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Badr EL HAJI

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Pr. TOUZANI Abdelfattah	***Faculté des sciences Dhar Al Mahraz Fès***	***Président***
Pr. MESKINE Driss	***Ecole Supérieure de Technologie d'Essaouira***	***Rapporteur***
Pr. EL MOUMNI Mostafa	***Faculté des sciences d'El Jadida***	***Rapporteur***
Pr. AZROUL Elhoussine	***Faculté des sciences Dhar Al Mahraz Fès***	***Rapporteur***
Pr. BENNOUNA Jaouad	***Faculté des sciences Dhar Al Mahraz Fès***	***Examinateur***
Pr. MEKKOUR Mounir	***Faculté des sciences Dhar Al Mahraz Fès***	***Examinateur***
Pr. RHOUDAF Mohamed	***Faculté des sciences de Meknés***	***Examinateur***
Pr. BENKIRANE Abdelmoujib	***Faculté des sciences Dhar Al Mahraz Fès***	***Directeur de thèse***

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Introduction Générale

Les équations aux dérivées partielles (EDP) ont une grande variété d'applications en dynamique des structures, mécanique des fluides que dans les théories de la gravité, de l'électromagnétisme (équations de Maxwell) ou des mathématiques financières (équation de Black-Scholes) et dans la mécanique quantique (les équations de Schrödinger et Heisenberg). Elles sont indispensable dans des domaines tels que la prévision météorologique, la synthèse d'images, sans oublier les équations les plus connus celle de la théorie de la relativité. Ces équations proviennent très souvent des bilans globaux traduisant des axiomes ainsi que des paramétrisations rendant compte des lois des phénomènes observés.

le but principal d'un mathématicien confronté à une équation aux dérivées partielles est de présenter un sens à cette équation dans des espaces fonctionnels convenable et plus précisément de prouver l'existence et l'unicité de la solution.Sans faire absence aux différentes difficultés liées au type de l'équation (elliptique, parabolique, hyperbolique, dégénéré, ...), à la régularité des données et aussi à la définition des espaces appropriés.

Ce travail a pour objectif l'étude de certains problèmes paraboliques et elliptiques dans les espaces de Sobolev et les espaces de Sobolev généralisés .

Le mémoire comporte trois parties, les troix premiers chapitres constituent la première partie, les deux chapitres qui suivent font l'objet de la deuxième et le dernier chapitre désigne la troisième .

Dans le premier chapitre de cette thèse, nous étudions l'existence de solutions de l'équation parabolique suivante:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} a(t,x,\nabla u) + H(t,x,u,\nabla u) = f & \text{dans} \quad (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{dans} \quad \Omega, \\ u(t,x) = 0 & \text{sur} \quad (0,T)\times\partial\Omega. \end{cases} \tag{0.0.1}$$

Avec Ω est un ouvert borné de $\mathbb{R}^N$, $T > 0$, $N \geq 2$, $2 < p < +\infty$, $u_0 \in L^\infty(\Omega)$ et $0 \leq f$ dans $L^r(0,T;L^q(\Omega))$,avec $\frac{p}{r} + \frac{N}{q} < p$, $r \geq \frac{p}{p-1}$, $q > 1$.

Une telle équation modélise des problèmes de la théorie des fluides non-Newtoniens et la théorie de la conduction thermique dans les matériaux conducteurs électriques, voir [59, 88] pour plus de détail. Dans le cas elliptique, une équation type est donnée par

$$\begin{cases} -\Delta_p u + B\frac{|\nabla u|^p}{u^m} = f & \text{dans} \quad \Omega, \\ u(x) = 0 & \text{sur} \quad \partial\Omega, \end{cases} \tag{0.0.2}$$

içi $m > 0$, $B > 0$ et f est une fonction positive (non identiquement nulle) dans $L^1(\Omega)$. Le problème est évidemment singulier puisqu'on demande à la solution d'être nulle sur la frontière de Ω .

Dans [21] l'existence d'une solution du problème (0.0.2) dans l'espace $H_0^1(\Omega)$ a été prouvé si $m < 2$ et pour une donnée f localement bornée.

Si $m < 1$ l'existence d'une solution dans $H_0^1(\Omega)$ a été prouvé dans [40] pour les données générales positives (non identiquement nulle),le cas $m = 1$ a été étudié dans [97] en supposant B assez petit. Des problèmes du type (0.0.2) avec des données qui changent de signe ont également été étudiés dans [81] dans le cas $m < 1$ (voir aussi [82] pour une étude plus approfondie concernant le cas fortement singulier).

Dans le cas parabolique, le problème comme

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u + B\frac{|\nabla u|^p}{u^m} = f & \text{dans} \quad (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{dans} \quad \Omega, \\ u(t,x) = 0 & \text{sur} \quad (0,T)\times\partial\Omega, \end{cases} \tag{0.0.3}$$

a été étudie dans le cas $p = 2$ et $m < 1$ (voir [98]).

Si $m = 1$ le problème singulier du type (0.0.3) a été traité dans [124, 125] pour des données strictement positives. Les problèmes dégénérés i.e. $(p > 2)$ ont été étudiés dans [128] dans le cas unidimensionnel [98] et dans [64] où les auteurs étudient l'existence de solutions pour une classe générale de problème singulier homogène $(f \neq 0)$.

Notre but est d'étudier l'existence de solutions pour une classe de problèmes paraboliques singulier

non homogène ($f \neq 0$) comme le problème (0.0.3) dans le cas $m = 1$. Nous nous occuperons principalement de l'étudier dans le cas $p > 2$.

Dans le chapitre 2, nous nous sommes intéressés à l'étude de certains problèmes quasi-linéaires elliptiques avec des termes d'ordre inférieur et des sources variables, ici nous sommes concernés par l'existence et les résultats de régularité d'une solution positive du problème suivant :

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) + u^s = \dfrac{f(x)}{u^\alpha} & \text{dans} \quad \Omega, \\ u \geq 0 & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega, \end{cases} \tag{0.0.4}$$

où Ω est un ouvert borné de $\mathbb{R}^N$ $(N > p)$, $0 < \alpha \leq 1$, $s \geq 1$ et f une fonction positive de $L^1(\Omega)$. La présence de terme d'ordre inférieur dans le problème elliptique de type (0.0.4) est très important dans les applications; habituellement, un terme d'ordre inférieur représente une absorption ou une réaction, selon son signe. Le problème (0.0.4) a été appliquée dans de nombreux domaines comme dans le contexte des catalyseurs chimiques hétérogènes, des fluides non Newtoniens ou celui de la théorie de la conduction thermique dans les matériaux conducteurs électriques, voir [101, 20] pour plus de détail. Dans ce travail, nous traitons des termes absorbants d'ordre zéro, qui a habituellement un effet de régularisation sur les solutions du problème (0.0.4). Nous renvoyons le lecteur à [106, 107] pour une autre approche utilisant les résultats sur les problèmes elliptiques et paraboliques dans le cadre des espaces Sobolev. Voir aussi (voir [44, 57]) . Dans [44] les auteurs étudient l'effet de régularisation du terme u^s sur la solution du problème classique suivant:

$$\begin{cases} -\Delta u = f(x) & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega, \end{cases} \tag{0.0.5}$$

lorsqu'on ajoute le terme u^s à côté gauche de(0.0.5), le problème devient :

$$\begin{cases} -\Delta u + u^s = f(x) & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega. \end{cases} \tag{0.0.6}$$

