جامعة سيدي محمد بن عبد الله بفاس
UNIVERSITÉ SIDI MOHAMED BEN ABDELLAH DE FES

كلية العلوم ظهر المهراز فاس
FACULTE DES SCIENCES DHAR EL MAHRAZ FES

N° d'Ordre : **44 /2021**

Formation Doctorale : Mathématiques et Application (MA)

Discipline : Mathématiques Appliquées

Spécialité : Probabilités et Statistique

Laboratoire : LPAIS

THESE DE DOCTORAT

Présentée par

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Mathematical Analysis of Some Stochastic Infectious Disease Models with White Noises and Lévy Jumps

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Summary

Epidemiology offers apt means and tools to specify the degree to which risk factors and intervention strategies influence public health in the case of an epidemic outbreak. These manners include surveys, data analysis, controls, and appropriate assessments. In this field, mathematical modeling plays an intrinsic role in elucidating the impact of communicable diseases' repercussions and prophesying likely future scenarios. The application of mathematical models to epidemiology requires characterizing the variables that affect the behavior of a disease in order to get a series of equations whose solution agrees with what is happening in reality. Accordingly, mathematical epidemiology represents the principal ground for decisions that aimed at preventing, dominating, or eradicating the disease.

Empirical records of many epidemics show that external fluctuations are significant factors in the spread process of infectious diseases. Undoubtedly, stochastic models are considered the most realistic and practical method to simulate and study the evolution of an epidemic under environmental disturbances. The analysis of these models requires many advanced analytical techniques due to their intricacy and difficulty. Focusing on this problematic, the present thesis aims to treat the evolutionary dynamics of some epidemic models with natural environmental noises and certain unexpected phenomena. Under a suitable hypothetical framework, we offer and develop new general and non-standard approaches to prove some interesting asymptotic properties of the perturbed models, namely: stability, ergodicity, periodicity, persistence in the mean, extinction of the disease, long-term character around the deterministic equilibriums. At the end of each section, some numerical simulations are presented to verify the theoretical results and clearly show the sharpness of the obtained conditions and thresholds.

In general, this thesis improves numerous studies existing in the field of stochastic mathematical epidemiology and provides alternative tools to treat and analyze more complex epidemiological and biological models.

Keywords: Epidemics; Infectious diseases; Environmental disturbances; Stochastic systems; Lévy noise; Asymptotic properties; Stability; Periodicity; Ergodicity; Persistence in the mean; Disease extinction; The long-time character around the deterministic equilibrium.

Résumé (French version)

L'épidémiologie offre des moyens et des outils appropriés pour spécifier dans quelle mesure les facteurs de risque et les stratégies d'intervention influencent la santé publique en cas d'épidémie. Ces méthodes comprennent des enquêtes, des analyses de données, des contrôles et des évaluations appropriées. Dans ce domaine, la modélisation mathématique joue un rôle intrinsèque dans l'élucidation de l'impact des répercussions des maladies transmissibles et la prédiction de scénarios futurs probables. L'application des modèles mathématiques à l'épidémiologie nécessite une caractérisation des variables qui affectent le comportement d'une maladie afin d'obtenir une série d'équations dont la solution concorde avec ce qui se passe dans la réalité. En conséquence, l'épidémiologie mathématique représente le principal motif de décision visant à prévenir, dominer ou éradiquer la maladie.

Les données empiriques de nombreuses épidémies montrent que les fluctuations externes sont des facteurs importants dans le processus de propagation des maladies infectieuses. Sans aucun doute, les modèles stochastiques sont considérés comme la méthode la plus réaliste et la plus pratique pour simuler et étudier l'évolution d'une épidémie en cas de perturbations environnementales. L'analyse de ces modèles nécessite de nombreuses techniques analytiques avancées en raison de leur complexité et de leur difficulté. En se concentrant sur cette problématique, la présente thèse vise à traiter la dynamique évolutive de certains modèles épidémiques avec des bruits environnementaux naturels et certains phénomènes inattendus. Dans un cadre hypothétique adapté, nous proposons et développons de nouvelles approches générales et non standard pour prouver certaines propriétés asymptotiques intéressantes des modèles perturbés, à savoir: stabilité, ergodicité, périodicité, persistance dans la moyenne, extinction de la maladie, caractère à long terme autour des équilibres déterministes. A la fin de chaque section, quelques simulations numériques sont présentées pour vérifier les résultats théoriques et montrer clairement la netteté des conditions et des seuils obtenus.

D'une manière générale, cette thèse améliore de nombreuses études existant dans le domaine de l'épidémiologie mathématique stochastique et fournit des outils alternatifs pour traiter et analyser des modèles épidémiologiques et biologiques plus complexes.

Mots clés: Épidémies; Maladies infectieuses; Perturbations environnementales; Systèmes stochastiques; Bruit de Lévy; Propriétés asymptotiques; Stabilité; Periodicité; Ergodicité; Persistance dans la moyenne; Extinction de la maladie; Le caractère long terme autour de l'équilibre déterministe.

Summary (Arabic version)

يقدم علم الأوبئة وسائل وأدوات مناسبة لتحديد الدرجة التي تؤثر بها عوامل الخطر واستراتيجيات التدخل على الصحة العامة في حالة تفشي وباء ما. تشمل هذه الأساليب استطلاعات وتحليل البيانات و تحديد الضوابط والتقييمات المناسبة. في هذا المجال، تلعب النمذجة الرياضياتية دورا جوهريا في توضيح تأثير تداعيات الأمراض المعدية والتنبؤ بالسيناريوهات المستقبلية المحتملة. يتطلب تطبيق النماذج الرياضياتية في علم الأوبئة توصيف المتغيرات التي تؤثر على انتشار الوباء من أجل الحصول على سلسلة من المعادلات التي يوافق حلها مع ما يحدث في الواقع. وفقا لذلك ، يمثل علم الأوبئة الرياضي الأساس الرئيسي للتنبؤ بسلوك الأوبئة واتخاذ القرارات التي تهدف إلى منع إنتشارها أو السيطرة عليها.

تظهر السجلات التجريبية للعديد من الأوبئة أن التقلبات الخارجية هي عوامل مهمة في عملية انتشار الأمراض المعدية. لا شك أن النماذج العشوائية تعتبر الطريقة الأكثر واقعية لمحاكاة ودراسة تطور الوباء في ظل الاضطرابات البيئية. يتطلب تحليل هذه النماذج العديد من التقنيات التحليلية المتقدمة نظراً لتعقيدها وصعوبة استخدامها. بالتركيز على هذه المشكلة، تهدف هذه الرسالة إلى معالجة الديناميكيات التطورية لبعض النماذج الوبائية المضطربة بضوضاء بيئي طبيعي و بعض الظواهر الغير متوقعة. في ظل إطار فرضي مناسب، نقدم ونطور مناهج عامة وغير إعتيادية لإثبات بعض الخصائص المقاربة للنماذج المضطربة: الاستقرار، الثبات، الدورية، الدوام المتوسطي، إختفاء الوباء، و السلوك طويل الأمد حول نقط التوازنات القطعية.

بشكل عام، تهدف هذه الأطروحة إلى تحسين العديد من الدراسات الموجودة في مجال علم الأوبئة الرياضي العشوائي وتوفر آليات بديلة لعلاج وتحليل نماذج وبائية وبيولوجية أكثر تعقيدا.

الكلمات المفتاحية: وباء؛ الأمراض المعدية؛ الاضطرابات البيئية؛ الانظمة العشوائية؛ ضوضاء ليفي؛ الخصائص المقاربة؛ الاستقرار؛ الثبات ؛ الدورية؛ الدوام المتوسطي؛ إختفاء الوباء؛ السلوك طويل الأمد حول نقط التوازنات القطعية.

Acknowledgements

After a half-decade of learning and research, I write these acknowledgments today as the final touch to my thesis. It has been a period of intense acquiring for me, not only in the scientific field but also on a personal level. Writing this thesis had an enormous positive impact on me. I would like to reflect on who has supported and aided me so much throughout this time. First of all, I would like to thank God who gave me the power to believe in myself and pursue my goals and dreams. Second, I must thank my parents for their love, wise advice, and support throughout my life. I would like to heartily thank my advisor Professor Driss Kiouach for his continued help and advice. I am extremely satisfied to work and be under his direction. In addition, I sincerely thank all committee members for reviewing my work and for their professional and neutral comments. Ultimately, I would like to thank my colleague and my friend Mr. Salim El Azami El-idrissi for his excellent cooperation and motivation.

Thank you very much, everyone!

Yassine Sabbar



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List of abbreviations

- SIR : Susceptible-Infected-Removed.
- SIRS : Susceptible-Infected-Removed-Susceptible.
- SIS : Susceptible-Infected-Susceptible.
- SIQS : Susceptible-Infected-Quarantined-Susceptible.
- SIQR : Susceptible-Infected-Quarantined-Removed.
- HBV : Hepatitis B Virus.
- SARS : Severe Acute Respiratory Syndrome.
- COVID-19 : Corona Virus Disease 2019.
- WHO : World Health Organization.
- ODE : Ordinary Differential Equation.
- SDE : Stochastic Differential Equation.
- SDE-J : Stochastic Differential Equation with Jumps.
- PDF : Probability Density Function.
- a.s. : almost surely.

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- Driss Kiouach and Yassine Sabbar, *The long-time behaviour of a stochastic SIR epidemic model with distributed delay and multidimensional Lévy jumps*, International Journal of Biomathematics, (Accepted paper, in press).
- Driss Kiouach, Yassine Sabbar and Salim El Azami El-idrissi, *New results on the asymptotic behavior of an SIS epidemiological model with quarantine strategy, stochastic transmission, and Lévy disturbance*, Mathematical Methods in the Applied Sciences, 2021.
- Driss Kiouach and Yassine Sabbar, *Nonlinear dynamical analysis of a stochastic SIRS epidemic system with vertical dissemination and switch from infectious to susceptible individuals*, Journal of Applied Nonlinear Dynamics, 2021, (Accepted paper, in press).
- Driss Kiouach and Yassine Sabbar, *Developing new techniques for obtaining the threshold of a stochastic SIR epidemic model with 3-dimensional Lévy process*, Journal of Applied Nonlinear Dynamics, 2021, (Accepted paper, in press).
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- Driss Kiouach and Yassine Sabbar, *Modeling the Impact of Media Intervention on Controlling the Diseases with Stochastic perturbations*, "2nd International Conference on Applied Mathematics", Fez, October 26-27, 2018.
- Driss Kiouach and Yassine Sabbar, *The threshold of a stochastic epidemic model with isolation and information intrusion*, "International Conference on Technology, Engineering and Mathematics" (TEM'18), Kenitra, Morocco, March 26-27, 2018.
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- Driss Kiouach and Yassine Sabbar, *A stochastic epidemic model incorporating social media coverage*, "1st International Scientific Day of Taroudant" October 6, 2018.

Prologue

Epidemiology represents the branch of medical science that studies and analyzes the reasons, distributions, and mechanisms of the diseases spread in a specific population. The idiom "epidemiology" is originated from the Greek words: "epi", "demos" and "logos", which literally means "study what happens to people" [1]. The incorporation of these three words indicates that the subject of the epidemiology applies exclusively to humans. Epidemiology assists in improving ways and methodologies used in clinical explorations, public health studies, and fundamental scientific research in medical investigations. Historically, the term "epidemiology" was first used to describe the characteristics of some epidemics in 1802 by the Spanish scientific Joaquin de Villalba (1752-1807) [30]. Since that date, epidemiological investigations focused on emerging diseases that prevailed in susceptible populations and that often caused numerous deaths. For instance, the "Spanish" influenza virus in 1918-1919 killed more than 50 million people, and every year there is seasonal influenza that caused an extra than 35 thousand deaths in the world. There are also many epidemics that persist in some populations for a long time and cause massive losses. Millions of people die each year from cholera, respiratory infections, measles, malaria, and dengue fever in diverse parts of the world. This is enormously happening in many countries with weak health care systems. In 2011, the World Health Organization (WHO) estimated that there were 1.5 million deaths from tuberculosis, 1.2 million from HIV/AIDS, and nearly half a million mortalities from malaria. The high epidemic mortality rate has significant implications for life expectancy and the economic system in the concerned countries. For example, the ongoing Coronavirus (COVID-19) caused a deep international crisis with 2.3 million deaths resulting in global health problems and changes in lifestyle. Currently, we are struggling to control the spread of this epidemic, which has the potential to be the deadliest disease in human history. In this regard, to face and manage such a crisis situation, various scientific methods and analytical studies are required. As a method and science, mathematical modeling is the appropriate tool. It is the art of translating real problems into flexible mathematical formulas whose theoretical and numerical analysis provides suggestions, answers, and constructive solutions. As a field of application, mathematical epidemiology plays the main role in the analysis of factors that may influence the prevalence of the disease.

The concept that the spread of epidemics obeys laws that can be formulated in a mathematical expression is not recent. The beginning started with Daniel Bernoulli in 1760 by publishing a scientific report in which he characterizes the impacts of Smallpox on life average employing a mathematical lifetime chart analysis. His study aims to show up the benefits of inoculation against the said epidemic [100]. Since then, many attempts have been made to describe the impact of some epidemics on the population, but the non-linear dynamics of its transmission were imperfectly understood until the twentieth century. In 1906, W. H. Hamer was the first to observe that a decreasing density of susceptible individuals can lead to the extinction of the measles epidemic. He proposed the first ordinary epidemiological system for modeling disease prevalence [100]. In the same vein, S. R. Ross in 1911 used mathematical modeling to discuss the effectiveness of different intervention methods against malaria [1]. In 1927, W.O. Kermack (public health doctor) and A. G. Mckendrick (biochemist) published a straightforward system of the spread of epidemics

by direct contact. At the time, they compared their model with actual data from the spread of the Bombay plague between 1905 and 1906 [15]. Their theory proved important during the elimination of Smallpox in the 1970s [34]. At the end of the twentieth century, mathematical epidemiology became more common in public health policy-making. Many contributions have been made to modeling some epidemics, and most of these studies have relied on various assumptions based on the deterministic formulation. Since the start of the 21st century, many epidemics appeared including the severe acute respiratory syndrome (SARS) epidemic in 2002 [14], influenza A virus sub-type H5N1 in 2005 [4], influenza A virus sub-type H1N1 in 2009 [88], Ebola epidemic in 2004 [17], and Coronavirus in 2019 [13]. That revived interest in mathematical modeling by improving the Kermack-McKendrick model and considering new hypotheses based on the characteristics of each epidemic.

The first generation of epidemic models was deterministic, and many issues were addressed, for example: could the epidemic situation worsen and affect a vast section of society? How many people are suffering from the epidemic? What are the effects of intervention strategies on the spread of the disease? Could complete extinction happen? When responding to these questions, the standard models have been refined and generalized in a variety of ways to make them more adapted to reality. Some of these enhancements and expansions have, for example, taking into account irregular contact between people and sudden environmental changes. The generation of formulations that take these assumptions into account is called stochastic epidemiological models. Naturally, both deterministic and stochastic epidemiological systems have a meaningful role to carry out. However, the focus of this thesis is on perturbed epidemiological models.

Substantially, in an ecosystem, epidemiological models are always perturbed by the external noise [3]. May [95] has shown that due to some environmental noises, the transmission rate, the birth rate, carrying capacity, and other parameters involved in the model should exhibit irregular and random fluctuation to a greater or lesser extent. Subsequently, lots of studies have inserted the white noise into the corresponding deterministic formulations to show the impacts of the environmental fluctuation on the population dynamics [16]. These systems are stochastic epidemic models with only white noise and hence its solution is continuous. However, when faced with sudden environmental shocks (earthquakes, hurricanes, epidemics, etc.), such disorders can break the continuity of the solution. Therefore, we should employ the stochastic differential equation with jumps to study the dynamic behavior of the epidemic model. Roughly speaking, the jump times are always random, and the expected time of jumps is analogous to Lévy jumps [37]. Therefore, these stochastic perturbations can be approximately modeled and simulated by Lévy jumps. Lévy processes are actually stochastic processes with stationary and independent increments. Their significance in applied probability theory stems from many important facts (see for example [27, 26, 36]).

This thesis aims to provide new ways to improve the analysis of some perturbed epidemiological models. Several generalizations, techniques, and approaches are proposed in order to predict and control the dynamics of infectious diseases under a suitable hypothetical framework. Thus, this thesis can be considered as a rich basis for outlook studies in the said field. Generally, this work contains six principal chapters. We begin with an introductory chapter (Chapter I) on the basic process of setting up an epidemic model under different hypotheses by using differential equations and some fundamentals classic notions in mathematical epidemiology. The chapter contains also all epidemic models treated during this thesis and provides a soft entry into epidemic modeling before presenting a more generalized context which required many stochastic analysis tools. These latter are introduced briefly in Chapter II. This auxiliary chapter provides a rigorous theoretical reference for important concepts and mathematical results used in this thesis. Our main results are offered from Chapter III which contains three sections about an SIR epidemic model with Lévy jumps. Notably, this epidemic model is represented by Itô-Lévy stochastic differential equations in order to simulate sudden and unexpected external phenomena. By adopting some innovative and ameliorated mathematical approach, we analyze the long-run characteristics of three categories of an SIR model with jumps. Chapter IV deals with the asymptotic properties of the HBV model with unexpected environmental disturbances. By adopting two types of Lévy perturbations, we

prove the existence of a unique ergodic stationary distribution by using the mutually exclusive possibilities. Furthermore, the threshold analysis is provided in the case of proportional Lévy noise. By integrating environmental disturbances and the various periods of immunity, in Chapter V, we study the dynamics of an epidemic model of the SIRS type. A rich analysis and a suitable generalization are offered to understand the characteristic of some epidemics with different periods of immunity and vertical transmission. It should be pointed that the study presented in this chapter can be generalized in the case of Lévy jumps by using the same techniques introduced in Chapters III and IV. Chapter VI extends previous chapters by treating the dynamic characteristics of a perturbed epidemic model with quarantine policy and Lévy disturbance. New general and non-standard results are proved in order to investigate some interesting asymptotic properties, namely: ergodicity, persistence in the mean, and extinction of the disease.

It is noteworthy that all studies described in the above-said chapters improve many existing studies in the field of mathematical epidemiology and provide new techniques to treat and analyze more general epidemic models. Furthermore, each chapter (Chapters IV-VI) represents the results previously published or submitted to indexed specialized journals.

Chapter I

Deterministic epidemiological system: general approach to modeling

Abstract

In this chapter, we introduce some classical deterministic epidemic models. These baseline systems will be extended to incorporate more realistic hypotheses about disease transmission, various control strategies, and the impact of demography. The principal intent of this chapter is to present all the epidemic models treated during this thesis, especially in Chapters III-VI.

1 Introduction

Infectious diseases are disorders caused by organisms like viruses, germs, bacteria or parasites. These organisms live in and on our bodies and move through many modes of transmission such as direct physical contact (*horizontal spread*), mother to baby (*vertical transmission*), airborne particles, water or food, or vectors of infection. The mathematical model of a contagious disease aims to characterize and describe its dissemination mechanisms, which can be illustrated mainly as follows: when the carrier of infection mingles with a host population of susceptible, the disease is transmitted to other individuals. An *infectious individual* is defined as one from whom the contagious agent can be naturally acquired. We define an *asymptomatic carrier* as a person with an infectious disease agent but displays no physical symptoms. The person who shelters an infectious agent and diagnosed as a disease case is called an *infected individual*. If the number of cases raises above the ordinary average within a short period of time, then an *epidemic* outbreak happens. Infected population regains their health and recovers from contagion, either via pharmaceutical treatment or owing to the behavior of the immune mechanism, and may get different phases of gained impregnability against reinfection according to the disease characteristics. If the infectious population decreases and disappears, the epidemic dies and becomes extinct. Otherwise, if new individuals insert into the host population from natural birth or migration, or if the contagion appears easy under the inducement conditions, the disease may persist over a lengthy period of time. We say in this case that the disease will be *endemic* in the population. A global *pandemic* happens if the epidemic propagates worldwide. For example, the growth of Coronavirus 2019 (COVID-19) disease, caused by Coronavirus-2 severe acute respiratory syndrome (SARS-Cov-2), is considered a global issue and a huge health crisis. This complicated disease first appeared in December 2019 in Wuhan (capital of Hubei in China), and since then, the world knew a terrifying increase in the number of cases due to the virus dissemination ability. The statistics witnessed this outbreak when the number of deaths and infected people surpassed 2,311,432 and 106,018,480 respectively at the time of writing this thesis. Unquestionably, these massive numbers

put the decision makers and health authorities in a hard position to find answers for the following inquiries:

- How dangerous will this pandemic be? The risk can be established by the accumulated number of infected individuals and the overall number of cases that may demand pharmaceutical treatment.
- How long will the disease last? will there be extinction or persistence?
- How efficient will lock-down, isolation, quarantine, or vaccination be?
- What are successful tools and measures to inhibit, control, and enucleate this endemic disease?

In order to help public health authorities make informed decisions, mathematical modeling is an important tool to obtain appropriate hypotheses and adequate answers to the above questions. The classical methods employing experimental and statistical approaches may not be sufficient, but the use of the dynamic models may provide an additional understanding of the transmission mechanisms of an epidemic. In general, the process of modelling and creating epidemiological systems involves the following steps:

1. Consider appropriate assumptions about the disease dissemination mechanisms based on the ready epidemiological data.
2. Set up an epidemiological model based on these assumptions. This generally begins by determining the sub-compartments of the model, drawing the switch diagram, and then deriving the mathematical equations.
3. Achieve a mathematical analysis of the constructed model to understand and establish all possible outcomes on the disease spread. This is done mainly by using existing mathematical approaches and theories and by performing numerical simulations to illustrate its validity.
4. Explain and interpret the theoretical findings according to the assumptions considered in stage 1.

Generally speaking, epidemiological models are a translation of our hypotheses on the disease characteristics, and an approximation of the real situation of its spread [40]. It should be noted that there is a continuous distinction in the epidemiological modeling between the proposed models depending on their ability to simulate reality and their ability to provide good forecasts about the future of the disease [31]. These epidemic models can be classified into deterministic and stochastic systems. The first class takes various forms of ordinary differential equations that describe the dynamic interactions between rates of change and different population sizes. In these models, each set of variable states is determined only by the parameters of the model and the initial state of the variables [34]. The second class is considered as a generalization of the previous one by taking into consideration the environmental perturbations and treating the disease infection as a stochastic process. These perturbed models are appropriate to deal with tiny communities and small population groups with a high number of infectious contacts [16]. Although this thesis deals with stochastic epidemic models, the presentation of the associated deterministic formulation is necessary. In what follows, we highlight some standard characteristics and theoretical results of this category.

2 Compartmental approach

In this section, we explain how to create a mathematical model for the spread process of an epidemic using a compartmental method. We first divide the host population into different classes (compartments) according to the characteristics of the individuals. In one of the simplest scenarios, we consider the following groups:

- Susceptible individuals who are healthy but can contract the infection. We denote the size of this class by S.
- Infected people who contracted the epidemic and are now sick with it. We denote the size of this class by I.
- Removed individuals who have recovered and can no longer contract the epidemic. This class is generally designated by R.

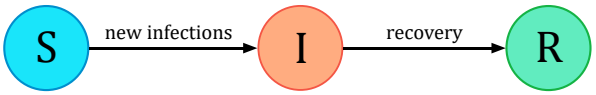


Figure I.1: Transfer diagram for a simple compartment epidemic model.

The schematic flow diagram of a straightforward SIR model is illustrated in Figure I.1. The major idea of epidemiological modeling is to show the evolution of the individuals' number in each of the three classes at a given time t. To establish this dynamic mathematically, we consider the small-time interval $[t, t+\delta t]$ and we extract the transition between the three classes via Figure I.1 such that the arrows indicate the direction of the transfer. We assume that the total change in each class is the number of the population entering the class minus the number leaving the same class during the time interval. By applying the same logic to each class, we obtain the following equations:

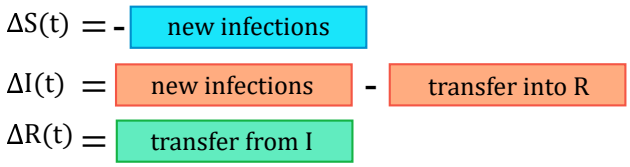


Figure I.2: The net changes between $S(t)$, $I(t)$, and $R(t)$.

We divide the two sides of these three equations by δt and let $\delta t \to 0$, then we obtain the following differential equations:

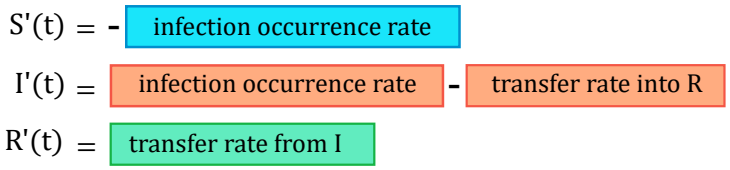


Figure I.3: Differential equations of the SIR epidemic model.

To derive the deterministic formulation in the form of a system of ordinary differential equations (ODEs), we make some assumptions depending on the epidemic characteristics. Then, we write the transfer rates on the right side as functions of $S(t)$, $I(t)$, and $R(t)$. As an illustrative example, in the next section, we consider a simple model that simulates the dynamics of susceptible, infectious, and removed individuals. This model was first suggested by Kermack and McKendrick in 1927. Since then, various studies have analyzed and improved it [20], and this thesis has the same objective.

3 The Kermack-McKendrick epidemic model

To show how the different rates in Figure I.3 depend on $S(t)$, $I(t)$ and $R(t)$, we assume the following assumptions about the transmission mechanism of an epidemic:

- The dissemination occurs horizontally by potential direct contact between individuals (horizontal transmission). Direct contact includes touching or sexual relations (see Figure I.4).
- The contact between the individuals is homogeneous and the population is well mixed.
- The infected individuals are also infectious.
- Recovered individuals acquire permanent immunity.

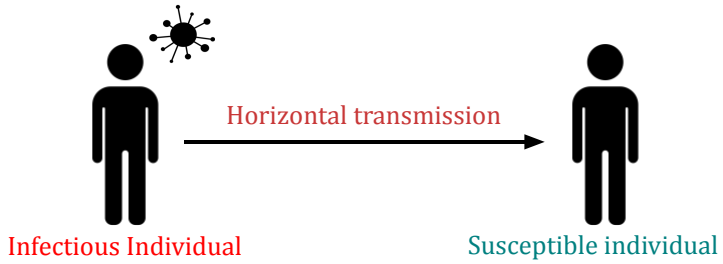


Figure I.4: Illustration of the horizontal transmission due to a potential direct contact between individuals.

To establish the differential equations of Kermack-McKendrick epidemic model, we determine the rate of change in each class. When a susceptible person enters into direct contact with an infectious one, that susceptible individual becomes infected and switched from the susceptible class into the infected one. The number of persons who become infected per unit of time is called *incidence*, and the change rate of the susceptible compartment is given by

$$\dot{S}(t) = -\text{incidence ratio}.$$

To present the incidence ratio expression, we consider that p is the probability that contact with a susceptible individual will lead to the infection and c the per capita contact rate. We define the function $F(t) = \beta I(t)$ as a force of infection, where $\beta = pc$ is the transmission rate. Then, the number of people who become infected per unit of time is equal to $F(t)S$. This number called the *mass action incidence* which is the simplest form of the various incidence expressions. With the use of this incidence rate style, we get the following differential equation for susceptible individuals: $\dot{S}(t) = -\beta S(t)I(t)$. After being infected, the individuals move to class $I(t)$. In addition, recovered or dead individuals leave the infected compartment with a constant recovery rate γ. Then, the differential equation for infected individuals is taking the following form: $\dot{I}(t) = \beta S(t)I(t) - \gamma I(t)$. People who leave the infected class enter the removed class. Then the third differential equation can be expressed as follows: $\dot{R}(t) = \gamma I(t)$. Finally, the total model is depicted in the diagram in Figure I.5 with the corresponding system of differential equations:

$$\begin{cases} \mathrm{d}S(t) = -\beta S(t)I(t)\mathrm{d}t, \\ \mathrm{d}I(t) = \big(\beta I(t)S(t) - \gamma I(t)\big)\mathrm{d}t, \\ \mathrm{d}R(t) = \gamma I(t)\mathrm{d}t. \end{cases} \tag{I.1}$$

To be well defined mathematically, this epidemic model is fitted with the following given initial values: $S(0) = S_0 > 0$, $I(0) = I_0 > 0$, and $R(0) = R_0 \geq 0$.

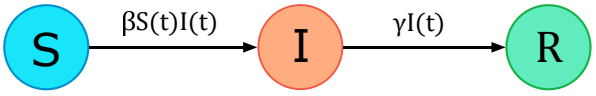


Figure I.5: Transfer diagram of the Kermack-McKendrick epidemic model.

In general, a differential equation model such as model (I.1) is mathematically well posed if, from any initial value, there is a unique global-in-time solution. Since the variables $S(t)$, $I(t)$ and

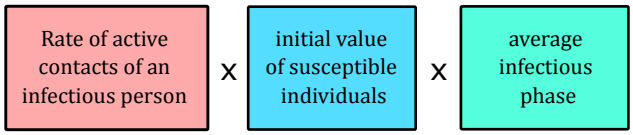
$R(t)$ denote the number of people, they are assumed to take positive values. From biological and mathematical considerations, it is necessary to prove that starting from positive initial conditions implies that the solution remains positive at all times. That is to say that the following biologically feasible region:

$$\mathbb{R}^3_+ = \Big\{(S,I,R) \in \mathbb{R}^3_+ | S > 0, I > 0, R > 0\Big\},$$

is positively invariant with respect to model (I.1). The deterministic epidemic model (I.1) called an SIR epidemic system (or Kermack-McKendrick model) and it is applied to model many diseases such as *measles*, *mumps*, *rubella*, and *pertussis*. In this model, the total population size $N(t) = S(t) + I(t) + R(t)$ remains constant since $\dot{N}(t) = 0$. From the first equation, we have $\dot{S}(t) \leq 0$. Then, $S(t)$ is decreasing and $S(t) \leq S_0$. In view of the second equation of (I.1), we get $\dot{I}(t) = \big(\beta S(t) - \gamma\big) I(t)$. Then, we can establish the following two cases:

- If $S_0 < \frac{\gamma}{\beta}$, we have $S(t) < S_0 < \frac{\gamma}{\beta}$, then $\dot{I}(t) < 0$ for all $t \geq 0$. Consequently, $I(t)$ strictly decreases and then no pandemic can happen in this state.
- If $S_0 > \frac{\gamma}{\beta}$, then $S(t) > \frac{\gamma}{\beta}$ for $t \in [0, \bar{t})$ where $\bar{t} > 0$. Consequently, $I(t)$ strictly increases for $t \in [0, \bar{t})$ and the pandemic occurs.

This shows the definition of the threshold which is the value to be exceeded for an epidemic to happen. In the same context, we consider the following expression $\beta \times S_0 \times \frac{1}{\gamma}$, which can be explained as



In epidemiology, this important term is called the basic reproduction number[a] $\mathcal{R}_0$ and is defined as the average number of secondary infections generated by a typical case of infection in a fully susceptible population during the middle infectious phase. It is used to measure the possibility of disease transmission. Comparing with the threshold phenomena mentioned above, we have the following conclusions:

- If $\mathcal{R}_0 < 1$, then the disease dies out.
- If $\mathcal{R}_0 > 1$, then the disease persists in the population.

Basically, the Kermack-McKendrick model (I.1) relies on the following assumption: there are no human demographic changes and the population is closed, i.e. no births, no deaths, and no migration to and from the host population. This assumption appears so restricted and not realistic. The demographics are one of the most significant factors and need to be included in mathematical modelling. In the next section, we consider an improved version of the system (I.1) that includes the population demographics.

4 An SIR model with vital dynamics

Epidemiological systems that do not take into account the births and deaths that occur in the population are called epidemiological systems without demography. This type of system is very useful for modeling diseases on a short time scale, especially for studying the transmission of childhood illnesses. However, there are many epidemics that have been spreading and developing

[a] In the literature, several techniques have been proposed for the calculation of $\mathcal{R}_0$ ([94], Chapter 5), but the most known is that of the next generation approach introduced by van den Driessche and Watmough in [32]. According to this method, $\mathcal{R}_0$ is the spectral radius ρ of the next generation matrix defined by $M = FV^{-1}$, where V and F are respectively the matrices expressing the infections transition and the emergence of new infected cases in the different contaminated compartments of the model.

for a long time, such as *tuberculosis*, *AIDS* and *hepatitis C*. In this state, the size of the host population can be modified, and the demographics cannot be neglected.

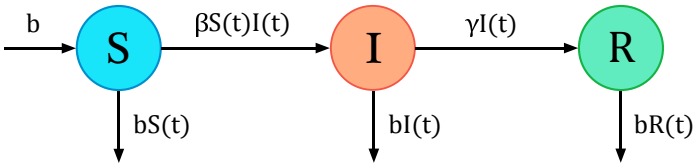


Figure I.6: Transfer diagram for an SIR epidemic model with demography.

To enrich the Kermack-McKendrick model (I.1) with the demographic assumption, we assume that the birth and death rates are the same and that there is no disease-related mortality. By denoting the demographic constant by b, the new transfer diagram is shown in Figure I.6, which leads to the following system of differential equations [100]:

$$\begin{cases} \mathrm{d}S(t) = \big(b - bS(t) - \beta S(t)I(t)\big)\mathrm{d}t, \\ \mathrm{d}I(t) = \big(\beta I(t)S(t) - (b+\gamma)I(t)\big)\mathrm{d}t, \\ \mathrm{d}R(t) = \big(\gamma I(t) - bR(t)\big)\mathrm{d}t. \end{cases} \tag{I.2}$$

Here, we notice that the size of the total population remains constant and is normalized to one. The invariant set of (I.2) is

$$\Gamma_1 = \Big\{(S,I,R) \in \mathbb{R}^3_+ |\ S + I + R = 1\Big\}.$$

Based on the explanation of the threshold phenomenon in the previous section, the basic reproduction number is expressed by $\mathcal{R}_0 = \frac{\beta}{b+\gamma}$. Since $N(t) = 1$, we can study, for example, two equations instead of three which simplifies the analysis. For this purpose, we consider the following two-dimensional system:

$$\begin{cases} \mathrm{d}S(t) = \big(b - bS(t) - \beta S(t)I(t)\big)\mathrm{d}t, \\ \mathrm{d}I(t) = \big(\beta I(t)S(t) - (b+\gamma)I(t)\big)\mathrm{d}t, \end{cases} \tag{I.3}$$

and the invariant set becomes $\Gamma_{0,1} = \{(S,I) \in \mathbb{R}^2_+ | 0 \leq S + I \leq 1\}$. In general, the long-term behaviour of the epidemic is established by analyzing the equilibriums of its model, or the states in which the solution does not change over time. There are two possible equilibriums for the model (I.3) which are obtained by setting $\dot{S}(t) = \dot{I}(t) = 0$ and getting the following equations:

$$b - bS(t) - \beta S(t)I(t) = 0 \quad \text{and} \quad \beta S(t)I(t) - (b+\gamma)I(t) = 0.$$

By solving this simple system, we found the following two possible states:

- Disease-free equilibrium $E^{\ominus} = (1,0)$ which is defined as the point at which no disease is present.
- Endemic equilibrium $E^{\circledast} = (S^{\circledast}, I^{\circledast})$, where $S^{\circledast} = \frac{b+\gamma}{\beta}$ and $I^{\circledast} = \frac{b[\beta-(b+\gamma)]}{\beta(b+\gamma)}$. In this case, the epidemic cannot be completely eliminated but remains in a population.

More precisely, we say that the model (I.3) has two possible cases:

- If $\mathcal{R}_0 \leq 1$, then $E^{\ominus} = (1,0)$ is the only equilibrium in $\Gamma_{0,1}$.
- If $\mathcal{R}_0 > 1$, then both $E^{\ominus}$ and the endemic equilibrium $E^{\circledast}$ exist in $\Gamma_{0,1}$.

The deterministic model (I.3) can be further improved by taking into account the movement of immigration, different demography rates, which makes the size of the total population variable [136]. Under these assumptions, we obtain a modified SIR epidemic model which is suitable for modelling many diseases such as *Morbilli* disease (*measles*) and *HBV*. We present this model in the next section.

5 An SIR model with varying total population size

To integrate the different demographic states and the movement of individuals into the SIR epidemic model, we assume the following additional assumptions:

- There is a recruitment rate $A > 0$ corresponding to natural births and immigration.
- All people are born sensitively to the infection.
- There is a disease-related mortality rate ($\alpha > 0$) in addition to the natural rate $\mu \equiv \mu_1 > 0$, where $\mu_2 = \mu_1 + \alpha$ is the general mortality rate.

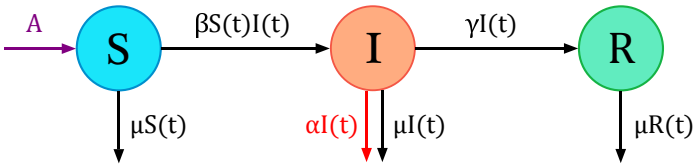


Figure I.7: Transfer diagram for an SIR model with varying total population and disease-caused death.

Under these assumptions, the SIR epidemic model with variable demography and non-constant population size is depicted in Figure I.7 with the following corresponding system of differential equations [94]:

$$\begin{cases} \mathrm{d}S(t) = \big(A - \mu S(t) - \beta S(t)I(t)\big)\mathrm{d}t, \\ \mathrm{d}I(t) = \big(\beta I(t)S(t) - (\mu + \alpha + \gamma)I(t)\big)\mathrm{d}t, \\ \mathrm{d}R(t) = \big(\gamma I(t) - \mu R(t)\big)\mathrm{d}t. \end{cases} \tag{I.4}$$

Conceptually, the *stability* of an equilibrium $x^\star$ means that solutions starting sufficiently close (nearby) to $x^\star$ remains close to the equilibrium $x^\star$ for all $t \geq 0$. An equilibrium is said to be *asymptotically stable* if it is stable and if every solution with an initial value sufficiently close to $x^\star$ approaches $x^\star$ as $t \to \infty$. That is to say, solutions that start close enough not only remain close enough but also eventually converge to the equilibrium. Therefore, asymptotic stability is a stronger condition than simple stability because it requires that trajectories satisfy more restrictive conditions. Moreover, the equilibrium $x^\star$ is said to be *globally asymptotically stable* if it is stable and if the solutions for all initial values approach $x^\star$ as $t \to \infty$. Otherwise, the equilibrium $x^\star$ that is not stable is said to be unstable. *Instability* of the equilibrium $x^\star$ means that there are solutions starting arbitrarily close to $x^\star$ that do not approach it.

When studying the asymptotic behavior of epidemics, the main issue is to analyze the steady states (equilibriums) of the model and their stability, which can mainly be determined by the basic reproduction ratio. In (I.4), the basic reproduction number is expressed by $\mathcal{R}_0 = \frac{\beta A}{\mu(\mu+\alpha+\gamma)}$ which is the threshold for an epidemic to persist or disappear. This model always has a disease-free equilibrium $E^\ominus = \left(\frac{A}{\mu}, 0, 0\right)$. When $\mathcal{R}_0 \leq 1$, the equilibrium $E^\ominus$ is globally asymptotically stable in

$$\Gamma_{SIR} = \left\{(S, I, R) \in \mathbb{R}^3_+ |\ S + I + R \leq \frac{A}{\mu}\right\}.$$

Therefore, the disease will extinct after some period of time. When $\mathcal{R}_0 > 1$, the equilibrium $E^\ominus$ is unstable and there is an endemic equilibrium $E^\circledast = \left(\frac{\mu+\alpha+\gamma}{\beta}, \frac{\mu}{\beta}(\mathcal{R}_0 - 1), \frac{\gamma}{\beta}(\mathcal{R}_0 - 1)\right)$ of the system (I.4) which is globally asymptotically stable; this means that the disease always remains in a population.

In the above-presented SIR model, it is supposed that the immunity received by recovery from the disease is permanent. This is not always right, as there may be a gradual lack of immunity with time. For example, the case of *seasonal influenza*. Temporary immunity can be described by an SIR epidemic model in which a transfer rate from R to S is added. We present this epidemic model in the next section.

6 An SIR model with temporary immunity

The SIR epidemic model can be extended to the case of provisional immunity, where members of the removed compartment may lack immunity over time and return to the susceptible class. This extension is called the SIRS epidemic model. By denoting the rate of the loss of immunity by k, the SIRS model can be depicted in Figure I.8 with the following corresponding system of differential equations [69]:

$$\begin{cases} \mathrm{d}S(t) = \big(A - \mu S(t) - \beta S(t)I(t) + kR(t)\big)\mathrm{d}t, \\ \mathrm{d}I(t) = \big(\beta I(t)S(t) - (\mu + \alpha + \gamma)I(t)\big)\mathrm{d}t, \\ \mathrm{d}R(t) = \big(\gamma I(t) - (\mu + k)R(t)\big)\mathrm{d}t. \end{cases} \tag{I.5}$$

Generally, for epidemic systems, there are different techniques for proving the global stability of equilibria. One of the most commonly used methods is the Lyapunov[a] function which is a scalar function that can be used to analyze the stability of an equilibrium point. We briefly state Lyapunov's theorem for the global stability of the equilibrium $x^\star$ (for detailed proof of Lyapunov's theorem see Theorem 4.2 of [52]).

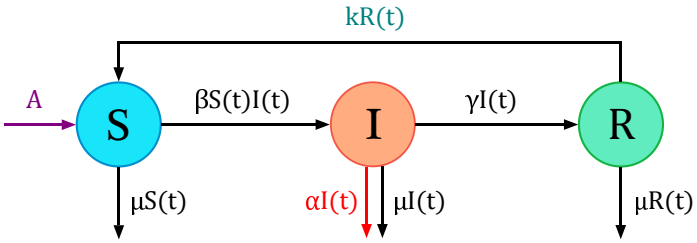


Figure I.8: Transfer diagram for an SIRS epidemic model with varying total population and disease-caused death.

Theorem 6.1 (**Lyapunov's Stability Theorem**). *If a function $\mathcal{V}(x)$ is globally positively definite*[b] *and radially unbounded*[c]*, and its time derivative is globally negative, $\dot{\mathcal{V}}(x) < 0$ for all $x \neq x^\star$, then the equilibrium $x^\star$ is globally asymptotically stable.*

If a function $\mathcal{V}(x)$ exists that verifies the conditions of the last theorem, then this function is called a *Lyapunov* stable. In Theorem 6.1, it is necessary to verify that the first derivative of the Lyapunov function is strictly negative. However, we can show only the non positivity due to the next theorem (more details, see corallary 4.2 of [52]).

Theorem 6.2 (**Krasovkii-LaSalle Theorem**). *Let $x^\star$ be an equilibrium of the autonomous system $\dot{x} = f(x)$. Assume that exists a continuously differentiable function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ such that*

- *$\mathcal{V}$ is positive definite on the entire space,*
- *$\mathcal{V}$ is radially unbounded,*
- *$\dot{\mathcal{V}}(x) \leq 0$ for all $x \in \mathbb{R}^n$.*

If the invariant set $\Gamma_V = \{x \in \mathbb{R}^n | \ \dot{V}(x) = 0\}$ contains only the equilibrium $x^\star$, then $x^\star$ is globally asymptotically stable.

It is very important to mention that there is no exact method to establish a Lyapunov function which is often difficult and complicated in terms of calculation. However, if a Lyapunov function

[a] Named after the Russian mathematician Aleksandr Mikhailovich Lyapunov.

[b] Let $\mathcal{V}$ be a continuous scalar function, that is, $\mathcal{V} : \mathbb{R}^n \mapsto \mathbb{R}$. The function $\mathcal{V}$ is termed positive definite on the entire space (global) if $\mathcal{V}(x^\star) = 0$ and $\mathcal{V}(x) > 0$ for $x \neq x^\star$.

[c] A scalar function $\mathcal{V}(x)$ such that $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ is termed radially unboundedly if $\mathcal{V}(x) \to \infty$ if $||x|| \to \infty$.

is found, it can directly prove the global stability of the equilibrium. For the model (I.5), we have that

$$\mathrm{d}N(t) = \big(A - \mu N(t) - \alpha I(t)\big)\mathrm{d}t.$$

Hence, the total population size N may vary in time and $\limsup_{t\to\infty} N(t) \leq \frac{A}{\mu}$. Consequently, it is sufficient to study the solutions in the positive invariant set

$$\Gamma_{SIRS} = \left\{(S, I, R) \in \mathbb{R}^3_+ |\ S + I + R \leq \frac{A}{\mu}\right\}.$$

In our case, the basic reproduction number of the model is $\mathcal{R}_0 = \frac{\beta A}{\mu(\mu+\alpha+\gamma)}$. According to the study presented in [69], the model (I.5) has the disease-free equilibrium $E^{\ominus} = \left(\frac{A}{\mu}, 0, 0\right)$ and the unique endemic equilibrium $E^{\circledast} = (S^{\circledast}, I^{\circledast}, R^{\circledast})$, where

$$\begin{aligned} S^{\circledast} &= \frac{A}{\mu\mathcal{R}_0}, \\ I^{\circledast} &= \frac{\mu(\mu+k)(\mu+\alpha+\gamma)(\mathcal{R}_0-1)}{\beta(\gamma\mu+(\mu+k)(\mu+\alpha))}, \\ R^{\circledast} &= \frac{\mu\gamma(\mu+\alpha+\gamma)(\mathcal{R}_0-1)}{\beta(\gamma\mu+(\mu+k)(\mu+\alpha))}. \end{aligned}$$

By using the following Lyapunov function

$$\mathcal{V}(S, I, R) = 0.5\left(\left(S - \frac{A}{\mu}\right) + I + R\right)^2 + \frac{(\alpha+2\mu)}{\beta} I + \frac{(\alpha+2\mu)}{2\gamma} R^2,$$

the authors proved that if $\mathcal{R}_0 \leq 1$, the disease-free equilibrium $E^{\ominus}$ of (I.5) is globally asymptotically stable in Γ_{SIRS}. When $\mathcal{R}_0 > 1$, they demonstrated that $E^{\circledast}$ is globally asymptotically stable in the interior of Γ_{SIRS} by employing the following Lyapunov function

$$\begin{aligned} \mathcal{V}(S, I, R) = 0.5&\Big((S - S^{\circledast}) + (I - I^{\circledast}) + (R - R^{\circledast})\Big)^2 \\ &+ \frac{(\alpha+2\mu)}{\beta}\left(I - I^{\circledast} - I^{\circledast}\ln\frac{I}{I^{\circledast}}\right) + \frac{(\alpha+2\mu)}{2\gamma}(R - R^{\circledast})^2. \end{aligned}$$

In the above Kermack - McKendrick model and its improved versions, the mechanism of infection is supposed to occur through direct contact between infectious and susceptible individuals. This is often called horizontal transmission. Another type of the spread is the vertical transmission, in which the infection is passed to a newborn directly from an infected mother (see Figure I.9). Examples of diseases that can be transmitted vertically include *HIV/AIDS*, *Chagas disease*, *syphilis*, *rubella*, and *herpes simplex virus*. Therefore, the consideration of dual transmission (horizontal and vertical) is very necessary in some cases. The SIRS epidemic model that considering the said hypothesis is presented in the following section.

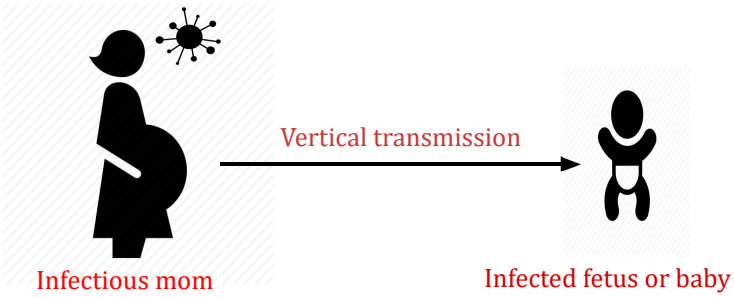


Figure I.9: Illustration of the vertical transmission from an infected mother to a newborn.

7 An SIRS model with vertical transmission

To model the vertical transmission mechanism of an epidemic, we assume that a fraction q of newborns in the infected population becomes infected at birth, and the remaining fraction $p = 1 - q$ is susceptible. In this case, we define the following notations:

- $b \times N$ is the complete number of newborns with a natural birth rate b.
- $q \times b \times I$ is the number of newborns infected at birth.
- $b \times N - q \times b \times I$ is the number of healthy but susceptible newborns.

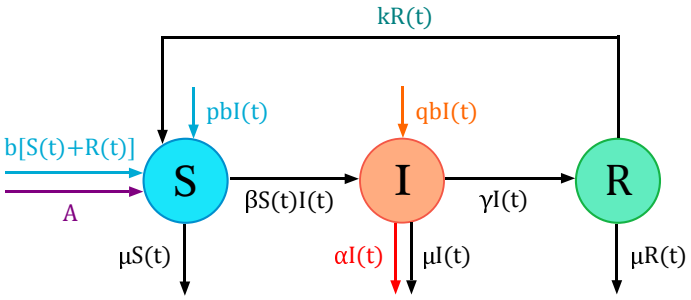


Figure I.10: Transfer diagram for an SIRS epidemic model with vertical dissemination.

We can consider the demography and immigration motion by adding the positive rate A of external healthy persons entering the susceptible population and the mortality rate μ. In Figure I.10, we illustrate the spread process of an epidemic with horizontal and vertical dissemination. Then, the resulting system is described by the following system of ordinary differential equations [173]:

$$\begin{cases} \mathrm{d}S(t) = \big(A + b\big(S(t) + R(t)\big) + pbI(t) - \mu S(t) - \beta S(t)I(t) + kR(t)\big)\mathrm{d}t, \\ \mathrm{d}I(t) = \big(\beta I(t)S(t) + qbI(t) - (\mu + \alpha + \gamma)I(t)\big)\mathrm{d}t, \\ \mathrm{d}R(t) = \big(\gamma I(t) - (\mu + k)R(t)\big)\mathrm{d}t. \end{cases} \tag{I.6}$$

In this model, the basic reproduction number is expressed by $\mathcal{R}_0 = \frac{\beta A}{(\mu - b)(\mu_2 + \gamma - qb)}$, where $\mu > b$ and $\mu_2 = \mu + \alpha$. As mentioned earlier, $\mathcal{R}_0$ is the threshold that determines whether the disease will die out or persist. When $\mathcal{R}_0 \leq 1$, there always exists a disease-free equilibrium $E^{\ominus} = \left(\frac{A}{\mu - b}, 0, 0\right)$ which is globally asymptotically stable. If $\mathcal{R}_0 > 1$, $E^{\ominus}$ is unstable and there is an endemic equilibrium $E^{\circledast} = (S^{\circledast}, I^{\circledast}, R^{\circledast})$ where

$$\begin{aligned} S^{\circledast} &= \frac{\mu_2 + \gamma - qb}{\beta}, \\ I^{\circledast} &= \frac{(\mu + k)(\mu - b)(\mu_2 + \gamma - qb)(\mathcal{R}_0 - 1)}{\beta((\mu - b)\gamma + (\mu + k)(\mu_2 - qb))}, \\ R^{\circledast} &= \frac{\gamma I^{\circledast}}{(\mu + k)}, \end{aligned}$$

which is globally asymptotically stable under a sufficient condition.

The SIRS model (I.6) integrates temporal immunity and vertical transmission is widely used to model many epidemics. But, for some diseases with bacterial agents such as *meningitis*, *plague*, and *venereal diseases*, the cure cannot generate immunity for a long time. Infected people can recover after certain treatments and return directly to the susceptible class due to the transient antibody [73]. By taking this new hypothesis into account in the model (I.6), we obtain a general SIRS model that considers different periods of immunity. We present this epidemic model in the following section.

8 An SIRS model with vertical transmission and various period of immunity

To approach the reality, the basic SIRS epidemic model will be enriched with the hypotheses of vertical transmission and transfer from infected to susceptible individuals. We assume that the recovery cannot produce immunity for a long time and the infected individuals may recover after some treatments and therapies and then go back directly to the susceptible compartment with a positive rate γ_1 [127]. Considering this new hypothesis, in Figure I.11, we give the transfer diagram for the general version of SIRS epidemic model. Thus, we get the following system of ODEs [58]:

$$\begin{cases} \mathrm{d}S(t) = \big(A + b\big(S(t)+I(t)\big) + pbI(t) - \mu S(t) - \beta S(t)I(t) + \gamma_1 I(t) + kR(t)\big)\mathrm{d}t, \\ \mathrm{d}I(t) = \big(\beta I(t)S(t) + qbI(t) - (\mu+\alpha+\gamma_1+\gamma_2)I(t)\big)\mathrm{d}t, \\ \mathrm{d}R(t) = \big(\gamma_2 I(t) - (\mu+k)R(t)\big)\mathrm{d}t. \end{cases} \tag{I.7}$$

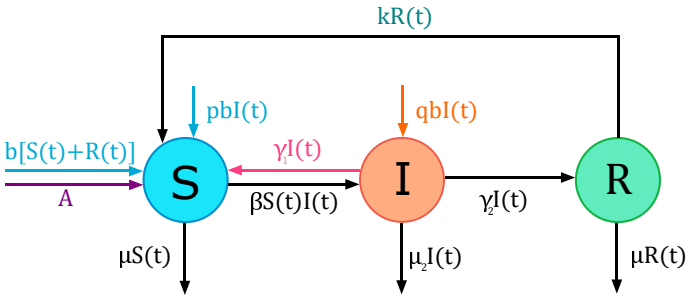


Figure I.11: Transfer diagram for an SIRS epidemic model with vertical transmission and various period of immunity.

According to the theory in [173, 73], the basic reproduction number of system (I.7) is

$$\mathcal{R}_0 = \frac{\beta A}{(\mu - b)(\mu_2 + \gamma_1 + \gamma_2 - qb)}.$$

If $\mathcal{R}_0 \leq 1$, the deterministic model (I.7) has only the disease-free equilibrium $E^{\ominus} = \left(\frac{A}{\mu - b}, 0, 0\right)$, which is asymptotically stable in Γ_{SIRS}. When $\mathcal{R}_0 > 1$, $E^{\ominus}$ becomes unstable and there exists a globally asymptotically stable endemic equilibrium $E^{\circledast} = (S^{\circledast}, I^{\circledast}, R^{\circledast})$ such that

$$\begin{aligned} S^{\circledast} &= \frac{(\mu_2 + \gamma_1 + \gamma_2 - qb)}{\beta}, \\ I^{\circledast} &= \frac{(\mathcal{R}_0 - 1)(\mu + k)(\mu - b)(\mu_2 + \gamma_1 + \gamma_2 - qb)}{\beta(\gamma_2(\mu - b) + (\mu + k)(\mu_2 - qb))}, \\ R^{\circledast} &= \frac{\gamma_2 I^*}{\mu + k}. \end{aligned}$$

As mentioned before, the incidence rate is the number of new infected situations per population in a given time phase. In many previous epidemic models, the bilinear incidence rate is frequently used (see e.g. [3, 71, 72]). However, there exist many forms of nonlinear incidence rate and each form presents some advantages as the following examples:

- $f(S,I) = \left(\beta - \frac{\beta_1 I}{v+I} SI\right)$, $(\beta > \beta_1,\ v > 0)$ [90];
- $f(S,I) = \frac{\beta SI}{1+kI^p}$, $(k > 0,\ 0 \leq p \leq 1)$ [68];
- $f(S,I) = \frac{\beta SI}{1+k_1 I+k_2 S}$, $(k_1 > 0,\ k_2 \geq 0)$ [50];
- $f(S,I) = \frac{\beta S^u I}{1+S^v}$, $(u \geq v)$ [138].

There have been many mathematical models [64, 65, 66, 89, 117] committed to studying the impacts of nonlinear transmission on the propagation of a disease. As a modified form, we can consider the following general functional response:

$$f(S,I) = \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I+\alpha_3 SI}, \tag{I.8}$$

where $\alpha_1,\ \alpha_2,\ \alpha_3 \geq 0$. It is necessary to mention that $1+\alpha_1 S+\alpha_2 I+\alpha_3 SI$ is a general form which represents mutual interference between S and I. In particular cases:

- when $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $f(S,I)$ becomes a bilinear mass-action function response (Namely, type **I** Holling functional response) [90];
- when $\alpha_2 = \alpha_3 = 0$, $f(S,I)$ becomes a saturated incidence rate (Namely, Holling type **II** functional response) [68];
- when $\alpha_3 = 0$, $f(S,I)$ becomes a Beddington-DeAnglis functional response (Namely, modified type **II** Holling functional response) [50];
- when $\alpha_3 = \alpha_1\alpha_2$, $f(S,I)$ becomes a Crowley-Martin functional response introduced in [138].

By considering the incidence (I.8) into the model (I.7), we obtain the following general SIRS epdiemic model:

$$\begin{cases} \mathrm{d}S(t) = \Big(A + b(S(t)+R(t)) - \frac{\beta S(t)I(t)}{\psi(S,I)} - \mu S(t) + pbI(t) + \gamma_1 I(t) + kR(t)\Big)\mathrm{d}t, \\ \mathrm{d}I(t) = \Big(\frac{\beta S(t)I(t)}{\psi(S,I)} + qbI(t) - (\mu_2+\gamma_1+\gamma_2)I(t)\Big)\mathrm{d}t, \\ \mathrm{d}R(t) = \big(\gamma_2 I(t) - (\mu+k)R(t)\big)\mathrm{d}t, \end{cases} \tag{I.9}$$

where $\psi(S,I) = 1+\alpha_1 S+\alpha_2 I+\alpha_3 SI$. The basic reproduction number of system (I.9) can be presented as follows:

$$\mathcal{R}_0 = \frac{\beta A}{\big((\mu-b)+\alpha_1 A\big)(\mu_2+\gamma_1+\gamma_2-qb)}.$$

As a real example, Hepatitis B remained enormous defiance and a great global health issue caused by the Hepatitis B virus (HBV) [134]. Chronic HBV can be transmitted by sexual contact, through the touch, by impregnation with polluted blood, or by the direct transmission of Hepatitis B from the mother to a fetus during pregnancy (vertical transmission) [174]. In the next section, we use the SIR epidemic system to model the spread of the said virus under the vaccination strategy.

9 The Hepatitis B epidemic model with successful vaccination

According to the recent statistics of the world health organization (WHO) [143], about 350 million people worldwide have been infected and carrying HBV. This serious infection is responsible for approximately 600,000 deaths each year [102]. Because of the high severity of HBV infection and the large number of deaths associated with it, it is compulsory to improve our control of this virus. Mathematical models are a vigorous tool to simulate and control the spread of the HBV infection. There exist many previous interesting works committed to studying Hepatitis B transmission. For example, Anderson and May [5] analyzed a straightforward mathematical model for illustrating the role of carrier individuals on the spread of HBV. In [106, 55], the authors developed the impact of vaccination and other controlling measures of HBV outbreak. They showed that the booster vaccine of Hepatitis B is very necessary and useful. By denoting the successful vaccination rate by θ, Khan et al. [56] formulated the characteristics of HBV disease transmission and proposed the following deterministic SIR model:

$$\begin{cases} \mathrm{d}S(t) = \big(A - \beta S(t)I(t) - (\mu+\theta)S(t)\big)\mathrm{d}t, \\ \mathrm{d}I(t) = \big(\beta S(t)I(t) - (\mu+r+\delta)I(t)\big)\mathrm{d}t, \\ \mathrm{d}R(t) = \big(\delta I(t) + \theta S(t) - \mu R(t)\big)\mathrm{d}t. \end{cases} \tag{I.10}$$

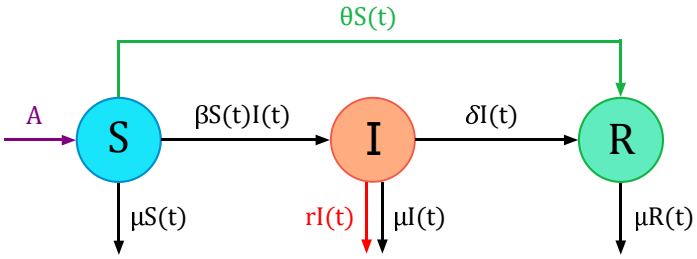


Figure I.12: Transfer diagram of Hepatitis B epidemic model.

The following region $\Gamma_{HBV} = \left\{(S,I,R) \in \mathbb{R}^3_+ : \ S+I+R \leq \frac{A}{\mu}\right\}$ is almost surely positively invariant by the system (I.10). If $\mathcal{R}_0 = \frac{\beta A}{(\mu+\theta)(\mu+r+\delta)} \leq 1$, then for any initial condition $(S(0),I(0),R(0)) \in \Gamma_{HBV}$ the disease-free equilibrium $E^{\ominus} = E^{\ominus}(S^0,0,R^0) = \left(\frac{A}{\mu+\theta},0,\frac{A\theta}{\mu(\mu+\theta)}\right)$ is globally asymptotically stable. If $\mathcal{R}_0 > 1$, then there exists a globally asymptotically stable the infected endemic equilibrium $E^{\circledast} = (S^{\circledast},I^{\circledast},R^{\circledast})$ such that

$$S^{\circledast} = \frac{\mu+r+\delta}{\beta}, \quad I^{\circledast} = (\mu+\theta)(\mathcal{R}_0-1) \ \text{ and } \ R^{\circledast} = \frac{\theta(\mu+r+\delta)+\beta\delta(\mu+\theta)(\mathcal{R}_0-1)}{\beta\mu}.$$

One intrusion method to mitigate the spread of infectious diseases is to quarantine some infections, in order to control the dissemination of the infection to susceptibles. The word quarantine originally corresponded to a period of forty days, which represent the length of time that arriving ships suspected of plague infection were constrained from intercourse with the shore in Mediterranean ports in the 15-19th centuries [41]. The word quarantine has evolved to mean forced isolation or stoppage of interactions with others. Over many centuries, quarantine has been implemented to reduce the transmission of human diseases like *plague*, *cholera*, *typhus*, *yellow fever*, *smallpox*, *Leprosy*, *diphtheria*, *tuberculosis*, *measles*, *mumps*, Ebola and Covid-19. In the following two sections, we examine the effects of quarantine in two endemic models for infectious diseases.

10 An epidemic model with permanent immunity and quarantine strategy

The study of infectious diseases has long been a subject where epidemiological issues are combined with financial and social problems. The rapid spread of COVID-19 these days shows that humanity stills suffer from epidemics that may lead to the collapse of medical and economic systems. By isolating infected individuals and quarantining the susceptible population at home, many countries have basically controlled the outbreak of COVID-19 [133, 54]. In order to analyze the impact of this strategy on the spread of epidemics and to predict their future behavior, we use different mathematical formulations according to their characteristics [118, 129]. In this section, we consider an SIQR epidemic model in the form of ordinary differential equations. These ODEs describe the evolution of susceptible $S(t)$, infected $I(t)$, isolated $Q(t)$ and recovered $R(t)$ individuals as time functions. The rates of change and the interactions between different population classes in our case are expressed by the following deterministic model [41]:

$$\begin{cases} \mathrm{d}S(t) = \big(A - \mu S(t) - \beta S(t)I(t)\big)\mathrm{d}t, \\ \mathrm{d}I(t) = \big(\beta S(t)I(t) - (\mu+\gamma+\delta+r_2)I(t)\big)\mathrm{d}t, \\ \mathrm{d}Q(t) = \big(\delta I(t) - (\mu+k+r_3)Q(t)\big)\mathrm{d}t, \\ \mathrm{d}R(t) = \big(\gamma I(t) + kQ(t) - \mu R(t)\big)\mathrm{d}t, \end{cases} \tag{I.11}$$

where $S(t)$, $I(t)$, $Q(t)$ and $R(t)$ denote the numbers of susceptible, infected but not quarantined, quarantined infected, and removed individuals at time t, respectively. The schematic flow diagram of the model (I.11) is illustrated in Figure I.13. The parameters can be summarized in the following list:

- A: the influx of people into the susceptible person compartment.
- μ: the natural death rate of the compartments S, I, Q and R.
- β: the transmission coefficient from compartment S to compartment I.
- γ: the recovery rate of infectious individuals.
- δ: is the isolation rate from I to Q.
- k: the recovery rate of isolated individuals.
- r_2: the disease-linked death rate of infectious individuals.
- r_3: the disease-linked death rate of isolated individuals.

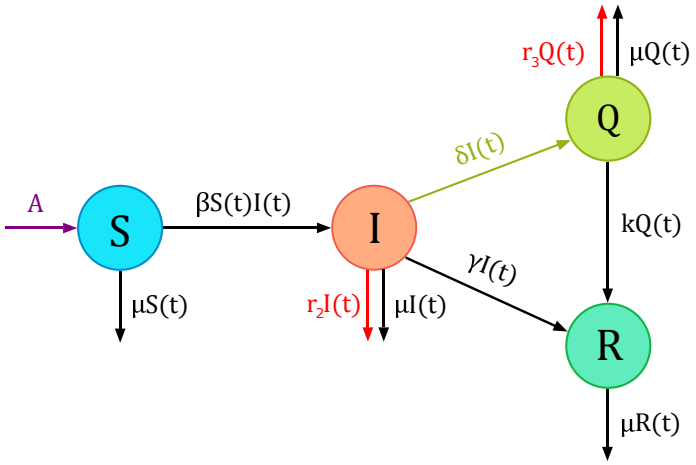


Figure I.13: Transfer diagram of SIQR epidemic model.

Consistently, we assume that all parameters are positive constants. In this model, the total population $N(t)$ of the model varies, because vulnerable parts of the population can be received through birth or immigrants, and individuals will die of natural and epidemic deaths. As in the previous epidemic models, the differential equation for N implies that solution (I.11) starting in the positive orthant $\mathbb{R}^4_+$ remain in the subset

$$\Gamma_{\text{SIQR}} = \left\{(S, I, Q, R) \in \mathbb{R}^+_4 : S + I + Q + R \leq \frac{A}{\mu}\right\}.$$

Then, it suffices to consider the solutions in the region Γ_{SIQR}. As before, the initial value problem is well-posed both mathematically and biologically in Γ_{SIQR}. The quarantine reproduction number of system (I.11) is $\mathcal{R}_0 = \frac{\beta A}{\mu(\mu+\delta+\gamma+r_2)}$. The deterministic SIQR model (I.11) always has the disease-free equilibrium $E^{\ominus}(S_0, 0, 0, 0) = (\frac{A}{\mu}, 0, 0, 0)$. If $\mathcal{R}_0 > 1$, then Γ_{SIQR} contains a unique positive endemic equilibrium $E^{\circledast} = (S^{\circledast}, I^{\circledast}, Q^{\circledast}, R^{\circledast})$, where

$$(S^{\circledast}, I^{\circledast}, Q^{\circledast}, R^{\circledast}) = \left(\frac{A}{\mu R_0}, \frac{\mu(R_0 - 1)}{\beta}, \frac{\delta I^*}{k + \mu + r_3}, \left(\gamma + \frac{k\delta}{k + \mu + r_3}\right)\frac{R_0 - 1}{\beta}\right).$$

The following two results describe the behavior of solutions of (I.11) in the invariant set Γ_{SIQR}.

- If $\mathcal{R}_0 \leq 1$, then Γ_{SIQR} is an asymptotic stability region for the disease-free equilibrium $E^{\ominus}$.
- If $\mathcal{R}_0 > 1$, then the region $\Gamma_{\text{SIQR}} - \{(S, I, Q, R) |\ I = 0\}$ is an asymptotic stability region for the endemic equilibrium $E^{\circledast}$.

11 An epidemic model with no full immunity and quarantine strategy

Lately, due to the passive effect of epidemics on population growth, controlling the dynamic behavior of these diseases and predicting what will happen has become a remarkable research topic. Hence, the establishment of mathematical formulations has become an important method to study the properties of epidemics. For more contagious diseases such as smallpox, measles, plague, mumps, and Ebola, the most efficient methods of intervention are to quarantine those who have already been infected, in order to reduce transmissions to susceptible individuals [120]. Moreover, in most of the previously mentioned epidemics, the infection does not grant immunity, and susceptible individuals become infected, and then some infected ones remain in class I throughout the period of infection before returning to the susceptible class, while other infected individuals are transferred to the quarantine class Q and they stay there until they are no longer contagious, at which time they return to S [120, 119, 121]. This process is modeled by an epidemic system called an SIQS model since one typical pathway is through S, then I, then Q, and then back to S, as shown in Figure I.14.

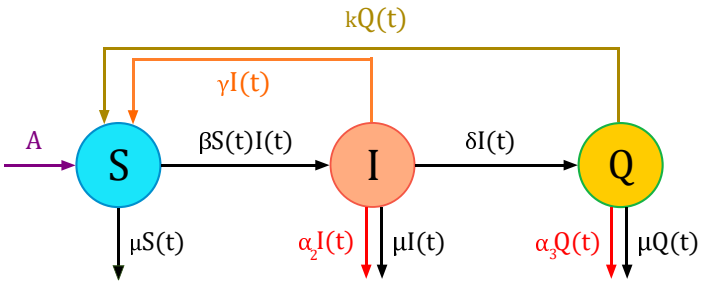


Figure I.14: Transfer diagram of SIS epidemic model with isolation.

The differential equations for this SIQS model are [153]:

$$\begin{cases} dS(t) = \big(A - \mu_1 S(t) - \beta S(t)I(t) + \gamma I(t) + kQ(t)\big)dt, \\ dI(t) = \big(\beta I(t)S(t) - (\mu_1 + \alpha_2 + \delta + \gamma)I(t)\big)dt, \\ dQ(t) = \big(\delta I(t) - (\mu_1 + \alpha_3 + k)Q(t)\big)dt, \end{cases} \tag{I.12}$$

where the parameters appearing in this system are described as follow:

- A is the recruitment rate of the susceptible individuals corresponding to new births.
- μ_1 is the natural death rate.
- δ is the isolation rate.
- r_2 is the disease-related mortality rate.
- r_3 is the death rate associated with the disease under isolation intervention. For simplicity, we denote $\mu_2 = \mu_1 + \alpha_2$ and $\mu_3 = \mu_1 + \alpha_3$ as a general mortality rates.
- γ and k are the rates which individuals recover and return to S from I and Q, respectively.
- β represents the transmission rate.

All parameters are usually assumed to be positive. The general transfer diagram for the model (I.12) is illustrated in Figure I.14. Herbert et al. [41] proved that the basic reproduction number of the deterministic model (I.12) is expressed by $\mathcal{R}_0 = \frac{\beta A}{\mu_1(\mu_2+\delta+\gamma)}$. This parameter is an essential quantity to predict whether a disease will persist or not. If $\mathcal{R}_0 \leq 1$, the model (I.12) has only the disease-free equilibrium $E^{\ominus} = (A/\mu_1, 0, 0)$ which is globally asymptotically stable, and if $\mathcal{R}_0 > 1$, $E^{\ominus}$ becomes unstable and there exists a global asymptotically stable endemic equilibrium $E^{\circledast} = (S^{\circledast}, I^{\circledast}, Q^{\circledast})$, where

$$S^{\circledast} = \frac{A}{\mu_1 \mathcal{R}_0}, \quad I^{\circledast} = \frac{A(1 - 1/\mathcal{R}_0)}{\mu_2(1 + \delta/(\mu_3 + k))}, \text{ and } Q^{\circledast} = \frac{\delta I^{\circledast}}{(\mu_3 + k)}.$$

Chapter II

Mathematical Preliminaries

Abstract

Modern probability has become an important tool for the formulation and the analysis of biological models which take into account the random aspect. This chapter presents some concepts of stochastic calculus that will be used in this thesis. Instead of giving a detailed theory, which can be found elsewhere (for example [93, 103, 51, 99, 104]), we have tried to collect only the definitions and results necessary to promote a full understanding of the main results.

1 Stochastic processes and their characteristics

The stochastic process is a mathematical object allowing to model the occurrence, at each time after a given initial moment, of a random and irregular phenomenon. The randomness and obvious lack of predictability are taken into account by introducing a measurable space $(\Omega, \mathcal{F})$ on which a probability measure $\mathbb{P}$ can be placed, where Ω is a sample description space and $\mathcal{F}$ is a σ-algebra. Conceptually, a collection $\{\mathcal{X}_t\}_{t\in I}$[b] of $\mathbb{R}^n$-valued random variables is called a stochastic process with index set I and state space $\mathbb{R}^n$. That is, for each time $t \geq 0$, $\omega \mapsto \mathcal{X}_t(\omega)$ is a measurable function from Ω to $\mathbb{R}^n$. In this chapter and for the purposes of this thesis, we let the parameter set I be $\mathbb{R}_+ = [0, \infty)$. For a fixed sample point $\omega \in \Omega$, the mapping $t \mapsto \mathcal{X}_t(\omega) \in \mathbb{R}^n$; $(t \geq 0)$ is the sample trajectory of the process $\mathcal{X}$ associated with ω. By abuse of notation, we often write a stochastic process $\mathcal{X} \equiv \{\mathcal{X}_t(\omega)\}_{t\geq 0}$ as $\{\mathcal{X}_t\}$ or $\mathcal{X}(\cdot)$. We therefore present the following definitions:

Definition 1.1. Let $\mathcal{X}$ be a $\mathbb{R}^n$-valued stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This process is said to be

- *Continuous* if for almost $\omega \in \Omega$, the function $t \mapsto \mathcal{X}_t(\omega)$ is continuous on $t \geq 0$. Hence, the process $\mathcal{X}$ is continuous if for almost all its paths are continuous. Furthermore, it is right continuous (resp. left continuous) if $t \mapsto \mathcal{X}_t(\omega)$ is right continuous (resp. left continuous) for almost $\omega \in \Omega$ and every $t \geq 0$.
- *Cadlag*[c] (right continuous and left limit) if it is right continuous and for almost all $\omega \in \Omega$, the left limit $\lim_{\substack{s\to t \\ s<t}} \mathcal{X}_s(\omega)$ exists and is finite for all $t > 0$.
- *Increasing* if $\mathcal{X}_s \leq \mathcal{X}_t$ for any $s < t$. Thus, a stochastic process is increasing if all its paths are increasing.

[b] The time parameter t may be either discrete or continuous, but in this thesis we will consider continuous-time stochastic processes.

[c] The ambiguous word "cadlag" is a French acronym for "continu à droite, limite à gauche" which simply means the right-continuous with left limits.

- *Integrable* if, for all $t \geq 0$, $\mathcal{X}_t$ is an integrable random variable. Moreover, it is square-integrable if $\mathbb{E}|\mathcal{X}_t|^2 < \infty$ for every $t \geq 0$.

- *Measurable* if the stochastic process regarded as a function of two variables (t, ω) from $\mathbb{R}_+ \times \Omega$ to $\mathbb{R}^n$ is measurable with respect to the σ-algebra $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$[a], where $\mathcal{B}(\mathbb{R}_+)$ is the family of all Borel subsets of $\mathbb{R}_+$.

- *Gaussian* if all its finite-dimensional distributions[b] are normal.

For two $\mathbb{R}^n$-valued stochastic processes $\mathcal{X}$ and $\mathcal{Y}$, there exist two different concepts of equality:

Definition 1.2. $\mathcal{Y}$ is a *version* or *modification* of $\mathcal{X}$ if, for every $t \geq 0$, we have $\mathcal{X}_t = \mathcal{Y}_t$ almost surely (for abbreviation, we write a.s.). That is to say, for each time $t \geq 0$, $\mathbb{P}\big(\{\omega : \mathcal{X}_t(\omega) = \mathcal{Y}_t(\omega)\}\big) = 1$.

Definition 1.3. $\mathcal{X}$ and $\mathcal{Y}$ are said to be stochastically *equivalent* or *indistinguishable* if for almost all $\omega \in \Omega$, $\mathcal{X}_t(\omega) = \mathcal{Y}_t(\omega)$ for all $t \geq 0$. That is, $\mathbb{P}\big(\{\omega : \mathcal{X}_t(\omega) = \mathcal{Y}_t(\omega) \text{ for all } t \geq 0\}\big) = 1$. This is the same as saying that they almost surely have the same sample paths.

2 Filtrations of $\mathcal{F}$

In the theory of stochastic calculus, *filtrations* are completely ordered collections of subsets that are used to model the information which is available at a given point of time. Therefore, it play an important role in the formalization of random processes. More formally, a filtration is a family $\{\mathcal{F}_t\}_{t\geq 0}$ of increasing sub-σ-algebras of $\mathcal{F}$ (i.e, $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for all $0 \leq t < s < \infty$). To further elaborate this notion, we let $\mathcal{F}_s$ and $\mathcal{F}_t$ be two σ-algebras on $(\Omega, \mathcal{F}, \mathbb{P})$, then $\mathcal{F}_s \subset \mathcal{F}_t$ if and only if $\mathcal{F}_t$ holds more information about the process than does $\mathcal{F}_s$.

Given a stochastic process $\mathcal{X}$, the elementary choice of a filtration is its natural (canonical) version, i.e., $\mathcal{F}_t^{\mathcal{X}} \triangleq \sigma\Big\{\mathcal{X}_s,\ 0 \leq s \leq t\Big\}$. That is to say, the smallest σ-algebra with respect to which $\mathcal{X}_s$ is measurable for every $s \in [0, t]$. As an interpretation, $\mathcal{F}_t^{\mathcal{X}}$ indicates all the information that can be extracted from the observation of $\mathcal{X}$ trajectories between 0 and t. Precisely, the fact that $A \in \mathcal{F}_t^{\mathcal{X}}$ means that by time t, an observer of $\mathcal{X}$ knows whether or not A has occurred. Note that $\mathcal{F}_\infty = \sigma\big(\bigcup_{t\geq 0} \mathcal{F}_t\big)$ denotes the σ-algebra generated by $\bigcup_{t\geq 0} \mathcal{F}_t$ that encodes the available information over all time t.

A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is said to be right continuous if $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \geq 0$. Roughly speaking, the right continuity of $\{\mathcal{F}_t\}_{t\geq 0}$ means that if the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ encodes the information available as time goes by, then $\mathcal{F}_{t^+} = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}$ allows an infinitesimal vision into the future at each $t \in [0, \infty)$.

A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is called a filtered probability space and expressed by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. From now on, in this thesis, we shall always work on a given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ such that $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null[c] sets). Of course, the completion of the filtration means that if an event is impossible, this impossibility is already known on the date 0.

To translate mathematically the non-anticipating property of a stochastic process, we propose the following definition.

Definition 2.1. A stochastic process $\{\mathcal{X}_t\}$ is said to be *adapted*[d] with respect to the information structure $\{\mathcal{F}_t\}_{t\geq 0}$ if, for each $t \geq 0$, $\mathcal{X}_t$ is an $\{\mathcal{F}_t\}$-measurable random variable.

[a] Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. The σ-algebra for the corresponding product space $\Omega_1 \times \Omega_2$ is called the product σ-algebra and is defined by $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma(\{B_1 \times B_2 : B_1 \in \mathcal{F}_1,\ B_2 \in \mathcal{F}_1\})$.

[b] For a stochastic process $\mathcal{X}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the joint distribution of a random vector $(\mathcal{X}(t_1), \ldots, \mathcal{X}(t_p))$ where $p \geq 1$ and $0 \leq t_1 < \cdots < t_p$, is called a finite-dimensional distribution of the process $\mathcal{X}$.

[c] A set $H \in \Omega$ is said to be $\mathbb{P}$-null if there exists a set $\tilde{H} \in \mathcal{F}$ such that $H \subset \tilde{H}$ and $\mathbb{P}(\tilde{H}) = 0$.

[d] An adapted process is also called a non-anticipating process.

An $\{\mathcal{F}_t\}$-adapted process is, therefore, a process whose value at any date t is revealed by the information carried by $\mathcal{F}_t$. Analogously, the stochastic process $\mathcal{X}$ is adapted to $\{\mathcal{F}_t\}$ if, for every $t \geq 0$, $\sigma(\mathcal{X}_s,\ 0 \leq s \leq t) \subseteq \mathcal{F}_t$. In this case, for any $t \geq 0$, $\mathcal{X}_t$ is known at time t when the information loaded by $\mathcal{F}_t$ is available and known. Furthermore, every process $\mathcal{X}$ is adapted to its natural filtration $\{\mathcal{F}_t^{\mathcal{X}}\}$.

Take into account the dynamic nature of the stochastic process with respect to filtration $\{\mathcal{F}_t\}_{t\geq 0}$, the notion of progressive measurability is required. Precisely, if we have a stochastic process, the following definition is appropriate.

Definition 2.2. A stochastic process $\mathcal{X}$ is said to be $\{\mathcal{F}_t\}$-*progressively measurable* if for every $T \geq 0$, $\{\mathcal{X}(t)\}_{0\leq t\leq T}$ regarded as function of (t,ω) from $[0,T]\times\Omega$ to $\mathbb{R}^n$ is $\mathcal{B}([0,T])\otimes\mathcal{F}_T$-measurable, where $\mathcal{B}([0,T])$ is the family of all Borel subsets of $[0,T]$.

Now, we introduce the concept of a predictable process which is a stochastic process whose value is knowable at an earlier time and the information about the process behavior is left-continuous. The notion of predictability is more facilely understood in the discrete-time case where $\mathcal{X}_k$ is $\mathcal{F}_k$-predictable if $\mathcal{X}_k$ is $\mathcal{F}_{k-1}$-measurable. We mention that every predictable process is adapted, but not conversely. For the continuous-time case, we let $\mathcal{S}$ denotes the smallest σ-algebra on $\mathbb{R}_+\times\Omega$ with respect to which every left continuous adapted process is measurable, then the following definition is adequate.

Definition 2.3. A continuous-time stochastic process $\mathcal{X}$ is said to be *predictable*[a] if the process regarded as a function of (t,ω) is $\mathcal{S}$-measurable.

3 Stopping times

In the last two subsections, we have introduced the required definitions of a stochastic process and filtration. Since we oftentimes monitor the stochastic processes at a random time, an additional concept about stopping time is required. Typically, we suppose that the stopping time is random variable which whose value is interpreted as the time at which a given stochastic process offers a specific behavior. As mentioned earlier, we are working on a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$.

Definition 3.1. A random variable $\tau : \Omega \to [0,\infty]$ (it can take the value ∞) is termed an $\{\mathcal{F}_t\}$-*stopping time* if $\{\tau \leq t\} \triangleq \{\omega :\ \tau(\omega) \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.

More generally, we can consider the stopping time as the first time that an adapted stochastic process reaches a certain value. we clarify this idea in the following theorem.

Theorem 3.2 (*Hitting times*). *Let $\mathcal{X}$ be an $\mathbb{R}^n$-valued cadlag $\{\mathcal{F}_t\}$-adapted process. Let also O an open subset of $\mathbb{R}^n$ and define*

$$\tau = \inf\{t \geq 0 : \mathcal{X}_t \notin O\},$$

with the hypothesis that the $\inf\emptyset = \infty$. *Then, τ is an $\{\mathcal{F}_t\}$-stopping time. It indicates the time of the first exist from O.*

In addition, let κ denotes another $\{\mathcal{F}_t\}$-stopping time, then $\tau_\kappa = \inf\{t \geq \kappa : \mathcal{X}_t \notin O\}$ is an $\{\mathcal{F}_t\}$-stopping time which is the first time of the exit from O after κ.

We indicate that any random time equal to a non-negative constant is a stopping time. Now, we briefly present some classic properties of stopping times.

Lemma 3.3. *Let τ be a stopping time, the set*

$$\mathcal{F}_\tau = \left\{A \in \mathcal{F} :\ A \cap \{\omega :\ \tau(\omega) \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\right\} \subset \mathcal{F}$$

is actually a sub-σ-algebra of $\mathcal{F}$ and τ is $\mathcal{F}_\tau$-measurable. Furthermore, for any $\{\mathcal{F}_t\}$-stopping times τ, ϑ, and $A \in \mathcal{F}_\vartheta$, we have $A \cap \{\vartheta \leq \tau\} \in \mathcal{F}_\tau$. Particularly, if $\vartheta \leq \tau$ on Ω, we get $\mathcal{F}_\vartheta \subseteq \mathcal{F}_\tau$.

[a] One can conclude that: left-continuous + non-anticipating $\Rightarrow$ predictable.

Lemma 3.4. *Let X denote a progressively measurable process and let τ be a stopping time, then $X^\tau \triangleq \{X_{\tau\wedge t}\}_{t\geq 0}$ is termed a stopping process of X.*

Eventually, we introduce some basic operations on stopping times such as taking the maximum or minimum of two times or, for right-continuous filtrations, taking the limit of a sequence of times.

Lemma 3.5.

- *If θ and ϑ are stopping times, then $\theta\vee\vartheta$ and $\theta\wedge\vartheta$ are also stopping times.*
- *Let θ_n be a sequence of stopping times converging to a limit θ. Assume that for each $\omega\in\Omega$, we have $\theta_n(\omega)\leq\theta(\omega)$ for large enough n. Then, θ is a stopping time.*
- *If θ_n is a sequence of stopping times, then $\sup_n \theta_n$ is a stopping time.*
- *Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of stopping times and the filtration is right-continuous, then $\liminf_{n\to\infty}\theta_n$ and $\limsup_{n\to\infty}\theta_n$ are stopping times.*

4 Continuous-time martingale

Now, we turn our attention to continuous-time martingales which are stochastic processes that remain the same on the average (i.e. the expected future value conditional on the present is equal to the current value). Notionally, an $\{\mathcal{F}_t\}$-adapted integrable process $\mathcal{M}\equiv\{\mathcal{M}_t\}_{t\geq 0}$ is called *continuous-time martingale*[a] with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if $\mathbb{E}(\mathcal{M}_t|\mathcal{F}_s)=\mathcal{M}_s$[b] a.s. for all $0\leq s<t<\infty$. If in the martingale property $\mathbb{E}(\mathcal{M}_t|\mathcal{F}_s)=\mathcal{M}_s$ "=" is replaced by "$\leq$" (resp. "$\geq$"), then $\mathcal{M}$ is called a supermartingale (resp. submartingale). Plainly, $\mathcal{M}$ is a submartingale if and only if $-\mathcal{M}$ is a supermartingale, and is a martingale if it is both a submartingale and a supermartingale.

In the following, we shall denote by $\mathcal{F}_\infty$ the σ-algebra $\bigvee_{t\geq 0}\mathcal{F}_t$ (the limit variable $\mathcal{M}_\infty$ is measurable with respect to $\mathcal{F}_\infty$), and we want to know whether the process indexed by $\mathbb{R}_+\cup\{+\infty\}$ obtained by adjoining $\mathcal{M}_\infty$ and $\mathcal{F}_\infty$ is still a (sub)martingale. The corresponding result is especially intersting for martinglaes.

Theorem 4.1. *For a right-continuous martingale $\mathcal{M}_t$, $t\in\mathbb{R}_+$, the following three conditions are equivalent,*

- *$\lim_{t\to\infty}\mathcal{M}_t$ exists in the L^1-sense;*
- *there existes a random variable $\mathcal{M}_\infty$ in L^1, such that $\mathcal{M}_t=\mathbb{E}(\mathcal{M}_\infty|\mathcal{F}_t)$;*
- *the family $\{\mathcal{M}_t, t\in\mathbb{R}_+\}$ is uniformly integrable.*

If these conditions hold, then $\mathcal{M}_\infty=\lim_{t\to\infty}\mathcal{M}_t$ a.s. Moreover, if for some $p>1$, the martingale is bounded in L^p, i.e. $\sup_t\mathbb{E}(|\mathcal{M}_t|^p)<\infty$, then the equivalent conditions above are satisfied and the convergence holds in the L^p-sense.

We now present the optional stopping theorem. If $\mathcal{M}$ is a uniformly integrable martingale, the $\mathcal{M}_\infty$ exists a.s. and if τ is a stopping time, we define $\mathcal{M}_\tau$ on $\{\tau=\infty\}$ by setting $\mathcal{M}_\tau=\mathcal{M}_\infty$.

Theorem 4.2. *If $\mathcal{M}$ is a right-continuous martingale and θ, ϑ are two bounded stopping times with $\theta\leq\vartheta$,*

$$\mathcal{M}_\theta=\mathbb{E}(\mathcal{M}_\vartheta|\mathcal{F}_\theta)\quad a.s.$$

If $\mathcal{M}$ is uniformly integrable, the family $\{\mathcal{M}_\theta\}$ where θ runs through the set of all stopping times is uniformly integrable and if $\theta\leq\vartheta$

$$\mathcal{M}_\theta=\mathbb{E}(\mathcal{M}_\vartheta|\mathcal{F}_\theta)=\mathbb{E}(\mathcal{M}_\infty|\mathcal{F}_\theta)\quad a.s.$$

[a] Notice that the definition of martingale makes sense only when the underlying filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and the probability measure $\mathbb{P}$ have been specified.

[b] Clearly, the martingale $\mathcal{M}$ has constant expectation: $\forall t\geq 0$, $\mathbb{E}(\mathcal{M}_t)=\mathbb{E}(\mathcal{M}_0)$.

Remark 4.3. The two statements are actually the same, as a martingale defined on an interval which is closed on the right is uniformly integrable.

In order to understand properly the role of martingales, we present some of their classes. One such category is that continuous, square-integrable martingales (i.e. $\mathbb{E}|\mathcal{M}_t|^2 < \infty$ for every $t \geq 0$). Hence, we give the following result.

Proposition 4.4. *Let $\mathcal{M}$ be a square-integrable continuous martingale. Then, there exist a unique continuous integrable increasing process denoted by $\{\langle \mathcal{M}, \mathcal{M}\rangle_t\}$ such that $\{\mathcal{M}_t^2 - \langle \mathcal{M}, \mathcal{M}\rangle_t\}$ is a continuous martingale vanishing at $t = 0$.*

We mention that the process $\{\langle \mathcal{M}, \mathcal{M}\rangle_t\}$ is called the quadratic variation of $\mathcal{M}$. Specifically, for any bounded stopping time τ, we have $\mathbb{E}\mathcal{M}_\tau^2 = \mathbb{E}\langle \mathcal{M}, \mathcal{M}\rangle_\tau$. Now, let $\tilde{\mathcal{M}} \equiv \{\tilde{\mathcal{M}}_t\}_{t\geq 0}$ be another square-integrable continuous martingale, then

$$\langle \mathcal{M}, \tilde{\mathcal{M}}\rangle_t = 0.5\left(\langle \mathcal{M} + \tilde{\mathcal{M}}, \mathcal{M} + \tilde{\mathcal{M}}\rangle_t - \langle \mathcal{M}, \mathcal{M}\rangle_t - \langle \tilde{\mathcal{M}}, \tilde{\mathcal{M}}\rangle_t\right).$$

We call this process the joint quadratic variation of $\mathcal{M}$ and $\tilde{\mathcal{M}}$. More generally, $\{\langle \mathcal{M}, \tilde{\mathcal{M}}\rangle_t\}$ is the unique continuous integrable process of finite variation such that $\{\mathcal{M}_t\tilde{\mathcal{M}}_t - \langle \mathcal{M}, \tilde{\mathcal{M}}\rangle_t\}$ is a continuous martingale. Identically, for any bounded stopping time τ, $\mathbb{E}\mathcal{M}_\tau\tilde{\mathcal{M}}_\tau = \mathbb{E}\langle \mathcal{M}, \tilde{\mathcal{M}}\rangle_\tau$.

For the purpose of this thesis, we need to generalize the martingale concept to that of local martingales. Extensively, localization is a method of extending a given property to a larger class of processes.

Definition 4.5. Let $\mathcal{P} \equiv \{\mathcal{P}_t\}_{t\geq 0}$ be a right continuous adapted process. Then, $\mathcal{P}$ is a *local martingale* if there exists a non-decreasing sequence of stopping times $\tau_k \uparrow \infty$ a.s. such that every $\{\mathcal{P}_{\tau_k \wedge t}\}_{t\geq 0}$ is a martingale.

It should be pointed out that every martingale is a local martingale, but not the other way around. Likewise, let $\mathcal{P}$ and $\tilde{\mathcal{P}}$ be two continuous local martingales, then their joint variation is the unique continuous adapted process of finite variation such that $\{\mathcal{P}_t\tilde{\mathcal{P}}_t - \langle \mathcal{P}, \tilde{\mathcal{P}}\rangle_t\}_{t\geq 0}$ is a continuous local martingale.

We now present the strong law of large numbers theorem for local martingales which is frequently used in this thesis.

Theorem 4.6 (Strong law of large numbers, [77]). *Let $\mathcal{P}$ be a continuous local martingale such that $\mathcal{P}_0 = 0$ a.s. Then*

$$\lim_{t\to\infty}\langle \mathcal{P}, \mathcal{P}\rangle_t = \infty \text{ a.s.} \quad \Rightarrow \quad \lim_{t\to\infty}\frac{\mathcal{P}_t}{\langle \mathcal{P}, \mathcal{P}\rangle_t} = 0 \text{ a.s.}$$

Furthermore

$$\limsup_{t\to\infty}\frac{\langle \mathcal{P}, \mathcal{P}\rangle_t}{t} < \infty \text{ a.s.} \quad \Rightarrow \quad \lim_{t\to\infty}\frac{\mathcal{P}_t}{t} = 0 \text{ a.s.}$$

In the following, we shall present a classical result known as Doob's martingale inequalities.

Theorem 4.7 (Doob's martingale inequalities, [144]). *Let $\{\mathcal{M}_t\}_{t\geq 0}$ be an $\mathbb{R}^n$-valued martingale. Let $[a, b]$ be a bounded interval in $\mathbb{R}_+$.*

- *If $p \geq 1$ and $\mathcal{M}_t \in \mathrm{L}^p(\Omega; \mathbb{R}^n)$, then*

$$\mathbb{P}\left\{\omega : \sup_{a\leq t\leq b}|\mathcal{M}_t(\omega)| \geq c\right\} \leq \frac{\mathbb{E}|\mathcal{M}_b|^p}{c^p}$$

  holds for all $c > 0$.

- *If $p > 1$ and $\mathcal{M}_t \in \mathrm{L}^p(\Omega;\mathbb{R}^n)$, then*

$$\mathbb{E}\left(\sup_{a\le t\le b}|\mathcal{M}_t|^p\right) \le \left(\frac{p}{p-1}\right)^p \mathbb{E}|\mathcal{M}_b|^p.$$

Finally, we present one more practical and useful convergence result.

