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Existence and Regularity of Solutions for some Elliptic Problems with Degenerate Coercivity

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Dedication

To my family and friends.

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Abstract

This thesis focuses on the investigation of some problems related to nonlinear elliptic partial differential equations of Dirichlet type. The existence and regularity solutions for such problems are treated in different spaces, such as classical Sobolev spaces and weighted Sobolev spaces. The present thesis provides findings regarding the existence and regularity of weak or entropy solutions, and it is consisting on four chapters.

- In the first chapter, we briefly present various tools and other preliminary results that will be useful in the following chapters.
- In chapter 2, we deal on the existence and regularity of an entropy solution of the Dirichlet problem

$$-\operatorname{div}[\omega(x)a(x,u,\nabla u)] = f - \operatorname{div} F, \qquad \text{in } \Omega,$$

  where $\omega \in \mathcal{A}_p$-weights, $f \in L^1(\Omega)$ and $F \in [L^{p'}(\Omega, \omega^{1-p'})]^N$.
- We established in chapter 3, the existence and some regularity results for solutions of unilateral problem associated with the elliptic problem

$$(P)\begin{cases} \operatorname{div} a(x,u,\nabla u) = \operatorname{div} F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

  in the framework of Sobolev spaces. We distinguished four cases in terms of the summability of the datum F.
  - If $F \in (L^m(\Omega))^N$ with $m > \frac{N}{p-1}$, then there exists a weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

  - If $F \in (L^m(\Omega))^N$ with $m = \frac{N}{p-1}$, then there exists an entropy solution $u \in \mathcal{T}_0^{1,p}(\Omega)$.
  - If $F \in (L^m(\Omega))^N$ with $\frac{Np'}{N-\theta(N-p)} \leq m < \frac{N}{p-1}$, then there exists a weak solution $u \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$, with $r = (1-\theta)(m(p-1))^*$.
  - If $F \in (L^m(\Omega))^N$ with $\max\left(p', \frac{Np'}{p((1-\theta)N+\theta)}\right) \leq m < \frac{Np'}{N-\theta(N-p)}$, then there exists an entropy solution solution $u \in W_0^{1,q}(\Omega) \cap L^r(\Omega)$, with $r = (1-\theta)(m(p-1))^*$ and $q = \frac{(1-\theta)Nm(p-1)}{N-\theta m(p-1)}$.

- Chapter 4 is dedicated to the study of the same unilateral problem as in the previous chapter, but in a broader context, that of weighted Sobolev spaces. We focus on the first two cases.

  - If $\frac{F}{\omega} \in (L^m(\Omega,\omega))^N$ with $m > \frac{N}{p-1}$, then there exists a weak solution $u \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$.
  - If $\frac{F}{\omega} \in (L^m(\Omega))^N$ with $m = \frac{N}{p-1}$, then there exists an entropy solution $u \in \mathcal{T}_0^{1,p}(\Omega,\omega)$.

Résumé

Les travaux présentés dans cette thèse portent sur l'étude de certains problèmes elliptiques d'équations aux dérivées partielles de type Dirichlet. L'existence et la régularité des solutions de tels problèmes sont traitées dans différents espaces, tels que les espaces de Sobolev classiques et les espaces de Sobolev avec poids. Notre travail fournit des résultats concernant l'existence et la régularité de solutions faibles ou entropiques pour certains problèmes elliptiques non linéaires, et il est composé de quatre chapitres.

- Dans le premier chapitre, nous présentons brièvement divers outils et autres résultats préliminaires qui seront utiles dans les chapitres suivants.

- Dans le chapitre 2, nous traitons l'existence et la régularité d'une solution entropique pour le problème de Dirichlet elliptique non linéaire

$$-\operatorname{div}[\omega(x)a(x,u,\nabla u)] = f - \operatorname{div} F, \qquad \text{dans } \Omega,$$

  où $\omega \in \mathcal{A}_p$-poids, $f \in L^1(\Omega)$ et $F \in [L^{p'}(\Omega, \omega^{1-p'})]^N$.

- Nous établissons au chapitre 3, l'existence et quelques résultats de régularité de solutions du problème unilatéral associé au problème elliptique

$$(P)\begin{cases} \operatorname{div} a(x,u,\nabla u) = \operatorname{div} F & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

  dans le cadre des espaces de Sobolev. Nous avons distingué quatre cas en termes de sommabilité de la donnée F.

- Si $F \in (L^m(\Omega))^N$ avec $m > \frac{N}{p-1}$, alors il existe une solution faible $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.
- Si $F \in (L^m(\Omega))^N$ avec $m = \frac{N}{p-1}$, alors il existe une solution entropique $u \in \mathcal{T}_0^{1,p}(\Omega)$.
- Si $F \in (L^m(\Omega))^N$ avec $\frac{Np'}{N-\theta(N-p)} \leq m < \frac{N}{p-1}$, alors il existe une solution faible $u \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$, avec $r = (1-\theta)(m(p-1))^*$.
- Si $F \in (L^m(\Omega))^N$ avec $\max\left(p', \frac{Np'}{p((1-\theta)N+\theta)}\right) \leq m < \frac{Np'}{N-\theta(N-p)}$, alors il existe une solution entropique $u \in W_0^{1,q}(\Omega) \cap L^r(\Omega)$, avec $r = (1-\theta)(m(p-1))^*$ et $q = \frac{(1-\theta)Nm(p-1)}{N-\theta m(p-1)}$.

• Le chapitre 4 est consacré à l'étude du même problème unilatéral traité au chapitre précédent, mais dans un contexte plus large, celui des espaces de Sobolev avec poids. Nous nous sommes concentrés sur les deux premiers cas.

- Si $\frac{F}{\omega} \in (L^m(\Omega, \omega))^N$ avec $m > \frac{N}{p-1}$, alors il existe une solution faible $u \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.
- Si $\frac{F}{\omega} \in (L^m(\Omega, \omega))^N$ aves $m = \frac{N}{p-1}$, alors il existe une solution entropique $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$.

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Notations

Ω	open bounded set of $\mathbb{R}^N$
$\partial\Omega$	boundary of Ω
$\lvert\cdot\rvert$ or meas	Lebesgue measure in $\mathbb{R}^N$
a.e.	almost everywhere (with respect to the Lebesgue measure)
X	a Banach space
X'	dual of X (i.e., space of linear and continuous functionals on X
p'	$\frac{p}{p-1}$
p^*	$\frac{Np}{N-p}$
$D_i u = \frac{\partial u}{\partial x_i}$	the partial derivative of u with respect to x_i
∇u	the gradient of u
$\operatorname{div} u$	divergence of u
$L^p(\Omega)$	space of functions f such that $\lvert f\rvert^p$ is Lebesgue integrable over $\Omega \subset \mathbb{R}^N$
$\lVert f\rVert_{L_p(\Omega)}$	$\left(\int_\Omega \lvert f(x)\rvert^p dx\right)^{\frac{1}{p}}$
$L^p(\Omega,\omega)$	weighted f such that $\lvert f\rvert^p$ is Lebesgue integrable over $\Omega \subset \mathbb{R}^N$
$\lVert f\rVert_{L_p(\Omega,\omega)}$	$\left(\int_\Omega \lvert f(x)\rvert^p \omega dx\right)^{\frac{1}{p}}$
$C_0^\infty(\Omega)$	the space of indefinitly defferentiable functions with compact support in Ω
$W^{1,p}(\Omega)$	$\left\{u \in L^p(\Omega); \nabla u \in (L^p(\Omega))^N\right\}$
$W_0^{1,p}(\Omega)$	closure of $C_0^\infty(\Omega)$ functions with compact support in Ω with respect to the $W^{1,p}(\Omega)$ norm
$W^{-1,p'}(\Omega)$	$(W_0^{1,p}(\Omega))'$
$W^{1,p}(\Omega,\omega)$	(Sobolev) space of functions with distributional gradient in $(L^p(\Omega))^N$
$T_k(s)$	$\max\{-k, \min\{k, s\}\}$

$\mathcal{T}_0^{1,p}(\Omega)$	$\{u : \Omega \longrightarrow \mathbb{R}$ measurable $: T_k(u) \in W_0^{1,p}(\Omega), \forall k > 0\}$
$\mathcal{T}_0^{1,p}(\Omega,\omega)$	$\{u : \Omega \longrightarrow \mathbb{R}$ measurable $: T_k(u) \in W_0^{1,p}(\Omega,\omega), \forall k > 0\}$
$\mu_u(t)$	distribution function of u
u^*	deacreasing rearrangement of u
$\hookrightarrow$	the symbol of the continuous imbedding
$\hookrightarrow\hookrightarrow$	the symbol of the compact imbedding

Publications

1. Y. Akdim, S. Lalaoui Rhali and Y. Oumouacha. *Solvability of nonlinear elliptic problems with degenerate coercivity in weighted sobolev space.* Accepted for publication in Nonlinear Dynamics and Systems Theory, 23(5), 2023.

2. Y. Akdim, S. Lalaoui Rhali and Y. Oumouacha. *Existence of solution for nonlinear anisotropic degenerated elliptic unilateral problems.* Asia Pacific Journal of Mathematics, 10, 2023.

Submited Work

1. Y. Akdim, S. Lalaoui Rhali and Y. Oumouacha. *On a class of nonlinear elliptic problems with obstacle in weighted Sobolev spaces.* submitted.

Introduction

A partial differential equation (PDE) is an equation involving an unknown function of two or more variables and certain of its partial derivatives. Unlike ordinary differential equations, which involve only one independent variable, PDEs describe complex relationships between multiple variables and their rates of change across space and time.

Over the past decades, there has been a resurgence of interest in the field of nonlinear partial differential equations. The practical uses of these equations, in the mathematical modeling of various physical phenomena within the field of modern applied science, are primarily responsible for this renewed interest. This list of applications encompasses various areas of study, such as the theory of fluid mechanics [9, 63, 65, 45, 56], image restoration [35, 10, 46, 48], elastic materials [71], biology [61], heat and mass transfer in nonhomogeneous media [8], and thermistor problems that describe the electrothermal behavior of organic semiconductor devices [31, 44].

Since the beginning of the 1950s, the investigation of nonlinear elliptic equations with second-order of Leray-Lions type has been the focus of extensive works. See for instance [57, 60, 69, 70] and the references therein.

In the pioneering paper [53], Leray and Lions have studied the following problem

$$Au = -\operatorname{div} a(x, u, \nabla u) = f, \tag{0.0.1}$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function that satisfies the coercivity condition

$$a(x, s, \xi).\xi \geq \alpha |\xi|^p. \tag{0.0.2}$$

In the variational case (i.e. when f is in the dual space $W^{-1,p'}(\Omega)$), they proved the exis-

tence of a weak solution for the Dirichlet problem associated to (0.0.1) as follows

$$\int_\Omega a(x,u,\nabla u)\nabla v dx = \int_\Omega f v dx, \quad \forall v \in W_0^{1,p}(\Omega).$$

If we consider the elliptic problem where the data are no longer in $W^{-1,p'}(\Omega))$, in the case of a datum f in $L^1(\Omega)$, we refer for example to [22]. Subsequently, existence results for the case where the second member is a measure were discussed in [18] (see [64, 23, 52] for further results in these directions).

In the work mentioned above, the functional framework in which the problem (0.0.1) have been studied is the standard Sobolev space. On the other hand, the recent development in modeling physical processes has sparked considerable interest in addressing the associated issues by exploiting the weighted Lebesgue and Sobolev spaces. More precisely, if the condition of ellipticity (0.0.2) is replaced by

$$a(x,s,\xi).\xi \geq \alpha\omega(x)|\xi|^p. \tag{0.0.3}$$

where ω is a function, positive almost everywhere (called weight function), then the difierential operator A becomes degenerate. In this case, it is brought to change the traditional approach by introducing a modifled version of the Sobolev spaces, called weighted Sobolev spaces $W^{1,p}(\Omega,\omega)$. It is important to acknowledge that the primary challenges encountered in the investigation of the degenerate problem manifest in the topological characteristics of weighted Sobolev spaces, namely in relation with compact embedding and the nature of convergence. The approach of degenerate problems was introduced initially by Murthy and Stampachia [59]. Several authors have addressed the extension of the situation of a linear degenerate elliptic operator of higher order, for instance see Kufner and Opic [50, 51].

The questions of the regularity and qualitative properties of solutions of such equations are very important for the applications and indispensable for understanding and controlling physical phenomena. Nowadays, we find in the literature numerous contributions devoted to this topic, see, for example, [1, 12, 27, 40] .

In the same way, nonlinear elliptic equations with principal part having degenerate coercivity has received a great deal of attention See [7, 16, 21, 20, 43], and [29, 30, 54]

for the parabolic case.

The work presented in this thesis has been motivated by the study of some Dirichlet problems associated with nonlinear elliptic equations that involves an elliptic operator in divergence form with degenerate coercivity

$$Au = -\operatorname{div} a(x, u, \nabla u).$$

where the Carathéodory function a satisfies the assumption

$$a(x, s, \xi) \cdot \xi \geq h^{p-1}(|s|)|\xi|^p, \tag{0.0.4}$$

where $h : \mathbb{R}^+ \to]\,0, \infty[$ is a decreasing continuous function such that its primitive H is unbounded

$$H(s) = \int_0^s h(t)dt, \tag{0.0.5}$$

One of the primary challenges associated with addressing these types of problems is the presence of the degenerate coercivity i.e. the function a under the assumption (0.0.4), which leads to a degeneracy phenomenon when the variable u becomes large (for an explicit example of the fact that A will not be coercive, see[62]). This implies that the differential operator A may not be coercive on $W^{1,p}(\Omega)$, therefore the standard Leray–Lions surjectivity theorem cannot be applied to prove the existence results even in the case in which the datum is very regular.

To overcome this problem, we will reason by approximation, cutting by means of truncatures the nonlinearity $a(x, t, \xi)$ in order to get a pseudomonotone and coercive differential operator (which thus have solution) and then we pass to the limit to obtain a solution.

The general strategy of the proofs of the existence and regularity results in this thesis can be resumed as follows:

- Prove that the truncated problem has a solution u_n;
- a priori estimates for the sequences (u_n);
- existence of a converging subsequence;
- passage to the limit as $n \to \infty$.

This thesis is composed by four chapters:

In Chapter 1, we present an overview of the real and functional analysis results that will be used frequently in this thesis. After recalling some basic results on classical Lebesgue and Sobolev spaces, we turn to the study of some properties of weighted Lebesgue and Sobolev spaces. We also recall some basic facts from integration theory such as the fundamental convergence theorems (Lebesgue's dominated convergence theorem, Fatou's Lemma, Vitali's Theorem,...). We end this chapter by stating some results on the theory of pseudo-monotone operators , and we recall the definition of decreasing rearrangement of a measurable function.

We deal in chapter 2 with a class of degenerate elliptic problems of the type

$$\begin{cases} -\operatorname{div}[\omega(x)a(x,u,\nabla u)] = f - \operatorname{div} F, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{0.0.6}$$

where

- $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathédory function satisfying the coercivity condition:

$$a(x,s,\xi).\xi \geq \frac{\alpha}{(1+|s|)^{\theta(p-1)}}|\xi|^p, \tag{0.0.7}$$

  for some $\alpha > 0$ and some real number θ such that $0 \leq \theta < 1$.

- ω belongs to a class of weights called $\mathcal{A}_p$-weights

- $f \in L^1(\Omega)$.

- $F \in [L^{p'}(\Omega, \omega^{1-p'})]^N$ with $(1/p + 1/p' = 1)$.

We intend to give an existence result of the entropy solution of Dirichlet problem (0.0.6) in the following sense

$$\int_\Omega \omega a(x,u,\nabla u)\cdot\nabla T_k(u-v)dx = \int_\Omega f T_k(u-v)dx + \int_\Omega F\cdot\nabla T_k(u-v)dx,$$

for every $k > 0$ and for every $v \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$. Moreover we established some regularity results on the entropy solution. The results of this chapter are presented in [6].

This work has been done in collaboration with Pr. Y. Akdim and Pr. S. Lalaoui Rhali, it has been published in Nonlinear Dynamics and Systems Theory Journal.

Chapter 3 focuses on the existence and regularity results for the unilateral problem associated to the equation

$$\begin{cases} \operatorname{div} a(x,u,\nabla u) = \operatorname{div} F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.8}$$

under the assumptions

$$a(x,s,\xi)\cdot\xi \geq h^{p-1}(|s|)|\xi|^p, \tag{0.0.9}$$

where $h:\mathbb{R}^+ \to]\,0,\infty[$ is a decreasing continuous function such that its primitive H is bounded

$$H(s) = \int_0^s h(t)dt. \tag{0.0.10}$$

Here $F \in (L^m(\Omega))^N$ for some $m > 1$. Let us consider the following problems

$$\begin{cases} u \geq \psi \text{ a.e. in } \Omega, \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla(u-v)dx \leq \int_\Omega F\cdot\nabla(u-v)dx, \\ \text{for all } v \in W_0^{1,p}(\Omega), \text{ with } v(x) \geq \psi(x) \text{ a.e. in } \Omega. \end{cases} \tag{0.0.11}$$

and

$$\begin{cases} u \geq \psi \text{ a.e. in } \Omega, \\ u \in \mathcal{T}_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-v)dx \leq \int_\Omega F\cdot\nabla T_k(u-v)dx, \\ \forall v \in W_0^{1,p}(\Omega), \text{ with } v(x) \geq \psi(x) \text{ a.e. in }. \end{cases} \tag{0.0.12}$$

Our main contributions can be summarized as follow

- if $m > \frac{N}{p-1}$, there exists a bounded solution u of (0.0.11) which belong to $W_0^{1,p}(\Omega)$.
- if $m = \frac{N}{p-1}$, there exists a solution u of (0.0.12)
- if $\frac{Np'}{N-\theta(N-p)} \leq m < \frac{N}{p-1}$, there exists a solution u of (0.0.11) such that

$u \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$, with $r = (1-\theta)(m(p-1))^*$.

- if $\max\left(p', \frac{Np'}{p((1-\theta)N+\theta)}\right) \leq m < \frac{Np'}{N-\theta(N-p)}$, there exists a solution u of (0.0.12) such that $u \in W_0^{1,q}(\Omega) \cap L^r(\Omega)$, with $r = (1-\theta)(m(p-1))^*$ and $q = \frac{(1-\theta)Nm(p-1)}{N-\theta m(p-1)}$.

Note that the content of this chapter is the subject of the paper [5]. This work has been done in collaboration with Pr. Y. Akdim and Pr. S. Lalaoui Rhali, it has been published in Asia Pacific Journal of Mathematics.

Chapter 4 is the twin chapter of the previous one. In the same direction, we present in this chapter the existence and regularity of solutions for the unilateral problem associated to the Dirichlet problem (0.0.8) in the framework of weighted Sobolev spaces. We show that

- if $\frac{F}{\omega} \in (L^m(\Omega,\omega))^N$ with $m > \frac{N}{p-1}$, then there exists a distributional solution $u \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$.

- if $\frac{F}{\omega} \in (L^m(\Omega,\omega))^N$ with $m = \frac{N}{p-1}$, there is an entropy solution $u \in \mathcal{T}_0^{1,p}(\Omega,\omega)$.

The results presented in this chapter are contained in [4]. This work has been done in collaboration with Pr. Y. Akdim and Pr. S. Lalaoui Rhali, it submitted for publication.

Chapter 1

Preliminaries

The aim of this chapter is to summarize and recall the notations and results of the real and functional analysis used throughout this work. First, we recall some functional spaces and their properties. These classes of spaces will be used, among other notions, to provide a framework for our elliptic problems.

Unless otherwise required, in this chapter Ω is a bounded open set of $\mathbb{R}^N$.

1.1 Functional spaces

1.1.1 Standard Lebesgue and Sobolev spaces

Let $p \in \mathbb{R}$ with $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ is defined as follows:

$$L^P(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u|^p dx < \infty \right\},$$

equipped with the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}.$$

For $p = \infty$, the space $L^\infty(\Omega)$ consists of all measurable functions that are essentially bounded on Ω. $L^\infty(\Omega)$ is endowed with the norm

$$\|u\|_{L^\infty(\Omega)} := \inf \{ C > 0; \ \ |u(x)| \leq C \ \ a.e. \ \ \text{in} \ \ \Omega \}.$$

Endowed with this norm $\|\cdot\|_{L^p(\Omega)}$, the space $L^p(\Omega)$ is a Banach space for any $1 \leq p \leq \infty$, reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$.

For all $1 < p < \infty$, the topological dual space of $L^p(\Omega)$ is isometrically identified with $L^{p'}(\Omega)$ where p' is the Hölder conjugate exponent of p, i.e.

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

While the dual of $L^1(\Omega)$ is $L^\infty(\Omega)$. but the dual of $L^\infty(\Omega)$ is strictly contains $L^1(\Omega)$.

We shall need the following inequalities in dealing with integral estimates. Assume that $1 < p < \infty$, then we recall:

- Young's inequality
$$ab \leqslant \frac{a^p}{p} + \frac{b^{p'}}{p'},$$
for a and b positive real numbers .

- Hölder's inequality
$$\int_\Omega uvdx \leqslant \|u\|_p \|v\|_{p'}, \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$$
this holds for functions $u \in L^p(\Omega), v \in L^{p'}(\Omega)$ with $1 < p < \infty$.

Fix $1 \leq p \leq \infty$. The Sobolev space $W^{1,p}(\Omega)$ consists of all locally integrable functions $u : \Omega \longrightarrow \mathbb{R}$ such that $u \in L^p(\Omega)$ and $\left(\frac{\partial u}{\partial x_i}\right)_{i=1,\ldots,N}$ exist in the weak sense and belong to $L^p(\Omega)$. It is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \begin{cases} \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max\left(\|u\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}\right), & p = \infty. \end{cases}$$

The Sobolev space $W^{1,p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$. It is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$.

Given $1 \leq p < \infty$, we denote by $W^{1,p}_0(\Omega)$ the closure of the set $\mathcal{C}^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(\Omega)}$. and we have

$$\left(W^{1,p}_0(\Omega)\right)' = W^{-1,p'}(\Omega).$$

In manipulating Sobolev spaces, we very often rely on certain so-called Sobolev embeddings. We recall one such injection given by Rellich's Theorem.

