geq 1}$ exists, then the continued fraction (2.8) converges to the finite value

$$f(x)=-\frac{Q_1^{**}(x)}{Q_0^{**}(x)}=\frac{1}{(x-q)\sum_{k=0}^{\alpha}q^{k-\alpha-1}}\frac{{}_2\phi_1(\frac{1}{x},0,\frac{q^2}{x};q,q^{\alpha+2})}{{}_2\phi_1(\frac{1}{x},0;\frac{q}{x};q,q^{\alpha+1})},$$

for every $x\notin \operatorname{supp} w$.

Remark. Let us conclude with some comments on the ergodicity of BDP associated with rates (2.9). Following [54], a BDP is said to be ergodic if $\lim_{t\to+\infty}P_{m,n}(t)=p_n$, for the stationary distribution p_n given by $p_n=\frac{\pi_n}{\sum_{l\geq 0}\pi_l}$.

For the BDP associated with rates (2.9), we have $\lim_{n\to+\infty}\frac{\lambda_n}{\mu_n}=\frac{1}{a}>1$, and hence $\sum_{n\geq 0}\pi_n=+\infty$. In particular, it will follow that $p_n=0$ for every n that will contradict $\sum_{n\geq 0}p_n=1$. Thus the BDP with rates (2.9) is not ergodic.

Chapter 3

Finite-dimensional completion for the matrix-valued truncated complex moment problem

This chapter is designed to demonstrate, first, that if a truncated complex matrix-valued sequence has a representing measure, then it serves as the initial data for an infinite complex matrix-valued sequence satisfying a specific finite-dimensional property. We show that finite-dimensional completion of a truncated data provides a necessary and sufficient condition, and hence a solution, for the matrix-valued truncated complex moment problem. As a consequence, we obtain a matrix generalization of Curto-Fialkow's result on flat positive extensions of moment matrices. In Section 3.1, we introduce some notation and useful facts needed in this Chapter. In Section 3.2, we introduce the finite-dimensional matrix-valued sequences and we show that every truncated matrix-valued moment sequence has a finite-dimensional completion. In the last section, we solve the matrix-valued moment problem for finite-dimensional matrix-valued sequences, and we prove that finite-dimensional completion is a necessary and sufficient condition for matrix-valued truncated complex moment problem. Further, we provide a matrix generalization of Curto-Fialkow's result on flat positive extension of moment matrices.

3.1 Preliminaries

Throughout this Chapter, we use the following notation. The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The symbols $\mathbb{Z}_+$, $\mathbb{N}$ and $\mathbb{R}_+$ stand for the sets of nonnegative integers, positive integers and nonnegative real numbers, respectively. For $z \in \mathbb{C}$, δ_z denotes as usual the Borel probability measure on $\mathbb{C}$ concentrated on $\{z\}$. We write $\mathbb{C}^{p\times q}$ for the $\mathbb{C}^*$-algebra of all complex $p \times q$-matrices with the usual matrix operations and $\mathcal{H}_p$ for the subspace of $p \times p$ Hermitian matrices over $\mathbb{C}$. For brevity, we will use the notation $\mathbb{C}^p = \{\mathbf{z} = (z_1, \ldots, z_p), z_1, \ldots, z_p \in \mathbb{C}\}$ in place of $\mathbb{C}^{p\times 1}$ equipped with the standard inner product $< \mathbf{z}, \mathbf{w} >:=< (z_1, \ldots, z_p); (w_1, \ldots, w_p) >= z_1\bar{w}_1 + \ldots + z_p\bar{w}_p$, the bar indicates the complex conjugate. The symbol $\mathbb{C}^p[z, \bar{z}]$ will designate the algebra of all polynomials in $z, \bar{z} \in \mathbb{C}$, with coefficients in $\mathbb{C}^p$. For every integer $m \geq 1$, we shall denote by $\mathbb{C}^p_m[z, \bar{z}]$ the subspace of $\mathbb{C}^p[z, \bar{z}]$ consisting of all polynomials p with $\deg(p) \leq m$, where $\deg(p)$ is the total degree of p. We also denote $\mathbb{C}^p_{+\infty}[z,\bar{z}] \equiv \mathbb{C}^p[z,\bar{z}]$. We will adopt in the sequel the usual lexicographic order to expand our complex polynomials.

Let $\mathcal{B}(\mathbb{C})$ denote the Borel σ-algebra of $\mathbb{C}$. A $\mathcal{H}_p$-valued measure μ on $\mathcal{B}(\mathbb{C})$ is a function $\mu : \mathcal{B}(\mathbb{C}) \to \mathcal{H}_p$ such that

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$$

for any sequence $E_1, E_2, \cdots \in \mathcal{B}(\mathbb{C})$ of pairwise disjoint sets. It is clear that the matrix measure μ has the form $(\mu_{ij})_{i,j=1}^p$, where $\{\mu_{ij}\}_{i,j\leq p}$ is a family of complex measures. A $\mathcal{H}_p$-valued measure μ is said to be positive if $\mu(E)$ is a positive hermitian matrix for all $E \in \mathcal{B}(\mathbb{C})$, or equivalently, for every $\mathbf{z} \in \mathbb{C}^p$, $< \mu(.)\mathbf{z}, \mathbf{z} >$ is a positive measure on $\mathcal{B}(\mathbb{C})$.

When $\mu = (\mu_{ij})_{i,j=1}^p$ is a positive $\mathcal{H}_p$-valued measure, diagonal entries μ_{ii} are positive finite measures and so is the associated tracial measure

$$\tau_\mu := \mu_{11} + \mu_{22} + \ldots + \mu_{pp}.$$

Moreover, $\mid \mu_{ij}(E) \mid \leq \tau_\mu(E)$ for all $E \in \mathcal{B}(\mathbb{C})$ and $i, j = 1, \ldots, p$. Indeed,

$$\mu(E) \equiv (\mu_{ij}(E))_{i,j=1}^p := \begin{pmatrix} \mu_{11}(E) & \mu_{12}(E) & \ldots & \mu_{1p}(E) \\ \mu_{21}(E) & \mu_{22}(E) & \ldots & \mu_{2p}(E) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{p1}(E) & \mu_{p2}(E) & \ldots & \mu_{pp}(E) \end{pmatrix} \geq 0$$

implies that $\mu_{ij}(E)\mu_{ji}(E) \leq \mu_{ii}(E)\mu_{jj}(E)$ and hence

$$\mid \mu_{ij}(E) \mid^2 = \mu_{ij}(E)\overline{\mu_{ij}(E)} = \mu_{ij}(E)\mu_{ji}(E) \leq \mu_{ii}(E)\mu_{jj}(E) \leq (\tau_\mu(E))^2.$$

Consequently, $\mu_{ij} << \tau_\mu$, that is, $\tau_\mu(E) = 0$ implies $\mu_{ij}(E) = 0$ for all $i, j = 1, \ldots, p$. It follows by the Radon-Nikodym Theorem [21], that there exist measurable functions f_{ij} (called Radon-Nikodym derivative of τ_μ with respect to μ_{ij}) such that

$$\mu_{ij}(E) = \int_E f_{ij} d\tau_\mu, \quad \text{for arbitrary } E \in \mathcal{B}(\mathbb{C}) \text{ and for all } i, j = 1, 2, \ldots, p. \tag{3.1}$$

The next theorem gives an important property of Radon-Nikodym derivatives that will be used in the sequel.

Theorem 3.1 ([13, Theorem 1.12])

Let μ be a positive matrix measure with Radon-Nikodym derivatives $f_{ij} \in \mathcal{L}_1(\tau_\mu)$ such that

$$\mu_{ij}(E) = \int_E f_{ij}(x) d\tau_\mu(x) \qquad \textit{for all } E \in \mathcal{B}(\mathbb{C}).$$

Then the matrix $(f_{ij}(x))$ is positive hermitian for τ_μ-almost all $x \in \mathbb{C}$.

For a measurable function $f : \mathbb{C} \to \mathbb{C}$, we define $\int f d\mu \in \mathcal{H}_p$ as follows

$$< \int f d\mu \, \mathbf{z}, \tilde{\mathbf{z}} >= \int f < d\mu \, \mathbf{z}, \tilde{\mathbf{z}} > \qquad \text{for all } \mathbf{z}, \tilde{\mathbf{z}} \in \mathbb{C}^p.$$

For column vector polynomials $P = \sum_{i+j=0}^{n} \mathbf{c}_{ij} \bar{z}^i z^j$ and $Q = \sum_{h+k=0}^{n} \mathbf{l}_{hk} \bar{z}^h z^k \in \mathbb{C}^p[z, \bar{z}]$, we define

$$\int P d\mu Q := \sum_{i+j=0}^{n} \sum_{h+k=0}^{n} \mathbf{l}_{hk}^* \int \bar{z}^{i+k} z^{j+h} d\mu \mathbf{c}_{ij}.$$

(A more detailed explanation of this definition can be found in [13].) Clearly, $\int P d\mu Q \in \mathbb{C}$ and

$$\int P d\mu P = \sum_{i+j=0}^{n} \sum_{h+k=0}^{n} \mathbf{c}_{hk}^* \int \bar{z}^{i+k} z^{j+h} d\mu \mathbf{c}_{ij}. \tag{3.2}$$

Therefore, it follows from (3.1) that

$$\int P d\mu P = \sum_{i+j=0}^{n} \sum_{h+k=0}^{n} \sum_{e,l=1}^{p} \int \overline{c_{(hk,e)} \bar{z}^h z^k} f_{el}(z, \bar{z}) c_{(ij,l)} \bar{z}^i z^j d\tau_\mu(z, \bar{z}) \tag{3.3}$$

for every $P = \sum_{i+j \le n} \mathbf{c}_{ij} \bar{z}^i z^j \in \mathbb{C}^p[z, \bar{z}]$, where $\mathbf{c}_{ij} = (c_{(ij,1)}, c_{(ij,2)}, \ldots, c_{(ij,p)}) \in \mathbb{C}^p$.

3.2 Finite dimensional completion

Consider next a $\mathcal{H}_p$-valued sequence $H^{(n)} \equiv \{H_{ij}\}_{i,j \in \mathbb{Z}_+; i+j \le 2n}$, where $n \in \mathbb{Z}_+ \cup \{+\infty\}$, for which the associated moment problem owns a solution, that is, there exists a positive $\mathcal{H}_p$-valued measure μ such that

$$H_{ij} = \int \bar{z}^i z^j d\mu \qquad \text{for all } i, j \in \mathbb{Z}_+ \text{ with } i + j \le 2n, \tag{3.4}$$

and let $P = \sum_{i+j \le m} \mathbf{c}_{ij} \bar{z}^i z^j \in \mathbb{C}^p[z, \bar{z}]$, where $\mathbf{c}_{ij} = (c_{(ij,1)}, c_{(ij,2)}, \ldots, c_{(ij,p)}) \in \mathbb{C}^p$, $m \in \mathbb{Z}_+$ and $m \le n$. From (3.2), (3.4) and (3.3), we derive

$$\int P d\mu P = \sum_{i+j=0}^{m} \sum_{h+k=0}^{m} \mathbf{c}_{hk}^* \int \bar{z}^{i+k} z^{j+h} d\mu(z, \bar{z}) \mathbf{c}_{ij} = \sum_{i+j=0}^{m} \sum_{h+k=0}^{m} \mathbf{c}_{hk}^* H_{i+k, j+h} \mathbf{c}_{ij}$$

and

$$\begin{aligned} \int P d\mu P &= \sum_{i+j=0}^{m} \sum_{h+k=0}^{m} \sum_{e,l=1}^{p} \int \overline{c_{(hk,e)} \bar{z}^h z^k} f_{el}(z, \bar{z}) c_{(ij,l)} \bar{z}^i z^j d\tau_\mu(z, \bar{z}) \\ &= \int \sum_{e,l=1}^{p} \overline{g_e(z, \bar{z})} f_{el}(z, \bar{z}) g_l(z, \bar{z}) d\tau_\mu, \end{aligned}$$

where $g_e(\bar{z}, z) := \sum_{i+j=0}^{m} c_{(ij,e)} \bar{z}^i z^j$. Using Theorem 3.1, we derive that $\int P d\mu P \geq 0$ and hence

$$\sum_{i+j=0}^{m} \sum_{h+k=0}^{m} \mathbf{c}_{hk}^* H_{i+k,j+h} \mathbf{c}_{ij} \geq 0 \quad \text{for all } \{\mathbf{c}_{ij}\}_{i,j \in \mathbb{Z}_+; i+j \leq m} \subset \mathbb{C}^p, m \in \mathbb{Z}_+ \text{ and } m \leq n. \quad (3.5)$$

Definition 3.1

A $\mathcal{H}_p$-valued sequence $H^{(n)} \equiv \{H_{ij}\}_{i,j \in \mathbb{Z}_+; i+j \leq 2n}$, where $n \in \mathbb{Z}_+ \cup \{+\infty\}$, is said to be positive if it satisfies (3.5).

It follows from the above discussion that the positivity property (3.5) is a necessary condition to solve the matrix-valued complex moment problem. Hence, it is natural to, and we will, restrict to positive $\mathcal{H}_p$-valued sequences.

Consequently, for every $N \in \mathbb{Z}_+ \cup \{+\infty\}$ with $N \leq n$, we define on $\mathbb{C}_N^p[\bar{z}, z]$ (where $\mathbb{C}_{+\infty}^p[\bar{z}, z] \equiv \mathbb{C}^p[\bar{z}, z]$) a positive hermitian form as follows

$$< \sum_{i+j=0}^{N} \mathbf{c}_{ij} \bar{z}^i z^j ; \sum_{h+k=0}^{N} \mathbf{d}_{hk} \bar{z}^h z^k > := \sum_{i+j=0}^{N} \sum_{h+k=0}^{N} \mathbf{d}_{hk}^* H_{i+k,j+h} \mathbf{c}_{ij}, \quad (3.6)$$

with associated semi-norm

$$\| \sum_{i+j=0}^{N} \mathbf{c}_{ij} \bar{z}^i z^j \|^2 = \sum_{i+j=0}^{N} \sum_{h+k=0}^{N} \mathbf{c}_{hk}^* H_{i+k,j+h} \mathbf{c}_{ij}.$$

(If $N = +\infty$, then $\sum_{i+j=0}^{N} \mathbf{c}_{ij} \bar{z}^i z^j \in \mathbb{C}_{+\infty}^p[\bar{z}, z] \equiv \mathbb{C}^p[\bar{z}, z]$, that is, c_{ij}'s are all zero but a finite number of them.) Note in passing that (3.6) yields

$$< e_j z^k ; e_i z^h >= e_i^* H_{hk} e_j \qquad \text{for all } h, k \in \mathbb{Z}_+ \text{ with } h + k \leq 2N,$$

where $\{e_i\}_{i=1}^p$ denote the standard orthonormal basis of $\mathbb{C}^p$, and hence

$$H_{hk} = (< e_j z^k, e_i z^h >)_{i,j=1}^p := \begin{pmatrix} < e_1 z^k, e_1 z^h > & < e_2 z^k, e_1 z^h > & \dots & < e_p z^k, e_1 z^h > \\ < e_1 z^k, e_2 z^h > & < e_2 z^k, e_2 z^h > & \dots & < e_p z^k, e_2 z^h > \\ \vdots & \vdots & \ddots & \vdots \\ < e_1 z^k, e_p z^h > & < e_2 z^k, e_p z^h > & \dots & < e_p z^k, e_p z^h > \end{pmatrix}. \quad (3.7)$$

Let now

$$\mathcal{I}_N := \{P \in \mathbb{C}_N^p[z, \bar{z}] \text{ such that } \|P\| = 0\} \quad (3.8)$$

be the null-space of the above seminorm. We equip the quotient space

$$V_N := \mathbb{C}_N^p[z, \bar{z}] / \mathcal{I}_N = \{[g]_N \mid g \in \mathbb{C}_N^p[z, \bar{z}]\}, \qquad \text{where } [g]_N = g + \mathcal{I}_N, \quad (3.9)$$

with an inner product induced from (3.6) by

$$< [f]_N, [g]_N >:=< f, g > \qquad \text{for all } f, g \in \mathbb{C}_N^p[z, \bar{z}]. \quad (3.10)$$

Remark.