Theorem 4.8 (Convergence theorem, [78]). *Let $\{A_t\}_{t\ge0}$ and $\{U_t\}_{t\ge0}$ be two continuous adapted increasing processes with $A_0 = U_0 = 0$ a.s. Let $\{\mathcal{P}_t\}_{t\ge0}$ be a continuous local martingale with $\mathcal{P}_0 = 0$ a.s. Let also ζ be a non-negative $\mathcal{F}_0$-measurable random variable. Define*

$$\mathcal{X}_t = \zeta + A_t - U_t + \mathcal{P}_t \quad \text{for all } t \ge 0.$$

If $\mathcal{X}_t$ is non-negative, then

$$\left\{\lim_{t\to\infty} A_t < \infty\right\} \subset \left\{\lim_{t\to\infty}\mathcal{X}_t \text{ exists and is finite}\right\} \cap \left\{\lim_{t\to\infty} U_t < \infty\right\} \text{ a.s.}$$

Particularly, if $\lim_{t\to\infty} A_t < \infty$ a.s., then for almost $\omega \in \Omega$, we have $\lim_{t\to\infty}\mathcal{X}_t(\omega)$ exists and is finite. In addition, we obtain $\lim_{t\to\infty} U_t(\omega) < \infty$.

5 Brownian motion

In this section, we present the Brownian motion[a]. We refer to [93] for a general introduction of this process, to [103] for a detailed description of the main properties of Brownian motion. Historically, Brownian motion has been exhibited to represent motions which evolve over time in a particularly disordered fashion, for example in physics to represent microscopic particles subjected to multiple shocks in their environment or in finance to represent very volatile stock market prices. Brownian motion plays a central role in the theory of stochastic processes. It appears in many theoretical as well as applied situations and it offers a fairly simple framework within which many calculations can be carried out. To characterize the motion mathematically, we give the following definition.



Robert Brown
(1773-1858)

Definition 5.1. A one-dimensional $\{\mathcal{F}_t\}$-*standard Brownian motion* ($\{\mathcal{F}_t\}$-SBM, for brevity) is a continuous $\{\mathcal{F}_t\}$-adapted process $\mathcal{W} \equiv \{\mathcal{W}_t\}_{t\ge0}$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ with the following properties:

1. $\mathbb{P}(\mathcal{W}_0 = 0) = 1$, that is, to establish the position of a Brownian particle, we start at $t = 0$, with the initial location specified as $\mathcal{W}_0 = 0$.

2. For $0 \le s < t < \infty$, the increment (displacement) $\mathcal{W}_t - \mathcal{W}_s$ is normally distributed with mean zero and variance $t - s$. Equivalently, $\mathcal{W}_t - \mathcal{W}_s \sim \sqrt{t-s}\, N(0,1)$.

3. For $0 \le s < t < \infty$, the increment $\mathcal{W}_t - \mathcal{W}_s$ is independent of $\mathcal{F}_s$. That is to say that for any $A \in \mathcal{B}(\mathbb{R})$, $B \in \mathcal{F}_s$,

$$\mathbb{P}\Big(\{\omega\in\Omega:\ \mathcal{W}_t(\omega) - \mathcal{W}_s(\omega) \in A \ \text{ and } \ \omega \in B\}\Big) = \mathbb{P}\big(\mathcal{W}_t(\omega) - \mathcal{W}_s(\omega) \in A\big) \times \mathbb{P}\big(B\big).$$

[a] In 1827, the first (heuristic) description of Brownian motion is due to the scottish botanist Robert Brown (who therefore gave it its name). He observes fine organic particles in suspension in a gas or a fluid and describes their particularly erratic movements.

Let $\mathcal{W}$ be an SBM and $0 \leq t_0 < t_1 < \cdots < \infty$, then the increments $\mathcal{W}_{t_i} - \mathcal{W}_{t_{i-1}}$, $1 \leq i \leq k$ are independent. Consequently, we say that $\mathcal{W}$ has *independent increments*. In addition, the distribution of $\mathcal{W}_{t_i} - \mathcal{W}_{t_{i-1}}$ depends only on the difference $t_i - t_{i-1}$, and in this case, we say that $\mathcal{W}$ has *stationary increments*. The peculiarity of an SBM $\mathcal{W}$ process lies in its ability to represent the cumulative impact of process noise. Explicitly, if $\mathcal{W}_s$ and $\mathcal{W}_t$ indicate the position of the process at times s and t, respectively, then the increment $\mathcal{W}_t - \mathcal{W}_s$ shows the net noise along the interval $]s, t]$.

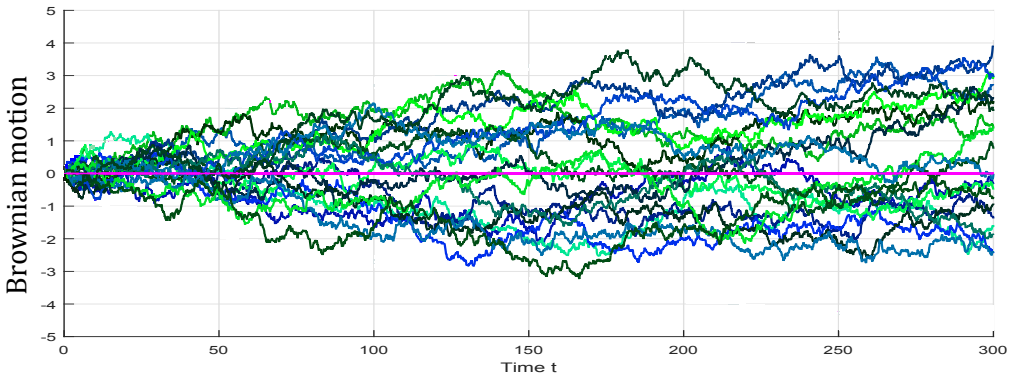


Figure II.1: Twenty sample paths of the standard Brownian motion $\mathcal{W}_t$ for $t \in [0, 300]$.

According to the definition of a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ for $0 \leq s < t$, we can state that there is at least as much information at time t as there is at time s. Therefore, the accumulated information is represented mathematically by the fact that $\mathcal{F}_s \subset \mathcal{F}_t$. Remark that $\{\mathcal{F}_t\}_{t\geq 0}$ is a part of the definition of $\mathcal{W}$, but we can define an SBM on $(\Omega, \mathcal{F}, \mathbb{P})$ without filtration. In this case, $\mathcal{W}$ satisfies the properties 1 and 2 but the property 3 is substituted by that it has the independent increments. In such a situation, one can define $\mathcal{F}_t^{\mathcal{W}} = \sigma(\mathcal{W}_s,\ 0 \leq s \leq t)$ for $t \geq 0$, as the σ-algebra generated by $\{\mathcal{W}_s |\ 0 \leq s \leq t\}$[a]. Evidently, $\mathcal{W}$ is an SBM with respect to the filtration $\{\mathcal{F}_t^{\mathcal{W}}\}_{t\geq 0}$. Furthermore, if $\mathcal{F}_t^{\mathcal{W}} \subset \mathcal{F}_t$, $t \geq 0$, and $\mathcal{W}_t - \mathcal{W}_s$ is independent of $\mathcal{F}_s$ $(0 \leq s < t)$, then $\mathcal{W}$ is a Brownian motion with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

It can also be noted that in Definition 5.1 we do not demand the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be complete and $\{\mathcal{F}_t\}_{t\geq 0}$ satisfy the usual conditions. Nevertheless, in this thesis we need to work on a complete probability space with a filtration verifying the said conditions. For this reason, we assume that $\mathcal{W}$ is a Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ indicate the completion[b] of $(\Omega, \mathcal{F}, \mathbb{P})$. Undoubtedly, $\mathcal{W}$ is a Brownian motion on the complete probability space $(\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$. In the same line, we use the notation $\mathfrak{N}$ to denote the collection of $\mathbb{P}$-null sets. Then, for all $t \geq 0$, $\bar{\mathcal{F}}_t = \sigma(\mathcal{F}_t^{\mathcal{W}} \cup \mathfrak{N})$ is the augmentation under $\mathbb{P}$ of the natural filtration $\{\mathcal{F}_t^{\mathcal{W}}\}$ generated by $\mathcal{W}$. Furthermore, $\{\bar{\mathcal{F}}_t\}_{t\geq 0}$ is a filtration on $(\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ verifying the usual conditions and $\mathcal{W}$ is an SBM process on $(\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with respect to $\{\bar{\mathcal{F}}_t\}_{t\geq 0}$.

Brownian motion has many salient features and important properties, and some of them are presented in the following:

- Symmetry property: the process $-\mathcal{W}$ is an SBM with respect to the same filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

[a] We often call $\{\mathcal{F}_t^{\mathcal{W}}\}_{t\geq 0}$ the natural filtration generated by $\mathcal{W}$.

[b] Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Define a class $\mathcal{N}_0$ of subsets of Ω as follows:

$$N \in \mathcal{N}_0 \text{ if and only if } \exists Z \in \mathcal{F} \text{ such that } N \subseteq Z \text{ and } \mathbb{P}(Z) = 0.$$

It is sometimes philosophically satisfying to be able to make precise the idea that "N in $\mathcal{N}_0$ is $\mathbb{P}$-measurable and $\mathbb{P}(N) = 0$". This is done as follows. For any subset F of Ω, we write $F \in \bar{\mathcal{F}}$ if $\exists E, G \in \mathcal{F}$ such that $E \subseteq F \subseteq G$ and $\mathbb{P}(G \setminus E) = 0$. It is very easy to show that $\bar{\mathcal{F}}$ is a σ-algebra on Ω and indeed that $\bar{\mathcal{F}} = \sigma(\mathcal{F}, \mathcal{N}_0)$. With obvious notation,we define for $F \in \bar{\mathcal{F}}$,

$$\bar{\mathbb{P}}(F) = \mathbb{P}(E) = \mathbb{P}(G),$$

it being easy to check that $\bar{\mathbb{P}}$ is well defined. Moreover, it is no problem to prove that $(\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ is a probability space, the completion of $(\Omega, \mathcal{F}, \mathbb{P})$.

- Scaling property: let $\alpha > 0$ and $\mathcal{X}_t = \frac{\mathcal{W}_{\alpha t}}{\sqrt{\alpha}}$ for all $t \geq 0$. Then, $\{\mathcal{X}_t\}_{t\geq 0}$ is an SBM with respect to the filtration $\{\mathcal{F}_{\alpha t}\}_{t\geq 0}$.
- For $t \geq 0$, it is noted that $\mathbb{E}(\mathcal{W}_t) = 0$, $\mathrm{Var}(\mathcal{W}_t) = \mathbb{E}(\mathcal{W}_t^2) = t$. Furthermore, $\mathcal{W}$ is a continuous square-integrable martingale and its quadratic variation $\langle \mathcal{W}, \mathcal{W}\rangle_t = t$ for all $t \geq 0$.
- The strong law of large numbers property: $\lim_{t\to\infty}(\mathcal{W}_t/t) = 0$ a.s.
- For almost any $\omega \in \Omega$, the sample path $\mathcal{W}_\cdot(\omega)$ of the SBM process $\mathcal{W}$ is nowhere differentiable.
- For almost any $\omega \in \Omega$, the sample path $t \mapsto \mathcal{W}_t(\omega)$ is locally Hölder continuous[a] with exponent $\bar{\alpha}$ if $0 < \bar{\alpha} < 0.5$; it is nowhere Hölder continuous for every $\bar{\alpha} > 0.5$.

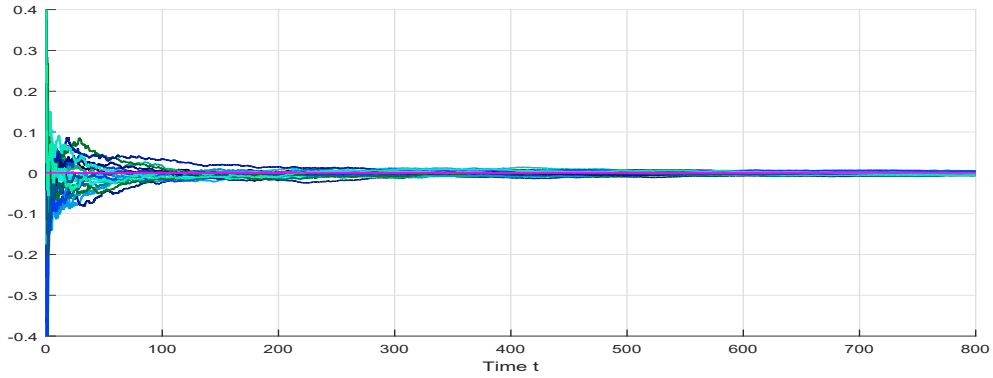


Figure II.2: An illustration of the strong law of large numbers property for Brownian motion.

For a n-dimensional standard Brownian motion, we have the following definition.

Definition 5.2. A n-dimensional process $\{\mathcal{W}_t = (\mathcal{W}_t^1, \ldots, \mathcal{W}_t^n)\}_{t\geq 0}$ is called a n-dimensional standard Brownian motion if any $\{\mathcal{W}_t^k\}$ is a one-dimensional standard Brownian motion, and $\{\mathcal{W}_t^1\}$,...,$\{\mathcal{W}_t^n\}$ are independent.

Readily, a n-dimensional standard Brownian motion is a n-dimensional continuous martingale associated with the following quadratic variations[b]

$$\langle \mathcal{W}^j, \mathcal{W}^k\rangle_t = \delta_{jk} t \quad \text{for } 1 \leq j,k \leq n.$$

Here, $\delta_{jk} = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases}$ denotes the Dirac delta function.

6 White noise process

Roughly speaking, a plausible mathematical interpretation for the "loud" phenomena is the so-termed one-dimensional white noise $\xi(t)$. In engineering and applied sciences, it is habitual to define it as a "derivative" (just notation) of $\mathcal{W}(t)$, i.e., $\xi(t) = \dot{\mathcal{W}}(t)$. Seemingly, the term "noise"$\mathrm{d}t$ is expressed as $\xi(t)\mathrm{d}t = \mathrm{d}\mathcal{W}(t)$. As was aforementioned, $\dot{\mathcal{W}}(t)$ does not exist and it has not an ordinary meaning; so, it can be seen as a non-standard derivative. To properly characterize the white noise $\xi(t)$, we present the following definition.

[a] As usual, the term "local" (or "locally") means that the definition should be restricted to any neighborhood. In this case, f is locally Hölder continuous if, for every interval (a,b) there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for every $x, y \in [a,b]$.

[b] This is described by the well-known Lévy theorem (see for example [93]).

Definition 6.1. Let $\mathcal{X} = \{\mathcal{X}_t\}_{t\geq 0}$ be a Gaussian stochastic process such that $\mathbb{E}\mathcal{X}_t^2 < \infty$, $t \geq 0$. The co-variance function of $\mathcal{X}$ is defined by $c(t,s) = \mathbb{E}\left(\{\mathcal{X}_t - \mathbb{E}\mathcal{X}_t\} \cdot \{\mathcal{X}_s - \mathbb{E}\mathcal{X}_s\}\right)$, where $t, s \geq 0$. If $c(t,s) = \mathfrak{h}(t-s)$ for some real-valued function $\mathfrak{h} : \mathbb{R} \to \mathbb{R}$ and if $\mathbb{E}(\mathcal{X}_t) = \mathbb{E}(\mathcal{X}_s)$ for all $t, s \geq 0$, then $\mathcal{X}_t$ is said to be stationary in the wide sense.

Formally, the *white noise* $\xi(t)$ is a Gaussian and wide-sense stationary process with $\mathfrak{h}(\cdot) = \delta_0$, where δ_0 is the Dirac point mass function.

7 Stochastic integration: Itô integral

Consider for example the dynamic of a given population N expressed by the following ordinary differential equation (ODE):

$$\begin{cases} \mathrm{dN}(t) = \mathcal{G}\mathrm{N}(t)\mathrm{d}t, \\ \mathrm{N}(0) = \mathrm{N}_0. \end{cases} \tag{II.1}$$

where $\mathcal{G}$ is the associated growth rate. Suppose that $\mathcal{G}$ is subject to some random fluctuations, then $\mathcal{G}$ can be rewrite as $\mathcal{G} + \sigma\text{noise}$, where σ is the intensity of the noise. Consequently, (II.1) can be transformed into the following perturbed ODE:

$$\mathrm{dN}(t) = \mathcal{G}\mathrm{N}(t)\mathrm{d}t + \sigma\mathrm{N}(t) \cdot \text{noise} \cdot \mathrm{d}t.$$

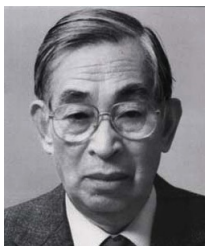
By using the integration, the last equality becomes

$$\mathrm{N}(t) = \mathrm{N}(0) + \int_0^t \mathcal{G}\mathrm{N}(s)\mathrm{d}s + \int_0^t \sigma\mathrm{N}(s) \cdot \text{noise} \cdot \mathrm{d}s.$$

But, how we can define the following expression $\displaystyle\int_0^t \sigma N(s) \cdot \text{noise} \cdot \mathrm{d}s$? To solve this problem, we consider the white noise $\xi(t) = \dot{\mathcal{W}}(t)$ as an elementary choice to present the noise. Hence,

$$\int_0^t \sigma N(s) \cdot \xi(s) \cdot \mathrm{d}s = \int_0^t \sigma N(s)\mathrm{d}\mathcal{W}(s). \tag{II.2}$$

Since the Brownian motion $\mathcal{W}$ is nowhere differentiable almost surely and whose paths are of infinite variation for almost every $\omega \in \Omega$, the integral (II.2) can not be defined in the ordinary way[a]. Hence, we need to define the integral for a large class of stochastic processes by making use of the stochastic characterization of $\mathcal{W}$. Notably, we will look at the Itô stochastic integral with respect to $\mathcal{W}$ which is firstly defined by Kiyosi Itô[b] in 1946. To this end, let $0 \leq a < b < \infty$ and denote $\mathcal{M}^2([a,b];\mathbb{R})$ the space of all real-valued measurable $\{\mathcal{F}_t\}$-adapted processes $g \equiv \{g(t)\}_{a\leq t\leq b}$ such that



Kiyosi Itô
(1915-2008)

$$||g||^2_{[a,b]} = \mathbb{E}\left(\int_a^b |g(t)|^2\mathrm{d}t\right) < \infty. \tag{II.3}$$

Clearly, under the metric (II.3), the space $\mathcal{M}^2([a,b];\mathbb{R})$ is complete. We say that two process $g, h \in \mathcal{M}^2([a,b];\mathbb{R})$ are equivalent, if $||g-h||^2_{[a,b]} = 0$. For a stochastic process $g \in \mathcal{M}^2([a,b];\mathbb{R})$, we aim to define the Itô integral $\mathcal{I}(g) = \int_a^b g(t)\mathrm{d}\mathcal{W}(t)$. The general idea is defining $I(h)$ for elementary process class h. Then, proving that each $g \in \mathcal{M}^2([a,b];\mathbb{R})$ can be approximated by simple process h's such that $\mathcal{I}(g)$ is the limit of $\mathcal{I}(h)$. A brief overview of this approach is presented in three steps.

[a] Brownian motion is not of bounded variation so we cannot apply the usual definitions of Riemann Stieltjes to evaluate the integrals along some realized path.

[b] Kiyosi Itô was a Japanese mathematician who made fundamental contributions to the theory of stochastic processes. He invented the concept of stochastic integral and is known as the founder of Itô calculus.

Step 1. A real-valued stochastic process $h \equiv \{h(t)\}_{a\le t\le b}$ is termed an elementary (step) process if there exists a partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a,b]$, and bounded random variables κ_i, $0 \le i \le k-1$ such that κ_i is $\mathcal{F}_{t_i}$-measurable and

$$h(t) = \kappa_0 \mathbb{1}_{[t_0,t_1]}(t) + \sum_{i=1}^{k-1} \kappa_i \mathbb{1}_{(t_i,t_{i+1}]}(t). \tag{II.4}$$

Let $\mathcal{M}_0^2([a,b];\mathbb{R})$ be the subspace of elementary processes in $\mathcal{M}^2([a,b];\mathbb{R})$ and define the stochastic integral with respect to the Brownian motion $\mathcal{W}$ of an elementary process h as

$$\mathcal{I}(h) = \int_a^b h(t)\mathrm{d}\mathcal{W}(t) = \sum_{i=0}^{k-1} \kappa_i \big[\mathcal{W}(t_{i+1}) - \mathcal{W}(t_i)\big].$$

Step 2. Clearly, the stochastic integral (Itô stochastic integral) $\mathcal{I}(h)$ is $\mathcal{F}_b$-measurable, κ_i is $\mathcal{F}_{t_i}$-measurable, and $\mathcal{W}(t_{i+1}) - \mathcal{W}(t_i)$ is independent of $\mathcal{F}_{t_i}$, $i = 0, 1, \ldots, k-1$. Moreover, $\mathcal{I}(h) \in \mathrm{L}^2(\Omega;\mathbb{R})$ since, if $h \in \mathcal{M}_0^2([a,b];\mathbb{R})$, then

$$\begin{aligned}
&\mathbb{E}\left(\int_a^b h(t)\mathrm{d}\mathcal{W}(t)\right) = 0,\\
&\mathbb{E}\left(\left|\int_a^b h(t)\mathrm{d}\mathcal{W}(t)\right|^2\right) = \mathbb{E}\left(\int_a^b |h(t)|^2 dt\right).
\end{aligned} \tag{II.5}$$

[The Itô isometry]

Additionally, if two processes $h_1, h_2 \in \mathcal{M}_0^2([a,b];\mathbb{R})$ and $m_1, m_2 \in \mathbb{R}$, then $m_1h_1 + m_2h_2 \in \mathcal{M}_0^2([a,b];\mathbb{R})$ and

$$\int_a^b \big(m_1h_1(t) + m_2h_2(t)\big)\mathrm{d}\mathcal{W}(t) = m_1\int_a^b h_1(t)\mathrm{d}\mathcal{W}(t) + m_2\int_a^b h_2(t)\mathrm{d}\mathcal{W}(t). \tag{II.6}$$

According to properties (II.5) and (II.6), we can extend the integral definition from simple processes to processes in $\mathcal{M}^2([a,b];\mathbb{R})$. The justification for this extension is presented in the next step.

Step 3. For any $g \in \mathcal{M}^2([a,b];\mathbb{R})$, there exists a sequence $\{h_n\}$ of elementary processes such that

$$\lim_{n\to\infty} \mathbb{E}\left(\int_a^b |g(t) - h_n(t)|^2\mathrm{d}t\right) = 0. \tag{II.7}$$

Via (II.5), (II.6) and (II.7) we get

$$\begin{aligned}
&\lim_{n_1,n_2\to\infty} \mathbb{E}\left(\left|\int_a^b h_{n_1}(t)\mathrm{d}\mathcal{W}(t) - \int_a^b h_{n_2}(t)\mathrm{d}\mathcal{W}(t)\right|^2\right)\\
&= \lim_{n_1,n_2\to\infty} \mathbb{E}\left(\left|\int_a^b \big(h_{n_1}(t) - h_{n_2}(t)\big)\mathrm{d}\mathcal{W}(t)\right|^2\right)\\
&= \lim_{n_1,n_2\to\infty} \mathbb{E}\left(\int_a^b |h_{n_1}(t) - h_{n_2}(t)|^2\mathrm{d}t\right) = 0.
\end{aligned}$$

Specifically, $\mathcal{I}(h_n)$ is a Cauchy sequence in $\mathrm{L}^2(\Omega;\mathbb{R})$, so that its limit exists and we define this limit as the Itô stochastic integral $\mathcal{I}(g)$. To summarize and conclude, the Itô stochastic integral of g with respect to $\mathcal{W}$ is

$$\mathcal{I}(g) = \int_a^b g(t)\mathrm{d}\mathcal{W}(t) = \lim_{n\to\infty}\int_a^b h_n(t)\mathrm{d}\mathcal{W}(t) \quad \text{in } \mathrm{L}^2(\Omega;\mathbb{R}),$$

where $\{h_n\}$ is a sequence of elementary processes verifying (II.7). This definition is independent of the particular sequence $\{h_n\}$.

Broadly speaking, the stochastic integral has many specific characteristics. We present here only some of its properties which are used during this thesis.

- $\mathcal{I}(g)$ is $\mathcal{F}_b$-measurable.
- $\mathcal{I}\big(c_1 g_1(t) + c_2 g_2(t)\big) = c_1\mathcal{I}(g_1) + c_2\mathcal{I}(g_2)$, for all $g_1, g_2 \in \mathcal{M}^2([a,b];\mathbb{R})$.
- $\mathbb{E}\mathcal{I}(g) = 0$ and $\mathbb{E}\big(\mathcal{I}(g)|\mathcal{F}_a\big) = 0$.
- $\mathbb{E}\left(|\mathcal{I}(g)|^2\right) = \mathbb{E}\left(\int_a^b |g(t)|^2 \mathrm{d}t\right)$.
- $\mathbb{E}\left(|\mathcal{I}(g)|^2 \,|\mathcal{F}_a\right) = \mathbb{E}\left(\int_a^b |g(t)|^2 \mathrm{d}t \Big| \mathcal{F}_a\right) = \int_a^b \mathbb{E}\left(|g(t)|^2 | \mathcal{F}_a\right) \mathrm{d}t$.
- If ζ is a real-valued, bounded $\mathcal{F}_a$-measurable random variable, then $\zeta g \in \mathcal{M}^2([a,b];\mathbb{R})$ and $\mathcal{I}(\zeta g) = \zeta \mathcal{I}(g)$.

Now, let $T > 0$. We define for $g \in \mathcal{M}^2([0,T];\mathbb{R})$, the indefinite Itô stochastic integral as

$$\mathcal{K}(t) = \int_0^t g(s)\mathrm{d}\mathcal{W}(s), \quad t \in [0,T],$$

where, by definition, $\mathcal{K}(0) = 0$. Evidently, $\{\mathcal{K}(t)\}_{t\in[0,T]}$ is a square integrable martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and

$$\mathbb{E}\big(\mathcal{K}(t)|\mathcal{F}_s\big) = \mathbb{E}\big(\mathcal{K}(s)|\mathcal{F}_s\big) + \mathbb{E}\left(\int_s^t g(r)\mathrm{d}\mathcal{W}(r)\Big|\mathcal{F}_s\right) = \mathcal{K}(s), \quad 0 \leq s < t \leq T.$$

Using Doob's martingale inequalities (Theorem 4.7), we obtain the following result:

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |\mathcal{K}(t)|^2\right) \leq 4\mathbb{E}\left(\int_0^T |g(s)|^2 \mathrm{d}s\right).$$

Furthermore, $\mathcal{K} = \{\mathcal{K}(t)\}_{t\in[0,T]}$ has a continuous version and its quadratic variation is given by

$$\langle \mathcal{K}, \mathcal{K}\rangle_t = \int_0^t |g(s)|^2 \mathrm{d}s, \quad 0 \leq t \leq T.$$

We will now extend the Itô stochastic integral to the multidimensional case. Let

$$\mathcal{W}(t) = \left\{(\mathcal{W}_1(t), \ldots, \mathcal{W}_m(t))^T\right\}_{t\geq 0}$$

be an m-dimensional standard Brownian motion. Let $g \in \mathcal{M}^2([0,T];\mathbb{R}^{n\times m})$ denote the collection of all $n\times m$-matrix-valued measurable $\{\mathcal{F}_t\}$-adapted process $g = \{(g_{ij}(t))_{n\times m}\}_{0\leq t\leq T}$ such that[a]

$$\mathbb{E}\left(\int_0^T |g(s)|^2 \mathrm{d}t\right) < \infty.$$

By using matrix representation, we define the multi-dimensional stochastic integral

$$\int_0^t g(s)\mathrm{d}\mathcal{W}(s) = \int_0^t \begin{pmatrix} g_{11}(s) & \ldots & g_{1m}(s) \\ \vdots & & \vdots \\ g_{n1}(s) & \ldots & g_{nm}(s) \end{pmatrix} \begin{pmatrix} \mathrm{d}\mathcal{W}_1(s) \\ \vdots \\ \mathrm{d}\mathcal{W}_m(s) \end{pmatrix}$$

to be the n-column-vector-valued process whose j'th component is the following sum of one-dimensional Itô integrals

$$\sum_{k=1}^m \int_0^t g_{jk}(s)\mathrm{d}\mathcal{W}_k(s).$$

[a] Here, $|A|$ presents the trace norm for matrix A, i.e., $|A| = \sqrt{\mathrm{trace}(A^T A)}$.

Let $\mathcal{M}^2(\mathbb{R}_+;\mathbb{R}^{n\times m})$ denote the family of all processes $g \in \mathfrak{L}^2(\mathbb{R}_+;\mathbb{R}^{n\times m})$[a] such that

$$\mathbb{E}\left(\int_0^T |g(t)|^2 \mathrm{d}t\right) < \infty \quad \text{for every } T > 0.$$

Markedly, if $g \in \mathcal{M}^2(\mathbb{R}_+;\mathbb{R}^{n\times m})$, then $\{g\}_{0\le t\le T} \in \mathcal{M}^2([0,T];\mathbb{R}^{n\times m})$ for every $T > 0$ and consequently the integral $\int_0^t g(s)\mathrm{d}\mathcal{W}(s)$ $t \ge 0$ is well defined. Now, our objective is to define this integral for all processes in $\mathfrak{L}^2(\mathbb{R}_+;\mathbb{R}^{n\times m})$. Consider $g \in \mathfrak{L}^2(\mathbb{R}_+;\mathbb{R}^{n\times m})$, we define for each integer $k \ge 1$, the following time

$$\tau_k = k \wedge \inf\left\{t \ge 0 : \quad \int_0^t |g(s)|^2 \mathrm{d}s \ge k\right\}.$$

Note that $\tau_k \uparrow \infty$ a.s. Furthermore, $\{g(t)\mathbb{1}_{[0,\tau_k]}(t)\}_{t\ge 0} \in \mathcal{M}^2(\mathbb{R}_+;\mathbb{R}^{n\times m})$ and the integral

$$\mathcal{I}_k(t) = \int_0^t g(s)\mathbb{1}_{[0,\tau_k]}(s)\mathrm{d}\mathcal{W}(s), \qquad t \ge 0$$

is well defined. For all $1 \le k \le \bar{m}$ and $t \ge 0$, $\mathcal{I}_{\bar{m}}(t\wedge\tau_k) = \mathcal{I}_k(t)$. Then, $\mathcal{I}_k(t) = \mathcal{I}_{\bar{m}}(t)$, for all $0 \le t \le \tau_k$. Generally, we define the indefinite stochastic integral $\mathcal{I}(t)$ by the following way $\mathcal{I}(t) = \mathcal{I}_k(t)$ on $0 \le t \le \tau_k$. Let $g \in \mathfrak{L}^2(\mathbb{R}_+;\mathbb{R}^{n\times m})$, then the indefinite Itô integral $\int_0^t g(s)\mathrm{d}\mathcal{W}(t)$ $(t \ge 0)$ of g with respect to $\mathcal{W}(t)$ is a $\mathbb{R}^n$-valued continuous local martingale.

Eventually, it is important to note that, in this thesis, all (three) of the following notations are equivalent and will be used frequently:

$$\int_0^t g\mathrm{d}\mathcal{W} \triangleq \int_0^t g(s)\mathrm{d}\mathcal{W}(s) \triangleq \int_0^t g(s,\omega)\mathrm{d}\mathcal{W}_s(\omega).$$

8 Itô's formula

To properly evaluate the stochastic Itô integrals, we will use what is called the *Itô rule* or *Itô formula* which is very important and plays a key role in stochastic analysis. Accordingly, let $\{\mathcal{W}(t)\}_{t\ge 0}$ be a one-dimensional SBM defined on the complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ and adapted to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$. In addition, let $f \in \mathfrak{L}^1(\mathbb{R}_+;\mathbb{R})$[b] and $g \in \mathfrak{L}^2(\mathbb{R}_+;\mathbb{R})$. Then, a one-dimensional continuous adapted process $\mathcal{X}(t)$ on $t \ge 0$ of the form

$$\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t f(s)\mathrm{d}s + \int_0^t g(s)\mathrm{d}\mathcal{W}(s), \tag{II.8}$$

is called an Itô process with the following stochastic differential representation

$$\mathrm{d}\mathcal{X}(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}\mathcal{W}(t).$$

Consequently, if $\mathcal{V} \in C^{1,2}(\mathbb{R}_+\times\mathbb{R};\mathbb{R})$[c], then $\mathcal{V}(t,\mathcal{X}(t))$ is again an Itô process with the stochastic differential given by

$$\begin{aligned}\mathrm{d}\mathcal{V}(t,\mathcal{X}(t)) = &\Big\{\mathcal{V}_t(t,\mathcal{X}(t)) + \mathcal{V}_x(\mathcal{X}(t),t)f(t) + 0.5\mathcal{V}_{xx}(t,\mathcal{X}(t))g^2(t)\Big\}\mathrm{d}t\\ &+ \mathcal{V}_x(t,\mathcal{X}(t))g(t)\mathrm{d}\mathcal{W}(t) \quad \text{a.s.}\end{aligned} \tag{II.9}$$

[a] $\mathfrak{L}^2(\mathbb{R}_+;\mathbb{R}^{n\times m})$ denotes the family of all $n\times m$-matrix-valued measurable $\{\mathcal{F}_t\}$-adapted processes $g = \{g(t)\}_{t\ge 0}$ such that $\int_0^T |g(t)|^2\mathrm{d}t < \infty$ a.s. for every $T > 0$.

[b] Generally, $\mathfrak{L}^1(\mathbb{R}_+;\mathbb{R}^n)$ denotes the family of all $\mathbb{R}^n$-valued measurable $\{\mathcal{F}_t\}$-adapted processes $f = \{f(t)\}_{t\ge 0}$ such that $\int_0^T |f(t)|\mathrm{d}t < \infty$ a.s. for every $T > 0$.

[c] Generally, $C^{1,2}(\mathbb{R}_+\times\mathbb{R}^n;\mathbb{R})$ denotes the family of all real-valued $\mathcal{V}(t,x)$ defined on $\mathbb{R}_+\times\mathbb{R}^n$ such that they are continuously twice differentiable in x and once in t.

where $\mathcal{V}_t = \frac{\partial \mathcal{V}}{\partial t}$, $\mathcal{V}_x = \frac{\partial \mathcal{V}}{\partial x}$ and $\mathcal{V}_{xx} = \frac{\partial^2 \mathcal{V}}{\partial x^2}$. Next, if $\mathcal{V} \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$, we set

$$\mathcal{V}_x = \left(\frac{\partial \mathcal{V}}{\partial x_1}, \ldots, \frac{\partial \mathcal{V}}{\partial x_n}\right) \quad \text{and} \quad \mathcal{V}_{xx} = \left(\frac{\partial^2 \mathcal{V}}{\partial x_j \partial x_k}\right)_{n\times n} = \begin{pmatrix} \frac{\partial^2 \mathcal{V}}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 \mathcal{V}}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 \mathcal{V}}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 \mathcal{V}}{\partial x_n \partial x_n} \end{pmatrix}.$$

In the multidimensional case, let $\mathcal{W}(t) = \big(\mathcal{W}_1(t), \ldots, \mathcal{W}_m(t)\big)^T$ $(t \geq 0)$, be an m-dimensional Brownian motion ($\mathcal{W}_k(t)$ are independent SBMs) and consider a n-dimensional continuous adapted process $\mathcal{X}(t) = \big(\mathcal{X}_1(t), \ldots, \mathcal{X}_n(t)\big)^T$ of the form

$$\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t f(s)\mathrm{d}s + \int_0^t g(s)\mathrm{d}\mathcal{W}(s),$$

where $f = (f_1, \ldots, f_n)^T \in \mathfrak{L}^1(\mathbb{R}_+; \mathbb{R}^n)$ and $g = (g_{jk})_{n\times m} \in \mathfrak{L}^2(\mathbb{R}_+; \mathbb{R}^{n\times m})$. Let also $\mathcal{V} \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R})$, then $\mathcal{V}(t, \mathcal{X}(t))$ is again an Itô process with the following stochastic equation

$$\begin{aligned} \mathrm{d}\mathcal{V}(t, \mathcal{X}(t)) = &\Big\{\mathcal{V}_t(t, \mathcal{X}(t)) + \mathcal{V}_x(t, \mathcal{X}(t))f(t) + 0.5\mathrm{trace}\Big(g^T(t)\mathcal{V}_{xx}(t, \mathcal{X}(t))g(t)\Big)\Big\}\mathrm{d}t \\ &+ \mathcal{V}_x(t, \mathcal{X}(t))g(t)\mathrm{d}\mathcal{W}(t) \quad \text{a.s.} \end{aligned} \tag{II.10}$$

As an application, and by using Itô formula, we introduce the following proprieties:

1. $\displaystyle\int_0^t \mathrm{d}\mathcal{W}(s) = \mathcal{W}(t), \text{ and } \int_0^t \mathcal{W}(s)\mathrm{d}\mathcal{W}(s) = \frac{1}{2}\mathcal{W}^2(t) - \frac{1}{2}t.$
2. $\displaystyle\int_0^t s\mathrm{d}\mathcal{W}(s) = t\mathcal{W}(t) - \int_0^t \mathcal{W}(s)\mathrm{d}s.$

Finally, let $[t, t + \mathrm{d}t]$ be an infinitesimal time interval, where both $\mathrm{d}t$ and $\sqrt{\mathrm{d}t}$ are positives. For $\mathrm{d}\mathcal{W}_t = \mathcal{W}_{t+\mathrm{d}t} - \mathcal{W}_t$, we get $\mathrm{d}\mathcal{W}_t \sim N(0, \mathrm{d}t)$.

9 Stochastic differential equations

Generally, a stochastic differential equation (SDE, for short) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process. SDEs are employed to simulate diverse phenomena such as unstable stock prices or biological systems subject to environmental fluctuations. Typically, SDEs contain a variable which represents random white noise calculated as the non-standard derivative of Brownian motion. However, other types of random behaviour are possible, such as Lévy jump processes (see for example [104]). A simple SDE is provided in the next example:

Example 9.1 (Brownian motion with constant drift μ and intensity σ). A Brownian motion (with drift) $\mathcal{X}(t)$ is the solution of the following SDE:

$$\mathrm{d}\mathcal{X}(t) = \mu\mathrm{d}t + \sigma\mathrm{d}\mathcal{W}(t),$$

with initial value $\mathcal{X}(0) = \mathcal{X}_0$. By direct integration, we get

$$\mathcal{X}(t) = \mathcal{X}_0 + \mu t + \sigma\mathcal{W}(t)$$

and hence $\mathcal{X}(t)$ is normally distributed with $\mathcal{X}_0 + \mu t$ and variance $\sigma^2 t$.

Globally, the aim of this section is to derive and examine the solution of a stochastic differential equation of the form

$$\mathrm{d}\mathcal{X}(t) = f(t, \mathcal{X}(t))\mathrm{d}t + g(t, \mathcal{X}(t))\mathrm{d}\mathcal{W}(t), \quad t \in [t_0, T],\ T > 0, \tag{II.11}$$

with initial value $\mathcal{X}(t_0) = \mathcal{X}_0$, where $\mathcal{X}_0$ is an $\mathcal{F}_{t_0}$-measurable $\mathbb{R}^n$-valued random variable such that $\mathbb{E}|\mathcal{X}_0|^2 < \infty$. Here, f and g are called the drift and diffusion coefficient functions, respectively. Generally speaking, we can view f as measuring short-time growth and g as representing short-term variability. When $g(t, \mathcal{X}(t)) \equiv 0$ in (II.11), the obtained expression is clearly the ordinary differential equation (ODE) (see [28]). Looking to details, we pursue to answer questions such as the following:

- Under what conditions is there a solution?
- If a solution exists, is it unique?
- If an exact solution is established, in what sense is it stable?

In order to resolve these issues, let $\mathcal{W}(t) = (\mathcal{W}_1(t), \ldots, \mathcal{W}_m(t))$ be an m-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let also $f : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be both Borel-measurable functions. Hence, the equation (II.11) is equivalent to the following stochastic integral representation:

$$\mathcal{X}(t) = \mathcal{X}_0 + \int_{t_0}^{t} f(s, \mathcal{X}(s))\mathrm{d}s + \int_{t_0}^{t} g(s, \mathcal{X}(s))\mathrm{d}\mathcal{W}(s), \quad t \in [t_0, T]. \tag{II.12}$$

Conceptually, an $\mathbb{R}^n$-dimensional stochastic process $\mathcal{X}(t) = \{\mathcal{X}(t)\}_{t\in[t_0,T]}$ is called a solution of (II.11) if

- $\mathcal{X}(t)$ is continuous and $\mathcal{F}_t$-adapted,
- $\{f(t, \mathcal{X}(t))\} \in \mathfrak{L}^1([t_0, T]; \mathbb{R}^n)$ and $\{g(t, \mathcal{X}(t))\} \in \mathfrak{L}^2(\mathbb{R}_+; \mathbb{R}^{n\times m})$,
- Equation (II.12) holds for every $t \in [t_0, T]$ almost surely.

A solution $\mathcal{X}(t)$ is said to be unique if any other solution $\tilde{\mathcal{X}}(t)$ is indistinguishable from $\mathcal{X}(t)$, that is

$$\mathbb{P}\Big(\mathcal{X}(t) = \tilde{\mathcal{X}}(t) \quad \text{for all } t \in [t_0, T]\Big) = 1.$$

Now, we present the conditions that guarantee the existence and uniqueness of the solution to equation (II.11).

Theorem 9.2 ([51]). *If the coefficient functions f, g of* SDE (II.11) *verify the conditions*

$$|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq \mathfrak{C}|x - y| \tag{II.13}$$
[Uniform Lipschitz condition]

for some positive constant $\mathfrak{C}$ and all $x, y \in \mathbb{R}^n$, $t \in [t_0, T]$; and

$$|f(t, x)|^2 + |g(t, x)|^2 \leq \bar{\mathfrak{C}}(1 + |x|^2) \tag{II.14}$$
[Linear growth condition]

for some positive constant $\bar{\mathfrak{C}}$ and all $(t, x) \in [t_0, T] \times \mathbb{R}^n$. Then, there exists a unique solution $\mathcal{X}(t)$ in $\mathcal{M}^2([t_0, T]; \mathbb{R}^n)$ of (II.11). *Furthermore, this solution verifies* $\sup_{t_0\leq t\leq T} \mathbb{E}|\mathcal{X}(t)|^2 < \infty$.

From the last theorem, $\mathcal{X}(t)$ is uniformly bounded in $\mathcal{M}^2([t_0, T]; \mathbb{R}^n)$ and if $\mathcal{X}(t)$, $\tilde{\mathcal{X}}(t)$ are both solutions of (II.11) with the same initial value, then

$$\mathbb{P}\left(\sup_{t_0\leq t\leq T} |\mathcal{X}(t) - \tilde{\mathcal{X}}(t)| = 0\right) = 1.$$

Consequently, the solution $\mathcal{X}(t)$ is unique in a pathwise sense.

The Lipschitz condition (II.13) ensures that f and g do not change faster than a linear function relative to changes in x. This implies the Lipschitz continuity of $f(t,\cdot)$ and $g(t,\cdot)$ for all $t \in [t_0, T]$. The linear growth condition (II.14) avails to bound f and g uniformly with respect to $t \in [t_0, T]$. This constraint on the growth of f and g ensures that almost surely the solution $\mathcal{X}(t)$ does not "explode" (i.e., tend to $+\infty$) in the interval $[t_0, T]$ regardless of the $\mathcal{X}_0$ value.

The above-mentioned solution is called a strong solution since it has a strong unique sample trajectory and because the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, the Brownian motion $\mathcal{W}(t)$, and even coefficient functions f and g are all given and specified precisely in advance. But, if just the coefficients $f(t,x)$ and $g(t,x)$ are preliminary specified and the pair of processes $(\tilde{\mathcal{X}}(t), \tilde{\mathcal{W}}(t))$ are defined on a constructed and appropriate filtered probability space such that (II.11) holds, then $\tilde{\mathcal{X}}(t)$ is called a weak solution. Specifically, the main difference between strong and weak solutions is that for strong solutions we are given Brownian motion defined on a given filtered probability space, while for weak solutions we are free to choose and construct a Brownian motion and an appropriate filtered probability space. If two weak solutions established under the said conditions are indistinguishable, then the pathwise uniqueness holds for (II.11). Moreover, if two solutions (weak or strong) of (II.11) have the same finite-dimensional probability distribution, then they are weakly unique. It should be noted that a strong solution to (II.11) is also a weak one, but the reverse is not generally true. According to the result presented in (Karatzas and Shreve [51], Corollary 5.3.23), the existence of the weak solution and the pathwise uniqueness implies the existence of a strong solution. Also, the pathwise uniqueness implies the weak uniqueness. Basically, in this thesis, we are always concerned with strong solutions.

The Lipschitz condition (II.13) and the linear growth condition (II.14) seem to be too restrictive hypotheses for the existence and uniqueness of the solutions to the SDE (II.11). The next theorem is a generalization of Theorem 9.2 in which the uniform Lipschitz condition (II.13) is replaced by the local Lipschitz condition.

Theorem 9.3. *Assume that the linear growth condition (II.14) holds, but the Lipschitz condition (II.13) is substituted with the following new local one: for every integer $\ell \geq 1$, there exists a positive constant $\mathfrak{C}_\ell$ such that, for all $x, y \in \mathbb{R}^n$, $t \in [t_0, T]$, $T > 0$ and with $\max\{|x|, |y|\} \leq \ell$,*

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \leq \mathfrak{C}_\ell |x - y|. \tag{II.15}$$
[Local Lipschitz condition]

Then, there exists a unique solution $\mathcal{X}(t)$ to SDE (II.11) in $\mathcal{M}^2([t_0, T]; \mathbb{R}^n)$.

The local Lipschitz condition (II.15) permits us to include numerous functions of f and g like the functions that are continuously differentiable in x on $[t_0, T] \times \mathbb{R}$. Likewise, linear growth can be replaced by a monotone condition as presented by the following theorem.

Theorem 9.4. *Assume that the local Lipschitz condition (II.15) holds, but the linear growth condition (II.14) is replaced with the following monotone condition: there exists a positive constant $\tilde{\mathfrak{C}}$ such that for all $(t,x) \in [t_0, T] \times \mathbb{R}^n$,*

$$x^T f(t,x) + 0.5|g(t,x)|^2 \leq \tilde{\mathfrak{C}}(1 + |x|^2). \tag{II.16}$$
[Monotone condition]

Then, there exists a unique solution $\mathcal{X}(t)$ to SDE (II.11) in $\mathcal{M}^2([t_0, T]; \mathbb{R}^n)$.

Generally, the monotone condition (II.16) guarantees the existence of the solution $\mathcal{X}(t)$ on the whole of the interval $[t_0, T]$. Furthermore, it should be emphasized that the linear growth condition (II.14) implies the monotone condition (II.16), but the reverse is not true. In the case of an autonomous[a] SDE, (II.11) is expressed as follows

$$\mathrm{d}\mathcal{X}(t) = f(\mathcal{X}(t))\mathrm{d}t + g(\mathcal{X}(t))\mathrm{d}\mathcal{W}(t), \quad t \in [t_0, T],\ T > 0, \tag{II.17}$$

[a] Here, both f and g are independent of the time t and $f(t, \mathcal{X}(t)) \equiv f(\mathcal{X}(t))$, $g(t, \mathcal{X}(t)) \equiv g(\mathcal{X}(t))$.

with initial value $\mathcal{X}(t_0) = \mathcal{X}_0$. If $\mathcal{X}_0$ is a random variable independent of $\mathcal{W}(t) - \mathcal{W}(t_0)$, then, under the conditions of (II.13) and (II.14), SDE (II.17) has a unique, continuous global[a] solution $\mathcal{X}(t)$ on $[t_0, \infty)$ such that

$$|f(x) - f(y)| + |g(x) - g(y)| \leq \mathfrak{C}|x - y|,$$
[Lipschitz condition]

is verified, and for a fixed $x \in \mathbb{R}^n$ value,

$$|f(x)|^2 + |g(x)|^2 \leq \mathfrak{C}^2(1 + |x|^2).$$
[Linear growth condition]

is satisfied.

Example 9.5 (Geometric Brownian motion). An autonomous geometric Brownian motion $\mathcal{X}(t)$ is the solution of an SDE with the drift constant μ and the diffusion coefficient $\sigma > 0$ which is expressed as follows:

$$\mathrm{d}\mathcal{X}(t) = \mu\mathcal{X}(t)\mathrm{d}t + \sigma\mathcal{X}(t)\mathrm{d}\mathcal{W}(t),$$

with initial value $\mathcal{X}(0) = \mathcal{X}_0 > 0$. The unique analytic solution is presented as following

$$\mathcal{X}(t) = e^{\log \mathcal{X}_0 + \bar{\mu}t + \sigma\mathcal{W}(t)} = \mathcal{X}_0 e^{\bar{\mu}t + \sigma\mathcal{W}(t)},$$

where $\bar{\mu} = \mu - 0.5\sigma^2$. Therefore, $\mathbb{E}\big(\mathcal{X}(t)\big) = \mathcal{X}_0 e^{\mu t}$ and $\mathrm{Var}\big(\mathcal{X}(t)\big) = \mathcal{X}_0^2 e^{2\mu t}\big(e^{\sigma^2 t} - 1\big)$.

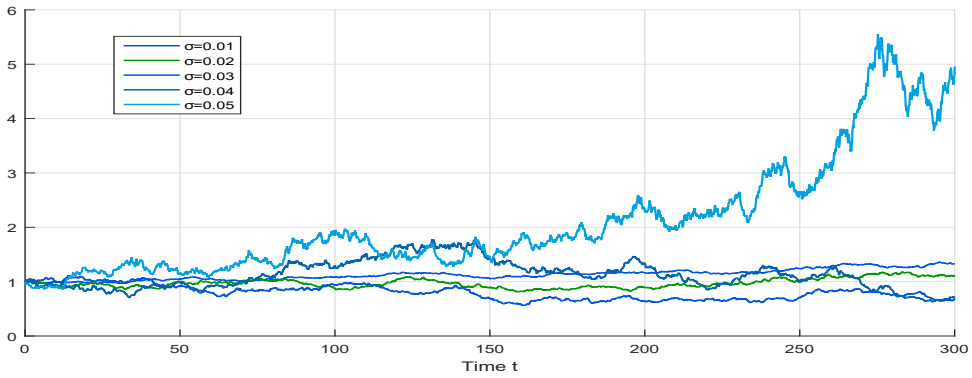


Figure II.3: Numerical simulations of the paths of the geometric Brownian motion with $\mu = 0.01$ and different values of intensities $\sigma = 0.01, 0.02, 0.03, 0.04, 0.05$.

Now, we will present some interesting properties of the solution $\mathcal{X}(t)$ to (II.11) under the hypotheses of Theorem 9.2. Specifically, we say that $\mathcal{X}(t)$, with $\mathcal{X}(t_0) = \mathcal{X}_0$, is stochastically bounded if for each $\epsilon > 0$ there exists a $\gamma_\epsilon = \gamma_\epsilon(t_0, \mathcal{X}_0) > 0$ such that

$$\inf_{t \in [t_0, T]} \mathbb{P}(|\mathcal{X}(t)| \leq \gamma_\epsilon) > 1 - \epsilon.$$

If γ_ϵ depends only on $\mathcal{X}_0$, then $\mathcal{X}(t)$ is said to be uniformly stochastically bounded. In addition, $\mathcal{X}(t)$ is continuous on $[t_0, T]$ if there is a constant $\mathfrak{b}$ such that

$$|\mathcal{X}(t) - \mathcal{X}(s)|^2 \leq \mathfrak{b}|t - s|, \quad t_0 \leq s, \ t \leq T,$$

where

$$\mathcal{X}(t) - \mathcal{X}(s) = \int_s^t f(u, \mathcal{X}(u))\mathrm{d}u + \int_s^t g(u, \mathcal{X}(u))\mathrm{d}\mathcal{W}(u).$$

Suppose that $\mathbb{E}(|\mathcal{X}_0|)^{2p} < \infty$, where p is a positive integer. Then, for the solution $\mathcal{X}(t)$ to the SDE (II.11) on $[t_0, T]$,

[a] Under assumptions of Theorem 9.2 on every finite sub-interval $[t_0, T] \subset [t_0, \infty)$, SDE $\mathrm{d}\mathcal{X}(t) = f(t, \mathcal{X}(t))\mathrm{d}t + g(t, \mathcal{X}(t))\mathrm{d}\mathcal{W}(t)$, $t \in [t_0, \infty)$ has a unique global solution $\mathcal{X}(t)$ defined on the entirety of $[t_0, \infty)$.

- $\mathbb{E}|\mathcal{X}(t)|^{2p} \leq \Big(1+\mathbb{E}(|\mathcal{X}_0|)^{2p}\Big)e^{\mathcal{M}(t-t_0)}$,
- $\mathbb{E}|\mathcal{X}(t)-\mathcal{X}_0|^{2p} \leq \bar{\mathcal{M}}\Big(1+\mathbb{E}(|\mathcal{X}_0|)^{2p}\Big)(t-t_0)^p e^{\mathcal{M}(t-t_0)}$, where $\mathcal{M}=2p(2p+1)\mathfrak{C}^2$ and $\bar{\mathcal{M}}$ are constants (dependent upon p, $\mathfrak{C}$, and $T-t_0$).
- $\mathbb{E}|X(t)-X(s)|^{2p} \leq \mathcal{M}_1|t-s|^p$, $t_0 \leq s,t \leq T$, and $\mathcal{M}_1$ constant.

Moreover, let $p \geq 2$, $\mathcal{X}(t_0)=\mathcal{X}_0 \in \mathrm{L}^p(\Omega;\mathbb{R}^n)$, and suppose that the linear growth condition (II.14) is verified. Then, the p-th-order moment

$$\mathbb{E}|\mathcal{X}(t)|^p \leq 2^{\frac{p-2}{2}}\Big(1+\mathbb{E}(|\mathcal{X}_0|)^p\Big)e^{p\alpha(t-t_0)}, \quad t\in[t_0,T],$$

where $\alpha=\mathfrak{C}+\mathfrak{C}^2(p-1)/2$.

10 SDEs and stability

Basically, the common difficulty with SDEs is that it may not be possible to establish the explicit form of the solution. It is therefore important to focus on the qualitative information that can be extracted about the solutions without actually solving the SDEs. In this regard, in what follows we will study the notion of the stability of a solution. That is, we will examine the question of whether small changes in the initial conditions or parameters of a dynamic system lead to low changes (stability) or large changes (instability) in the solution. When we try to switch the main principles of Lyapunov stability for deterministic ODEs to their stochastic version, some compulsory queries present themselves.

- How to define stochastic stability?
- How to define Lyapunov functions and what properties must a stochastic Lyapunov function have?

To respond to these questions, let us first present the SDE (II.11), where

$$\mathrm{d}\mathcal{X}(t)=f(t,\mathcal{X}(t))\mathrm{d}t+g(t,\mathcal{X}(t))\mathrm{d}\mathcal{W}(t), \quad t\geq t_0. \tag{II.18}$$

Here, we assume that the assumptions of Theorem 9.2 are satisfied and almost surely $\mathcal{X}(t_0)=\mathcal{X}_0\in\mathbb{R}^n$. For any $\mathcal{X}_0$ independent of $\mathcal{W}(t)$, $t\geq t_0$, Equation (II.18) has a unique global solution $\mathcal{X}(t)\equiv X_t(t_0,\mathcal{X}_0)$ with continuous and finite moments sample paths. Suppose that

$$f(t,0)=0 \ \text{ and } \ g(t,0)=0 \quad \text{for all } t\geq t_0. \tag{II.19}$$

Hence, equation (II.18) has the solution $\mathcal{X}(t)\equiv\mathbf{0}$ corresponding to the initial value $\mathcal{X}(t_0)=0$. This solution is called the trivial solution or equilibrium position.

Moreover, let $0<\mathfrak{s}\leq\infty$ and $\mathcal{S}_\mathfrak{s}=\{\mathcal{X}\in\mathbb{R}^n: \ |\mathcal{X}|<\mathfrak{s}\}$. Denote by $C^{1,2}(\mathbb{R}_+\times\mathcal{S}_\mathfrak{s};\mathbb{R}_+)$ the family of all non-negative functions $\mathcal{V}(t,\mathcal{X}(t))$ defined on $\mathbb{R}_+\times\mathcal{S}_\mathfrak{s}$ such that they are continuously twice differentiable in $\mathcal{X}$ and once in t. Then, the differential operator $\mathcal{L}$ associated with equation (II.18) is

$$\mathcal{L}=\frac{\partial}{\partial t}+\sum_{i=1}^n f_i(t,x)\frac{\partial}{\partial x_i}+0.5\sum_{i,k=1}^n\Big\{g(t,x)g^T(t,x)\Big\}_{ik}\frac{\partial^2}{\partial x_i\partial x_k}.$$

If $\mathcal{L}$ applies on the function $\mathcal{V}\in C^{1,2}(\mathbb{R}_+\times\mathcal{S}_\mathfrak{s};\mathbb{R}_+)$, then

$$\mathcal{L}\mathcal{V}(t,x)=\mathcal{V}_t(t,x)+\mathcal{V}_x(t,x)f(t,x)+0.5trace\Big\{g^T(t,x)\mathcal{V}_{xx}(t,x)g(t,x)\Big\}.$$

Note that if $x(t)\in\mathcal{S}_\mathfrak{s}$, we have via Itô's formula that

$$\mathrm{d}\mathcal{V}(t,x(t))=\mathcal{L}\mathcal{V}(t,x(t))\mathrm{d}t+\mathcal{V}_x(t,x(t))g(t,x(t))\mathrm{d}\mathcal{W}(t). \tag{II.20}$$

In the stochastic case, the presence of the term $d\mathcal{W}(t)$ in (II.20) implies that the stability of $\mathcal{X}(t)$ to (II.18) requires that $\mathbb{E}\big(d\mathcal{V}(t)\big) \leq 0$. This condition is verified if $\mathcal{LV}\big(t,\mathcal{X}(t)\big) \leq 0$ for all $t \geq 0$ since $\mathbb{E}\big(d\mathcal{V}(t)\big) = \mathbb{E}\Big(\mathcal{LV}(\mathcal{X}(t))dt\Big)$. So, the stability condition $\dfrac{d\mathcal{V}}{dt} \leq 0$ of the deterministic case is substituted by $\mathcal{LV}(t,\mathcal{X}(t)) \leq 0$. Hence, the function $\mathcal{V}$ serves as the stochastic Lyapunov version to SDE (II.18)

10.1 Stability in probability

Under the aforementioned assumptions, our discussion of the stochastic stability of the equilibriums should now be framed in terms of probability.

Definition 10.1. The trivial solution $\mathcal{X}(t) \equiv 0$ (under the hypothesis (II.19)) is termed stochastically stable (stable in probability) if for every $\epsilon > 0$,

$$\lim_{\mathcal{X}_0 \to 0} \mathbb{P}\left(\sup_{t \geq t_0} |\mathcal{X}_t(t_0, \mathcal{X}_0)| \geq \epsilon\right) = 0,$$

otherwise, the said solution is stochastically unstable. The equilibrium solution is stochastically asymptotically stable if it is stochastically stable and

$$\lim_{\mathcal{X}_0 \to 0} \mathbb{P}\left(\lim_{t \to \infty} \mathcal{X}_t(t_0, \mathcal{X}_0) = 0\right) = 1.$$

The equilibrium solution is globally stochastically asymptotically stable or stochastically asymptotic stable in the large if it is stochastically stable and, for all $\mathcal{X}_0 \in \mathbb{R}^n$

$$\mathbb{P}\left(\lim_{t \to \infty} \mathcal{X}_t(t_0, \mathcal{X}_0) = 0\right) = 1.$$

Recall that we always assume that the assumptions of the existence an uniqueness of the solution are fulfilled and, moreover, $f(t,0) = 0$, $g(t,0) = 0$. To generalize and extend the Lyapunov method for studying the stability of ODEs to that of SDEs, we present the following theorems.

Theorem 10.2 (stochastic stability). *If there exists a positive-definite*[a] *stochastic Lyapunov function* $\mathcal{V}(t,x) \in C^{1,2}([t_0,\infty) \times \mathcal{S}_{\mathfrak{s}}; \mathbb{R}_+)$ *such that* $\mathcal{LV}(t,x) \leq 0$ *for all* $(t,x) \in [t_0,\infty) \times \mathcal{S}_{\mathfrak{s}}$, *then the trivial of equation (II.18) is stochastically stable.*

Theorem 10.3 (Stochastic asymptotically stability). *If there exists a positive-definite decrescent*[b] *function* $\mathcal{V}(t,x) \in C^{1,2}([t_0,\infty)\times\mathcal{S}_{\mathfrak{s}}; \mathbb{R}_+)$ *such that* $\mathcal{LV}(t,x)$ *is negative-definite*[c], *then the trivial solution of equation (II.18) is stochastically asymptotically stable.*

Theorem 10.4 (Global asymptotic stability). *If there exists a positive-definite decrescent radially unbounded function* $\mathcal{V}(t,x) \in C^{1,2}([t_0,\infty) \times \mathbb{R}^n; \mathbb{R}_+)$ *such that* $\mathcal{LV}(t,x)$ *is negative-definite, then the trivial solution of equation (II.18) is stochastically asymptotically stable in the large.*

10.2 Almost sure exponential stability

Now, we present the formal definition of the almost sure exponential stability.

Definition 10.5. The trivial solution of equation (II.18) (under the hypothesis (II.19)) is said to be almost surely exponentially stable if

$$\limsup_{t \to \infty} \frac{1}{t} \log |\mathcal{X}_t(t_0, \mathcal{X}_0)| < 0 \quad \text{a.s.} \tag{II.21}$$

for all $\mathcal{X}_0 \in \mathbb{R}^n$. The left-hand side of the last inequality is termed the sample Lyapunov exponents of the solution $\mathcal{X}$.

[a] A continuous function $\mathcal{V}(t,x)$ defined on $[t_0,\infty) \times \mathcal{S}_{\mathfrak{s}}$ is termed positive-definite if $\mathcal{V}(t,0) \equiv 0$ and for some $\mu \in \mathfrak{K}$ ($\mathfrak{K}$ is the family of all continuous nondecreasing functions $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(0) = 0$ and $\mu(r) > 0$ if $r > 0$), $\mathcal{V}(t,x) \geq \mu(|x|)$ for all $t \in [t_0,\infty)$ and $x \in \mathcal{S}_{\mathfrak{s}}$.

[b] A continuous positive function $\mathcal{V}$ is termed decrescent if for some $\mu \in \mathfrak{K}$, $\mathcal{V}(t,x) \leq \mu(|x|)$ for all $(t,x) \in [t_0,\infty) \times \mathcal{S}_{\mathfrak{s}}$.

[c] A function $\mathcal{V}$ i aid to be negative-definite if $-\mathcal{V}$ is positive-definite

Generally, the almost sure exponential stability means that almost all sample paths of the solution will tend to the equilibrium state $\mathcal{X} = 0$ exponentially fast if and only if the sample Lyapunov exponents are negative. Under this standing hypothesis, we have the following useful theorem.

Theorem 10.6 (Almost sure exponential stability). *Assume that there exists a function $\mathcal{V}(t,\mathcal{X}) \in C^{1,2}([t_0,\infty)\times\mathbb{R}^n;\mathbb{R}_+)$, and constants $p>0$, $\mathfrak{C}_1>0$, $\mathfrak{C}_2\in\mathbb{R}$, $\mathfrak{C}_3\geq 0$, such that for all $x\neq 0$ and $t\geq t_0$,*

- $\mathfrak{C}_1|x|^p \leq \mathcal{V}(t,x)$,
- $\mathcal{L}\mathcal{V}(t,x) \leq \mathfrak{C}_2\mathcal{V}(t,x)$,
- $|\mathcal{V}_x(t,x)g(t,x)|^2 \geq \mathfrak{C}_3\mathcal{V}^2(t,x)$.

Then

$$\limsup_{t\to\infty}\frac{1}{t}\log|\mathcal{X}_t(t_0,\mathcal{X}_0)| \leq \frac{2\mathfrak{C}_2-\mathfrak{C}_3}{2p} \quad \text{a.s.}$$

for all $\mathcal{X}_0\neq 0$ in $\mathbb{R}^n$. Especially, if $2\mathfrak{C}_2-\mathfrak{C}_3<0$, the trivial solution of SDE (II.18) is almost surely exponentially stable.

10.3 Moment exponential stability

In this subsection, we will present the pth moment exponential stability for SDE (II.18), where $p>0$. We begin with the following definition.

Definition 10.7. The trivial solution of SDE (II.18) is said to be pth moment exponentially stable if there is a pair of positive constants $\mathfrak{d}_1$ and $\mathfrak{d}_2$ such that

$$\mathbb{E}(|\mathcal{X}_t(t_0,\mathcal{X}_0)|^p) \leq \mathfrak{d}_1|\mathcal{X}_0|^p e^{-\mathfrak{d}_2(t-t_0)} \quad \text{on} \quad t\geq t_0$$

for all $\mathcal{X}_0\in\mathbb{R}^n$. In particular, when $p=2$, it is usually said to be exponentially stable in mean square.

Evidently, the pth moment exponential stability shows that the pth moment of the solution will tend to 0 exponentially fast. That is,

$$\limsup_{t\to\infty}\frac{1}{t}\log\Big(\mathbb{E}(|\mathcal{X}_t(t_0,\mathcal{X}_0)|^p)\Big)<0.$$

Globally speaking, the pth moment exponential stability and the almost sure exponential stability do not imply each other and extra conditions are required in order to establish one from the other. The next result gives the hypotheses under which the pth moment exponential stability implies the almost sure exponential stability.

Theorem 10.8. *Assume that there is a positive constant $\mathfrak{C}$ such that*

$$x^T f(t,x)\vee|g(t,x)|^2 \leq \mathfrak{C}|\mathcal{X}|^2 \quad \text{for all } (t,x)\in[t_0,\infty)\times\mathbb{R}^n.$$

Then, the pth moment exponential stability of the trivial solution of equation (II.18) implies the almost sure exponential stability.

Now, we will offer a sufficient criterion for the pth moment exponential stability via a Lyapunov function.

Theorem 10.9. *Suppose that there is a function $\mathcal{V}(t,x)\in C^{2,1}([t_0,\infty)\times\mathbb{R}^n;\mathbb{R}_+)$, and positive constants $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$, such that*

$$\mathfrak{p}_1|x|^p \leq \mathcal{V}(t,x) \leq \mathfrak{p}_2|x|^p,$$
$$\mathcal{L}\mathcal{V}(t,x) \leq -\mathfrak{p}_3\mathcal{V}(t,x),$$

for all $(t,x)\in[t_0,\infty)\times\mathbb{R}^n$. Then

$$\mathbb{E}(|\mathcal{X}_t(t_0,\mathcal{X}_0)|^p) \leq \frac{\mathfrak{p}_2}{\mathfrak{p}_1}|\mathcal{X}_0|^p e^{-\mathfrak{p}_3(t-t_0)} \quad \text{on } t\geq t_0,$$

for all $\mathcal{X}_0\in\mathbb{R}^n$. That is, the trivial solution of SDE (II.18) is pth moment exponentially stable and the pth moment Lyapunov exponent should not be greater than $-\mathfrak{p}_3$.

11 Ergodic stationary distribution and periodicity

Let $\mathcal{X} \equiv \{\mathcal{X}_t\}_{t\geq 0}$ be a stochastic process with values in $\mathbb{R}^n$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The σ-algebra $\mathcal{F}_t = \sigma(\mathcal{X}_s,\ 0 \leq s \leq t)$, $t \geq 0$, is the history of the process up to and including time t. More accurately, $\mathcal{F}_t$ records the information ready from our observation of $\mathcal{X}_s$ for all times $0 \leq s < t$. $\mathcal{X}$ is called a Markov process if the Markov property

$$\mathbb{P}(\mathcal{X}_t \in B|\mathcal{F}_s) = \mathbb{P}(\mathcal{X}_t \in B|\mathcal{X}_s)$$

holds a.s. for all $0 \leq s \leq t < \infty$ and all Borel sets $B \subset \mathbb{R}$. Clearly, given the value of $\mathcal{X}_s$, one can prophesy the probabilities of outlook values $\mathcal{X}_t$ as well as if you knew the whole history of the process ahead of time s. The process only knows $\mathcal{X}_s$ and is not conscious of how it got there so that the after time depends on the past only through the present, that is, once the present is recognized, the past and future are independent.

The transition probability of the Markov process is a function $\mathbb{P}(s, x; t, B)$ defined on $0 \leq s \leq t < \infty$, $x \in \mathbb{R}$, with the following properties:

1. For every $0 \leq s \leq t < \infty$, $\mathbb{P}\big(s, \mathcal{X}_s; t, B\big) = \mathbb{P}\big(\mathcal{X}_t \in B|\mathcal{X}_s\big)$.
2. For every $0 \leq s \leq t < \infty$ and $x \in \mathbb{R}$, $\mathbb{P}\big(s, x; t, \cdot\big)$ is a probability measure on the family of Borel sets $\mathcal{B}$.
3. For every $0 \leq s \leq t < \infty$ and $B \in \mathcal{B}$, $\mathbb{P}(s, \cdot; t, B)$ is Borel measurable.
4. For every $0 \leq s \leq r \leq t < \infty$ and $x \in \mathbb{R}$, and $B \in \mathcal{B}$, [a]

$$\mathbb{P}(s, x; t, B) = \int_{\mathbb{R}} \mathbb{P}(r, y; t, B)\mathbb{P}(s, x; r, dy).$$

[Chapman-Kolmogorov equation]

In terms of the previous transition probability, the Markov property turns as $\mathbb{P}(\mathcal{X}_t \in B|\mathcal{F}_s) = \mathbb{P}(s, \mathcal{X}_s; t, B)$; and hence we write $\mathbb{P}(\mathcal{X}_t \in B|\mathcal{X}_s = x) = \mathbb{P}(s, x; t, B)$ which is the probability that the process will be in set B at time t given that the process was in state x at time $s \leq t$. A Markov process $\mathcal{X}$ is termed homogeneous (with respect to t) if its transition probability $\mathbb{P}(s, x; t, B)$ is stationary, that is,

$$\mathbb{P}(s + u, x; t + u, B) = \mathbb{P}(s, x; t, B), \quad 0 \leq s \leq t < \infty, \quad x \in \mathbb{R}, \quad \text{and} \quad B \in \mathcal{B}.$$

In this case, the transition probability is a function of x, B, and the difference $t-s$ since $\mathbb{P}(s, x; t, A) = \mathbb{P}(0, x; t - s, B)$. Therefore, we can purely write $\mathbb{P}(0, x; t, B) = \mathbb{P}(x; t, B)$. Generally and without losing generality, let $\mathbb{P}(x; t, \cdot)$ denote the probability measure defined as follows

$$\mathbb{P}(x; t, B) = \mathbb{P}(\mathcal{X}(t) \in B|\mathcal{X}(0) = x), \quad \text{for any Borel set } B \subset \mathbb{R}^n.$$

If there is a probability measure $\pi(\cdot)$ on the measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that

$$\lim_{t\to\infty} \mathbb{P}(x; t, B) = \pi(B) \quad \text{for all } x \in \mathbb{R}^n.$$

Then, the stochastic differential equation (II.17) has a stationary distribution $\pi(\cdot)$[b]. With reference to the book of Khasminskii [57], we have the following lemma, which is very helpful to prove the

[a] This property means that a single-step transition probability can be expressed in terms of a combination of two-step transition probabilities with respect to an arbitrary intermediate time r.

[b] A stationary distribution for a Markov process is a probability measure π over a state space $\mathbb{R}^n$ that satisfies

$$\int_{\mathbb{R}^n} \mathbb{P}(x; t, B)\pi(\mathrm{d}x) = \pi(B).$$

A stationary distribution of a Markov process is a probability distribution that remains unchanged as time progresses.

result related to the stationary distribution. Let $\mathcal{X}$ be a homogeneous Markov process in $\mathbb{R}^n$, and is described by the following stochastic differential equation:

$$\mathrm{d}\mathcal{X}(t) = f(\mathcal{X}(t))\mathrm{d}t + \sum_{r=1}^{k} g_r(\mathcal{X}(t))\mathrm{d}\mathcal{W}_r(t).$$

The diffusion matrix is defined as follows:

$$M(x) = \big(a_{ij}(x)\big), \qquad a_{ij}(x) = \sum_{r=1}^{k} g_r^i(x) g_r^j(x).$$

Lemma 11.1. *The Markov process $\mathcal{X}$ has a unique ergodic[a] stationary distribution π if there exists a bounded open domain $D \subset \mathbb{R}^n$ with regular boundary such that its closure $\bar{D} \subset \mathbb{R}^n$, and the following hypotheses hold*

- *In the open domain D and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $M(x)$ is bounded away from zero.*
- *If $x \in \mathbb{R}^n \setminus D$, the mean time τ at which a path issuing from x reaches the set D is finite, and $\sup_{x \in K} \mathbb{E}\tau < \infty$ for every compact subset $K \subset \mathbb{R}^n$.*

Moreover, if f is a function integrable with respect to the measure π, then

$$\mathbb{P}\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f\big(\mathcal{X}_t\big)\mathrm{d}t = \int_{\mathbb{R}^n} f(x)\pi(\mathrm{d}x)\right\} = 1,$$

for all $x \in \mathbb{R}^n$.

Dissimilar from the above result, in this thesis, we use the Feller property and mutually exclusive possibilities to derive the condition for existence of the ergodic stationary distribution, which can close the gap left by using the Khasminskii method.

The definition of Markov processes, is much too general for many applications. However, many of the processes which we study also satisfy the much stronger Feller property. The definition of Feller processes involves putting continuity constraints on the transition function, for which it is necessary to restrict attention to processes lying in a topological space $(\mathbb{E}, \mathcal{T}_{\mathbb{E}})$. It will be assumed that $\mathbb{E}$ is locally compact[b], Hausdorff[c], and has a countable base[d] (lccb, for short).

Given a topological space $\mathbb{E}$, $C_0(\mathbb{E})$ denotes the continuous real-valued functions vanishing at infinity[e]. That is, $f : \mathbb{E} \to \mathbb{R}$ is in $C_0(\mathbb{E})$ if it is continuous and, for any $\epsilon > 0$, the set $\{x : \ |f(x)| \geq \epsilon\}$ is compact.

[a] Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and ϕ_t, $t \in \mathbb{R}$ a group of measurable transformations of Ω. We say that ϕ_t is measure preserving if $\mu(\phi_t^{-1}(A)) = \mu(A)$ for all $t \in \mathbb{R}$ and all $A \in \mathcal{F}$. Now, let ϕ_t be a group of measure preserving transformations of $(\Omega, \mathcal{F}, \mu)$. Then, for any $f \in L^1(\mu)$ the limit

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\phi_s(x))\mathrm{d}s = f^*(x)$$

exists μ-a.s. The limit $f^*(x)$ is ϕ_t invariant, and $\int_\Omega f\mathrm{d}\mu = \int_\Omega f^*\mathrm{d}\mu$. The group of transformation ϕ_t is said to be ergodic if f^* is constant μ-a.s. and in that case $f^*(x) = \int f\mathrm{d}\mu$.

[b] Let $\mathbb{E}$ be a topological space. Most commonly $\mathbb{E}$ is called locally compact if every point x of $\mathbb{E}$ has a compact neighbourhood.

[c] In topology, a Hausdorff space (separated space) is a topological space where for any two distinct points there exist neighbourhoods of each which are disjoint from each other.

[d] A base for the topology $\mathcal{T}_{\mathbb{E}}$ of a topological space $(\mathbb{E}, \mathcal{T}_{\mathbb{E}})$ is a family B of open subsets of $\mathbb{E}$ such that every open set of the topology is equal to a union of some sub-family of B.

[e] A function is said to vanish at infinity if its values approach 0 as the input grows without bounds.

Definition 11.2 (Feller property). Let $\mathbb{E}$ be an lccb space. Then, a transition function $\{\mathbb{P}_t\}_{t\geq 0}$ is feller if, for all $f \in C_0(\mathbb{E})$,

- $\mathbb{P}_t f \in C_0(\mathbb{E})$.
- $t \mapsto \mathbb{P}_t f$ is continuous with respect to the norm topology on $C_0(\mathbb{E})$.
- $\mathbb{P}_0 f = f$.

A Markov process $\mathcal{X}$ whose transition function is Feller is a Feller process.

Proposition 11.3 ([12]). *Let $\mathcal{X}$ be a process that starts with initial value x. $\mathcal{X}$ is said to be a Feller process if, for any fixed $t \geq 0$ and any bounded, continuous and $\mathcal{F}$-measurable function $h : \mathbb{R}^n \to \mathbb{R}$, $\mathbb{E}\big[h(\mathcal{X}_t)\big]$ depends continuously upon x.*

Now, we present a lemma which gives mutually exclusive possibilities for the existence of an ergodic stationary distribution to the system (I.2).

Lemma 11.4 (*Mutually exclusive possibilities lemma*, [128]). *Let $\mathcal{X}$ be a stochastic Feller process with values in $\mathbb{R}^n$, then either an ergodic probability measure exists, or*

$$\lim_{t\to\infty} \sup_{\hat{\nu}} \frac{1}{t} \int_0^t \int_{\mathbb{R}^n} \mathbb{P}\big(x; s, \Sigma\big)\hat{\nu}(\mathrm{d}x)\mathrm{d}s = 0, \tag{II.22}$$

for any compact set $\Sigma \subset \mathbb{R}^n$, where the supremum is taken over all initial distributions $\hat{\nu}$ on $\mathbb{R}^n$ and $\big(x; s, \Sigma\big)$ is the probability for $\mathcal{X} \in \Sigma$ with $\mathcal{X}(0) = x \in \mathbb{R}^n$.

In the natural world, due to individual life cycles and seasonal variation, birth rate, and other parameters always present periodic changes for population models. For the stochastic biological system, the existence of a periodic solution is a very important dynamical behavior. Let us present the definition of a periodic solution.

Definition 11.5. A stochastic process $\mathcal{X}(t) = \mathcal{X}(t, \omega)$ $(-\infty < t < +\infty)$ is said to be periodic with period θ if for every finite sequence of numbers $t_1, t_2, \ldots, t_n$ the joint distribution of random variables $\mathcal{X}(t_1 + h), \ldots, \mathcal{X}(t_n + h)$ is independent of h, where $h = k\theta$ $(k = \pm 1, \pm 2, \ldots)$.

A Marcov process $\mathcal{X}(t)$ is θ-periodic if and only if its transition probability function (if it exists) is θ-periodic and the function $\mathbb{P}_0(t, A) = \mathbb{P}\{\mathcal{X}(t) \in A\}$ satisfies the equation

$$\mathbb{P}_0(s, A) = \int_{\mathbb{R}^n} \mathbb{P}_0(s, \mathrm{d}x)\mathbb{P}(s, x; s + \theta, A) \equiv \mathbb{P}_0(s + \theta, A).$$

Consider the following equation

$$\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t f(s, \mathcal{X}(s))\mathrm{d}s + \sum_{r=1}^{k} \int_0^t g_r(s, \mathcal{X}(s))\mathrm{d}\mathcal{W}_r(s), \quad \mathcal{X} \in \mathbb{R}^n. \tag{II.23}$$

Lemma 11.6. *Suppose that the coefficients of (II.23) are θ-periodic in t and satisfies the following condition:*

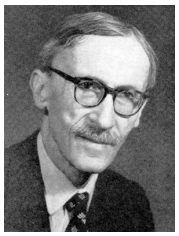
$$|f(s, x) - f(s, y)| + \sum_{r=1}^{k} |g_r(s, x) - g_r(s, y)| \leq \mathcal{M}|x - y|, \qquad |f(s, x)| + \sum_{r=1}^{k} |g_r(s, x)| \leq \mathcal{M}(1 + |x|),$$

in every cylinder $I \times D$, where $\mathcal{M}$ is a constant. Furthermore, we suppose that there exists a function $\mathcal{V}(t, x) \in \mathfrak{C}^2$ in $\mathbb{R}^n$ which is θ-periodic in t, and satisfies the following conditions

$$\inf_{|x|>R} \mathcal{V}(t, x) \to \infty \ \text{ as } R \to \infty \quad \text{and} \quad \mathcal{L}\mathcal{V}(t, x) \leq -1 \quad \text{outside some compact set.}$$

Then, there exists a solution of (II.23) which is a θ-periodic Markov process.

12 Lévy processes



Paul Lévy
(1886-1971)

Theoretical and empirical studies show that for the analysis of some epidemics, it is essential to take into account the possibility of an almost instantaneous movement of large amplitude (jump) in the number of infected or deaths. To model mathematically these phenomena, the jumps processes offer a suitable context to describe and predict the epidemics future. In this section, we define a Lévy process, named after the French mathematician Paul Lévy[a], and we present some indication of how rich a category of processes they form. Specifically, we give some basic concepts and necessary results for the applied calculus of jump-diffusion used in this thesis. Since there are sundry good books and documents which offer a detailed theory, we will just briefly summarize it here. For complete knowledge, we refer the reader to the following books [6, 37, 104]. Now, we begin by making the following definition.

12.1 Poisson random measure and Lévy processes

Definition 12.1. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space. An $\{\mathcal{F}_t\}$-adapted process $\mathcal{Z}(t) \equiv \{\mathcal{Z}_t\}_{t\geq 0}$ in $\mathbb{R}$ is called a *Lévy process* if

1. $\mathcal{Z}(0) = 0$ a.s.
2. $\mathcal{Z}(t)$ has independent increments (i.e. $\mathcal{Z}(t+s)-\mathcal{Z}(t)$ is independent of $\mathcal{F}_t = \sigma(\mathcal{Z}(u),\ \ 0 \leq u \leq t)$.
3. $\mathcal{Z}(t)$ has stationary increments (i.e. the distribution of $\mathcal{Z}_{t+\epsilon} - \mathcal{Z}_t$ depends on ϵ, but not on t).
4. $\mathcal{Z}(t)$ is stochastically continuous (i.e, for all $t \geq 0$ and all $\kappa > 0$, $\mathbb{P}\big(|\mathcal{Z}(t+\epsilon) - \mathcal{Z}(t)| > \kappa\big) \to 0$ as $\epsilon \to 0$).

Let $\mathcal{Z}(t)$ be a Lévy process. Then, $\mathcal{Z}(t)$ has a cadlag version (right continuous with left limits) which is also a Lévy process. In addition, let $\mathcal{B}_0$ be the family of Borel sets $D \subset \mathbb{R}$ whose closure $\bar{D}$ does not contain 0. By denoting $\Delta\mathcal{Z}(t) = \mathcal{Z}(t) - \mathcal{Z}(t^-)$ the jump of $\mathcal{Z}(t)$ at $t > 0$, we can associate the counting measure $\mathcal{N}$ to $\mathcal{Z}(t)$ in the following way: for all $D \in \mathcal{B}_0$, we put

$$\mathcal{N}(t, D) = \mathcal{N}(t, D, \omega) = \sum_{0<s\leq t} \mathbb{1}_D\big(\Delta\mathcal{Z}(s)\big). \tag{II.24}$$

[The number of jumps of size $\Delta\mathcal{Z}(t) \in D$ which occur before or at time t]

Generally, (II.24) is the *Poisson random measure* which indicates the counting measure of jumps in D up to the time t. As the path is cadlag, for all $D \in \mathcal{B}_0$, we have $\mathcal{N}(t, D) < \infty$ a.s.

Certainly, a Brownian motion $\{\mathcal{W}(t)\}_{t\geq 0}$ has stationary and independent increments. Hence, $\mathcal{W}(t)$ is a (continuous) Lévy process. Another significant example is the following.

Example 12.2 (Poisson Process). The Poisson process $\Theta(t)$ with associated intensity $\gamma > 0$ is a Lévy process taking values in $\mathbb{N} \cup \{0\}$ such that

$$\mathbb{P}\big(\Theta(t) = n\big) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, 2, \ldots$$

Let us return to the definition offered in (II.24). The set function $D \mapsto \mathcal{N}(t, D, \omega)$ is a σ-finite measure on $\mathcal{B}_0$ for each fixed t, ω with the associated differential form $\mathcal{N}(t, \mathrm{d}u)$. Furthermore, for each fixed ω, the set function $(a, b] \times D \mapsto \mathcal{N}(b, D, \omega) - \mathcal{N}(a, D, \omega)$, $(a, b] \subset [0, \infty)$, $D \in \mathcal{B}_0$, defines a σ-finite measure with the associated differential form $\mathcal{N}(\mathrm{d}t, \mathrm{d}u)$. Now, we consider the mean

[a] Paul Pierre Lévy was a French mathematician who was active especially in probability theory, introducing fundamental concepts such as Lévy processes, Lévy measures, and Lévy distribution.

measure $\nu(D) = \mathbb{E}\big(\mathcal{N}(1, D)\big)$. This set function also defines a σ-finite measure on $\mathcal{B}_0$ and is called the *Lévy measure* associated to $\mathcal{Z}(t)$. By fixing D in $\mathcal{B}_0$, the process

$$\Theta_D(t) \triangleq \Theta_D(t, \omega) \triangleq \mathcal{N}(t, D, \omega),$$

is a Poisson process with the intensity $\lambda = \nu(D)$.

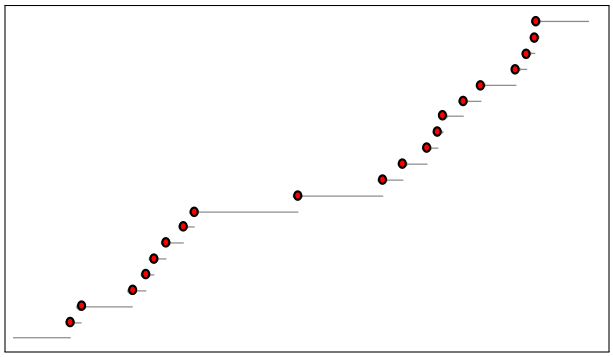


Figure II.4: Illustration of the Poisson process.

Example 12.3 (Compound Poisson process). Let $Q(n)$, $n \in \mathbb{N}$ be a sequence of i.i.d.[a] random variables in $\mathbb{R}$ with common finite distribution $\mu_{Q(1)} = \mu_Q$. Let also $\Theta(t)$ be a Poisson process with intensity $\lambda > 0$, independent of all the $Q(n)$'s. The *compound Poisson* process $Q^\sigma(t)$ is expressed by

$$Q^\sigma(t) = Q(1) + \cdots + Q\big(\Theta(t)\big), \quad t > 0.$$

An increment of this process is defined by

$$Q^\sigma(s) - Q^\sigma(t) = \sum_{k=\Theta(t+1)}^{\Theta(s)} Q(k), \quad s > t. \tag{II.25}$$

Remark that (II.25) is independent of $Q(1), \ldots, Q\big(\Theta(t)\big)$, and its distribution depends only on the difference $(s - t)$. Consequently, $Q^\sigma(t)$ is a Lévy process. In order to determine the Lévy measure ν of $Q^\sigma(t)$, we consider $D \in \mathrm{B}_0$, then

$$\nu(D) = \mathbb{E}\big(\mathcal{N}(1, D)\big) = \underbrace{\mathbb{E}\left(\sum_{0<s\leq 1} \mathbb{1}_D\big(\Delta Q^\sigma(s)\big)\right)}_{\mathbb{E}(\text{(number of jumps)} \times \mathbb{1}_D\text{(jump)})} = \mathbb{E}\big(\Theta(1)\mathbb{1}_D(Q)\big) = \lambda \mu_Q(D).$$

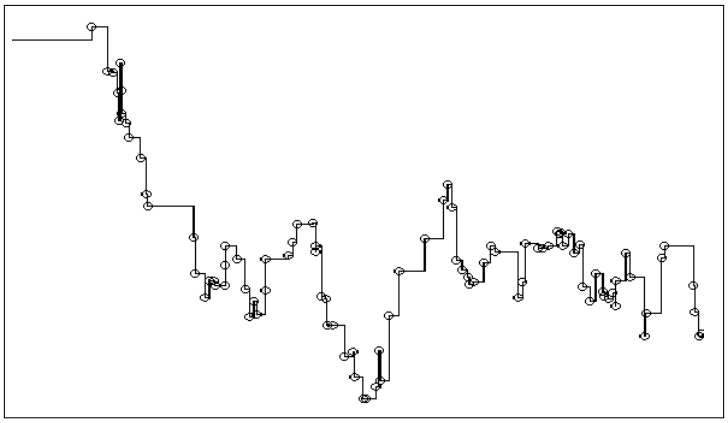


Figure II.5: Illustration of the compound Poisson process.

[a] A collection of random variables is independent and identically distributed (i.i.d.) if each random variable has the same probability distribution as the others and all are mutually independent.

Roughly speaking, the previous example shows that a Lévy process can be represented by a compound Poisson process if and only if its Lévy measure is finite. But, there exist many Lévy processes with infinite Lévy measure ν with the fact that $\int_{|u|\leq r} |u|\nu(\mathrm{d}u) = \infty$, $r > 0$. Moreover, it is easy to remark that for any fixed r the process

$$\mathcal{M}^{(k)}(t) \triangleq \int_{\frac{1}{k}\leq u\leq r} u\big[\mathcal{N}(t,\mathrm{d}u) - t\nu(\mathrm{d}u)\big], \quad k = 1, 2, \ldots$$

are L^2-martingales and they converge in L^2 to a martingale $\mathcal{M}(t)$ denoted by

$$\mathcal{M}(t) = \int_{|u|\leq r} u\big[\mathcal{N}(t,\mathrm{d}u) - t\nu(\mathrm{d}u)\big].$$

12.2 Itô-Lévy decomposition

The following results gives a general description of Lévy processes.

Proposition 12.4 (**Itô-Lévy decomposition**). *Let $\mathcal{Z}(t)$ be a Lévy process. Then, $\mathcal{Z}(t)$ admits the following unique representation*

$$\mathcal{Z}(t) = \alpha t + \sigma\mathcal{W}(t) + \int_{|u|<r} u\widetilde{\mathcal{N}}(t,\mathrm{d}u) + \int_{|u|\geq r} u\mathcal{N}(t,\mathrm{d}u). \tag{II.26}$$

Here, $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{R}$, $r \in [0,\infty]$, and

$$\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) = \mathcal{N}(\mathrm{d}t,\mathrm{d}u) - \nu(\mathrm{d}u)\mathrm{d}t$$

is the compensated Poisson random measure of $\mathcal{Z}(\cdot)$. It is assumed that the standard Brownian motion $\mathcal{W}(t)$ is independent of $\widetilde{\mathcal{N}}$. Note that for each $D \in \mathcal{B}_0$, the process $\mathcal{M}(t) \triangleq \widetilde{\mathcal{N}}(t,D)$ is a martingale. Specifically, if $\alpha = 0$ and $r = \infty$, we call $\mathcal{Z}(t)$ a Lévy martingale. Furthermore, if $\mathbb{E}|\mathcal{Z}(t)| < \infty$ for all $t > 0$, then we can choose $r = \infty$. Hence, $\mathcal{Z}(t)$ can be written in the compact form

$$\mathcal{Z}(t) = \alpha t + \sigma\mathcal{W} + \int_{\mathbb{R}} u\widetilde{\mathcal{N}}(t,\mathrm{d}u).$$

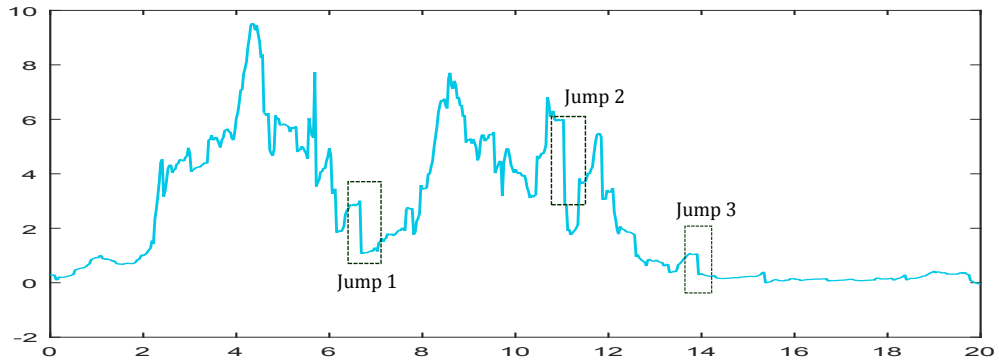


Figure II.6: An illustration of the Itô-Lévy jumps process paths.

Theorem 12.5 (**Lévy-Khintchine formula**). *Let $\mathcal{Z}(t)$ be a Lévy process with Lévy measure ν. Then,*

$$\int_{\mathbb{R}} \min(1,|u|^2)\nu(\mathrm{d}u) < \infty \quad \textit{and} \quad \mathbb{E}\left(e^{i\xi\mathcal{Z}(t)}\right) = e^{t\psi(\xi)}, \quad \xi \in \mathbb{R},$$

where,

$$\psi(\xi) = -\frac{1}{2}\sigma^2\xi^2 + i\alpha\xi + \int_{|u|<r}\left\{e^{i\xi u} - 1 - i\xi u\right\}\nu(\mathrm{d}u) + \int_{|u|\geq r}\left\{e^{i\xi u} - 1\right\}\nu(\mathrm{d}u).$$

Now, let $\mathfrak{D}^{sk}$ be the space of a cadlag adapted processes with the Skorohod topology[a]. Let also $\mathfrak{D}^{ucp}$ denote the space of a cadlag adapted processes endowed with the associated topology given by the uniform convergence on compacts in probability[b] (ucp, for short). By considering an elementary process $h(t)$ of the form (II.4) and a cadlag process $\mathcal{X}$, we define

$$\mathfrak{I}_{\mathcal{X}}(h) \triangleq \int_0^t h(s)\mathrm{d}\mathcal{X}(s) \triangleq \kappa_0\mathcal{X}_0 + \sum_i \kappa_i \left(\mathcal{X}_{t_{i+1}\wedge t} - \mathcal{X}_{t_i\wedge t}\right), \quad t > 0,$$

where $\kappa_i \in \mathcal{F}_{t_i}$ and $t_0 \le t_1 \le \cdots \le t_{n+1} < \infty$. Let $\mathcal{X}$ be a semimartingale[c]. Then, the mapping $\mathfrak{I}_{\mathcal{X}}$ can be extended to a continuous linear map $\mathfrak{I}_{\mathcal{X}} : \mathfrak{D}^{ucp} \to \mathfrak{D}^{sk}$. By using this construction, we can define stochastic integrals of the form $\int_0^t h(s)\mathrm{d}\mathcal{Z}(s)$ for all $h \in \mathfrak{D}^{ucp}$. From the decomposition (II.26), we obtain the generalized stochastic integrals of the following form

$$\mathcal{X}(t,\omega) = \mathcal{X}_0 + \int_0^t f(s,\omega)\mathrm{d}s + \int_0^t g(s,\omega)\mathrm{d}\mathcal{W}(s) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \eta(s,u,\omega)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u),$$

where the integrands are $\{\mathcal{F}_t\}$-predictable and satisfy the following assumption

$$\int_0^t \left\{|f(s,\omega)| + g^2(s,\omega) + \int_{\mathbb{R}\setminus\{0\}} \eta^2(s,u,\omega)\nu(\mathrm{d}u)\right\}\mathrm{d}s < \infty \quad \text{a.s. for all } t > 0.$$

Then, the differential notation *Itô-Lévy* process is defined as follows:

$$\mathrm{d}\mathcal{X}(t,\omega) = f(t,\omega)\mathrm{d}t + g(t,\omega)\mathrm{d}\mathcal{W}(t) + \int_{\mathbb{R}\setminus\{0\}} \eta(t,u,\omega)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \tag{II.27}$$

Obviously, if $\mathbb{E}\left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} \eta^2(t,u,\omega)\nu(\mathrm{d}u)\mathrm{d}t\right) < \infty$, then the process $\mathcal{M}(t) = \int_0^t \int_{\mathbb{R}\setminus\{0\}} \eta(t,u,\omega)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)$, $0 \le t \le T$ is a martingale. Moreover, if $\int_0^T \int_{\mathbb{R}\setminus\{0\}} \eta^2(t,u,\omega)\nu(\mathrm{d}u)\mathrm{d}t < \infty$ a.s., then the process $\mathcal{M}(t)$ is a local martingale for all $0 \le t \le T$.