Dans [45] les auteurs ont étudié le problème suivant:

$$\begin{cases} -\Delta u = \frac{f(x)}{u^\gamma} & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega. \end{cases} \tag{0.0.7}$$

récemment dans [67] les auteurs ont étudié la régularité de la solution du problème suivant :

$$\begin{cases} -\Delta u + u^s = \frac{f(x)}{u^\gamma} & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega. \end{cases} \tag{0.0.8}$$

Le but du deuxième chapitre est de donner un aperçu des résultats prouvés pour le problème (0.0.4) et des questions liés à notre problème, certaines méthodes classiques ne peuvent pas être directement appliquées à notre problème. Nous appliquons la méthode de régularisation et le théorème du point fixe de Schauder. Nous utilisons une fonction teste appropriée ainsi qu'un argument de compacité pour surmonter les difficultés découlant de la non linéarité de l'opérateur différentiel et des conditions d'absorption, puis nous prouvons l'existence et la régularité des solutions.

Dans le chapitre 3 nous avons étudié un cas plus général du problème (0.0.4), il s'agit du problème suivant:

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) + \psi(u) = g(u)f(x) & \text{dans} \quad \Omega \\ u > 0 & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega. \end{cases} \tag{0.0.9}$$

Le terme d'absorption $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ est une fonction croissante, continue sur $\mathbb{R}$ vérifiant les hypothèses ci-dessous:

$$\psi(t)t \geq 0 \text{ pour tout } t \in \mathbb{R}, \tag{0.0.10}$$

$$\int_0 (\psi(t)t)^{-\frac{1}{2}} = \infty, \tag{0.0.11}$$

$$\psi(t) \geq t^s \text{ pour tout } t \geq 0 \text{ et pour certain } s \geq 1. \tag{0.0.12}$$

Le terme source singulier $g : \mathbb{R}^* \longrightarrow \mathbb{R}^+$ est une fonction décroissante continue et bornée sur $\mathbb{R}^*$, telle que:

$$\lim_{t \longrightarrow 0} g(t) = +\infty, \tag{0.0.13}$$

et telle que

$$\text{il existe } c > 0, \delta^{'} > \delta > 0 \text{ and } \alpha, \eta \in (0,1] \text{ s.t. } \begin{cases} g(t) \leq \frac{c}{t^\eta} & \text{si} \quad t < \delta \\ g(t) \leq \frac{c}{t^\alpha} & \text{si} \quad t > \delta^{'}. \end{cases} \tag{0.0.14}$$

La deuxième partie de cette thèse est consacrée à l'étude de quelques types de problèmes elliptiques non-linéaire dans l'espace d'Orlicz-Sobolev avec poids.

Le chapitre 4 sera consacré à l'étude d'une solution entropique pour le problème elliptique non-linéaire suivant :

$$\begin{cases} A(u) = f(x) & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega, \end{cases} \tag{0.0.15}$$

Avec $A(u) = -\text{div}\ (\rho(x)a(x,u,\nabla u))$, Ω est un ouvert borné de $\mathbb{R}^N$, $N \geq 2$. $a(x,u,\nabla u) = (a_i(x,u,\nabla u))_{1\leq i\leq N}$, $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ est une fonction de Carathéodoré (c.à.d. mesurable par rapport à x pour tout (s,ξ) et continue par rapport à (s,ξ) pour presque tout x dans Ω) telle que pour tout ξ, ξ' dans $\mathbb{R}^N$, $\xi \neq \xi'$,

$$|a_i(x,s,\xi)| \leq |\phi_i(x)| + K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|s|)) + K_i\overline{M}^{-1}M(c_1\,|\xi|), \tag{0.0.16}$$

$$\Big(a(x,s,\xi) - a(x,s,\xi')\Big)(\xi - \xi') \geq 0, \tag{0.0.17}$$

$$a(x,s,\xi)\xi \geq \alpha M(\lambda_1|\xi|), \tag{0.0.18}$$

avec c_1, c_2, λ_1 et K_i appartient à $\mathbb{R}_+$. M, P sont deux N-fonctions tels que $P \prec\prec M$. de plus $\overline{M}$, $\overline{P}$ soient les fonctions complémentaires de M and P respectivement, ρ est une fonction poids sur Ω (qui est, mesurable et positive sur Ω p.p) et ϕ_0, $\phi_i \in E_{\overline{M}}(\Omega,\rho)$.

$$f \in L^1(\Omega), \tag{0.0.19}$$

Il est bien connu que dans les espaces d'Orlicz-Sobolev avec poids généralisent de nombreux espaces classiques, comme les espaces de Lebesgue, Lebesgue avec poids ou les espaces d'Orlicz, voir [104]. Ces espaces ont de nombreuses applications dans divers domaines comme les EDP, les fluides électrorhéologiques et la restauration d'images, voir [69, 58, 90]. Le but de ce chapitre, est de traiter une classe de problèmes pour lesquels les méthodes des opérateurs monotones classique (développée par Minty [100], Browder [55], Brézis [54] et Lions [91] dans $W_0^{1,p}(\Omega)$) ne s'appliquent pas. La raison est que $a(x,u,\nabla u)$ ne vérifie pas nécessairement la condition de monotonie stricte:

$$\Big(a(x,s,\xi) - a(x,s,\xi')\Big)(\xi - \xi') > 0, \text{ pour tout } \xi,\xi' \in \mathbb{R}^N,\ (\xi \neq \xi')$$

d'un opérateur de type Leray-Lions, mais plutôt une condition de monotonie plus large:

$$\Big(a(x,s,\xi) - a(x,s,\xi')\Big)(\xi - \xi') \geq 0, \text{ pour tout } \xi,\xi' \in \mathbb{R}^N.$$

Nous montrons en effet dans ce chapitre l'existence de solutions pour le problème (0.0.15) sous la condition de monotonie large, sans faire appel à la convergence presque partout des gradients des solutions

approchées. Le principal outil de notre démonstration étant le lemme de Minty [103])

Le chapitre 5 de cette thèse a comme but d'étudier l'existence de solution pour le problème de Dirichlet non-linéaire suivant:

$$\begin{cases} A(u) = f(x) - \text{div } F(u) & \text{dans} \quad \Omega, \\ u = 0 & \text{sur} \quad \partial\Omega, \end{cases} \tag{0.0.20}$$

Avec $f \in L^1(\Omega), F \in (C^0(\mathbb{R}))^N$ et $A(u) = -\text{div } (\rho(x)a(x,u,\nabla u))$, Ω est un ouvert borné de $\mathbb{R}^N$, $N \geq 2$.

