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Derivations-like maps with applications on path algebras

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DERIVATIONS-LIKE MAPS WITH APPLICATIONS ON PATH ALGEBRAS

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Dédicace

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Résumé

La présente thèse présente nos contributions au domaine des dérivations et les applications associées sur les algèbres de chemin. Dans cette thèse, nous avons étendu, introduit et développé des concepts et techniques intéressants.

Le premier objectif de cette thèse est d'étudier et de caractériser certaines applications liées aux dérivations sur les algèbres de chemin afin de généraliser des résultats existants et d'en introduire de nouveaux dans ce domaine.

Le deuxième objectif est d'introduire de nouvelles perspectives liées aux dérivations pour résoudre une variante d'une conjecture et étudier des propriétés catégorielles.

Mots clés: Dérivations généralisées de Lie, σ-dérivations de Jordan, σ-dérivations de Lie, $\mathcal{G}_n$-dérivations, Dérivations locales, σ-dérivations généralisées, Conjecture de Lvov-Kaplansky, Algèbres de chemin, Algèbres du carquois liées, Carquois.

Summary

The present thesis presents our contributions to the domain of derivations and related maps on path algebras. In this thesis, we have extended, introduced, and developed interesting concepts and techniques.

The first aim of this thesis is to investigate and characterize some related maps to derivations on path algebras to generalize existing results and introduce new ones to this domain.

The second aim is to introduce new perspectives related to derivations to solve a variant of a conjecture and investigate categorical properties.

Keywords: Lie generalized derivations, Jordan σ-derivations, Lie σ-derivations, $\mathcal{G}_n$-derivations, Local derivations, Generalized σ-derivations, Lvov-Kaplansky conjecture, Path algebras, Bound quiver algebras, Quivers.

Articles involved in this thesis

There are five published papers involved in this thesis cited as follows:

1. A. Adrabi, D. Bennis, and B. Fahid: Lie generalized derivations on bound quiver algebras. *Communications in Algebra.* 2021; 49:1950–1965
2. A. Adrabi, D. Bennis, and B. Fahid: Jordan (Lie) σ-derivations on path algebras. *FILOMAT.* 2022; 36:6231–6243
3. A. Adrabi, D. Bennis, and B. Fahid: Jordan $\mathcal{G}_n$-Derivations on Path Algebras. *Communications of the Korean Mathematical Society.* 2022; 37:957–967
4. A. Adrabi, D. Bennis, and B. Fahid: On Local (like) Derivations on Path Algebras. *Acta Mathematica Vietnamica.* 2023; 48:387–399
5. A. Adrabi, D. Bennis, and B. Fahid: Categorical Properties of Generalized σ-Derivations on Modules. *Communications in Algebra.* 2023; 51:1950–1965

Contents

CHAPTER 1

Introduction (Version Française)

If I have seen further, it is by standing on the shoulders of giants.

Issac Newton

Cette thèse présente nos contributions au domaine des dérivations en étudiant certaines sortes des dérivations sur une structure algébrique appelée algèbres de chemin. Précisément, cette thèse traite des généralisations des dérivations sur des algèbres de chemin appelées dérivations généralisées de Lie, σ-dérivations de Jordan, σ-dérivations de Lie, $\mathcal{G}_n$-dérivations de Jordan, dérivations locales et σ-dérivations généralisées. Rappelons qu'une application additive $d : R \to R$ est une dérivation sur un anneau R s'elle satisfait la règle de Leibniz; c'est-à-dire $d(xy) = d(x)y + xd(y)$ pour chaque élément x et y en R. Une dérivation d sur R est dite intérieure s'il existe un élément a en R tel que $d(x) = [a, x]$ pour chaque élément x en R, où $[a, x] = ax - xa$ est appelé le commutateur de a et x.

D'après les articles [12] et [59], l'origine de la notion des dérivations abstraites remonte aux années trente du XIXe siècle avec Teichmüller [132], Hasse [62] et Jacobson [69]. Cependant, il semble difficile d'affirmer avec certitude lequel d'entre eux est le fondateur de ce domaine (voir également Ritt [123], Kaplansky [75] et Kolchin [88]). Dans les années cinquante, la notion des dérivations abstraites a connu un développement fulgurant après les travaux de Kaplansky [76] et Posner [121] ce qui a pousser de nombreux auteurs à caractériser les dérivations dans différents contextes comme Baclawski [15], Benkovič [18], Bennis et Fahid [27] et d'autres [41, 45, 67, 71, 72, 76, 79, 125, 131, 141].

Dans [63], Herstein a introduit la notion des dérivations de Jordan sur un anneau associatif R comme des dérivations sur l'anneau de Jordan de R. C'est-à-dire une application additive $f : R \to R$ qui satisfait $f(x \circ y) = f(x) \circ y + x \circ f(y)$ pour chaque élement x et y en R où $x \circ y = xy + yx$ est appelé l'anticommutateur de x et y. Il a montré que toute dérivation de Jordan sur un anneau premier avec une caractéristique différente de deux est une dérivation. La notion des dérivations de Jordan a été étudiée dans des nombreux contextes par des différents auteurs comme Cusack [44], Brešar [33, 34], Benkovič [19, 20], Lie et Wei [101] et d'autres [6, 7, 46, 52, 53, 55, 81, 95, 102–104, 107, 109, 127, 137, 140, 142].

Dans [112], Martindale III a commencé l'investigation des dérivations de Lie sur un anneau primitif[1] R de caractéristique différente de deux et contenant un idempotent non-trivial. C'est-à-dire une application additive $f : R \to R$ qui satisfait $f([x, y]) = [f(x), y] + [x, f(y)]$ pour chaque élement x et y en R. Il a montré que toute dérivation de Lie sur R est d'une forme standard (ou propre). En d'autres termes, chaque dérivation de Lie $f : R \to R$ est la somme d'une dérivation de R dans un anneau primitif $\overline{R}$ qui

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[1] Un anneau R est dit primitif à gauche s'il possède un simple R-module fidèle à gauche.

contient R et une application additive de R dans le centre[2] de $\overline{R}$ qui s'annule sur les commutateurs de R. Ce qui a pousser d'autres auteurs à étudier la question de savoir quand chaque dérivation de Lie est d'une forme standard. Voir, par exemple, Brešar [35], Cheung [40], Benkovič [21, 23, 25] et d'autres [17, 47, 54, 84, 108, 115, 116, 126, 136, 137, 144].

Dans [120], Ore a introduit la notion des σ-dérivations en équipant l'anneau des polynômes $F[X]$ comme un F-module à droite avec la multiplication $Xa = \overline{a}X + a'$ pour tout élement a en F, où F est un corp non nécessairement commutatif. L'élément $\overline{a}$ est appelé le conjugué de a, a' est appelé la dérivée de a. En équipant $F[X]$ avec la multiplication de polynômes, $F[X]$ devient un anneau non commutatif, appelé l'extension d'Ore. Par conséquent, il s'ensuit que pour chaque élement a et b en F, nous avons $\overline{a+b} = \overline{a}+\overline{b}$, $\overline{ab} = \overline{a}\overline{b}$, $(a+b)' = a' + b'$ et $(ab)' = a'b + \overline{a}b'$. Donc, une σ-dérivation (skew-derivation) d d'un anneau R dans lui-même est une application additive associée à un endomorphisme σ de R dans lui-même qui satisfait les deux dernières égalités. C'est-à-dire une application additive $d : R \to R$ est appelée σ-dérivation si elle satisfait $d(xy) = d(x)y + \sigma(x)d(y)$ pour chaque élement x et y en R. La notion des σ-dérivations peut être obtenue à partir d'une autre construction comme dans Cohn [42]. De nombreux auteurs s'intéressent à l'étude de la notion des σ-dérivations et de ses extensions, nous citons Jacobson [70], Smits [129], Leroy et Matczuk [98], Montgomery et Paul Smith [117], Kharchenko et Popov [78], Bergen et Grzeszczuk [29], Mamouni, Oukhtite et Zerra [111] et d'autres [8, 9, 16, 30, 31, 38, 50, 61, 65, 68, 77, 87, 89, 91, 93, 94, 97, 99, 110, 113, 114, 130, 134, 139, 141, 143].

Étant donné que les algèbres de chemin ont diverses intersections avec de nombreuses structures algébriques bien connues en fonction des carquois qui leur sont associés, cela donne que les algèbres de chemin généralisent et/ou construisent des algèbres matricielles triangulaires supérieures, des algèbres matricielles, des algèbres d'extension à un point, des algèbres du carquois liées, des algèbres de chemin de Leavitt, des algèbres d'incidence, des algèbres associatives libres et autres. Plusieurs auteurs sont intéressés et motivés pour étudier et caractériser les dérivations et les applications liées aux dérivations en utilisant l'aspect combinatoire des algèbres de chemin comme Guo et Li [58], Li et Wei [105], Hou [66], Li et Wei [101], Lopatkin [106]. Par conséquent, la présente thèse s'inscrit dans la même direction de recherche consistant à étudier les dérivations et les applications en relation avec eux sur les algèbres de chemin. Ainsi, cette thèse comporte cinq chapitres qui couvrent cinq articles publiés (i.e. [1–5]).

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[2] Le centre d'un anneau R est le sous-ensemble $\{x : xy = yx, \forall y \in R\}$.

Dans la suite de ce chapitre, K désigne un corps de caractéristique différente de 2 et A une algèbre associative unitaire sur K.

Dans Chapitre 3, nous étudions les dérivations généralisées de Lie sur les algèbres du carquois liées. Ce type des applications linéaires a été introduit par Bennis, Ebrahimi Vishki, Fahid, Khadem-Maboudi et Mokhtari dans [26] comme contrepartie généralisée des dérivations de Lie. Rappelons qu'une application linéaire $f : A \to A$ est appelée d-dérivation généralisée de Lie si elle satisfait:

$$f([x, y]) = [f(x), y] + [x, d(y)], \tag{1.1}$$

pour chaque élement x et y en A, où $d : A \to A$ est une application linéaire. Ainsi, lorsque $f = d$, f n'est rien d'autre que la dérivation de Lie classique. Dans Section 3.1, nous donnons quelques préliminaires sur les algèbres de chemin, les algèbres de carquois liées et les algèbres triangulaires. Dans Section 3.2, nous étudions la propriété des dérivations généralisées de Lie sur les algèbres du carquois liées. Théorème 3.2.2 énonce des assertions équivalentes sur les dérivations généralisées de Lie. Pour fournir une caractérisation complète des dérivations généralisées de Lie sur les algèbres du carquois liée, nous n'avons qu'à nous concentrer sur la caractérisation des dérivations sur les algèbres du carquois liées. Cependant, contrairement au cas des algèbres de chemin [58, Théorème 2.3], la caractérisation des dérivations sur une algèbre du carquois liées $\mathfrak{Q}$ dépend aussi de la forme d'une base de $\mathfrak{Q}$ vue comme un espace vectoriel. L'investigation de cette question nous a conduit à établir une autre caractérisation d'une dérivation sur une algèbre de chemin basée uniquement sur la détermination des images des sommets et des arêtes plutôt que sur la détermination des images de tous les chemins comme cela est fait dans [58, Théorème 2.3] (voir Proposition 3.2.3). Cela aide à déterminer la forme d'une dérivation sur une algèbre du carquois liée (voir Proposition 3.2.4). Dans Corollaire 3.2.5, nous montrons que, lorsque nous ajoutons une condition sur la base d'une algèbre du carquois liée, alors nous pouvons établir une condition équivalente sur la forme de l'application linéaire (voir Proposition 3.2.4) pour qu'elle devienne une dérivation (voir aussi Exemples 3.2.6 et 3.2.7). Dans Section 3.3, nous nous intéressons à la propriété d'unicité de la décomposition en forme propre énoncée dans Théorème 3.2.2. Une étude approfondie de cette question nous amène à étudier la décomposition des dérivations sur une algèbre du carquois liée en une somme directe de dérivations sur des algèbres générées par des composantes connexes (voir Lemme 3.3.3). Théorème 3.3.4 énonce des assertions équivalentes sur l'unicité de la décomposition des dérivations généralisées de Lie.

Dans Chapitre 4, nous étudions les σ-dérivations de Jordan et les σ-dérivations de Lie sur les algèbres de chemin, cette recherche est motivée par les articles de Benkovič [22, 24]. Rappelons qu'une application linéaire $f : A \to A$ est appelée une σ-dérivation de Jordan si elle satisfait:

$$f(x \circ y) = f(x)y + \sigma(x)f(y) + f(y)x + \sigma(y)f(x), \tag{1.2}$$

pour chaque élement x et y en A. Aussi, une application linéaire $f : A \to A$ est appelée une σ-dérivation de Lie si elle satisfait:

$$f([x, y]) = f(x)y + \sigma(x)f(y) - f(y)x - \sigma(y)f(x), \tag{1.3}$$

pour chaque x et y en A. Les σ-dérivations de Jordan et les σ-dérivations de Lie sont respectivement des généralisations des dérivations de Jordan et des dérivations de Lie. Dans les articles [22, 24], Benkovič a supposé la condition de fidélité. Dans le cas des algèbres de chemin, Li et Wei [101] ont montré que la condition de fidélité peut être ignorée lorsque les algèbres de chemin peuvent être considérées comme des extensions à un point. Par conséquent, nous nous inspirons des études [22, 24, 101, 102] pour étudier les σ-dérivations de Jordan et les σ-dérivations de Lie sur les algèbres de chemin. A savoir, on confirme la remarque de Li et Wei [101] et on la prouve sur toute algèbre de chemin associée à un carquois fini et acyclique. Dans Section 4.1, nous inspectons la propriété de fidélité et la propriété loyale sur les algèbres de chemin, et nous énonçons deux résultats liés à ces propriétés sous certaines conditions (voir Théorème 4.1.4 et Proposition 4.1.5). Dans Section 4.2, nous étudions les σ-dérivations de Jordan sur les algèbres de chemin, et nous montrons que toute σ-dérivation de Jordan sur une algèbre de chemin est une σ-dérivation (voir Théorème 4.2.2). Dans Section 4.3, nous étudions les σ-dérivations de Lie sur les algèbres de chemin, et nous énonçons que chaque σ-dérivation de Lie sur une algèbre de chemin est de forme standard (voir Théorème 4.3.1). Notez que, lorsque σ est un automorphisme intérieur sur une algèbre de chemin, le problème de l'étude des σ-dérivations de Jordan et des σ-dérivations de Lie sur les algèbres de chemin se réduit à l'étude des dérivations de Jordan et les dérivations de Lie, respectivement, comme indiqué dans [22, Proposition 2.4] et [24, Proposition 2.3].

Dans Chapitre 5, nous introduisons et étudions la notion des $\mathcal{G}_n$-dérivations de Jordan sur les algèbres de chemin qui généralise la notion des $\{g, h\}$-dérivations de Jordan. Rappelons qu'une application linéaire $f : A \to A$ est appelée une $\{g, h\}$-dérivation de Jordan si elle satisfait:

$$f(x \circ y) = g(x) \circ y + x \circ h(y), \tag{1.4}$$

pour chaque élement x et y en A. Pour $g = f$, une $\{g, h\}$-dérivation de Jordan n'est qu'une dérivation généralisée de Jordan, et pour $g = h = f$, ce n'est rien d'autre que la dérivation classique de Jordan (voir Brešar [36] pour plus de détails). De la discussion au début de [36, Section 2], il s'est avéré que les $\{g, h\}$-dérivations sont en fait un cas particulier de dérivations généralisées (voir [37] pour plus de détails sur les dérivations généralisées). L'objectif principal de [36] est d'étudier quand une $\{g, h\}$-dérivation de Jordan est une $\{g, h\}$-dérivation sur des algèbres tensorielles (voir aussi [90]). Dans Section 5.1, nous définissons la notion des $\mathcal{G}_n$-dérivations de Jordan et nous étudions sa relation avec la notion des $\{g, h\}$-dérivations de Jordan. Dans Section 5.2, Théorème 5.2.3 montre qu'une $\mathcal{G}_2$-dérivation de Jordan sur une algèbre de chemin KQ est une $\mathcal{G}_2$-dérivation si et seulement si $g_1(1) \in Z(KQ)$ ou $g_2(1) \in Z(KQ)$. Théorème 5.2.4 montre que, pour tout $n > 2$, toute $\mathcal{G}_n$-dérivation de Jordan sur KQ est une $\mathcal{G}_n$-dérivation. A savoir, pour chaque $n > 2$, $\mathcal{G}_n$-dérivations de Jordan sur KQ sont $\mathcal{G}_n$-dérivations contrairement au cas $n = 2$, il existe des $\mathcal{G}_2$-dérivations de Jordan qui ne sont pas des $\mathcal{G}_2$-dérivations comme on le verra dans Exemple 5.2.1, qui donne que $\mathcal{G}_n$-derivations de Jordan généralise naturellement $\mathcal{G}_2$-dérivations de Jordan (i.e. $\{g, h\}$-dérivations de Jordan). Dans Section 5.3, nous présentons nos investigations sur une variante de la conjecture de Lvov-Kaplansky. Rappelons la question suivante connue sous le nom de conjecture de Lvov-Kaplansky (voir Filippov, Kharchenko et Shestakov [49]):

Question 1.1. Soit $\zeta(x_1, \ldots, x_n)$ un polynôme multilinéaire sur un corps $\mathbb{F}$. L'ensemble des valeurs de ζ sur l'algèbre matricielle $M_m(\mathbb{F})$ est-il un espace vectoriel?

Le lecteur est renvoyé à [74] pour plus d'informations sur les résultats récents et importants sur ce sujet. Notre enquête est motivée par le travail effectué dans Fagundes [48], Wang [138] et Quispe Urure et Franca [122] sur des algèbres matricielles triangulaires supérieures particulières. En fait, puisque les algèbres matricielles triangulaires supérieures sont des algèbres de chemin associées à des carquois linéaires (voir Birkenmeier, Park et Rizvi [32]), nous allons pousser la question plus loin dans une autre direction et demander:

Question 1.2. Soit $\zeta(x_1, \ldots, x_n)$ un polynôme multilinéaire sur K. L'ensemble des valeurs de ζ sur KQ est-il un espace vectoriel?

Théorème 5.3.1 répond positivement à Question 1.2 et donc il généralise le travail effectué pour les algèbres matricielles triangulaires supérieures. Nous donnons aussi quelques exemples qui appliquent Théorème 5.3.1 sur certains cas particulièrement importants.

Dans Chapitre 6, nous étudions les dérivations locales et les dérivations locale généralisées sur les algèbres de chemin associées aux carquois finis et acycliques. Rappelons qu'une application linéaire $\delta : A \to A$ est appelée une dérivation locale si elle satisfait:

$$\delta(x) = d_x(x), \tag{1.5}$$

pour chaque x élement en A où d_x est une dérivation sur A. Une application linéaire $\Delta : A \to A$ est appelée une dérivation locale généralisée sur A si elle satisfait:

$$\Delta(x) = D_x(x), \tag{1.6}$$

pour chaque élement x en A où D_x est une dérivation généralisée sur A. La notion des dérivations locales a été introduite indépendamment par Kadison dans [73] et par Larson et Sourour dans [92]. À savoir, Kadison dans [73] a initié le problème de montrer quand chaque dérivation locale est une dérivation. Par la suite, il y a eu de nombreuses études sur les dérivations locales et d'autres applications locales comme les dérivations locales de Jordan et les dérivations locales de Lie sur diverses algèbres. Voir par exemple [14, 39, 82, 119, 135, 145] et les références qui s'y trouvent. Puisque les algèbres d'incidence finies associées aux posets et les algèbres matricielles triangulaires supérieures sont des cas particuliers d'algèbres de chemin (voir [13, 128] pour la relation entre les algèbres de chemin et les algèbres matricielles triangulaires supérieures, et Section 6.1 pour la relation entre les algèbres de chemin et les algèbres d'incidence), il est naturel de demander d'étendre l'étude des dérivations locales au contexte des algèbres de chemins. Dans Section 6.1, nous étudions la relation entre les algèbres de chemin et les algèbres d'incidence (finitaire). Puisque tout poset fini $(P, \leq)$ a un carquois acyclique fini qui le représente, il s'ensuit que toute algèbre d'incidence finie associée à un poset est une algèbre de chemin (voir Ancykutty [10] et Spiegel et O'Donnell [131]). Dans la même section, nous donnons un exemple des algèbres de chemin qui ne sont pas des algèbres d'incidence (voir Exemple 6.1.4). Dans Section 6.2, nous montrons que toute dérivation locale sur une algèbre de chemin est une dérivation (voir Théorème 6.2.3), et nous étudions les dérivations locales de Jordan et les dérivations locales de Lie sur les algèbres de chemin. Enfin, dans Section

6.3, nous prouvons que toute dérivation locale généralisée sur une algèbre de chemin est une dérivation généralisée (voir Théorème 6.3.4). Nous terminons Section 6.3 en étudiant les dérivations locales généralisées de Lie sur les algèbres de chemin.

Dans Chapitre 7, motivés par le travail effectué de Nakajima dans [118], et le travail effectué par Lee et Liu dans [96], nous introduisons la notion des σ-dérivations généralisées dans le contexte des modules comme suit: Pour un élément m dans un S/R-module M et un automorphisme σ sur l'algèbre S, une application R-module $f_m^\sigma : S \to M$ est appelée σ-dérivation généralisée, si f_m^σ satisfait:

$$f_m^\sigma(xy) = f_m^\sigma(x)y + \sigma(x)f_m^\sigma(y) + \sigma(x)ma,$$

pour chaque élement x et y en S, où R est un anneau commutatif. Il est clair que les σ-dérivations généralisées généralisent et étendent les dérivations généralisées, il suffit de prendre $\sigma = \mathrm{Id}$ alors les σ-dérivations généralisées coïncident avec les dérivations généralisées. Par conséquent, nous étudions les propriétés homologiques des σ-dérivations généralisées, le premier résultat principal de Section 7.1 généralise Nakajima [118, Théorème 2.4] tel que la séquence suivante:

$$0 \to M \xrightarrow{\psi_M} \mathrm{gDer}(S, M, \sigma) \xrightarrow{\phi_M} \mathrm{Der}(S, M, \sigma) \to 0,$$

est divisé et exact (i.e split exact), où σ est un automorphisme sur S et M est un S/R-bimodule (voir Théorème 7.1.3). Le deuxième résultat principal de Section 7.1 indique que les modules des σ-dérivations externes généralisées et des σ-dérivations externes sont isomorphes (voir Théorème 7.1.4). Autrement dit, on a l'isomorphisme suivant

$$\mathrm{gDer}(S, M, \sigma)/\mathrm{gInn}(S, M, \sigma) \cong \mathrm{Der}(\sigma, S, M)/\mathrm{Inn}(\sigma, S, M),$$

où σ est un automorphisme sur S et M est un S/R-bimodule. Dans Section 7.2, nous construisons deux catégories notées $\mathcal{A}_S$ et $\mathcal{D}_S$. La catégorie $\mathcal{A}_S$ est une catégorisation du groupe d'automorphismes $\mathrm{Aut}(S)$ d'un anneau S tel que les objets de $\mathcal{A}_S$ sont des automorphismes de S, et il existe un morphisme entre deux objets de $\mathcal{A}_S$ s'ils sont conjugués dans $\mathrm{Aut}(S)$ comme un groupe. La catégorie $\mathcal{D}_S$ est définie comme étant une catégorie de foncteurs, dont les objets sont des foncteurs $\mathrm{gDer}(S, -, \sigma)$ avec σ est un automorphisme sur S. Nous prouvons que les catégories $\mathcal{A}_S$ et $\mathcal{D}_S$ sont équivalentes. On définit un produit tensoriel $- \otimes_n^p -$ avec p est un nombre premier et n est un entier positif, et on donne des conditions nécessaires et suffisantes dans lesquelles $- \otimes_n^p -$ est un associatif bifoncteur (voir Lemme 7.2.4). Le résultat principal de Section 7.2 indique que la catégorie $(\mathcal{D}_S, \otimes_n^p)$ est semi-monoïdal sous certaines conditions nécessaires et suffisantes (voir Théorème 7.2.6). Dans Section 7.3, nous appliquons les principaux résultats sur les algèbres de chemin des carquois acycliques (infinis) pour donner les conditions nécessaires et suffisantes lorsqu'une dérivation généralisée est une dérivation intérieure généralisée (voir Théorème 7.3.5 et Théorème 7.3.11).

CHAPTER 2

Introduction (English Version)

> If I have seen further, it is by standing on the shoulders of giants.
>
> *Issac Newton*

This thesis presents our contributions to the domain of derivations by studying sorts of derivations on an algebraic structure called path algebras. Precisely, this thesis treats generalizations of derivations on path algebras called Lie generalized derivations, Jordan σ-derivations, Lie σ-derivations, Jordan $\mathcal{G}_n$-derivations, local derivations and generalized σ-derivations. Recall that an additive map $d : R \to R$ is a derivation on a ring R if it satisfies Leibniz' rule; that is $d(xy) = d(x)y + xd(y)$ for every x and y in R. A derivation d on R is called inner if there exists an element a in R such that $d(x) = [a, x]$ for every x in R, where $[a, x] = ax - xa$ is called the commutator of a and x.

From the papers [12] and [59], the origin of the notion of abstract derivations dates back to the thirties of the nineteenth century with Teichmüller [132], Hasse [62] and Jacobson [69]. However, it seems difficult to affirm with certainty which of them the founder of this domain (see also Ritt [123], Kaplansky [75] and Kolchin [88]). In the fifties, the notion of abstract derivations has got an over tremendous development after the works of Kaplansky [76] and Posner [121] which led many authors to characterize derivations in different contexts as Baclawski [15], Benkovič [18], Bennis and Fahid [27] and others [41, 45, 67, 71, 72, 76, 79, 125, 131, 141].

In [63], Herstein introduced the notion Jordan derivations on an associative ring R as derivations on the Jordan ring of R. That is an additive map $f : R \to R$ that satisfies $f(x \circ y) = f(x) \circ y + x \circ f(y)$ for every x and y in R where $x \circ y = xy + yx$ is called anticommutator of x and y. He showed that every Jordan derivation on a prime ring with characteristic different from two is a derivation. The notion Jordan derivation has been investigated in many contexts by different authors as Cusack [44], Brešar [33, 34], Benkovič [19, 20], Li and Wei [101] and others [6, 7, 46, 52, 53, 55, 81, 95, 102–104, 107, 109, 127, 137, 140, 142].

In [112], Martindale III started the investigation of Lie derivations on a primitive[1] ring R with characteristic different from two and contains a nontrivial idempotent. That is, an additive map $f : R \to R$ that satisfies $f([x, y]) = [f(x), y] + [x, f(y)]$ for every x and y in R. He showed that every Lie derivation on R is of a standard form (or proper). In other words, every Lie derivation $f : R \to R$ is sum of a derivation from R into a primitive ring $\overline{R}$ that contains R and an additive map from R into the center[2] of $\overline{R}$ that vanishes on the commutators of R. which led other authors to study the question

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[1] A ring R is said to be left-primitive if it has a simple faithful left R-module.

[2] The center of a ring R is the subset $\{x : xy = yx, \forall y \in R\}$.

of when every Lie derivation is of a standard form. See, for example, Brešar [35], Cheung [40], Benkovič [21, 23, 25] and others [17, 47, 54, 84, 108, 115, 116, 126, 136, 137, 144].

In [120], Ore introduced the notion of σ-derivations by equipping the polynomial ring $F[X]$ as a right F-module with the multiplication $Xa = \overline{a}X + a'$ for every a in F, where F is a field not necessary commutative. The element $\overline{a}$ is called the conjugate of a, a' is called the derivative of a. By equipping $F[X]$ with the multiplication-like of polynomials, $F[X]$ becomes a non-commutative ring, which is called Ore extension. Hence, it follows that for every a and b in F, we have $\overline{a+b} = \overline{a} + \overline{b}$, $\overline{ab} = \overline{a}\overline{b}$, $(a+b)' = a' + b'$ and $(ab)' = a'b + \overline{a}b'$. Therefore, an σ-derivation (skew-derivation) d from a ring R into itself is an additive map associated with an endomorphism σ from R into itself that satisfies the last two equalities. That is an additive map $d : R \to R$ is called σ-derivation if it satisfies $d(xy) = d(x)y + \sigma(x)d(y)$ for every x and y in R. The notion of σ-derivations can be obtained from another construction as in Cohn [42]. Many authors are interested in studying the notion of σ-derivations and its extensions, we cite Jacobson [70], Smits [129], Leroy and Matczuk [98], Montgomery and Paul Smith [117], Kharchenko and Popov [78], Bergen and Grzeszczuk [29], Mamouni, Oukhtite, and Zerra [111] and others [8, 9, 16, 30, 31, 38, 50, 61, 65, 68, 77, 87, 89, 91, 93, 94, 97, 99, 110, 113, 114, 130, 134, 139, 141, 143].

Since path algebras have various intersections with many well-known algebraic structures depending on the quivers associated with them, it yields that path algebras generalize and/or construct upper triangular matrix algebras, matrix algebras, one-point extension algebras, bound quiver algebras, Leavitt path algebras, incidence algebras, free associative algebras, and others. Several authors are interested and motivated to study and characterize derivations and related maps to derivations by using the combinatorial aspect of path algebras as Guo and Li [58], Li and Wei [105], Hou [66], Li and Wei [101], Lopatkin [106]. Therefore, the present thesis fall in the same research direction of studying derivations and related maps to them on path algebras. Hence, this thesis has five chapters that cover five published papers (i.e. [1–5]).

In the rest of this chapter, K denotes a field of characteristic different than 2 and A a unital associative algebra over K.

In Chapter 3, we study Lie generalized derivations on bound quiver algebras. This kinds of linear maps have been introduced in Bennis, Ebrahimi Vishki, Fahid, Khadem-Maboudi, and Mokhtari [26] as a generalized counterpart of Lie derivations. Recall a linear map $f : A \to A$ is called Lie generalized d-derivation if it satisfies:

$$f([x, y]) = [f(x), y] + [x, d(y)], \tag{2.1}$$

for every x and y in A, where $d : A \to A$ is a linear map. Thus, when $f = d$, f is nothing but the classical Lie derivation. In Section 3.1, we give some preliminaries on path algebras, bound quiver algebras, and triangular algebras. In Section 3.2, we study the properness of Lie generalized derivations on bound quiver algebras. Theorem 3.2.2 states equivalent assertions about Lie generalized derivations. To provide a complete characterization of Lie generalized derivations on bound quiver algebras, we only need to focus on characterizing derivations on bound quiver algebras. However, unlike the path algebra case [58, Theorem 2.3], the characterization of derivations on a bound quiver algebra $\mathfrak{Q}$ depends also on the form of a basis of $\mathfrak{Q}$ seen as a vector space. The investigation of this question lead us to establish another characterization of a derivation on a path algebra based only on the determination of images of vertices and edges rather than the determination of images of all paths as done in [58, Theorem 2.3] (see Proposition 3.2.3). This helps in determining the form of a derivation on a bound quiver algebra (see Proposition 3.2.4). In Corollary 3.2.5, we show that, when we add a condition on a basis of a bound quiver algebra, then we can establish an equivalent condition on the form of the linear map (see Proposition 3.2.4) so that it becomes a derivation (see also Examples 3.2.6 and 3.2.7). In Section 3.3, we are interested in the uniqueness property of the proper form decomposition stated in Theorem 3.2.2. A deep study of this question leads us to investigate the decomposition of derivations over a bound quiver algebra to a direct sum of derivations on algebras generated by connected components (see Lemma 3.3.3). Theorem 3.3.4 states equivalent assertions about the uniqueness of the decomposition of Lie generalized derivations.

In Chapter 4, we investigate Jordan σ-derivations and Lie σ-derivations on path algebras, this investigation is motivated by Benkovič's papers [22, 24]. Recall that a linear map $f : A \to A$ is called a Jordan σ-derivation if it satisfies:

$$f(x \circ y) = f(x)y + \sigma(x)f(y) + f(y)x + \sigma(y)f(x), \tag{2.2}$$

for every x and y in A. Also, a linear map $f : A \to A$ is called a Lie σ-derivation if it satisfies:

$$f([x, y]) = f(x)y + \sigma(x)f(y) - f(y)x - \sigma(y)f(x), \tag{2.3}$$

for every x and y in A. Jordan σ-derivations and Lie σ-derivations are generalizations of Jordan derivations and Lie derivations, respectively. In papers [22, 24], Benkovič assumed a faithfulness condition. In the case of path algebras, Li and Wei [101] showed that the condition of faithfulness can be ignored when path algebras can be viewed as one-point extensions. Therefore, we are inspired from the studies [22, 24, 101, 102] to investigate Jordan σ-derivations and Lie σ-derivations on path algebras. Namely, we confirm the remark of Li and Wei [101] and we prove it on any path algebra associated with a finite and acyclic quiver. In Section 4.1, we inspect the faithfulness property and the loyal property on path algebras, and we state two results related to these properties under some conditions (see Theorem 4.1.4 and Proposition 4.1.5). In Section 4.2, we investigate Jordan σ-derivations on path algebras, and we show that every Jordan σ-derivation on a path algebra is a σ-derivation (see Theorem 4.2.2). In Section 4.3, we study Lie σ-derivations on path algebras, and we state that every Lie σ-derivation on a path algebra is of a standard form (see Theorem 4.3.1). Note that, when σ is an inner automorphism on a path algebra, the problem of studying Jordan σ-derivations and Lie σ-derivations on path algebras is reduced to the study of Jordan derivations and Lie derivations, respectively, as stated in [22, Proposition 2.4] and [24, Proposition 2.3].