12.3 The Itô-Lévy formula

We consider the process $\mathcal{X}$ expressed by (II.27) and let $\mathcal{V} : \mathbb{R}^2 \to \mathbb{R}$ be a C^2-function such that $\mathcal{Y}(t) \equiv \mathcal{V}(t,\mathcal{X}(t))$. Then, $\mathcal{Y}(t)$ is again an Itô-Lévy process and

$$\begin{aligned}\mathrm{d}\mathcal{Y}(t) = {} & \frac{\partial\mathcal{V}}{\partial t}(t,\mathcal{X}(t))\mathrm{d}t + \frac{\partial\mathcal{V}}{\partial x}(t,\mathcal{X}(t))\Big\{f(t,\omega)\mathrm{d}t + g(t,\omega)\mathrm{d}\mathcal{W}(t)\Big\} + 0.5g^2(t,\omega)\frac{\partial^2\mathcal{V}}{\partial x^2}(t,\mathcal{X}(t))\mathrm{d}t \\ & + \int_{\mathbb{R}\setminus\{0\}} \left\{\mathcal{V}\big(t,\mathcal{X}(t) + \eta(t,u,\omega)\big) - \mathcal{V}(t,\mathcal{X}(t)) - \frac{\partial\mathcal{V}}{\partial x}(t,\mathcal{X}(t))\eta(t,u,\omega)\right\}\nu(\mathrm{d}u)\mathrm{d}t \\ & + \int_{\mathbb{R}\setminus\{0\}} \big\{\mathcal{V}\big(t,\mathcal{X}(t^-) + \eta(t,u,\omega)\big) - \mathcal{V}(t,\mathcal{X}(t^-))\big\}\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).\end{aligned}$$

Example 12.6 (The geometric Itô-Lévy process). In this example, we consider the following Itô-Lévy stochastic differential equation

$$\mathrm{d}\mathcal{X}(t) = \alpha\mathcal{X}(t)\mathrm{d}t + \sigma\mathcal{X}(t)\mathrm{d}\mathcal{W}(t) + \int_{\mathbb{R}\setminus\{0\}} \eta(t,u)\mathcal{X}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),$$

[a] A Skorohod topology on D^{sk} is defined by introducing the Skorohod metric d^{sk} defined by

$$d^{sk}(f,g) = \inf_{\tau}\sup_{t\ge 0}\{|f(t) - g(\tau(t))| + |\tau(t) - t|\},$$

where τ moves over all strictly increasing, continuous mappings of $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\tau(0) = 0$.

[b] Here, $H_n \overset{ucp}{\to} H$ if for all $t > 0$, $\sup_{0\le s\le t} |H_n(s) - H(s)| \to 0$ in probability.

[c] A stochastic process $\mathcal{X}(t)$ is called semimartingale, if it can be decomposed as $\mathcal{X} = \mathcal{Y}(t) + \mathcal{A}(t)$, where $\mathcal{Y}$ is a local martingale and $\mathcal{A}$ is cadlag adapted process of locally bounded variation.

where α, σ are constants and $\eta > -1$. Then, $\mathcal{X}(t)$ can never jump to 0 or a negative value. In order to establish explicitly the solution $\mathcal{X}(t)$ of this Lévy SDE, we reformulate it as follows:

$$\frac{\mathrm{d}\mathcal{X}(t)}{\mathcal{X}(t^-)} = \alpha \mathrm{d}t + \sigma \mathrm{d}\mathcal{W}(t) + \int_{\mathbb{R}\setminus\{0\}} \eta(t,u)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).$$

By setting $\mathcal{Y}(t) = \ln \mathcal{X}(t)$ and using Itô's formula, we obtain

$$\mathrm{d}\mathcal{Y}(t) = \left(\alpha - 0.5\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}\mathcal{W}(t) + \int_{\mathbb{R}\setminus\{0\}} \{\ln(1+\eta(t,u)) - \eta(t,u)\}\, \nu(\mathrm{d}u)\mathrm{d}t + \int_{\mathbb{R}\setminus\{0\}} \ln(1+\eta(t,u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).$$

Therefore

$$\begin{aligned}\mathcal{Y}(t) = \mathcal{Y}(0) &+ \left(\alpha - 0.5\sigma^2\right)t + \sigma\mathcal{W}(t) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \{\ln(1+\eta(t,u)) - \eta(t,u)\}\, \nu(\mathrm{d}u)\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}\setminus\{0\}} \ln(1+\eta(t,u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

Consequently

$$\begin{aligned}\mathcal{X}(t) = \mathcal{X}(0)\exp\Bigg\{&\left(\alpha - 0.5\sigma^2\right)t + \sigma\mathcal{W}(t) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \{\ln(1+\eta(t,u)) - \eta(t,u)\}\, \nu(\mathrm{d}u)\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}\setminus\{0\}} \ln(1+\eta(t,u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\Bigg\}.\end{aligned}$$

Next, we formulate the corresponding multi-dimensional version of latest result. Let $\mathcal{X}(t) \in \mathbb{R}^n$ be an Itô-Lévy process of the following form:

$$\mathrm{d}\mathcal{X}(t) = f(t,\omega)\mathrm{d}t + g(t,\omega)\mathrm{d}\mathcal{W}(t) + \int_{\mathbb{R}^n\setminus\{0\}} \eta(t,u,\omega)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \tag{II.28}$$

where $f : [0,T]\times\Omega \to \mathbb{R}^n$, $g : [0,T]\times\Omega \to \mathbb{R}^{n\times m}$ and $\eta : [0,T]\times R^n \times \Omega \to \mathbb{R}^{n\times \ell}$ are adapted processes such that the integrals exist. Here, $\mathcal{W}(t)$ is an m-dimensional Brownian motion and

$$\begin{aligned}\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)^T &= \left(\widetilde{\mathcal{N}}_1(\mathrm{d}t,\mathrm{d}u_1),\ldots,\widetilde{\mathcal{N}}_\ell(\mathrm{d}t,\mathrm{d}u_\ell)\right) \\ &= \left(\mathcal{N}_1(\mathrm{d}t,\mathrm{d}u_1) - \nu_1(\mathrm{d}u_1)\mathrm{d}t,\ldots,\mathcal{N}_\ell(\mathrm{d}t,\mathrm{d}u_\ell) - \nu_\ell(\mathrm{d}u_\ell)\mathrm{d}t\right),\end{aligned}$$

where $\{\mathcal{N}_j\}$ are independent Poisson random measures with Lévy measures ν_j. Specifically, when written in detail the component number i of $\mathcal{X}(t)$ in (II.28), then $\mathcal{X}_i(t)$ has the following form

$$\mathrm{d}\mathcal{X}_i(t) = f_i(t,\omega)\mathrm{d}t + \sum_{j=1}^m g_{ij}(t,\omega)\mathrm{d}\mathcal{W}_j(t) + \sum_{j=1}^\ell \int_{\mathbb{R}} \eta_{ij}(t,u_j,\omega)\widetilde{\mathcal{N}}_j(\mathrm{d}t,\mathrm{d}u_j), \quad 1 \le i \le n.$$

Let $\mathcal{V} \in C^{1,2}([0,T]\times\mathbb{R}^n;\mathbb{R})$ and let $\mathcal{Y}(t) = \mathcal{V}(t,\mathcal{X}(t))$. Then

$$\begin{aligned}\mathrm{d}\mathcal{Y}(t) = &\frac{\partial\mathcal{V}}{\partial t}\mathrm{d}t + \sum_{i=1}^n \frac{\partial\mathcal{V}}{\partial x_i}\left(f_i\mathrm{d}t + g_i\mathrm{d}\mathcal{W}(t)\right) + 0.5\sum_{i,j=1}^n \left(gg^T\right)_{ij}\frac{\partial^2\mathcal{V}}{\partial x_i\partial x_j}\mathrm{d}t \\ &+ \sum_{k=1}^\ell \int_{\mathbb{R}\setminus\{0\}} \left\{\mathcal{V}\left(t,\mathcal{X}(t^-) + \eta^{(k)}(t,u_k)\right) - \mathcal{V}(t,\mathcal{X}(t^-)) - \sum_{i=1}^n \frac{\partial\mathcal{V}}{\partial x_i}(t,\mathcal{X}(t^-))\eta_i^{(k)}(t,u_k)\right\}\nu_k(\mathrm{d}u_k)\mathrm{d}t \\ &+ \sum_{k=1}^\ell \int_{\mathbb{R}\setminus\{0\}} \left\{\mathcal{V}\left(t,\mathcal{X}(t^-) + \eta^{(k)}(t,u_k)\right) - \mathcal{V}(t,\mathcal{X}(t^-))\right\}\widetilde{\mathcal{N}}_k(\mathrm{d}t,\mathrm{d}u_k),\end{aligned}$$

where $\eta^{(k)} \in \mathbb{R}^n$ is column number k of the $n\times\ell$ matrix $\eta = [\eta]_{ik}$ and $\eta_i^{(k)} = \eta_{ik}$ is the coordinate number i of $\eta^{(k)}$.

Generally, the Itô-Lévy isometries state the following results:

- $\mathbb{E}\left[\left(\int_0^T g(s,\omega)\mathrm{d}\mathcal{W}(s)\right)^2\right] = \mathbb{E}\left[\int_0^T g^2(s,\omega)ds\right]$.
[The Itô isometry]

- $\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}\setminus\{0\}} \eta(s,u,\omega)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\right)^2\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} \eta^2(s,u,\omega)\nu(\mathrm{d}u)\mathrm{d}s\right]$.
[The Lévy isometry]

If the last quantity is finite, then $\mathcal{M}(t) = \int_0^T \int_{\mathbb{R}\setminus\{0\}} \eta(s,u,\omega)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)$ is a martingale for $t \geq T$ with
[The martingale property]
the following quadratic variation

$$\langle \mathcal{M},\mathcal{M}\rangle_t = \int_0^T \int_{\mathbb{R}\setminus\{0\}} \eta^2(s,u,\omega)\nu(\mathrm{d}u)\mathrm{d}s.$$

12.4 Itô-Lévy stochastic differential equations

Theorem 12.7. *Consider the following Lévy SDE in $\mathbb{R}^n$:*

$$\mathrm{d}\mathcal{X}(t) = f(t,\mathcal{X}(t))\mathrm{d}t + g(t,\mathcal{X}(t))\mathrm{d}\mathcal{W}(t) + \int_{\mathbb{R}^n\setminus\{0\}} \eta(t,\mathcal{X}(t^-),u)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \quad \mathcal{X}(0) = \mathcal{X}_0 \in \mathbb{R}^n,$$

where $f : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $g : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $\eta : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n\times\ell}$ verify the following assumptions

$$|f(t,x)-f(t,y)|^2 + ||g(t,x)-g(t,y)||^2 + \sum_{k=1}^{\ell}\int_{\mathbb{R}\setminus\{0\}} |\eta^{(k)}(t,x,u_k)-\eta^{(k)}(t,y,u_k)|^2\nu_k(\mathrm{d}u_k) \leq \hat{\mathfrak{C}}_1|x-y|^2$$
[Lipschitz continuity condition]
(II.29)

for some finite constant $\hat{\mathfrak{C}}_1$ and for all $x,y \in \mathbb{R}^n$, and

$$|f(t,x)|^2 + ||g(t,x)||^2 + \int_{\mathbb{R}\setminus\{0\}} \sum_{k=1}^{\ell} |\eta_k(t,x,u)|^2\nu_k(\mathrm{d}u_k) \leq \hat{\mathfrak{C}}_2\left(1+|x|^2\right) \tag{II.30}$$
[At most linear growth condition]

for some finite constant $\hat{\mathfrak{C}}_2$ and for all $x \in \mathbb{R}^n$. Then, there exists a unique cadlag adapted solution $\mathcal{X}(t)$ and $\mathbb{E}\left[|\mathcal{X}(t)|^2\right] < \infty$ for all $t \geq 0$.

Remark 12.8. Solutions of Lévy SDE in the time homogeneous case, i.e. when $f(t,x) = f(x)$, $g(t,x) = g(x)$ and $\eta(t,x,u) = \eta(x,u)$, are called Lévy jump diffusions.

Now, we compare the following two one-dimensional stochastic differential equations driven by a Brownian motion $(\mathcal{W}_t)_{t\geq 0}$ and a Poisson process $(\mathcal{N}_t)_{t\geq 0}$

$$\begin{aligned}
\mathcal{X}_1(t) &= \mathcal{X}_1(0) + \int_0^t f_1(s,\mathcal{X}_1(s))\mathrm{d}s + \int_0^t g_1(s,\mathcal{X}_1(s))\mathrm{d}\mathcal{W}(s) + \int_0^t\int_{\mathbb{R}^n\setminus\{0\}} \eta(s,\mathcal{X}_1(s^-),u)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u),\\
\mathcal{X}_2(t) &= \mathcal{X}_2(0) + \int_0^t f_2(s,\mathcal{X}_2(s))\mathrm{d}s + \int_0^t g_2(s,\mathcal{X}_2(s))\mathrm{d}\mathcal{W}(s) + \int_0^t\int_{\mathbb{R}^n\setminus\{0\}} \eta(s,\mathcal{X}_2(s^-),u)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u),
\end{aligned} \tag{II.31}$$

where $\mathcal{X}_1(0)$ and $\mathcal{X}_2(0)$ are the initial states of these two SDEs with initial time $t \in [0,T]$. The main objective of the next theorem is to find a necessary and sufficient condition of the above coefficients that ensures

$$\mathcal{X}_1(0) \leq \mathcal{X}_2(0) \quad \Rightarrow \mathcal{X}_1(s) \leq \mathcal{X}_2(s), \qquad \forall s \in [t,T], \quad \mathbb{P}-a.s., \quad \forall t \leq T. \tag{II.32}$$

Theorem 12.9 (Stochastic comparaison Theorem, [109]). *We assume (II.29) and (II.30) are verified. Then the following conditions are equivalent*

1. *(II.32) holds for SDEs (II.31);*
2. *For each $(t,x) \in [0,T] \times \mathbb{R}$, the coefficients f_i, g_i and η_i, $i = 1,2$ satisfy:*

$$\begin{aligned} f_1(t,x) &\leq f_2(t,x), \\ g_1(t,x) &= g_2(t,x), \\ \eta_1(t,x,u) &= \eta_2(t,x,u), \quad \nu(\mathrm{d}u) - a.s. \end{aligned}$$

For the situation without jumps, we have

Corollary 12.10. *We assume (II.29) and (II.30) as well as $\eta_1 \equiv \eta_2 \equiv 0$. Then, the following conditions are equivalent:*

1. *(II.32) holds for SDEs (II.31);*
2. *For each $(t,x) \in [0,T] \times \mathbb{R}$, the coefficients f_i and g_i $i = 1,2$ satisfy:*

$$\begin{aligned} f_1(t,x) &\leq f_2(t,x), \\ g_1(t,x) &= g_2(t,x). \end{aligned}$$

13 Some definitions and useful results

Definition 13.1. We consider a population and $\mathcal{X}(t)$ its size at time t.

- The population is said to go to extinction if $\lim_{t\to\infty} \mathcal{X}(t) = 0$ a.s.
- The population is said to be non-persistent in the mean if $\lim_{t\to\infty} \langle \mathcal{X}(t) \rangle = 0$ a.s., where

$$\langle \psi(t) \rangle = \frac{1}{t} \int_0^t \psi(s) \mathrm{d}s.$$

- The population is said to be exponentially extinct if $\limsup_{t\to\infty} t^{-1} \ln \mathcal{X}(t) < 0$ a.s.
- The population is said to be weakly persistent if $\limsup_{t\to\infty} \mathcal{X}(t) > 0$ a.s.
- The population is said to be weakly persistent in the mean if $\limsup_{t\to\infty} \langle \mathcal{X}(t) \rangle > 0$ a.s.
- The population is said to be strongly persistent in the mean if $\liminf_{t\to\infty} \langle \mathcal{X}(t) \rangle > 0$ a.s.
- The population is said to be stochastically permanent if for any $\epsilon > 0$, there exists a pair of positive constants $\beta_1 = \beta_1(\epsilon)$ and $\beta_2 = \beta_2(\epsilon)$ such that for any initial value $\mathcal{X}_0 \in \mathbb{R}_+$, we have

$$\limsup_{t\to\infty} \mathbb{P}\left(|\mathcal{X}(t)| > \beta_1\right) \leq \epsilon \quad \text{[Stochastically ultimate boundedness]} \qquad \text{and} \qquad \liminf_{t\to\infty} \mathbb{P}\left(|\mathcal{X}(t)| \geq \beta_2\right) \geq 1 - \epsilon. \quad \text{[Stochastic persistence]}$$

Obviously, it can be seen from above definitions that the exponentially extinction implies the extinction and the non-persistence in the mean. In addition, the strong persistence in the mean implies the weak persistence and the weak persistence in the mean. But, the reverse of the above reasoning is not generally true. Furthermore, a stochastic epidemic model is termed stochastically permanent, if it is both stochastically ultimate bounded and persistent.

Lemma 13.2. *Let $\mathcal{X} \in C(\Omega \times [0,+\infty), \mathbb{R}_+)$. We suppose that $1 + \eta_i(u) > 0$ $(i = 1,2,\dots)$, for all u in a measurable subset $Z \subset (0,\infty)$ satisfying $\nu(Z) < \infty$. Then, we have the following conclusions.*

- *If there exist* $T > 0$, $\mathfrak{C}_0 > 0$, $\mathfrak{C}$, $\mathcal{M}_i$ *such that when* $t \geq T$,

$$\ln \mathcal{X}(t) \leq \mathfrak{C}t - \mathfrak{C}_0 \int_0^t \mathcal{X}(s)\mathrm{d}s + \sum_{i=1}^n \sigma_i \mathcal{W}_i(t) + \sum_{i=1}^n \mathcal{M}_i \int_0^t \int_{\mathcal{U}} \ln\big(1+\eta_i(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) \quad a.s.,$$

then

$$\begin{cases} \limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathcal{X}(s)\mathrm{d}s \leq \frac{\mathfrak{C}}{\mathfrak{C}_0} \quad a.s. \ \text{if} \ \mathfrak{C} \geq 0, \\ \lim_{t\to\infty} \mathcal{X}(t) = 0 \quad a.s. \ \text{if} \ \mathfrak{C} < 0. \end{cases}$$

- *If there exist* $T > 0$, $\mathfrak{C}_0 > 0$, $\mathfrak{C} > 0$, $\mathcal{M}_i$ *such that when* $t \geq T$,

$$\ln \mathcal{X}(t) \geq \mathfrak{C}t - \mathfrak{C}_0 \int_0^t \mathcal{X}(s)\mathrm{d}s + \sum_{i=1}^n \sigma_i \mathcal{W}_i(t) + \sum_{i=1}^n \mathcal{M}_i \int_0^t \int_{\mathcal{U}} \ln\big(1+\eta_i(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) \quad a.s.,$$

then $\liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathcal{X}(s)\mathrm{d}s \leq \frac{\mathfrak{C}}{\mathfrak{C}_0}$ *a.s.*

The following elementary inequalities will be used frequently in the sequel.

- **Hölder's inequality** If $1 < p < \infty$ and q is given by $1/p+1/q = 1$, $\mathbb{E}(|X|^p) < \infty$, and $\mathbb{E}(|Y|^q) < \infty$, then

$$|\mathbb{E}(XY)| \leq \mathbb{E}(|XY|) \leq \big(\mathbb{E}|X|^p\big)^{\frac{1}{p}}\big(\mathbb{E}|Y|^q\big)^{\frac{1}{q}}.$$

  The Hölder inequality gives $\mathbb{E}(|X|) \leq \mathbb{E}(|X|^p)^{1/p}$ for $Y = 1$ and Cauchy-Schwarz inequality for $p = q = 2$.

- **Minkovski's inequality** If $1 \leq p < \infty$ and X, $Y \in L_p$, then $X + Y \in L_p$ and

$$\big(\mathbb{E}|X+Y|^p\big)^{\frac{1}{p}} \leq \big(\mathbb{E}|X|^p\big)^{\frac{1}{p}} + \big(\mathbb{E}|Y|^p\big)^{\frac{1}{p}}.$$

- **Chebyshev's inequality**

$$\mathbb{P}\big\{\omega : \ |X(\omega)| \geq \mathfrak{C}\big\} \leq \mathfrak{C}^{-p}\mathbb{E}|X|^p$$

  if $\mathfrak{C} > 0$, $p > 0$, and $X \in \mathrm{L}^P(\Omega;\mathbb{R}^n)$.

- **Burkholder-Davis-Gundy inequality** Let $g \in \mathfrak{L}^2(\mathbb{R}_+;\mathbb{R}^{n\times m})$. For any $t \geq 0$, define

$$\mathfrak{X}(t) = \int_0^t g(s)\mathrm{d}\mathcal{W}(s) \quad \text{and} \quad \mathfrak{A}(t) = \int_0^t |g(s)|^2\mathrm{d}s.$$

  Then for every $p > 0$, there exist positive constants $\mathfrak{C}_p$ and $\mathfrak{C}_p$, such that

$$\mathfrak{C}_p\mathbb{E}|\mathfrak{A}(t)|^{\frac{p}{2}} \leq \mathbb{E}\left(\sup_{0\leq s\leq t}|\mathfrak{X}(s)|^p\right) \leq \mathfrak{C}_p\mathbb{E}|\mathfrak{A}(t)|^{\frac{p}{2}} \quad \text{for all } t \geq 0.$$

- **Gronwall's inequality** Let $T > 0$ and $\mathfrak{C} \geq 0$. Let $f(\cdot)$ be a Borel measurable bounded nono-negative function on $[0,T]$, and let $g(\cdot)$ be a non-negative integrable function on $[0,T]$. If

$$f(t) \leq \mathfrak{C} + \int_0^t f(s)g(s)\mathrm{d}s \quad \text{for all } 0 \leq t \leq T,$$

  then

$$f(t) \leq \mathfrak{C}\exp\left(\int_0^t g(s)\mathrm{d}s\right) \quad \text{for all } 0 \leq t \leq T.$$

The basics of the stochastic analysis and jump-diffusion theory are presented, followed by some classical tools that are will be used in the following chapters to analyze the various disturbed epidemic models. It should be emphasized that this chapter is just a collection of the definitions and notations necessary to comprehend the key results of this thesis. For more details and specifics, we refer the reader to this book [51].

Now, we are in the position to present the main contributions and results of this thesis. We will apply the theories of this chapter to analyze mathematically the perturbed version of the epidemic models presented in Chapter I.

Chapter III

Some new results on the dynamics of a stochastic SIR epidemic model

Abstract

Recently, emerging epidemics like COVID-19 and its variants require predictive mathematical models to implement suitable responses in order to limit their negative and profound impact on society. The SIR system is a straightforward mathematical formulation to model the dissemination of many infectious diseases. The present chapter reports novel theoretical and analytical results for a perturbed version of an SIR epidemic model. Notably, our epidemic model is represented by Itô-Lévy stochastic differential equations in order to simulate sudden and unexpected external phenomena. By adopting some innovative and ameliorated mathematical approaches, we analyze the long-run characteristics of three categories of an SIR model with jumps. Initially, we suppose that the stochastic disturbances associated with the disease transmission coefficient and the mortality rate are presented with two perturbations: standard Gaussian white and Lévy noises, respectively. We investigate some attractive asymptotic properties of this new perturbed model, namely: persistence in the mean, ergodicity, and extinction of the disease. Next, we consider an SIR epidemic model driven by a multidimensional Lévy jump process. We aim to develop a mathematical method for deriving the stochastic threshold which can determine the existence of a unique ergodic stationary distribution or the extinction of the epidemic. Our method differs from previous approaches by the use of the comparison theorem, mutually exclusive possibilities lemma, and some alternative techniques of the stochastic differential systems. Ultimately, we consider the case of an SIR model with the distributed latency period and Lévy jumps. We aim to study the dynamic behavior characteristics of this perturbed delayed model. Within this scope, we offer sufficient conditions for two epidemiological properties: extinction and persistence of the epidemic. One of the most intriguing results is that the dynamics of the stochastic model are closely related to the intensities of white noises and Lévy jumps, which can give us a good insight into the evolution of the epidemic in some unexpected situations. Broadly speaking, our work complements the results of some previous investigations and provides a new way to predict and analyze the dynamic behavior of epidemics with distributed delay. For illustrative purposes, numerical examples are presented for each perturbed model in order to check the theoretical studies.

1 Dynamic characterization of a stochastic SIR infectious disease model with dual perturbation

1.1 Introduction

As mentioned previously, the dissemination of infectious diseases may subject to some uncertainties and stochastic phenomena due to fluctuations in the natural environment. Many studies pointed

out that stochastic differential equations can be applied to formulate the perturbed SIR epidemic model and explore the dynamics of infectious diseases [2, 150]. By assuming that the disease transmission is subject to some small random fluctuations and the contact rate β is disturbed by the white noise, the SIR model (I.4) is transformed into the following SDEs:

$$\begin{cases} \mathrm{d}S(t) = \big(A - \mu_1 S(t) - \beta S(t)I(t)\big)\mathrm{d}t - \sigma S(t)I(t)\mathrm{d}\mathcal{W}(t), \\ \mathrm{d}I(t) = \big(\beta S(t)I(t) - (\mu_2 + \gamma)I(t)\big)\mathrm{d}t + \sigma S(t)I(t)\mathrm{d}\mathcal{W}(t), \\ \mathrm{d}R(t) = \big(\gamma I(t) - \mu_1 R(t)\big)\mathrm{d}t, \end{cases} \tag{III.1}$$

where $\mathcal{W}(t)$ is the standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions containing all the random variables that we meet in this section. σ is the intensity of environmental white noise. In the following, we present some previous results on the dynamics of the model (III.1):

- Tornatore et al. [136] proved that $E^{\ominus}$ of the deterministic system (I.4) is globally asymptotically stable under a suitable condition, and they established a threshold that determines the dynamics of the stochastic system (III.1).
- Ji et al. [47] proved the global stability of the stochastic system (III.1) by using the Lyapunov functional method, and they studied the asymptotic character of the solution around the endemic point $E^{\circledast}$ of (I.4).
- Ji and Jiang [43] investigated the threshold behavior of the model (III.1) which determines the extinction or the persistence of the epidemic.

Besides the above-mentioned bilinear perturbation, the deterministic system (I.4) can be perturbed by assuming that the white noise is directly proportional to $S(t)$, $I(t)$ and $R(t)$. More formally, the perturbed version of (I.4) can be described by the following SDEs:

$$\begin{cases} \mathrm{d}S(t) = \big(A - \mu_1 S(t) - \beta S(t)I(t)\big)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t), \\ \mathrm{d}I(t) = \big(\beta S(t)I(t) - (\mu_2 + \gamma)I(t)\big)\mathrm{d}t + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t), \\ \mathrm{d}R(t) = \big(\gamma I(t) - \mu_1 R(t)\big)\mathrm{d}t + \sigma_3 R(t)\mathrm{d}\mathcal{W}_3(t), \end{cases} \tag{III.2}$$

where $\mathcal{W}_i(t)$ $(i = 1, 2, 3)$ are the independent standard Brownian motions and σ_i $(i = 1, 2, 3)$ are the intensities of environmental white noises. Many previous studies have analyzed the dynamics of the stochastic model (III.2). For example:

- Ji et al. [46] investigated the asymptotic behavior of the model (III.2) around the disease-free equilibrium $E^{\ominus}$ of (I.4).
- Lin et al. [76] analyzed the long-time behavior of the stochastic model (III.2). Precisely, they discussed the convergence of the solution densities to a singular measure.

However, epidemic models just perturbed by the Gaussian white noise can not successfully characterize the situation when the population suffers sudden catastrophic disturbance such as earthquakes, hurricanes, and floods. The severe environmental change can affect the dynamical behavior of the epidemic significantly [62, 60]. Therefore, it is necessary to employ the discontinuous Lévy process to analyze and control infectious diseases during these phenomena. According to the Lévy-Itô decomposition [23] (see also Chapter II), a Lévy process is decomposed into the sum of a linear drift, a Brownian motion, and a superposition of centered Poisson processes with different jump sizes. If the SIR model is governed by Lévy jumps, then the deterministic model (I.4) can be described by the following SDEs with jumps:

$$\begin{cases} \mathrm{d}S(t) = \big(A - \mu_1 S(t) - \beta S(t)I(t)\big)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \displaystyle\int_{\mathcal{U}} \eta_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u), \\ \mathrm{d}I(t) = \big(\beta S(t)I(t) - (\mu_2 + \gamma)I(t)\big)\mathrm{d}t + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \displaystyle\int_{\mathcal{U}} \eta_2(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u), \\ \mathrm{d}R(t) = \big(\gamma I(t) - \mu_1 R(t)\big)\mathrm{d}t + \sigma_3 R(t)\mathrm{d}\mathcal{W}_3(t) + \displaystyle\int_{\mathcal{U}} \eta_3(u)R(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u), \end{cases} \tag{III.3}$$

where $S(t^-)$, $I(t^-)$ and $R(t^-)$ are the left limits of $S(t)$, $I(t)$ and $R(t)$, respectively. $\mathcal{W}_i(t)$ $(i = 1, 2, 3)$ are independent Brownian motions and $\sigma_i > 0$ $(i = 1, 2, 3)$ are their intensities. $\mathcal{N}$ is a Poisson counting measure with compensating martingale $\widetilde{\mathcal{N}}$ and characteristic measure ν on a measurable subset $\mathcal{U}$ of $(0, \infty)$ satisfying $\nu(\mathcal{U}) < \infty$. $\mathcal{W}_i(t)$ $(i = 1, 2, 3)$ are independent of $\mathcal{N}$. We assume that ν is a Lévy measure such that $\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u) = \mathcal{N}(\mathrm{d}t, \mathrm{d}u) - \nu(\mathrm{d}u)\mathrm{d}t$ and the functions $\eta_i : \mathcal{U} \to \mathbb{R}$ $(i = 1, 2, 3)$ are bounded and continuous. The following references are the two works that have studied the dynamics of the model (III.3):

- Zhang and Wang [154] analyzed the asymptotic behavior of the stochastic solution of the model (III.3) around the equilibriums $E^{\ominus}$ and $E^{\circledast}$.
- Zhou and Zhang [172] investigated the effect of the jumps on the dynamics of the model (III.3), and they obtained the stochastic threshold which determines the extinction or persistence of the disease.

Generally speaking, there are two popular ways to introduce stochastic factors into the epidemic models. The first is to presume that the spread of the epidemic is subject to some small and standard random fluctuations that can be described by Gaussian white noise. The other is to assume that the model is affected by massive disturbances caused by sudden environmental shocks. Mathematically, these huge random perturbations can be described by Lévy noise. In this section, we present a new stochastic SIR epidemic model with dual perturbations. We merge the stochastic transmission with a discontinuous perturbed mortality rate. We suppose that the transmission of the epidemic may subject to some small uncertainties, and the mortality rate can be affected by huge environmental phenomena. This new perturbation allows us to examine the effects of both Gaussian white noise and Lévy noise on the transmission dynamics of infectious diseases during environmental shocks. Motivated by the above discussion, the stochastic variability in the mortality rate μ_1 and the epidemic transmission rate β are presented as follows:

$$\mu_1 \rightsquigarrow \mu_1 - \sigma_1 \dot{\mathcal{W}}_1(t) - \dot{\mathcal{J}}(t) \quad \text{and} \quad \beta \rightsquigarrow \beta + \sigma_2 \dot{\mathcal{W}}_2(t),$$

where $\mathcal{W}_i(t)$ $(i = 1, 2)$ are independent Brownian motions with intensities σ_i $(i = 1, 2)$, and $\mathcal{J}(t) = \int_0^t \int_{\mathcal{U}} \eta(u)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u)$. The perturbed version corresponding to the system (I.4) can be expressed by the following form:

$$\begin{cases} \mathrm{d}S(t) = \big(A - \mu_1 S(t) - \beta S(t)I(t)\big)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \displaystyle\int_{\mathcal{U}} \eta(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u) - \sigma_2 S(t)I(t)\mathrm{d}\mathcal{W}_2(t), \\ \mathrm{d}I(t) = \big(\beta S(t)I(t) - (\mu_2 + \gamma)I(t)\big)\mathrm{d}t + \sigma_1 I(t)\mathrm{d}\mathcal{W}_1(t) + \displaystyle\int_{\mathcal{U}} \eta(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u) + \sigma_2 S(t)I(t)\mathrm{d}\mathcal{W}_2(t), \\ \mathrm{d}R(t) = \big(\gamma I(t) - \mu_1 R(t)\big)\mathrm{d}t + \sigma_1 R(t)\mathrm{d}\mathcal{W}_1(t) + \displaystyle\int_{\mathcal{U}} \eta(u)R(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u). \end{cases} \tag{III.4}$$

In this section, we devote our main attention to study the asymptotic proprieties of the new stochastic epidemic model (III.4) that includes both white noise and Lévy jumps. The principal contributions are as follows:

- We prove the existence of a unique ergodic stationary distribution to the model (III.4) by employing the Feller property and mutually exclusive possibilities lemma 11.4. In this sense, we propose an alternative method to establish the explicit threshold expression without using the invariant probability distribution of an auxiliary process, which can close the gap left by using the method presented in [163] and [162].
- We prove that the persistence of the disease occurs under the same condition of the ergodicity of the model (III.4). As a result, we treat a problem that is intentionally ignored in literature; it is not biologically reasonable to consider two distinct thresholds for the persistence and the existence of stationary distribution of (III.4).
- We give sufficient conditions for the stochastic extinction of the disease.

1.2 Main results

Firstly, we introduce some notations and lemmas which are used in our main results. To properly study our model (III.4), we have the following fundamental assumptions on the jump-diffusion coefficients:

- $(\mathcal{H}_1)$ We assume that the jump coefficient $\eta(u)$ satisfy $\int_{\mathcal{U}} \eta^2(u)\nu(\mathrm{d}u) < \infty$.
- $(\mathcal{H}_2)$ For all $u \in \mathcal{U}$, we assume that $1+\eta(u) > 0$ and $\int_{\mathcal{U}} \big(\eta(u) - \ln(1+\eta(u))\big)\nu(\mathrm{d}u) < \infty$.
- $(\mathcal{H}_3)$ We suppose that $\int_{\mathcal{U}} \Big(\ln(1+\eta(u))\Big)^2 \nu(\mathrm{d}u) < \infty$.
- $(\mathcal{H}_4)$ We suppose that $\int_{\mathcal{U}} \Big((1+\eta(u))^2 - 1\Big)^2 \nu(\mathrm{d}u) < \infty$.
- $(\mathcal{H}_5)$ Assume that for some $p > 1$, $\chi_2 = \mu_1 - \frac{(2p-1)}{2}\sigma_1^2 - \frac{1}{2p}\ell_p > 0$, where

$$\ell_p = \int_{\mathcal{U}} \Big((1+\eta(u))^{2p} - 1 - 2p\eta(u)\Big)\nu(\mathrm{d}u) < \infty.$$

Remark 1.1. Biologically, if $\eta(u) > 0$ the Lévy jumps increase the quantity of the host population. Otherwise, if $-1 < \eta(u) < 0$, the number of individuals is minimized gradually.

Remark 1.2. The assumptions $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ mean biologically that the intensity of Lévy jumps cannot exceed the environmental carrying capacity.

In view of the epidemiological interpretation and the dynamical behavior, whether the stochastic model is well-posed is the first concern thing. Therefore, to analyze the stochastic model (III.4), the first problem to be checked is the existence of a unique global positive solution, that is, there is no explosion in finite time under any positive initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$. It is known that there exists a unique global solution to the stochastic models for any given initial value if the coefficients verify the local Lipschitz and the linear growth conditions. But, the coefficients of the model (III.4) do not verify the linear growth condition, which may let the solution to explode at a finite time. The following theorem assures the well-posedness of the stochastic model (III.4).

Theorem 1.3. *Let $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. For any initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$, there exists a unique positive solution $(S(t), I(t), R(t))$ of the system (III.4) on $t \geq 0$, and the solution will stay in $\mathbb{R}^3_+$ almost surely.*

The proof is somehow standard and classic (see for example [171]), so we omit it.
In the following, we always presume that the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_5)$ hold.

Lemma 1.4. *Let $(S(t), I(t), R(t))$ be the solution of (III.4) with initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$ and $N(t) = S(t) + I(t) + R(t)$. Then*

$$\mathbb{E}\big(N^{2p}(t)\big) \leq \big(N(0)\big)^{2p} e^{\{-p\chi_2 t\}} + \frac{2\chi_1}{\chi_2}, \quad \text{and} \quad \limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}\Big(N^{2p}(s)\Big)\mathrm{d}s \leq \frac{2\chi_1}{\chi_2},$$

where $\chi_1 = \sup_{x>0}\Big\{Ax^{2p-1} - \frac{\chi_2}{2}x^{2p}\Big\}$.

Proof. Making use of Itô's lemma, we obtain

$$\begin{aligned}
\mathrm{d}(N(t))^{2p} &= 2p(N(t))^{2p-1}(A - \mu_1 N(t) - rI(t))\mathrm{d}t + p(2p-1)\sigma_1^2(N(t))^{2p}\mathrm{d}t \\
&\quad + \int_{\mathcal{U}} (N(t))^{2p}\big((1+\eta(u))^{2p} - 1 - 2p\eta(u)\big)\nu(\mathrm{d}u)\mathrm{d}t + 2p\sigma_1(N(t))^{2p}\mathrm{d}\mathcal{W}_1(t)] \\
&\quad + \int_{\mathcal{U}} (N(t^-))^{2p}\big((1+\eta(u))^{2p} - 1\big)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u).
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{d}(N(t))^{2p} &\le 2p(N(t))^{2p-1}\big(A-\mu_1 N(t)\big)\mathrm{d}t + p(2p-1)\sigma_1^2(N(t))^{2p}\mathrm{d}t + 2p\sigma_1(N(t))^{2p}\mathrm{d}\mathcal{W}_1(t)\\
&\quad + \int_{\mathcal{U}}(N(t))^{2p}\big((1+\eta(u))^{2p}-1-2p\eta(u)\big)\nu(\mathrm{d}u)\mathrm{d}t\\
&\quad + \int_{\mathcal{U}}(N(t^-))^{2p}\big((1+\eta(u))^{2p}-1\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\
&= 2p\Bigg\{A(N(t))^{2p-1} - \overbrace{\bigg(\mu_1 - \frac{(2p-1)}{2}\sigma_1^2 - \frac{1}{2p}\int_{\mathcal{U}}\big((1+\eta(u))^{2p}-1-2p\eta(u)\big)\nu(\mathrm{d}u)\bigg)}^{\triangleq\chi_2}(N(t))^{2p}\Bigg\}\mathrm{d}t\\
&\quad + 2p\sigma_1(N(t))^{2p}\mathrm{d}\mathcal{W}_1(t) + \int_{\mathcal{U}}(N(t^-))^{2p}\big((1+\eta(u))^{2p}-1\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}$$

We choose neatly $p>1$ such that $\chi_2>0$, we have

$$\begin{aligned}
\mathrm{d}(N(t))^{2p} &\le 2p\Big\{\chi_1 - \frac{\chi_2}{2}(N(t))^{2p}\Big\}\mathrm{d}t + 2p\sigma_1(N(t))^{2p}\mathrm{d}\mathcal{W}_1(t)\\
&\quad + \int_{\mathcal{U}}(N(t^-))^{2p}\big((1+\eta(u))^{2p}-1\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathrm{d}\Big((N(t))^{2p}e^{p\chi_2 t}\Big) &= p\chi_2(N(t))^{2p}e^{p\chi_2 t} + e^{p\chi_2 t}\mathrm{d}(N(t))^{2p}\\
&\le 2p\chi_1 e^{p\chi_2 t}\mathrm{d}t + e^{p\chi_2 t}2p\sigma_1(N(t))^{2p}\mathrm{d}\mathcal{W}_1(t)\\
&\quad + \int_{\mathcal{U}}(N(t^-))^{2p}\big((1+\eta(u))^{2p}-1\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}$$

Then, and by taking integration and expectation on both sides, we get

$$\mathbb{E}\big(N^{2p}(t)\big) \le (N(0))^{2p}e^{-p\chi_2 t} + 2p\chi_1\int_0^t e^{-p\chi_2(t-s)}\mathrm{d}s \le (N(0))^{2p}e^{-p\chi_2 t} + \frac{2\chi_1}{\chi_2}.$$

Obviously, we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\big(N^{2p}(s)\big)\mathrm{d}s \le (N(0))^{2p}\limsup_{t\to+\infty}\frac{1}{t}\int_0^t e^{-p\chi_2 s}\mathrm{d}s + \frac{2\chi_1}{\chi_2} = \frac{2\chi_1}{\chi_2}.$$

■

Now, we consider the following subsystem

$$\begin{cases}
\mathrm{d}X(t) = \big(A-\mu_1 X(t)\big)\mathrm{d}t + \sigma_1 X(t)\mathrm{d}\mathcal{W}_1(t) + \displaystyle\int_{\mathcal{U}}\eta(u)X(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \quad \forall t>0,\\
X(0) = N(0) > 0.
\end{cases} \tag{III.5}$$

Remark 1.5. The solution of (III.5) is unique and positive. The proof is presented in Example 1.5, pp. 7 of [104].

Lemma 1.6. *[171] Let $(S(t),I(t),R(t))$ be the positive solution of system (III.4) with any given initial condition $(S(0),I(0),R(0))\in\mathbb{R}^3_+$. Let also $X(t)$ be the positive solution of equation (III.5) with any given initial value $X(0)=N(0)\in\mathbb{R}_+$. Then*

- $\lim\limits_{t\to\infty}\frac{X(t)}{t}=0$, $\lim\limits_{t\to\infty}\frac{X^2(t)}{t}=0$, $\lim\limits_{t\to\infty}\frac{S(t)}{t}=0$, $\lim\limits_{t\to\infty}\frac{I(t)}{t}=0$, and $\lim\limits_{t\to\infty}\frac{R(t)}{t}=0$ a.s.
- $\lim\limits_{t\to\infty}\frac{1}{t}\int_0^t X(s)\mathrm{d}\mathcal{W}_1(s)=0$, $\lim\limits_{t\to\infty}\frac{1}{t}\int_0^t X^2(s)\mathrm{d}\mathcal{W}_1(s)=0$, $\lim\limits_{t\to\infty}\frac{1}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_1(s)=0$,
  $\lim\limits_{t\to\infty}\frac{1}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_2(s)=0$ $\lim\limits_{t\to\infty}\frac{1}{t}\int_0^t I(s)\mathrm{d}\mathcal{W}_1(s)=0$, and $\lim\limits_{t\to\infty}\frac{1}{t}\int_0^t R(s)\mathrm{d}\mathcal{W}_1(s)=0$ a.s.

- $\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta(u)X(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)=0$, $\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathcal{U}}X^2(s^-)\Big((1+\eta(u))^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)=0$,
  $\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)=0$, $\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)=0$,
  and $\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta(u)R(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)=0$ a.s.

Remark 1.7. For a detailed proof and generalized version of the last lemma, we refer the reader to the proof of Lemma 2.5 in Chapter VI.

Remark 1.8. In the absence of Lévy jumps (see for example [139]), the stationary distribution expression is used to estimate the time averages of the auxiliary process solution by employing the ergodic theorem [93]. Unluckily, the said expression is still unknown in the case of the Lévy noise. This issue is implicitly mentioned in [163, 162] as an open question, and the authors presented the threshold analysis of their model with an unknown stationary distribution formula. In this thesis, we propose an alternative method to establish the exact expression of the threshold parameter without having recourse to the use of ergodic theorem. This new idea that we propose is presented in the following lemma.

Lemma 1.9. *Let $X(t)$ be the solution of system (III.5) with an initial value $X(0)\in\mathbb{R}_+$ and assume that $\chi_3=2\mu_1-\sigma_1^2-\int_{\mathcal{U}}\eta^2(u)\nu(du)>0$. Then*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X(s)\mathrm{d}s=\frac{A}{\mu_1}\ \text{a.s.}\qquad\text{and}\qquad\lim_{t\to\infty}\frac{1}{t}\int_0^t X^2(s)\mathrm{d}s=\frac{2A^2}{\mu_1\chi_3}\ \text{a.s.}$$

Proof. Integrating from 0 to t on both sides of (III.5) yields

$$\frac{X(t)-X(0)}{t}=A-\frac{\mu_1}{t}\int_0^t X(s)\mathrm{d}s+\frac{\sigma_1}{t}\int_0^t X(s)\mathrm{d}\mathcal{W}_1(s)+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta(u)X(t^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

Clearly, we can derive that

$$\frac{1}{t}\int_0^t X(s)\mathrm{d}s=\frac{A}{\mu_1}-\frac{X(t)-X(0)}{\mu_1 t}+\frac{\sigma_1}{\mu_1 t}\int_0^t X(s)\mathrm{d}\mathcal{W}_1(s)+\frac{1}{\mu_1 t}\int_0^t\int_{\mathcal{U}}\eta(u)X(t^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

From Lemma 1.6, one can derive that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X(s)\mathrm{d}s=\frac{A}{\mu_1}\ \text{a.s.}$$

Applying the generalized Itô's formula to model (III.5) leads to

$$\begin{aligned}\mathrm{d}X^2(t)=&\left(2X(t)\Big(A-\mu_1X(t)\Big)+\sigma_1^2X^2(t)+\int_{\mathcal{U}}X^2(t)\Big((1+\eta(u))^2-1-2\eta(u)\Big)\nu(\mathrm{d}u)\right)\mathrm{d}t\\&+2\sigma_1X^2(t)\mathrm{d}\mathcal{W}_1(t)+\int_{\mathcal{U}}X^2(t^-)\Big((1+\eta(u))^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).\end{aligned}$$

Integrating both sides of the last equation from 0 to t, yields

$$\begin{aligned}X^2(t)-X^2(0)=&2A\int_0^t X(s)\mathrm{d}s-\overbrace{\left(2\mu_1-\sigma_1^2-\int_{\mathcal{U}}\Big((1+\eta(u))^2-1-2\eta(u)\Big)\nu(\mathrm{d}u)\right)}^{\triangleq\chi_3>0}\int_0^t X^2(s)\mathrm{d}s\\&+2\sigma_1\int_0^t X^2(s)\mathrm{d}\mathcal{W}_1(s)+\int_0^t\int_{\mathcal{U}}X^2(s^-)\Big((1+\eta(u))^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

Therefore

$$\frac{1}{t}\int_0^t X^2(s)\mathrm{d}s = \frac{2A}{\chi_3 t}\int_0^t X(s)\mathrm{d}s + \frac{(X^2(0)-X^2(t))}{\chi_3 t} + \frac{2\sigma_1}{\chi_3 t}\int_0^t X^2(s)\mathcal{W}_1(s) + \frac{1}{\chi_3 t}\int_0^t\int_{\mathcal{U}} X^2(s^-)\Big((1+\eta(u))^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

From Lemma 1.6, we can easily verify that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X^2(s)\mathrm{d}s = \frac{2A^2}{\mu_1\chi_3} \quad \text{a.s.}$$

■

The aim of the following theorem is to give the condition for the persistence in the mean of the disease and the ergodicity of the stochastic model (III.4). Our approach is inspired by the work of Zhao and Yuan [162]. We suppose that $\chi_3 > 0$ and we define the following parameter:

$$\mathcal{R}_0^s = \left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)^{-1}\left(\frac{\beta A}{\mu_1} - \frac{A^2\sigma_2^2}{\mu_1\chi_3} - \int_{\mathcal{U}}\Big(\eta(u)-\ln(1+\eta(u))\Big)\nu(\mathrm{d}u)\right).$$

Theorem 1.10. *If $\mathcal{R}_0^s > 1$, then for any value $(S(0),I(0),R(0)) \in \mathbb{R}^3_+$, the disease is persistent in the mean. Furthermore, under the same condition, the stochastic system (III.4) admits a unique stationary distribution and it has the ergodic property.*

Proof. On the one hand, based on the model (III.4), we get

$$\mathrm{d}\Big(S(t)+I(t)\Big) = \Big(A-\mu_1 S(t)-(\mu_2+\gamma)I(t)\Big)\mathrm{d}t + \sigma_1\Big(S(t)+I(t)\Big)\mathrm{d}\mathcal{W}_1(t) + \int_{\mathcal{U}}\eta(u)\left(S(t^-)+I(t^-)\right)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).$$

Taking integral on both sides of the last equation from 0 to t, we see that

$$\frac{1}{t}\Big(S(t)+I(t)-S(0)-I(0)\Big) = A - \frac{\mu_1}{t}\int_0^t S(s)\mathrm{d}s - \frac{(\mu_2+\gamma)}{t}\int_0^t I(s)\mathrm{d}s + \frac{\sigma_1}{t}\int_0^t (S(s)+I(s))\mathrm{d}\mathcal{W}_1(s) + \frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta(u)\left(S(s^-)+I(s^-)\right)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

Then, one can obtain that

$$\frac{1}{t}\int_0^t S(s)\mathrm{d}s = \frac{A}{\mu_1} - \frac{(\mu_2+\gamma)}{\mu_1 t}\int_0^t I(s)\mathrm{d}s + \Phi_1(t), \tag{III.6}$$

where

$$\Phi_1(t) = \frac{\sigma_1}{\mu_1 t}\int_0^t (S(s)+I(s))\mathrm{d}\mathcal{W}_1(s) - \frac{1}{\mu_1 t}\Big(S(t)+I(t)-S(0)-I(0)\Big) + \frac{1}{\mu_1 t}\int_0^t\int_{\mathcal{U}}\eta(u)\left(S(s^-)+I(s^-)\right)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

On the other hand, applying Itô's formula to the second equation of (III.4), we get

$$\mathrm{d}\ln I(t) = \left(\beta S(t) - \frac{\sigma_2^2}{2}S^2(t) - \left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}}\eta(u)-\ln(1+\eta(u))\nu(\mathrm{d}u)\right)\mathrm{d}t + \sigma_1\mathrm{d}\mathcal{W}_1(t) + \sigma_2 S(t)\mathrm{d}\mathcal{W}_2(t) + \int_{\mathcal{U}}\ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \tag{III.7}$$

Integrating (III.7) from 0 to t and then dividing t on both sides, we have

$$\frac{1}{t}(\ln I(t) - \ln I(0)) = \frac{\beta}{t}\int_0^t S(s)\mathrm{d}s - \frac{\sigma_2^2}{2t}\int_0^t S^2(s)\mathrm{d}s - \left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}} \eta(u) - \ln(1+\eta(u))\nu(\mathrm{d}u)$$
$$+ \frac{\sigma_1}{t}\mathcal{W}_1(t) + \frac{\sigma_2}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_2(s) + \frac{1}{t}\int_0^t\int_{\mathcal{U}} \ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

From (III.6), we obtain

$$\frac{1}{t}(\ln I(t) - \ln I(0)) = \frac{\beta A}{\mu_1} - \frac{\beta(\mu_2+\gamma)}{\mu_1 t}\int_0^t I(s)\mathrm{d}s + \beta\Phi_1(t) - \frac{\sigma_2^2}{2t}\int_0^t S^2(s)\mathrm{d}s - \left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right)$$
$$- \int_{\mathcal{U}} \eta(u) - \ln(1+\eta(u))\nu(\mathrm{d}u) + \frac{\sigma_1}{t}\mathcal{W}_1(t) + \frac{\sigma_2}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_2(s)$$
$$+ \frac{1}{t}\int_0^t\int_{\mathcal{U}} \ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

By the stochastic comparison theorem, we have $N(t) \le S(t)$ a.s. and

$$\frac{1}{t}\big(\ln I(t) - \ln I(0)\big) \ge \left(\frac{\beta A}{\mu_1} - \frac{\sigma_2^2}{2t}\int_0^t X^2(s)\mathrm{d}s - \left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}} \eta(u) - \ln(1+\eta(u))\nu(\mathrm{d}u)\right)$$
$$+ \beta\Phi_1(t) - \frac{\beta(\mu_2+\gamma)}{\mu_1 t}\int_0^t I(s)\mathrm{d}s + \frac{\sigma_1}{t}\mathcal{W}_1(t) + \frac{\sigma_2}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_2(s)$$
$$+ \frac{1}{t}\int_0^t\int_{\mathcal{U}} \ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

Hence, we further get

$$\frac{\beta(\mu_2+\gamma)}{\mu_1 t}\int_0^t I(s)\mathrm{d}s \ge \left(\frac{\beta A}{\mu_1} - \frac{\sigma_2^2}{2t}\int_0^t X^2(s)\mathrm{d}s - \left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}} \eta(u) - \ln(1+\eta(u))\nu(\mathrm{d}u)\right)$$
$$+ \beta\Phi_1(t) - \frac{1}{t}(\ln I(t) - \ln I(0)) + \frac{\sigma_1}{t}\mathcal{W}_1(t) + \frac{\sigma_2}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_2(s)$$
$$+ \frac{1}{t}\int_0^t\int_{\mathcal{U}} \ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

By assumption $(\mathcal{H}_3)$, Lemma 1.6 and the large number theorem for martingales, we can easily verify that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(s)\mathrm{d}s \ge \frac{\mu_1}{\beta(\mu_2+\gamma)}\left(\frac{\beta A}{\mu_1} - \frac{A^2\sigma_2^2}{\mu_1\chi_3} - \left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}} \eta(u) - \ln(1+\eta(u))\nu(\mathrm{d}u)\right)$$
$$= \frac{\mu_1}{\beta(\mu_2+\gamma)}\left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right)\left(\mathcal{R}_0^s - 1\right) > 0 \ \text{ a.s.}$$

This shows that the system (III.4) is persistent in the mean with probability one.

In the following, we will discuss the existence of a unique ergodic stationary distribution of the positive solutions to the system (III.4). Similar to the proof of Lemma 3.2 in [135], we briefly verify the Feller property of the SDE model (III.4). The main purpose of the next analysis is to prove that (II.22) is impossible. Same as the above, we have

$$\mathrm{d}\left\{\ln I(t) - \frac{\beta}{\mu_1}\Big(X(t) - S(t)\Big)\right\} = \left(\beta S(t) - \left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right) - \frac{\sigma_2^2}{2}S^2(t) - \int_{\mathcal{U}}\Big(\eta(u) - \ln(1+\eta(u))\Big)\nu(\mathrm{d}u)\right)\mathrm{d}t$$
$$- \frac{\beta}{\mu_1}\Big(-\mu_1(X(t) - S(t)) + \beta S(t)I(t)\Big)\mathrm{d}t + \sigma_1\mathrm{d}\mathcal{W}_1(t) + \sigma_2 S(t)\mathrm{d}\mathcal{W}_2(t)$$
$$- \frac{\beta\sigma_1}{\mu_1}(X(t) - S(t))\mathrm{d}\mathcal{W}_1(t) - \frac{\beta\sigma_2}{\mu_1}S(t)I(t)\mathrm{d}\mathcal{W}_2(t)$$
$$- \frac{\beta}{\mu_1}\int_{\mathcal{U}}\eta(u)(X(t^-) - S(t^-))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) + \int_{\mathcal{U}}\ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).$$

Hence

$$\begin{aligned}
\mathrm{d}\left\{\ln I(t) - \frac{\beta}{\mu_1}\big(X(t)-S(t)\big)\right\} &= \left(\beta X(t) - \left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right) - \frac{\sigma_2^2}{2}S^2(t) - \int_{\mathcal{U}}\Big(\eta(u)-\ln(1+\eta(u))\Big)\nu(\mathrm{d}u)\right)\mathrm{d}t \\
&\quad - \frac{\beta^2}{\mu_1}S(t)I(t)\mathrm{d}t + \sigma_1\mathrm{d}\mathcal{W}_1(t) + \sigma_2 S(t)\mathrm{d}\mathcal{W}_2(t) - \frac{\beta\sigma_1}{\mu_1}(X(t)-S(t))\mathrm{d}\mathcal{W}_1(t) \\
&\quad - \frac{\beta\sigma_2}{\mu_1}S(t)I(t)\mathrm{d}\mathcal{W}_2(t) - \frac{\beta}{\mu_1}\int_{\mathcal{U}}\eta(u)(X(t^-)-S(t^-))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) \\
&\quad + \int_{\mathcal{U}}\ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \qquad \text{(III.8)}
\end{aligned}$$

Integrating from 0 to t on both sides of (III.8) yields

$$\begin{aligned}
&\ln\frac{I(t)}{I(0)} - \frac{\beta}{\mu_1}\Big(X(t)-S(t)\Big) + \frac{\beta}{\mu_1}\Big(X(0)-S(0)\Big) \\
&= \int_0^t\left(\beta X(s) - \frac{\sigma_2^2}{2}S^2(s) - \left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}}\Big(\eta(u)-\ln(1+\eta(u))\Big)\nu(\mathrm{d}u)\right)\mathrm{d}s \\
&\quad - \frac{\beta^2}{\mu_1}\int_0^t S(s)I(s)\mathrm{d}s + \sigma_1\mathcal{W}_1(t) + \sigma_2\int_0^t S(s)\mathrm{d}\mathcal{W}_2(s) - \frac{\beta\sigma_1}{\mu_1}\int_0^t(X(s)-S(s))\mathrm{d}\mathcal{W}_1(s) \\
&\quad - \frac{\beta\sigma_2}{\mu_1}\int_0^t S(s)I(s)\mathrm{d}\mathcal{W}_2(s) - \frac{\beta}{\mu_1}\int_0^t\int_{\mathcal{U}}\eta(u)(X(s^-)-S(s^-))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) + \int_0^t\int_{\mathcal{U}}\ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).
\end{aligned}$$

Then, we get

$$\begin{aligned}
\int_0^t \beta S(s)I(s)\mathrm{d}s &= \frac{\mu_1}{\beta}\int_0^t\left(\beta X(s) - \frac{\sigma_2^2}{2}S^2(s) - \left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}}\Big(\eta(u)-\ln(1+\eta(u))\Big)\nu(\mathrm{d}u)\right)\mathrm{d}s \\
&\quad + \Big(X(t)-S(t)\Big) - \Big(X(0)-S(0)\Big) - \frac{\mu_1}{\beta}\ln\frac{I(t)}{I(0)} + \frac{\sigma_1\mu_1}{\beta}\mathcal{W}_1(t) + \frac{\sigma_2\mu_1}{\beta}\int_0^t S(s)\mathrm{d}\mathcal{W}_2(s) \\
&\quad - \sigma_1\int_0^t(X(s)-S(s))\mathrm{d}\mathcal{W}_1(s) - \sigma_2\int_0^t S(s)I(s)\mathrm{d}\mathcal{W}_2(s) \\
&\quad - \int_0^t\int_{\mathcal{U}}\eta(u)(X(s^-)-S(s^-))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) + \frac{\mu_1}{\beta}\int_0^t\int_{\mathcal{U}}\ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).
\end{aligned}$$

From Lemma 1.6, one can derive that

$$\lim_{t\to\infty}\frac{X(t)}{t} = 0,\quad \lim_{t\to\infty}\frac{S(t)}{t} = 0 \text{ and } \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta(u)(X(s^-)-S(s^-))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) = 0 \text{ a.s.}$$

Moreover,

$$\lim_{t\to\infty}\frac{\mu_1\sigma_2}{\beta t}\int_0^t S(u)\mathrm{d}\mathcal{W}_2(s) = 0 \text{ and } \lim_{t\to\infty}\frac{\sigma_1}{t}\int_0^t\Big(X(u)-S(u)\Big)\mathrm{d}\mathcal{W}_1(s) = 0 \quad \text{a.s.}$$

Application of the strong law of large numbers and assumption $(\mathcal{H}_3)$ shows that

$$\lim_{t\to\infty}\frac{\mu_1\sigma_1\mathcal{W}_1(t)}{\beta t} = 0 \text{ and } \lim_{t\to\infty}\frac{\mu_1}{\beta t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) = 0 \quad \text{a.s.}$$

Applying similar arguments to those in Lemma 2.5 of [63], we obtain

$$\lim_{t\to\infty}\frac{\sigma_2}{t}\int_0^t S(s)I(s)\mathrm{d}\mathcal{W}_2(s) = 0 \quad \text{a.s.}$$

Since $\limsup_{t\to\infty}\frac{1}{t}\ln\frac{I(t)}{I(0)} \le \limsup_{t\to\infty}\frac{1}{t}\ln\frac{N(t)}{I(0)} \le 0$ a.s. (see Lemma 2.1. of [26]), one can derive that

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\int_0^t \beta S(s)I(s)\mathrm{d}s\\
&\ge \frac{\mu_1}{\beta}\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left(\beta X(s)-\frac{\sigma_2^2}{2}X^2(s)-\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)-\int_{\mathcal{U}}\Big(\eta(u)-\ln(1+\eta(u))\Big)\nu(\mathrm{d}u)\right)\mathrm{d}s\\
&= \frac{\mu_1}{\beta}\lim_{t\to\infty}\frac{1}{t}\int_0^t\left(\beta X(s)-\frac{\sigma_2^2}{2}X^2(s)-\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)-\int_{\mathcal{U}}\Big(\eta(u)-\ln(1+\eta(u))\Big)\nu(\mathrm{d}u)\right)\mathrm{d}s. \qquad \text{(III.9)}
\end{aligned}$$

Now, from Lemma 1.9, it follows that

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\int_0^t \beta S(s)I(s)\mathrm{d}s &\ge \frac{\mu_1}{\beta}\times\left(\frac{\beta A}{\mu_1}-\frac{A^2\sigma_2^2}{\mu_1\chi_3}-\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)-\int_{\mathcal{U}}\Big(\eta(u)-\ln(1+\eta(u))\Big)\nu(\mathrm{d}u)\right)\\
&= \frac{\mu_1}{\beta}\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)\Big(\mathcal{R}_0^s-1\Big) > 0 \quad \text{a.s.} \qquad \text{(III.10)}
\end{aligned}$$

To continue our analysis, we need to set the following subsets:

$$\begin{aligned}
\Omega_1 &= \{(t,\omega)\in\mathbb{R}_+\times\Omega|\ S(t,\omega)\ge\epsilon,\ \text{and},\ I(t,\omega)\ge\epsilon\},\\
\Omega_2 &= \{(t,\omega)\in\mathbb{R}_+\times\Omega|\ S(t,\omega)\le\epsilon\},\\
\Omega_3 &= \{(t,\omega)\in\mathbb{R}_+\times\Omega|\ I(t,\omega)\le\epsilon\},
\end{aligned}$$

where $\epsilon>0$ is a positive constant to be determined later. Therefore, by (III.10), we get

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\Big(\beta S(s)I(s)1_{\Omega_1}\Big)\mathrm{d}s &\ge \liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\Big(\beta S(s)I(s)\Big)\mathrm{d}s-\limsup_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\Big(\beta S(s)I(s)1_{\Omega_2}\Big)\mathrm{d}s\\
&\quad-\limsup_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\Big(\beta S(s)I(s)1_{\Omega_3}\Big)\mathrm{d}s\\
&\ge \frac{\mu_1}{\beta}\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)\Big(\mathcal{R}_0^s-1\Big)-\beta\epsilon\limsup_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\Big(I(s)\Big)\mathrm{d}s\\
&\quad-\beta\epsilon\limsup_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\Big(S(s)\Big)\mathrm{d}s.
\end{aligned}$$

Thus, one can see that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\Big(\beta S(s)I(s)1_{\Omega_1}\Big)\mathrm{d}s \ge \frac{\mu_1}{\beta}\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)\Big(\mathcal{R}_0^s-1\Big)-\frac{2A\beta\epsilon}{\mu_1}.$$

We can choose $\epsilon\le\frac{\mu_1^2}{4\beta^2A}\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)\Big(\mathcal{R}_0^s-1\Big)$, and then we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\Big(\beta S(s)I(s)1_{\Omega_1}\Big)\mathrm{d}s \ge \frac{\mu_1}{2\beta}\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)\Big(\mathcal{R}_0^s-1\Big)>0. \qquad \text{(III.11)}$$

Let $p>1$ such that $\chi_2>0$ and q is given by $\frac{1}{q}+\frac{1}{p}=1$. By utilizing the Young inequality $xy\le\frac{x^p}{p}+\frac{y^q}{q}$ for all $x,y>0$, we get

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\Big(\beta S(s)I(s)1_{\Omega_1}\Big)\mathrm{d}s &\le \liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\Big(p^{-1}(\eta\beta S(s)I(s))^p+q^{-1}\eta^{-q}1_{\Omega_1}\Big)\mathrm{d}s\\
&\le \liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\Big(q^{-1}\eta^{-q}1_{\Omega_1}\Big)\mathrm{d}s+p^{-1}(\eta\beta)^p\limsup_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\Big(N^{2p}(s)\Big)\mathrm{d}s,
\end{aligned}$$

where η is a positive constant satisfying

$$\eta^p \leq \frac{p\mu_1\chi_2\beta^{-(p+1)}}{8\chi_1}\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)\left(\mathcal{R}_0^s-1\right).$$

By Lemma 1.4 and (III.11), we deduce that

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}(1_{\Omega_1})ds &\geq q\eta^q\left(\frac{\mu_1}{2\beta}\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)\left(\mathcal{R}_0^s-1\right)-\frac{2\chi_1\eta^p\beta^p}{p\chi_2}\right)\\ &\geq \frac{\mu_1 q\eta^q}{4\beta}\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)\left(\mathcal{R}_0^s-1\right)>0. \end{aligned} \tag{III.12}$$

Setting

$$\begin{aligned}\Omega_4 &= \{(t,\omega)\in\mathbb{R}_+\times\Omega|\ S(t,\omega)\geq\zeta,\ \text{or},\ I(t,\omega)\geq\zeta\},\\ \Sigma &= \{(t,\omega)\in\mathbb{R}_+\times\Omega|\ \epsilon\leq S(t,\omega)\leq\zeta,\ \text{and},\ \epsilon\leq I(t,\omega)\leq\zeta\},\end{aligned}$$

where $\zeta>\epsilon$ is a positive constant to be explained in the following. By using the Markov's inequality, we can observe that

$$\int_\Omega \mathbb{1}_{\Omega_4}(t,\omega)d\mathbb{P}(\omega)\leq\mathbb{P}(S(t)\geq\zeta)+\mathbb{P}(I(t)\geq\zeta)\leq\frac{1}{\zeta}\mathbb{E}[S(t)+I(t)]\leq\frac{1}{\zeta}\left(\frac{2A}{\mu_1}+N(0)\right).$$

Choosing

$$\frac{1}{\zeta}\leq\frac{\mu_1 q\eta^q}{8\beta}\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)\left(\mathcal{R}_0^s-1\right)\left(\frac{2A}{\mu_1}+N(0)\right)^{-1}.$$

We thus obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}[1_{\Omega_4}]ds\leq\frac{\mu_1 q\eta^q}{8\beta}\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)\left(\mathcal{R}_0^s-1\right).$$

According to (III.12), one can derive that

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}[1_\Sigma]ds &\geq \liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}[1_{\Omega_1}]ds-\limsup_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}[1_{\Omega_4}]ds\\ &\geq\frac{\mu_1 q\eta^q}{8\beta}\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)\left(\mathcal{R}_0^s-1\right)>0.\end{aligned}$$

Based on the above analysis, we have determined a compact domain $\Sigma\subset\mathbb{R}_+^3$ such that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{P}\Big(s,\big(S(0),I(0),R(0)\big),\Sigma\Big)ds\geq\frac{\mu_1 q\eta^q}{8\beta}\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)\left(\mathcal{R}_0^s-1\right)>0.$$

Applying similar arguments to those in Theorem 3.1 in [135] or Corollary 4.4 in [57], we show the uniqueness of the ergodic stationary distribution of our model (III.4). This completes the proof. ■

Now, we will give the result on the disease extinction. Define

$$\hat{\mathcal{R}}_0^s=\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)^{-1}\left(\frac{\beta A}{\mu_1}-\frac{\sigma_2^2A^2}{2\mu_1^2}-\int_{\mathcal{U}}\Big(\eta(u)-\ln(1+\eta(u))\Big)\nu(\mathrm{d}u)\right).$$

Theorem 1.11. *Let $(S(t),I(t),R(t))$ be the solution of system (III.4) with positive initial value. If*

$$\hat{\mathcal{R}}_0^s<1\ \text{ and }\ \sigma_2^2\leq\frac{\mu_1\beta}{A}, \tag{III.13}$$

or

$$\frac{\beta^2}{2\sigma_2^2}-\left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right)-\int_{\mathcal{U}}\Big(\eta(u)-\ln(1+\eta(u))\Big)\nu(du)<0. \tag{III.14}$$

Then, the disease dies out exponentially with probability one.

Proof. By Itô's formula for all $t \geq 0$, we have

$$\mathrm{d}\ln I(t) = \left(\beta S(t) - \frac{\sigma_2^2}{2}S^2(t) - \left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}}\Big(\eta(u) - \ln(1+\eta(u))\Big)\nu(\mathrm{d}u)\right)\mathrm{d}t$$
$$+ \sigma_1 \mathrm{d}\mathcal{W}_1(t) + \sigma_2 S(t)\mathrm{d}\mathcal{W}_2(t) + \int_{\mathcal{U}} \ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u). \tag{III.15}$$

Integrating (III.15) from 0 to t and then dividing t on both sides, we get

$$\frac{\ln I(t)}{t} = \frac{\beta}{t}\int_0^t S(s)\mathrm{d}s - \frac{\sigma_2^2}{2t}\int_0^t S^2(s)\mathrm{d}s - \left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}}\Big(\eta(u) - \ln(1+\eta(u))\Big)\nu(\mathrm{d}u) + \Phi_2(t), \tag{III.16}$$

where

$$\Phi_2(t) = \frac{\ln I(0)}{t} + \frac{\sigma_1}{t}\mathcal{W}_1(t) + \frac{\sigma_2}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_2(s) + \frac{1}{t}\int_0^t\int_{\mathcal{U}} \ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u).$$

Obviously, we know that

$$\frac{1}{t}\int_0^t S^2(s)\mathrm{d}s \geq \left(\frac{1}{t}\int_0^t S(s)\mathrm{d}s\right)^2.$$

Therefore, and by using (III.6), we derive

$$\begin{aligned}\frac{\ln I(t)}{t} &\leq \frac{\beta}{t}\int_0^t S(s)\mathrm{d}s - \frac{\sigma_2^2}{2}\left(\frac{1}{t}\int_0^t S(s)\mathrm{d}s\right)^2 - \left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}}\Big(\eta(u) - \ln(1+\eta(u))\Big)\nu(\mathrm{d}u) + \Phi_2(t)\\ &\leq \beta\left(\frac{A}{\mu_1} - \frac{(\mu_2+\gamma)}{\mu_1 t}\int_0^t I(s)\mathrm{d}s + \Phi_1(t)\right) - \frac{\sigma_2^2}{2}\left(\frac{A}{\mu_1} - \frac{(\mu_2+\gamma)}{\mu_1 t}\int_0^t I(s)\mathrm{d}s + \Phi_1(t)\right)^2\\ &\quad - \left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}}\Big(\eta(u) - \ln(1+\eta(u))\Big)\nu(\mathrm{d}u) + \Phi_2(t).\end{aligned}$$

Hence, one can see that

$$\begin{aligned}\frac{\ln I(t)}{t} &\leq \frac{\beta A}{\mu_1} - \frac{A^2\sigma_2^2}{2\mu_1^2} - \left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}}\Big(\eta(u) - \ln(1+\eta(u))\Big)\nu(\mathrm{d}u)\\ &\quad - \frac{(\mu_2+\gamma)}{\mu_1}\left(\beta - \frac{A\sigma_2^2}{\mu_1}\right)\frac{1}{t}\int_0^t I(s)\mathrm{d}s - \frac{\sigma_2^2}{2}\left(\frac{(\mu_2+\gamma)}{\mu_1 t}\int_0^t I(u)\mathrm{d}u\right)^2 + \Phi_2(t) + \Phi_3(t),\end{aligned} \tag{III.17}$$

where

$$\Phi_3(t) = \beta\Phi_1(t) - \frac{\sigma_2^2}{2}\Phi_1^2(t) - \frac{\sigma_2^2 A\Phi_1(t)}{\mu_1} + \sigma_2^2\Phi_1(t)\frac{(\mu_2+\gamma)}{\mu_1 t}\int_0^t I(s)\mathrm{d}s.$$

Based on Lemma 1.6, one has

$$\lim_{t\to\infty}\Phi_2(t) = \lim_{t\to\infty}\Phi_3(t) = 0 \quad \text{a.s.}$$

Taking the superior limit on both sides of (III.17), then by condition (III.13), we arrive at

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \leq \left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right)\left(\hat{\mathcal{R}}_0^s - 1\right) < 0 \quad \text{a.s.}$$

Now, if the condition (III.14) is satisfied, then from (III.16) we get

$$\frac{\ln I(t)}{t} = \frac{\beta}{t}\int_0^t S(s)\mathrm{d}s - \frac{\sigma_2^2}{2t}\int_0^t S^2(s)\mathrm{d}s - \left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}}\eta(u) - \ln(1+\eta(u))\nu(\mathrm{d}u)$$
$$+ \frac{\ln I(0)}{t} + \frac{\sigma_1}{t}\mathcal{W}_1(t) + \frac{\sigma_2}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_2(s) + \frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

Then

$$\begin{aligned}\frac{\ln I(t)}{t} &= \frac{\beta^2}{2\sigma_2^2} - \left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}}\eta(u) - \ln(1+\eta(u))\nu(\mathrm{d}u) - \frac{1}{t}\int_0^t\left(\frac{\sigma_2^2}{2}\left(S(s)-\frac{\beta}{\sigma_2^2}\right)^2\right)\mathrm{d}s\\ &\quad + \frac{\ln I(0)}{t} + \frac{\sigma_1}{t}\mathcal{W}_1(t) + \frac{\sigma_2}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_2(s) + \frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\ &\le \frac{\beta^2}{2\sigma_2^2} - \left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}}\eta(u) - \ln(1+\eta(u))\nu(\mathrm{d}u) + \frac{\ln I(0)}{t}\\ &\quad + \frac{\sigma_1}{t}\mathcal{W}_1(t) + \frac{\sigma_2}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_2(s) + \frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

By the large number theorem for martingales, Lemma 1.6 and the condition (III.14), our desired result holds true. This completes the proof. ∎

1.3 Numerical examples

In this subsection, we will validate our theoretical results with the help of numerical simulation examples taking parameters from the theoretical data mentioned in Table III.1. We numerically simulate the solution of system (III.4) with initial value $(S(0), I(0), R(0)) = (1, 0.5, 0.4)$. The used numerical method is described in Appendix A.

Parameters	Description	Value
A	The recruitment rate	0.09
μ_1	The natural mortality rate	0.05
β	The transmission rate	0.08
γ	The recovered rate	0.01
μ_2	The general mortality rate	0.09

Table III.1: Some theoretical parameter values of the model (III.4).

Example 1.12 (Stationary distribution and permanence). We choose the stochastic fluctuations intensities $\sigma_1 = 0.02$ and $\sigma_2 = 0.03$, and we assume that $\eta(u) = 0.05$, $\mathcal{U} = (0,\infty)$ and $\nu(\mathcal{U}) = 1$. Other parameters are given in Table III.1. Then, $\mathcal{R}_0^s = 1.3961 > 1$. From Figure III.1, we show the existence of the unique stationary distributions for S, I and R of model (III.4) at $t = 300$, where the smooth curves are the probability density functions of $S(t)$, $I(t)$ and $R(t)$, respectively. It can be obviously observed that the solution of the stochastic model (III.4) persists in the mean under the same condition (see Figure III.2).

Example 1.13 (Extinction). We choose the white noise intensities $\sigma_1 = 0.1$ and $\sigma_2 = 0.3$ to ensure that the condition (III.14) of Theorem 1.11 is satisfied. We can conclude that for any initial value, $I(t)$ obeys

$$\limsup_{t\to\infty}\frac{1}{t}\ln\frac{I(t)}{I(0)} \le \frac{\beta^2}{2\sigma_2^2} - \left(\mu_2+\gamma+\frac{\sigma_1^2}{2}\right) - \int_{\mathcal{U}}\Big(\eta(u) - \ln(1+\eta(u))\Big)\nu(\mathrm{d}u) = -0.0862 < 0 \quad \text{a.s.}$$

That is, $I(t)$ will tend to zero exponentially with probability one (see Figure III.3 (a)). To verify that the condition (III.13) is satisfied, we change σ_1 to 0.01, σ_2 to 0.02, and β to 0.05 and keep other

parameters unchanged. Then we have

$$\hat{\mathcal{R}}_0^s = \left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right)^{-1} \left(\frac{\beta A}{\mu_1} - \frac{\sigma_2^2 A^2}{2\mu_1^2} - \int_{\mathcal{U}} \Big(\eta(u) - \ln(1+\eta(u))\Big)\nu(du)\right) = 0.8792 < 1,$$

and

$$\sigma_1^2 - \frac{\mu_1 \beta}{A} = -0.0277 < 0.$$

Therefore, the condition (III.13) of Theorem 1.11 is satisfied. We can conclude that for any initial value, $I(t)$ obeys

$$\limsup_{t\to\infty} \frac{1}{t} \ln \frac{I(t)}{I(0)} \leq (\hat{\mathcal{R}}_0^s - 1)\left(\mu_2 + \gamma + \frac{\sigma_1^2}{2}\right) = -0.0121 < 0 \quad \text{a.s.}$$

That is, $I(t)$ will tend to zero exponentially with probability one (see Figure III.3 (b)).

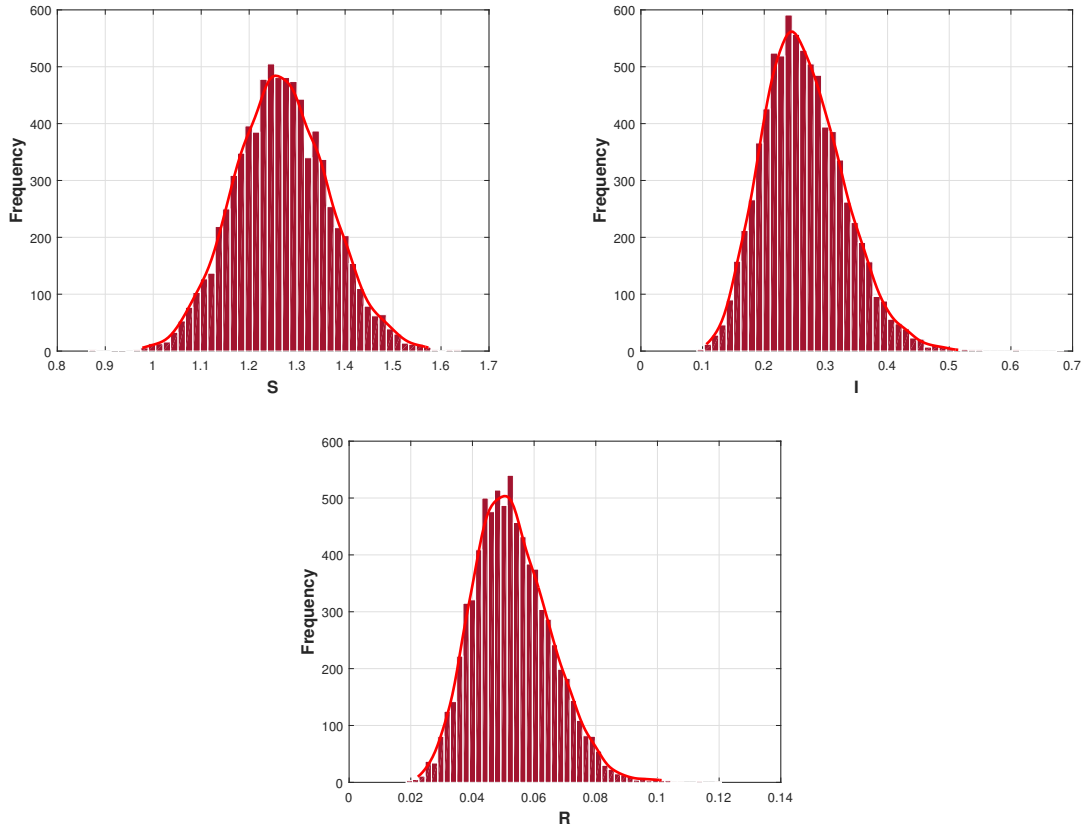


Figure III.1: Histogram of the probability density function for S, I, and R population at $t = 300$ for the stochastic model (III.4), the smoothed curves are the probability density functions of $S(t)$, $I(t)$ and $R(t)$, respectively.

1.4 Discussion

Generally speaking, the dissemination of the epidemic diseases presents a global issue that concerns decision-makers to elude deaths and deterioration of economies. Many scientists are motivated to understand and suggest the ways for diminishing the epidemic dissemination. The first generation proposed the deterministic models that showed a lack of realism due to the neglecting of environmental perturbations. Recent studies present a deep understanding of the diseases outbreak process by taking into account their random aspect. The contribution of this section is the first work that combines two different disturbances: white and Lévy noises. This original idea generalizes the existing works [136, 47, 173, 73, 127, 154, 172]. Our work based on the following new techniques:

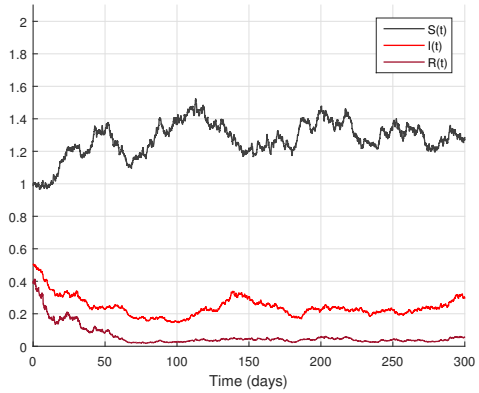
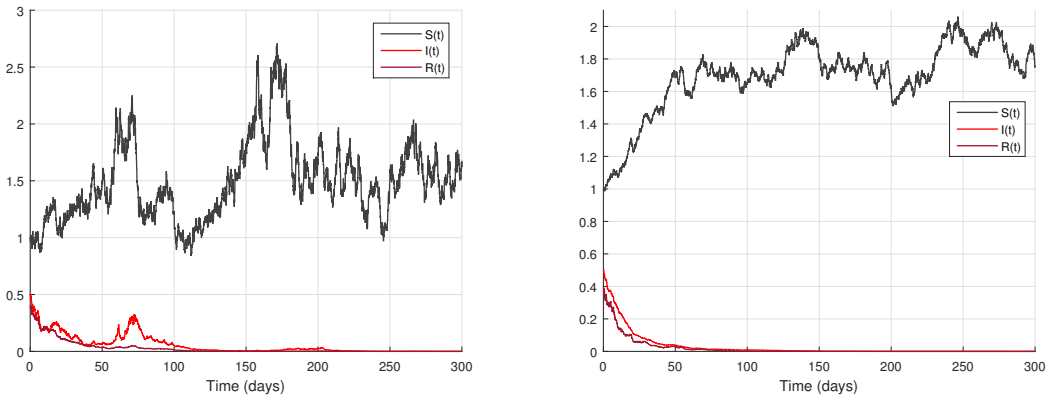


Figure III.2: The paths of $S(t)$, $I(t)$ and $R(t)$ for the stochastic model (III.4) with initial values $(S(0), I(0), R(0)) = (1, 0.5, 0.4)$.



(a) (b)

Figure III.3: The paths of $S(t)$, $I(t)$ and $R(t)$ for the stochastic model (III.4) with initial values $(S(0), I(0), R(0)) = (1, 0.5, 0.4)$.

1. The calculation of the solution averages of the auxiliary equation (III.5) instead of the classic method based on the explicit form of the stationary distribution to the model (III.5).
2. The investigation of the persistence condition with a new approach based on the stochastic comparison theorem.
3. The use of Feller property and mutually exclusive possibilities lemma for proving the ergodicity of the model (III.4).

Based on the above techniques, our analysis leads to the following main results:

1. In Theorem 1.10, we proved that the persistence in the mean of the disease occurs under the same condition of the existence of a unique ergodic stationary distribution.
2. In Theorem 1.11, we showed that the extinction of the disease in the stochastic system (III.4) occurs if one of the conditions (III.13) and (III.14) holds. It should be noted that these conditions are sufficient for the extinction of the epidemic.

Comparing our study with previous research, our theoretical analysis leads to establishing a new appropriate condition for the persistence and the existence of ergodic stationary distribution for the model (III.4). But, some interesting topics merit further investigation. On the one hand, one may propose an improved method to obtain the global threshold between the existence of the unique ergodic stationary distribution (persistence) and the extinction of a disease. On the other hand, it is necessary to check that the method utilized in this section can be also applied to analyze other stochastic realistic models such as SIRS, SEIR, and single-species population models [142, 19, 18, 140].

2 Threshold analysis of a stochastic SIR epidemic model with 3-dimensional Lévy process

2.1 Introduction

The intent of this section is to deal with the perturbed version of the SIR epidemic model (I.4) driven by a 3-dimensional Lévy. In the following, we give the threshold for the ergodicity and the extinction of the disease, but before stating the principal result, we first rewrite the stochastic model which takes the following form:

$$\begin{cases} \mathrm{d}S(t) = \big(A - \mu_1 S(t) - \beta S(t)I(t)\big)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \displaystyle\int_{\mathcal{U}} \eta_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}I(t) = \big(\beta S(t)I(t) - (\mu_2+\gamma)I(t)\big)\mathrm{d}t + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \displaystyle\int_{\mathcal{U}} \eta_2(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}R(t) = \big(\gamma I(t) - \mu_1 R(t)\big)\mathrm{d}t + \sigma_3 R(t)\mathrm{d}\mathcal{W}_3(t) + \displaystyle\int_{\mathcal{U}} \eta_3(u)R(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \end{cases} \tag{III.18}$$

In [172], the authors proved that the parameter

$$\mathcal{T}_0^s = \Big(\mu_2+\gamma\Big)^{-1}\left(\frac{\beta A}{\mu_1} - \frac{\sigma_2^2}{2} - \int_{\mathcal{U}}\Big(\eta_2(u) - \ln(1+\eta_2(u))\Big)\nu(\mathrm{d}u)\right)$$

is the threshold of the stochastic model (III.18) (under the same hypothetical framework of this present analysis, see the subsection 2.2.1). More specifically, if $\mathcal{T}_0^s < 1$, the epidemic eventually vanishes with probability one; while if $\mathcal{T}_0^s > 1$, the disease persists almost surely.

As far as we know, no previous research has investigated the ergodicity of the stochastic system (III.18). It is of interest to study the long term behavior of the stochastic epidemic model (III.18) which provides a link between mathematical study, actual diseases, and public health planning. Our contribution aims to develop a mathematical method to study the ergodicity of the model (III.18) as an important asymptotic property which means that the stochastic model has a unique stationary distribution that predicts the survival of the infected population in the future. Based on the works [162] and [163], we employe the Feller property, the mutually exclusive possibilities lemma, and the stochastic comparison theorem to prove that $\mathcal{T}_0^s$ is the threshold between the existence of the ergodic stationary distribution and the extinction. It should be noted that the approach used to prove the ergodicity is different from the Khasminskii method widely used in the literature (see for example [150, 161, 75]), and the method used to prove the extinction is different from that used in [172].

2.2 Main results

2.2.1 Well-posedness of the stochastic model (III.18)

For the purpose of well analyzing our model (III.18), it necessary that we make the following standard assumptions:

- $(\mathcal{H}_1)$ We assume that $\displaystyle\int_{\mathcal{U}} \eta_i^2(u)\nu(\mathrm{d}u) < \infty$ $(i = 1,2,3)$.
- $(\mathcal{H}_2)$ We assume that $1+\eta_i(u) > 0$, and $\displaystyle\int_{\mathcal{U}}\Big(\eta_i(u) - \ln(1+\eta_i(u))\Big)\nu(\mathrm{d}u) < \infty$ $(i = 1,2,3)$.
- $(\mathcal{H}_3)$ We suppose that exists a constant $\kappa > 0$, such that $\displaystyle\int_{\mathcal{U}}\Big(\ln(1+\eta_i(u))\Big)^2\nu(\mathrm{d}u) \le \kappa < \infty$.
- $(\mathcal{H}_4)$ We assume that for some $p > 1$, $\chi_2 = \mu_1 - \dfrac{(2p-1)}{2}\max\{\sigma_1^2,\sigma_2^2\} - \dfrac{1}{2p}\ell_p > 0$, where

$$\ell_p = \int_{\mathcal{U}}\Big(\big(1+\eta_1(u)\vee\eta_2(u)\big)^{2p} - 1 - \eta_1(u)\wedge\eta_2(u)\Big)\nu(\mathrm{d}u) < \infty.$$

Theorem 2.1. *Under $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, for any initial value $\big(S(0), I(0), R(0)\big) \in \mathbb{R}^3_+$, there exists a unique positive solution $\big(S(t), I(t), R(t)\big)$ of the system (III.18) on $t \geq 0$, and the solution will stay in $\mathbb{R}^3_+$ almost surely.*

Proof. By assumption $(\mathcal{H}_1)$ and the fact that the drift and the diffusion are locally Lipschitz, the model (III.18) admits a unique local solution $(S(t), I(t), R(t))$ on $[0, \tau_e)$ for any initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$, where τ_e is the explosion time. To show that this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $\epsilon_0 > 0$ be sufficiently large, such that $S(0)$, $I(0)$, $R(0)$ lie within the interval $\left[\frac{1}{\epsilon_0}, \epsilon_0\right]$. For each integer $\epsilon \geq \epsilon_0$, we define the following stopping time:

$$\tau_\epsilon = \inf\left\{t \in [0, \tau_e) : \ \ S(t) \notin \left(\frac{1}{\epsilon}, \epsilon\right) \ \text{ or } \ I(t) \notin \left(\frac{1}{\epsilon}, \epsilon\right) \ \text{ or } \ R(t) \notin \left(\frac{1}{\epsilon}, \epsilon\right)\right\},$$

where throughout this section, we set $\inf \emptyset = \infty$. Evidently, τ_ϵ is increasing as $\epsilon \to \infty$. Set $\tau_\infty = \lim\limits_{\epsilon \to \infty} \tau_\epsilon$, whence $\tau_\infty \leq \tau_e$. If we can prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and the solution $(S(t), I(t), R(t)) \in \mathbb{R}^3_+$ for all $t \geq 0$ almost surely. If this statement is false, then there exist a pair of positive constants $T > 0$ and $k \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} \geq k$. Hence, there is an integer $\epsilon \geq \epsilon_0$ such that

$$\mathbb{P}\{\tau_\epsilon \leq T\} \geq k \quad \text{for all} \quad \epsilon \geq \epsilon_0. \tag{III.19}$$

Define a C^2-function $\mathcal{V} : \mathbb{R}^3_+ \to [0, \infty)$ by

$$\mathcal{V}(S, I, R) = \left(S - m - m \ln \frac{S}{m}\right) + (I - 1 - \ln I) + (R - 1 - \ln R),$$

where $m > 0$ is a positive constant to be determined later. Then, by Itô's formula, we obtain that

$$\begin{aligned}
\mathrm{d}\mathcal{V}(S, I, R) = {} & \mathcal{L}\mathcal{V}(S, I, R)\mathrm{d}t + \left(1 - \frac{m}{S}\right)\sigma_1 S \mathrm{d}\mathcal{W}_1(t) + \left(1 - \frac{1}{I}\right)\sigma_2 I \mathrm{d}\mathcal{W}_2(t) + \left(1 - \frac{1}{R}\right)\sigma_3 R \mathrm{d}\mathcal{W}_3(t) \\
& + \int_{\mathcal{U}} \Big(m\eta_1(u)S(t^-) - m\ln(1 + \eta_1(u))\Big)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u) + \int_{\mathcal{U}} \Big(\eta_2(u)I(t^-) - \ln(1 + \eta_2(u))\Big)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u) \\
& + \int_{\mathcal{U}} \Big(\eta_3(u)R(t^-) - \ln(1 + \eta_3(u))\Big)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u),
\end{aligned}$$

where,

$$\begin{aligned}
\mathcal{L}\mathcal{V}(S, I, R) = {} & A - \mu_1 S - \frac{mA}{S} + m\beta I + m\mu_1 - (\mu_2 + \gamma)I - \beta S + (\mu_2 + \gamma) + \gamma I - \mu_1 R - \gamma\frac{I}{R} + \mu_1 + \frac{m\sigma_1^2}{2} \\
& + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \int_{\mathcal{U}} \Big(m\eta_1(u) - m\ln(1 + \eta_1(u))\Big)\nu(\mathrm{d}u) + \int_{\mathcal{U}} \Big(\eta_2(u) - \ln(1 + \eta_2(u))\Big)\nu(\mathrm{d}u) \\
& + \int_{\mathcal{U}} \Big(\eta_3(u) - \ln(1 + \eta_3(u))\Big)\nu(\mathrm{d}u).
\end{aligned}$$

Then

$$\mathcal{L}\mathcal{V}(S, I, R) \leq A - \mu_2 I + m\beta I + \mu_1 + m\mu_1 + \mu_2 + \gamma + \frac{m\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \mathcal{J}_1,$$

where

$$\begin{aligned}
\mathcal{J}_1 = {} & \int_{\mathcal{U}} \Big(m\eta_1(u) - m\ln(1 + \eta_1(u))\Big)\nu(\mathrm{d}u) + \int_{\mathcal{U}} \Big(\eta_2(u) - \ln(1 + \eta_2(u))\Big)\nu(\mathrm{d}u) \\
& + \int_{\mathcal{U}} \Big(\eta_3(u) - \ln(1 + \eta_3(u))\Big)\nu(\mathrm{d}u).
\end{aligned}$$

Given the fact that $x - \ln(1+x) \geq 0$ for all $x > 1$ and the hypothesis $(\mathcal{H}_2)$, we conclude that $\mathcal{J}_1$ is positive and finite. To simplify, we choose $m = \frac{\mu_2}{\beta}$, then we obtain

$$\mathcal{L}\mathcal{V}(S,I,R) \leq A - \mu_2 I + m\beta I + \mu_1 + m\mu_1 + \mu_2 + \gamma + \frac{m\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \mathcal{J}_1 \equiv \mathcal{J}_2.$$

Therefore,

$$\begin{aligned}\int_0^{\tau_\epsilon \wedge T} \mathrm{d}\mathcal{V}(S(t),I(t),R(t)) \leq \int_0^{\tau_\epsilon \wedge T} \mathcal{J}_2 \mathrm{d}t + \int_0^{\tau_\epsilon \wedge T} \int_{\mathcal{U}} \Big(\eta_1(u)S(t^-) - m\ln(1+\eta_1(u))\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) \\ + \int_0^{\tau_\epsilon \wedge T} \int_{\mathcal{U}} \Big(\eta_2(u)I(t^-) - \ln(1+\eta_2(u))\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) \\ + \int_0^{\tau_\epsilon \wedge T} \int_{\mathcal{U}} \Big(\eta_3(u)R(t^-) - \ln(1+\eta_3(u))\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).\end{aligned}$$

Taking expectation yields

$$\mathbb{E}V(S(\tau_\epsilon \wedge T), I(\tau_\epsilon \wedge T), R(\tau_\epsilon \wedge T)) \leq \mathcal{V}(S(0),I(0),R(0)) + \mathcal{J}_2 T.$$

Extending ϵ to ∞ leads to the contradiction $\infty > V(S(0),I(0),R(0)) + \mathcal{J}_2 T = \infty$. Thus, $\tau_\infty = \infty$ a.s. which completes the proof of the theorem. ∎

2.2.2 Threshold analysis of the model (III.18)

In this subsection, we always presume that the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ hold. The aim of the following theorem is to determine the threshold for the stochastic model (III.18).