If $1 \leq p < N$, the Sobolev conjugate p^* of p is

$$p^* = \frac{Np}{N-p}.$$

Note that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}, \quad p^* > p.$$

For every $u \in W_0^{1,p}(\Omega)$, there exists a positive constant C depending only on N and p such that

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)};$$

this inequality is called Sobolev's inequality.

Theorem 1.1.1. *[38] **(Rellich-Kondrachov)** Suppose that Ω is of class C^1 and $p < N$. Then we have the following compact embedding*

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega), \quad \textit{for each } 1 \leq q < p^*.$$

In particular, $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ for every p.

1.1.2 Weighted Sobolev spaces

In this section, we provide brief facts about the weighted Sobolev space as well as some $\mathcal{A}_p$-weight features. Let $\omega = \omega(x)$ be a weight function, that is a locally integrable function on $\mathbb{R}^N$, such that $0 < \omega < \infty$. By integration, each weight ω generates a measure on the measurable subsets of $\mathbb{R}^N$. This measure is denoted by μ and is defined as follows:

$$\mu(\mathcal{S}) = \int_{\mathcal{S}} \omega(x)dx, \tag{1.1.1}$$

for every measurable set $\mathcal{S} \subset \mathbb{R}^N$.

Definition 1.1.1. *Let ω be a weight. For $1 \leq p < \infty$, the weighted Lebesgue space $L^p(\Omega, \omega)$ is the set of all measurable functions u on Ω such that*

$$L^p(\Omega,\omega) = \left\{ u = u(x);\ \ u\omega^{\frac{1}{p}} \in L^p(\Omega) \right\}.$$

$L^p(\Omega,\omega)$ is a Banach space (uniformly convex and hence reflexive if $p > 1$) endowed with the norm

$$\|u\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x) dx \right)^{1/p}.$$

We would point out that the dual of the weighted Lebesgue space $L^p(\Omega,\omega)$ is

$$(L^p(\Omega,\omega))' = L^{p'}(\Omega,\omega^*), \qquad \text{with } \omega^* = \omega^{1-p'}.$$

In order to define the weighted Sobolev space, we assume additionally that the weight ω satisfies the two following conditions

$$\omega^{1-p'} \in L^1_{loc}(\Omega), \tag{1.1.2}$$

Under the assumption (1.1.2), we have $L^p(\Omega,\omega) \hookrightarrow L^1_{loc}(\Omega)$. Indeed, it follows immediately from the Hölder's inequality, since for $u \in L^p(\Omega,\omega)$ and for every Q compact of Ω we have

$$\int_Q |u|^p dx = \int_Q |u|^p \omega^{\frac{1}{p}} \omega^{-\frac{1}{p}} dx \leq \|u\|_{L^p(Q,\omega)} \left(\int_Q \omega^{1-p'} \right)^{\frac{1}{p'}}.$$

Therefore, it make sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 1.1.2. *Let ω be a weight function and $1 < p < \infty$. The weighted Sobolev space $W^{1,p}(\Omega,\omega)$ is defined as the set of all functions $u \in L^p(\Omega,\omega)$ with weak derivatives*

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega,\omega), \qquad \textit{for all } i = 1,\dots,N.$$

The norm of u in $W^{1,p}(\Omega,\omega)$ is given by

$$\|u\|_{W^{1,p}(\Omega,\omega)} = \left(\int_\Omega |u|^p \omega(x) \mathrm{d}x + \int_\Omega |\nabla u|^p \omega(x) \mathrm{d}x \right)^{\frac{1}{p}}. \tag{1.1.3}$$

Thanks to the condition (1.1.2), $W^{1,p}(\Omega,\omega)$ is a uniformly convex (and hence reflexive) Banach space.
If we additionally suppose that also

$$\omega \in L^1_{loc}(\Omega), \tag{1.1.4}$$

then $C_0^\infty(\Omega)$ is a subset of $W^{1,p}(\Omega,\omega)$, and we can introduce the space $W_0^{1,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (1.1.3). Note that the expression

$$\|u\|_{W_0^{1,p}(\Omega,\omega)} = \left(\int_\Omega |\nabla u|^p \omega(x)\mathrm{d}x\right)^{\frac{1}{p}}. \tag{1.1.5}$$

is a norm on $W_0^{1,p}(\Omega,\omega)$ and is equivalent to the norm (1.1.3). Moreover $W_0^{1,p}(\Omega,\omega)$ is a reflexive Banach space.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega,\omega)$ is equivalent to $W^{-1,p'}(\Omega,\omega^*)$.

1.1.3 $\mathcal{A}_p$-weights

We briefly recall some fundamentals on the class of $\mathcal{A}_p$ weights, that was introduced by B. Muckenhoupt .

Definition 1.1.3. *[58] Let ω be a weight and $1 \le p < \infty$. We say that ω belongs to the $\mathcal{A}_p$-weights classes, if there exists a positive constant $C_{\omega,p}$ such that, for every ball $\mathcal{B} \subset \mathbb{R}^N$,*

$$\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} \omega(x)\mathrm{d}x\right)\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} \omega^{1/(1-p)}(x)\mathrm{d}x\right)^{p-1} \le C \quad \text{if } p>1,$$
$$\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} \omega(x)\mathrm{d}x\right)\left(\operatorname*{ess\,sup}_{x\in\mathcal{B}} \frac{1}{\omega(x)}\right) \le C \quad \text{if } p=1,$$

where $|.|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$.

As an example of an $\mathcal{A}_p$-weight, the function $\omega(x) = |x|^\alpha, x \in \mathbb{R}^N$, is in $\mathcal{A}_p$ if and only if, $-N < \alpha < N(p-1)$.

Remark that if $\omega \in \mathcal{A}_p$, then it enjoys the two properties (1.1.2) and (1.1.4). As a result, the definition of the weighted Sobolev space and its features discussed previously remain

valid only if $\omega \in \mathcal{A}_p$.

Lemma 1.1.1. *[47]. Let $\mathcal{B}$ be a ball in $\mathbb{R}^N$ and $\mathcal{S}$ be a measurable subset of $\mathcal{B}$. If $\omega \in \mathcal{A}_p$, $1 < p < \infty$, then*

$$\left(\frac{|\mathcal{S}|}{|\mathcal{B}|}\right)^p \leq C_{\omega,p}\frac{\mu(\mathcal{S})}{\mu(\mathcal{B})}.$$

Remark 1.1.1. *If $\mu(\mathcal{S}) = 0$ then $|\mathcal{S}| = 0$. Thus, for every sequence (u_n) in $\mathcal{B}$ that converges μ-a.e. to some u, we have $u_n \longrightarrow u$ a.e.*

Theorem 1.1.2. *[39](**The weighted Sobolev inequality**). Let $\omega \in \mathcal{A}_p$ with $1 < p < \infty$. There exist positive constants C_Ω and δ such that for all $u \in C_0^\infty(\Omega)$ and all η satisfying $1 \leq \eta \leq \frac{N}{N-1} + \delta$,*

$$\|u\|_{L^{\eta p}(\Omega,\omega)} \leq C_\Omega \||\nabla u|\|_{L^p(\Omega,\omega)}. \tag{1.1.6}$$

Theorem 1.1.3. *[36]. Let Ω be a bounded smooth domain. For $\omega \in \mathcal{A}_p$, we have the compact embedding*

$$W_0^{1,p}(\Omega,\omega) \hookrightarrow\hookrightarrow L^p(\Omega,\omega). \tag{1.1.7}$$

1.1.4 Other spaces

In what follows, $T_k(x)$ denotes the truncation function defined for $k > 0$, and x in $\mathbb{R}$ by

$$T_k(x) = \begin{cases} k & \text{if } x > k, \\ x & \text{if } -k \leq x \leq k, \\ -k & \text{if } x < -k. \end{cases}$$

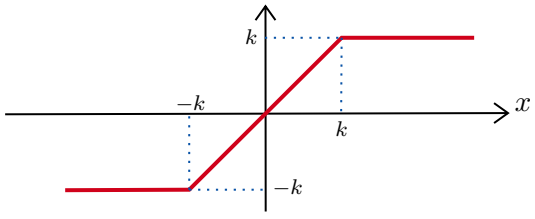


Figure 1.1: truncation function

Remark that:

1. T_k is a Lipschitz continuous function satisfying $|T_k(x)| \leq k$ and $|T_k(x)| \leq |x|$.

2. In all cases, there holds $T_k(u) \in L^\infty(\Omega)$ with $\|T_k(u)\|_{L^\infty(\Omega)} \leq k$.

3. If $u \in W_0^{1,p}(\Omega)$ then $T_k(u) \in W_0^{1,p}(\Omega)$ since $T_k(0) = 0$.

We define the spaces:

$$\mathcal{T}_0^{1,p}(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable, such that } : T_k(u) \in W_0^{1,p}(\Omega) \text{ for any } k > 0\right\};$$

$$\mathcal{T}_0^{1,p}(\Omega,\omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable, such that } : T_k(u) \in W_0^{1,p}(\Omega,\omega) \text{ for any } k > 0\right\}.$$

We recall the following result (see [15], Lemma 2.1):

Lemma 1.1.2. *For every $u \in \mathcal{T}_0^{1,p}(\Omega)$ there exists a unique measurable map $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}, \quad \text{a.e. in } \Omega, \text{ for every } k > 0.$$

Moreover, if $u \in W_0^{1,p}(\Omega)$, then v coincides with the usual distributional gradient ∇u.

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$.

Weighted Marcinkiewicz spaces

Definition 1.1.4. *Let $0 < r < \infty$ and let ω be a weighted function. The weighted Marcinkiewicz space $\mathcal{M}^r(\Omega,\omega)$ is defined as the set of all measurable functions $f : \Omega \to \mathbb{R}$ such that the function*

$$\Gamma_f(k) = \mu(\{x \in \Omega : |f(x)| > k\}), \quad \text{for } k > 0,$$

satisfies, for some positive constant C, an estimate of the form $\Gamma_f(k) \leq Ck^{-r}$.

Remark 1.1.2. *It follows from [42] that if $1 \leq q < p$ and $\Omega \subset \mathbb{R}^N$ is a bounded measurable set. Then*

$$L^p(\Omega,\omega) \subset \mathcal{M}^p(\Omega,\omega) \text{ and } \mathcal{M}^p(\Omega,\omega) \subset L^q(\Omega,\omega).$$

1.2 Integration and convergence theorems

We recall here some well known theorems proved in Lebesgue spaces.

Definition 1.2.1. *Let u, u_n be measurable real-valued functions on Ω.*

1. *Suppose that u, u_n are in $L^p(\Omega)$. The sequence (u_n) converges in $L^p(\Omega)$ to u if*

$$\lim_{n\to\infty} \|u_n - u\|_{L^p(\Omega)} = 0.$$

2. *The sequence (u_n) is said to converge almost everywhere on Ω to u if*

$$meas\left(\{x \in \Omega : u_n(x) \text{ does not converge to } u(x)\}\right) = 0.$$

and we write $u_n \to u$ a.e. in Ω.

3. *The sequence (u_n) is said to converge in measure to u , if for all $\varepsilon > 0$*

$$\lim_{n\to\infty} \text{meas}\left(\{x \in \Omega : |u_n(x) - u(x)| > \varepsilon\}\right) = 0.$$

4. *We say that a sequence (u_n) is Cauchy in measure if for every $\varepsilon > 0$*

$$\lim_{n,m\to\infty} \text{meas}\left(\{x \in \Omega : |u_n(x) - u_m(x)| > \varepsilon\}\right) = 0.$$

Theorem 1.2.1. *[14] Let (u_n) be a sequence of measurable functions which is Cauchy in measure. Then there is a subsequence which converges almost everywhere and in measure to a measurable function u.*

Lemma 1.2.1. ***(Fatou's Lemma)*** *Let (u_n) be a sequence of nonnegative measurable functions on Ω, then*

$$\int_\Omega (\liminf_{n\to\infty} u_n)dx \leq \liminf_{n\to\infty} \left(\int_\Omega u_n dx\right).$$

The following theorems give us some sufficient conditions for the convergence in $L^p(\Omega)$.

Theorem 1.2.2. ***(Lebesgue's dominated convergence theorem)***
Let $p \in [1, \infty[$ and (u_n) be a sequence of functions of $L^p(\Omega)$ such that

i) $u_n \longrightarrow u$ a.e. in Ω;

ii) there exists a function $v \in L^p(\Omega)$ satisfying $|u_n(x)| \leq v$ a.e. in Ω.

Then $u \in L^p(\Omega)$ and $u_n \longrightarrow u$ in $L^p(\Omega)$.

Definition 1.2.2. ***(Equi-integrability)***

A family $\mathfrak{F}$ of $L^p(\Omega)$ functions $(1 \leq p < \infty)$ is said to be equi-integrable if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for all $\mathcal{A} \subseteq \Omega$ with $meas(\mathcal{A}) < \delta$ and for every $f \in \mathfrak{F}$,

$$\int_{\mathcal{A}} |f|^p dx < \varepsilon.$$

Theorem 1.2.3. ***(Vitali's Theorem)*** *Let $p \in [1, \infty[$ and (u_n) be a sequence of functions of $L^p(\Omega)$ such that*

i) $u_n \longrightarrow u$ a.e. in Ω;

ii) the sequence (u_n) is equi-integrable.

Then u_n converges to u in $L^p(\Omega)$.

Theorem 1.2.4. *If $u \in L^p(\Omega)$, then $T_k(u) \to u$ in $L^p(\Omega)$ strongly when $k \to +\infty$. And if $u \in W^{1,p}(\Omega)$, then $T_k(u) \to u$ in $W^{1,p}(\Omega)$ strongly.*

1.3 Decreasing rearrangement

Let u be a real-valued measurable function defined in Ω. One can define the distribution function $\mu_u(t)$ of u as:

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}|, \quad t \geq 0.$$

The decreasing rearrangement u^* of u is then defined as the generalized inverse function of μ_u:

$$u^*(s) = \inf \{t \geq 0 : \mu_u(t) \leq s\} \qquad \text{for } s \in [0, |\Omega|]$$

We recall the following proprieties (see [68])

$$u^*(0) = \operatorname{ess\,sup} |u| \text{ and } u^*(+\infty) = 0 \tag{1.3.1}$$

and

$$u^*(\mu(t)) \leq t, \qquad \text{for every nonnegative } \ t \tag{1.3.2}$$

If f is any continuous increasing map from $[0,\infty]$ into $[0,\infty]$ such that $f(0)=0$, then (see [16])

$$\int_\Omega f(|u(x)|)dx = \int_0^\infty f\left(u^*(t)\right)dt.$$

More properties of decreasing rearrangement can be found, for instance, in [13, 49]

1.4 Monotones operators

Let X be a reflexive and separable Banach space and A a mapping from X to X'.

Definition 1.4.1. *Let $A : X \to X'$*

1. *A is monotone if*
$$\langle A(u)-A(v), u-v\rangle \geq 0, \quad \forall u,v \in X.$$

2. *A is hemi-continuous if for all $u,v,w \in X$, the mapping $t \to \langle A(u+tv), w\rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$.*

3. *The operator A is said to be coercive if*
$$\frac{< A(v), v >}{\|v\|_X} \longrightarrow +\infty \quad \text{as} \quad \|v\|_X \longrightarrow +\infty, v \in X.$$

Definition 1.4.2. *i) We say that the operator A is sense 1 pseudo-monotone if*

$$\left.\begin{array}{r} u_n \rightharpoonup u \text{ in } X \text{ weakly} \\ A\left(u_n\right) \rightharpoonup \psi \text{ in } X' \text{ weakly} \\ \limsup_{n\to+\infty} \left\langle A\left(u_n\right), u_n\right\rangle \leq \langle \psi, u\rangle \end{array}\right\} \Longrightarrow \psi = A(u) \text{ and } \left\langle A\left(u_n\right), u_n\right\rangle \to \langle A(u), u\rangle$$

ii) We say that A is sense 2 pseudo-monotone if

$$\left.\begin{array}{r} u_n \rightharpoonup u \text{ in } X \text{ weakly} \\ \limsup_{n\to+\infty} \left\langle A\left(u_n\right), u_n - u\right\rangle \leq 0 \end{array}\right\} \Longrightarrow \forall v \in X, \liminf_{n\to+\infty} \left\langle A\left(u_n\right), u_n - v\right\rangle \geq \langle A(u), u-v\rangle$$

The two definitions of pseudo-monotonicity are basically equivalent. More precisely,

if A is bounded, then it is sense 1 pseudo-monotone if and only if it is sense 2 pseudo-monotone.

Theorem 1.4.1. *[55](Surjectivity Theorem) Let $A : X \longrightarrow X'$ be a bounded, pseudomonotone and coercive operator. Then A is surjective, that is, for every f in V' there exists u in V such that $A(u) = f$.*

Chapter 2

Solvability of nonlinear elliptic problems with degenerate coercivity in weighted Sobolev spaces

Let Ω be a bounded smooth subset of $\mathbb{R}^N$ with $N \geq 2$. We are interested in proving the existence of entropy solutions for the following elliptic Dirichlet problem

$$(P)\begin{cases} -\operatorname{div}[\omega(x)a(x,u,\nabla u)] = f - \operatorname{div} F, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

Here $a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathédory function (that is, measurable with respect to x for every $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, and continuous with respect to (s,ξ) for almost every $x \in \Omega$) satisfying the following conditions:

$$a(x,s,\xi).\xi \geq \frac{\alpha}{(1+|s|)^{\theta(p-1)}}|\xi|^p, \tag{2.0.1}$$

with $1 < p < \infty$, for some $\alpha > 0$ and some real number θ such that

$$0 \leq \theta < 1. \tag{2.0.2}$$

As far as the datum f and F are concerned, we will assume that f belongs to the space $L^1(\Omega)$, and $\frac{F}{\omega} \in [L^{p'}(\Omega,\omega)]^N$ with $(1/p + 1/p' = 1)$.

Problems like (P) have been studied by many authors in the in the non-weighted case. In [52], Leone and Porretta studied the nonlinear elliptic problem

$$Bu = f(x) - \operatorname{div}(F) \quad in \ \ \Omega$$

in the setting of Sobolev spaces where $Bu = -\operatorname{div}(a(x,u,\nabla u))$ is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ to $W^{-1.p'}(\Omega)$, they showed the existence of entropy solutions (see also [28], [72]). In addition, Alvino et al. proved in [7] the existence of entropy solutions for the nonlinear elliptic equations $-\operatorname{div}(a(x,u,\nabla u)) = f$, under assumption (2.0.1).

Notice that the existence of a weak solution for the problem (P) has been obtained by Cavalheiro in [33] under the following condition

$$a(x,s,\xi).\xi \geq \alpha|\xi|^p, \tag{2.0.3}$$

and by assuming that $f/\omega \in L^{p'}(\Omega,\omega)$. Also he discussed in [34] the existence of entropy solutions when $f \in L^1(\Omega)$.

Our objective in this work is to study the problem (P) when the operator satisfies assumption (2.0.1) instead of (2.0.3). The main difficulties that arise in our study are due, on the one hand to the fact that the differential operator $A(u) = -\operatorname{div}(\omega(x)a(x,u,\nabla u))$, which is well defined between $W_0^{1,p}(\Omega,\omega)$ and its dual $W^{-1,p'}(\Omega,\omega^{1-p'})$ may not be coercive on $W_0^{1,p}(\Omega,\omega)$ when u is large. This implies that the classical methods used in order to prove the existence of a solution for problem (P) cannot be applied even if the datum is very regular. On the other hand f belongs only to $L^1(\Omega)$, so that proving existence of a weak solution (i.e., in the distribution meaning) seems to be an arduous task. To overcome it we use in this paper the framework of entropy solutions.This notion was introduce in [15] and then used by many authors to study elliptic equations (see [3],[7],[32],[52]).

2.1 Existence of an entropy solution

2.1.1 Entropy solutions

We first give the definition of an entropy solution of problem (P). For a given constant $k > 0$, the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ is defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\ sign(s), & \text{if } |s| > k. \end{cases}$$

We denote by $\mathcal{T}_0^{1,p}(\Omega, \omega)$ the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that for every $k > 0$ the truncated function $T_k(u)$ belongs to $W_0^{1,p}(\Omega, \omega)$.

We give the following Lemma which is a generalization of Lemma 1.1.2 in weighted Sobolev spaces.

Lemma 2.1.1. *Let $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$, then there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}. \tag{2.1.1}$$

If $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$, the weak gradient ∇u of u is defined as the unique function v which satisfies (2.1.1).

Definition 2.1.1. *A function $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$ is an entropy solution of the problem (P) if*

$$\int_\Omega \omega a(x, u, \nabla u) \cdot \nabla T_k(u - v)dx = \int_\Omega f T_k(u - v)dx + \int_\Omega F \cdot \nabla T_k(u - v)dx, \tag{2.1.2}$$

for every $k > 0$ and for every $v \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.

2.1.2 Basic assumptions

Let Ω be an open bounded smooth domain of $\mathbb{R}^N (N \geq 2)$, $p > 1$ and $\omega \in \mathcal{A}_p$.
Let $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodry function (that is, $a(., s, \xi)$ is measurable on Ω for every (t, ξ) in $\mathbb{R} \times \mathbb{R}^N$, and $a(x, ., .)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω). Assume that:

$$a(x, s, \xi) \cdot \xi \geq b(|s|)|\xi|^p, \tag{2.1.3}$$

for almost every x in Ω and for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where

$$b(s) = \frac{\alpha}{(1+s)^{\theta(p-1)}}, \tag{2.1.4}$$

for some $0 \leq \theta < 1$ and some $\alpha > 0$;

$$|a(x,s,\xi)| \leq h_0(x) + h_1(x)|s|^{p-1} + h_2(x)|\xi|^{p-1}, \tag{2.1.5}$$

for almost every x in Ω and for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where h_0, h_1 and h_2 are non-negative functions, with $h_0 \in L^{p'}(\Omega,\omega)$ and $h_1, h_2 \in L^{\infty}(\Omega)$;

$$[a(x,s,\xi) - a(x,s,\xi')] \cdot (\xi - \xi') > 0, \tag{2.1.6}$$

for almost every x in Ω and for every $s \in \mathbb{R}$, for every ξ, ξ' in $\mathbb{R}^N$ with $\xi \neq \xi'$. As regards the source term, we assume that

$$f \in L^1(\Omega), \tag{2.1.7}$$

and

$$\frac{F}{\omega} \in [L^{p'}(\Omega,\omega)]^N. \tag{2.1.8}$$

2.1.3 Main Result

The main result proved in this chapter is the following.