In Chapter 5, we introduce and study the notion of Jordan $\mathcal{G}_n$-derivations on path algebras which generalizes the notion of Jordan $\{g, h\}$-derivations. Recall that a linear map $f : A \to A$ is called a Jordan $\{g, h\}$-derivation if is satisfies:

$$f(x \circ y) = g(x) \circ y + x \circ h(y), \tag{2.4}$$

for every x and y in A. For $g = f$, a Jordan $\{g, h\}$-derivation is just a Jordan generalized derivation, and for $g = h = f$, it is nothing but the classical Jordan derivation (see Brešar [36] for further details). From the discussion at the beginning of [36, Section 2], it turned out that $\{g, h\}$-derivations are in the fact a particular case of generalized derivations (see [37] for more details about generalized derivations). The main aim in [36] is to investigate when a Jordan $\{g, h\}$-derivation is a $\{g, h\}$-derivation on tensor algebras (see also [90]). In Section 5.1, we define the notion of Jordan $\mathcal{G}_n$-derivations and we investigate its relation with the notion of Jordan $\{g, h\}$-derivations. In Section 5.2, Theorem 5.2.3 shows that a Jordan $\mathcal{G}_2$-derivation on a path algebra KQ is a $\mathcal{G}_2$-derivation if and only if $g_1(1) \in Z(KQ)$ or $g_2(1) \in Z(KQ)$. Theorem 5.2.4 shows that, for every $n > 2$, any Jordan $\mathcal{G}_n$-derivation on KQ is a $\mathcal{G}_n$-derivation. Namely, for every $n > 2$, Jordan $\mathcal{G}_n$-derivations on KQ are $\mathcal{G}_n$-derivations unlike the case $n = 2$, there exist some Jordan $\mathcal{G}_2$-derivations which are not

$\mathcal{G}_2$-derivations as will be shown in Example 5.2.1, which yields that Jordan $\mathcal{G}_n$-derivations generalize naturally Jordan $\mathcal{G}_2$-derivations (i.e. Jordan $\{g,h\}$-derivations). In Section 5.3, we present our investigations on a variant of Lvov-Kaplansky conjecture. Recall the following question known as Lvov-Kaplansky conjecture (see Filippov, Kharchenko, and Shestakov [49]):

Question 2.1. Let $\zeta(x_1,\ldots,x_n)$ be a multi-linear polynomial over a field $\mathbb{F}$. Is the set of values of ζ on the matrix algebra $M_m(\mathbb{F})$ a vector space?

The reader is referred to [74] for more information about recent and important results on this subject. Our investigation is motivated by the work done in Fagundes [48], Wang [138] and Quispe Urure and Franca [122] on particular upper triangular matrix algebras. In fact, since upper triangular matrix algebras are path algebras associated with line quivers (see Birkenmeier, Park, and Rizvi [32]), we will push the question further in another direction and ask:

Question 2.2. Let $\zeta(x_1,\ldots,x_n)$ be a multi-linear polynomial over K. Is the set of values of ζ on KQ a vector space?

Theorem 5.3.1 answers Question 2.2 positively and so it generalizes the work done for upper triangular matrix algebras. We give also some examples which apply Theorem 5.3.1 on some particular important cases.

In Chapter 6, we study local derivations and local generalized derivations on path algebras associated with finite and acyclic quivers. Recall that a linear map $\delta : A \to A$ is called a local derivation if is satisfies:

$$\delta(x) = d_x(x), \tag{2.5}$$

for every x in A where d_x is a derivation on A. A linear map $\Delta : A \to A$ is called a local generalized derivation on A if it satisfies:

$$\Delta(x) = D_x(x), \tag{2.6}$$

for every x in A where D_x is a generalized derivation on A. The notion of local derivations was introduced independently by Kadison in [73] and by Larson and Sourour in [92]. Namely, Kadison in [73] has initiated the problem of showing when every local derivation is a derivation. Subsequently, there have been many studies on local derivations and other local maps as local Jordan derivations and local Lie derivations on various algebras see for example [14, 39, 82, 119, 135, 145] and the references therein. Since finite incidence algebras associated with posets, and upper triangular matrix algebras are particular cases of path algebras (see [13, 128] for the relation between path algebras and upper triangular matrix algebras, and Section 6.1 for the relation between path algebras and incidence algebras), it is natural to ask of extending the study of local-like derivations to the context of path algebras. In Section 6.1, we study the relationship between path algebras and (finitary) incidence algebras. Since every finite poset $(P, \leq)$ has a finite acyclic quiver that represent it, it follows that every finite incidence algebra associated with a poset is a path algebra (see Ancykutty [10] and Spiegel and O'Donnell [131]). In the same section, we give an example of some path algebras which are not incidence algebras (see Example 6.1.4). In Section 6.2, we show that every local derivation on a path algebra is a derivation (see Theorem 6.2.3), and we investigate local Jordan derivations and local Lie derivations on path algebras. Finally, in Section 6.3, we prove that every local generalized derivation on a path algebra is a generalized derivation (see Theorem 6.3.4). We end Section 6.3 by studying local Lie generalized derivations on path algebras.

In Chapter 7, Motivated by the work done by Nakajima in [118], and the work done by Lee and Liu in [96], we introduce the notion of generalized σ-derivations on the context of modules as follows: For an element m in S/R-module M and an automorphism σ on algebra S, an R-module map $f_m^\sigma : S \to M$ is called a generalized σ-derivation, if f_m satisfies:

$$f_m^\sigma(xy) = f_m^\sigma(x)y + \sigma(x)f_m^\sigma(y) + \sigma(x)my,$$

for every x and y in S, where R is a commutative ring. It is clear that generalized σ-derivations generalize and extend generalized derivations, it suffice to take $\sigma = \mathrm{Id}$ then generalized σ-derivations coincide with generalized derivations. Therefore, we study homological properties of generalized σ-derivations, the first main result of Section 7.1 generalizes Nakajima [118, Theorem 2.4] such that the following sequence:

$$0 \to M \xrightarrow{\psi_M} \mathrm{gDer}(S, M, \sigma) \xrightarrow{\phi_M} \mathrm{Der}(S, M, \sigma) \to 0,$$

is split exact, where σ is an automorphism on S and M is an S/R-bimodule (see Theorem 7.1.3). The second main result of Section 7.1 states that modules of generalized outer σ-derivations and outer σ-derivations are isomorphic (see Theorem 7.1.4). In other words, we have the following isomorphism

$$\mathrm{gDer}(S, M, \sigma)/\mathrm{gInn}(S, M, \sigma) \cong \mathrm{Der}(\sigma, S, M)/\mathrm{Inn}(\sigma, S, M),$$

where σ is an automorphism on S and M is an S/R-bimodule. In Section 7.2, we construct two categories denoted by $\mathcal{A}_S$ and $\mathcal{D}_S$. The category $\mathcal{A}_S$ is a categorification of the automorphism group $\mathrm{Aut}(S)$ of a ring S such that objects of $\mathcal{A}_S$ are automorphisms of S, and there exists a morphism between two objects in $\mathcal{A}_S$ if they are conjugated in $\mathrm{Aut}(S)$ as a group. The category $\mathcal{D}_S$ is defined to be a functor category, which its objects are functors $\mathrm{gDer}(S, -, \sigma)$ with σ is an automorphism on S. We prove that categories $\mathcal{A}_S$ and $\mathcal{D}_S$ are equivalent. We define a tensor product $- \otimes_n^p -$ with p is a prime number and n is a positive integer, and we give necessary and sufficient conditions in which $- \otimes_n^p -$ is an associative bifunctor (see Lemma 7.2.4). The main result of Section 7.2 states that the category $(\mathcal{D}_S, \otimes_n^p)$ is semi-monoidal under some necessary and sufficient conditions (see Theorem 7.2.6). In Section 7.3, we apply main results on path algebras of (infinite) acyclic quivers to give necessary and sufficient conditions when a generalized derivation is a generalized inner derivation (see Theorem 7.3.5 and Theorem 7.3.11).

CHAPTER 3

Lie generalized derivations on bound quiver algebras

‘Obvious’ is the most dangerous word in Mathematics.

Eric Temple Bell

The following chapter presents the published paper titled “**Lie generalized derivations on bound quiver algebras**”[1] on Communications in Algebra.

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[1] A. Adrabi, D. Bennis, and B. Fahid: Lie generalized derivations on bound quiver algebras. *Communications in Algebra.* 2021; 49:1950–1965.

Abstract

In this chapter, we investigate Lie generalized derivations on bound quiver algebras associated with a finite acyclic quiver. The first main result shows that every bound quiver algebra associated with a finite acyclic quiver has the properness Lie generalized derivation property. The second main result investigates the uniqueness property. Namely, among other things, we show that a bound quiver algebra associated with a finite acyclic quiver E has the uniqueness Lie generalized derivation property if and only if the quiver E does not contain any isolated vertex.

Bound quiver algebras and particularly path algebras have attracted the intention of many researchers. They serve among several things to provide combinatorial characterizations of algebraic notions. Namely, they have been used in the theory of functional identities as a natural rich source of examples as well as to provide a combinatorial characterization of algebraic properties related to linear maps on rings such as derivations, generalized derivations and Lie derivations. In this chapter, we deal with Lie generalized derivations on bound quiver algebras.

Throughout this chapter K will denote a field with characteristic different than two, A will be a unital associative algebra over K with the center $Z(A)$. Let us recall the definition of generalized derivation given in Gölbaşi and Kaya [56]: A linear map $D : A \to A$ is called a right (resp., left) generalized derivation if there exists a linear map $d : A \to A$ such that $D(xy) = D(x)y + xd(y)$ (resp., $D(xy) = d(x)y + xD(y)$) for all x and y in A. Thus, D is called a generalized derivation associated with d if D is both a left and a right generalized derivation. Any derivation D on A (i.e., a linear map that satisfies $D(xy) = D(x)y+xD(y)$ for all x and y in A) is a generalized derivation. A Lie generalized derivation f is said to be proper if it can be written as the sum of a generalized derivation $D : A \to A$ and a linear map $l : A \to A$ with the set of images is a subset of $Z(A)$ and it vanishes on all commutators of A. In this chapter, we generalize the study done for Lie derivations in [101] by studying the properness Lie generalized derivation property and the uniqueness Lie generalized derivation property. We say that an algebra A has the properness Lie generalized derivation property if every Lie generalized derivation on A is proper. Also, we say that A has the uniqueness Lie generalized derivation property if every Lie generalized derivation on A has a unique proper form decomposition. Notice that a sum of a generalized inner derivation and a Lie derivation is a generalized Lie derivation (see Chapter Introduction). This kind of generalized Lie derivations were first discussed in Benkovič [21] and then characterized in Bennis, Vishki, Fahid, and Bahmani [28]. They are called strong generalized Lie derivations.

3.1 Preliminaries

We assume some familiarity with the basic notions of path algebras and bound quiver algebras (for more details, see Schiffler [128]). Nevertheless, we need to fix some conventions and notations used throughout the chapter. We will recall certain useful results.

In the sequel, $E = (E^0, E^1, s, t)$ designates a quiver, where E^0 and E^1 are sets of vertices and edges of E, respectively, and the pairs of maps s and t from E^1 into E^0 determine the edges of E. Moreover, E will be always a finite quiver; that is, E^0 and E^1 are both finite sets. For any edge $e \in E^1$, $s(e)$ is called the source of e and $t(e)$ is called the target of e. A non-trivial path $p \in E$ consists of an ordered list $e_1, \dots, e_l$ with $e_i \in E^1$ and $1 \leq i \leq l$ such that $t(e_i) = s(e_{i+1})$ for each $1 \leq i < l$. We write $p = e_1 \cdots e_l$. The integer $l \geq 0$ is called the length of p and it is denoted by $\ell(p)$. By convention, a vertex can be seen as a path with length zero, it is also called a trivial path. The notions of source and target are extended to paths as follows: For every path $p = e_1 \cdots e_l$ we define $s(p)$ to be $s(e_1)$ and $t(p)$ to be $t(e_l)$. The set of all paths in E will be denoted by $\mathcal{P}$. If $s(p) = t(p)$ and $\ell(p) > 0$, then p is called a directed cycle. If $s(p) \neq t(p)$, then p is called an acyclic path. The set of acyclic paths will be denoted by $\mathcal{P}_A$. If E has no directed cycles, then it is called acyclic. Two paths are called parallel if they start at same source and end at the same range.

In this chapter, E is always a finite acyclic quiver and KE is the associated path algebra over the field K. Recall that a two-sided ideal J of KE is called admissible, if there exists $m \geq 2$ such that

$$(R_E)^m \subseteq J \subseteq (R_E)^2,$$

where R_E is the radical of KE (generated by all the edges of E). If J is an admissible ideal of KE, the pair (E, J) is called a bound quiver and the quotient algebra KE/J is called the algebra of the bound quiver (E, J) or, simply, a bound quiver algebra. It is well-known that the admissible ideals are generated by a finite set of particular combinations called relations. Recall that a relation σ on E over K is a linear combination of parallel paths with length at least two. Thus, if J is an admissible ideal of KE, then there is a set of relations ρ on E over K such that J is generated by ρ. So, we may denote the bound quiver (E, J) by (E, ρ).

In this chapter, we set ρ as a set of relations on E over K that generates an admissible ideal J (i.e., $J =< \rho >$) and we denote by $\mathfrak{Q}$ the associated bound quiver algebra; that is, $\mathfrak{Q} = KE/J$. Since the quiver E is finite and acyclic, J could be a zero ideal and in this case $\mathfrak{Q} = KE$. So our results hold as well for path algebra associated with a finite acyclic quiver.

For an element $x \in KE$, we write $\overline{x}$ the corresponding element in $\mathfrak{Q}$. It is proved in [100, Lemma 2.3] that if $p_1, p_2 \in \mathcal{P}$, $p_1, p_2 \notin < \rho >$ and $\overline{p_1} = \overline{p_2} \in \mathfrak{Q}$, then p_1 and p_2 are parallel. Thus, the maps $\overline{s}$ and $\overline{r}$ defined on $\mathfrak{Q}$ by $\overline{s}(\overline{p}) := \overline{s(p)}$ and $\overline{r}(\overline{p}) := \overline{t(p)}$, for every $p \in \mathcal{P}$, are well defined. If no ambiguity arises, we will denote $\overline{s}$ simply by s and also $\overline{r}$ simply by r.

We also need to recall a construction of a particular basis of $\mathfrak{Q}$ seen as a vector space which will be useful throughout the chapter. Denote by $\mathcal{P}_{\geq 2}$ the set of all paths in E with a length at least 2. By [100, Proposition 2.1], if X is a basis of $< \rho >$, then there is a subset H of $X \cup \mathcal{P}_{\geq 2}$ which contains X, such that $\mathcal{P}' = E^0 \cup E^1 \cup (H \backslash X)$ is a basis of a complement space of $< \rho >$ in KE. Thus, $\mathcal{B} = \{\overline{p} : p \in \mathcal{P}'\}$ is a basis of $\mathfrak{Q}$ as a vector space. In this chapter, whenever we consider a basis X of $< \rho >$, we mean a basis with the form described above. Also, whenever we consider an element $\overline{b}$ of $\mathcal{B}$, b is understood to be in $\mathcal{P}'$. Denote the set of cosets of non-trivial paths in $\mathcal{P}'$ by $\mathcal{B}_A$ and the set of cosets of vertices in $\mathcal{P}'$ by $\mathcal{B}_V$. Whenever we consider $\overline{p} \in \mathcal{B}_A$ (resp., $\overline{v} \in \mathcal{B}_V$), then, without explicit mention, p is a non-trivial path in $\mathcal{P}'$ (resp., v is a vertex in $\mathcal{P}'$). If E has $n \geq 1$ connected component $C_1, \ldots, C_n$, then we denote the set of cosets of non-trivial paths in C_i contained in $\mathcal{P}'$ by $\mathcal{B}_A^i$, the set of cosets of vertices in C_i contained in $\mathcal{P}'$ by $\mathcal{B}_V^i$ and their union by $\mathcal{B}^i$. Thus, as above, whenever we consider $\overline{p} \in \mathcal{B}_A^i$ (resp., $\overline{v} \in \mathcal{B}_V^i$), then, without explicit mention, p is a non-trivial path of C_i in $\mathcal{P}'$ (resp., v is a vertex of C_i in $\mathcal{P}'$).

Corrales García, Martín Barquero, Martín González, Siles Molina, and Solanilla Hernández [43] described the centers of path algebras, Cohn algebras and Leavitt path algebras. The following lemma describes the center of a bound quiver algebra.

Lemma 3.1.1. *Suppose that E has n connected components $C_1, \ldots, C_n$ where $n \in \mathbb{N}^*$. Let $\overline{1_{C_i}} \in \mathfrak{Q}$ where 1_{C_i} is the sum of all the vertices in C_i. Then, the center of $\mathfrak{Q}$ is equal to $\bigoplus_{i=1}^n K\overline{1_{C_i}}$. Consequently, it is isomorphic to K^n.*

Proof. We can assume that the connected components are ordered by their number of vertices in the sense that C_i has more vertices than C_{i+1}. If C_1 has one vertex, then $\mathfrak{Q}$ is isomorphic to K^n. So, without loss of generality, we assume that C_1 contains more than one vertex. Then, $\overline{1_{C_1}} \in Z(\mathfrak{Q})$ since $\overline{1_{C_1}p} = \overline{p1_{C_1}} = \overline{0}$ for every element $\overline{p} \in \mathcal{B} \backslash \mathcal{B}^1$ and $\overline{1_{C_1}q} = \sum \overline{v_{C_1}q} = \overline{s(q)q} = \overline{qt(q)} = \overline{q} \sum \overline{v_{C_1}} = \overline{q1_{C_1}}$ for every element $\overline{q} \in \mathcal{B}^1$. Therefore, $\bigoplus_{i=1}^n K\overline{1_{C_i}} \subset Z(\mathfrak{Q})$. Suppose that the other inclusion does not hold. Then, there exists an element $\overline{x} \in Z(\mathfrak{Q}) \backslash \bigoplus_{i=1}^n K\overline{1_{C_i}}$, which can be written as $\overline{x} = \sum_i (\sum a_{C_i}\overline{v_{C_i}} + \sum b_{C_i}\overline{p_{C_i}})$, where a_{C_i} and $b_{C_i} \in K$, $\overline{v_{C_i}} \in \mathcal{B}_V^i$ and $\overline{p_{C_i}} \in \mathcal{B}_A^i$. By taking an arbitrary element $\overline{v} \in \mathcal{B}_V$, we obtain $a_{C_i,0}\overline{v} + b_{C_i,0}\overline{p_{C_i,0}} = \overline{xv} = \overline{vx} = a_{C_i,0}\overline{v} + b_{C_i,1}\overline{p_{C_i,1}}$ where $a_{C_i,0}, b_{C_i,0}$ and $b_{C_i,1}$ are in K, $\overline{p_{C_i,0}}$ and $\overline{p_{C_i,1}}$ are in $\mathcal{B}_A^i$. Since $\overline{p_{C_i,0}} \neq \overline{p_{C_i,1}}$, we obtain that all b_{C_i} are zero.

So, $\overline{x}$ can be written as $\overline{x} = \sum_i \sum a_{C_i}\overline{v_{C_i}}$. By taking an arbitrary element $\overline{e} \in \mathcal{B}_A^1$, we obtain $a_{C_1,0}\overline{e} = \overline{xe} = \overline{ex} = a_{C_1,1}\overline{e}$ where $a_{C_1,0}$ and $a_{C_1,1}$ are in K. Thus, $a_{C_1,0} = a_{C_1,1}$. By repeating the same calculation we deduce that $a_{C_1,0} = a_{C_1,j}$ for every j. Therefore, $\overline{x} = a_{C_1,0}\overline{1_{C_1}} + \sum_{i\geq 2}\sum a_{C_i}\overline{v_{C_i}}$. By the same reasoning, we obtain that $\overline{x}$ is a linear combination of $\overline{1_{C_i}}$, absurd. □

It is known that any bound quiver algebra of a finite acyclic quiver has a triangular matrix algebra representation. Explicitly, it can be seen as a one-point extension (see Li and Wei [101]). Let us recall this fact. First, recall, for two unital associative algebras A and B over K and an (A, B)-bimodule M, the triangular matrix algebra $\mathcal{T}_K = \mathrm{Tri}(A, M, B)$ is the associative algebra of formal triangular matrices

$$\mathcal{T}_K = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \middle| a \in A, b \in B, m \in M \right\}.$$

endowed with matrix addition and matrix multiplication (see Birkenmeier, Park, and Rizvi [32]). The center of $\mathcal{T}_K$ is

$$Z(\mathcal{T}_K) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| a \in Z(A), b \in Z(B), am = mb, \forall m \in M \right\}.$$

[32, Theorem 5.1.4] states that a unital algebra $\mathcal{A}$ has a triangular matrix representation if there exists a non-trivial idempotent $x \in \mathcal{A}$, such that $(1 - x)\mathcal{A}x = 0$. Namely, in this case, $\mathcal{A}$ is isomorphic to $\mathrm{Tri}(A, M, B)$, where $A = x\mathcal{A}x$, $M = x\mathcal{A}(1 - x)$ and $B = (1 - x)\mathcal{A}(1 - x)$. Particularly, if A is isomorphic to K, then $\mathrm{Tri}(A, M, B)$ is called a one-point extension of B by M. Therefore, every bound quiver algebra $\mathfrak{Q}$ is a one-point extension. Indeed, since E is acyclic, it admits a source s (i.e., there is no edge e such that $s = t(e)$). Then, we take the corresponding element $\overline{s} \in \mathfrak{Q}$ as a non-trivial idempotent. Hence, we have $(\overline{1} - \overline{s})\mathfrak{Q}\overline{s} = \overline{0}$, $A = \overline{s}\mathfrak{Q}\overline{s} \cong K$, $M = \overline{s}\mathfrak{Q}(\overline{1} - \overline{s})$ and $B = (\overline{1} - \overline{s})\mathfrak{Q}(\overline{1} - \overline{s})$. So, $\mathfrak{Q}$ has a triangular matrix representation which is a one-point extension.

The following result characterizes when a triangular algebra has the properness Lie generalized derivation property. It is shown that this condition is linked with the smallest sub-algebra of A containing all commutators and idempotents of A. This set will be denoted by $W(A)$. Recall that, from [101, Lemma 4.3], $W(\mathfrak{Q}) = \mathfrak{Q}$.

Lemma 3.1.2 ([26, Corollary 3.15])**.** *Let A and B be unital algebras over the field K and M be an (A, B)-bimodule. Then, the triangular algebra Tri(A, M, B) has the properness*

Lie generalized derivation property if the following two conditions are satisfied:

1. *The algebras A and B have the properness Lie generalized property.*
2. $W(A) = A$ *and* $W(B) = B$.

3.2 The properness Lie generalized derivation property on bound quiver algebras

The aim of this section is to give a complete characterization of Lie generalized derivations on bound quiver algebras. Before we state the main result of this section, we give the following lemma.

We recall that a linear map $I : A \to A$ is called a generalized inner derivation if there are fixed elements m and n in A such that $I(x) = mx + xn$ for every $x \in A$.

Lemma 3.2.1. *Let $D : \mathfrak{Q} \to \mathfrak{Q}$ be a linear map. Then, the following assertions are equivalents:*

1. *D is a generalized derivation.*
2. *D is a sum of a derivation d on $\mathfrak{Q}$ and a generalized inner derivation I on $\mathfrak{Q}$ of the form $I(\overline{x}) = \overline{mx}$ for every $\overline{x} \in \mathfrak{Q}$, where $\overline{m} \in Z(\mathfrak{Q})$.*

Proof. (1) $\Longrightarrow$ (2). Let D be a generalized derivation associated with a linear map d. Thus, we have $D(\overline{x}) = D(\overline{1})\overline{x} + d(\overline{x})$ for every $\overline{x} \in \mathfrak{Q}$. It follows that

$$\begin{aligned}
D(\overline{xy}) &= D(\overline{x})\overline{y} + \overline{x}d(\overline{y}) \\
&= (D(\overline{1})\overline{x} + d(\overline{x}))\overline{y} + \overline{x}d(\overline{y}) \\
&= D(\overline{1})\overline{xy} + d(\overline{x})\overline{y} + \overline{x}d(\overline{y}), \text{ and} \\
D(\overline{xy}) &= d(\overline{x})\overline{y} + \overline{x}D(\overline{y}) \\
&= d(\overline{x})\overline{y} + \overline{x}(D(\overline{1})\overline{y} + d(\overline{y})) \\
&= d(\overline{x})\overline{y} + \overline{x}D(\overline{1})\overline{y} + \overline{x}d(\overline{y}).
\end{aligned}$$

Hence, $\overline{x}D(\overline{1})\overline{y} = D(\overline{1})\overline{xy}$ for all $\overline{x}$ and $\overline{y}$ in $\mathfrak{Q}$. In particular, if we set $\overline{y} = \overline{1}$, we get $\overline{x}D(\overline{1}) = D(\overline{1})\overline{x}$ for every $\overline{x} \in \mathfrak{Q}$. Then, $D(\overline{1}) \in Z(\mathfrak{Q})$. We define an inner generalized derivation I by $I(\overline{x}) = \overline{mx}$ for every $\overline{x} \in \mathfrak{Q}$ where $\overline{m} = D(\overline{1})$. Therefore, since $D(\overline{1})\overline{xy} + d(\overline{x})\overline{y} + \overline{x}d(\overline{y}) = D(\overline{xy}) = D(\overline{1})\overline{xy} + d(\overline{xy})$ for all $\overline{x}$ and $\overline{y}$ in $\mathfrak{Q}$, we conclude that d is a

derivation.
(2) $\Longrightarrow$ (1). Let D be a linear map that satisfies the hypotheses. Then, we have for all $\overline{x}$ and $\overline{y}$ in $\mathfrak{Q}$

$$\begin{aligned}
D(\overline{xy}) &= I(\overline{xy}) + d(\overline{xy}) \\
&= \overline{mxy} + d(\overline{xy}) \\
&= \overline{mxy} + d(\overline{x})\overline{y} + \overline{x}d(\overline{y}) \\
&= (\overline{mx} + d(\overline{x}))\overline{y} + \overline{x}d(\overline{y}) \\
&= D(\overline{x})\overline{y} + \overline{x}d(\overline{y}) \\
&= \overline{xmy} + d(\overline{x})\overline{y} + \overline{x}d(\overline{y}) \\
&= d(\overline{x})\overline{y} + \overline{x}(\overline{my} + d(\overline{y})) \\
&= d(\overline{x})\overline{y} + \overline{x}D(\overline{y}).
\end{aligned}$$

Hence, D is a generalized derivation. □

Now, we are in a position to give the main result of this section.

Theorem 3.2.2. *Let $T : \mathfrak{Q} \to \mathfrak{Q}$ be a linear map. Then, the following assertions are equivalents:*

1. *T is a Lie generalized derivation.*
2. *T is a proper Lie generalized derivation.*
3. *T is a strong generalized Lie derivation such that $T = I + L$ with L is a Lie derivation on $\mathfrak{Q}$ and I is a generalized inner derivation on $\mathfrak{Q}$ of the form $I(\overline{x}) = \overline{mx}$ for every $\overline{x} \in \mathfrak{Q}$, where $\overline{m} \in Z(\mathfrak{Q})$.*
4. *There exist $\overline{m}$ and $\overline{b_v}$ in $Z(\mathfrak{Q})$ and a derivation d on $\mathfrak{Q}$ such that T has the following form:*
    - *(a) $T(\overline{v}) = \overline{mv} + \overline{b_v} + d(\overline{v})$ for every $\overline{v} \in \mathcal{B}_V$, and*
    - *(b) $T(\overline{p}) = \overline{mp} + d(\overline{p})$ for every $\overline{p} \in \mathcal{B}_A$.*

Proof. (1) $\Longrightarrow$ (2). This is proved similarly to [101, Theorem 4.4]. We set $\mathfrak{Q}_0 = \mathfrak{Q}$. We know that $\mathfrak{Q}_0$ is isomorphic to $\mathrm{Tri}(K, M_1, \mathfrak{Q}_1)$, where, for a source s_0 in E, $M_1 = \overline{s_0}\mathfrak{Q}_0(\overline{1} - \overline{s_0})$ and $\mathfrak{Q}_1 = (\overline{1} - \overline{s_0})\mathfrak{Q}_0(\overline{1} - \overline{s_0})$. By Lemma 3.1.2, to prove that $\mathfrak{Q}_0$ has the properness Lie generalized derivation property, we need to show that $\mathfrak{Q}_1$ has the properness Lie generalized derivation property, $W(K) = K$ and $W(\mathfrak{Q}_1) = \mathfrak{Q}_1$. With

[101, Lemma 4.3], we have $W(K) = K$ and $W(\mathfrak{Q}_1) = \mathfrak{Q}_1$. Since $\mathfrak{Q}_1$ is also a bound quiver algebra, $\mathfrak{Q}_1$ is isomorphic to $\mathrm{Tri}(K, M_2, \mathfrak{Q}_2)$. Again, with [101, Lemma 4.3], we have $W(\mathfrak{Q}_2) = \mathfrak{Q}_2$. We repeat this process finite-times until we get $\mathfrak{Q}_n \cong K$ for some positive integer n. This is possible due to the fact that E is finite and acyclic. Therefore, by applying Lemma 3.1.2 recursively from $\mathfrak{Q}_{n-1}$ to $\mathfrak{Q}_0$, we conclude that $\mathfrak{Q}_0$ has the properness Lie generalized derivation property. Hence, every Lie generalized derivation T on $\mathfrak{Q}$ is proper.

(2) $\Longrightarrow$ (3). Let T be a proper Lie generalized derivation. Then, by Lemma 3.2.1, T can be written as $T = D + l = I + d + l$, where D is a generalized derivation on $\mathfrak{Q}$, I is a generalized inner derivation on $\mathfrak{Q}$, d is a derivation on $\mathfrak{Q}$ and $l : \mathfrak{Q} \to \mathfrak{Q}$ is a linear map such that its set of images is a subset of $Z(\mathfrak{Q})$ and it vanishes on all commutators of $\mathfrak{Q}$. Since the sum $d + l$ is a Lie derivation, T is the sum of a generalized inner derivation and a Lie derivation. Therefore, T is a strong generalized Lie derivation.

(3) $\Longrightarrow$ (4). By the result [101, Theorem 4.4], every Lie derivation on $\mathfrak{Q}$ is proper. Then, a Lie derivation L can be written as $L = d + l$, where d is a derivation on $\mathfrak{Q}$ and $l : \mathfrak{Q} \to \mathfrak{Q}$ is a linear map such that its set of images is a subset of $Z(\mathfrak{Q})$ and it vanishes on all commutators of $\mathfrak{Q}$. Therefore, we have $T(\overline{x}) = \overline{mx} + d(\overline{x}) + l(\overline{x})$ for every $\overline{x} \in \mathfrak{Q}$. On the one hand, for every $\overline{p} \in \mathcal{B}_A$, we have $T(\overline{p}) = \overline{mp} + d(\overline{p})$; indeed, this is due the fact that $\overline{p}$ can be written as $\overline{p} = [\overline{p_1}, \overline{p_2}]$ where $\overline{p_1}$ and $\overline{p_2}$ are in $\mathcal{B}$, hence $l(\overline{p}) = \overline{0}$. Also, for every $\overline{v} \in \mathcal{B}_V$, we have $T(\overline{v}) = \overline{mv} + \overline{b_v} + d(\overline{v})$, where $\overline{b_v} = l(\overline{v})$.

(4) $\Longrightarrow$ (1). We suppose that T satisfies the hypotheses. Let $L : \mathfrak{Q} \to \mathfrak{Q}$ be a linear map defined by $L(\overline{v}) = \overline{b_v} + d(\overline{v})$ and $L(\overline{p}) = d(\overline{p})$ for every $\overline{v} \in \mathcal{B}_V$ and $\overline{p} \in \mathcal{B}_A$. We claim that T is a Lie generalized L-derivation. Indeed, we only need to verify it in the three cases:

(a) The elements $\overline{v}, \overline{w}$ are in $\mathcal{B}_V$.

(b) The elements $\overline{p}, \overline{q}$ are in $\mathcal{B}_A$.

(c) The element $\overline{v}$ is in $\mathcal{B}_V$ and the element $\overline{p}$ is in $\mathcal{B}_A$.