Theorem 2.2. *The parameter $\mathcal{T}_0^s$ is the threshold of the stochastic model (III.18). That is to say that:*

1. *If $\mathcal{T}_0^s > 1$, then the stochastic system (III.18) admits a unique stationary distribution and it has the ergodic property for any initial value $(S(0),I(0),R(0)) \in \mathbb{R}_+^3$.*
2. *If $\mathcal{T}_0^s < 1$, then the epidemic dies out exponentially with probability one.*

Before proving this main theorem, we present some useful lemmas. Firstly, we consider the following subsystem

$$\begin{cases} \mathrm{d}\psi(t) = (A - \mu_1\psi(t))\mathrm{d}t + \sigma_1\psi(t)\mathrm{d}\mathcal{W}_1(t) + \displaystyle\int_{\mathcal{U}} \eta_1(u)\psi(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) \quad \forall t > 0, \\ \psi(0) = S(0) > 0. \end{cases} \tag{III.20}$$

Lemma 2.3. *[171] Let $(S(t),I(t),R(t))$ be the positive solution of the system (III.18) with any given initial condition $(S(0),I(0),R(0)) \in \mathbb{R}_+^3$. Let also $\psi(t) \in \mathbb{R}_+$ be the solution of the equation (III.20) with any given initial value $\psi(0) = S(0) \in \mathbb{R}_+$. Then*

1. $\lim\limits_{t\to\infty} \frac{1}{t}\psi(t) = 0$, $\lim\limits_{t\to\infty} \frac{1}{t}S(t) = 0$, *and* $\lim\limits_{t\to\infty} \frac{1}{t}I(t) = 0$ a.s.
2. $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t \psi(s)\mathrm{d}\mathcal{W}_1(s) = 0$, $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_1(s) = 0$, *and* $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t I(s)\mathrm{d}\mathcal{W}_1(s) = 0$ a.s.
3. $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathcal{U}} \eta_1(u)\psi(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) = 0$, $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathcal{U}} \eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) = 0$, *and* $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathcal{U}} \eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) = 0$ a.s.

Lemma 2.4. *Let $\psi(t)$ be the solution of the system (III.20) with an initial value $\psi(0) \in \mathbb{R}_+$. Then,*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \psi(s)\mathrm{d}s = \frac{A}{\mu_1} \quad \text{a.s.}$$

Proof. Integrating from 0 to t on both sides of (III.20) yields

$$\frac{\psi(t)-\psi(0)}{t} = A - \frac{\mu_1}{t}\int_0^t \psi(s)\mathrm{d}s + \frac{\sigma_1}{t}\int_0^t \psi(s)\mathrm{d}\mathcal{W}_1(s) + \frac{1}{t}\int_0^t\int_{\mathcal{U}} \eta_1(u)\psi(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

Clearly, we can derive that

$$\frac{1}{t}\int_0^t \psi(s)\mathrm{d}s = \frac{A}{\mu_1} - \frac{\psi(t)-\psi(0)}{\mu_1 t} + \frac{\sigma_1}{\mu_1 t}\int_0^t \psi(s)\mathrm{d}\mathcal{W}_1(s) + \frac{1}{\mu_1 t}\int_0^t\int_{\mathcal{U}} \eta_1(u)\psi(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

According to Lemma 2.3 and the large number theorem for martingales, we can easily verify that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \psi(s)\mathrm{d}s = \frac{A}{\mu_1} \quad \text{a.s.}$$

■

Lemma 2.5. *Let $(S(t), I(t), R(t))$ be the solution of (III.18) with initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$, then for any $p > 1$ such that $\chi_2 > 0$, it holds that*

1. $\mathbb{E}[S(t)+I(t)]^{2p} \le \big[S(0)+I(0)\big]^{2p} e^{\{-p\chi_2 t\}} + \dfrac{2\chi_1}{\chi_2}$;
2. $\limsup\limits_{t\to\infty} \dfrac{1}{t}\displaystyle\int_0^t \mathbb{E}\Big[\big((S(s)+I(s)\big)^{2p}\Big]\mathrm{d}s \le \dfrac{2\chi_1}{\chi_2}$, *where* $\chi_1 = \sup\limits_{x>0}\Big\{Ax^{2p-1} - \dfrac{\chi_2}{2}x^{2p}\Big\}$.

The demonstration of this result is similar to the proof of Lemma 1.4 and so it is omitted here.

Lemma 2.6. *[163] Let $h(t) > 0$, $k(t) \ge 0$ and $G(t)$ be functions on $[0,+\infty)$, $c \ge 0$ and $d > 0$ be constants, such that $\lim\limits_{t\to\infty} \frac{G(t)}{t} = 0$ and*

$$\ln h(t) \le ct + k(t) - d\int_0^t h(s)ds + G(t).$$

If $k(t)$ is a non-decreasing function, then

$$\limsup_{t\to\infty} \frac{1}{t}\Big(- k(t) + d\int_0^t h(s)ds\Big) \le c.$$

Proof of Theorem 2.2. We begin by proving the first point.

1. Similarly to the proof of Lemma 3.2. in [135], we briefly verify the Feller property of the SDE model (III.18). The main purpose of the next analysis is to prove that (II.22) is impossible for system (III.18). Applying Itô's formula gives

$$\begin{aligned}
\mathrm{d}\Big\{\ln I(t) - \frac{\beta}{\mu_1}\big(\psi(t)-S(t)\big)\Big\} = &\left(\beta S(t) - \left(\mu_2+\gamma+\frac{\sigma_2^2}{2}\right) - \int_{\mathcal{U}}\Big(\eta_2(u) - \ln(1+\eta_2(u))\Big)\nu(\mathrm{d}u)\right)\mathrm{d}t \\
&- \frac{\beta}{\mu_1}\Big(- \mu_1\big(\psi(t)-S(t)\big) + \beta S(t)I(t)\Big)\mathrm{d}t + \sigma_2\mathrm{d}W_2(t) \\
&- \frac{\beta}{\mu_1}(\psi(t)-S(t))\mathrm{d}\mathcal{W}_1(t) + \int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) \\
&- \frac{\beta}{\mu_1}\int_{\mathcal{U}}\eta_1(u)\big(\psi(t^-)-S(t^-)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{d}\left\{\ln I(t)-\frac{\beta}{\mu_1}\big(\psi(t)-S(t)\big)\right\}&=\left(\beta\psi(t)-\left(\mu_2+\gamma+\frac{\sigma_2^2}{2}\right)-\int_{\mathcal{U}}\Big(\eta_2(u)-\ln(1+\eta_2(u))\Big)\nu(\mathrm{d}u)\right)\mathrm{d}t\\&\quad-\frac{\beta^2 S(t)I(t)}{\mu_1}\mathrm{d}t+\sigma_2\mathrm{d}W_2(t)-\frac{\beta}{\mu_1}(\psi(t)-S(t))\mathrm{d}\mathcal{W}_1(t)\\&\quad+\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)-\frac{\beta}{\mu_1}\int_{\mathcal{U}}\eta_1(u)\big(\psi(t^-)-S(t^-)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).\end{aligned}\tag{III.21}$$

Integrating from 0 to t on both sides of (III.21) yields

$$\begin{aligned}\ln\frac{I(t)}{I(0)}-\frac{\beta}{\mu_1}(\psi(t)-S(t))+\frac{\beta}{\mu_1}(\psi(0)-S(0))&=\int_0^t\beta\psi(s)\mathrm{d}s-\left(\mu_2+\gamma+\frac{\sigma_2^2}{2}\right)t\\&\quad+t\int_{\mathcal{U}}\Big(\eta_2(u)-\ln(1+\eta_2(u))\Big)\nu(\mathrm{d}u)\\&\quad-\frac{\beta^2}{\mu_1}\int_0^t S(s)I(s)\mathrm{d}s+\sigma_2\mathcal{W}_2(t)\\&\quad-\frac{\beta}{\mu_1}\int_0^t(\psi(s)-S(s))\mathrm{d}\mathcal{W}_1(s)\\&\quad+\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\&\quad-\frac{\beta}{\mu_1}\int_0^t\int_{\mathcal{U}}\eta_1(u)\big(\psi(s^-)-S(s^-)\big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

Then, we have

$$\begin{aligned}\int_0^t\beta S(s)I(s)\mathrm{d}s&=\frac{\mu_1}{\beta}\int_0^t\beta\psi(s)\mathrm{d}s-\frac{\mu_1}{\beta}\left(\left(\mu_2+\gamma+\frac{\sigma_2^2}{2}\right)+\int_{\mathcal{U}}\Big(\eta_2(u)-\ln(1+\eta_2(u))\Big)\nu(\mathrm{d}u)\right)t\\&\quad+(\psi(t)-S(t))-(\psi(0)-S(0))-\frac{\mu_1}{\beta}\ln\frac{I(t)}{I(0)}+\frac{\mu_1\sigma_2}{\beta}\mathcal{W}_2(t)\\&\quad-\int_0^t(\psi(s)-S(s))\mathrm{d}\mathcal{W}_1(s)+\frac{\mu_1}{\beta}\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\&\quad-\int_0^t\int_{\mathcal{U}}\eta_1(u)\big(\psi(s^-)-S(s^-)\big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}\tag{III.22}$$

For all $t\geq 0$, we let

$$\begin{aligned}\mathcal{M}_1(t)&=\frac{\mu_1\sigma_2}{\beta}\mathcal{W}_2(t)-\int_0^t(\psi(s)-S(s))\mathrm{d}\mathcal{W}_1(s)+\frac{\mu_1}{\beta}\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\&\quad-\int_0^t\int_{\mathcal{U}}\eta_1(u)\big(\psi(s^-)-S(s^-)\big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

We know that $\mathcal{Z}(t)=\left(\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\right)_{t\geq 0}$ is a local martingale with quadratic variation

$$\langle\mathcal{Z}(t),\mathcal{Z}(t)\rangle_t=\left(\int_{\mathcal{U}}\big(\ln(1+\eta_2(u))\big)^2\nu(\mathrm{d}u)\right)t.$$

By using strong law of large numbers, Lemma 2.3 and the condition $(\mathcal{H}_3)$, we get $\lim_{t\to\infty}\frac{\mathcal{M}_1(t)}{t}=0$, a.s. From the system (III.18), we obtain

$$\begin{aligned}\mathrm{d}(S(t)+I(t))&=\big(A-\mu_1 S(t)-(\mu_2+\gamma)I(t)\big)\mathrm{d}t+\sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t)+\sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t)\\&\quad+\int_{\mathcal{U}}\Big(\eta_1(u)S(t^-)+\eta_2(u)I(t^-)\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}\tag{III.23}$$

Applying Itô's formula to the equality (III.23) gives that

$$\begin{aligned}
\mathrm{d}\ln\left(\frac{1}{S(t)+I(t)}\right) &= \left(\frac{-A}{S(t)+I(t)} + \frac{\mu_1 S(t) + (\mu_2+\gamma)I(t)}{S(t)+I(t)} + \frac{\sigma_1^2 S^2(t) + \sigma_2^2 I^2(t)}{2(S(t)+I(t))^2}\right)\mathrm{d}t \\
&\quad - \int_{\mathcal{U}} \left(\ln\frac{(1+\eta_1(u))S(t) + (1+\eta_2(u))I(t)}{S(t)+I(t)} - \frac{\eta_1(u)S(t)+\eta_2(u)I(t)}{S(t)+I(t)}\right)\nu(\mathrm{d}u) \\
&\quad - \frac{\sigma_1 S(t)}{S(t)+I(t)}\mathrm{d}\mathcal{W}_1(t) - \frac{\sigma_2 I(t)}{S(t)+I(t)}\mathrm{d}\mathcal{W}_2(t) \\
&\quad - \int_{\mathcal{U}} \left(\ln\frac{(1+\eta_1(u))S(t^-) + (1+\eta_2(u))I(t^-)}{S(t^-)+I(t^-)}\right)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}$$

Integrating, we get

$$\ln\left(\frac{1}{S(t)+I(t)}\right) = \ln\left(\frac{1}{S(0)+I(0)}\right) - A\int_0^t \frac{1}{S(s)+I(s)}\mathrm{d}s + \mathcal{M}_2(t) + \mathcal{M}_3(t),$$

where

$$\begin{aligned}
\mathcal{M}_2(t) &= \int_0^t \frac{\mu_1 S(s) + (\mu_2+\gamma)I(s)}{S(s)+I(s)}\mathrm{d}s + \int_0^t \frac{\sigma_1^2 S^2(s) + \sigma_2^2 I^2(s)}{2(S(s)+I(s))^2}\mathrm{d}s \\
&\quad - \int_0^t \int_{\mathcal{U}} \left(\ln\frac{(1+\eta_1(u))S(s) + (1+\eta_2(u))I(s)}{S(s)+I(s)} - \frac{\eta_1(u)S(s)+\eta_2(u)I(s)}{S(s)+I(s)}\right)\nu(\mathrm{d}u)\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}_3(t) &= -\int_0^t \frac{\sigma_1 S(s)}{S(s)+I(s)}\mathrm{d}\mathcal{W}_1(s) - \int_0^t \frac{\sigma_2 I(s)}{S(s)+I(s)}\mathrm{d}\mathcal{W}_1(s) \\
&\quad - \int_0^t \int_{\mathcal{U}} \left(\ln\frac{(1+\eta_1(u))S(s^-) + (1+\eta_2(u))I(s^-)}{S(s^-)+I(s^-)}\right)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).
\end{aligned}$$

By lemmas 2.6 and 2.3, we get

$$\limsup_{t\to\infty}\frac{1}{t}\left(\int_0^t \frac{A}{S(s)+I(s)}\mathrm{d}s - \mathcal{M}_2(t)\right) \le 0 \quad \text{a.s.}$$

Then

$$\limsup_{t\to\infty}\frac{1}{t}\ln\big(S(t)+I(t)\big) \le 0 \quad \text{a.s.}$$

Hence

$$\limsup_{t\to\infty}\frac{1}{t}\ln\frac{I(t)}{I(0)} \le \limsup_{t\to\infty}\frac{1}{t}\ln\frac{\big(S(t)+I(t)\big)}{I(0)} \le 0 \quad \text{a.s.}$$

Thus, it follows from (III.22) that

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\int_0^t \beta S(s)I(s)\mathrm{d}s &\ge \frac{\mu_1}{\beta}\left(\liminf_{t\to\infty}\frac{1}{t}\int_0^t \beta\psi(s)\mathrm{d}s - \left(\left(\mu_2+\gamma+\frac{\sigma_2^2}{2}\right) + \int_{\mathcal{U}}\Big(\eta_2(u) - \ln(1+\eta_2(u))\Big)\nu(\mathrm{d}u)\right)\right) \\
&= \frac{\mu_1}{\beta}\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t \beta\psi(s)\mathrm{d}s - \left(\left(\mu_2+\gamma+\frac{\sigma_2^2}{2}\right) + \int_{\mathcal{U}}\Big(\eta_2(u) - \ln(1+\eta_2(u))\Big)\nu(\mathrm{d}u)\right)\right) \\
&= \frac{\mu_1(\mu_2+\gamma)}{\beta}\Big(\mathcal{T}_0^s - 1\Big) > 0 \quad \text{a.s.}
\end{aligned}$$

By employing Lemma 2.5, the remainder of this demonstration is quite similar to that of Theorem 1.10.

2. Now, we will prove that if $\mathcal{T}_0^s < 1$, we have extinction of the disease. By applying Itô's formula and using Lemma 2.4, we get that

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\ln\frac{I(t)}{I(0)} &= \beta\limsup_{t\to\infty}\int_0^t S(s)\mathrm{d}s - \left((\mu_2+\gamma)+\frac{\sigma_2^2}{2}+\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)\right)\\ &\leq \beta\lim_{t\to\infty}\int_0^t \psi(s)\mathrm{d}s - \left((\mu_2+\gamma)+\frac{\sigma_2^2}{2}+\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)\right)\\ &= (\mu_2+\gamma)\left(\mathcal{T}_0^s-1\right) < 0 \quad \text{a.s.}\end{aligned}$$

This completes the proof.

■

2.3 Numerical example

In this subsection, we will validate our theoretical result with the help of numerical simulations taking parameters from the theoretical data mentioned in Table III.2. We numerically simulate the solution of the system (III.18) with initial value $(S(0), I(0), R(0)) = (0.4, 0.3, 0.1)$. For the purpose of showing the effects of the perturbations on the disease dynamics, we have realized the simulations many times.

Parameters	Description	Value
A	The recruitment rate	0.09
μ_1	The natural mortality rate	0.05
β	The transmission rate	0.06
γ	The recovered rate	0.01
μ_2	The general mortality	0.09

Table III.2: Some theoretical parameter values of the model (III.18).

We have chosen the stochastic fluctuations intensities $\sigma_1 = 0.02$, $\sigma_2 = 0.08$ and $\sigma_3 = 0.01$. Furthermore, we assume that $\eta_1(u) = 0.05$, $\eta_2(u) = 0.02$, $\eta_3(u) = 0.01$, $\mathcal{U} = (0,\infty)$ and $\nu(\mathcal{U}) = 1$. Then, $\mathcal{T}_0^s = 1.0460 > 1$. From Figure III.4, we show the existence of the unique stationary distributions for $S(t)$, $I(t)$ and $R(t)$ of model (III.18) at $t = 700$, where the smooth curves are the probability density functions of $S(t)$, $I(t)$ and $R(t)$, respectively. Now, we choose $A = 0.08$, then $\mathcal{T}_0^s = 0.9260 < 1$. That is, $I(t)$ will tend to zero exponentially with probability one (see Figure III.5).

2.4 Discussion

Eventually, this subsection presents new techniques to analyze the threshold of a stochastic SIR epidemic model with Lévy jumps. This short study is based on the following new techniques:

1. The calculation of the temporary average of a solution of (III.20) instead of the classic method based on the explicit form of the stationary distribution to (III.20).

2. The use of Feller property and mutually exclusive possibilities lemma for proving the ergodicity of the model (III.18).

According to the above techniques, our analysis leads to establish the threshold parameter for the existence of an ergodic stationary distribution and the extinction of the disease.

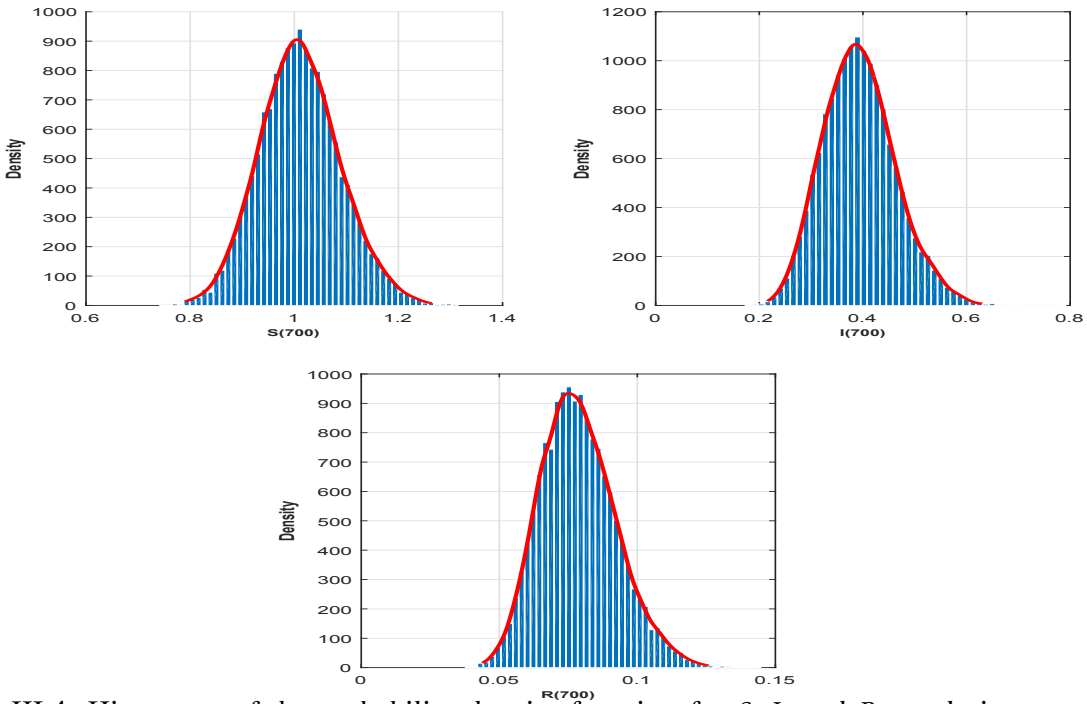


Figure III.4: Histogram of the probability density function for S, I, and R population at $t = 700$ for the stochastic model (III.18), the smoothed curves are the probability density functions of $S(t)$, $I(t)$ and $R(t)$, respectively.

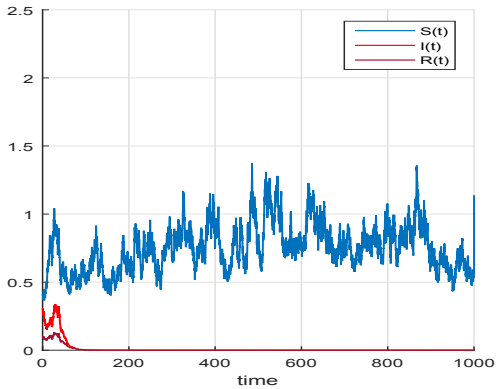


Figure III.5: The paths of $S(t)$, $I(t)$ and $R(t)$ for the stochastic (III.18) when $\mathcal{T}_0^s = 0.9260 < 1$.

3 The long-time behaviour of a stochastic SIR epidemic model with distributed delay and 3-multidimensional Lévy jumps

3.1 Introduction

In the SIR model, we often assume that recovered individuals can get continuous immunity. Many studies have paid close attention to the characteristics of the long-term epidemics immune response [40, 97, 148]. To confer the realistic aspect of the epidemic model and make it biologically reasonable, numerous scholars considered the SIR epidemic model with time delay because an individual may not be infectious until some time after becoming infected [67]. In the above-mentioned works, the time delay is assumed to be single-valued. The constant delay may be considered if the variation of the time is known exactly, which is not real for many biological reasons [92]. Considering the variable infectivity in the time interval yields a model with a distributed delay [10]. Therefore, it is more realistic to introduce a continuously distributed delay in the biological modeling [132, 126]. Analyzing the characteristics of the SIR model with a

distributed time delay still a rich subject that may deliver new comprehension of the epidemics propagation which motivates our work. According to the approach of Muroya et al. [101], we consider the delay kernel $\mathcal{G} : [0,\infty) \to [0,\infty)$ as a normalized L^1-function, i.e, $\int_0^\infty \mathcal{G}(s)ds = 1$. The average delay for the kernel $\mathcal{G}$ can be presented by the following quantity $\int_0^\infty s\mathcal{G}(s)ds < \infty$. Hence, the incidence rate at time τ can be presented as the following form: $\beta S(\tau)\int_{-\infty}^{\tau} \mathcal{G}(\tau - s)I(s)ds$, where β denotes the transmission rate, $S(t)$ and $I(t)$ represent the fractions of susceptible and infective individuals at time t. The SIR epidemic model with distributed delay can be expressed as follows [82]:

$$\begin{cases} dS(t) = \left(A - \mu_1 S(t) - \beta S(t)\displaystyle\int_{-\infty}^{t} \mathcal{G}(t-s)I(s)ds\right)dt, \\ dI(t) = \left(\beta S(t)\displaystyle\int_{-\infty}^{t} \mathcal{G}(t-s)I(s)ds - (\mu_2+\gamma)I(t)\right)dt, \\ dR(t) = \big(\gamma I(t) - \mu_3 R(t)\big)dt, \end{cases} \tag{III.24}$$

where $R(t)$ is the fraction number of recovered populations at time t. The remaining parameters appearing in this system are described as follows:

- A is the recruitment rate of susceptible individuals corresponding to births and immigration.
- μ_1, μ_3 are the natural death rates associated respectively to the susceptible and recovered populations, μ_2 is a general mortality rate including the effect of the disease fatality.
- γ is the rate of individuals leaving I to R (recovered rate).

The threshold number of the deterministic system (III.24) is $\mathcal{T}^\star = \dfrac{\beta A}{\mu_1(\mu_2+\gamma)}$ which determines the persistence ($\mathcal{T}^\star > 1$) or the extinction ($\mathcal{T}^\star < 1$) of the epidemic. Many studies showed that the deterministic epidemic model (III.24) is suitable to describe the transmission process of some known epidemics such as Rubella, Whooping cough, Measles and Smallpox. Due to many biological and mathematical considerations [82], in this section, we consider the delay kernel with Gamma distribution $\mathcal{G}(s) = \dfrac{s^n \pi_\star^{n+1} e^{-\pi_\star s}}{n!}$, $s \in (0,\infty)$, where the constant $\pi_\star > 0$ is the rate of exponential fading memory, which means the retrogradation of the past memories effect. As a special case, we treat our model with a low kernel function $\mathcal{G}$ with $n = 0$. By letting $\mathcal{D}(t) = \int_{-\infty}^{t} \pi_\star e^{-\pi_\star(t-s)} I(s)ds$ and using the linear chain approach, system (III.24) can be transformed into the following equivalent system:

$$\begin{cases} dS(t) = \big(A - \mu_1 S(t) - \beta S(t)\mathcal{D}(t)\big)dt, \\ dI(t) = \big(\beta S(t)\mathcal{D}(t) - (\mu_2+\gamma)I(t)\big)dt, \\ dR(t) = \big(\gamma I(t) - \mu_3 R(t)\big)dt, \\ d\mathcal{D}(t) = \pi_\star\big(I(t) - \mathcal{D}(t)\big)dt. \end{cases} \tag{III.25}$$

Although the use of deterministic models can explain and simulate some phenomena in real life, such models do not actually consider the effect of the natural stochasticity, and we plainly wish to learn how randomness affects our epidemic models [147, 152, 149, 123, 9, 110, 115]. Generally, one of the ordinary extensions from the deterministic SIR model to the stochastic version is to incorporate environmental white noises, which appear from an almost continuous series of small variations on the model parameters [24, 160, 158, 124, 125, 111, 122, 96]. Therefore, the stochastic delayed SIR epidemic can be an accurate tool to predict the long-run dynamics of infectious epidemics [79, 85, 136, 47, 173, 73, 127, 53, 42]. In [82], the authors inserted the stochastic perturbation in the model (III.24) by assuming that the white noise is directly proportional to the variable S and they

obtained the following stochastic system:

$$\begin{cases} \mathrm{d}S(t) = \big(A - \mu_1 S(t) - \beta S(t)\mathcal{D}(t)\big)\mathrm{d}t + \sigma S(t)\mathrm{d}\mathcal{W}(t), \\ \mathrm{d}I(t) = \big(\beta S(t)\mathcal{D}(t) - (\mu_2 + \gamma)I(t)\big)\mathrm{d}t, \\ \mathrm{d}R(t) = \big(\gamma I(t) - \mu_3 R(t)\big)\mathrm{d}t, \\ \mathrm{d}\mathcal{D}(t) = \pi_\star\big(I(t) - \mathcal{D}(t)\big)\mathrm{d}t. \end{cases} \tag{III.26}$$

where $\mathcal{W}(t)$ is a standard Brownian motion with associated intensity $\sigma > 0$. Specifically, they proved the existence and uniqueness of an ergodic stationary distribution to the model (III.26). Then, they established sufficient conditions for the extinction of a disease that spreads according to this model. On the basis of these findings, one question was catches our attention. It is possible to develop and generalize the stochastic model proposed in [82]? So, the objective of this work is to expound on this problem and provide a suitable analytical context. Specifically, we aim to describe the strong fluctuations by considering a general version of the dynamical model (III.26). It is clear that the population systems may suffer certain sudden environmental catastrophes, such as earthquakes, floods, droughts, etc [156, 157, 159]. For example, the recent massive explosion in the port city of Beirut. The impact of this unexpected disaster has been extremely devastating, especially when it has occurred simultaneously with the COVID-19 pandemic. This led to a sudden worsening of the health situation and a jump increase in the number of deaths. Mathematically, we use the Lévy process to describe the phenomena that cause a big jump to occur occasionally [37, 130, 131, 154, 172]. By considering this type of random perturbations, the model (III.26) becomes the following system of stochastic differential equations with Lévy jumps (SDE-Js for short):

$$\begin{cases} \mathrm{d}S(t) = \big(A - \mu_1 S(t) - \beta S(t)\mathcal{D}(t)\big)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \displaystyle\int_{\mathcal{U}} \eta_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u), \\ \mathrm{d}I(t) = \big(\beta S(t)\mathcal{D}(t) - (\mu_2 + \gamma)I(t)\big)\mathrm{d}t + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \displaystyle\int_{\mathcal{U}} \eta_2(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u), \\ \mathrm{d}R(t) = \big(\gamma I(t) - \mu_3 R(t)\big)\mathrm{d}t + \sigma_3 R(t)\mathrm{d}\mathcal{W}_3(t) + \displaystyle\int_{\mathcal{U}} \eta_3(u)R(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u), \\ \mathrm{d}\mathcal{D}(t) = \pi_\star\big(I(t) - \mathcal{D}(t)\big)\mathrm{d}t + \sigma_4 \mathcal{D}(t)\mathrm{d}\mathcal{W}_4(t) + \displaystyle\int_{\mathcal{U}} \eta_4(u)\mathcal{D}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u), \end{cases} \tag{III.27}$$

where $S(t^-)$, $I(t^-)$, $R(t^-)$ and $\mathcal{D}(t^-)$ are the left limits of $S(t)$, $I(t)$, $R(t)$ and $\mathcal{D}(t)$, respectively. $\mathcal{W}_i(t)$ $(i = 1, 2, 3, 4)$ are independent Brownian motions and $\sigma_i > 0$ $(i = 1, 2, 3, 4)$ are their intensities. $\mathcal{N}$ is a Poisson counting measure with compensating martingale $\widetilde{\mathcal{N}}$ and characteristic measure ν on a measurable subset $\mathcal{U}$ of $(0, \infty)$ satisfying $\nu(\mathcal{U}) < \infty$. $\mathcal{W}_i(t)$ $(i = 1, 2, 3, 4)$ are independent of $\mathcal{N}$. We assumed that ν is a Lévy measure such that $\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u) = \mathcal{N}(\mathrm{d}t, \mathrm{d}u) - \nu(\mathrm{d}u)\mathrm{d}t$ and we suppose that the function $\eta_i : \mathcal{U} \to \mathbb{R}$ is bounded and continuous. Since the compartment $R(t)$ does not appear in the equations of $S(t)$, $I(t)$ and $\mathcal{D}(t)$, it is sufficient to analyze the dynamic behavior of the following SDE-J model:

$$\begin{cases} \mathrm{d}S(t) = \big(A - \mu_1 S(t) - \beta S(t)\mathcal{D}(t)\big)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \displaystyle\int_{\mathcal{U}} \eta_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u), \\ \mathrm{d}I(t) = \big(\beta S(t)\mathcal{D}(t) - (\mu_2 + \gamma)I(t)\big)\mathrm{d}t + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \displaystyle\int_{\mathcal{U}} \eta_2(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u), \\ \mathrm{d}\mathcal{D}(t) = \pi_\star\big(I(t) - \mathcal{D}(t)\big)\mathrm{d}t + \sigma_4 \mathcal{D}(t)\mathrm{d}\mathcal{W}_4(t) + \displaystyle\int_{\mathcal{U}} \eta_4(u)\mathcal{D}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u). \end{cases} \tag{III.28}$$

In this study, we develop a new analysis to deal with stochastic models with jumps in epidemiology. Our main goal is to investigate sufficient conditions of the stochastic extinction and persistence in the mean. These two important properties are sufficient to predict and analyze the dynamics of a given epidemic. We apply a new approach to estimate the values of the averages $t^{-1}\displaystyle\int_0^t \psi(s)\mathrm{d}s$ and $t^{-1}\displaystyle\int_0^t \psi^2(s)\mathrm{d}s$, where $\psi(t)$ is the positive solution of the following subsystem:

$$\begin{cases} \mathrm{d}\psi(t) = \big(A - \mu_1 \psi(t)\big)\mathrm{d}t + \sigma_1 \psi(t)\mathrm{d}\mathcal{W}_1(t) + \displaystyle\int_{\mathcal{U}} \eta_1(u)\psi(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u), \quad \forall t > 0, \\ \psi(0) = S(0) > 0. \end{cases} \tag{III.29}$$

Our approach allows us to close the gap left by using the classical method presented for example in [163]. Furthermore, we give an optimal sufficient condition for the stochastic extinction. For the purpose of well understanding the dynamics of the delayed model (III.28), we give a sufficient condition of the disease persistence. The analysis in this section seems to be promising to investigate other related stochastic delayed models with Lévy noises in epidemiology and even in biology.

3.2 Main results

In mathematical epidemiology, we are generally interested in two things, the first is to know when the epidemic will die out, and the second is when it will continue and persist. In this section, we will try our best to find sufficient conditions for these two interesting asymptotic proprieties in terms of model parameters and intensities of noises. For the sake of notational simplicity, we define

- $\bar{\sigma} \triangleq \max\left\{\sigma_1^2, \sigma_2^2, \sigma_4^2\right\}$ and $\vartheta \triangleq \min\left\{\mu_1, \mu_2 + \gamma - \pi_\star, \pi_\star\right\}$.
- $\bar{\eta}(u) \triangleq \max\{\eta_1(u), \eta_2(u), \eta_4(u)\}$ and $\underline{\eta}(u) \triangleq \min\{\eta_1(u), \eta_2(u), \eta_4(u)\}$.
- $\hat{\zeta}_p(u) \triangleq \left[1 + \bar{\eta}(u)\right]^p - 1 - p\bar{\eta}(u)$ and $\check{\zeta}_p(u) \triangleq \left[1 + \underline{\eta}(u)\right]^p - 1 - p\underline{\eta}(u)$.
- $\xi(u) \triangleq \max\left\{\hat{\zeta}_p(u), \check{\zeta}_p(u)\right\}$ and $\ell_p \triangleq \int_{\mathcal{U}} \xi(u)\nu(\mathrm{d}u)$.

To properly study the long-term of the perturbed model (III.28), we have the following hypotheses on the jump-diffusion coefficients:

- ($\mathcal{H}_1$): We assume that the jump coefficients $\lambda_i(u)$ in (I.2) satisfy $\int_{\mathcal{U}} \lambda_i^2(u)\nu(\mathrm{d}u) < \infty$ $(i = 1, 2, 4)$.
- ($\mathcal{H}_2$): For all $i = 1, 2, 4$, we assume that $1 + \lambda_i(u) > 0$ and $\int_{\mathcal{U}} \Big(\lambda_i(u) - \ln\left(1 + \lambda_i(u)\right)\Big)\nu(\mathrm{d}u) < \infty$.
- ($\mathcal{H}_3$): For $i = 1, 2, 4$, we assume that $\int_{\mathcal{U}} \Big[\ln(1 + \eta_i(u))\Big]^2 \nu(\mathrm{d}u) < \infty$.
- ($\mathcal{H}_4$): For $i = 1, 2, 4$, we assume that $\int_{\mathcal{U}} \Big[\left(1 + \bar{\eta}(u)\right)^2 - 1\Big]^2 \nu(\mathrm{d}u) < \infty$.
- ($\mathcal{H}_5$): We suppose that there exists some real number $p > 2$ such that

$$\chi_{1,p} = \vartheta - \frac{(p-1)}{2}\bar{\sigma} - \frac{1}{p}\ell_p > 0.$$

For the convenience of discussion in the stochastic model (III.28), we introduce two lemmas which will be used in our analysis.

Lemma 3.1 ([63]). *We assume that the conditions ($\mathcal{H}_4$) and ($\mathcal{H}_5$) hold. Let $(S(t), I(t), \mathcal{D}(t))$ be the positive solution of the system (III.28) with any given initial condition $(S(0), I(0), \mathcal{D}(0)) \in \mathbb{R}^3_+$. Let also $\psi(t) \in \mathbb{R}_+$ be the solution of the equation (III.29) with any given initial value $\psi(0) = S(0) \in \mathbb{R}_+$. Then*

- $\lim_{t\to\infty} t^{-1}\,\psi(t) = 0$, $\lim_{t\to\infty} t^{-1}\,\psi^2(t) = 0$, $\lim_{t\to\infty} t^{-1}\,S(t) = 0$, $\lim_{t\to\infty} t^{-1}\,I(t) = 0$, and $\lim_{t\to\infty} t^{-1}\,\mathcal{D}(t) = 0$ a.s.
- $\lim_{t\to\infty} t^{-1} \int_0^t \psi(s)\mathrm{d}\mathcal{W}_1(s) = 0$, $\lim_{t\to\infty} t^{-1} \int_0^t \psi^2(s)\mathrm{d}\mathcal{W}_1(s) = 0$, $\lim_{t\to\infty} t^{-1} \int_0^t S(s)\mathrm{d}\mathcal{W}_1(s) = 0$, $\lim_{t\to\infty} t^{-1} \int_0^t I(s)\mathrm{d}\mathcal{W}_2(s) = 0$, and $\lim_{t\to\infty} t^{-1} \int_0^t \mathcal{D}(s)\mathrm{d}\mathcal{W}_4(s) = 0$ a.s.

- $\lim_{t\to\infty} t^{-1}\int_0^t\int_{\mathcal{U}} \eta_1(u)\psi(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)=0$, $\lim_{t\to\infty} t^{-1}\int_0^t\int_{\mathcal{U}} \Big((1+\eta_1(u))^2-1\Big)\psi^2(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)=0$,
  $\lim_{t\to\infty} t^{-1}\int_0^t\int_{\mathcal{U}} \eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)=0$, $\lim_{t\to\infty} t^{-1}\int_0^t\int_{\mathcal{U}} \eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)=0$,
  and $\lim_{t\to\infty} t^{-1}\int_0^t\int_{\mathcal{U}} \eta_4(u)\mathcal{D}(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)=0$ a.s.

Remark 3.2. By using the same approach adopted in Lemma 2.5 of [63], we can easily prove the last result. Note that the hypothesis $(\mathcal{H}_5)$ is an ameliorated version of it corresponding hypothesis frequently used in many previous works, for example, [27, 26, 36]. Therefore, the adoption of $\chi_{1,p}$ in this section raises the optimality of our calculus and results.

Lemma 3.3. *Assume that the conditions $(\mathcal{H}_4)$ and $(\mathcal{H}_5)$ hold. Let $\psi(t)$ be the solution of (III.29) with an initial value $\psi(0)\in\mathbb{R}_+$. Then, $\chi_2=2\mu_1-\sigma_1^2-\int_{\mathcal{U}}\eta_1^2(u)\nu(\mathrm{d}u)>0$, and*

- $\lim_{t\to\infty} t^{-1}\int_0^t \psi(s)\mathrm{d}s=\dfrac{A}{\mu_1}$ a.s.
- $\lim_{t\to\infty} t^{-1}\int_0^t \psi^2(s)\mathrm{d}s=\dfrac{2A^2}{\mu_1\chi_2}$ a.s.

Proof. Integrating from 0 to t on both sides of (III.29) yields

$$\frac{\psi(t)-\psi(0)}{t}=A-\frac{\mu_1}{t}\int_0^t\psi(s)\mathrm{d}s+\frac{\sigma_1}{t}\int_0^t\psi(s)\mathrm{d}\mathcal{W}_1(s)+t^{-1}\int_0^t\int_{\mathcal{U}}\eta_1(u)\psi(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

Clearly, we can derive that

$$t^{-1}\int_0^t\psi(s)\mathrm{d}s=\frac{A}{\mu_1}-\frac{\psi(t)-\psi(0)}{\mu_1 t}+\frac{\sigma_1}{\mu_1 t}\int_0^t\psi(s)\mathrm{d}\mathcal{W}_1(s)+\frac{1}{\mu_1 t}\int_0^t\int_{\mathcal{U}}\eta_1(u)\psi(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

By Lemma 3.1, we get

$$\lim_{t\to\infty} t^{-1}\int_0^t\psi(s)\mathrm{d}s=\frac{A}{\mu_1}\ \text{a.s.}$$

Now, applying the generalized Itô's formula to model (III.29) leads to

$$\begin{aligned}\mathrm{d}\psi^2(t)=&\Big(2\psi(t)\Big(A-\mu_1\psi(t)\Big)+\sigma_1^2\psi^2(t)+\int_{\mathcal{U}}\psi^2(t)\Big((1+\eta_1(u))^2-1-2\eta_1(u)\Big)\nu(\mathrm{d}u)\Big)\mathrm{d}t\\&+2\sigma_1\psi^2(t)\mathrm{d}\mathcal{W}_1(t)+\int_{\mathcal{U}}\psi^2(t^-)\Big((1+\eta_1(u))^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).\end{aligned}$$

Integrating both sides of the last expression from 0 to t and then dividing by t, yields

$$\begin{aligned}\frac{\psi^2(t)-\psi^2(0)}{t}=&2A\times\frac{1}{t}\int_0^t\psi(s)\mathrm{d}s-\overbrace{\Big(2\mu_1-\sigma_1^2-\int_{\mathcal{U}}\eta_1^2(u)\nu(\mathrm{d}u)\Big)}^{\equiv\chi_2}\times\frac{1}{t}\int_0^t\psi^2(s)\mathrm{d}s\\&+2\sigma_1\times\frac{1}{t}\int_0^t\psi^2(s)\mathrm{d}\mathcal{W}_1(s)+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\psi^2(s^-)\Big((1+\eta_1(u))^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\chi_2}{t}\int_0^t\psi^2(s)\mathrm{d}s=&\frac{2A}{t}\int_0^t\psi(s)\mathrm{d}s+\frac{\big(\psi^2(0)-\psi^2(t)\big)}{t}+\frac{2\sigma_1}{t}\int_0^t\psi^2(s)\mathrm{d}\mathcal{W}_1(s)\\&+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\psi^2(s^-)\Big((1+\eta_1(u))^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}\tag{III.30}$$

Clearly, $\chi_2 \neq 0$, because if it is not the case, we will obtain by letting t go to infinity in (III.30) $\frac{2A^2}{\mu} = 0$, which is obviously impossible. So, and using Lemma 3.1, we can easily verify that

$$\lim_{t\to\infty} t^{-1} \int_0^t \psi^2(s)\mathrm{d}s = \frac{2A^2}{\mu_1\chi_2} \quad \text{a.s.}$$

and since $t^{-1}\int_0^t \psi^2(s)\mathrm{d}s > 0$ for all $t > 0$, we can conclude also that $\frac{2A^2}{\mu_1\chi_2} > 0$ and then $\chi_2 > 0$. Hence the proof is completed. ∎

We are now in the position to state and prove the main results of this section. In the following, we always presume that the hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_5)$ hold.

3.2.1 Stochastic extinction of the epidemic

In this subsection, we give a sufficient condition for the stochastic extinction of the disease in the system (III.28). For brevity and simplicity in writing the next result, we adopt the following notations:

- $\Upsilon \triangleq \min\{\mu_2+\gamma, \pi_\star\}\left(\sqrt{\mathcal{T}^\star}-1\right)\mathbb{1}_{\{\mathcal{T}^\star\leq 1\}} + \max\{\mu_2+\gamma, \pi_\star\}\left(\sqrt{\mathcal{T}^\star}-1\right)\mathbb{1}_{\{\mathcal{T}^\star>1\}}$.
- $\eta \triangleq \sigma_1^2 + \int_{\mathcal{U}} \eta_1^2(u)\nu(\mathrm{d}u)$ and $\Sigma \triangleq \left(2\left(\sigma_2^{-2}+\sigma_4^{-2}\right)\right)^{-1}$.
- $\bar{\aleph}(u) \triangleq \Big(\ln\left(1+\eta_2(u)\wedge\eta_4(u)\right) - \eta_2(u)\wedge\eta_4(u)\Big) \times \mathbb{1}_{\{\eta_2(u)\wedge\eta_4(u)>0\}}$.
- $\underline{\aleph}(u) \triangleq \Big(\ln\left(1+\eta_2(u)\vee\eta_4(u)\right) - \eta_2(u)\vee\eta_4(u)\Big) \times \mathbb{1}_{\{\eta_2(u)\vee\eta_4(u)\leq 0\}}$.
- $\aleph(u) \triangleq \bar{\aleph}(u) + \underline{\aleph}(u)$ and $\Pi \triangleq \int_{\mathcal{U}} \aleph(u)\nu(\mathrm{d}u)$.
- $\Theta \triangleq \Upsilon + \Pi - \Sigma + \pi_\star\left(\frac{\mathcal{T}^\star\eta}{\chi_2}\right)^{\frac{1}{2}}$.
- For any vector $x \in \mathbb{R}^n$, we denote its transpose by x^T.

Theorem 3.4. *Let us denote by $(S(t), I(t), \mathcal{D}(t))$ the solution of the stochastic system (III.28) that starts from a given initial data $(S(0), I(0), \mathcal{D}(0)) \in \mathbb{R}^3_+$. Under the hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_5)$, we have*

$$\limsup_{t\to\infty} t^{-1}\ln\left(\frac{1}{\mu_2+\gamma}I(t) + \frac{\sqrt{\mathcal{T}^\star}}{\pi_\star}\mathcal{D}(t)\right) \leq \Theta \quad \text{a.s.}$$

Notably, if $\Theta < 0$, then the epidemic will go to zero exponentially with probability one. Consequently,

$$\lim_{t\to\infty} I(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} \mathcal{D}(t) = 0 \ \text{ a.s.}$$

Proof. Our proof starts with the use of Theorem 1.4 in [11] to establish that there is a left eigenvector of the following matrix

$$\mathfrak{M}_0 = \begin{pmatrix} 0 & \frac{\beta A}{\mu_1(\mu_2+\gamma)} \\ 1 & 0 \end{pmatrix}$$

corresponding to $\sqrt{\mathcal{T}^\star}$. This vector will be denoted by $(e_1, e_2) = (1, \sqrt{\mathcal{T}^\star})$. Then, $\sqrt{\mathcal{T}^\star}(e_1, e_2) = (e_1, e_2)\mathfrak{M}_0$. On the other hand, we define a $\mathfrak{C}^2$-function $\mathcal{M} : \mathbb{R}^2_+ \to \mathbb{R}_+$ by

$$\mathcal{M}(I(t), \mathcal{D}(t)) = \omega_1 I(t) + \omega_2 \mathcal{D}(t),$$

where $\omega_1 = \frac{e_1}{\mu_2+\gamma}$ and $\omega_2 = \frac{e_2}{\pi_\star}$. By applying the generalized Itô's formula with Lévy jumps we obtain

$$\begin{aligned}\mathrm{d}\ln\mathcal{M}(I(t),\mathcal{D}(t)) &= \mathcal{L}\ln\mathcal{M}(I(t),\mathcal{D}(t))\mathrm{d}t + \frac{1}{\omega_1 I(t)+\omega_2\mathcal{D}(t)}\Big\{\omega_1\sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \omega_2\sigma_4\mathcal{D}(t)\mathrm{d}\mathcal{W}_3(t)\Big\}\\ &\quad + \int_{\mathcal{U}}\ln\left(1+\frac{\omega_1\eta_2(u)I(t^-)+\omega_2\eta_4(u)\mathcal{D}(t^-)}{\omega_1 I(t^-)+\omega_2\mathcal{D}(t^-)}\right)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}\ln\mathcal{M}(I(t),\mathcal{D}(t)) &= \frac{1}{\omega_1 I(t)+\omega_2\mathcal{D}(t)}\Big\{\omega_1\Big(\beta S(t)\mathcal{D}(t)-(\mu_2+\gamma)I(t)\Big)+\omega_2\pi_\star\Big(I(t)-\mathcal{D}(t)\Big)\Big\}\\ &\quad - \frac{1}{2(\omega_1 I(t)+\omega_2\mathcal{D}(t))^2}\Big\{\omega_1^2\sigma_2^2 I^2(t)+\omega_2^2\sigma_4^2\mathcal{D}^2(t)\Big\}\\ &\quad + \int_{\mathcal{U}}\left[\ln\left(1+\frac{\omega_1\eta_2(u)I(t)+\omega_2\eta_4(u)\mathcal{D}(t)}{\omega_1 I(t)+\omega_2\mathcal{D}(t)}\right)-\frac{\omega_1\eta_2(u)I(t)+\omega_2\eta_4(u)\mathcal{D}(t)}{\omega_1 I(t)+\omega_2\mathcal{D}(t)}\right]\nu(\mathrm{d}u).\end{aligned}$$

Moreover, it is easy to show the following inequality

$$\left(\frac{1}{\sigma_2^2}+\frac{1}{\sigma_4^2}\right)\times\Big(\omega_1^2\sigma_2^2 I^2(t)+\omega_2^2\sigma_4^2\mathcal{D}^2(t)\Big)\geq\left(\frac{1}{\sigma_2}\omega_1\sigma_2 I(t)+\frac{1}{\sigma_4}\omega_2\sigma_4\mathcal{D}(t)\right)^2.$$

In order to find an optimal and good majorization, we adopt the fact that

$$\int_{\mathcal{U}}\left[\ln\left(1+\frac{\omega_1\eta_2(u)I(t)+\omega_2\eta_4(u)\mathcal{D}(t)}{\omega_1 I(t)+\omega_2\mathcal{D}(t)}\right)-\frac{\omega_1\eta_2(u)I(t)+\omega_2\eta_4(u)\mathcal{D}(t)}{\omega_1 I(t)+\omega_2\mathcal{D}(t)}\right]\nu(\mathrm{d}u)\leq\Pi. \tag{III.31}$$

By using the last two results, we get

$$\begin{aligned}\mathcal{L}\ln\mathcal{M}(I(t),\mathcal{D}(t)) &\leq \frac{\omega_1\beta\mathcal{D}(t)}{\omega_1 I(t)+\omega_2\mathcal{D}(t)}\left(S(t)-\frac{A}{\mu_1}\right)+\Pi-\Sigma\\ &\quad + \frac{1}{\omega_1 I(t)+\omega_2\mathcal{D}(t)}\left\{\omega_1\left(\frac{\beta A}{\mu_1}\mathcal{D}(t)-(\mu_2+\gamma)I(t)\right)+\omega_2\pi_\star\Big(I(t)-\mathcal{D}(t)\Big)\right\}.\end{aligned}$$

By the stochastic comparison theorem, we have

$$\begin{aligned}\mathcal{L}\ln\mathcal{M}(I(t),\mathcal{D}(t)) &\leq \frac{\omega_1\beta\mathcal{D}(t)}{\omega_1 I(t)+\omega_2\mathcal{D}(t)}\left(\psi(t)-\frac{A}{\mu_1}\right)+\Pi-\Sigma\\ &\quad + \frac{1}{\omega_1 I(t)+\omega_2\mathcal{D}(t)}\left\{\frac{e_1}{\mu_2+\gamma}\left(\frac{\beta A}{\mu_1}\mathcal{D}(t)-(\mu_2+\gamma)I(t)\right)+\frac{e_2}{\pi_\star}\Big(\pi_\star I(t)-\pi_\star\mathcal{D}(t)\Big)\right\}.\end{aligned}$$

Then, we obtain that

$$\begin{aligned}&\mathcal{L}\ln\mathcal{M}(I(t),\mathcal{D}(t))\\ &\leq \frac{\omega_1\beta}{\omega_2}\left|\psi(t)-\frac{A}{\mu_1}\right|+\Pi-\Sigma+\frac{1}{\omega_1 I(t)+\omega_2\mathcal{D}(t)}(e_1,e_2)\Big(\mathfrak{M}_0(I(t),\mathcal{D}(t))^T-(I(t),\mathcal{D}(t))^T\Big)\\ &= \frac{\omega_1\beta}{\omega_2}\left|\psi(t)-\frac{A}{\mu_1}\right|+\Pi-\Sigma+\frac{1}{\omega_1 I(t)+\omega_2\mathcal{D}(t)}\big(\sqrt{\mathcal{T}^\star}-1\big)\big(e_1 I(t)+e_2 D(t)\big)\\ &= \frac{\omega_1\beta}{\omega_2}\left|\psi(t)-\frac{A}{\mu_1}\right|+\Pi-\Sigma+\frac{1}{\omega_1 I(t)+\omega_2\mathcal{D}(t)}\big(\sqrt{\mathcal{T}^\star}-1\big)\big(\omega_1(\mu_2+\gamma)I(t)+\pi_\star\omega_2\mathcal{D}(t)\big)\\ &\leq \Upsilon+\Pi-\Sigma+\frac{\omega_1\beta}{\omega_2}\left|\psi(t)-\frac{A}{\mu_1}\right|.\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}\mathrm{d}\ln\mathcal{M}(I(t),\mathcal{D}(t)) \leq{}& \big(\Upsilon+\Pi-\Sigma\big)\mathrm{d}t+\frac{\omega_1\beta}{\omega_2}\left|\psi(t)-\frac{A}{\mu_1}\right|\mathrm{d}t\\ &+\frac{1}{\omega_1 I(t)+\omega_2 D(t)}\Big\{\omega_1\sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t)+\omega_2\sigma_4 D(t)\mathrm{d}\mathcal{W}_4(t)\Big\}\\ &+\int_{\mathcal{U}}\ln\big(1+\eta(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),\end{aligned}$$

where $\eta(u)=\max\{\eta_2(u),\eta_4(u)\}$. Now, by integrating both sides of the last inequality and dividing by t, we find immediately that

$$\begin{aligned}t^{-1}\ \ln\mathcal{M}(I(t),\mathcal{D}(t)) \leq{}& t^{-1}\ \ln\mathcal{M}(I(0),\mathcal{D}(0))+\Upsilon+\Pi-\Sigma\\ &+\frac{\omega_1\beta}{\omega_2 t}\int_0^t\left|\psi(s)-\frac{A}{\mu_1}\right|\mathrm{d}s+t^{-1}\ \mathcal{J}_3(t)+t^{-1}\ \mathcal{J}_4(t),\end{aligned}\tag{III.32}$$

where

$$\begin{aligned}\mathcal{J}_3(t)&=\int_0^t\frac{\sigma_2\omega_1 I(s)}{\omega_1 I(s)+\omega_2 D(s)}\mathrm{d}\mathcal{W}_2(s)+\int_0^t\frac{\sigma_4\omega_2 D(s)}{\omega_1 I(s)+\omega_2 D(s)}\mathrm{d}\mathcal{W}_4(s),\\ \mathcal{J}_4(t)&=\int_0^t\int_{\mathcal{U}}\ln\big(1+\eta(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

It is easy to check that $\mathcal{J}_3(t)$ is a local martingale with finite quadratic variation, and by the hypothesis $(\mathcal{H}_3)$ we can affirm that $\mathcal{J}_4(t)$ is also a local martingale with finite quadratic variation. By the strong law of large numbers for local martingales [93], we get

$$\lim_{t\to\infty}\ t^{-1}\ \mathcal{J}_3(t)=0\quad\text{a.s}\quad\text{and}\quad\lim_{t\to\infty}\ t^{-1}\ \mathcal{J}_4(t)=0\quad\text{a.s.}$$

Now, by using the Holder's inequality, we deduce that

$$t^{-1}\ \int_0^t\left|\psi(s)-\frac{A}{\mu_1}\right|\mathrm{d}s\leq\ t^{-\frac{1}{2}}\left(\int_0^t\left(\psi(s)-\frac{A}{\mu_1}\right)^2\mathrm{d}s\right)^{\frac{1}{2}}=\left(t^{-1}\ \int_0^t\left(\psi^2(s)-\frac{2A}{\mu_1}\psi(s)+\left(\frac{A}{\mu_1}\right)^2\right)\mathrm{d}s\right)^{\frac{1}{2}}.$$

It follows from Lemme 3.3 that

$$\lim_{t\to\infty}t^{-1}\ \int_0^t\left|\psi(s)-\frac{A}{\mu_1}\right|\mathrm{d}s\leq\left(\frac{2A^2}{\mu_1\chi_2}-\frac{2A^2}{\mu_1^2}+\frac{A^2}{\mu_1^2}\right)^{\frac{1}{2}}=\left(\frac{A^2\Big(\sigma_1^2+\int_{\mathcal{U}}\eta_1^2(u)\nu(\mathrm{d}u)\Big)}{\mu_1^2\chi_2}\right)^{\frac{1}{2}}.$$

Taking the superior limit on both sides of (III.32) leads to

$$\limsup_{t\to\infty}\ t^{-1}\ \ln\mathcal{M}(I(t),\mathcal{D}(t))\leq\Upsilon+\Pi-\Sigma+\frac{\omega_1\beta}{\omega_2}\left(\frac{A^2\Big(\sigma_1^2+\int_{\mathcal{U}}\eta_1^2(u)\nu(\mathrm{d}u)\Big)}{\mu_1^2\chi_2}\right)^{\frac{1}{2}}\quad\text{a.s.}$$

Which implies,

$$\limsup_{t\to\infty}\ t^{-1}\ \ln\mathcal{M}(I(t),\mathcal{D}(t))\leq\Upsilon+\Pi-\Sigma+\pi_\star\left(\frac{\mathcal{T}^\star\eta}{\chi_2}\right)^{\frac{1}{2}}=\Theta\quad\text{a.s.}$$

That is to say, if $\Theta<0$, then $\limsup_{t\to\infty}\ t^{-1}\ \ln I(t)<0$, and $\limsup_{t\to\infty}\ t^{-1}\ \ln\mathcal{D}(t)<0$ a.s., which implies in turn that the disease will die out with probability one and this completes the proof. ■

Remark 3.5. When the jumps coefficients $\eta_i(u)$ $(i=1,2,4)$ and the white intensities σ_i $(i=2,4)$ degenerate to zero, our results in Theorem 3.4 coincide with Theorem 2.3 in [82]. Therefore, our results generalize the consequence of the mentioned paper.

3.2.2 Persistence in mean of the epidemic

The study of the persistence in the mean is a significant characteristic to know more about epidemic dynamics. For this reason, in this subsection, we will give the condition for the disease persistence. For simplicity of notation, we define the following quantity

$$\widetilde{\mathcal{T}^\star} = \beta\left(\frac{A}{\mu_1+\bar{\sigma}_1}\right)\left((\mu_2+\gamma+\bar{\sigma}_2)+\beta\left(\frac{A}{\mu_1+\bar{\sigma}_1}\right)\frac{\bar{\sigma}_4}{\pi_\star}\right)^{-1},$$

where $\bar{\sigma}_i \triangleq 0.5\sigma_i^2 + \int_{\mathcal{U}}\Big(\eta_i(u)-\ln(1+\eta_i(u)\Big)\nu(\mathrm{d}u)$, $i=1,2,4$.

Theorem 3.6. *Let $(S(t),I(t),\mathcal{D}(t))$ be the solution of (III.28) with any positive initial data. The stochastic model (III.28) has the following property: if $\widetilde{\mathcal{T}^\star}>1$ holds, then the disease $I(t)$ persists in the mean almost surely.*

Proof. Begin by considering the following function

$$\mathcal{Z}(S(t),I(t),\mathcal{D}(t)) = -\mathfrak{C}_1\ln S(t) - \ln I(t) - \mathfrak{C}_2\ln\mathcal{D}(t) + \mathfrak{C}_3\mathcal{D}(t).$$

where $\mathfrak{C}_i$, $(i=1,2,3)$ are positive constants to be determined in the following. From Itô's formula and system (III.28), we have

$$\begin{aligned}\mathrm{d}\mathcal{Z}(S(t),I(t),\mathcal{D}(t)) &= \mathcal{L}\mathcal{Z}(S(t),I(t),\mathcal{D}(t))\mathrm{d}t - \mathfrak{C}_1\sigma_1\mathrm{d}\mathcal{W}_1(t) - \sigma_2\mathrm{d}\mathcal{W}_2(t) - \mathfrak{C}_2\sigma_4\mathrm{d}\mathcal{W}_4(t)\\ &\quad + \mathfrak{C}_3\sigma_4 D(t)\mathrm{d}\mathcal{W}_4(t) - \int_{\mathcal{U}}\mathfrak{C}_1\ln\big(1+\eta_1(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) - \int_{\mathcal{U}}\ln\big(1+\eta_2(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\ &\quad - \int_{\mathcal{U}}\mathfrak{C}_2\ln\big(1+\eta_4(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) + \int_{\mathcal{U}}\mathfrak{C}_3\eta_4(u)\mathcal{D}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}\mathcal{Z}(S(t),I(t),\mathcal{D}(t)) &= -\frac{\mathfrak{C}_1}{S(t)}\Big(A-\mu_1S(t)-\beta S(t)\mathcal{D}(t)\Big) + \frac{\mathfrak{C}_1\sigma_1^2}{2} - \frac{1}{I(t)}\big(\beta S(t)\mathcal{D}(t)-(\mu_2+\gamma)I(t)\big)\\ &\quad + \frac{\sigma_2^2}{2} - \frac{\mathfrak{C}_2\pi_\star}{\mathcal{D}(t)}(I(t)-\mathcal{D}(t)) + \frac{\mathfrak{C}_2\sigma_4^2}{2} + \mathfrak{C}_3\pi_\star\big(I(t)-\mathcal{D}(t)\big)\\ &\quad + \int_{\mathcal{U}}\mathfrak{C}_1\big(\eta_1(u)-\ln(1+\eta_1(u))\big)\nu(\mathrm{d}u) + \int_{\mathcal{U}}\big(\eta_2(u)-\ln(1+\eta_2(u))\big)\nu(\mathrm{d}u)\\ &\quad + \int_{\mathcal{U}}\mathfrak{C}_2\big(\eta_4(u)-\ln(1+\eta_4(u))\big)\nu(\mathrm{d}u).\end{aligned}$$

We then find that

$$\begin{aligned}\mathcal{L}\mathcal{Z}(S(t),I(t),\mathcal{D}(t)) &= -\frac{\beta S(t)\mathcal{D}(t)}{I(t)} - \frac{\mathfrak{C}_1A}{S(t)} - \frac{\mathfrak{C}_2\pi_\star I(t)}{\mathcal{D}(t)} + (\mathfrak{C}_1\beta-\mathfrak{C}_3\pi_\star)\mathcal{D}(t)\\ &\quad + \mathfrak{C}_1(\mu_1+\bar{\sigma}_1) + \mathfrak{C}_2(\pi_\star+\bar{\sigma}_4) + (\mu_2+\gamma+\bar{\sigma}_2) + \mathfrak{C}_3\pi_\star I(t)\\ &\le -3\big(\beta A\pi_\star\mathfrak{C}_1\mathfrak{C}_2\big)^{\frac{1}{3}} + (\mathfrak{C}_1\beta-\mathfrak{C}_3\pi_\star)\mathcal{D}(t) + \mathfrak{C}_1(\mu_1+\bar{\sigma}_1)\\ &\quad + \mathfrak{C}_2(\pi_\star+\bar{\sigma}_4) + (\mu_2+\gamma+\bar{\sigma}_2) + \mathfrak{C}_3\pi_\star I(t).\end{aligned}$$

By choosing

$$\begin{aligned}\mathfrak{C}_1 &= \beta\Big(\frac{A}{\mu_1+\bar{\sigma}_1}\Big)^2(\pi_\star+\bar{\sigma}_4)/A\pi_\star,\\ \mathfrak{C}_2 &= \beta\Big(\frac{A}{\mu_1+\bar{\sigma}_1}\Big)/(\pi_\star+\bar{\sigma}_4),\\ \mathfrak{C}_3 &= \mathfrak{C}_1\beta/\pi_\star,\end{aligned}$$

we may actually obtain that

$$\begin{aligned}\mathcal{LZ}(S(t),I(t),\mathcal{D}(t)) &\leq -\beta\Big(\frac{A}{\mu_1+\bar{\sigma}_1}\Big)+(\mu_2+\gamma+\bar{\sigma}_2)+\beta\Big(\frac{A}{\mu_1+\bar{\sigma}_1}\Big)\frac{\bar{\sigma}_4}{\pi_\star}+\mathfrak{C}_1\beta I(t)\\ &= -\beta\Big(\frac{A}{\mu_1+\bar{\sigma}_1}\Big)\Big(1-\frac{1}{\widetilde{\mathcal{T}^\star}}\Big)+\mathfrak{C}_1\beta I(t).\end{aligned}$$

Hence, we get

$$\begin{aligned}\mathrm{d}\mathcal{Z}(S(t),I(t),\mathcal{D}(t)) \leq &\Big(-\beta\Big(\frac{A}{\mu_1+\bar{\sigma}_1}\Big)\Big(1-\frac{1}{\widetilde{\mathcal{T}^\star}}\Big)+\mathfrak{C}_1\beta I(t)\Big)\mathrm{d}t-\mathfrak{C}_1\sigma_1\mathrm{d}\mathcal{W}_1(t)-\sigma_2\mathrm{d}\mathcal{W}_2(t)\\ &-\mathfrak{C}_2\sigma_4\mathrm{d}\mathcal{W}_4(t)+\mathfrak{C}_3\sigma_4 D(t)\mathrm{d}\mathcal{W}_4(t)-\int_{\mathcal{U}}\mathfrak{C}_1\ln\big(1+\eta_1(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\ &-\int_{\mathcal{U}}\ln\big(1+\eta_2(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)-\int_{\mathcal{U}}\mathfrak{C}_2\ln\big(1+\eta_4(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\ &+\int_{\mathcal{U}}\mathfrak{C}_3\eta_4(u)\mathcal{D}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).\end{aligned}$$

Integrating from 0 to t and dividing by t on both sides of the last inequality, yields

$$\begin{aligned}t^{-1}\ \mathcal{Z}(S(t),I(t),\mathcal{D}(t)) \leq\ & t^{-1}\ \mathcal{Z}(S(0),I(0),\mathcal{D}(0))-\beta\Big(\frac{A}{\mu_1+\bar{\sigma}_1}\Big)\Big(1-\frac{1}{\widetilde{\mathcal{T}^\star}}\Big)\\ &+\mathfrak{C}_1\beta\ t^{-1}\int_0^t I(s)\mathrm{d}s+\ t^{-1}\ \mathcal{J}_5(t)+\ t^{-1}\ \mathcal{J}_6(t),\end{aligned}$$

where

$$\begin{aligned}\mathcal{J}_5(t) &= -\Big(\mathfrak{C}_1\sigma_1\mathcal{W}_1(t)+\sigma_2\mathcal{W}_2(t)+\mathfrak{C}_2\sigma_4\mathcal{W}_4(t)\Big)+\mathfrak{C}_3\sigma_4\int_0^t D(s)\mathrm{d}\mathcal{W}_4(s),\\ \mathcal{J}_6(t) &= -\int_0^t\int_{\mathcal{U}}\mathfrak{C}_1\ln\big(1+\eta_1(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)-\int_0^t\int_{\mathcal{U}}\ln\big(1+\eta_2(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\ &\quad-\int_0^t\int_{\mathcal{U}}\mathfrak{C}_2\ln\big(1+\eta_4(u)\big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)+\int_0^t\int_{\mathcal{U}}\mathfrak{C}_3\eta_4(u)\mathcal{D}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).\end{aligned}$$

By using, the strong law of large numbers for local martingales and Lemma 3.1, we can obtain

$$\lim_{t\to\infty}\ t^{-1}\ \mathcal{J}_5(t)=0\quad\text{a.s}\quad\text{and}\quad\lim_{t\to\infty}\ t^{-1}\ \mathcal{J}_6(t)=0\quad\text{a.s}$$

Therefore

$$\liminf_{t\to\infty}\ t^{-1}\ \int_0^t I(s)\mathrm{d}s\geq\frac{1}{\mathfrak{C}_1}\Big(\frac{A}{\mu_1+\bar{\sigma}_1}\Big)\Big(1-\frac{1}{\widetilde{\mathcal{T}^\star}}\Big)>0\quad\text{a.s.}$$

This shows that the disease persists in the mean as claimed. ■

Remark 3.7. Persistence in the mean is an important concept in mathematical epidemiology. It captures the long-term survival of the disease even when the population size is quite low at $t=0$. Moreover, the persistence of the model refers to a situation where the disease is endemic in a population.

3.3 Numerical simulations

This section is devoted to illustrate our theoretical results by employing numerical simulations. In the three following examples, we apply the algorithm presented in [112] to discretize the disturbed system (III.28). Using the software **M**atlab2015b and the parameter values listed in Table III.3, we numerically simulate the solution of the system (III.28) with the initial value $(S(0),I(0),\mathcal{D}(0))=(0.6,0.3,0.05)$.

Parameters	Numerical values			
A	0.9	0.3	0.6	0.6
μ_1	0.3	0.3	0.4	0.4
β	0.07	1.3	0.35	0.8
γ	0.05	0.05	0.2	0.3
μ_2	0.5	0.5	0.3	0.3
$\pi_\star$	0.09	0.09	0.7	0.2
σ_1	0.15	0.15	0.2	0.169
σ_2	0.25	0.25	0.15	0.15
σ_4	0.27	0.27	0.13	0.13
η_1	0.2	0.2	0.5	0.5
η_2	0.23	0.23	0.3	0.3
η_4	0.1	0.1	0.7	0.7
	Figure III.6	Figure III.7	Figure III.8	Figure III.9

Table III.3: Nominal values of the system parameters and disturbances intensities adopted in the different simulation examples .

3.3.1 The stochastic extinction case

In order to exhibit the strong random fluctuations effect on epidemic dynamics, we present in Figure III.6, the trajectories of the stochastic solution $(S(t), I(t), \mathcal{D}(t))$. We assume that $\mathcal{U} = (0, \infty)$ and $\nu(\mathcal{U}) = 1$, then by using the parameters listed in Table III.3, we must check the existence of p such that $\chi_{1,p} > 0$. By simple calculation, we easily get $\chi_{1,p} = 0.0206$ for $p = 2.1$. Then, the condition $(\mathcal{H}_5)$ is satisfied. With the chosen parameters, we can obtain the following values:

Expression	Value
$\mathcal{T}^\star \triangleq \beta A(\mu_1(\mu_2+\gamma))^{-1}$	0.3818
$\Upsilon \triangleq \min\{\mu_2+\gamma, \pi_\star\}(\sqrt{\mathcal{T}^\star}-1)$	-0.0344
$\Pi \triangleq \int_{\mathcal{U}} \aleph(u)\nu(du)$	-0.0047
$\Sigma \triangleq \left(2(\sigma_2^{-2}+\sigma_4^{-2})\right)^{-1}$	0.0168
$\eta \triangleq \sigma_1^2 + \int_{\mathcal{U}} \eta_1^2(u)\nu(du)$	0.0625
$\chi_2 = 2\mu_1 - \sigma_1^2 - \int_{\mathcal{U}} \eta_1^2(u)\nu(du)$	0.5375
$\Theta \triangleq \Upsilon + \Pi - \sigma + \pi_\star\sqrt{\mathcal{T}^\star}\eta^{\frac{1}{2}}\chi_2^{-\frac{1}{2}}$	-0.0369

Table III.4: Some expressions and their corresponding values.

From Table III.4, we have $\Theta < 0$, then the condition of Theorem 3.4 is verified. That is to say that the epidemic dies out exponentially almost surely which is exactly illustrated in Figure III.6.

3.3.2 The stochastic persistence case

Consider the system (III.28) with parameters appearing in Table III.3. Then, we obtain the following values:

Expression	Value
$\bar{\sigma}_1 \triangleq 0.5\sigma_1^2 + \int_{\mathcal{U}} \left(\eta_1(u) - \ln(1+\eta_1(u)\right)\nu(du)$	0.0289
$\bar{\sigma}_2 \triangleq 0.5\sigma_2^2 + \int_{\mathcal{U}} \left(\eta_2(u) - \ln(1+\eta_2(u)\right)\nu(du)$	0.0542
$\bar{\sigma}_4 \triangleq 0.5\sigma_4^2 + \int_{\mathcal{U}} \left(\eta_4(u) - \ln(1+\eta_4(u)\right)\nu(du)$	0.0411
$\widetilde{\mathcal{T}^\star} \triangleq \beta\left(\frac{A}{\mu_1+\bar{\sigma}_1}\right)\left((\mu_2+\gamma+\bar{\sigma}_2) + \beta\left(\frac{A}{\mu_1+\bar{\sigma}_1}\right)\frac{\bar{\sigma}_4}{\pi_\star}\right)^{-1}$	1.0344

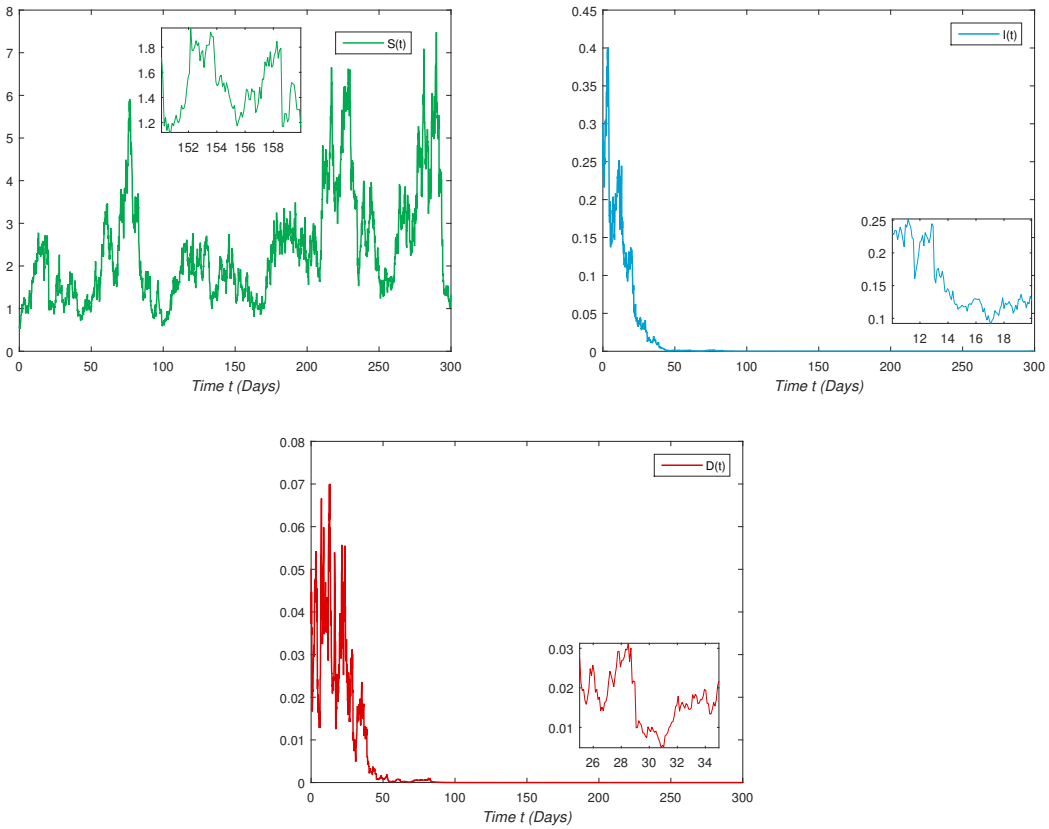


Figure III.6: The paths of $S(t)$, $I(t)$ and $\mathcal{D}(t)$ for the stochastic model (III.28) when $\Theta = -0.0369 < 0$.

Table III.5: Some expressions and their corresponding values.

Therefore, $\widetilde{\mathcal{T}^\star} > 1$. From Figure III.7, we observe the persistence of the epidemic $I(t)$ in this case, which agree well with Theorem 3.6. Furthermore, the solutions $S(t)$ and $D(t)$ are persistent which implies the non-extinction of the stochastic model (III.28).

3.3.3 The Lévy jumps effect on the epidemic dynamics

To find out the effect of white noise and jumps intensities on epidemic dynamics, in this example, we will compare the trajectories of the following systems:

- The deterministic model (III.26) ($\sigma_i = 0$ and $\eta_i = 0$, $i = 1, 2, 4$).
- The stochastic version of (III.26) with degenerate diffusion [82] ($\sigma_2 = \sigma_4 = 0$ and $\eta_i = 0$, $i = 1, 2, 4$).
- The SDE-J system (III.28) ($\sigma_i \neq 0$ and $\eta_i \neq 0$, $i = 1, 2, 4$).

We take the values appearing in Table III.3 which are the same as those used in [82]. For the rest of parameters, we choose $\sigma_1 = 0.2$, $\sigma_2 = 0.15$, $\sigma_4 = 0.13$, $\eta_1 = 0.5$, $\eta_2 = 0.3$, and $\eta_4 = 0.7$. For the sake of a comparison, we choose the following initial value$(S(0), I(0), \mathcal{D}(0)) = (0.2, 0.3, 0.4)$ used in [82]. We see from Figure (III.8) that the effects of Lévy jumps lead to the extinction of the disease while the deterministic model (III.26) and the perturbed model driven by degenerate diffusion both predict persistence. Therefore, we say that the jumps have negative effects on the prevalence of epidemics. This means that jumps can change the asymptotic behavior of the epidemic model significantly. To examine the effect of jumps intensities on dynamical system (III.26) in the case of persistence, we shall decrease the intensity σ_1 to 0.169 and take other parameter as in the last column of Table III.3. From Figure III.9, we observe the persistence of the epidemic in all cases with a greater variation in the case of Lévy jumps. Plainly, environmental factors and unexpected phenomena have significant impacts on the spread of epidemics.

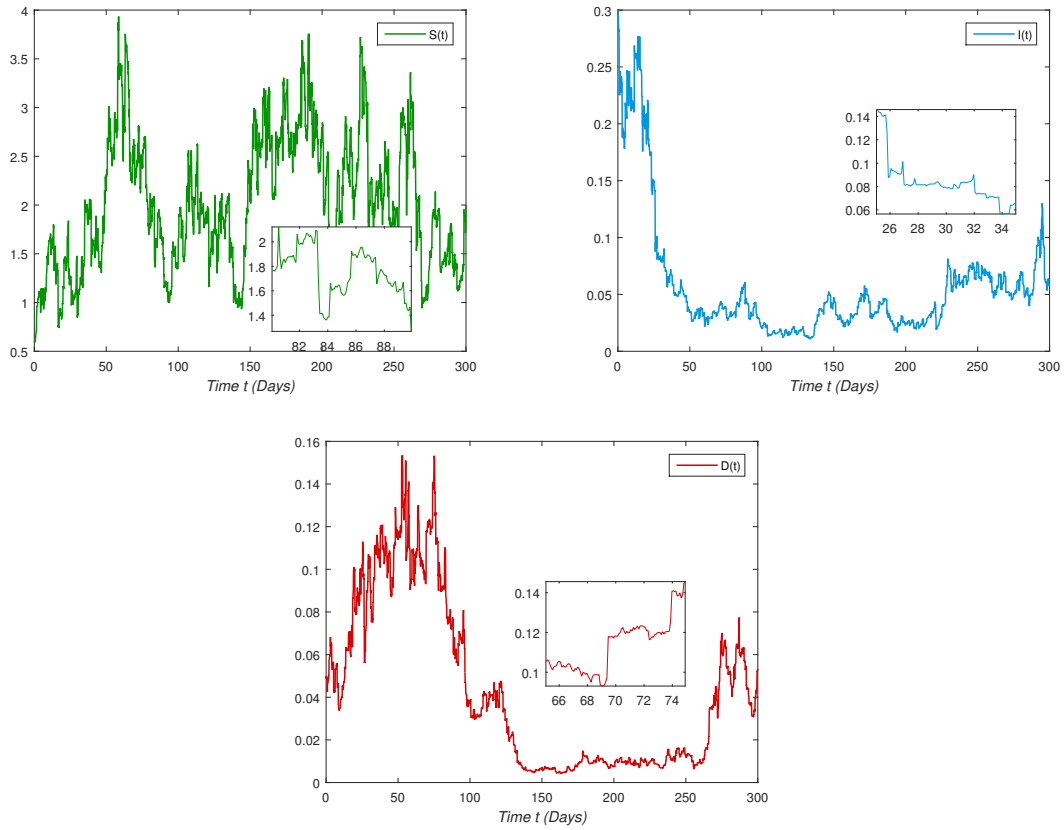


Figure III.7: The paths of $S(t)$, $I(t)$ and $\mathcal{D}(t)$ for the stochastic model (III.28) when $\widetilde{\mathcal{T}^\star} = 1.0344 > 1$.

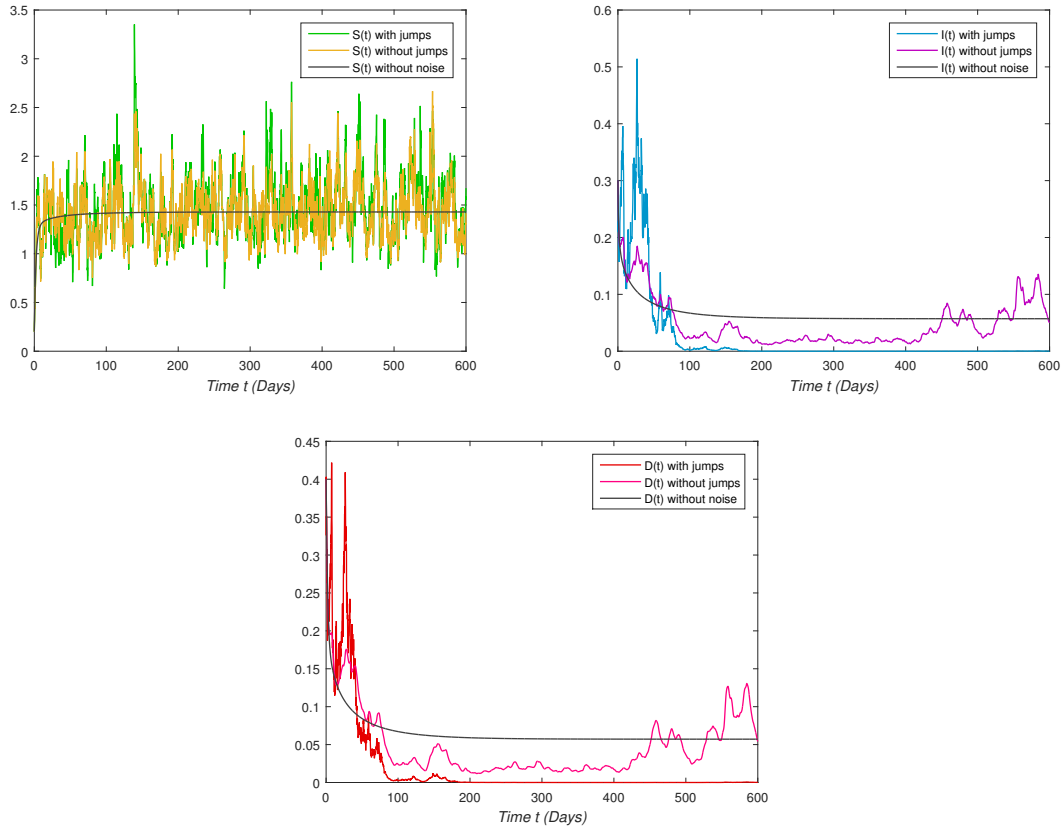


Figure III.8: The paths of $S(t)$, $I(t)$ and $\mathcal{D}(t)$ for the deterministic model (III.25), the model (III.26) with degenerate diffusion, and the SDE-J (III.28).

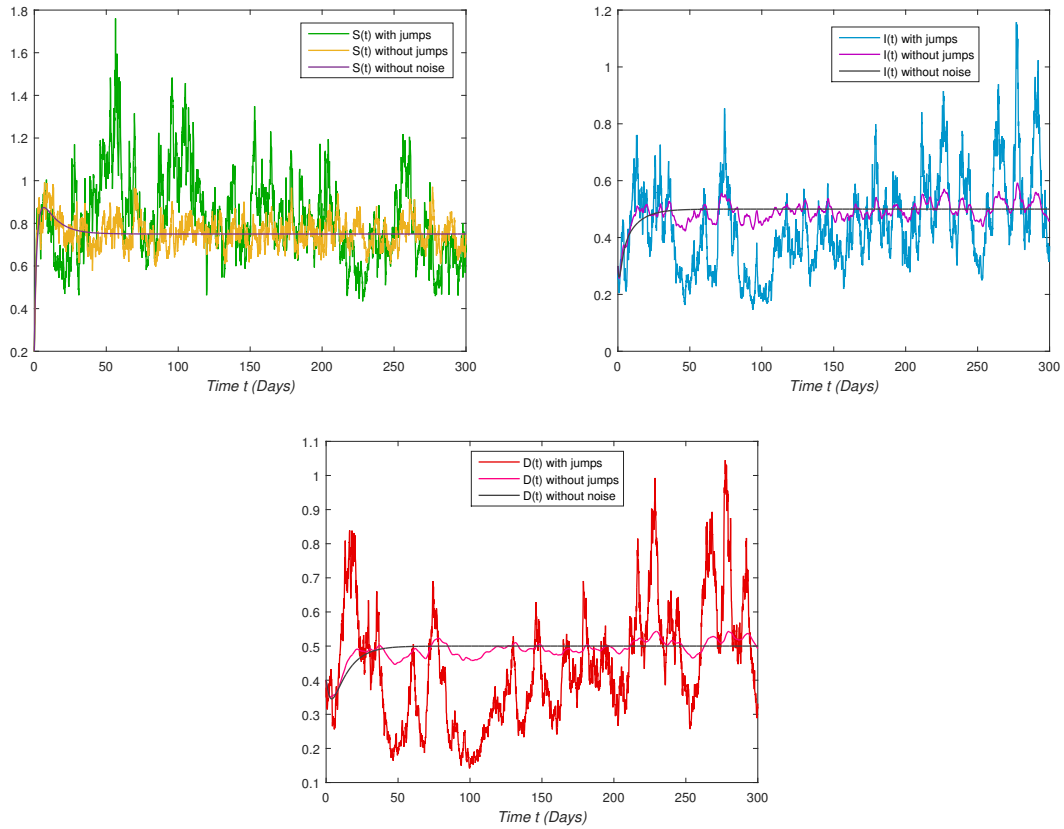


Figure III.9: The paths of $S(t)$, $I(t)$ and $\mathcal{D}(t)$ associated respectively to the models (III.25), (III.26) and (III.28).

3.4 Discussion

This section takes into account these two factors. Specifically, we have analyzed a delayed SIR epidemic model that incorporates proportional Lévy jumps. For analytical reasons, we have employed the linear chain approach to transform the model with a weak kernel case (III.24) into the equivalent system (III.25). After proving the well-posedness of this perturbed model, we have analyzed its long-term behavior. Under some hypotheses, the main epidemiological findings of our study are presented as follows:

1. We have given sufficient condition for the extinction of the epidemic.
2. We have established sufficient condition for the persistence in the mean of the epidemic.

Compared to the existing literature, the novelty of our work lies in new mathematical analysis techniques and improvements which are summarized in the following items:

1. Our work is distinguished from previous works [27, 26, 36] by the use of the expression $\chi_{1,p}$ which boosts the optimality of our calculus and results.
2. Our study offers an alternative method to the gap mentioned in (Theorem 2.2, [163]). Without using the explicit formula of the distribution stationary $f(\cdot)$ of ψ (which still up to now unknown), we calculate the following time averages:

$$\lim_{t\to\infty} t^{-1} \int_0^t \psi(s)\mathrm{d}s \quad \text{and} \quad \lim_{t\to\infty} t^{-1} \int_0^t \psi^2(s)\mathrm{d}s \quad \text{a.s.}$$

3. In order to find an optimal and good majorization, we have considered the inequality (III.31) in our analysis without eliminated it (since $\ln(1+x)-x \leq 0$ for all $x > -1$) which differs from the calculus presented in [26].

Generally speaking, our theoretical results indicate that the conditions of extinction and persistence are mainly depending on the magnitude of the noise intensities as well as the system parameters. From numerical simulations, we remark that Lévy jumps affect significantly the long-run behavior of an epidemic. Eventually, we point out that this chapter extends the study presented in [82] to the case of Lévy jumps and delivers some new insights for understanding the propagation of diseases with distributed delay. Furthermore, the method developed in this chapter can be used to investigate a class of related stochastic models driven by Lévy noise.

Chapter IV

Asymptotic analysis of the stochastic Hepatitis B virus model with unexpected environmental disturbances

Abstract

The Hepatitis B virus (HBV) represents a rich subject that sparks the interest of epidemiological researchers. This chapter aims to examine the long-run properties of the HBV system with unexpected environmental disturbances. We begin with an imprecise HBV model perturbed by Lévy noise. The uncertainty property is justified by the fact that the parameters associated with the HBV model are not certain, but the range to which it belongs can readily be determined. The resulting model is mathematically and biologically well-posed. Under an appropriate assumption, we prove the existence of a unique ergodic stationary distribution by using the mutually exclusive possibilities lemma demonstrated by Stettner in 1986. Furthermore, in the case of the proportional Lévy process with specified parameters, we present the threshold expression which determines the permanence or the extinction of HBV. Basically, we find that Lévy random perturbations can help to reduce the spread of HBV. That is, compared with the stochastic model, the deterministic formulation overestimates the spreading capacity of disease.

1 Ergodic stationary distribution of an imprecise HBV epidemic model with Lévy jumps

1.1 Introduction

The deterministic HBV model (I.10) can be improved by taking into account the unpredictable biological conditions [168, 70, 137, 98]. Also, environmental fluctuations have important effects on the growth and propagation of an epidemic disease [59, 61]. Khan et al. [56] discussed the dynamic of a stochastic HBV epidemic model with varying population size. They supposed that the effect of randomly fluctuating environment is manifest themselves mainly as fluctuations in the HBV transmission rate. To confer the realistic aspect to our study and make it biologically reasonable, in this section, we extend the work of Khan et al. [56] to the case of Lévy noise perturbation. We take into consideration the effects due to some unexpected and severe environmental disturbances (tsunami, floods, earthquakes, hurricanes, whirlwinds, etc.) on the disease outbreak [7, 172]. Thus,

we consider the following model:

$$\begin{cases} dS(t) = \Big(A - \beta S(t)I(t) - (\mu+\theta)S(t)\Big)dt - \sigma S(t)I(t)d\mathcal{W}(t) - \displaystyle\int_{\mathcal{U}} \eta(u)S(t^-)I(t^-)\widetilde{\mathcal{N}}(dt,du), \\ dI(t) = \Big(\beta S(t)I(t) - (\mu+\delta+r)I(t)\Big)dt + \sigma S(t)I(t)d\mathcal{W}(t) + \displaystyle\int_{\mathcal{U}} \eta(u)S(t^-)I(t^-)\widetilde{\mathcal{N}}(dt,du), \\ dR(t) = \Big(\delta I(t) + \theta S(t) - \mu R(t)\Big)dt, \end{cases} \tag{IV.1}$$

where $S(t^-)$ and $I(t^-)$ are the left limits of $S(t)$ and $I(t)$, respectively. $\mathcal{W}(t)$ is a real-valued Brownian motion with intensity $\sigma > 0$. $\widetilde{\mathcal{N}}$, ν, η and $\mathcal{U}$ are defined in the previous chapters such that $\nu(\mathcal{U}) < \infty$. In system (IV.1), we assume that model parameters are precisely known and constant. However, this hypothesis may not be validated due to the lack of data and errors of measurements. It is more realistic to study HBV dynamics with interval-valued parameters. Recently, Pal et al. [105] used interval-valued parameters to analyze the prey-predator model due to the lack of precise biological data such as prey and predator population growth rates. The same logic was applied for epidemic models. In [108], the authors treated a cholera epidemic model with uncertain parameters. They investigated the stability condition of equilibrium points. Bao et al. [8] studied a stochastic SIRS model that includes Lévy jumps and interval parameters. They established the stochastic threshold which determines the extinction and persistence in the mean of disease. In [29], the authors studied an imprecise SIR epidemic model. They solved the optimal control problem.

In this section, we consider the HBV epidemic model with stochastic transmissions and lévy noise. To make our model more realistic, we consider imprecise biological parameters. To the best of our knowledge, the existence of a stationary distribution of system (IV.1) with imprecise parameters remains not proved. In the next subsection, we propose a solution to the mentioned problem by considering an original method different from the Lyapunov approach described in [57]. Before proving the existence of unique a stationary distribution in subsection 1.2.3, we demonstrate the well-posedness of the model (IV.1) with interval-valued parameters in subsection 1.2.2. Some simulation examples are proposed in subsection 1.3 to illustrate our theoretical study.

1.2 Main results

1.2.1 Imprecise stochastic HBV model

Before showing the main result of this section, we first present some definitions of interval numbers and interval-valued function who are used in our study. Then, we construct the imprecise stochastic HBV model.