Theorem 2.1.1. *Assume that the Carathéodry function a satisfies (2.1.3)-(2.1.6). Then there exists an entropy solution u of the problem (P).*

Proposition 2.1.1. *The entropy solution u of the problem (P) satisfies*

$$u \in \mathcal{M}^r(\Omega,\omega), \quad \text{with } r = \eta(p-1)(1-\theta), \tag{2.1.9}$$

$$|\nabla u| \in \mathcal{M}^s(\Omega,\omega), \quad \text{with } s = \frac{p\eta(p-1)(1-\theta)}{r+\theta(p-1)+1}, \tag{2.1.10}$$

where η is the constant given in the weighted Sobolev inequality (1.1.6).

2.2 Useful Lemmas

Lemma 2.2.1. *[2]. Let ω be a weight function on Ω and $1 < p < \infty$. Let $g \in L^p(\Omega, \omega)$ and $g_n \in L^p(\Omega, \omega)$ with $\|g_n\|_{L^p(\Omega,\omega)} \leq C$, where C is a constant. If $g_n(x) \to g(x)$ a.e. in Ω, then*

$$g_n \rightharpoonup g, \quad \text{weakly in } L^p(\Omega, \omega).$$

Lemma 2.2.2. *Let $\omega \in \mathcal{A}_p$ with $1 < p < \infty$, and assume that (2.1.3)-(2.1.6) hold. Let $\{u_n\}$ be a sequence in $W_0^{1,p}(\Omega, \omega)$ such that*

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega, \omega) \text{ and } \mu\text{-a.e. in } \Omega, \tag{2.2.1}$$

and

$$\int_\Omega \omega \left[a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)\right] \nabla(u_n - u) dx \longrightarrow 0. \tag{2.2.2}$$

Then $u_n \to u$ in $W_0^{1,p}(\Omega, \omega)$.

Proof. We set

$$D_n = \omega \left[a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)\right] \nabla(u_n - u).$$

From (2.1.6), it is clear that D_n is a positive function, and (2.2.2) implies that

$$D_n \longrightarrow 0 \text{ strogly in } L^1(\Omega, \omega).$$

Since $u_n \to u$ μ-a.e. in Ω then $u_n \to u$ a.e. in Ω, and since $D_n \to 0$ a.e. in Ω, there exists a subset B in Ω with measure zero such that for all $x \in \Omega \backslash B$,

$$|u(x)| < \infty, \quad |\nabla u(x)| < \infty, \quad h_0(x) < \infty, \quad u_n \to u, \quad D_n \to 0.$$

On the other hand, using (2.1.3) and (2.1.5) we have for every $x \in \Omega\backslash B$,

$$\begin{aligned}
D_n(x) =& \omega(x)\left[a(x,u_n(x),\nabla u_n(x)) - a(x,u_n(x),\nabla u(x))\right]\cdot\nabla(u_n(x)-u(x))\\
=& \omega(x)a(x,u_n(x),\nabla u_n(x))\cdot\nabla u_n(x) + \omega(x)a(x,u_n(x),\nabla u(x))\cdot\nabla u(x)\\
&- \omega(x)a(x,u_n(x),\nabla u_n(x))\cdot\nabla u(x) - \omega(x)a(x,u_n(x),\nabla u(x))\cdot\nabla u_n(x)\\
\geq& \omega(x)b(|u_n(x)|)|\nabla u_n(x)|^p + \omega(x)b(|u_n(x)|)|\nabla u(x)|^p\\
&- \omega(x)\left(h_0(x) + h_1(x)|u_n(x)|^{p-1} + h_2(x)|\nabla u_n(x)|^{p-1}\right)|\nabla u(x)|\\
&- \omega(x)\left(h_0(x) + h_1(x)|u_n(x)|^{p-1} + h_2(x)|\nabla u(x)|^{p-1}\right)|\nabla u_n(x)|.
\end{aligned}$$

Since $u_n(x) \to u(x)$ then $|u_n(x)|$ is bounded, there exist two constants $C_1(x)$ and $C_2(x)$ which depend on x but do not depend on n such that

$$D_n(x) \geq C_1(x)|\nabla u_n(x)|^p - C_2(x)\left[1 + |\nabla u_n(x)|^{p-1} + |\nabla u_n(x)|\right]. \tag{2.2.3}$$

Thus, $|\nabla u_n(x)|$ is bounded uniformly with respect to n, indeed, from (2.2.3)

$$D_n(x) \geq |\nabla u_n(x)|^p\left[C_1(x) - \frac{C_2(x)}{|\nabla u_n(x)|^p} - \frac{C_2(x)}{|\nabla u_n(x)|} - \frac{C_2(x)}{|\nabla u_n(x)|^{p-1}}\right].$$

If $|\nabla u_n(x)| \to \infty$ (for a subsequence), then $D_n(x) \to \infty$ which gives a contradiction.

Let now χ be a cluster point of $\nabla u_n(x)$. We have $|\chi| < \infty$ and since the function $a(x,\cdot,\cdot)$ is continuous, we obtain

$$\left[a\left(x,u(x),\chi\right) - a(x,u(x),\nabla u(x))\right]\cdot(\chi - \nabla u(x)) = 0, \tag{2.2.4}$$

then together with (2.1.6), we conclude that $\chi = \nabla u(x)$. The uniqueness of the cluster point implies

$$\nabla u_n(x) \to \nabla u(x), \quad \forall x \in \Omega\backslash B,$$

that is

$$\nabla u_n \to \nabla u, \quad \text{a.e. in } \Omega.$$

Using (2.1.6), we can prove that the sequence $\omega a(x,u_n,\nabla u_n)$ is bounded in $\left(L^{p'}(\Omega,\omega^{1-p'})\right)^N$.

Therefore, since $a\left(x, u_n, \nabla u_n\right) \to a(x, u, \nabla u)$ a.e. in Ω. Lemma 2.2.1 implies

$$\omega a(x, u_n, \nabla u_n) \rightharpoonup \omega a(x, u, \nabla u) \quad \text{in } \left(L^{p'}(\Omega, \omega^{1-p'})\right)^N.$$

We set $\bar{y}_n = \omega a(x, u_n, \nabla u_n) \cdot \nabla u_n$ and $\bar{y} = \omega a(x, u, \nabla u) \cdot \nabla u$. As in [26] we get

$$\bar{y}_n \to \bar{y} \text{ in } L^1(\Omega).$$

By (2.1.3) we have

$$b(|u_n|)|\nabla u_n|^{p(x)} \leq a\left(x, u_n, \nabla u_n\right) \nabla u_n.$$

Let $z_n = \omega|\nabla u_n|^p, z = \omega|\nabla u|^p, y_n = \frac{\bar{y}_n}{b(|u_n|)}$, and $y = \frac{\bar{y}}{b(|u|)}$. Then by Fatou's lemma, it follows that

$$\int_\Omega 2y dx \leq \liminf_{n\to\infty} \int_\Omega y + y_n - |z_n - z|\, dx;$$

i.e., $0 \leq -\limsup_{n\to\infty} \int_\Omega |z_n - z|\, dx$.

Then

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_n - z|\, dx \leq \limsup_{n\to\infty} \int_\Omega |z_n - z|\, dx \leq 0,$$

this implies

$$\nabla u_n \to \nabla u \quad \text{in } \left(L^p(\Omega, \omega)\right)^N.$$

Hence $u_n \to u$ in $W_0^{1,p}(\Omega, \omega)$, which completes the present proof. □

Lemma 2.2.3. *Let $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$, where $\omega \in \mathcal{A}_p$, $1 < p < \infty$. Let $1 \leq \lambda < p$, and suppose that u satisfies*

$$\int_{\{|u|<k\}} |\nabla u|^p \omega \mathrm{d}x \leq M k^\lambda, \quad \forall k > 0. \tag{2.2.5}$$

Then u belongs to $\mathcal{M}^r(\Omega, \omega)$, with $r = \eta(p - \lambda)$ (where $1 \leq \eta \leq \frac{N}{N-1} + \delta$). More precisely, there exists $C > 0$ such that

$$\Gamma_k(u) \leq C M^\eta k^{-r}.$$

Proof. For $0 < \varepsilon \leq k$, we have $\{x \in \Omega : |u| \geq \varepsilon\} = \{x \in \Omega : |T_k(u)| \geq \varepsilon\}$.

Thus

$$\begin{aligned}\mu(\{x\in\Omega:|u|>\varepsilon\}) &= \mu(\{x\in\Omega:|T_k(u)|\geq\varepsilon\})\\ &= \int_{\{|T_k(u)|\geq\varepsilon\}}\omega(x)\mathrm{d}x\\ &= \frac{1}{\varepsilon^{\eta p}}\int_{\{|T_k(u)|\geq\varepsilon\}}\varepsilon^{\eta p}\omega(x)\mathrm{d}x\\ &\leq \frac{1}{\varepsilon^{\eta p}}\int_{\{|T_k(u)|\geq\varepsilon\}}|T_k(u)|^{\eta p}\omega(x)\mathrm{d}x\\ &\leq \frac{1}{\varepsilon^{\eta p}}\left\|T_k(u)\right\|^{\eta p}_{L^{\eta p}(\Omega,\omega)}.\end{aligned}$$

Using Theorem 1.1.2 and inequality (2.2.5), we get

$$\|T_k(u)\|_{L^{\eta p}(\Omega,\omega)}\leq C_\Omega\left\||\nabla T_k(u)\right\|_{L^p(\Omega,\omega)}\leq C_\Omega(Mk^\lambda)^{1/p},$$

which implies that

$$\mu(\{x\in\Omega:|u|>\varepsilon\})\leq\frac{1}{\varepsilon^{\eta p}}C(Mk^\lambda)^\eta.$$

Therefore, by taking $\varepsilon=k$, we have

$$\mu(\{x\in\Omega:|u|>k\})\leq\frac{1}{k^{\eta p}}C(Mk^\lambda)^\eta=CM^\eta k^{-\eta(p-\lambda)}=CM^\eta k^{-r}.$$

□

Lemma 2.2.4. *Assume that the hypothesis in Lemma 2.2.3 hold true. Then $|\nabla u|\in\mathcal{M}^s(\Omega,\omega)$, where $s=pr/(r+\lambda)$ (with r as in Lemma 2.2.3). More precisely, there exists $C>0$ such that*

$$\Gamma_k(|\nabla u|)\leq CM^{(r+\eta\lambda)/(r+\lambda)}k^{-s}.$$

Proof. We set for every $k,\rho>0$, the function

$$\Psi(k,\rho)=\mu\left(\{x\in\Omega:|\nabla u|^p>\rho,|u|>k\}\right).$$

It is clear that the function $\rho \to \Psi(k,\rho)$ is decreasing. Thus for $k>0$ and $\rho>0$ we obtain

$$\Psi(0,\rho) \leq \frac{1}{\rho}\int_0^\rho \Psi(0,s)\mathrm{d}s \leq \Psi(k,0) + \frac{1}{\rho}\int_0^\rho (\Psi(0,s)-\Psi(k,s))\mathrm{d}s. \tag{2.2.6}$$

From Lemma 2.2.3, we have

$$\Psi(k,0) \leq CM^\eta k^{-r}. \tag{2.2.7}$$

Now, observe that

$$\begin{aligned}\Psi(0,s)-\Psi(k,s) &= \mu\left(\{x\in\Omega : |\nabla u|^p > s, |u|>0\}\right) - \mu\left(\{x\in\Omega : |\nabla u|^p > s, |u|>k\}\right)\\ &= \mu\left(\{x\in\Omega : |\nabla u|^p > s, |u|\leq k\}\right).\end{aligned}$$

Hence, we get by using Proposition 6.24 in [41] that

$$\begin{aligned}\int_0^\infty (\Psi(0,s)-\Psi(k,s))ds &= \int_0^\infty \mu\left(\{x\in\Omega : |\nabla u|^p > s, |u|\leq k\}\right) ds\\ &= \int_{\{|u|<k\}} |\nabla u(x)|^p \omega(x) dx\\ &\leq Mk^\lambda.\end{aligned} \tag{2.2.8}$$

Going back to (2.2.6) and using (2.2.7) and (2.2.8) we obtain

$$\Psi(0,\rho) \leq CM^\eta k^{-r} + \frac{1}{\rho}Mk^\lambda. \tag{2.2.9}$$

A minimization of the right-hand side of (2.2.9) in k gives

$$\Psi(0,\rho) \leq CM^{(r+\eta\lambda)/(r+\lambda)}\rho^{-r/(r+\lambda)}.$$

Setting $\rho = h^p$ we obtain

$$\Psi(0,h^p) = \mu\left(\{x\in\Omega : |\nabla u| > h\}\right) \leq CM^{(r+\eta\lambda)/(r+\lambda)}h^{-rp/(r+\lambda)} = CM^{(r+\eta\lambda)/(r+\lambda)}h^{-s},$$

where $s = rp/(r+\lambda)$. □

Let $n \in \mathbb{N}$, and define, for u in $W_0^{1,p}(\Omega,\omega)$, the differential operator

$$A_n(u) = -\operatorname{div}[\omega(x)a(x,T_n(u),\nabla u)].$$

Lemma 2.2.5. *The operator A_n maps $W_0^{1,p}(\Omega,\omega)$ into its dual $W^{-1,p'}(\Omega,\omega^{1-p'})$. Moreover A_n is bounded, pseudomonotone and coercive in the following sense*

$$\frac{< A_n v, v >}{\|v\|_{W_0^{1,p}(\Omega,\omega)}} \longrightarrow +\infty \quad \textit{if} \quad \|v\|_{W_0^{1,p}(\Omega,\omega)} \longrightarrow +\infty, v \in W_0^{1,p}(\Omega,\omega).$$

Proof. By (2.1.5), we deduce that for every u in $W_0^{1,p}(\Omega,\omega)$

$$\begin{aligned}\int_\Omega \Big(\omega|a(x,T_n(u),\nabla u)|\Big)^{p'} \omega^{1-p'} dx & \\ &\leq \int_\Omega \Big(h_0 + h_1|T_n(u)|^{p/p'} + h_2|\nabla u|^{p/p'}\Big)^{p'} \omega dx \\ &\leq C\left[\|h_0\|^{p'}_{L'(\Omega,\omega)} + \Big(C_\Omega\|h_1\|^{p'}_{L^\infty(\Omega)} + \|h_2\|^{p'}_{L^\infty(\Omega)}\Big)\|u\|^p_{W_0^{1,p}(\Omega,\omega)}\right].\end{aligned}$$

which mean that $\omega a(x,T_n(u),\nabla u)$ belongs to $\big(L^{p'}(\Omega,\omega^{1-p'})\big)^N$.
Therefore

$$A_n(u) \in W^{-1,p'}(\Omega,\omega^{1-p'}), \quad \forall n \in \mathbb{N}.$$

Thanks to Hölder's inequality and (2.1.5), we have for all $u, v \in W_0^{1,p}(\Omega,\omega)$,

$$|\langle A_n u, v\rangle| \leq \left(C\left[\|h_0\|^{p'}_{L'(\Omega,\omega)} + \Big(C_\Omega\|h_1\|^{p'}_{L^\infty(\Omega)} + \|h_2\|^{p'}_{L^\infty(\Omega)}\Big)\|u\|^p_{W_0^{1,p}(\Omega,\omega)}\right]\right)^{1/p'} \|v\|_{W_0^{1,p}(\Omega,\omega)}.$$

Thus A_n is bounded from $W_0^{1,p}(\Omega,\omega)$ to $W^{-1,p'}(\Omega,\omega^{1-p'})$.

For the coercivity, by using (2.1.3), we get for every $u \in W_0^{1,p}(\Omega,\omega)$

$$\begin{aligned}\langle A_n u, u\rangle &= \int_\Omega \omega a(x,T_n(u),\nabla u)\cdot\nabla u dx \\ &\geq \int_\Omega \omega b(|T_n(u)|)|\nabla u|^p dx \\ &\geq b(n)\|u\|^p_{W_0^{1,p}(\Omega,\omega)}.\end{aligned}$$

Hence, the operator A_n is coercive.

It remain to show that A_n is pseudomonotone. We thus consider a sequence $u_j \in W_0^{1,p}(\Omega,\omega)$ such that

$$\begin{cases} u_j \rightharpoonup u, & \text{in } W_0^{1,p}(\Omega,\omega), \\ A_n u_j \rightharpoonup \psi^n, & \text{in } W^{-1,p'}(\Omega,\omega^{1-p'}), \\ \limsup_{j\to\infty} \langle A_n u_j, u_j\rangle & \leq \langle \psi^n, u\rangle. \end{cases} \tag{2.2.10}$$

We shall prove that

$$\psi^n = A_n u \text{ and } \langle A_n u_j, u_j\rangle \to \langle A_n u, u\rangle.$$

Firstly, since $W_0^{1,p}(\Omega,\omega) \hookrightarrow\hookrightarrow L^p(\Omega,\omega)$, then

$$u_j \to u \quad \text{in } L^p(\Omega,\omega) \text{ for a subsequence denoted again } (u_j)_j.$$

Due to the boundedness of the sequence $(u_j)_j$ in $W_0^{1,p}(\Omega,\omega)$, and using the growth assumption (2.1.5), we have that $\omega a(x, T_n(u_j), \nabla u_j)$ is bounded in $\left(L^{p'}(\Omega,\omega^{1-p'})\right)^N$. Therefore there exists a function $\phi \in L^{p'}(\Omega,\omega^{1-p'})$ such that

$$\omega a(x, T_n(u_j), \nabla u_j) \rightharpoonup \phi \text{ weakly in } \left(L^{p'}(\Omega,\omega^{1-p'})\right)^N. \tag{2.2.11}$$

It is clear that, for all $v \in W_0^{1,p}(\Omega,\omega)$

$$\begin{aligned} \langle \psi^n, v\rangle &= \lim_{j\to+\infty} \langle A_n u_j, v\rangle \\ &= \lim_{j\to+\infty} \int_\Omega \omega a(x, T_n(u_j), \nabla u_j)\cdot\nabla v dx \\ &= \int_\Omega \phi\cdot\nabla v dx. \end{aligned} \tag{2.2.12}$$

Hence, by hypotheses we have

$$\begin{aligned} \limsup_{j\to+\infty} < A_n u_j, u_j > &= \limsup_{j\to+\infty} \int_\Omega \omega a(x, T_n(u_j), \nabla u_j)\cdot\nabla u_j dx \\ &\leq \int_\Omega \phi\cdot\nabla u dx. \end{aligned} \tag{2.2.13}$$

On the other hand, by (2.1.6)

$$[a(x,T_n(u_j),\nabla u_j)-a(x,T_n(u_j),\nabla u))]\cdot\nabla(u_j-u)>0.$$

Hence

$$\begin{aligned}\int_\Omega \omega a(x,T_n(u_j),\nabla u_j)\cdot\nabla u_j dx > &\int_\Omega \omega a(x,T_n(u_j),\nabla u_j)\cdot\nabla u dx\\ &+\int_\Omega \omega a(x,T_n(u_j),\nabla u)\cdot(\nabla u_j-\nabla u)dx.\end{aligned}\tag{2.2.14}$$

Using (2.2.11), we get

$$\liminf_{j\to+\infty}\int_\Omega \omega a(x,T_n(u_j),\nabla u_j)\cdot\nabla u_j dx\geq\int_\Omega \phi\cdot\nabla u.\tag{2.2.15}$$

By using (2.2.13) and (2.2.15), we get

$$\lim_{j\to\infty}\int_\Omega \omega a(x,T_n(u_j),\nabla u_j)\cdot\nabla u_j dx=\int_\Omega \phi\cdot\nabla u dx.\tag{2.2.16}$$

By means of (2.2.12) and (2.2.16), we obtain

$$\langle A_n u_j,u_j\rangle\to\langle\psi^n,u\rangle \text{ as } j\longrightarrow+\infty.$$

On the other hand, by (2.2.16) and the fact that $\omega a(x,T_n(u_j),\nabla u)\to\omega a(x,T_n(u),\nabla u)$ strongly in $\left(L^{p'}(\Omega,\omega^{1-p'})\right)^N$ it can be easily seen that

$$\lim_{j\to+\infty}\int_\Omega (\omega a(x,T_n(u_j),\nabla u_j)-\omega a(x,T_n(u_j),\nabla u))\cdot(\nabla u_j-\nabla u)\,dx=0,$$

and so, by virtue of Lemma 2.2.2

$$\nabla u_j\to\nabla u \text{ a.e. in } \Omega,$$

we deduce then that $\omega a(x,T_n(u_j),\nabla u_j)$ converges to $\omega a(x,T_n(u),\nabla u)$ weakly in $\left(L^{p'}(\Omega,\omega^{1-p'})\right)^N$. Thus implies that $\psi^n=A_n u$. □

2.3 Proof of the existence result

The proof of the existence result will be presented in several steps.

2.3.1 Approximating problem

We consider the sequence of approximate problems

$$(P_n)\begin{cases} A_n(u_n) = f_n - \operatorname{div} F & in\ \ \Omega, \\ u_n = 0 & on\ \ \partial\Omega, \end{cases}$$

where (f_n) is a sequence of functions in $C_0^\infty(\Omega)$ which is strongly convergent to f in $L^1(\Omega)$ such that $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$.

Since the source term $f_n - \operatorname{div}(F)$ belongs to the dual space $W^{-1,p'}(\Omega, \omega^{1-p'})$, in view of Lemma 2.2.5 the operator A_n is pseudomonotone, and by using Theorem 2.7 in [55], there exists at least one solution $u_n \in W_0^{1,p}(\Omega, \omega)$ of problem (P_n) in the sense that

$$\int_\Omega \omega a(x, T_n(u_n), \nabla u_n).\nabla v dx = \int_\Omega f_n v dx + \int_\Omega F.\nabla v dx, \tag{2.3.1}$$

for every $v \in W_0^{1,p}(\Omega, w)$.