Case (a): Let $\overline{v}$ and $\overline{w}$ in $\mathcal{B}_V$. Then, we have $[\overline{v}, \overline{w}] = \overline{vw} - \overline{wv} = \overline{0}$. So, $T([\overline{v}, \overline{w}]) = \overline{0}$ and,

$$\begin{aligned}
[T(\overline{v}), \overline{w}] + [\overline{v}, L(\overline{w})] &= [\overline{mv} + \overline{b_v} + d(\overline{v}), \overline{w}] + [\overline{v}, \overline{b_w} + d(\overline{w})] \\
&= [\overline{mv} + \overline{b_v}, \overline{w}] + [d(\overline{v}), \overline{w}] + [\overline{v}, \overline{b_w}] + [\overline{v}, d(\overline{w})] \\
&= d(\overline{v})\overline{w} - \overline{w}d(\overline{v}) + \overline{v}d(\overline{w}) - d(\overline{w})\overline{v} \\
&= \overline{0}.
\end{aligned}$$

Hence, $T([\overline{v},\overline{w}])=[T(\overline{v}),\overline{w}]+[\overline{v},L(\overline{w})]$ for all $\overline{v}$ and $\overline{w}$ in $\mathcal{B}_V$.
Case (b): Let $\overline{p}$ and $\overline{q}$ in $\mathcal{B}_A$. Then, $T([\overline{p},\overline{q}])=T(\overline{pq})-T(\overline{qp})=\overline{mpq}+d(\overline{pq})-\overline{mqp}-d(\overline{qp})=\overline{m}[\overline{p},\overline{q}]+d([\overline{p},\overline{q}])$ and,

$$\begin{aligned}[T(\overline{p}),\overline{q}]+[\overline{p},L(\overline{q})]&=[\overline{mp}+d(\overline{p}),\overline{q}]+[\overline{p},d(\overline{q})]\\&=[\overline{mp},\overline{q}]+[d(\overline{p}),\overline{q}]+[\overline{p},d(\overline{q})]\\&=\overline{m}[\overline{p},\overline{q}]+d([\overline{p},\overline{q}]).\end{aligned}$$

Hence, $T([\overline{p},\overline{q}])=[T(\overline{p}),\overline{q}]+[\overline{p},L(\overline{q})]$ for all $\overline{p}$ and $\overline{q}$ in $\mathcal{B}_A$.
Case (c): Let $\overline{v}\in\mathcal{B}_V$ and $\overline{p}\in\mathcal{B}_A$. Then, $T([\overline{v},\overline{p}])=T(\overline{vp})-T(\overline{pv})=\overline{mvp}+d(\overline{vp})-\overline{mpv}-d(\overline{pv})=\overline{m}[\overline{v},\overline{p}]+d([\overline{v},\overline{p}])$ and,

$$\begin{aligned}[T(\overline{v}),\overline{p}]+[\overline{v},L(\overline{p})]&=[\overline{mv}+\overline{b_v}+d(\overline{v}),\overline{p}]+[\overline{v},d(\overline{p})]\\&=[\overline{mv}+\overline{b_v},\overline{p}]+[d(\overline{v}),\overline{p}]+[\overline{v},d(\overline{p})]\\&=\overline{m}[\overline{v},\overline{p}]+d([\overline{v},\overline{p}]).\end{aligned}$$

Hence, $T([\overline{v},\overline{p}])=[T(\overline{v}),\overline{p}]+[\overline{v},L(\overline{p})]$ for every $\overline{v}\in\mathcal{B}_V$ and $\overline{p}\in\mathcal{B}_A$. Since the commutator is anticommutative, we have $T([\overline{p},\overline{v}])=-T([\overline{v},\overline{p}])=-\overline{m}[\overline{v},\overline{p}]-d([\overline{v},\overline{p}])=\overline{m}[\overline{p},\overline{v}]+d([\overline{p},\overline{v}])$ and,

$$\begin{aligned}[T(\overline{p}),\overline{v}]+[\overline{p},L(\overline{v})]&=[\overline{mp}+d(\overline{p}),\overline{v}]+[\overline{p},\overline{b_v}+d(\overline{v})]\\&=[\overline{mp},\overline{v}]+[d(\overline{p}),\overline{v}]+[\overline{p},d(\overline{v})]\\&=\overline{m}[\overline{p},\overline{v}]+d([\overline{p},\overline{v}]).\end{aligned}$$

Hence, $T([\overline{p},\overline{v}])=[T(\overline{p}),\overline{v}]+[\overline{p},L(\overline{v})]$ for every $\overline{v}\in\mathcal{B}_V$ and $\overline{p}\in\mathcal{B}_A$. Finally, the result holds. □

To provide a complete characterization of Lie generalized derivations on bound quiver algebras, we only need by the fourth assertion of Theorem 3.2.2 to characterize derivations on bound quiver algebras. Knowing that a characterization of derivations on path algebras is already established in [58, Theorem 2.3], it is natural to try first to extend it to the case of bound quiver algebras. However, the approach adopted in [58, Theorem 2.3] seems to be complicated in the case of bound quiver algebras. This leads us to improve [58, Theorem 2.3]. In fact, we observed that a derivation on a path algebra can be determined once we know only the images of vertices and edges as shown in the following result.

Proposition 3.2.3. *A linear map $d: KE\to KE$ is a derivation if and only if d can be* *expressed in the form:*

1. *For every vertex v,*

$$d(v) = \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=v}} c_q^v q + \sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=v}} c_q^v q, \tag{3.1}$$

where all c_q^v are coefficients in K satisfying $c_q^{s(q)} + c_q^{t(q)} = 0$ for every $q \in \mathcal{P}_A$.

2. *For every edge e,*

$$d(e) = \sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=s(e)}} c_q^{s(e)} qe + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=s(e),\\ t(q)=t(e)}} c_q^e\, q + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=t(e)}} c_q^{t(e)} eq, \tag{3.2}$$

where all c_q^e are coefficients in K and the coefficients $c_q^{s(e)}$ and $c_q^{t(e)}$ are those obtained in the expression (3.1).

3. *For every non-trivial path $p = e_1 \cdots e_n \in \mathcal{P}_A$, where $e_1, \ldots, e_n$ are edges in E,*

$$\begin{aligned} d(p) = {} & \sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=s(p)}} c_q^{s(p)} qp + \sum_{1\le i\le n} \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=s(e_i),\\ t(q)=t(e_i)}} c_q^{e_i} e_1 \cdots e_{i-1} q e_{i+1} \cdots e_n \\ & + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=t(p)}} c_q^{t(p)}\, pq, \end{aligned} \tag{3.3}$$

where the coefficients $c_q^{s(p)}$ and $c_q^{t(p)}$ are those obtained in the expression (3.1), and the coefficients $c_q^{e_i}$ are those obtained in the expression (3.2).

Proof. We only need to prove the “the only if” part. The converse implication is a straightforward calculations. Thus, let d be a derivation on KE. For each vertex v in E^0, let $d(v) = \sum_{q\in\mathcal{P}} c_q^v q$ with $c_q^v \in K$. We have

$$\begin{aligned} \sum_{q\in\mathcal{P}} c_q^v q = d(v) = d(v^2) & \\ & = d(v)v + vd(v) \\ & = (\sum_{q\in\mathcal{P}} c_q^v q)v + v(\sum_{q\in\mathcal{P}} c_q^v q) \\ & = 2c_v^v v + \sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=v}} c_q^v q + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=v}} c_q^v q. \end{aligned}$$

Hence, $c_v^v = 0$ and $d(v) = \sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=v}} c_q^v q + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=v}} c_q^v q$. Now, let v and u be two vertices in E^0 such that $v \neq u$, then we have

$$\begin{aligned}
0 = d(uv) &= d(u)v + ud(v) \\
&= (\sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=u}} c_q^u q + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=u}} c_q^u q)v + u(\sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=v}} c_q^v q + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=v}} c_q^v q) \\
&= \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=u}} c_q^u qv + \sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=v}} c_q^v uq \\
&= \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=u,\\ t(q)=v}} (c_q^u + c_q^v)q.
\end{aligned}$$

Hence, $c_q^{s(q)} + c_q^{t(q)} = 0$ for every $q \in \mathcal{P}_A$. Next, let p be a non-trivial path in $\mathcal{P}_A$. If $d(p) = \sum_{q\in\mathcal{P}} c_q^p q$, then in the case of p is an edge in E, we obtain

$$\begin{aligned}
d(p) &= d(s(p)p) \\
&= d(s(p))p + s(p)d(p) \\
&= d(s(p))p + s(p)d(pt(p)) \\
&= d(s(p))p + s(p)d(p)t(p) + pd(t(p)) \\
&= (\sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=s(p)}} c_q^{s(p)} q + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=s(p)}} c_q^{s(p)} q)p + s(p)d(p)t(p) \\
&\quad + p(\sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=t(p)}} c_q^{t(p)} q + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=t(p)}} c_q^{t(p)} q) \\
&= \sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=s(p)}} c_q^{s(p)} qp + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=s(p)\\ t(q)=t(p)}} c_q^p q + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=t(p)}} c_q^{t(p)} pq.
\end{aligned}$$

Hence, $d(p) = \sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=s(p)}} c_q^{s(p)} qp + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=s(p)\\ t(q)=t(p)}} c_q^p q + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=t(p)}} c_q^{t(p)} pq$. In the other case, p can be expressed as $p = e_1 \cdots e_n$, where $e_1, \ldots, e_n$ are edges in E such that $n = \ell(p) \geq 2$. So, it follows that

$$d(p) = \sum_{i=1}^{n} e_1 \cdots e_{i-1} d(e_i) e_{i+1} \cdots e_n$$

$$
\begin{aligned}
&= d(e_1)e_2\cdots e_n + \sum_{i=2}^{n-1} e_1\cdots e_{i-1}d(e_i)e_{i+1}\cdots e_n + e_1\cdots e_{i-1}d(e_n)\\
&= \sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=s(e_1)}} c_q^{s(e_1)} q e_1\cdots e_n + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=s(e_1)\\ t(q)=t(e_1)}} c_q^{e_1} q e_2\cdots e_n\\
&\quad + \sum_{i=2}^{n-1} \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=s(e_i)\\ t(q)=t(e_i)}} c_q^{e_i} e_1\cdots e_{i-1} q e_{i+1}\cdots e_n\\
&\quad + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=s(e_n)\\ t(q)=t(e_n)}} c_q^{e_n} e_1\cdots e_{n-1} q + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=t(e_n)}} c_q^{t(e_n)} e_1\cdots e_n q.
\end{aligned}
$$

Hence,

$$
d(p) = \sum_{\substack{q\in\mathcal{P}_A,\\ t(q)=s(p)}} c_q^{s(p)} qp + \sum_{i=1}^{n} \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=s(e_i),\\ t(q)=t(e_i)}} c_q^{e_i} e_1\cdots e_{i-1} q e_{i+1}\cdots e_n + \sum_{\substack{q\in\mathcal{P}_A,\\ s(q)=t(p)}} c_q^{t(p)} pq.
$$

Therefore, d satisfies (3.1) and (3.3). ☐

By following the same argument as in the proof of Proposition 3.2.3, we can get the explicit form of derivations on bound quiver algebras. However, to prove the converse implication as in Proposition 3.2.3, we need more information on a basis of the bound quiver algebra seen as a vector space. In Corollary 3.2.5, we show that, for a particular case of bases of a bound quiver algebra, we can establish equivalent conditions.

Proposition 3.2.4. *Let d be a derivation on $\mathfrak{Q}$. Then, d can be expressed in the form:*

1. *For every vertex v,*

$$
d(\overline{v}) = \sum_{\substack{\overline{q}\in\mathcal{B}_A,\\ s(\overline{q})=\overline{v}}} c_{\overline{q}}^{\overline{v}}\overline{q} + \sum_{\substack{\overline{q}\in\mathcal{B}_A,\\ t(\overline{q})=\overline{v}}} c_{\overline{q}}^{\overline{v}}\overline{q}, \tag{3.4}
$$

where all $c_{\overline{q}}^{\overline{v}}$ are coefficients in K satisfying $c_{\overline{q}}^{s(\overline{q})} + c_{\overline{q}}^{t(\overline{q})} = 0$ for every $\overline{q} \in \mathcal{B}_A$.

2. *For every edge e,*

$$
d(\overline{e}) = \sum_{\substack{\overline{q}\in\mathcal{B}_A,\\ t(\overline{q})=s(\overline{e})}} c_{\overline{q}}^{s(\overline{e})}\overline{qe} + \sum_{\substack{\overline{q}\in\mathcal{B}_A,\\ s(\overline{q})=s(\overline{e}),\\ t(\overline{q})=t(\overline{e})}} c_{\overline{q}}^{\overline{e}}\overline{q} + \sum_{\substack{\overline{q}\in\mathcal{B}_A,\\ s(\overline{q})=t(\overline{e})}} c_{\overline{q}}^{t(\overline{e})}\overline{eq}, \tag{3.5}
$$

where all $c_{\overline{q}}^{\overline{e}}$ *are coefficients in* K *and the coefficients* $c_{\overline{q}}^{s(\overline{e})}$ *and* $c_{\overline{q}}^{t(\overline{e})}$ *are those obtained in the expression (3.4).*

3. *For every non-trivial path* $p = e_1 \cdots e_n \in \mathcal{P}_A$*, where* $e_1, \ldots, e_n$ *are edges in* E*,*

$$d(\overline{p}) = \sum_{\substack{\overline{q}\in\mathcal{B}_A,\\ t(\overline{q})=s(\overline{p})}} c_{\overline{q}}^{s(\overline{p})}\overline{qp} + \sum_{1\leq i\leq n}\sum_{\substack{\overline{q}\in\mathcal{B}_A,\\ s(\overline{q})=s(\overline{e_i}),\\ t(\overline{q})=t(\overline{e_i})}} c_{\overline{q}}^{\overline{e_i}}\overline{e_1}\cdots\overline{e_{i-1}qe_{i+1}}\cdots\overline{e_n} + \sum_{\substack{\overline{q}\in\mathcal{B}_A,\\ s(\overline{q})=t(\overline{p})}} c_{\overline{q}}^{t(\overline{p})}\overline{pq}, \tag{3.6}$$

where the coefficients $c_{\overline{q}}^{s(\overline{p})}$ *and* $c_{\overline{q}}^{t(\overline{p})}$ *are those obtained in the expression (3.4), and the coefficients* $c_{\overline{q}}^{\overline{e_i}}$ *are those obtained in the expression (3.5).*

As mentioned before, we see in the following result that if we add some restrictions on the set of relations of the ideal $< \rho >$, we can get a complete characterization of a derivation on bound quiver algebras.

Corollary 3.2.5. *Assume that* ρ *is a subset of* $\mathcal{P}_{\geq 2}$*. A linear map* d *on* $\mathfrak{Q}$ *is a derivation if and only if it satisfies the following conditions:*

1. d *can be expressed as in Proposition 3.2.4,*

2. *If* $p = e_1 \cdots e_i \cdots e_n \in < \rho >$ *and there exists a parallel path* $q \in \mathcal{P}_A$ *to* e_i *such that* $\overline{e_1 \cdots e_{i-1} q e_{i+1} \cdots e_n}$ *is in* $\mathcal{B}_A$*, then* $c_{\overline{q}}^{\overline{e_i}} = 0$*.*

Proof. ($\Longrightarrow$) We only need to prove the second condition. Let d be a derivation on $\mathfrak{Q}$, we claim that if $p = e_1 \cdots e_n \in < \rho >$ and there exists a parallel path $q \in \mathcal{P}_A$ to some edge e_i such that $\overline{e_1 \cdots e_{i-1} q e_{i+1} \cdots e_n}$ is in $\mathcal{B}_A$. Then, $c_{\overline{q}}^{\overline{e_i}} = 0$. Indeed, $d(\overline{p}) = \overline{0}$ since $p \in < \rho >$. Hence, by Proposition 3.2.4, we obtain

$$\begin{aligned} d(\overline{p}) &= \sum_{\substack{\overline{q}\in\mathcal{B}_A,\\ t(\overline{q})=s(\overline{p})}} c_{\overline{q}}^{s(\overline{p})}\overline{qp} + \sum_{1\leq i\leq n}\sum_{\substack{\overline{q}\in\mathcal{B}_A,\\ s(\overline{q})=s(\overline{e_i}),\\ t(\overline{q})=t(\overline{e_i})}} c_{\overline{q}}^{\overline{e_i}}\overline{e_1}\cdots\overline{e_{i-1}qe_{i+1}}\cdots\overline{e_n} + \sum_{\substack{\overline{q}\in\mathcal{B}_A,\\ s(\overline{q})=t(\overline{p})}} c_{\overline{q}}^{t(\overline{p})}\overline{pq} \\ &= \sum_{1\leq i\leq n}\sum_{\substack{\overline{q}\in\mathcal{B}_A,\\ s(\overline{q})=s(\overline{e_i}),\\ t(\overline{q})=t(\overline{e_i})}} c_{\overline{q}}^{\overline{e_i}}\overline{e_1}\cdots\overline{e_{i-1}qe_{i+1}}\cdots\overline{e_n}. \end{aligned}$$

This is due to the fact that pq and qp are in $< \rho >$. Thus, if the element $\overline{e_1}\cdots\overline{e_{i-1}qe_{i+1}}\cdots\overline{e_n} \in \mathcal{B}_A$, then $c_{\overline{q}}^{\overline{e_i}} = 0$, as desired.

($\Longleftarrow$) Let d be a linear map on $\mathfrak{Q}$. So, the second condition guarantees that for every path $p = e_1 \cdots e_n \in < \rho >$, we have

$$d(\overline{p}) = \sum_{i=1}^{n} \overline{e_1} \cdots \overline{e_{i-1}} d(\overline{e_i}) \overline{e_{i+1}} \cdots \overline{e_n} = \overline{0}.$$

The rest of calculations is similar to the ones done in the proof of the converse implication of Proposition 3.2.3. $\square$

Example 3.2.6. Let E be the following quiver:

$$\begin{tikzcd} E: & v_1 \arrow[r, bend left, "e_1"] \arrow[r, bend right, "e_2"'] & v_2 \arrow[r, "e_3"] & v_3 \end{tikzcd} .$$

Consider the bound quiver (E, J) associated with the set of relations $\rho = \{e_2e_3\}$. Thus, a basis of $\mathfrak{Q} = KE/J$ can be described as the union of the sets $\mathcal{B}_V = \{\overline{v_1}, \overline{v_2}, \overline{v_3}\}$ and $\mathcal{B}_A = \{\overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{e_1e_3}\}$. Hence, $\mathfrak{Q}$ is isomorphic as a vector space to K^7. It follows from Corollary 3.2.5 that a linear map on $\mathfrak{Q}$ is a derivation if and only if it can be presented by the following matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c^{\overline{v_1}}_{\overline{e_1}} & -c^{\overline{v_1}}_{\overline{e_1}} & 0 & c^{\overline{e_1}}_{\overline{e_1}} & 0 & 0 & 0 \\
c^{\overline{v_1}}_{\overline{e_2}} & -c^{\overline{v_1}}_{\overline{e_2}} & 0 & c^{\overline{e_1}}_{\overline{e_2}} & c^{\overline{e_2}}_{\overline{e_2}} & 0 & 0 \\
0 & c^{\overline{v_2}}_{\overline{e_3}} & -c^{\overline{v_2}}_{\overline{e_3}} & 0 & 0 & c^{\overline{e_3}}_{\overline{e_3}} & 0 \\
c^{\overline{v_1}}_{\overline{e_1e_3}} & 0 & -c^{\overline{v_1}}_{\overline{e_1e_3}} & c^{\overline{v_2}}_{\overline{e_3}} & 0 & -c^{\overline{v_1}}_{\overline{e_1}} & c^{\overline{e_1}}_{\overline{e_1}} + c^{\overline{e_3}}_{\overline{e_3}}
\end{pmatrix}$$

When the considered relations in ρ are not simply paths but rather linear combinations of parallel paths, then the characterization of a derivation on bound quiver algebras could be different of the one established in Corollary 3.2.5 as shown by the following example in which we use the same quiver E of Example 3.2.6 but with different set of relations.

Example 3.2.7. Let (E, ρ) be a bound quiver associated with the set of relations $\rho = \{e_1e_3 - e_2e_3\}$, where E is the quiver considered in Example 3.2.7. Then, a linear map d is a derivation on $\mathfrak{Q}$ if and only if it can be presented by the following matrix:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c^{\overline{v_1}}_{\overline{e_1}} & -c^{\overline{v_1}}_{\overline{e_1}} & 0 & c^{\overline{e_1}}_{\overline{e_1}} & c^{\overline{e_2}}_{\overline{e_1}} & 0 & 0 \\ c^{\overline{v_1}}_{\overline{e_2}} & -c^{\overline{v_1}}_{\overline{e_2}} & 0 & c^{\overline{e_1}}_{\overline{e_2}} & c^{\overline{e_2}}_{\overline{e_2}} & 0 & 0 \\ 0 & c^{\overline{v_2}}_{\overline{e_3}} & -c^{\overline{v_2}}_{\overline{e_3}} & 0 & 0 & c^{\overline{e_3}}_{\overline{e_3}} & 0 \\ c^{\overline{v_1}}_{\overline{e_1 e_3}} & 0 & -c^{\overline{v_1}}_{\overline{e_1 e_3}} & c^{\overline{v_2}}_{\overline{e_3}} & c^{\overline{v_2}}_{\overline{e_3}} & -c^{\overline{v_1}}_{\overline{e_1}} - c^{\overline{v_1}}_{\overline{e_2}} & c^{\overline{e_1}}_{\overline{e_1}} + c^{\overline{e_3}}_{\overline{e_3}} + c^{\overline{e_1}}_{\overline{e_2}} \end{pmatrix}$$

with $c^{\overline{e_1}}_{\overline{e_1}} + c^{\overline{e_1}}_{\overline{e_2}} = c^{\overline{e_2}}_{\overline{e_2}} + c^{\overline{e_2}}_{\overline{e_1}}$.

Proof. ($\Longrightarrow$) If d is a derivation on $\mathfrak{Q}$, then from Proposition 3.2.4,

$$\begin{aligned} d(\overline{v_1}) &= c^{\overline{v_1}}_{\overline{e_1}}\overline{e_1} + c^{\overline{v_1}}_{\overline{e_2}}\overline{e_2} + c^{\overline{v_1}}_{\overline{e_1 e_3}}\overline{e_1 e_3}. \\ d(\overline{v_2}) &= -c^{\overline{v_1}}_{\overline{e_1}}\overline{e_1} - c^{\overline{v_1}}_{\overline{e_2}}\overline{e_2} + c^{\overline{v_2}}_{\overline{e_3}}\overline{e_3}. \\ d(\overline{v_3}) &= -c^{\overline{v_2}}_{\overline{e_3}}\overline{e_3} - c^{\overline{v_1}}_{\overline{e_1 e_3}}\overline{e_1 e_3}. \\ d(\overline{e_1}) &= c^{\overline{e_1}}_{\overline{e_1}}\overline{e_1} + c^{\overline{e_1}}_{\overline{e_2}}\overline{e_2} + c^{\overline{v_2}}_{\overline{e_3}}\overline{e_1 e_3}. \\ d(\overline{e_2}) &= c^{\overline{e_2}}_{\overline{e_1}}\overline{e_1} + c^{\overline{e_2}}_{\overline{e_2}}\overline{e_2} + c^{\overline{v_2}}_{\overline{e_3}}\overline{e_2 e_3} \\ &= c^{\overline{e_2}}_{\overline{e_1}}\overline{e_1} + c^{\overline{e_2}}_{\overline{e_2}}\overline{e_2} + c^{\overline{v_2}}_{\overline{e_3}}\overline{e_1 e_3}. \\ d(\overline{e_3}) &= c^{\overline{v_2}}_{\overline{e_1}}\overline{e_1 e_3} + c^{\overline{v_2}}_{\overline{e_2}}\overline{e_2 e_3} + c^{\overline{e_3}}_{\overline{e_3}}\overline{e_3} \\ &= -c^{\overline{v_1}}_{\overline{e_1}}\overline{e_1 e_3} - c^{\overline{v_1}}_{\overline{e_2}}\overline{e_1 e_3} + c^{\overline{e_3}}_{\overline{e_3}}\overline{e_3} \\ &= (-c^{\overline{v_1}}_{\overline{e_2}} - c^{\overline{v_1}}_{\overline{e_1}})\overline{e_1 e_3} + c^{\overline{e_3}}_{\overline{e_3}}\overline{e_3}. \\ d(\overline{e_1 e_3}) &= c^{\overline{e_1}}_{\overline{e_1}}\overline{e_1 e_3} + c^{\overline{e_1}}_{\overline{e_2}}\overline{e_2 e_3} + c^{\overline{e_3}}_{\overline{e_3}}\overline{e_1 e_3} \\ &= (c^{\overline{e_1}}_{\overline{e_1}} + c^{\overline{e_3}}_{\overline{e_3}} + c^{\overline{e_1}}_{\overline{e_2}})\overline{e_1 e_3}. \end{aligned}$$

Also, since $\overline{e_1 e_3} - \overline{e_2 e_3} = \overline{0}$,

$$\begin{aligned} d(\overline{e_1 e_3} - \overline{e_2 e_3}) &= d(\overline{e_1})\overline{e_3} + \overline{e_1}d(\overline{e_3}) - d(\overline{e_2})\overline{e_3} - \overline{e_2}d(\overline{e_3}) \\ &= c^{\overline{e_1}}_{\overline{e_1}}\overline{e_1 e_3} + c^{\overline{e_1}}_{\overline{e_2}}\overline{e_2 e_3} + c^{\overline{e_3}}_{\overline{e_3}}\overline{e_1 e_3} - c^{\overline{e_2}}_{\overline{e_1}}\overline{e_1 e_3} - c^{\overline{e_2}}_{\overline{e_2}}\overline{e_2 e_3} - c^{\overline{e_3}}_{\overline{e_3}}\overline{e_2 e_3} \\ &= c^{\overline{e_1}}_{\overline{e_1}}\overline{e_1 e_3} + c^{\overline{e_1}}_{\overline{e_2}}\overline{e_1 e_3} + c^{\overline{e_3}}_{\overline{e_3}}\overline{e_1 e_3} - c^{\overline{e_2}}_{\overline{e_1}}\overline{e_1 e_3} - c^{\overline{e_2}}_{\overline{e_2}}\overline{e_1 e_3} - c^{\overline{e_3}}_{\overline{e_3}}\overline{e_1 e_3} \\ &= (c^{\overline{e_1}}_{\overline{e_1}} - c^{\overline{e_2}}_{\overline{e_2}} + c^{\overline{e_1}}_{\overline{e_2}} - c^{\overline{e_2}}_{\overline{e_1}})\overline{e_1 e_3} \\ &= \overline{0}. \end{aligned}$$

Therefore, $c^{\overline{e_1}}_{\overline{e_1}} + c^{\overline{e_1}}_{\overline{e_2}} = c^{\overline{e_2}}_{\overline{e_2}} + c^{\overline{e_2}}_{\overline{e_1}}$.

($\Longleftarrow$) Now, let d be a linear map presented by the matrix M. The condition $c^{\overline{e_1}}_{\overline{e_1}}+c^{\overline{e_1}}_{\overline{e_2}} = c^{\overline{e_2}}_{\overline{e_2}} + c^{\overline{e_2}}_{\overline{e_1}}$ yields to

$$\begin{aligned}
d(\overline{e_2})\overline{e_3} + \overline{e_2}d(\overline{e_3}) &= c^{\overline{e_2}}_{\overline{e_1}}\overline{e_1e_3} + c^{\overline{e_2}}_{\overline{e_2}}\overline{e_2e_3} + c^{\overline{e_3}}_{\overline{e_3}}\overline{e_2e_3} \\
&= (c^{\overline{e_2}}_{\overline{e_1}} + c^{\overline{e_2}}_{\overline{e_2}} + c^{\overline{e_3}}_{\overline{e_3}})\overline{e_1e_3} \\
&= (c^{\overline{e_1}}_{\overline{e_1}} + c^{\overline{e_1}}_{\overline{e_2}} + c^{\overline{e_3}}_{\overline{e_3}})\overline{e_1e_3} \\
&= d(\overline{e_1})\overline{e_3} + \overline{e_1}d(\overline{e_3})
\end{aligned}$$

Hence, $d(\overline{e_1})\overline{e_3} + \overline{e_1}d(\overline{e_3}) - d(\overline{e_2})\overline{e_3} - \overline{e_2}d(\overline{e_3}) = \overline{0}$. The rest of calculations is similar to the ones done in the proof of the converse implication of Proposition 3.2.3. □

3.3 The uniqueness Lie generalized derivation property on bound quiver algebras

In general, a bound quiver algebra does not have the uniqueness Lie generalized derivation property. For example, if E has an isolated vertex t, then we obtain $(\overline{1} - \overline{t})\mathfrak{Q}\overline{t} = 0$, $\overline{t}\mathfrak{Q}(\overline{1} - \overline{t}) = \overline{0}$, $\overline{t}\mathfrak{Q}\overline{t} \cong K$ and $B = (\overline{1} - \overline{t})\mathfrak{Q}(\overline{1} - \overline{t})$. Hence, $\mathfrak{Q}$ is isomorphic to the triangular algebra $\mathrm{Tri}(K, 0, B)$ which can be identified with the direct product of algebras $K \times B$. So, the center of $\mathfrak{Q}$ can be described by $Z(\mathfrak{Q}) = \{(\alpha, \overline{\beta}) | \alpha \in K, \overline{\beta} \in Z(B)\}$. Let T be a Lie generalized derivation on $\mathfrak{Q}$ which it is not a Lie derivation. By Theorem 3.2.2, T can be written as follows:

$$\begin{aligned}
T(a, \overline{b}) &= D(a, \overline{b}) + l(a, \overline{b}) \\
&= (\alpha, \overline{\beta})(a, \overline{b}) + d(a, \overline{b}) + l(a, \overline{b})
\end{aligned}$$

where $0 \neq (\alpha, \overline{\beta}) \in Z(\mathfrak{Q})$, D is a generalized derivation on $\mathfrak{Q}$, d is a derivation on $\mathfrak{Q}$ and $l : \mathfrak{Q} \to \mathfrak{Q}$ is a linear map such that its set of images is a subset of $Z(\mathfrak{Q})$ and it vanishes on all commutators of $\mathfrak{Q}$. Let $\lambda \neq 0$ be an element in K. Then, we have

$$\begin{aligned}
T(a, \overline{b}) &= (\alpha + (\lambda - \lambda), \overline{\beta})(a, \overline{b}) + d(a, \overline{b}) + l(a, \overline{b}) \\
&= (\alpha + \lambda, \overline{\beta})(a, \overline{b}) - (\lambda, 0)(a, \overline{b}) + d(a, \overline{b}) + l(a, \overline{b}).
\end{aligned}$$

Let D_λ be a linear map defined by $D_\lambda(a, \overline{b}) = (\alpha + \lambda, \overline{\beta})(a, \overline{b}) + d(a, \overline{b})$ and let l_λ be a linear map defined by $l_\lambda(a, \overline{b}) = l(a, \overline{b}) - (\lambda, 0)(a, \overline{b})$ for every $(a, \overline{b}) \in \mathfrak{Q}$. By Lemma 3.2.1, D_λ is a generalized derivation on $\mathfrak{Q}$, $D \neq D_\lambda$, and by elementary calculations we deduce that $l_\lambda : \mathfrak{Q} \to \mathfrak{Q}$ is a linear map such that its set of images is a subset of $Z(\mathfrak{Q})$ and it vanishes on all commutators of $\mathfrak{Q}$ with $l_\lambda \neq l$. This shows that the proper form of Lie generalized derivation T is not unique.

Following the example above, the following natural question arises: Which kind of bound quiver algebra has the uniqueness Lie generalized derivation property?

The aim of this section is to answer this question.