Definition 1.1 ([105]). An interval number Z is defined as $Z = [\hat{z}, \check{z}] = \{x | \hat{z} \le x \le \check{z}, x \in \mathbb{R}\}$ where $\mathbb{R}$ is the set of all real numbers and $\hat{z}$, $\check{z}$ are the lower and upper limits of the interval numbers respectively. Furthermore, any real number z can be represented in term of interval number as $[z, z]$.

Definition 1.2 ([105]). An interval-valued function for the interval $[x, y]$ can be represented by the following function:

$$\psi(p) = x^{(1-p)}y^p \quad \text{for} \quad p \in [0,1].$$

Theorem 1.3. *The following stochastic differential equation with interval-valued parameters*

$$\begin{cases} dS(t) = \Big(\overline{A} - \overline{\beta} S(t)I(t) - (\overline{\mu}+\overline{\theta})S(t)\Big)dt - \overline{\sigma} S(t)I(t)d\mathcal{W}(t) - \displaystyle\int_{\mathcal{U}} \eta(u)S(t^-)I(t^-)\widetilde{\mathcal{N}}(dt,du), \\ dI(t) = \Big(\overline{\beta} S(t)I(t) - (\overline{\mu}+\overline{\delta}+\overline{r})I(t)\Big)dt + \overline{\sigma} S(t)I(t)d\mathcal{W}(t) + \displaystyle\int_{\mathcal{U}} \eta(u)S(t^-)I(t^-)\widetilde{\mathcal{N}}(dt,du), \\ dR(t) = \Big(\overline{\delta} I(t) + \overline{\theta} S(t) - \overline{\mu} R(t)\Big)dt, \end{cases}$$

where $\overline{A} \in [\hat{A}, \check{A}]$, $\overline{\beta} \in [\hat{\beta}, \check{\beta}]$, $\overline{\mu} \in [\hat{\mu}, \check{\mu}]$, $\overline{\theta} \in [\hat{\theta}, \check{\theta}]$, $\overline{\delta} \in [\hat{\delta}, \check{\delta}]$, $\overline{r} \in [\hat{r}, \check{r}]$ *and* $\overline{\sigma} \in [\hat{\sigma}, \check{\sigma}]$, *is provided an interval-valued functional form of parameters by the following stochastic differential equation:*

$$\begin{cases} \mathrm{d}S(t) = \Big((\hat{A})^{1-p}(\check{A})^{p} - (\hat{\beta})^{1-p}(\check{\beta})^{p} S(t)I(t) - \big((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p}\big)S(t)\Big)\mathrm{d}t - \mathrm{d}\mathcal{Z}(t, \\ \mathrm{d}I(t) = \Big((\hat{\beta})^{1-p}(\check{\beta})^{p} S(t)I(t) - ((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\delta})^{1-p}(\check{\delta})^{p} + (\hat{r})^{1-p}(\check{r})^{p})I(t)\Big)\mathrm{d}t + \mathrm{d}\mathcal{Z}(t), \\ \mathrm{d}R(t) = \Big((\hat{\delta})^{1-p}(\check{\delta})^{p} I(t) + (\hat{\theta})^{1-p}(\check{\theta})^{p} S(t) - (\hat{\mu})^{1-p}(\check{\mu})^{p} R(t)\Big)\mathrm{d}t, \end{cases} \tag{IV.2}$$

where

$$\mathrm{d}\mathcal{Z}(t) = (\hat{\sigma})^{1-p}(\check{\sigma})^{p} S(t)I(t)\mathrm{d}\mathcal{W}(t) + \int_{\mathcal{U}} \eta(u)S(t^{-})I(t^{-})\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u), \qquad \text{for } p \in [0,1].$$

The proof is similar to that in [105] and hence is omitted here.

1.2.2 Well-posedness of the stochastic model (IV.2)

The theorem 1.4 is a prerequisite for analyzing the long-run behavior of the model (IV.2). From epidemiological considerations, it is reasonable to suppose that the intensity of Lévy jumps cannot exceed environmental carrying capacity. Hence, we impose the following standard assumption:

Assumption 1. *The function* $\eta(u)$ *is bounded and* $\left|\left(\frac{(\hat{A})^{1-p}(\check{A})^{p}}{(\hat{\mu})^{1-p}(\check{\mu})^{p}}\right)\eta(u)\right| \leq \Gamma < 1$, $u \in \mathcal{U}$.

By using the Lyapunov analysis method (as mentioned in [8]), we shall verify that the solution of the system (IV.2) is global and positive.

Theorem 1.4. *For any initial value* $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$, *there exists a unique positive solution* $(S(t), I(t), R(t))$ *of the system (IV.2) on* $t \geq 0$, *and the solution will remain in* $\mathbb{R}^3_+$ *with probability one. That is to say, the solution* $(S(t), I(t), R(t)) \in \mathbb{R}^3_+$ *for all* $t \geq 0$ *almost surely.*

Proof. Since the coefficients of the system (IV.2) satisfy the local Lipschitz condition, then for any initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$ there is a unique local solution $(S(t), I(t), R(t))$ on $[0, \tau_e)$, where τ_e is the explosion time. To show that the solution is global, we only need to prove that $\tau_e = \infty$ a.s. Let $\epsilon_0 > 0$ be sufficiently large such that S_0, I_0, R_0 liying within the interval $\left[\frac{1}{\epsilon_0}, \epsilon_0\right]$. For each integer $\epsilon \geq \epsilon_0$, we define the following stopping time:

$$\tau_\epsilon = \inf\left\{t \in [0, \tau_e) : \min\{S(t), I(t), R(t)\} \leq \frac{1}{\epsilon} \text{ or } \max\{S(t), I(t), R(t)\} \geq \epsilon\right\},$$

where throughout this section we set $\inf \emptyset = \infty$ (as usual, $\emptyset$ denotes the empty set). Clearly, τ_ϵ is increasing as $\epsilon \to \infty$. Set $\tau_\infty = \lim_{\epsilon \to \infty} \tau_\epsilon$ whence $\tau_\infty \leq \tau_e$. If we can prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and the solution $(S(t), I(t), R(t)) \in \mathbb{R}^3_+$ for all $t \geq 0$ almost surely. Specifically, to complete the proof, we only need to prove that $\tau_\infty = \infty$ a.s. If this statement is false, then there exists a pair of positive constants $T > 0$ and $k \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > k.$$

Hence, there is an integer $\epsilon_1 \geq \epsilon_0$ such that

$$\mathbb{P}\{\tau_\epsilon \leq T\} \geq k \quad \text{for all} \quad \epsilon \geq \epsilon_1. \tag{IV.3}$$

For $t \geq \tau_\epsilon$ and each ϵ,

$$\begin{aligned} \mathrm{d}(S + I + R) &= \Big((\hat{A})^{1-p}(\check{A})^{p} - (\hat{\mu})^{1-p}(\check{\mu})^{p}(S + I + R) - (\hat{r})^{1-p}(\check{r})^{p} I\Big)\mathrm{d}t \\ &\leq \Big((\hat{A})^{1-p}(\check{A})^{p} - (\hat{\mu})^{1-p}(\check{\mu})^{p}(S + I + R)\Big)\mathrm{d}t. \end{aligned}$$

Then

$$
\begin{aligned}
S(t)+I(t)+R(t) &\leq \frac{(\hat{A})^{1-p}(\check{A})^{p}}{(\hat{\mu})^{1-p}(\check{\mu})^{p}} + e^{-(\hat{\mu})^{1-p}(\check{\mu})^{p}t}\left(S(0)+I(0)+R(0)-\frac{(\hat{A})^{1-p}(\check{A})^{p}}{(\hat{\mu})^{1-p}(\check{\mu})^{p}}\right),\\
&\leq \begin{cases} \frac{(\hat{A})^{1-p}(\check{A})^{p}}{(\hat{\mu})^{1-p}(\check{\mu})^{p}}, & \text{if } S(0)+I(0)+R(0) \leq \frac{(\hat{A})^{1-p}(\check{A})^{p}}{(\hat{\mu})^{1-p}(\check{\mu})^{p}},\\ S(0)+I(0)+R(0), & \text{if } S(0)+I(0)+R(0) > \frac{(\hat{A})^{1-p}(\check{A})^{p}}{(\hat{\mu})^{1-p}(\check{\mu})^{p}}, \end{cases}\\
&\equiv \mathfrak{C}.
\end{aligned}
$$

From mathematical and biological considerations, we can study the disease dynamics of the model (IV.2) in the following bounded set

$$
\Delta = \left\{(S,I,R)\in\mathbb{R}^3_+ : \quad S+I+R \leq \frac{(\hat{A})^{1-p}(\check{A})^{p}}{(\hat{\mu})^{1-p}(\check{\mu})^{p}} \text{ a.s.}\right\}.
$$

Therefore, the region Δ is almost surely positively invariant set by the system (IV.2). Now, we define the following Lyapunov C^2-function $V:\mathbb{R}^3_+\to\bar{\mathbb{R}}_+$ by

$$
\mathcal{V}(S,I,R) = (S-1-\ln S)+(I-1-\ln I)+(R-1-\ln R).
$$

For $0\leq t\leq \tau_\epsilon\wedge T$, using Itô's formula, we obtain that

$$
\begin{aligned}
\mathrm{d}\mathcal{V}(S,I,R) &= \mathcal{L}\mathcal{V}(S,I,R)\mathrm{d}t - (\hat{\sigma})^{1-p}(\check{\sigma})^{p}S\mathrm{d}\mathcal{W}(t) + (\hat{\sigma})^{1-p}(\check{\sigma})^{p}I\mathrm{d}\mathcal{W}(t)\\
&\quad - \int_{\mathcal{U}}\Big(\eta(u)S(t^-)I(t^-)+\ln(1-\eta(u)I(t^-))\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\\
&\quad + \int_{\mathcal{U}}\Big(\eta(u)S(t^-)I(t^-)-\ln(1+\eta(u)S(t^-))\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),
\end{aligned}
\tag{IV.4}
$$

where $\mathcal{L}$ is the differential operator, and

$$
\begin{aligned}
\mathcal{L}\mathcal{V}(S,I,R) &= \left(1-\frac{1}{S}\right)\left((\hat{A})^{1-p}(\check{A})^{p}-(\hat{\beta})^{1-p}(\check{\beta})^{p}SI-\left((\hat{\mu})^{1-p}(\check{\mu})^{p}+(\hat{\theta})^{1-p}(\check{\theta})^{p}\right)S\right)\\
&\quad+\left(1-\frac{1}{I}\right)\left((\hat{\beta})^{1-p}(\check{\beta})^{p}SI-((\hat{\mu})^{1-p}(\check{\mu})^{p}+(\hat{\delta})^{1-p}(\check{\delta})^{p}+(\hat{r})^{1-p}(\check{r})^{p})I\right)\\
&\quad+\left(1-\frac{1}{R}\right)\left((\hat{\delta})^{1-p}(\check{\delta})^{p}I+(\hat{\theta})^{1-p}(\check{\theta})^{p}S-(\hat{\mu})^{1-p}(\check{\mu})^{p}R\right)\\
&\quad+\frac{1}{2}(\hat{\sigma})^{2-2p}(\check{\sigma})^{2p}I^2+\frac{1}{2}(\hat{\sigma})^{2-2p}(\check{\sigma})^{2p}S^2\\
&\quad-\int_{\mathcal{U}}(\ln(1-\eta(u)I)+\eta(u)I)\nu(\mathrm{d}u)-\int_{\mathcal{U}}(\ln(1+\eta(u)S)-\eta(u)S)\nu(\mathrm{d}u)\\
&\leq (\hat{A})^{1-p}(\check{A})^{p}+(\hat{\beta})^{1-p}(\check{\beta})^{p}\mathfrak{C}+(\hat{\theta})^{1-p}(\check{\theta})^{p}+(\hat{\sigma})^{2-2p}(\check{\sigma})^{2p}\mathfrak{C}^2+(\hat{r})^{1-p}(\check{r})^{p}\\
&\quad+3(\hat{\mu})^{1-p}(\check{\mu})^{p}+(\hat{\delta})^{1-p}(\check{\delta})^{p}+\int_{\mathcal{U}}\mathcal{H}_1\nu(\mathrm{d}u)+\int_{\mathcal{U}}\mathcal{H}_2\nu(\mathrm{d}u),
\end{aligned}
$$

where

$$
\mathcal{H}_1 = -\ln(1-\eta(u)I)-\eta(u)I \text{ and } \mathcal{H}_2 = -\ln(1+\eta(u)S)+\eta(u)S.
$$

By Assumption 1, we have $1-\eta(u)I>0$. In addition, by Taylor-Lagrange's formula, we show that

$$
\mathcal{H}_1 = \eta(u)I-\eta(u)I+\frac{\eta^2(u)I^2}{2(1-\kappa\eta(u)I)^2}\leq\frac{\Gamma^2}{2(1-\Gamma)^2},\quad \kappa\in(0,1).
\tag{IV.5}
$$

Similarly, we get

$$
\mathcal{H}_2 = -\eta(u)S+\eta(u)S+\frac{\eta^2(u)S^2}{2(1+\kappa\eta(u)S)^2}\leq\frac{\Gamma^2}{2(1-\Gamma)^2},\quad \kappa\in(0,1).
\tag{IV.6}
$$

Therefore,

$$\begin{aligned}\mathcal{L}\mathcal{V}(S,I,R) &\leq (\hat{A})^{1-p}(\check{A})^{p} + (\hat{\beta})^{1-p}(\check{\beta})^{p}\mathfrak{C} + (\hat{\theta})^{1-p}(\check{\theta})^{p} + (\hat{\sigma})^{2-2p}(\check{\sigma})^{2p}\mathfrak{C}^{2} + (\hat{r})^{1-p}(\check{r})^{p} \\ &\quad + 3(\hat{\mu})^{1-p}(\check{\mu})^{p} + \Big((\hat{\beta})^{1-p}(\check{\beta})^{p} + (\hat{\delta})^{1-p}(\check{\delta})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p}\Big) + \frac{\Gamma^{2}}{(1-\Gamma)^{2}}\nu(\mathcal{U}) \\ &\equiv \widetilde{\mathfrak{C}},\end{aligned}$$

where $\widetilde{\mathfrak{C}}$ is a positive constant. Integrating both sides of (IV.4) from 0 to $\tau_\epsilon \wedge T$, and taking expectation, we get

$$\mathbb{E}\mathcal{V}\Big(S(\tau_\epsilon \wedge T), I(\tau_\epsilon \wedge T), R(\tau_\epsilon \wedge T)\Big) \leq \mathcal{V}(S(0), I(0), R(0)) + \widetilde{\mathfrak{C}}T.$$

Setting $\Omega_\epsilon = \{\tau_\epsilon \leq T\}$ for $\epsilon \geq \epsilon_0$ and by (IV.3), we have $\mathbb{P}(\Omega_\epsilon) \geq k$. For $\omega \in \Omega_\epsilon$, there is some component of $S(\tau_\epsilon, \omega)$, $I(\tau_\epsilon, \omega)$ and $R(\tau_\epsilon, \omega)$ equals either ϵ or $\frac{1}{\epsilon}$. Hence, $\mathcal{V}(S(\tau_\epsilon), I(\tau_\epsilon), R(\tau_\epsilon))$ is not less than $\epsilon - 1 - \ln\epsilon$ or $\frac{1}{\epsilon} - 1 - \ln\frac{1}{\epsilon}$. Consequently,

$$\begin{aligned}\mathcal{V}(S(0), I(0), R(0)) + \widetilde{\mathfrak{C}}T &\geq \mathbb{E}\Big(\mathbb{1}_{\Omega_\epsilon}\mathcal{V}(S(\tau_\epsilon, \omega), I(\tau_\epsilon, \omega), R(\tau_\epsilon, \omega))\Big) \\ &\geq k\left((\epsilon - 1 - \ln\epsilon) \wedge \left(\frac{1}{\epsilon} - 1 - \ln\frac{1}{\epsilon}\right)\right).\end{aligned}$$

Extending ϵ to ∞ leads to the contradiction. Thus, $\tau_\infty = \infty$ a.s. which completes the proof of the theorem. ∎

1.2.3 Existence and uniqueness of a stationary distribution to the system (IV.2)

Our aim in this subsection is to give the appropriate condition for the SDE model (IV.2) has a unique ergodic stationary distribution. To this end, we use Lemma 11.4 (Chapter 2) and we introduce the following notation:

$$\mathcal{R}_0^s = \frac{1}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\delta})^{1-p}(\check{\delta})^{p} + (\hat{r})^{1-p}(\check{r})^{p})}\left(\frac{(\hat{\beta}\hat{A})^{1-p}(\check{\beta}\check{A})^{p}}{(\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p}} - \frac{\widetilde{\sigma}^{2}(\hat{A})^{2-2p}(\check{A})^{2p}}{2(\hat{\mu})^{2-2p}(\check{\mu})^{2p}}\right),$$

where $\widetilde{\sigma}^{2} = (\hat{\sigma})^{2-2p}(\check{\sigma})^{2p} + \displaystyle\int_{\mathcal{U}} \frac{\eta^{2}(u)}{(1-\Gamma)^{2}}\nu(\mathrm{d}u)$. Furthermore, we let

$$\begin{aligned}\Omega_1 &= \{(t,\omega) \in \mathbb{R}_+ \times \Omega |\ S(t,\omega) \geq \epsilon, \text{ and, } I(t,\omega) \geq \epsilon\}, \\ \Omega_2 &= \{(t,\omega) \in \mathbb{R}_+ \times \Omega |\ S(t,\omega) \leq \epsilon\}, \\ \Omega_3 &= \{(t,\omega) \in \mathbb{R}_+ \times \Omega |\ I(t,\omega) \leq \epsilon\}, \\ \Omega_4 &= \{(t,\omega) \in \mathbb{R}_+ \times \Omega |\ S(t,\omega) \geq \zeta, \text{ or, } I(t,\omega) \geq \zeta\}, \\ \Omega_5 &= \{(t,\omega) \in \mathbb{R}_+ \times \Omega |\ \epsilon \leq S(t,\omega) \leq \zeta, \text{ and, } \epsilon \leq I(t,\omega) \leq \zeta\},\end{aligned}$$

where $\epsilon < \zeta$ are positive constants to be determined later.

Theorem 1.5. *If $\mathcal{R}_0^s > 1$, the stochastic system (IV.2) admits a unique stationary distribution and it has the ergodic property for any initial value $(S_0, I_0, R_0) \in \Delta$.*

Proof. The following proof is divided into three steps:

Step I. Similar to the proof of Lemma 3.2 in [135] or Theorem 2.5 in [162], we briefly verify the Feller property of the SDE model (IV.2). The main purpose of the next steps is to prove that (II.22) is impossible.

Step II. Define

$$\mathcal{V}(t) = \ln I(t) + \frac{(\hat{\beta})^{1-p}(\check{\beta})^{p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})}S(t).$$

Applying Itô's formula gives

$$
\begin{aligned}
\mathrm{d}\mathcal{V}(t) = \Bigg(& (\hat{\beta})^{1-p}(\check{\beta})^{p} S(t) - ((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\delta})^{1-p}(\check{\delta})^{p} + (\hat{r})^{1-p}(\check{r})^{p}) \\
& - \frac{1}{2}(\hat{\sigma})^{2-2p}(\check{\sigma})^{2p} S^{2}(t) + \int_{\mathcal{U}} \Big(\ln(1+\eta(u)S(t^{-})) - \eta(u)S(t^{-}) \Big) \nu(\mathrm{d}u) \Bigg) \mathrm{d}t \\
& + (\hat{\sigma})^{1-p}(\check{\sigma})^{p} S(t)\mathrm{d}\mathcal{W}(t) + \int_{\mathcal{U}} \ln(1+\eta(u)S(t^{-}))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) \\
& + \frac{(\hat{\beta}\hat{A})^{1-p}(\check{\beta}\check{A})^{p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})} \mathrm{d}t - \frac{(\hat{\beta})^{2-2p}(\check{\beta})^{2p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})} S(t)I(t)\mathrm{d}t \\
& - \frac{(\hat{\beta})^{1-p}(\check{\beta})^{p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})} ((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p}) S(t)\mathrm{d}t \\
& - \frac{(\hat{\beta})^{1-p}(\check{\beta})^{p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})} (\hat{\sigma})^{1-p}(\check{\sigma})^{p} S(t)I(t)\mathrm{d}\mathcal{W}(t) \\
& - \frac{(\hat{\beta})^{1-p}(\check{\beta})^{p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})} \int_{\mathcal{U}} \eta(u)S(t^{-})I(t^{-})\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \qquad \text{(IV.7)}
\end{aligned}
$$

Noting that $0 < S < \frac{(\hat{A})^{1-p}(\check{A})^{p}}{(\hat{\mu})^{1-p}(\check{\mu})^{p}}$, the equality (IV.7) can be rewritten as follows

$$
\begin{aligned}
\mathrm{d}\mathcal{V}(t) \geq \Bigg(& \frac{(\hat{\beta}\hat{A})^{1-p}(\check{\beta}\hat{A})^{p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})} - ((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\delta})^{1-p}(\check{\delta})^{p} + (\hat{r})^{1-p}(\check{r})^{p}) \\
& - \frac{(\hat{\sigma}\hat{A})^{2-2p}(\check{\sigma}\hat{A})^{2p}}{2(\hat{\mu})^{2-2p}(\check{\mu})^{2p}} - \frac{(\hat{\beta})^{2-2p}(\check{\beta})^{2p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})} S(t)I(t) \Bigg) \mathrm{d}t \\
& + \int_{\mathcal{U}} \Big(\ln(1+\eta(u)S(t^{-})) - \eta(u)S(t^{-}) \Big) \nu(\mathrm{d}u)\mathrm{d}t \\
& + (\hat{\sigma})^{1-p}(\check{\sigma})^{p} S(t)\mathrm{d}\mathcal{W}(t) + \int_{\mathcal{U}} \ln(1+\eta(u)S(t^{-}))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) \\
& - \frac{(\hat{\beta})^{1-p}(\check{\beta})^{p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})} (\hat{\sigma})^{1-p}(\check{\sigma})^{p} S(t)I(t)\mathrm{d}\mathcal{W}(t) \\
& - \frac{(\hat{\beta})^{1-p}(\check{\beta})^{p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})} \int_{\mathcal{U}} \eta(u)S(t^{-})I(t^{-})\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \qquad \text{(IV.8)}
\end{aligned}
$$

Integrating the inequality (IV.8) from 0 to t leads to

$$
\begin{aligned}
\mathcal{V}(t) - \mathcal{V}(0) \geq & \frac{(\hat{\beta}\hat{A})^{1-p}(\check{\beta}\check{A})^{p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})} - ((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\delta})^{1-p}(\check{\delta})^{p} + (\hat{r})^{1-p}(\check{r})^{p}) \\
& - \frac{(\hat{\sigma}\hat{A})^{2-2p}(\check{\sigma}\hat{A})^{2p}}{2(\hat{\mu})^{2-2p}(\check{\mu})^{2p}} - \frac{(\hat{\beta})^{2-2p}(\check{\beta})^{2p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})} \int_{0}^{t} S(s)I(s)\mathrm{d}s \\
& + \int_{0}^{t} \int_{\mathcal{U}} \Big(\ln(1+\eta(u)S(s^{-})) - \eta(u)S(s^{-}) \Big) \nu(\mathrm{d}u)\mathrm{d}s \\
& + \mathcal{K}_{1}(t) + \mathcal{K}_{2}(t) + \mathcal{K}_{3}(t) + \mathcal{K}_{4}(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{K}_{1}(t) &= \int_{0}^{t} (\hat{\sigma})^{1-p}(\check{\sigma})^{p} S(s)\mathrm{d}\mathcal{W}(s), \\
\mathcal{K}_{2}(t) &= \frac{-(\hat{\beta})^{1-p}(\check{\beta})^{p}(\hat{\sigma})^{1-p}(\check{\sigma})^{p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})} \int_{0}^{t} S(s)I(s)\mathrm{d}\mathcal{W}(s), \\
\mathcal{K}_{3}(t) &= \int_{0}^{t} \int_{\mathcal{U}} \ln(1+\eta(u)S(s))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u), \\
\mathcal{K}_{4}(t) &= \frac{-(\hat{\beta})^{1-p}(\check{\beta})^{p}}{((\hat{\mu})^{1-p}(\check{\mu})^{p} + (\hat{\theta})^{1-p}(\check{\theta})^{p})} \int_{0}^{t} \int_{\mathcal{U}} \eta(u)S(s)I(s)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).
\end{aligned}
$$

The quadratic variation of $\mathcal{K}_1$ is defined by $\langle \mathcal{K}_1, \mathcal{K}_1 \rangle_t = \int_0^t (\hat{\sigma})^{2-2p} (\check{\sigma})^{2p} S^2(s) \mathrm{d}s$. Therefore, we get

$$\begin{aligned} \limsup_{t\to\infty} \frac{\langle \mathcal{K}_1, \mathcal{K}_1 \rangle_t}{t} &= (\hat{\sigma})^{2-2p} (\check{\sigma})^{2p} \limsup_{t\to\infty} \frac{1}{t} \int_0^t S^2(s) \mathrm{d}s \\ &\leq (\hat{\sigma})^{2-2p} (\check{\sigma})^{2p} \frac{(\hat{A})^{2-2p} (\check{A})^{2p}}{(\hat{\mu})^{2-2p} (\check{\mu})^{2p}} < \infty \ \text{ a.s.} \end{aligned}$$

Similarly, we have

$$\limsup_{t\to\infty} \frac{\langle \mathcal{K}_2, \mathcal{K}_2 \rangle_t}{t} = \frac{(\hat{\beta})^{2-2p} (\check{\beta})^{2p} (\hat{\sigma})^{2-2p} (\check{\sigma})^{2p}}{((\hat{\mu})^{1-p} (\check{\mu})^p + (\hat{\theta})^{1-p} (\check{\theta})^p)^2} \limsup_{t\to\infty} \frac{1}{t} \int_0^t S^2(s) I^2(s) \mathrm{d}s < \infty \ \text{ a.s.}$$

By the Assumption 1, we deduce that

$$\ln(1-\Gamma) \leq \ln(1+\eta(u)S(s)) \leq \ln(1+\Gamma).$$

Then

$$\begin{aligned} \limsup_{t\to\infty} \frac{\langle \mathcal{K}_3, \mathcal{K}_3 \rangle_t}{t} &= \limsup_{t\to\infty} \frac{1}{3} \int_0^t \int_{\mathcal{U}} (\ln(1+\eta(u)S(s))^2 \nu(\mathrm{d}u)\mathrm{d}s \\ &\leq \max\left\{ (\ln(1+\Gamma))^2, (\ln(1-\Gamma))^2 \right\} \nu(\mathcal{U}) < \infty \ \text{ a.s.} \end{aligned}$$

and

$$\limsup_{t\to\infty} \frac{\langle \mathcal{K}_4, \mathcal{K}_4 \rangle_t}{t} \leq \frac{(\hat{\beta A})^{2-2p} (\check{\beta A})^{2p}}{\left((\hat{\mu})^{1-p} (\check{\mu})^p + (\hat{\theta})^{1-p} (\check{\theta})^p\right) \left((\hat{\mu})^{1-p} (\check{\mu})^p\right)^2} \nu(\mathcal{U}) < \infty \ \text{ a.s.}$$

According to the strong law of large numbers for local martingales, one can conclude that

$$\lim_{t\to\infty} \frac{1}{t} \mathcal{K}_i(t) = 0 \ \text{ a.s.}, \quad i = 1,2,3,4. \tag{IV.9}$$

By using (IV.6) and Assumption 1, we get

$$\frac{1}{t} \int_0^t \int_{\mathcal{U}} \Big(\ln(1+\eta(u)S(s)) - \eta(u)S(s) \Big) \nu(\mathrm{d}u)\mathrm{d}s \geq -\frac{1}{2} \frac{(\hat{A})^{2-2p} (\check{A})^{2p}}{(\hat{\mu})^{2-2p} (\check{\mu})^{2p}} \int_{\mathcal{U}} \frac{\eta^2(u)}{(1-\Gamma)^2} \nu(\mathrm{d}u).$$

Let

$$\widetilde{\sigma}^2 = (\hat{\sigma})^{2-2p} (\check{\sigma})^{2p} + \int_{\mathcal{U}} \frac{\eta^2(u)}{(1-\Gamma)^2} \nu(\mathrm{d}u).$$

Therefore

$$\begin{aligned} \liminf_{t\to\infty} \frac{1}{t} \int_0^t (\hat{\beta})^{1-p} (\check{\beta})^p S(s) I(s) \mathrm{d}s \geq \frac{((\hat{\mu})^{1-p} (\check{\mu})^p + (\hat{\theta})^{1-p} (\check{\theta})^p)}{(\hat{\beta})^{1-p} (\check{\beta})^p} \Bigg(& \frac{(\hat{\beta}\hat{A})^{1-p} (\check{\beta}\check{A})^p}{((\hat{\mu})^{1-p} (\check{\mu})^p + (\hat{\theta})^{1-p} (\check{\theta})^p)} \\ & - ((\hat{\mu})^{1-p} (\check{\mu})^p + (\hat{\delta})^{1-p} (\check{\delta})^p + (\hat{r})^{1-p} (\check{r})^p) - \widetilde{\sigma}^2 \frac{(\hat{A})^{2-2p} (\hat{A})^{2p}}{2((\hat{\mu})^{2-2p} (\check{\mu})^{2p})} \Bigg). \end{aligned}$$

Thus we can derive that

$$\begin{aligned} \liminf_{t\to\infty} \frac{1}{t} \int_0^t (\hat{\beta})^{1-p} (\check{\beta})^p S(s) I(s) \mathrm{d}s \geq \frac{((\hat{\mu})^{1-p} (\check{\mu})^p + (\hat{\theta})^{1-p} (\check{\theta})^p)}{(\hat{\beta})^{1-p} (\check{\beta})^p} \Big(& (\hat{\mu})^{1-p} (\check{\mu})^p + (\hat{\delta})^{1-p} (\check{\delta})^p \\ & + (\hat{r})^{1-p} (\check{r})^p \Big) \Big(\mathcal{R}_0^s - 1 \Big) > 0 \quad \text{a.s.} \end{aligned} \tag{IV.10}$$

Step III. To continue our analysis, we use the definition of Ω_1, Ω_2 and Ω_3. Then, it follows from (IV.10) that

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\Big((\hat{\beta})^{1-p}(\check{\beta})^p S(s)I(s)1_{\Omega_1}\Big)\mathrm{d}s \geq \liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\Big((\hat{\beta})^{1-p}(\check{\beta})^p S(s)I(s)\Big)\mathrm{d}s\\ - \limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\Big((\hat{\beta})^{1-p}(\check{\beta})^p S(s)I(s)1_{\Omega_2}\Big)\mathrm{d}s\\ - \limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\Big((\hat{\beta})^{1-p}(\check{\beta})^p S(s)I(s)1_{\Omega_3}\Big)\mathrm{d}s\\ \geq \frac{((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\theta})^{1-p}(\check{\theta})^p)}{(\hat{\beta})^{1-p}(\check{\beta})^p}\Big((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\delta})^{1-p}(\check{\delta})^p\\ + (\hat{r})^{1-p}(\check{r})^p\Big)\Big(\mathcal{R}_0^s - 1\Big) - \frac{2(\hat{\beta}\hat{A})^{1-p}(\check{\beta}\check{A})^p\epsilon}{(\hat{\mu})^{1-p}(\check{\mu})^p}.\end{aligned}$$

We can choose

$$\epsilon \leq \frac{((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\theta})^{1-p}(\check{\theta})^p)((\hat{\mu})^{1-p}(\check{\mu})^p)}{4(\hat{\beta})^{2-2p}(\check{\beta})^{2p}(\hat{A})^{1-p}(\check{A})^p}\Big((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\delta})^{1-p}(\check{\delta})^p + (\hat{r})^{1-p}(\check{r})^p\Big)\Big(\mathcal{R}_0^s - 1\Big),$$

then we obtain

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\Big((\hat{\beta})^{1-p}(\check{\beta})^p S(s)I(s)1_{\Omega_1}\Big)\mathrm{d}s \geq \frac{((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\theta})^{1-p}(\check{\theta})^p)}{2(\hat{\beta})^{1-p}(\check{\beta})^p}\Big((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\delta})^{1-p}(\check{\delta})^p\\ + (\hat{r})^{1-p}(\check{r})^p\Big)\Big(\mathcal{R}_0^s - 1\Big) > 0 \quad \text{a.s.}\end{aligned}$$

Let a and b two real numbers greater than 1 such that $\frac{1}{a} + \frac{1}{b} = 1$. By utilizing Young inequality $xy \leq \frac{x^a}{a} + \frac{y^b}{b}$ for all $x,y > 0$, we get

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\Big((\hat{\beta})^{1-p}(\check{\beta})^p S(s)I(s)1_{\Omega_1}\Big)\mathrm{d}s \leq \liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\Big(a^{-1}\Big(\varpi(\hat{\beta})^{1-p}(\check{\beta})^p S(s)I(s)\Big)^a + b^{-1}\varpi^{-b}1_{\Omega_1}\Big)\mathrm{d}s\\ \leq a^{-1}\Big(\varpi(\hat{\beta})^{1-p}(\check{\beta})^p\Big)^a\Big(\frac{(\hat{A})^{1-p}(\check{A})^p}{(\hat{\mu})^{1-p}(\check{\mu})^p}\Big)^{2a}\\ + \liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}(b^{-1}\varpi^{-b}1_{\Omega_1})\mathrm{d}s, \qquad \text{(IV.11)}\end{aligned}$$

where ϖ is a positive constant satisfying

$$\begin{aligned}\varpi^a \leq \frac{a}{4}\Big((\hat{\beta})^{1-p}(\check{\beta})^p\Big)^{-(a+1)}\Big(\frac{(\hat{A})^{1-p}(\check{A})^p}{(\hat{\mu})^{1-p}(\check{\mu})^p}\Big)^{-2a}\Big((\hat{\mu})^{1-p}(\check{\mu})^p\\ + (\hat{\theta})^{1-p}(\check{\theta})^p\Big)\Big((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\delta})^{1-p}(\check{\delta})^p + (\hat{r})^{1-p}(\check{r})^p\Big)\Big(\mathcal{R}_0^s - 1\Big).\end{aligned}$$

From (IV.11), we deduce that

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}(1_{\Omega_1})\mathrm{d}s\\ &\geq \frac{((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\theta})^{1-p}(\check{\theta})^p)b\varpi^b}{4(\hat{\beta})^{1-p}(\check{\beta})^p}\Big((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\delta})^{1-p}(\check{\delta})^p + (\hat{r})^{1-p}(\check{r})^p\Big)\Big(\mathcal{R}_0^s - 1\Big) > 0 \quad \text{a.s.} \qquad \text{(IV.12)}\end{aligned}$$

By using the Markov's inequality, we can observe that

$$\int_\Omega \mathbb{1}_{\Omega_4}(t,\omega)\mathrm{d}\mathbb{P}(\omega) \leq \mathbb{P}(S(t) \geq \zeta) + \mathbb{P}(I(t) \geq \zeta) \leq \frac{1}{\zeta}\mathbb{E}(S(t) + I(t)) \leq \frac{1}{\zeta}\frac{(\hat{A})^{1-p}(\check{A})^p}{(\hat{\mu})^{1-p}(\check{\mu})^p}.$$

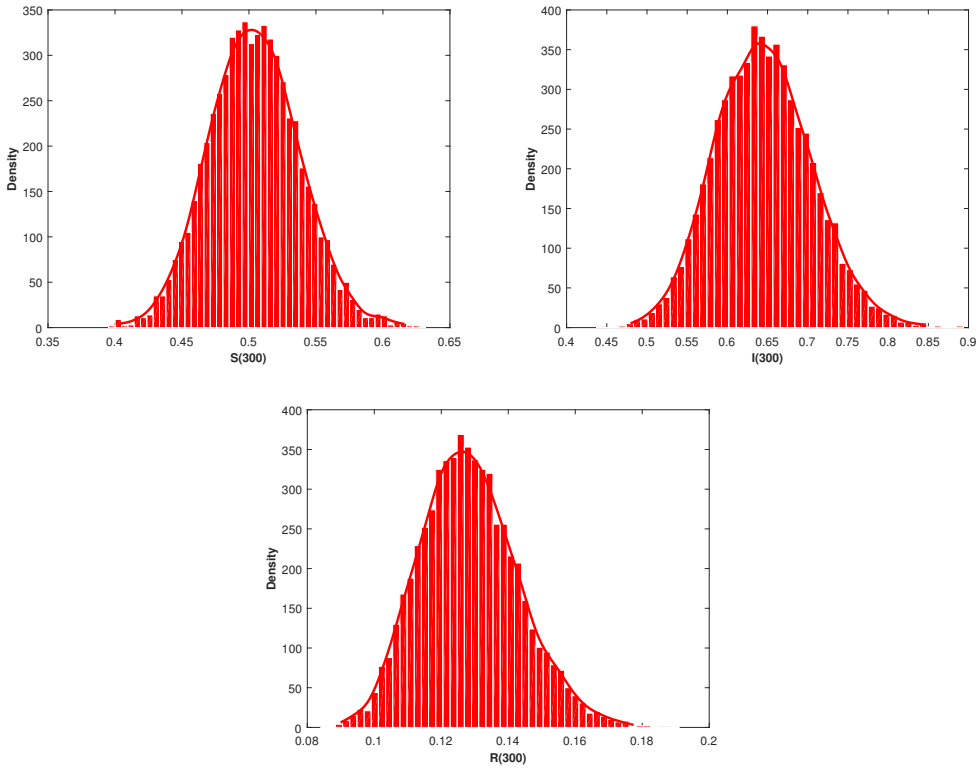


Figure IV.1: Histogram of the probability density function for S, I, and R population at $t = 300$ for the stochastic model (IV.2), the smoothed curves are the probability density functions of $S(t)$, $I(t)$ and $R(t)$, respectively. Here, we get $\mathcal{R}_0^s = 2.1848 > 1$ for $p = 0$.

Choosing

$$\frac{1}{\zeta} \leq \frac{((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\theta})^{1-p}(\check{\theta})^p)b\varpi^b}{8(\hat{\beta}\hat{A})^{1-p}(\check{\beta}\check{A})^p}\Big((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\delta})^{1-p}(\check{\delta})^p + (\hat{r})^{1-p}(\check{r})^p\Big)\Big(\mathcal{R}_0^s - 1\Big).$$

We thus obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}(1_{\Omega_4})\mathrm{d}s \leq \frac{((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\theta})^{1-p}(\check{\theta})^p)b\varpi^b}{8(\hat{\beta})^{1-p}(\check{\beta})^p}\Big((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\delta})^{1-p}(\check{\delta})^p + (\hat{r})^{1-p}(\check{r})^p\Big)\Big(\mathcal{R}_0^s - 1\Big).$$

According to (IV.12), one can derive that

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}(1_{\Omega_5})\mathrm{d}s &\geq \liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}(1_{\Omega_1})\mathrm{d}s - \limsup_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}(1_{\Omega_4})\mathrm{d}s\\ &\geq \frac{((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\theta})^{1-p}(\check{\theta})^p)b\varpi^b}{8(\hat{\beta})^{1-p}(\check{\beta})^p}\Big((\hat{\mu})^{1-p}(\check{\mu})^p\\ &\quad + (\hat{\delta})^{1-p}(\check{\delta})^p + (\hat{r})^{1-p}(\check{r})^p\Big)\Big(\mathcal{R}_0^s - 1\Big) > 0 \quad \text{a.s.}\end{aligned}$$

Consequently, we have determined a compact domain $\Omega_5 \subset \mathbb{R}^3_+$ such that

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}\Big(u,(S(0),I(0),R(0)),\Omega_5\Big)\mathrm{d}s &\geq \frac{((\hat{\mu})^{1-p}(\check{\mu})^p + (\hat{\theta})^{1-p}(\check{\theta})^p)b\varpi^b}{8(\hat{\beta})^{1-p}(\check{\beta})^p}\Big((\hat{\mu})^{1-p}(\check{\mu})^p\\ &\quad + (\hat{\delta})^{1-p}(\check{\delta})^p + (\hat{r})^{1-p}(\check{r})^p\Big)\Big(\mathcal{R}_0^s - 1\Big) > 0 \quad \text{a.s.}\end{aligned}$$

Applying similar arguments to those in [135], we show the uniqueness of the ergodic stationary distribution of our model (IV.2), denoted by $\pi(\cdot)$. This completes the proof. ∎

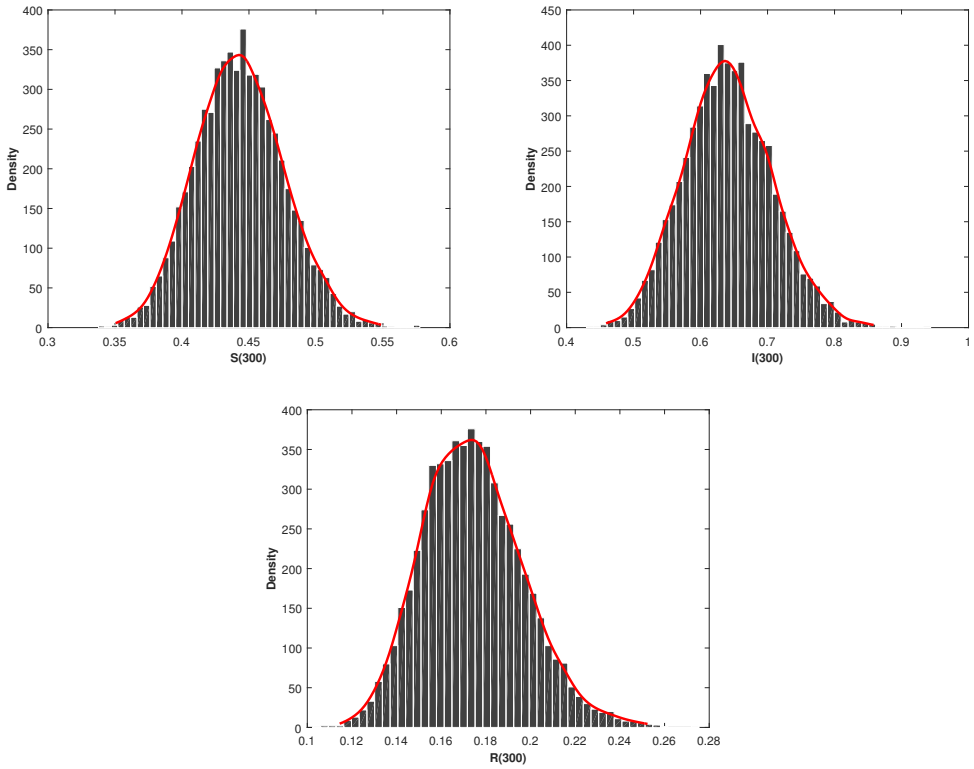


Figure IV.2: Histogram of the probability density function for S, I, and R population at $t = 300$ for the stochastic model (IV.2), the smoothed curves are the probability density functions of $S(t)$, $I(t)$ and $R(t)$, respectively. Here, we get $\mathcal{R}_0^s = 1.99 > 1$ for $p = 0.5$.

1.3 Numerical simulations

In this subsection, in order to show different dynamical results of the stochastic model (IV.1) under imprecise parameter values, we present some numerical simulations. The parameters values are given in Table IV.1.

Notation	Value	Source	Notation	Value
$\hat{A}$	0.75	Assumed	$\check{A}$	1.2
$\hat{\mu}$	0.09	[56]	$\check{\mu}$	0.2
$\hat{\beta}$	0.5	Assumed	$\check{\beta}$	0.8
$\hat{\delta}$	0.3	[56]	$\check{\delta}$	0.5
$\hat{r}$	0.1	[56]	$\check{r}$	0.3
$\hat{\theta}$	0.2	[56]	$\check{\theta}$	0.3
$\hat{\sigma}$	0.08	Theoretical	$\check{\sigma}$	0.1

Table IV.1: Parameters value used in numerical simulations.

For the purpose of showing the effects of imprecise parameters and Lévy noise on HBV dynamics, we have realized the simulation many times. We assume that $\eta(u) = 0.003$, $\mathcal{U} = (0, \infty)$ and $\nu(\mathcal{U}) = 1$. Noticing that the Assumption 1, is always held with parameters value in Table IV.1. From Figures IV.1, IV.2 and IV.3, we show the existence of the unique stationary distributions for $S(t)$, $I(t)$ and $R(t)$ of model (IV.2) at $t = 300$, where the smooth curves are the probability density functions of $S(t)$, $I(t)$ and $R(t)$, respectively. That is to say that the solution of the SDE model (IV.2) persists strongly. Furthermore, different values of the parameter imprecision p can crucially affect the persistence of HBV.

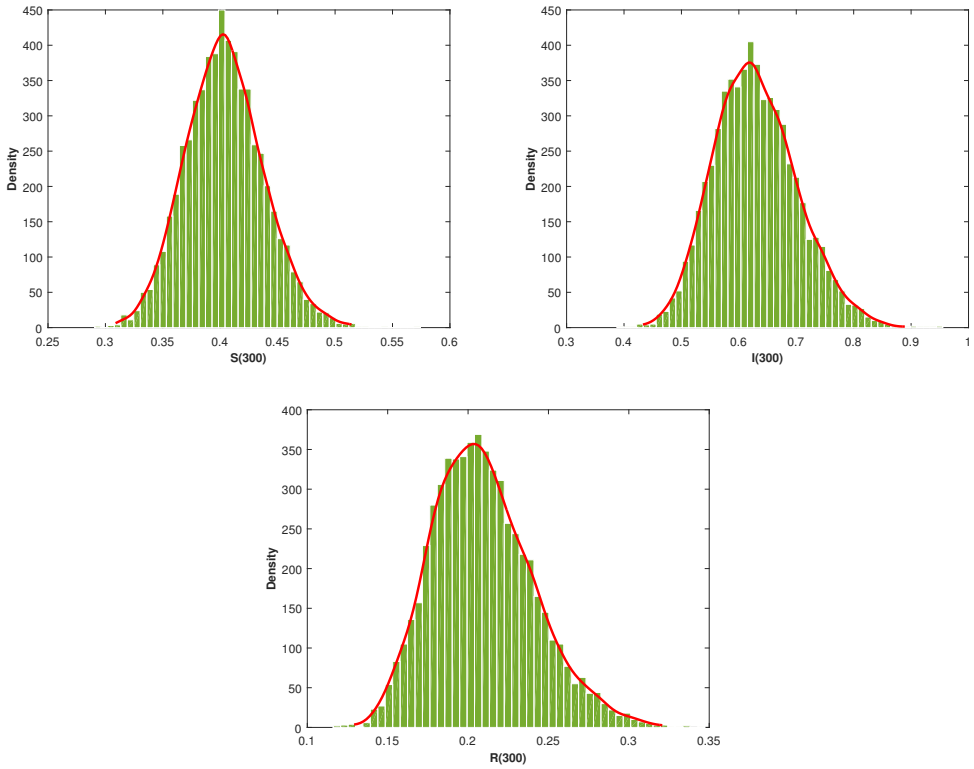


Figure IV.3: Histogram of the probability density function for S, I, and R population at $t = 300$ for the stochastic model (IV.2), the smoothed curves are the probability density functions of $S(t)$, $I(t)$ and $R(t)$, respectively. Here, we get $\mathcal{R}_0^s = 1.7398$ for $p = 1$.

1.4 Discussion

In the study of the dynamics of stochastic systems, the ergodicity is one of the most important and significant characteristics. For this purpose, we have used the mutually exclusive possibilities lemma to establish the sharp and optimal condition for the existence of the stationary distribution without employing the classical Lyapunov method. To ensure the realistic aspect of our model, we replaced constant parameters in the model (IV.1) by imprecise ones. Based on Theorem 4.2 in [57], for any π-integrable function $g : \mathbb{R}_+ \to \mathbb{R}$:

$$\mathbb{P}\left(\lim_{t\to+\infty} \frac{1}{t}\int_0^t g(X(s))\,\mathrm{d}s = \int_{\mathbb{R}_+} g(x)\pi(x)\,dx\right) = 1.$$

The ergodic property for HBV means that the stochastic model has a unique stationary distribution which predicts the survival of the infected population in the future. That means the HBV persists for all time regardless of the initial conditions [35]. Furthermore, the ergodic property grants a reason why the integral average of a solution of the system (IV.2) converges to a fixed point whilst the system may fluctuate around as time goes by.

2 Threshold analysis of the stochastic HBV epidemic model with Lévy Jumps

2.1 Introduction

In reality, the spread of a disease is characterized by randomness due to the unpredictable character of human contacts. From biological and mathematical perspectives, our main concern is to improve the model of Khan et al. [56], and include the unexpected stochastically effects by using the approach mentioned in [171]. We assume that the Lévy perturbations are directly proportional to

$S(t)$, $I(t)$, and $R(t)$. Then, we obtain the following stochastic HBV epidemic model:

$$\begin{cases} \mathrm{d}S(t) = \big(A - \beta S(t)I(t) - (\mu+\theta)S(t)\big)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \displaystyle\int_{\mathcal{U}} \eta_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}I(t) = \big(\beta S(t)I(t) - (\mu+\delta+r)I(t)\big)\mathrm{d}t + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \displaystyle\int_{\mathcal{U}} \eta_2(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}R(t) = \big(\delta I(t) + \theta S(t) - \mu R(t)\big)\mathrm{d}t + \sigma_3 R(t)\mathrm{d}\mathcal{W}_3(t) + \displaystyle\int_{\mathcal{U}} \eta_3(u)R(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \end{cases} \tag{IV.13}$$

In this section, we will establish the threshold parameter which determines the extinction and the persistence in the mean of the HBV. This threshold value coincides with the deterministic threshold $\mathcal{R}_0 = \frac{\beta A}{(\mu+\theta)(\mu+\delta+r)}$, in the absence of white noise and Lévy jumps. For the purpose of well analyzing our model (IV.13), it necessary that we make the following standard assumptions.

- **$(\mathcal{A}_1)$** We assume that the jump coefficients satisfy $\displaystyle\int_{\mathcal{U}} \eta_i^2(u)\nu(\mathrm{d}u) < \infty$ $(i = 1, 2, 3)$.
- **$(\mathcal{A}_2)$** For all $u \in \mathcal{U}$, we assume that $1 + \eta_i(u) > 0$, $(i = 1, 2, 3)$ and

$$\int_{\mathcal{U}} \big[\eta_i(u) - \ln(1+\eta_i(u))\big]\nu(\mathrm{d}u) < \infty.$$

For the proof of Lemmas 2.3 and 2.4, we add the following supplementary assumptions:

- **$(\mathcal{A}_3)$** We suppose that exists a constant $\kappa > 0$, such that $\displaystyle\int_{\mathcal{U}} \big[\ln(1+\eta_i(u))\big]^2\nu(\mathrm{d}u) < \kappa$.
- **$(\mathcal{A}_4)$** There exists $\varrho > 1$ such that $\mu - \frac{1}{2}(\varrho-1)\widetilde{\sigma}^2 - \frac{1}{\varrho}\lambda > 0$, where $\widetilde{\sigma}^2 = \sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2$, and

$$\lambda_p = \int_{\mathcal{U}} \Big\{\big[1 + \eta_1(u) \vee \eta_2(u) \vee \eta_3(u)\big]^p - 1 - p\big[\eta_1(u) \wedge \eta_2(u) \wedge \eta_3(u)\big]\Big\}\nu(\mathrm{d}u).$$

Remark 2.1. The condition **$(\mathcal{A}_1)$** and **$(\mathcal{A}_2)$** are necessary to prove the existence and uniqueness of the solution. Based on biological considerations, we need to add the conditions **$(\mathcal{A}_3)$** and **$(\mathcal{A}_4)$** to guarantee that the sufficient small intensity of Lévy jumps cannot exceed the environmental carrying capacity.

Since S, I and R in system (IV.13) denote the population densities, they should be nonnegative. To analyze the asymptotic properties of system (IV.13), the first step is to show that the solution of the system is unique, positive and global. Under assumptions **$(\mathcal{A}_1)$**-**$(\mathcal{A}_4)$**, we present the following result:

Theorem 2.2. *For any initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$, there exists a unique positive solution $(S(t), I(t), R(t))$ of system (IV.13) on $t \geq 0$, and the solution will remain in $\mathbb{R}^3_+$ with probability one.*

The proof of the last theorem is standard (see for example [58]). Hence, we also omit it here. Now, we list two results used in our analysis.

Lemma 2.3 ([172]). *Let $(S(t), I(t), R(t))$ be the solution of model (IV.13) with any positive initial value. Then,*

- $\displaystyle\lim_{t\to\infty} \frac{S(t)+I(t)+R(t)}{t} = 0$ a.s.
- $\displaystyle\lim_{t\to\infty} \frac{S(t)}{t} = 0$, $\displaystyle\lim_{t\to\infty} \frac{I(t)}{t} = 0$, *and* $\displaystyle\lim_{t\to\infty} \frac{R(t)}{t} = 0$ a.s.

Lemma 2.4 ([172]). *Let $(S(t), I(t), R(t))$ be the solution of model (IV.13) with any positive initial value. Then,*

$$\lim_{t\to\infty}\frac{\int_0^t\int_{\mathcal{U}}\eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0,\qquad \lim_{t\to\infty}\frac{\int_0^t\int_{\mathcal{U}}\eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0,$$
$$\text{and}\qquad \lim_{t\to\infty}\frac{\int_0^t\int_{\mathcal{U}}\eta_3(u)R(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0\quad \text{a.s.}$$

The proof of Lemmas 2.3 and 2.4 is almost the same as those in Lemmas 2.1 and 2.2 in [172]. Hence we omit it here.

2.2 Main results

2.2.1 The stochastic extinction

In this section, our main concern is to determine the condition for the extinction of the HBV epidemic. We define the threshold of our stochastic epidemic model (IV.13) as follows:

$$\mathcal{R}_0^{\mathrm{i}}=\frac{1}{(\mu+\delta+r)}\left(\frac{\beta A}{\mu+\theta}-\frac{\sigma_2^2}{2}-\int_{\mathcal{U}}[\eta_2(u)-\ln(1+\eta_2(u))]\nu(\mathrm{d}u)\right).$$

Theorem 2.5. *Let assumptions $(\mathcal{A}_1)$-$(\mathcal{A}_4)$ hold and let $(S(t), I(t), R(t))$ be the solution of system (IV.13) with any initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$. If $\mathcal{R}_0^{\mathrm{i}} < 1$, then*

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}\le(\mu+\delta+r)\left(\mathcal{R}_0^{\mathrm{i}}-1\right)<0\quad\text{a.s.}$$

That is to say, the HBV epidemic dies out exponentially with probability one. Furthermore, we have that

$$\lim_{t\to\infty}\langle S(t)\rangle=\frac{A}{\mu+\theta}\quad\text{a.s.}\qquad\text{and}\qquad\lim_{t\to\infty}\langle R(t)\rangle=\frac{\theta A}{\mu(\mu+\theta)}\quad\text{a.s.}$$

Proof. Integrating from 0 to t on both sides of the first and second equations of the proposed stochastic HBV model (IV.13), leads to the following equations

$$\frac{S(t)-S(0)}{t}=A-(\mu+\theta)\langle S(t)\rangle-\beta\langle S(t)I(t)\rangle+\frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_1(s)+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u),$$
$$\frac{I(t)-I(0)}{t}=\beta\langle S(t)I(t)\rangle-(\mu+\delta+r)\langle I(t)\rangle+\frac{\sigma_2}{t}\int_0^t I(s)\mathrm{d}\mathcal{W}_2(s)+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

Obviously, we obtain

$$\langle S(t)\rangle=\frac{A}{\mu+\theta}-\frac{\mu+\delta+r}{\mu+\theta}\langle I(t)\rangle+\psi_1(t),\tag{IV.14}$$

where $\psi_1(t)$ is defined by the following equation

$$\psi_1(t)=-\frac{1}{\mu+\theta}\left\{\frac{S(t)-S(0)}{t}+\frac{I(t)-I(0)}{t}-\frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_1(s)-\frac{\sigma_2}{t}\int_0^t I(s)\mathrm{d}\mathcal{W}_2(s)\right.$$
$$\left.-\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)-\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\right\}.$$

By Lemmas 2.3 and 2.4, we have

$$\lim_{t\to\infty}\psi_1(t)=0 \qquad \text{a.s.}$$

Making use Itô's formula to (IV.13), one can get that

$$\begin{aligned}\mathrm{d}\ln I(t)=&\left(\beta S(t)-(\mu+\delta+r)-\frac{1}{2}\sigma_2^2-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)\right)\mathrm{d}t\\&+\sigma_2\mathrm{d}\mathcal{W}_2(t)+\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).\end{aligned}\tag{IV.15}$$

Integrating (IV.15) from 0 to t on both sides, we get

$$\begin{aligned}\frac{\ln I(t)}{t}=&\beta\langle S(t)\rangle-(\mu+\delta+r)+\frac{\sigma_2\mathcal{W}_2(t)}{t}+\frac{\ln I(0)}{t}-\frac{1}{2}\sigma_2^2-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)\\&+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).\end{aligned}$$

Substituting the result (IV.14) into the previous equation, implies that

$$\begin{aligned}\frac{\ln I(t)}{t}&=\beta\left(\frac{A}{\mu+\theta}-\frac{\mu+\delta+r}{\mu+\theta}\langle I(t)\rangle+\psi_1(t)\right)-(\mu+\delta+r)-\frac{1}{2}\sigma_2^2+\frac{\ln I(0)}{t}\\&\quad-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)+\frac{\sigma_2\mathcal{W}_2(t)}{t}+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\&\le\frac{\beta A}{\mu+\theta}+\beta\psi_1(t)-(\mu+\delta+r_2)-\frac{1}{2}\sigma_2^2+\frac{\ln I(0)}{t}-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)\\&\quad+\frac{\sigma_2\mathcal{W}_2(t)}{t}+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

Let $\mathcal{M}_1(t)=\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)$. According to the hypothesis $(\mathcal{A}_3)$, we have

$$\langle\mathcal{M}_1,\mathcal{M}_1\rangle(t)=t\int_{\mathcal{U}}[\ln(1+\eta_2(u))]^2\nu(\mathrm{d}u)<\kappa t.$$

By the large number theorem for martingales, we get

$$\lim_{t\to\infty}\frac{\mathcal{M}_1(t)}{t}=0\quad\text{and}\quad\lim_{t\to\infty}\frac{\mathcal{W}_2(t)}{t}=0\qquad\text{a.s.}$$

Consequently, if the condition $\mathcal{R}_0^{\mathrm{i}}<1$ holds, we have

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}\le(\mu+\delta+r_2)\left(\mathcal{R}_0^{\mathrm{i}}-1\right)<0\quad\text{a.s.}$$

Therefore,

$$\lim_{t\to\infty}I(t)=0\quad\text{a.s.}\tag{IV.16}$$

By (IV.14) and (IV.16), one can see that

$$\lim_{t\to\infty}\langle S(t)\rangle=\frac{A}{\mu+\theta}\quad\text{a.s.}\tag{IV.17}$$

Next, we shall show that $\lim_{t\to\infty}\langle R(t)\rangle=\frac{\theta A}{\mu(\mu+\theta)}$ a.s. From system (IV.13), we have

$$\frac{R(t)-R(0)}{t}=\delta\langle I(t)\rangle+\theta\langle S(t)\rangle-\mu\langle R(t)\rangle+\frac{\sigma_4}{t}\int_0^t R(s)\mathrm{d}\mathcal{W}_3(s)+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_3(u)R(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

Making use of Lemmas 2.3 - 2.4, (IV.17) and (IV.16), we get the desired result. ∎

2.2.2 The disease persistence

In this section, we shall establish the condition for the persistence of the HBV epidemic.

Theorem 2.6. *Let assumptions $(\mathcal{A}_1)$-$(\mathcal{A}_4)$ hold and let $(S(t), I(t), R(t))$ be the solution of system (IV.13) with any initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$. If $\mathcal{R}_0^{\mathrm{i}} > 1$, then*

$$\lim_{t\to\infty} \langle I(t)\rangle = \widetilde{I} > 0 \quad \text{a.s.} \quad \lim_{t\to\infty} \langle S(t)\rangle = \widetilde{S} > 0 \quad \text{a.s.} \quad \text{and} \quad \lim_{t\to\infty} \langle R(t)\rangle = \widetilde{R} > 0 \quad \text{a.s.}$$

where

$$\begin{aligned}
\widetilde{I} &= \frac{1}{\beta}(\mu+\theta)\big(\mathcal{R}_0^{\mathrm{i}} - 1\big),\\
\widetilde{S} &= \frac{A}{\mu+\theta} - \frac{1}{\beta}(\mu+\delta+r)\big(\mathcal{R}_0^{\mathrm{i}} - 1\big),\\
\widetilde{R} &= \frac{\theta A}{\mu+\theta} + \left(\frac{\delta(\mu+\theta) - \theta(\mu+\delta+r)}{\beta}\right)\big(\mathcal{R}_0^{\mathrm{i}} - 1\big).
\end{aligned}$$

That is to say, the HBV epidemic will prevail if $\mathcal{R}_0^{\mathrm{i}} > 1$.

Proof. According to the Itô's formula, one can see that

$$\begin{aligned}
\frac{\ln I(t)}{t} &= \beta\left(\frac{A}{\mu+\theta} - \frac{\mu+\delta+r}{\mu+\theta}\langle I(t)\rangle + \psi_1(t)\right) - (\mu+\delta+r) - \frac{1}{2}\sigma_2^2 + \frac{\ln I(0)}{t} + \frac{\sigma_2 \mathcal{W}_2(t)}{t}\\
&\quad - \int_{\mathcal{U}} \eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u) + \frac{1}{t}\int_0^t \int_{\mathcal{U}} \ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u)\\
&= (\mu+\delta+r)\Big(\mathcal{R}_0^{\mathrm{i}} - 1\Big) - \beta\frac{\mu+\delta+r}{\mu+\theta}\langle I(t)\rangle + \beta\psi_1(t) + \frac{1}{t}\int_0^t \int_{\mathcal{U}} \ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u)\\
&\quad + \frac{\ln I(0)}{t} + \frac{\sigma_2 \mathcal{W}_2(t)}{t}.
\end{aligned}$$

By using Lemmas 2.3 - 2.4, we get

$$\lim_{t\to\infty} \langle I(t)\rangle = \frac{(\mu+\theta)}{\beta}\big(\mathcal{R}_0^{\mathrm{i}} - 1\big) > 0 \quad \text{a.s.}$$

From (IV.14), on can conclude that

$$\lim_{t\to\infty} \langle S(t)\rangle = \frac{A}{\mu+\theta} - \frac{\mu+\delta+r}{\mu+\theta}\lim_{t\to\infty} \langle I(t)\rangle.$$

Consequently,

$$\lim_{t\to\infty} \langle S(t)\rangle = \frac{A}{\mu+\theta} - \frac{1}{\beta}(\mu+\delta+r)\big(\mathcal{R}_0^{\mathrm{i}} - 1\big) > 0 \quad \text{a.s.}$$

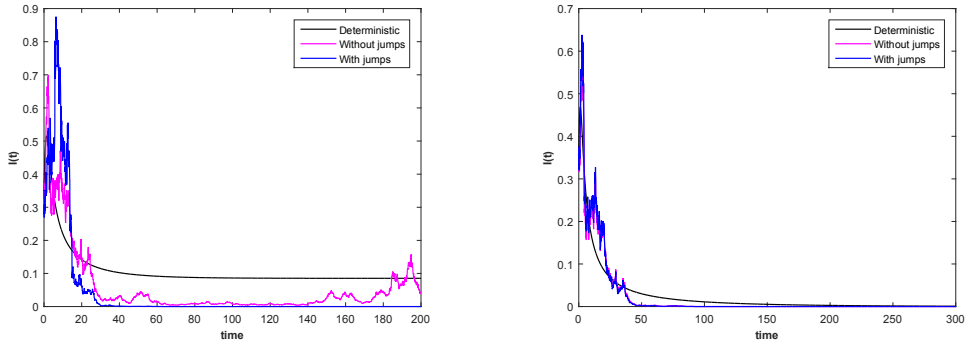
On the other hand, we have

$$\frac{R(t) - R(0)}{t} = \delta\langle I(t)\rangle + \theta\langle S(t)\rangle - \mu\langle R(t)\rangle \frac{\sigma_3}{t}\int_0^t R(s)\mathrm{d}\mathcal{W}_3(s) + \frac{1}{t}\int_0^t \int_{\mathcal{U}} \eta_3(u) R(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u).$$

Thus,

$$\lim_{t\to\infty} \langle R(t)\rangle = \frac{\theta A}{\mu(\mu+\theta)} + \left(\frac{\delta(\mu+\theta) - \theta(\mu+\delta+r)}{\beta\mu}\right)\big(\mathcal{R}_0^{\mathrm{i}} - 1\big) > 0 \quad \text{a.s.}$$

This completes the proof. ∎



(a) Case 1. (b) Case 2.

Figure IV.4: Numerical simulation of the paths $I(t)$ for the HBV epidemic model (IV.13) (with jumps), the trajectories of $I(t)$ for the system (IV.13) when $\eta_i = 0$ (without jumps) and the solution $I(t)$ of the corresponding deterministic HBV system.

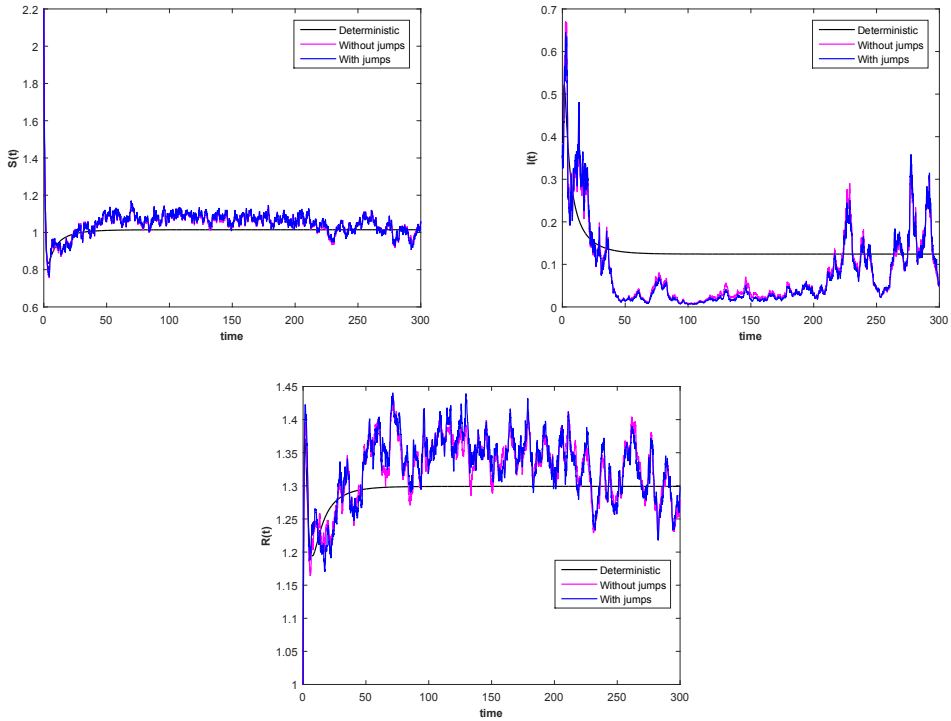


Figure IV.5: Numerical simulation of the paths $(S(t), I(t), R(t))$ for the HBV epidemic model (IV.13).

2.3 Numerical simulations

In this subsection, we shall use Euler numerical approximation to illustrate the rigor of our analytical results. The two examples are given below concern the results obtained in Theorems 2.5 and 2.6. Moreover, we numerically simulate the solution of a corresponding deterministic HBV system for the comparison.

Example 2.7. Choose $A = 1$, $\beta = 0.65$, $\mu = 0.4$, $\theta = 0.5$, $\delta = 0.1$, $r = 0.2$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\sigma_3 = 0.15$, $\eta_1 = 0.01$, $\eta_2 = 0.1$, $\eta_3 = 0.03$, $\mathcal{U} = (0, \infty)$ and $\nu(\mathcal{U}) = 1$. Then, $\mathcal{R}_0 = 1.0317 > 1$, $\mathcal{R}_0^s = 1.0032 > 1$ and $\mathcal{R}_0^{\dot{\imath}} = 0.9965 < 1$. The computer simulations illustrated by Figure IV.4 (a), support the result of Theorem 2.5. That is to say, the HBV in system (IV.13) (with jumps) dies out exponentially with probability one, although the disease in system (IV.13) without jumps persists. If we decrease β to 0.0.62, we get $\mathcal{R}_0 = 0.9841 < 1$, $\mathcal{R}_0^s = 0.9556 < 1$ and $\mathcal{R}_0^{\dot{\imath}} = 0.9489 < 1$. By Theorem 2.5, the HBV will tend to zero exponentially with probability one (see Figure IV.5 (b)).

Example 2.8. Let $\beta = 0.69$ and other parameters be the same as the previous example. Then, $\mathcal{R}_0 = 1.0952 > 1$, $\mathcal{R}_0^s = 1.0667 > 1$ and $\mathcal{R}_0^{\dot{\imath}} = 1.0600 > 1$. We can conclude, by Theorem 2.6, the solution $(S(t), I(t), R(t))$ of (IV.13) obeys that

$$\lim_{t\to\infty} \langle I(t)\rangle = 0.078 > 0 \quad \text{a.s.}$$
$$\lim_{t\to\infty} \langle S(t)\rangle = 1.050 > 0 \quad \text{a.s.}$$
$$\lim_{t\to\infty} \langle R(t)\rangle = 1.332 > 0 \quad \text{a.s.}$$

This means that HBV infection persists almost surely. The computer simulations showed in Figure IV.5 support the result 2.6 clearly.

2.4 Discussion

By comparing Figures IV.4 and IV.5, one can observe that Lévy jumps has a great effect on the extinction and persistence of the epidemic. A larger noise is more likely to suppress the spread of HBV. Furthermore, in the case of environmental disturbances, the dangers of diseases outbreak increase exponentially. It is crucial to encourage vaccination against transmitted of HBV.

3 Sharp conditions for the existence of a unique stationary distribution of HBV model with Lévy noise perturbation

3.1 Introduction

The threshold concept for the stochastic models generalizes the well-known one in the deterministic setting and provides important information about parameter relationships and the occurrence of disease outbreaks. Under appropriate assumptions, in the previous section, we have established that

$$\mathcal{R}_0^{\dot{\imath}} = \frac{1}{(\mu+\delta+r)}\left(\frac{\beta A}{\mu+\theta} - \frac{\sigma_2^2}{2} - \int_{\mathcal{U}} [\eta_2(u) - \ln(1+\eta_2(u))]\nu(du)\right),$$

is the stochastic threshold of the model (IV.13). If $\mathcal{R}_0^{\dot{\imath}} \le 1$, then

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le (\mu+\delta+r)\left(\mathcal{R}_0^J - 1\right) < 0 \quad \text{a.s.}$$

Moreover, we showed that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(s)\mathrm{d}s = \frac{A}{\mu+\theta} \quad \text{a.s.} \qquad \text{and} \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t R(s)\mathrm{d}s = \frac{\theta A}{\mu(\mu+\theta)} \quad \text{a.s.}$$

That is to say that the HBV epidemic dies out exponentially with probability 1. Otherwise, when $\mathcal{R}_0^{\mathrm{i}} > 1$, the HBV epidemic will prevail almost surely. Our former results lead to establishing that in the case of environmental disturbances, the danger of the HBV increase exponentially and it is important to apply appropriate strategies against its huge propagation. In this section, we extend our previous analysis by analyzing the ergodicity of the model (IV.13). The ergodic stationary distribution is an important dynamical property in the field of stochastic analysis [57]. It indicates the weak stability and offers a rigorous description of the stochastic persistence, which allows us a deeper grasp of the endemic state [35]. To the best of our knowledge, there are no results relative to the existence of a unique stationary distribution of model (IV.13) have been reported, and the effects of Lévy jumps on the existence of the ergodic stationary distribution are still unknown. The classical approach used to prove the ergodicity is the Lyapunov function analysis method proposed by Khasminskii [57]. However, in most cases, this approach can derive sufficient conditions only under additional assumptions that are not optimal (see for example [84]). In this section, we use a different method [162] to study this problem which gives sharp sufficient conditions by using the mutually exclusive possibilities lemma demonstrated in [128]. By assuming the assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ hold, we present the following result.

Lemma 3.1. *Assume that for some $p > 2$, $\chi_2 = \min\{\theta, \delta + r\} - \frac{(2p-1)}{2}\max\left\{\sigma_1^2, \sigma_2^2\right\} - \frac{1}{2p}\ell > 0$, where*

$$\ell_p = \int_{\mathcal{U}} \Big((1+\eta_1(u) \vee \eta_2(u))^{2p} - 1 - 2p\eta_1(u) \wedge \eta_2(u)\Big)\nu(\mathrm{d}u).$$

Let $(S(t), I(t), R(t))$ be the solution of (IV.13) with initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$. Then

1. $\mathbb{E}\big((S(t)+I(t))^{2p}(t)\big) \leq (S(0)+I(0))^{2p} e^{\{-p\chi_2 t\}} + \dfrac{2\chi_1}{\chi_2}$,
2. $\limsup\limits_{t\to+\infty} \dfrac{1}{t}\displaystyle\int_0^t \mathbb{E}\big((S(s)+I(s))^{2p}\big)\mathrm{d}s \leq \dfrac{2\chi_1}{\chi_2}$ *a.s., where* $\chi_1 = \sup\limits_{x>0}\left\{Ax^{2p-1} - \frac{\chi_2}{2}x^{2p}\right\}$.

Now, we consider the following stochastic subsystem:

$$\begin{cases} \mathrm{d}\mathcal{X}(t) = \big(A - (\mu+\theta)\mathcal{X}(t)\big)\mathrm{d}t + \sigma_1\mathcal{X}(t)\mathrm{d}\mathcal{W}_1(t) + \displaystyle\int_{\mathcal{U}} \eta_1(u)\mathcal{X}(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u) \quad \forall t > 0 \\ \mathcal{X}(0) = S(0). \end{cases} \tag{IV.18}$$

Lemma 3.2. *Let $\mathcal{X}(t)$ be the solution of system (IV.18) with an initial value $X(0) \in \mathbb{R}_+$. Then, the system (IV.18) admits a unique ergodic stationary distribution, denoted by $\pi(\cdot)$.*

Proof. Let $f(t) = \frac{1}{\mathcal{X}(t)}$. By using the Itô's lemma to (IV.18), we get

$$\begin{aligned} \mathrm{d}f(t) &= \left(\frac{-1}{\mathcal{X}^2(t)}\big(A - (\mu+\theta)\mathcal{X}(t)\big) + \frac{1}{\mathcal{X}(t)}\sigma_1^2 + \frac{1}{\mathcal{X}(t)}\int_{\mathcal{U}}\left(\frac{1}{1+\eta_1(u)} - 1 + \eta_1(u)\right)\nu(\mathrm{d}u)\right)\mathrm{d}t \\ &\quad - \frac{\sigma_1}{\mathcal{X}(t)}\mathrm{d}\mathcal{W}_1(t) + \frac{1}{X(t^-)}\int_{\mathcal{U}}\left(\frac{1}{1+\eta_1(u)} - 1\right)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u) \\ &= f(t)(\chi_3 - Af(t)) - \sigma_1 f(t)\mathrm{d}\mathcal{W}_1(t) + \int_{\mathcal{U}}\left(\frac{1}{1+\eta_1(u)} - 1\right)f(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u), \end{aligned} \tag{IV.19}$$

where

$$\chi_3 = (\mu+\theta) + \sigma_1^2\int_{\mathcal{U}}\left(\frac{1}{1+\eta_1(u)} - 1 + \eta_1(u)\right)\nu(\mathrm{d}u).$$

Plainly, the equation (IV.19) is the logistic growth model driven by lévy jumps. Simply, we verify that

$$\begin{aligned} \Lambda &= \chi_3 - \frac{\sigma_1^2}{2} - \int_{\mathcal{U}}\left(\frac{1}{1+\eta_1(u)} - 1 + \ln\frac{1}{1+\eta_1(u)}\right)\nu(\mathrm{d}u) \\ &= (\mu+\theta) + \frac{\sigma_1^2}{2} + \int_{\mathcal{U}}(\eta_1(u) - \ln(1+\eta_1(u))\nu(\mathrm{d}u) \\ &> 0. \end{aligned}$$

Tong et al. [135] proved that if $\Lambda > 0$, then there is a unique stationary distribution to equation (IV.19). By virtue of Theorem 1 and Remark 2 in [151] or Theorem 2.2 in [163], we deduce that the model (IV.18) admits a unique ergodic stationary distribution with support $\mathbb{R}_+$, denoted by $\pi(\cdot)$. ■

Remark 3.3. Based on Lemma 3.2 and the Theorem 3.3.1 in [113], for any π-integrable function: $g : \mathbb{R}_+ \to \mathbb{R}$

$$\mathbb{P}\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t g(X(s))\mathrm{d}s = \int_{\mathbb{R}_+} g(x)\pi(x)\mathrm{d}x\right) = 1.$$

Lemma 3.4. *[93] Assume that for some $p > 1$ such that $\chi_2 > 0$. Let $(S(t), I(t), R(t))$ be the positive solution of the system (IV.13) with any given initial condition $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$. Let also $X(t) \in \mathbb{R}_+$ be the solution of the equation (IV.18) with any given initial value $X(0) = S(0) \in \mathbb{R}_+$. Then*

1. $\lim_{t\to\infty}\frac{X(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{S(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{I(t)}{t} = 0 \quad$ a.s.
2. $\lim_{t\to\infty}\frac{\int_0^t\int_{\mathcal{U}}\eta_1(u)X(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0, \quad \lim_{t\to\infty}\frac{\int_0^t\int_{\mathcal{U}}\eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0,$
   $\lim_{t\to\infty}\frac{\int_0^t\int_{\mathcal{U}}\eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t} = 0$ a.s.

3.2 Main results

Assume that for some $p > 1$ and $\chi_2 > 0$. The aim of the following theorem is to give the condition of the unique stationary distribution. For simplicity, we define the parameter:

$$\mathcal{T}_0^{\mathrm{i}} = \frac{1}{(\mu+\delta+r)}\left(\beta\int_{\mathbb{R}^+} x\pi(x)\mathrm{d}x - \frac{\sigma_2^2}{2} - \int_{\mathcal{U}}\Big(\eta_2(u) - \ln(1+\eta_2(u))\Big)\nu(\mathrm{d}u)\right).$$

Theorem 3.5. *If $\mathcal{T}_0^{\mathrm{i}} > 1$, then the stochastic system (IV.13) admits a unique stationary distribution and it has the ergodic property for any initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$.*

Remark 3.6. Motivated by the approach used in the proof of Theorem 1.5, we readily prove the ergodicity of the stochastic model. To avoid redundancy, we omit the detailed demonstration. From Remark 1.8, we can conclude that

$$\int_{\mathbb{R}^+} x\pi(x)\mathrm{d}x = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathcal{X}(s)\mathrm{d}s = \frac{A}{\mu+\theta}.$$

Consequently $\mathcal{T}_0^{\mathrm{i}} = \mathcal{R}_0^{\mathrm{i}}$ which is the global threshold of the stochastic HBV model (IV.13).

3.3 Discussion

In the latter two sections, our contribution is that, epidemiologically, we generally answer this question: How do unexpected environment fluctuations affect the threshold of the HBV model? We find that these perturbations reduce the deterministic threshold of an HBV model. Put differently, the threshold of the stochastic HBV model is less than the threshold of the corresponding deterministic version. This means the Lévy stochastic perturbation of the transmission rate can suppress the spread of the disease. This is consistent with some existing studies.

Chapter V

Analysis of a stochastic SIRS epidemic model with vertical transmission and transfer from infectious to susceptible

Abstract

By integrating environmental disturbances and various periods of immunity, we analyze the dynamics of an epidemic model of the SIRS type. Initially, we consider a perturbed system with constant population size. This model is discussed in two cases: autonomous and non-autonomous. We establish sufficient conditions for extinction and persistence in the mean of a disease. In addition, we prove the global stability of the system under a suitable condition of random intensity. In the case of the time-dependent system, we show that there exists at least one positive periodic solution. In the second section, we aim to generalize the nonlinear incidence rate of a stochastic SIRS epidemic model. Our principal system is enriched with the hypothesis of varying population size. Under some conditions imposed on the intensity of the white noise perturbation, the global stability of the system is proven. Furthermore, the threshold of our model which determines the extinction and persistence of the disease is established. In the third section, we propose a new stochastic SIRS epidemic model by simultaneously introducing the stochastic transmission and the proportional perturbation. For the permanence case, the ergodicity of the perturbed system is proved by employing an alternative approach different from the Lyapunov method. Under the similar condition of the existence of the unique ergodic stationary distribution, the persistence in the mean of the epidemic is shown. When the basic reproduction number $\mathcal{R}_0 > 1$, the asymptotic behavior of the solution around the endemic equilibrium of the deterministic model is provided. For the extinction case, sufficient conditions for the disappearance of the epidemic are obtained. When $\mathcal{R}_0 \leq 1$, the analysis of the asymptotic behavior around the disease-free equilibrium of the deterministic model is also investigated. In order to make the readers understand our study better, we present some numerical simulations to illustrate our theoretical results.

1 Dynamical behavior of autonomous and non-autonomous SIRS epidemic model with vertical transmission and transfer from infectious to susceptible

1.1 Introduction

In this section, we treat an SIRS epidemic model with vertical transmission, transfer from infectious to susceptible, and constant population size which takes the following form:

$$\begin{cases} \mathrm{d}S(t) = \Big(b(S(t)+R(t)) - \beta S(t)I(t) - bS(t) + pbI(t) + \gamma_1 I(t) + kR(t)\Big)\mathrm{d}t, \\ \mathrm{d}I(t) = \Big(\beta I(t)S(t) - bI(t) - \gamma_1 I(t) - \gamma_2 I(t) + \underbrace{q}_{q=1-p} bI(t)\Big)\mathrm{d}t, \\ \mathrm{d}R(t) = \Big(\gamma_2 I(t) - (b+k)R(t)\Big)\mathrm{d}t. \end{cases} \tag{V.1}$$

The parameters involved in system (V.1) are all positive constants. Due to homogeneously mixed contact between individuals, we use the mass action rate βSI as an incidence rate of our epidemic model and we assume that the total population size $N(t) = S(t) + I(t) + R(t)$ is normalized to one. By using the same arguments presented in [73], we can easily establish that the basic reproduction number of system (V.1) is $\mathcal{R}_0 = \dfrac{\beta}{pb + \gamma_1 + \gamma_2}$. If $\mathcal{R}_0 \leq 1$, the deterministic model (V.1) has only the disease-free equilibrium $E^{\ominus}(1,0,0)$ which is asymptotically stable in the case of $\mathcal{R}_0 < 1$, and if $\mathcal{R}_0 > 1$, $E^{\ominus}$ becomes unstable and there exists an endemic equilibrium $E^{\circledast} = (S^{\circledast}, I^{\circledast}, R^{\circledast})$ which is globally asymptotically stable.

The deterministic model constructed above can be improved by taking into account the unpredictable biological conditions [59]. In the real world, biological phenomena are often affected by the environmental noise, and the nature of epidemic growth is inherently random due to the unpredictability of person-person contacts in the case of horizontal transmission, or in the other case of vertical transmission (mother-fetus) [38]. Consequently, there are many authors who studied stochastic epidemic models. They introduced random effects into models by different techniques (see e.g., [97]). In particular, the following three techniques are often used. The first one is parameters perturbation of the model [68, 165, 167]; the second one is the environmental noise that is proportional to the variables of the model [19, 107, 33]; and the last one is the vigor of the positive equilibrium of the deterministic model [87]. In this section, we assume that the transmission rate involved in the system (V.1) is not absolutely constant and may fluctuate around some average values. Both from a biological and mathematical perspective, the principal main of our study is to investigate how the stochastic character of human transmission affects the spread of diseases by studying the dynamical behavior of our stochastic model. We assume that the contact rate is perturbed by Gaussian white noise, which is presented by $\beta + \sigma\dot{\mathcal{W}}(t)$, where $\mathcal{W}(t)$ is a standard Brownian motion with intensity $\sigma > 0$. Then, we obtain the following autonomous SIRS epidemic model with the perturbation stochastic:

$$\begin{cases} \mathrm{d}S(t) = \Big(b(S(t)+R(t)) - \beta S(t)I(t) - bS(t) + pbI(t) + \gamma_1 I(t) + kR(t)\Big)\mathrm{d}t - \sigma S(t)I(t)\mathrm{d}\mathcal{W}(t), \\ \mathrm{d}I(t) = \Big(\beta S(t)I(t) - (pb + \gamma_1 + \gamma_2)I(t)\Big)\mathrm{d}t + \sigma S(t)I(t)\mathrm{d}\mathcal{W}(t), \\ \mathrm{d}R(t) = \Big(\gamma_2 I(t) - (b+k)R(t)\Big)\mathrm{d}t. \end{cases} \tag{V.2}$$

In the previous perturbed epidemic system, we only considered the constant parameters and neglected the time-dependent factor. However, the time-dependent factor plays a very crucial role in the dissemination of infectious diseases and the fluctuation has often observed in the incidence of many epidemics [39]. Recently, a few authors considered the seasonal fluctuations in epidemic models, such as [49, 45, 86]. For this reason, we also assume that all coefficients in system (V.2)

are T-periodic functions and can be presented by the following non-autonomous model:

$$\begin{cases} \mathrm{d}S(t) = \Big(b(t)(S(t)+R(t)) - \beta(t)S(t)I(t) - b(t)S(t) + p(t)b(t)I(t) + \gamma_1(t)I(t) + k(t)R(t)\Big)\mathrm{d}t \\ \qquad\qquad -\sigma(t)S(t)I(t)\mathrm{d}\mathcal{W}(t), \\ \mathrm{d}I(t) = \Big(\beta(t)S(t)I(t) - (p(t)b(t) + \gamma_1(t) + \gamma_2(t))I(t)\Big)\mathrm{d}t + \sigma(t)S(t)I(t)\mathrm{d}\mathcal{W}(t), \\ \mathrm{d}R(t) = \Big(\gamma_2(t)I(t) - (b(t)+k(t))R(t)\Big)\mathrm{d}t. \end{cases} \tag{V.3}$$

With reference to the existing results in the literature, our contributions are as follow:

- For the autonomous system (V.2), we give the sufficient conditions for the persistence and the extinction of a disease. Then, we give a condition of the moment exponential stability of $E^{\ominus}$.
- For the non-autonomous system (V.3), we establish sufficient conditions for the persistence and the extinction of a disease with seasonal variations. Then, we prove the existence of a positive periodic solution.

For simplicity, we use the following notations:

- If $f(t)$ is a bounded function on $[0,\infty)$, we define $\check{f} = \max\limits_{t\in[0,T]} f(t)$ and $\hat{f} = \min\limits_{t\in[0,T]} f(t)$.
- We consider a bounded set Δ as follows:

$$\Delta = \Big\{x = (x_1,x_2,x_3) :\ x_1 > 0,\ x_2 > 0,\ x_3 > 0 \ \text{ and } \ x_1+x_2+x_3 = 1\Big\}.$$

On the existence and uniqueness of a positive solution of systems (V.2) and (V.3), we present the following results:

Theorem 1.1. *For any initial value in Δ, there is a unique positive solution $(S(t),I(t),R(t))$ of system (V.2) on $t \geq 0$, and the solution will remain in $\mathbb{R}^3_+$ with probability one. That is to say, the solution $(S(t),I(t),R(t)) \in \mathbb{R}^3_+$ for all $t \geq 0$ almost surely.*

The proof of Theorem 1.1 is standard and hence is omitted here. In addition, from biological consideration, we can study the disease dynamics of model (V.2) in the bounded set Δ.

Corollary 1.2. *Consider the model (V.3), for any given initial value in Δ, there is a unique positive solution $(S(t),I(t),R(t))$ on $t \geq 0$ and $\mathbb{P}((S(t),I(t),R(t)) \in \Delta) = 1$.*

1.2 Main results

1.2.1 Dynamical behavior of autonomous system (V.2)

In this section, our main concern is to study the stochastic dynamics of the autonomous epidemic model (V.2). Given that the value of the deterministic threshold $\mathcal{R}_0$ characterizes the dynamical behaviors of the system (V.1) and guarantees persistence or extinction of the disease. Similarly, we define the threshold of our stochastic SIRS epidemic model (V.2) as follows:

$$\mathcal{R}_0^s = \frac{\beta}{pb + \gamma_1 + \gamma_2 + \frac{\sigma^2}{2}}.$$

To analyze the dynamic of system (V.2), it is sufficient to study the following system:

$$\begin{cases} \mathrm{d}I(t) = (\beta(1-I(t)-R(t))I(t) - (pb+\gamma_1+\gamma_2)I(t))\,\mathrm{d}t + \sigma(1-I(t)-R(t))I(t)\mathrm{d}\mathcal{W}(t), \\ \mathrm{d}R(t) = (\gamma_2 I(t) - (b+k)R(t))\,\mathrm{d}t. \end{cases} \tag{V.4}$$

Obviously, the region $\Gamma_0 = \big\{(I(t),R(t)) \in \mathbb{R}^2_+ :\ I(t)+R(t) < 1 \text{ for all } t \geq 0\big\}$ is a positively invariant domain corresponding to the model (V.4).

Theorem 1.3. *Let $(I(t), R(t))$ be the solution of system (V.4) with any initial value in Γ_0. If one of the following conditions is satisfied,*
(a) $\sigma^2 \le \beta$ *and* $\mathcal{R}_0^s < 1$,
(b) $\sigma^2 > \beta$ *and* $\frac{\beta^2}{2\sigma^2} - (pb + \gamma_1 + \gamma_2) < 0$,
then, $I(t)$ tend to zero exponentially a.s.

Proof. It follows from Itô's formula that

$$\begin{aligned} \mathrm{d}\ln I(t) &= \left(\beta(1 - I(t) - R(t)) - (pb + \gamma_1 + \gamma_2) - \frac{\sigma^2}{2}(1 - I(t) - R(t))^2\right)\mathrm{d}t + \sigma(1 - I(t) - R(t))\mathrm{d}\mathcal{W}(t) \\ &\le \left(\beta(1 - I(t) - R(t)) - (pb + \gamma_1 + \gamma_2) - \frac{\sigma^2}{2}(1 - 2I(t) - 2R(t))\right)\mathrm{d}t + \sigma(1 - I(t) - R(t))\mathrm{d}\mathcal{W}(t) \\ &\le \left(\beta - \left(pb + \gamma_1 + \gamma_2 + \frac{\sigma^2}{2}\right) - (\beta - \sigma^2)(I(t) + R(t))\right)\mathrm{d}t + \sigma(1 - I(t) - R(t))\mathrm{d}\mathcal{W}(t). \end{aligned}$$

Let $\mathcal{M}(t) = \int_0^t \sigma(1 - I(s) - R(s))\mathrm{d}\mathcal{W}(s)$. It is a real-valued continuous local martingale, and its quadratic variation is

$$\langle \mathcal{M}, \mathcal{M}\rangle_t = \int_0^t \sigma^2(1 - I(s) - R(s))^2\mathrm{d}s \le \sigma^2 t,$$

which implies $\lim_{t\to\infty} \frac{M(t)}{t} = 0$ a.s. according to the strong law of large numbers for martingales.
If $\sigma^2 \le \beta$, then

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le \left(pb + \gamma_1 + \gamma_2 + \frac{\sigma^2}{2}\right)(\mathcal{R}_0^s - 1) < 0 \ \text{ a.s.}$$

While if $\sigma^2 > \beta$, then

$$\mathrm{d}\ln I(t) \le \left(\frac{\beta^2}{2\sigma^2} - (pb + \gamma_1 + \gamma_2)\right)\mathrm{d}t + \sigma(1 - I(t) - R(t))\mathrm{d}\mathcal{W}(t).$$

Integrating it from 0 to t and taking the limit superior of both sides, we obtain

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le \frac{\beta^2}{2\sigma^2} - (pb + \gamma_1 + \gamma_2) < 0 \ \text{ a.s.}$$

This completes the proof. ∎

Lemma 1.4. *Consider the equation*

$$x(t) = x(t_0) + \int_{t_0}^t f(s, x(s))\mathrm{d}s + \int_{t_0}^t g(s, x(s))\mathrm{d}\mathcal{W}(s), \quad x \in \mathbb{R}^n, \quad t_0 \in \mathbb{R}.$$

Suppose that:

1. *The functions f and g satisfy the local Lipschitz and linear growth conditions;*
2. $\sup_{t\ge 0}\{\mathbb{E}|x(t)|^p\} < \infty$, *where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$.*

Then almost every sample path of $\int_{t_0}^t g(s, x(s))\mathrm{d}\mathcal{W}(s)$ is uniformly continuous on $t \ge 0$.

Theorem 1.5. *Suppose that one of the conditions in Theorem 1.3 holds. Then, the solution $(S(t), I(t), R(t))$ converge almost surely to the disease-free equilibrium $(1, 0, 0)$ in $\bar{\Delta}$.*

The following lemma is needed for the proof of Theorem 1.5.