By choosing $v = T_k(u_n)$ as a test function in (2.3.1), we obtain

$$\int_\Omega \omega a(x, T_n(u_n), \nabla u_n).\nabla T_k(u_n) dx = \int_\Omega f_n T_k(u_n) dx + \int_\Omega F.\nabla T_k(u_n) dx. \tag{2.3.2}$$

Using (2.0.1), we have for $n > k$

$$\begin{aligned} \int_\Omega \omega a(x, T_n(u_n), \nabla u_n).\nabla T_k(u_n) dx &= \int_\Omega \omega a(x, T_k(u_n), \nabla T_k(u_n)).\nabla T_k(u_n) dx \\ &\geq \int_\Omega b(|T_k(u_n)|)|\nabla T_k(u_n)|^p \omega dx \\ &\geq b(k) \int_\Omega |\nabla T_k(u_n)|^p \omega dx. \end{aligned}$$

We have also

$$\left|\int_\Omega f_n T_k(u_n)dx\right| \leq \int_\Omega |f_n||T_k(u_n)|dx \leq k\|f_n\|_{L^1(\Omega)} \leq k\|f\|_{L^1(\Omega)},$$

and by using Hölder and Young's inequality, we get

$$\begin{aligned}\left|\int_\Omega \langle F, \nabla T_k(u_n)\rangle dx\right| &\leq \int_\Omega \left|\frac{F}{\omega}\right| |\nabla T_k(u_n)|\,\omega dx \\ &\leq \left(\int_\Omega \left|\frac{F}{\omega}\right|^{p'} \omega dx\right)^{1/p'} \left(\int_\Omega |\nabla T_k(u_n)|^p \omega dx\right)^{1/p} \\ &\leq \frac{b(k)}{2}\int_\Omega |\nabla T_k(u_n)|^p \omega dx + C_1 \int_\Omega \left|\frac{F}{\omega}\right|^{p'} \omega dx.\end{aligned}$$

Hence in (2.3.2), we get

$$b(k)\int_\Omega |\nabla T_k(u_n)|^p \omega dx \leq k\|f\|_{L^1(\Omega)} + \frac{b(k)}{2}\int_\Omega |\nabla T_k(u_n)|^p \omega dx + C_1 \int_\Omega \left|\frac{F}{\omega}\right|^{p'} \omega dx$$

then

$$\begin{aligned}\int_\Omega |\nabla T_k(u_n)|^p \omega dx &\leq \frac{2}{b(k)}\left(k\|f\|_{L^1(\Omega)} + C_1 \left\|\frac{F}{\omega}\right\|^{p'}_{L^{p'}(\Omega,\omega)}\right) \\ &\leq \frac{2}{\alpha}(1+k)^{\theta(p-1)}\left(k\|f\|_{L^1(\Omega)} + C_1 \left\|\frac{F}{\omega}\right\|^{p'}_{L^{p'}(\Omega,\omega)}\right) \\ &\leq C(1+k)^{\theta(p-1)}k \\ &\leq C(1+k)^{\theta(p-1)+1}.\end{aligned}$$

Thus we have the estimate

$$\int_\Omega |\nabla T_k(u_n)|^p \omega dx \leq C(1+k)^{\theta(p-1)+1}. \tag{2.3.3}$$

2.3.2 Locally convergence of u_n in μ-measure

Combining the previous estimation with lemma 2.2.3, we conclude that $u_n \in \mathcal{M}^r(\Omega,\omega)$ with $r = \eta(p-1)(1-\theta)$, we also have that $\mu(\{x \in \Omega : |u_n(x)| > k\})$ is bounded uniformly

in n for every $k > 0$, that is,

$$\mu(\{x \in \Omega : |u_n(x)| > k\}) \leq Ck^{-r}, \tag{2.3.4}$$

and by lemma 2.2.4 we have that the sequence $\{|\nabla u_n|\}$ is bounded in $\mathcal{M}^s(\Omega, \omega)$ with $s = \frac{pr}{r+\theta(p-1)+1}$. Our objectif now is to prove that $u_n \to u$ locally in $\mu-$measure. For that, we will use same reasoning as in the proof of Theorem 2.11 of [32] (see also Theorem 6.1 of [15]). Let $\rho > 0$, we have

$$\{|u_n - u_m| > \rho\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > \rho\},$$

so that

$$\mu(\{|u_n - u_m| > \rho\}) \leq \mu(\{|u_n| > k\}) + \mu(\{|u_m| > k\}) + \mu(\{|T_k(u_n) - T_k(u_m)| > \rho\}). \tag{2.3.5}$$

Fix $\varepsilon > 0$, from (2.3.4) there exists $k_\varepsilon = k$ such that

$$\mu(\{|u_n| > k\}) + \mu(\{|u_m| > k\}) \leq \frac{\varepsilon}{2}.$$

Since $\{\nabla T_k(u_n)\}$ is bounded in $L^p_{loc}(\Omega, \omega)$ for all $k > 0$ and $T_k(u_n)$ belongs to $W^{1,p}_0(\Omega, \omega)$, we can assume that $\{T_k(u_n)\}$ is a Cauchy sequence in $L^q(\Omega \cap B_R, \omega)$ for any $q < p\eta = p*$ and any $R > 0$ and

$$T_k(u_n) \to T_k(u) \quad \text{ in } L^p_{loc}(\Omega, \omega) \text{ and a.e.}$$

Then

$$\begin{aligned}\mu(\{|T_k(u_n) - T_k(u_m)| > \rho\} \cap B_R) &= \int_{\{|T_k(u_n)-T_k(u_m)|>\rho\}\cap B_R} \omega dx \\ &\leq k^{-q} \int_{\Omega\cap B_R} |T_k(u_n) - T_k(u_m)|^q \omega dx \leq \frac{\varepsilon}{2},\end{aligned}$$

for all $n, m \geq n_0(k, \rho, R)$. It follows that u_n is a Cauchy sequence in μ-measure. As a consequence, there exist a function u and a subsequence, still denoted by u_n, such that:

$$u_n \to u \quad \mu\text{-a.e. in } \Omega, \tag{2.3.6}$$

then by Remark 1.1.1, one has

$$u_n \to u \quad \text{a.e. in } \Omega. \tag{2.3.7}$$

Using (2.3.6) and (2.3.3) we have

$$\begin{aligned} &T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega,\omega) \text{ for every } k>0, \\ &T_k(u_n) \to T_k(u) \text{ strongly in } L^p(\Omega,\omega) \text{ and } \mu \text{ -a.e. in } \Omega \text{ for every } k>0. \end{aligned} \tag{2.3.8}$$

Hence, $T_k(u) \in W_0^{1,p}(\Omega,\omega)$.

Furthermore, by the weak lower semicontinuity of the norm $W_0^{1,p}(\Omega,\omega)$, estimate (2.3.3) still holds for u, that is

$$\int_\Omega |\nabla T_k(u)|^p \omega dx \le C(1+k)^{\theta(p-1)+1}, \qquad \forall k>0.$$

Applying again Lemma 2.2.3 and Lemma 2.2.4, we find that $u \in \mathcal{M}^r(\Omega,\omega)$ and $|\nabla u| \in \mathcal{M}^s(\Omega,\omega)$.

2.3.3 Strong convergence of truncations

Our aim now is to prove that

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega,\omega) \text{ for all } k>0. \tag{2.3.9}$$

For $n>k$, we write

$$\begin{aligned} I(n) &= \int_\Omega \omega[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\cdot\nabla(T_k(u_n)-T_k(u))dx \\ &= \int_\Omega \omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla(T_k(u_n)-T_k(u))dx \\ &\quad - \int_\Omega \omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla(T_k(u_n)-T_k(u))dx \\ &= I_1(n) - I_2(n). \end{aligned}$$

Remark that assumption (2.1.3) implies that $a(x,s,0)=0$, therefore

$$\begin{aligned} I_1(n) &= \int_{\{|u_n|<k\}} \omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))dx \\ &= \int_\Omega \omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))dx \\ &\quad - \int_{\{|u_n|\geq k\}} \omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))dx. \end{aligned}$$

Observing that $\nabla T_k(u_n)=0$ on the set $\{|u_n|\geq k\}$, we obtain

$$\begin{aligned} I(n) &= \int_\Omega \omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))dx \\ &\quad + \int_{\{|u_n|\geq k\}} \omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u)dx \\ &\quad - \int_\Omega \omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla(T_k(u_n)-T_k(u))dx. \end{aligned}$$

We take $T_k(u_n)-T_k(u)$ as test function in (2.3.1) and we get

$$\begin{aligned} &\int_\Omega \omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))dx \\ &\quad = \int_\Omega f_n(T_k(u_n)-T_k(u))dx + \int_\Omega F\cdot\nabla(T_k(u_n)-T_k(u))dx. \end{aligned}$$

By the almost convergence of u_n and using the strong convergence of f_n in $L^1(\Omega)$, we obtain

$$\lim_{n\to\infty}\int_\Omega f_n(T_k(u_n)-T_k(u))dx=0.$$

Also since F belongs to $\left(L^{p'}(\Omega,\omega^{1-p'})\right)^N$ and by (2.3.8) we obtain

$$\lim_{n\to\infty}\int_\Omega F\cdot\nabla(T_k(u_n)-T_k(u))dx=0.$$

Therefore

$$\lim_{n\to\infty}\int_\Omega \omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))dx=0. \tag{2.3.10}$$

Using the growth assumption (2.1.5), we have for every u in $W_0^{1,p}(\Omega,\omega)$

$$\begin{aligned}\int_\Omega \Big(\omega|a(x,T_n(u),\nabla u)|\Big)^{p'}\omega^{1-p'}dx &\\ &\le \int_\Omega \Big(h_0+h_1\,|T_n(u)|^{p/p'}+h_2\,|\nabla u|^{p/p'}\Big)^{p'}\omega dx\\ &\le C\left[\int_\Omega h_0^{p'}\omega dx+\int_\Omega h_1^{p'}|u|^p\omega dx+\int_\Omega h_2^{p'}\,|\nabla u|^p\,\omega dx\right]\\ &\le C\left[\|h_0\|^{p'}_{L^{p'}(\Omega,\omega)}+\|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega |u|^p\omega dx+\|h_2\|^{p'}_{L^\infty(\Omega)}\int_\Omega|\nabla u|^p\omega dx\right]\\ &\le C\left[\|h_0\|^{p'}_{L'(\Omega,\omega)}+\Big(C_\Omega\|h_1\|^{p'}_{L^\infty(\Omega)}+\|h_2\|^{p'}_{L^\infty(\Omega)}\Big)\,\|u\|^p_{W_0^{1,p}(\Omega,\omega)}\right].\end{aligned}$$

which mean that $\omega(x)|a(x,T_n(u),\nabla u)|$ is bounded $L^{p'}(\Omega,\omega^{1-p'})$. Therefore it converges weakly to some g in $L^{p'}(\Omega,\omega^{1-p'})$ and we have

$$\lim_{n\to\infty}\int_{\{|u_n|\ge k\}}\omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u)dx=\int_{\{|u|\ge k\}}g\cdot\nabla T_k(u)=0. \tag{2.3.11}$$

By virtue of Vitali's theorem, we obtain

$$\omega(x)a(x,T_n(u_n),\nabla T_k(u))\to\omega(x)a(x,u,\nabla T_k(u)) \text{ strongly in } \Big(L^{p'}(\Omega,\omega^{1-p'})\Big)^N.$$

It follows from (2.3.8) that

$$\lim_{n\to\infty}\int_\Omega\omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla(T_k(u_n)-T_k(u))dx=0. \tag{2.3.12}$$

Bringing together (2.3.10)-(2.3.12), we conclude that

$$\lim_{n\to\infty} I(n)=0. \tag{2.3.13}$$

Now we can apply Lemma 3.2 in [2] to get (2.3.9). This convergence implies that, for every fixed $k>0$, we have

$$\omega(x)a(x,T_k(u_n),\nabla T_k(u_n))\to\omega(x)a(x,T_k(u),\nabla T_k(u)) \quad \text{in } (L^{p'}(\Omega,\omega^{1-p'}))^N.$$

2.3.4 Passage to the limit

Finally, we need to show that u is an entropy solution of the Dirichlet problem (P). Let us take $\psi_n = T_k(u_n - v)$ as test function in (2.3.1), with $v \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$. We obtain

$$\int_\Omega \omega a\left(x, T_n(u_n), \nabla u_n\right) \cdot \nabla \psi_n dx = \int_\Omega f_n \psi_n dx + \int_\Omega F \cdot \nabla \psi_n dx.$$

If $M = k + \|v\|_{L^\infty(\Omega)}$ and $n > M$, then

$$\begin{aligned}\int_\Omega \omega a\left(x, T_n(u_n), \nabla u_n\right) \cdot \nabla T_k\left(u_n - v\right) dx &= \int_\Omega \omega a(x, T_n(u_n), \nabla T_M(u_n)) \cdot \nabla T_k(u_n - v) dx \\ &= \int_\Omega \omega a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_k(u_n - v) dx.\end{aligned}$$

Thus, we can write

$$\int_\Omega \omega a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_k(u_n - v) dx = \int_\Omega f_n T_k(u_n - v) dx + \int_\Omega F \cdot \nabla T_k(u_n - v) dx. \tag{2.3.14}$$

Passing to the limit when n goes to ∞, we obtain

$$\int_\Omega \omega a(x, u, \nabla u) \cdot \nabla T_k(u - v) dx = \int_\Omega f T_k(u - v) dx + \int_\Omega F \cdot \nabla T_k(u - v) dx.$$

which means that u is an entropy solution of the problem (P).

2.4 Application

We put ourselves in the situation $N = 2$, $p = 3$. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

- The weight function $\omega(x, y) = (x^2 + y^2)^{-1/2}$ is such that $\omega \in \mathcal{A}_3$.
- The functions f and F are given by:

$$\begin{aligned} f(x, y) &= \frac{\cos(xy)}{(x^2 + y^2)^{1/3}}, \quad f \in L^1(\Omega), \\ F(x, y) &= \left(\left(x^2 + y^2\right)\sin(xy), \left(x^2 + y^2\right)^{-1/3}\cos(xy)\right), \quad F \in \left[L^{\frac{3}{2}}(\Omega, \omega^{-\frac{1}{2}})\right]^2. \end{aligned}$$

- The Carathéodory function a is defined as follows

$$\begin{aligned} a\colon \Omega \times \mathbb{R} \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ ((x,y),s,\xi) &\longmapsto \frac{\xi}{\sqrt{1+|s|}}. \end{aligned}$$

Therefore, by virtue of Theorem 2.1.1, the problem

$$\begin{cases} -\operatorname{div}\left[\omega(x,y)a((x,y),u,\nabla u)\right] = f(x,y) - \operatorname{div} F(x,y) & \text{in } \Omega \\ u(x,y) = 0, & \text{on } \partial\Omega \end{cases}$$

has an entropy solution.

Chapter 3

Existence of solutions for some unilateral problems with degenerate coercivity

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, and $|F| \in L^m(\Omega)$ for some $m \geq 1$. Consider the nonlinear Dirichlet problem

$$Au = -\operatorname{div} F, \tag{3.0.1}$$

where $Au = -\operatorname{div} a(\cdot, u, \nabla u)$ is a nonlinear elliptic differential operator of monotone type defined on $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ with p being a real such that $1 < p < N$.

Our objective is to give a generalization of results obtained in [16] to the unilateral problem. By using a priori estimates and some rearrangement techniques, we derive our desired result.

In the coercive case, problem (3.0.1) has been studied in [24], [25], where the authors proved the existence and regularity of solutions. We refer to the references therein, for more results in different particular cases.

In a recent work [16], Benkirane and Youssfi considered the Dirichlet problem (3.0.1) in the noncoercive case, more precisely when the Carathéodory function $a(x, s, \xi)$ satisfies

the following degenerate coercivity condition

$$a(x,s,\xi)\cdot\xi \geq h^{p-1}(|s|)|\xi|^p, \tag{3.0.2}$$

with h is a continuous decreasing function such that its primitive H is unbounded. The authors discussed the regularity of solutions in terms of the summability of the datum F.

The existence results for similar problems to (3.0.1) with datum f rather than $\operatorname{div} F$ have been obtained by several authors. In this direction, we cite [7] where the authors studied the Dirichlet problem

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.0.3}$$

They proved the existence and some regularity of solutions under various assumptions on the summability of the function f, see also [21]. Recently, in [11] Ayadi and Souilah studied the obstacle problem associated to (3.0.3) using the penalization method.

In [19], Boccardo and Cirmi showed the existence and uniqueness of the solution of the obstacle problem associated to the Dirichlet problem

$$\begin{cases} -\operatorname{div} a(x,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.0.4}$$

where the data f belongs to $L^1(\Omega)$ (see also [20]).

In the present paper we deal with the existence and regularity results for the unilateral problem associated to the equation (3.0.1) under the condition of degenerate coercivity (3.0.2). Mainly, we will prove that the results obtain in [16] remain true in the case of unilateral problems. Due to the lack of coercivity of the nonlinear elliptic operator A, the results are obtained by means of approximation through suitable coercive problems.

To state our main results, let us first introduce the fundamental assumptions. Then we distinguish several cases according to m to establish a piori estimates needed.

3.1 Basic assumptions and main results

We propose through this chapter to discuss the existence of solutions for the unilateral problem associated to the Dirichlet problem

$$\begin{cases} \operatorname{div} a(x,u,\nabla u) = \operatorname{div} F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.1}$$

where ψ is a measurable function on Ω such that

$$\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

and p is a real number such that $1 < p < N$.
$\mathcal{K}_\psi$ is the convex set defined by

$$\mathcal{K}_\psi = \left\{ v \in W_0^{1,p}(\Omega) : v(x) \geq \psi(x) \text{ in } \Omega \right\}.$$

$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (i.e., $a(\cdot, s, \xi)$ is measurable on Ω, for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and $a(x,\cdot,\cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$, a.e. $x \in \Omega$), such that the following assumptions holds for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, for every $\xi \neq \eta \in \mathbb{R}^N$:

$$a(x,s,\xi) \cdot \xi \geq h^{p-1}(|s|)|\xi|^p, \tag{3.1.2}$$

$$|a(x,s,\xi)| \leq \beta \left(j(x) + |s|^{p-1} + |\xi|^{p-1} \right), \tag{3.1.3}$$

$$(a(x,s,\xi) - a(x,s,\eta)) \cdot (\xi - \eta) > 0, \tag{3.1.4}$$

with $h : \mathbb{R}^+ \to]0,\infty[$ being a decreasing continuous function such that its primitive H is unbounded

$$H(s) = \int_0^s h(t)dt, \tag{3.1.5}$$

$j(x)$ is a positive function lying in $L^{p'}(\Omega)$ and β is a positive constant.

As regards the datum, we suppose that $F \in (L^m(\Omega))^N$ for some $m > 1$. This allowed us to discuss four cases; namely $m > \frac{N}{p-1}$, $m = \frac{N}{p-1}$, $\tilde{m} < m < m = \frac{N}{p-1}$ and $\bar{m} \leq m \leq \tilde{m}$

where $\tilde{m}$ and $\bar{m}$ are defined later.

Our first contribution concerns the existence of bounded solutions with the data having high summability. We suppose that $m > \frac{N}{p-1}$.

Theorem 3.1.1. *Assume that* (3.1.2)-(3.1.5) *hold. Let* $|F|$ *belong to* $L^m(\Omega)$*, with*

$$m > \frac{N}{p-1}. \tag{3.1.6}$$

Then there exists a function $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *which is solution of the unilateral problem*

$$\begin{cases} u \in \mathcal{K}_\psi, \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla(u-v)dx \le \int_\Omega F\cdot\nabla(u-v)dx, \\ \forall v \in \mathcal{K}_\psi. \end{cases} \tag{3.1.7}$$

Our second contribution deals on the limit case where $m = \frac{N}{p-1}$. The solutions we get do not belong in general to $W_0^{1,p}(\Omega)$. We will introduce a different formulation of the unilateral problem, along with a new definition of the gradient for a measurable function.

For a fixed $k \ge 0$ and $s \in \mathbb{R}$, we recall the definition of the usual truncation function $T_k(s)$

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ k\frac{s}{|s|} & \text{if } |s| > k, \end{cases}$$

and we denote by $\mathcal{T}_0^{1,p}(\Omega)$ the space of the measurable function u such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$. Note that the gradient of functions in $\mathcal{T}_0^{1,p}(\Omega)$ is defined in Lemma 1.1.2.

The existence of solutions for the unilateral problem associated with the case $m = \frac{N}{p-1}$ is given in the following result.

Theorem 3.1.2. *Assume that* (3.1.2)-(3.1.5) *hold. Let* $|F| \in L^m(\Omega)$ *with*

$$m = \frac{N}{p-1}. \tag{3.1.8}$$

Assume in addition that

$$\lim_{t\to\infty} \frac{1}{t\,h(t)} = 0. \tag{3.1.9}$$

Then there exists a measurable function u which is solution of the unilateral problem

$$\begin{cases} u \geq \psi \text{ a.e. in } \Omega, \\ u \in \mathcal{T}_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-v)dx \leq \int_\Omega F\cdot\nabla T_k(u-v)dx, \\ \forall v \in \mathcal{K}_\psi \cap L^\infty(\Omega). \end{cases} \tag{3.1.10}$$

We investigate in the third contribution the case where we weak the summability of $|F|$, we obtain unbounded solutions. In the interest of simplification, we assume that the function h in (3.1.2) is defined as follows

$$h(s) = \frac{1}{(1+|s|)^\theta}, \qquad \text{with } 0 \leq \theta < 1. \tag{3.1.11}$$

Let us set

$$\tilde{m} = \frac{Np'}{N-\theta(N-p)}.$$

We deal first on the case where $\tilde{m} \leq m < \frac{N}{p-1}$.

Theorem 3.1.3. *Assume that (3.1.3), (3.1.4) and (3.1.2) (with (3.1.11)) hold. Let $|F|$ belong to $L^m(\Omega)$, with*

$$\tilde{m} \leq m < \frac{N}{p-1}. \tag{3.1.12}$$

Then there exists a function $u \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$, with

$$r = (1-\theta)(m(p-1))^*, \tag{3.1.13}$$

which is a solution of the unilateral problem (3.1.7).

As before we define

$$\bar{m} = \max\left(p', \frac{Np'}{p((1-\theta)N+\theta)}\right).$$

Our objectif now is to discuss the case where $\bar{m} \leq m < \tilde{m}$.