Lemma 3.3.1. *Assume that the quiver associated with $\mathfrak{Q}$ is connected and it is not a one vertex. Let D be a generalized derivation on $\mathfrak{Q}$ with $D(\overline{x}) \in Z(\mathfrak{Q})$ for every $\overline{x} \in \mathfrak{Q}$. Then, D is zero.*

Proof. Let D be a generalized derivation that satisfies $D(\overline{x}) \in Z(\mathfrak{Q})$ for every $\overline{x} \in \mathfrak{Q}$. Let s be a source in E, then $\mathfrak{Q}$ is isomorphic to the triangular algebra $\operatorname{Tri}(K, M_s, \mathfrak{Q}_s)$ where $K \cong \overline{s}\mathfrak{Q}\overline{s}$, $M_s = \overline{s}\mathfrak{Q}(\overline{1} - \overline{s})$ and $\mathfrak{Q}_s = (\overline{1} - \overline{s})\mathfrak{Q}(\overline{1} - \overline{s})$. Hence, by Lemma 3.1.1, Lemma 3.2.1 and [40, Lemma 5], the form of D is as follows

$$D\begin{pmatrix} a & \overline{m} \\ 0 & \overline{b} \end{pmatrix} = \alpha \begin{pmatrix} a & \overline{m} \\ 0 & \overline{b} \end{pmatrix} + \begin{pmatrix} p_{K,s}(a) & a\overline{n} - \overline{nb} + f_s(\overline{m}) \\ 0 & p_{\mathfrak{Q}_s}(\overline{b}) \end{pmatrix} \in Z(\mathfrak{Q})$$

for every $\begin{pmatrix} a & \overline{m} \\ 0 & \overline{b} \end{pmatrix} \in \mathfrak{Q}$, where $\alpha \in K$, $\overline{n} \in M_s$, $p_{K,s}$ is a derivation on K, $p_{\mathfrak{Q}_s}$ is a derivation on $\mathfrak{Q}_s$ and $f_s : M_s \to M_s$ is a linear map which satisfies $f_s(a\overline{m}) = p_{K,s}(a)\overline{m} + af_s(\overline{m})$ and $f_s(\overline{mb}) = \overline{m}p_{\mathfrak{Q}_s}(\overline{b}) + f_s(\overline{m})\overline{b}$. Due to the fact that every derivation on a field is zero, the form of D is reduced as follows

$$D\begin{pmatrix} a & \overline{m} \\ 0 & \overline{b} \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha\overline{m} + a\overline{n} - \overline{nb} + f_s(\overline{m}) \\ 0 & \alpha\overline{b} + p_{\mathfrak{Q}_s}(\overline{b}) \end{pmatrix} \in Z(\mathfrak{Q}).$$

By hypotheses, we have $\alpha a \in Z(K) = K$, $\alpha\overline{b} + p_{\mathfrak{Q}_s}(\overline{b}) \in Z(\mathfrak{Q}_s)$, $\alpha\overline{m} + a\overline{n} - \overline{nb} + f_s(\overline{m}) = \overline{0}$ and $\alpha a\overline{c} = \overline{c}(\alpha\overline{b} + p_{\mathfrak{Q}_s}(\overline{b}))$ for every $\overline{c} \in M_s$. In particular, for $\overline{0} \neq \overline{c} \in M_s$, $a = 1$ and $\overline{b} = \overline{0}$, we get $\alpha\overline{c} = 0$, then $\alpha = 0$ since M_s is a non-zero vector space over K. Therefore, D is a derivation. It follows from [101, Corollary 3.5] that D is zero. ☐

Proposition 3.3.2. *Assume that the quiver associated with $\mathfrak{Q}$ is connected and it is not a one vertex. Then, $\mathfrak{Q}$ has the uniqueness Lie generalized derivation property.*

Proof. Let T be a generalized derivation on $\mathfrak{Q}$. Let D_1 and D_2 be two generalized derivations on $\mathfrak{Q}$ and $l_1 : \mathfrak{Q} \to \mathfrak{Q}$ and $l_2 : \mathfrak{Q} \to \mathfrak{Q}$ be two linear maps such that their sets of images are subsets of $Z(\mathfrak{Q})$ and they vanish on all commutators of $\mathfrak{Q}$. If $T = D_1+l_1 = D_2+l_2$, then $D_1 - D_2 = l_2 - l_1$. Clearly, the set of images of $D_1 - D_2$ is a subset of $Z(\mathfrak{Q})$. Therefore, by Lemma 3.3.1, we have $D_1 - D_2 = 0$. Consequently, $D_1 = D_2$ and $l_1 = l_2$. Thus, $\mathfrak{Q}$ has the uniqueness Lie generalized derivation property. □

To generalize Proposition 3.3.2 to bound quivers with more than one connected component, we suppose that the bound quiver (E, I) associated with $\mathfrak{Q}$ has n connected component, where n is a positive integer greater than one. So, $\mathfrak{Q}$ is isomorphic to $\bigoplus_{i=1}^n K(C_i, \rho_i)$, where C_i is the ith connected component of E and ρ_i is defined as follows:

$$\rho_i = \begin{cases} \rho \cap KC_i, & \text{if } \rho \cap KC_i \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, let ϕ be a homomorphism from $KE = \bigoplus_{i=1}^n KC_i$ into $\bigoplus_{i=1}^n K(C_i, \rho_i)$ which sends each element $x_i \in KC_i$ to $x_i + < \rho_i > \in K(C_i, \rho_i)$, then $KE/\ker(\phi)$ is isomorphic to $\bigoplus_{i=1}^n K(C_i, \rho_i)$. Since the kernel of ϕ is defined by $\ker(\phi) = \{(x_1, \ldots, x_n) : x_i \in < \rho_i >\} = \bigoplus_{i=1}^n < \rho_i >$ and, for each $1 \leq k, j \leq n$, we have $< \rho_k > \cap < \rho_j > = \{0\}$, the equality $\bigoplus_{i=1}^n < \rho_i > = < \cup_i^n \rho_i > = < \rho >$ holds. Therefore, we get the desired result.

This leads us to write every derivation on $\mathfrak{Q}$ as a direct sum of derivations on $K(C_i, \rho_i)$ as follows.

Lemma 3.3.3. *Let d be a derivation on $\mathfrak{Q}$. Then, d can be written as $d = \bigoplus_{i=1}^n d_i$, where d_i is a derivation on $K(C_i, \rho_i)$ for every $1 \leq i \leq n$.*

Proof. First, we need to show that every $K(C_i, \rho_i)$ is invariant by d. Let C_{i_0} be a fixed connected component in E where $1 \leq i_0 \leq n$. For every vertex $v \in C_{i_0}$, we have by Proposition 3.2.4

$$d(\overline{v}) = \sum_{\substack{\overline{q} \in \mathcal{B}_A, \\ s(\overline{q}) = \overline{v}}} c_{\overline{q}}^{\overline{v}} \overline{q} + \sum_{\substack{\overline{q} \in \mathcal{B}_A, \\ t(\overline{q}) = \overline{v}}} c_{\overline{q}}^{\overline{v}} \overline{q}.$$

Since every element in the linear combination $d(\overline{v})$ starts or ends at $\overline{v}$, $d(\overline{v})$ is in $K(C_{i_0}, \rho_{i_0})$. Now, for every path $p \in C_{i_0}$, we have by Proposition 3.2.4

$$d(\overline{p}) = \sum_{\substack{\overline{q} \in \mathcal{B}_A, \\ t(\overline{q}) = s(\overline{p})}} c_{\overline{q}}^{s(\overline{p})} \overline{qp} + \sum_{\substack{\overline{q} \in \mathcal{B}_A, \\ s(\overline{q}) = s(\overline{p}), \\ t(\overline{q}) = t(\overline{p})}} c_{\overline{q}}^{\overline{p}} \overline{q} + \sum_{\substack{\overline{q} \in \mathcal{B}_A, \\ s(\overline{q}) = t(\overline{p})}} c_{\overline{q}}^{t(\overline{p})} \overline{pq}.$$

Since every element in the linear combination $d(\overline{p})$ starts or ends at $s(\overline{p})$ or $t(\overline{p})$, $d(\overline{p})$ is in $K(C_{i_0}, \rho_{i_0})$. Therefore, the space $K(C_{i_0}, \rho_{i_0})$ is invariant by d and the restriction $d_i = d|_{K(C_i,\rho_i)}$ on $K(C_i, \rho_i)$ defines a derivation on $K(C_i, \rho_i)$. Moreover, we get $d = \bigoplus_{i=1}^n d_i$. □

Lemma 3.2.1 shows that every generalized derivation D on $\mathfrak{Q}$ can be written as a sum of a derivation d on $\mathfrak{Q}$ and a generalized inner derivation I of the form $I(\overline{x}) = \overline{mx}$ for every $\overline{x} \in \mathfrak{Q}$, where $\overline{m} \in Z(\mathfrak{Q})$. By the definition of I, the space $K(C_i, \rho_i)$ is invariant by I and since $K(C_i, \rho_i)$ is invariant by d, we get that $K(C_i, \rho_i)$ is invariant by D, this yields to $D = \bigoplus_{i=1}^n D_i$, where D_i is a generalized derivation on $K(C_i, \rho_i)$.

Now, we can give the main result of this section.

Theorem 3.3.4. *Let $\mathcal{I}$ be the set of all generalized inner derivations I of the form $I(\overline{x}) = \overline{mx}$ for every $\overline{x} \in \mathfrak{Q}$ with $\overline{m} \in Z(\mathfrak{Q})$ and $\mathcal{L}$ be the set of all linear maps from $\mathfrak{Q}$ into itself such that their sets of images are subsets of $Z(\mathfrak{Q})$ and they vanish on all commutators of $\mathfrak{Q}$. Then, the following assertions are equivalents:*

1. *The bound quiver algebra $\mathfrak{Q}$ has the uniqueness Lie generalized derivation property.*
2. *The associated quiver with $\mathfrak{Q}$ does not contain any isolated vertex.*
3. *$\mathcal{I} \cap \mathcal{L} = \{0\}$.*

Proof. (1) $\implies$ (2). By contrapositive, we suppose that (E, ρ) contains an isolated vertex t. So, $\mathfrak{Q}$ is isomorphic to $K \oplus (\bigoplus_{i=2}^n K(C_i, \rho_i))$. Let T be a Lie generalized derivation on $\mathfrak{Q}$, then T can be expressed as $T = \bigoplus_{i=1}^n D_i + l$, D_1 is a generalized derivation on K, D_i is a generalized derivation on $K(C_i, \rho_i)$ for each $i = 2, \dots, n$ and $l \in \mathcal{L}$. By using the same reasoning as we did in the example at beginning of this section, we obtain that $T = D_\lambda + l_\lambda$ for some $0 \neq \lambda \in K$, where D_λ is a generalized derivation on $\mathfrak{Q}$ defined by $D_\lambda = D + \phi_\lambda$ and $l_\lambda \in \mathcal{L}$ defined by $l_\lambda = l - \phi_\lambda$, with $\phi_\lambda \in \mathcal{I} \cap \mathcal{L}$ defined by $\phi_\lambda(\overline{x}) = \lambda\overline{tx}$ for all $\overline{x} \in \mathfrak{Q}$. It follows that $D \neq D_\lambda$ and $l \neq l_\lambda$. Consequently, $\mathfrak{Q}$ does not have the uniqueness Lie generalized property.

(2) $\implies$ (3). By contradiction, we suppose that the associated bound quiver with $\mathfrak{Q}$ does not contain any isolated vertex and $\mathcal{I} \cap \mathcal{L} \neq \{0\}$. Let $0 \neq I \in \mathcal{I} \cap \mathcal{L}$, by hypothesis, we have $I(\overline{x}) = \overline{mx} \in Z(\mathfrak{Q})$ for every $\overline{x} \in \mathfrak{Q}$ where $\overline{m} \in Z(\mathfrak{Q})$. On the one hand, we have I vanishes on all commutators of $\mathfrak{Q}$. In particularly, I vanishes on the set $\mathcal{B}_A$, since for every element $\overline{p} \in \mathcal{B}_A$ can be written as $[s(\overline{p}), \overline{p}]$. On the other hand, we have for every $\overline{v} \in \mathcal{B}_V$, $I(\overline{v}) = \overline{mv} = \lambda_{\overline{v}}\overline{v} \in Z(\mathfrak{Q})$ for some $\lambda_{\overline{v}} \in K$. This can be true only if $\lambda_{\overline{v}} = 0$, this

is due the fact that by hypotheses E does not contain any isolated vertex, which yields by Lemma 3.1.1 to $\overline{v} \notin Z(\mathfrak{Q})$ for all $\overline{v} \in \mathcal{B}_V$. Therefore, I vanishes on $\mathcal{B}_V$, hence I is zero, contradiction.

(3) $\Longrightarrow$ (1). By contradiction, we suppose that $\mathcal{I} \cap \mathcal{L} = \{0\}$ and $\mathfrak{Q}$ does not have the uniqueness Lie generalized derivation property. Let T be a Lie generalized derivation on $\mathfrak{Q}$ such that $T = I_1 + d_1 + l_1 = I_2 + d_2 + l_2$ and $I_1 - I_2 + d_1 - d_2 = l_1 - l_2 \neq 0$, where I_1 and $I_2 \in \mathcal{I}$, l_1 and $l_2 \in \mathcal{L}$ and d_1 and d_2 are two derivations on $\mathfrak{Q}$. Let $\overline{v}$ be an element in $\mathcal{B}_V$, by Proposition 3.2.4 the linear combination $(d_1 - d_2)(\overline{v})$ is in $< \mathcal{B}_A >$, hence we obtain $(d_1 - d_2)(\overline{v}) = \overline{0}$. Suppose that E does not contain any isolated vertex. So, for any element $\overline{v} \in \mathcal{B}_V$, $(I_1 - I_2)(\overline{v}) = (l_1 - l_2)(\overline{v}) = \alpha_{\overline{v}}\overline{v} \in Z(\mathfrak{Q})$ for some $\alpha_{\overline{v}} \in K$, by Lemma 3.1.1, this is true only if $\alpha_{\overline{v}}$ is zero. Therefore, we have $(l_1 - l_2)(\overline{b}) = \overline{0}$ for every element $\overline{b} \in \mathcal{B}$, a contradiction. Now, suppose that E contains an isolated vertex. In the first case, we assume that for every isolated vertex u in E we have $(I_1 - I_2)(\overline{u}) = (l_1 - l_2)(\overline{u}) = \overline{0}$. Then, for any element $\overline{v} \in \mathcal{B}_V$ associated with a non-isolated vertex v in E, $(I_1 - I_2)(\overline{v}) = (l_1 - l_2)(\overline{v}) = \alpha_{\overline{v}}\overline{v} \in Z(\mathfrak{Q})$ for some $\alpha_{\overline{v}} \in K$. Again by Lemma 3.1.1, this is true only if $\alpha_{\overline{v}}$ is zero. Thus, we get a contradiction. In the second case, we assume that there is an isolated vertex u in E such that $(I_1 - I_2)(\overline{u}) \neq \overline{0}$, then $(I_1 - I_2)(\overline{p}) = \overline{0}$ for every $\overline{p} \in \mathcal{B}_A$. Indeed, if $(I_1 - I_2)(\overline{p}) \neq \overline{0}$, then $(I_1 - I_2)(s(\overline{p})) \neq \overline{0} \in Z(\mathfrak{Q})$, which yields by Lemma 3.1.1 a contradiction. Therefore, $0 \neq I_1 - I_2 \in \mathcal{I} \cap \mathcal{L}$, contradiction. □

Notice that Theorem 3.3.4 shows that $\mathfrak{Q}$ has the uniqueness Lie generalized derivation property if and only if the space of all Lie generalized derivations on $\mathfrak{Q}$ can be written as a direct sum of the space of all derivations on $\mathfrak{Q}$, $\mathcal{I}$ and $\mathcal{L}$.

Since an upper triangular matrix algebra is isomorphic to the path algebra KA_n, where A_n is the finite linear quiver, we deduce, from Theorem 3.3.4, that every upper triangular matrix algebra with a dimension great that one has the uniqueness Lie generalized derivation property.

CHAPTER 4

Jordan (Lie) σ-derivations on path algebras

> I am still learning.
>
> *Michelangelo*

The following chapter presents the published paper titled "**Jordan (Lie) σ-derivations on path algebras**"[1] on FILOMAT.

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[1] A. Adrabi, D. Bennis, and B. Fahid: Jordan (Lie) σ-derivations on path algebras. *FILOMAT*. 2022; 36:6231–6243.

Abstract

In this chapter, we investigate Jordan σ-derivations and Lie σ-derivations on path algebras. This work is motivated by the one of Benkovič done on triangular algebras and the study of Jordan derivations and Lie derivations on path algebras done by Li and Wei. Namely, main results state that every Jordan σ-derivation is a σ-derivation and every Lie σ-derivation is of a standard form on a path algebra when the associated quiver is acyclic and finite.

Throughout this chapter K denotes a field of characteristic different then two. Let A be a unital algebra over K and let σ be an automorphism on A. The set of all σ-derivations on A is denoted by $\mathrm{Der}_\sigma(A)$ and the set of all inner σ-derivations on A is denoted by $\mathrm{Inn}_\sigma(A)$. We denote the set of all Jordan σ-derivations on A by $\mathrm{Jor}_\sigma(A)$, and the set of all Lie σ-derivations on A by $\mathrm{Lie}_\sigma(A)$.

In the sequel, $E = (E^0, E^1, s, t)$ denotes a finite acyclic quiver, where E^0 and E^1 are sets of vertices and edges of E, respectively, and maps $s, t : E^1 \to E^0$ determine the edges of E. We denote by KE the path algebra over K associated with E and by $\mathcal{P}$ the set of all paths in E. Also, we denote by $\mathcal{P}_A$ the set of all non-trivial acyclic paths in E (for more details, see Schiffler [128]). However, it is important to notice that in our paper the product of two paths in E is defined as follows: A non-trivial path $p = e_1 \cdots e_n$ in E is a sequence of edges such that $t(e_i) = s(e_{i+1})$ for every $1 \le i < n$, and the product of two paths $p = e_1 \cdots e_n$ and $q = f_1 \cdots f_m$ in E is defined by

$$pq = \begin{cases} e_1 \cdots e_n f_1 \cdots f_m, & \text{if } t(e_n) = s(f_1), \\ 0, & \text{otherwise.} \end{cases}$$

There are some authors (see for instance [101, 102]) who prefer to define the product of paths in the opposite way as follows: A non-trivial path $p = e_n \cdots e_1$ in E is a sequence of edges such that $s(e_{i+1}) = t(e_i)$ for every $1 \le i < n$, and the product of two paths $p = e_n \cdots e_1$ and $q = f_m \cdots f_1$ in E is define by

$$pq = \begin{cases} e_n \cdots e_1 f_m \cdots f_1, & \text{if } s(e_1) = t(f_m), \\ 0, & \text{otherwise.} \end{cases}$$

The relationship between path algebras and their opposite path algebras was discussed in details by [57, Section 1].

4.1 The faithfulness property on path algebras

In this section, we investigate the faithfulness property and the loyal property on path algebras, and we give a construction of a non-trivial idempotent $\mathfrak{e}$ in a path algebra KE such that the bimodule $\mathfrak{e}KE(1-\mathfrak{e})$ is a left faithful $\mathfrak{e}KE\mathfrak{e}$-module as well a right faithful $(1-\mathfrak{e})KE(1-\mathfrak{e})$-module under some constraints.

Recall that a triangular algebra A is a unital algebra that contain a non-trivial idempotent $\mathfrak{e}$ such that $\mathfrak{e}A(1-\mathfrak{e})=0$. Hence, it can be written as $A=\mathfrak{e}A\mathfrak{e}+\mathfrak{e}A(1-\mathfrak{e})+(1-\mathfrak{e})A(1-\mathfrak{e})$ or in a matrix form

$$\begin{pmatrix} \mathfrak{e}A\mathfrak{e} & \mathfrak{e}A(1-\mathfrak{e}) \\ 0 & (1-\mathfrak{e})A(1-\mathfrak{e}) \end{pmatrix}.$$

When $\mathfrak{e}A\mathfrak{e}$ is isomorphic to a field K, it is called a one-point extension algebra rather than a triangular algebra. Since for a path algebra KE over a field K, the subspace $sKEs$ is isomorphic to K for any source s of E, thus one can view KE as a one-point extension with $sKE(1-s)$ is a vector space over K (see [4, Preliminaries]). Hence, $sKE(1-s)$ is faithful as a left K-module.

Recall that a (A,B)-bimodule M is called loyal if $aMb=\{0\}$ implies that $a=0$ or $b=0$ for every a in A and b in B. We have the following immediate result:

Lemma 4.1.1. *Let KE be a path algebra which admits a source s. Then, $sKE(1-s)$ is a loyal $(K,(1-s)KE(1-s))$-bimodule if and only if E has only one source.*

Since $sKE(1-s)$ is a K-vector space, it is evident that to check $sKE(1-s)$ is a loyal, it suffices to show that it is faithful as a right $(1-s)KE(1-s)$-module. Here, we give some examples and counterexamples.

Example 4.1.2. Let E be the following quiver:

$$\begin{tikzcd} s \arrow[r, "e_1"] & v_1 \arrow[r, "e_2"] & v_2 \arrow[r, "e_3"] \arrow[dr, "e_5"] & v_3 \arrow[r, "e_4"] & v_4 \\ & & & v_5 & \end{tikzcd}$$

Since E has only one source, it follows by Lemma 4.1.1 that $sKE(1-s)$ is a loyal $(K,(1-s)KE(1-s))$-bimodule.

Example 4.1.3. Let E be the following quiver:

$$\begin{tikzcd} s \arrow[r, "e_1"] & v_1 \arrow[r, "e_2"] & v_2 & t \arrow[l, "e_3"'] \end{tikzcd}$$

Since E has two sources s and t, it follows by Lemma 4.1.1 that $sKE(1-s)$ is not a loyal $(K,(1-s)KE(1-s))$-bimodule. This can be checked by a straightforward calculations. Without loss of generality, choose the source s as a non-trivial idempotent. Then, we have $sKE(1-s)t=sKEt=\{0\}$, but $t\neq 0$, hence $sKE(1-s)$ is not a right faithful $(1-s)KE(1-s)$-module, so it is not a loyal $(K,(1-s)KE(1-s))$-bimodule.

One may ask what will happen if we choose either the vertex v_1 or the vertex v_2 instead of the source s or the source t in Example 4.1.3. To investigate this case, we recall first the definition of generalized matrix algebra. Let A and B be two K-algebras, M a (A, B)-bimodule, N a (B, A)-bimodule, and $\Phi_{MN} : M \otimes_B N \to A$ and $\Psi_{NM} : N \otimes_A M \to B$ two bimodule homomorphisms, called the pairings, satisfying the following commutative diagrams:

$$\begin{tikzcd} M \otimes_B N \otimes_A M \arrow[r, "\Phi_{MN}\otimes Id_M"] \arrow[d, "Id_M\otimes\Psi_{NM}"] & A \otimes_A M \arrow[d, "\cong"'] \\ M \otimes_B B \arrow[r, "\cong"] & M \end{tikzcd} , \quad \begin{tikzcd} N \otimes_A M \otimes_B N \arrow[r, "\Psi_{NM}\otimes Id_N"] \arrow[d, "Id_N\otimes\Phi_{MN}"] & B \otimes_B N \arrow[d, "\cong"'] \\ N \otimes_A A \arrow[r, "\cong"] & N. \end{tikzcd}$$

Then, the set

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} = \left\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} \middle| a \in A, m \in M, n \in N, b \in B \right\}$$

forms an K-algebra under matrix-like addition and matrix-like multiplication. There is no constraint condition concerning bimodules M and N. Such a K-algebra is called a generalized matrix algebra. Let $\mathcal{A}$ be a unital K-algebra with a non-trivial idempotent $\mathfrak{e}$, then $\mathcal{A}$ is isomorphic to the generalized matrix algebra

$$\begin{pmatrix} \mathfrak{e}\mathcal{A}\mathfrak{e} & \mathfrak{e}\mathcal{A}(1-\mathfrak{e}) \\ (1-\mathfrak{e})\mathcal{A}\mathfrak{e} & (1-\mathfrak{e})\mathcal{A}(1-\mathfrak{e}) \end{pmatrix},$$

without the assumption of the bimodule $(1-\mathfrak{e})\mathcal{A}\mathfrak{e}$ equals to zero as in triangular algebras.

Now, choose v_1 in Example 4.1.3 as a non-trivial idempotent. Then, we have

$$KE \cong \begin{pmatrix} v_1 K v_1 & K\{e_2\} \\ K\{e_1\} & K\{s, t, v_2, e_3\} \end{pmatrix} \cong \begin{pmatrix} K & K\{e_2\} \\ K\{e_1\} & K\{s, t, v_2, e_3\} \end{pmatrix}.$$

And, neither $K\{e_2\}$ is a right faithful $K\{s, t, v_2, e_3\}$-module nor $K\{e_1\}$ is a left faithful $K\{s, t, v_2, e_3\}$-module even for Example 4.1.2. Then, by choosing a vertex which is neither a source nor a sink, we obtain the following result.

Theorem 4.1.4. *There is no quiver such that the* $(K, (1-v)KE(1-v))$*-bimodule* $vKE(1-v)$ *or the* $((1-v)KE(1-v), K)$*-bimodule* $(1-v)KEv$ *is loyal, when* v *is a vertex which is neither a source nor a sink.*

Proof. Let E be a quiver and v be a vertex which is neither a source nor a sink. Since $vKE(1-v)$ and $(1-v)KEv$ are vector spaces over K, it follows that we only need to show that $vKE(1-v)$ is not a right faithful $(1-v)KE(1-v)$-module and $(1-v)KEv$ is not a left faithful $(1-v)KE(1-v)$-module. By hypotheses all sources and sinks are in $(1-v)KE(1-v)$, therefore by choosing any source s, we obtain that $vKE(1-v)s = 0$, also by choosing any sink t, we obtain $t(1-v)KEv = 0$. Finally, bimodules $vKE(1-v)$ and $(1-v)KEv$ are not loyal. □

Now, we aim to construct a non-trivial idempotent $\mathfrak{e}$ in KE such that the bimodule $\mathfrak{e}KE(1-\mathfrak{e})$ is faithful as a left $\mathfrak{e}KE\mathfrak{e}$-module and also as a right $(1-\mathfrak{e})KE(1-\mathfrak{e})$-module. To this end, we state the following proposition. Recall that a vertex is called isolated if there is no edge that starts or ends at it.

Proposition 4.1.5. *Let $\mathfrak{e}$ be the sum of all sources in E. Then, the bimodule $\mathfrak{e}KE(1-\mathfrak{e})$ is faithful as a left $\mathfrak{e}KE\mathfrak{e}$-module and also as a right $(1-\mathfrak{e})KE(1-\mathfrak{e})$-module if and only if E does not contain isolated vertices.*

Proof. Let $\mathfrak{e}$ be the sum of all sources in E. Assume that the bimodule $\mathfrak{e}KE(1-\mathfrak{e})$ is faithful as a left $\mathfrak{e}KE\mathfrak{e}$-module and also as a right $(1-\mathfrak{e})KE(1-\mathfrak{e})$-module, hence $\mathfrak{e}KE(1-\mathfrak{e})$ contains all paths that start from all sources in E. Then, for every source s in E, we have $s\mathfrak{e}KE(1-\mathfrak{e}) \neq 0$, and for every vertex v in E not a source, we have $\mathfrak{e}KE(1-\mathfrak{e})v \neq 0$. Hence, E does not contain an isolated vertex.

Now, assume that E contains an isolated vertex w, then $w\mathfrak{e}KE(1-\mathfrak{e}) = \mathfrak{e}KE(1-\mathfrak{e})w = 0$. Hence, $\mathfrak{e}KE(1-\mathfrak{e})$ is not a faithful module as a left module nor as a right module. □

In general, the bimodule $\mathfrak{e}KE(1-\mathfrak{e})$ is not a loyal bimodule even if E is a connected quiver. Indeed, in Example 4.1.3, we set $\mathfrak{e} = s+t$, then it follows that $\mathfrak{e}KE(1-\mathfrak{e}) = K\{e_1, e_1e_2, e_3\}$, and for the elements t and v_1, we have $t\mathfrak{e}KE(1-\mathfrak{e})v_1 = 0$. So, $\mathfrak{e}KE(1-\mathfrak{e})$ is not a loyal bimodule.

4.2 Jordan σ-derivations on path algebras

In this section, we study Jordan σ-derivations on path algebras. The main result of this section states that every Jordan σ-derivation is a σ-derivation.

The following lemma is useful throughout the paper, it states that an automorphism on a path algebra cannot translate vertices back and forth on the same non-trivial path.

Lemma 4.2.1. *Let σ be an automorphism on KE. Then,*

1. *For a non-trivial path* $p \in \mathcal{P}_A$*, we have* $\sigma(t(p))(KE)s(p) = \{0\}$*.*
2. *For a vertex* $v \in E^0$*, we have* $\sigma(v)(\mathcal{P}_A)v = \{0\}$*.*

Proof. Assume by contradiction that $\sigma(t(p))(KE)s(p) \neq \{0\}$. Then, there exists a path $k \in \mathcal{P}$ such that $\sigma(t(p))kp \neq 0$. Hence, $\sigma(\sigma(t(p))kp)\sigma(t(p))kp = \sigma(\sigma(t(p))kp)kp \neq 0$ with the length of all paths in the linear combination $\sigma(\sigma(t(p))kp)kp$ is greater than the length of p. By repeating the same reasoning recursively, we obtain a contradiction since E is a finite and acyclic quiver.

Now, assume by contradiction that $\sigma(v)kv \neq 0$ for some $k \in \mathcal{P}_A$. Then, we have $\sigma(\sigma(v)kv)kv \neq 0$. By the same reasoning as we did before, we obtain a contradiction. □

The main theorem of this section shows that every Jordan σ-derivation on a path algebra is a σ-derivation without assuming the faithfulness property of the bimodule $sKE(1-s)$, where s is a source in E. A similar result has established in [22, Theorem 3.1] for triangular algebras with the faithfulness condition. But before that, we construct a new Jordan σ-derivation g_f on KE from an arbitrary Jordan σ-derivation f on KE which is constructed by a similar reasoning as in [22, Lemma 3.3] on triangular algebras. Let σ be an automorphism on KE, f be a Jordan σ-derivation on KE, and a_f be an element in KE defined as follows

$$a_f = \sum_{u \in E^0} \sigma(u)f(u)(1-u) - \sigma(1-u)f(u)u.$$

Let d_f be an inner σ-derivation on KE defined by $d_f(x) = \sigma(x)a_f - a_f x$ for every $x \in KE$. Then, we have

$$d_f(v) = \sigma(v)f(v)(1-v) + \sigma(1-v)f(v)v - \sum_{\substack{u \in E^0 \\ u \neq v}} \sigma(v)f(u)u - \sum_{\substack{u \in E^0 \\ u \neq v}} \sigma(u)f(u)v,$$

for every $v \in E^0$. Since $f(v) = f(v)v + \sigma(v)f(v)$ for every vertex $v \in E^0$, it follows that $\sigma(v)f(v)v = 0$ and $f(v)(1-v) = \sigma(v)f(v)(1-v)$ for every vertex $v \in E^0$. Hence, for every vertex $v \in E^0$, we obtain

$$\begin{aligned} f(v) &= f(v)v + f(v)(1-v) \\ &= \sigma(v)f(v)v + \sigma(1-v)f(v)v + f(v)(1-v) \\ &= \sigma(1-v)f(v)v + \sigma(v)f(v)(1-v). \end{aligned}$$

Therefore, we define g_f on KE by $g_f = f - d_f$, then g_f is a Jordan σ-derivation on KE, and it satisfies the following equality:

$$g_f(v) = \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)f(u)u + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)f(u)v, \tag{4.1}$$

for every $v \in E^0$. Since for every vertex $w \neq v$ where v is a fixed vertex we have

$$\sigma(v)g_f(v)w = \sigma(v)f(w)w \text{ and } \sigma(w)g_f(v)v = \sigma(w)f(w)v,$$

it follows that the equality (4.1) can be written as

$$g_f(v) = \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)g_f(v)u + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)g_f(v)v, \tag{4.2}$$

for every $v \in E^0$. Note that non-trivial paths are nilpotents in KE.

The main result of this section is stated as follows.