(i) For every positive integer $N < n$,

$$\mathcal{I}_N \subset \mathcal{I}_{N+1} \text{ and } \dim V_N \leq \dim V_{N+1}. \tag{3.11}$$

(ii) For every positive integer $N < n$ and every $f, g, fg \in \mathbb{C}_N[z, \bar{z}]$,

$$f \in \mathcal{I}_N \Longrightarrow fg \in \mathcal{I}_N. \tag{3.12}$$

(iii) If $H^{(n)}$ has a representing measure μ, then (3.6) can be reformulated as

$$< P, Q >:= \int P d\mu Q \qquad \text{for } P, Q \in \mathbb{C}_N^p[z, \bar{z}]. \tag{3.13}$$

Next we give a dimensional stability property for the quotient spaces $\{V_N\}_{N=0}^n$.

Proposition 3.1

Let $H^{(n+1)} = \{H_{ij}\}_{i+j \leq 2n+2}$, where $n \in \mathbb{Z}_+ \cup \{+\infty\}$, be a positive $\mathcal{H}_p$-valued sequence and let $\{V_N\}_{N \leq n+1}$ be the associated quotient spaces given as in (3.9). If $\dim V_s = \dim V_{s+1}$ for some positive integer $s < n$, then $\dim V_s = \dim V_{s+1} = \ldots = \dim V_n$.

Proof. The case when $s = n - 1$ is trivial. Let us assume that $s \leq n - 2$ and take an arbitrary $\bar{z}^h z^k \in \mathbb{C}_{s+2}[z, \bar{z}]$. Without loss of generality, we may suppose that $h > 0$. Since $\dim V_s = \dim V_{s+1}$, then there exists a polynomial $g \in \mathbb{C}_s[z, \bar{z}]$ such that $\bar{z}^{h-1} z^k - g \in \mathcal{I}_{s+1} \subset \mathcal{I}_{s+2}$. Using (3.12), we get $\bar{z}^h z^k - \bar{z} g \in \mathcal{I}_{s+2}$ and because $\bar{z} g \in \mathbb{C}_{s+1}[z, \bar{z}]$, we obtain

$$\dim V_{s+2} = \dim V_{s+1} (= \dim V_s).$$

By repeating this reasoning we obtain $\dim V_s = \dim V_{s+1} = \ldots = \dim V_n$.

Proposition 3.1 and Inequality (3.11) give rise to the next definition.

Definition 3.2

Let $H^{(n)} = \{H_{ij}\}_{i+j \leq 2n}$, where $n \in \mathbb{Z}_+ \cup \{+\infty\}$, be a positive $\mathcal{H}_p$-valued sequence and let $\{V_N\}_{N \leq n}$ be the associated quotient spaces given as in (3.9). The sequence $H^{(n)}$ is said to be finite-dimensional if there exists a positive integer $r < n$ such that

$$\dim V_0 < \ldots < \dim V_{r-1} < \dim V_r = \dim V_{r+1} = \cdots = \dim V_n. \tag{3.14}$$

The integer r will be called the rank of $H^{(n)}$ and we write $r = rank(H^{(n)})$.

Clearly, a $\mathcal{H}_p$-valued sequence $H^{(n)}$ is finite-dimensional with $r = rank(H^{(n)})$ if and only if

$$\forall f \in \mathbb{C}_n[z, \bar{z}] \quad \exists \phi(f) \in \mathbb{C}_r[z, \bar{z}] \text{ such that } f - \phi(f) \in \mathcal{I}_n. \tag{3.15}$$

The following technical lemma is needed in the proof of Theorem 3.2.

Lemma 3.1

Let $g, f \in \mathbb{C}_n[z, \bar{z}]$ be such that $gf \in \mathbb{C}_n[z, \bar{z}]$ and let $\phi(f)$ be as in (3.15). Then

$$[g\phi(f)]_n = [\phi(gf)]_n = [\phi(g)f]_n.$$

Proof. Since $f - \phi(f) \in \mathcal{I}_n$, then

$$[g\phi(f)]_n = [g(\phi(f) - f) + gf]_n = [gf]_n = [\phi(gf)]_n.$$

Next, we show that the finite dimensional property (3.14) provides an interesting necessary condition for the truncated matrix-valued complex moment problem. We start by recalling the Tchakaloff's theorem version for matrix-valued moment sequences from [58, Theorem 2]. It states that if a truncated $\mathcal{H}_p$-valued sequence $H^{(n)} \equiv \{H_{ij}\}_{i+j\leq 2n}$, where $n \in \mathbb{Z}_+$, has a representing measure then there exists a finite atomic representing measure Ξ such that

$$card\ supp\ \Xi \leq \frac{p^2(n+2)(n+1)}{2}.$$

In fact, the (finite atomic) measure Ξ gives rise to an infinite $\mathcal{H}_p$-sequence $\tilde{H} \equiv \{\tilde{H}_{ij}\}_{i+j\in\mathbb{Z}_+}$ defined by

$$\tilde{H}_{ij} = \int \bar{z}^i z^j d\Xi \qquad \text{for all } i, j \in \mathbb{Z}_+, \tag{3.16}$$

in which $\tilde{H}$ is a $\mathcal{H}_p$-valued moment sequence extension of $H^{(n)}$,

$$\tilde{H}_{ij} = H_{ij} \qquad \text{for all } i + j \leq 2n. \tag{3.17}$$

Since the $\mathcal{H}_p$-valued measure Ξ is positive, so is the associated $\mathcal{H}_p$-valued moment sequence $\tilde{H}$, see the discussion right before Definition 3.1.

The following proposition establishes a connection between the truncated matrix-valued moment problem and finite-dimensional sequences.

Proposition 3.2

Let n be a positive integer and let $H^{(n)} \equiv \{H_{ij}\}_{i+j\leq 2n}$ be a given truncated $\mathcal{H}_p$-sequence. If $H^{(n)}$ has a representing measure, then $H^{(n)}$ admits a finite-dimensional extension $\tilde{H} \equiv \{\tilde{H}_{ij}\}_{i,j\in\mathbb{Z}_+}$.

Proof. Let $\tilde{H}$ be the $\mathcal{H}_p$-valued sequence extension of $H^{(n)}$ given in (3.16) and let Ξ be the associated finite atomic representing measure, write $supp\Xi = \{\lambda_1, \lambda_2, \dots, \lambda_d\} \subset \mathbb{C}$. Setting

$$
\begin{aligned}
P &:= \prod_{i=1}^{d}(z-\lambda_i) = z^d - \sum_{i=1}^{d-1} a_i z^i \in \mathbb{C}[z,\bar{z}],\\
Q &:= C\bar{z}^{n'} z^{n-d} P = C\bar{z}^{n'} z^{n} - \sum_{i=1}^{d-1} C a_i \bar{z}^{n'} z^{n-d+i} \in \mathbb{C}^p_{n+n'}[z,\bar{z}],\\
\text{and } G_{n+n'-1} &:= \sum_{i=1}^{d-1} C a_i \bar{z}^{n'} z^{n-d+i} \in \mathbb{C}^p_{n+n'-1}[z,\bar{z}],
\end{aligned}
$$

where $n-d, n' \in \mathbb{Z}_+$ and $C \in \mathbb{C}^p$.

Since $\tilde{H}$ is positive, we can define quotient spaces $\{V_N\}_{N\in\mathbb{Z}_+}$ as in (3.9). By using (3.13), (3.2) and $P_{|supp\Xi} = 0$, we derive

$$
\|Q\|^2 = \int Q d\Xi Q = C \int (\bar{z}^{n'} z^{n-d} P)(\overline{\bar{z}^{n'} z^{n-d} P}) d\Xi \, C^* = 0
$$

and hence $Q = C\bar{z}^{n'} z^n - G_{n+n'-1} \in \mathcal{I}_{n'+n}$. As a result, for every integers $n, n' \in \mathbb{Z}_+$, with $n \geq d$, and any column $C \in \mathbb{C}^p$ there exists $G_{n+n'-1} \in \mathbb{C}^p_{n+n'-1}[z,\bar{z}]$ such that

$$
[C\bar{z}^{n'} z^n]_{n'+n} = [G_{n+n'-1}]_{n'+n}.
$$

In a similar manner we derive, for every $C \in \mathbb{C}^p$ and all integers $m', m-d \in \mathbb{Z}_+$,

$$
[C\bar{z}^m z^{m'}]_{m+m'} = [G'_{m+m'-1}]_{m+m'} \qquad \text{for some } G'_{m+m'-1} \in \mathbb{C}^p_{m+m'-1}[z,\bar{z}].
$$

Hence, if $n \geq d$ or $m \geq d$ (in particular, if $n+m \geq 2d$), then there exists $G \in \mathbb{C}^p_{n+m-1}[z,\bar{z}]$ such that $[C\bar{z}^m z^n]_{m+n} = [G]_{m+n}$, where $C \in \mathbb{C}^p$. Thus, for every $K \in \mathbb{C}^p_k[z,\bar{z}]$, with $k \geq 2d$, there exists $G \in \mathbb{C}^p_{k-1}[z,\bar{z}]$ such that $[K]_k = [G]_k$. This implies $\dim V_{k-1} = \dim V_k$ for all integer $k \geq 2d$. Hence, by Proposition 3.1, $\tilde{H}$ is finite-dimensional, as desired.

3.3 A solution to the matrix-valued truncated complex moment problem

We showed in Proposition 3.2 that the finite-dimensional extension is a necessary condition for the truncated matrix-valued moment problem. In other words, if a truncated $\mathcal{H}_p$-sequence has a representing measure, then it is an initial data of a finite-dimensional $\mathcal{H}_p$-sequence. Thus, a natural question arises: whether a given finite-dimensional $\mathcal{H}_p$-sequence has a representing measure?

The next theorem furnish an answer to the above question for infinite $\mathcal{H}_p$-valued sequences, which are finite-dimensional. The truncated case will be treated in Theorem 3.4.

Theorem 3.2 *Let $\tilde{H} \equiv \{\tilde{H}_{ij}\}_{i+j\in\mathbb{Z}_+}$ be a finite-dimensional $\mathcal{H}_p$-sequence with $r = rank(\tilde{H})$ and let $d = \dim V_r$. Then $\tilde{H}$ has a unique representing measure which is d-atomic.*

Proof. Let ϕ be as in (3.15) and define on V_r the linear operator

$$\tau : V_r \ni [f]_r \to [\phi(zf)]_r \in V_r, \tag{3.18}$$

which is well-defined. Indeed, let $f \in \mathcal{I}_r$ by using (3.10), (3.15), (3.6) and the Cauchy-Schwarz inequality we derive

$$\begin{aligned} \| \tau([f]_r) \|^2 &=< [\phi(zf)]_r, [\phi(zf)]_r >=< \phi(zf), \phi(zf) >=< zf, \phi(zf) > \\ &=< f, \bar{z}\phi(zf) > \quad \leq \quad \| f \| \| \bar{z}\phi(zf) \| = 0, \end{aligned}$$

and hence $\tau([f]_r) = 0$.

We claim that the operator τ is normal. For every $f, h \in \mathbb{C}_r[z, \bar{z}]$, we have

$$< \tau([f]_r), [h]_r >=< [\phi(zf)]_r, [h]_r >=< \phi(zf), h >=< zf, h >=< f, \bar{z}h >=< [f]_r, [\phi(\bar{z}h)]_r >$$

thus the adjoint operator τ^* is given by

$$\tau^*([f]) = \phi([\bar{z}f]) \qquad \text{for } f \in \mathbb{C}_r[z, \bar{z}].$$

Moreover,

$$\begin{aligned} < \tau^*\tau([f]_r), [h]_r > &=< \tau([f]_r), \tau([h]_r) >=< [\phi(zf)]_r, [\phi(zh)]_r >=< zf, zh > \\ &=< \bar{z}zf, h >=< z\bar{z}f, h >=< \bar{z}f, \bar{z}h >=< \tau^*([f]_r), \tau^*([h]_r) > \\ &=< \tau\tau^*([f]_r), [h]_r > \end{aligned}$$

and hence $\tau^*\tau = \tau\tau^*$, that is, the operator τ is normal.

Let us denote by $\mathcal{P}(V_r)$ the set of all orthogonal projections on V_r and let $E : \mathcal{B}(\mathbb{C}) \to \mathcal{P}(V_r)$ be the (unique) spectral measure of τ, then

$$\tau^{*n}\tau^m = \int \bar{z}^n z^m dE(z, \bar{z}) \qquad \text{for } n, m \in \mathbb{Z}_+. \tag{3.19}$$

We denote by $(e_1, e_2, \ldots, e_p)$ the canonical basis of $\mathbb{C}^p$. It is clear that $\mu_{ij}(.) =< E(.)e_j, e_i >$ is a complex measure on $\mathcal{B}(\mathbb{C})$.

We now consider the $\mathcal{H}_p$-valued measure

$$\mu(.) \equiv (\mu_{ij}(.))_{i,j=1}^p := \begin{pmatrix} < E(.)e_1, e_1 > & < E(.)e_2, e_1 > & \ldots & < E(.)e_p, e_1 > \\ < E(.)e_1, e_2 > & < E(.)e_2, e_2 > & \ldots & < E(.)e_p, e_2 > \\ \vdots & \vdots & \ddots & \vdots \\ < E(.)e_1, e_p > & < E(.)e_2, e_p > & \ldots & < E(.)e_p, e_p > \end{pmatrix},$$

The measure $\mu(.)$ is positive, since $C^*\mu(B)C =< E(B)\sum_{i=1}^p c_i e_i, \sum_{i=1}^p c_i e_i > \geq 0$ whenever

$B \in \mathcal{B}(\mathbb{C})$ and $C = (c_1, c_2, \ldots, c_p)^T \in \mathbb{C}^p$. Moreover, for $n, m \in \mathbb{Z}_+$, we have

$$\begin{aligned}
\tilde{H}_{nm} &= (< e_j z^m, e_i z^n >)_{i,j=1}^p, && \text{according to (3.7)},\\
&= (< (\tau^*)^n \tau^m e_j, e_i >)_{i,j=1}^p\\
&= (\int \bar{z}^n z^m < dE(z,\bar{z}) e_j, e_i >)_{i,j=1}^p, && \text{by using (3.19)},\\
&= \int \bar{z}^n z^m d\mu,
\end{aligned}$$

and then μ is a representing measure of $\tilde{H}$.

It remains to show that $card\ supp(\mu) = \dim V_r (= d)$. Let $\mathfrak{B}(V_r)$ denote the $\mathbb{C}^*$-algebra of all bounded linear operator on V_r and define the map

$$\Pi : V_r \ni [p]_r \to p(\tau) \in \mathfrak{B}(V_r),$$

which is well defined by using Lemma 3.1. Indeed, if $p \in \mathcal{I}_r$ and $f, h \in \mathbb{C}_r[z, \bar{z}]$, we have

$$\begin{aligned}
< \Pi([p]_r)([f]_r), [h]_r > &=< p(\tau)([f]_r), [h]_r >=< [\phi(pf)]_r, [h]_r >=< \phi(pf), h >\\
&=< fp, h >=< p, \bar{f}h >=< [p]_r, [\phi(\bar{f}h)]_r >= 0
\end{aligned}$$

and hence $\Pi([p]_r) = 0$.

Also, since $\Pi([p]_r)([1]_r) = p(\tau)[1]_r = [p]_r$, then

$$\Pi([p]_r) = 0 \Longrightarrow [p]_r = 0_{V_r}.$$

Thus, the linear map Π is injective. Moreover, we have

$$\Pi([\bar{p}]_r) = \bar{p}(\tau) = (p(\tau))^* = (\Pi([p]_r))^* \qquad \text{for } p \in \mathbb{C}_r[z, \bar{z}].$$

Note that $\Pi(V_r) = \mathcal{A} := \{p(\tau) \mid p \in \mathbb{C}[z, \bar{z}]\}$ is a unital commutative sub-$\mathbb{C}^*$-algebra of $\mathfrak{B}(V_r)$. Therefore,

$$\Pi : V_r \ni [p]_r \longrightarrow p(\tau) \in \mathcal{A}$$

is a linear *-isomorphism and hence $\dim \mathcal{A} = \dim V_r = d$. Thus, $\mathcal{A}$ has precisely d characters, say $\varphi_1, \varphi_2, \ldots, \varphi_d$, which are pairwise different. Consequently, the spectrum of τ is

$$\sigma(\tau) = \{\varphi_i(\tau) : i = 1, 2, \ldots, d\}.$$

We end our proof by showing that $card\ \sigma(\tau) = d$. If not, then $\varphi_i(\tau) = \varphi_j(\tau)$ for some $i \neq j$, which implies that

$$\varphi_i(p(\tau)) = p(\varphi_i(\tau)) = p(\varphi_j(\tau)) = \varphi_j(p(\tau)) \qquad \text{for all } p(\tau) \in \mathcal{A}$$

this yields $\varphi_i = \varphi_j$, which is a contradiction.