Theorem 3.1.4. *Assume that (3.1.3), (3.1.4) and (3.1.2) (with (3.1.11)) hold.*
Let $|F| \in L^m(\Omega)$ with

$$\bar{m} \leq m < \tilde{m}. \tag{3.1.14}$$

Then there exists a measurable function u wich is solution of the unilateral problem (3.1.10). *Moreover, u belongs to $W_0^{1,q}(\Omega) \cap L^r(\Omega)$, with*

$$r = (1-\theta)(m(p-1))^*, \tag{3.1.15}$$

and

$$q = \frac{(1-\theta)Nm(p-1)}{N-\theta m(p-1)}. \tag{3.1.16}$$

3.2 A priori estimates

To prove existence results stated in the previous section we introduce the following approximating problems. Let $n \in \mathbb{N}$

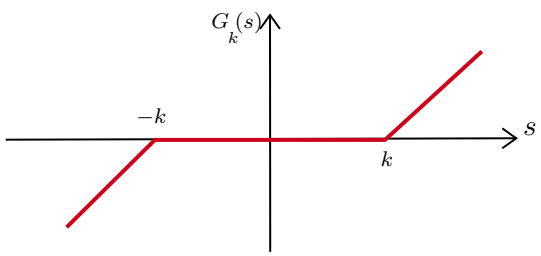
$$\begin{cases} u_n \in \mathcal{K}_\psi, \\ \langle A_n u_n, u_n - v\rangle \leq \displaystyle\int_\Omega F \cdot \nabla(u_n - v)dx, \\ \forall v \in \mathcal{K}_\psi, \end{cases} \tag{3.2.1}$$

where $A_n u_n = -\operatorname{div} a(x, T_n(u_n), \nabla u_n)$. For a fixed n, thanks to the hypotheses (3.1.2)-(3.1.4) A_n is a nonlinear operator of Leray-Lions type, and since $\operatorname{div} F$ belongs to $W^{-1,p'}(\Omega)$, by well-known results (see [55]) there exists at least a solution u_n of the approximate problem (3.2.1).

Lemma 3.2.1. *Let $|F|$ be in $L^m(\Omega)$ with $m > \frac{N}{p-1}$ and let u_n be a solution of the approximate problem* (3.2.1). *Then the sequence (u_n) is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Proof. Let us start with the estimate in $L^\infty(\Omega)$. Define, for s in $\mathbb{R}$

$$G_k(s) = s - T_k(s).$$



For $\varepsilon > 0$ and $k > \|\psi\|_\infty$, let $v = u_n - T_\varepsilon(G_k(u_n))$

$$v = u_n - T_\varepsilon(u_n - T_k(u_n) = \begin{cases} u_n, & \text{if } |u_n| \le k \\ u_n - (|u_n| - k)\,\mathrm{sign}(u_n), & \text{if } k < |u_n| \le k + \varepsilon \\ u_n - \varepsilon\,\mathrm{sign}(u_n), & \text{if } |u_n| > k + \varepsilon. \end{cases}$$

It is clear that v belongs to $W_0^{1,p}(\Omega)$, and $v \ge \psi$. So the function v belongs to $\mathcal{K}_\psi$ and can be used as a test function in (3.2.1), giving

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_\varepsilon(G_k(u_n))dx \le \int_\Omega F \cdot \nabla T_\varepsilon(G_k(u_n))dx. \tag{3.2.2}$$

using assumption (3.1.2), it follows that

$$h^{p-1}(k+\varepsilon) \int_{A_k} |\nabla u_n|^p dx \le \int_{A_k} F \cdot \nabla u_n dx, \tag{3.2.3}$$

where we have set

$$A_k = \{x \in \Omega : k < |u_n| \le k + \varepsilon\}.$$

By Hölder's inequality we have

$$h^p(k+\varepsilon) \int_{A_k} |\nabla u_n|^p dx \le \int_{A_k} |F|^{p'} dx,$$

using again Hölder's inequality in the right-hand side, we obtain

$$h^p(k+\varepsilon) \int_{A_k} |\nabla u_n|^p dx \le |A_k|^{1-\frac{p'}{m}} \left[\int_{A_k} |F|^m dx\right]^{\frac{p'}{m}}.$$

We can pass to the limit as k goes to 0 to get

$$h^p(t) \left[-\frac{d}{dt}\int_{\{|u_n|>t\}} |\nabla u_n|^p dx\right] \le \left[-\frac{d}{dt}\int_{\{|u_n|>t\}} |F|^m dx\right]^{\frac{p'}{m}} \left(-\mu'_{u_n}(t)\right)^{1-\frac{p'}{m}}. \tag{3.2.4}$$

Recall the Fleming–Rishel formula and the isoperimetric inequality (see [67])

$$N C_N^{1/N} \mu_{u_n}(t)^{1-1/N} \le -\frac{d}{dt}\int_{\{|u_n|>t\}} |\nabla u_n| dx, \tag{3.2.5}$$

where C_N is the measure of the unit ball in $\mathbb{R}^N$.

Hölder's inequality implies that

$$-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|dx \leq \left[-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|^p dx\right]^{\frac{1}{p}}\left(-\mu'_{u_n}(t)\right)^{1-\frac{1}{p}}. \tag{3.2.6}$$

From (3.2.4), (3.2.5) and (3.2.6), we obtain

$$h(t) \leq \frac{1}{NC_N^{\frac{1}{N}}}\left[-\frac{d}{dt}\int_{\{|u_n|>t\}}|F|^m dx\right]^{\frac{p'}{pm}}\frac{\left(-\mu'_{u_n}(t)\right)^{1-\frac{p'}{pm}}}{\left(\mu_n(t)\right)^{\frac{N-1}{N}}}. \tag{3.2.7}$$

Integrating both sides of (3.2.7) between 0 and τ

$$\int_0^\tau h(t)dt \leq \frac{1}{NC_N^{\frac{1}{N}}}\int_0^\tau\left[-\frac{d}{dt}\int_{\{|u_n|>t\}}|F|^m dx\right]^{\frac{p'}{pm}}\frac{\left(-\mu'_{u_n}(t)\right)^{1-\frac{p'}{pm}}}{\left(\mu_n(t)\right)^{\frac{N-1}{N}}}dt,$$

then using Hölder's inequality, we get

$$H(\tau) \leq \frac{1}{NC_N^{\frac{1}{N}}}\left(\int_0^\tau\left[-\frac{d}{dt}\int_{\{|u_n|>t\}}|F|^m dx\right]dt\right)^{\frac{p'}{pm}}\left(\int_0^\tau\frac{\left(-\mu_{u_n}(t)\right)}{\left(\mu_n(t)\right)^{\frac{N-1}{N}\frac{pm}{pm-p'}}}dt\right)^{\frac{pm-p'}{pm}}.$$

Making a change of variables ($\mu_n(t)=\sigma$), the above inequality becomes

$$H(\tau) \leq \frac{1}{NC_N^{\frac{1}{N}}}\|F\|^{\frac{p'}{p}}_{(L^m(\Omega))^N}\left[\int_{\mu_n(\tau)}^{|\Omega|}\frac{d\sigma}{\sigma^{\frac{N-1}{N}\frac{pm}{pm-p'}}}\right]^{1-\frac{p'}{pm}}.$$

The definition of the rearrangement and the inequality (1.3.2) allow us to have

$$H(u_n^*(v)) \leq \frac{1}{NC_N^{\frac{1}{N}}}\|F\|^{\frac{p'}{p}}_{(L^m(\Omega))^N}\left[\int_{v}^{|\Omega|}\frac{d\sigma}{\sigma^{\frac{N-1}{N}\frac{pm}{pm-p'}}}\right]^{1-\frac{p'}{pm}}. \tag{3.2.8}$$

Therefore, by (1.3.1) we get

$$H(\|u_n\|_{L^\infty(\Omega)}) \leq \frac{1}{NC_N^{\frac{1}{N}}}\|F\|^{\frac{p'}{p}}_{(L^m(\Omega))^N}\left[\int_{0}^{|\Omega|}\frac{d\sigma}{\sigma^{\frac{N-1}{N}\frac{pm}{pm-p'}}}\right]^{1-\frac{p'}{pm}}. \tag{3.2.9}$$

Since the integral on the right hand is conergent (due to $m > \frac{N}{p-1}$), and the function H is unbounded ensures that the norm of u_n in L^∞ is uniformly bounded, indeed one has

$$\|u_n\|_\infty \leq H^{-1}\left(\frac{1}{NC_N^{\frac{1}{N}}}\|F\|^{p'}_{(L^m(\Omega))^N}\left(\int_0^{|\Omega|}\frac{d\sigma}{\sigma^{\frac{N-1}{N}\frac{pm}{pm-p'}}}\right)^{1-\frac{p'}{pm}}\right) := c_\infty. \tag{3.2.10}$$

Now we prove that the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$.

Let us consider ψ as test function in (3.2.1), then using (3.1.2) and (3.1.3) we get

$$\begin{aligned} h^{p-1}(c_\infty)\int_\Omega |\nabla u_n|^p dx - \beta\int_\Omega (j(x)+|T_n(u_n)|^{p-1}+|\nabla u_n|^{p-1}).\nabla\psi \\ \leq \int_\Omega F\cdot\nabla(u_n-\psi)dx. \end{aligned} \tag{3.2.11}$$

The second term on the left-hand-side of (3.2.11), is evaluated by using Young's inequality with $\nu > 0$. We will denote by $C_i, i = 1, 2\ldots$ some generic constants

$$\begin{aligned} \beta\int_\Omega (j(x)+|T_n(u_n)|^{p-1}+|\nabla u_n|^{p-1}).\nabla\psi \\ &\leq C_1 + \beta\int_\Omega |\nabla u_n|^{p-1}.\nabla\psi \\ &\leq C_1 + \beta\nu\int_\Omega |\nabla u_n|^p dx + C_2(\nu)\int_\Omega |\nabla\psi|^p dx \\ &\leq C_3(\nu) + \beta\nu\int_\Omega |\nabla u_n|^p dx. \end{aligned} \tag{3.2.12}$$

Concerning the term on the right-hand-side of (3.2.11)

$$\begin{aligned} \int_\Omega F\cdot\nabla(u_n-\psi)dx &\leq \int_\Omega |F|\cdot|\nabla u_n|dx + \int_\Omega |F|\cdot|\nabla\psi|dx \\ &\leq C_4(\nu)\int_\Omega |F|^{p'}dx + \frac{1}{p}\int_\Omega |\nabla\psi|^p dx + \nu\int_\Omega |\nabla u_n|^p dx. \end{aligned} \tag{3.2.13}$$

Combining (3.2.11), (3.2.12) and (3.2.13), we deduce that

$$\left(h^{p-1}(c_\infty) - \nu(\beta+1)\right)\int_\Omega |\nabla u_n|^p dx \leq C_5(\nu),$$

we can now choose ν such that $h^{p-1}(c_\infty) = 2\nu(\beta+1)$, to get

$$\frac{1}{2}h^{p-1}(c_\infty)\int_\Omega |\nabla u_n|^p dx \leq C_6.$$

The last inequality tells us that the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$.

□

Lemma 3.2.2. *Assume that m satisfies (3.1.12), let $|F|$ belong to $L^m(\Omega)$, and let u_n be a solution of (3.2.1). Then the norms of u_n in $L^r(\Omega)$ (where r is defined by (3.1.13))and in $W_0^{1,p}(\Omega)$ are bounded by a constants which depend on $\theta, m, N, |\Omega|$ and the norm of F in $L^m(\Omega)$.*

Proof. Let us start with the estimate in $L^r(\Omega)$. For $k \in \mathbb{N}$ with $k > \|\psi\|_{L^\infty(\Omega)}$,
let $v = u_n - T_1(G_k(u_n))$ (which is an admissible test function). Choosing v as test function in (3.2.1), we have

$$\int_\Omega a(x, T_n(u_n), \nabla u_n)\cdot\nabla T_1(G_k(u_n))dx \leq \int_\Omega F\cdot\nabla T_1(G_k(u_n))dx, \tag{3.2.14}$$

which gives

$$\int_{B_k} a(x, T_n(u_n), \nabla u_n)\cdot\nabla u_n dx \leq \int_{B_k} F\cdot\nabla u_n dx, \tag{3.2.15}$$

where we have set

$$B_k = \{x\in\Omega : k < |u_n| \leq k+1\}.$$

Hence, using (3.1.2) and Hölder's inequality, we get

$$\begin{aligned}\int_{B_k} h^{p-1}(|u_n|)|\nabla u_n|^p dx &\leq \int_{B_k} F\cdot\nabla u_n dx \\ &\leq \left[\int_{B_k}|F|^{p'}dx\right]^{\frac{1}{p'}}\left[\int_{B_k}|\nabla u_n|^p dx\right]^{\frac{1}{p}},\end{aligned}$$

and since h is decreasing, we obtain

$$\int_{B_k}|\nabla u_n|^p dx \leq (2+k)^{\theta p}\int_{B_k}|F|^{p'}dx. \tag{3.2.16}$$

If γ is a positive number (to be fixed later).

Let $p^* = \frac{pN}{N-p}$ be the Sobolev conjugate exponent of p, by Sobolev's inequality and the previous inequality we can write

$$\begin{aligned}\left[\int_\Omega |u_n|^{\gamma p^*} dx\right]^{\frac{p}{p^*}} &\leq C \int_\Omega |\nabla(|u_n|^\gamma)|^p dx \\ &= C \int_\Omega |u_n|^{p(\gamma-1)} |\nabla u_n|^p dx \\ &= C \sum_{k=0}^{\infty} \int_{B_k} |u_n|^{p(\gamma-1)} |\nabla u_n|^p dx \\ &\leq C \sum_{k=0}^{\infty} (1+k)^{p(\gamma-1)} \int_{B_k} |\nabla u_n|^p dx \\ &\leq C \sum_{k=0}^{\infty} (1+k)^{p(\gamma-1)} (2+k)^{\theta p} \int_{B_k} |F|^{p'} dx \\ &\leq C \sum_{k=0}^{\infty} (2+k)^{p(\theta+\gamma-1)} \int_{B_k} |F|^{p'} dx.\end{aligned} \tag{3.2.17}$$

Therefore, using the fact that on the set B_k we have $|u_n| > k$ and then Hölder's inequality, we get

$$\begin{aligned}\left[\int_\Omega |u_n|^{\gamma p^*} dx\right]^{\frac{p}{p^*}} &\leq C \sum_{k=0}^{\infty} \int_{B_k} |F|^{p'} (2+|u_n|)^{p(\theta+\gamma-1)} dx \\ &= C \int_\Omega |F|^{p'} (2+|u_n|)^{p(\theta+\gamma-1)} dx \\ &\leq C \left[\int_\Omega |F|^m dx\right]^{\frac{p'}{m}} \left[\int_\Omega (2+|u_n|)^{\frac{pm(\theta+\gamma-1)}{m-p'}} dx\right]^{1-\frac{p'}{m}} \\ &\leq C \|F\|^{p'}_{(L^m(\Omega))^N} \left[1 + \int_\Omega |u_n|^{\frac{pm(\theta+\gamma-1)}{m-p'}} dx\right]^{1-\frac{p'}{m}}.\end{aligned} \tag{3.2.18}$$

We now choose γ such that

$$\frac{pm(\theta+\gamma-1)}{m-p'} = \gamma p^*,$$

namely

$$\gamma = \frac{(1-\theta)(m(p-1))^*}{p^*}.$$

It turns out that $\gamma p^* = r$, so that from (3.2.18) we have

$$\left[\int_\Omega |u_n|^r dx\right]^{\frac{p}{p^*}} \leq C\|F\|^{p'}_{(L^m(\Omega))^N}\left[1+\int_\Omega |u_n|^r\, dx\right]^{1-\frac{p'}{m}},$$

since $1-\dfrac{p'}{m} < \dfrac{p}{p^*}$ (due to the fact that $m < \frac{N}{p-1}$), one has

$$\int_\Omega |u_n|^r dx \leq C,$$

therefore, u_n is bounded in $L^r(\Omega)$.

Now we turn to the estimates in $W_0^{1,p}(\Omega)$. We write

$$\int_\Omega |\nabla(|u_n|^\gamma)|^p dx = C\int_\Omega |u_n|^{p(\gamma-1)}|\nabla u_n|^p dx.$$

It is clear that $\gamma \geq 1$ since $m \geq \frac{p'N}{N-\theta(N-p)}$, thus

$$\int_{\{|u_n|>1\}} |\nabla u_n|^p dx \leq C\int_\Omega |u_n|^{p(\gamma-1)}|\nabla u_n|^p dx.$$

Hence, from (3.2.17) and (3.2.18) we obtain

$$\int_{\{|u_n|>1\}} |\nabla u_n|^p dx \leq C.$$

On the other hand, inequality (3.2.16), written for $k = 0$, and then using the Hölder inequality, we get

$$\int_{\{|u_n|\leq 1\}} |\nabla u_n|^p dx \leq 2^{\theta p}\|F\|^{p'}_{(L^m(\Omega))^N}|\Omega|^{1-\frac{p'}{m}}.$$

We conclude that

$$\int_\Omega |\nabla u_n|^p dx \leq C,$$

which completes the proof of Lemma 3.2.2. □

Lemma 3.2.3. *Assume that m satisfies (3.1.14), let $|F|$ belong to $L^m(\Omega)$, and let u_n be a solution of (3.2.1). Then, there exist two positive constants c_1, c_2 depending on Ω, N, F, θ*

and m, such that, for any $n \in \mathbb{N}$,

$$\|u_n\|_{L^r(\Omega)} \leq c_1, \tag{3.2.19}$$

and

$$\|u_n\|_{W_0^{1,q}(\Omega)} \leq c_2, \tag{3.2.20}$$

where r and q are defined by (3.1.15) and (3.1.16) respectively. Moreover, the sequence $(T_k(u_n))$ is bounded in $W_0^{1,p}(\Omega)$ for every $k > 0$.

Proof. Let $k \in \mathbb{N}$ and $\tau \geq 0$. If we take $u_n - T_1(G_k(u_n))$ as a test function in (3.2.1), and use hypothesis (3.1.2), we obtain

$$\int_{B_k} h^{p-1}(|u_n|)|\nabla u_n|^p dx \leq \int_{B_k} F \cdot \nabla u_n dx, \tag{3.2.21}$$

where we have set

$$B_k = \{x \in \Omega : k < |u_n| \leq k+1\},$$

hence, using Hölder's inequality, we get

$$\begin{aligned}\int_{B_k} h^{p-1}(|u_n|)|\nabla u_n|^p dx &\leq \int_{B_k} F \cdot \nabla u_n dx \\ &\leq \left[\int_{B_k} |F|^{p'} dx\right]^{\frac{1}{p'}} \left[\int_{B_k} |\nabla u_n|^p dx\right]^{\frac{1}{p}},\end{aligned}$$

and since h is decreasing, we obtain

$$\int_{B_k} |\nabla u_n|^p dx \leq (2+k)^{\theta p} \int_{B_k} |F|^{p'} dx. \tag{3.2.22}$$

On the other hand, let $\gamma > 0$ be a real number which will be chosen later

$$\begin{aligned}\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^\gamma} dx &= \sum_{k=0}^{+\infty} \int_{B_k} \frac{|\nabla u_n|^p}{(1+|u_n|)^\gamma} dx \\ &\leq \sum_{k=0}^{+\infty} \frac{1}{(1+k)^\gamma} \int_{B_k} |\nabla u_n|^p dx,\end{aligned}$$

using (3.2.22), we get

$$\begin{aligned}\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^\gamma}dx &\le \sum_{k=0}^{\infty}\frac{(2+k)^{\theta p}}{(1+k)^\gamma}\int_{B_k}|F|^{p'}dx\\ &\le 2^{\theta p}\sum_{k=0}^{\infty}(1+k)^{\theta p-\gamma}\int_{B_k}|F|^{p'}dx\\ &\le 2^{\theta p}\sum_{k=0}^{\infty}\int_{B_k}|F|^{p'}(1+|u_n|)^{\theta p-\gamma}dx\\ &= 2^{\theta p}\int_\Omega |F|^{p'}(1+|u_n|)^{\theta p-\gamma}dx\\ &\le 2^{\theta p}\|F\|^{p'}_{(L^m(\Omega))^N}\left[\int_\Omega (1+|u_n|)^{\frac{(\theta p-\gamma)m}{m-p'}}dx\right]^{1-\frac{p'}{m}}.\end{aligned} \tag{3.2.23}$$

Let r and q be as in (3.1.15) and (3.1.16). Then one can check that $r = q^* = Nq/(N-q)$. Therefore, using the Sobolev inequality and (3.2.23), we have

$$\begin{aligned}\left[\int_\Omega |u_n|^r dx\right]^{\frac{q}{r}} &\le c\int_\Omega |\nabla u_n|^q dx = c\int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^{\frac{\gamma q}{p}}}(1+|u_n|)^{\frac{\gamma q}{p}}dx\\ &\le c\left[\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^\gamma}dx\right]^{\frac{q}{p}}\left[\int_\Omega (1+|u_n|)^{\frac{\gamma q}{p-q}}dx\right]^{\frac{p-q}{p}}\\ &\le c2^{\theta q}\|F\|^{\frac{qp'}{p}}_{(L^m(\Omega))^N}\left[\int_\Omega (1+|u_n|)^{\frac{(\theta p-\gamma)m}{m-p'}}dx\right]^{(1-\frac{p'}{m})\frac{q}{p}}\left[\int_\Omega (1+|u_n|)^{\frac{\gamma q}{p-q}}dx\right]^{\frac{p-q}{p}}.\end{aligned} \tag{3.2.24}$$

We choose now γ such that

$$\frac{\lambda q}{p-q} = r,$$

that is

$$\gamma = \frac{pN - m(p-1)(N-\theta(N-p))}{N-m(p-1)}. \tag{3.2.25}$$

Thanks to the choice of γ in (3.2.25), we get

$$\frac{(\theta p-\gamma)m}{m-p'} = r.$$

Consequently, we get

$$\begin{aligned}\left[\int_\Omega |u_n|^r dx\right]^{\frac{q}{r}} &\leq c2^{\theta q}\|F\|^{\frac{qp'}{p}}_{(L^m(\Omega))^N}\left[\int_\Omega (1+|u_n|)^r dx\right]^{1-\frac{p'q}{mp}} \\ &\leq c2^{\theta q}\|F\|^{\frac{qp'}{p}}_{(L^m(\Omega))^N}\left[|\Omega|+\int_\Omega |u_n|^r dx\right]^{1-\frac{p'q}{mp}}.\end{aligned} \tag{3.2.26}$$

Since $m < \frac{N}{p-1}$, the last exponent in (3.2.26) is smaller than $\frac{q}{r}$.
Therefore, we obtain

$$\left[\int_\Omega |u_n|^r dx\right]^{\frac{q}{r}} \leq c2^{\theta q}\|F\|^{\frac{qp'}{p}}_{(L^m(\Omega))^N}\left[|\Omega|+\int_\Omega |u_n|^r dx\right]^{\frac{q}{r}}. \tag{3.2.27}$$

Thus, we conclude the boundedness of u_n in $L^r(\Omega)$.
Going back to (3.2.24), this in turn implies an estimate for the norm of u_n in $W_0^{1,q}(\Omega)$.