Theorem 4.2.2. *Every Jordan σ-derivation on KE is a σ-derivation.*

Proof. Let f be a Jordan σ-derivation on KE. According to the discussion above, we may assume that f is the sum of an inner σ-derivation on KE and a Jordan σ-derivation g_f on KE satisfies the equality (4.2). Let v and w be two different vertices. Then, we have

$$\begin{aligned}
0 = g_f(v\circ w) &= g_f(v)w + \sigma(v)g_f(w) + g_f(w)v + \sigma(w)g_f(v)\\
&= \Bigg(\sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)g_f(v)u + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)g_f(v)v\Bigg)w\\
&\quad + \sigma(v)\Bigg(\sum_{\substack{u\in E^0\\ u\neq w}} \sigma(w)g_f(w)u + \sum_{\substack{u\in E^0\\ u\neq w}} \sigma(u)g_f(w)w\Bigg)\\
&\quad + \Bigg(\sum_{\substack{u\in E^0\\ u\neq w}} \sigma(w)g_f(w)u + \sum_{\substack{u\in E^0\\ u\neq w}} \sigma(u)g_f(w)w\Bigg)v\\
&\quad + \sigma(w)\Bigg(\sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)g_f(v)u + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)g_f(v)v\Bigg)\\
&= \sigma(v)g_f(v)w + \sigma(v)g_f(w)w + \sigma(w)g_f(w)v + \sigma(w)g_f(v)v.
\end{aligned}$$

By multiplying the last line by $\sigma(v)$ (resp. $\sigma(w)$)) from the left and by w (resp. v) from the right, it yields to

$$\sigma(v)g_f(v)w + \sigma(v)g_f(w)w = \sigma(w)g_f(v)v + \sigma(w)g_f(w)v = 0. \tag{4.3}$$

Hence, we deduce

$$\begin{aligned}
0 &= g_f(v)w + \sigma(v)g_f(w) \\
&= \Bigg(\sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)g_f(v)u + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)g_f(v)v\Bigg)w \\
&\quad + \sigma(v)\Bigg(\sum_{\substack{u\in E^0\\ u\neq w}} \sigma(w)g_f(w)u + \sum_{\substack{u\in E^0\\ u\neq w}} \sigma(u)g_f(w)w\Bigg) \\
&= \sigma(v)g_f(v)w + \sigma(v)g_f(w)w.
\end{aligned}$$

Therefore, we obtain $0 = g_f(vw) = g_f(v)w + \sigma(v)g_f(w)$. When vertices v and w are equal, it follows immediately that $g_f(v) = g_f(v)v + \sigma(v)g_f(v)$ for every vertex v. For a non-trivial path $p \in \mathcal{P}_A$, by the equality (4.2) and Lemma 4.2.1, we have:

$$\begin{aligned}
g_f(p) &= g_f(s(p)\circ p) \\
&= g_f(s(p))p + \sigma(s(p))g_f(p) + g_f(p)s(p) + \sigma(p)g_f(s(p)) \\
&= \sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)g_f(s(p))p + \sigma(s(p))g_f(p) + g_f(p)s(p) \qquad (4.4) \\
&= g_f(t(p)\circ p) \\
&= g_f(t(p))p + \sigma(t(p))g_f(p) + g_f(p)t(p) + \sigma(p)g_f(t(p)) \\
&= \sigma(t(p))g_f(p) + g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)g_f(t(p))u. \qquad (4.5)
\end{aligned}$$

By substituting (4.5) in (4.4) and using Lemma 4.2.1, we obtain

$$\begin{aligned}
g_f(p) = &\sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)g_f(s(p))p \\
&+ \sigma(s(p))\Bigg(\sigma(t(p))g_f(p) + g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)g_f(t(p))u\Bigg)
\end{aligned}$$

$$+\Big(\sigma(t(p))g_f(p)+g_f(p)t(p)+\sum_{\substack{u\in E^0\\ u\neq t(p)}}\sigma(p)g_f(t(p))u\Big)s(p)$$
$$=\sum_{\substack{u\in E^0\\ u\neq s(p)}}\sigma(u)g_f(s(p))p+\sigma(s(p))g_f(p)t(p)+\sum_{\substack{u\in E^0\\ u\neq t(p)}}\sigma(p)g_f(t(p))u.$$

Therefore, we obtain $g_f(p)=g_f(s(p))p+\sigma(s(p))g_f(p)=g_f(p)t(p)+\sigma(p)g_f(t(p))$. Let v be a vertex and p a non-trivial path such that $pv=vp=0$. Then, we have

$$\begin{aligned}g_f(p)v+\sigma(p)g_f(v)&=\Big(\sum_{\substack{u\in E^0\\ u\neq s(p)}}\sigma(u)g_f(s(p))p+\sigma(s(p))g_f(p)t(p)+\sum_{\substack{u\in E^0\\ u\neq t(p)}}\sigma(p)g_f(t(p))u\Big)v\\&\quad+\sigma(p)\Big(\sum_{\substack{u\in E^0\\ u\neq v}}\sigma(v)g_f(v)u+\sum_{\substack{u\in E^0\\ u\neq v}}\sigma(u)g_f(v)v\Big)\\&=\sigma(p)\Big(\sigma(t(p))g_f(t(p))v+\sigma(t(p))g_f(v)v\Big).\end{aligned}$$

By equality (4.3), we obtain $g_f(p)v+\sigma(p)g_f(v)=0$. Then, $g_f(pv)=g_f(p)v+\sigma(p)g_f(v)$. Similarly, we have

$$\begin{aligned}g_f(v)p+\sigma(v)g_f(p)&=\Big(\sum_{\substack{u\in E^0\\ u\neq v}}\sigma(v)g_f(v)u+\sum_{\substack{u\in E^0\\ u\neq v}}\sigma(u)g_f(v)v\Big)p\\&\quad+\sigma(v)\Big(\sum_{\substack{u\in E^0\\ u\neq s(p)}}\sigma(u)g_f(s(p))p+\sigma(s(p))g_f(p)t(p)+\sum_{\substack{u\in E^0\\ u\neq t(p)}}\sigma(p)g_f(t(p))u\Big)\\&=\Big(\sigma(v)g_f(v)s(p)+\sigma(v)g_f(s(p))s(p)\Big)p\end{aligned}$$

By equality (4.3), we obtain $g_f(v)p+\sigma(v)g_f(p)=0$. Then, $g_f(vp)=g_f(v)p+\sigma(v)g_f(p)$. Now, let p and q be two non-trivial paths in $\mathcal{P}_A$. On the first hand, we assume that $p\circ q\neq 0$. Without loss of generality we suppose that $pq\neq 0$, then we have

$$\begin{aligned}g_f(pq)&=g_f(p\circ q)\\&=g_f(p)q+\sigma(p)g_f(q)+g_f(q)p+\sigma(q)g_f(p)\\&=g_f(p)q+\sigma(p)g_f(q)\\&\quad+\Big(\sum_{\substack{u\in E^0\\ u\neq s(q)}}\sigma(u)g_f(s(q))q+\sigma(s(q))g_f(p)t(q)+\sum_{\substack{u\in E^0\\ u\neq t(q)}}\sigma(q)g_f(t(p))u\Big)p\end{aligned}$$

$$
\begin{aligned}
&+\sigma(q)\Bigg(\sum_{\substack{u\in E^0\\ u\neq s(p)}}\sigma(u)g_f(s(p))p+\sigma(s(p))g_f(p)t(p)+\sum_{\substack{u\in E^0\\ u\neq t(p)}}\sigma(p)g_f(t(p))u\Bigg)\\
&=g_f(p)q+\sigma(p)g_f(q).
\end{aligned}
$$

This is due to Lemma 4.2.1. On the other hand, we assume that $p\circ q=0$, it follows that $pq=qp=0$. Then, we have

$$
\begin{aligned}
g_f(p)q+\sigma(p)g_f(q)&=\Bigg(\sum_{\substack{u\in E^0\\ u\neq s(p)}}\sigma(u)g_f(s(p))p+\sigma(s(p))g_f(p)t(p)+\sum_{\substack{u\in E^0\\ u\neq t(p)}}\sigma(p)g_f(t(p))u\Bigg)q\\
&\quad+\sigma(p)\Bigg(\sum_{\substack{u\in E^0\\ u\neq s(q)}}\sigma(u)g_f(s(q))q+\sigma(s(q))g_f(p)t(q)+\sum_{\substack{u\in E^0\\ u\neq t(q)}}\sigma(q)g_f(t(q))u\Bigg)\\
&=\sigma(p)g_f(t(p))q+\sigma(p)g_f(s(q))q\\
&=\sigma(p)\Big(\sigma(t(p))g_f(t(p))s(q)+\sigma(t(p))g_f(s(q))s(q)\Big)q.
\end{aligned}
$$

Hence, by equality (4.3), we obtain $g_f(pq)=g_f(p)q+\sigma(p)g_f(q)=0$. Which yields that g_f is a σ-derivation on KE, thus f is a σ-derivation on KE. The proof is completed. $\square$

In the next example, we will fix an automorphism σ on KE and we will show that for an arbitrary Jordan σ-derivation on KE, it is a σ-derivation. We express every linear map f on KE by

$$f(p)=\sum_{q\in\mathcal{P}}c_q^p q\quad(\forall p\in\mathcal{P}).$$

This is true due to the fact that the set $\mathcal{P}$ is finite, and it is a basis of KE as a K-vector space.

Example 4.2.3. Let E be the following quiver:

$$\begin{tikzcd}
v_1 \arrow[r, bend left, "e_2"] \arrow[r, "e_1"'] & v_3 & v_2. \arrow[l, bend right, "e_4"'] \arrow[l, "e_3"]
\end{tikzcd}$$

Let σ be an automorphisms on KE defined as follows: $\sigma(v_1)=v_1+e_1+e_2$, $\sigma(v_2)=v_2+e_3+e_4$, $\sigma(v_3)=v_3-e_1-e_2-e_3-e_4$, $\sigma(e_1)=e_2$, $\sigma(e_2)=e_1$, $\sigma(e_3)=e_3$ and $\sigma(e_4)=e_4$. Let f be a Jordan σ-derivation on KE. We only need to compute the image of the elements of the basis $\mathcal{P}$. The computation is as follows:

$$f(v_1)=2^{-1}f(v_1\circ v_1)$$

$$\begin{aligned}
&= 2^{-1}(f(v_1)v_1 + \sigma(v_1)f(v_1) + f(v_1)v_1 + \sigma(v_1)f(v_1))\\
&= f(v_1)v_1 + \sigma(v_1)f(v_1)\\
&= f(v_1)v_1 + (v_1 + e_1 + e_2)f(v_1)\\
&= 2c^{v1}_{v_1}v_1 + c^{v1}_{e_1}e_1 + c^{v1}_{e_2}e_2 + c^{v1}_{v_3}(e_1 + e_2).
\end{aligned}$$

Hence, we get $f(v_1) = c^{v1}_{e_1}e_1 + c^{v1}_{e_2}e_2$. Similarly, we obtain $f(v_2) = c^{v2}_{e_3}e_3 + c^{v2}_{e_4}e_4$. We have

$$\begin{aligned}
f(v_3) &= 2^{-1}f(v_3 \circ v_3)\\
&= 2^{-1}(f(v_3)v_3 + \sigma(v_3)f(v_3) + f(v_3)v_3 + \sigma(v_3)f(v_3))\\
&= f(v_3)v_3 + \sigma(v_3)f(v_3)\\
&= f(v_3)v_3 + (v_3 - e_1 - e_2 - e_3 - e_4)f(v_3)\\
&= 2c^{v3}_{v_3}v_3 + c^{v3}_{e_1}e_1 + c^{v3}_{e_2}e_2 + c^{v3}_{e_3}e_3 + c^{v3}_{e_4}e_4 - c^{v3}_{v_3}(e_1 + e_2 + e_3 + e_4).
\end{aligned}$$

Hence, we get $f(v_3) = c^{v3}_{e_1}e_1 + c^{v3}_{e_2}e_2 + c^{v3}_{e_3}e_3 + c^{v3}_{e_4}e_4$. We compute the images of all edges as follows

$$\begin{aligned}
f(e_1) &= f(v_1 \circ e_1)\\
&= f(v_1)e_1 + \sigma(v_1)f(e_1) + f(e_1)v_1 + \sigma(e_1)f(v_1)\\
&= (v_1 + e_1 + e_2)f(e_1) + f(e_1)v_1\\
&= 2c^{e1}_{v_1}v_1 + c^{e1}_{e_1}e_1 + c^{e1}_{e_2}e_2 + c^{e1}_{v_3}(e_1 + e_2)
\end{aligned}$$

Hence, we get $f(e_1) = c^{e1}_{e_1}e_1 + c^{e1}_{e_2}e_2$. Similarly, we obtain $f(e_2) = c^{e2}_{e_1}e_1 + c^{e2}_{e_2}e_2$, $f(e_3) = c^{e3}_{e_3}e_3 + c^{e3}_{e_4}e_4$ and $f(e_4) = c^{e4}_{e_3}e_3 + c^{e4}_{e_4}e_4$. By straightforward verification, we deduce that f is a σ-derivation on KE.

Denote by $\mathrm{Id}([A, A])$ the ideal of A generated by all commutators in A. As in [22], an algebra A is not of a triangular form if for each idempotent $\mathfrak{e}$ in A the condition $(1-\mathfrak{e})A\mathfrak{e} = \{0\}$ implies that $\mathfrak{e}A(1-\mathfrak{e}) = \{0\}$. In the next theorem, we assume that $\mathfrak{e}A(1-\mathfrak{e})$ is faithful as a left $\mathfrak{e}A\mathfrak{e}$-module and also as a right $(1-\mathfrak{e})A(1-\mathfrak{e})$-module.

Theorem 4.2.4 ([22, Theorem 4.1])**.** *Let A be a 2-torsion free triangular matrix algebra. Let us assume that one of the following statements holds:*

1. $\mathfrak{e}A\mathfrak{e}$ *is not of a triangular form,*
2. $(1-\mathfrak{e})A(1-\mathfrak{e})$ *is not of a triangular form,*
3. $\mathfrak{e}A\mathfrak{e} = Id([\mathfrak{e}A\mathfrak{e}, \mathfrak{e}A\mathfrak{e}])$,

4. $(1-\mathfrak{e})A(1-\mathfrak{e}) = \mathit{Id}([(1-\mathfrak{e})A(1-\mathfrak{e}), (1-\mathfrak{e})A(1-\mathfrak{e})])$,

5. $\mathfrak{e}A(1-\mathfrak{e})$ *is a loyal* $(\mathfrak{e}A\mathfrak{e}, (1-\mathfrak{e})A(1-\mathfrak{e}))$*-bimodule.*

Then, any Jordan σ-derivation on A is a σ-derivation.

Here, an immediate consequence of Lemma 4.1.1, Proposition 4.1.5, and Theorem 4.2.2.

Corollary 4.2.5. *In the case when $A = KE$ with E is a quiver without isolated vertices, the faithfulness constraint for Theorem 4.2.4 is unnecessary and one of the conditions* (1)*,* (2) *or* (5) *is always satisfied.*

4.3 Lie σ-derivations on path algebras

In this section, we study Lie σ-derivations on path algebras. The main result of this section states that every Lie σ-derivation is of a standard form. Also, we show that $\mathrm{Lie}_\sigma(KE) = \mathrm{Der}(KE)_\sigma \oplus \mathrm{L}_\sigma(KE)$, where $\mathrm{L}_\sigma(KE)$ is the set of all maps that vanish on all commutators of KE and their values are in the σ-center of KE.

Recall a σ-centre $Z_\sigma(A)$ of A is the set defined by $Z_\sigma(A) = \{\lambda \in A : \sigma(x)\lambda = \lambda x, \forall x \in A\}$, where σ is an automorphism on A. A more detailed discussion about σ-centres was provided in [24, Section 2].

The main result of this section shows that every Lie σ-derivation a standard form without assuming the faithfulness property of the bimodule $sKE(1-s)$ with s is a source in E. A similar result established in [24, Theorem 3.5] for triangular algebras with the condition of faithfulness. As in the previous section, we construct a new Lie σ-derivation g_f on KE from an arbitrary Lie σ-derivation f on KE, and it is done by a similar reasoning as in [24, Lemma 3.2] on triangular algebras. Let σ be an automorphism on KE, f be a Lie σ-derivation on KE, and a_f be an element in KE defined as follows

$$a_f = \sum_{u \in E^0} \sigma(u)f(u)(1-u) - \sigma(1-u)f(u)u.$$

Let d_f be an inner σ-derivation on KE defined by $d_f(x) = \sigma(x)a_f - a_f x$ for every $x \in KE$.

Then, we have

$$d_f(v) = \sigma(v)f(v)(1-v) + \sigma(1-v)f(v)v - \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)f(u)u - \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)f(u)v,$$

for every $v \in E^0$. We define g_f on KE by

$$g_f = f - d_f, \tag{4.6}$$

hence g_f is a Lie σ-derivation on KE satisfies the following equality:

$$g_f(v) = \sigma(v)f(v)v + \sigma(1-v)f(v)(1-v) + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)f(u)u + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)f(u)v, \tag{4.7}$$

for every $v \in E^0$. Now, let δ_f be a linear map on KE defined as follows:

$$\delta_f(v) = \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)f(u)u + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)f(u)v, \tag{4.8a}$$

$$\delta_f(p) = g_f(p), \tag{4.8b}$$

for every $v \in E^0$ and $p \in \mathcal{P}_A$. We claim that δ_f is a σ-derivation on KE. Indeed, let $v \in E^0$. Then, we have

$$\begin{aligned}\delta_f(v)v + \sigma(v)\delta_f(v) &= \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(u)f(u)v + \sum_{\substack{u\in E^0\\ u\neq v}} \sigma(v)f(u)u\\ &= \delta_f(v).\end{aligned}$$

Let v and w be two different vertices. Then, we have

$$\begin{aligned}0 &= \delta_f(v)w + \sigma(v)\delta_f(w)\\ &= \sigma(v)f(w)w + \sigma(v)f(v)w\\ &= \delta_f(vw).\end{aligned}$$

This is due to the fact that

$$\begin{aligned}0 &= g_f([v,w])\\ &= g_f(v)w + \sigma(v)g_f(w) - g_f(w)v - \sigma(w)g_f(v)\end{aligned}$$

$$\begin{aligned}&= \sigma(1-v)f(v)w + \sigma(v)f(w)w + \sigma(v)f(w)(1-w) + \sigma(v)f(v)w \\ &\quad - \sigma(1-w)f(w)v - \sigma(w)f(v)v - \sigma(w)f(v)(1-v) - \sigma(w)f(w)v. \qquad (4.9)\end{aligned}$$

Hence, by multiplying (4.9) from the left by $\sigma(v)$ and from right by w, we obtain

$$\sigma(v)f(w)w + \sigma(v)f(v)w = 0, \qquad (4.10)$$

for every two different vertices v and w. Let p be a non-trivial path in $\mathcal{P}_A$. Then, we have

$$\begin{aligned}\delta_f(s(p))p + \sigma(s(p))\delta_f(p) &= \sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)f(u)p + \sigma(s(p))g_f(p), \\ \delta_f(p)t(p) + \sigma(p)\delta_f(t(p)) &= g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)f(u)u.\end{aligned}$$

Also, we have

$$\begin{aligned}g_f(p) &= g_f([s(p),p]) \\ &= g_f(s(p))p + \sigma(s(p))g_f(p) - g_f(p)s(p) - \sigma(p)g_f(s(p)) \qquad (4.11) \\ &= g_f([p,t(p)]) \\ &= g_f(p)t(p) + \sigma(p)g_f(t(p)) - g_f(t(p))p - \sigma(t(p))g_f(p). \qquad (4.12)\end{aligned}$$

Then, by multiplying (4.11) from the left by $\sigma(t(p))$ and multiplying (4.12) from the right by $s(p)$, and by applying Lemma 4.2.1, we obtain

$$g_f(p)s(p) = \sigma(t(p))g_f(p) = 0.$$

Also, we have by equality (4.7):

$$\begin{aligned}g_f(s(p))p &= \sigma(s(p))f(s(p))p + \sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)f(u)p, \\ \sigma(p)g_f(t(p)) &= \sigma(p)f(t(p))t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)f(u)u, \\ \sigma(p)g_f(s(p)) &= \sigma(s(p))g_f(s(p))p = \sigma(s(p))f(s(p))p, \\ g_f(t(p))p &= \sigma(p)g_f(t(p))t(p) = \sigma(p)f(t(p))t(p).\end{aligned}$$

Hence, we deduce that

$$g_f(p) = \sum_{\substack{u\in E^0\\ u\neq s(p)}} \sigma(u)f(u)p + \sigma(s(p))g_f(p) = g_f(p)t(p) + \sum_{\substack{u\in E^0\\ u\neq t(p)}} \sigma(p)f(u)u. \tag{4.13}$$

Which yields by equality (4.7) to

$$\begin{aligned}\delta_f(p) &= \delta_f(s(p))p + \sigma(s(p))\delta_f(p)\\ &= \delta_f(p)t(p) + \sigma(p)\delta_f(t(p)).\end{aligned}$$

Let v be a vertex in E^0 and p be a non-trivial path in $\mathcal{P}_A$ such that $[v,p]=0$. Then, we have

$$\begin{aligned}0 &= \delta_f(v)p + \sigma(v)\delta_f(p)\\ &= \sigma(v)f(s(p))p + \sigma(v)f(v)p\\ &= \delta_f(p)v + \sigma(p)\delta_f(v)\\ &= \sigma(p)f(v)v + \sigma(p)f(t(p))v.\end{aligned}$$

Hence, by equality (4.10), we obtain $\delta_f(vp) = \sigma(v)f(v)p + \sigma(v)f(s(p))p = 0$ and $\delta_f(pv) = \sigma(p)f(v)v + \sigma(p)f(t(p))v = 0$. Let p and q be non-trivial paths in $\mathcal{P}_A$ such that $pq \neq 0$. By Lemma 4.2.1 and the equality (4.13), we have $\delta_f(q)p = g_f(q)p = \sigma(s(q))g_f(q)p = 0$ and $\sigma(q)\delta_f(p) = \sigma(q)g_f(p) = \sigma(q)g_f(p)t(p) = 0$. Thus, we obtain

$$\delta_f(pq) = \delta_f(p)q + \sigma(p)\delta_f(q).$$

Let p and q be non-trivial paths in $\mathcal{P}_A$ such that $pq = qp = 0$. Then, we have

$$\begin{aligned}0 = \delta_f(pq) &= \delta_f(p)q + \sigma(p)\delta_f(q)\\ &= \sigma(s(p))\delta_f(p)q + \sigma(p)\delta_f(s(q))q.\end{aligned}$$

This due to the fact that

$$\begin{aligned}0 &= g_f([p,q])\\ &= g_f(p)q + \sigma(p)g_f(q) - g_f(q)p - \sigma(q)g_f(p)\\ &= \sigma(s(p))\delta_f(p)q + \sigma(p)\delta_f(s(q))q - \sigma(s(q))\delta_f(q)p - \sigma(q)\delta_f(s(p))p.\end{aligned}$$

Which yields to $\sigma(s(p))\delta_f(p)q + \sigma(p)\delta_f(s(q))q = 0$. Finally, we obtain that δ_f is a σ-

derivation on KE and the claim is proved. Therefore, g_f is the sum of δ_f and the linear map l_f on KE defined by

$$l_f(v) = \sigma(v)f(v)v + \sigma(1-v)f(v)(1-v), \text{ and } l_f(p) = 0, \tag{4.14}$$

for every vertex v and a non-trivial path p. Now, we are in a position to state the main result of this section.

Theorem 4.3.1. *Every Lie σ-derivation on KE is of a standard form.*

Proof. Let f be a Lie σ-derivation on KE. According to equality (4.6), we may assume that f is a sum of a Lie σ-derivation g_f on KE and an inner σ-derivation d_f on KE. And, by the discussion above, we assume that g_f is a sum of σ-derivation δ_f on KE and linear map l_f on KE defined as in (4.14). Since l_f vanishes on commutators of KE by construction, we only need to show that $l_f(v) \in Z_\sigma(KE)$ for every vertex v in E to prove that f is of a standard form.

Let $v \in E^0$ be a fixed vertex, from the hypotheses, we obtain that $l_f = g_f - \delta_f$ is a Lie σ-derivation on KE. Hence, for every non-trivial path $p \in \mathcal{P}_A$, we have $l_f([v,p]) = l_f(v)p - \sigma(p)l_f(v) = 0$. Which yields to $\sigma(p)l_f(v) = l_f(v)p$ for every $p \in \mathcal{P}_A$. Let $v \neq u \in E^0$, then we have

$$0 = l_f([v,u]) = l_f(v)u + \sigma(v)l_f(u) - l_f(u)v - \sigma(u)l_f(v). \tag{4.15}$$

Multiplying (4.15) from the left by $\sigma(u)$, we obtain

$$0 = \sigma(u)f(v)u - \sigma(u)f(v)(1-v). \tag{4.16}$$

And, multiplying (4.15) from the right by u, we obtain

$$0 = \sigma(1-v)f(v)u - \sigma(u)f(v)u. \tag{4.17}$$

Hence, we have $\sigma(u)f(v)(1-v) = \sigma(1-v)f(v)u$, which yields to $\sigma(u)l_f(v) = l_f(v)u$. For the case when $v = u$, we have $\sigma(v)l_f(v) = \sigma(v)f(v)v = l_f(v)v$. Therefore, $\sigma(u)l_f(v) = l_f(v)u$ for every vertex $u \in E^0$. Finally, we deduce that $l_f(v) \in Z_\sigma(KE)$ for every vertex $v \in E^0$. □

Denote by $W(A)$ the algebra generated by idempotents and commutators of A. In the next proposition, we assume that $\mathfrak{e}A(1-\mathfrak{e})$ is faithful as a left $\mathfrak{e}A\mathfrak{e}$-module and also as a right $(1-\mathfrak{e})A(1-\mathfrak{e})$-module, where $\mathfrak{e}$ is a non-trivial idempotent in A.

Proposition 4.3.2 ([24, Corollary 4.4]). *Let A be a 2-torsion free triangular algebra such that $A = W(A)$. Then, any Lie σ-derivation d of A is of the form $d = \Delta + \gamma$, where $\Delta : A \to A$ is a σ-derivation and $\gamma : A \to Z_\sigma(A)$ is a linear map the vanishes on $[A, A]$.*

One can state a similar result to Corollary 4.2.5. Since every non-trivial path in E can be viewed as a commutator and all vertices are idempotents, we have the following consequence of Theorem 4.3.1 and Proposition 4.1.5.

Corollary 4.3.3. *In the case when $A = KE$ with E is a quiver without isolated vertices, then the faithfulness constraint for Proposition 4.3.2 is unnecessary.*

In the next example, we use a quiver from [51], which we can define on it an automorphism does not map a vertex to vertex.

Example 4.3.4. Let $K = F_5$ and let E be the following quiver $v_1 \xrightarrow{\quad e \quad} v_2$ and σ an automorphism on KE defined as in [51, Page 1398], i.e.:

$$\sigma(v_1) = v_1 + e, \sigma(v_2) = v_2 - e \text{ and } \sigma(e) = e.$$

Then, the σ-centre of KE is $Z_\sigma(KE) = \{0, v_1+v_2+4e, 2v_1+2v_2+3e, 3v_1+3v_2+2e, 4v_1+4v_2+e\}$. Let f be the Lie σ-derivation on KE defined by

$$f(v_1) = v_1 + v_2 + e, f(v_2) = 4v_1 + 4v_2 + 4e \text{ and } f(e) = 3e.$$

Then, f can be written as a sum of a σ-derivation d on KE defined by

$$d(v_1) = 2e, d(v_2) = 3e \text{ and } d(e) = 4e,$$

and a linear map l on KE that vanishes on all commutators of KE and its values are in σ-center of KE, which is defined by

$$l(v_1) = v_1 + v_2 + 4e, l(v_2) = 4v_1 + 4v_2 + e \text{ and } l(e) = 0.$$

Denote the set of all maps that vanishes on all commutators and their values are in the σ-center of A by $L_\sigma(A)$. Notice that every element of $L_\sigma(A)$ is a Lie σ-derivation on A. Indeed, let l be an element in $L_\sigma(A)$, thus by definition, we have $l([x, y]) = 0$ for every x and y in A. And, we have

$$l(x)y + \sigma(x)l(y) - l(y)x - \sigma(y)l(x) = 0,$$

due to the fact that $\mathrm{Im}(l)$ is a subset of σ-center of A. Hence, we obtain the next result.

Corollary 4.3.5. *The following sequence is exact and split as K-vector spaces.*

$$0 \longrightarrow L_\sigma(KE) \xrightarrow{\varphi} Lie_\sigma(KE) \xrightarrow{\psi} Der_\sigma(KE) \longrightarrow 0, \tag{4.18}$$

where φ is a canonical inclusion and $\psi : f \mapsto f - l_f$ with l_f is the associated map with f that vanishes on all commutators and its values are in the σ-center of KE as defined in (4.14).

Proof. On the first hand, the map φ is K-linear by definition. And, on the other hand, by Theorem 4.3.1, ψ is an epimorphism and $\mathrm{Im}(\varphi) = \mathrm{Ker}(\psi)$. Thus, the sequence (4.18) is exact. To show it splits, define $\overline{\psi} : \mathrm{Der}_\sigma(KE) \to \mathrm{Lie}_\sigma(KE)$ to be the canonical injection. Then, for every derivation d in $\mathrm{Der}_\sigma(KE)$, we have $\psi\overline{\psi}(d) = \psi(d) = d - l_d = d$, due to the fact that $l_d = 0$. Therefore, $\psi\overline{\psi}$ is the identity on $\mathrm{Der}_\sigma(KE)$ and the sequence (4.18) is split. $\square$

CHAPTER 5

Jordan $\mathcal{G}_n$-derivations on path algebras

> Just because we can't find a solution, it doesn't mean there isn't one.
>
> *Andrew Wiles*

The following chapter presents the published paper titled “**Jordan $\mathcal{G}_n$-derivations on path algebras**”[1] on Communications of the Korean Mathematical Society.

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[1] A. Adrabi, D. Bennis, and B. Fahid: Jordan $\mathcal{G}_n$-Derivations on Path Algebras. *Communications of the Korean Mathematical Society.* 2022; 37:957–967.

Abstract

Recently, Brešar's Jordan $\{g, h\}$-derivations has been investigated on triangular algebras. As a first aim of this chapter, we extend this study to an interesting general context. Namely, we introduce the notion of Jordan $\mathcal{G}_n$-derivations, with $n \geq 2$, which is a natural generalization of Jordan $\{g, h\}$-derivations. Then, we study this notion on path algebras. We prove that, when $n > 2$, every Jordan $\mathcal{G}_n$-derivation on a path algebra is a $\{g, h\}$-derivation. However, when $n = 2$, we give an example showing that this implication does not hold true in general. So, we characterize when it holds. As a second aim, we give a positive answer to a variant of Lvov-Kaplansky conjecture on path algebras. Namely, we show that the set of values of a multi-linear polynomial on a path algebra KE is either $\{0\}$, KE or the space spanned by paths of a length greater than or equal to 1.

Throughout this chapter, K will denote a field with characteristic zero, A will be a K-algebra with the center $Z(A)$.

5.1 Definitions

One can naturally continue the way started by Brešar and introduce a rather general case of Jordan $\{g,h\}$-derivations using the following notations: denote $x_1 \circ x_2$ by $\circ_2 x_i$ for all $x_1, x_2 \in A$ and $(\circ_{n-1} x_i) \circ x_n$ by $\circ_n x_i$ for all $x_1, \dots, x_n \in A$ with $n \geq 2$. By convention, we set by $\circ_0 x_i = \frac{1}{2}$ and $\circ_1 x_i = x_1$ for all $x_1 \in A$. Whence, the generalization of Jordan $\{g,h\}$-derivations is stated as follows: let $\mathcal{G}_n = \{g_i\}_{1 \leq i \leq n}$ be a finite family of linear maps on A with $n \geq 2$. We say that a linear map $f : A \to A$ is a Jordan $\mathcal{G}_n$-derivation, if for every n-tuple $(x_1, \dots, x_n) \in A^n$

$$f(\circ_n x_i) = \sum_{j=1}^{n} (((\circ_{j-1} x_i \circ g_{\sigma(j)}(x_j)) \circ x_{j+1}) \cdots) \circ x_n, \qquad \forall \sigma \in S_n, \tag{5.1}$$

where S_n is the symmetric group of degree n. Also, following Brešar's approach, we consider the following notion: we say that a linear map $f : A \to A$ is a $\mathcal{G}_n$-derivation on A, if for every $(x_1, \dots, x_n) \in A^n$,

$$f(\prod_{i=1}^{n} x_i) = \sum_{i=1}^{n} x_1 \cdots x_{i-1} g_{\sigma(i)}(x_i) x_{i+1} \cdots x_n, \qquad \forall \sigma \in S_n. \tag{5.2}$$

However, following similar argument done by Brešar, we deduce that $\mathcal{G}_n$-derivations and $\{g,h\}$-derivations are the same. In fact, let f be a $\mathcal{G}_n$-derivation with $n \geq 2$. Then, by taking $x_2 = \cdots = x_n = 1$ in (5.2), we obtain

$$f(x_1) = g_{\sigma(1)}(x_1) + x_1 g_{\sigma(2)}(1) + x_1(\sum_{i=3}^{n} g_{\sigma(i)}(1)), \quad \forall \sigma \in S_n. \tag{5.3}$$

And taking $x_1 = x_3 = \cdots = x_n = 1$ in (5.2), we obtain

$$f(x_2) = g_{\sigma(1)}(1) x_2 + g_{\sigma(2)}(x_2) + x_2(\sum_{i=3}^{n} g_{\sigma(i)}(1)), \quad \forall \sigma \in S_n. \tag{5.4}$$

Comparing both expressions, we see that every $g_i(1)$ lies in $Z(A)$. Setting $\lambda = f(1) = \sum_{i=1}^{n} g_i(1)$, we then infer from (5.2) and (5.3) that, for all $i \in \{1, \dots, n\}$, $f(x) - \lambda x = g_i(x) - g_i(1)x$. If we set $d(x) = f(x) - \lambda x$ for all $x \in A$, then d is a derivation. Thus, every $\mathcal{G}_n$-derivation f can be written as

$$f(x) = \lambda x + d(x). \tag{5.5}$$

Therefore, every $\mathcal{G}_n$-derivation can be viewed as a generalized derivation on A. Conversely, if a linear map f has the form as in (5.5) and $\lambda = \sum_i^n \lambda_i$, where $\lambda_i \in Z(A)$. Then, f is a $\mathcal{G}_n$-derivation on A where each g_i is defined by

$$g_i(x) = \lambda_i x + d(x) \quad (x \in A). \tag{5.6}$$

In this context, it is natural to ask whether a Jordan $\mathcal{G}_n$-derivation is nothing but a Jordan $\mathcal{G}_2$-derivation?

In order to answer this question, we investigate Jordan $\mathcal{G}_n$-derivations on path algebras associated with a finite acyclic quiver. Thus, we assume some familiarity with basic notions of path algebras (for more details, see [128]).