Notez qu'aucune hypothèse de croissance n'est supposée sur la fonction F,ce qui implique que le terme $divF(u)$ peut ne pas avoir de sens, même au sens des distribution. $a(x,u,\nabla u) = (a_i(x,u,\nabla u))_{1\leq i\leq N}$, $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ est une fonction de Carathéodoré (c.à.d. mesurable par rapport à x pour tout (s,ξ) et continue par rapport à (s,ξ) pour presque tout x dans Ω) telle que pour tout ξ, ξ' dans $\mathbb{R}^N$, $\xi \neq \xi'$, telle que pour tout ξ, ξ' dans $\mathbb{R}^N$, $\xi \neq \xi'$,

$$|a_i(x,s,\xi)| \leq |\phi_i(x)| + K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|s|)) + K_i\overline{M}^{-1}M(c_1|\xi|), \tag{0.0.21}$$

$$\Big(a(x,s,\xi) - a(x,s,\xi')\Big)(\xi - \xi') \geq 0, \tag{0.0.22}$$

$$a(x,s,\xi)\xi \geq M(\lambda_1|\xi|), \tag{0.0.23}$$

avec c_1, c_2, λ_1 et K_i appartient à $\mathbb{R}_+$. M, P sont deux N-fonctions tels que $P \prec\prec M$. de plus $\overline{M}$, $\overline{P}$ soient les fonctions complémentaires de M and P respectivement, ρ est une fonction poids sur Ω (qui est, mesurable et positive sur Ω p.p) et ϕ_0, $\phi_i \in E_{\overline{M}}(\Omega,\rho)$.

La troisième partie de ce mémoire à pour but le traitement d'un problème elliptique non-linéaire (l'existence d'une solution renormalisée) dans l'espace de Musielak-Orlics-Sobolev.

Le dernier chapitre de ce travail a pour objectif d'étudier l'existence de solution renormalisée du problème aux limites non linéaire suivant:

$$\begin{cases} A(u) - \text{div } \Phi(u) + g(x,u,\nabla u) = f \quad in \quad \Omega \\ u \equiv 0, \quad on \quad \partial\Omega \end{cases} \tag{0.0.24}$$

dans le cadre des espaces de Musielak-Orlicz. Où Ω est un ouvert borné de $\mathbb{R}^N, N \geq 2$, $A(u) = -\text{ div } a(x,u,\nabla u)$ est un opérateur de Leray-Lions définie de l'espace $W_0^1 L_\varphi(\Omega)$ vers son dual $W^{-1}L_\psi(\Omega)$, et $\Phi \in \mathrm{C}^0\left(\mathrm{R}, \mathrm{R}^N\right)$. Avec a est une fonction vérifiant les conditions suivantes :

$$a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N \text{est une function de Carathéodory.} \tag{0.0.25}$$

Il existe deux fonctions de Musielak-Orlicz φ et γ tel que $\gamma \prec\prec \varphi$, Soit $d(\cdot) \in E_\psi(\Omega)$ une fonction positive et ν, β sont des constantes positives tel que pour $x \in \Omega$ p.p, et pour toutes $s \in \mathbb{R}, \xi \in \mathbb{R}^N$

$$|a(x,s,\xi)| \leq \beta \left(d(x) + \psi_x^{-1}\gamma(x,\nu|s|) + \psi_x^{-1}\varphi(x,\nu|\xi|)\right); \tag{0.0.26}$$

$$\left(a(x,s,\xi) - a\left(x,s,\xi'\right)\right)\left(\xi - \xi'\right) > 0; \tag{0.0.27}$$

$$a(x,s,\xi).\xi \geq \alpha\varphi(x,|\xi|). \tag{0.0.28}$$

De plus, soit $g(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ est une fonction de Carathéodory tel que pour $x \in \Omega$ p.p. et pour toutes $s \in \mathbb{R}, \xi \in \mathbb{R}^N$, verifiant les conditions:

$$|g(x,s,\xi)| \leq c(x) + b(|s|)\varphi(x,|\xi|); \tag{0.0.29}$$

$$g(x,s,\xi)s \geq 0; \tag{0.0.30}$$

avec $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ est une fonction positive continue de $L^1\left(\mathbb{R}^+\right)$ et $c(\cdot) \in L^1(\Omega)$

Le côté droit de (6.1.1) et $\Phi : \mathbb{R} \to \mathbb{R}^N$ vérifient:

$$f \in W^{-1}E_\psi(\Omega); \tag{0.0.31}$$

$$\Phi \in \mathcal{C}^0\left(\mathbb{R}, \mathbb{R}^N\right). \tag{0.0.32}$$

Notez qu'aucune hypothèse de croissance n'est supposée sur la fonction Φ, ce qui implique que le terme $-\text{div } \Phi(u)$ peut ne pas avoir de sens, même au sens des distribution.

Notre but dans ce chapitre est de montrer l'existence d'une solution renormalisée au problem (6.1.1) dans l'espace de Musielak-Orlicz, dans le cas où la fonction de Musielak-Orlicz φ ne satisfait pas la condition Δ_2, tandis que le côté droit est dans $W^{-1}E_\psi(\Omega)$, $\Phi \in \mathcal{C}^0\left(\mathbb{R}, \mathbb{R}^N\right)$. et le terme non-linéaire $g(x,s,\xi)$ ayant une condition de croissance par rapport au gradient.

Articles published or accepted

(1) **On the existence of solution for degenerate parabolic equations with singular terms** , by Abdelmoujib Benkirane, Badr El Haji and Mostafa El Moumni, **published in " Pure and Applied Mathematics Quarterly, Vol. 14, No. 3-4 (2018), pp. 591-606."**

(2) **On a nonlinear elliptic problems having large monotonocity with L^1-data in weighted Orlicz-Sobolev spaces,** by Badr El Haji , Mostafa El Moumni and Khaled Kouhaila, **published in " Moroccan J. of Pure and Appl. Anal. (MJPAA) Volume 5(1), 2019, Pages 104-116"**

(3) **Existence of renormalized solution for nonlinear elliptic boundary value problem without Δ_2 -condition,** by Nourdine EL AMARTY, Badr EL HAJI and Mostafa EL MOUMNI, **published in SeMA Journal (2020), DOI: https://doi.org/10.1007/s40324-020-00224-z**

(4) **Existence of entropy solution for nonlinear elliptic problem having large monotonocity in weighted orlicz-sobolov spaces,** by Badr El Haji, Mostafa El Moumni and Khaled Kouhaila, **accepted for publication in Le matematiche**

(5) **On the existence solutions for some Nonlinear elliptic problem,** Abdelmoujib BENKIRANE, Badr EL HAJI and Mostafa EMOUMNI, **accepted for publicatin in Boletim da Sociedade Paranaense de Matemática,**

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(2) **Existence results for strongly nonlinear elliptic problems with sign condition and measure data in Musielak-Orlicz spaces**, Abdelmoujib BENKIRANE, Badr EL HAJI and Mostafa EL MOUMNI **submitted in Complex Variables and Elliptic Equations,**

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Part I

Parabolic and elliptic equations with singular terms

Chapter 1

On the existence of solution for degenerate parabolic equations with singular terms

Abstract

This chapter is devoted to study the existence of solutions to the parabolic problems whose simplest model is the following

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u \ (:= \operatorname{div}(|\nabla u|^{p-2}\nabla u)) + B\frac{|\nabla u|^p}{u} = f & \text{in} \quad (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \\ u(t,x) = 0 & \text{on} \quad (0,T)\times\partial\Omega, \end{cases}$$

with $f \in L^1(\Omega)$ and $\phi \in C^0(\mathbb{R},\mathbb{R}^N)$, also we will give some regularity results for these solutions.