Since $supp(\mu) = \sigma(\tau)$, then $card\ supp(\mu) = card\ \sigma(\tau) = d$, as desired. This complete the proof of the theorem.

By combining the results in Proposition 3.2 and Theorem 3.2, we obtain our main theorem.

Theorem 3.3

Let $n \in \mathbb{Z}_+$ and let $H^{(n)} \equiv \{H_{ij}\}_{i+j\leq 2n}$ be a given truncated $\mathcal{H}_p$-valued sequence. Then the following assertions are equivalent:

(i) *$H^{(n)}$ has a representing measure.*

(ii) *$H^{(n)}$ has a d-atomic representing measure, where $d \in \mathbb{Z}_+$.*

(iii) *$H^{(n)}$ admits a finite-dimensional extension $\tilde{H} \equiv \{\tilde{H}_{ij}\}_{i,j\in\mathbb{Z}_+}$ with $r = rank(\tilde{H})$ and $d = \dim V_r$.*

Proof. $(ii) \Rightarrow (i)$ is trivial.
$(i) \Rightarrow (iii)$ follows from Proposition 3.2.
$(iii) \Rightarrow (ii)$ Theorem 3.2 yields $\tilde{H}$ admits a d-atomic representing measure, and so is $H^{(n)}$.

The next result extends the flat extension theorem given in [24, 25] to matrix-valued moment problem.

Theorem 3.4

Let $r \in \mathbb{Z}_+$ and let $H^{(r)} \equiv \{H_{ij}\}_{i+j\leq 2r}$ be a finite-dimensional $\mathcal{H}_p$-valued sequence. Then $H^{(r)}$ has a unique representing measure, which is $rank(H^{(r)})$-atomic.

Proof. We will show that $H^{(r)}$ admits a unique finite-dimensional extension $\tilde{H} \equiv \{\tilde{H}_{ij}\}_{i,j\in\mathbb{Z}_+}$ and hence $rank(H^{(r)}) = rank(\tilde{H})$ according to Proposition 3.1. Thus, Theorem 3.2 yields $\tilde{H}$, and in force $H^{(r)}$, has a unique representing measure, which is $rank(H^{(r)})$-atomic.

First, let us define a one-step positive extension of $H^{(r)}$, say $\tilde{H}^{(r+1)} \equiv \{\tilde{H}_{ij}\}_{i+j\leq 2r+2}$, such that $\dim V_{r+1} = \dim V_r (= \dim V_{r-1})$. For $i, j, h, k \in \mathbb{Z}_+$ such that $i + j, k + h \leq r$ and for every $C, D \in \mathbb{C}^p$, the truncated $\mathcal{H}_p$-valued sequence $\tilde{H}^{(r+1)}$ will verify, and is defined by,

$$\begin{aligned}
D^*\tilde{H}_{i+k,j+h}C &= D^*H_{i+k,j+h}C = < C\bar{z}^i z^j, D\bar{z}^h z^k >, \\
D^*\tilde{H}_{i+k+1,j+h}C &= < \bar{z}\phi(C\bar{z}^i z^j), D\bar{z}^h z^k >, \\
D^*\tilde{H}_{i+k,j+h+1}C &= < z\phi(C\bar{z}^i z^j), D\bar{z}^h z^k >, \\
D^*\tilde{H}_{i+k+1,j+h+1}C &= < z\phi(C\bar{z}^i z^j), z\phi(D\bar{z}^h z^k) >, \\
D^*\tilde{H}_{i+k+2,j+h}C &= < \bar{z}\phi(C\bar{z}^i z^j), z\phi(D\bar{z}^h z^k) >, \\
D^*\tilde{H}_{i+k,j+h+2}C &= < z\phi(C\bar{z}^i z^j), \bar{z}\phi(D\bar{z}^h z^k) > .
\end{aligned}$$

Thus, for every $\{c_{nm}\}_{n+m\leq r+1}$,

$$\begin{aligned}
\sum_{i+j=0}^{r+1}\sum_{h+k=0}^{r+1}\mathbf{c}_{hk}^*\tilde{H}_{i+k,j+h}\mathbf{c}_{ij} &= \sum_{i+j=0}^{r}\sum_{h+k=0}^{r}\mathbf{c}_{hk}^*\tilde{H}_{i+k,j+h}\mathbf{c}_{ij} + \sum_{i+j=r+1}\sum_{h+k=0}^{r}\mathbf{c}_{hk}^*\tilde{H}_{i+k,j+h}\mathbf{c}_{ij} \\
&\quad + \sum_{i+j=0}^{r}\sum_{h+k=r+1}\mathbf{c}_{hk}^*\tilde{H}_{i+k,j+h}\mathbf{c}_{ij} + \sum_{i+j=r+1}\sum_{h+k=r+1}\mathbf{c}_{hk}^*\tilde{H}_{i+k,j+h}\mathbf{c}_{ij} \\
&= \sum_{i+j=0}^{r}\sum_{h+k=0}^{r}\mathbf{c}_{hk}^*\tilde{H}_{i+k,j+h}\mathbf{c}_{ij} + \sum_{h+k=0}^{r}\mathbf{c}_{hk}^*\tilde{H}_{r+1+k,h}c_{r+1,0} \\
&\quad + \sum_{i=0}^{r}\sum_{h+k=0}^{r}\mathbf{c}_{hk}^*\tilde{H}_{i+k,r+1-i+h}c_{i,r+1-i} + \sum_{i+j=0}^{r}c_{r+1,0}^*\tilde{H}_{i,r+1+j}\mathbf{c}_{ij} \\
&\quad + \sum_{i+j=0}^{r}\sum_{h=0}^{r}c_{h,r+1-h}^*\tilde{H}_{i+r+1-h,j+h}\mathbf{c}_{ij} \\
&\quad + \sum_{i=0}^{r+1}\sum_{h=0}^{r+1}c_{h,r+1-h}^*\tilde{H}_{i+r+1-h,h+r+1-i}c_{i,r+1-i}
\end{aligned}$$

this yields

$$\begin{aligned}
\sum_{i+j=0}^{r+1}\sum_{h+k=0}^{r+1}\mathbf{c}_{hk}^*\tilde{H}_{i+k,j+h}\mathbf{c}_{ij} &= \sum_{i+j=0}^{r}\sum_{h+k=0}^{r} < \bar{z}^i z^j\mathbf{c}_{ij}, \bar{z}^h z^k\mathbf{c}_{hk} > \\
&\quad + \sum_{h+k=0}^{r} < \bar{z}\phi(c_{r+1,0}\bar{z}^r), \mathbf{c}_{hk}\bar{z}^h z^k > \\
&\quad + \sum_{i=0}^{r}\sum_{h+k=0}^{r} < z\phi(\bar{z}^i z^{r-i}c_{i,r+1-i}), \phi(\bar{z}^h z^k\mathbf{c}_{hk}) > \\
&\quad + \sum_{i+j=0}^{r} < \bar{z}^i z^j\mathbf{c}_{ij}, \bar{z}\phi(\bar{z}^r c_{r+1,0}) > \\
&\quad + \sum_{i+j=0}^{r}\sum_{h=0}^{r} < \bar{z}^i z^j\mathbf{c}_{ij}, z\phi(\bar{z}^h z^{r-h}c_{h,r+1-h}) > \\
&\quad + \sum_{i=0}^{r}\sum_{h=0}^{r} < z\phi(\bar{z}^i z^{r-i}c_{i,r+1-i}), z\phi(\bar{z}^h z^{r-h}c_{h,r+1-h}) > \\
&\quad + \sum_{h=0}^{r} < \bar{z}\phi(\bar{z}^r c_{r+1,0}), z\phi(\bar{z}^h z^{r-h}c_{h,r+1-h}) > \\
&\quad + \sum_{i=0}^{r} < z\phi(\bar{z}^i z^{r-i}c_{i,r+1-i}), \bar{z}\phi(\bar{z}^r c_{r+1,0}) > \\
&\quad + < \bar{z}\phi(\bar{z}^r c_{r+1,0}), \bar{z}\phi(\bar{z}^r c_{r+1,0}) > .
\end{aligned}$$

Setting $g = \sum_{i+j=0}^{r}\mathbf{c}_{ij}\bar{z}^i z^j + \sum_{i=0}^{r} z\phi(c_{i,r+1-i}\bar{z}^i z^{r-i}) + \bar{z}\phi(c_{r+1,0}\bar{z}^r) \in \mathbb{C}_r^p[z,\bar{z}]$, we get

$$\sum_{i+j=0}^{r+1}\sum_{h+k=0}^{r+1}\mathbf{c}_{hk}^*\tilde{H}_{i+k,j+h}\mathbf{c}_{ij} =< g, g >\geq 0$$

and hence $\tilde{H}^{(r+1)}$ is the unique one-step positive extension of $H^{(r)}$. By repeating this process, we get an infinite $\mathcal{H}_p$-valued sequence $\tilde{H} \equiv \{\tilde{H}_{ij}\}_{i,j\in\mathbb{Z}_+}$, which is the unique positive extension of $H^{(r)}$. Thus, one can define $\{V_N\}_{N\in\mathbb{N}}$ as in (3.9). Since $\dim V_r = \dim V_{r-1}$, Proposition 3.1 yields $\tilde{H}$ is finite-dimensional. Therefore, $H^{(r)}$ admits a unique finite-dimensional extension $\tilde{H} \equiv \{\tilde{H}_{ij}\}_{i,j\in\mathbb{Z}_+}$, as desired.

Chapter 4

On the joint numerical radius of generalized spherical Aluthge transforms of operators

Let $\mathcal{H},\mathcal{K}$ denote two Hilbert spaces, and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the Banach space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. In the case $\mathcal{H}=\mathcal{K}$, the linear space $\mathcal{B}(\mathcal{H},\mathcal{K})$ is simply denoted $\mathcal{B}(\mathcal{H})$ and is a Banach algebra under the usual operator norm. We write $\langle .,.\rangle$ to denote the inner product on $\mathcal{H}$ and for $h\in\mathcal{H}$, the norm of h is given by $\|h\|=\sqrt{<h,h>}$. For $T\in\mathcal{B}(\mathcal{H})$, we denote by $\mathcal{R}(T)$, $\ker(T)$ and T^* the range of T, the null space of T and the adjoint operator of T respectively. The polar decomposition of $T\in\mathcal{B}(\mathcal{H})$ is given by $T=U|T|$, where $|T|=\sqrt{T^*T}$ and U is a partial isometry satisfying $\ker U=\ker T$ and $\ker U^*=\ker T^*$. The Aluthge transform of T, defined by the expression $\hat{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$.

In the sequel, we will focus on the spherical generalized Aluthge transform of $d-$tuples of operators on $\mathcal{H}$. We are essentially concerned with generalisations of numerical range operator inequalities for single operators. Our goal is to provide appropriate extensions in the setting of $d-$tuples of numerical range inequalities of an operator and its Aluthge transform satisfied by single operators.

We start with some definitions and notations. A $d-$tuple $\mathbf{T}=(T_1,...,T_d)\in\mathcal{B}(\mathcal{H},\mathcal{H}^d)$ of bounded operators on a Hilbert space $\mathcal{H}$ is said to be a commuting $d-$tuple of operators if $T_nT_m=T_mT_n$ for every $1\le n,m\le d$. The joint norm $\|\mathbf{T}\|$ of $\mathbf{T}$, the spectral radius $r(\mathbf{T})$ of $\mathbf{T}$, the joint numerical range $W(\mathbf{T})$ of $\mathbf{T}$, and the numerical radius $w(\mathbf{T})$ of $\mathbf{T}$, are defined as follows

$$\begin{aligned}
\|\mathbf{T}\| &= \sup\left\{\left(\sum_{k=1}^{d}\|T_kx\|^2\right)^{\frac{1}{2}} : x\in\mathcal{H}, \|x\|=1\right\};\\
r(\mathbf{T}) &= \sup\{\|\lambda\| : \lambda=(\lambda_1,...,\lambda_d)\in\sigma_T(\mathbf{T})\};\\
W(\mathbf{T}) &= \{(\langle T_1x,x\rangle,...,\langle T_dx,x\rangle) : x\in\mathcal{H}, \|x\|=1\};\\
w(\mathbf{T}) &= \sup\{\|\lambda\| : \lambda=(\lambda_1,...,\lambda_d)\in W(\mathbf{T})\}.
\end{aligned}$$

Where $\sigma_T(\mathbf{T})$ denotes the usual Taylor joint spectrum of $\mathbf{T}$, defined for commuting $d-$tuples. For a detailed study on joint spectral theory of commuting $d-$tuples, we refer to [103].

It is well known that the numerical radius defines an equivalent operator norm on $\mathcal{B}(\mathcal{H},\mathcal{H}^d)$

for every $d \geq 1$ and that for $\mathbf{T} = (T_1, ..., T_d) \in \mathcal{B}(\mathcal{H}, \mathcal{H}^d)$, we have

$$\frac{1}{2\sqrt{d}}\|\mathbf{T}\| \leq w(\mathbf{T}) \leq \|\mathbf{T}\|.$$

See [77] for example.

As extension of the canonical polar decomposition of a single operator, the spherical polar decomposition of a $d-$tuple $\mathbf{T}$ of operators on $\mathcal{H}$ is given by the expression

$$\mathbf{T} = (T_1, ..., T_d) = (V_1, ..., V_d)P = (V_1P, ..., V_dP),$$

where $P = \sqrt{\mathbf{T}^*.\mathbf{T}} = \sqrt{T_1^*T_1 + ... + T_d^*T_d}$ and $\mathbf{V} = (V_1, ..., V_d)$ is a joint partial isometry on $\mathcal{H}$ subject to the condition $\ker \mathbf{V} = \bigcap_{i=1}^d \ker V_i = \bigcap_{i=1}^d \ker T_i = \ker P$. In particular, $\mathbf{V}^*.\mathbf{V} = \sum_{k=1}^{d} V_k^*V_k$ is the orthogonal projection onto the initial space $(\ker \mathbf{V})^\perp = \mathcal{R}(P)$. Analogously, for $0 \leq s \leq 1$, the generalized spherical Aluthge transform $\widehat{\mathbf{T}}^{\mathbf{s}}$ of $\mathbf{T}$ is defined by

$$\widehat{\mathbf{T}}^s = (P^sV_1P^{1-s}, ..., P^sV_dP^{1-s}) = P^s(V_1, ..., V_d)P^{1-s}.$$

When $s = \frac{1}{2}$, we have $\widehat{\mathbf{T}}^{\frac{1}{2}} = \widehat{\mathbf{T}} = (\sqrt{P}V_1\sqrt{P}, ..., \sqrt{P}V_d\sqrt{P})$ is the classical spherical Aluthge transform of $\mathbf{T}$. Also under the convention $P^0 = I$, we get $\widehat{\mathbf{T}}^0 = \mathbf{T}$ and $\widehat{\mathbf{T}}^1 = \widehat{\mathbf{T}}^D = P\mathbf{V} = (PV_1, ..., PV_d)$ is the usual spherical Duggal transform of $\mathbf{T}$.

Our investigations hereafter are mostly devoted to the numerical range of generalized spherical Aluthge transforms. We focus primarily on generalizations of several known operator inequalities for single operators which have been obtained recently for spherical Aluthge transforms to the case of generalized spherical Aluthge transforms of $d-$tuples.