In the sequel, $E = (E^0, E^1, s, t)$ designates a finite acyclic quiver, where E^0 and E^1 are sets of vertices and edges of E, respectively, and the maps s and t from E^1 into E^0 determine the edges of E. We denote by KE the path algebra over K associated with E.

5.2 Main results

In this section, the set $\mathcal{G}_n$ will be a fixed family $\{g_i\}_{1\leq i\leq n}$ of linear maps on KE, where $n \geq 2$. We show when every Jordan $\mathcal{G}_n$-derivation on path algebras is a $\mathcal{G}_n$-derivation. We will see that for every $n > 2$ this implication holds, however for the case $n = 2$, it does not as shown by the following example.

Example 5.2.1. Let E be the following quiver: $v_2 \xleftarrow{e_1} v_1 \xrightarrow{e_2} v_3$ and let f be a Jordan $\mathcal{G}_2$-derivation on KE defined by:

$$\begin{aligned}
&f(v_1) = 2v_1, && g_1(v_1) = v_1 + e_1 + e_2, && g_2(v_1) = v_1 - e_1 - e_2,\\
&f(v_2) = 2v_2, && g_1(v_2) = v_2 + e_1, && g_2(v_2) = v_2 - e_1,\\
&f(v_3) = 2v_3, && g_1(v_3) = v_3 + e_2, && g_2(v_3) = v_3 - e_2,\\
&f(e_1) = 2e_1, && g_1(e_1) = e_1, && g_2(e_1) = e_1,\\
&f(e_2) = 2e_2, && g_1(e_2) = e_2, && g_2(e_2) = e_2.
\end{aligned}$$

By elementary calculations, we have $g_1(v_1)v_1 + v_1 g_2(v_1) \neq f(v_1^2)$, hence f is not a $\mathcal{G}_2$-derivation on KE.

To prove the main results of this section, we need the following lemma.

Lemma 5.2.2. *For every Jordan $\mathcal{G}_n$-derivation f on KE with $n \geq 2$, $f(1)$ is in $Z(KE)$. Moreover, if $n > 2$, then $g_i(1)$ is in $Z(KE)$ for all $i \in \{1, \ldots, n\}$.*

Proof. Assume f to be a Jordan $\mathcal{G}_n$-derivation on KE with $n \geq 2$. Let z be a non-trivial idempotent in KE. Then, we have

$$\begin{aligned} 0 &= f((((z \circ (1-z)) \circ 1) \cdots) \circ 1) \\ &= (((g_1(z) \circ (1-z)) \circ 1) \cdots) \circ 1 + ((((z \circ g_2(1-z)) \circ 1) \cdots) \circ 1 + 0 \\ &= 2^{n-2}(g_1(z) \circ (1-z) + z \circ g_2(1-z)) \\ &= g_1(z) \circ (1-z) + z \circ g_2(1-z) \\ &= 2g_1(z) - g_1(z)z - zg_1(z) + zg_2(1) - zg_2(z) + g_2(1)z - g_2(z)z \end{aligned} \tag{5.7}$$

Multiplying (5.7) by z from the left, we obtain

$$0 = zg_1(z) - zg_1(z)z + zg_2(1) - zg_2(z) + zg_2(1)z - zg_2(z)z \tag{5.8}$$

Multiplying (5.7) by z from the right, we obtain

$$0 = g_1(z)z - zg_1(z)z + zg_2(1)z - zg_2(z)z + g_2(1)z - g_2(z)z \tag{5.9}$$

By comparing the equalities (5.8) and (5.9), we get

$$zg_1(z) + zg_2(1) - zg_2(z) = g_1(z)z + g_2(1)z - g_2(z)z. \tag{5.10}$$

Similarly, by the definition of Jordan $\mathcal{G}_n$-derivations, we obtain

$$zg_{\sigma(1)}(z) + zg_{\sigma(2)}(1) - zg_{\sigma(2)}(z) = g_{\sigma(1)}(z)z + g_{\sigma(2)}(1)z - g_{\sigma(2)}(z)z, \tag{5.11}$$

for every $\sigma \in S_n$. Therefore, we have

$$zg_2(z) + zg_1(1) - zg_1(z) = g_2(z)z + g_1(1)z - g_1(z)z. \tag{5.12}$$

It follows from (5.10) and (5.12) that

$$z(g_1(1) + g_2(1)) = (g_1(1) + g_2(1))z. \tag{5.13}$$

Since every element $s(p)+p$ is a non-trivial idempotent in KE with p is a non-trivial path in E, $g_1(1)+g_2(1)$ commutes with all paths in KE. Thus, $g_1(1)+g_2(1) \in Z(KE)$. Hence by the definition of Jordan $\mathcal{G}_n$-derivations, we conclude that $g_{\sigma(1)}(1)+g_{\sigma(2)}(1) \in Z(KE)$ for all $\sigma \in S_n$. Now, assume that $n > 2$, then it follows that $g_1(1)+g_2(1)$, $g_3(1)+g_2(1)$ and $g_1(1)+g_3(1)$ are in $Z(KE)$. Since $Z(KE)$ is a group, we have $g_1(1)+g_2(1)-g_3(1)-g_2(1) = g_1(1)-g_3(1) \in Z(KE)$. Therefore, $g_1(1)-g_3(1)+g_1(1)+g_3(1) = 2g_1(1) \in Z(KE)$. So, $g_1(1) \in Z(KE)$. By similar reasoning, we obtain that all $g_i(1)$ are in $Z(KE)$. □

We start with the first main result which treats the case $n = 2$.

Theorem 5.2.3. *Every Jordan $\mathcal{G}_2$-derivation f on KE is a $\mathcal{G}_2$-derivation if and only if $g_1(1) \in Z(KE)$ or $g_2(1) \in Z(KE)$.*

Proof. It is clear that if f is a $\mathcal{G}_2$-derivation, then $g_1(1) \in Z(KE)$ and $g_2(1) \in Z(KE)$. So, it remains to prove the converse implication. Let f be a Jordan $\mathcal{G}_2$-derivation on KE, then we have

$$f(x \circ y) = g_1(x) \circ y + x \circ g_2(y) \quad (x, y \in KE). \tag{5.14}$$

Take $y = 1$ in (5.14), then we obtain

$$f(x) = g_1(x) + x \circ g_2(\frac{1}{2}) \quad (x \in KE). \tag{5.15}$$

Similarly, take $x = 1$, then we obtain

$$f(y) = g_2(y) + y \circ g_1(\frac{1}{2}) \quad (y \in KE). \tag{5.16}$$

Without loss of generality, suppose that $g_1(1) \in Z(KE)$. It follows by Lemma 5.2.2, that $g_2(1) \in Z(KE)$. Therefore, the equalities (5.15) and (5.16) become $f(x) = g_1(x) + g_2(1)x$ and $f(y) = g_2(y) + g_1(1)y$ for all $x, y \in KE$, respectively. For all $x, y \in KE$, we have

$$\begin{aligned} f(x \circ y) &= g_1(x) \circ y + x \circ g_2(y) \\ &= (f(x) - g_2(1)x) \circ y + x \circ (f(y) - g_1(1)y) \\ &= f(x) \circ y + x \circ (f(y) - f(1)y). \end{aligned}$$

Hence, f is a Jordan generalized derivation on KE. Therefore, by the discussion in [4, Preliminaries] and [101, Proposition 3.7], f is a generalized derivation with $f(1) = g_1(1) + g_2(1)$. Hence, it follows that f is a $\mathcal{G}_2$-derivation. □

In the rest of this chapter, $\mathcal{P}$ will denote the set of all paths in E including vertices.

Note that $\mathcal{P}$ is a basis of KE as a K-vector space. Now, for the case where $n > 2$, we have the following second main result.

Theorem 5.2.4. *Every Jordan $\mathcal{G}_n$-derivation on KE with $n > 2$ is a $\mathcal{G}_n$-derivation.*

Proof. Let f be a Jordan $\mathcal{G}_n$-derivation on KE with $n > 2$. Then, for every path $p \in \mathcal{P}$, we have

$$\begin{aligned} f(p) &= \frac{1}{2^{n-1}} f(((p \circ 1) \cdots) \circ 1) \\ &= \frac{1}{2^{n-1}} \big(((g_1(p) \circ 1) \cdots) \circ 1 + \cdots + ((p \circ 1) \cdots) \circ g_n(1)\big) \\ &= \frac{1}{2^{n-1}} \big(2^{n-1} g_1(p) + 2^{n-1} (\sum_{i=2}^{n} g_i(1)) p\big) \\ &= g_1(p) + (\sum_{i=2}^{n} g_i(1)) p. \end{aligned} \tag{5.17}$$

And,

$$\begin{aligned} f(p) &= \frac{1}{2^{n-1}} f(((p \circ 1) \cdots) \circ 1) \\ &= \frac{1}{2^{n-1}} \big(((g_2(p) \circ 1) \cdots) \circ 1 + \cdots + ((p \circ 1) \cdots) \circ g_n(1)\big) \\ &= \frac{1}{2^{n-1}} \big(2^{n-1} g_2(p) + 2^{n-1} (\sum_{\substack{i=1 \\ i \neq 2}}^{n} g_i(1)) p\big) \\ &= g_2(p) + (\sum_{\substack{i=1 \\ i \neq 2}}^{n} g_i(1)) p. \end{aligned} \tag{5.18}$$

This is due to the fact that by Lemma 5.2.2, all $g_i(1) \in Z(KE)$. We claim that f is a Jordan generalized derivation, we only need to check it on every element in $\mathcal{P}$. Let x and y be two elements in $\mathcal{P}$. Then, we have

$$\begin{aligned} f(x \circ y) &= \frac{1}{2^{n-2}} f((((x \circ y) \circ 1) \cdots) \circ 1) \\ &= \frac{1}{2^{n-2}} \big((((g_1(x) \circ y) \circ 1) \cdots) \circ 1 + (((x \circ g_2(y)) \circ 1) \cdots) \circ 1\big) \\ &\quad + \frac{1}{2^{n-2}} \big((((x \circ y) \circ g_3(1)) \cdots) \circ 1 + \cdots + (((x \circ y) \circ 1) \cdots) \circ g_n(1)\big) \\ &= g_1(x) \circ y + x \circ g_2(y) + (x \circ y)(\sum_{i=3}^{n} g_i(1)). \end{aligned} \tag{5.19}$$

It follows by (5.17) and (5.18) that

$$
\begin{aligned}
(5.19) &= (f(x) - (\sum_{i=2}^{n} g_i(1))x) \circ y + x \circ (f(y) - (\sum_{\substack{i=1\\ i\neq 2}}^{n} g_i(1))y) \\
&\quad + (x \circ y)(\sum_{i=3}^{n} g_i(1)) \\
&= f(x) \circ y + x \circ (f(y) - (\sum_{i=1}^{n} g_i(1))y).
\end{aligned}
$$

Hence, f is a Jordan generalized derivation on KE. Therefore, by the discussion in [4, Preliminaries] and [101, Proposition 3.7], f is a generalized derivation with $f(1) = \sum_{i=1}^{n} g_i(1)$ and $g_i(1) \in Z(KE)$. Hence, it follows that f is a $\mathcal{G}_n$-derivation. $\square$

5.3 Application on a variant of Lvov-Kaplansky conjecture

In this section, we investigate a variant of Lvov-Kaplansky conjecture (see Question 2.2 in the introduction). Our main result is as follows.

In the proof, we denote the length of a path p in E by $\ell(p)$ (i.e. the number of edges in the path p). By convention, we set the length of vertices to zero.

Theorem 5.3.1. *Let $\zeta(x_1, \dots, x_n) = \sum_{\sigma\in S_n} c_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ be a multi-linear polynomial over K, with $c_\sigma \in K$. Then, the set of values of ζ on KE is either $\{0\}$, KE or the space spanned by paths of a length greater than or equal to 1.*

Proof. We prove the result by recurrence on the length l of the longest path in E. Let V_j be the space spanned by paths in E with a length greater than or equal to $j \in \mathbb{N}$. It follows that $V_0 = KE$ and $V_{l+k+1} = \{0\}$ for all $k \in \mathbb{N}$, since there is no path with a length greater than l. Now, define I_p to be

$$
I_p = \{(x_1, \dots, x_n) \in (\mathcal{P} \cup \{1\})^n : \exists \sigma \in S_n, \prod_{i=1}^{n} x_{\sigma(i)} = p\}, \tag{5.20}
$$

where $p \in \mathcal{P}$. Let $\zeta(x_1, \dots, x_n) = \sum_{\sigma\in S_n} c_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ be a multi-linear polynomial over K, where $c_\sigma \in K$. Let d_σ be a $\mathcal{G}_n$-derivation on KE with $g_i = \frac{c_\sigma}{n} I$, where I is the identity map on KE. Then, ζ can be written as

$$\zeta(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} c_\sigma \prod_{i=1}^{n} x_{\sigma(i)} = \sum_{\sigma \in S_n} d_\sigma (\prod_{i=1}^{n} x_{\sigma(i)}),$$

for every $(x_1, \ldots, x_n) \in (KE)^n$. Let $p \in \mathcal{P}$ with $\ell(p) = 0$. Assume that there exists an element $x \in I_p$ such that $\zeta(x) \neq 0$. Then, $\sum_{\sigma \in S_n} c_\sigma \neq 0$. Hence, we have

$$\zeta(x) = (\sum_{\sigma \in S_n} d_\sigma)(p) = \alpha_p p,$$

for every $p \in \mathcal{P}$ and for every $x \in I_p$, where $\alpha_p \in K^*$. Therefore, by linearity, the set of values of ζ on KE is KE itself. Now, to prove the set of values of ζ on KE is V_j, where $0 < j \leq l$, we assume that, for every $q \in \mathcal{P}$ with $\ell(q) < j$, and for every $y \in I_q$, we have $\zeta(y) = 0$ and there exists $x_0 \in I_{p_0}$ for some $p_0 \in \mathcal{P}$ with $\ell(p_0) = j$ such that $\zeta(x_0) \neq 0$. Then, there exists a subset $S_{x_0} = \{\sigma \in S_n : \prod_{i=1}^{n} x_{\sigma(i),0} \neq 0\}$ of S_n such that $\sum_{\sigma \in S_{x_0}} c_\sigma \neq 0$. Hence, we have

$$\zeta(x) = (\sum_{\sigma \in S_{x_0}} d_\sigma)(p) = \alpha_p p,$$

for every $p \in \mathcal{P}$ with $\ell(p) \geq j$ and for every $x \in I_p$ with the components of x has a similar decomposition of sub-paths of p as x_0 of p_0, where $\alpha_p \in K^*$. Therefore, the set of values of ζ on KE is V_j. Otherwise, if $\zeta(y) = 0$ for every $q \in \mathcal{P}$ and every $y \in I_q$, then the set of values of ζ on KE is $\{0\}$. □

We end this section with the following examples. We assume in these examples that KE has some paths of length greater than or equal to 2 and $K = \mathbb{C}$ or $K = \mathbb{R}$.

Example 5.3.2. Consider the multi-linear polynomial $\zeta(x_1, x_2, x_3) = (x_1 \circ x_2) \circ x_3$ over K. Then, the set of values of ζ on KE is KE itself. This is due the fact that all coefficients are positive. Therefore, for every $p \in \mathcal{P}$, we have $\zeta(p, 1, 1) = \alpha_p p$, as desired.

In the following example, we use the notation of the proof of Theorem 5.3.1.

Example 5.3.3. Consider the multi-linear polynomial

$$\zeta(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4 - x_1 x_2 x_4 x_3 - x_2 x_1 x_3 x_4 + x_2 x_1 x_4 x_3$$

over K. Then, the set of values of ζ on KE is the space spanned by all paths of a length greater than or equal to 2. This can be checked by choosing a path $p_0 = e_1 \cdots e_l$ in $\mathcal{P}$ with

$\ell(p) \geq 2$ and $x_0 = (t(e_1), e_1, t(e_1), e_2 \cdots e_l)$. Hence, $\zeta(x_0) = -p_0$. Therefore, by similar decomposition of all paths with a length greater than or equal to 2 as the decomposition done for p_0 into sub-paths in x_0, we obtain the desired result.

Recall the following definition of Lie polynomials of order 3.

Definition 5.3.4 ([11, Definition 4])**.** A non-zero multi-linear Lie polynomial ζ of degree 3 is a polynomial over K that can be written in the form

$$\zeta(x_1, x_2, x_3) = c_1[[x_1, x_2], x_3] + c_2[[x_1, x_3], x_2],$$

where c_1 and c_2 are not both 0 and $c_i \in K$.

Example 5.3.5. Let ζ be the Lie polynomial of the order 3 defined as:

$$\begin{aligned}\zeta(x_1, x_2, x_3) &= [[x_1, x_2], x_3] + [[x_1, x_3], x_2] \\ &= x_1x_2x_3 + x_1x_3x_2 - 2x_2x_1x_3 + x_2x_3x_1 - 2x_3x_1x_2 + x_3x_2x_1.\end{aligned}$$

Then, the set of values of ζ on KE is the space spanned by all paths with a length greater than or equal to 1. Indeed, for every $p \in \mathcal{P}$ with $\ell(p) = 0$, and every $x \in I_p$, $\zeta(x) = 0$, where I_p is defined as in (5.20). Now, for an edge p_0 in $\mathcal{P}$, we have $x_0 = (p_0, t(p_0), t(p_0)) \in I_{p_0}$ and $\zeta(x_0) = 2p_0 \neq 0$. Hence, for every path p with $\ell(p) > 0$, we have

$$\zeta(p, t(p), t(p)) = 2p.$$

By linearity, we deduce that the set of values of ζ on KE is the space spanned by paths with length at least one.

For the definition of Lie polynomials of order 4, we have the following definition.

Definition 5.3.6 ([11, Definition 5])**.** A non-zero multi-linear Lie polynomial ζ of degree 4 is a polynomial over K that can be written in the form

$$\begin{aligned}\zeta(x_1, x_2, x_3, x_4) = {} & c_1[[[x_1, x_2], x_3], x_4] + c_2[[[x_1, x_2], x_4], x_3] + c_3[[[x_1, x_3], x_2], x_4] \\ & + c_4[[[x_1, x_3], x_4], x_2] + c_5[[[x_1, x_4], x_2], x_3] + c_6[[[x_1, x_4], x_3], x_2],\end{aligned}$$

where c_i are not all 0 and $c_i \in K$.

Example 5.3.7. Let ζ be the Lie polynomial of the order 4 defined as:

$$\zeta(x_1, x_2, x_3, x_4) = [[[x_1, x_2], x_4], x_3] + [[[x_1, x_3], x_4], x_2] - 2[[[x_1, x_4], x_2], x_3]$$

$$\begin{aligned}
&= x_1x_2x_4x_3 + x_1x_3x_4x_2 - 2x_1x_4x_2x_3 - x_2x_1x_3x_4 + x_2x_1x_4x_3 \\
&+ x_2x_3x_1x_4 - x_2x_4x_1x_3 - x_2x_4x_3x_1 - x_3x_1x_2x_4 + x_3x_1x_4x_2 \\
&- x_3x_2x_1x_4 + 2x_3x_2x_4x_1 - x_3x_4x_1x_2 - x_3x_4x_2x_1 + x_4x_1x_2x_3 \\
&- x_4x_1x_3x_2 + x_4x_2x_1x_3 + x_4x_3x_1x_2.
\end{aligned}$$

By the same reasoning as in the previous example, we choose an edge p_0 in $\mathcal{P}$, we have $x_0 = (s(p_0), p, t(p_0), t(p_0)) \in I_{p_0}$ and $\zeta(x_0) = p_0 \neq 0$. Hence, we conclude that the set of values of ζ on KE is the space spanned by paths with length at least one.

Since 2×2-upper triangular matrix algebra $T_2(K)$ is isomorphic to path algebra associated with the line quiver $E_2 : v_1 \xrightarrow{e} v_2$, we have the following result:

Corollary 5.3.8 ([138, Theorem 1.1])**.** *Let K be a field with characteristic zero. Let $\zeta(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} c_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ be a multi-linear polynomial over K, with $c_\sigma \in K$. Then, the image of ζ on KE_2 is KE_2, Ke or $\{0\}$.*

By similar reasoning, when K is a field with characteristic zero, the main result [48, Theorem 3] is generalized from strictly upper triangular matrix algebras to upper triangular matrix algebras $T_m(K) \cong KE_m$, where $m \geq 2$ and E_m is the line quiver $v_1 \xrightarrow{e_1} v_2 \cdots v_{m-1} \xrightarrow{e_{m-1}} v_m$.

CHAPTER 6

On local (like) derivations on path algebras

> Without knowledge action is useless and knowledge without action is futile.
>
> *Abu Bakr*

The following chapter presents the published paper titled "**On Local (like) Derivations on Path Algebras**"[1] on Acta Mathematica Vietnamica.

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[1] A. Adrabi, D. Bennis, and B. Fahid: On Local (like) Derivations on Path Algebras. *Acta Mathematica Vietnamica.* 2023; 48:387–399.

Abstract

In this paper, we investigate local derivations and local generalized derivations on path algebras associated with finite acyclic quivers. We show that every local derivation on a path algebra is a derivation, and every local generalized derivation on a path algebra is a generalized derivation. Also, we apply main results on several related maps to local derivations. The established results generalize several ones on some known algebras such as incidence algebras.

Throughout this paper K is a field of characteristic different than 2 and A is a unital algebra over K. We assume some familiarity with the basic notions of path algebras (for more details, see [128]). In the sequel, $Q = (Q_0, Q_1, s, t)$ designates a finite acyclic quiver, where Q_0 and Q_1 are sets of vertices and edges of Q, respectively, and the maps $s, t : Q_1 \to Q_0$ determine the edges of Q. We denote by KQ the path algebra over K associated with Q. Also, we denote by $\mathcal{P}Q$ the set of all paths in Q, and we denote by $\mathcal{P}Q_A$ the set of all non-trivial acyclic paths in Q.

6.1 Path algebras and (finitary) incidence algebras

In this section, we discuss the relationship between path algebras and (finitary) incidence algebras. This section is aimed to be self-contained to expose the relationship between path algebras and (finitary) incidence algebras. We start by recalling the definition of pre-ordered sets and partially ordered sets.

We recall the definition of incidence algebras of partially ordered set.

Definition 6.1.1 ([131, Definition 1.2.1])**.** The incidence algebra $I(X, R)$ of a locally finite partially ordered set X over a unital commutative ring R is the set:

$$I(X, R) = \{f : X \times X \to R | f(x, y) = 0 \text{ if } x \not\leq y\},$$

with operations given by

$$\begin{aligned} (f + g)(x, y) &= f(x, y) + g(x, y), \\ (f \cdot g)(x, y) &= \sum_{x \leq z \leq y} f(x, z)g(z, y), \text{ and} \\ (r \cdot f)(x, y) &= r \cdot f(x, y), \end{aligned}$$

for f and g in $I(X, R)$ with r in R and x, y and z in X.

The unity element 1 of $I(X, R)$ is given by $1(x, y) = \delta_{x,y}$ for $x \leq y$, where $\delta_{x,y}$ is the Kronecker delta. If x and y in X with $x \leq y$, let $e_{xy} : X \times X \to R$ be defined by

$$e_{xy}(a, b) = \begin{cases} 1 & \text{if } x = a \text{ and } y = b, \\ 0 & \text{otherwise,} \end{cases}$$

where a and b are in X. Then, we have $e_{xy}e_{wz} = \delta_{y,w}e_{xz}$, and we write e_x for e_{xx}. Moreover, the set $\{e_{xy} | x \leq y\}$ is a linear basis of $I(X, R)$. Some authors define incidence algebras on more general settings (see [131]). For example, X being a pre-ordered set which can be illustrated by the work of [119], where they investigated local derivations on finite incidence algebras defined as follows: Let X is a pre-ordered set such that $\text{card}(X) = n$ with n is a positive natural number, then the algebra $I(X, R)$ can be seen as a sub-algebra of the full matrix algebra $M_n(R)$ and $I(X, R)$ is called a finite incidence algebra over R. In other words, let n be a fixed positive natural number and $I_n = \{1, \ldots, n\}$, and let ρ be a reflexive and transitive relation on I_n. Then, the set

$$M_n(R)_\rho = \{M \in M_n(R) : M_{ij} = 0, \forall (i, j) \notin \rho\},$$

where M_{ij} is the (i, j)-coefficient in the matrix M with (i, j) is in $I_n \times I_n$. Then, $M_n(R)_\rho$ is a finite incidence algebra.

An algebra B is called a Kadison algebra if every local derivation on B is a derivation on B. In [119], Nowicki and Nowosad state that every local derivation on a finite incidence algebra over R is a derivation as follows:

Theorem 6.1.2 ([119, Theorem 3]). *Let R be a unital commutative ring. Then, every algebra of the form $M_n(R)_\rho$ is a Kadison algebra.*

Therefore, a path algebra over K associated with a acyclic quiver without parallel paths is a particular case of finite incidence algebras (see [10] for more details). Hence, we have the following corollary.

Corollary 6.1.3. *For every finite acyclic quiver Q without parallel paths, the path algebra KQ is a Kadison algebra.*

When a quiver Q has parallel paths, the path algebra KQ is no longer a finite incidence algebra as showed by the next example.

Example 6.1.4. Let Q be the following quiver:

$$\begin{tikzcd} Q : v \arrow[r, bend left, "e"] \arrow[r, bend right, "f"'] & u . \end{tikzcd}$$

Then, KQ is isomorphic to sub-algebra of $M_3(K)$ generated by the set

$$S = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Hence, there is no reflexive and transitive relation ρ on the set $I_3 = \{1, 2, 3\}$ such that $M_3(K)_\rho$ is isomorphic to KS. Therefore, KS is not a finite incidence algebra.

In [80], Khripchenko and Novikov generalized the notion of incidence algebras to finitary incidence algebras as follows: Let $(X, \leq)$ be a partially ordered set and denote by $MI(X, R)$ the R-module of formal sums:

$$a = \sum_{x \leq y} a(x, y) e_{xy}, \tag{6.1}$$

where $a(x, y)$ is in R. The sum (6.1) is called finitary series if for every (x, y) in $X \times X$ such that $x < y$ there exists a finite number of elements (u, v) in $X \times X$ satisfy $x \leq u < v \leq y$ and $a(u, v) \neq 0$. The set of finitary series is a sub-module of $MI(X, R)$ and it is denoted by $FI(X, R)$. By [80, Theorem 1], $FI(X, R)$ is an associated unital algebra over R.

In [82], Khrypchenko studied local derivation on finitary incidence algebras and he stated the following main result.

Theorem 6.1.5 ([82, Theorem 2.7])**.** *Let R be a unital commutative ring. Then, $FI(P, R)$ is a Kadison algebra.*

But, we still unable to consider Example 6.1.4 as a finitary incidence algebra even for finitary incidence algebras of quasiorders (i.e. pre-ordered sets) in [83]. Therefore, we get rich examples of path algebras that are not finitary incidence algebras due to the fact of parallel paths in quivers.

We end this section by the following facts.

1. Finite incidence algebras of posets are a particular case of finitary incidence algebras,
2. Finite incidence algebras and finitary incidence algebras are particular cases of finitary incidence algebras of quasiorders,
3. Path algebras without parallel paths are the same as finite incidence algebras of posets,
4. Path algebras with parallel paths are not finitary incidence algebras of quasiorders, and
5. (Finite) Incidence algebras are not path algebras in general.

6.2 Local derivations on path algebras

In this section, we investigate local derivations on path algebras associated with finite acyclic quivers. The main result of this section shows that every local derivation on a path algebra is a derivation. We also discuss local Jordan derivations and local Lie derivations on path algebras.

The following result is needed all along this paper.

Proposition 6.2.1 ([4, Propostion 3.3]). *A linear map $d : KQ \to KQ$ is a derivation if and only if d can be expressed in the form:*

1. *For every vertex v,*

$$d(v) = \sum_{\substack{q \in \mathcal{P}Q_A, \\ s(q)=v}} c_q^v q + \sum_{\substack{q \in \mathcal{P}Q_A, \\ t(q)=v}} c_q^v q, \tag{6.2}$$

where all c_q^v are coefficients in K satisfying $c_q^{s(q)} + c_q^{t(q)} = 0$ for every $q \in \mathcal{P}Q_A$.

2. *For every non-trivial path $p = e_1 \cdots e_n \in \mathcal{P}Q_A$, where $e_1, \ldots, e_n$ are edges in Q,*

$$\begin{aligned} d(p) = & \sum_{\substack{q \in \mathcal{P}Q_A, \\ t(q)=s(p)}} c_q^{s(p)} qp + \sum_{1 \leq i \leq n} \sum_{\substack{q \in \mathcal{P}Q_A, \\ s(q)=s(e_i), \\ t(q)=t(e_i)}} c_q^{e_i} e_1 \cdots e_{i-1} q e_{i+1} \cdots e_n \\ & + \sum_{\substack{q \in \mathcal{P}Q_A, \\ s(q)=t(p)}} c_q^{t(p)} pq. \end{aligned} \tag{6.3}$$

where all c_q^e are coefficients in K.

Before we state the main result of this section, we state first the following lemma.

Lemma 6.2.2. *Let δ be a local derivation on KQ. Then, there exists an inner derivation d_a on KQ such that $\delta - d_a$ vanishes on the set of vertices.*

Proof. Let δ be a local derivation on KQ and $(d_i)_{i \in KQ}$ be the family of derivations associated with δ. Then, we have

$$0 = \delta(1) = \delta(\sum_{v \in Q_0} v) = \sum_{v \in Q_0} d_v(v) = \sum_{v \in Q_0} (d_v(v)v + vd_v(v)).$$

Define a in KQ as follows

$$a=\sum_{v\in Q_0} d_v(v)v=-\sum_{v\in Q_0} vd_v(v).$$

And, define the inner derivation d_a on KQ by $d_a(x)=ax-xa$ for every x in KQ. Therefore, the map $\overline{\delta}=\delta-d_a$ is a local derivation on KQ, and $\overline{\delta}$ vanishes on the set of vertices as desired. $\square$

Notice that we can write the second term in the equality (7.16) as follows

$$\sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} c_q^p q=\sum_{1\leq i\leq n}\ \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(e_i),\\ t(q)=t(e_i)}} c_q^{e_i} e_1\cdots e_{i-1}qe_{i+1}\cdots e_n.$$

We use this short notation in the rest the paper. Now, we are in a position to state the main result of this section.

Theorem 6.2.3. *Every path algebra is a Kadison algebra.*

Proof. Let δ be a local derivation on KQ and $(d_i)_{i\in KQ}$ be the family of derivations associated with δ. By Lemma 6.2.2, we assume that δ vanishes on the set of vertices. Let p be a non-trivial path in $\mathcal{P}Q_A$. It follows by Proposition 6.2.1 that

$$\begin{aligned}
\delta(s(p)+p)&=d_{s(p)+p}(s(p)+p)\\
&=\sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p)}} a_q^{s(p)}s(p)q+\sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=s(p)}} a_q^{s(p)}qs(p)\\
&\quad+\sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=s(p)}} a_q^{s(p)}qp+\sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} a_q^{t(p)}pq+\sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} a_q^p q\\
&=\delta(t(p)+p)\\
&=d_{t(p)+p}(t(p)+p)\\
&=\sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} b_q^{t(p)}t(p)q+\sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=t(p)}} b_q^{t(p)}qt(p)\\
&\quad+\sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=s(p)}} b_q^{s(p)}qp+\sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} b_q^{t(p)}pq+\sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} b_q^p q\\
&=d_p(p)
\end{aligned}$$

$$= \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=s(p)}} c_q^{s(p)} qp + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} c_q^{t(p)} pq + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} c_q^{p} q,$$

where coefficients a_q^e, b_q^e and c_q^e are in K. Hence, we obtain

$$\delta(p) = \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} c_q^{p} q.$$

for every non-trivial path p in $\mathcal{P}Q_A$. To show that δ is a derivation, it suffices to show that for every non-trivial path p in $\mathcal{P}Q_A$ with a length greater than or equal to 2 and a vertex v such that $p = p_1vp_2$ where p_1 and p_2 are non-trivial paths in $\mathcal{P}Q_A$, we have $\delta(p) = \delta(p_1)p_2 + p_1\delta(p_2)$. Let $x = p_1 + p_2 - p - v$ and $y = p_1 + p_2 - p$. Then, we have

$$\begin{aligned}
\delta(y) &= \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} b_q^{p_1} q + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} b_q^{p_2} q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} b_q^{p} q\\
&= d_x(x)\\
&= \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=v}} c_q^{s(p)} qp_1 + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v}} c_q^{v} p_1q + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^{p_1} q\\
&\quad + \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=v}} c_q^{v} qp_2 + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} c_q^{t(p)} p_2q + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^{p_2} q\\
&\quad - \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=s(p)}} c_q^{s(p)} qp - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} c_q^{t(p)} pq - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} c_q^{p} q\\
&\quad - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v}} c_q^{v} vq - \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=v}} c_q^{v} qv.
\end{aligned}$$

It yields to

$$\sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} b_q^{p_1} q = \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^{p_1} q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^{v} qv. \tag{6.4}$$

$$\sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} b_q^{p_2}q = \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^{p_2}q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^{v}vq. \tag{6.5}$$

$$\sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} b_q^{p}q = \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} c_q^{p}q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^{v}\,p_1q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^{v}qp_2. \tag{6.6}$$

By multiplying the equality (6.4) from the right by p_2 and the equality (6.5) from the left by p_1, we obtain

$$\sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} b_q^{p_1}qp_2 = \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^{p_1}qp_2 - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^{v}qp_2. \tag{6.7}$$

$$\sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} b_q^{p_2}p_1q = \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^{p_2}p_1q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^{v}p_1q. \tag{6.8}$$

Therefore, by substituting (6.7) and (6.8) in (6.6), we obtain

$$\begin{aligned}\sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} b_q^{p}q &= \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} b_q^{p_1}qp_2 + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} b_q^{p_2}p_1q\\ &= \delta(p_1)p_2 + p_1\delta(p_2).\end{aligned}$$

This is due the fact that

$$\begin{aligned}0 &= s(p)(d_x(p) - d_x(p_1)p_2 - p_1d_x(p_2))t(p)\\ &= \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} c_q^{p}q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^{p_2}p_1q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^{p_1}qp_2.\end{aligned}$$

Finally, δ is a derivation on KQ, the proof is completed. □

We apply the previous result to local Jordan derivations and local Lie derivations. A linear map J on A is called a local Jordan derivation if for every $x \in A$, there exists a Jordan derivation J_x on A such that $J(x) = J_x(x)$ (see [145]). Similarly, a linear map L

on A is called a local Lie derivation on A if for every $y \in A$ there exists a Lie derivation on A such that $L(y) = L_y(y)$ (see [39]).