1.1 Introduction

We are going to study the following parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} a(t,x,\nabla u) + H(t,x,u,\nabla u) = f & \text{in} \quad (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \\ u(t,x) = 0 & \text{on} \quad (0,T)\times\partial\Omega. \end{cases} \tag{1.1.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$, $T > 0$, $N \geq 2$, $2 < p < +\infty$, $u_0 \in L^\infty(\Omega)$ and $0 \leq f$ in $L^r(0,T;L^q(\Omega))$, with $\frac{p}{r} + \frac{N}{q} < p$, $r \geq \frac{p}{p-1}$, $q > 1$. Such equation arises in the theory of non-Newtonian

fluids and also the theory of heat conduction in electrically conducting materials, see [59, 88] for detailed discussion.

In the elliptic case, Consider the equation

$$\begin{cases} -\Delta_p u + B\frac{|\nabla u|^p}{u^m} = f & \text{in} \quad \Omega, \\ u(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1.2}$$

here $m > 0$, $B > 0$ and f is a nonnegative (not identically zero) function in $L^1(\Omega)$. The problem is obviously singular as we ask the solution to vanish at the boundary of Ω. In [21] the existence of a finite energy (i.e., in $H_0^1(\Omega)$) solution to problem (1.1.2) has been proved if $m < 2$ and for data f locally bounded away from zero. The case of a possibly degenerate datum f has been also considered. If $m < 1$ the existence of a solution in $H_0^1(\Omega)$ was proved in [40] for general nonnegative (not identically zero) data, while the case $m = 1$ was faced in [97] provided B was small enough. Problems as in (1.1.2) with possibly changing-sign data have also been considered in [81] in the case $m < 1$ (see also [82] for further considerations concerning the strongly singular case).

In the parabolic case, problems as

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u + B\frac{|\nabla u|^p}{u^m} = f & \text{in} \quad (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \\ u(t,x) = 0 & \text{on} \quad (0,T)\times\partial\Omega, \end{cases} \tag{1.1.3}$$

have been considered in the case $p = 2$ and $m < 1$ (see [98]). If $m = 1$ singular problems as (1.1.3) have been considered in [124, 125] for smooth strictly positive data, while degenerate problems (i.e. $p > 2$) were studied in [128] in the one dimensional case and in [64], the authors study the existence of solutions for a general class of singular homogeneous ($f = 0$).

The aim of this chapter is to study existence of solutions for a class of singular non-homogeneous ($f \neq 0$) parabolic problems as (1.1.3) in the limit case $m = 1$. We will mainly be concerned with the case $p > 2$.

1.2 Essential Assumptions.

From now on, we will set $Q = (0,T)\times\Omega$. Let us spend a few words on how positive constant will be denoted hereafter. If no otherwise specified, we will write C, C' and C'' to denote any positive constant

(possibly different) which only depends on the data, that is on quantities that are fixed in the assumptions (f, N, Ω, T, B, α, p, β, and so on. . .). But they will never depend on the indexes of the sequences we will often introduce. For the sake of simplicity we will often use the simplified notation

$$\int_Q f = \int_Q f(t,x)dtdx$$

when referring to integrals when no ambiguity on the variable of integration is possible. For a fixed $k > 0$, we define the truncation functions $T_k : \mathbb{R} \to \mathbb{R}$ and $G_k : \mathbb{R} \to \mathbb{R}$ as follows

$$T_k(s) = \max(-k, \min(s,k)) \text{ and } G_k(s) = s - T_k(s) = (|s| - k)^+ \text{sign}(s).$$

Let us state our main assumptions. Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 2$, $T > 0$. The function

$$0 \leq f \in L^r(0,T; L^q(\Omega)), \tag{1.2.1}$$

with $\frac{p}{r} + \frac{N}{q} < p, \quad r \geq p', \quad q > 1$, satisfies

$$m_\omega(f) = \text{ess inf}\,\{f(x,t) : x \in \omega, t \in (0,T)\} > 0, \forall \omega \subset\subset \Omega.$$

Moreover, we consider an initial datum $u_0(x)$ which is a function in $L^\infty(\Omega)$ such that $u_0 \geq c > 0$ almost everywhere on Ω, and we suppose that

$$m_\omega(u_0) = \text{ess inf}\,\{u_0(x) : x \in \omega\} > 0, \forall \omega \subset\subset \Omega. \tag{1.2.2}$$

Let $a : (0,T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory vector-valued function such that

$$a(t,x,\xi)\xi \geq \alpha|\xi|^p, \tag{1.2.3}$$

$$|a(x,s,\xi)| \leq \beta|\xi|^{p-1}, \tag{1.2.4}$$

$$(a(t,x,\xi) - a(t,x,\eta))(\xi - \eta) > 0, \tag{1.2.5}$$

for a.e. $(t,x) \in (0,T) \times \Omega$, for every $\xi \neq \eta \in \mathbb{R}^N$ and α, β are positive constants and $p > 2$.

Furthermore, let $H(t,x,s,\xi) : (0,T) \times \Omega \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that

$$0 \leq H(t,x,s,\xi) \leq B\frac{|\xi|^p}{s}, \tag{1.2.6}$$

for a.e. $(t,x) \in (0,T) \times \Omega$, for every $s > 0, \xi \in \mathbb{R}^N$, and B is a positive constant.

Consider problem (5.1.1). Here is the meaning of weak solution for such a problem.

Definition 1.2.1 *A weak solution to problem (5.1.1) is a function* $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap C(0,T;L^1(\Omega))$ *such that for every* $\omega \subset\subset \Omega$ *there exists* c_ω *such that* $u \geq c_\omega > 0$ *in* $(0,T)\times\omega$*; furthermore, we have that*

$$-\int_Q \langle u, \varphi' \rangle - \int_\Omega u_0\varphi(0) + \int_Q a(t,x,\nabla u)\nabla\varphi + \int_Q H(t,x,u,\nabla u)\varphi = \int_Q f\varphi,$$

for every $\varphi \in C_c^1([0,T)\times\Omega)$*, that is, for every* C^1 *function which vanishes in a neighborhood of* $\{T\}\times\Omega$ *and of* $(0,T)\times\partial\Omega$*.*

Note that under assumption $(1.2.6)$, the function $H(t,x,u,\nabla u)$ belongs to $L^1_{loc}(Q)$ thanks to the property of local positivity required on u.

Our main result is the following:

Theorem 1.2.1 *Assume that* (1.2.1)–(1.2.6) *hold. If* $p > 2$ *there exists a weak solution to problem* (5.1.1)*.*

Before the proof we recall some technical tools we will use. The first one is a well known consequence of the Gagliardo-Nirenberg inequality which is valid on any cylinder of the type $Q = (0,T)\times\Omega$ with bounded Ω (see for instance [105] , Lecture II).

Lemma 1.2.1 *Let* $v \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^\beta(\Omega))$*, with* $p \geq 1$*,* $\beta \geq 1$*. Then* $v \in L^\sigma(Q)$ *with* $\sigma = p\frac{N+\beta}{N}$ *and*

$$\int_Q |v|^\sigma \leq C\|v\|^{\frac{\beta p}{N}}_{L^\infty(0,T;L^\beta(\Omega))} \int_Q |\nabla v|^p. \tag{1.2.7}$$

Finally, we will need the following local version of a lemma by Stampacchia (see [120]).