This Chapter is organized as follows. In Section 4.1, we prove inequalities involving the joint operator norm and the joint numerical radius in the case of the generalized spherical Aluthge transform. Mainly, we extend two recent results of K. Feki and T. Yamazaki [35, Theorem 2.1 and Theorem 2.2] by showing that for an arbitrary $d-$tuple $\mathbf{T}$ and $0 \leq s \leq 1$, we have

$$\|\widehat{\mathbf{T}}^s\| \leq \|\mathbf{T}\| \quad \text{and} \quad w(\widehat{\mathbf{T}}^s + \widehat{\mathbf{T}}^{1-s}) \leq w(\mathbf{T}) + w(\mathbf{T}^D) \leq 2w(\mathbf{T}).$$

In addition, we provide a simple proof of the recent result, [3, Theorem 2.6] that improves [35, Theorem 3.1]. More precisely, if $\mathbf{T} = (T_1, ..., T_d) \in \mathcal{B}(\mathcal{H})^d$ and $0 < s < 1$, then

$$w(\mathbf{T}) \leq \inf_{0 \leq s \leq 1}\left(\frac{1}{4}(\|\mathbf{T}\|^{2s} + \|\mathbf{T}\|^{2(1-s)}) + \frac{1}{2}w(\widehat{\mathbf{T}}^s)\right).$$

We use the previous extension to obtain further results extending known ones as for Corollary 4.1, Corollary 4.2 and Proposition 4.5 below. As an application of our calculations, we retrieve some recent results from [35, 107] by using alternative proofs. Section 4.2 is devoted to the link between the existence of nontrivial joint invariant subspaces for the generalized Aluthge transform and for the original $d-$tuple of bounded operators.

4.1 Numerical range and generalized spherical Aluthge transforms

In a proof relying on classical operator inqualities, K. Feki and T. Yamazaki show in [35, Theorem 2.1], that for an arbitrary $d-$tuple of operators $\mathbf{T} = (T_1, ..., T_d)$, the next inequality holds

$$\|\widehat{\mathbf{T}}\| \leq \|\mathbf{T}\|. \tag{4.1}$$

We extend (4.1) to generalized spherical Aluthge transforms, using a short proof.

Proposition 4.1

Let $\mathbf{T} = (T_1, ..., T_d)$ be a $d-$tuple of operators. Then, for every $0 \leq s \leq 1$, we have

$$\|\widehat{\mathbf{T}}^s\| \leq \|\mathbf{T}\|.$$

Proof . We have $\|\mathbf{V}\| = \sup\limits_{\|x\|=1} (\sum\limits_{k=1}^{d} \|V_k x\|^2)^{\frac{1}{2}} \leq 1$ and since P is selfadjoint, we obtain $\|P^a\| = \|P\|^a = \|\mathbf{T}\|^a$ for every $a > 0$. Thus

$$\|\widehat{\mathbf{T}}^s\| = \|P^s \mathbf{V} P^{1-s}\| \leq \|P^s\| \|\mathbf{V}\| \|P^{1-s}\| \leq \|P^s\| \|P^{1-s}\| = \|\mathbf{T}\|.$$

We also mention the next inequality stated in [35, Theorem 2.2]. For $\mathbf{T} \in \mathcal{B}(\mathcal{H})^d$, we have

$$w(\widehat{\mathbf{T}}) \leq \frac{1}{2}(w(\mathbf{T}) + w(\mathbf{T}^D)). \tag{4.2}$$

We have the next extension of (4.2).

Theorem 4.1

Let $\mathbf{T} = (T_1, ..., T_d) \in \mathcal{B}(\mathcal{H})^d$ be a $d-$tuple of operators and $0 \leq s \leq 1$. Then

$$w(\widehat{\mathbf{T}}^s + \widehat{\mathbf{T}}^{1-s}) \leq w(\mathbf{T}) + w(\mathbf{T}^D).$$

The following observation from [108] is to be used in our proof. For $T \in \mathcal{B}(\mathcal{H})$, we have

$$w(T) = \sup_{\theta \in \mathbb{R}} \|\Re(e^{i\theta} T)\|, \quad \text{where } \Re(T) = \frac{T + T^*}{2}. \tag{4.3}$$

Also, we will use two known results.

Theorem 4.2 ([97, Theorem 4])

Let $T \in \mathcal{B}(\mathcal{H})$. Then

$$\overline{W(T)} = \bigcap_{\mu \in \mathbb{C}} \{z : |z - \mu| \leq \|T - \mu I\|\},$$

where $\overline{W(T)}$ means the closure of the numerical range of T.

Lemma 4.1 ([37, Theorem 3.12.1]) *Let S_1 and S_2 be positive operators and $Q \in \mathcal{B}(\mathcal{H})$. For every $0 \leq \alpha \leq 1$, we have*

$$\|S_1^\alpha Q S_2^{1-\alpha} + S_1^{1-\alpha} Q S_2^\alpha\| \leq \|S_1 Q + Q S_2\|.$$

Proof of Theorem 4.1. We notice first that since $P \geq 0$, we have $P + \epsilon I$ is invertible for every $\epsilon > 0$. Let us then denote $V_{i_\epsilon} = V_i - \mu(P + \epsilon I)^{-1}$ for $i = 1, ..., d$. From Lemma 4.1, we derive

$$\|P^s V_{i_\epsilon} P^{1-s} + P^{1-s} V_{i_\epsilon} P^s\| \leq \|P V_{i\epsilon} + V_{i_\epsilon} P\|.$$

Moreover,

$$P^s V_{i_\epsilon} P^{1-s} = P^s V_i P^{1-s} - \mu P^s (P + \epsilon I)^{-1} P^{1-s} = P^s V_i P^{1-s} - \mu P (P + \epsilon I)^{-1}$$

and similarly

$$P^{1-s} V_{i_\epsilon} P^s = P^{1-s} V_i P^s - \mu P (P + \epsilon I)^{-1}.$$

Now, by using Lebesgue's Dominated Convergence Theorem, we obtain

$$\lim_{\epsilon \to 0} P^s (P + \epsilon I)^{-1} P^{1-s} = \lim_{\epsilon \to 0} P^{1-s} (P + \epsilon I)^{-1} P^s = \lim_{\epsilon \to 0} P (P + \epsilon I)^{-1} = I.$$

This yields

$$\|P^s V_i P^{1-s} + P^{1-s} V_i P^s - 2\mu I\| \leq \|P V_i + V_i P - 2\mu I\|.$$

Therefore, from Theorem 4.2, it follows that

$$\overline{W(P^s V_i P^{1-s} + P^{1-s} V_i P^s)} \subset \overline{W(P V_i + V_i P)} \subset \{\overline{W(P V_i)} + \overline{W(V_i P)}\}.$$

Taking the supremum, we deduce that

$$w(\widehat{\mathbf{T}}^s + \widehat{\mathbf{T}}^{1-s}) \leq w(\mathbf{T}) + w(\mathbf{T}^D).$$

Recall that two $d-$tuples of operators $\mathbf{T} = (T_1, ..., T_d)$ and $\mathbf{S} = (S_1, ..., S_d)$, are said to be criss-cross commuting if, for every $1 \leq i, j, k \leq d$ we have $T_i S_j T_k = T_k S_j T_i$ and $S_i T_j S_k = S_k T_j S_i$. It is not difficult to see that if $\mathbf{T} = (T_1, ..., T_d)$ and $\mathbf{S} = (S_1, ..., S_d)$, are criss-cross commuting, then $\mathbf{ST} = (S_1 T_1, ..., S_d T_d)$ is a commuting $d-$tuple if and only if $\mathbf{TS} = (T_1 S_1, ..., T_d S_d)$ is a commuting $d-$tuple. See [9] for further properties on criss-cross commuting $d-$tuples.

The next proposition written in [11, Lemma 2.6] for commuting pairs $(d = 2)$ is valid for arbitrary commuting $d-$tuples.

Proposition 4.2

Let $\mathbf{T}=(T_1,...,T_d)=(V_1P,\cdots,V_dP)$ be a commuting $d-$tuple of operators on $\mathcal{H}$. We have the following

- $\mathbf{V}=(V_1,\cdots,V_d)$ *and* $\mathbf{P}=(P,\cdots,P)$ *are criss-cross commuting;*
- *For every $0\leq s\leq 1$, the generalized spherical Aluthge transform $\widehat{\mathbf{T}}^s$ is a commuting $d-$tuple.*

In the following, we recover [35, Theorem 2.3] in an alternative way.

Proposition 4.3

Let $\mathbf{T}$ be a $d-$commuting tuple. Then

$$w(\mathbf{T}^D)\leq w(\mathbf{T}).$$

Proof. Since $\sum\limits_{k=1}^{d}(V_k^*V_k)$ is a projection onto $\overline{\mathcal{R}(P)}$, we have

$$\begin{aligned}
|\langle P\mathbf{V}x,x\rangle| &= |\langle\sum_{i=1}^{d}PV_ix,x\rangle| \\
&= |\langle\sum_{k=1}^{d}(\ V_k^*V_k\ )\sum_{i=1}^{d}PV_ix,x\rangle| \\
&= |\sum_{k=1}^{d}\sum_{i=1}^{d}\langle V_kPV_ix,V_kx\rangle| \\
&= |\sum_{k=1}^{d}\sum_{i=1}^{d}\langle V_iPV_kx,V_kx\rangle| \quad (\text{ by proposition 4.2}) \\
&\leq \sum_{k=1}^{d}\|V_k\|^2|\sum_{i=1}^{d}\langle V_iPy_k,y_k\rangle| \quad (\text{ where } y_k=\frac{V_kx}{\|V_k\|}) \\
&\leq w(\mathbf{V}P).
\end{aligned}$$

By taking the supremum over all $x\in\mathcal{H}$ with $\|x\|=1$, we get $w(\mathbf{T}^D)\leq w(\mathbf{T})$ as required. The next extension of [35, Theorem 3.1] to the setting of generalized spherical Aluthge transforms can be found in [3, Theorem 2.6]. We provide a new short proof.

Theorem 4.3

Let $\mathbf{T}=(T_1,...,T_d)\in\mathcal{B}(\mathcal{H})^d$ and $0<s<1$. Then

$$w(\mathbf{T})\leq\inf_{0\leq s\leq 1}(\frac{1}{4}(\|\mathbf{T}\|^{2s}+\|\mathbf{T}\|^{2(1-s)})+\frac{1}{2}w(\widehat{\mathbf{T}}^s)).$$

We start by showing the next auxiliary result, needed in the proof of Theorem 4.3.

Lemma 4.2

Let $\mathbf{T}=(T_1,...,T_d)=(V_1P,...,V_dP)\in\mathcal{B}(\mathcal{H})^d$*. For every* $\theta\in\mathbb{R}$*, we have*

$$\Re(\,e^{i\theta}V_iP\,)\leq\frac{1}{4}(e^{-i\theta}P^s+V_iP^{1-s})(e^{i\theta}P^s+P^{1-s}V_i^*).$$

Proof. For $i=1,\dots,d$ and $\theta\in\mathbb{R}$

$$\begin{aligned}4\Re(e^{i\theta}V_iP)-(e^{-i\theta}P^s+V_iP^{1-s})(e^{i\theta}P^s+P^{1-s}V_i^*) &= -P^{2s}+e^{-i\theta}PV_i^*+e^{i\theta}V_iP-V_iP^{2(1-s)}V_i^*\\ &= -(e^{-i\theta}P^s-V_iP^{1-s})(e^{i\theta}P^s-P^{1-s}V_i^*\,)\\ &\leq 0,\end{aligned}$$

as desired.

Proof of Theorem 4.3.

$$\begin{aligned}\|\Re(e^{i\theta}\mathbf{T})\| &\leq \frac{1}{4}\|\sum_{i=1}^{d}(e^{-i\theta}P^s+V_iP^{1-s})(e^{i\theta}P^s+P^{1-s}V_i^*)\| \quad (\text{ Lemma 4.2 })\\ &= \frac{1}{4}\|\sum_{i=1}^{d}(e^{i\theta}P^s+P^{1-s}V_i^*)(e^{-i\theta}P^s+V_iP^{1-s})\| \quad (\|X^*X\|=\|XX^*\|)\\ &= \frac{1}{4}\|\sum_{i=1}^{d}P^{2s}+P^{1-s}V_i^*V_iP^{1-s}+2\Re(e^{i\theta}P^sV_iP^{1-s})\|\\ &\leq \frac{1}{4}\|P^{2s}\|+\frac{1}{4}\|P^{2(1-s)}\|\|\sum_{i=1}^{d}V_i^*V_i\|+\frac{1}{2}\|\sum_{i=1}^{d}\Re(e^{i\theta}P^sV_iP^{1-s})\|\\ &= \frac{1}{4}\|T\|^{2s}+\frac{1}{4}\|T\|^{2(1-s)}\|\sum_{i=1}^{d}V_i^*V_i\|+\frac{1}{2}\|\sum_{i=1}^{d}\Re(e^{i\theta}P^sV_iP^{1-s})\|\\ &\leq \frac{1}{4}\|T\|^{2s}+\frac{1}{4}\|T\|^{2(1-s)}+\frac{1}{2}w(P^sV_iP^{1-s}).\end{aligned}$$

We conclude by taking the supremum over all $\theta\in\mathbb{R}$, and by using Expression 4.3 in the above inequality.

Remark. In Theorem 4.3, we denote $\phi(s)=\frac{1}{4}(\|\mathbf{T}\|^{2s}+\|\mathbf{T}\|^{2(1-s)})+\frac{1}{2}w(\widehat{\mathbf{T}}^s)$. Is it true that $\phi(\frac{1}{2})=\inf_{s\in[0,1]}\phi(s)$?

Using the same proof as in the single case provided in [108], it is not difficult to see that $w(\widehat{\mathbf{T}})\leq\|\mathbf{T}^2\|^{\frac{1}{2}}$. We derive the next consequence of Theorem 4.3.

Corollary 4.1

Let $\mathbf{T}=(T_1,...,T_d)$ *be a commuting* d*-tuple of operators on* $\mathcal{H}$*. For* $s=\frac{1}{2}$*, we have*

1) $w(\widehat{\mathbf{T}})\leq\ w(\mathbf{T})\ \leq\frac{1}{2}(\|\mathbf{T}\|+w(\widehat{\mathbf{T}}))$ *[[35], Theorem 3.1]*

$\leq\frac{1}{2}(\|\mathbf{T}\|+\|\mathbf{T}^2\|^{\frac{1}{2}})$ *[[60], Theorem 1]*

$\leq\|\mathbf{T}\|.$

2) $If\ \widehat{\mathbf{T}} = 0\ or\ \mathbf{T}^2 = 0,\ then\ \ w(\mathbf{T}) = \frac{1}{2}\|\mathbf{T}\|.$

Recall that an operator $T \in \mathcal{H}$ is said to be a normaloid if $\|T\| = r(T)$. It is known that T is a normaloid if and only if $\|T\| = w(T)$. For further results on normaloid operators, see [37] for example. The following corollary we deal with normaloid Aluthge transforms of commuting $d-$tuple of operators.

Corollary 4.2

Let $\mathbf{T} = (T_1, ..., T_d)$ *be a commuting* $d-$*tuple of operators. We have*

- $\mathbf{T}$ *is a normaloid,* $\iff$ $\widehat{\mathbf{T}}$ *is normaloid.*
- *If* $\mathbf{T}$ *is a normaloid, then* $w(\mathbf{T}) = w(\widehat{\mathbf{T}})$.

F. Kittanneh established in [41] that the numerical range of a bounded operator satisfies $w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|$. In [30] S. S. Dragomir proved that $w^2(T) \leq \frac{1}{2}(w(T^2) + \|T\|^2)$. Also the following sharper inequality was obtained recently by S. Bag, P. Bhunia and K. Paul in [15].

Proposition 4.4 ([15, Theorem 2.5])

Let $T \in \mathcal{B}(\mathcal{H})$. *Then*

$$w^2(T) \leq \frac{1}{2}\|T\|\|\hat{T}\| + \frac{1}{4}\|T^*T + TT^*\| \leq \|T\|^2.$$

Our next result generalizes Proposition 4.1 for $d-$tuples.

Proposition 4.5

Let $\mathbf{T} = (T_1, ..., T_d) = (V_1P, ..., V_dP) \in \mathcal{B}(\mathcal{H})^d$. *Then*

$$w^2(\mathbf{T}) \leq \frac{1}{2}w(\mathbf{T}^2) + \frac{1}{4}\|\mathbf{T}^*\mathbf{T} + \mathbf{T}\mathbf{T}^*\| \leq \frac{1}{2}\|\mathbf{T}\|\|\widehat{\mathbf{T}}^s\| + \frac{1}{4}\|\mathbf{T}^*\mathbf{T} + \mathbf{T}\mathbf{T}^*\| \leq \|\mathbf{T}\|^2. \quad (4.4)$$

Proof. By writing $\Re(e^{i\theta}\mathbf{T}) = \frac{1}{2}(e^{i\theta}\mathbf{T} + e^{-i\theta}\mathbf{T}^*)$, it follows that

$$\begin{aligned} 4\|(\Re(e^{i\theta}\mathbf{T}))^2\| &= \|\sum_{k=1}^{d} e^{2i\theta}T_k^2 + e^{-2i\theta}T_k^{*2} + T_k^*T_k + T_kT_k^*\| \\ &\leq \|2\Re(e^{i2\theta}\mathbf{T}^2)\| + \|\mathbf{T}^*\mathbf{T} + \mathbf{T}\mathbf{T}^*\| \\ &\leq 2w(\mathbf{T}^2) + \|\mathbf{T}^*\mathbf{T} + \mathbf{T}\mathbf{T}^*\|. \end{aligned}$$

Now, by taking the supremum over all $\theta \in \mathbb{R}$, and then using Expression (4.3), we obtain the first inequality.