To prove the last part of Lemma 3.2.3, we take $u_n - T_k(u_n - \psi)$ as a test function in (3.2.1) to get

$$\int_\Omega a(x, T_n(u_n), \nabla u_n)\cdot\nabla T_k(u_n-\psi)dx \leq \int_\Omega F\cdot\nabla T_k(u_n-\psi)dx. \tag{3.2.28}$$

Since we are on the set $\{|u_n - \psi| \leq k\}$, we set $M = k + \|\psi\|_{L^\infty(\Omega)}$.
For $n > M$, we have

$$\begin{aligned}\int_\Omega a(x, T_n(u_n), \nabla u_n)\cdot\nabla T_k(u_n-\psi)dx &= \int_\Omega a(x, T_M(u_n), \nabla u_n)\cdot\nabla T_k(u_n-\psi)dx \\ &= \int_{\{|u_n-\psi|\leq k\}} a(x, T_M(u_n), \nabla u_n)\cdot\nabla(u_n-\psi)dx.\end{aligned}$$

Using (3.1.2) in (3.2.28), we obtain

$$\begin{aligned}h^{p-1}(M)\int_{\{|u_n-\psi|\leq k\}} |\nabla u_n)|^p dx - \int_{\{|u_n-\psi|\leq k\}} a(x, T_M(u_n), \nabla u_n)\cdot\nabla\psi dx \\ \leq \int_{\{|u_n-\psi|\leq k\}} F\cdot\nabla(u_n-\psi)dx.\end{aligned} \tag{3.2.29}$$

and therefore, using (3.1.3), Höder's and Young's inequalities, one can estimate the second

term on the left-hand side

$$\begin{aligned}\int_{\{|u_n-\psi|\le k\}} & a(x,T_M(u_n),\nabla u_n)\cdot\nabla\psi dx \\ &\le \int_{\{|u_n-\psi|\le k\}} \beta(j(x)+|T_M(u_n)|^{p-1}+|\nabla u_n|^{p-1})\cdot|\nabla\psi|dx \\ &\le C+\int_{\{|u_n-\psi|\le k\}}|\nabla u_n|^{p-1}\cdot|\nabla\psi|dx \\ &\le C+\frac{1}{p'}\int_{\{|u_n-\psi|\le k\}}|\nabla u_n|^p.\end{aligned}$$

For the term in the right-hand side, since $|F|$ belongs to $L^{p'}(\Omega)$ and using Höder's and Young's inequalities, we obtain

$$\begin{aligned}\int_{\{|u_n-\psi|\le k\}} F\cdot\nabla(u_n-\psi)dx &\le \int_{\{|u_n-\psi|\le k\}}|F||\nabla u_n|dx+\int_{\{|u_n-\psi|\le k\}}|F||\nabla\psi|dx \\ &\le C+\frac{1}{p}\int_{\{|u_n-\psi|\le k\}}|\nabla u_n|^p.\end{aligned}$$

Therefore

$$\int_{\{|u_n-\psi|\le k\}}|\nabla u_n|^p\le C,$$

replacing k with $k+\|\psi\|_\infty$ in the last inequality, we get

$$\int_{\{|u_n|\le k\}}|\nabla u_n|^p\le\int_{\{|u_n-\psi|\le k\}}|\nabla u_n|^p\le C. \tag{3.2.30}$$

So that the Lemma 3.2.3 is completely proved. □

3.3 Proof of the results

3.3.1 Proof of theorem 3.1.1 (case $m>\frac{N}{p-1}$)

The Lemma 3.2.1 guarantees the existence of a subsequence, still denoted by (u_n), and a measurable function u such that

$$u_n\to u \quad \text{and a.e. in } \Omega, \tag{3.3.1}$$

and

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega). \tag{3.3.2}$$

Since $u_n(x) \geq \psi(x)$ a.e. in Ω, $\forall n \in \mathbb{N}$

$$u \geq \psi \quad \text{a.e. in } \Omega. \tag{3.3.3}$$

Moreover, we shall prove that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{3.3.4}$$

In order to get (3.3.4), it is sufficient to show that

$$u_n \to u \quad \text{strongly in } W_0^{1,p}(\Omega). \tag{3.3.5}$$

We write

$$\begin{aligned} &\int_\Omega (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u) dx \\ &\quad = \int_\Omega a(x,u_n,\nabla u_n) \cdot (\nabla u_n - \nabla u) dx - \int_\Omega a(x,u_n,\nabla u) \cdot (\nabla u_n - \nabla u) dx. \end{aligned} \tag{3.3.6}$$

For $n > c_\infty$, if we take u as test function in (3.2.1), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n) \cdot \nabla(u_n - u) dx \leq \int_\Omega F \cdot \nabla(u_n - u) dx. \tag{3.3.7}$$

Since F belongs at least to $\left(L^{p'}(\Omega)\right)^N$, the term -div F is in $W^{-1,p'}(\Omega)$, so that by (3.3.2) we obtain

$$\lim_{n\to\infty} \int_\Omega F \cdot (\nabla u_n - \nabla u)\, dx = 0, \tag{3.3.8}$$

this and (3.3.7) yield

$$\lim_{n\to\infty} \int_\Omega a\,(x,u_n,\nabla u_n) \cdot (\nabla u_n - \nabla u)\, dx \leq 0. \tag{3.3.9}$$

In view of the growth assumption (3.1.3) and Vitali's theorem we have

$$a(x, u_n, \nabla u) \to a(x, u, \nabla u) \quad \text{strongly in } \left(L^{p'}(\Omega)\right)^N. \tag{3.3.10}$$

It follows by (3.3.2) that

$$\lim_{n\to\infty} \int_\Omega a(x, u_n, \nabla u) \cdot (\nabla u_n - \nabla u)\, dx = 0 \tag{3.3.11}$$

Therefore, we obtain

$$\lim_{n\to\infty} \int_\Omega \left(a\left(x, u_n, \nabla u_n\right) - a\left(x, u_n, \nabla u\right)\right) \cdot (\nabla u_n - \nabla u)\, dx = 0. \tag{3.3.12}$$

Using Lemma 5 in [26], we get (3.3.5).

Also we have

$$\nabla u_n \to \nabla u \quad \text{strongly in } \left(L^p(\Omega)\right)^N \text{ and a.e. in } \Omega. \tag{3.3.13}$$

We can now pass to the limit. Let $v \in \mathcal{K}_\psi$, choosing v as test function in (3.2.1). For every $n > c_\infty$ one has

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla(u_n - v) dx \leq \int_\Omega F \cdot \nabla(u_n - v) dx. \tag{3.3.14}$$

The right-hand side converges, as n tends to infinity, to

$$\int_\Omega F \cdot \nabla(u - v) dx. \tag{3.3.15}$$

In view of Fatou's Lemma, we obtain

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla u dx \leq \liminf_{n\to\infty} \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n dx. \tag{3.3.16}$$

Thanks to (3.3.13) and Vitali's theorem, we get

$$a(x, u_n, \nabla u_n) \to a(x, u, \nabla u) \quad \text{strongly in } \left(L^{p'}(\Omega)\right)^N,$$

thus

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v dx \longrightarrow \int_\Omega a(x,u,\nabla u)\cdot\nabla v dx. \tag{3.3.17}$$

Consequently, by combining (3.3.15)–(3.3.17), it is then possible to pass to the limit as n tends to infinity in (4.3.10) to obtain

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla(u-v)dx \leq \int_\Omega F\cdot\nabla(u-v)dx.$$

3.3.2 Proof of theorem 3.1.2 (case $m = \frac{N}{p-1}$)

Under the hypotheses of Theorem 3.1.2, we have the following results.

Lemma 3.3.1. *Let u_n be a solution of (3.2.1). Then there exists a measurable function u such that*

$$u_n \to u \quad \textit{in measure}, \tag{3.3.18}$$

$$u_n \to u \quad \textit{a.e. in } \Omega, \tag{3.3.19}$$

and

$$T_k(u_n) \rightharpoonup T_k(u) \quad \textit{weakly in } W_0^{1,p}(\Omega) \textit{ for every } k>0. \tag{3.3.20}$$

Proof. Firstly, we show that (u_n) is a Cauchy sequence in measure.
Let $\eta \geq 0$, if we use as test function $v = u_n - \eta T_k(u_n)$ in (3.2.1) which is an admissible test function for η small enough, we get

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n)dx \leq \int_\Omega F\cdot\nabla T_k(u_n)dx.$$

For $n>k$, we obtain

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dx \leq \int_\Omega F\cdot\nabla T_k(u_n)dx,$$

using (3.1.2), we get

$$\int_\Omega h^{p-1}(|T_k(u_n)|)|\nabla T_k(u_n)|^p dx \leq \int_\Omega F\cdot\nabla T_k(u_n)dx.$$

Since h is a deacrising function and using Hölder's inequality, it follows that

$$h^{p-1}(k)\int_\Omega |\nabla T_k(u_n)|^p dx \leq \left[\int_\Omega |F|^{p'}dx\right]^{\frac{1}{p'}}\left[\int_\Omega |\nabla T_k(u_n)|^p dx\right]^{\frac{1}{p}},$$

hence

$$\int_\Omega |\nabla T_k(u_n)|^p dx \leq \frac{1}{h^p(k)}\|F\|^{p'}_{L^{p'(\Omega)}}. \tag{3.3.21}$$

We use now the Sobolev inequality with exponents p and p^* ($p^* = \frac{Np}{N-p}$), so there exists a constant C which does not depend on n such that

$$\|T_k(u_n)\|^p_{L^{p^*}(\Omega)} \leq C\|\nabla T_k(u_n)\|^p_{L^p(\Omega)},$$

hence, by (3.3.21) we deduce

$$\int_\Omega |T_k(u_n)|^{p^*}dx \leq \frac{C}{h^{p^*}(k)}\|F\|^{\frac{p^*}{p-1}}_{L^{p'(\Omega)}}.$$

Since the set $\{x\in\Omega : |u_n(x)| > k\} \subset \Omega$, we have

$$|\{|u_n| > k\}| \leq \frac{C}{k^{p^*}h^{p^*}(k)}\|F\|^{\frac{p^*}{p-1}}_{L^{p'(\Omega)}}. \tag{3.3.22}$$

For t and $\varepsilon > 0$ we can write

$$\{|u_n - u_m| > t\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > t\}. \tag{3.3.23}$$

Since $\lim_{k\to\infty} \dfrac{1}{k\,h(k)} = 0$, for sufficiently large k, by inequality (3.3.22)

$$|\{|u_n| > k\}| < \frac{\varepsilon}{3} \quad \text{and} \quad |\{|u_m| > k\}| < \frac{\varepsilon}{3}. \tag{3.3.24}$$

By (3.3.21), the sequence $\{|T_k(u_n)|\}$ is bounded in $W^{1,p}_0(\Omega)$, we can assume that it is a Cauchy sequence in measure. Thus the existence of some N such that for $n, m \geq N$, we have

$$|\{|T_k(u_n) - T_k(u_m)| > t\}| < \frac{\varepsilon}{3}. \tag{3.3.25}$$

Therefore

$$|\{|u_n - u_m| > t\}| < \varepsilon, \quad \forall n, m \geq N. \tag{3.3.26}$$

Then (u_n) is a Cauchy sequence in measure, thus there exists a subsequence still denoted (u_n) which converge almost everywhere to some measurable function u, and from (3.3.21) it follows that

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega) \text{ for every } k > 0.$$

□

Lemma 3.3.2. *Let u_n be a solution of (3.2.1). Then*

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega. \tag{3.3.27}$$

To prove the previous lemma, it suffices to show that

$$\nabla u_n \longrightarrow \nabla u \quad \text{in measure .} \tag{3.3.28}$$

For this purpose, we will need the following estimates:

Lemma 3.3.3. *There exists a constant $C > 0$ such that, for every $k > 0$, we have*

$$|\{|u| > k\}| \leq \frac{C}{k^{p^*} h^{p^*}(k)} \|F\|_{L^{p'}(\Omega)}^{\frac{p^*}{p-1}}, \tag{3.3.29}$$

$$|\{|\nabla u_n| > k\}| \leq \frac{C}{k^{p^*} h^{p^*}(k)} \|F\|_{L^{p'}(\Omega)}^{\frac{p^*}{p-1}} + \frac{1}{k^p h^p(k)} \|F\|_{L^{p'}(\Omega)}^{p'}, \tag{3.3.30}$$

and

$$|\{|\nabla u| > k\}| \leq \frac{C}{k^{p^*} h^{p^*}(k)} \|F\|_{L^{p'}(\Omega)}^{\frac{p^*}{p-1}} + \frac{1}{k^p h^p(k)} \|F\|_{L^{p'}(\Omega)}^{p'}. \tag{3.3.31}$$

Proof. The proof is entirely similar to the corresponding one of Lemma 3 in [16]. □

Proof of Lemma 3.3.2. Let $\lambda > 0$, we set for some $k > 0$ and $\varepsilon > 0$

$$E_1 = \{x \in \Omega : |u_n| > k\} \cup \{x \in \Omega : |u| > k\} \cup \{x \in \Omega : |\nabla u_n| > k\} \cup \{x \in \Omega : |\nabla u| > k\},$$

$$E_2 = \{x \in \Omega : |u_n - u| > \varepsilon\},$$

$$E_3 = \{x \in \Omega : |u_n - u| \leq \varepsilon, |u_n| \leq k, |u| \leq k, |\nabla u_n| \leq k, |\nabla u| \leq k, |\nabla u_n - \nabla u| \geq \lambda\}.$$

We have that:

$$\{x \in \Omega : |\nabla u_n - \nabla u| \geq \lambda\} \subset E_1 \cup E_2 \cup E_3. \tag{3.3.32}$$

For fixed $\sigma > 0$, we will prove that there exists N such that

$$|\{x \in \Omega : |\nabla u_n - \nabla u| \geq \lambda\}| \leq \sigma \quad \text{for every } n \geq N. \tag{3.3.33}$$

By virtue of the inequalities (3.3.22), (3.3.29), (3.3.30) and (3.3.31) and using the fact that

$$\lim_{k\to\infty} \frac{1}{k\, h(k)} = 0,$$

we can conclude that there exists some k_σ such that

$$|E_1| \leq \frac{\sigma}{3} \quad \text{for all } n \text{ and } k \geq k_\sigma. \tag{3.3.34}$$

On the other hand, the sequence u_n is convergent in measure to u, this guarantees that there exists N_1 such that

$$|E_2| \leq \frac{\sigma}{3}, \quad \text{for all } n \geq N_1. \tag{3.3.35}$$

Now, we prove that there exists N_2 depending on k and σ, and verifying

$$|E_3| \leq \frac{\sigma}{3}, \quad \text{for all } n \geq N_2. \tag{3.3.36}$$

In order to derive (3.3.36) we need the following standard lemma:

Lemma 3.3.4. *Let (X, T, m) be a measurable space, such that $|X| < +\infty$. Let γ be a measurable function $\gamma : X \to [0, +\infty[$ such that $|\{x \in X : \gamma(x) = 0\}| = 0$. Then for any $\sigma > 0$ there exists $\delta > 0$ such that:*

$$\int_A \gamma \mathrm{d}m \leq \delta \Rightarrow |A| \leq \sigma.$$

Assumption (3.1.4) implies that there exists a real valued function $\gamma : \Omega \to [0, +\infty[$ such that

$$|\{x \in \Omega : \gamma(x) = 0\}| = 0, \tag{3.3.37}$$

and

$$(a(x,s,\xi)-a(x,s,\eta))\cdot(\xi-\eta)\geq\gamma(x)\quad\text{ a.e. in }\Omega, \tag{3.3.38}$$

for every $s\in\mathbb{R},\xi,\eta\in\mathbb{R}^N$ such that $|s|\leq k$, $|\xi|\leq k$, $|\eta|\leq k$ and $|\xi-\eta|\geq\lambda$.

Thus, we get

$$\int_{E_3}\gamma(x)dx\leq\int_{E_3}(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u))\cdot(\nabla u_n-\nabla u)dx.$$

In E_3, we have $|u|\leq k$ and $|u_n-u|\leq\varepsilon$, so that

$$\begin{aligned}\int_{E_3}\gamma(x)dx&\leq\int_{E_3}(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla T_k(u)))\cdot\nabla T_\varepsilon(u_n-T_k(u))dx\\&\leq\int_\Omega(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla T_k(u)))\cdot\nabla T_\varepsilon(u_n-T_k(u))dx\\&=\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\nabla T_\varepsilon\left(u_n-T_k(u)\right)dx\\&\quad-\int_\Omega a\left(x,u_n,\nabla T_k(u)\right)\cdot\nabla T_\varepsilon\left(u_n-T_k(u)\right)dx.\end{aligned}$$

We evaluate the term

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_\varepsilon(u_n-T_k(u))dx. \tag{3.3.39}$$

Let $v=u_n-\tau T_\varepsilon(u_n-T_k(u))$ where $\tau\geq0$. For τ small enough, v is an admissible test function in (3.2.1). It follows that

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_\varepsilon(u_n-T_k(u))dx\leq\int_\Omega F\cdot\nabla T_\varepsilon(u_n-T_k(u))dx,$$

thus, for $n>k+\varepsilon$ we have

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_\varepsilon(u_n-T_k(u))dx\leq\int_\Omega F\cdot\nabla T_\varepsilon(u_n-T_k(u))dx,$$

passing to the limit as n tends to infinity

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_\varepsilon(u_n-T_k(u))dx&\leq\int_\Omega F\cdot\nabla T_\varepsilon(u-T_k(u))dx\\&=\int_{k<|u|\leq k+\varepsilon}F\cdot\nabla T_{k+\varepsilon}(u)dx,\end{aligned}$$

therfore

$$\lim_{\varepsilon\to 0^+}\lim_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_\varepsilon(u_n-T_k(u))dx\le 0. \tag{3.3.40}$$

Concerning the second integral

$$\int_\Omega a(x,u_n,\nabla T_k(u_n))\cdot\nabla T_\varepsilon(u_n-T_k(u))dx. \tag{3.3.41}$$

Since we have the inclusion

$$\{x\in\Omega:|u_n-T_k(u)|\le\varepsilon\}\subset\{x\in\Omega:|u_n|\le k+\varepsilon\},$$

we can write

$$\begin{aligned}\int_\Omega a(x,u_n,&\nabla T_k(u_n))\cdot\nabla T_\varepsilon(u_n-T_k(u))dx\\ &=\int_{\{|u_n-T_k(u)|\le\varepsilon\}} a(x,T_{k+\varepsilon}(u_n),\nabla T_k(u_n))\cdot\nabla(T_{k+\varepsilon}(u_n)-T_k(u))dx.\end{aligned}$$

By (3.3.20) and using Vitali's theorem we can conclude that $\forall h,k>0$

$$a(x,T_h(u_n),\nabla T_k(u))\to a(x,T_h(u),\nabla T_k(u))\quad\text{ strongly in }(L^{p'}(\Omega))^N.$$

Therefore

$$\begin{aligned}\lim_{n\to\infty}&\int_{\{|u_n-T_k(u)|\le\varepsilon\}} a(x,T_{k+\varepsilon}(u_n),\nabla T_k(u_n))\cdot\nabla(T_{k+\varepsilon}(u_n)-T_k(u))dx\\ &=\int_{\{|u-T_k(u)|<\varepsilon\}} a(x,T_{k+\varepsilon}(u),\nabla T_k(u))\cdot\nabla(T_{k+\varepsilon}(u)-T_k(u))dx\\ &=\int_{\{k<|u|\le k+\varepsilon\}} a(x,u,\nabla T_k(u))\cdot\nabla T_{k+\varepsilon}(u)dx\\ &=\int_{\{k<|u|\le k+\varepsilon\}} a(x,u,0)\cdot\nabla T_{k+\varepsilon}(u)dx\\ &=0.\end{aligned}$$

Hence, there exists N_2 (depending on k and σ) such that

$$\int_{E_3} \gamma(x)dx \leq \frac{\delta}{3} \quad \text{for every } n \geq N_2.$$

By Lemma 3.3.3, we conclude that

$$|E_3| \leq \frac{\sigma}{3}, \quad \text{for all } n \geq N_2. \tag{3.3.42}$$

We take $N = \max(N_1, N_2)$, then

$$|\{x \in \Omega : |\nabla u_n - \nabla u| \geq \lambda\}| \leq \sigma \quad \text{for every } n \geq N.$$

this completes the proof of Lemma 3.3.2. □

We can now pass to the limit in (3.2.1) to obtain a solution of (3.1.10).
Let $v \in \mathcal{K}_\psi \cap L^\infty(\Omega)$; the function $u_n - T_k(u_n - v)$ is an admissible test function in (3.2.1). This choice yields

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n - v)dx \leq \int_\Omega F \cdot \nabla T_k(u_n - v)dx, \tag{3.3.43}$$

Taking into account the fact that the integral in (3.3.43) is on the subset of the set

$$\{x \in \Omega, |u_n| \leq k + \|v\|_{L^\infty(\Omega)}\},$$

we set $M = k + \|v\|_{L^\infty(\Omega)}$ and let $n > M$, we can rewrite (3.3.43) as

$$\int_{\{|u_n - v| \leq k\}} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_k(u_n - v)dx \leq \int_{\{|u_n - v| \leq k\}} F \cdot \nabla T_k(u_n - v)dx. \tag{3.3.44}$$

The right-hand side of (3.3.44) converges to

$$\int_{\{|u - v| \leq k\}} F \cdot \nabla(u - v)dx. \tag{3.3.45}$$

In view of Fatou's Lemma and (3.3.18) and (3.3.27), we obtain

$$\int_{\{|u-v|\leq k\}} a(x,u,\nabla u)\cdot\nabla u dx \leq \liminf_{n\to\infty}\int_{\{|u_n-v|\leq k\}} a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla u_n dx. \quad (3.3.46)$$