Lemma 1.2.2 *Let* $\omega(h,r)$ *be a function defined on* $[0,+\infty)\times[0,1]$*, which is nonincreasing in* h *and nondecreasing in* r*; suppose that there exist constants* $k_0 \geq 0$*,* $M,\rho,\sigma > 0$*, and* $\eta > 1$ *such that*

$$\omega(h,r) \leq \tfrac{M\omega(k,R)^\eta}{(h-k)^\rho(R-r)^\sigma,}$$

for all $h > k \geq k_0$ *and* $0 \leq r < R \leq 1$*. Then, for every r in* $(0,1)$*, there exists* $d > 0$*, given by*

$$d^\rho = \tfrac{M2^{\frac{\eta(\rho+\sigma)}{\eta-1}}\omega(k_0,1)^{\eta-1}}{(1-r)^\sigma}$$

such that

$$\omega(d,r) = 0.$$

Finally, let us state the following result that will be very useful in the sequel; its proof relies on an easy application of Egorov theorem.

Lemma 1.2.3 *Let $Q = \Omega\times]0,T[$, where Ω is an open and bounded subset of $\mathbb{R}^N$ and $T > 0$. Let ρ_ε be a sequence of $L^1(Q)$ functions that converges to ρ weakly in $L^1(Q)$, and let σ_ε be a sequence of functions in $L^\infty(Q)$ that is bounded in $L^\infty(Q)$ and converges to σ almost everywhere on Q. Then*

$$\lim_{\varepsilon\to 0}\int_Q \rho_\varepsilon\sigma_\varepsilon dxdt = \int_Q \rho\sigma dxdt$$

1.3 Proof of Main result.

Our strategy in order to prove Theorem 6.4.1 will rely on an approximation argument. The next subsection will introduce our approximating problems.

1.3.1 The approximating problems

We consider the approximating problems

$$\begin{cases} \frac{\partial u_n}{\partial t} - \text{div } a(t,x,\nabla u_n) + H(t,x,u_n,\nabla u_n) = f & \text{in} \quad Q, \\ u_n(0,x) = u_0(x) + \frac{1}{n} & \text{in} \quad \Omega, \\ u_n(t,x) = \frac{1}{n} & \text{on} \quad (0,T)\times\partial\Omega. \end{cases} \tag{1.3.1}$$

A weak solution to this problem is a function u_n such that $u_n \geq \frac{1}{n}$ a.e. in Q.

$u_n - \frac{1}{n} \in L^p([0,T];W_0^{1,p}(\Omega)) \cap C(0,T;L^1(\Omega))$ and $\frac{\partial u_n}{\partial t} \in L^1(\Omega) + L^{p'}([0,T];W^{-1,p'}(\Omega))$ and such that

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, v\rangle + \int_Q a(t,x,\nabla u_n)\nabla v + \int_Q H(t,x,u_n,\nabla u_n)v = \int_Q fv \tag{1.3.2}$$

for any $v \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$. A nonnegative weak solution u_n to problem (1.3.1) does exist. In fact, problem (1.3.1) is equivalent to

$$\begin{cases} \frac{\partial v_n}{\partial t} - \text{div } a(t,x,\nabla v_n) + H(t,x,v_n + \frac{1}{n},\nabla v_n) = f & \text{in} \quad Q, \\ v_n(0,x) = u_0(x) & \text{in} \quad \Omega, \\ v_n(t,x) = 0 & \text{on} \quad (0,T)\times\partial\Omega. \end{cases} \tag{1.3.3}$$

where $v_n = u_n - \frac{1}{n}$.

To prove that a solution of (1.3.3) exists, we first extend $H(t,x,s,\xi)$ to be zero for $s \leq 0$, and, for $\varepsilon > 0$, we consider the problem

$$\begin{cases} \frac{\partial v_{n,\varepsilon}}{\partial t} - \operatorname{div} a(t,x,\nabla v_{n,\varepsilon}) + \frac{T_\varepsilon(v_{n,\varepsilon})}{\varepsilon} H(t,x,v_{n,\varepsilon}+\frac{1}{n},\nabla v_{n,\varepsilon}) = f & \text{in} \quad Q, \\ v_{n,\varepsilon}(0,x) = u_0(x) & \text{in} \quad \Omega, \\ v_{n,\varepsilon}(t,x) = 0 & \text{on} \quad (0,T)\times\partial\Omega. \end{cases} \tag{1.3.4}$$

A nonnegative solution $v_{n,\varepsilon}$ to problem (1.3.4) exists by the results proven in [63]. Then, if we take a sequence of values $\varepsilon \downarrow 0$, one can prove that the sequence $\{v_{n,\varepsilon}\}_\varepsilon$ converges strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ to some function v_n. Then one can pass to the limit for $\varepsilon \downarrow 0$ in the first two terms of (1.3.4), in the sense of distributions. As far as the third term is concerned, we observe that, on the set $\{v_n > 0\}$, the function $\frac{T_\varepsilon(v_{n,\varepsilon})}{\varepsilon}$ converges a.e. to 1, while on the set $\{v_n = 0\}$ (where we cannot identify the limit of $\frac{T_\varepsilon(v_{n,\varepsilon})}{\varepsilon}$, but where $\nabla v_n = 0$ a.e. by Stampacchia's result contained in [120]) the term $H(t,x,v_{n,\varepsilon}+\frac{1}{n},\nabla v_{n,\varepsilon})$ converges to $H(t,x,v_n+\frac{1}{n},\nabla v_{n,})$ a.e., which is zero a.e. on this set, since $H(t,x,\frac{1}{n},0) = 0$ a.e. by assumption (1.2.6). Therefore, v_n is a weak solution of (1.3.3).

1.3.2 A priori estimates

A standard argument allows us to show that some basic estimates on the approximating solutions hold. We collect them in the following :

Lemma 1.3.1 *Let $p \geq 2$, and let u_n be a solution to problem* (1.3.1)*. Then,*

$$\|u_n\|_{L^p(0,T;W^{1,p}(\Omega))} \leq C,\ \|u_n\|_{L^\infty(Q))} \leq C \text{ and } \int_Q H(t,x,u_n,\nabla u_n) \leq C. \tag{1.3.5}$$

Moreover, there exists a function u in $L^p(0,T;W_0^{1,p}(\Omega))$ such that (up to subsequences) $u_n - \frac{1}{n}$ converges to u weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ and a.e. on Q. Finally,

$$\nabla u_n \to \nabla u, \text{ a.e. on } Q.$$

Proof 1.3.1 *The proof of the first two estimates is quite standard and can be deduced for instance as in [42] (see also [41]) using the fact that the lower order term is positive. In order to get* (1.3.5) *one can use $\frac{1}{\varepsilon}T_\varepsilon(u_n - \frac{1}{n})$ as test function in* (1.3.1) *. Integrating by parts, dropping nonnegative terms and letting ε go to zero one gets, by Fatou's lemma*

$$\int_Q H(t,x,u_n,\nabla u_n) \leq \int_\Omega (u_0(x)+\frac{1}{n}) + \int_Q f,$$

which implies (1.3.5). *The almost everywhere convergence of the gradients of* u_n *is a consequence of* (1.3.5) *and of a result in [[41], Theorem 3.3.].*

1.3.3 Concluding the proof of Theorem 6.4.1

We wish to pass to the limit in the weak formulation of (1.3.1).