Lemma 1.6. *([25]) Let f be a non-negative function defined on $[0,\infty)$ that is integrable on $[0,\infty)$ and uniformly continuous on $[0,\infty)$. Then, $\lim_{t\to\infty} f(t) = 0$.*

Proof of Theorem 1.5. From the first equation of the system (V.2), we can write

$$\mathrm{d}\,(1 - S(t)) = \Big(- b\,(1 - S(t)) + \beta S(t)I(t) + bI(t) - (pb + \gamma_1)I - kR(t)\Big)\mathrm{d}t + \sigma S(t)I(t)\mathrm{d}\mathcal{W}(t).$$

Or in integrated form as

$$1 - S(t) = 1 - S(0) - \int_0^t \Big[b\,(1 - S(s)) + (pb + \gamma_1)I(s) + kR(s)\Big]\mathrm{d}s + \int_0^t \Big[\beta S(s)I(s) + bI(s)\Big]\mathrm{d}s + \mathcal{M}(t), \tag{V.5}$$

where $\mathcal{M}(t) = \sigma \int_0^t S(s)I(s)\mathrm{d}\mathcal{W}(s)$. Clearly, $\mathcal{M}(t)$ is a continuous local martingale with $\mathcal{M}(0) = 0$. Define

$$X(t) = X(0) - U(t) + A(t) + \mathcal{M}(t),$$

where

$$\begin{aligned} X(0) &= 1 - S(0), \\ U(t) &= \int_0^t \Big(b\,(1 - S(s)) + (pd + \gamma_1)I(s) + kR(s)\Big)\mathrm{d}s, \\ A(t) &= \int_0^t \Big(\beta S(s)I(s) + bI(s)\Big)\mathrm{d}s. \end{aligned}$$

By (V.5), we have $X(t) \geq 0$ a.s. for all $t \geq 0$. It is clear that $A(t)$ and $U(t)$ are continuous adapted increasing processes on $t \geq 0$ with $A(0) = U(0) = 0$. From Theorem 1.1, we have in Δ

$$\lim_{t\to\infty} \int_0^t \Big(\beta S(s)I(s) + bI(s)\Big)\mathrm{d}s \leq \lim_{t\to\infty} \int_0^t (\beta + b)I(s)\mathrm{d}s \leq \int_0^\infty (\beta + b)\mathfrak{C}_1 e^{-\mathfrak{C}_2 s}\mathrm{d}s < \infty.$$

Consequently, we get

$$\lim_{t\to\infty} (1 - S(t)) < \infty \quad \text{a.s.} \tag{V.6}$$

Now, we prove that $S(t)$ is uniformly continuous. Since $x(t) = (S(t), I(t), R(t)) \in \Delta$, the coefficients of (V.2) satisfy the local Lipschitz and linear growth conditions. Moreover, we have

$$\sup_{t\geq 0}\Big\{\mathbb{E}\Big(\sqrt{S^2(t) + I^2(t) + R^2(t)}\Big)^p\Big\} \leq 1.$$

From Lemma 1.4, $x(t)$ is uniformly continuous, and hence $S(t)$ is uniformly continuous. By applying Lemma 1.6 to (V.6), we get $\lim_{t\to\infty} S(t) = 1$ a.s. Since the solution is positive and $S(t) + I(t) + R(t) = 1$, one can conclude that $(S(t), I(t), R(t))$ converges almost surely to the disease-free equilibrium $(1, 0, 0)$ in $\bar{\Delta}$. ∎

Theorem 1.7. *Let $(I(t), R(t))$ be a solution of system (V.4) with any initial value in Γ_0. If $\mathcal{R}_0^s > 1$, then*

$$\begin{aligned} \liminf_{t\to\infty} \langle I(t)\rangle &\geq \frac{b + k}{\beta(b + k + \gamma_2)}\left(pb + \gamma_1 + \gamma_2 + \frac{\sigma^2}{2}\right)(\mathcal{R}_0^s - 1) > 0 \quad \text{a.s.} \\ \liminf_{t\to\infty} \langle R(t)\rangle &\geq \frac{\gamma_2}{\beta(b + k + \gamma_2)}\left(pb + \gamma_1 + \gamma_2 + \frac{\sigma^2}{2}\right)(\mathcal{R}_0^s - 1) > 0 \quad \text{a.s.} \end{aligned}$$

Proof. Integrating the second equation of system (V.4) yields

$$\frac{R(t)-R(0)}{t} = \gamma_2\langle I(t)\rangle - (b+k)\langle R(t)\rangle.$$

Then,

$$\langle R(t)\rangle = \frac{1}{b+k}\left(\gamma_2\langle I(t)\rangle - \frac{R(t)-R(0)}{t}\right). \tag{V.7}$$

On the other hand, as the proof of Theorem 1.3, we can conclude that

$$\begin{aligned}
\mathrm{d}\ln I(t) &= \left(\beta(1-I(t)-R(t)) - (pb+\gamma_1+\gamma_2) - \frac{\sigma^2}{2}(1-I(t)-R(t))^2\right)\mathrm{d}t + \mathrm{d}\mathcal{M}(t)\\
&\geq \left(\beta(1-I(t)-R(t)) - (pb+\gamma_1+\gamma_2) - \frac{\sigma^2}{2}\right)\mathrm{d}t + \mathrm{d}\mathcal{M}(t)\\
&= \left(\beta - \left(pb+\gamma_1+\gamma_2+\frac{\sigma^2}{2}\right) - \beta(I(t)+R(t))\right)\mathrm{d}t + \mathrm{d}\mathcal{M}(t).
\end{aligned}$$

Integrating it from 0 to t and dividing by t on both sides, we get

$$\begin{aligned}
\frac{\ln I(t)}{t} &\geq \left(\beta - \left(pb+\gamma_1+\gamma_2+\frac{\sigma^2}{2}\right)\right) - \beta\langle I(t)\rangle - \beta\langle R(t)\rangle + \frac{\mathcal{M}(t)}{t} + \frac{\ln I(0)}{t}\\
&\geq \left(\beta - \left(pb+\gamma_1+\gamma_2+\frac{\sigma^2}{2}\right)\right) - \beta\langle I(t)\rangle - \frac{\beta}{b+k}\left(\gamma_2\langle I(t)\rangle - \frac{R(t)-R(0)}{t}\right) + \frac{\mathcal{M}(t)}{t} + \frac{\ln I(0)}{t}\\
&= \left(\beta - \left(pb+\gamma_1+\gamma_2+\frac{\sigma^2}{2}\right)\right) - \beta\left(1+\frac{\gamma_2}{b+k}\right)\langle I(t)\rangle + \frac{\beta(R(t)-R(0))}{t(b+k)} + \frac{\mathcal{M}(t)}{t} + \frac{\ln I(0)}{t}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\liminf_{t\to\infty}\langle I(t)\rangle &\geq \frac{b+k}{\beta(b+k+\gamma_2)}\left(\beta - \left(pb+\gamma_1+\gamma_2+\frac{\sigma^2}{2}\right)\right)\\
&= \frac{b+k}{\beta(b+k+\gamma_2)}\left(pb+\gamma_1+\gamma_2+\frac{\sigma^2}{2}\right)\left(\mathcal{R}_0^s - 1\right) > 0 \quad \text{a.s.}
\end{aligned}$$

Finally, it follows from (V.7) that

$$\liminf_{t\to\infty}\langle R(t)\rangle \geq \frac{\gamma_2}{\beta(b+k+\gamma_2)}\left(pb+\gamma_1+\gamma_2+\frac{\sigma^2}{2}\right)\left(\mathcal{R}_0^s - 1\right) > 0 \quad \text{a.s.}$$

This finishes the proof. ∎

In order to obtain the stability conditions, we have the following inequalities:

Lemma 1.8. *Let $h \geq 2$ and $\epsilon, x, y > 0$. Then*

$$\begin{aligned}
x^{h-1}y &\leq \frac{(h-1)\epsilon}{h}x^h + \frac{1}{h\epsilon^{h-1}}y^h,\\
x^{h-2}y^2 &\leq \frac{(h-2)\epsilon}{h}x^h + \frac{2}{h\epsilon^{\frac{h-2}{2}}}y^h.
\end{aligned}$$

Now, we get the sufficient conditions for the moment exponential stability which are given by the following theorem:

Theorem 1.9. *Let $h \geq 2$. If the condition $\sigma^2 < \dfrac{2(pb+\gamma_1+\gamma_2-\beta)}{h-1}$ holds, the disease-free equilibrium P^0 of the system (V.2) is h^{th} moment exponentially stable in Δ.*

Proof. Let $h \geq 2$. Now, we consider the following Lyapunov function

$$\mathcal{V}_2(X) = \lambda_1 (1-S)^h + \frac{1}{h} I^h + \lambda_2 R^h, \qquad \text{for all } X = (S, I, R) \in \Delta.$$

where λ_i, $(i = 1, 2)$ are real positive constants to be determined later. It is easy to check that the first condition of Theorem 10.9 is true. Furthermore

$$\begin{aligned}\mathcal{L}\mathcal{V}_2 =& -h\lambda_1 (1-S)^h b + h\lambda_1 (1-S)^{h-1} \beta IS - h\lambda_1 (pb+\gamma_1) (1-S)^{h-1} I \\ &+ h\lambda_1 b (1-S)^{h-1} I - h\lambda_1 k (1-S)^{h-1} R + [\beta S - (pb+\gamma_1+\gamma_2)] I^h \\ &+ h\lambda_2\gamma_2 R^{h-1} I - h\lambda_2 (b+k) R^h + \frac{\lambda_1 h(h-1)}{2} (1-S)^{h-2} \sigma^2 S^2 I^2 \\ &+ \frac{(h-1)}{2} \sigma^2 I^h S^2.\end{aligned}$$

In Δ, we have

$$\begin{aligned}\mathcal{L}\mathcal{V}_2 \leq& -h\lambda_1 (1-S)^h b + \left[h\lambda_1 (\beta + b)\right] (1-S)^{h-1} I + \frac{h(h-1)\sigma^2}{2} \lambda_1 (1-S)^{h-2} I^2 \\ &- \left((pb+\gamma_1+\gamma_2-\beta) - \frac{(h-1)\sigma^2}{2} \right) I^h + h\lambda_2\gamma_2 R^{h-1} I - h\lambda_2 (b+k) R^h.\end{aligned}$$

By using Lemma 1.8, we get

$$\begin{aligned} I (1-S)^{h-1} &\leq \frac{(h-1)\epsilon}{h} (1-S)^h + \frac{1}{h\epsilon^{h-1}} I^h, \\ I^2 (1-S)^{h-2} &\leq \frac{(h-2)\epsilon}{h} (1-S)^h + \frac{2}{h\epsilon^{\frac{h-2}{2}}} I^h, \\ I R^{h-1} &\leq \frac{(h-1)\epsilon}{h} R^h + \frac{1}{h\epsilon^{h-1}} I^h. \end{aligned}$$

Then

$$\begin{aligned}\mathcal{L}\mathcal{V}_2 \leq& -\left(hb - \left((b+\beta)(h-1) + \frac{\sigma^2 (h-1)(h-2)}{2} \right) \epsilon \right) \lambda_1 (1-S)^h \\ &+ \left[\left((b+\beta)\epsilon^{1-h} + \frac{(h-1)\sigma^2}{2} \epsilon^{\frac{2-h}{h}} \right) \lambda_1 + \gamma_2 \epsilon^{1-h} \lambda_2 - (pb+\gamma_1+\gamma_2-\beta) - \frac{(h-1)\sigma^2}{2} \right] I^h \\ &- \left(h(b+k) - (h-1)\gamma_2 \epsilon \right) \lambda_2 R^h.\end{aligned}$$

We chose ϵ sufficiently small such that the coefficients of $(1-S)^h$ and R^h be negative, and as $(pb+\gamma_1+\gamma_2-\beta) - \frac{(h-1)\sigma^2}{2} > 0$, we can choose λ_1 and λ_2 positive such the coefficient of I^h be negative. According to Theorem 10.9, the proof is completed. ∎

Under Theorem 1.9, we have in the case $h = 2$ the following corollary.

Corollary 1.10. *If the condition $\sigma^2 < 2(pb+\gamma_1+\gamma_2-\beta)$ holds, the disease free equilibrium P^0 of the system (V.2) is globally asymptotically stable in Δ.*

Remark 1.11. The global stability signifies that the dynamic system will reach to the equilibrium value from any initial value. Biologically, the global stability of free disease equilibrium may be interpreted as the inescapable fate of the epidemic situation regardless of its initial situation. But this inevitability happens as long as the community strictly follows the associated mathematical model of an epidemic. Furthermore, the h^{th} moment exponential stability means that the h^{th} moment of the state will tend to the equilibrium exponentially fast.

1.2.2 Dynamical behavior of the non-autonomous system (V.3)

To study the dynamic of system (V.3), it is sufficient to analyze the following system:

$$\begin{cases} \mathrm{d}I(t) = (\beta(t)(1-I(t)-R(t))I(t) - (p(t)b(t)+\gamma_1(t)+\gamma_2(t))I(t))\,\mathrm{d}t + \sigma(t)(1-I(t)-R(t))I(t)\mathrm{d}\mathcal{W}(t), \\ \mathrm{d}R(t) = (\gamma_2(t)I(t) - (b(t)+k(t))R(t))\,\mathrm{d}t. \end{cases} \tag{V.8}$$

Similarly, we define the following threshold of our stochastic epidemic model (V.8)

$$\langle \mathcal{R}_0^s \rangle_T = \frac{\langle \beta \rangle_T}{\langle pb + \gamma_1 + \gamma_2 + \sigma^2/2 \rangle_T}.$$

Theorem 1.12. *Let $(I(t), R(t))$ be a solution of system (V.8) with any initial value in Γ_0. If one of the following conditions is satisfied,*
(a) $\sigma^2(t) \le \beta(t)$ *for all* $t \ge 0$, *and* $\langle \mathcal{R}_0^s \rangle_T < 1$,
(b) $\sigma^2(t) > \beta(t)$ *for all* $t \ge 0$, *and* $\left\langle \frac{\beta^2}{2\sigma^2} - (pb+\gamma_1+\gamma_2) \right\rangle_T < 0$,
then $I(t)$ tends to zero exponentially a.s.

Proof. It follows from Itô's formula that

$$\begin{aligned} \mathrm{d}\ln I(t) &= \left(\beta(t)(1-I(t)-R(t)) - (p(t)b(t)+\gamma_1(t)+\gamma_2(t)) - \frac{\sigma^2(t)}{2}(1-I(t)-R(t))^2\right)\mathrm{d}t \\ &\quad + \sigma(t)(1-I(t)-R(t))\mathrm{d}\mathcal{W}(t) \\ &\le \left(\beta(t)(1-I(t)-R(t)) - (p(t)b(t)+\gamma_1(t)+\gamma_2(t)) - \frac{\sigma^2(t)}{2}(1-2I(t)-2R(t))\right)\mathrm{d}t \\ &\quad + \sigma(t)(1-I(t)-R(t))\mathrm{d}\mathcal{W}(t) \\ &\le \left(\beta(t) - \left(p(t)b(t)+\gamma_1(t)+\gamma_2(t)+\frac{\sigma^2(t)}{2}\right) - (\beta(t)-\sigma^2(t))(I(t)+R(t))\right)\mathrm{d}t \\ &\quad + \sigma(t)(1-I(t)-R(t))\mathrm{d}\mathcal{W}(t). \end{aligned}$$

Let $\mathcal{M}(t) := \int_0^t \sigma(s)(1-I(s)-R(s))\mathrm{d}\mathcal{W}(s)$. It is a real-valued continuous local martingale, and its quadratic variation is $\langle \mathcal{M}, \mathcal{M} \rangle_t = \int_0^t \sigma^2(s)(1-I(s)-R(s))^2\mathrm{d}s \le \check{\sigma}^2 t$, which implies $\lim_{t\to\infty} \frac{\mathcal{M}(t)}{t} = 0$ a.s. according to the strong law of large numbers for martingales.

1. If $\sigma^2(t) \le \beta(t)$ for all $t \ge 0$. Then

$$\begin{aligned} \limsup_{t\to\infty} \frac{\ln I(t)}{t} &\le \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(\beta(s) - \left(p(s)b(s)+\gamma_1(s)+\gamma_2(s)+\frac{\sigma^2(s)}{2}\right)\right)\mathrm{d}s \\ &= \frac{1}{T}\int_0^T \left(\beta(s) - \left(p(s)b(s)+\gamma_1(s)+\gamma_2(s)+\frac{\sigma^2(s)}{2}\right)\right)\mathrm{d}s \\ &= \langle pb+\gamma_1+\gamma_2+\sigma^2/2\rangle_T \left(\langle \mathcal{R}_0^s\rangle_T - 1\right) < 0 \quad \text{a.s.} \end{aligned}$$

2. If $\sigma^2(t) > \beta(t)$, then

$$\mathrm{d}\ln I(t) \le \left(\frac{\beta^2(t)}{2\sigma^2(t)} - (p(t)b(t)+\gamma_1(t)+\gamma_2(t))\right)\mathrm{d}t + \sigma(t)(1-I(t)-R(t))\mathrm{d}\mathcal{W}(t).$$

Integrating this from 0 to t and taking the limit superior of both sides, we obtain

$$\begin{aligned} \limsup_{t\to\infty} \frac{\ln I(t)}{t} &\le \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(\frac{\beta^2(s)}{2\sigma^2(s)} - (p(s)b(s)+\gamma_1(s)+\gamma_2(s))\right)\mathrm{d}s \\ &= \frac{1}{T}\int_0^T \left(\frac{\beta^2(s)}{2\sigma^2(s)} - (p(s)b(s)+\gamma_1(s)+\gamma_2(s))\right)\mathrm{d}s \\ &= \left\langle \frac{\beta^2}{2\sigma^2} - (pb+\gamma_1+\gamma_2)\right\rangle_T < 0 \quad \text{a.s.} \end{aligned}$$

This completes the proof. ∎

Theorem 1.13. *Let $(I(t), R(t))$ be a solution of system (V.8) with any initial value in Γ_0. If $\langle \mathcal{R}_0^s \rangle_T > 1$, then*

$$\liminf_{t\to\infty} \langle I(t) \rangle \geq \frac{\hat{b}+\hat{k}}{\check{\beta}(\hat{b}+\hat{k}+\check{\gamma}_2)} \langle pb+\gamma_1+\gamma_2+\sigma^2/2 \rangle_T \left(\langle \mathcal{R}_0^s \rangle_T - 1 \right) > 0 \quad a.s.$$

$$\liminf_{t\to\infty} \langle R(t) \rangle \geq \frac{\hat{\gamma}_2(\hat{b}+\hat{k})}{\check{\beta}(\check{b}+\check{k})(\hat{b}+\hat{k}+\check{\gamma}_2)} \langle pb+\gamma_1+\gamma_2+\sigma^2/2 \rangle_T \left(\langle \mathcal{R}_0^s \rangle_T - 1 \right) > 0 \quad a.s.$$

Proof. Integrating the second equation of system (V.8) yields

$$\frac{R(t)-R(0)}{t} = \langle \gamma_2 I \rangle_t - \langle (b+k)R \rangle_t \leq \check{\gamma}_2 \langle I \rangle_t - (\hat{b}+\hat{k}) \langle R \rangle_t,$$

then,

$$\langle R \rangle_t \leq \frac{1}{\hat{b}+\hat{k}} \left(\check{\gamma}_2 \langle I \rangle_t - \frac{R(t)-R(0)}{t} \right).$$

On the other hand, we can conclude that

$$\begin{aligned}
\mathrm{d}\ln I(t) &= \left(\beta(t)(1-I(t)-R(t)) - (p(t)b(t)+\gamma_1(t)+\gamma_2(t)) - \frac{1}{2}\sigma^2(t)(1-I(t)-R(t))^2 \right) \mathrm{d}t + \mathrm{d}\mathcal{M}(t) \\
&\geq \left(\beta(t)(1-I(t)-R(t)) - (p(t)b(t)+\gamma_1(t)+\gamma_2(t)) - \frac{\sigma^2(t)}{2} \right) \mathrm{d}t + \mathrm{d}\mathcal{M}(t) \\
&= \left(\beta(t) - \left(p(t)b(t)+\gamma_1(t)+\gamma_2(t)+\frac{\sigma^2(t)}{2} \right) - \beta(t)(I(t)+R(t)) \right) \mathrm{d}t + \mathrm{d}\mathcal{M}(t),
\end{aligned}$$

where $\mathcal{M}(t)$ is defined as in the proof of Theorem 1.12. Integrating from 0 to t and dividing by t on both sides, we get

$$\begin{aligned}
\frac{\ln I(t)}{t} &\geq \langle \beta - \left(pb+\gamma_1+\gamma_2+\sigma^2/2 \right) \rangle_t - \check{\beta} \langle I \rangle_t - \check{\beta} \langle R \rangle_t + \frac{\mathcal{M}(t)}{t} + \frac{\ln I(0)}{t} \\
&\geq \langle \beta - (pb+\gamma_1+\gamma_2+\sigma^2/2) \rangle_t - \check{\beta} \langle I \rangle_t - \frac{\check{\beta}}{\hat{b}+\hat{k}} \left(\check{\gamma}_2 \langle I \rangle_t - \frac{R(t)-R(0)}{t} \right) + \frac{\mathcal{M}(t)}{t} + \frac{\ln I(0)}{t} \\
&= \langle \beta - (pb+\gamma_1+\gamma_2+\sigma^2/2) \rangle_t - \check{\beta} \left(1 + \frac{\check{\gamma}_2}{\hat{b}+\hat{k}} \right) \langle I \rangle_t + \frac{\check{\beta}(R(t)-R(0))}{t(\hat{b}+\hat{k})} + \frac{\mathcal{M}(t)}{t} + \frac{\ln I(0)}{t}.
\end{aligned}$$

Therefore,

$$\liminf_{t\to\infty} \langle I \rangle_t \geq \frac{\hat{b}+\hat{k}}{\check{\beta}(\hat{b}+\hat{k}+\check{\gamma}_2)} \langle pb+\gamma_1+\gamma_2+\sigma^2/2 \rangle_T \left(\langle \mathcal{R}_0^s \rangle_T - 1 \right) > 0 \quad \text{a.s.}$$

Finally, it follows from the system (V.8) that

$$(\check{b}+\check{k}) \langle R \rangle_t \geq \langle (b+k)R \rangle_t = \langle \gamma_2 I \rangle_t - \frac{R(t)-R(0)}{t} \geq \hat{\gamma}_2 \langle I \rangle_t - \frac{R(t)-R(0)}{t}.$$

Then

$$\liminf_{t\to\infty} \langle R \rangle_t \geq \frac{\hat{\gamma}_2(\hat{b}+\hat{k})}{\check{\beta}(\check{b}+\check{k})(\hat{b}+\hat{k}+\check{\gamma}_2)} \langle pb+\gamma_1+\gamma+\sigma^2/2 \rangle_T \left(\langle \mathcal{R}_0^s \rangle_T - 1 \right) > 0 \quad \text{a.s.}$$

This finishes the proof. ∎

Now, by constructing a suitable function $\mathcal{V}$, we will show the existence of a positive T-periodic solution of system (V.8).

Theorem 1.14. *If $\langle \mathcal{R}_0^s \rangle_T > 1$, then there exists a T-periodic solution of system (V.8).*

Proof. It is easy to see that the coefficients of system (V.8) satisfy the local Lipschitz. To prove the periodicity by using Lemma 11.6, we construct a $T-$periodic C^2-function $\mathcal{V}(t,X)$ and a closed set $U \subset \Gamma_0$. We define a function $\mathcal{V} : [0,\infty) \times \Gamma_0 \to \mathbb{R}$ by

$$\mathcal{V}(t,I,R) = \mathcal{V}_1(I,R) + \mathcal{V}_2(I,R) + \mathcal{V}_3(t),$$

where

$$\begin{aligned}
\mathcal{V}_1(I,R) &= M\left(-\ln I + \frac{\check{\beta}}{\hat{b}+\hat{k}}R\right),\\
\mathcal{V}_2(I,R) &= -\ln R - \ln(1-I-R),\\
\mathcal{V}_3(t) &= M\psi(t),
\end{aligned}$$

and $M > 0$ satisfies the following condition

$$-M\left\langle pb + \gamma_1 + \gamma_2 + \frac{\sigma^2}{2}\right\rangle_T \left(\langle \mathcal{R}_0^s\rangle_T - 1\right) + 2\check{b} + \check{k} + \check{\beta} + \frac{\check{\sigma}^2}{2} \le -2. \tag{V.9}$$

We let

$$\begin{cases} \dot{\psi}(t) = -\left\langle \beta - \left(pb + \gamma_1 + \gamma_2 + \frac{\sigma^2}{2}\right)\right\rangle_T + \left(\beta(t) - \left(p(t)b(t) + \gamma_1(t) + \gamma_2(t) + \frac{\sigma^2(t)}{2}\right)\right),\\ \psi(0) = 0. \end{cases} \tag{V.10}$$

It is easy to prove that $\psi(t)$ is T-periodic function. In fact, integrating (V.10) from t to $t+T$, we obtain

$$\psi(t+T) - \psi(t) = \int_t^{t+T} \dot{\psi}(s)\mathrm{d}s = 0.$$

Hence, the function $\mathcal{V}(t,I,R)$ is also T-periodic function. Making use of Itô's formula, we get

$$\begin{aligned}
\mathcal{L}\mathcal{V}_1(I,R) = M\Bigg(&-\Big(\beta(1-I(t)-R(t)) - (p(t)b(t) + \gamma_1(t) + \gamma_2(t))\\
&- \frac{\sigma^2(t)}{2}(1-I(t)-R(t))^2\Big) + \frac{\check{\beta}\gamma_2(t)I(t)}{\hat{b}+\hat{k}} - \frac{\check{\beta}}{\hat{b}+\hat{k}}(b(t)+k(t))R(t)\Bigg)\\
\le M\Bigg(&-\left(\beta(t) - \left((p(t)b(t) + \gamma_1(t) + \gamma_2(t) + \frac{\sigma^2(t)}{2}\right)\right) + \check{\beta}\left(1 + \frac{\check{\gamma}_2}{\hat{b}+\hat{k}}\right)I(t)\Bigg),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}\mathcal{V}_2(I,R) = &-\frac{\gamma_2(t)I(t)}{R(t)} + (b(t)+k(t)) - \frac{b(t)}{1-I(t)-R(t)} + b(t) + \beta I(t)\\
&+ \frac{bI(t)}{1-I(t)-R(t)} - \frac{(p(t)b(t)+\gamma_1(t))I(t)}{1-I(t)-R(t)} - \frac{k(t)R(t)}{1-I(t)-R(t)} + \frac{\sigma^2(t)I^2(t)}{2}\\
\le &-\frac{\hat{\gamma}_2 I(t)}{R(t)} + 2\check{b} + \check{k} + \check{\beta} + \frac{\check{\sigma}^2}{2}.
\end{aligned}$$

Thus, one can see that

$$\begin{aligned}
\mathcal{L}\mathcal{V}(t,I,R) &\le M\left(-\left\langle \beta - \left(pb + \gamma_1 + \gamma_2 + \frac{\sigma^2}{2}\right)\right\rangle_T + \check{\beta}\left(1 + \frac{\check{\gamma}_2}{\hat{b}+\hat{k}}\right)I(t)\right) - \frac{\hat{\gamma}_2 I(t)}{R(t)} + 2\check{b} + \check{k} + \check{\beta} + \frac{\check{\sigma}^2}{2}\\
&= M\left(-\left\langle pb + \gamma_1 + \gamma_2 + \frac{\sigma^2}{2}\right\rangle_T\left(\langle \mathcal{R}_0^s\rangle_T - 1\right) + \check{\beta}\left(1 + \frac{\check{\gamma}_2}{\hat{b}+\hat{k}}\right)I(t)\right) - \frac{\hat{\gamma}_2 I(t)}{R(t)} + 2\check{b} + \check{k} + \check{\beta} + \frac{\check{\sigma}^2}{2}.
\end{aligned}$$

Define a bounded closed set $U = \left\{(I,R) \in \Gamma_0 : \quad I \geq \epsilon_1 \quad \text{and} \quad R \geq \epsilon_2\right\}$, where $0 < \epsilon_1, \epsilon_2 < 1$ verify the following conditions

$$M\check{\beta}\left(1 + \frac{\check{\gamma}_2}{\hat{b}+\hat{k}}\right)\epsilon_1 \leq 1, \tag{V.11}$$

$$\epsilon_2 = \epsilon_1^2, \tag{V.12}$$

$$M\check{\beta}\left(1 + \frac{\check{\gamma}_2}{\hat{b}+\hat{k}}\right) - \frac{\hat{\gamma}_2}{\epsilon_1} \leq 1. \tag{V.13}$$

For convenience, we divide U^c into two domains

$$U_\epsilon^1 = \left\{(I,R) \in \Gamma_0 : \quad 0 < I < \epsilon_1\right\},$$
$$U_\epsilon^2 = \left\{(I,R) \in \Gamma_0 : \quad \epsilon_1 \leq I < 1,\ 0 < R < \epsilon_2\right\}.$$

Clearly, $U^c = \Gamma_0 \backslash U = \bigcup_{i=1}^{2} U_\epsilon^i$. Now we prove that $\mathcal{LV}(t,y,z) \leq -1$ on $[0,\infty)\times U^c$, which is equivalent to showing it on the above two domains.

Case 1. When $(t,I,R) \in [0,\infty)\times U_\epsilon^1$, we have

$$\mathcal{LV}(t,I,R) \leq -M\left\langle pb + \gamma_1 + \gamma_2 + \frac{\sigma^2}{2}\right\rangle_T \left(\langle \mathcal{R}_0^s\rangle_T - 1\right) + M\check{\beta}\left(1 + \frac{\check{\gamma}_2}{\hat{b}+\hat{k}}\right)\epsilon_1 + 2\check{b} + \check{k} + \check{\beta} + \frac{\check{\sigma}^2}{2}.$$

In view of (V.11), we get $LV \leq -1 \quad$ for all $(t,I,R) \in [0,\infty)\times U_\epsilon^1$.

Case 2. Similarly, when $(t,I,R) \in [0,\infty)\times U_\epsilon^2$, we get

$$\mathcal{LV}(t,I,R) \leq -M\left\langle pb + \gamma_1 + \gamma_2 + \frac{\sigma^2}{2}\right\rangle_T \left(\langle \mathcal{R}_0^s\rangle_T - 1\right) + M\check{\beta}\left(1 + \frac{\check{\gamma}_2}{\hat{b}+\hat{k}}\right) - \hat{\gamma}_2\frac{\epsilon_1}{\epsilon_2} + 2\check{b} + \check{k} + \check{\beta} + \frac{\check{\sigma}^2}{2}.$$

In accordance with (V.12) and (V.13), we get $\mathcal{LV} \leq -1 \quad$ for all $(t,I,R) \in [0,\infty)\times U_\epsilon^2$. Consequently, we deduce that $\mathcal{LV} \leq -1$ for all $(t,I,R) \in [0,\infty)\times U^c$. Therefore, the system (V.8) has a T-periodic solution according to Lemma 11.6. ■

1.3 Numerical simulations

In this section, we present some numerical simulation results to show the effect of the white noise on the dynamics of SIRS models (V.2) and (V.3) by using the Milstein's method. For this purpose, the numerical scheme for stochastic autonomous model (V.4) is given by:

$$\begin{cases} I_{j+1} = I_j + I_j\left[\beta(1-I_j-R_j) - (pb+\gamma_1+\gamma_2)\right]\Delta t + \sigma I_j(1-I_j-R_j)\sqrt{\Delta t}\epsilon_j \\ \qquad\quad + \frac{\sigma^2}{2}I_j(1-I_j-R_j)(1-2I_j-R_j)(\epsilon_j^2-1)\Delta t, \\ R_{j+1} = R_j + \left[\gamma_2 I_j - (b+k)R_j\right]\Delta t, \end{cases}$$

where ϵ_j, $j = 1,2,...,n$, are j-th realization of the Gaussian random normal variate $\mathrm{N}(0,1)$. Likewise for the non-autonomous model (V.8), we obtain the following discretized equations:

$$\begin{cases} I_{j+1} = I_j + I_j\left[\beta_{j\Delta t}(1-I_j-R_j) - (p_{j\Delta t}b_{j\Delta t} + \gamma_{1\,j\Delta t} + \gamma_{2\,j\Delta t})\right]\Delta t \\ \qquad\quad + \sigma_{j\Delta t}I_j(1-I_j-R_j)\sqrt{\Delta t}\epsilon_j + \frac{1}{2}\sigma_{j\Delta t}^2 I_j(1-I_j-R_j)(1-2I_j-R_j)(\epsilon_j^2-1)\Delta t, \\ R_{j+1} = R_j + \left[\gamma_{2\,j\Delta t}I_j - (b_{j\Delta t}+k_{j\Delta t})R_j\right]\Delta t. \end{cases}$$

Moreover, we numerically simulate the solution of the corresponding deterministic system (V.1) for comparison.

Parameters	Description	Value
β	The transmission rate	0.041
b	The demography rate	0.3
γ_1	The transfer rate from I to S	0.01
γ_2	The transfer rate from I to R	0.03
p	The rate of the offspring of infected parents that are susceptible	0.01
k	The rate which individuals recover and return to S from R	0.02

Table V.1: The theoretical parameter values of the model (V.1).

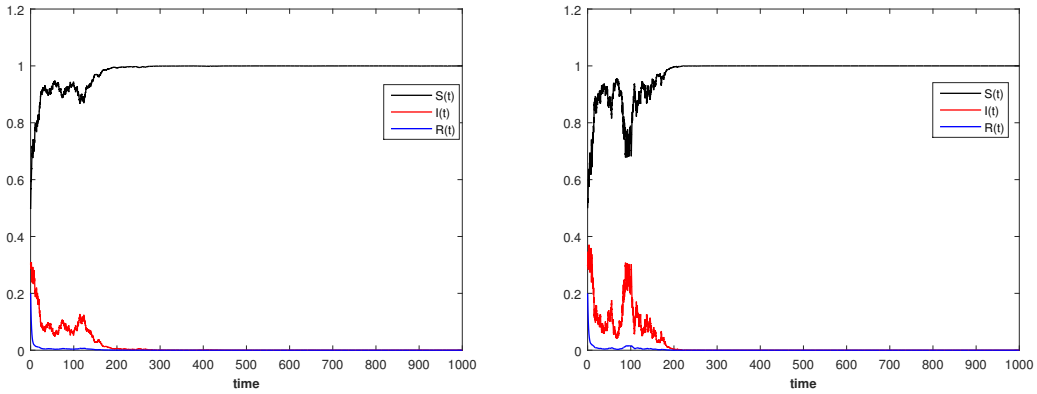


Figure V.1: Compute simulation of the paths $(S(t), I(t), R(t))$ for the SIRS epidemic model (V.2).

Example 1.15. (Stochastic dynamics of the autonomous model (V.2)) For the deterministic model (V.1) and its stochastic description (V.2), the parameters are taken from Table V.1. In this case, $\mathcal{R}_0 = 1.0199 > 1$, then the model (V.1) admits a unique endemic equilibrium $E^{\circledast}$ which is globally stable for any initial values $(S(0), I(0), R(0))$. For the corresponding stochastic model (V.2), we choose $\sigma = 0.1$, then we have $\sigma^2 \leq \beta$ and

$$\mathcal{R}_0^s = \frac{\beta}{(pb + \gamma_1 + \gamma_2 + \sigma^2/2)} = 0.9071 < 1.$$

Therefore, from Theorem 1.3, $I(t)$ almost surely tends to zero exponentially with probability one

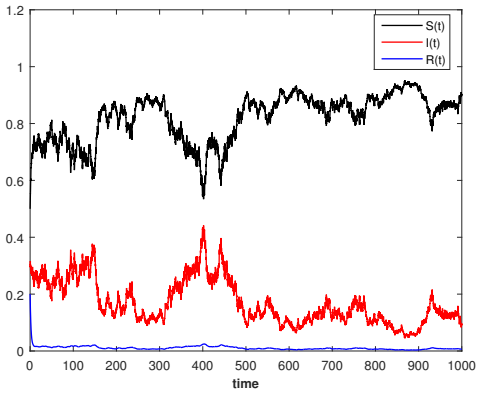


Figure V.2: Compute Simulations for the SDE model (V.2) with the parameters in Example 1.15

(see FigureV.1 - left). To see the effect of fluctuations on the system dynamics (V.2) with large noise intensity, we increase σ to 0.5, than we get $\sigma^2 > \beta$ and $\frac{\beta^2}{2(pb+\gamma_1+\gamma_2)} - \sigma^2 = -0.0402 < 0$. Therefore, $I(t)$ will tend to zero exponentially with probability one (see Figure V.1). When we increase the value of β to 0.05, we obtain $\mathcal{R}_0^s = 1.1062 > 1$, then according to Theorem 1.7, the solution of the stochastic

system (V.2) is persistent in the mean (see Figure V.2 - right). For a clear comparison with the deterministic solution of the epidemic model (V.1), the disease will prevail with $\mathcal{R}_0 = 1.2438 > 1$.

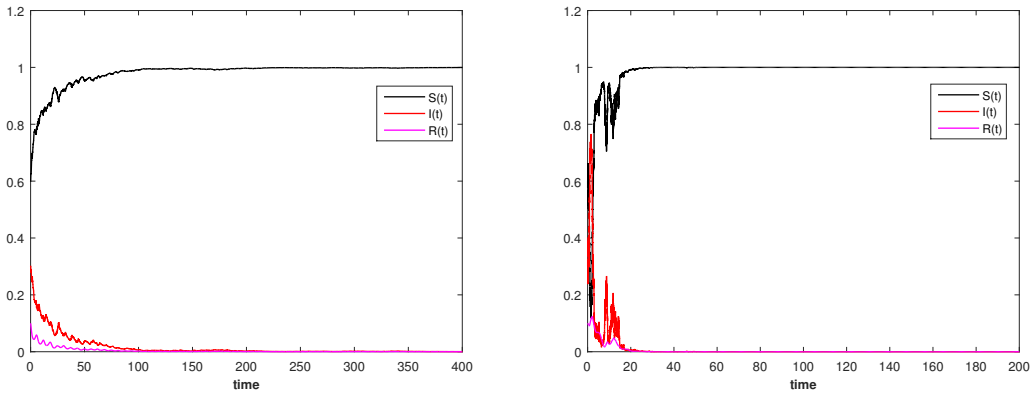


Figure V.3: Compute simulation of the paths $(S(t), I(t), R(t))$ for the SIRS epidemic model (V.3).

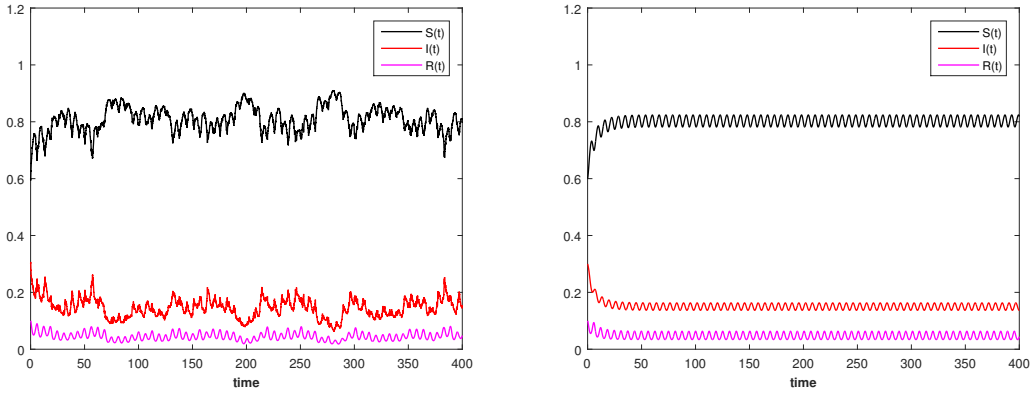


Figure V.4: Compute simulation of the paths $(S(t), I(t), R(t))$ for the SIRS epidemic model (V.3).

Example 1.16. (Stochastic dynamics of the non-autonomous model (V.3)) Assume that the parameters of the non-autonomous system (V.3) are taken from Table V.2

Function	Expression			
$\beta(t)$	$0.4 + 0.1\cos(t)$	$0.6 + 0.35\cos(t)$	$0.5 + 0.1\cos(t)$	$0.5 + 0.1\cos(t)$
$b(t)$	$0.08 + 0.02\sin(t)$	$0.08 + 0.02\sin(t)$	$0.08 + 0.02\sin(t)$	$0.08 + 0.02\sin(t)$
$\gamma_1(t)$	$0.2 + 0.02\sin(t)$	$0.2 + 0.02\sin(t)$	$0.2 + 0.02\sin(t)$	$0.2 + 0.02\sin(t)$
$\gamma_2(t)$	$0.2 + 0.03\sin(t)$	$0.2 + 0.03\sin(t)$	$0.2 + 0.03\sin(t)$	$0.2 + 0.03\sin(t)$
$p(t)$	$0.03 + 0.01\sin(t)$	$0.03 + 0.01\sin(t)$	$0.03 + 0.01\sin(t)$	$0.03 + 0.01\sin(t)$
$k(t)$	$0.6 + 0.5\sin(t)$	$0.6 + 0.5\sin(t)$	$0.6 + 0.5\sin(t)$	$0.6 + 0.5\sin(t)$
$\sigma(t)$	$0.1 + 0.05\cos(t)$	1	$0.1 + 0.05\cos(t)$	0
	Figure V.3 - left	Figure V.3 - right	Figure V.4 - left	Figure V.4 - right

Table V.2: Some periodic functions used to simulate the paths of non-autonomous model (V.3).

For the corresponding stochastic model (V.3), we get $\langle \mathcal{R}_0^s \rangle_T = 0.9466 < 1$, where $T = 2\pi$. Obviously, $\sigma^2(t) \leq \beta(t)$ for all $t \geq 0$ (see the first column of Table V.2). So, the condition (a) of Theorem 1.12 is satisfied. That is, $I(t)$ will tend to zero exponentially with probability one (see Figure V.3 - left). To verify that the condition (b) is satisfied, we choose $\beta(t) = 0.6 + 0.35\cos(t)$ and $\sigma(t) = 1$. Clearly,

$\sigma^2(t) > \beta(t)$ for all $t \geq 0$, and $\left\langle \frac{\beta^2}{2\sigma^2} - (pb + \gamma_1 + \gamma_2) \right\rangle_T = -4.0151 < 0$. Therefore, the disease dies out with probability one (see Figure V.3 - right). That is to say, large noises can lead the disease to extinction. In order to illustrate the conclusions of Theorem 1.13, in Figure V.4 (left), we choose the values of the system parameters from Table V.2 (if $\sigma(t) = 0$, see Figure V.4 right). Other values of the system parameters are unchanged. Direct calculation shows that $\langle \mathcal{R}_0 \rangle_T = 2.0018 > 1$ and $\langle \mathcal{R}_0^s \rangle_T = 1.1805 > 1$. According to Theorem 1.13, the solution of stochastic non-autonomous system (V.3) is persistent. From Theorem 1.14 it follows that the system (V.3) has a positive periodic solution.

1.4 Discussion

Generally, in this study, we have shown that the large noise will exponentially suppress the disease, and if the noise is sufficiently small, the conditions $\mathcal{R}_0^s < 1$ (autonomous case) or $\langle \mathcal{R}_0^s \rangle_T < 1$ (non-autonomous case) are sufficient for guaranteeing the extinction of the disease. So, if the noise is sufficiently small, $\mathcal{R}_0^s$ or $\langle \mathcal{R}_0^s \rangle_T$ can be considered as the thresholds of the systems (V.2) and (V.3) respectively, and their values determine the persistence or extinction of the infectious disease. Therefore, an interesting open problem is the possibility of establishing the extinction of the models (V.2) and (V.3) only when $\mathcal{R}_0^s < 1$ and $\langle \mathcal{R}_0^s \rangle_T < 1$, respectively.

2 Stability and threshold of a stochastic SIRS epidemic model with vertical transmission, general functional response and transfer from infectious to susceptible

2.1 Introduction

The main of this study is to investigate how the stochastic character of human transmission affect disease's spreading through studying the dynamical behavior of our stochastic model. We assume that the contact rate β is perturbed by Gaussian white noise, which is presented by $\beta + \sigma \dot{\mathcal{W}}(t)$, where $\mathcal{W}(t)$ is a standard Brownian motion with intensity $\sigma > 0$. Then, we obtain the following SIRS epidemic model with stochastic perturbation and general functional response

$$\begin{cases} \mathrm{d}S(t) = \left[A + b(S(t) + R(t)) - \frac{\beta S(t) I(t)}{\psi(S,I)} - \mu S(t) + pbI(t) + \gamma_1 I(t) + kR(t)\right] \mathrm{d}t - \frac{\sigma S(t) I(t)}{\psi(S,I)} \mathrm{d}\mathcal{W}(t), \\ \mathrm{d}I(t) = \left[\frac{\beta S(t) I(t)}{\psi(S,I)} + qbI(t) - (\mu + \gamma_1 + \gamma_2 + a) I(t)\right] \mathrm{d}t + \frac{\sigma S(t) I(t)}{\psi(S,I)} \mathrm{d}\mathcal{W}(t), \\ \mathrm{d}R(t) = [\gamma_2 I(t) - (\mu + k) R(t)] \, \mathrm{d}t, \end{cases} \tag{V.14}$$

where $\psi(S,I) = 1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI$ ($\alpha_1, \alpha_2, \alpha_3 \geq 0$), $p = 1 - q$ and $\mu > b$. There are many authors who studied stochastic epidemic models. They introduced random effects into models by different techniques (see e.g. [56, 7, 116, 21, 135, 57, 145, 21, 91, 141]). Furthermore, in the study of the dynamical behavior of the epidemic models, we are interested in two situations, one is when the disease goes to extinction, the other is when the disease prevails. Thus many authors have studied this interesting topic. For example, Ji et al. [44] investigated the threshold of SIR epidemic models with stochastic perturbation. Liu et al. [83] established a interesting results on the threshold behavior in a stochastic SIQR epidemic model with standard incidence and regime switching. Zhao and Jiang [165] investigated the threshold of a stochastic SIRS epidemic model with saturated incidence and then considered the threshold of a stochastic SIS epidemic model with vaccination [166, 164]. furthermore, they studied the threshold of a stochastic SIRS epidemic model with varying population size [167]. Zhao and Yuan [170] studied the threshold behavior of a stochastic SIVR epidemic model with standard incidence and imperfect vaccine. In case that the disease goes extinct, they showed that the disease-free equilibrium is almost surely stable by using the nonnegative semimartingale convergence theorem. C. Zhu et al. [173] studied the threshold of a stochastic SIRS model with vertical transmission and saturated incidence.

In this work, we consider a stochastic SIRS model with general incidence rate. This generalization is the main difficulty to be overcome in establishing the conditions of the threshold of SIRS model (V.19), which, to the best of our knowledge, has never been examined in the previously known literature. In addition, we prove the global stability of the stochastic SIRS model (V.19) with stochastic perturbations, which has not been proven in the previous papers. Define a bounded set Δ as follows:

$$\Delta = \left\{ x = (x_1, x_2, x_3) : \ x_1 > 0, \ x_2 > 0, \ x_3 > 0 \ \text{ and } \ x_1 + x_2 + x_3 < \frac{A}{\mu - b} \right\}.$$

Since $S(t)$, $I(t)$ and $R(t)$ represent the number of the susceptible, the infected and the recovered individuals at time t, respectively, they should be non-negative. So, the first step of our study is to prove that the system (V.14) has a unique global positive solution.

Theorem 2.1 ([58]). *For any initial value in Δ, there is the unique positive solution $(S(t), I(t), R(t))$ of system (V.14) on $t \geq 0$, and the solution will remain in $\mathbb{R}^3_+$ with probability one, that is to say, the solution $(S(t), I(t), R(t)) \in \mathbb{R}^3_+$ for all $t \geq 0$ almost surely.*

From the Theorem 2.1, we can conclude the following corollary:

Corollary 2.2. *Δ is an almost surely positively invariant domain for the system (V.14).*

2.2 Main results

2.2.1 Moment exponential stability

Theorem 2.3. *Let $h \geq 2$. If*

$$\frac{1}{2}(h-1)\left(\frac{A}{(\mu - b) + \alpha_1 A}\right)^2 \sigma^2 < (\mu + \gamma_1 + \gamma_2 + a - qb) - \frac{\beta A}{(\mu - b) + \alpha_1 A}$$

holds, the disease-free equilibrium $E^{\ominus}$ of the system (V.14) is h^{th} moment exponentially stable in Δ.

Proof. Let $h \geq 2$. Considering the Lyapunov function

$$\mathcal{V}(S, I, R) = \lambda_1 \left(\frac{A}{\mu - b} - S\right)^h + \frac{1}{h} I^h + \lambda_2 R^h, \qquad \text{for all} \quad (S, I, R) \ \text{ in } \ \Delta,$$

where λ_i, $i = 1, 2$ are real positive constants to be determined later. Then, we have

$$\begin{aligned}
\mathcal{L}\mathcal{V}(S, I, R) = &- h\lambda_1 \left(\frac{A}{\mu - b} - S\right)^h (\mu - b) + h\lambda_1 \left(\frac{A}{\mu - b} - S\right)^{h-1} \frac{\beta IS}{\psi(S, I)} - h\lambda_1 (pb + \gamma_1) \left(\frac{A}{\mu - b} - S\right)^{h-1} I \\
&- h\lambda_1 (b + k) \left(\frac{A}{\mu - b} - S\right)^{h-1} R + \left(\frac{\beta S}{\psi(S, I)} - (\mu + \gamma_1 + \gamma_2 + a - qb)\right) I^h + h\lambda_2 \gamma_2 R^{h-1} I \\
&- h\lambda_2 (\mu + k) R^h + \frac{\lambda_1 h(h-1)}{2} \left(\frac{A}{\mu - b} - S\right)^{h-2} \frac{\sigma^2 S^2 I^2}{\psi^2(S, I)} + \frac{(h-1)}{2} \frac{\sigma^2 I^h S^2}{\psi^2(S, I)}.
\end{aligned}$$

In Δ, we have

$$\begin{aligned}
\mathcal{L}\mathcal{V}(S, I, R) \leq &- h\lambda_1 \left(\frac{A}{\mu - b} - S\right)^h (\mu - b) + \left[h\lambda_1 \frac{\beta A}{(\mu - b) + \alpha_1 A}\right] \left(\frac{A}{\mu - b} - S\right)^{h-1} I + \frac{h(h-1)\sigma^2 A^2}{2[(\mu - b) + \alpha_1 A]^2} \lambda_1 \\
&\times \left(\frac{A}{\mu - b} - S\right)^{h-2} I^2 - \left(\left(\mu + \gamma_1 + \gamma_2 + a - qb - \frac{\beta A}{(\mu - b) + \alpha_1 A}\right) - \frac{(h-1)}{2} \frac{A^2 \sigma^2}{((\mu - b) + \alpha_1 A)^2}\right) I^h \\
&+ h\lambda_2 \gamma_2 R^{h-1} I - h\lambda_2 (\mu + k) R^h.
\end{aligned}$$

By using Lemma 1.8, we get

$$I\left(\frac{A}{\mu-b}-S\right)^{h-1} \leq \frac{(h-1)\epsilon}{h}\left(\frac{A}{\mu-b}-S\right)^{h}+\frac{1}{h\epsilon^{h-1}}I^{h},$$
$$I^{2}\left(\frac{A}{\mu-b}-S\right)^{h-2} \leq \frac{(h-2)\epsilon}{h}\left(\frac{A}{\mu-b}-S\right)^{h}+\frac{1}{h\epsilon^{\frac{h-2}{2}}}I^{h},$$
$$IR^{h-1} \leq \frac{(h-1)\epsilon}{h}R^{h}+\frac{1}{h\epsilon^{h-1}}I^{h}.$$

Then

$$\begin{aligned}\mathcal{LV}(S,I,R) \leq &-\left[h(\mu-b)-\left(\frac{A\beta(h-1)}{(\mu-b)+\alpha_1 A}+\frac{\sigma^2A^2(h-1)(h-2)}{2[(\mu-b)+\alpha_1A]^2}\right)\epsilon\right]\lambda_1\left(\frac{A}{(\mu-b)}-S\right)^{h}\\&-\left[\left(\mu+\gamma_1+\gamma_2+a-qb-\frac{\beta A}{(\mu-b)+\alpha_1A}\right)-\frac{(h-1)}{2}\frac{A^2\sigma^2}{[(\mu-b)+\alpha_1A]^2}\right.\\&\left.+\left(\frac{\beta A}{(\mu-b)+\alpha_1A}\epsilon^{1-h}+\frac{(h-1)\sigma^2A^2}{[(\mu-b)+\alpha_1A]^2}\epsilon^{\frac{2-h}{h}}\right)\lambda_1+\gamma_2\epsilon^{1-h}\lambda_2\right]I^{h}\\&-\left(h(\mu+k)-(h-1)\gamma_2\epsilon\right)\lambda_2R^{h}.\end{aligned}$$

We chose ϵ sufficiently small such that the coefficients of $\left(\frac{A}{\mu-b}-S\right)^{h}$ and R^h be negative, and as $\left(\mu+\gamma_1+\gamma_2+a-qb-\frac{\beta A}{(\mu-b)+\alpha_1A}\right)-\frac{(h-1)}{2}\frac{A^2\sigma^2}{[(\mu-b)+\alpha_1A]^2}>0$, we can choose λ_1 and λ_2 positive such the coefficient of I^h be negative. This ends the proof. ∎

Under Theorem 2.3, we have in the case of $h=2$ the following corollary:

Corollary 2.4. *If the condition* $\frac{1}{2}(\frac{A}{(\mu-b)+\alpha_1A})^2\sigma^2<(\mu+\gamma_1+\gamma_2+a-qb)-\frac{\beta A}{(\mu-b)+\alpha_1A}$ *holds, the disease free* $E^{\ominus}$ *of the system (V.14) is globally asymptotically stable in* Δ.

2.2.2 Persistence in the mean

In this subsection, our main concern is to determine the conditions for the spread and disappearance of an infectious disease. Given that the value of the deterministic threshold $\mathcal{R}_0$ characterizes the dynamical behaviors of system (I.9) and guarantees the persistence or theextinction of the disease. Similarly, we define the following threshold of our stochastic SIRS epidemic model (V.14) as follows

$$\mathcal{R}_0^s=\mathcal{R}_0\left(1-\frac{\sigma^2A}{2\beta[(\mu-b)+\alpha_1A]}\right)=\frac{\beta A}{((\mu-b)+\alpha_1A)(\mu_2+\gamma_1+\gamma_2-qb)}\left(1-\frac{\sigma^2A}{2\beta[(\mu-b)+\alpha_1A]}\right).$$

Theorem 2.5. *If* $\mathcal{R}_0^s>1$, *then the solution* $(S(t),I(t),R(t))$ *of system (V.14) with positive initial value is persistent in the mean. Moreover, we have*

$$\liminf_{t\to\infty}\langle I(t)\rangle \geq \mathfrak{C}_*>0,$$
$$\liminf_{t\to\infty}\langle R(t)\rangle \geq \frac{\gamma_2\mathfrak{C}_*}{(\mu+k)}>0,$$
$$\liminf_{t\to\infty}\left\langle\frac{A}{\mu-b}-S\right\rangle \geq \frac{[(\mu-b+a)(\mu+k)+(\mu-b)\gamma_2]\mathfrak{C}_*}{(\mu-b)(\mu+k)}>0,$$

where

$$\mathfrak{C}_*=\frac{(\mathcal{R}_0^s-1)(\mu_2+\gamma_1+\gamma_2-qb)\beta^{-1}(\mu-b)(\mu+k)((\mu-b)+\alpha_1A)}{(k+b)(\mu-b)(\mu-b+a)+(\mu-b)\left(\beta A-(pb+\gamma_1)(\mu-b)\right)+A(\mu+k)\left(\alpha_2(\mu-b)+\alpha_3A\right)}.$$

Proof. We consider a function $\mathcal{V}^\star$ defined by

$$\mathcal{V}^\star(S,I,R) = \omega_1(S+I+R) + \omega_2 S + \ln I,$$

where $\omega_1$and ω_2 are real positive constants to be chosen in the follwing. Using Itô's formula, we have

$$\begin{aligned}
\mathrm{d}\mathcal{V}^\star(S,I,R) =& \omega_1\Big(A-(\mu-b)S-(\mu-b+a)I-(\mu-b)R\Big)\mathrm{d}t + \omega_2\Bigg(A-(\mu-b)S-\frac{\beta IS}{\psi(S,I)}\\
&+(pb+\gamma_1)I+(b+k)R\Bigg)\mathrm{d}t - \omega_2\frac{\sigma SI}{\psi(S,I)}\mathrm{d}\mathcal{W}(t) + \Bigg(-(\mu+\gamma_1+\gamma_2+a-qb)\\
&+\frac{\beta S}{\psi(S,I)} - \frac{1}{2}\frac{\sigma^2S^2}{\psi^2(S,I)}\Bigg)\mathrm{d}t + \frac{\sigma S}{\psi(S,I)}\mathrm{d}\mathcal{W}(t).
\end{aligned}$$

Since $(S(t),I(t),R(t)) \in \Delta$, we get $-\dfrac{\beta SI}{\psi(S,I)} \geq -\dfrac{\beta AI}{\mu-b}$, and

$$\begin{aligned}
\frac{\beta S}{\psi(S,I)} &= \frac{\beta A}{(\mu-b)+\alpha_1 A} - \frac{\beta(\mu-b)}{(\mu-b)+\alpha_1 A}\frac{\left(\frac{A}{\mu-b}-S\right)}{\psi(S,I)} - \frac{\beta\alpha_2 A}{(\mu-b)+\alpha_1 A}\frac{I}{\psi(S,I)} - \frac{\beta\alpha_3 A}{(\mu-b)+\alpha_1 A}\frac{SI}{\psi(S,I)}\\
&\geq \frac{\beta A}{(\mu-b)+\alpha_1 A} - \frac{\beta(\mu-b)}{(\mu-b)+\alpha_1 A}\left(\frac{A}{\mu-b}-S\right) - \frac{\beta A}{(\mu-b)+\alpha_1 A}\left(\alpha_2+\alpha_3\frac{A}{(\mu-b)}\right)I.
\end{aligned}$$

Inject these tow inequalities in expression of $\mathrm{d}\mathcal{V}^\star$, we get

$$\begin{aligned}
\mathrm{d}\mathcal{V}^\star(S,I,R) \geq& \omega_1\Big(A-(\mu-b)s-(\mu-b+a)I-(\mu-b)R\Big)\mathrm{d}t\\
&+\omega_2\Bigg(A-(\mu-b)S-\frac{\beta AI}{\mu-b}+(pb+\gamma_1)I+(b+k)R\Bigg)\mathrm{d}t - \omega_2\frac{\sigma SI}{\psi(S,I)}\mathrm{d}\mathcal{W}\\
&+\Bigg(-(\mu+\gamma_1+\gamma_2+a-qb)+\frac{\beta A}{(\mu-b)+\alpha_1 A} - \frac{\beta(\mu-b)}{(\mu-b)+\alpha_1 A}\left(\frac{A}{\mu-b}-S\right)\\
&-\frac{\beta A}{(\mu-b)+\alpha_1 A}\left(\alpha_2+\alpha_3\frac{A}{(\mu-b)}\right)I\Bigg)\mathrm{d}t - \frac{\sigma^2S^2}{2\psi^2(S,I)}\mathrm{d}t + \frac{\sigma S}{\psi(S,I)}\mathrm{d}\mathcal{W}(t).
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{d}\mathcal{V}^\star(S,I,R) \geq& \left((\omega_1+\omega_2)(\mu-b) - \frac{\beta(\mu-b)}{(\mu-b)+\alpha_1 A}\right)\left(\frac{A}{(\mu-b)}-S\right)\mathrm{d}t + [\omega_2(b+k)-\omega_1(\mu-b))]R\mathrm{d}t\\
&-\left(\omega_1(\mu-b+a)+\omega_2\left(\frac{\beta A}{\mu-b}-(pb+\gamma_1)\right)+\frac{\beta A}{(\mu-b)+\alpha_1 A}(\alpha_2+\alpha_3\frac{A}{\mu-b})\right)I\mathrm{d}t\\
&+\left(-(\mu+\gamma_1+\gamma_2+a-qb)+\frac{\beta A}{(\mu-b)+\alpha_1 A} - \frac{\sigma^2A^2}{2[(\mu-b)+\alpha_1 A]^2}\right)\mathrm{d}t\\
&-(\omega_2 I-1)\frac{\sigma S}{\psi(S,I)}\mathrm{d}\mathcal{W}(t).
\end{aligned}$$

In order to eliminate $\left(\frac{A}{\mu}-S\right)$ and R from the last inequality, we choose ω_1 and ω_2 as follows

$$\omega_1 = \frac{\beta(b+k)}{(\mu+k)((\mu-b)+\alpha_1 A)} \quad \text{and} \quad \omega_2 = \frac{\beta(\mu-b)}{(\mu+k)((\mu-b)+\alpha_1 A)}.$$

By integration, we get

$$\begin{aligned}
&\mathcal{V}^\star(S(t),I(t),R(t))\\
&\geq -\frac{(k+b)(\mu-b)(\mu-b+a)+(\mu-b)\left(\beta A-(pb+\gamma_1)(\mu-b)\right)+A(\mu+k)\left(\alpha_2(\mu-b)+\alpha_3 A\right)}{\beta^{-1}(\mu-b)(\mu+k)((\mu-b)+\alpha_1 A)}\\
&\times\int_0^t I(s)\mathrm{d}s + (\mathcal{R}_0^s-1)(\mu+\gamma_1+\gamma_2+a-qb)t + \mathcal{V}^\star(S(0),I(0),R(0)) - \int_0^t(\omega_2 I-1)\frac{\sigma S}{\psi(S,I)}\mathrm{d}\mathcal{W}(t).
\end{aligned}$$

Hence,

$$\ln I(t) \geq -\frac{(k+b)(\mu-b)(\mu-b+a)+(\mu-b)\left(\beta A-(pb+\gamma_1)(\mu-b)\right)+A(\mu+k)\left(\alpha_2(\mu-b)+\alpha_3 A\right)}{\beta^{-1}(\mu-b)(\mu+k)((\mu-b)+\alpha_1 A)}$$
$$\times\int_0^t I(s)\mathrm{d}s+(\mathcal{R}_0^s-1)(\mu+\gamma_1+\gamma_2+a-qb)t+\mathcal{K}(t),$$

where

$$\mathcal{K}(t)=\mathcal{V}^\star(S(0),I(0),R(0))-(\omega_1+\omega_2)S-\omega_1 I-\omega_1 R-\int_0^t(\omega_2 I(s)-1)\frac{\sigma S}{\psi(S,I)}\mathrm{d}\mathcal{W}(t).$$

Thus, the strong law of large number for martingales implies that $\lim_{t\to\infty}\frac{\mathcal{K}(t)}{t}=0$ a.s. Consequently

$$\liminf_{t\to\infty}\langle I(t)\rangle \geq \frac{(\mathcal{R}_0^s-1)(\mu+r_2+\delta+\gamma)\beta^{-1}(\mu-b)(\mu+k)((\mu-b)+\alpha_1 A)}{(k+b)(\mu-b)(\mu-b+a)+(\mu-b)\left(\beta A-(pb+\gamma_1)(\mu-b)\right)+A(\mu+k)\left(\alpha_2(\mu-b)+\alpha_3 A\right)}$$
$$:=\mathfrak{C}_*.$$

Next, the third equation of system (V.14) gives

$$\frac{R(t)-R(0)}{t}=\gamma_2\langle I(t)\rangle+(\mu+k)\langle R(t)\rangle.$$

Implies that $\langle R(t)\rangle=\frac{\gamma_2}{(\mu+k)}\langle I(t)\rangle+\mathcal{K}_1(t)$ where $\mathcal{K}_1(t)=\frac{R(0)-R(t)}{(\mu+k)t}$ and $\lim_{t\to\infty}\mathcal{K}_1(t)=0$ a.s. Then

$$\liminf_{t\to\infty}\langle R(t)\rangle\geq\frac{\gamma_2\mathfrak{C}_*}{(\mu+k)}.$$

Finally, it follow from the system (V.14) that

$$\mathrm{d}(S+I+R)=\left((\mu-b)\left(\frac{A}{\mu-b}-S\right)-(\mu-b+a)I-(\mu-b)R\right)\mathrm{d}t.$$

Therefore

$$\left\langle\frac{A}{\mu-b}-S(t)\right\rangle=\frac{\mu-b+a}{\mu-b}\langle I(t)\rangle+\langle R(t)\rangle+\mathcal{K}_2(t),$$

where $\mathcal{K}_2(t)=\frac{N(t)-N(0)}{\mu t}$ and $\lim_{t\to\infty}\mathcal{K}_2(t)=0$ a.s. Then

$$\liminf_{t\to\infty}\left\langle\frac{A}{\mu-b}-S(t)\right\rangle\geq\frac{[(\mu-b+a)(\mu+k)+(\mu-b)\gamma_2]\mathfrak{C}_*}{(\mu-b)(\mu+k)}\quad\text{a.s.}$$

■

2.2.3 Stochastic extinction

In this subsection, we investigate the conditions for the extinction of the disease.

Theorem 2.6. *Let $(S(t),I(t),R(t))$ be the solution of system (V.14) with initial value $(S(0),I(0),R(0))$ in Δ. Then*

1. $\limsup_{t\to\infty}\frac{\ln I(t)}{t}\leq\frac{\beta^2}{2\sigma^2}-(\mu+\gamma_1+\gamma_2+a-qb)<0$ a.s. if $\sigma^2>\frac{\beta^2}{2(\mu+\gamma_1+\gamma_2+a-qb)}$ holds, (V.15)

2. $\limsup_{t\to\infty}\frac{\ln I(t)}{t}\leq(\mathcal{R}_0^s-1)(\mu+\gamma_1+\gamma_2+a-qb)<0$ a.s. if $\mathcal{R}_0^s<1$ and $\sigma^2\leq\frac{\beta[(\mu-b)+\alpha_1 A]}{A}$ hold. (V.16)

Proof. It follows from Itô's formula that

$$\mathrm{d}\ln I(t) = \left(\frac{\beta S}{\psi(S(t),I(t))} - (\mu+\gamma_1+\gamma_2+a-qb) - \frac{\sigma^2}{2}\frac{S^2(t)}{\psi^2(S(t),I(t))}\right)\mathrm{d}t + \frac{\sigma S(t)}{\psi(S(t),I(t))}\mathrm{d}\mathcal{W}(t).$$

Integrating this from 0 to t and dividing by t on both sides, we have

$$\frac{\ln I(t)}{t} = \frac{1}{t}\int_0^t \left(\frac{\beta S(s)}{\psi(S(s),I(s))} - (\mu+\gamma_1+\gamma_2+a-qb) - \frac{\sigma^2}{2}\frac{S^2(s)}{\psi^2(S(s),I(s))}\right)\mathrm{d}s + \frac{\ln I(0)}{t} + \frac{\mathcal{M}(t)}{t}, \quad \text{(V.17)}$$

where $\mathcal{M}(t) = \displaystyle\int_0^t \frac{\sigma S(s)}{\psi(S(s),I(s))}\mathrm{d}\mathcal{W}(s)$ is a local martingale with the quadratic variation that verifies $\langle \mathcal{M},\mathcal{M}\rangle_t \le \frac{\sigma^2 A^2 t}{\mu-b}$. An application of the strong law of large numbers for local martingales leads to

$$\lim_{t\to\infty}\frac{\mathcal{M}(t)}{t} = 0 \quad \text{a.s.}$$

From Equation (V.17), we have

$$\begin{aligned}\frac{\ln I(t)}{t} &= \frac{1}{t}\int_0^t \left(-\frac{\sigma^2}{2}\left(\frac{S(s)}{\psi(S(s),I(s))} - \frac{\beta}{\sigma^2}\right)^2 - (\mu+\gamma_1+\gamma_2+a-qb) + \frac{\beta^2}{2\sigma^2}\right)\mathrm{d}s + \frac{\ln I(0)}{t} + \frac{M(t)}{t}\\ &\le \left(\frac{\beta^2}{2\sigma^2} - (\mu+\gamma_1+\gamma_2+a-qb)\right) + \frac{\ln I(0)}{t} + \frac{\mathcal{M}(t)}{t}.\end{aligned}$$

Taking the limit superior of both sides, we obtain the desired assertion if the condition (a) is satisfied.
Now, and from the last inequality, one can establish that

$$\begin{aligned}\frac{\ln I(t)}{t} &\le \left(\frac{\beta A}{(\mu-b)+\alpha_1 A} - (\mu+\gamma_1+\gamma_2+a-qb) - \frac{\sigma^2}{2}\frac{A^2}{[(\mu-b)+\alpha_1 A]^2}\right) + \frac{\ln I(0)}{t} + \frac{\mathcal{M}(t)}{t}\\ &\le (\mathcal{R}_0^s - 1)(\mu+\gamma_1+\gamma_2+a-qb) + \frac{\ln I(0)}{t} + \frac{\mathcal{M}(t)}{t}.\end{aligned}$$

If the condition (b) is satisfied, and by taking the limit superior of both sides, we obtain the desired assertion (V.16). This finishes the proof. ∎

2.3 Numerical simulations

In this subsection, we present the numerical simulations to support the above theoretical results, illustrating extinction and persistence in mean of disease. We mainly use Milstein's method to discrete the system (V.14). Moreover, we numerically simulate the solution of a corresponding deterministic system for comparison.

Parameters	Value		
A	0.4	0.2	0.2
β	0.2	0.52	0.8
μ	0.3	0.3	0.3
b	0.2	0.2	0.2
γ_1	0.01	0.01	0.1
γ_2	0.03	0.03	0.5
q	0.1	0.01	0.01
a	0.1	0.1	0.1
k	0.2	0.2	0.2
α_1	0.05	0.05	0.05
α_2	0.02	0.02	0.02
α_3	0.01	0.01	0.01
σ	0.5	0.1	0.1
	Figure V.5 - left	Figure V.5 - right	Figure V.6

Table V.3: The theoretical parameter values of the model (V.14).

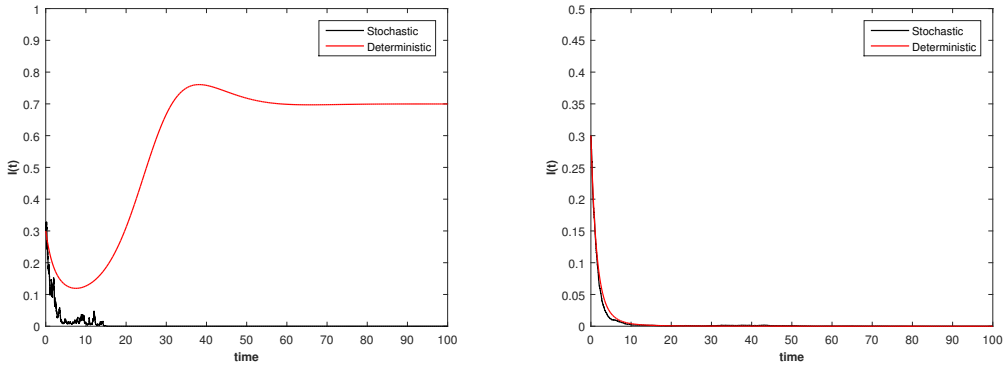


Figure V.5: Compute simulation of the paths $I(t)$ for the SIRS epidemic model (V.14), and $I(t)$ of the corresponding deterministic version.

Example 2.7. For our model (V.14), the parameters are taken from Table V.3 (first column). If $\sigma = 0$, we get $\mathcal{R}_0 = 1.5873 > 1$, thus the deterministic version admits a unique endemic equilibrium $E^{\circledast}$ which is globally stable for any initial values $(S(0), I(0), R(0))$. For the corresponding stochastic version, we choose $\sigma = 0.5$, then we have $\frac{\beta^2}{2(\mu+\gamma_1+\gamma_2+a-qb)} - \sigma^2 = -0.2024 < 0$. Therefore, the condition (a) of Theorem 2.6 is satisfied. We can conclude that for any initial value, $I(t)$ obeys

$$\limsup_{t\to\infty} \frac{1}{t} \ln \frac{I(t)}{I(0)} \leq \frac{\beta^2}{2\sigma^2} - (\mu + \gamma_1 + \gamma_2 + a - qb) = -0.34 \quad \text{a.s.}$$

That is, $I(t)$ will tend to zero exponentially with probability one (see Figure V.5 - left).

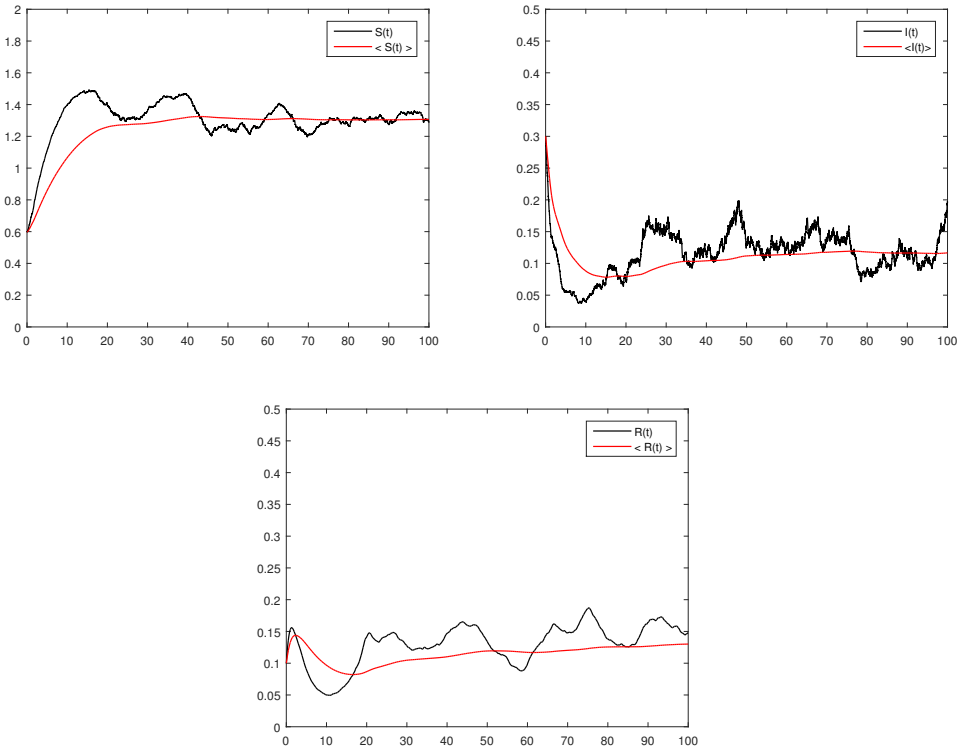


Figure V.6: Compute simulation of the paths $(S(t), I(t), R(t))$ for the SIRS epidemic model (V.14).

Example 2.8. We choose the parameter values from Table V.3 (second column), then we have $\mathcal{R}_0^s = 0.9210 < 1$, and $\sigma^2 - \frac{\beta(\mu-b)}{A} = -0.25 < 0$. Then, the condition (b) of Theorem 2.6 is satisfied. We can conclude that for any initial value, $I(t)$ obeys

$$\limsup_{t\to\infty} \frac{1}{t} \ln \frac{I(t)}{I(0)} \leq (\mathcal{R}_0^s - 1)(\mu + \gamma_1 + \gamma_2 + a - qb) = -0.015 \quad \text{a.s.}$$

That is, $I(t)$ will tend to zero exponentially with probability one (see Figure V.5 - right).

Example 2.9. We choose the parameter values from Table V.3 (third column), then we get

$$\mathcal{R}_0^s = 1.4409 > 1.$$

According to Theorem 2.5, the solution of the stochastic system (V.14) is persistent in the mean (see Figure V.6).

2.4 Discussion

In the above study, we have considered a stochastic SIRS epidemic model with vertical transmission and transfer from infectious to susceptible. We have obtained sufficient conditions for the stochastic stability of disease-free equilibrium by using a suitable Lyapunov function and other techniques of the stochastic analysis. Furthermore, we have showed that the disease is persistent when the basic reproduction number $\mathcal{R}_0^s > 1$. Finally, we have given two sufficient conditions for the extinction of the disease with probability one. From Theorem 2.6, we have showed that the large noise will exponentially suppress the disease, and if the noise is sufficiently small, the condition $\mathcal{R}_0^s < 1$ is sufficient for guaranteeing the extinction of the disease. So, if the noise is sufficiently small, $\mathcal{R}_0^s$ can be considered as the threshold of system (V.14) and its value determines the persistence and extinction of the infectious disease. Besides the white noise perturbation, epidemic models may be disturbed by telegraph noise which can lead to the system to switch from one environmental regime to another. The telegraph noise can be illustrated as a switching between two or more sub-regimes of different environments. For example, the growth rate for some fish in dry season will be much different from it in rainy season. The switching between environmental regimes is often memory-less and the waiting time for the next switching follows the exponential distribution. Therefore, this study can be extend to the case of regime switching.

3 Analysis of a stochastic SIRS epidemic system with vertical dissemination, switch from infectious to susceptible individuals, stochastic transmission and proportional perturbation

3.1 Introduction

The deterministic model (I.7) provides a great view on the spread of the epidemic, but in the real and complex world, the population dynamics of infectious diseases are naturally affected by random fluctuations and perturbations [38, 97, 59, 114, 169, 62]. Therefore, the stochastic SIRS epidemic can be an accurate tool to predict the long-run of infectious epidemics [165, 19, 167, 87, 45]. For example, in the previous section, we assumed that the disease transmission coefficient β is subject to the environmental white noise, that is $\beta \to \beta + \sigma\dot{\mathcal{W}}(t)$ where $\mathcal{W}(t)$ is a standard Brownian motion and $\sigma > 0$ is the intensity of environmental white noise. Then, the system (I.7) becomes the following stochastic differential equation:

$$\begin{cases} \mathrm{d}S(t) = \big(A + b(S(t)+R(t)) - \beta S(t)I(t) - \mu_1 S(t) + pbI(t) + \gamma_1 I(t) + kR(t)\big)\mathrm{d}t - \sigma S(t)I(t)\mathrm{d}\mathcal{W}(t), \\ \mathrm{d}I(t) = \big(\beta S(t)I(t) + qbI(t) - (\mu_2 + \gamma_1 + \gamma_2)I(t)\big)\mathrm{d}t + \sigma S(t)I(t)\mathrm{d}\mathcal{W}(t), \\ \mathrm{d}R(t) = \big(\gamma_2 I(t) - (\mu + k)R(t)\big)\mathrm{d}t. \end{cases} \tag{V.18}$$

In the same section, we investigated the conditions for the moment exponential stability of the equilibrium $E^{\ominus}$. Then, we established sufficient conditions for the extinction and persistence of the model (V.18) with a general incidence rate.

Besides the parameter perturbation method, there is another type of the stochastic disturbances. One can include the effect of environmental fluctuations by assuming the noise is proportional to the variables $S(t)$, $I(t)$, and $R(t)$. Many authors have considered the proportional perturbation into epidemic models and have studied their dynamics [19, 155, 22]. In this section, we simultaneously consider the stochastic transmission and the proportional perturbation in the SIRS epidemic system. The combination of these dual fluctuations makes the transmission dynamics more biologically realistic in a very noisy environment with non-constant population size $N(t) = S(t) + I(t) + R(t)$. The perturbed model can be described by the following stochastic differential equation:

$$\begin{cases} \mathrm{d}S(t) = \big(A + b(S(t)+R(t)) - \beta S(t)I(t) - \mu_1 S(t) + pbI(t) + \gamma_1 I(t) + kR(t)\big)\mathrm{d}t \\ \qquad\qquad +\sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) - \sigma_4 S(t)I(t)\mathrm{d}\mathcal{W}_4(t), \\ \mathrm{d}I(t) = \big(\beta S(t)I(t) + qbI(t) - (\mu_2+\gamma_1+\gamma_2)I(t)\big)\mathrm{d}t + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \sigma_4 S(t)I(t)\mathrm{d}\mathcal{W}_4(t), \\ \mathrm{d}R(t) = \big(\gamma_2 I(t) - (\mu_3+k)R(t)\big)\mathrm{d}t + \sigma_3 R(t)\mathrm{d}\mathcal{W}_3(t), \end{cases} \tag{V.19}$$

where $\mathcal{W}_i(t)$ $(i = 1,2,3,4)$ represent independent Brownian motions defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with positive intensities $\sigma_i > 0$ $(i = 1,2,3,4)$. Here, $\mu_2 = \mu + \alpha_2$ and $\mu_3 = \mu + \alpha_3$ are the general mortality rates, where α_2 and α_3 are the mortality rates associated to the disease.

This section aims to analyze the asymptotic behavior of the epidemic system (V.19). We will give a sharp condition for the existence of the unique ergodic stationary distribution by using a new method different from the Lyapunov method described in [57]. Our method allows combining the persistence and the ergodicity under the same optimal condition which is not provided with the classical method. As a result, we treat a problem that is ignored in literature; it is not biologically reasonable to consider two distinct conditions for the persistence and the existence of the stationary distribution. Another approach to study the asymptotic behavior of the stochastic solution was considered by [48, 47, 146]. They investigated the asymptotic behavior around the endemic and disease-free equilibria by measuring the mean value of the oscillation between the solution and the deterministic equilibria. But, to the best of our knowledge, this approach has not been obtained theoretically until now in the case of the stochastic system (V.19). Thus, it is significant to study the dynamical behaviors around the deterministic equilibria $E^{\ominus}$ and $E^{\circledast}$ in this general context. Under some restricted conditions of white noise intensities, we prove that the time average of the distance (or mean distance) between the stochastic solution and the deterministic equilibrium is eventually small. Furthermore, we focus on sufficient conditions for the extinction case in order to give a general view of the dynamics of an epidemic modeled by the system (V.19).

The following lemma aims to show the existence and uniqueness of the global positive solution, which is necessary for analyzing the long-term behavior of the model (V.19). Since the proof is standard (see [58]), we present it without proof.

Lemma 3.1. *For any initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$, there exists a unique positive solution $(S(t), I(t), R(t))$ of the system (V.19) on $t \geq 0$, and the solution will remain in $\mathbb{R}^3_+$ with probability one.*

By using Lemma 3.1, we obtain the following important result.

Lemma 3.2. *Let $(S(t), I(t), R(t))$ be the solution of (V.19) with initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$, then for any $p_1 \in \left[\frac{1}{2}, \frac{1}{2} + \frac{\mu_1 - b}{\sigma^2}\right)$, we have the following estimations:*

$$\mathbb{E}[N^{2p_1}(t)] \leq \frac{2C_1(2p_1)}{C_2(2p_1)} + N^{2p_1}(0)e^{\{-p_1 C_2(2p_1)t\}} \quad \text{and} \quad \limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}[N^{2p_1}(u)]\mathrm{d}u \leq \frac{2C_1(2p_1)}{C_2(2p_1)},$$

where $\sigma^2 = \max\{\sigma_1^2, \sigma_2^2, \sigma_3^2\}$, $C_2(2p_1) = (\mu_1 - b) - \frac{(2p_1-1)}{2}\sigma^2$ and $C_1(2p_1) = \sup_{x>0}\left\{Ax^{2p_1-1} - \frac{C_2(2p_1)}{2}x^{2p_1}\right\}$.

Proof. From system (I.2), we have

$$\mathrm{d}N(t) = \Big(A - (\mu_1 - b)N(t) - \alpha_2 I(t) - \alpha_3 R(t)\Big)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \sigma_3 R(t)\mathrm{d}\mathcal{W}_3(t).$$

Making use of Itô's formula, we obtain

$$\begin{aligned}\mathrm{d}N^{2p_1}(t) &= 2p_1N^{2p_1-1}(t)\Big(A-(\mu_1-b)N(t)-\alpha_2I(t)-\alpha_3R(t)\Big)\mathrm{d}t\\&\quad+\frac{2p_1(2p_1-1)}{2}N^{2p_1-2}(t)\Big(\sigma_1^2S^2(t)+\sigma_2^2I^2(t)+\sigma_3^2R^2(t)\Big)\mathrm{d}t\\&\quad+2p_1N^{2p_1-1}(t)\Big(\sigma_1S(t)\mathrm{d}\mathcal{W}_1(t)+\sigma_2I(t)\mathrm{d}\mathcal{W}_2(t)+\sigma_3R(t)\mathrm{d}\mathcal{W}_3(t)\Big).\end{aligned}$$

Then

$$\begin{aligned}\mathrm{d}N^{2p_1}(t) &\le 2p_1\Big[AN^{2p_1-1}(t)-\Big((\mu_1-b)-\frac{(2p_1-1)}{2}\sigma^2\Big)N^{2p_1}(t)\Big]\mathrm{d}t\\&\quad+2p_1N^{2p_1-1}(t)\Big(\sigma_1S(t)\mathrm{d}\mathcal{W}_1(t)+\sigma_2I(t)\mathrm{d}\mathcal{W}_2(t)+\sigma_3R(t)\mathrm{d}\mathcal{W}_3(t)\Big)\\&\le 2p_1\Big[C_1(2p_1)-\frac{C_2(2p_1)}{2}N^{2p_1}(t)\Big]dt\\&\quad+2p_1N^{2p_1-1}(t)\Big(\sigma_1S(t)\mathrm{d}\mathcal{W}_1(t)+\sigma_2I(t)\mathrm{d}\mathcal{W}_2(t)+\sigma_3R(t)\mathrm{d}\mathcal{W}_3(t)\Big),\end{aligned}\tag{V.20}$$

where $C_1(2p_1)=\sup_{x>0}\{Ax^{2p_1-1}-\frac{C_2(2p_1)}{2}x^{2p_1}\}$ and $C_2(2p_1)=(\mu_1-b)-\frac{(2p_1-1)}{2}\eta^2$. On the other hand, we have

$$\mathrm{d}\Big[N^{2p_1}(t)e^{p_1C_2(2p_1)t}\Big]=p_1C_2(2p_1)N^{2p_1}(t)e^{p_1C_2(2p_1)t}+e^{p_1C_2(2p_1)t}\mathrm{d}N^{2p_1}(t).$$

In view of (V.20), we get

$$\begin{aligned}&\mathrm{d}\Big[N^{2p_1}(t)e^{p_1C_2(2p_1)t}\Big]\\&\le 2p_1e^{p_1C_2(2p_1)t}\Big[C_1(2p_1)\mathrm{d}t+N^{2p_1-1}(t)\Big(\sigma_1S(t)\mathrm{d}\mathcal{W}_1(t)+\sigma_2I(t)\mathrm{d}\mathcal{W}_2(t)+\sigma_3R(t)\mathrm{d}\mathcal{W}_3(t)\Big)\Big].\end{aligned}\tag{V.21}$$

Integrating (V.21) from 0 to t, then by taking the expectation, we get

$$\mathbb{E}[N^{2p_1}(t)]\le N^{2p_1}(0)e^{-p_1C_2(2p_1)t}+2p_1C_1(2p_1)\int_0^t e^{-p_1C_2(2p_1)(t-u)}\mathrm{d}u\le N^{2p_1}(0)e^{-p_1C_2(2p_1)t}+\frac{2C_1(2p_1)}{C_2(2p_1)}.$$

Obviously, we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}[N^{2p_1}(u)]\mathrm{d}u\le N^{2p_1}(0)\limsup_{t\to\infty}\frac{1}{t}\int_0^t e^{-p_1C_2(2p_1)u}\mathrm{d}u+\frac{2C_1(2p_1)}{C_2(2p_1)}=\frac{2C_1(2p_1)}{C_2(2p_1)}.$$

This completes the proof of Lemma 3.1. ∎

Now, we consider the following auxiliary stochastic differential equation

$$\mathrm{d}\psi(t)=\Big(A-(\mu_1-b)\psi(t)\Big)\mathrm{d}t+\sigma_1S(t)\mathrm{d}\mathcal{W}_1(t)+\sigma_2I(t)\mathrm{d}\mathcal{W}_2(t)+\sigma_3R(t)\mathrm{d}\mathcal{W}_3(t),\tag{V.22}$$

with the initial condition $\psi(0)=N(0)>0$.

Remark 3.3. Note the positivity of the solutions $(S(t),I(t),R(t))$ and $\psi(t)$, then $N(t)\le\psi(t)$ a.s. holds due to the stochastic comparison theorem.

Lemma 3.4 ([63]). *Let $(S(t),I(t),R(t))$ be the positive solution of the system (V.19) with any given initial condition $(S(0),I(0),R(0))\in\mathbb{R}^3_+$. Let also $\psi(t)\in\mathbb{R}_+$ be the solution of the equation (V.22) with any given initial value $\psi(0)\in\mathbb{R}_+$. Then*

$$\lim_{t\to\infty}\frac{\psi(t)}{t}=0,\quad\lim_{t\to\infty}\frac{\psi^2(t)}{t}=0,\quad\lim_{t\to\infty}\frac{S(t)}{t}=0,\quad\lim_{t\to\infty}\frac{I(t)}{t}=0,\quad\text{and}\quad\lim_{t\to\infty}\frac{R(t)}{t}=0\quad a.s.$$

Moreover, if $\mu_1 - b > \frac{\sigma^2}{2}$, *then*

$$\lim_{t\to\infty} \frac{\int_0^t S(u)\mathrm{d}\mathcal{W}_1(u)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t S(u)\mathrm{d}\mathcal{W}_4(u)}{t} = 0,$$
$$\lim_{t\to\infty} \frac{\int_0^t I(u)\mathrm{d}\mathcal{W}_2(u)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t R(u)\mathrm{d}\mathcal{W}_3(u)}{t} = 0 \quad \text{a.s.}$$

Similar to the proof of the last Lemma (see Lemma 2.2 in [93]), we can easily prove the next result by using Burkholder-Davis-Gundy inequality [93], Doob's martingale inequality [93], and Borel-Cantelli Lemma [93].

Lemma 3.5 ([63]). *Let* $(S(t), I(t), R(t))$ *be the positive solution of the system (V.19) with any given initial condition* $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$ *and* $\psi(t) \in \mathbb{R}_+$ *be the solution of the equation (V.22) with any given initial value* $\psi(0) \in \mathbb{R}_+$. *If* $\mu_1 - b > \frac{\sigma^2}{2}$, *then*

$$\lim_{t\to\infty} \frac{\int_0^t S^2(u)\mathrm{d}\mathcal{W}_1(u)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t S^2(u)\mathrm{d}\mathcal{W}_4(u)}{t} = 0,$$
$$\lim_{t\to\infty} \frac{\int_0^t I^2(u)\mathrm{d}\mathcal{W}_2(u)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t R^2(u)\mathrm{d}\mathcal{W}_3(u)}{t} = 0 \quad \text{a.s.}$$

Remark 3.6. In the following, we always assume that $\mu_1 - b > \frac{\sigma^2}{2}$. This assumption is a standard key parametric condition for the proofs of our results.

Now, we are in a position to show and prove our main results, which is given in the following subsection.

3.2 Main results

3.2.1 Ergodicity and persistence in the mean

Before establishing the condition for the existence of the stationary distribution to system (V.19), we start with the following lemma.