Since $T_M(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, so as a consequence of (3.1.3), $|a(x,T_M(u_n),\nabla T_M(u_n))|$ is bounded in $L^{p'}(\Omega)$. Thus together with (3.3.19) and (3.3.27), $a(x,T_M(u_n),\nabla T_M(u_n))$ is weakly convergent to $a(x,T_M(u),\nabla T_M(u))$ in $(L^{p'}(\Omega))^N$. Therefore

$$\int_{\{|u_n-v|\leq k\}} a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla v dx \longrightarrow \int_{\{|u-v|\leq k\}} a(x,u,\nabla u)\cdot\nabla v dx. \quad (3.3.47)$$

From (3.3.45)–(3.3.47), it is then possible to pass to the limit as n tends to infinity in (3.3.44) to obtain

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-v)dx \leq \int_\Omega F\cdot\nabla T_k(u-v)dx.$$

3.3.3 Proof of theorem 3.1.3

In order to prove Theorem 3.1.3 we need the following convergence result

Lemma 3.3.5. *let u_n be a solution of* (3.2.1). *Then*

$$T_k(u_n)\longrightarrow T_k(u) \quad \textit{strongly in } W_0^{1,p}(\Omega) \textit{ for all } k>0. \quad (3.3.48)$$

Proof. From Lemma 3.2.2, we have that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, so that

$$T_k(u_n)\rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega). \quad (3.3.49)$$

Observe that for $n > k$, one has

$$\begin{aligned}
&\int_\Omega \left(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\right)\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx\\
&\quad=\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx\\
&\qquad-\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx\\
&\quad=\int_{\{|u_n|<k\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx\\
&\qquad-\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx\\
&\quad=\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx\\
&\qquad-\int_{\{|u_n|\geq k\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx\\
&\qquad-\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx\\
&\quad=\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx\\
&\qquad-\int_{\{|u_n|\geq k\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u)dx\\
&\qquad-\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx.
\end{aligned}$$

We shall prove that the previous integral converges to zero. Indeed, on one hand, by choosing $v = u_n - (T_k(u_n) - T_k(u))$ as a test function in (3.2.1), which is an admissible test function, we obtain

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))dx \leq \int_\Omega F\cdot\nabla(T_k(u_n)-T_k(u))dx. \tag{3.3.50}$$

Thanks to (3.3.49), we have

$$\lim_{n\to\infty}\int_\Omega F\cdot\nabla(T_k(u_n)-T_k(u))dx = 0.$$

Therefore,

$$\lim_{n\to\infty}\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))dx \leq 0. \tag{3.3.51}$$

On the other hand, by the growth assumption (3.1.3), we get

$$\int_\Omega |a(x,T_n(u_n),\nabla u_n)|^{p'}dx \le \beta c\int_\Omega (j^{p'}(x)+|u_n|^p+|\nabla u_n|^p)dx$$
$$\le \beta c\left(\|j\|^{p'}_{L^{p'}(\Omega)}+\|u_n\|^p_{L^p(\Omega)}+\|u_n\|^p_{W_0^{1,p}(\Omega)}\right).$$

therefore the sequence $\{a(x,T_n(u_n),\nabla u_n)\}$ is bounded in $(L^{p'}(\Omega))^N$. Then, it converges weakly to some l in $(L^{p'}(\Omega))^N$ and we obtain

$$\lim_{n\to\infty}\int_{\{|u_n|\ge k\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u)dx=\int_{\{|u|\ge k\}} l\cdot\nabla T_k(u)dx=0. \tag{3.3.52}$$

By virtue of Lemma 3.2.2 and Vitali's theorem, we obtain

$$a(x,T_n(u_n),\nabla T_k(u))\to a(x,u,\nabla T_k(u)) \text{ strongly in } \left(L^{p'}(\Omega)\right)^N,$$

it follows from (3.3.49) that

$$\lim_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx=0, \tag{3.3.53}$$

which with (3.3.51) and (3.3.52) allow us to get

$$\lim_{n\to\infty}\int_\Omega \left(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\right)\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx=0. \tag{3.3.54}$$

Now we can apply Lemma 5 of [26] to conclude (3.3.48) □

From the precedent lemma, we also have

$$\nabla u_n\longrightarrow\nabla u \quad \text{a.e. in } \Omega. \tag{3.3.55}$$

We are now in position to prove Theorem 3.1.3. Choosing $v\in\mathcal{K}_\psi$ as test function in (3.2.1), we get

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(u_n-v)dx\le\int_\Omega F\cdot\nabla(u_n-v)dx. \tag{3.3.56}$$

The right-hand side of (3.3.56) converges as n tends to infinity to

$$\int_\Omega F \cdot \nabla(u-v)dx. \tag{3.3.57}$$

In view of Fatou's Lemma, we obtain

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla u dx \leq \liminf_{n\to\infty}\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n dx. \tag{3.3.58}$$

Combining (3.3.55) and the assumptions on the function $a(x,s,\xi)$, we have

$$a(x,T_n(u_n),\nabla u_n) \rightharpoonup a(x,u,\nabla u) \quad \text{weakly in } (L^{p'}(\Omega))^N.$$

Thus

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla v dx \longrightarrow \int_\Omega a(x,u,\nabla u)\cdot\nabla v dx. \tag{3.3.59}$$

Then we passing to the limit thanks to the previous results, we prove the Theorem 3.1.3.

3.3.4 Proof of theorem 3.1.4

In virtue of Lemma 3.2.3, there exists a subsequence, still denoted by u_n, which is weakly convergent to some function u in $W_0^{1,q}(\Omega)\cap L^r(\Omega)$. Moreover,

$$u_n \to u \quad \text{a.e. in } \Omega. \tag{3.3.60}$$

From (3.2.30), $T_k(u_n)$ belongs to $W_0^{1,p}(\Omega)$ for every $k>0$, which implies together with (3.3.60) that

$$T_k(u_n)\to T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega). \tag{3.3.61}$$

Lemma 3.3.6. *Let u_n be a sequence of solutions of the problems (3.2.1) with the same assumptions as in the statement of Theorem 3.1.4. Then there exists a subsequence, denoted by u_n such that*

$$\nabla u_n \longrightarrow \nabla u \quad \textit{a.e. in } \Omega. \tag{3.3.62}$$

Proof. Fix λ such that $1 < \lambda < \frac{q}{p}$, define

$$I_{n,\Omega} = \int_\Omega \{(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))(\nabla u_n - \nabla u)\}^\lambda dx. \tag{3.3.63}$$

We shall prove that the integral $I_{n,\Omega}$ converges to zero. We split it on the sets

$$C_k = \{x \in \Omega : |u(x)| > k\} \quad \text{and} \quad \bar{C}_k = \{x \in \Omega : |u(x)| \leq k\},$$

to get

$$\begin{aligned} I_{n,\Omega} =& \int_{C_k} \{(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))(\nabla u_n - \nabla u)\}^\lambda dx \\ &+ \int_{\bar{C}_k} \{(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))(\nabla u_n - \nabla u)\}^\lambda dx \\ =& I_{n,C_k} + I_{n,\bar{C}_k}. \end{aligned}$$

Let $\gamma = \frac{q}{p\lambda}$, using assumption (3.1.3) then Hölder's inequality with exponents γ and γ' in the first integral I_{n,C_k}, one has

$$\begin{aligned} I_{n,C_k} \leq& c \left(\int_\Omega j^{p'}(x) dx \right)^\lambda |C_k|^{1-\lambda} \\ &+ c \left(\int_{C_k} (|u_n| + |\nabla u_n| + |\nabla u|)^q dx \right)^{\frac{1}{\gamma}} |C_k|^{1-\frac{1}{\gamma}}. \end{aligned}$$

thus, by means of estimate (3.2.19), we get

$$I_{n,C_k} \leq c(|C_k|^{1-\lambda} + |C_k|^{1-\frac{1}{\gamma}}),$$

we thus obtain

$$\lim_{k\to\infty} \lim_{n\to\infty} I_{n,C_k} = 0. \tag{3.3.64}$$

Concerning the second integral $I_{n,\bar{C}_k}$, we have

$$\begin{aligned} I_{n,\bar{C}_k} &= \int_{\bar{C}_k} \{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla T_k(u)))(\nabla u_n - \nabla T_k(u))\}^\lambda dx \\ &\leq \int_\Omega \{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla T_k(u)))(\nabla u_n - \nabla T_k(u))\}^\lambda dx \\ &= J_{n,\Omega}. \end{aligned}$$

Again, we split the integral $J_{n,\Omega}$ on the sets

$$D_{k,l} = \{x \in \Omega : |u_n - T_k(u)| > l\}, \quad \bar{D}_{k,l} = \{x \in \Omega : |u_n - T_k(u) \leq l\}, \quad (l \in \mathbb{N}),$$

obtaining

$$\begin{aligned} J_{n,\Omega} = &\int_{D_{k,l}} \{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla T_k(u)))(\nabla u_n - \nabla T_k(u))\}^\lambda dx \\ &+ \int_{\bar{D}_{k,l}} \{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla T_k(u)))(\nabla u_n - \nabla T_k(u))\}^\lambda dx \\ &= J_{n,D_{k,l}} + J_{n,\bar{D}_{k,l}}. \end{aligned}$$

the measure of the set $D_{k,l}$ tends to zero as l tends to ∞ uniformly in n and k

$$\lim_{l\to\infty} \lim_{k\to\infty} \lim_{n\to\infty} J_{n,D_{k,l}} = 0. \tag{3.3.65}$$

Since on $\bar{D}_{k,l}$ we can write $\nabla(u_n - T_k(u)) = \nabla T_l(u_n - T_k(u))$, we have

$$J_{n,\bar{D}_{k,l}} = \int_\Omega \{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla T_k(u))) \cdot \nabla T_l(u_n - T_k(u))\}^\lambda dx$$

therefore, using the Hölder inequality, we get

$$\begin{aligned} J_{n,\bar{D}_{k,l}} &\leq |\Omega|^{1-\lambda} \left[\int_\Omega (a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla T_k(u))) \cdot \nabla T_l(u_n - T_k(u)) dx\right]^\lambda \\ &= |\Omega|^{1-\lambda} (J_{n,\bar{D}_{k,l},1} - J_{n,\bar{D}_{k,l},2})^\lambda, \end{aligned}$$

where

$$J_{n,\bar{D}_{k,l},1} = \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_l(u_n - T_k(u))dx,$$

and

$$J_{n,\bar{D}_{k,l},2} = \int_\Omega a(x, T_n(u_n), \nabla T_k(u)) \cdot \nabla T_l(u_n - T_k(u))dx.$$

The integral in $J_{n,\bar{D}_{k,l},2}$ is on the set $\{|u_n - T_k(u)| \leq l\}$, which is a subset of the set $\{|u_n| \leq l + k\}$; hence, if we take $n \geq l + k := M$, we get

$$J_{n,\bar{D}_{k,l},2} = \int_\Omega a(x, T_M(u_n), \nabla T_k(u)) \cdot \nabla T_l(u_n - T_k(u))dx.$$

Using the almost everywhere convergence (3.3.60) and the Vitali theorem we get

$$a(x, T_M(u_n), \nabla T_k(u)) \to a(x, T_M(u), \nabla T_k(u)) \quad \text{strongly in } (L^{p'}(\Omega))^N.$$

As consequence of (3.3.60) and(3.3.61), we have

$$\nabla T_l(u_n - T_k(u)) \rightharpoonup \nabla T_l(u - T_k(u)) \quad \text{weakly in } (L^{p'}(\Omega))^N, \tag{3.3.66}$$

so that

$$\lim_{n\to\infty} J_{n,\bar{D}_{k,l},2} = \int_\Omega a(x, T_M(u), \nabla T_k(u)) \cdot \nabla T_l(u - T_k(u))dx = 0. \tag{3.3.67}$$

To evaluate the integral $J_{n,\bar{D}_{k,l},1}$. Let $\tau \geq 0$, we choose $u_n - \tau T_l(u_n - T_k(u))$ as test function in (3.2.1), to get

$$J_{n,\bar{D}_{k,l},1} \leq \int_\Omega F \cdot \nabla T_l(u_n - T_k(u))dx.$$

using (3.3.66)

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega F \cdot \nabla T_l(u_n - T_k(u))dx &= \int_\Omega F \cdot \nabla T_l(u - T_k(u))dx \\ &= \int_{\{|u|>k\}} F \cdot \nabla T_l(u)dx.\end{aligned}$$

Therefore

$$\lim_{l\to\infty}\lim_{k\to\infty}\lim_{n\to\infty} J_{n,\bar{D}_{k,l},1} = 0. \tag{3.3.68}$$

Gathering results (3.3.64), (3.3.65), (3.3.68) and (3.3.67), we obtain

$$\lim_{n\to\infty} I_n = 0.$$

Since the integrand function in I_n is non-negative, we have

$$\|\{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla u))(\nabla u_n - \nabla u)\}^{\lambda}\|_{L^1(\Omega)} \to 0,$$

thus,there exist a subsequence still denoted by u_n,

$$(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla u))(\nabla u_n - \nabla u) \to 0.$$

Under our assumption on the function $a(x,s,\xi)$ and the previous limit, we conclude (3.3.62) as in [26]. □

Let $v \in \mathcal{K}_\psi \cap L^\infty(\Omega)$, and choose $u_n - T_k(u_n - v)$ as test function in (3.2.1). We get

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(u_n - v)dx \le \int_\Omega F\cdot\nabla(u_n - v)dx. \tag{3.3.69}$$

Finally, we can pass to the limit in (3.3.69), this completes the proof of theorem 3.1.4.

Chapter 4

On a class of nonlinear elliptic problems with obstacle in weighted Sobolev spaces

In the previous chapter, we have studied the obstacle problem associated to the Dirichlet problem

$$\begin{cases} \operatorname{div} a(x,u,\nabla u) = \operatorname{div} F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.0.1}$$

in the setting of standard Sobolev spaces $W^{1,p}(\Omega)$ where Ω be an open bounded of $\mathbb{R}^N$, $N \geq 2$ and $1 < p < N$. Now, in this chapter will also address the same obstacle problem but in a generalizied framework, that of weighted Sobolev spaces $W^{1,p}(\Omega,\omega)$. To achieve our objective we assume that

- **Assumption (A_1):** We assume that the expression:

$$|||u||| = \left(\int_\Omega |\nabla u|^p \omega dx\right)^{\frac{1}{p}},$$

  is a norm defined on $W_0^{1,p}(\Omega,\omega)$ equivalent to the norm (1.1.3), and there exist a

weight w on Ω and a parameter $q \in [p^*, \infty[$, such that the Hardy inequality

$$\left(\int_\Omega |u|^q w dx\right)^{\frac{1}{q}} \leq c \left(\int_\Omega |\nabla u|^p \omega dx\right)^{\frac{1}{p}}, \tag{4.0.2}$$

holds for every $u \in W_0^{1,p}(\Omega, \omega)$ with a constant $c > 0$ independent of u. Moreover, the imbedding

$$W_0^{1,p}(\Omega, \omega) \hookrightarrow L^q(\Omega, w), \tag{4.0.3}$$

is compact.

We suppose also that

$$\omega^{-s} \in L^1(\Omega) \quad \text{with } s \in \left[\frac{N}{p}, \infty\right[\cap \left[\frac{1}{p-1}, \infty\right[. \tag{4.0.4}$$

This assumption implies that the imdedding

$$W_0^{1,p}(\Omega, \omega) \hookrightarrow L^p(\Omega) \tag{4.0.5}$$

is compact (see [37]).

Let A be a nonlinear operator from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$ defined by

$$A(u) = -\operatorname{div}(a(x, u, \nabla u))$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following conditions:

- **Assumption (A_2)** For all $s \in \mathbb{R}$, $\xi, \mu \in \mathbb{R}^N$ and for almost every $x \in \Omega$,

$$a(x, s, \xi) \cdot \xi \geq \omega(x) b^{p-1}(s) |\xi|^p \tag{4.0.6}$$

$$|a(x, s, \xi)| \leq \omega^{\frac{1}{p}}(x) \gamma(|s|) \left(h(x) + \omega^{\frac{1}{p'}}(x) |\xi|^{p-1}\right) \tag{4.0.7}$$

$$(a(x, s, \xi) - a(x, s, \mu)) \cdot (\xi - \mu) > 0, \text{ whenever } \xi \neq \mu, \tag{4.0.8}$$

where:

  - $b(s) = \dfrac{\alpha}{(1 + |s|)^\theta}$, for some $0 \leq \theta < 1$ and some $\alpha > 0$;

– $\gamma : [0, +\infty[\to]0, +\infty[$ is a continuous function;

– h is a non negative function lying in $L^{p'}(\Omega)$.

4.1 Obstacle problem

Given an obstacle $\psi : \Omega \to \bar{\mathbb{R}}$ which is a measurable function such that

$$\psi^+ \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega).$$

Let us define

$$\mathcal{K}_\psi = \{v \in W_0^{1,p}(\Omega, \omega) : v(x) \geq \psi(x) \text{ a.e. in } \Omega\}.$$

The obstacle problem (A, ψ, F) associated with the problem (4.0.1) can be formulated as follows

$$\begin{cases} u \geq \psi \text{ a.e in } \Omega, \\ \displaystyle\int_\Omega a(x, u, \nabla u) \cdot \nabla(u - v)dx \leq \int_\Omega F \cdot \nabla(u - v)dx, \\ \forall v \in \mathcal{K}_\psi. \end{cases} \tag{4.1.1}$$

Our contribution in this chapter focuses on the cases $m > \frac{N}{p-1}$ and $m = \frac{N}{p-1}$, provided $-\operatorname{div} F$ belongs to $W^{-1,p'}(\Omega, \omega^*)$. The last fact is true if $m > \frac{N}{p-1}$. In the case where $m = \frac{N}{p-1}$, the formulation (4.1.1) is not valid. To address this challenge, it is necessary to offer a general formulation of the obstacle problem (A, ψ, F) by using the truncation function. For $k > 0$ and for $s \in \mathbb{R}$, we denote by $T_k(s)$ the usual truncation defined by

$$T_k(s) = \begin{cases} -k & \text{if } s < -k \\ s & \text{if } |s| \leq k \\ k & \text{if } s > k \end{cases}$$

Definition 4.1.1. *An entropy solution of the obstacle problem* (A, ψ, F) *is a measurable fuction* u *satisfying*

$$\begin{cases} u \geq \psi \text{ a.e. in } \Omega, \\ T_k(u) \in W_0^{1,p}(\Omega, \omega), \quad \text{for all } k > 0, \\ \displaystyle\int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - v) dx \leq \int_\Omega F \cdot \nabla T_k(u - v) dx, \\ \forall v \in \mathcal{K}_\psi \cap L^\infty(\Omega). \end{cases} \tag{4.1.2}$$

We have the following results:

Theorem 4.1.1. *Let*

$$\frac{F}{\omega} \in (L^m(\Omega, \omega))^N, \quad \text{with } m > \frac{N}{p-1}.$$

Assuming that (4.0.6) – (4.0.8) *hold true. Then there exists at least a function*

$$u \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$$

which is solution of the unilateral problem (4.1.1) .

Theorem 4.1.2. *Let*

$$\frac{F}{\omega} \in (L^{\frac{N}{p-1}}(\Omega, \omega))^N.$$

Assuming that (4.0.6) – (4.0.8) *hold true. Then the unilateral problem* (4.1.2) *has at least one solution.*

4.2 A priori estimates

Both the two existence results will be proved by approximation. Accordingly, we consider the sequence of problems

$$\begin{cases} u_n \geq \psi \text{ a.e in } \Omega, \\ \langle A_n u_n, u_n - v \rangle \leq \displaystyle\int_\Omega F \cdot \nabla(u_n - v) dx, \\ \forall v \in W_0^{1,p}(\Omega, \omega) \text{ with } v \geq \psi \text{ a.e in } \Omega, \end{cases} \tag{4.2.1}$$

where $A_n u_n = -\operatorname{div} a(x, T_n(u_n), \nabla u_n)$.

Lemma 4.2.1. *([17] Lemma A.1)*
Under the assumptions $(\mathbf{A1}) - (\mathbf{A1})$*, the operator* A_n *maps* $W_0^{1,p}(\Omega, \omega)$ *into its dual* $W^{-1,p'}(\Omega, \omega^{1-p'})$*. Moreover,* A_n *is bounded, coercive and pseudomonotone.*

Since $m > p'$ the term $\operatorname{div} F$ is in the dual space $W^{-1,p'}(\Omega, \omega^*)$. Therefore, there exist at least a function u_n solution of (4.2.1) (see[53]).

In the following, we will prove some a priori estimates on the solutions of approximating problems (4.2.1).
To prove the L^∞ a priori estimate, we will need the following well-known Stampacchia's lemma (see [66], Lemma 4.1).