For the second inequality, we have

$$\begin{aligned} w(\mathbf{T}^2)^2 = \sup_{\|x\|\leq 1} |<\mathbf{T}^2x, x>|^2 &= \sup_{\|x\|\leq 1} \sum_{k=1}^{d} |<V_kP^{1-s}P^sV_kP^{1-s}P^sx, x>|^2 \\ &= \sup_{\|x\|\leq 1} \sum_{k=1}^{d} |<P^sV_kP^{1-s}P^sx, P^{1-s}V_k^*x>^2| \\ &\leq \sup_{\|x\|\leq 1} \sum_{k=1}^{d} (\|\widehat{\mathbf{T}}^s\|\|P^s\|\|P^{1-s}\|)^2\|V_k^*x\|^2 \\ &\leq \|\widehat{\mathbf{T}}^s\|^2\|\mathbf{T}\|^2 \text{ (since } \sum_{k=1}^{d}\|V_k^*x\|^2 \leq 1). \end{aligned}$$

The last inequality is obvious and the proof is complete.

In [3, Theorem 3.1], it is shown that $w^2(\mathbf{T}) \leq \frac{1}{2}\|\mathbf{T}^*\mathbf{T} + \|\mathbf{T}\|^2 I\| \leq \|\mathbf{T}\|^2$. Noticing that $\mathbf{T}^*\mathbf{T} + \mathbf{T}\mathbf{T}^* \leq \mathbf{T}^*\mathbf{T} + \|\mathbf{T}\|^2 I$, we can recover the previous inequality from the next result.

Proposition 4.6

Let $\mathbf{T} = (T_1, ..., T_d) = (V_1P, ..., V_dP) \in \mathcal{B}(\mathcal{H})^d$. *Then*

$$w^2(\mathbf{T}) \leq \frac{1}{2}\|\mathbf{T}^*\mathbf{T} + \mathbf{T}\mathbf{T}^*\| \leq \|\mathbf{T}\|^2. \tag{4.5}$$

Proof. Thanks to Proposition 4.5, it suffices to see that $w(\mathbf{T}^2) \leq \frac{1}{2}\|\mathbf{T}^*\mathbf{T} + \mathbf{T}\mathbf{T}^*\|$. Indeed, using polarisation formula for the inner product on $x \in \mathcal{H}$, we obtain

$$<\mathbf{T}^2x, x>= \sum_i <T_ix, T_i^*x>= \sum_i \frac{1}{4}[\|(T_i + T_i^*)x\|^2 - \|(T_i - T_i^*)x\|^2],$$

for every $x \in \mathcal{H}$. In addition

$$\begin{aligned} \|(T_i + T_i^*)x\|^2 - \|(T_i - T_i^*)x\|^2 &=< ((T_i + T_i^*)^2 - (T_i - T_i^*)^2)x, x> \\ &= 2 < (T_iT_i^* + T_i^*T_i)x, x> . \end{aligned}$$

It follows that

$$<\mathbf{T}^2x, x>= \frac{1}{2} < (T_iT_i^* + T_i^*T_i)x, x>,$$

and the required result derives by taking the supremum for $\|x\| = 1$.

4.2 Joint invariant subspace

In [51], the authors prove that an operator $T \in \mathcal{B}(\mathcal{H})$ has a nontrivial invariant subspace if and only if $\hat{T}$ does. We extend in this section this result to the more general setting of commuting $d-$tuples of bounded operators and for generalized Aluthge transforms. Recall first the next definition.

Definition 4.1

Let $\mathbf{T} = (T_1, ..., T_d)$ *be a* $d-$*tuple of operators on a Hilbert space* $\mathcal{H}$*. A closed subspace* M *in* $\mathcal{H}$ *is said to be a joint invariant subspace of* $\mathbf{T}$ *if* $T_i(M) \subset M$ *for every* $1 \le i \le d$*. We denote by* $JLat(\mathbf{T})$ *the lattice of all joint invariant subspaces of* $\mathbf{T}$*.*

We have the following extension of [51].

Proposition 4.7

Let $\mathbf{T} = (T_1, ..., T_d)$ *be a commuting d-tuple of operators and* $0 \le s \le 1$*. Then*

$$JLat(\mathbf{T}) \neq \{0, \mathcal{H}\} \iff JLat(\widehat{\mathbf{T}}^s) \neq \{0, \mathcal{H}\},$$

where 0 *in the trivial null space.*

Proof. ($\Rightarrow$) Let $\mathcal{M} \in JLat(\widehat{\mathbf{T}}^s)$ be nontrivial and denote $\mathcal{N} = span_j V_j P^{1-s}(\mathcal{M})$. First, if $\mathcal{N} = 0$, then $\mathcal{M} \in JLat(\mathbf{T})$. Otherwise, we will show that $\mathcal{N} \in JLat(\mathbf{T})$. From $P^s V_j P^{1-s}\mathcal{M} \subset \mathcal{M}$ for every $j = 1, \cdots, d$, we derive that

$$T_i\mathcal{N} = V_i P\mathcal{N} = V_i P span_j(V_j P^{1-s}\mathcal{M}) = V_i P^{1-s} span_j(P^s V_j P^{1-s}\mathcal{M}) \subset V_i P^{1-s}\mathcal{M} \subset \mathcal{N},$$

and then $\mathcal{N} \in JLat(\mathbf{T})$.

($\Leftarrow$) Let $\mathcal{M} \in JLat(\mathbf{T})$ be nontrivial and denote $\mathcal{N} = P^s(\mathcal{M})$. Again, if $\mathcal{N} = 0$, then $\mathcal{M} \in JLat(\widehat{\mathbf{T}}^s)$, and if $\mathcal{N} \neq 0$, we will get $\mathcal{N} \in JLat(\widehat{\mathbf{T}}^s)$. Indeed, for $j = 1, \cdots, d$, we have

$$P^s V_j P^{1-s}(\mathcal{N}) = P^s V_j P\mathcal{M} \subset P^s\mathcal{M} = \mathcal{N}.$$

Finally $JLat(\mathbf{T}) \neq \{0, \mathcal{H}\} \iff JLat(\widehat{\mathbf{T}}^s) \neq \{0, \mathcal{H}\}$.

From the proof of the previous result, if $\ker(\mathbf{T}) =: \cap_{i \le d} \ker(T_i) = 0$, then the mappings

$$\begin{aligned} \phi : \quad & \mathcal{M} \in JLat(\mathbf{T}) \to \mathcal{N} = P^s(\mathcal{M}) \in JLat(\widehat{\mathbf{T}}^s), \ and \\ \psi : \quad & \mathcal{N} \in JLat(\widehat{\mathbf{T}}^s) \to \mathcal{M} = span_j V_j P^{1-s}(\mathcal{N}) \in JLat(\mathbf{T}) \end{aligned}$$

are one to one. We are however not able to show that $JLat(\mathbf{T})$ and $JLat(\widehat{\mathbf{T}}^s)$ are isomorphic in general. It is also known that even in the single case, that there are operators T such that $Lat(T)$ and $Lat(\widehat{T})$ are not isomorphic. It is then natural to ask.

Let $\mathbf{T}$ be a commuting $d-$tuple. Is there any relation between $JLat(\mathbf{T})$ and $JLat(\widehat{\mathbf{T}}^s)$? Our second question concerns the lattice of joint hyper invariant subspaces. More precisely, M is said to be hyper invariant subspaces for T if it is invariant under all operators commuting with T. The lattice of hyper invariant subspaces of T is denoted $HLat(T)$. We also denote $JHLat(\mathbf{T})$ for the lattice of joint hyper invariant subspaces of $\mathbf{T}$. In contrast with the lattice of invariant subspaces, even for a single operator, it is not true that $HLat(T) \neq \{0, \mathcal{H}\} \iff HLat(\widehat{T}^s) \neq \{0, \mathcal{H}\}$ as shown in [51, Example 1.7]. In the other direction if T is a quasi-affine transformation (T is one to one and has closed range), then $HLat(T) \neq \{0, \mathcal{H}\} \iff HLat(\widehat{T}^s) \neq \{0, \mathcal{H}\}$. See [51, Example 1.7 Theorem 2.5]. The next question arises naturally.

Let $\mathbf{T}$ be a $d-$tuple of quasi-affine transformations, do we have

$$JHLat(\mathbf{T}) \neq \{0, \mathcal{H}\} \iff JHLat(\widehat{\mathbf{T}}^s) \neq \{0, \mathcal{H}\}?$$

The answer is clearly affirmative if $\mathbf{T}$ is a commuting tuples of quasi-affine transformations since in this case we have $JHLat(\mathbf{T}) = HLat(T_i)$ for every $1 \leq i \leq d$.

4.3 Additional remarks and comments

We end this section with a recent extension of generalized Aluthge transform that fits with the previous context. To be precise, M. Bakherad and K. Shebrawi introduced in [91] the $(f, g)-$Aluthge transform as $\Delta_{f,g}(T) = f(|T|)Ug(|T|)$ for $T \in \mathcal{B}(\mathcal{H})$, and where f and g are both non-negative functions such that $f(x)g(x) = x$ for all $x \geq 0$. It is clear that for $A = g(|T|)$ and $B = f(|T|)U$, we have $AB = T$ and $BA = \Delta_{f,g}(T)$. Thus the $AB - BA$ approach applies to find common spectral properties in this setting. Several interesting numerical range operator inequalities involving convex functions have been obtained in [91]. The notion of $(f, g)-$spherical Aluthge transform of $\mathbf{T} = (T_1, ..., T_d) = (V_1P, \cdots, V_dP) \in \mathcal{B}(\mathcal{H})^d$ has been defined in [109], by

$$\Delta_{f,g}(\mathbf{T}) = f(P)(V_1, \cdots, V_d)g(P).$$

The authors in [109], extend various results on generalized Aluthge transform to the promising class consisting in $(f, g)-$spherical Aluthge transforms. This transformation is very large and has wide range of application, it will be useful to develop adequate techniques to tackle more problems on related inequalities.

Chapter 5

Wandering subspace property and Beurling-type theorem for doubly commuting $n-$tuples

Let $\mathcal{H}, \mathcal{K}$ denote two Hilbert spaces, and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ be the Banach space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. In the case $\mathcal{H} = \mathcal{K}$, the linear space $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is simply denoted $\mathcal{B}(\mathcal{H})$ and is a Banach algebra under the usual operator norm. For $T \in \mathcal{B}(\mathcal{H})$, we write $\mathcal{R}(T)$, $\ker(T)$, and T^* for the range of T, the null space of T and the adjoint operator of T respectively. Furthermore, it is common to express $H_\infty(T) = \bigcap\limits_{k=0}^{\infty} T^k(\mathcal{H})$ for the generalized range of T. We will say that an operatopr T is analytic if $H_\infty(T) = \{0\}$. Here $T^0 = I_{\mathcal{H}}$, stands for the identity operator on $\mathcal{H}$. The operator $T \in \mathcal{B}(\mathcal{H})$ is bounded below if there is $c > 0$ satisfying $\|x\| \leq c\|Tx\|$ for every $x \in \mathcal{H}$. It is clear that T is bounded below, if and only if T is left invertible, which is also equivalent to T is one to one with closed range. In the particular case $c \geq 1$ the operator T said to be expansive. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is an invariant subspace for T if $T(\mathcal{M}) \subseteq \mathcal{M}$; and $\mathcal{M}$ is a reducing subspace for T if it is invariant for both T and T^*. For any given subspace E on $\mathcal{H}$, we set $[E]_T = \bigvee\limits_{k\geq 0} T^k(E)$ for the smallest closed $T-$invariant subspace of $\mathcal{H}$ containing E.

The main purpose of this Chapter to investigate the link between each of the wandering subspace property, Beurling-type theorem and Wold-type decomposition of a doubly commuting $n-$tuple and of its operator coordinates. We target to unify various known results in this topic concerning wandering subspace property, Beurling-type theorem and Wold-type decomposition of a doubly commuting $n-$tuple. In particular, we aim to extend Sarkar's work on Wold-type decomposition for $n-$tuples of doubly commuting isometries to the more general class of $n-$tuples of doubly commuting operators.

This Chapter is organized as follows. Section 5.3 is devoted to wandering subspace property for doubly commuting $n-$tuples. It is shown that a doubly commuting n-tuples satisfies wandering subspace property if and only if all its operator coordinates satisfy wandering subspace property. Section 5.4 is devoted to Beurling-type theorem for doubly commuting $n-$tuples. We treat the problem of wandering subspace property on doubly commuting invariant subspaces. We show that if a commuting $n-$tuple satisfies Beurling-type theorem, then the coordinates admit the wandering subspace property on doubly commuting joint invariant subspaces. In section 5.5, following the ideas of Sarkar [20], we extend Wold-type decomposition of doubly commuting $n-$tuples of isometries to the class of doubly commuting $n-$tuples of expansive operators. We show in particular, that a doubly commuting $n-$tuples possesses Wold-type decomposition if and only it the coordinates does. We end this section with some application to doubly commuting $n-$tuples of isometries satisfying wandering subspace property.

5.1 Wandering subspace property and Beurling-type theorem

The notion of wandering subspaces for an arbitrary operator was introduced by Halmos in [42], where its connections with invariant subspaces of the unilateral and bilateral shifts was investigated. More precisely

Definition 5.1

Let T be a bounded operator on $\mathcal{H}$. A closed subspace $\mathcal{W}$ is called wandering subspace (for T) if

$$\mathcal{W} \perp T^k(\mathcal{W}), \text{ for every } k \geq 1.$$

A wandering subspace $\mathcal{W}$ is said to be generating if

$$\mathcal{H} = [\mathcal{W}]_T.$$

For any operator T with non dense range, $\ker(T^*)$ is a wandering subspace. Indeed, $\ker(T^*) \perp \mathcal{R}(T)$ and $T^k(\ker(T^*)) \subset \mathcal{R}(T)$, for every $k \geq 1$. On the other hands, it is clear from the definition, that if $\mathcal{W}$ is a generating wandering subspace for T, then $\mathcal{W} \perp \mathcal{R}(T)$ and hence $\mathcal{W} \subset \ker(T^*)$. In particular any operator with dense range has no generating wandering subspace.

An operator T with a generating wandering subspace, is said to possess the wandering subspace property. That is $\mathcal{H} = [\mathcal{W}]_T$ for some wandering subspace $\mathcal{W}$. It is easy to check that an operator T with wandering subspace property has a unique generating wandering subspace. In fact, $\mathcal{W} = \ker(T^*) = [\mathcal{H} \ominus T\mathcal{H}]$ and hence necessarily, $\mathcal{H} = [\mathcal{H} \ominus T\mathcal{H}]_T$.

It is worth noticing that the restriction of an operator possessing the wandering subspace property does not need to share this property. See [85] for example. However, we have next probably well known result that will be useful in the sequel. We include its proof for completeness.