Claim 1 : A key tool in order to pass to the limit will be the following one.

Lemma 1.3.2 *Let* $p > 2$, *and let* u_n *be a weak solution of problem* (1.3.1). *Then, for any* $\omega \subset\subset \Omega$, *there exists a constant* c_ω *such that*

$$u_n \geq c_\omega > 0, \text{ in } (0,T)\times\omega, \text{ for every n in } \mathbb{N}.$$

In order to simplify notations, we henceforth write $a(\nabla u_n)$ instead of $a(t,x,\nabla u_n)$ and $H(u_n,\nabla u_n)$ instead of $H(t,x,u_n,\nabla u_n)$.

1.3.4 proof

We divide the proof into three steps.

Step 1. There is no loss of generality in assuming that the constant B which appears in (1.2.6) satisfies $B > \max(\alpha, p-1)$.

We use $v = -u_n^{-B}\psi$ in (1.3.2), for any nonnegative $\psi(t,x) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ which is zero in a neighborhood of $(0,T)\times\partial\Omega$, in order to obtain

$$-\int_0^T \langle \frac{\partial u_n}{\partial t}, u_n^{-B}\psi\rangle + B\int_Q a(\nabla u_n)\nabla u_n u_n^{-B-1}\psi + \int_Q a(\nabla u_n)\nabla\psi u_n^{-B} - \int_Q H(u_n,\nabla u_n)u_n^{-B}\psi$$
$$= -\int_Q f u_n^{-B}\psi \leq 0,$$

from which taking into account the assumptions (1.2.3), (1.2.6) and $B > \alpha$, one obtains

$$-\int_0^T \langle \frac{\partial u_n}{\partial t}, u_n^{-B}\psi\rangle + \int_Q a(\nabla u_n)\nabla\psi u_n^{-B} \leq 0.$$

Therefore, if we set $u_n = \frac{B+1-p}{p-1}\omega_n^{-\frac{p-1}{B+1-p}}$ and $\gamma = \frac{(p-1)(B-1)}{B+1-p}$, we have

$$\int_0^T \langle \frac{\partial \omega_n}{\partial t}, \omega_n^{\gamma-1}\psi\rangle + C\int_Q a(-\omega_n^{\frac{-B}{B+1-p}}\nabla\omega_n)\nabla\psi\omega_n^{\frac{B(p-1)}{B+1-p}} \quad \leq \quad 0, \tag{1.3.6}$$

for every nonnegative $\psi(t,x) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ which is zero in a neighborhood of $(0,T)\times\partial\Omega$ and for some positive constant C depending only on B and p. Observe that $B > p-1$ and $p > 2$ imply that $\gamma > p-1 > 1$. Moreover, observe that

$$\omega_n(t,x) = cn^{\frac{B+1-p}{p-1}} \text{ on } (0,T)\times\partial\Omega, \qquad \omega_n(0,x) = c(u_0 + \tfrac{1}{n})^{-\frac{B+1-p}{p-1}} = \omega_{0n}$$

for some positive constant c. In particular, since u_0 is bounded away from zero, then ω_{0n} is bounded in $L^\infty(\Omega)$ and the values of ω_n blow up on the boundary as n goes to infinity. We look for an a priori local bound on the L^∞ norm of ω_n.

Step 2. Local L^∞ bound for ω_n.

Without loss of generality we assume that $0 \in \Omega$; we will prove that the bound holds true in a ball B_ρ centered at zero of radius ρ with $0 < r < \rho < R \leq 1$, then a standard covering argument will allow us to conclude.

We fix $k > \|\omega_{0n}\|_{L^\infty(B_R)}$ and we define the sets

$$A_{K,\rho}(t) = \{x \in B_\rho : \omega_n(t,x) > k\}, \qquad A_{K,\rho} = \{(t,x) \in (0,T)\times B_\rho : \omega_n(t,x) > k\}$$

We consider a cut-off function $\varphi(x) \in C_c^\infty(B_R)$ such that

$$0 \leq \varphi(x) \leq 1, \qquad \varphi(x) \equiv 1 \text{ in } B_r, \qquad |\nabla\varphi| \leq \tfrac{C}{R-r},$$

and use $\psi(t,x) = G_k(\omega_n(t,x))\varphi^\delta(x)$ as test function in (1.3.6) where $\delta = \frac{p(\gamma+1)}{\gamma-p+1}$. Integrating between 0 and $t < T$, and using the assumptions (1.2.3) and (1.2.4), we obtain

$$\begin{aligned}\int_0^t \langle \frac{\partial \omega_n}{\partial t}, \omega_n^{\gamma-1}G_k(\omega_n)\varphi^\delta\rangle + \int_0^t\int_{A_{k,R(\tau)}} |\nabla G_k(\omega_n)|^p\varphi^\delta \leq \\ \frac{C}{R-r}\int_0^t\int_{A_{k,R(\tau)}} |\nabla G_k(\omega_n)|^{p-1}\varphi^{\delta-1}G_k(\omega_n).\end{aligned} \tag{1.3.7}$$

We use Young's inequality in order to absorb the term $|\nabla G_k(\omega_n)|^{p-1}$ in the right hand side; using the definition of δ, we get

$$\int_0^t \langle \frac{\partial \omega_n}{\partial t}, \omega_n^{\gamma-1} G_k(\omega_n)\varphi^\delta \rangle + \int_0^t \int_{A_{k,R(\tau)}} |\nabla G_k(\omega_n)|^p \varphi^\delta \leq \frac{C}{(R-r)^p} \int_{A_{k,R}} G_k(\omega_n)^p \varphi^{\frac{p^2}{\gamma-p+1}}.$$

Notice that, since $0 \leq \varphi \leq 1$,

$$\begin{aligned} |\nabla(G_k(\omega_n)\varphi^\delta)|^p &\leq C(|\nabla G_k(\omega_n)|^p \varphi^{\delta p} + \tfrac{G_k(\omega_n)^p \varphi^{\delta p - p}}{(R-r)^p}) \\ &\leq C(|\nabla G_k(\omega_n)|^p \varphi^{\delta} + \tfrac{G_k(\omega_n)^p \varphi^{\delta - p}}{(R-r)^p}) \end{aligned}$$

so that, observing that $\delta - p = \frac{p^2}{\gamma-p+1}$, we finally get

$$\int_0^t \langle \frac{\partial \omega_n}{\partial t}, \omega_n^{\gamma-1} G_k(\omega_n)\varphi^\delta \rangle + \int_0^t \int_{A_{k,R(\tau)}} |\nabla G_k(\omega_n)\varphi^\delta|^p \leq \frac{C}{(R-r)^p} \int_{A_{k,R}} G_k(\omega_n)^p \varphi^{\frac{p^2}{\gamma-p+1}}.$$