Lemma 3.7. *Let* $\psi(t) \in \mathbb{R}_+$ *be the solution of the equation (V.22) with any given initial value* $\psi(0) = N(0) \in \mathbb{R}_+$. *Then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \psi(u)\mathrm{d}u = \frac{A}{\mu_1 - b} \quad \text{a.s.}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \psi^2(u)\mathrm{d}u \le \frac{2A^2}{(\mu_1 - b)(2(\mu_1 - b) - \sigma^2)} \quad \text{a.s.}$$

Proof. We integrate (V.22) from 0 to t on both sides, then

$$\psi(t) - \psi(0) = At + (\mu_1 - b)\int_0^t \psi(u)\mathrm{d}u + \sigma_1 \int_0^t S(u)\mathrm{d}\mathcal{W}_1(u) + \sigma_2 \int_0^t I(u)\mathrm{d}\mathcal{W}_2(u) + \sigma_3 \int_0^t R(u)\mathrm{d}\mathcal{W}_3(u).$$

Therefore, we get

$$\frac{1}{t}\int_0^t \psi(u)\mathrm{d}u = \frac{A}{\mu_1 - b} - \frac{\psi(t) - \psi(0)}{(\mu_1 - b)t} + \frac{\sigma_1}{(\mu_1 - b)t}\int_0^t S(u)\mathrm{d}\mathcal{W}_2(u)$$
$$+ \frac{\sigma_2}{(\mu_1 - b)t}\int_0^t I(u)\mathrm{d}\mathcal{W}_2(u) + \frac{\sigma_3}{(\mu_1 - b)t}\int_0^t R(u)\mathrm{d}\mathcal{W}_3(u).$$

By using Lemma 3.4, we get

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \psi(u)\mathrm{d}u = \frac{A}{\mu_1 - b} \quad \text{a.s.} \tag{V.23}$$

By Itô's formula for all $t \geq 0$, we have

$$\begin{aligned}
\mathrm{d}\psi^2(t) &= \Big(2\psi(t)\big(A - (\mu_1 - b)\psi(t)\big) + \sigma_1^2 S^2(t) + \sigma_2^2 I^2(t) + \sigma_3^2 R^2(t)\Big)\mathrm{d}t \\
&\quad + 2\psi(t)\Big(\sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \sigma_3 R(t)\mathrm{d}\mathcal{W}_3(t)\Big) \\
&\leq \Big(2\psi(t)\big(A - (\mu_1 - b)\psi(t)\big) + \sigma^2\psi^2(t)\Big)\mathrm{d}t \\
&\quad + 2\psi(t)\Big(\sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \sigma_3 R(t)\mathrm{d}\mathcal{W}_3(t)\Big).
\end{aligned} \tag{V.24}$$

Obviously, we obtain

$$\begin{aligned}
\frac{1}{t}\int_0^t \psi^2(u)\mathrm{d}u &\leq \frac{2A}{(2(\mu_1 - b) - \sigma^2)}\frac{1}{t}\int_0^t \psi(u)\mathrm{d}u - \frac{\psi^2(t) - \psi^2(0)}{(2(\mu_1 - b) - \sigma^2)t} + \frac{\sigma_1}{(2(\mu_1 - b) - \sigma^2)t}\int_0^t \psi(u)S(u)\mathrm{d}\mathcal{W}_1(u) \\
&\quad + \frac{\sigma_2}{(2(\mu_1 - b) - \sigma^2)t}\int_0^t \psi(u)I(u)\mathrm{d}\mathcal{W}_2(u) + \frac{\sigma_3}{(2(\mu_1 - b) - \sigma^2)t}\int_0^t \psi(u)R(u)\mathrm{d}\mathcal{W}_3(u).
\end{aligned}$$

Using the same method as that used in the demonstration of Lemma 2.5. in [63], we obtain

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\int_0^t \psi(u)S(u)\mathrm{d}\mathcal{W}_1(u) &= 0 \\
\lim_{t\to\infty}\frac{1}{t}\int_0^t \psi(u)I(u)\mathrm{d}\mathcal{W}_2(u) &= 0 \\
\lim_{t\to\infty}\frac{1}{t}\int_0^t \psi(u)R(u)\mathrm{d}\mathcal{W}_3(u) &= 0 \quad \text{a.s.}
\end{aligned}$$

Therefore, we can see that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \psi^2(u)\mathrm{d}u \leq \frac{2A^2}{(\mu_1 - b)(2(\mu_1 - b) - \sigma^2)} \quad \text{a.s.} \tag{V.25}$$

■

For simplicity, we define

$$\mathcal{R}_0^s = \frac{1}{\Big(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\Big)}\left(\frac{\beta A}{(\mu_1 - b)} - \frac{\sigma_4^2 A^2}{(\mu_1 - b)(2(\mu_1 - b) - \sigma^2)}\right).$$

Theorem 3.8. *If $\mathcal{R}_0^s > 1$, then the stochastic system (V.19) admits a unique stationary distribution and it has the ergodic property for any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$.*

Proof. Similar to the proof of Lemma 3.2 in [135], we briefly verify the Feller property of the stochastic model (V.19). The main purpose of this demonstration is to prove that (II.22) is impossible. We define

$$\mathcal{V}(t) = \ln I(t) - m\big(\psi(t) - S(t)\big),$$

where m is a positive constant to be determined later. Applying Itô's formula gives

$$\begin{aligned}
\mathrm{d}\mathcal{V}(t) &= \left(\beta S(t) - (\mu_2 + \gamma_1 + \gamma_2 - qb) - \frac{\sigma_4^2}{2}S^2(t) - \frac{\sigma_2^2}{2}\right)\mathrm{d}t + \sigma_4 S(t)\mathrm{d}\mathcal{W}_4(t) + \sigma_2\mathrm{d}\mathcal{W}_2(t) \\
&\quad - m\Big(-(\mu_1 - b)\big(\psi(t) - S(t)\big) + \beta S(t)I(t) - (pb + \gamma_1)I(t) - (b + k)R(t)\Big)\mathrm{d}t \\
&\quad - m\sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) - m\sigma_3 R(t)\mathrm{d}\mathcal{W}_3(t) + m\sigma_4 S(t)I(t)\mathrm{d}\mathcal{W}_4(t).
\end{aligned}$$

We choose $m = \frac{\beta}{\mu_1 - b}$, then we have

$$\begin{aligned} d\mathcal{V}(t) = &\left(\beta S(t) - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right) - \frac{\sigma_4^2}{2}S^2(t)\right)dt + \beta\Big(\psi(t) - S(t)\Big)dt \\ &+ \frac{\beta(pb + \gamma_1)}{\mu_1 - b}I(t)dt + \frac{\beta(b+k)}{\mu_1 - b}R(t)dt - \frac{\beta^2}{\mu_1 - b}S(t)I(t)dt + \sigma_4 S(t)d\mathcal{W}_4(t) \\ &+ \sigma_2 d\mathcal{W}_2(t) - m\sigma_2 I(t)d\mathcal{W}_2(t) - m\sigma_3 R(t)d\mathcal{W}_3(t) + m\sigma_4 S(t)I(t)d\mathcal{W}_4(t). \end{aligned}$$

Consequently,

$$\begin{aligned} d\mathcal{V}(t) \geq &\left(\beta\psi(t) - \frac{\sigma_4^2}{2}S^2(t) - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right)\right)dt - \frac{\beta^2}{\mu_1 - b}S(t)I(t)dt + \sigma_4 S(t)d\mathcal{W}_4(t) \\ &+ \sigma_2 d\mathcal{W}_2(t) - m\sigma_2 I(t)d\mathcal{W}_2(t) - m\sigma_3 R(t)d\mathcal{W}_3(t) + m\sigma_4 S(t)I(t)d\mathcal{W}_4(t). \end{aligned} \tag{V.26}$$

Integrating from 0 to t on both sides of (V.26) yields

$$\begin{aligned} \mathcal{V}(t) - \mathcal{V}(0) \geq &\int_0^t \left(\beta\psi(u) - \frac{\sigma_4^2}{2}S^2(u) - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right)\right)du \\ &- \frac{\beta}{\mu_1 - b}\int_0^t \beta S(u)I(u)du + \sigma_4\int_0^t S(u)d\mathcal{W}_4(u) + \sigma_2\mathcal{W}_2(t) \\ &- m\sigma_2\int_0^t I(u)d\mathcal{W}_2(u) - m\sigma_3\int_0^t R(u)d\mathcal{W}_3(u) + m\sigma_4\int_0^t S(u)I(u)d\mathcal{W}_4(u). \end{aligned}$$

Then, we get

$$\begin{aligned} \int_0^t \beta S(u)I(u)du \geq &\frac{\mu_1 - b}{\beta}\int_0^t \left(\beta\psi(u) - \frac{\sigma_1^2}{2}S^2(u) - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right)\right)du + \Big(\psi(t) - S(t)\Big) \\ &- \Big(\psi_0 - S_0\Big) - \frac{\mu_1 - b}{\beta}\ln\frac{I(t)}{I(0)} + \frac{\sigma_4}{m}\int_0^t S(u)d\mathcal{W}_4(u) + \frac{\sigma_2}{m}\mathcal{W}_2(t) \\ &- \sigma_2\int_0^t I(u)d\mathcal{W}_2(u) - \sigma_3\int_0^t R(u)d\mathcal{W}_3(u) + \sigma_4\int_0^t S(u)I(u)d\mathcal{W}_4(u). \end{aligned} \tag{V.27}$$

From Lemma 3.4, one can derive that

$$\lim_{t\to\infty}\frac{\psi(t)}{t} = 0 \text{ and } \lim_{t\to\infty}\frac{S(t)}{t} = 0 \quad \text{a.s.}$$

Moreover,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(u)d\mathcal{W}_4(u) = 0 \ \lim_{t\to\infty}\frac{1}{t}\int_0^t I(u)d\mathcal{W}_2(u) = 0 \text{ and } \lim_{t\to\infty}\frac{1}{t}\int_0^t R(u)d\mathcal{W}_3(u) = 0 \quad \text{a.s.}$$

Applications of the strong law of large numbers shows that

$$\lim_{t\to\infty}\frac{\mathcal{W}_2(t)}{t} = 0 \quad \text{a.s.}$$

Applying similar arguments to those in (Theorem 3.1, [25]), we obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(u)I(u)d\mathcal{W}_4(u) = 0 \quad \text{a.s.}$$

Since $\limsup_{t\to\infty}\frac{1}{t}\ln\frac{I(t)}{I(0)} \leq \limsup_{t\to\infty}\frac{1}{t}\ln\frac{N(t)}{I(0)} \leq 0$, we get

$$\begin{aligned} \liminf_{t\to\infty}\frac{1}{t}\int_0^t \beta S(u)I(u)du &\geq \frac{(\mu_1 - b)}{\beta}\liminf_{t\to\infty}\frac{1}{t}\int_0^t \left(\beta\psi(u) - \frac{\sigma_1^2}{2}\psi^2(u) - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right)\right)du \\ &= \frac{(\mu_1 - b)}{\beta}\lim_{t\to\infty}\frac{1}{t}\int_0^t \left(\beta\psi(u) - \frac{\sigma_1^2}{2}\psi^2(u) - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right)\right)du. \end{aligned} \tag{V.28}$$

By Lemma 3.7, we then have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \beta S(u)I(u)\mathrm{d}u \geq \frac{(\mu_1-b)}{\beta}\times\left(\frac{\beta A}{(\mu_1-b)}-\frac{A^2\sigma_1^2}{(\mu_1-b)(2(\mu_1-b)-\sigma^2)}-\left(\mu_2+\gamma_1+\gamma_2-qb+\frac{\sigma_2^2}{2}\right)\right)$$
$$=\frac{(\mu_1-b)}{\beta}\left(\mu_2+\gamma_1+\gamma_2-qb+\frac{\sigma_2^2}{2}\right)(\mathcal{R}_0^s-1)>0 \quad \text{a.s.}$$

The remaining of the proof follows the lines of demonstration of Theorem 1.10. ■

In the following result, we give the condition for persistence in the mean of the disease.

Theorem 3.9. *If $\mathcal{R}_0^s > 1$, then for any value $(S(0),I(0),R(0)) \in \mathbb{R}^3_+$, the disease $I(t)$ is persistent in the mean.*

Proof. By the model (V.19), we have

$$\begin{aligned}\mathrm{d}(S(t)+I(t)+R(t)) &= \big(A-(\mu_1-b)(S(t)+I(t)+R(t))-\alpha_2 I(t)-\alpha_3 R(t)\big)\mathrm{d}t\\ &\quad+\sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t)+\sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t)+\sigma_3 R(t)\mathrm{d}\mathcal{W}_3(t).\end{aligned} \tag{V.29}$$

Integrating (V.29) from 0 to t, and then dividing t on both sides, we get

$$\begin{aligned}&\frac{1}{t}\Big((S(t)+I(t)+R(t))-(S(0)+I(0)+R(0))\Big)\\ &=A-(\mu_1-b)\frac{1}{t}\int_0^t S(u)\mathrm{d}u-(\mu_2-b)\frac{1}{t}\int_0^t I(u)\mathrm{d}u-(\mu_3-b)\frac{1}{t}\int_0^t R(u)\mathrm{d}u\\ &\quad+\frac{\sigma_1}{t}\int_0^t S(u)\mathrm{d}\mathcal{W}_1(u)+\frac{\sigma_2}{t}\int_0^t I(u)\mathrm{d}\mathcal{W}_2(u)+\frac{\sigma_3}{t}\int_0^t R(u)\mathrm{d}\mathcal{W}_3(u).\end{aligned}$$

Taking the integration for the third equation of model (V.19) yields

$$R(t)-R(0)=\gamma_2\int_0^t I(u)\mathrm{d}u-(\mu_1+k)\int_0^t R(u)\mathrm{d}u+\sigma_3\int_0^t R(u)\mathrm{d}\mathcal{W}_3(u). \tag{V.30}$$

Dividing t on both sides of equation (V.30), we have

$$\begin{aligned}\frac{1}{t}\int_0^t R(u)\mathrm{d}u &= \frac{\gamma_2}{(\mu_1+k)}\frac{1}{t}\int_0^t I(u)\mathrm{d}u+\frac{\sigma_3}{(\mu_1+k)}\frac{1}{t}\int_0^t R(u)\mathrm{d}\mathcal{W}_3(u)\\ &\quad-\frac{1}{(\mu_1+k)t}(R(t)-R(0)).\end{aligned}$$

Then, one can obtain that

$$\frac{1}{t}\int_0^t S(u)\mathrm{d}u=\frac{A}{(\mu_1-b)}-\frac{1}{t}\left(\frac{(\mu_2-b)}{(\mu_1-b)}+\frac{\gamma_2}{(\mu_1+k)}\right)\int_0^t I(u)\mathrm{d}u+\Phi_1(t), \tag{V.31}$$

where

$$\begin{aligned}\Phi_1(t) &= \frac{\sigma_3}{(\mu_1+k)t}\int_0^t R(u)\mathrm{d}\mathcal{W}_3(u)-\frac{1}{(\mu_1+k)t}(R(t)-R(0))\\ &\quad+\frac{1}{(\mu_1-b)t}\left(\frac{\sigma_1}{t}\int_0^t S(u)\mathrm{d}\mathcal{W}_1(u)+\frac{\sigma_2}{t}\int_0^t I(u)\mathrm{d}\mathcal{W}_2(u)+\frac{\sigma_3}{t}\int_0^t R(u)\mathrm{d}\mathcal{W}_3(u)\right)\\ &\quad-\frac{1}{(\mu_1-b)t}\Big((S(t)+I(t)+R(t))-(S(0)+I(0)+R(0))\Big).\end{aligned}$$

Applying Itô's formula to the second equation of (V.19), we get

$$\mathrm{d}\ln I(t)=\left(\beta S(t)-\left(\mu_2+\gamma_1+\gamma_2-qb+\frac{\sigma_2^2}{2}\right)-\frac{\sigma_4^2}{2}S^2(t)\right)\mathrm{d}t+\sigma_4 S(t)\mathrm{d}\mathcal{W}_4(t)+\sigma_2\mathrm{d}\mathcal{W}_2(t). \tag{V.32}$$

Integrating (V.32) from 0 to t and then dividing t on both sides, we have

$$\frac{1}{t}(\ln I(t) - \ln(0)) = \frac{\beta}{t}\int_0^t S(u)\mathrm{d}u - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right) - \frac{\sigma_4^2}{2t}\int_0^t S^2(u)\mathrm{d}u + \frac{\sigma_4}{t}\int_0^t S(u)\mathrm{d}\mathcal{W}_4(u) + \frac{\sigma_2}{t}\mathcal{W}_2(t).$$

From (V.31), we get

$$\frac{1}{t}(\ln I(t) - \ln I(0)) = \frac{\beta A}{(\mu_1 - b)} - \frac{\beta}{t}\left(\frac{(\mu_2 - b)}{(\mu_1 - b)} + \frac{\gamma_2}{(\mu_1 + k)}\right)\int_0^t I(u)\mathrm{d}u + \beta\Phi_1(t) - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right) - \frac{\sigma_4^2}{2t}\int_0^t S^2(u)\mathrm{d}u + \frac{\sigma_4}{t}\int_0^t S(u)\mathrm{d}\mathcal{W}_4(u) + \frac{\sigma_2}{t}\mathcal{W}_2(t).$$

By Lemma 3.7 and Remark 3.3, we obtain

$$\frac{1}{t}(\ln I(t) - \ln I(0)) \geq \left(\frac{\beta A}{(\mu_1 - b)} - \frac{\sigma_4^2}{2t}\int_0^t \psi^2(u)\mathrm{d}u - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right)\right) + \beta\Phi_1(t) - \frac{\beta}{t}\left(\frac{(\mu_2 - b)}{(\mu_1 - b)} + \frac{\gamma_2}{(\mu_1 + k)}\right)\int_0^t I(u)\mathrm{d}u + \frac{\sigma_4}{t}\int_0^t S(u)\mathrm{d}\mathcal{W}_4(u) + \frac{\sigma_2}{t}\mathcal{W}_2(t).$$

Hence, we further have

$$\frac{\beta}{t}\left(\frac{(\mu_2 - b)}{(\mu_1 - b)} + \frac{\gamma_2}{(\mu_1 + k)}\right)\int_0^t I(u)\mathrm{d}u \geq \left(\frac{\beta A}{(\mu_1 - b)} - \frac{\sigma_4^2}{2t}\int_0^t \psi^2(u)\mathrm{d}u - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right)\right) + \beta\Phi_1(t) - \frac{1}{t}(\ln I(t) - \ln I(0)) + \frac{\sigma_4}{t}\int_0^t S(u)\mathrm{d}\mathcal{W}_4(u) + \frac{\sigma_2}{t}\mathcal{W}_2(t).$$

Based on Lemma 3.4, we obtain

$$\lim_{t\to\infty}\Phi_1(t) = 0, \quad \lim_{t\to\infty}\frac{1}{t}(\ln I(t) - \ln I(0)) = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{\sigma_4}{t}\int_0^t S(u)\mathrm{d}\mathcal{W}_4(u) = 0 \quad \text{a.s.}$$

Applications of the strong law of large numbers shows that

$$\lim_{t\to\infty}\frac{\sigma_2\mathcal{W}_2(t)}{t} = 0 \quad \text{a.s.}$$

Then, we conclude that

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(u)\mathrm{d}u \\
&\geq \frac{1}{\beta}\left(\frac{(\mu_2 - b)}{(\mu_1 - b)} + \frac{\gamma_2}{(\mu_1 + k)}\right)^{-1}\left(\frac{\beta A}{(\mu_1 - b)} - \frac{2A^2}{(\mu_1 - b)(2(\mu_1 - b) - \sigma^2)} - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right)\right) \\
&= \frac{1}{\beta}\left(\frac{(\mu_2 - b)}{(\mu_1 - b)} + \frac{\gamma_2}{(\mu_1 + k)}\right)^{-1}\left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right)(\mathcal{R}_0^s - 1) > 0 \ \text{ a.s.}
\end{aligned}$$

This shows that the system (V.19) is persistent in the mean with probability one. This completes the proof. ∎

3.2.2 Asymptotic behavior around the endemic equilibrium

When $\mathcal{R}_0 = \dfrac{\beta A}{(\mu_1 - b)(\mu_2 + \gamma_1 + \gamma_2 - qb)} > 1$, the endemic equilibrium $E^{\circledast}$ of the deterministic system is globally asymptotically stable. However, the stochastic model (V.19) does not have the endemic equilibrium $E^{\circledast}$. In this subsection, we show that the solution of the stochastic model (V.19) oscillates around $E^{\circledast}$ under certain conditions. We prove that the distance between the solution $X(t) = (S(t), I(t), R(t))$ and the endemic equilibrium $E^{\circledast}$ has the following form:

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \|X(u) - E^{\circledast}\|^2 \mathrm{d}u \le C, \quad \text{a.s.}$$

where C is a positive constant. For simplicity, we introduce the following notations:

$$\begin{aligned}
r_1 &= \gamma_2\big(2(\mu_1 - b) + \alpha_2 + \gamma_2\big)\big(2(\mu_1 - b) + \alpha_3\big) + \gamma_2(b + k)\big(2(\mu_1 - b) + \alpha_2\big),\\
r_2 &= \gamma_2\beta\big(2(\mu_1 - b) + \alpha_3\big),\\
r_3 &= \gamma_2\beta(b + k),\\
r_4 &= \alpha_2\beta(b + k),\\
s_1 &= \big(r_1 + r_3\big)\big((\mu_1 - b) - \sigma_1^2\big),\\
s_2 &= r_2\big((\mu_2 - b + \gamma_2) - \sigma_2^2\big) + r_3\big((\mu_2 - b) - \sigma_2^2\big),\\
s_3 &= r_3\big((\mu_3 - b) - \sigma_3^2\big) + r_4\big((\mu_3 + k) - \sigma_3^2\big),\\
s_4 &= \frac{r_1}{2}\sigma_2^2 I^* + \frac{r_2}{2}\sigma_1^2 S^{*2} + \frac{r_2}{2}\sigma_2^2 I^{*2} + r_3\sigma_1^2 S^{*2} + r_3\sigma_2^2 I^{*2} + r_3\sigma_3^2 R^{*2} + r_3\sigma_4^2 R^{*2},\\
\hat{\mathcal{S}} &= s_4 + \frac{r_1\sigma_4^2 I^*}{(\mu_1 - b)(2(\mu_1 - b) - \sigma^2)}.
\end{aligned}$$

Theorem 3.10. *Assume that $(S(t), I(t), R(t))$ is the solution of the model (V.19) with the initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$. If $\mathcal{R}_0 > 1$, $s_1 > 0$, $s_2 > 0$, and $s_3 > 0$, then*

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \Big(s_1(S(u) - S^*)^2 + s_2(I(u) - I^*)^2 + s_3(R(u) - R^*)^2\Big)\mathrm{d}u \le \hat{\mathcal{S}} \quad \text{a.s.}$$

Proof. We define the following non-negative function

$$f(S, I, R) = r_1 f_1(I) + r_2 f_2(S, I) + r_3 f_3(S, I, R) + r_4 f_4(R),$$

where

$$\begin{aligned}
f_1(I) &= I - I^* - I^* \ln \frac{I}{I^*},\\
f_2(S, I) &= \frac{1}{2}(S - S^* + I - I^*)^2,\\
f_3(S, I, R) &= \frac{1}{2}(S - S^* + I - I^* + R - R^*)^2,\\
f_4(R) &= \frac{1}{2}(R - R^*)^2.
\end{aligned}$$

Then, by Itô's formula, we obtain

$$\mathrm{d}f_1(I) = \mathcal{L}f_1 \mathrm{d}t + \sigma_2 (I - I^*)\, \mathrm{d}\mathcal{W}_2(t) + \sigma_4 S (I - I^*)\, \mathrm{d}\mathcal{W}_4(t),$$

where

$$\mathcal{L}f_1 = \beta (S - S^*)(I - I^*) + \frac{\sigma_2^2}{2} I^* + \frac{\sigma_4^2}{2} S^2 I^*.$$

Similarly, by Itô's formula, we arrive at

$$\mathrm{d}f_2(S,I) = \mathcal{L}f_2\mathrm{d}t + (S - S^* + I - I^*)(\sigma_1 S\mathrm{d}\mathcal{W}_1(t) + \sigma_2 I\mathrm{d}\mathcal{W}_2(t)),$$

where

$$\begin{aligned}\mathcal{L}f_2 &= (S - S^* + I - I^*)\big(-(\mu_1 - b)(S - S^*) + (b + k)(R - R^*) \\ &\quad - (\mu_2 - b + \gamma_2)(I - I^*)\big) + \frac{\sigma_1^2}{2}S^2 + \frac{\sigma_2^2}{2}I^2 \\ &\le (S - S^* + I - I^*)\big(-(\mu_1 - b)(S - S^*) + (b + k)(R - R^*) \\ &\quad - (\mu_2 - b + \gamma_2)(I - I^*)\big) + \sigma_1^2(S - S^*)^2 + \sigma_1^2 S^{*2} + \sigma_2^2(I - I^*)^2 + \sigma_2^2 I^{*2}.\end{aligned}$$

In a similar way, we get

$$\mathrm{d}f_3(S,I,R) = \mathcal{L}f_3\mathrm{d}t + (S - S^* + I - I^* + R - R^*)(\sigma_1 S\mathrm{d}\mathcal{W}_1(t) + \sigma_2 I\mathrm{d}\mathcal{W}_2(t) + \sigma_3 R\mathrm{d}\mathcal{W}_3(t)),$$

where

$$\begin{aligned}\mathcal{L}f_3 &= (S - S^* + I - I^* + R - R^*)\big(-(\mu_1 - b)(S - S^*) - (\mu_2 - b)(I - I^*) \\ &\quad - (\mu_3 - b)(R - R^*)\big) + \frac{\sigma_1^2}{2}S^2 + \frac{\sigma_2^2}{2}I^2 + \frac{\sigma_3^2}{2}R^2 \\ &\le (S - S^* + I - I^* + R - R^*)\big(-(\mu_1 - b)(S - S^*) - (\mu_2 - b)(I - I^*) \\ &\quad - (\mu_3 - b)(R - R^*)\big) + \sigma_1^2(S - S^*)^2 + \sigma_1^2 S^{*2} + \sigma_2^2(I - I^*)^2 \\ &\quad + \sigma_2^2 I^{*2} + \sigma_3^2(R - R^*)^2 + \sigma_3^2 R^{*2}.\end{aligned}$$

Using Itô's formula, it follows that

$$\mathrm{d}f_4(R) = \mathcal{L}f_4\mathrm{d}t + \sigma_4(R - R^*)R\mathrm{d}\mathcal{W}_4(t),$$

where

$$\begin{aligned}\mathcal{L}f_4 &= (R - R^*)\big(\gamma_2(I - I^*) - (\mu_3 + k)(R - R^*)\big) + \frac{\sigma_3^2}{2}R^2 \\ &\le (R - R^*)\big(\gamma_2(I - I^*) - (\mu_3 + k)(R - R^*)\big) + \sigma_3^2(R - R^*)^2 + \sigma_3^2 R^{*2}.\end{aligned}$$

Therefore, we can see that

$$\begin{aligned}\mathrm{d}f(S,I,R) &= r_1\mathrm{d}f_1(I) + r_2\mathrm{d}f_2(S,I) + r_3\mathrm{d}f_3(S,I,R) + r_4\mathrm{d}f_4(R) \\ &= \mathcal{L}f\mathrm{d}t + r_1\left[\sigma_2\,(I - I^*)\,\mathrm{d}\mathcal{W}_2(t) + \sigma_4 S\,(I - I^*)\,\mathrm{d}\mathcal{W}_4(t)\right] \\ &\quad + r_2\left[(S - S^* + I - I^*)(\sigma_1 S\mathrm{d}\mathcal{W}_1(t) + \sigma_2 I\mathrm{d}\mathcal{W}_2(t))\right] \\ &\quad + r_3\left[(S - S^* + I - I^* + R - R^*)(\sigma_1 S\mathrm{d}\mathcal{W}_1(t) + \sigma_2 I\mathrm{d}\mathcal{W}_2(t)\right. \\ &\quad \left. + \sigma_3 R(t)\mathrm{d}\mathcal{W}_3(s))\right] + r_4\sigma_4(R - R^*)R\mathrm{d}\mathcal{W}_3(t), \qquad \text{(V.33)}\end{aligned}$$

where

$$
\begin{aligned}
\mathcal{L}f &= r_1\mathcal{L}f_1 + r_2\mathcal{L}f_2 + r_3\mathcal{L}f_3 + r_4\mathcal{L}f_4 \\
&\le r_1\beta(S - S^* + I - I^*) - r_2(\mu_1 - b)(S - S^*)^2 + r_2(b + k)(S - S^*)(R - R^*) \\
&\quad - r_2(\mu_2 - b + \gamma_2)(S - S^*)(I - I^*) - r_2(\mu_1 - b)(S - S^*)(I - I^*) \\
&\quad + r_2(b + k)(I - I^*)(R - R^*) - r_2(\mu_2 - b + \gamma_2)(I - I^*)^2 \\
&\quad - r_3(\mu_1 - b)(S - S^*)^2 - r_3(\mu_2 - b)(S - S^*)(I - I^*) \\
&\quad - r_3(\mu_3 - b)(S - S^*)(R - R^*) - r_3(\mu_1 - b)(S - S^*)(I - I^*) \\
&\quad - r_3(\mu_2 - b)(I - I^*)^2 - r_3(\mu_3 - b)(I - I^*)(R - R^*) \\
&\quad - r_3(\mu_1 - b)(R - R^*)(S - S^*) - r_3(\mu_2 - b)(I - I^*)(R - R^*) \\
&\quad - r_3(\mu_3 - b)(R - R^*)^2 + r_4\gamma_2(R - R^*)(I - I^*) \\
&\quad - r_4(\mu_3 + k)(R - R^*)^2 + r_1\frac{\sigma_2^2}{2}I^* + r_1\frac{\sigma_4^2}{2}S^2I^* + r_2\sigma_1^2(S - S^*)^2 \\
&\quad + r_2\sigma_1^2S^{*2} + r_2\sigma_2^2(I - I^*)^2 + r_2\sigma_2^2I^{*2} + r_3\sigma_1^2(S - S^*)^2 + r_3\sigma_1^2S^{*2} \\
&\quad + r_3\sigma_2^2(I - I^*)^2 + r_3\sigma_2^2I^{*2} + r_3\sigma_3^2(R - R^*)^2 + r_3\sigma_3^2R^{*2}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\mathcal{L}f &\le r_1\Big[\beta\,(I - I^*)\,(S - S^*) + \frac{\sigma_2^2}{2}I^* + \frac{\sigma_4^2}{2}S^2I^*\Big] \\
&\quad + r_2\Big[(S - S^* + I - I^*)\big(-(\mu_1 - b)(S - S^*) + (b + k)(R - R^*) \\
&\quad + (\mu_2 - b + \gamma_1 + \gamma_2)(I - I^*)\big) + \sigma_1^2(S - S^*)^2 + \sigma_1^2S^{*2} + \sigma_2^2(I - I^*)^2 + \sigma_2^2I^{*2}\Big] \\
&\quad + r_3\Big[(S - S^* + I - I^* + R - R^*)\big(-(\mu_1 - b)(S - S^*) + (\mu_2 - b)(I - I^*) \\
&\quad + (\mu_3 - b)(R - R^*)\big) + \sigma_1^2(S - S^*)^2 + \sigma_1^2S^{*2} + \sigma_2^2(I - I^*)^2 + \sigma_2^2I^{*2} \\
&\quad + \sigma_3^2(R - R^*)^2 + \sigma_3^2R^{*2}\Big] + r_4\Big[(R - R^*)\big(\gamma_2(I - I^*) - (\mu_3 + k)(R - R^*)\big) \\
&\quad + \sigma_3^2(R - R^*)^2 + \sigma_3^2R^{*2}\Big].
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
\mathcal{L}f &\le -\Big(\big(r_1 + r_3\big)\big((\mu_1 - b) - \sigma_1^2\big)\Big)(S - S^*)^2 \\
&\quad - \Big(r_2\big((\mu_2 - b + \gamma_2) - \sigma_2^2\big) + r_3\big((\mu_2 - b) - \sigma_2^2\big)\Big)(I - I^*)^2 \\
&\quad - \Big(r_3\big((\mu_3 - b) - \sigma_3^2\big) + r_4\big((\mu_3 + k) - \sigma_3^2\big)\Big)(R - R^*)^2 \\
&\quad + \frac{r_1}{2}\sigma_2^2I^* + \frac{r_2}{2}\sigma_1^2S^{*2} + \frac{r_2}{2}\sigma_2^2I^{*2} + r_3\sigma_1^2S^{*2} \\
&\quad + r_3\sigma_2^2I^{*2} + r_3\sigma_3^2R^{*2} + r_3\sigma_4^2R^{*2} + r_1\frac{\sigma_4^2}{2}S^2I^* \\
&= -s_1(S - S^*)^2 - s_2(I - I^*)^2 - s_3(R - R^*)^2 + s_4 + r_1\frac{\sigma_4^2}{2}S^2I^*.
\end{aligned}
$$

Integrating both sides of (V.33) from 0 to t, yields

$$
0 \le f(S(t), I(t), R(t)) = f(S(0), I(0), R(0)) + \int_0^t \mathcal{L}f(S(u), I(u), R(u))\mathrm{d}u + \mathcal{Z}(t),
$$

where

$$\begin{aligned}\mathcal{Z}(t) = {} & r_1\sigma_2\int_0^t (I(u)-I^*)\,\mathrm{d}\mathcal{W}_2(u) + r_1\sigma_4\int_0^t S(u)\,(I(u)-I^*)\,\mathrm{d}\mathcal{W}_4(u)\\ & + r_2\sigma_1\int_0^t (S(u)-S^*+I(u)-I^*)S(u)\mathrm{d}\mathcal{W}_1(u)\\ & + r_2\sigma_2\int_0^t (S(u)-S^*+I(u)-I^*)I(u)\mathrm{d}\mathcal{W}_2(u))\\ & + r_3\sigma_1\int_0^t (S(u)-S^*+I(u)-I^*+R(u)-R^*)S(u)\mathrm{d}\mathcal{W}_1(u)\\ & + r_3\sigma_2\int_0^t (S(u)-S^*+I(u)-I^*+R(u)-R^*)I(u)\mathrm{d}\mathcal{W}_2(u)\\ & + r_3\sigma_3\int_0^t (S(u)-S^*+I(u)-I^*+R(u)-R^*)R(u)\mathrm{d}\mathcal{W}_3(u)\\ & + r_4\int_0^t \sigma_4(R(u)-R^*)R(u)\mathrm{d}\mathcal{W}_3(u).\end{aligned}$$

It then follows that

$$\begin{aligned}0 \le \frac{1}{t}f(S(t),I(t),R(t)) \le {} & \frac{1}{t}f(S(0),I(0),R(0))\\ & - \frac{1}{t}\int_0^t \Big(s_1(S(u)-S^*)^2 - s_2(I(u)-I^*)^2 - s_3(R(u)-R^*)^2\Big)\mathrm{d}u\\ & + s_4 + \frac{r_1\sigma_4^2 I^*}{2t}\int_0^t S^2(s)\mathrm{d}s + \frac{\mathcal{Z}(t)}{t}.\end{aligned}$$

Applying similar arguments to those in (Theorem 3.1, [25], page 1610), we obtain

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\int_0^t S(u)I(u)\mathrm{d}\mathcal{W}_i(u) = 0 \quad (i=1,2,3,4) \quad \text{a.s.}\\ &\lim_{t\to\infty}\frac{1}{t}\int_0^t S(u)R(u)\mathrm{d}\mathcal{W}_i(u) = 0 \quad (i=1,2,3) \quad \text{a.s.}\\ &\lim_{t\to\infty}\frac{1}{t}\int_0^t I(u)R(u)\mathrm{d}\mathcal{W}_i(u) = 0 \quad (i=1,2,3) \quad \text{a.s.}\end{aligned} \tag{V.34}$$

Therefore, by Lemmas 3.4 and 3.5, we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \Big(s_1(S(u)-S^*)^2 + s_2(I(u)-I^*)^2 + s_3(R(u)-R^*)^2\Big)\mathrm{d}u \le s_4 + \frac{r_1\sigma_4^2 I^*}{2}\limsup_{t\to\infty}\frac{1}{t}\int_0^t S^2(u)\mathrm{d}u \quad \text{a.s.}$$

By the stochastic comparison theorem, one has

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \Big(s_1(S(u)-S^*)^2 + s_2(I(u)-I^*)^2 + s_3(R(u)-R^*)^2\Big)\mathrm{d}u \le s_4 + \frac{r_1\sigma_4^2 I^*}{2}\lim_{t\to\infty}\frac{1}{t}\int_0^t \psi^2(u)\mathrm{d}u \quad \text{a.s.}$$

Hence, by Lemma 3.7, we get

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \Big(s_1(S(s)-S^*)^2 + s_2(I(s)-I^*)^2 + s_3(R(s)-R^*)^2\Big)\mathrm{d}s \le \hat{\mathcal{S}} \quad \text{a.s.}$$

The proof is completed. ∎

3.2.3 Sufficient conditions for the extinction of the disease

In this subsection, we will establish the sufficient conditions for the extinction of the disease in the stochastic model (V.19). Define

$$\hat{\mathcal{R}}_0^s = \frac{1}{\left(\mu_2+\gamma_1+\gamma_2-qb+\frac{\sigma_2^2}{2}\right)}\left(\frac{\beta A}{(\mu_1-b)} - \frac{\sigma_4^2A^2}{2(\mu_1-b)^2}\right).$$

Theorem 3.11. *Let $(S(t), I(t), R(t))$ be the solution of system (V.19) with positive initial value. If*

$$\hat{\mathcal{R}}_0^s < 1 \ \text{ and } \ \sigma_4^2 \le \frac{(\mu_1 - b)\beta}{A}, \tag{V.35}$$

or

$$\frac{\beta^2}{2\sigma_4^2} - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right) < 0. \tag{V.36}$$

Then, the disease dies out exponentially with probability one. That is to say,

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} < 0 \quad a.s. \tag{V.37}$$

Proof. By Itô's formula for all $t \ge 0$, we have

$$\begin{aligned} \mathrm{d}\ln I(t) = &\left(\beta S(t) - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right) - \frac{\sigma_4^2}{2}S^2(t)\right)\mathrm{d}t \\ &+ \sigma_4 S(t)\mathrm{d}\mathcal{W}_1(t) + \sigma_2 \mathrm{d}\mathcal{W}_2(t). \end{aligned} \tag{V.38}$$

Integrating (V.38) from 0 to t and then dividing t on both sides, we get

$$\frac{\ln I(t)}{t} = \frac{\beta}{t}\int_0^t S(u)\mathrm{d}u - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right) - \frac{\sigma_4^2}{2t}\int_0^t S^2(u)\mathrm{d}u + \Phi_2(t), \tag{V.39}$$

where

$$\Phi_2(t) = \frac{\sigma_4}{t}\int_0^t S(u)\mathrm{d}\mathcal{W}_4(u) + \frac{\sigma_2 \mathcal{W}_2(t)}{t} - \frac{\ln I(0)}{t}.$$

Obviously, we know that

$$\frac{1}{t}\int_0^t S^2(u)\mathrm{d}u \ge \left(\frac{1}{t}\int_0^t S(u)\mathrm{d}u\right)^2.$$

Therefore, from (V.31), we derive

$$\begin{aligned} \frac{\ln I(t)}{t} &\le \frac{\beta}{t}\int_0^t S(u)\mathrm{d}u - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right) - \frac{\sigma_4^2}{2}\left(\frac{1}{t}\int_0^t S(u)\mathrm{d}u\right)^2 + \Phi_2(t) \\ &= \beta\left(\frac{A}{(\mu_1 - b)} - \frac{1}{t}\left(\frac{(\mu_2 - b)}{(\mu_1 - b)} + \frac{\gamma_2}{(\mu_1 + k)}\right)\int_0^t I(u)\mathrm{d}u + \Phi_1(t)\right) - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right) \\ &\quad - \frac{\sigma_4^2}{2}\left(\frac{A}{(\mu_1 - b)} - \frac{1}{t}\left(\frac{(\mu_2 - b)}{(\mu_1 - b)} + \frac{\gamma_2}{(\mu_1 + k)}\right)\int_0^t I(u)\mathrm{d}u + \Phi_1(t)\right)^2 + \Phi_2(t). \end{aligned}$$

Hence one can see that

$$\begin{aligned} \frac{\ln I(t)}{t} \le &\frac{\beta A}{(\mu_1 - b)} - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right) - \frac{A^2\sigma_4^2}{2(\mu_1 - b)^2} \\ &- \left(\frac{(\mu_2 - b)}{(\mu_1 - b)} + \frac{\gamma_2}{(\mu_1 + k)}\right)\left(\beta - \frac{A\sigma_4^2}{(\mu_1 - b)}\right)\frac{1}{t}\int_0^t I(u)\mathrm{d}u \\ &- \frac{\sigma_4^2}{2t^2}\left(\left(\frac{(\mu_2 - b)}{(\mu_1 - b)} + \frac{\gamma_2}{(\mu_1 + k)}\right)\int_0^t I(u)\mathrm{d}u\right)^2 + \Phi_2(t) + \Phi_3(t), \end{aligned} \tag{V.40}$$

where

$$\Phi_3(t) = \beta\Phi_1(t) - \frac{\sigma_4^2}{2}\Phi_1^2(t) - \frac{\sigma_4^2 A\Phi_1(t)}{(\mu_1 - b)} + \sigma_4^2\Phi_1(t)\left(\frac{(\mu_2 - b)}{(\mu_1 - b)} + \frac{\gamma_2}{(\mu_1 + k)}\right)\int_0^t I(u)\mathrm{d}u.$$

An application of the large number theorem for martingales and Lemma 3.4, one has

$$\lim_{t\to\infty}\frac{\Phi_2(t)}{t} = \lim_{t\to\infty}\frac{\Phi_3(t)}{t} = 0 \quad \text{a.s.}$$

Taking the superior limit on both sides of (V.40), then by condition (V.35), we arrive at

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \leq \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2}{2}\right)\left(\hat{\mathcal{R}}_0^s - 1\right) < 0 \quad \text{a.s.}$$

Now, if the condition (V.36) is satisfied, then

$$\frac{\ln I(t)}{t} \leq \frac{\beta^2}{2\sigma_4^2} - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right) + \frac{\sigma_4}{t}\int_0^t S(u)\mathrm{d}\mathcal{W}_4(u) + \frac{\sigma_2\mathcal{W}_2(t)}{t} - \frac{\ln I(0)}{t}.$$

By lemma 3.4, large number theorem for martingales and the condition (V.36), our desired result (V.37) holds true. This completes the proof. ∎

3.2.4 Asymptotic behavior around the disease-free equilibrium

When $\mathcal{R}_0 \leq 1$, the disease-free equilibrium $E^{\ominus}$ of the deterministic model is asymptotically stable, which indicates that the epidemic will die out. We know that $E^{\ominus}$ isn't an equilibrium of the stochastic model (V.19), it is of great interests to show how the stochastic solution of the model (V.19) spirals closely around the equilibrium $E^{\ominus}$. In the next Theorem, we prove that the distance between the solution $X(t) = (S(t), I(t), R(t))$ and the disease-free equilibrium E^0 has the following form:

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \|X(u) - E^{\ominus}\|^2\mathrm{d}u \leq C, \quad \text{a.s.}$$

where C is a positive constant. For simplicity, we introduce the following notations:

$$\begin{aligned}
v_1 &= \frac{\mu_1 - b}{2} - \sigma_1^2,\\
v_2 &= \frac{\mu_1 - b}{2} + \alpha_2 + \gamma_2 - \frac{\sigma_2^2}{2} - \frac{\gamma_2^2}{2(\mu_3 - b)},\\
v_3 &= \frac{\mu_3}{2} + \frac{b}{2} + k - \frac{\sigma_3^2}{2} - \frac{(b+k)^2}{(\mu_1 - b)},\\
v_4 &= \min\{v_1, v_2, v_3\},\\
\hat{\mathcal{T}} &= \sigma_1^2\left(\frac{A}{\sqrt{v_4}(\mu_1 - b)}\right)^2.
\end{aligned}$$

Theorem 3.12. *Let $(S(t), I(t), R(t))$ be the solution of the model (V.19) with initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$. Suppose that $\mathcal{R}_0 \leq 1$, $v_1 > 0$, $v_2 > 0$ and $v_3 > 0$, then*

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \left(\left(S(u) - \frac{A}{\mu_1 - b}\right)^2 + I^2(u) + R^2(u)\right)\mathrm{d}u \leq \hat{\mathcal{T}} \quad \text{a.s.}$$

Proof. We define the following function

$$g(S,I,R) = g_1(S,I) + u_1 g_2(I) + g_3(R),$$

where

$$g_1(S,I) = \frac{1}{2}\left(S - \frac{A}{\mu_1 - b} + I\right), \qquad g_2(I) = I, \qquad g_3(R) = \frac{1}{2}R^2,$$

and u_1 is a positive constant to be determined later. Then the function g is positive definite. By Itô's formula for all $t \geq 0$, we have

$$\mathrm{d}g_1(S,I) = \mathcal{L}g_1 \mathrm{d}t + \left(S - \frac{A}{\mu_1 - b} + I\right)\left(\sigma_1 S \mathrm{d}\mathcal{W}_1(t) + \sigma_2 I \mathrm{d}\mathcal{W}_2(t)\right),$$

where

$$\begin{aligned}
\mathcal{L}g_1 &= \left(S - \frac{A}{\mu_1 - b} + I\right)(A - (\mu_1 - b)S + (b+k)R - (\mu_2 - b + \gamma_2)I) + \frac{\sigma_1^2}{2}S^2 + \frac{\sigma_2^2}{2}I^2 \\
&= \left(S - \frac{A}{\mu_1 - b} + I\right)\left(-(\mu_1 - b)\left[S - \frac{A}{\mu_1 - b}\right] + (b+k)R - (\mu_2 - b + \gamma_2)I\right) \\
&\quad + \frac{\sigma_1^2}{2}\left(S - \frac{A}{\mu_1 - b} + \frac{A}{\mu_1 - b}\right)^2 + \frac{\sigma_2^2}{2}I^2 \\
&= -(\mu_1 - b)\left[S - \frac{A}{\mu_1 - b}\right]^2 - (\mu_2 - b + \gamma_2)I(t)\left[S - \frac{A}{\mu_1 - b}\right] \\
&\quad + \frac{(b+k)}{\sqrt{\mu_1 - b}}\sqrt{\mu_1 - b}R\left[S - \frac{A}{\mu_1 - b}\right] - (\mu_1 - b)\left[S - \frac{A}{\mu_1 - b}\right]I \\
&\quad - (\mu_2 - b + \gamma_2)I^2 + \frac{(b+k)}{\sqrt{\mu_1 - b}}\sqrt{\mu_1 - b}RI + \frac{\sigma_1^2}{2}\left(S - \frac{A}{\mu_1 - b} + \frac{A}{\mu_1 - b}\right)^2 + \frac{\sigma_2^2}{2}I^2.
\end{aligned}$$

Then we employ the following inequalities $(x+y)^2 \leq 2x^2 + 2y^2$ and $2xy \leq x^2 + y^2$ to estimate the cross terms. The above inequality can be rewritten as

$$\begin{aligned}
\mathcal{L}g_1 &\leq -(\mu_1 - b)\left[S - \frac{A}{\mu_1 - b}\right]^2 - \left(2(\mu_1 - b) + \alpha_2 + \gamma_2\right)I\left[S - \frac{A}{\mu_1 - b}\right] \\
&\quad - (\mu_2 - b + \gamma_2)I^2 + \frac{(b+k)^2}{2(\mu_1 - b)}R^2 + \frac{(\mu_1 - b)}{2}I^2 + \frac{(b+k)^2}{2(\mu_1 - b)}R^2 \\
&\quad + \frac{\mu_1 - b}{2}\left[S - \frac{A}{\mu_1 - b}\right]^2 + \sigma_1^2\left(S - \frac{A}{\mu_1 - b}\right)^2 + \sigma_1^2\left(\frac{A}{\mu_1 - b}\right)^2 + \frac{\sigma_2^2}{2}I^2.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\mathcal{L}g_1 &\leq -\left(\frac{\mu_1 - b}{2} - \sigma_1^2\right)\left[S - \frac{A}{\mu_1 - b}\right]^2 - \left(2(\mu_1 - b) + \alpha_2 + \gamma_2\right)I\left[S - \frac{A}{\mu_1 - b}\right] \\
&\quad - \left(\frac{\mu_1 - b}{2} + \alpha_2 + \gamma_2 - \frac{\sigma_2^2}{2}\right)I^2 + \frac{(b+k)^2}{(\mu_1 - b)}R^2 + \sigma_1^2\left(\frac{A}{\mu_1 - b}\right)^2.
\end{aligned}$$

Again, applying Itô's formula, one can obtain

$$\mathrm{d}g_2 = \mathcal{L}g_2 \mathrm{d}t + \sigma_2 I \mathrm{d}\mathcal{W}_2(t) + \sigma_4 SI \mathrm{d}\mathcal{W}_4(t),$$

$$\begin{aligned}
\mathcal{L}g_2 &= \beta SI - (\mu_2 + \gamma_1 + \gamma_2 - qb)I \\
&\leq \beta\left(S - \frac{A}{\mu_1 - b}\right)I + \left(\frac{A}{\mu_1 - b} - (\mu_2 + \gamma_1 + \gamma_2 - qb)\right)I.
\end{aligned}$$

Recalling that $\mathcal{R}_0 \leq 1$, then we obtain

$$\mathcal{L}g_2 \leq \beta \left(S - \frac{A}{\mu_1 - b}\right) I(t).$$

Identically, one can get that

$$\mathrm{d}g_3(R) = \mathcal{L}g_3 \mathrm{d}t + \sigma_3 R^2 \mathrm{d}\mathcal{W}_3(t),$$

where

$$\begin{aligned}
\mathcal{L}g_3(R) &= \gamma_2 RI - (\mu_3 + k)R^2 + \frac{\sigma_3^2}{2}R^2 \\
&\leq \frac{\gamma_2^2 I^2}{2(\mu_3 - b)} + \frac{\mu_3 - b}{2}R^2 - (\mu_3 + k)R^2 + \frac{\sigma_3^2}{2}R^2 \\
&= \frac{\gamma_2^2 I^2}{2(\mu_3 - b)} - \left(\frac{\mu_3}{2} + \frac{b}{2} + k - \frac{\sigma_3^2}{2}\right)R^2.
\end{aligned}$$

By setting $u_1 = \frac{1}{\beta}\Big(2(\mu_1 - b) + \alpha_2 + \gamma_1\Big)$, we get

$$\begin{aligned}
\mathcal{L}g \leq &- \left(\frac{\mu_1 - b}{2} - \sigma_1^2\right)\left(S - \frac{A}{\mu_1 - b}\right)^2 - \left(\frac{\mu_1 - b}{2} + \alpha_2 + \gamma_2 - \frac{\sigma_2^2}{2} - \frac{\gamma_2^2}{2(\mu_3 - b)}\right) I^2 \\
&- \left[\frac{\mu_3}{2} + \frac{b}{2} + k - \frac{\sigma_3^2}{2} - \frac{(b+k)^2}{(\mu_1 - b)}\right] R^2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathrm{d}g(S,I,R) = &\ \mathcal{L}g\mathrm{d}t + \sigma_1\Big(S - \frac{A}{\mu_1 - b} + I\Big)S\mathrm{d}\mathcal{W}_1(t) + \sigma_2\Big(S - \frac{A}{\mu_1 - b} + I\Big)I\mathrm{d}\mathcal{W}_2(t) \\
&+ \sigma_2 I\mathrm{d}\mathcal{W}_2(t) + \sigma_4 SI\mathrm{d}\mathcal{W}_4(t) + \sigma_3 R^2\mathrm{d}\mathcal{W}_3(t) \\
\leq &- \left(\frac{\mu_1 - b}{2} - \sigma_1^2\right)\left(S - \frac{A}{\mu_1 - b}\right)^2 - \left(\frac{\mu_1 - b}{2} + \alpha_2 + \gamma_2 - \frac{\sigma_2^2}{2} - \frac{\gamma_2^2}{2(\mu_3 - b)}\right) I^2 \\
&- \left(\frac{\mu_3}{2} + \frac{b}{2} + k - \frac{\sigma_3^2}{2} - \frac{(b+k)^2}{(\mu_1 - b)}\right) R^2 + \sigma_1^2\left(\frac{A}{\mu_1 - b}\right)^2 + \sigma_1\left(S - \frac{A}{\mu_1 - b} + I\right) S\mathrm{d}\mathcal{W}_1(t) \\
&+ \sigma_2\left(S - \frac{A}{\mu_1 - b} + I\right) I\mathrm{d}\mathcal{W}_2(t) + \sigma_2 I\mathrm{d}\mathcal{W}_2(t) + \sigma_4 SI\mathrm{d}\mathcal{W}_4(t) + \sigma_3 R^2\mathrm{d}\mathcal{W}_3(t).
\end{aligned}$$

Integrating both sides of the last inequality from 0 to t, yields

$$\begin{aligned}
0 \leq g(S(t),I(t),R(t)) - g(S(0),I(0),R(0)) \leq &-v_1\int_0^t \left(S(u) - \frac{A}{\mu_1 - b}\right)^2 \mathrm{d}u - v_2\int_0^t I^2(u)\mathrm{d}u \\
&- v_3\int_0^t R^2(u)\mathrm{d}u + \sigma_1^2\left(\frac{A}{\mu_1 - b}\right)^2 t + \mathcal{W}(t),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{W}(t) = &\ \sigma_1\int_0^t \left(S(u) - \frac{A}{\mu_1 - b} + I(u)\right) S(u)\mathrm{d}\mathcal{W}_1(u) + \sigma_2\int_0^t \left(S(u) - \frac{A}{\mu_1 - b} + I(u)\right) I(u)\mathrm{d}\mathcal{W}_2(u) \\
&+ \sigma_2\int_0^t I(u)\mathrm{d}\mathcal{W}_2(u) + \sigma_4\int_0^t S(u)I(u)\mathrm{d}\mathcal{W}_4(u) + \sigma_3\int_0^t R^2(u)\mathrm{d}\mathcal{W}_3(u).
\end{aligned}$$

Employing Lemmas 3.4, 3.5 and (V.34) yields

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \left[v_1\left(S(u) - \frac{A}{\mu_1 - b}\right)^2 + v_2 I^2(u) + v_3 R^2(u)\right]\mathrm{d}u \leq \sigma_1^2\left(\frac{A}{\mu_1 - b}\right)^2 \quad \text{a.s.}$$

Then, we get

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\left[\left(S(u)-\frac{A}{\mu_1-b}\right)^2+I^2(u)+R^2(u)\right]du\le\hat{\mathcal{T}}\quad\text{a.s.}$$

This completes the proof. ∎

3.3 Numerical simulations

In this subsection, in order to show different dynamical results of the stochastic model (V.19) under theoretical parameter values (see Table V.4), we present some numerical simulations. We use the Euler-Maruyama method to simulate the trajectories of the stochastic model (V.19).

Parameters	Value		
A	0.2	0.2	0.2
β	0.03	0.022	0.03
μ_1	0.05	0.05	0.05
μ_2	0.09	0.09	0.09
μ_3	0.0055	0.0055	0.0055
b	0.01	0.01	0.01
γ_1	0.01	0.01	0.01
γ_2	0.02	0.02	0.02
q	0.2	0.2	0.2
k	0.04	0.04	0.04
σ_1	0.028	0.028	0.028
σ_2	0.03	0.03	0.3
σ_3	0.025	0.025	0.0287
σ_4	0.3	0.3	0.2
	Figures V.7 and V.8	Figure V.9 (left)	Figures V.9 (right) and V.10

Table V.4: The theoretical parameter values of the model (V.19).

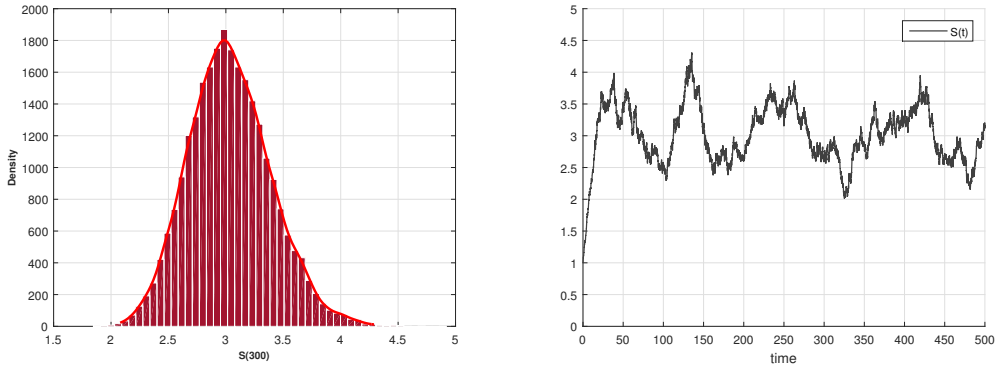
Example 3.13 (The permanence case). For the purpose of showing the effects of the perturbations on the disease dynamics, we have realized the simulation many times. We have chosen the parameter value from Table V.4 (second column). Then, $2(\mu_1-b)-\sigma^2=0.0791>0$ and $\mathcal{R}_0^s=1.6242>1$. From figure V.7, we show the existence of the unique stationary distributions for $S(t)$, $I(t)$ and $R(t)$ of the model (V.19), where the smooth curves are the probability density functions of $S(t)$, $I(t)$ and $R(t)$, respectively. It can be obviously observed that the solution of the SDE model (V.19) persists in the mean. For the deterministic system, we obtain $\mathcal{R}_0=1.6949>1$. Then, the deterministic endemic equilibrium $E^{\circledast}=(2.95,0.9567,0.2733)$ is globally asymptotically stable. Then, we have

$$\begin{aligned}\sigma_1^2-(\mu_1-b)&=-0.0392<0,\\ \sigma_2^2-(\mu_2-b)&=-0.0791<0,\\ \sigma_3^2-(\mu_3-b)&=-0.0193<0.\end{aligned}$$

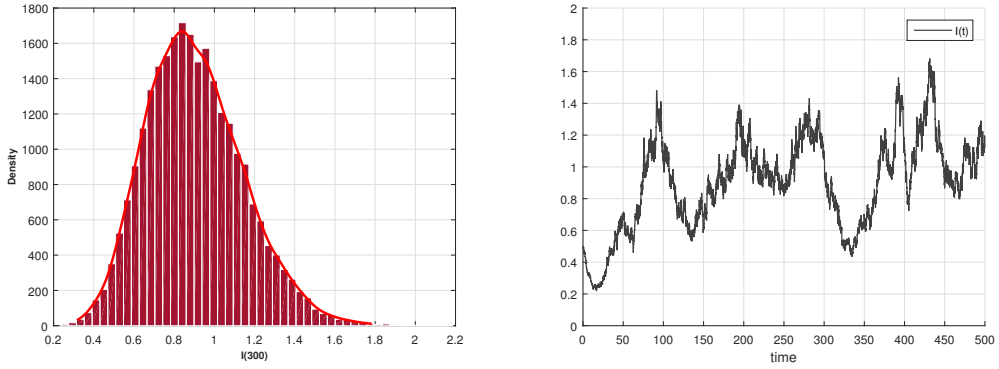
Therefore, the conditions of Theorem 3.10 are satisfied. As expected, the stochastic solution is oscillating around the endemic equilibrium E^* for a long time (see Figure V.8).

Example 3.14 (The extinction case). To verify that the condition (V.35) is satisfied, we decrease β to 0.022 and keep other parameters unchanged. Then, we have $\hat{\mathcal{R}}_0^s=0.9132<1$ and $\sigma_4^2-\frac{(\mu_1-b)\beta}{A}=-0.0040<0$. Therefore, the condition (V.35) of Theorem 1.1 is satisfied. We can conclude that for any initial value, $I(t)$ obeys

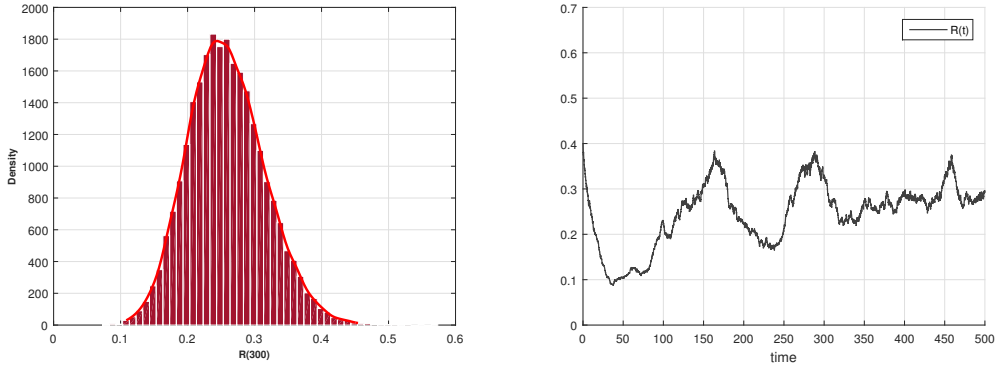
$$\limsup_{t\to\infty}\frac{1}{t}\ln\frac{I(t)}{I(0)}\le(\hat{\mathcal{R}}_0^s-1)\left(\mu_2+\gamma_1+\gamma_2-qb+\frac{\sigma_2^2}{2}\right)=-0.0082<0\quad\text{a.s.}$$



(a) The left figure is the stationary distribution for S(t), the right picture is the trajectory of S(t).



(b) The left figure is the stationary distribution for I(t), the right picture is the trajectory of I(t).



(c) The left figure is the stationary distribution for R(t), the right picture is the trajectory of R(t).

Figure V.7: The numerical illustration of obtained results in Theorems 1.1 and 3.9.

That is, $I(t)$ will tend to zero exponentially with probability one (see Figure V.9 - Right). Now, we choose $\beta = 0.03$ and the white noise intensities $\sigma_1 = 0.028$, $\sigma_2 = 0.3$, $\sigma_3 = 0.0287$ and $\sigma_4 = 0.2$ to ensure that the condition (V.36) of Theorem (3.11) is satisfied. We can conclude that for any initial value, $I(t)$ obeys

$$\limsup_{t\to\infty} \frac{1}{t} \ln \frac{I(t)}{I(0)} \leq \frac{\beta^2}{2\sigma_4^2} - \left(\mu_2 + \gamma_1 + \gamma_2 - qb + \frac{\sigma_2^2}{2}\right) = -0.2980 < 0 \quad \text{a.s.}$$

That is, $I(t)$ will tend to zero exponentially with probability one (see Figure V.9 - left). For the deterministic system , we obtain $\mathcal{R}_0 = 0.9322 < 1$. Then, the deterministic disease-free equilibrium $E^{\ominus} = (5, 0, 0)$ is globally asymptotically stable. Then, we have

$$\begin{aligned}
\sigma_1^2 - \frac{\mu_1 - b}{2} &= -0.0192 < 0, \\
\sigma_2^2 + \frac{\gamma_2^2}{(\mu_3 - b)} - (\mu_1 - b) + 2(\alpha_2 + \gamma_2) &= -0.1191 < 0, \\
\sigma_3^2 + \frac{2(b+k)^2}{(\mu_1 - b)} - \mu_3 + b + 2k &= -0.0043 < 0.
\end{aligned}$$

Then, the conditions of Theorem 3.12 are verified and the stochastic solution is oscillating around the deterministic disease-free equilibrium $E^{\ominus}$ for a long time (see Figure V.10).

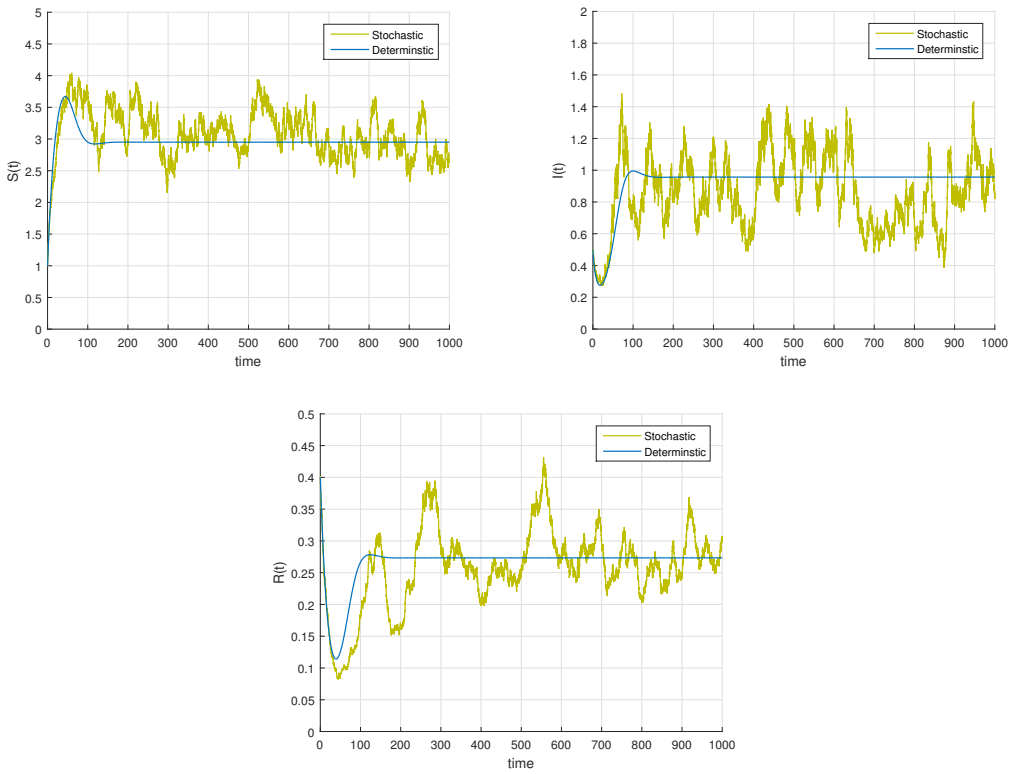


Figure V.8: The numerical simulation of solution (S(t),I(t),R(t)).

The outbreak of epidemics has caused severe damage and loss to populations. Many scientific papers are devoted to analyzing the outbreak of the epidemics for a long time, and many solutions and suggestions have been used to control the dissemination of infectious diseases. However, the effects of environmental disturbances on the epidemic cannot be ignored. In our present study, we proposed and analyzed a new stochastic version of the SIRS epidemic model by considering the bi-linear and proportional perturbations. This is a general and well-established way of introducing stochastic environmental noise into realistic dynamic models. Our study led to five main results:

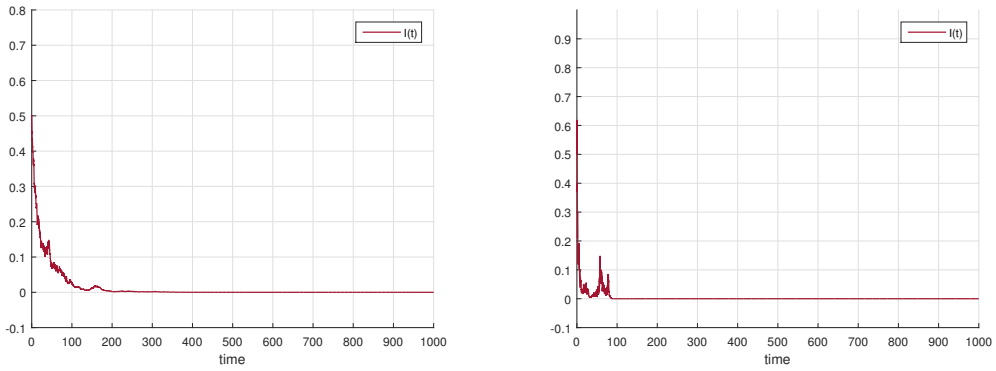


Figure V.9: The numerical simulation of I(t).

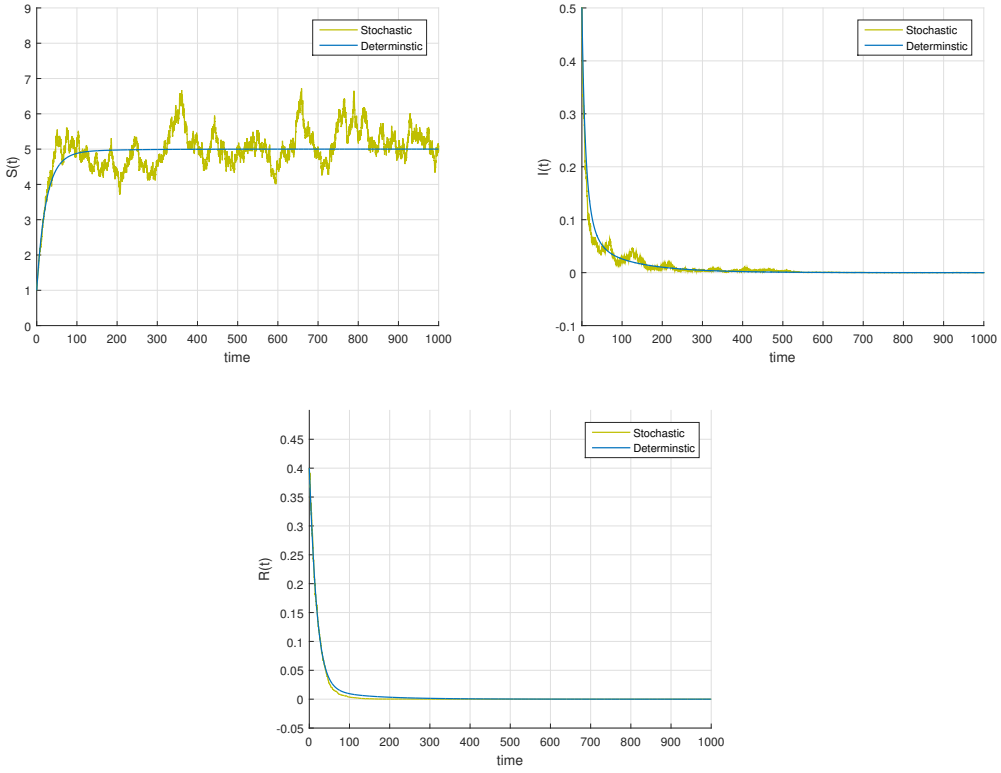


Figure V.10: The numerical simulation of solution (S(t),I(t),R(t)).

1. In Theorem 3.8, we gave a systematic approach to the analysis of the existence of a unique ergodic stationary distribution when $\mathcal{R}_0^s > 1$ by using the property of the Feller process and mutually exclusive possibilities lemma. From a biological viewpoint, the ergodicity implies that the epidemic will be prevalent and persistent in the long term.

2. In Theorem 3.9, we checked that the persistence in the mean of the disease occurs under the same condition $\mathcal{R}_0^s > 1$ by using a new simple method. Although the solution is unbounded, we have estimated the time average of $\psi^2(t)$, where $\psi(t)$ is the solution of equation (V.22).

3. In Theorem 3.10, we obtained sufficient conditions to guarantee that the solution of the stochastic system fluctuates around the endemic deterministic equilibrium of system (I.1). The present study is the first attempt, to the best of our knowledge, to analyze the asymptotic behavior of the SIRS epidemic model (V.19).

4. In Theorem 3.11, we showed that the extinction of a disease in the stochastic system (V.19) occurs if one of the conditions (V.35) and (V.36) holds. It should be noted that these conditions are sufficient for the extinction of the epidemic.

5. In Theorem 3.12, we proved that under some conditions, the solution of system (I.2) will oscillate around the disease-free deterministic equilibrium, and the intensity of fluctuations is related to the intensity η_1, but do not relate to the intensities of the other white noises.

To illustrate duly our theoretical results, we have performed some numerical simulations by using MATLAB software. From Figure V.7, we have shown the existence of a unique stationary distribution which means that the disease persists for all time regardless of the initial conditions. Furthermore, the ergodic property grants a reason why the integral average of a solution of system (V.19) converges to a fixed point whilst the system may fluctuate around as time goes by. Note that, in general, the endemic equilibrium is not a steady-state of the stochastic model (I.2). This latter will fluctuate around the deterministic endemic equilibrium which is illustrated clearly in Figure V.8. Moreover, in a biological and epidemiological view, the disease will persist when the basic reproduction number is higher than one and s_1, s_2 and s_3 are positive. In the case of extinction, in Figure V.10, we have plainly estimated the average oscillation around the deterministic free-disease equilibrium to exhibit whether the disease will die out. In a biological interpretation, one expects any solution to be close to the deterministic equilibrium most of the time if the stochastic effects are slight and the basic reproduction number is less than one.

3.4 Discussion

Our study improved many existing results. Precisely, we obtained sharper results for a more general stochastic SIRS model by using new techniques and approaches. Furthermore, our work can be applied to treat more biological and ecological systems such as the heroin model and to analyze the impact of predation in the spread of infectious diseases. However, our paper brings more challenges to propose an improved method to obtain the global threshold between the existence of the unique ergodic stationary distribution (persistence) and extinction. We seek in our future works to treat this interesting problem.

Chapter VI

Dynamic characteristics of a perturbed epidemic model with quarantine policy and Lévy disturbance

Abstract

The spread of infectious diseases is a major challenge in our contemporary world, especially after the recent outbreak of Coronavirus disease 2019 (COVID-19). The quarantine strategy is one of the important intervention measures to control the spread of an epidemic by greatly minimizing the likelihood of contact between infected and susceptible individuals. This chapter treats two kinds of epidemic models with quarantine. We start with the threshold analysis of a stochastic SIQR model with Lévy jumps. Then, we analyze the impact of various stochastic disturbances on the epidemic dynamics during the quarantine period. For this purpose, we present an SIQS epidemic model that incorporates the stochastic transmission and the Lévy noise in order to simulate both small and massive perturbations. Under appropriate conditions, some interesting asymptotic properties are proved, namely: ergodicity, persistence in the mean, and extinction of the disease. The theoretical results show that the dynamics of the perturbed model are determined by parameters that are closely related to the stochastic noises. Our work improves many existing studies in the field of mathematical epidemiology and provides new techniques to predict and analyze the dynamic behavior of epidemics.

1 The threshold of a stochastic SIR epidemiological model with quarantine and Lévy jumps

1.1 Introduction

As stated before, the effects due to unexpected environmental shocks have been neglected in the deterministic formulation. The proposed solution to this issue is introducing a jump process into the underlying population dynamics. Many works have introduced Lévy jumps process into their models. Many works have introduced Lévy jumps process into their models. For example, Bao et al. [7] produced pioneering works on this approach. They first studied a stochastic Lotka-Volterra population systems with Lévy jumps. From then on, many interesting studies on the epidemic models with Lévy jumps have been reported (see e.g. [80, 81]). Motivated by the above mentioned works, in this contribution, we consider the following SIQR epidemic model with both white noise

and Lévy jumps perturbations:

$$\begin{cases} \mathrm{d}S(t) = \Big(A - \mu S(t) - \beta S(t)I(t)\Big)\mathrm{d}t + \sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \displaystyle\int_{\mathcal{U}} \eta_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}I(t) = \Big(\beta S(t)I(t) - (\mu+\gamma+\delta+r_2)I(t)\Big)\mathrm{d}t + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \displaystyle\int_{\mathcal{U}} \eta_2(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}Q(t) = \Big(\delta I(t) - (\mu+k+r_3)Q(t)\Big)\mathrm{d}t + \sigma_3 Q(t)\mathrm{d}\mathcal{W}_3(t) + \displaystyle\int_{\mathcal{U}} \eta_3(u)Q(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}R(t) = \Big(\gamma I(t) + kQ(t) - \mu R(t)\Big)\mathrm{d}t + \sigma_4 R(t)\mathrm{d}\mathcal{W}_4(t) + \displaystyle\int_{\mathcal{U}} \eta_4(u)R(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u), \end{cases} \tag{VI.1}$$

where $S(t^-)$, $I(t^-)$, $Q(t^-)$ and $R(t^-)$ are the left limits of $S(t)$, $I(t)$, $Q(t)$ and $R(t)$, respectively. $\mathcal{N}$ is a Poisson counting measure with the compensated process $\widetilde{\mathcal{N}}$ and Lévy measure ν on a measurable subset $\mathcal{U}$ of $(0,\infty)$ satisfying $\nu(\mathcal{U}) < \infty$. Principally, we have that $\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) = \mathcal{N}(\mathrm{d}t,\mathrm{d}u) - \nu(\mathrm{d}u)\mathrm{d}t$ and we assume that the independent Brownian motions $\mathcal{W}_i(t)$ $(i = 1,2,3,4)$ are independent of $\mathcal{N}$. The functions $\eta_i : \mathcal{U} \to \mathbb{R}$ are bounded and continuous.

The general aim of this short section is to investigate the asymptotic properties of the stochastic epidemic model (VI.1) and to establish the threshold which determines the extinction and the persistence in the mean of the disease. This threshold value coincides with the deterministic threshold in absence of white noise and Lévy jumps. For the purpose of well analyzing our model (VI.1), it necessary that we make the following assumptions:

- **$(\mathcal{A}_1)$** We assume that the jump coefficients $\eta_i(u)$ in (VI.1) verify $\displaystyle\int_{\mathcal{U}} \eta_i^2(u)\nu(\mathrm{d}u) < \infty$ $(i = 1,2,3)$.
- **$(\mathcal{A}_2)$** For all $u \in Z$, $1 + \eta_i(u) > 0$, $i = 1,2,3,4$ and $\displaystyle\int_{\mathcal{U}} \Big(\eta_i(u) - \ln(1+\eta_i(u))\Big)\nu(\mathrm{d}u) < \infty$.
- **$(\mathcal{A}_3)$** We suppose that exists a constant $\kappa > 0$, such that $\displaystyle\int_{\mathcal{U}} \Big(\ln(1+\eta_i(u))\Big)^2 \nu(\mathrm{d}u) < \kappa$.
- **$(\mathcal{A}_4)$** There exists $\varrho > 1$ such that $\mu - \dfrac{1}{2}(\varrho-1)\widetilde{\sigma}^2 - \dfrac{1}{\varrho}\lambda_p > 0$, where $\widetilde{\sigma}^2 = \max\{\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\}$, and

$$\lambda_p = \int_{\mathcal{U}} \Big\{\Big(1 + \max\{\eta_1(u),\eta_2(u),\eta_3(u),\eta_4(u)\}\Big)^p - 1 - p\Big(\min\{\eta_1(u),\eta_2(u),\eta_3(u),\eta_4(u)\}\Big)\Big\}\nu(\mathrm{d}u).$$

Remark 1.1. Generally, the assumptions **$(\mathcal{A}_1)$** - **$(\mathcal{A}_4)$** mean that the intensities of Lévy jumps are not very large.

Under assumptions **$(\mathcal{A}_1)$** - **$(\mathcal{A}_2)$**, we check the well-posedness of the perturbed system (VI.1).

Theorem 1.2. *For any initial value $(S(0),I(0),Q(0),R(0)) \in \mathbb{R}^4_+$, there exists the unique positive solution $(S(t),I(t),Q(t),R(t))$ of system (VI.1) on $t \geq 0$, and the solution will remain in $\mathbb{R}^4_+$ with probability one. That is to say, the solution $(S(t),I(t),Q(t),R(t)) \in \mathbb{R}^4_+$ for all $t \geq 0$ almost surely.*

Now, we list some former results used in the previous chapters. Under the assumptions **$(\mathcal{A}_1)$** - **$(\mathcal{A}_4)$**, we have the following lemmas:

Lemma 1.3 ([171]). *Let $(S(t),I(t),Q(t),R(t))$ be the solution of model (VI.1) with any positive initial value. Then*

- $\displaystyle\lim_{t\to\infty} \frac{S(t)+I(t)+Q(t)+R(t)}{t} = 0$ *a.s.*
- $\displaystyle\lim_{t\to\infty} \frac{S(t)}{t} = 0$, $\displaystyle\lim_{t\to\infty} \frac{I(t)}{t} = 0$, $\displaystyle\lim_{t\to\infty} \frac{Q(t)}{t} = 0$ *and* $\displaystyle\lim_{t\to\infty} \frac{R(t)}{t} = 0$ a.s.

Lemma 1.4 ([171]). *Let $(S(t),I(t),Q(t),R(t))$ be the solution of model (VI.1) with any positive initial value. Then*

- $\lim_{t\to\infty}\frac{\int_0^t S(s)\mathrm{d}\mathcal{W}_1(s)}{t}=0,\quad \lim_{t\to\infty}\frac{\int_0^t I(s)\mathrm{d}\mathcal{W}_2(s)}{t}=0,\quad \lim_{t\to\infty}\frac{\int_0^t Q(s)\mathrm{d}\mathcal{W}_3(s)}{t}=0,$
  $\lim_{t\to\infty}\frac{\int_0^t R(s)\mathrm{d}\mathcal{W}_4(s)}{t}=0$ a.s.
- $\lim_{t\to\infty}\frac{\int_0^t\int_{\mathcal{U}}\eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0,\quad \lim_{t\to\infty}\frac{\int_0^t\int_{\mathcal{U}}\eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0,$
  $\lim_{t\to\infty}\frac{\int_0^t\int_{\mathcal{U}}\eta_3(u)Q(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0,\quad \lim_{t\to\infty}\frac{\int_0^t\int_{\mathcal{U}}\eta_4(u)R(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0$ a.s.

1.2 Main results

In this subsection, our main concern is to determine the condition for the extinction and persistence of an infectious disease under quarantine strategies. Given that the value of the deterministic threshold $\mathcal{R}_0=\frac{\beta A}{\mu(\mu+\gamma+\delta+r_2)}$ characterizes the dynamical behavior of the deterministic system and guarantees persistence or extinction of the disease, similarly we define the threshold of our stochastic SIQR epidemic model (VI.1) as follows:

$$\mathcal{R}_0^{\mathrm{i}}=\frac{1}{(\mu+\gamma+\delta+r_2)}\left(\frac{\beta A}{\mu}-\frac{\sigma_2^2}{2}-\int_{\mathcal{U}}\Big(\eta_2(u)-\ln(1+\eta_2(u))\Big)\nu(\mathrm{d}u)\right).$$

Theorem 1.5. *Let $(S(t),I(t),Q(t),R(t))$ be the solution of system (VI.1) with any initial positive value. If $\mathcal{R}_0^{\mathrm{i}}<1$, then*

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}\le(\mu+\gamma+\delta+r_2)\Big(\mathcal{R}_0^{\mathrm{i}}-1\Big)<0\quad a.s.$$

Furthermore, we have that

$$\lim_{t\to\infty}\langle S(t)\rangle=\frac{A}{\mu}\ \text{ a.s.}\qquad \lim_{t\to\infty}\langle Q(t)\rangle=0\ \text{ a.s.}\qquad \text{and}\qquad \lim_{t\to\infty}\langle R(t)\rangle=0\ \text{ a.s.}$$

Proof. Integrating from 0 to t on both sides of the first and second equations of system (VI.1), gives

$$\begin{aligned}\frac{S(t)-S(0)}{t}&=A-\mu\langle S(t)\rangle-\beta\langle S(t)I(t)\rangle+\frac{\sigma_1}{t}S(s)\mathrm{d}\mathcal{W}_1(s)+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u),\\ \frac{I(t)-I(0)}{t}&=\beta\langle S(t)I(t)\rangle-(\mu+\delta+\gamma+r_2)\langle I(t)\rangle+\frac{\sigma_2}{t}I(s)\mathrm{d}\mathcal{W}_2(s)+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

Noticeably, we get

$$\langle S(t)\rangle=\frac{A}{\mu}-\frac{\mu+\delta+\gamma+r_2}{\mu}\langle I(t)\rangle+\psi_1(t),\tag{VI.2}$$

where

$$\begin{aligned}\psi_1(t)=-\frac{1}{\mu}\Bigg\{&\frac{S(t)-S(0)}{t}+\frac{I(t)-I(0)}{t}-\frac{\sigma_1}{t}S(s)\mathrm{d}\mathcal{W}_1(s)-\frac{\sigma_2}{t}I(s)\mathrm{d}\mathcal{W}_2(s)\\&-\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)-\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\Bigg\}.\end{aligned}$$

By Lemmas 1.3 and 1.4, we have $\lim_{t\to\infty}\psi_1(t) = 0$ a.s. Making use Itô's formula to (VI.1), one can conclude that

$$\begin{aligned} \mathrm{d}\ln I(t) =& \left(\beta S(t) - (\mu+\delta+\gamma+r_2) - \frac{1}{2}\sigma_2^2 - \int_{\mathcal{U}}\Big(\eta_2(u) - \ln(1+\eta_2(u))\Big)\nu(\mathrm{d}u)\right)\mathrm{d}t \\ &+ \sigma_2\mathrm{d}\mathcal{W}_2(t) + \int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \end{aligned}$$

Integrating from 0 to t on both sides, we get

$$\begin{aligned} \frac{\ln I(t)}{t} =& \beta\langle S(t)\rangle - (\mu+\delta+\gamma+r_2) - \frac{1}{2}\sigma_2^2 - \int_{\mathcal{U}}\Big(\eta_2(u) - \ln(1+\eta_2(u))\Big)\nu(\mathrm{d}u) \\ &+ \frac{\sigma_2\mathcal{W}_2(t)}{t} + \frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) + \frac{\ln I(0)}{t}. \end{aligned}$$

Substituting the result (VI.2) into the previous equation, implies that

$$\begin{aligned} \frac{\ln I(t)}{t} =& \beta\left(\frac{A}{\mu} - \frac{\mu+\delta+\gamma+r_2}{\mu}\langle I(t)\rangle + \psi_1(t)\right) - (\mu+\delta+\gamma+r_2) - \frac{1}{2}\sigma_2^2 + \frac{\ln I(0)}{t} \\ &- \int_{\mathcal{U}}\Big(\eta_2(u) - \ln(1+\eta_2(u))\Big)\nu(\mathrm{d}u) + \frac{\sigma_2\mathcal{W}_2(t)}{t} + \frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) \\ \leq& \frac{\beta A}{\mu} - \beta\psi_1(t) - (\mu+\delta+\gamma+r_2) - \frac{1}{2}\sigma_2^2 + \frac{\ln I(0)}{t} - \int_{\mathcal{U}}\eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u) \\ &+ \frac{\sigma_2\mathcal{W}_2(t)}{t} + \frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u). \end{aligned}$$

Let $\mathcal{K}_1(t) = \int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)$. According to the hypothesis $(\mathcal{A}_3)$, we have

$$\langle\mathcal{K}_1,\mathcal{K}_1\rangle(t) = t\int_{\mathcal{U}}[\ln(1+\eta_2(u))]^2\nu(\mathrm{d}u) < \kappa t.$$

By the strong law of large numbers for martingales, we get $\lim_{t\to\infty}\frac{\mathcal{K}_1(t)}{t} = 0$ and $\lim_{t\to\infty}\frac{\mathcal{W}_2(t)}{t} = 0$ a.s. Consequently, if the condition $\mathcal{R}_0^{\mathrm{i}} < 1$ holds, we have

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \leq (\mu+\gamma+\delta+r_2)\left(\mathcal{R}_0^{\mathrm{i}} - 1\right) < 0 \quad \text{a.s.}$$

Therefore $\lim_{t\to\infty} I(t) = 0$ a.s. Consequently, from (VI.2), one can see that $\limsup_{t\to\infty}\langle S(t)\rangle = \frac{A}{\mu}$ a.s. Next, we shall show that $\lim_{t\to\infty}\langle Q(t)\rangle = 0$ and $\lim_{t\to\infty}\langle R(t)\rangle = 0$ a.s. From system (VI.1), we have

$$\frac{Q(t)-Q(0)}{t} = \delta\langle I(t)\rangle - (\mu+k+r_3)\langle Q(t)\rangle + \frac{\sigma_3}{t}\int_0^t Q(s)\mathrm{d}\mathcal{W}_3(s) + \frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_3(u)Q(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u),$$

and

$$\frac{R(t)-R(0)}{t} = \gamma\langle I(t)\rangle + k\langle Q(t)\rangle - \mu\langle R(t)\rangle + \frac{\sigma_4}{t}\int_0^t R(s)\mathrm{d}\mathcal{W}_4(s) + \frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_4(u)R(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

Making use of Lemmas 1.3 and 1.4, we get the desired result. ∎

Now, we will establish the condition for persistence of the disease in the next theorem.

Theorem 1.6. *Let $(S(t),I(t),Q(t),R(t))$ be the solution of system (VI.1) with any initial positive value. If $\mathcal{R}_0^{\mathrm{i}} > 1$, then*

$$\lim_{t\to\infty}\langle I(t)\rangle = \widetilde{I} > 0 \quad \text{a.s.} \quad \lim_{t\to\infty}\langle S(t)\rangle = \widetilde{S} > 0 \quad \text{a.s.} \quad \lim_{t\to\infty}\langle Q(t)\rangle = \widetilde{Q} > 0 \quad \text{a.s.} \quad \lim_{t\to\infty}\langle R(t)\rangle = \widetilde{R} > 0 \quad \text{a.s.}$$

where $\widetilde{I} = \frac{\mu}{\beta}\left(\mathcal{R}_0^{\mathrm{i}} - 1\right)$, $\widetilde{S} = \frac{A}{\mu} - \frac{(\mu+\delta+\gamma+r_2)\left(\mathcal{R}_0^{\mathrm{i}}-1\right)}{\beta}$, $\widetilde{Q} = \frac{\delta\mu}{\beta(\mu+k+r_3)}\left(\mathcal{R}_0^{\mathrm{i}} - 1\right)$, $\widetilde{R} = \frac{(\gamma+k\delta)\mu}{\beta(\mu+k+r_3)}\left(\mathcal{R}_0^{\mathrm{i}} - 1\right)$. That is to say, the epidemic model (VI.1) will prevail if $\mathcal{R}_0^{\mathrm{i}} > 1$.

Proof. According to the Itô's formula, one can see that

$$\begin{aligned}\frac{\ln I(t)}{t} &= \beta\left(\frac{A}{\mu} - \frac{\mu+\delta+\gamma+r_2}{\mu}\langle I(t)\rangle + \psi_1(t)\right) - (\mu+\delta+\gamma+r_2) - \frac{1}{2}\sigma_2^2 + \frac{\ln I(0)}{t}\\ &\quad - \int_{\mathcal{U}} \eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u) + \frac{\sigma_2 \mathcal{W}_2(t)}{t} + \frac{1}{t}\int_0^t \int_{\mathcal{U}} \ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\ &= (\mu+\gamma+\delta+r_2)\left(\mathcal{R}_0^{\mathrm{i}} - 1\right) - \beta\frac{\mu+\delta+\gamma+r_2}{\mu}\langle I(t)\rangle + \beta\psi_1(t) + \frac{\ln I(0)}{t} + \frac{\sigma_2\mathcal{W}_2(t)}{t}\\ &\quad + \frac{1}{t}\int_0^t \int_{\mathcal{U}} \ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

Using Lemmas 1.3 - 1.4, we get

$$\lim_{t\to\infty}\langle I(t)\rangle = \frac{\mu}{\beta}\left(\mathcal{R}_0^{\mathrm{i}} - 1\right) > 0 \quad \text{a.s.}$$

From (VI.2), on can conclude that

$$\lim_{t\to\infty}\langle S(t)\rangle = \frac{A}{\mu} - \frac{\mu+\delta+\gamma+r_2}{\mu}\lim_{t\to\infty}\langle I(t)\rangle.$$

Consequently,

$$\lim_{t\to\infty}\langle S(t)\rangle = \frac{A}{\mu} - \frac{(\mu+\delta+\gamma+r_2)\left(\mathcal{R}_0^{\mathrm{i}} - 1\right)}{\beta} > 0 \quad \text{a.s.}$$

On the other hand, we establish that

$$\lim_{t\to\infty}\langle Q(t)\rangle = \frac{\delta\mu}{\beta(\mu+k+r_3)}\left(\mathcal{R}_0^{\mathrm{i}} - 1\right) > 0 \quad \text{a.s.}$$

and

$$\lim_{t\to\infty}\langle R(t)\rangle = \frac{(\gamma+k\delta)\mu}{\beta(\mu+k+r_3)}\left(\mathcal{R}_0^{\mathrm{i}} - 1\right) > 0 \quad \text{a.s.}$$

This completes the proof. ∎

1.3 Numerical simulations

In this subsection, we shall use Euler numerical approximation to illustrate the rigor of our analytical results. The two examples are given below concern the results obtained in Theorems 1.5 and 1.6.

Parameters	Description	Numerical values	
A	Recruitment rate of S	4	4
μ	Natural mortality rate	0.2	0.2
β	The transmission rate	0.0176 - 0.016	0.021
γ	Recovered rate for I	0.02	0.02
δ	Quarantine rate of I	0.01	0.01
k	Recovered rate for Q	0.01	0.01
r_2	Disease-related death rate of I	0.1	0.1
r_3	Disease-related death rate of Q	0.2	0.2
σ_1	Intensity of $\mathcal{W}_1(t)$	0.11	0.11
σ_2	Intensity of $\mathcal{W}_2(t)$	0.35	0.35
σ_3	Intensity of $\mathcal{W}_3(t)$	0.12	0.12
σ_4	Intensity of $\mathcal{W}_4(t)$	0.18	0.18
η_1	Jump intensity of S	0.02	0.2
η_2	Jump intensity of I	0.12	0.12
η_3	Jump intensity of R	0.01	0.01
η_4	Jump intensity of Q	0.02	0.02
		Figure VI.1	Figures VI.2 - VI.3

Table VI.1: Nominal values of the system parameters and disturbances intensities adopted in the different simulation examples .

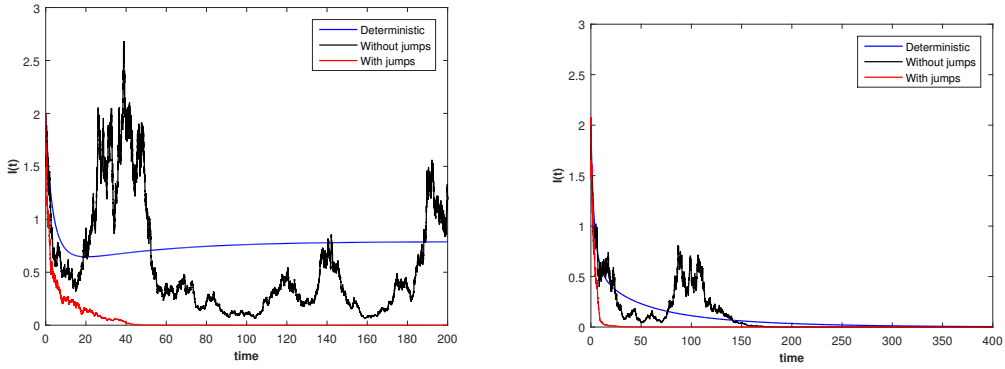


Figure VI.1: Compute simulation of the paths $I(t)$ for the SIQR epidemic model (VI.1) (with jump), the trajectory of $I(t)$ for the system (without jump, i.e. $\eta_i = 0$) and the solution $I(t)$ of the corresponding deterministic system. The left image for $\beta = 0.176$ and the right one for $\beta = 0.16$.

Example 1.7. By taking parameters from the theoretical data mentioned in the Table VI.1, we obtain

$$\mathcal{R}_0 = \frac{\beta A}{\mu(\mu+\gamma+\delta+r_2)} = 1.067 > 1,$$
$$\mathcal{R}_0^s = \frac{1}{\mu+\gamma+\delta+r_2}\left(\frac{\beta A}{\mu} - \frac{\sigma_2^2}{2}\right) = 1.0054 > 1,$$
$$\mathcal{R}_0^i = \frac{1}{\mu+\gamma+\delta+r_2}\left(\frac{\beta A}{\mu} - \frac{\sigma_2^2}{2} - \int_{\mathcal{U}}[\eta_2(u) - \ln(1+\eta_2(u))]\nu(du)\right) = 0.998 < 1.$$

The computer simulations shown in Figure VI.1 - (left) support the result of Theorem 1.5. That is to say, the disease in system (VI.1) dies out exponentially with probability one. Although the disease in stochastic (without jumps) and deterministic systems is persistent. If we decrease β to 0.016, we get

$$\mathcal{R}_0 = \frac{\beta A}{\mu(\mu+\gamma+\delta+r_2)} = 0.9696 < 1,$$
$$\mathcal{R}_0^s = \frac{1}{\mu+\gamma+\delta+r_2}\left(\frac{\beta A}{\mu} - \frac{\sigma_2^2}{2}\right) = 0.908 < 1,$$
$$\mathcal{R}_0^i = \frac{1}{\mu+\gamma+\delta+r_2}\left(\frac{\beta A}{\mu} - \frac{\sigma_2^2}{2} - \int_{\mathcal{U}}[\eta_2(u) - \ln(1+\eta_2(u))]\nu(du)\right) = 0.901 < 1.$$

By the Theorem 1.5, the disease will tend to zero exponentially with probability one (see Figure VI.1 - right).

Example 1.8. Let $\beta = 0.021$ and other parameters be presented in Table VI.1. Then

$$\mathcal{R}_0 = \frac{\beta A}{\mu(\mu+\gamma+\delta+r_2)} = 1.2727 > 1,$$
$$\mathcal{R}_0^s = \frac{1}{\mu+\gamma+\delta+r_2}\left(\frac{\beta A}{\mu} - \frac{\sigma_2^2}{2}\right) = 1.2655 > 1,$$
$$\mathcal{R}_0^i = \frac{1}{\mu+\gamma+\delta+r_2}\left(\frac{\beta A}{\mu} - \frac{\sigma_2^2}{2} - \int_{\mathcal{U}}[\eta_2(u) - \ln(1+\eta_2(u))]\nu(du)\right) = 1.2588 > 1.$$

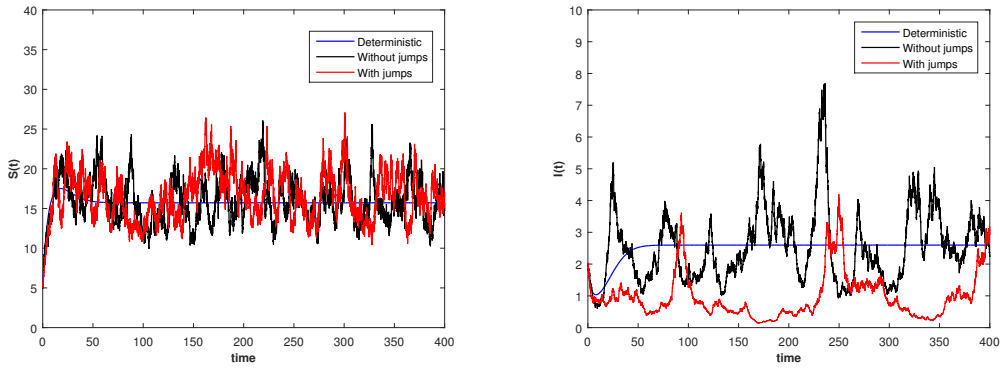


Figure VI.2: Compute simulation of the paths $(S(t), I(t))$ for the SIQR epidemic model (VI.1).