As consequences of Theorem 6.2.3, we obtain the following corollaries.

Corollary 6.2.4. *Every local Jordan derivation on KQ is a derivation.*

Proof. The proof follows from [101, Theorem 3.4] and Theorem 6.2.3. □

Corollary 6.2.5. *Every local Lie derivation on KQ is a Lie derivation.*

Proof. Let L be a local Lie derivation on KQ. By [101, Theorem 4.4], every Lie derivation f on KQ is a sum of a derivation and a linear map on KQ that has values in the center of KQ and vanishes on all commutators of KQ. Define the map $l : x \mapsto \sum_{v \in Q_0} l_v(vx)$ for every $x \in KQ$, where l_v is the center valued map that vanishes on all commutators of KQ and associated with the Lie derivation L_v. By construction l is a linear map, center valued and vanishes on all commutators of KQ. Denote $L - l$ by $\overline{L}$, then for every path p in $\mathcal{P}Q$, we have $\overline{L}(p) = d_p(p)$, where d_p is the associated derivation to the Lie derivation L_p. Hence, $\overline{L}$ is a local derivation on KQ and therefore, By Theorem 6.2.3, it is a derivation. Finally, L is a Lie derivation on KQ. □

6.3 Local generalized derivations on path algebras

In this section, we investigate local generalized derivations on path algebras. We show that every local generalized derivation on a path algebra is a generalized derivation. We apply the main result of this section on other local maps.

We start by recalling the following result.

Lemma 6.3.1 ([104, Proposition 2.1])**.** *Let $f : A \to M$ be a generalized derivation with an associated linear map d. Then d is a derivation and $f(x) = f(1)x + d(x)$ for all x in A.*

Recall that a generalized inner derivation on A is a linear map $I : A \to A$ such that there exists two fixed elements m and n satisfying $I(x) = mx + xn$ for every x in A. Then, we have the following lemma.

Lemma 6.3.2. *Let Δ be a local generalized derivation on KQ. Then, there exists a generalized inner derivation I on KQ such that $\Delta - I$ vanishes on the set of vertices.*

Proof. Let Δ be a local derivation on KQ and $(D_i)_{i\in KQ}$ be the family of generalized derivations associated with Δ. By Lemma 6.3.2, we have $\Delta(v) = D_v(v) = D_v(1)v + d_v(v)$ for every vertex v with d_v is a derivation on KQ. Define a, b_1 and b_2 in KQ as follows

$$a = \sum_{v\in Q_0} D_v(1)v, b_1 = \sum_{v\in Q_0} d_v(v)v, \text{ and } b_2 = \sum_{v\in Q_0} vd_v(v).$$

Hence, the linear map $I(x) = (a + b_1)x + xb_2$ for every x in KQ is a generalized inner derivation on KQ and $\overline{\Delta} = \Delta - I$ is a local derivation on KQ that vanishes on the set of vertices as desired. □

Notice that the linearity of local derivations prevent arbitrary families of derivation to be a local derivation. One can use Theorem 6.2.3 to prove that every local generalized derivation is a generalized derivation under some certain conditions as the next remark shows.

Remark 6.3.3. Let Δ be a local generalized derivation on KQ and $(D_i)_{i\in KQ}$ be the family of generalized derivation associated with Δ. Assume that there exists a local derivation δ on KQ associated with the family of derivations $(d_i)_{i\in KQ}$ where d_i is an associated derivation with D_i for every i in KQ. Then, by Theorem 6.2.3, δ is a derivation on KQ. Hence, the linear map $\Delta_1 = \Delta - \delta$ is a local generalized derivation on KQ defined as follows:

$$\Delta_1(x) = D_x(1)x \ (\forall x \in KQ).$$

By Lemma 6.3.2, there exists a generalized inner derivation I such that $\overline{\Delta} = \Delta_1 - I$ vanishes on the set of vertices. Let p be a non-trivial path in $\mathcal{P}Q_A$ then it follows that

$$\overline{\Delta}(s(p) + p) = D_{s(p)+p}(1)(s(p) + p) = \overline{\Delta}(p) = D_p(1)p.$$

Hence, we obtain $\overline{\Delta}(p) = 0$ for every path p in $\mathcal{P}Q$. Therefore, Δ is a generalized derivation on KQ.

Following the terminology used in Section 6.1, we call a generalized Kadison algebra an algebra on which every local generalized derivation is a generalized derivation. The main result of this section is stated as follows.

Theorem 6.3.4. *Every path algebra is a generalized Kadison algebra.*

Proof. Let Δ be a local derivation on KQ and $(D_i)_{i\in KQ}$ be the family of derivations associated with Δ. By Lemma 6.3.2, we assume that Δ vanishes on the set of vertices.

Let p be a non-trivial path in $\mathcal{P}Q_A$. It follows by Proposition 6.2.1 that

$$\begin{aligned}
\Delta(s(p)+p) &= D_{s(p)+p}(s(p)+p) \\
&= \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p)}} a_q^{s(p)} s(p)q + \left(D_{s(p)+p}(1) + \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=s(p)}} a_q^{s(p)} q \right) s(p) \\
&\quad + \left(D_{s(p)+p}(1) + \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=s(p)}} a_q^{s(p)} q \right) p + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} a_q^{t(p)}\, pq \\
&\quad + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} a_q^{p} q \\
&= \Delta(t(p)+p) \\
&= D_{t(p)+p}(t(p)+p) \\
&= \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} b_q^{t(p)} t(p)q + \left(D_{t(p)+p}(1) + \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=t(p)}} b_q^{t(p)} q \right) t(p) \\
&\quad + \left(D_{t(p)+p}(1) + \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=s(p)}} b_q^{s(p)} q \right) p + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} b_q^{t(p)}\, pq \\
&\quad + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} b_q^{p} q \\
&= D_p(p) \\
&= \left(D_p(1) + \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=s(p)}} c_q^{s(p)} q \right) p + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} c_q^{t(p)}\, pq + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} c_q^{p} q,
\end{aligned}$$

where coefficients a_q^e, b_q^e and c_q^e are in K. Hence, we obtain

$$\Delta(p) = D_p(p) = s(p)D_p(1)p + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} c_q^{p} q. \tag{6.9}$$

for every non-trivial path p in $\mathcal{P}Q_A$. To show that Δ is a generalized derivation, it suffices to show that for every non-trivial path p in $\mathcal{P}Q_A$ with a length greater than or equal to 2 and a vertex v such that $p = p_1 v p_2$ where p_1 and p_2 are non-trivial paths in $\mathcal{P}Q_A$, we have $\Delta(p) = \Delta(p_1)p_2 + p_1 d(p_2)$ with d is a linear map on KQ. Let $x = p_1 + p_2 - p - v$ and $y = p_1 + p_2 - p$. Then, we have

$$\begin{aligned}
\Delta(y) &= s(p)D_{p_1}(1)p_1 + vD_{p_2}(1)p_2 - s(p)D_p(1)p + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} b_q^{p_1} q \\
&\quad + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} b_q^{p_2} q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} b_q^{p} q \\
&= D_x(x) \\
&= D_x(1)(p_1 + p_2 - p - v) + \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=v}} c_q^{s(p)} q p_1 + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v}} c_q^{v}\, p_1 q + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^{p_1} q \\
&\quad + \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=v}} c_q^{v} q p_2 + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} c_q^{t(p)}\, p_2 q + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^{p_2} q \\
&\quad - \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=s(p)}} c_q^{s(p)} q p - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} c_q^{t(p)}\, p q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} c_q^{p} q \\
&\quad - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v}} c_q^{v} v q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=v}} c_q^{v} q v.
\end{aligned}$$

It yields to $vD_x(1)v = 0$ and

$$s(p)D_{p_1}(1)p_1 + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} b_q^{p_1} q = s(p)D_x(1)p_1 + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^{p_1} q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^{v} q v. \tag{6.10}$$

$$vD_{p_2}(1)p_2 + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} b_q^{p_2} q = \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^{p_2} q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^{v} v q. \tag{6.11}$$

$$\begin{aligned} s(p)D_p(1)p + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} b_q^p q = s(p)D_x(1)p &+ \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} c_q^p q \\ &- \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^v p_1 q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^v q p_2. \end{aligned} \tag{6.12}$$

By multiplying the equality (6.10) from the right by p_2 and the equality (6.11) from the left by p_1, we obtain

$$\begin{aligned} s(p)D_{p_1}(1)p_1p_2 + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} b_q^{p_1} q p_2 = s(p)D_x(1)p_1p_2 &+ \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^{p_1} q p_2 \\ &- \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^v q p_2. \end{aligned} \tag{6.13}$$

$$p_1D_{p_2}(1)p_2 + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} b_q^{p_2} p_1 q = \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^{p_2} p_1 q - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^v p_1 q. \tag{6.14}$$

Therefore, by substituting (6.13) and (6.14) in (6.12), we obtain

$$\begin{aligned} s(p)D_p(1)p + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} b_q^p q &= s(p)D_{p_1}(1)p_1p_2 + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} b_q^{p_1} q p_2 \\ &\quad + p_1D_{p_2}(1)p_2 + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} b_q^{p_2} p_1 q \\ &= \Delta(p_1)p_2 + p_1\Delta(p_2). \end{aligned}$$

This is due the fact that

$$\begin{aligned} 0 &= s(p)\left(D_x(p) - D_x(p_1)p_2 - p_1(D_x(p_2) - D_x(1)p_2)\right)t(p) \\ &= s(p)D_x(1)p + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=t(p)}} c_q^p q - s(p)D_x(1)p_1p_2 - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p),\\ t(q)=v}} c_q^{p_1} q p_2 - \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v,\\ t(q)=t(p)}} c_q^{p_2} p_1 q. \end{aligned}$$

Hence, Δ is a derivation on KQ, therefore it is a generalized derivation on KQ. $\square$

Recall that a linear map f is called a Jordan generalized derivation on A if there exists a linear map d on A such that

$$f(x \circ y) = f(x) \circ y + x \circ d(y),$$

for every x and y in A. A linear map f is called a generalized Jordan derivation on A if there exists a linear map d on A such that

$$f(x \circ y) = f(x)y + f(y)x + xd(y) + yd(x),$$

for every x and y in A. We call a linear map J on A a local Jordan generalized derivation if for every $x \in A$, there exists a Jordan generalized derivation J_x on A such that $J(x) = J_x(x)$. Similarly, we call a linear map J on A a local generalized Jordan derivation on A is for every $y \in A$, there exists a generalized Jordan derivation J_y on A such that $J(y) = J_y(y)$.

By using the first statement of [101, Proposition 3.7] and Theorem 6.3.4, we obtain the following fact.

Corollary 6.3.5. *Every local Jordan generalized derivation on KQ is a generalized derivation.*

Also, by using the second statement of [101, Proposition 3.7] and Theorem 6.3.4, we obtain the following fact.

Corollary 6.3.6. *Every local generalized Jordan derivation on KQ is a generalized derivation.*

A linear map L is said to be a local Lie generalized derivation on A if for every $x \in A$, there exists a Lie generalized derivation L_x on A such that $L(x) = L_x(x)$.

Corollary 6.3.7. *Every local Lie generalized derivation on KQ is a Lie generalized derivation.*

Proof. The proof is similar to the proof of Corollary 6.2.5. $\square$

CHAPTER 7

Categorical properties of generalized σ-derivations on modules

> The beginning is the most important part of the work.
>
> Plato

The following chapter presents the published paper titled “**Categorical Properties of Generalized σ-Derivations on Modules**”[1] on Communications in Algebra.

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[1] A. Adrabi, D. Bennis, and B. Fahid: Categorical Properties of Generalized σ-Derivations on Modules. *Communications in Algebra.* 2023; 51:1950–1965.

Abstract

In this paper, we investigate the notion of generalized σ-derivations on modules with σ is an automorphism on an algebra S over a commutative ring R. Which is an extension of the generalized derivations on modules introduced by Nakajima. Namely, we study homological properties of generalized σ-derivations. Also, we equip a category of functors from the category of S/R-modules to the category of R-modules with a tensor product where each functor sends an S/R-module to an R-module of generalized σ-derivations with σ is an automorphism on an algebra S. We show that category is semi-monoidal. As an application, we characterize when a generalized derivation on a path algebra of an acyclic quiver is a generalized inner derivation.

Throughout this paper, K denotes a field with characteristic different than two, R a commutative ring, A an algebra over K and S an algebra over R. Recall that an S/R-bimodule M is a left and a right S-module such that $x(my) = (xm)y$, $r(xm) = x(rm)$ and $rm = mr$ for every x and y in S, r in R, and m in M. An R-module map $d : S \to M$ is called a derivation if d satisfies $d(xy) = d(x)y + xd(y)$ for every x and y in S, and d is called an inner derivation if there exists an element m in M such that $d(x) = mx - xm$ for every x in S.

We denote the set of generalized σ-derivations by $\mathrm{gDer}(S, M, \sigma)$. In the case when the element $-m$ is equal to 0, $f^\sigma := f^\sigma_0$ is called a σ-derivation, and the set of σ-derivations is denoted by $\mathrm{Der}(S, M, \sigma)$. For elements m and n in M and an automorphism σ on S, an R-module map $f^\sigma_{-m-n} : S \to M$ is called a generalized inner σ-derivation, if f^σ_{-m-n} is satisfies:

$$f^\sigma_{-m-n}(x) = mx + \sigma(x)n,$$

for every x in S. The set of generalized inner σ-derivations is denoted by $\mathrm{gInn}(S, M, \sigma)$. In the case when m and n are equal, f^σ_{-m-n} is called inner σ-derivation, and the set of inner σ-derivations is denoted by $\mathrm{Inn}(S, M, \sigma)$.

7.1 Homological properties of generalized σ-derivations

In this section, we study homological properties of generalized σ-derivations. We focus on the relationship between R-modules $\mathrm{gDer}(S, M, \sigma)$ and $\mathrm{Der}(S, M, \sigma)$.

The following result generalizes [118, Lemma 2.1] in the context of generalized σ-derivations.

Lemma 7.1.1. *Let f^σ_m be a generalized σ-derivation. Then, there exists a σ-derivation $d_{f^\sigma_m} : S \to M$ such that $f^\sigma_m(xy) = f^\sigma_m(x)y + \sigma(x)d_{f^\sigma_m}(y)$. If $\{m \in M | Sm = 0\} = 0$, then $d_{f^\sigma_m}$ is uniquely determined by f^σ_m.*

Proof. Define a map $d_{f^\sigma_m} : S \to M$ by $d_{f^\sigma_m} = f^\sigma_m + m_l$; that is, $d_{f^\sigma_m}(x) = f^\sigma_m(x) + mx$ for every x in S. Then, we have

$$d_{f^\sigma_m}(xy) = f^\sigma_m(xy) + mxy$$

$$\begin{aligned}
&= f_m^\sigma(x)y + \sigma(x)f_m^\sigma(y) + \sigma(x)my + mxy \\
&= (f_m^\sigma(x) + mx)y + \sigma(x)(f_m^\sigma(y) + my) \\
&= d_{f_m^\sigma}(x)y + \sigma(x)d_{f_m^\sigma}(y),
\end{aligned}$$

for every x and y in S. Moreover, if $\{m \in M | Sm = 0\} = 0$, assume that $f_m^\sigma(xy) = f_m^\sigma(x)y + xd_{f_m^\sigma}(y) = f_m^\sigma(x)y + xd(y)$, where $d_{f_m^\sigma}$ and d are two σ-derivations, then we obtain $S(d_{f_m^\sigma} - d)(y) = 0$ for every y in S. Therefore, we have $d_{f_m^\sigma} = d$. □

Similarly to [118, Remark 2.3], for a generalized σ-derivation f_m^σ, the map $d : S \to M$ defined by $d(x) = f_m^\sigma(x) + \sigma(x)m$ for every x in S is a σ-derivation, but we have $f_m^\sigma(xy) \neq f_m^\sigma(x)y + \sigma(x)d(y)$ for every x and y in S.

Remark 7.1.2. Let $d^\sigma : S \to M$ be a σ-derivation and $l_m : S \to M$ be an R-linear map defined by $l_m(x) = mx$ for every x in S. Then, $f_{-m}^\sigma := d^\sigma + l_m$ is a generalized σ-derivation. Indeed, we have

$$\begin{aligned}
f_{-m}^\sigma(xy) &= d^\sigma(xy) + mxy \\
&= d^\sigma(x)y + \sigma(x)d^\sigma(y) + mxy + \sigma(x)my - \sigma(x)my \\
&= (d^\sigma(x) + mx)y + \sigma(x)(d^\sigma(y) + my) - \sigma(x)my \\
&= (d^\sigma(x) + l_m(x))y + \sigma(x)(d^\sigma(y) + l_m(y)) - \sigma(x)my.
\end{aligned}$$

Assume that S contains the unity element 1. Then, we can write a generalized σ-derivation f_m^σ as

$$f_m^\sigma(x) = f_m^\sigma(1)x + d_{f_m^\sigma}(x) \text{ and } d_{f_m^\sigma}(1) = 0, \tag{7.1}$$

for every x in S, where $d_{f_m^\sigma}$ is the associated σ-derivation with f_m^σ.

Now, if f_m^σ and g_n^σ are generalized σ-derivations, then $f_m^\sigma + g_n^\sigma$ and rf_{rm}^σ for every r in R are generalized σ-derivations. Thus, the set $\mathrm{gDer}(S, M, \sigma)$ is an R-module. The following theorem states the relationship between R-modules $\mathrm{gDer}(S, M, \sigma)$ and $\mathrm{Der}(S, M, \sigma)$.

Theorem 7.1.3. *Let M be an S/R-bimodule. Then, the following sequence is split exact as R-modules.*

$$0 \to M \xrightarrow{\psi_M} gDer(S, M, \sigma) \xrightarrow{\phi_M} Der(S, M, \sigma) \to 0, \tag{7.2}$$

where $\psi_M(m) = l_{-m}^\sigma$, $\phi_M(f_m^\sigma) = f_m^\sigma + l_{-m}^\sigma$ where $l_{-m}^\sigma(x) = mx$ for every x in S.

Proof. Since $\mathrm{gDer}(S,M,\sigma)$ is an R-module, ψ_M and ϕ_M are R-module maps. Then by Lemma 7.1.1 and Remark 7.1.2, ϕ_M is an epimorphism and $\ker\phi_M = \mathrm{Im}\psi_M$. Define an R-linear map $\overline{\phi}_M : \mathrm{Der}(S,M,\sigma) \to \mathrm{gDer}(S,M,\sigma)$ by $\overline{\phi}_M(d^\sigma) = d^\sigma$ for every d^σ in $\mathrm{Der}(S,M,\sigma)$. Thus, $\overline{\phi}_M \circ \phi_M$ is the identity map on $\mathrm{Der}(S,M,\sigma)$ and therefore (7.2) is split exact. ☐

The sequence (7.2) gives a functorial relation between functors $\mathrm{Der}(S,-,\sigma)$ and $\mathrm{gDer}(S,-,\sigma)$ from the category of S/R-bimodules to the category of R-modules as follows: Let $\alpha : M \to N$ be a homomorphism of S/R-bimodule. Then, α induces an R-module map

$$\begin{aligned}\alpha_* : \mathrm{gDer}(S,M,\sigma) &\longrightarrow \mathrm{gDer}(S,N,\sigma)\\ f_m^\sigma &\longmapsto (\alpha f_m^\sigma)^\sigma_{\alpha(m)}\end{aligned}$$

Generally, let σ,τ and ζ be three automorphisms on S such that $\tau = \zeta^{-1}\sigma\zeta$. Let $\alpha : M \to N$ be a homomorphism of S/R-bimodule. Then, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{gDer}(S,M,\sigma) \arrow[rr, "\alpha_*^\sigma"] \arrow[dr, "\zeta_M"] \arrow[dd, "\phi_M^\sigma"'] & & \mathrm{gDer}(S,N,\sigma) \arrow[dr, "\zeta_N"] \arrow[dd, dashed, "\phi_N^\sigma"] & \\
& \mathrm{gDer}(S,M,\tau) \arrow[rr, "\alpha_*^\tau"] \arrow[dd, "\phi_M^\tau"] & & \mathrm{gDer}(S,N,\tau) \arrow[dd, "\phi_N^\tau"] \\
\mathrm{Der}(S,M,\sigma)\oplus M \arrow[rr, dashed, "\overline{\alpha_*^\sigma}"] \arrow[dr, "\overline{\zeta_M}"'] & & \mathrm{Der}(S,N,\sigma)\oplus N \arrow[dr, dashed, "\overline{\zeta_N}"] & \\
& \mathrm{Der}(S,M,\tau)\oplus M \arrow[rr, "\overline{\alpha_*^\tau}"] & & \mathrm{Der}(S,N,\tau)\oplus N
\end{tikzcd}$$

Maps are defined as follows:

$$\begin{aligned}
&\alpha_*^x : \mathrm{gDer}(S,M,x) \longrightarrow \mathrm{gDer}(S,N,x), && f_m^x \longmapsto (\alpha f_m^x)^x_{\alpha(m)}; && (7.3)\\
&\overline{\alpha_*^x} : \mathrm{Der}(S,M,x)\oplus M \longrightarrow \mathrm{Der}(S,N,x)\oplus N, && d^x\oplus m \longmapsto (\alpha d^x)^x \oplus \alpha(m); && (7.4)\\
&\zeta_Y : \mathrm{gDer}(S,Y,\sigma) \longrightarrow \mathrm{gDer}(S,Y,\tau), && f_y^\sigma \longmapsto (l_Y^\zeta f_y^\sigma \zeta)^\tau_y; && (7.5)\\
&\overline{\zeta}_Y : \mathrm{Der}(S,Y,\sigma)\oplus Y \longrightarrow \mathrm{Der}(S,Y,\tau)\oplus Y, && d^\sigma \oplus y \longmapsto (l_Y^\zeta d^\sigma \zeta)^\tau \oplus y; && (7.6)\\
&\phi_Y^x : \mathrm{gDer}(S,Y,x) \longrightarrow \mathrm{Der}(S,Y,x)\oplus Y, && f_y^x \longmapsto (f_y^x + l_y)^x \oplus y; && (7.7)
\end{aligned}$$

for each $x = \alpha, \tau$; $Y = M, N$; where $l_y : Y \to Y$, $l_y(a) = ya$ for every a in Y and R-linear maps $l_Y^\zeta : Y \to Y$, $l_Y^\zeta(sy) = \zeta^{-1}(s)y$, $l_Y^\zeta(ys) = y\zeta^{-1}(s)$ and $l_Y^\zeta(y) = y$ for every s in S and y in Y. Therefore, the top square diagram gives a natural transformation which will be discussed in detail in Section 7.2, and the front (also, back) square diagram gives the following natural transformation:

$$\Phi^x : \mathrm{gDer}(S,-,x) \longrightarrow \mathrm{Der}(S,-,x) \oplus F$$

where $x = \sigma, \tau$ and F is a forgetful functor from the category of S/R-modules to the category of R-modules. Thus, functors $\mathrm{gDer}(S,-,x)$ and $\mathrm{Der}(S,-,x) \oplus F$ are naturally equivalent, where $x = \sigma, \tau$.

Now, for an automorphism σ on S, we have the functor $\mathrm{gInn}(S,-,\sigma)$ is subfuctor of $\mathrm{gDer}(S,-,\sigma)$. Then, we obtain the commutative diagram of functors

$$\begin{tikzcd}
\mathrm{gInn}(S,-,\sigma) \arrow[r, "\Phi^{\sigma}_{\mathrm{res}}"] \arrow[d, "I"] & \mathrm{Inn}(S,-,\sigma) \arrow[d, "I_1"] \\
\mathrm{gDer}(S,-,\sigma) \arrow[r, "\Phi^{\sigma}"] & \mathrm{Der}(S,-,\sigma) \oplus F
\end{tikzcd}$$

where $\Phi^{\sigma}_{\mathrm{res}}$ is a restriction of Φ^{σ} and I, I_1 are canonical injections. Recall that a derivation is called outer if it is not an inner derivation. We extend this notion to our context. So, a (generalized) σ-derivation is called a (generalized) outer σ-derivation if it is not a (generalized) inner σ-derivation. The next main result of this section shows that every generalized outer σ-derivation is an outer σ-derivation.

Theorem 7.1.4. *Let σ be an automorphism on S. Then, we have*

$$gDer(S,M,\sigma)/gInn(S,M,\sigma) \cong Der(S,M,\sigma)/Inn(S,M,\sigma).$$

Proof. By a similar reasoning as in [118, Theorem 2.8], we obtain the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "i_0"] & \mathrm{gInn}(S,M,\sigma) \arrow[r] \arrow[d, "i"] & \mathrm{Inn}(S,M,\sigma) \arrow[r] \arrow[d, "i_1"] & 0 \\
0 \arrow[r] & M \arrow[r, "\psi_M"] & \mathrm{gDer}(S,M,\sigma) \arrow[r, "\phi_M"] & \mathrm{Der}(S,M,\sigma) \arrow[r] & 0
\end{tikzcd}$$

where i_0, i_1 and i are canonical injections. Since the category of R-modules is abelian, it follows by [124, Corollay 6.12 (Snake Lemma)] that

$$0 \longrightarrow \mathrm{gDer}(S,M,\sigma)/\mathrm{gInn}(S,M,\sigma) \longrightarrow \mathrm{Der}(S,M,\sigma)/\mathrm{Inn}(S,M,\sigma) \longrightarrow 0$$

as desired. □

In the case where S has the unity element 1, the above theorem is straightforward fact from the equality (7.1).

7.2 Category of functors gDer$(S, -, \sigma)$

In this section, we study the category of functors gDer$(S, -, \sigma)$ from the category of S/R-modules to the category of R-modules equipped with a tensor product.

First, we construct the category $\mathcal{D}_S$ with objects gDer$(S, -, \sigma)$ for every automorphism σ of S as its class of objects, and morphisms between objects are natural transformations:

$$N_\zeta : \mathrm{gDer}(S, -, \sigma) \to \mathrm{gDer}(S, -, \tau)$$

that makes the following diagram commutes:

$$\begin{tikzcd}
\mathrm{gDer}(S, M, \sigma) \arrow[r, "\alpha_*^\sigma"] \arrow[d, "\zeta_M"] & \mathrm{gDer}(S, N, \sigma) \arrow[d, "\zeta_N"] \\
\mathrm{gDer}(S, M, \tau) \arrow[r, "\alpha_*^\tau"] & \mathrm{gDer}(S, N, \tau),
\end{tikzcd} \tag{7.8}$$

as defined in (7.5) and (7.6) (i.e. the top square diagram). Now, we categorify the set of automorphisms Aut(S) by letting automorphisms be its class of objects, and there exists a morphism between two automorophisms σ and τ if there exists an automorphsim ζ such that $\tau = \zeta^{-1}\sigma\zeta$. We denote this category by $\mathcal{A}_S$. Let $\mathit{\Delta}_S$ be a functor from the category $\mathcal{A}_S$ to the category $\mathcal{D}_S$ defined by

$$\begin{aligned}
\mathit{\Delta}_S : \mathcal{A}_S &\longrightarrow \mathcal{D}_S \\
\sigma &\longmapsto \mathrm{gDer}(S, -, \sigma)
\end{aligned}$$

Indeed, each automrphism σ is sent to the functor gDer$(S, -, \sigma)$, and every morphism a_ζ^σ : $\sigma \to \tau$ is sent to the natural transformation $\mathit{\Delta}_S(a_\zeta^\sigma)$ from gDer$(S, -, \sigma)$ to gDer$(S, -, \tau)$. For morphisms $a_\zeta^\sigma : \sigma \to \tau$ and $a_\eta^\tau : \tau \to \mu$, we have $\mu = \eta^{-1}\zeta^{-1}\sigma\zeta\eta$, so by definition the big square commutes

$$\begin{tikzcd}
\mathrm{gDer}(S,M,\sigma) \arrow[r, "\alpha_*^{\sigma}"] \arrow[d, "\zeta_M"] & \mathrm{gDer}(S,N,\sigma) \arrow[d, "\zeta_N"] \\
\mathrm{gDer}(S,M,\tau) \arrow[r, "\alpha_*^{\tau}"] \arrow[d, "\eta_M"] & \mathrm{gDer}(S,N,\tau) \arrow[d, "\eta_N"] \\
\mathrm{gDer}(S,M,\mu) \arrow[r, "\alpha_*^{\mu}"] & \mathrm{gDer}(S,N,\mu)
\end{tikzcd}$$

which proves the composition of natural transformations. The associativity and identity are straightforward. Note that by definition, for every pair of objects σ and τ in $\mathcal{A}_S$, there is only one and unique morphism between them which it is an isomorphism when it exists. Therefore, $\mathcal{A}_S$ is a groupoid category. Similarly, the category $\mathcal{D}_S$ is a groupoid category by construction.

Remark 7.2.1. Categories $\mathcal{A}_S$ and $\mathcal{D}_S$ are equivalent. Indeed, let Λ_S be a functor from $\mathcal{D}_S$ to $\mathcal{A}_S$ defined by $\Lambda_S(\mathrm{gDer}(S,-,\sigma)) = \sigma$ and $\Lambda_S(N_\zeta) = l_\zeta$, where $N_\zeta : \mathrm{gDer}(S,-,\sigma) \to \mathrm{gDer}(S,-,\tau)$ and $l_\zeta : \sigma \to \tau$. Then, $\Delta_S \circ \Lambda_S \cong Id_{\mathcal{A}_S}$ and $\Lambda_S \circ \Delta_S \cong Id_{\mathcal{D}_S}$.

Let n be a positive integer and p a prime number, and let σ be a fixed element in $\mathrm{Aut}(S)$. For every object τ in $\mathcal{A}_S$, define $\sigma \otimes_n^p - : \mathcal{A}_S \to \mathcal{A}_S$ to be

$$\begin{aligned} \sigma \otimes_n^p \tau &:= (\sigma \otimes_n^p -)(\tau) \\ &= \sigma^{-1}\tau^{\overline{n}}\sigma \end{aligned}$$

where $\overline{n} = n$ modulo p. And, for every morphism $l_\mu : \tau_1 \to \tau_2$ in $\mathrm{Aut}(S)$ (i.e. $\mu^{-1}\tau_1\mu = \tau_2$), define $\sigma \otimes_n^p l_\mu$ to be

$$\sigma \otimes_n^p l_\mu := (\sigma \otimes_n^p -)(l_\mu) : \sigma^{-1}\tau_1^{\overline{n}}\sigma \to \sigma^{-1}\tau_2^{\overline{n}}\sigma.$$

Indeed, $\sigma \otimes_n^p l_\mu$ is well defined due to the fact that

$$(\sigma^{-1}\mu^{-1}\sigma)(\sigma^{-1}\tau_1^{\overline{n}}\sigma)(\sigma^{-1}\mu\sigma) = \sigma^{-1}\tau_2^{\overline{n}}\sigma. \tag{7.9}$$

Remark 7.2.2. All morphisms between objects in $\mathcal{A}_S$ are isomorphims. Indeed, let $l_\zeta : \sigma \to \tau$. By definition we have $\zeta^{-1}\sigma\zeta = \tau$, it follows that $\zeta\tau\zeta^{-1} = \sigma$. Denote ζ^{-1} by $\overline{\zeta}$, hence the morphism $l_{\overline{\zeta}} : \tau \to \sigma$ is well defined and morphisms

$$l_\zeta l_{\overline{\zeta}} : \sigma \to \sigma, \quad l_{\overline{\zeta}} l_\zeta : \tau \to \tau,$$

are identity maps on σ and τ, respectively.