Since $\delta > p-1$, we can use again Young's inequality to have

$$\begin{aligned} &\int_0^t \langle \frac{\partial \omega_n}{\partial t}, \omega_n^{\gamma-1} G_k(\omega_n)\varphi^\delta \rangle + \int_0^t \int_{A_{k,R(\tau)}} |\nabla G_k(\omega_n)\varphi^\delta|^p \\ &\qquad \leq \frac{1}{2(\gamma+1)T} \int_{A_{k,R}} G_k(\omega_n)^{\gamma+1}\varphi^\delta + \frac{C}{(R-r)^{\frac{(\gamma+1)p}{\gamma+1-p}}} |A_{k,R}| \\ &\qquad \leq \frac{1}{2(\gamma+1)} \sup_t \int_{A_{k,R}(t)} G_k(\omega_n)^{\gamma+1}\varphi^\delta \, dx + \frac{C}{(R-r)^{\frac{(\gamma+1)p}{\gamma+1-p}}} |A_{k,R}|. \end{aligned} \tag{1.3.8}$$

Now we deal with the time derivative part. Let us define, for $s \geq 0$,

$$\Psi_k(s) = \int_0^s G_k(\sigma)\sigma^{\gamma-1} \, d\sigma$$

Then it is easy to check that

$$\Psi_k(s) \geq \frac{1}{\gamma+1} G_k(s)^{\gamma+1}$$

that $k > \|\omega_0\|_{L^\infty(B_R)}$, we obtain

$$\int_0^t \langle \frac{\partial \omega_n}{\partial t}, \omega_n^{\gamma-1} G_k(\omega_n)\varphi^\delta \rangle = \int_{B_R} \Psi_k(\omega_n(t,x))\varphi^\delta \geq \frac{1}{\gamma+1} \int_{B_R} G_k(\omega_n)^{\gamma+1}\varphi^\delta.$$

Therefore we can use (1.3.8) in order to deduce

$$\frac{1}{\gamma+1} \int_{B_R} G_k(\omega_n)^{\gamma+1}\varphi^\delta \leq \frac{1}{2(\gamma+1)} \sup_t \int_{A_{k,R}} G_k(\omega_n)^{\gamma+1}\varphi^\delta \, dx + \frac{C}{(R-r)^{\frac{(\gamma+1)p}{\gamma+1-p}}} |A_{k,R}|.$$

and we can take the spermium over $t \in (0,T)$ on the left in order to get

$$\sup_t \int_{A_{k,R}(t)} G_k(\omega_n)^{\gamma+1}\varphi^\delta \, dx \le \frac{C}{(R-r)^{\frac{(\gamma+1)p}{\gamma+1-p}}}|A_{k,R}|.$$

Gathering together all these facts, and using again that $\varphi^\delta \ge \varphi^{\delta(\gamma+1)}$, we end up with the following estimate

$$\sup_t \int_{A_{k,R(t)}} G_k(\omega_n\varphi^\delta)^{\gamma+1} \, dx + \int_{A_{k,R}} |\nabla G_k(\omega_n)\varphi^\delta|^p \, dx \le \frac{C}{(R-r)^{\frac{(\gamma+1)p}{\gamma+1-p}}}|A_{k,R}|.$$

We are now in the position to apply the Gagliardo-Nirenberg inequality (1.2.7) to the function $G_k(\omega_n)\varphi^\delta$, with $\beta = \gamma + 1$; recalling that $\varphi \equiv 1$ on B_r, we obtain

$$\int_{A_{k,R}} G_k(\omega_n)^{p\frac{N+\gamma+1}{N}} \, dx \le \frac{C}{(R-r)^{\frac{(\gamma+1)p(N+p)}{(\gamma+1-p)N}}}|A_{k,R}|^{1+\frac{p}{N}}.$$

Stampacchia's procedure is now quite standard. For $h > k$, one obtains

$$\int_{A_{k,R}} G_k(\omega_n)^{p\frac{N+\gamma+1}{N}} \, dx \ge \int_{A_{h,r}} G_h(\omega_n)^{p\frac{N+\gamma+1}{N}} \, dx \ge (h-k)^{p\frac{N+\gamma+1}{N}}|A_{h,r}|,$$

that is,

$$|A_{h,r}| \le \frac{C|A_{k,R}|^{1+\frac{p}{N}}}{(h-k)^{p\frac{N+\gamma+1}{N}}(R-r)^{\frac{(\gamma+1)p(N+p)}{(\gamma+1-p)N}}}.$$

Therefore, if we choose $\omega(h,r) = |A_{h,r}|$, we can apply Lemma 6.3.8 in order to deduce that, for every fixed $\rho \in (r,R)$, $|A_{h,r}| = 0$ if h is larger than some constant C_ρ. It follows that

$$\omega_n \le C_\rho, \text{ a.e. on } (0,T)\times B_\rho, \text{ for every } n. \tag{1.3.9}$$

Step 3. End of the proof. Recalling that $B > p-1$ and the definition of ω_n we use (1.3.9) to have, a.e. on $(0,T)\times B_\rho$

$$u_n = \tfrac{B+1-p}{p-1}\omega_n^{-\frac{p-1}{B+1-p}} \ge \tfrac{B+1-p}{p-1}C_\rho^{-\frac{p-1}{B+1-p}} = c_\rho > 0.$$

As we said, by means of a standard covering procedure this estimate can be proven to hold on any set of the form $(0,T)\times\omega$, with $\omega \subset\subset \Omega$.

Claim 2 : Passing to the limit. In order to pass to the limit as n tends to infinity, we need the following result.

Proposition 1.3.1 *We have*

$$u_n \longrightarrow u \textit{ strongly in } L^p(0,T;W^{1,p}(\omega)),$$

for every open set $\omega \subset\subset \Omega$.

Proof of Proposition 1.3.1

The sequence $\{\frac{\partial u_n}{\partial t}\}$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$. Using the Aubin-Simon compactness argument (see Corollary 4 in [119]), we deduce that, up to a subsequence,

$$u_n \longrightarrow u \text{ in } L^p(Q),$$

for some u in $L^p(0,T;W^{1,p}(\Omega)) \cap L^\infty(Q)$. We will prove that, for every open set $\omega \subset\subset \Omega$,

$$u_n \longrightarrow u \text{ in } L^p(0,T;W^{1,p}(\omega)), \tag{1.3.10}$$

We now introduce a classical regularization u_ν of the function u with respect to time (see [92]). For every $\nu \in \mathbb{N}$, we define u_ν as the solution of the Cauchy problem

$$\begin{cases} \frac{1}{\nu}\frac{\partial u_\nu}{\partial t} + u_\nu = u \\ u_\nu(0) = u_{0,\nu} \end{cases}$$

where $u_{0,\nu} \in W_0^{1,p}(\Omega)) \cap L^\infty(\Omega)$, satisfies $u_{0,\nu} \longrightarrow u_0$ strongly in $L^1(\Omega)$, $*-$ weakly in $L^\infty(\Omega)$ and $\lim_{\nu\to+\infty} \frac{1}{\nu}\|u_{0,\nu}\|_{W_0^{1,p}(\Omega)} = 0$.