Lemma 5.1

Let $T \in \mathcal{B}(\mathcal{H})$ possessing the wandering subspace property and let E be a reducing subspace for T. Then $T_{|E}$ owns wandering subspace property and

$$E = [E \ominus T(E)]_T.$$

Proof. First, for E a reducing subspace for T, we have $(T_{|E})^* = T^*_{|E}$ and then

$$\ker(T_{|E})^* = \ker(T^*_{|E}) = \ker(T^*) \cap E.$$

Suppose that T possesses the wandering subspace property. For every $x \in \mathcal{H}$, we write $x = \sum_{n\in\mathbb{N}} T^n x_n$, with $(x_n)_{n\in\mathbb{N}}$ is sequence of elements in $\ker(T^*)$. From the orthogonal decomposition $\mathcal{H} = E \oplus E^\perp$ in reducing subspaces, we have $x_n = y_n + z_n$, for every $n \geq 0$, with $y_n \in E$ and $z_n \in E^\perp$. We deduce in particular that $0 = T^* x_n = T^* y_n + T^* z_n \in E \oplus E^\perp$, and hence $T^* y_n = T^* z_n \in E \cap E^\perp = \{0\}$. It follows that $y_n \in \ker(T_{|E})^*$ and $z_n \in \ker(T_{|E^\perp})^*$. Now for every $x \in E$, from

$$x = \sum_{n\in\mathbb{N}} T^n y_n + \sum_{n\in\mathbb{N}} T^n z_n \in E \oplus E^\perp,$$

we obtain

$$x - \sum_{n\in\mathbb{N}} T^n y_n = \sum_{n\in\mathbb{N}} T^n z_n \in E \cap E^\perp = \{0\}.$$

Thus $T_{|E}$ has the wandering subspace property with

$$\mathcal{W}_E = \mathcal{W} \cap E = (\mathcal{H} \ominus T(\mathcal{H})) \cap E = E \ominus T_{|E}(E) = E \ominus T(E).$$

Another important property in this context that attracted deep attention is the Beurling-type theorem for T. An operator T admits Beurling-type theorem if its the restriction to any closed $T-$invariant subspace has wandering subspace property. Equivalently, if $\mathcal{M} = [\mathcal{M} \ominus T\mathcal{M}]_T$ for every closed $T-$invariant subspace $\mathcal{M}$. The previous definition was motivated by the celebrated Beurling-type theorem for M_z the usual shift operator on the Hardy space of analytic functions on the unit disc. It states that, $\mathcal{M} = [\mathcal{M} \ominus z\mathcal{M}]_{M_z}$ for every closed M_z-invariant subspace $\mathcal{M}$.

In the same manner as for wandering subspace property in Lemma 5.1, we have the next result on Beurling-type theorem for restrictions.

Lemma 5.2

Let $T \in \mathcal{B}(\mathcal{H})$ satisfying Beurling-type theorem and let E be an invariant subspace for T. Then $T_{|E}$ satisfies Beurling-type theorem.

In the two recent decades, the study of the class of operators satisfying Beurling-type theorem has experienced growing interest. We mention in particular [2, 14, 83, 100] including the case of the shift operator on Hilbert space of analytic functions. For arbitrary bounded linear operators on abstract Hilbert spaces, most contributions focused on left invertible operators satisfying some operator inequalities.

We assemble next some of the most significant contributions in the last decades on left invertible operators.

Proposition 5.1

Let $T \in \mathcal{L}(\mathcal{H})$ be an analytic left invertible operator such that one of the following inequalities is satisfied for all $x, y \in \mathcal{H}$,

- *[83, S. Richter. 1988]* $\|T^2x\|^2 + \|x\|^2 \leq 2\|Tx\|^2$;
- *[93, S. Shimorin. 2001]* $\|Tx + y\|^2 \leq 2(\|x\|^2 + \|Ty\|^2)$;
- *[74, A. Olofsson. 2005]* $\exists\, c_p, c > 0$ *with* $\sum_{p\geq 2} \frac{1}{c_p} = \infty$ *such that*

$$\|T^p x\|^2 \leq c_p(\|Tx\|^2 - \|x\|^2) + c\|x\|^2;$$

- *[50, K. Izuchi et all. 2010]* T *is a contraction such that,*

$$\|Tx\|^2 + \|T^{*2}Tx\|^2 \leq 2\|T^*Tx\|^2 \text{ and } \lim_{k\to\infty} \|T^{*k}x\| = 0.$$

Then T satisfies Beurling-type theorem.

A more recent development in this problem has been initiated by the last author and all. It concerns operators with closed range, where the Moore Penrose inverse replaces the left inverse, See [31, 33, 34] and [32] for further information.

5.2 Wold-type decomposition

An operator V is an isometry if $V^*V = I_{\mathcal{H}}$ and is unitary if $VV^* = V^*V = I_{\mathcal{H}}$. That is both V and V^* are isometric. A completely non isometric operator (resp. completely non unitary) is such that, there is no reducing subspace $\mathcal{M}$ such that $V_{|\mathcal{M}}$ is isometric (resp. $V_{|\mathcal{M}}$ is unitary).

The classical Wold decomposition theorem states that if V is an isometry on a Hilbert space $\mathcal{H}$, then $\mathcal{H}$ is the direct sum of two reducing subspaces for V,

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s,$$

such that $V_{|\mathcal{H}_u} \in \mathcal{L}(\mathcal{H}_u)$ is unitary and $V_{|\mathcal{H}_s} \in \mathcal{L}(\mathcal{H}_s)$ is unitarily equivalent to a unilateral shift.

This decomposition is unique and the canonical subspaces are defined by

$$\mathcal{H}_u := \bigcap_{n=1}^{\infty} V^n\mathcal{H} \quad \text{and} \quad \mathcal{H}_s := \bigoplus_{n=0}^{\infty} V^n\mathcal{W},$$

where $\mathcal{W} = \ker(V^*) = \mathcal{H} \ominus V\mathcal{H}$. It follows immediately from Wold-type decomposition, that an isometry has the wandering subspace property if and only if it has no unitary part ($\mathcal{H}_u = \{0\}$) and equivalenty if V is analytic. Moreover, since the restriction of an isometry to any invariant subspace is also isometric, it is immediate that, an anlytic isometry satisfies Beurling-type theorem.

Because of the wide range of applications for Wold-type decomposition of an isometry, the existence of a similar decomposition for more general classes of operators became central in many recent papers. The next formulation of Wold-type decomposition for arbitrary contractions is considered to be the dominant definition by several authors. It can be found in [75], for example.

Definition 5.2

A contraction $T \in \mathcal{L}(\mathcal{H})$ is said to have a Wold-type decomposition provided that $\mathcal{H}$ admits an orthogonal decomposition $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{cnu}$ in two reducing subspaces such that $T_u = T_{|\mathcal{H}_u}$ is unitary and $T_{cnu} = T_{|\mathcal{H}_{cnu}}$is a completely non unitary contraction satisfying $\lim_{n\to\infty} T^n_{cnu}x = 0$ for every $x \in \mathcal{H}$.

The main purpose of investigations on Wold-type decomposition is to relax the condition T is isometric to weaker assumptions in order to obtain the decomposition given in Definition 5.2. Some deep results are obtained for operators close to isomtries, that is when the operator T is assumed to satisfy operator inequalities as in Proposition 5.1. See [50, 74, 83, 93] and [33] for example.

An easy observation gives the next lemma for Wold-type decomposition on restrictions to reducing subspaces, in the line of Lemma 5.1 and Lemma 5.2. The proof is similar and is omitted.

Lemma 5.3

If $T \in \mathcal{B}(\mathcal{H})$ admits Wold-type decomposition and E is a reducing subspace for T. Then $T_{|E}$ admits Wold-type decomposition.

5.3 Wandering subspace property for doubly commuting $n-$tuples

Let $\mathbf{T} = (T_1, ..., T_n) \in \mathcal{B}(\mathcal{H}, \mathcal{H}^{\mathbf{n}})$ be an $n-$tuple of operators. The $n-$tuple $\mathbf{T}$ is commuting if $T_iT_j = T_jT_i$, for every i, j and $\mathbf{T}$ is said to be doubly commuting if moreover $T_i^*T_j = T_jT_i^*$, for every $i \neq j$.

Through this Chapter, we will use the next notation. For $n \geq 2$, we write $I_n = \{1, 2, ..., n\}$, $\mathbf{k} = (k_1, ..., k_n) \in \mathbb{N}^n$, $\Lambda = \{\lambda_1, \cdots, \lambda_p\} \subset I_n$ and $\Lambda^* = I_n \setminus \Lambda$, the complement set of Λ in I_n. We pose $\mathbf{T}_\Lambda = (T_{\lambda_1}, ..., T_{\lambda_i}) \in \mathcal{B}(\mathcal{H}, \mathcal{H}^{\mathbf{i}})$ and $\mathbf{T}_\Lambda^{\mathbf{k}} = T_{\lambda_1}^{k_1} ... T_{\lambda_i}^{k_i} \in \mathcal{B}(\mathcal{H})$. Also, for $i \in I_n$, we will write $i^* = \{i\}^* = I_n \setminus \{i\}$ and then $\mathbf{T}_{i^*} = (T_1, \cdots, T_{i-1}, T_{i+1}, \cdots, T_n) \in \mathcal{B}(\mathcal{H}, \mathcal{H}^{\mathbf{n-1}})$.

In the remainder of this Chapter, $\mathbf{T} = (T_1, ..., T_n)$ is a commuting $n-$tuple. A closed subspace $\mathcal{M}$ in $\mathcal{H}$ is said to be joint $\mathbf{T}-$invariant (resp. joint $\mathbf{T}-$reducing) if $\mathcal{M}$ is invariant (resp. reducing) under T_i for every $i = 1, ..., n$. As before, a closed subspace $\mathcal{W}$ in $\mathcal{H}$ is a wandering subspace for the $n-$tuple $\mathbf{T}$ if

$$\mathcal{W} \perp \mathbf{T}^{\mathbf{k}}(\mathcal{W}), \text{ for every } \mathbf{k} \in \mathbb{N}^n \backslash \{(0, ..., 0)\},$$

and is a generating wandering subspace for the $n-$tuple $\mathbf{T}$ if

$$\mathcal{H} = \bigvee_{\mathbf{k} \in \mathbb{N}^n} \mathbf{T}^{\mathbf{k}} \mathcal{W}.$$

Clearly, a generating wandering subspace for a commuting $n-$tuple $\mathbf{T}$ is unique when exists, and

$$\mathcal{W} = \mathcal{H} \ominus \mathbf{T}(\mathcal{H}) = \mathcal{H} \ominus (\bigvee_{i \in I_n} T_i(\mathcal{H})) = \bigcap_{i \in I_n} (\mathcal{H} \ominus T_i\mathcal{H}).$$

Under the notation, $\mathcal{W}_i = \mathcal{H} \ominus T_i\mathcal{H} = \ker(T_i^*)$, it will come that

$$\mathcal{W} = \mathcal{H} \ominus \mathbf{T}\mathcal{H} = \bigcap_{i \in I_n} \mathcal{W}_i = \bigcap_{i \in I_n} \ker(T_i^*) = \ker(\mathbf{T}^*).$$

As extension of the single case, we will say that

- A commuting $n-$tuple $\mathbf{T}$ has the wandering subspace property if $\mathbf{T}$ admits a generating wandering subspace $\mathcal{W}$. In which case $\mathcal{W} = \bigcap_{i \in I_n} \mathcal{W}_i$.
- $\mathbf{T}$ satisfies Beurling-type theorem, if the restriction of $\mathbf{T}$ to any (joint) invariant subspace has the wandering subspace property.

Combining Lemma 5.1 and [20, Theorem 2.3], we obtain the following improved version of Theorem 2.3 in [20].

Proposition 5.2

Let $\mathbf{T} = (T_1, ..., T_n)$ *be a doubly commuting* $n-$*tuple. For an integer* $p \leq n$ *and* $\Lambda = \{\lambda_1, \cdots, \lambda_p\} \subset I_n$*. If* T_i *admits the wandering subspace property for every* $i \in I_n$*, then* $\mathbf{T}_\Lambda$ *admits the wandering subspace property, for every* $\Lambda \subset I_n$ *and*

$$\mathcal{H} \ominus \mathbf{T}_\Lambda \mathcal{H} = \bigvee_{\mathbf{k} \in \mathbb{N}^{n-p}} \mathbf{T}^{\mathbf{k}}_{\Lambda^*}(\mathcal{W}).$$

Proof. It suffices to see that the subspace $\mathcal{H} \ominus \mathbf{T}_\Lambda \mathcal{H}$ is $\mathbf{T}_{\Lambda^*}-$reducing and that

$$(\mathcal{H} \ominus \mathbf{T}_\Lambda \mathcal{H}) \ominus \mathbf{T}_{\Lambda^*}(\mathcal{H} \ominus \mathbf{T}_\Lambda \mathcal{H}) = \mathcal{H} \ominus \mathbf{T}\mathcal{H}.$$

We derive the next result.

Theorem 5.1

Let $\mathbf{T} = (T_1, ..., T_n)$ *be a doubly commuting* $n-$*tuple. The following are equivalent*

1. $\mathbf{T}$ *admits the wandering subspace property;*
2. T_i *admits the wandering subspace property, for every* $i \in I_n$*;*
3. $\mathbf{T}_\Lambda$ *admits the wandering subspace property, for every* $\Lambda \subset I_n$ *;*
4. *There exist* $\Lambda \subset I_n$*; such that* $\mathbf{T}_\Lambda$ *admits the wandering subspace property and* $\mathcal{H} \ominus \mathbf{T}_\Lambda \mathcal{H} = \bigvee_{\boldsymbol{k} \in \mathbb{N}^{n-p}} \mathbf{T}^{\mathbf{k}}_{\Lambda^*}(\mathcal{W})$.

Proof. $(1) \Rightarrow (2)$. Suppose $\mathbf{T}$ admits the wandering subspace property and write $\mathcal{H} = \bigvee_{\mathbf{k} \in \mathbb{N}^n} \mathbf{T}^{\mathbf{k}} \mathcal{W}$. For $i \in I_n$, denote $\mathcal{V}_{i^*} = \bigvee_{\mathbf{k} \in \mathbb{N}^{n-1}} \mathbf{T}^{\mathbf{k}}_{i^*}(\mathcal{W})$. Since $\mathbf{T}$ is a doubly commuting $n-$tuple, we have

$$T_i^* \mathcal{V}_{i^*} = \bigvee_{\mathbf{k} \in \mathbb{N}^{n-1}} T_i^* \mathbf{T}^{\mathbf{k}}_{i^*}(\mathcal{W}) = \bigvee_{\mathbf{k} \in \mathbb{N}^{n-1}} \mathbf{T}^{\mathbf{k}}_{i^*} T_i^*(\mathcal{W}) \subset \{0\}.$$

It follows that $\mathcal{V}_{i^*} \subset \ker(T_i^*)$, and hence that $\mathcal{V}_{i^*} \perp T_i^n(\mathcal{V}_{i^*})$ for every $n \geq 1$. On the other hand, since

$$\bigvee_{n \in \mathbb{N}} T_i^n \mathcal{V}_{i^*} = \bigvee_{n \in \mathbb{N}} \bigvee_{\mathbf{k} \in \mathbb{N}^{n-1}} T_i^n \mathbf{T}^{\mathbf{k}}_{i^*}(\mathcal{W}) = \bigvee_{\mathbf{l} \in \mathbb{N}^n} \mathbf{T}^{\mathbf{l}} \mathcal{W} = \mathcal{H}.$$

we obtain, T_i admits the wandering subspace property.

$(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ are immediate from Proposition 5.2.

$(4) \Rightarrow (1)$. We have $\mathcal{W} = \cap_{j \in I_n} \mathcal{W}_j$ and $\mathcal{W} \perp \mathbf{T}^{\mathbf{k}}(\mathcal{W})$ for every nonzero $\mathbf{k} \in \mathbb{N}^n$. Also, for $\Lambda \subset I_n$ such that (3) holds, we have

$$\mathcal{H} = \bigvee_{\mathbf{i} \in \mathbb{N}^p} T_\Lambda^{\mathbf{i}} \mathcal{V}_\Lambda = \bigvee_{\mathbf{i} \in \mathbb{N}^p} \bigvee_{\mathbf{k} \in \mathbb{N}^{n-p}} T_\Lambda^{\mathbf{i}} \mathbf{T}^{\mathbf{k}}_{\Lambda^*}(\mathcal{W}) = \bigvee_{\mathbf{l} \in \mathbb{N}^n} \mathbf{T}^{\mathbf{l}} \mathcal{W}.$$

Remark. Notice that in Theorem 5.1, when $\mathbf{T}$ admits the wandering subspace property for some doubly commuting $n-$tuple, then T_i admits the wandering subspace property and the identity $\ker(T_i^*) = \mathcal{V}_{i^*} = \bigvee_{\mathbf{k}\in\mathbb{N}^{n-1}} \mathbf{T}_{i^*}^{\mathbf{k}}(\mathcal{W})$ is automatically full filled for every $i \in I_n$. For the converse, it suffice to require that $\mathcal{V}_{i^*} = \bigvee_{\mathbf{k}\in\mathbb{N}^{n-1}} \mathbf{T}_{i^*}^{\mathbf{k}}(\mathcal{W})$ is a generating wandering subspace of T_i, for some $i \in I_n$.