To show that $\sigma \otimes_n^p -$ is a functor, we only need to check the other conditions. Let $l_{\mu_1} : \tau_1 \to \tau_2$ and $l_{\mu_2} : \tau_2 \to \tau_3$ be two morphisms in $\mathcal{A}_S$. Since we have $\mu_2^{-1}\mu_1^{-1}\tau_1\mu_1\mu_2 = \tau_3$, similarly to the equality (7.9), it implies that $\sigma \otimes_n^p l_{\mu_2}l_{\mu_1} : \sigma \otimes_n^p \tau_1 \to \sigma \otimes_n^p \tau_3$ is well defined. Also, we have $\sigma \otimes_n^p l_{\mu_1} : \sigma \otimes_n^p \tau_1 \to \sigma \otimes_n^p \tau_2$ and $\sigma \otimes_n^p l_{\mu_2} : \sigma \otimes_n^p \tau_2 \to \sigma \otimes_n^p \tau_3$, which gives the following equalities:

$$(\sigma^{-1}\mu_1^{-1}\sigma)(\sigma^{-1}\tau_1^{\overline{n}}\sigma)(\sigma^{-1}\mu_1\sigma) = \sigma^{-1}\tau_2^{\overline{n}}\sigma$$
$$(\sigma^{-1}\mu_2^{-1}\sigma)(\sigma^{-1}\tau_2^{\overline{n}}\sigma)(\sigma^{-1}\mu_2\sigma) = \sigma^{-1}\tau_3^{\overline{n}}\sigma.$$

By substitution, we get $\sigma^{-1}\mu_2^{-1}\mu_1^{-1}\tau_1^{\overline{n}}\mu_1\mu_2\sigma = \sigma^{-1}\tau_3^{\overline{n}}\sigma$. Therefore, the equality $\sigma \otimes_n^p l_{\mu_2}l_{\mu_1} = (\sigma \otimes_n^p l_{\mu_2})(\sigma \otimes_n^p l_{\mu_1})$ holds. For the identity morphism $1_\tau : \tau \to \tau$, we have $\sigma \otimes_n^p 1_\tau : \sigma \otimes_n^p \tau \to \sigma \otimes_n^p \tau$. From the fact that there is a unique morphism from an object to itself, it yields that $\sigma \otimes_n^p 1_\tau = 1_{\sigma \otimes_n^p \tau}$. Thus, $\sigma \otimes_n^p -$ is a functor on $\mathcal{A}_S$. Now, for a fixed object τ in $\mathcal{A}_S$, define $- \otimes_n^p \tau : \mathcal{A}_S \to \mathcal{A}_S$ by

$$\begin{aligned}\sigma \otimes_n^p \tau &:= (- \otimes_n^p \tau)(\sigma)\\ &= \sigma^{-1}\tau^{\overline{n}}\sigma,\end{aligned}$$

for every object σ in $\mathcal{A}_S$. For every morphism $l_\zeta : \sigma_1 \to \sigma_2$ in $\mathcal{A}_S$ (i.e. $\zeta^{-1}\sigma_1\zeta = \sigma_2$), define $l_\zeta \otimes_n^p \tau$ to be

$$l_\zeta \otimes_n^p \tau := (- \otimes_n^p \tau)(l_\zeta) : \sigma_1 \otimes_n^p \tau \to \sigma_2 \otimes_n^p \tau.$$

Indeed, $l_\zeta \otimes_n^p \tau$ is well defined due to the fact that

$$(\sigma_2^{-1}\sigma_1)(\sigma_1^{-1}\tau^{\overline{n}}\sigma_1)(\sigma_1^{-1}\sigma_2) = \sigma_2^{-1}\tau^{\overline{n}}\sigma_2. \tag{7.10}$$

The rest is straightforward calculations as above. Finally, we obtain that $- \otimes_n^p - : \mathcal{A}_S \times \mathcal{A}_S \to \mathcal{A}_S$ is a bifunctor.

From now on, we assume that n is a fixed positive integer. Recall the definition of semi-monoidal categories.

Definition 7.2.3 ([64, Definition 1]). A semi-monoidal category is a category $\mathcal{C}$ with a bifunctor $- \otimes - : \mathcal{C} \times \mathcal{C}$ that is associative up to an object-indexed family of natural isomorphisms $\mathfrak{a}_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ satisfying MacLane's pentagon condition

$$(1_W \otimes \mathfrak{a}_{X,Y,Z})\mathfrak{a}_{W,X\otimes Y,Z}(\mathfrak{a}_{W,X,Y} \otimes 1_Z) = \mathfrak{a}_{W,X,Y\otimes Z}\mathfrak{a}_{W\otimes X,Y,Z}.$$

A functor between semi-monoidal categories that (strictly) preserves the tensor is a (strict) semi-monoidal functor.

Lemma 7.2.4. *The bifunctor* $-\otimes_n^p-$ *is associative if and only if every automorphism* σ *in* $Aut(S)$ *is conjugated to* $\sigma^{\overline{n}}$*.*

Proof. Let σ,τ and η be three objects in $\mathcal{A}_S$. Thus, we have

$$\begin{aligned}(\sigma\otimes_n^p\tau)\otimes_n^p\eta &= (\sigma^{-1}\tau^{\overline{n}}\sigma)\otimes_n^p\eta\\ &= (\sigma^{-1}\tau^{\overline{n}}\sigma)^{-1}\eta^{\overline{n}}(\sigma^{-1}\tau^{\overline{n}}\sigma).\end{aligned}$$

Also, we have

$$\begin{aligned}\sigma\otimes_n^p(\tau\otimes_n^p\eta) &= \sigma^{-1}(\tau\otimes_n^p\eta)^{\overline{n}}\sigma\\ &= \sigma^{-1}(\tau^{-1}\eta^{\overline{n}}\tau)^{\overline{n}}\sigma\\ &= \sigma^{-1}(\tau^{-1}\eta^{\overline{n}^2}\tau)\sigma.\end{aligned}$$

Hence, $(\sigma\otimes_n^p\tau)\otimes_n^p\eta$ and $\sigma\otimes_n^p(\tau\otimes_n^p\eta)$ are isomorphic if and only if there exists an automorphism ζ such that

$$\zeta^{-1}(\sigma^{-1}(\tau^{\overline{n}})^{-1}\sigma)\eta^{\overline{n}}(\sigma^{-1}\tau^{\overline{n}}\sigma)\zeta=\sigma^{-1}(\tau^{-1}\eta^{\overline{n}^2}\tau)\sigma. \tag{7.11}$$

In other words, every automorphism $\eta^{\overline{n}}$ is conjugated to $\eta^{\overline{n}^2}$. Since p is a prime number, it follows that $\overline{n}$ has an inverse element denoted by $\overline{n}^{-1}$. Thus, for every automorphism δ in Aut(S), we substitute η by $\delta^{\overline{n}^{-1}}$ in the equality (7.11), hence we obtain

$$\tau\sigma\zeta^{-1}(\sigma^{-1}(\tau^{\overline{n}})^{-1}\sigma)\delta(\sigma^{-1}\tau^{\overline{n}}\sigma)\zeta\sigma^{-1}\tau^{-1}=\delta^{\overline{n}}. \tag{7.12}$$

Therefore, the equality (7.12) holds if and only if every automorphism δ in Aut(S) is conjugated to $\delta^{\overline{n}}$. □

Proposition 7.2.5. *The category* $(\mathcal{A}_S,\otimes_n^p)$ *is a semi-monoidal category if and only if every automorphism* σ *in* $Aut(S)$ *is conjugated to* $\sigma^{\overline{n}}$*.*

Proof. By Lemma 7.2.4, we need to check MacLane's pentagon to prove that $(\mathcal{A}_S,\otimes_n^p)$ is a semi-monoidal. In other words, we only need to show that the following diagram commutes:

$$\begin{tikzcd}
& (\sigma \otimes_n^p \tau) \otimes_n^p (\eta \otimes_n^p \theta) \arrow[dr, "f_2"] & \\
((\sigma \otimes_n^p \tau) \otimes_n^p \eta) \otimes_n^p \theta \arrow[ur, "f_1"] \arrow[d, "f_3"'] & & \sigma \otimes_n^p (\tau \otimes_n^p (\eta \otimes_n^p \theta)) \\
(\sigma \otimes_n^p (\tau \otimes_n^p \eta)) \otimes_n^p \theta \arrow[rr, "f_4"'] & & \sigma \otimes_n^p ((\tau \otimes_n^p \eta) \otimes_n^p \theta) \arrow[u, "f_5"']
\end{tikzcd}$$

Using the fact that $\mathcal{A}_S$ is a groupoid, the problem is reduced to maps between objects (i.e. to show the objects are conjugated). Let σ, τ, η and θ be objects in $\mathcal{A}_S$. It is clear that the morphism f_3 exists. For the rest of morphisms, we have

$$\begin{aligned}
((\sigma \otimes_n^p \tau) \otimes_n^p \eta) \otimes_n^p \theta &= ((\sigma \otimes_n^p \tau) \otimes_n^p \eta)^{-1}\theta^{\overline{n}}((\sigma \otimes_n^p \tau) \otimes_n^p \eta), \\
(\sigma \otimes_n^p (\tau \otimes_n^p \eta)) \otimes_n^p \theta &= (\sigma \otimes_n^p (\tau \otimes_n^p \eta))^{-1}\theta^{\overline{n}}(\sigma \otimes_n^p (\tau \otimes_n^p \eta)), \\
\sigma \otimes_n^p ((\tau \otimes_n^p \eta) \otimes_n^p \theta) &= \sigma^{-1}((\tau \otimes_n^p \eta) \otimes_n^p \theta)^{\overline{n}}\sigma \\
&= \sigma^{-1}((\tau \otimes_n^p \eta)^{-1}\theta^{\overline{n}}(\tau \otimes_n^p \eta))^{\overline{n}}\sigma \\
&= \sigma^{-1}((\tau \otimes_n^p \eta)^{-1}\theta^{\overline{n}^2}(\tau \otimes_n^p \eta))\sigma, \\
\sigma \otimes_n^p (\tau \otimes_n^p (\eta \otimes_n^p \theta)) &= \sigma^{-1}(\tau \otimes_n^p (\eta \otimes_n^p \theta))^{\overline{n}}\sigma \\
&= \sigma^{-1}(\tau^{-1}(\eta \otimes_n^p \theta)^{\overline{n}}\tau)^{\overline{n}}\sigma \\
&= \sigma^{-1}(\tau^{-1}(\eta^{-1}\theta^{\overline{n}}\eta)^{\overline{n}}\tau)^{\overline{n}}\sigma \\
&= \sigma^{-1}(\tau^{-1}(\eta^{-1}\theta^{\overline{n}^3}\eta)\tau)\sigma. \\
(\sigma \otimes_n^p \tau) \otimes_n^p (\eta \otimes_n^p \theta) &= (\sigma^{-1}\tau^{\overline{n}}\sigma) \otimes_n^p (\eta^{-1}\theta^{\overline{n}}\eta) \\
&= (\sigma^{-1}(\tau^{\overline{n}})^{-1}\sigma)(\eta^{-1}\theta^{\overline{n}}\eta)^{\overline{n}}(\sigma^{-1}\tau^{\overline{n}}\sigma) \\
&= (\sigma^{-1}(\tau^{\overline{n}})^{-1}\sigma)(\eta^{-1}\theta^{\overline{n}^2}\eta)(\sigma^{-1}\tau^{\overline{n}}\sigma).
\end{aligned}$$

Therefore, morphisms f_1, f_2, f_4 and f_5 exist if and only if every automorphism σ in $\mathrm{Aut}(S)$ is conjugated to $\sigma^{\overline{n}}$. □

By a similar reasoning, let n be a positive integer and p a prime number, and let σ be a fixed automorphism in $\mathrm{Aut}(S)$. Then, for every automorphism τ in $\mathrm{Aut}(S)$, define $\mathrm{gDer}(S,-,\sigma) \otimes_n^p - : \mathcal{D}_S \to \mathcal{D}_S$ by

$$\begin{aligned}
\mathrm{gDer}(S,-,\sigma) \otimes_n^p \mathrm{gDer}(S,-,\tau) &:= (\mathrm{gDer}(S,-,\sigma) \otimes_n^p -)(\mathrm{gDer}(S,-,\tau)) \\
&= \mathrm{gDer}(S,-,\sigma^{-1}\tau^{\overline{n}}\sigma).
\end{aligned}$$

Let τ_1 and τ_2 be two automorphisms in $\mathrm{Aut}(S)$ such that there exists an automorphism μ that satisfies $\mu^{-1}\tau_1\mu = \tau_2$. Let $N_\mu : \mathrm{gDer}(S,-,\tau_1) \to \mathrm{gDer}(S,-,\tau_2)$ be a natural transformation. Then, the following natural transformation is well defined:

$$\begin{aligned}
\mathrm{gDer}(S,-,\sigma)\otimes_n^p N_\mu &:= (\mathrm{gDer}(S,-,\sigma)\otimes_n^p -)(N_\mu)\\
&= \mathrm{gDer}(S,-,\sigma)\otimes_n^p \mathrm{gDer}(S,-,\tau_1) \to \mathrm{gDer}(S,-,\sigma)\otimes_n^p \mathrm{gDer}(S,-,\tau_2)\\
&= \mathrm{gDer}(S,-,\sigma^{-1}\tau_1^{\overline{n}}\sigma) \to \mathrm{gDer}(S,-,\sigma^{-1}\tau_2^{\overline{n}}\sigma).
\end{aligned}$$

Indeed, due to the fact that $(\sigma^{-1}\mu^{-1}\sigma)(\sigma^{-1}\tau_1^{\overline{n}}\sigma)(\sigma^{-1}\mu\sigma) = \sigma^{-1}\tau_2^{\overline{n}}\sigma$, it yields that for every homomorphism $\alpha : M \to N$ of S/R-bimodule, we have a commutative diagram as in (7.8), hence it is a natural transformation, as desired. Also, we obtain that $-\otimes_n^p \mathrm{gDer}(S,-,\sigma) : \mathcal{D}_S \to \mathcal{D}_S$ is a functor. Therefore, $-\otimes_n^p - : \mathcal{D}_S \times \mathcal{D}_S \to \mathcal{D}_S$ is a bifunctor.

Now, we are in a position to state the main result of this section.

Theorem 7.2.6. *The category $(\mathcal{D}_S, \otimes_n^p)$ is a semi-monoidal and the functor Δ_S is a strict semi-monoidal functor if and only if every automorphism σ in $Aut(S)$ is conjugated to $\sigma^{\overline{n}}$.*

Proof. The proof of semi-monoidal property is word by word as in Lemma 7.2.4 and Proposition 7.2.5. Thus, by replacing every object σ in $\mathcal{A}_S$ by the corresponding object $\mathrm{gDer}(S,-,\sigma)$ in $\mathcal{D}_S$. To show that Δ_S is a strict semi-monoidal functor, let σ and τ two objects in $\mathcal{A}_S$. Then, we have

$$\begin{aligned}
\Delta_S(\sigma\otimes_n^p\tau) &= \Delta_S(\sigma^{-1}\tau^{\overline{n}}\sigma)\\
&= \mathrm{gDer}(S,-,\sigma^{-1}\tau^{\overline{n}}\sigma)\\
&= \mathrm{gDer}(S,-,\sigma)\otimes_n^p \mathrm{gDer}(S,-,\tau)\\
&= \Delta_S(\sigma)\otimes_n^p\Delta_S(\tau).
\end{aligned}$$

Therefore, we have Δ_S preserves $\otimes_n^p$. To show associativity, let σ, τ and η be three objects in $\mathcal{A}_S$. Then, we have

$$\begin{aligned}
\Delta_S((\sigma\otimes_n^p\tau)\otimes_n^p\eta) &= \mathrm{gDer}(S,-,(\sigma^{-1}\tau^{\overline{n}}\sigma)^{-1}\eta^{\overline{n}}(\sigma^{-1}\tau^{\overline{n}}\sigma))\\
&= (\mathrm{gDer}(S,-,\sigma)\otimes_n^p \mathrm{gDer}(S,-,\tau))\otimes_n^p \mathrm{gDer}(S,-,\eta), \qquad (7.13)\\
\Delta_S(\sigma\otimes_n^p(\tau\otimes_n^p\eta)) &= \mathrm{gDer}(S,-,\sigma^{-1}(\tau^{-1}\eta^{\overline{n}^2}\tau)\sigma)\\
&= \mathrm{gDer}(S,-,\sigma)\otimes_n^p(\mathrm{gDer}(S,-,\tau)\otimes_n^p \mathrm{gDer}(S,-,\eta)). \qquad (7.14)
\end{aligned}$$

Therefore, there is an isomorphism between (7.13) and (7.14) if and only if every automorphism σ in $\mathrm{Aut}(S)$ is conjugated to $\sigma^{\overline{n}}$. Finally, $\varDelta_S$ is a strict semi-monoidal functor if and only if hypothesis is satisfied. □

Note that for both categories $(\mathcal{A}_S, \otimes^p_n)$ and $(\mathcal{D}_S, \otimes^p_n)$, the left unitary constraint as in [133, Definition 1.2.1] (i.e. morphisms $\mathrm{Id} \otimes^p_n \sigma \to \sigma$) exists for every σ in $\mathcal{A}_S$, similarly, for $\mathcal{D}_S$, but not the right unitary constraint. Also, it might to note that by construction, every element of categories $(\mathcal{A}_S, \otimes^p_n)$ and $(\mathcal{D}_S, \otimes^p_n)$ is self-similar as in [64, Definition 4].

7.3 Application to path algebras

In this section, we apply previous results to the vector space $\mathrm{gDer}(KQ)$ over K and its subvector spaces, where KQ is a path algebra over the field K.

We assume some familiarity with the basic notions of path algebras (for more details, see [128]). In the sequel, $Q = (Q_0, Q_1, s, t)$ denoted an (infinite) acyclic quiver, where Q_0 and Q_1 are sets of vertices and edges of Q, respectively, and the pairs of maps s and t from Q_1 into Q_0 determine the edges of Q. We denote by KQ the path algebra over K associated with Q. Recall, a path which contains at least one edge is called a non-trivial path in quiver Q, otherwise, it is called a trivial path (i.e. the elements of the set Q_0). We denote by $\mathcal{P}Q$ the set of all paths in Q which it is a linear basis of KQ. Also, we denote by $\mathcal{P}Q_A$ the set of all non-trivial acyclic paths in Q.

Denote $\mathrm{gDer}(KQ, KQ, \mathrm{Id})$ by $\mathrm{gDer}(KQ)$, $\mathrm{gInn}(KQ, KQ, \mathrm{Id})$ by $\mathrm{gInn}(KQ)$, $\mathrm{Der}(KQ, KQ, \mathrm{Id})$ by $\mathrm{Der}(KQ)$ and $\mathrm{Inn}(KQ, KQ, \mathrm{Id})$ by $\mathrm{Inn}(KQ)$.

Remark 7.3.1. By the proof of Theorem 7.1.4, we obtain the following commutative diagram with the rows are split exact:

$$\begin{tikzcd}
0 \arrow[r] & KQ \arrow[r] \arrow[d] & \mathrm{gInn}(KQ) \arrow[r] \arrow[d] & \mathrm{Inn}(KQ) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & KQ \arrow[r] & \mathrm{gDer}(KQ) \arrow[r] & \mathrm{Der}(KQ) \arrow[r] & 0.
\end{tikzcd}$$

7.3.1 Generalized inner derivations on path algebras of finite quivers

In this subsection, we investigate generalized inner derivations on path algebras of finite and acyclic quivers. First, we recall the characterization of derivations on KQ.

Lemma 7.3.2 ([4, Propostion 3.3])**.** *Let Q be a finite and acyclic quiver. A linear map $d : KQ \to KQ$ is a derivation if and only if d can be expressed in the form:*

1. *For every vertex v,*

$$d(v) = \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=v}} c_q^v q + \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=v}} c_q^v q, \tag{7.15}$$

   where all c_q^v are coefficients in K satisfying $c_q^{s(q)} + c_q^{t(q)} = 0$ for every q in $\mathcal{P}Q_A$.

2. *For every non-trivial path $p = p_1 \cdots p_n$ in $\mathcal{P}Q_A$, where $p_1, \ldots, p_n$ are edges in Q,*

$$\begin{aligned} d(p) = &\sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=s(p)}} c_q^{s(p)} qp + \sum_{1\leq i\leq n} \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=s(p_i),\\ t(q)=t(p_i)}} c_q^{p_i} p_1 \cdots p_{i-1} q p_{i+1} \cdots p_n \\ &+ \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} c_q^{t(p)} pq. \end{aligned} \tag{7.16}$$

   where all $c_q^{p_i}$ are coefficients in K.

The following proposition characterizes inner derivations depending on the second term in the equality (7.16).

Proposition 7.3.3. *Let Q be a finite acyclic quiver and $d : KQ \to KQ$ be a derivation. If for every non-trivial path p in $\mathcal{P}Q_A$, we have $s(p)d(p)t(p) = 0$. Then, d is an inner derivation on KQ.*

Proof. Let d be a derivation that satisfies the hypotheses, hence by Lemma 7.3.2, we obtain that, for every path p in $\mathcal{P}Q$, we have

$$d(p) = \sum_{\substack{q\in\mathcal{P}Q_A,\\ t(q)=s(p)}} c_q^{s(p)} qp + \sum_{\substack{q\in\mathcal{P}Q_A,\\ s(q)=t(p)}} c_q^{t(p)} pq$$

where all c_q^e are coefficients in K, this is due to the fact that Q is acyclic. Let $C = C_1 \cup C_2$ be a subset of KQ, where C_1 and C_2 are defined as follows:

$$C_1 = \bigcup_{v\in Q_0} \{c_q^v qv \in KQ | C_q^v qv \text{ in } d(v) \text{ as a linear combination}\},$$

$$C_2 = \bigcup_{v\in Q_0} \{-c_q^v vq \in KQ | C_q^v vq \text{ in } d(v) \text{ as a linear combination}\}.$$

Let $a = \sum_{c \in C} c$, then $d_a(x) = ax - xa$ is an inner derivation, and it is equal to d. ☐

As an immediate consequence, every derivation on KQ is a sum of an inner derivation and a derivation that vanishes on all vertices. Which is similar to [60, Lemma, page 116].

Corollary 7.3.4. *Let Q be a finite acyclic quiver. Then, every derivation d on KQ can be written as $d = d_a + \delta$, where d_a an inner derivation on KQ with a in KQ and δ is a derivation on KQ such that $\delta(v) = 0$ for every vertex v in Q_0.*

Now, we are in a position to state the main result of this subsection.

Theorem 7.3.5. *Let Q be a finite acyclic quiver and δ be a derivation on KQ such that δ vanishes on the set of vertices Q_0. Then, we have δ is an inner derivation if and only if*

$$(\forall p \in \mathcal{P}Q_A)(\forall q \in \mathcal{P}Q_A)\ q \text{ is parallel to } p \implies \delta(q) = c_p q, \tag{7.17}$$

where c_p is the coefficient of p in $\delta(p)$.

Proof. Assume that δ is not an inner derivation. Then, by hypotheses and Lemma 7.3.2, there is no linear combination a in KQ of vertices such that $\delta(x) = ax - xa$ for all x in KQ. Hence, there exists a non-trivial path p such that $\delta(p) \neq \alpha p$ for all α in K, or there exists two parallel non-trivial paths p and q in $\mathcal{P}Q$ such that $c_p \neq c_q$, where c_p and c_q are coefficients of p and q in $\delta(p)$ and $\delta(q)$, respectively. Therefore, the implication (7.17) does not hold. Conversely, if the implication (7.17) is not satisfied, then δ is not an inner derivation. ☐

Hence, from [60], we have the following corollary.

Corollary 7.3.6 ([60, Corollary, page 111])**.** *Let Q be a finite acyclic quiver without parallel paths. Then, $HH^1(KQ) = 0$ if and only if Q is a tree.*

By using Remark 7.3.1 and Corollary 7.3.6, we obtain the following result.

Corollary 7.3.7. *Let Q be a finite acyclic quiver without parallel paths. Then, every generalized derivation on KQ is a generalized inner derivation on KQ if and only if Q is a tree.*

Proof. By Corollary 7.3.6, we have that every derivation on KQ is an inner derivation on KQ. Hence, by Remark 7.3.1 and [124, Proposition 2.72 (Five Lemma)], we obtain the desired result. ☐

7.3.2 Generalized inner derivations on path algebras of infinite quivers

In this subsection, we investigate generalized derivations on path algebras of infinite acyclic quivers with infinite set of vertices and finite number of edges between every two vertices. With this setting, we obtain path algebras without the unity element 1.

In the case when Q is an infinite acyclic quiver with the set of vertices is infinite, Theorem 7.3.5 does not hold. As an example, let Q be an infinite line quiver

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \xrightarrow{e_3} v_4 \xrightarrow{e_4} \cdots$$

and δ a derivation on KQ that vanishes on the set of vertices and for every edge e in Q_1, we have $\delta(e) = e$. Then, δ is not an inner derivation, this is due to the fact that infinite sum is not allowed by axioms of algebras (i.e. the sum of all vertices in Q_0). But, for every finite sub-quiver of Q, Theorem 7.3.5 holds. This leads us to define the map Γ as follows: Let $\mathfrak{Q}_{\mathrm{o}}$ be the power set of the set Q_0 and $\mathfrak{Q}$ be the set of all sub-quivers of Q. Define $\Gamma : \mathfrak{Q}_{\mathrm{o}} \to \mathfrak{Q}$ to be a map that sends a subset S_0 of Q_0 to a sub-quiver that contains all paths (i.e. trivial and non-trivial paths) between vertices S_0.

Example 7.3.8. Let Q be the following quiver:

$$\begin{tikzcd} Q: & v_1 \arrow[r, bend left, "e_1"] \arrow[r, bend right, "e_2"'] & v_2 \arrow[r, "e_3"] & v_3 \arrow[r, "e_4"] & v_4 . \end{tikzcd}$$

Then, for $\{v_1, v_3\}$ in $\mathfrak{Q}_{\mathrm{o}}$, we have $\Gamma(\{v_1, v_3\}) =$ $\begin{tikzcd} v_1 \arrow[r, bend left, "e_1"] \arrow[r, bend right, "e_2"'] & v_2 \arrow[r, "e_3"] & v_3 \end{tikzcd}$. Now, if Q is the following quiver:

$$\begin{tikzcd} & v_5 & & & \\ Q: & v_1 \arrow[u] \arrow[r] & v_2 \arrow[d] & v_3 \arrow[l] \arrow[r] & \cdots . \\ & & v_4 & & \end{tikzcd}$$

Then, for $\{v_1, v_3\}$ in $\mathfrak{Q}_{\mathrm{o}}$, we have $\Gamma(\{v_1, v_3\}) =$ $\begin{tikzcd} v_1 \arrow[r] & v_2 & v_3 \arrow[l] \end{tikzcd}$.

Denote the property: An infinite quiver with infinite set of vertices and a finite number of edges between every two vertices by $(*)$. With this setting, the sets $\mathrm{Inn}(KQ)$ and gInn(KQ) are strictly included in Der(KQ) and gDer(KQ), respectively.

The following definition is one of key ingredients of the main result of this subsection. Recall, the support of a map f on A is the subset supp(f) of A defined as $\mathrm{supp}(f) = \{x \in A | f(x) \neq 0\}$.

Definition 7.3.9. The disconnected support of a linear map f on KQ, denoted by $\mathrm{supp}_d(f)$, it is a subset of $\mathcal{P}Q \cap \mathrm{supp}(f)$ such that for every element p in $\mathrm{supp}_d(f)$, there is no element q in $\mathrm{supp}_d(f)$ that starts or ends at $s(p)$ and $t(p)$, respectively.

Here is an example of a disconnected support.

Example 7.3.10. Let Q be an infinite line quiver that starts at vertex s. Let d be an inner derivation on KQ defined by $d(x) = sx - xs$ for every x in KQ. Then, we have for every non-trivial path p that starts at s, $d(p) \neq 0$. Hence, the sets $\{p\}$ are disconnected supports of d, where $sp \neq 0$.

The next theorem gives necessary and sufficient conditions for a derivation on a path algebra of a quiver that satisfies $(*)$ to be an inner derivation.

Theorem 7.3.11. *Let Q be a quiver that satisfies $(*)$ and d is a derivation on KQ. Then, d is an inner derivation if and only if*

1. *d has a finite disconnected support, and*
2. *One of the following assertions holds:*
   - *(a) For every non-trivial path p in $\mathcal{P}Q_A$, we have $s(p)d(p)t(p) = 0$,*
   - *(b) d vanishes on the set of vertices Q_0 and*

     $$(\forall p \in \mathcal{P}Q_A)(\forall q \in \mathcal{P}Q_A)\ q \text{ is parallel to } p \implies d(q) = c_p q,$$

     where c_p is the coefficient of p in $d(p)$,
   - *(c) d can be written as sum of two derivations as in (a) and (b).*

Proof. Assume that d is an inner derivation. Then, by the definition of inner derivations, d must satisfy (1). Let $S_0 = \{v \in Q_0 | \exists p \in \mathcal{P},\ vd(p) \neq 0 \text{ or } d(p)v \neq 0\}$ and $\tilde{Q} = \Gamma(S_0)$. It yields that $d(K\tilde{Q}) \neq 0$ and $d(KQ) \subset K\tilde{Q}$, this is due to Lemma 7.3.2. Let $\tilde{d}$ be the restriction and co-restriction of d on $K\tilde{Q}$, it follows that $\tilde{d}$ is an inner derivation on $K\tilde{Q}$. Hence, by Lemma 7.3.2, $\tilde{d}$ can be written as sum of a derivation $\tilde{d}_1$ as in (a) and a derivation $\tilde{\delta}$ that vanishes on the set of vertices. Since by Proposition 7.3.3, d_1 is an inner derivation on $K\tilde{Q}$, it yields that $\tilde{\delta}$ is an inner derivation on $K\tilde{Q}$. Therefore, by Theorem 7.3.5, $\tilde{\delta}$ satisfies the property

$$(\forall p \in \mathcal{P}\tilde{Q}_A)(\forall q \in \mathcal{P}\tilde{Q}_A)\ q \text{ is parallel to } p \implies \tilde{\delta}(q) = c_p q.$$

By extending $\tilde{\delta}$ to KQ, we obtain a derivation that satisfies (b). The converse implication is straightforward verification by using Lemma 7.3.2, Proposition 7.3.3 and Theorem 7.3.5. □

The set of derivations with infinite support is not closed under addition. As an example, let d be a derivation with infinite support, then also $-d$ is a derivation with infinite support. Thus, $d + (-d)$ is equal to the zero derivation, however the zero derivation has a finite support. Therefore, $\mathrm{Der}(KQ)$ cannot be written as a directed sum of the set of derivations with finite support and the set of derivations with infinite support.

Again, by using Remark 7.3.1 and Corollary 7.3.11, we obtain the following result.

Corollary 7.3.12. *Let Q be a quiver that satisfies $(*)$ without parallels paths and D is a generalized derivation on KQ. Then, every generalized derivation on KQ is a generalized inner derivation on KQ.*

CHAPTER 8

Perspectives

> Adventure is worthwhile.
>
> Aesop

By reading the introduction of this thesis, one can have many ideas for many research questions. It is nearly impossible to focus on one direction without a constant temptation by other directions. So, we decide to focus on something else related to derivations called Hopf actions. A classical examples of actions are group actions on algebras in which a group acts on an algebra by automorphisms, also Lie algebra actions on algebras in which a Lie algebra acts by derivations on an algebra.

Recently, some authors start to investigate Hopf actions on path algebras as in [85, 86]. So, our perspectives is two folds: The first one is study Hopf actions on path algebras that act as derivations-like, and the second is to use the language of tensor categories as toolbox to prove results.

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Résume

La présente thèse présente nos contributions au domaine des dérivations et les applications associées sur les algèbres de chemin. Dans cette thèse, nous avons étendu, introduit et développé des concepts et techniques intéressants. Le premier objectif de cette thèse est d'étudier et de caractériser certaines applications liées aux dérivations sur les algèbres de chemin afin de généraliser des résultats existants et d'en introduire de nouveaux dans ce domaine. Le deuxième objectif est d'introduire de nouvelles perspectives liées aux dérivations pour résoudre une variante d'une conjecture et étudier des propriétés catégorielles.

Mots-clefs: Lie generalized derivations, Jordan G_n-derivations, Generalized σ-derivations, Lvov-Kaplansky conjecture, Path algebras.

Abstract

The present thesis presents our contributions to the domain of derivations and related maps on path algebras. In this thesis, we have extended, introduced, and developed interesting concepts and techniques. The first aim of this thesis is to investigate and characterize some related maps to derivations on path algebras to generalize existing results and introduce new ones to this domain. The second aim is to introduce new perspectives related to derivations to solve a variant of a conjecture and investigate categorical properties.

Keywords : Lie generalized derivations, Jordan G_n-derivations, Generalized σ-derivations, Lvov-Kaplansky conjecture, Path algebras.