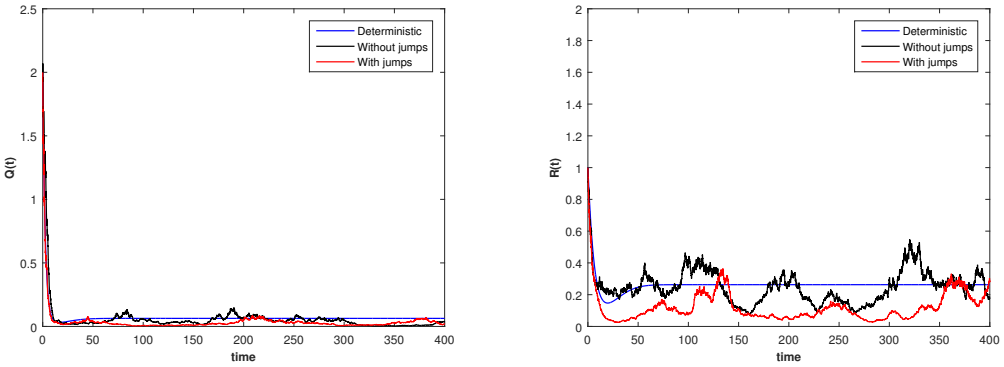


Figure VI.3: Compute simulation of the paths $(Q(t), R(t))$ for the SIQR epidemic model (VI.1)

By Theorem 1.6, we can conclude that the mean solution of $(S(t), I(t), Q(t), R(t))$ of (VI.1) obeys that

$$\begin{aligned}
\lim_{t\to\infty} \langle I(t)\rangle &= 2.548 > 0 \quad \text{a.s.}\\
\lim_{t\to\infty} \langle S(t)\rangle &= 18 > 0 \quad \text{a.s.}\\
\lim_{t\to\infty} \langle Q(t)\rangle &= 0.127 > 0 \quad \text{a.s.}\\
\lim_{t\to\infty} \langle R(t)\rangle &= 0.387 > 0 \quad \text{a.s.}
\end{aligned}$$

This means the disease persists almost surely. The computer simulations are shown in Figure VI.2 and Figure VI.3 support these results clearly.

1.4 Discussion

Ultimately, this short section is concerned with the persistence and extinction of a stochastic SIQR epidemic model with Lévy jumps. We have shown that a Lévy-jumps noise (e.g., tsunami, volcanoes, avian influenza, hurricanes, earthquakes, toxic pollutants, etc.) in reality could mitigate the epidemic in the case of non-persistence. Generally, our results showed that noises have important effects on the dynamic of the disease. Some interesting topics deserve further investigation. We may analyze more realistic but complex models, such as considering the effects of regime-switching with Lévy jumps on the stochastic system (VI.1). We leave these works as our future works.

2 New and general results on the asymptotic behavior of an SIS epidemiological model with quarantine strategy, stochastic transmission, and Lévy disturbance

2.1 Introduction

As previously reported, the spread of infectious diseases can undergo random disturbances and stochastic phenomena due to environmental fluctuations [127, 73]. Since disturbed models can describe many practical problems very well, many types of stochastic differential equations have been used to analyze various epidemic models in recent years [173, 47]. There are two common ways to introduce stochastic factors into epidemic systems. The first one is to assume that the transmission of the diseases is subject to some small random fluctuations which can be described by the Gaussian white noise [156, 157]. The other one is to admit that the model parameters are affected by massive environmental perturbations like the climate changes, earthquakes, hurricanes, floods, etc [154, 172]. For a better explaination to these phenomena, the use of a compensated Poisson process into the population dynamics provides an appropriate and more realistic context [159]. Considering these two types of random disturbances, many works have analyzed the asymptotic behaviors of various epidemic models, including persistence in the mean, extinction, and ergodicity [163, 154, 172]. These interesting researches have served an important role in the stochastic modeling of epidemics. But, all these models have considered either the standard white Gaussian noise or the Lévy jumps. In this work, we combine these two perturbations by treating an SIQS epidemic model that simultaneously includes the stochastic transmission and the discontinuous Lévy process. This original idea extends the study presented in [159] and gives us a general view of the disease dynamics under different scenarios of random perturbations.

The threshold analysis of perturbed epidemic systems is very important for understanding and controlling of the disease spread. In our case, the deterministic SIQS model will be perturbed not only by white noise but also by Lévy jumps, which makes its analysis more complicated and needs some new techniques and methods. During this study, we aim to develop a mathematical approach to prove the existence of a unique ergodic stationary distribution and persistence in the mean of the new perturbed model. Without using the classical Lyapunov method presented in [57], we obtain sufficient conditions for the ergodicity by employing the Feller property and mutually exclusive possibilities lemma. Under the same conditions, we demonstrate that the persistence in the mean of the disease occurs. To analyze properly our new model, we study the stochastic extinction case.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, and containing all the random variables that will be meted in this section. We merge the stochastic transmission with a discontinuous perturbed mortality rates. The random variability in the epidemic transmission β and the mortality rates μ_i $(i = 1, 2, 3)$ are presented by a decomposition of usual white noise and the Lévy-Itô process, respectively. Under these assumptions, the evolution of an epidemic during the quarantine strategy is modeled by the following system of stochastic differential equations:

$$\begin{cases} \mathrm{d}S(t) = \big(A - \mu_1 S(t) - \beta S(t)I(t) + \gamma I(t) + kQ(t)\big)\mathrm{d}t + \mathrm{d}\mathcal{P}_1(t), \\ \mathrm{d}I(t) = \big(\beta S(t)I(t) - (\mu_2 + \delta + \gamma)I(t)\big)\mathrm{d}t + \mathrm{d}\mathcal{P}_2(t), \\ \mathrm{d}Q(t) = \big(\delta I(t) - (\mu_3 + k)Q(t)\big)\mathrm{d}t + \mathrm{d}\mathcal{P}_3(t), \end{cases} \tag{VI.3}$$

where

$$\begin{aligned} \mathrm{d}\mathcal{P}_1(t) &= \sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \int_{\mathcal{U}} \eta_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u) - \sigma_\beta S(t)I(t)\mathrm{d}\mathcal{W}_\beta(t), \\ \mathrm{d}\mathcal{P}_2(t) &= \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \int_{\mathcal{U}} \eta_2(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u) + \sigma_\beta S(t)I(t)\mathrm{d}\mathcal{W}_\beta(t), \\ \mathrm{d}\mathcal{P}_3(t) &= \sigma_3 Q(t)\mathrm{d}\mathcal{W}_3(t) + \int_{\mathcal{U}} \eta_3(u)Q(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u). \end{aligned}$$

Here, $\mathcal{W}_\beta(t)$ and $\mathcal{W}_i(t)$ $(i = 1,2,3)$ are the mutually independent Brownian motions defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the positive intensities σ_β and σ_i $(i = 1,2,3)$. $S(t^-)$, $I(t^-)$ and $Q(t^-)$ are the left limits of $S(t)$, $I(t)$ and $Q(t)$, respectively. $\mathcal{N}$ is a Poisson counting measure with compensating martingale $\widetilde{\mathcal{N}}$ and characteristic measure ν on a measurable subset Z of $(0,\infty)$ satisfying $\nu(Z) < \infty$. It assumed that ν is a Lévy measure such that $\widetilde{\mathcal{N}}(dt, du) = \mathcal{N}(dt, du) - \nu(du)dt$. We also assume that $\mathcal{W}_i(t)$ $(i = 1,2,3,\beta)$ are independent of $\mathcal{N}$. The functions $\eta_i : \mathcal{U} \to \mathbb{R}$ $(i = 1,2,3)$ are bounded and continuous.
For the sake of notational simplicity, we define

- $\bar{\eta}(u) = \max\{\eta_1(u), \eta_2(u), \eta_3(u)\}$ and $\underline{\eta}(u) = \min\{\eta_1(u), \eta_2(u), \eta_3(u)\}$.
- $\hat{\rho}_{n,p}(u) = \big[1 + \bar{\eta}(u)\big]^{np} - 1 - np\bar{\eta}(u)$ and $\check{\rho}_{n,p}(u) = \big[1 + \underline{\eta}(u)\big]^{np} - 1 - np\underline{\eta}(u)$.
- $\ell_{n,p} = \int_{\mathcal{U}} \big[\hat{\rho}_{n,p}(u) \vee \check{\rho}_{n,p}(u)\big]\nu(du)$ and $\bar{\sigma} = \max\{\sigma_1^2, \sigma_2^2, \sigma_3^2\}$.

To properly study our model (VI.3), we have the following fundamental assumptions on the jump-diffusion coefficients:

- **($\mathbf{A}_1$)** We assume that $\int_{\mathcal{U}} \eta_i^2(u)\nu(du) < \infty$, $\{i = 1,2,3\}$.
- **($\mathbf{A}_2$)** For all $u \in \mathcal{U}$, $1 + \eta_i(u) > 0$ and $\int_{\mathcal{U}} \big[\eta_i(u) - \ln(1 + \eta_i(u))\big]\nu(du) < \infty$, $\{i = 1,2,3\}$.
- **($\mathbf{A}_3$)** We suppose that $\int_{\mathcal{U}} \big[\ln(1 + \eta_i(u))\big]^2\nu(du) < \infty$, $\{i = 1,2,3\}$.
- **($\mathbf{A}_4$)** We suppose that $\int_{\mathcal{U}} \big[(1 + \bar{\eta}(u))^2 - 1\big]^2\nu(du) < \infty$.
- **($\mathbf{A}_5$)** We suppose that for each positive integer n there is some real number $p > 1$ for which

$$\gamma_{n,p} = \mu_1 - \frac{(np-1)}{2}\bar{\sigma} - \frac{1}{np}\ell_{n,p} > 0.$$

2.2 Main results

2.2.1 Generalization of some former results

In view of the biological interpretation, the question of whether the stochastic model is well-posed is the first concern. Therefore, to analyze the stochastic model (VI.3), it is necessary to verify the existence of a unique global positive solution, that is, there is no explosion in finite time for any positive initial value $(S(0), I(0), Q(0)) \in \mathbb{R}^3_+$. The following lemma assures the well-posedness of the stochastic model (VI.3).

Lemma 2.1. *Let assumptions ($\mathbf{A}_1$) and ($\mathbf{A}_2$) hold. For any initial value $Y(0) = (S(0), I(0), Q(0)) \in \mathbb{R}^3_+$, there exists a unique positive solution $Y(t) = (S(t), I(t), Q(t))$ of system (VI.3) on $t \geq 0$, and this solution will stay in $\mathbb{R}^3_+$ almost surely.*

The proof is somehow standard and classic (see for example [58, 171]), so we omit it here.

In the following, we always presume that the assumptions ($\mathbf{A}_1$) - ($\mathbf{A}_5$) hold. For reference purposes, we will prepare several useful lemmas.

Lemma 2.2. *Let n be a positive integer and let $Y(t)$ denotes the solution of system (VI.3) that starts from a given point $Y(0) \in \mathbb{R}^3_+$. Then, for any $p > 1$ that satisfies $\gamma_{n,p} > 0$, we have*

- $\mathbb{E}\big[N^{np}(t)\big] \leq N^{np}(0) \times e^{-\frac{np\gamma_{n,p}}{2}t} + \dfrac{2\bar{\delta}}{\gamma_{n,p}}$.

- $\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\big[N^{np}(s)\big]\mathrm{d}s \le \frac{2\bar{\delta}}{\gamma_{n,p}}$ *a.s.*

where $\bar{\delta} = \sup_{x>0}\{Ax^{np-1} - \frac{\gamma_{n,p}}{2}x^{np}\}$ *and* $N(t) = S(t) + I(t) + Q(t)$.

Proof. Making use of Itô's lemma to $N^{np}(t)$, we obtain

$$\begin{aligned}
\mathrm{d}N^{np}(t) = &\Big\{npN^{np-1}(t)\Big(A - \mu_1 N(t) - r_2 I(t) - r_3 Q(t)\Big) + \frac{np}{2}(np-1)N^{np-2}(t)\Big(\sigma_1^2 S^2(t) + \sigma_2^2 I^2(t) + \sigma_3^2 Q^2(t)\Big)\\
&+ \int_{\mathcal{U}} N^{np}(t)\Bigg[\bigg(1 + \eta_1(u)\frac{S(t)}{N(t)} + \eta_2(u)\frac{I(t)}{N(t)} + \eta_3(u)\frac{Q(t)}{N(t)}\bigg)^{np} - 1\\
&- np\bigg(\eta_1(u)\frac{S(t)}{N(t)} + \eta_2(u)\frac{I(t)}{N(t)} + \eta_3(u)\frac{Q(t)}{N(t)}\bigg)\Bigg]\nu(\mathrm{d}u)\Big\}\mathrm{d}t\\
&+ npN^{np-1}(t)\Big(\sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \sigma_3 Q(t)\mathrm{d}\mathcal{W}_3(t)\Big)\\
&+ \int_{\mathcal{U}} N^{np}(t^-)\Bigg[\bigg(1 + \eta_1(u)\frac{S(t^-)}{N(t^-)} + \eta_2(u)\frac{I(t^-)}{N(t^-)} + \eta_3(u)\frac{Q(t^-)}{N(t^-)}\bigg)^{np} - 1\Bigg]\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{d}N^{np}(t) \le &\Big(npN^{np-1}(t)\Big(A - \mu_1 N(t)\Big) + \frac{np}{2}(np-1)N^{np}(t)\bar{\sigma} + N^{np}(t)\int_{\mathcal{U}}\big[\hat{\rho}_{n,p}(u)\vee\check{\rho}_{n,p}(u)\big]\nu(\mathrm{d}u)\Big)\mathrm{d}t\\
&+ npN^{np-1}(t)\Big(\sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \sigma_3 Q(t)\mathrm{d}\mathcal{W}_3(t)\Big)\\
&+ \int_{\mathcal{U}} N^{np}(t^-)\Big((1+\bar{\eta}(u))^{np} - 1\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u). \qquad \text{(VI.4)}
\end{aligned}$$

Rewriting the above inequality, one can see that

$$\begin{aligned}
\mathrm{d}N^{np}(t) \le &\, np\Big\{AN^{np-1}(t) - \Big(\mu_1 - \frac{(np-1)}{2}\bar{\sigma} - \frac{1}{np}\int_{\mathcal{U}}\big[\hat{\rho}_{n,p}(u)\vee\check{\rho}_{n,p}(u)\big]\nu(\mathrm{d}u)\Big)N^{np}(t)\Big\}\mathrm{d}t\\
&+ npN^{np-1}(t)\Big(\sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \sigma_3 Q(t)\mathrm{d}\mathcal{W}_3(t)\Big)\\
&+ \int_{\mathcal{U}} N^{np}(t^-)\Big((1+\bar{\eta}(u))^{np} - 1\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}$$

We choose neatly $p > 1$ such that $\gamma_{n,p} = \mu_1 - \frac{(np-1)}{2}\bar{\sigma} - \frac{1}{np}\int_{\mathcal{U}}\big[\hat{\rho}_{n,p}(u)\vee\check{\rho}_{n,p}(u)\big]\nu(\mathrm{d}u) > 0$. Therefore

$$\begin{aligned}
\mathrm{d}N^{np}(t) \le &\, np\Big\{\delta - \frac{\gamma_{n,p}}{2}N^{np}(t)\Big\}\mathrm{d}t + npN^{np-1}(t)\Big(\sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \sigma_3 Q(t)\mathrm{d}\mathcal{W}_3(t)\Big)\\
&+ \int_{\mathcal{U}} N^{np}(t^-)\Big((1+\bar{\eta}(u))^{np} - 1\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathrm{d}N^{np}(t)\times e^{\frac{np\gamma_{n,p}}{2}t} &= p\gamma_{n,p}\, N^{np}(t)\times e^{\frac{np\gamma_{n,p}}{2}t} + e^{\frac{np\gamma_{n,p}}{2}t}\mathrm{d}N^{np}(t)\\
&\le np\bar{\delta}e^{\frac{np\gamma_{n,p}}{2}t} + e^{\frac{np\gamma_{n,p}}{2}t}\Big[npN^{np-1}(t)\Big(\sigma_1 S(t)\mathrm{d}\mathcal{W}_1(t) + \sigma_2 I(t)\mathrm{d}\mathcal{W}_2(t) + \sigma_3 Q(t)\mathrm{d}\mathcal{W}_3(t)\Big)\\
&\quad + \int_{\mathcal{U}} N^{np}(t^-)\Big((1+\bar{\eta}(u))^{np} - 1\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)\Big].
\end{aligned}$$

Then, by taking the integration and the expectations, we get

$$\mathbb{E}\big[N^{np}(t)\big] \le N^{np}(0)\times e^{-\frac{np\gamma_{n,p}}{2}t} + np\delta\int_0^t e^{-\frac{np}{2}\gamma_{n,p}(t-s)}\mathrm{d}s \le N^{np}(0)e^{-\frac{np\gamma_{n,p}}{2}t} + \frac{2\bar{\delta}}{\gamma_{n,p}}.$$

Obviously, we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\big[N^{np}(s)\big]\mathrm{d}s \le N^{np}(0)\times\limsup_{t\to\infty}\frac{1}{t}\int_0^t e^{-\frac{np\gamma_{n,p}}{2}s}\mathrm{d}s+\frac{2\bar{\delta}}{\gamma_{n,p}}=\frac{2\bar{\delta}}{\gamma_{n,p}}.$$

This completes the proof. ∎

Remark 2.3. Throughout this remark, $\tilde{X}$ is standing for the sum $\eta_1(u)S+\eta_2(u)I+\eta_3(u)Q$, where $u\in\mathcal{U}$. In the study of stochastic biological models driven by Lévy jumps (see for example, [172, 171, 27, 26, 36]), the following quantity

$$\int_{\mathcal{U}} N^{np}\left[\left(1+\frac{\tilde{X}}{N}\right)^{np}-1-np\frac{\tilde{X}}{N}\right]\nu(\mathrm{d}u),$$

is widely majorazed by

$$\int_{\mathcal{U}} N^{np}\Big((1+\bar{\eta}(u))^{np}-1-np\underline{\eta}(u)\Big)\nu(\mathrm{d}u).$$

However, the last estimation can be ameliorated by considering the following inequality

$$\int_{\mathcal{U}} N^{np}\left[\left(1+\frac{\tilde{X}}{N}\right)^{np}-1-np\frac{\tilde{X}}{N}\right]\nu(\mathrm{d}u)\le\int_{\mathcal{U}} N^{np}\big[\hat{\rho}_{n,p}(u)\vee\check{\rho}_{n,p}(u)\big]\nu(\mathrm{d}u),\tag{VI.5}$$

which is established from the observation that the function

$$g(x)=(1+x)^{np}-1-npx,\quad n,p\ge 1,$$

is decreasing for $x\in(-1,0)$ and increasing for $x\ge 0$. Needless to say, this last fact makes necessarily $g(a)\vee g(b)$ as the highest value of g on any interval $[a,b]\subset(-1,\infty)$. The adoption of the inequality (VI.5) in our calculus, especially in (VI.4), (VI.7) and (VI.12), will improve many classical results presented in the above mentioned papers.

Remark 2.4. Lemma 2.2 takes into consideration the stochastic transmission and the effect of Lévy jumps, and this makes it clearly an extended version of Lemma 2.3 presented in [162].

Lemma 2.5. *Consider the initial value problem*

$$\begin{cases}\mathrm{d}X(t)=\big(A-\mu_1X(t)\big)\mathrm{d}t+\bar{\mathcal{P}}_1(t)+\bar{\mathcal{P}}_2(t)+\mathcal{P}_3(t),\\ X(0)=N(0)\in\mathbb{R}_+,\end{cases}\tag{VI.6}$$

where

$$\bar{\mathcal{P}}_1(t)=\sigma_1S(t)\mathrm{d}\mathcal{W}_1(t)+\int_{\mathcal{U}}\eta_1(u)S(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),$$
$$\bar{\mathcal{P}}_2(t)=\sigma_2I(t)\mathrm{d}\mathcal{W}_2(t)+\int_{\mathcal{U}}\eta_2(u)I(t^-)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).$$

Let us denote by $Y(t)=(S(t),I(t),Q(t))$ and $X(t)$ the positive solutions of systems (VI.3) and (VI.6) respectively. Then

- $\lim\limits_{t\to\infty}\dfrac{X^n(t)}{t}=0$ *a.s.*, $\quad\forall n\in\{1,2,\cdots\}$.

- $\lim\limits_{t\to\infty}\dfrac{\int_0^t X(s)S(s)\mathrm{d}\mathcal{W}_1(s)}{t}=0,\ \lim\limits_{t\to\infty}\dfrac{\int_0^t X(s)I(s)\mathrm{d}\mathcal{W}_2(s)}{t}=0,\ \textit{and}\ \lim\limits_{t\to\infty}\dfrac{\int_0^t X(s)Q(s)\mathrm{d}\mathcal{W}_3(s)}{t}=0\quad a.s.$
- $\lim\limits_{t\to\infty}\dfrac{\int_0^t\int_{\mathcal{U}}\left((1+\bar{\eta}(u))^2-1\right)X^2(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0\quad a.s.$

Proof. Our approach to demonstrate this lemma is mainly adapted from [171]. The proof falls naturally into three steps.

Step 1. Applying the generalized Itô's formula [27] to $\mathcal{K}(X)=X^{np}$, where n is a fixed integer number, we derive

$$\begin{aligned}\mathrm{d}\mathcal{K}(X)\le{}&\mathcal{L}\mathcal{K}\mathrm{d}t+npX^{np-1}\Big(\sigma_1S\mathrm{d}\mathcal{W}_1(t)+\sigma_2I\mathrm{d}\mathcal{W}_2(t)+\sigma_3Q\mathrm{d}\mathcal{W}_3(t)\Big)\\&+\int_{\mathcal{U}}X^{np}(t^-)\Big((1+\bar{\eta}(u))^{np}-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u),\end{aligned}\tag{VI.7}$$

where

$$\mathcal{L}\mathcal{K}\le npX^{np-2}\Big[AX-\Big(\mu_1-\frac{(np-1)}{2}\bar{\sigma}-\frac{1}{np}\ell_{n,p}\Big)X^2\Big].$$

Choose a positive constant $p>1$ such that $\gamma_{n,p}=\mu_1-\frac{(np-1)}{2}\bar{\sigma}-\frac{1}{np}\ell_{n,p}>0$. Then

$$\begin{aligned}\mathrm{d}\mathcal{K}(X)\le{}&\Big(npX^{np-2}\big(AX-\gamma_{n,p}X^2\big)\Big)\mathrm{d}t+npX^{np-1}\Big(\sigma_1S\mathrm{d}\mathcal{W}_1(t)+\sigma_2I\mathrm{d}\mathcal{W}_2(t)+\sigma_3Q\mathrm{d}\mathcal{W}_3(t)\Big)\\&+\int_{\mathcal{U}}X^{np}(t^-)\Big((1+\bar{\eta}(u))^{np}-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).\end{aligned}\tag{VI.8}$$

For any constant m satisfying $m\in(0,np\gamma_{n,p})$, one can see that

$$\begin{aligned}\mathrm{d}e^{ms}\mathcal{K}(X(s))\le{}&\mathcal{L}\big(e^{mt}\mathcal{K}(X(t))\big)+npe^{mt}X^{np-1}(t)\Big(\sigma_1S(t)\mathrm{d}\mathcal{W}_1(t)+\sigma_2I(t)\mathrm{d}\mathcal{W}_2(t)+\sigma_3Q(t)\mathrm{d}\mathcal{W}_3(t)\Big)\\&+e^{mt}\int_{\mathcal{U}}X^{np}(t^-)\Big((1+\bar{\eta}(u))^{np}-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).\end{aligned}$$

Integrating both sides of the last inequality from 0 to t, we get

$$\begin{aligned}\int_0^t\mathrm{d}e^{ms}\mathcal{K}(X(s))\le{}&\int_0^t\Big(me^{ms}\mathcal{K}(X(s))+e^{ms}\mathcal{L}\big(\mathcal{K}(X(s))\big)\Big)\mathrm{d}s\\&+np\int_0^te^{ms}X^{np-1}(s)\Big(\sigma_1S(s)\mathrm{d}\mathcal{W}_1(s)+\sigma_2I(s)\mathrm{d}\mathcal{W}_2(s)+\sigma_3Q(s)\mathrm{d}\mathcal{W}_3(s)\Big)\\&+\int_0^te^{ms}\int_{\mathcal{U}}X^{np}(s^-)\Big((1+\bar{\eta}(u))^{np}-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

Taking expectation on both sides yields that

$$\mathbb{E}e^{mt}\mathcal{K}(X(t))\le\mathcal{K}(X(0))+\mathbb{E}\Big\{\int_0^t\Big(me^{ms}\mathcal{K}(X(s))+e^{ms}\mathcal{L}\big(\mathcal{K}(X(s))\big)\Big)\mathrm{d}s\Big\}.$$

In view of (VI.8), we can see that

$$me^{mt}\mathcal{K}(X(t))+e^{mt}\mathcal{L}\big(\mathcal{K}(X)\big)\le npe^{mt}\bar{H},$$

where $\bar{H}=\sup\limits_{X>0}\Big\{X^{np-2}\Big[-\Big(\gamma_{n,p}-\frac{m}{np}\Big)X^2+AX\Big]+1\Big\}$. Then, we have

$$\mathbb{E}e^{mt}\mathcal{K}(X(s))\le\mathcal{K}(X(0))+\frac{np\bar{H}}{m}e^{mt}.$$

Therefore, we get

$$\limsup_{t\to\infty} \mathbb{E}\big[X^{np}(t)\big] \le \frac{np\bar{H}}{m} \quad \text{a.s.}$$

Consequently, there exists a positive constant $\bar{M}$ such that for all $t \ge 0$,

$$\mathbb{E}\big[X^{np}(t)\big] \le \bar{M}. \tag{VI.9}$$

Step 2. Integrating from 0 to t after applying the famous Burkholder-Davis-Gundy inequality [93] to (VI.8), allows us to conclude that for an arbitrarily small positive constant z, $m = 1, 2, ...,$

$$\mathbb{E}\Big[\sup_{mz\le t\le (m+1)z} X^{np}(t)\Big] \le \mathbb{E}\Big[X(mz)\Big]^{np} + \Big(z_1 z + z_2 z^{\frac{1}{2}}\Big(np\bar{\sigma} + \int_{\mathcal{U}} \big((1+\bar{\eta}(u))^{np} - 1\big)^2 \nu(\mathrm{d}u)\Big)\Big) \times \Big[\sup_{mz\le t\le (m+1)z} X^{np}(t)\Big],$$

where z_1 and z_2 are positive constants. Specially, we select $z > 0$ such that

$$z_1 z + z_2 z^{\frac{1}{2}}\Big(np\bar{\sigma} + \int_{\mathcal{U}} \big((1+\bar{\eta}(u))^{np} - 1\big)^2 \nu(\mathrm{d}u)\Big) \le \frac{1}{2}.$$

Then

$$\mathbb{E}\Big[\sup_{mz\le t\le (m+1)z} X^{np}(t)\Big] \le 2\bar{M}.$$

Let $\bar{\epsilon} > 0$ be arbitrary. By employing Chebyshev's inequality, we derive

$$\mathbb{P}\Big\{\sup_{mz\le t\le (m+1)z} X^{np}(t) > (mz)^{1+\bar{\epsilon}}\Big\} \le \frac{\mathbb{E}\Big[\sup_{mz\le t\le (m+1)z} X^{np}(t)\Big]}{(mz)^{1+\bar{\epsilon}}} \le \frac{2\bar{M}}{(mz)^{1+\bar{\epsilon}}}.$$

Making use of the Borel-Cantelli lemma gives that for almost all $\omega \in \Omega$

$$\sup_{mz\le t\le (m+1)z} X^{np}(t) \le (mz)^{1+\bar{\epsilon}}, \tag{VI.10}$$

verifies for all but finitely many m. Consequently, there exists a positive constant $m_0(\omega)$ such that $m_0 \le m$ and (VI.10) holds for almost all $\omega \in \Omega$. In other words, for almost all $\omega \in \Omega$, if $m_0 \le m$ and $mz \le t \le (m+1)z$,

$$\frac{\ln X^{np}(t)}{\ln t} \le \frac{(1+\bar{\epsilon})\ln(mz)}{\ln(mz)} = 1 + \bar{\epsilon}.$$

Because $\bar{\epsilon}$ is arbitrarily small, then

$$\limsup_{t\to\infty} \frac{\ln X^{n}(t)}{\ln t} \le \frac{1}{p} \quad \text{a.s.}$$

Therefore, for any small $\bar{v} \in (0, 1 - 1/p)$, there is a constant $\bar{V} = \bar{V}(\omega)$, for which if $t \ge \bar{V}$ then

$$\ln X^{n}(t) \le \Big(\frac{1}{p} + \bar{v}\Big)\ln t.$$

Hence

$$\limsup_{t\to\infty} \frac{X^{n}(t)}{t} \le \limsup_{t\to\infty} \frac{t^{\frac{1}{p}+\bar{v}}}{t} = 0.$$

This together with the positivity of the solution implies

$$\lim_{t\to\infty}\frac{X^n(t)}{t}=0\quad\text{a.s.}$$

Step 3. Now, we define

$$\mathcal{I}_1(t)=\frac{1}{t}\int_0^t X(s)S(s)\mathrm{d}\mathcal{W}_1(s),\qquad \mathcal{I}_2(t)=\frac{1}{t}\int_0^t X(s)I(s)\mathrm{d}\mathcal{W}_2(s),$$
$$\mathcal{I}_3(t)=\frac{1}{t}\int_0^t X(s)Q(s)\mathrm{d}\mathcal{W}_3(s),\qquad \mathcal{I}_4(t)=\frac{1}{t}\int_0^t\int_{\mathcal{U}} X^2(s^-)\big(\big(1+\bar{\eta}\big)^2-1\big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).$$

In view of the Burkholder-Davis-Gundy inequality, we find that for $\bar{p}>2$,

$$\mathbb{E}\bigg[\sup_{m\le t\le(m+1)}|\mathcal{I}_1(t)|^{\bar{p}}\bigg]\le C_{\bar{p}}\mathbb{E}\bigg[\int_0^t X^4(s)\mathrm{d}s\bigg]^{\frac{\bar{p}}{2}}\le C_{\bar{p}}\bigg[\mathbb{E}\int_0^t X^4(s)\mathrm{d}s\bigg]^{\frac{\bar{p}}{2}}\le C_{\bar{p}}\bigg[\mathbb{E}\int_0^t |X^4(s)|\mathrm{d}s\bigg]^{\frac{\bar{p}}{2}},\tag{VI.11}$$

where $C_{\bar{p}}=\Big[\frac{\bar{p}^{\bar{p}+1}}{2(\bar{p}-1)^{\bar{p}-1}}\Big]^{\bar{p}/2}>0$. Similarly to the previous case, we find

$$\mathbb{E}\bigg[\sup_{m\le t\le(m+1)}|\mathcal{I}_4(t)|^{\bar{p}}\bigg]\le C_{\bar{p}}\bigg(\int_{\mathcal{U}}\Big(\big(1+\bar{\eta}\big)^2-1\Big)^2\nu(\mathrm{d}u)\bigg)^{\frac{\bar{p}}{2}}\bigg[\mathbb{E}\int_0^t|X^4(s)|\mathrm{d}s\bigg]^{\frac{\bar{p}}{2}}.$$

Via (VI.9) and (VI.11), one can see that

$$\mathbb{E}\Big[\sup_{m\le t\le(m+1)}|\mathcal{I}_1(t)|^{\bar{p}}\Big]\le 2^{1+\frac{\bar{p}}{2}}\bar{M}C_{\bar{p}}m^{\frac{\bar{p}}{2}}.$$

For any arbitrary positive constant $\tilde{\epsilon}$, and by making use of Chebyshev's inequality, we obtain

$$\mathbb{P}\Big\{\sup_{m\le t\le(m+1)}|\mathcal{I}_1(t)|^{\bar{p}}>\bar{p}^{1+\tilde{\epsilon}+\frac{\bar{p}}{2}}\Big\}\le\frac{\mathbb{E}\Big[\sup_{m\le t\le(m+1)}|\mathcal{I}_1(m+1)|^{\bar{p}}\Big]}{\bar{p}^{1+\tilde{\epsilon}+\frac{\bar{p}}{2}}}\le\frac{2^{1+\frac{\bar{p}}{2}}\bar{M}C_{\bar{p}}}{\bar{p}^{1+\tilde{\epsilon}}},\quad m=1,2,\ldots$$

Using the Borel-Cantelli lemma, one has

$$\frac{\ln|\mathcal{I}_1(t)|^{\bar{p}}}{\ln t}\le\frac{\big(1+\tilde{\epsilon}+\frac{\bar{p}}{2}\big)\ln m}{\ln m}=1+\tilde{\epsilon}+\frac{\bar{p}}{2}.$$

Taking the limit superior on both sides of the last inequality and applying the arbitrariness of $\tilde{\epsilon}$, we deduce

$$\limsup_{t\to\infty}\frac{\ln|\mathcal{I}_1(t)|}{\ln t}\le\frac{1}{2}+\frac{1}{\bar{p}}\quad\text{a.s.}$$

That is to say, for any positive constant $\bar{\tau}\in\big(0,\frac{1}{2}-\frac{1}{\bar{p}}\big)$, there exists a constant $\bar{T}=\bar{T}(\omega)$ such that for all $t\ge\bar{T}$,

$$\ln|\mathcal{I}_1(t)|\le\Big(\frac{1}{2}+\frac{1}{\bar{p}}+\bar{\tau}\Big)\ln t.$$

Dividing both sides of the last inequality by t and taking the limit superior, we have

$$\limsup_{t\to\infty}\frac{|\mathcal{I}_1(t)|}{t}\le\limsup_{t\to\infty}\frac{t^{\frac{1}{2}+\frac{1}{\bar{p}}+\bar{\tau}}}{t}=0.$$

Combining it with $\liminf_{t\to\infty}\frac{|\mathcal{I}_1(t)|}{t}\ge 0$, one has $\lim_{t\to\infty}\frac{|\mathcal{I}_1(t)|}{t}=\lim_{t\to\infty}\frac{\mathcal{I}_1(t)}{t}=0$ a.s.
In the same way, we prove that

$$\lim_{t\to\infty}\frac{\mathcal{I}_2(t)}{t}=0,\quad\lim_{t\to\infty}\frac{\mathcal{I}_3(t)}{t}=0,\quad\lim_{t\to\infty}\frac{\mathcal{I}_4(t)}{t}=0\quad\text{a.s.}$$

This completes the proof. ∎

Remark 2.6. The positivity of the solutions $X(t)$ and $Y(t)$ together with the stochastic comparison theorem [93], leads to the fact that $N(t) \leq X(t)$ a.s. which in turn implies that

$$\lim_{t\to\infty}\frac{S^n(t)}{t}=0,\quad \lim_{t\to\infty}\frac{I^n(t)}{t}=0,\quad \lim_{t\to\infty}\frac{Q^n(t)}{t}=0,\quad \text{and even}\quad \lim_{t\to\infty}\frac{N^n(t)}{t}=0\ \text{ a.s.}$$

Remark 2.7. By comparing our findings with those of Lemmas 3.3 and 3.4 in [171], one can conclude that the new result 2.5 presents a modified and generalized version to these lemmas, which will be necessary to prove Lemma 2.10.

Lemma 2.8. *Let $Y(0) \in \mathbb{R}^3_+$ be a positive given value. If $Y(t)$ denotes the positive solution of system (VI.3) that starts from $Y(0)$, then*

- $\lim\limits_{t\to\infty}\dfrac{\int_0^t S(s)\mathrm{d}\mathcal{W}_\beta(s)}{t}=0,\quad \lim\limits_{t\to\infty}\dfrac{\int_0^t S(s)\mathrm{d}\mathcal{W}_1(s)}{t}=0,\quad \lim\limits_{t\to\infty}\dfrac{\int_0^t I(s)\mathrm{d}\mathcal{W}_2(s)}{t}=0,$
  and $\lim\limits_{t\to\infty}\dfrac{\int_0^t Q(s)\mathrm{d}\mathcal{W}_3(s)}{t}=0$ a.s.
- $\lim\limits_{t\to\infty}\dfrac{\int_0^t\int_{\mathcal{U}}\eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0,\ \lim\limits_{t\to\infty}\dfrac{\int_0^t\int_{\mathcal{U}}\eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0,$
  and $\lim\limits_{t\to\infty}\dfrac{\int_0^t\int_{\mathcal{U}}\eta_3(u)Q(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)}{t}=0$ a.s.

Remark 2.9. The last lemma is easily demonstrated by using an analysis similar to that in the proof of Lemma 2.5.

Lemma 2.10. *Let $X(t) \in \mathbb{R}_+$ be the solution of the equation (VI.6) with any given initial value $X(0) = N(0) \in \mathbb{R}_+$. Suppose that $\chi = 2\mu_1 - \bar{\sigma} - \int_{\mathcal{U}}\left[\bar{\eta}^2(u) \vee \underline{\eta}^2(u)\right]\nu(\mathrm{d}u) > 0$, then*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X(s)\mathrm{d}s=\frac{A}{\mu_1}\ \ a.s.$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X^2(s)\mathrm{d}s\leq\frac{2A^2}{\mu_1\chi}\ \ a.s.$$

Proof. Integrating from 0 to t on both sides of (VI.6) yields

$$\begin{aligned}\frac{X(t)-X(0)}{t}=A-\frac{\mu_1}{t}\int_0^t X(s)\mathrm{d}s+\frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_1(s)+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\+\frac{\sigma_2}{t}\int_0^t I(s)\mathrm{d}\mathcal{W}_2(s)+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\+\frac{\sigma_3}{t}\int_0^t Q(s)\mathrm{d}\mathcal{W}_3(s)+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_3(u)Q(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

Clearly, we can derive that

$$\begin{aligned}\frac{1}{t}\int_0^t X(s)\mathrm{d}s=\frac{A}{\mu_1}-\frac{X(t)-X(0)}{\mu_1 t}+\frac{\sigma_1}{\mu_1 t}\int_0^t S(s)\mathrm{d}\mathcal{W}_1(s)+\frac{1}{\mu_1 t}\int_0^t\int_{\mathcal{U}}\eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\+\frac{\sigma_2}{\mu_1 t}\int_0^t I(s)\mathrm{d}\mathcal{W}_2(s)+\frac{1}{\mu_1 t}\int_0^t\int_{\mathcal{U}}\eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\+\frac{\sigma_3}{\mu_1 t}\int_0^t Q(s)\mathrm{d}\mathcal{W}_3(s)+\frac{1}{\mu_1 t}\int_0^t\int_{\mathcal{U}}\eta_3(u)Q(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

According to Lemma 2.8, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X(s)\mathrm{d}s = \frac{A}{\mu_1}\ \text{a.s.}$$

Now, applying the generalized Itô's formula to equation (VI.6) leads to

$$\begin{aligned}\mathrm{d}X^2(t) \le{}& \Big(2X(t)\Big(A-\mu_1X(t)\Big)+\bar{\sigma}X^2(t)+\int_{\mathcal{U}}X^2(t)\big[\bar{\eta}^2(u)\vee\underline{\eta}^2(u)\big]\nu(\mathrm{d}u)\Big)\mathrm{d}t\\ &+2X(t)\Big(\sigma_1S(t)\mathrm{d}\mathcal{W}_1(t)+\sigma_2I(t)\mathrm{d}\mathcal{W}_2(t)+\sigma_3Q(t)\mathrm{d}\mathcal{W}_3(t)\Big)\\ &+\int_{\mathcal{U}}X^2(t^-)\Big(\big(1+\bar{\eta}(u)\big)^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u).\end{aligned}\tag{VI.12}$$

Integrating both sides from 0 to t, yields

$$\begin{aligned}X^2(t)-X^2(0)\le{}& 2A\int_0^t X(s)\mathrm{d}s-\Big(2\mu_1-\bar{\sigma}-\int_{\mathcal{U}}\big[\bar{\eta}^2(u)\vee\underline{\eta}^2(u)\big]\nu(\mathrm{d}u)\Big)\int_0^t X^2(s)\mathrm{d}s\\ &+2\sigma_1\int_0^t X(s)S(s)\mathrm{d}\mathcal{W}_1(s)+2\sigma_2\int_0^t X(s)I(s)\mathrm{d}\mathcal{W}_2(s)+2\sigma_3\int_0^t X(s)Q(s)\mathrm{d}\mathcal{W}_3(s)\\ &+\int_0^t\int_{\mathcal{U}}X^2(s^-)\Big(\big(1+\bar{\eta}(u)\big)^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

Let $\chi = 2\mu_1-\bar{\sigma}-\int_{\mathcal{U}}\big[\bar{\eta}^2(u)\vee\underline{\eta}^2(u)\big]\nu(\mathrm{d}u) > 0$. Therefore

$$\begin{aligned}\frac{1}{t}\int_0^t X^2(s)\mathrm{d}s \le{}& \frac{2A}{\chi t}\int_0^t X(s)\mathrm{d}s+\frac{X^2(0)-X^2(t)}{\chi t}+\frac{2\sigma_1}{\chi t}\int_0^t X(s)S(s)\mathrm{d}\mathcal{W}_1(s)\\ &+\frac{2\sigma_2}{\chi t}\int_0^t X(s)I(s)\mathrm{d}\mathcal{W}_2(s)+\frac{2\sigma_3}{\chi t}\int_0^t X(s)Q(s)\mathrm{d}\mathcal{W}_3(s)\\ &+\frac{1}{\chi t}\int_0^t\int_{\mathcal{U}}X^2(s)\Big(\big(1+\bar{\eta}(u)\big)^2-1\Big)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

By Lemma 2.5 and assumptions **(A$_4$)-(A$_5$)**, we can easily verify that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X^2(s)\mathrm{d}s \le \frac{2A^2}{\mu_1\chi}\ \text{a.s.}$$

■

2.2.2 Ergodicity and persistence in the mean

In the following, we aim to give the condition for the ergodicity the persistence of the disease. We suppose that $\chi > 0$ and we define the parameter:

$$\mathcal{R}_0^s = \Big(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\Big)^{-1}\left(\frac{\beta A}{\mu_1}-\frac{A^2\sigma_\beta^2}{\mu_1\chi}-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)\right).$$

For simplicity, we introduce the following notations:

- $M_1 = \dfrac{\mu_1^2}{4\beta^2A}\left(\mu_2+\delta+\gamma+\dfrac{\sigma_2^2}{2}\right)\Big(\mathcal{R}_0^s-1\Big)$.
- $M_2 = \dfrac{p\mu_1\gamma_{2,p}\beta^{-(p+1)}}{8\bar{\delta}}\left(\mu_2+\delta+\gamma+\dfrac{\sigma_2^2}{2}\right)\Big(\mathcal{R}_0^s-1\Big)$.
- $M_3 = \dfrac{\mu_1 q}{8\beta}\left(\mu_2+\delta+\gamma+\dfrac{\sigma_2^2}{2}\right)\left(\dfrac{2A}{\mu_1}+N(0)\right)^{-1}\Big(\mathcal{R}_0^s-1\Big)$.

Theorem 2.11. *If $\mathcal{R}_0^s > 1$, the stochastic system (VI.3) admits a unique stationary distribution and it has the ergodic property for any initial value $Y(0) \in \mathbb{R}^3_+$.*

Proof. Applying the generalized Itô's formula to $\ln I - \frac{\beta}{\mu_1}(X - S)$, we easily derive

$$\begin{aligned}
\mathrm{d}\left\{\ln I(t) - \frac{\beta}{\mu_1}\Big(X(t) - S(t)\Big)\right\} \geq \left(\beta X(t) - (\mu_2 + \delta + \gamma) - \frac{\sigma_2^2}{2} - \frac{\sigma_\beta^2}{2}S^2(t) - \int_{\mathcal{U}} \eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u)\right)\mathrm{d}t \\
- \frac{\beta^2}{\mu_1}S(t)I(t)\mathrm{d}t + \sigma_2 \mathrm{d}\mathcal{W}_2(t) + \int_{\mathcal{U}} \ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}u) + \sigma_\beta S(t)\mathrm{d}\mathcal{W}_\beta(t) \\
- \frac{\beta}{\mu_1}\sigma_\beta S(t)I(t)\mathrm{d}\mathcal{W}_\beta(t) - \frac{\beta}{\mu_1}\bar{\mathcal{P}}_2(t) - \frac{\beta}{\mu_1}\mathcal{P}_3(t). \qquad \text{(VI.13)}
\end{aligned}$$

Integrating from 0 to t on both sides of (VI.13) yields

$$\begin{aligned}
&\ln\frac{I(t)}{I(0)} - \frac{\beta}{\mu_1}\Big(X(t) - S(t)\Big) + \frac{\beta}{\mu_1}\Big(X(0) - S(0)\Big) \\
&\geq \int_0^t \left(\beta X(s) - (\mu_2 + \delta + \gamma) - \frac{\sigma_2^2}{2} - \frac{\sigma_\beta^2}{2}S^2(s) - \int_{\mathcal{U}} \eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u)\right)\mathrm{d}s \\
&\quad - \frac{\beta^2}{\mu_1}\int_0^t S(s)I(s)\mathrm{d}s + \sigma_2 W_2(t) + \int_0^t\int_{\mathcal{U}} \ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u) + \sigma_\beta\int_0^t S(s)\mathrm{d}\mathcal{W}_\beta(s) \\
&\quad - \frac{\beta\sigma_\beta}{\mu_1}\int_0^t S(s)I(s)\mathrm{d}\mathcal{W}_\beta(s) - \frac{\beta\sigma_2}{\mu_1}\int_0^t I(s)\mathrm{d}\mathcal{W}_2(s) - \frac{\beta}{\mu_1}\int_0^t\int_{\mathcal{U}} \eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u) \\
&\quad - \frac{\beta\sigma_3}{\mu_1}\int_0^t Q(s)\mathrm{d}\mathcal{W}_3(s) - \frac{\beta}{\mu_1}\int_0^t\int_{\mathcal{U}} \eta_3(u)Q(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u).
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^t \beta S(s)I(s)\mathrm{d}s \\
&\geq \frac{\mu_1}{\beta}\int_0^t \left(\beta X(s) - (\mu_2 + \delta + \gamma) - \frac{\sigma_2^2}{2} - \frac{\sigma_\beta^2}{2}S^2(s) - \int_{\mathcal{U}} \eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u)\right)\mathrm{d}s \\
&\quad - \frac{\mu_1}{\beta}\ln\frac{I(t)}{I(0)} + \Big(X(t) - S(t)\Big) - \Big(X(0) - S(0)\Big) + \frac{\mu_1}{\beta}\sigma_2 W_2(t) + \frac{\mu_1}{\beta}\int_0^t\int_{\mathcal{U}} \ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u) \\
&\quad + \frac{\mu_1\sigma_\beta}{\beta}\int_0^t S(s)\mathrm{d}\mathcal{W}_\beta(s) - \sigma_\beta\int_0^t S(s)I(s)\mathrm{d}\mathcal{W}_\beta(s) - \sigma_2\int_0^t I(s)\mathrm{d}\mathcal{W}_2(s) - \int_0^t\int_{\mathcal{U}} \eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u) \\
&\quad - \beta\sigma_3\int_0^t Q(s)\mathrm{d}\mathcal{W}_3(s) - \int_0^t\int_{\mathcal{U}} \eta_3(u)Q(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u). \qquad \text{(VI.14)}
\end{aligned}$$

From Remark 2.6 and Lemma 2.8, one can derive that

$$\lim_{t\to\infty}\frac{X(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{S(t)}{t} = 0, \quad \text{and} \quad \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathcal{U}} (\eta_2(u)I(s^-) + \eta_3(u)Q(s^-))\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u) \quad \text{a.s.}$$

Moreover,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_\beta(s) = 0, \quad \frac{1}{t}\int_0^t I(s)\mathrm{d}\mathcal{W}_2(s) = 0 \ \text{ and } \ \lim_{t\to\infty}\frac{1}{t}\int_0^t Q(s)\mathrm{d}\mathcal{W}_3(s) = 0 \quad \text{a.s.}$$

Application of the strong law of large numbers and assumption **(A$_3$)** shows that

$$\lim_{t\to\infty}\frac{\mathcal{W}_2(t)}{t} = 0 \ \text{ and } \ \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathcal{U}} \ln(1+\eta_2(u))\tilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u) = 0 \quad \text{a.s.}$$

Applying similar arguments to those in the proof of Lemma 2.5, we obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(s)I(s)\mathrm{d}\mathcal{W}_\beta(s) = 0 \quad \text{a.s.}$$

Since $\limsup_{t\to\infty}\frac{1}{t}\ln\frac{I(t)}{I(0)} \le \limsup_{t\to\infty}\frac{1}{t}\ln\frac{N(t)}{I(0)} \le 0$ a.s., one can derive that

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\int_0^t \beta S(s)I(s)\mathrm{d}s\\
&\ge \frac{\mu_1}{\beta}\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left(\beta X(s)-(\mu_2+\delta+\gamma)-\frac{\sigma_2^2}{2}-\frac{\sigma_\beta^2}{2}X^2(s)-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)\right)\mathrm{d}s\\
&= \frac{\mu_1}{\beta}\lim_{t\to\infty}\frac{1}{t}\int_0^t \beta X(s)\mathrm{d}s-\frac{\sigma_\beta^2}{2}\lim_{t\to\infty}\frac{1}{t}\int_0^t X^2(s)\mathrm{d}s-\left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u).
\end{aligned}$$

From Lemma 2.10, it follows that

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\int_0^t \beta S(s)I(s)\mathrm{d}s &\ge \frac{\mu_1}{\beta}\times\left(\frac{\beta A}{\mu_1}-\frac{A^2\sigma_\beta^2}{\mu_1\chi}-\left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)\right)\\
&= \frac{\mu_1}{\beta}\left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)\left(\mathcal{R}_0^s-1\right)>0 \quad \text{a.s.} \qquad \text{(VI.15)}
\end{aligned}$$

To continue our analysis, we need to set the following subsets:

- $\Omega_1=\{(t,\omega)\in\mathbb{R}_+\times\Omega|\ S(t,\omega)\ge\epsilon,\ \text{and},\ I(t,\omega)\ge\epsilon\}$,
- $\Omega_2=\{(t,\omega)\in\mathbb{R}_+\times\Omega|\ S(t,\omega)\le\epsilon\}$,
- $\Omega_3=\{(t,\omega)\in\mathbb{R}_+\times\Omega|\ I(t,\omega)\le\epsilon\}$,

where $\epsilon>0$ is a positive constant to be determined later. Therefore, by (VI.15), we get

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\left(\beta S(s)I(s)\mathbb{1}_{\Omega_1}\right)\mathrm{d}s\ge\frac{\mu_1}{\beta}\left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)\left(\mathcal{R}_0^s-1\right)-\frac{2A\beta\epsilon}{\mu_1}.$$

We can choose $\epsilon\le M_1$, and then we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\left(\beta S(s)I(s)\mathbb{1}_{\Omega_1}\right)\mathrm{d}s\ge\frac{\mu_1}{2\beta}\left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)\left(\mathcal{R}_0^s-1\right)>0. \qquad \text{(VI.16)}$$

Let $q=a_0>1$ be a positive integer and $1<p=\frac{a_0}{a_0-1}$ such that $\gamma_{2,p}>0$ and $\frac{1}{q}+\frac{1}{p}=1$. By utilizing the Young inequality $xy\le\frac{x^p}{p}+\frac{y^q}{q}$ for all $x,y>0$, we get

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\left(\beta S(s)I(s)\mathbb{1}_{\Omega_1}\right)\mathrm{d}s &\le \liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\left(p^{-1}(\eta\beta S(s)I(s))^p+q^{-1}\eta^{-q}\mathbb{1}_{\Omega_1}\right)\mathrm{d}s\\
&\le \liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\left(q^{-1}\eta^{-q}\mathbb{1}_{\Omega_1}\right)\mathrm{d}s+p^{-1}(\eta\beta)^p\limsup_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\left[N^{2p}(s)\right]\mathrm{d}s,
\end{aligned}$$

where η is a positive constant satisfying $\eta^p\le M_2$. By Lemma 2.2 and (VI.16), we deduce that

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}\left[\mathbb{1}_{\Omega_1}\right]\mathrm{d}s &\ge q\eta^q\left(\frac{\mu_1}{2\beta}\left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)\left(\mathcal{R}_0^s-1\right)-\frac{2\eta^p\beta^p\gamma_{2,p}}{p\bar{\delta}}\right)\\
&\ge \frac{\mu_1 q\eta^q}{4\beta}\left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)\left(\mathcal{R}_0^s-1\right)>0. \qquad \text{(VI.17)}
\end{aligned}$$

Setting

$$\begin{aligned}
\Omega_4&=\{(t,\omega)\in\mathbb{R}_+\times\Omega|\ S(t,\omega)\ge\zeta,\ \text{or},\ I(t,\omega)\ge\zeta\},\\
\Sigma&=\{(t,\omega)\in\mathbb{R}_+\times\Omega|\ \epsilon\le S(t,\omega)\le\zeta,\ \text{and},\ \epsilon\le I(t,\omega)\le\zeta\},
\end{aligned}$$

where $\zeta>\epsilon$ is a positive constant to be explained in the following. By using the Markov's inequality, we can observe that

$$\int_\Omega\mathbb{1}_{\Omega_4}(t,\omega)\mathrm{d}\mathbb{P}(\omega)\le\mathbb{P}(S(t)\ge\zeta)+\mathbb{P}(I(t)\ge\zeta)\le\frac{1}{\zeta}\mathbb{E}[S(t)+I(t)]\le\frac{1}{\zeta}\left(\frac{2A}{\mu_1}+N(0)\right).$$

Choosing $\frac{1}{\zeta} \leq M_3\eta^q$. We thus obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[\mathbb{1}_{\Omega_4}]\mathrm{d}s \leq \frac{\mu_1 q\eta^q}{8\beta}\Big(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\Big)\Big(\mathcal{R}_0^s-1\Big).$$

According to (VI.17), one can derive that

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[\mathbb{1}_{\Sigma}]\mathrm{d}s &\geq \liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[\mathbb{1}_{\Omega_1}]\mathrm{d}s - \limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}[\mathbb{1}_{\Omega_4}]\mathrm{d}s\\ &\geq \frac{\mu_1 q\eta^q}{8\beta}\Big(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\Big)\Big(\mathcal{R}_0^s-1\Big) > 0.\end{aligned}$$

Based on the above analysis, we have determined a compact domain $\Sigma \subset \mathbb{R}^3_+$ such that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}\Big(s, Y(0), \Sigma\Big)\mathrm{d}s \geq \frac{\mu_1 q\eta^q}{8\beta}\Big(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\Big)\Big(\mathcal{R}_0^s-1\Big) > 0.$$

Applying similar arguments to those in Theorem 5.1 of [57], we show the uniqueness of the ergodic stationary distribution of our model (VI.3). This completes the proof. ∎

Theorem 2.12. *If $\mathcal{R}_0^s > 1$, then for any value $Y(0) \in \mathbb{R}^3_+$, the disease is persistent in the mean.*

Proof. From model (VI.3) it yields

$$\mathrm{d}(S(t)+I(t)+Q(t)) = \big(A-\mu_1 S(t)-\mu_2 I(t)-\mu_3 Q(t)\big)\mathrm{d}t + \bar{\mathcal{P}}_1(t)+\bar{\mathcal{P}}_2(t)+\mathcal{P}_3(t). \tag{VI.18}$$

Integrating (VI.18) from 0 to t, and then dividing t on both sides, we get

$$\begin{aligned}&\frac{1}{t}\Big((S(t)+I(t)+Q(t))-(S(0)+I(0)+Q(0))\Big)\\ &= A-\frac{\mu_1}{t}\int_0^t S(s)\mathrm{d}s-\frac{\mu_2}{t}\int_0^t I(s)\mathrm{d}s-\frac{\mu_3}{t}\int_0^t Q(s)\mathrm{d}s+\frac{\sigma_1}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_1(s)\\ &\quad+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)+\frac{\sigma_2}{t}\int_0^s I(s)\mathrm{d}\mathcal{W}_2(s)+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)\\ &\quad+\frac{\sigma_3}{t}\int_0^t Q(s)\mathrm{d}\mathcal{W}_3(s)+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\eta_3(u)Q(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

Taking the integration for the third equation of model (VI.3) yields

$$Q(t)-Q(0) = \delta\int_0^t I(s)\mathrm{d}s-(\mu_3+k)\int_0^t Q(s)\mathrm{d}s+\sigma_3\int_0^t Q(s)\mathrm{d}\mathcal{W}_3(s)+\int_0^t\int_{\mathcal{U}}\eta_3(u)Q(t^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u). \tag{VI.19}$$

Dividing t on both sides of equation (VI.19), we have

$$\begin{aligned}\frac{1}{t}\int_0^t Q(s)\mathrm{d}s &= \frac{\delta}{(\mu_3+k)}\frac{1}{t}\int_0^t I(s)\mathrm{d}s+\frac{\sigma_3}{(\mu_3+k)}\frac{1}{t}\int_0^t Q(s)\mathrm{d}\mathcal{W}_3(s)\\ &\quad+\frac{1}{(\mu_3+k)}\int_0^t\int_{\mathcal{U}}\eta_3(u)Q(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)-\frac{1}{(\mu_3+k)t}(Q(t)-Q(0)).\end{aligned}$$

Then, one can obtain that

$$\frac{1}{t}\int_0^t S(s)\mathrm{d}s = \frac{A}{\mu_1}-\frac{1}{t}\Big(\frac{\mu_2}{\mu_1}+\frac{\delta\mu_3}{\mu_1(\mu_3+k)}\Big)\int_0^t I(s)\mathrm{d}s+\Phi_1(t), \tag{VI.20}$$

where

$$\Phi_1(t) = \frac{\sigma_3\mu_3}{\mu_1(\mu_3+k)t}\int_0^t Q(s)\mathrm{d}\mathcal{W}_3(s) + \frac{\mu_3}{\mu_1(\mu_3+k)t}\int_0^t\int_{\mathcal{U}}\eta_3(u)Q(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) - \frac{1}{(\mu_3+k)t}(Q(t)-Q(0))$$
$$+ \frac{\sigma_1}{\mu_1 t}\int_0^t S(s)\mathrm{d}\mathcal{W}_1(s) + \frac{1}{\mu_1 t}\int_0^t\int_{\mathcal{U}}\eta_1(u)S(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) + \frac{\sigma_2}{\mu_1 t}\int_0^t I(s)\mathrm{d}\mathcal{W}_2(s)$$
$$+ \frac{1}{\mu_1 t}\int_0^t\int_{\mathcal{U}}\eta_2(u)I(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) + \frac{\sigma_3}{\mu_1 t}\int_0^t Q(s)\mathrm{d}\mathcal{W}_3(s) + \frac{1}{\mu_1 t}\int_0^t\int_{\mathcal{U}}\eta_3(u)Q(s^-)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)$$
$$- \frac{1}{\mu_1 t}\Big((S(t)+I(t)+Q(t)) - (S(0)+I(0)+Q(0))\Big).$$

Applying Itô's formula to the second equation of (VI.3), we get

$$\mathrm{d}\ln I(t) = \left(\beta S(t) - (\mu_2+\delta+\gamma) - \frac{\sigma_2^2}{2} - \frac{\sigma_\beta^2}{2}S^2(t) - \int_{\mathcal{U}}\eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u)\right)\mathrm{d}t$$
$$+ \sigma_2\mathrm{d}\mathcal{W}_2(t) + \int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u) + \sigma_\beta S(t)\mathrm{d}\mathcal{W}_\beta(t). \tag{VI.21}$$

Integrating (VI.21) from 0 to t and then dividing t on both sides, we have

$$\frac{1}{t}(\ln I(t) - \ln I(0)) = \frac{\beta}{t}\int_0^t S(s)\mathrm{d}s - (\mu_2+\delta+\gamma) - \frac{\sigma_2^2}{2} - \frac{\sigma_\beta^2}{2t}\int_0^t S^2(s)\mathrm{d}s - \int_{\mathcal{U}}\eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u)$$
$$+ \sigma_2\frac{W_2(t)}{t} + \frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) + \frac{\sigma_\beta}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_\beta(s).$$

From (VI.20), we get

$$\frac{1}{t}(\ln I(t) - \ln I(0)) = \frac{\beta A}{\mu_1} - \frac{\beta}{t}\left(\frac{\mu_2}{\mu_1} + \frac{\delta\mu_3}{\mu_1(\mu_3+k)}\right)\int_0^t I(s)\mathrm{d}s + \beta\Phi_1(t) - (\mu_2+\delta+\gamma)$$
$$- \frac{\sigma_2^2}{2} - \frac{\sigma_\beta^2}{2}\int_0^t S^2(s)\mathrm{d}s - \int_{\mathcal{U}}\eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u)$$
$$+ \sigma_2\frac{W_2(t)}{t} + \frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) + \frac{\sigma_\beta}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_\beta(s).$$

Since $S(t) \le X(t)$ a.s., we obtain

$$\frac{1}{t}(\ln I(t) - \ln I(0)) \ge \frac{\beta A}{\mu_1} - \frac{\beta}{t}\left(\frac{\mu_2}{\mu_1} + \frac{\delta\mu_3}{\mu_1(\mu_3+k)}\right)\int_0^t I(s)\mathrm{d}s + \beta\phi_1(t) - (\mu_2+\delta+\gamma)$$
$$- \frac{\sigma_2^2}{2} - \frac{\sigma_\beta^2}{2}\int_0^t X^2(s)\mathrm{d}s - \int_{\mathcal{U}}\eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u)$$
$$+ \sigma_2\frac{W_2(t)}{t} + \frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) + \frac{\sigma_\beta}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_\beta(s).$$

Hence, we further have

$$\frac{\beta}{t}\left(\frac{\mu_2}{\mu_1} + \frac{\delta\mu_3}{\mu_1(\mu_3+k)}\right)\int_0^t I(s)\mathrm{d}s \ge -\frac{1}{t}(\ln I(t) - \ln I(0)) + \frac{\beta A}{\mu_1} + \beta\phi_1(t) - (\mu_2+\delta+\gamma)$$
$$- \frac{\sigma_2^2}{2} - \frac{\sigma_\beta^2}{2}\int_0^t X^2(s)\mathrm{d}s - \int_{\mathcal{U}}\eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u)$$
$$+ \sigma_2\frac{W_2(t)}{t} + \frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u) + \frac{\sigma_\beta}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_\beta(s).$$

By assumption (A_3), Lemmas 2.8 - 2.10, and the large number theorem for martingales, we can easily verify that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(s)\mathrm{d}s \geq \frac{1}{\beta}\left(\frac{\mu_2}{\mu_1}+\frac{\delta\mu_3}{\mu_1(\mu_3+k)}\right)^{-1}\left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)(\mathcal{R}_0^s-1)>0 \ \text{ a.s.}$$

This shows that the system (VI.3) is persistent in the mean with probability one. This completes the proof. ∎

2.2.3 The extinction of the disease

Now, we will give the result on the extinction of the disease. Define

$$\hat{\mathcal{R}}_0^s=\left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)^{-1}\left(\frac{\beta A}{\mu_1}-\frac{\sigma_\beta^2A^2}{2\mu_1^2}-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)\right).$$

Theorem 2.13. *Let $Y(t)$ be the solution of system (VI.3) with initial value $Y(0)\in\mathbb{R}_+^3$. If*

$$\hat{\mathcal{R}}_0^s<1 \ \text{ and } \ \sigma_\beta^2\leq\frac{\mu_1\beta}{A}, \tag{VI.22}$$

or

$$\frac{\beta^2}{2\sigma_\beta^2}-\left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)<0, \tag{VI.23}$$

then the disease dies out exponentially with probability one. That is to say,

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}<0 \quad a.s. \tag{VI.24}$$

Proof. By Itô's formula for all $t\geq 0$, we have

$$\begin{aligned}\mathrm{d}\ln I(t)=&\left(\beta S(t)-(\mu_2+\delta+\gamma)-\frac{\sigma_2^2}{2}-\frac{\sigma_\beta^2}{2}S^2(t)-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)\right)\mathrm{d}t\\&+\sigma_2\mathrm{d}\mathcal{W}_2(t)+\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}u)+\sigma_\beta S(t)\mathrm{d}\mathcal{W}_\beta(t).\end{aligned} \tag{VI.25}$$

Integrating (VI.25) from 0 to t and then dividing t on both sides, we get

$$\frac{\ln I(t)}{t}=\frac{\beta}{t}\int_0^t S(s)\mathrm{d}s-\left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)-\frac{\sigma_\beta^2}{2t}\int_0^t S^2(s)\mathrm{d}s+\Phi_2(t), \tag{VI.26}$$

where

$$\Phi_2(t)=\frac{\sigma_\beta}{t}\int_0^t S(s)\mathrm{d}\mathcal{W}_\beta(s)-\frac{\sigma_2W_2(t)}{t}+\frac{1}{t}\int_0^t\int_{\mathcal{U}}\ln(1+\eta_2(u))\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}u)-\frac{\ln I(0)}{t}.$$

Obviously, we know that

$$\frac{1}{t}\int_0^t S^2(s)\mathrm{d}s\geq\left(\frac{1}{t}\int_0^t S(s)\mathrm{d}s\right)^2.$$

Therefore, from (VI.20), we derive

$$\begin{aligned}
&\frac{\ln I(t)}{t}\\
&\leq \frac{\beta}{t}\int_0^t S(s)\mathrm{d}s - \left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right) - \int_{\mathcal{U}} \eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u) - \frac{\sigma_\beta^2}{2}\left(\frac{1}{t}\int_0^t S(s)\mathrm{d}s\right)^2 + \Phi_2(t)\\
&= \beta\left(\frac{A}{\mu_1} - \frac{1}{t}\left(\frac{\mu_2}{\mu_1}+\frac{\delta\mu_3}{\mu_1(\mu_3+k)}\right)\int_0^t I(s)\mathrm{d}s + \phi_1(t)\right) - \left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right) - \int_{\mathcal{U}} \eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u)\\
&\quad - \frac{\sigma_\beta^2}{2}\left(\frac{A}{\mu_1} - \frac{1}{t}\left(\frac{\mu_2}{\mu_1}+\frac{\delta\mu_3}{\mu_1(\mu_3+k)}\right)\int_0^t I(s)\mathrm{d}s + \phi_1(t)\right)^2 + \Phi_2(t).
\end{aligned}$$

Hence, one can see that

$$\begin{aligned}
\frac{\ln I(t)}{t} &\leq \frac{\beta A}{\mu_1} - \left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right) - \int_{\mathcal{U}} \eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u) - \frac{A^2\sigma_\beta^2}{2\mu_1^2}\\
&\quad - \left(\frac{\mu_2}{\mu_1}+\frac{\delta\mu_3}{\mu_1(\mu_3+k)}\right)\left(\beta - \frac{A\sigma_\beta^2}{\mu_1}\right)\frac{1}{t}\int_0^t I(s)\mathrm{d}s\\
&\quad - \frac{\sigma_\beta^2}{2t^2}\left(\left(\frac{\mu_2}{\mu_1}+\frac{\delta\mu_3}{\mu_1(\mu_3+k)}\right)\int_0^t I(s)\mathrm{d}s\right)^2 + \Phi_2(t) + \Phi_3(t),
\end{aligned} \tag{VI.27}$$

where

$$\Phi_3(t) = \beta\Phi_1(t) - \frac{\sigma_\beta^2}{2}\Phi_1^2(t) - \frac{\sigma_\beta^2 A\Phi_1(t)}{\mu_1} + \sigma_\beta^2\Phi_1(t)\left(\frac{\mu_2}{\mu_1}+\frac{\delta\mu_3}{\mu_1(\mu_3+k)}\right)\int_0^t I(s)\mathrm{d}s.$$

Based on Lemma 2.8, one has

$$\lim_{t\to\infty}\frac{\Phi_2(t)}{t} = \lim_{t\to\infty}\frac{\Phi_3(t)}{t} = 0 \quad \text{a.s.}$$

Taking the superior limit on both sides of (VI.27), then by condition (VI.22), we arrive at

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \leq \left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)\left(\hat{\mathcal{R}}_0^s - 1\right) < 0 \quad \text{a.s.}$$

Now, from (VI.26), we have

$$\begin{aligned}
\frac{\ln I(t)}{t} &= \frac{\beta}{t}\int_0^t S(s)\mathrm{d}s - \left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right) - \int_{\mathcal{U}} \eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u) - \frac{\sigma_\beta^2}{2t}\int_0^t S^2(s)\mathrm{d}s + \Phi_2(t)\\
&= \frac{\beta^2}{2\sigma_\beta^2} - \left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right) - \int_{\mathcal{U}} \eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u) - \frac{\sigma_\beta^2}{2}\frac{1}{t}\int_0^t \left(S(s)\mathrm{d}s - \frac{\beta}{\sigma_\beta^2}\right)^2 ds + \Phi_2(t)\\
&\leq \frac{\beta^2}{2\sigma_\beta^2} - \left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right) - \int_{\mathcal{U}} \eta_2(u) - \ln(1+\eta_2(u))\nu(\mathrm{d}u)) + \Phi_2(t).
\end{aligned}$$

By the large number theorem for martingales, Lemma 2.8 and the condition (VI.23), our desired result (VI.24) holds true. This completes the proof. ∎

2.3 Numerical examples

In this section, we will validate our theoretical results with the help of numerical simulation examples taking parameters from the theoretical data mentioned in the Table VI.2. We numerically simulate the solution of system (VI.3) with the initial values $(S(0), I(0), Q(0)) = (0.5, 0.3, 0.1)$. The unit of time is one day.

Parameters	Numerical values		
A	0.1	0.1	0.1
μ_1	0.05	0.05	0.05
μ_2	0.09	0.09	0.09
μ_3	0.052	0.052	0.052
β	0.075	0.05	0.05
γ	0.01	0.01	0.01
δ	0.03	0.03	0.03
k	0.04	0.04	0.04
σ_1	0.01	0.01	0.01
σ_2	0.03	0.12	0.01
σ_3	0.07	0.12	0.12
σ_β	0.02	0.1	0.02
η_1	0.01	0.2	0.2
η_2	0.02	0.12	0.12
η_3	0.05	0.01	0.01
	Figures VI.4 and VI.5	Figure VI.6 (a)	Figure VI.6 (b)

Table VI.2: Nominal values of the system parameters and disturbances intensities adopted in the different simulation examples.

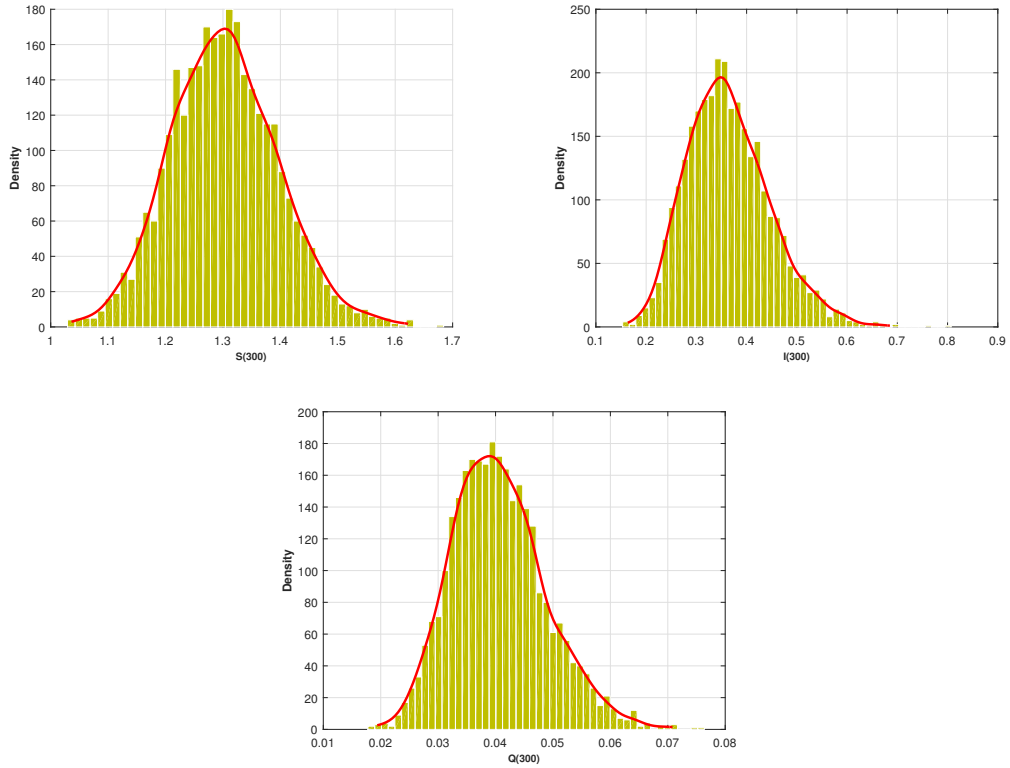


Figure VI.4: Histogram of the probability density function for S, I, and Q population at $t = 300$ for the stochastic model (VI.3), the smoothed curves are the probability density functions of $S(t)$, $I(t)$ and $Q(t)$, respectively.

Example 2.14. We assume that $\mathcal{U} = (0, \infty)$ and $\nu(\mathcal{U}) = 1$. Then, $\mathcal{R}_0^s = 1.1756 > 1$. From Figure VI.4, we show the existence of the unique stationary distributions for $S(t)$, $I(t)$ and $Q(t)$ of the model (VI.3) at $t = 300$, where the smooth curves are the probability density functions of $S(t)$, $I(t)$ and $Q(t)$, respectively. It can be obviously observed that the solution of the stochastic model (VI.3) persists in the mean (see Figure VI.5).

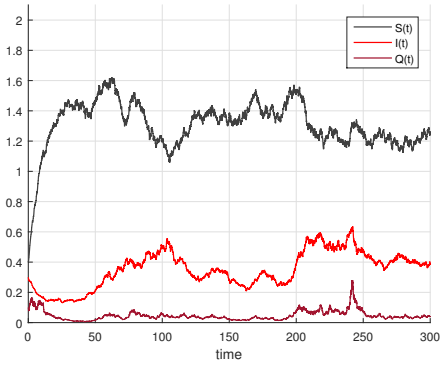
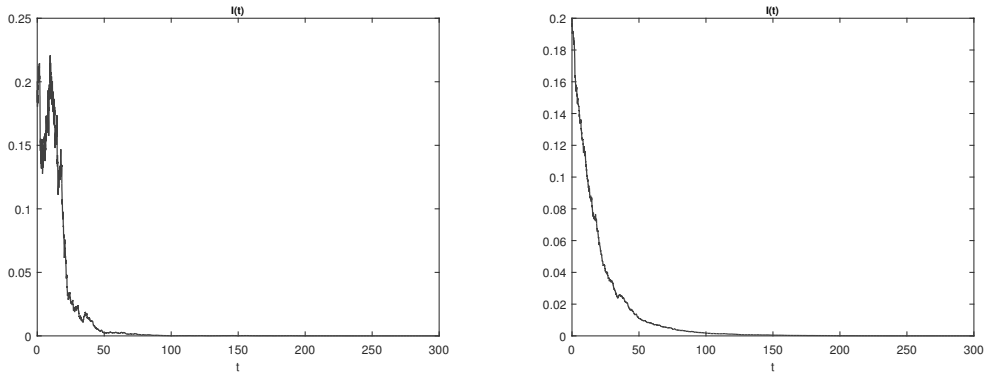


Figure VI.5: The paths of $S(t)$, $I(t)$ and $Q(t)$ for the stochastic model (VI.3) with initial values $(S(0), I(0), Q(0)) = (0.5, 0.3, 0.1)$.



(a) (b)

Figure VI.6: The numerical simulation of $I(t)$ in the system (VI.3).

Example 2.15. Now, we choose the white noise intensities $\sigma_2 = 0.12$ and $\sigma_\beta = 0.1$ to ensure that the condition (VI.23) of theorem (2.13) is satisfied. We can conclude that for any initial value, $I(t)$ obeys

$$\limsup_{t\to\infty} \frac{1}{t} \ln \frac{I(t)}{I(0)} \leq \frac{\beta^2}{2\sigma_\beta^2} - \left(\mu_2 + \delta + \gamma + \frac{\sigma_2^2}{2}\right) - \int_{\mathcal{U}} \eta_2(u) - \ln(1+\eta_2(u))\nu(du) = -0.1374 < 0 \quad \text{a.s.}$$

That is, $I(t)$ will tend to zero exponentially with probability one (see Figure VI.6 (a)). To verify that the condition (VI.22) is satisfied, we change σ_2 to 0.01, σ_β to 0.02 and β to 0.05 and keep other parameters unchanged. Then we have

$$\hat{\mathcal{R}}_0^s = \left(\mu_2 + \delta + \gamma + \frac{\sigma_2^2}{2}\right)^{-1} \left(\frac{\beta A}{\mu_1} - \frac{\sigma_\beta^2 A^2}{2\mu_1^2} - \int_{\mathcal{U}} \eta_2(u) - \ln(1+\eta_2(u))\nu(du)\right) = 0.7650 < 1,$$

and

$$\sigma_\beta^2 - \frac{\mu_1 \beta}{A} = -0.0249 < 0.$$

Therefore, the condition (VI.22) of Theorem 2.13 is satisfied. We can conclude that for any initial value, $I(t)$ obeys

$$\limsup_{t\to\infty} \frac{1}{t} \ln \frac{I(t)}{I(0)} \leq \left(\mu_2 + \delta + \gamma + \frac{\sigma_2^2}{2}\right)\left(\hat{\mathcal{R}}_0^s - 1\right) = -0.0306 < 0 \quad \text{a.s.}$$

That is, $I(t)$ will tend to zero exponentially with probability one (see Figure VI.6 (b)).

2.4 Discussion

In this latest study, we proposed a new version of a perturbed SIS epidemiological model with a quarantine strategy. This model simultaneously takes into account random transmission and the effects of jumps. We have addressed possible scenarios of the pandemic spread during unforeseen climate changes or environmental shocks. Compared with the existing literature, the novelty of our study manifested in new analysis techniques and improvements which are summarized in the following items:

- Our study is distinguished from previous works [172, 171, 27, 26, 36] by improving the majorization of the following quantity

$$\int_{\mathcal{U}} N^{np}(t)\left(\left(1+\frac{\tilde{X}}{N}\right)^{np}-1-np\frac{\tilde{X}}{N}\right)\nu(\mathrm{d}u),$$

  which raises the optimality of our calculus and results.

- Our results in Lemmas 2.5 and 2.8 provide an extended and generalized version of classical lemmas 3.3 and 3.4 presented in [171] which are widely used in the literature.

- Our study provides an improved threshold

$$\mathcal{R}_0^s=\left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)^{-1}\left(\frac{\beta A}{\mu_1}-\frac{A^2\sigma_\beta^2}{\mu_1\chi}-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)\right),$$

  by taking into consideration the Remark 2.3. This parameter is a sufficient condition for the existence of a unique ergodic stationary distribution and persistence of the disease under some assumptions. The last two asymptotic properties are proven in Theorems 2.11 and 2.12, by using a new approach based on Lemma 2.10 and the mutually exclusive possibilities lemma.

- Our study offers an alternative method to the gap mentioned in (Theorem 2.2, [163]). Without using the explicit formula of the distribution stationary $\pi(\cdot)$ of X (which still up to now unknown), we gave the expression of the ergodicity and persistence threshold.

- For the case of non-persistence, in Theorem 2.13, we proved that the following parameter

$$\hat{\mathcal{R}}_0^s=\left(\mu_2+\delta+\gamma+\frac{\sigma_2^2}{2}\right)^{-1}\left(\frac{\beta A}{\mu_1}-\frac{A^2\sigma_\beta^2}{2\mu_1^2}-\int_{\mathcal{U}}\eta_2(u)-\ln(1+\eta_2(u))\nu(\mathrm{d}u)\right),$$

  is a sufficient conditions for the disappearance of the disease.

Eventually, we point out that the obtained results extend and generalize many previous works (for example, [156, 157, 159]), by analyzing the dynamics of the SIQS epidemic models with two disturbances. We believe that this contribution can be a rich basis for future studies.

Conclusions and outlook

The aim of this thesis was to study duly some famous stochastic epidemic models. To this end, a novel analyzing framework was developed in order to answer several questions and problematics. This framework utilized many new techniques different from those used frequently in the literature. This concluding chapter summarizes the work achieved within this thesis, highlights the key findings, and discusses how they relate to the thesis aims and the global context.

- **Problematic context of the various studies conducted in this thesis and our proposed solutions**
  The study of the stochastic epidemic models requires numerous innovative analytical techniques due to their intricacy and difficulty. The analyzing work in this thesis aimed to provide creative insight by addressing the gaps mentioned in several recent papers.

  - ▸ **Combination of two types of disturbances**
    We have begun by supposing that the stochastic disturbances associated to the disease transmission coefficient and the mortality rate are presented with two perturbations: standard Gaussian white noise and Lévy jumps, respectively. This novel idea was not considered formerly. The adoption of this dual perturbation demands some new techniques which are presented in the first section of Chapter III. In the case of the SIRS system, we have simultaneously considered the stochastic transmission and the proportional perturbation. The combination of these dual fluctuations makes the transmission dynamics more biologically realistic in a very noisy environment with non-constant population size. A global analysis of the long-run behavior of the epidemic model was provided in Section 3 of Chapter V. Since the isolation of individuals is one of the most efficient strategies, in Section 2 of Chapter VI, we have considered a general form of an SIQS epidemic model with proportional Lévy jumps and random transmission in order to simulate both slight and massive perturbations. Under appropriate conditions, some attractive asymptotic properties have proved, namely: ergodicity, persistence in the mean, and extinction of the disease. Our work introduced in the said section improves many existing studies in the field of mathematical epidemiology and provides original techniques to predict and analyze the dynamic behavior of epidemics.
  - ▸ **Treatment of some open problems**
    In the obscurity of Lévy jumps, the stationary distribution expression is used to estimate the time averages of the auxiliary process solution by employing the ergodic theorem. Unluckily, the said expression is still unknown in the case of the Lévy noise. This issue is implicitly mentioned as a direct question in many papers [163, 162]. In this thesis, we have proposed an alternative method to establish the exact expression of the threshold parameter without having recourse to the use of the ergodic theorem. This creative idea that we propose is detailed in Section 1 of Chapter III, Section 2.1 of Chapter III, Section 1.1 of Chapter IV, and Section 2 of Chapter VI.
  - ▸ **Generalization of some classic results**
    In Section 2 of Chapter VI, we have proved a general version of some habitual results

[27, 26, 36]. Our obtained findings provide an extended and generalized version of the techniques which are widely used in the literature. Furthermore, in Section 2 of Chapter V, we have generalized the incidence rate of a stochastic SIRS with some real hypotheses. We have obtained sufficient conditions for the stochastic stability of disease-free equilibrium by using a suitable Lyapunov function and other techniques of the stochastic analysis. In addition, we have studied the conditions of the persistence in the mean and extinction of a disease.

- **Improvement of the some hypotheses and assumptions**
  In this thesis, we have ameliorated some assumptions on Lévy intensities which raise the optimality of our calculus and results. Especially in Section 2.1 of Chapter III, Section 3 of Chapter III, and Section 2 of Chapter VI.
- **Amelioration of some thresholds**
  Initially, we have proposed an alternative method to establish the explicit threshold expression without using the invariant probability distribution of an auxiliary process, which can close the gap left by using the classical Lyapunov method. In addition, we have employed an alternative method to prove the existence of unique stationary distribution (see Section 1 of Chapter III, Section 2.1 of Chapter III, Section 1.1 of Chapter IV, and Section 2 of Chapter VI). Eventually, we have improved the conditions of the extinction of the SIR epidemic model with delay.
- **Extension of some epidemic model to the Lévy case**
  During this thesis, we have extended many epidemic models to a more general context. The SIR, HBV, SIQR models are treated by considering the effects of Lévy noises, and some interesting asymptotic properties are established.

- **General summary and future works**
  By considering previous mathematical analyses, knowledge gaps were identified. This highlights the importance of the work within this thesis, as it addresses some of these gaps, furthering our understanding of transmissible diseases under random fluctuations. So, given the capabilities of the findings of this thesis, we consider several proposals to continue with this work in the future:

  - **Analysis of epidemic models under regime switching_** It should be noted that the epidemic models may be perturbed by color noise which can cause the system to convert (transfer) from one environmental regime to another one. Generally, the switching between the regimes is often memoryless and the waiting time for the next switching follows the exponential distribution. Consequently, the regime-switching can be modeled by operating a continuous-time Markov chain with values in a finite state space. We treat this interesting approach in our forthcoming works.
  - **Analysis of a hybrid switching of epidemic models**
    Besides a white noise, another two driving processes are taken into account: a stationary Lévy process and a continuous-time finite-state Markov chain. The combination of these two perturbations provides a hybrid system. This fascinating topic deserves further consideration in the future.
  - **Analysis of more complex and general biological models**
    In fact, it is necessary to check that the methods utilized in this thesis can be also applied to analyze other stochastic realistic models like chemostat and single-species population models. We leave these investigations for our future works.

Appendix A

Numerical solution

Stochastic differential equations with an explicit solution are only exceptional cases. Therefore, numerical methods for the discretization of the said equations are required in order to approximate their solutions. This approximation is termed a numerical solution. The general aim of this appendix is to present some numerical methods used to simulate the sample paths of the solution in the previous chapters. For this purpose, we illustrate the numerical solution of the following stochastic differential equation:

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + g(X(t))\mathrm{d}\mathcal{W}(t), \qquad t \in [0,T], \tag{A.1}$$

where, as habitual, $\mathcal{W}$ is the standard Brownian motion. To guarantee the existence and uniqueness of a strong solution, we suppose that the functions $f(x)$ and $g(x)$ verify the Lipschitz and linear growth conditions. We start with the standard method in the field of numerical analysis for the stochastic differential equations.

1 The Euler Approximation

Generally, a numerical solution $\mathcal{X}^{(n)} \equiv (\mathcal{X}^{(n)}(t), \quad t \in [0,T])$ of the SDE (A.1) is a stochastic process that approximates its solution. This approximation is characterized by a partition τ_n of $[0,T]$:

$$\tau_n : \quad 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T,$$

with the associated mesh $\delta_n = \mathrm{mesh}(\tau_n) = \max_{i=1,\ldots,n}(t_i - t_{i-1})$. The process $\mathcal{X}^{(n)}$ is calculated only at the points t_i of τ_n. Actually, we are interested in solutions $\mathcal{X}$ with continuous sample paths, so we assume that $\mathcal{X}^{(n)}(t)$ on (t_{i-1}, t_i) is derived by simple linear interpolation of the points $(t_{i-1}, \mathcal{X}^{(n)}_{t_{i-1}})$ and $(t_i, \mathcal{X}^{(n)}_{t_i})$. Consequently, a numerical approximation scheme determines $\mathcal{X}^{(n)}$ at the points t_i with the following form:

The Euler approximation scheme

Denote, as habitual, $\Delta_i = t_i - t_{i-1}$ and $\Delta_i\mathcal{W} = \mathcal{W}(t_i) - \mathcal{W}(t_{i-1})$, $i = 1,\ldots,n$. Then

$$\begin{aligned}
\mathcal{X}^{(n)}_0 &= \mathcal{X}(0),\\
\mathcal{X}^{(n)}(t_1) &= \mathcal{X}^{(n)}(0) + f(\mathcal{X}^{(n)}(0))\Delta_1 + g(\mathcal{X}^{(n)}(0))\Delta_1\mathcal{W},\\
\mathcal{X}^{(n)}(t_2) &= \mathcal{X}^{(n)}(t_1) + f(\mathcal{X}^{(n)}(t_1))\Delta_2 + g(\mathcal{X}^{(n)}(t_1))\Delta_2\mathcal{W},\\
&\vdots\\
\mathcal{X}^{(n)}(t_{n-1}) &= \mathcal{X}^{(n)}(t_{n-2}) + f(\mathcal{X}^{(n)}(t_{n-2}))\Delta_{n-1} + g(\mathcal{X}^{(n)}(t_{n-2}))\Delta_{n-1}\mathcal{W},\\
\mathcal{X}^{(n)}(T) &= \mathcal{X}^{(n)}(t_{n-1}) + f(\mathcal{X}^{(n)}(t_{n-1}))\Delta_n + g(\mathcal{X}^{(n)}(t_{n-1}))\Delta_n\mathcal{W}.
\end{aligned}$$

In practice, we can usually choose equidistant points t_i such that $\delta_n = \text{mesh}(\tau_n) = T/n$ and

$$\mathcal{X}^{(n)}(iT/n) = \mathcal{X}^{(n)}(i-1)T/n) + f(\mathcal{X}^{(n)}(i-1)T/n))\delta_n + g(\mathcal{X}^{(n)}(i-1)T/n))\Delta_i\mathcal{W}, \qquad i = 1, \ldots, n.$$

2 The Milstein Approximation

Consider the following stochastic differential

$$\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t f(\mathcal{X}(s))\mathrm{d}s + \int_0^t g(\mathcal{X}(s))\mathrm{d}\mathcal{W}(s), \qquad t \in [0, T].$$

For the points t_i of the partition τ_n, we consider the difference $\mathcal{X}(t_i) - \mathcal{X}(t_{i-1})$ and derive:

$$\mathcal{X}(t_i) = \mathcal{X}(t_{i-1}) + \int_{t_{i-1}}^{t_i} f(\mathcal{X}(s))\mathrm{d}s + \int_{t_{i-1}}^{t_i} g(\mathcal{X}(s))\mathrm{d}\mathcal{W}(s), \qquad i = 1, \ldots, n. \tag{A.2}$$

Remark that the Euler approximation is based on a discretization of the integrals in (A.2) such that

$$\int_{t_{i-1}}^{t_i} f(\mathcal{X}(s))\mathrm{d}s \approx f(\mathcal{X}(t_{i-1}))\Delta_i, \quad \text{and} \quad \int_{t_{i-1}}^{t_i} g(\mathcal{X}(s))\mathrm{d}\mathcal{W}(s) \approx g(\mathcal{X}(t_{i-1}))\Delta_i\mathcal{W},$$

and then substitute $\mathcal{X}(t_i)$ with $\mathcal{X}^{(n)}(t_i)$:

$$\mathcal{X}^{(n)}(t_i) = \mathcal{X}^{(n)}(t_{i-1}) + f(\mathcal{X}^{(n)}(t_{i-1}))\Delta_i + g(\mathcal{X}^{(n)}(t_{i-1}))\Delta_i\mathcal{W}, \qquad i = 1, \ldots, n.$$

However, the Milstein approximation employs the so-termed Taylor-Itô expansion of (A.2). The general idea based on the applying of the Itô to the integrands $f(\mathcal{X}(s))$ and $g(\mathcal{X}(s))$ in (A.2). Notice that the Milstein scheme is the Euler approximation with an extra correction term containing the squared increments of Brownian motion.

The Milstein approximation scheme

$\mathcal{X}^{(n)}(0) = \mathcal{X}(0)$ and for $i = 1, \ldots, n$,

$$\begin{aligned}\mathcal{X}^{(n)}(t_i) = \mathcal{X}^{(n)}(t_{i-1}) &+ f(\mathcal{X}^{(n)}(t_{i-1}))\Delta_i + g(\mathcal{X}^{(n)}(t_{i-1}))\Delta_i\mathcal{W} \\ &+ 0.5g(\mathcal{X}^{(n)}(t_{i-1}))\dot{g}(\mathcal{X}^{(n)}(t_{i-1}))\Big((\Delta_i\mathcal{W})^2 - \Delta_i\Big).\end{aligned}$$

In order to measure the quality of the approximation of $\mathcal{X}$ and compare the two previous schemes, we have the following results

- The Euler approximation converses strongly with order 0.5.
- The milstein approximation converges strongly with order 1.

Consequently, the milstein leads to substantial improvement of the quality of the numerical solution.

3 The Euler approximation for jump-diffusion SDE

In this section, we briefly illustrate the discrete time approximation for the following autonomous jump-diffusion stochastic differential equation

$$\mathrm{d}\mathcal{X}(t) = f(\mathcal{X}(t))\mathrm{d}t + g(\mathcal{X}(t))\mathrm{d}\mathcal{W}(t) + \int_{\mathcal{U}} \eta(\mathcal{X}(t^-), u)\mathcal{N}_\nu(\mathrm{d}t, \mathrm{d}u), \tag{A.3}$$

for $t \in [0, T]$, with $\mathcal{X}(0) \in \mathbb{R}$. Here $\mathcal{W}$ is a Brownian motion and $\mathcal{N}_\nu(\mathrm{d}t, \mathrm{d}u)$ is a $\mathcal{F}_t$-adapted Poisson measure. The associated mark space $\mathcal{U} \subset (0, \infty)$ is 1-dimensional and the intensity jump measure is $\nu(\mathrm{d}u)\mathrm{d}t = \lambda\mathcal{V}\mathrm{d}u\mathrm{d}t$, where $\mathcal{V}(\cdot)$ is a given probability density function. The Jump-diffusion SDE (A.3) can be expressed in integral form as:

$$\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t f(\mathcal{X}(s))\mathrm{d}s + \int_0^t g(\mathcal{X}(s))\mathrm{d}\mathcal{W}(s) + \sum_{i=1}^{\mathcal{N}_\nu(t)} \eta(\mathcal{X}(\tau_i^-), \xi_i),$$

where $\{(\tau_i, \xi_i),\ i \in \{1, 2, \ldots, \mathcal{N}(t)\}\}$ is the double sequence of jump times and marks generated by the $\mathcal{N}_\nu$. As above-mentioned approximations, we consider the time discretization $0 = t_0 <_1 < \cdots < t_{n_T} = T$, and let for all $t \in [0, T]$ the index $n_t = \max\{n \in \{0, 1, \ldots\} : \ t_n \leq t\}$ be the last point discretization before t. In practice, we can choose $t_n = n\Delta$, for $n \in \{0, 1, \ldots, T/\Delta\}$, where $\Delta \in (0, 1)$ is the time step size. The simplest Euler scheme of (A.3) is described in the following box.

The Euler scheme for jump-diffusion SDE

$$\begin{aligned}\mathcal{X}_{n+1} &= \mathcal{X}_n + f(\mathcal{X}_n)\Delta + g(\mathcal{X}_n)\Delta\mathcal{W}_n + \int_{t_n}^{t_{n+1}} \int_{\mathcal{U}} \eta(\mathcal{X}_n, u)\mathcal{N}_\nu(\mathrm{d}t, \mathrm{d}u) \\ &= \mathcal{X}_n + f(\mathcal{X}_n)\Delta + g(\mathcal{X}_n)\Delta\mathcal{W}_n + \sum_{i=\mathcal{N}_\nu(t_n)+1}^{\mathcal{N}_\nu(t_{n+1})} \eta(\mathcal{X}_n, \xi_i),\end{aligned}$$

for $n \in \{0, 1, \ldots, n_T - 1\}$.

The numerical analysis of stochastic differential equations with jumps is a relatively new area of applied probability theory. A detailed of adapted and improved numerical techniques are given in the following interesting paper [74].

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