We derive an extended version of Theorem 1 in [96].

Corollary 5.1

Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting $n-$tuple on $\mathcal{H}$. The following are equivalent

1. *$\mathbf{T}$ admits the wandering subspace property;*
2. *T_i admits the wandering subspace property and $\mathcal{H} \ominus T_i\mathcal{H} = \bigvee_{\mathbf{k}\in\mathbb{N}^{n-1}} \mathbf{T}_{i^*}^{\mathbf{k}}(\mathcal{W})$, for every $i \in I_n$;*
3. *There exists $i \in I_n$ such that T_i admits the wandering subspace property and $\mathcal{H} \ominus T_i\mathcal{H} = \bigvee_{\mathbf{k}\in\mathbb{N}^{n-1}} \mathbf{T}_{i^*}^{\mathbf{k}}(\mathcal{W})$.*

Remark. It is natural to ask, as in the case of a single operator or as in previous section, if we can obtain similar results for Beurling-type theorem. More precisely, for a given doubly commuting $n-$tuple $\mathbf{T} = (T_1, ..., T_n)$, is it true that $\mathbf{T}$ satisfies Beurling-type theorem if and only if T_i satisfies Beurling-type theorem for every $i \in I_n$. This is definitively not true as stated in the next theorem for doubly commuting pairs of shifts.

Theorem 5.2 ([49, Theorem 17])

There exists a nontrivial function $f \in H^2(\mathbb{D}^2)$ such that $[f]_{M_{(z_1,z_2)}} \ominus (z_1[f]_{M_{(z_1,z_2)}} + z_2[f]_{M_{(z_1,z_2)}})$ does not generate $[f]_{M_{(z_1,z_2)}}$.

The main obstacle to obtaining $[f]_{M_{(z_1,z_2)}} \ominus (z_1[f]_{M_{(z_1,z_2)}} + z_2[f]_{M_{(z_1,z_2)}})$ generates $[f]_{M_{(z_1,z_2)}}$ in the previous result is the fact that, the restriction of the doubly commuting pair $M_{(z_1,z_2)}$ to $[f]_{M_{(z_1,z_2)}}$ is not necessarily doubly commuting. This motivates the next well known definition

Definition 5.3

A joint invariant subspace E of a doubly commuting $n-$tuple $\boldsymbol{T} = (T_1, \cdots, T_n)$ is said to be doubly commuting if $\boldsymbol{T}_{|E} = (T_{1|E}, \cdots, T_{n|E})$ is a doubly commuting $n-$tuples.

Taking into consideration the previous remark, we give the following extension of Proposition 5.1.

Theorem 5.3

Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting $n-$tuple of analytic left invertible operators such that each T_i satisfies one of the following inequalities for all $x, y \in \mathcal{H}$:

1. $\|T_i^2 x\|^2 + \|x\|^2 \leq 2\|T_i x\|^2$;
2. $\|T_i x + y\|^2 \leq 2(\|x\|^2 + \|T_i y\|^2)$;
3. *There exists some positive constants c_p, c with $\sum_{p\geq 2} \frac{1}{c_p} = \infty$ such that:*
$$\|T_i^p x\|^2 \leq c_p(\|T_i x\|^2 - \|x\|^2) + c\|x\|^2;$$
4. *T_i is a contraction such that,*
$$\|T_i x\|^2 + \|T_i^{*2} T_i x\|^2 \leq 2\|T_i^* T_i x\|^2 \text{ and } \lim_{k\to\infty} \|T_i^{*k} x\| = 0.$$

Then for every doubly commuting invariant subspace M, we have $\mathbf{T}_{|M}$ has the wandering subspace property.

Proof. The assumptions in the previous theorem imply that, for every $\mathbf{T}-$invariant subspace, $T_{i|M}$ satisfies the wandering subspace property for every $i \in I_n$ and Theorem 5.1 allows to conclude.

The next lemma sheds further light on the impact of wandering space property on double commutativity of commuting $n-$tuples. We have the next technical lemma of independent interest.

Lemma 5.4

Let S, T be commuting operators. The following are equivalent

1. $S^*T = TS^*$.
2. *T commutes with S^*S and $\ker(S^*)$ is T-reducing.*
3. *S commutes with T^*T and $\ker(T^*)$ is S-reducing.*

Proof. $(1) \Rightarrow (2)$ is trivial. For $(2) \Rightarrow (1)$, since $\ker(S^*)$ reduces T, we obtain $S^*Tx = TS^*x = 0$ for every $x \in \ker(S^*)$. From the orthogonal decomposition $\mathcal{H} = \ker(S^*) \oplus \overline{R(S)}$, it suffices to show that $TS^*Sx = S^*TSx$, for every $x \in \mathcal{H}$. Indeed

$$TS^*Sx = S^*STx = S^*TSx.$$

$(1) \iff (3)$ holds by passing to adjoint.

As an immediate consequence, we have the next useful result

Proposition 5.3

Let $\mathbf{T} = (T_1, ..., T_n)$ be a commuting $n-$tuple of operators. The following are equivalent

1. *$\mathbf{T}$ is doubly commuting;*
2. *T_i commutes with $T_j^*T_j$ and $\mathcal{W}_i$ is T_j-reducing for all $1 \leq i \neq j \leq n$;*
3. *T_i commutes with $T_j^*T_j$ and $\mathcal{W}_i$ is T_j-reducing for all $1 \leq i < j \leq n$.*

For $\Lambda \subset I_n$, we denote by $\mathcal{W}_\Lambda = \bigcap_{i\in\Lambda} \mathcal{W}_i$, where by convention $\mathcal{W}_\emptyset = \mathcal{H}$. Clearly, if $(T_1, ..., T_n)$ is a doubly commuting $n-$tuple, we have $\mathcal{W}_\Lambda$ reduces T_j for every $j \in \Lambda^* = I_n \setminus \Lambda$. Moreover, the following expression, to be used in the sequel, holds

$$\mathcal{W}_\Lambda \ominus T_j\mathcal{W}_\Lambda = \mathcal{W}_\Lambda \cap (T_j\mathcal{W}_\Lambda)^\perp = \mathcal{W}_\Lambda \cap \mathcal{W}_j. \tag{5.1}$$

We give in the next corollary an immediate extension of [80, Theorem 3].

Corollary 5.2

Let $\mathbf{T} = (T_1, ..., T_n)$ be a commuting $n-$tuple of left invertible operators on $\mathcal{H}$, such that each of T_i with $i \in I_n$ satisfies Olofsson's, Richter's or Shimorin's inequality, then

1. *$\mathbf{T}$ is doubly commuting, and*
2. *T_i is analytic for every $i \in I_n$,*

if and only if

(a) *$\mathbf{T}$ has the wandering subspace property,*

(b) *T_j commutes with $T_i^*T_i$ for every $i < j$, and*

(c) *$[\mathcal{W}]_{\mathbf{T}_{i^*}} \subset \mathcal{W}_i$.*

Proof. Under any of the previously mentioned inequalities and the assumption (2) in Corollary 5.2, we derive that T_i admits wandering subspace property for every $i \in I_n$. If in addition $\mathbf{T}$ is doubly commuting, we use Corollary 5.1 to show that (a) and (c) holds and Lemma 5.3 to show (b) is valid. The reverse implication runs in the same manner.

The case of the shift operator on weighted Hilbert spaces. Let $\omega = (\omega_\mathbf{k})_{\mathbf{k}\in\mathbb{N}^n}$ be a sequence of nonzero real numbers and let l^2_ω be the Hilbert space of formal series in n variables given by:

$$l^2_\omega = \{f(\mathbf{z}) = \sum_{\mathbf{k}} a_\mathbf{k}\mathbf{z}^\mathbf{k} : \sum_{\mathbf{k}} |a_\mathbf{k}|^2\omega_\mathbf{k}^2 < \infty\}.$$

Here for $\mathbf{z} = (z_1, \cdots, z_n)$ and $\mathbf{k} = (k_1, \cdots, k_n)$, we pose $\mathbf{z}^\mathbf{k} = \prod_{i\in I_n} z_i^{k_i}$. A complete survey of such spaces and their shift operator is provided by A. Shields in the exhaustive survey [92], in one variable setting.

Assume that for every $i \in I_n$, the shift $M_{z_i}(f) = z_i f$ is bounded on l^2_ω. The multi-variable shift is the $n-$tuple given by

$$\mathbf{M_z} = (M_{z_1}, \cdots, M_{z_n}).$$

Since $M_{z_i}M_{z_j}\mathbf{z^k} = M_{z_j}M_{z_i}\mathbf{z^k} = z_iz_j\mathbf{z^k}$, for every $\mathbf{k} \in \mathbb{N}^n$, we derive that $\mathbf{M_z}$ is a commuting $n-$tuple. Moreover, for every $i \in I_n$, and $\mathbf{k} = (k_1, \cdots, k_n)$ we have

$$M_{z_i}^*(\mathbf{z^k}) = \begin{cases} \frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}-\epsilon_\mathbf{i}}}\mathbf{z}^{\mathbf{k}-\epsilon_\mathbf{i}} & if \quad k_i \neq 0, \\ 0 & if \quad k_i = 0. \end{cases}$$

where ϵ_i is the $n-$tuple with 1 in the i^{th} entry and zeros elsewhere.
It follows that

$$(M_{z_j}M_{z_i}^* - M_{z_i}^*M_{z_j})(\mathbf{z^k}) = \begin{cases} (\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}-\epsilon_\mathbf{i}}} - \frac{\omega_{\mathbf{k}+\epsilon_\mathbf{j}}}{\omega_{\mathbf{k}-\epsilon_\mathbf{i}+\epsilon_\mathbf{j}}})(\mathbf{z}^{\mathbf{k}-\epsilon_\mathbf{i}+\epsilon_\mathbf{j}}) & if \quad k_i \neq 0 \\ 0 & if \quad k_i = 0 \end{cases}$$

and hence $\mathbf{M_z}$ is doubly commuting, if and only if$f$

$$\omega_{\mathbf{k}}\omega_{\mathbf{k}+\epsilon_j-\epsilon_i} = \omega_{\mathbf{k}-\epsilon_i}\omega_{\mathbf{k}+\epsilon_j} \quad \text{for every} \;\; i \neq j.$$

Remark. A large family of doubly commuting multi-variable shifts is provided by multi-indexed weights given by $\omega_{\mathbf{k}} = \prod\limits_{i\in I_n} \beta_{k_i}$ for $\mathbf{k} = (k_1, \cdots, k_n)$, where $(\beta_k)_k$ is a scalar sequence of real numbers. In particular, $\mathbf{M_z}$ is doubly commuting on the Hardy space, the Bergman space and the Dirichlet space of the polydisc. Now, using Theorem 5.1, it follows that th $n-$tuple $\mathbf{M_z}$ has wandering subspace property on these spaces.

Taking in count Theorem 5.3, if a doubly commuting operator $\mathbf{M_z}$ is such that every $i \in I_n$ one of the conditions is satisfied,

- $\omega_{\mathbf{k}+2\epsilon_i} + \omega_{\mathbf{k}}^2 - 2\omega_{\mathbf{k}+\epsilon_i}^2 \leq 0,$
- There exists some positive constants c_p, c with $\sum_{p\geq 2}\frac{1}{c_p} = \infty$ such that: $\omega_{\mathbf{k}+p\epsilon_i}^2 \leq c_p(\omega_{\mathbf{k}+\epsilon_i}^2 - \omega_{\mathbf{k}}^2) + c\omega_{\mathbf{k}}^2,$
- $\omega_{\mathbf{k}+\epsilon_i}^2 + \frac{\omega_{\mathbf{k}+\epsilon_i}}{\omega_{\mathbf{k}-\epsilon_i}} \leq 2\frac{\omega_{\mathbf{k}+\epsilon_i}}{\omega_{\mathbf{k}}},$

then $\mathbf{M_z}$ has the wandering subspace property.

5.4 Generating wandering subspaces for invariant subspaces of Hilbert spaces of analytic functions over the polydisc

We devote this section to the wandering subspace properties for invariant subspaces in Hilbert spaces of analytic functions over the polydisc containing the ring of all polynomials. Denote $\mathbb{D}^n = \{\mathbf{z} = (z_1, \cdots, z_n) : z_1, \cdots, z_n \in \mathbb{D}\}$ for the polydisc and let $\mathcal{O}(\mathbb{D}^n)$ be the

Frechet algebra of all analytic functions over $\mathbb{D}^n$. A Hilbert space $\mathcal{H}(\mathbb{D}^n)$ of analytic functions over $\mathbb{D}^n$ is a subspace of $\mathcal{O}(\mathbb{D}^n)$ containing all polynomials and such that

$$M_{z_i} : f(z_1, \cdots, z_n) \to z_i f(z_1, \cdots, z_n),$$

is bounded for every $i \in I_n$. The $n-$shift is the commuting $n-$tuple given by, $\mathbf{M_z} = (M_{z_1}, \cdots, M_{z_n})$. If S is a joint invariant subspace of $\mathbf{M_z}$, it is clear that $\mathbf{M_{z}}_{|S} = (M_{z_1|S}, \cdots, M_{z_n|S})$ is a commuting $n-$tuple but may fail to be doubly commuting. See [70], where a characterisation is provided for invariant subspaces S in the Hardy space over the bidisc such that $\mathbf{M_z}_{|S}$ is doubly commuting. This motivates the next definition from [20, 70].

The validity of the wandering subspace property for doubly commuting invariant subspace is stated as follows

Theorem 5.4

Let E be a doubly commuting invariant subspace of a doubly commuting $n-$tuple $\mathbf{T} = (T_1, ..., T_n)$ satisfying Beurling-type theorem , then $T_{i|E}$ satisfies the wandering subspace property for every $i \in I_n$.

Proof. We have $\mathbf{T}_{|E}$ satisfies the wandering subspace property. We use Theorem 5.1 to conclude.

We deduce the following corollary.

Corollary 5.3 ([20, Theorem 3.1, Theorem 3.2] and [80, Theorem 5])

Suppose E is a is doubly commuting closed joint $\mathbf{M_z}$-invariant subspace of $\mathcal{H}(\mathbb{D}^n)$, where $\mathcal{H}(\mathbb{D}^n)$ is the Hardy space, the Bergman space or the Dirichlet space over the polydisc. Then $(\mathbf{M_z}_{|E})_\Lambda$ has the wandering subspace property for any non-empty subset $\Lambda \subset I_n$.

Proof. Theorem 5.4 allows to conclude, since the Hardy space, the Dirichlet and the Bergman space all satisfy Beurling-type theorem.

The next result extends Corollary 5.3, since all weights associated with the Hardy space, the Dirichlet and the Bergman space are covered by Remark 5.3.

Corollary 5.4

Let $\mathcal{H}(\mathbb{D}^n) = l^2_\omega$ be a Hilbert space of analytic function, where $\omega_\mathbf{k} = \prod\limits_{i \in I_n} \beta_{k_i}$ for $\mathbf{k} = (k_1, \cdots, k_n)$ and where $(\beta_k)_k$ is a scalar sequence of real numbers. If E is a is doubly commuting closed joint $\mathbf{M_z}$-invariant subspace of $\mathcal{H}(\mathbb{D}^n)$. Then $(\mathbf{M_z}_{|E})_\Lambda$ has the wandering subspace property for any non-empty subset $\Lambda \subset I_n$.

Remark. It is easy to see that additional results can be extended by using Theorem 5.4 as [88, Theorem 2.3] in the complex field case. It is also clear that one may include various results on operator satisfying suitable inequalities of left invertible operator close to isometries.

We have the following natural question motivated by Corollary 5.3.

Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting $n-$tuple satisfying Beurling-type theorem, and E be a doubly commuting invariant subspace of $\mathbf{T}$. Is it true that $T_{i|E}$ satisfies Beurling-type theorem on doubly commuting invariant subspace for every $i \in I_n$?

5.5 Wold-type decomposition for doubly commuting $n-$tuples

We provide now an extended version of Theorem 3.1 in [87] on doubly commuting isometries to the more general setting of doubly commuting $n-$tuples of operators with Wold-type decomposition.