Lemma 4.2.2. *Let* φ *be a positive, decreasing real function such that if* $h > k$*, we have*

$$\varphi(h) \leq \frac{C}{(h-k)^v} (\varphi(k))^\lambda, \quad \lambda > 1, v > 0,$$

C *is a positive constant. Then, there exists a constant* l *such that*

$$\varphi(l) = 0.$$

Lemma 4.2.3. *Let* $\dfrac{F}{\omega} \in (L^m(\Omega, \omega))^N$*, with* $m > \dfrac{N}{p-1}$ *and let* u_n *be a solution of (4.2.1). Then, there exist two positive constants* c_1*,* c_2*, depending on* N*,* $|\Omega|$*, and* $\|F\|_{L^m(\Omega)}$ *such that, for any* $n \in \mathbb{N}$*,*

$$\|u_n\|_{L^\infty(\Omega)} \leq c_1, \tag{4.2.2}$$

$$\|u_n\|_{W_0^{1,p}(\Omega,\omega)} \leq c_2. \tag{4.2.3}$$

Proof. Define, for s in $\mathbb{R}$ and for k positive

$$G_k(s) = s - T_k(s),$$

and

$$B(s) = \int_0^s \frac{1}{(1+|t|)^\theta} dt.$$

Let $v = u_n - G_k(B(u_n))$, where $k \geq \|\psi\|_{L^\infty(\Omega)}$. It is easy to see that $v \geq \psi$ a.e. in Ω, as it belongs to $\mathcal{K}_\psi$. By using v as test function, we get

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla G_k(B(u_n)) dx \leq \int_\Omega F \cdot \nabla G_k(B(u_n)) dx.$$

Which is equivalent to

$$\int_{A_k^n} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n b(u_n) dx \leq \int_{A_k^n} F \cdot \nabla B(u_n) dx,$$

where $A_k^n = \{x \in \Omega : |B(u_n)| > k\}$.
Using assumption (4.0.6) in the left-hand side and Hölder's inequality in the right-hand side of the above inequality, it follows that

$$\int_{A_k^n} \omega b^{p-1}(|T_n(u_n)|)|\nabla u_n|^p b(u_n) dx \leq \left(\int_{A_k^n} \left| \frac{F}{\omega} \right|^{p'} \omega dx \right)^{\frac{1}{p'}} \left(\int_{A_k^n} \nabla |B(u_n)|^p \omega dx \right)^{\frac{1}{p}}.$$

Since b is decreasing and $|T_n(u_n)| \leq |u_n|$, we get

$$\int_{A_k^n} |\nabla B(u_n)|^p \omega dx \leq \int_{A_k^n} \left| \frac{F}{\omega} \right|^{p'} \omega dx.$$

By using (4.0.2), we have

$$\int_{A_k^n} |G_k(B(u_n))|^p \omega dx \leq \int_{A_k^n} \left| \frac{F}{\omega} \right|^{p'} \omega dx.$$

Observe that $A_h^n \subset A_k^n$ for every $h > k$, and we have

$$h - k \leq |G_k(B(u_n))|.$$

Using Hölder's inequality, we get

$$(h-k)^p \int_{A_h^n} \omega dx \leq \left(\int_{A_k^n} \left| \frac{F}{\omega} \right|^m \omega dx \right)^{\frac{p'}{m}} \left(\int_{A_k^n} \omega dx \right)^{\frac{m-p'}{m}}.$$

$$\mu(A_h^n) \leq \frac{C \left\| \frac{F}{\omega} \right\|_{L^m(\Omega,\omega)}^{p'}}{\alpha(h-k)^q} \mu(A_k^n)^{\frac{m-p'}{m}\frac{q}{p}},$$

μ is the measure generated by the weight Ω on measurable subsets of $\mathbb{R}^N$.
Since $\frac{m-p'}{m}\frac{q}{p} > 1$, thanks to by Satmpacchia's Lemma 4.2.2, it follows that $\{B(u_n)\}$ is uniformly bounded. The properties of the function B,in particular the fact that

$$\lim_{s\to+\infty} B(s) = +\infty, \lim_{s\to-\infty} B(s) = -\infty,$$

yield a bound for u_n in $L^\infty(\Omega)$, so that there exists a constant c_1 such that

$$\|u_n\|_{L^\infty(\Omega)} \leq c_1.$$

In order to get an estimation of u_n in $W_0^{1,p}(\Omega,\omega)$, let us choose ψ as test funtion in (4.2.1). Using the hypothesis (4.0.6) and (4.0.7) one obtains, if $\|u_n\|_{L^\infty(\Omega)} \leq c_1$,

$$\begin{aligned} b^{p-1}(c_1)\int_\Omega |\nabla u_n|^p \omega dx - \gamma_{c_1}\int_\Omega h(x)|\nabla\psi|\omega^{\frac{1}{p}}dx - \gamma_{c_1}\int_\Omega |\nabla u_n|^{p-1}|\nabla\psi|\omega dx \\ \leq \int_\Omega F\cdot\nabla(u_n-\psi)dx. \end{aligned} \tag{4.2.4}$$

where $\gamma_{c_1} = \max_{|t|\leq c_1}\gamma(t)$.
By using Hölder's and Young's inequalities we obtain for a positive constant λ, we have

$$-\gamma_{c_1}\int_\Omega h(x)|\nabla\psi|\omega^{\frac{1}{p}}dx - \gamma_{c_1}\int_\Omega |\nabla u_n|^{p-1}|\nabla\psi|\omega dx \geq C_1(\lambda) - \lambda\gamma_{c_1}\int_\Omega |\nabla u_n|^p\omega dx,$$

and

$$\int_\Omega F\cdot\nabla(u_n-\psi)dx \leq C_2(\lambda) + \lambda\int_\Omega |\nabla u_n|^p\omega dx.$$

By invoking the last two inequalities in (4.2.4), we get

$$(b^{p-1}(c_1) - \lambda(\gamma_{c_1}+1))\int_\Omega |\nabla u_n|^p\omega dx \leq C.$$

We deduce then (4.2.3). □

Lemma 4.2.4. *Let* $\frac{F}{\omega} \in (L^m(\Omega,\omega))^N$, $m = \frac{N}{p-1}$ *and let* u_n *be a solution of (4.2.1). Then, there exists a positive constants* C *(which does not depend on* n*) such that*

$$\int_\Omega |\nabla T_k(u_n)|^p \omega dx \leq C(k). \tag{4.2.5}$$

Proof. Let $v = u_n - \eta T_k(u_n)$ where $\eta \geq 0$. Since $v \in W_0^{1,p}(\Omega)$ and for η small enough, we have $v \geq \psi$, thus v is an admissible test function in the approximation problem (4.2.1). Then for $n > k$

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx \leq \int_\Omega F \cdot \nabla T_k(u_n) dx.$$

By the assumption (4.0.6) and the fact that the function b is decreasing, we get

$$b^{p-1}(k) \int_\Omega |\nabla T_k(u_n))|^p \omega dx \leq \int_\Omega F \cdot \nabla T_k(u_n) dx.$$

Using Hölder's inequality, we obtain

$$\int_\Omega |\nabla T_k(u_n))|^p \omega dx \leq \frac{1}{b^p(k)} \int_\Omega \left|\frac{F}{\omega}\right|^{p'} \omega dx.$$

□

4.3 Proof of the main results

4.3.1 Proof of Theorem 4.1.1 (case $m > \frac{N}{p-1}$)

The previous lemma 4.2.3 guarantees the existence of a subsequence, still denoted by $\{u_n\}$, and a measurable function u such that

$$u_n \to u \quad \text{and a.e. in } \Omega, \tag{4.3.1}$$

and

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega,\omega). \tag{4.3.2}$$

Since $u_n(x) \geq \psi(x)$ a.e. in Ω, $\forall n \in \mathbb{N}$

$$u \geq \psi \quad \text{a.e. in } \Omega. \tag{4.3.3}$$

Moreover, we shall prove that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{4.3.4}$$

Step 1: Strong convegence of u_n in $W_0^{1,p}(\Omega, \omega)$

Lemma 4.3.1. *Let u_n be a solution of the approximate problem (4.2.1), then*

$$u_n \to u \quad \textit{strongly in } W_0^{1,p}(\Omega, \omega). \tag{4.3.5}$$

By the compact imbedding (4.0.3), we can deduce that

$$u_n \to u \text{ a.e. in } \Omega. \tag{4.3.6}$$

We now shall prove that the weak convergence is strong.

For $n > c_1$, using u as a test function, we get

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot (\nabla u_n - \nabla u) dx \leq \int_\Omega F \cdot (\nabla u_n - \nabla u) dx.$$

Since $|F| \in L^{p'}(\Omega, \omega^{1-p'})$, by (4.3.2) we get

$$\lim_{n\to\infty} \int_\Omega F \cdot (\nabla u_n - \nabla u)\, dx = 0.$$

Thus, we have

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot (\nabla u_n - \nabla u) dx \leq 0. \tag{4.3.7}$$

On the other hand, we can write

$$\begin{aligned}&\int_\Omega (a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u))\cdot(\nabla u_n-\nabla u)dx\\&\quad=\int_\Omega a(x,u_n,\nabla u_n)\cdot(\nabla u_n-\nabla u)dx-\int_\Omega a(x,u_n,\nabla u)\cdot(\nabla u_n-\nabla u)dx.\end{aligned}$$

Thanks to (4.3.6) and (4.0.7), we can use Vitali's theorem to obtain

$$a(x,u_n,\nabla u)\to a(x,u,\nabla u)\quad \text{ strongly in } (L^{p'}(\Omega,\omega^{1-p'}))^N.$$

Then, it follows from (4.3.2) that

$$\lim_{n\to\infty}\int_\Omega a(x,u_n,\nabla u)\cdot(\nabla u_n-\nabla u)dx=0. \tag{4.3.8}$$

Bringing (4.3.7), (4.3.8) and (4.0.6) together, we conclude that

$$\lim_{n\to\infty}\int_\Omega (a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u))\cdot(\nabla u_n-\nabla u)dx=0. \tag{4.3.9}$$

Now we can apply Lemma 5 of [26] to conclude the strong convergence of u_n in $W_0^{1,p}(\Omega,\omega)$ which in turn implies (4.3.5).

u is a solution of the obstacle problem (4.1.1)

By taking $v\in\mathcal{K}_\psi$ as a test function in the approximate problem (4.2.1). For every $n>c_1$ one has

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla(u_n-v)dx\le\int_\Omega F\cdot\nabla(u_n-v)dx. \tag{4.3.10}$$

and by passing to the limit as n tend to infinity, we obtain

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla(u-v)dx\le\int_\Omega F\cdot\nabla(u-v)dx. \tag{4.3.11}$$

which conclude the proof of the Theorem 4.1.1.

4.3.2 Proof of Theorem 4.1.2 (case $m = \frac{N}{p-1}$)

Step 1: (u_n) **is a Cauchy sequence in measure**

Proposition 4.3.1. *Let u_n be a solution of approximating problem (4.2.1). Then there exists a measurable function u and a subsequence of u_n such that*

$$T_k(u_n) \to T_k(u) \quad \textit{strongly in} \quad W_0^{1,p}(\Omega, \omega)$$

Proof. Firsty, we prove that (u_n) is a Cauchy sequence in measure in a ball $\mathcal{B}_r$.

$$\begin{aligned} k \ meas(\{|u_n| > k\} \cap \mathcal{B}_r) &= \int_{\{|u_n|>k\}\cap\mathcal{B}_r} |T_k(u_n)|dx \\ &\leq \int_{\mathcal{B}_r} |T_k(u_n)|dx \\ &\leq \left(\int_\Omega |T_k(u_n)|^p \omega dx\right)^{\frac{1}{p}} \left(\int_{\mathcal{B}_r} \omega^{1-p'}\omega dx\right)^{\frac{1}{p'}} \\ &\leq \frac{C}{b(k)}. \end{aligned}$$

Therfore

$$meas(\{|u_n| > k\} \cap \mathcal{B}_r) \leq \frac{C}{kb(k)}. \tag{4.3.12}$$

For all $\rho > 0$, we have

$$(\{|u_n - u_m| > \rho\} \cap \mathcal{B}_r) \subset (\{|u_n| > k\} \cap \mathcal{B}_r) \cup (\{|u_m| > k\} \cap \mathcal{B}_r) \cup (\{|T_k(u_n) - T_k(u_m)| > \rho\} \cap \mathcal{B}_r),$$

which implies that

$$\begin{aligned} meas(\{|u_n - u_m| > \rho\} \cap \mathcal{B}_r) \leq\ & meas(\{|u_n| > k\} \cap \mathcal{B}_r) + meas(\{|u_m| > k\} \cap \mathcal{B}_r) \\ & + meas(\{|T_k(u_n) - T_k(u_m)| > \rho\} \cap \mathcal{B}_r). \end{aligned}$$

From (4.3.12), we have that $meas(\{|u_n| > k\} \cap \mathcal{B}_r) \longrightarrow 0$ as k tends to ∞. As a consequence, for $\varepsilon > 0$, there exists k_0 such that for every $k \geq k_0$ we have

$$meas(\{|u_n| > k\} \cap \mathcal{B}_r) < \frac{\varepsilon}{3} \quad \text{and} \quad meas(\{|u_m| > k\} \cap \mathcal{B}_r) < \frac{\varepsilon}{3} \tag{4.3.13}$$

On the other hand, since $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega,\omega)$ there exists some $v_k \in W_0^{1,p}(\Omega,\omega)$, such that

$$T_k(u_n) \rightharpoonup v_k \text{ weakly in } W_0^{1,p}(\Omega,\omega),$$

by the compact embedding (4.0.5), we get

$$T_k(u_n) \to v_k \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega.$$

Therefore, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω. Then there exists N_0 such that

$$meas(\{|T_k(u_n) - T_k(u_m)| > \rho\} \cap \mathcal{B}_r) < \frac{\varepsilon}{3}, \quad \forall n, m \geq N_0. \tag{4.3.14}$$

As a result of (4.3.13) and (4.3.14), for some k_0 large enough, we have

$$meas(\{|u_n - u_m| > \rho\} \cap \mathcal{B}_r) < \varepsilon \quad \text{for all } n, m \geq N_0.$$

This proves that $(u_n)_n$ is a Cauchy sequence in measure in B_R, thus converges almost everywhere to some measurable function u. Then

$$\begin{aligned} &T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega,w), \\ &T_k(u_n) \to T_k(u) \quad \text{strongly in } L^p(\Omega) \text{ and a.e. in } \Omega. \end{aligned} \tag{4.3.15}$$

Step 2: Strong convergence of truncations

In this step, we prove the strong convergence of truncations

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega,\omega).$$

In order to prove this convergence, we will show the following

$$\lim_{n\to\infty} \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \cdot \nabla T_k(u_n - u)) dx = 0.$$

Let $h > k$, and set $\varphi = u_n - T_h(u_n) + T_k(u_n) - T_k(u)$. For η small enough, we take

$v = u_n - \eta T_{2k}(\varphi)$ as a test function in (4.2.1), we obtain

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot T_{2k}(\varphi) dx \leq \int_\Omega F \cdot \nabla T_{2k}(\varphi) dx. \tag{4.3.16}$$

Splitting the integral on the left hand side $|u_n| \leq k$ and $|u_n| > k$, we can write

$$\begin{aligned}\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_{2k}(\varphi) dx = &\int_{\{|u_n| \leq k\}} a(x, T_k(u_n), \nabla u_n) \cdot \nabla T_{2k}(\varphi) dx \\ &+ \int_{\{|u_n| > k\}} a(x, T_k(u_n), \nabla u_n) \cdot \nabla T_{2k}(\varphi) dx.\end{aligned}$$

- If $|u_n| \leq k$. Then $\varphi = u_n - T_k(u)$.

$$\begin{aligned}&\int_{\{|u_n| \leq k\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_{2k}(\varphi) dx \\ &\qquad = \int_{\{|u_n| \leq k\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_{2k}(u_n - T_k(u)) dx \\ &\qquad = \int_{\{|u_n| \leq k\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla (T_k(u_n) - T_k(u)) dx \\ &\qquad = \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla (T_k(u_n) - T_k(u)) dx\end{aligned}$$

- If $|u_n| > k$ since u_n, $T_k(u_n)$ and $T_k(u)$ are in $W_0^{1,p}(\Omega, \omega)$ if $|\varphi| \leq 2k$, we obtain

$$\nabla T_{2k}(\varphi) = \nabla(\varphi) = \nabla u_n - \nabla T_h(u_n) - \nabla T_k(u).$$

There are two case as follows:

  - If $k < |u_n| < h$, we have $\nabla T_h(u_n) = \nabla u_n$. Then

$$\nabla T_{2k}(\varphi) = -\nabla T_k(u)$$

  - If $h < |u_n| \leq M$, $M = 4k + h$, we have $\nabla \varphi_n = 0$ for $|u_n| > M$. we have that $\nabla T_h(u_n) = 0$. Then

$$\nabla T_{2k}(\varphi) = \nabla(u_n - T_k(u)),$$

since $a(x,s,\xi)\cdot\xi \geq 0$, in both cases we obtain

$$a(x,T_k(u_n),\nabla u_n)\cdot\nabla T_{2k}(\varphi) \geq -|a(x,T_k(u_n),\nabla u_n)||\nabla T_k(u)|$$

Therefore we have

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla(T_k(u_n)-T_k(u))dx \\ \leq \int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_{2k}(\varphi)dx \\ + \int_{\{|u_n|>k\}} -|a(x,T_k(u_n),\nabla u_n)||\nabla T_k(u)|dx.\end{aligned}$$

Hence

$$\begin{aligned}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)))\cdot\nabla(T_k(u_n)-T_k(u))dx \\ \leq \int_{\{|u_n|>k\}} |a(x,T_k(u_n),\nabla u_n)||\nabla T_k(u)|dx \\ + \int_\Omega F\cdot\nabla T_{2k}(\varphi)dx \\ - \int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla(T_k(u_n)-T_k(u))dx.\end{aligned} \tag{4.3.17}$$

Considering now the test function $v = u_n - \eta T_{2k}(u_n - T_h(u_n))$ in (4.2.1), we can proceed as in the proof of the lemma 4.2.4 to get

$$\int_\Omega |\nabla T_{2k}(u_n - T_h(u_n))|^p \omega dx \leq c(k).$$

where $c(k)$ does not depend on h.
Using the fact that

$$T_{2k}(u_n - T_h(u_n)) \rightharpoonup T_{2k}(u - T_h(u)) \quad \text{weakly in } W_0^{1,p}(\Omega,\omega),$$

we get

$$\int_\Omega |\nabla T_{2k}(u - T_h(u))|^p \omega dx \leq c(k).$$

$$\begin{aligned}\left|\int_\Omega F\cdot\nabla T_{2k}(u-T_h(u))dx\right| &\le \int_\Omega |F|\,|\nabla T_{2k}\,(u-T_h(u))|\,dx\\ &= \int_{\{h<|u|<2k+h\}} |F||\nabla u|dx\\ &\le \left(\int_{\{|u|\ge h\}} |\frac{F}{\omega}|^{p'}\omega\mathrm{d}x\right)^{1/p'}\left(\int_{\{h<|u|<2k+h\}}|\nabla u|^p\omega dx\right)^{1/p} \quad \text{where}\\ &= \left(\int_{\{|u|\ge h\}} |\frac{F}{\omega}|^{p'}\omega\mathrm{d}x\right)^{1/p'}\left(\int_\Omega |\nabla T_{2k}\,(u-T_h(u))|^p\,\omega dx\right)^{1/p}\\ &= C\left(\int_{\{|u|\ge h\}} |\frac{F}{\omega}|^{p'}\omega dx\right)^{1/p'},\end{aligned}$$

C depends on k but not on h.

Therefore, we have

$$\lim_{h\to\infty}\int_\Omega F\cdot\nabla T_{2k}(u-T_h(u))dx=0. \tag{4.3.18}$$

Since $|a(x,T_k(u_n),\nabla u_n)|$ is bounded in $L^{p'}(\Omega,\omega)$ and $\chi_{\{|u_n|>k\}}|\nabla T_k(u)|\to 0$ in $L^p(\Omega,\omega)$ as n tends to infinity, we have

$$\lim_{n\to\infty}\int_{\{|u_n|>k\}}|a(x,T_k(u_n),\nabla u_n)||\nabla T_k(u)|dx=0. \tag{4.3.19}$$

It is clear that

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla(T_k(u_n)-T_k(u))dx \underset{n\to\infty}{\longrightarrow} = 0. \tag{4.3.20}$$

Furthermore, we have that

$$T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\rightharpoonup T_{2k}(u-T_h(u)) \quad \text{weakly in } W_0^{1,p}(\Omega,\omega).$$

Hence, by (4.3.18),(4.3.19) and (4.3.20), passing to the limit in (4.3.17), we have

$$\lim_{n\to\infty}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\cdot\nabla(T_k(u_n)-T_k(u))dx=0.$$

We deduce that

$$T_k\,(u_n)\to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega,\omega). \tag{4.3.21}$$

□

Step3: u is a solution of the obstacle problem (4.1.2)

let $v \in \mathcal{K}_\psi \cap L^\infty(\Omega)$ and let us take $u_n - T_k(u_n - v)$ as test function in (4.2.1). We obtain

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n - v) dx \leq \int_\Omega F \cdot \nabla T_k(u_n - v) dx, \tag{4.3.22}$$

Thanks to (4.3.21), we have $\nabla u_n \to \nabla u$ almost everywhere in Ω. Then we can pass to the limit in (4.3.22).

Conclusion and Perspectives

In this thesis, we focused on the existence and regularity of solutions for certain classes of nonlinear elliptic equations with degenerate coercivity. More precisely, we have proved the existence of entropy solutions in the weighted Sobolev space for the Dirichlet problem

$$(\mathcal{P}_1)\begin{cases} -\text{ div }[\omega(x)a(x,u,\nabla u)] = f(x) - \text{ div }(F), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 2$ and ω is an $\mathcal{A}_p$-weight function. We established also some existence and regularity results of solution for the unilateral problems associated to the Dirichlet problem

$$(\mathcal{P}_2)\begin{cases} \text{div } a(x,u,\nabla u) = div(F), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

in both classical Sobolev spaces and weighted Sobolev spaces.
This work raises a number of questions that deserve to be addressed in the future, for example we hope to:

- Extend the results obtained about $(\mathcal{P}_1)$ to
  - The parabolic case.
  - Anisotropic Sobolev spaces.
- Discuss the existence of the solution for the unilateral problem associated to $(\mathcal{P}_2)$ in the cases
  - $\tilde{m} = \frac{Np'}{N-\theta(N-p)} \leq m < \frac{N}{p-1}$.

– $\bar{m} = \max\left(p', \frac{Np'}{p((1-\theta)N+\theta)}\right) \leq m < \frac{Np'}{N-\theta(N-p)} = \tilde{m}$.

- Investigate the problem $(\mathcal{P}_2)$ in weighted Sobolev spaces with variable-exponent.

- Treat $(\mathcal{P}_1)$ in the general case of Stefan-type problems

$$(\mathcal{P}_3)\begin{cases}\beta(u) - \operatorname{div}[\omega(x)\, a(x,u,\nabla u) - F(u)] \ni f, & \text{in } \Omega,\\ u(x) = 0, & \text{on } \partial\Omega.\end{cases}$$

We also aim to investigate the mean field games (MFG) system with p-Laplacian, instead of the Laplacian Δ

$$\begin{cases}-u_t - \nu\Delta u + H(x,Du) = F(x,m) \quad in(0,T)\times\Omega,\\ m_t - \nu\Delta m - div(mH_p(x,Du) = 0 \quad in(0,T)\times\Omega.\end{cases}$$

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