Theorem 5.5

Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting $n-$tuple of left invertible operators on a Hilbert space $\mathcal{H}$ such that T_i admits Wold-type decomposition, for every $i \in I_n$. Then there exists 2^n-joint $(T_1, ..., T_n)-$reducing subspaces $\{\mathcal{H}_\Lambda : \Lambda \subseteq I_n\}$ (some of them may be trivial) such that

$$\mathcal{H} = \bigoplus_{\Lambda \subseteq I_n} \mathcal{H}_\Lambda,$$

where

$$\mathcal{H}_\Lambda = \bigvee_{\mathbf{k}\in\mathbb{N}^p} \mathbf{T}_\Lambda^{\mathbf{k}} \left(\bigcap_{m\in\mathbb{N}^{n-p}} T_{\Lambda^*}^{m} \mathcal{W}_\Lambda \right), \tag{5.2}$$

with $\mathcal{H}_\emptyset = \bigcap_{\mathbf{k}\in\mathbb{N}^n} \mathbf{T}^{\mathbf{k}}\mathcal{H} = \mathcal{H}_\infty(\mathbf{T})$ and $\mathcal{H}_{I_n} = \{0\}$.
Furthermore, for each $\Lambda \subseteq I_n$, $T_i|\mathcal{H}_\Lambda$ is unitary if $i \in \Lambda^*$.

Proof. We will use mathematical induction to prove the assertion. We shall first prove our assertion when $n = 2$. Suppose that $\mathbf{T} = (T_1, T_2)$ is a pair of doubly commuting of left invertible operators such that T_1 and T_2 admit Wold-type decomposition. We will have

$$\mathcal{H} = \mathcal{H}_\infty(T_1) \oplus [\mathcal{W}_1]_{T_1}.$$

Since $T_1^*T_2 = T_2T_1^*$, we obtain $\mathcal{H}_\infty(T_1)$ and $[\mathcal{W}_1]_{T_1}$ are reducing subspaces for T_2, and then

$$\mathcal{H}_\infty(T_1) \cap \mathcal{W}_2 = \bigcap_{k\geq 0} T_1^k(\mathcal{W}_2) \text{ and } [\mathcal{W}_1]_{T_1} \cap \mathcal{W}_2 = [\mathcal{W}_1 \cap \mathcal{W}_2]_{T_1}.$$

We use Lemma 5.3 with $T_{2|\mathcal{H}_\infty(T_1)}$ and $T_{2|[\mathcal{W}_1]_{T_1}}$, to derive the following

$$\mathcal{H}_\infty(T_1) = (\mathcal{H}_\infty(T_1))_\infty(T_2) \oplus [\mathcal{H}_\infty(T_1) \cap \mathcal{W}_2]_{T_2} = \mathcal{H}_\infty(\mathbf{T}) \oplus [(\mathcal{W}_2)_\infty(T_1)]_{T_2};$$

and

$$[\mathcal{W}_1]_{T_1} = ([\mathcal{W}_1]_{T_1})_\infty(T_2) \oplus [[\mathcal{W}_1]_{T_1} \cap \mathcal{W}_2]_{T_2} = ([\mathcal{W}_1]_{T_1})_\infty(T_2) \oplus [\mathcal{W}_1 \cap \mathcal{W}_2]_{\mathbf{T}}.$$

This yields

$$\mathcal{H} = \mathcal{H}_\infty(\mathbf{T}) \oplus [(\mathcal{W}_2)_\infty(T_1)]_{T_2} \oplus ([\mathcal{W}_1]_{T_1})_\infty(T_2) \oplus [\mathcal{W}_1 \cap \mathcal{W}_2]_{\mathbf{T}}.$$

The result is then true for $n = 2$, where we have

$\mathcal{H}_{\{\emptyset\}}(\mathbf{T}) = \mathcal{H}_\infty(\mathbf{T}) = \bigcap_{k_1,k_2\in\mathbb{N}} T_1^{k_1}T_2^{k_2}(\mathcal{W}_1 \cap \mathcal{W}_2)$,

$\mathcal{H}_{\{1\}}(\mathbf{T}) = [(\mathcal{W}_1)_\infty(T_2)]_{T_1} = \bigvee_{k_1\in\mathbb{N}} T_1^{k_1}\left(\bigcap_{k_2\in\mathbb{N}} T_2^{k_2}\mathcal{W}_1\right)$,

$\mathcal{H}_{\{2\}}(\mathbf{T}) = [(\mathcal{W}_2)_\infty(T_1)]_{T_2} = \bigvee_{k_2\in\mathbb{N}} T_2^{k_2}\left(\bigcap_{k_1\in\mathbb{N}} T_1^{k_1}\mathcal{W}_2\right)$,

and $\mathcal{H}_{\{1,2\}}(\mathbf{T}) = [\mathcal{W}_1 \cap \mathcal{W}_2]_{\mathbf{T}} = \bigvee_{k_1,k_2\in\mathbb{N}^2} T_1^{k_1}T_2^{k_2}(\mathcal{W}_1 \cap \mathcal{W}_2)$.

Now, let $p < n$ and assume that the statement is true for any doubly commuting $p-$tuple. Clearly any doubly commuting $(p+1)$-tuple $\mathbf{T} = (T_1, ..., T_p, T_{p+1})$ can be regarded as a doubly commuting pair by writing, $\mathbf{T} = (\mathbf{S}, T_{p+1})$, where $\mathbf{S}$ is the doubly commuting p-tuple given by $\mathbf{S} = (T_1, ..., T_p)$, $T_{p+1}\mathbf{S} = (T_{p+1}T_1, ..., T_{p+1}T_p)$ and $\mathbf{S}T_{p+1} = (T_1T_{p+1}, ..., T_pT_{p+1})$.

As in the first step, we write $\mathcal{H} = \mathcal{H}_\infty(T_{p+1}) \oplus [\mathcal{W}_{p+1}]_{T_{p+1}}$, and again since $\mathcal{H}_\infty(T_{p+1})$ and $[\mathcal{W}_{p+1}]_{T_{p+1}}$ are $\mathbf{S}-$ reducing, we use Lemma 5.3 with $\mathbf{S}_{|\mathcal{H}_\infty(T_{p+1})}$ and $\mathbf{S}_{|[\mathcal{W}_{p+1}]_{T_{p+1}}}$ together with the induction hypothesis, to obtain

$$\mathcal{H} = \mathcal{H}_\infty(T_{p+1}) \oplus [\mathcal{W}_{p+1}]_{T_{p+1}} = \left(\bigoplus_{\Lambda\subseteq I_p} (\mathcal{H}_\infty(T_{p+1}))_\Lambda\right) \bigoplus \left(\bigoplus_{\Lambda\subseteq I_p} ([\mathcal{W}_{p+1}]_{T_{p+1}})_\Lambda\right),$$

In addition, for $\Lambda = \{\lambda_1, \cdots, \lambda_q\} \subset I_p$, we have

$$\begin{aligned}(\mathcal{H}_\infty(T_{p+1}))_\Lambda &= \bigvee_{\mathbf{k}\in\mathbb{N}^q} \mathbf{T}_\Lambda^{\mathbf{k}}\left(\bigcap_{k_2\in\mathbb{N}^{p-q}} \mathbf{T}_{\Lambda^*}^{k_2} \bigcap_{m\in\mathbb{N}} T_{p+1}^m \mathcal{W}_\Lambda\right) \\ &= \bigvee_{\mathbf{k}\in\mathbb{N}^q} \mathbf{T}_\Lambda^{\mathbf{k}}\left(\bigcap_{(m,k_2)\in\mathbb{N}^{p+1-q}} \mathbf{T}_{\Lambda^*\cup\{p+1\}}^{(m,k_2)} \mathcal{W}_\Lambda\right),\end{aligned}$$

and

$$
\begin{aligned}
([\mathcal{W}_{p+1}]_{T_{p+1}})_\Lambda &= \left(\bigvee_{\mathbf{k}\in\mathbb{N}^q} \mathbf{T}_\Lambda^{\mathbf{k}} \bigcap_{k_2\in\mathbb{N}^{p-q}} \mathbf{T}_{\Lambda^*}^{k_2} \left([\mathcal{W}_{p+1}]_{T_{p+1}}\right)\right) \\
&= \bigvee_{\mathbf{k}\in\mathbb{N}^q} \mathbf{T}_\Lambda^{\mathbf{k}} \bigcap_{k_2\in\mathbb{N}^{p-q}} \mathbf{T}_{\Lambda^*}^{k_2} \left(\bigvee_{m\in\mathbb{N}} T_{p+1}^m(\mathcal{W}_\Lambda)\right) \\
&= \bigvee_{\mathbf{k}\in\mathbb{N}^q} \mathbf{T}_\Lambda^{\mathbf{k}} \bigvee_{m\in\mathbb{N}} T_{p+1}^m \bigcap_{k_2\in\mathbb{N}^{p-q}} \mathbf{T}_{\Lambda^*}^{k_2} \left(\bigvee_{m\in\mathbb{N}} T_{p+1}^m(\mathcal{W}_\Lambda)\right) \\
&= \bigvee_{(m,\mathbf{k})\in\mathbb{N}^{q+1}} \mathbf{T}_\Lambda^{(m,\mathbf{k})} \left(\bigcap_{k_2\in\mathbb{N}^{p-q}} \mathbf{T}_{\Lambda^*}^{k_2}\left(\mathcal{W}_\Lambda\right)\right)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathcal{H} &= \bigoplus_{\Lambda\subseteq I_{p+1}} \bigvee_{\mathbf{k}\in\mathbb{N}^q} \mathbf{T}_\Lambda^{\mathbf{k}} \left(\bigcap_{(m,k_2)\in\mathbb{N}^{p+1-q}} \mathbf{T}_{\Lambda^*\cup\{p+1\}}^{(m,k_2)} \mathcal{W}_\Lambda \right) \oplus \\
&\quad \bigoplus_{\Lambda\subseteq I_{p+1}} \bigvee_{(m,\mathbf{k})\in\mathbb{N}^{q+1}} \mathbf{T}_\Lambda^{(m,\mathbf{k})} \left(\bigcap_{k_2\in\mathbb{N}^{p+1-q}} \mathbf{T}_{\Lambda^*}^{k_2}\left(\mathcal{W}_\Lambda\right)\right) \\
&= \bigoplus_{\Lambda\subseteq I_{p+1}} \bigvee_{\mathbf{k}\in\mathbb{N}^{p+1}} \mathbf{T}_\Lambda^{\mathbf{k}} \left(\bigcap_{m\in\mathbb{N}^{n-p}} T_{\Lambda^*}^m \mathcal{W}_\Lambda \right).
\end{aligned}
$$

This completes the proof.

Remark. Recall that if S, T are unitarily equivalent and S admits Wold-type decomposition, then so does T. Two commuting $n-$tuples $\mathbf{T} = (T_1, ..., T_n)$ and $\mathbf{S} = (S_1, ..., S_n)$ are unitarily equivalent when there is an unitary transformation U such that $U^*T_iU = S_i$ for every $i \in I_n$. Using the previous theorem, it is not difficult to see that, if $\mathbf{S}$ and $\mathbf{T}$ are unitarily doubly commuting $n-$tuples of left invertible operators, then $\mathbf{T}$ admits Wold-type decomposition if and only if $\mathbf{S}$ admit Wold-type decomposition.

A first application of Theorem 5.5 provides a Wold-type decomposition for doubly commuting $n-$tuples.

Theorem 5.6

Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting $n-$tuple of left invertible operators on a Hilbert space $\mathcal{H}$ such that T_i admits Wold-type decomposition, for every $i \in I_n$. Then $\mathcal{H}$ decomposes in two reducing subspaces $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Where the restriction of $\mathbf{T}$ to $\mathcal{H}_1$ is unitary and its restriction to $\mathcal{H}_2$ has wandering subspace property. In particular $\mathbf{T}$ admits a Von Neumann-Wold decompossition.

Proof. We use the previous theorem to write $\mathcal{H} = \bigoplus_{\Lambda\subseteq I_n} \mathcal{H}_\Lambda$. Now for $\mathcal{H}_1 = \mathcal{H}_\emptyset$ and $\mathcal{H}_2 = \bigoplus_{\emptyset\neq\Lambda\subseteq I_n} \mathcal{H}_\Lambda$, the required result is fulfilled.

In the case if doubly commuting $n-$tuples of isometries, we derive the next simple caracterisation of doubly commuting $n-$tuples of isometries that admits wandering subspace property.

Corollary 5.5

Let $\mathbf{T} = (T_1, ..., T_n)$ *be a doubly commuting* $n-$*tuple of isometric operators. The following are equivalent*

1. $\mathbf{T}$ *admits wandering subspace property,*
2. $\prod\limits_{i \in I_n} T_i$ *admits wandering subspace property.*

Proof. $(1) \Rightarrow (2)$. Suppose that $\mathbf{T}$ admits wandering subspace property and write $\mathcal{H} = \bigoplus_{\mathbf{k} \in \mathbb{N}^n} \mathbf{T}^{\mathbf{k}} \mathcal{W}$. Let $\mathbb{N}^n_{\Pi} = \{(k, k \cdots, k), k \in \mathbb{N}\}$ and $\mathcal{W}_{\Pi} = \bigoplus_{\mathbf{k} \in \mathbb{N}^n \setminus \mathbb{N}^n_{\Pi}} \mathbf{T}^{\mathbf{k}} \mathcal{W}$. We have $T_i^k \mathcal{W}_{\Pi} \perp T_i^l \mathcal{W}_{\Pi}$ for every $k \neq l$ and $\mathcal{H} = \bigoplus_{k \in \mathbb{N}} \prod_{i \in I_n} T_i^k \mathcal{W}_{\Pi}$. Thus $\prod_{i \in I_n} T_i$ satisfies wandering subspace property.

$(2) \Rightarrow (1)$. Derives from [78, Proposition 2.1]

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Résumé

Dans la première partie de cette dissertation, nous étudions le processus de naissance et de mort avec des taux $\lambda_n = q^n(1 - aq^{n+1}), \mu_n = aq^n(1 - q^n)$ et $0 < a, q < 1$. Nous démontrons que les polynômes orthogonaux correspondants généralisent les petits polynômes de q-Laguerre. De plus, nous obtenons quelques formules pour la solution minimale et la convergence de fractions continues associées aux petits polynômes de q-Laguerre orthogonaux.

En outre, nous abordons le problème des moments complexes tronqués à valeurs matricielles. Nous montrons que l'achèvement de dimension finie des données tronquées fournit une condition nécessaire et suffisante, et donc une solution, pour le problème des moments complexes tronqués à valeurs matricielles.

Dans la deuxième partie, nous examinons la classe des n-uplets d'opérateurs bornés qui commutent deux à deux $\mathbf{T} = (\mathrm{T}_{\mathbf{1}}, \dots, \mathrm{T}_{\boldsymbol{n}})$ et ses applications. Nous généralisons et affinons plusieurs inégalités impliquant le rayon numérique conjoint et la norme d'opérateur conjointe de **T**. En outre, nous montrons que **T** satisfait la propriété d'espace errant, le théorème de Beurling ou admet une décomposition de type Wold si et seulement si ses coordonnées le font. Finalement, nous présentons une décomposition explicite de type Wold pour les n-uplets d'opérateurs inversibles à gauche.

Mots-clefs (5) : Problème des moments, processus de naissance et de mort, opérateur n-uplets, décomposition de Wold, opérateut d'Aluthge,

Abstract

In the first part of this dissertation, we study birth and death process with rates $\lambda_n = q^n(1 - aq^{n+1}), \mu_n = aq^n(1 - q^n)$ where $0 < a, q < 1$. We show that the corresponding orthogonal polynomials generalize little q-Laguerre polynomials. We also give the minimal solution of the three-term recurrence relation, and we obtain some formulas for the convergent of continued fractions associated with the little q-Laguerre orthogonal polynomials.

Furthermore, we examine the matrix-valued truncated complex moment problem. We show that finite-dimensional completion of a truncated data provides a necessary and sufficient condition, and hence a solution, for the matrix-valued truncated complex moment problem.

In the second part, we explore the class of n-tuples of bounded operators that pairwise commute, denoted as $\mathbf{T} = (\mathrm{T}_{\mathbf{1}}, \dots, \mathrm{T}_{\boldsymbol{n}})$, and its applications. We generalize and refine several inequalities involving the joint numerical radius and joint operator norm of **T**. Additionally, we demonstrate that **T** satisfies the wandering subspace property, Beurling's theorem, or admits a Wold-type decomposition if and only if it the coordinates does. Finally, we provide an explicit Wold-type decomposition for n-tuples of left-invertible operators

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Keywords (5) : Moment problem, birth and death process, operator n-tuples, Wold decomposition, Aluthge operator.