Then one has (see [92]) :

$$u_\nu \in L^p(0,T;W_0^{1,p}(\Omega)),\ \frac{\partial u_\nu}{\partial t} \in L^p(0,T;W_0^{1,p}(\Omega)) \text{ and } \|u_\nu\|_{L^\infty(Q)} \leq \|u\|_{L^\infty(Q)},$$

and as ν tends to infinity,

$$u_\nu \longrightarrow u \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)). \tag{1.3.11}$$

Let $\varphi_\lambda(s) = se^{\lambda s^2}$ (with λ to be chosen later). We will denote by $\varepsilon(\nu,n)$ any quantity such that

$$\lim_{\nu\to+\infty} \limsup_{n\to\infty} |\varepsilon(\nu,n)| = 0.$$

For $0 \leq \phi \in C_c^\infty(\Omega)$, we have that

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, \varphi_\lambda(u_n - u_\nu)\phi \rangle \geq \varepsilon(\nu, n). \tag{1.3.12}$$

Now, using (1.3.12) and $\varphi_\lambda(u_n - u_\nu)\phi$ as test function in (1.3.2), we obtain

$$\begin{aligned}\int_Q a(\nabla u_n)\nabla(u_n - u_\nu)\varphi_\lambda'(u_n - u_\nu)\phi + \int_Q a(\nabla u_n)\nabla\phi\varphi_\lambda(u_n - u_\nu) + \int_Q H(u_n, \nabla u_n)\varphi_\lambda(u_n - u_\nu)\phi \\ \leq \int_Q f\varphi_\lambda(u_n - u_\nu)\phi - \varepsilon(\nu, n).\end{aligned}$$

Moreover, if $\omega \subset\subset \Omega$ is such that $\text{supp}\,\phi \subset \omega$ since $u_n \to u$ weakly in $L^p(0,T; W_0^{1,p}(\Omega))$ and $\varphi_\lambda(u_n - u_\nu)\phi$ converges to $\varphi_\lambda(u - u_\nu)\phi$ $*$-weakly in $L^\infty(Q)$, so that, by Lemma 1.2.3, we have

$$\int_Q f\varphi_\lambda(u_n - u_\nu)\phi - \int_Q a(\nabla u_n)\nabla\phi\varphi_\lambda(u_n - u_\nu) = \varepsilon(\nu, n).$$

If c_ω is the constant given by Lemma 6.3.3, we have, recalling that $\text{supp}\,\phi \subset \omega$

$$\begin{aligned}\Big| \int_Q H(u_n, \nabla u_n)\varphi_\lambda(u_n - u_\nu)\phi \Big| &\leq B\int_{\omega\times(0,T)} \frac{|\nabla u_n|^p}{u_n}|\varphi_\lambda(u_n - u_\nu)|\phi \\ &\leq \frac{B}{c_\omega}\int_Q |\nabla u_n|^p|\varphi_\lambda(u_n - u_\nu)|\phi.\end{aligned}$$

Thus,

$$\int_Q a(\nabla u_n)\nabla(u_n - u_\nu)\varphi_\lambda'(u_n - u_\nu)\phi - \frac{B}{c_\omega}\int_Q |\nabla u_n|^p|\varphi_\lambda(u_n - u_\nu)|\phi \leq \varepsilon(\nu, n). \tag{1.3.13}$$

Then we can write

$$\begin{aligned}\int_Q a(\nabla u_n)\nabla(u_n - u_\nu)\varphi_\lambda'(u_n - u_\nu)\phi &= \int_Q (a(\nabla u_n) - a(\nabla u_\nu))\nabla(u_n - u_\nu)\varphi_\lambda'(u_n - u_\nu)\phi \\ &\quad + \int_Q a(\nabla u_\nu)\nabla(u_n - u_\nu)\varphi_\lambda'(u_n - u_\nu)\phi \\ &= \int_Q (a(\nabla u_n) - a(\nabla u_\nu))\nabla(u_n - u_\nu)\varphi_\lambda'(u_n - u_\nu)\phi + \varepsilon(\nu, n).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_Q |\nabla u_n|^p|\varphi_\lambda(u_n - u_\nu)|\phi &\leq \alpha^{-1}\int_Q a(\nabla u_n)\nabla u_n|\varphi_\lambda(u_n - u_\nu)|\phi \\ &= \alpha^{-1}\int_Q (a(\nabla u_n) - a(\nabla u_\nu))\nabla(u_n - u_\nu)|\varphi_\lambda(u_n - u_\nu)|\phi \\ &\quad + \int_Q a(\nabla u_\nu)\nabla(u_n - u_\nu)|\varphi_\lambda(u_n - u_\nu)|\phi + \int_Q a(\nabla u_n)\nabla u_\nu|\varphi_\lambda(u_n - u_\nu)|\phi \\ &= \alpha^{-1}\int_Q (a(\nabla u_n) - a(\nabla u_\nu))\nabla(u_n - u_\nu)|\varphi_\lambda(u_n - u_\nu)|\phi + \varepsilon(\nu, n).\end{aligned}$$

Therefore, from (5.2.3) we obtain

$$\int_Q (a(\nabla u_n) - a(\nabla u_\nu))\nabla(u_n - u_\nu)[\varphi'_\lambda(u_n - u_\nu) - \frac{B}{\alpha c_\omega}|\varphi_\lambda(u_n - u_\nu)|]\phi \leq \varepsilon(\nu, n).$$

Choosing λ large enough so that $\varphi'_\lambda(s) - \frac{B}{\alpha c_\omega}|\varphi_\lambda(s)| \geq \frac{1}{2}$ for every $s \in \mathbb{R}$, we deduce that

$$\int_Q (a(\nabla u_n) - a(\nabla u_\nu))\nabla(u_n - u_\nu)\phi \leq \varepsilon(\nu, n).$$

From here it is standard (see for example [99]) to prove that $u_n - u_\nu$ tends to zero strongly in $L^p(0, T; W^{1,p}(\omega))$. Recalling (1.3.11), we thus have that (1.3.10) holds.

Using Proposition 1.3.1 we can prove the (local) strong convergence of the lower order terms.

Lemma 1.3.3 *We have*

$$H(u_n, \nabla u_n) \to H(u, \nabla u) \text{ locally strongly in } L^1(Q).$$

Proof of Lemma 1.3.3

Gathering together the results of Proposition 1.3.1, Lemma 1.3.1, and Lemma 6.3.3, we can apply Lebesgue's dominated convergence theorem to prove that

$$\frac{|\nabla u_n|^p}{u_n} \to \frac{|\nabla u|^p}{u}, \text{ locally strongly in } L^1(Q).$$

It is then straightforward to conclude using (1.2.6) and Vitali's theorem.

Thanks to all the results proved so far we can pass to the limit in (1.3.1), to have that u is a solution of (5.1.1) in the sense of Definition 6.4.1, thus concluding the proof of Theorem 6.4.1.

Remark 1.3.1 *The hypothesis $p > 2$ seems necessary to obtain the boundlessness of Lemma 1.3.2, we can not say if we can do without it or not.*

Chapter 2

On the existence solutions for some Nonlinear elliptic problem

Abstract

In the present paper, we study the existence and regularity of positive solutions for the following boundary value problem : $-\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + u^s = \dfrac{f}{u^\alpha}$ in Ω and $u = 0$ on $\partial\Omega$, where Ω is an open and bounded subset of $\mathbb{R}^N$ $(N > p > 1)$, $0 < \alpha \leq 1$, $s \geq 1$ and f is a nonnegative function that belongs to some Lebesgue space.

2.1 Introduction and Main Result

In this paper, we are concerned with the existence and regularity results for the positive solution to the following problem :

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + u^s = \dfrac{f(x)}{u^\alpha} & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$ $(N > p)$, $0 < \alpha \leq 1$, $s \geq 1$ and f be in $L^1(\Omega)$ function.

Problem (2.1.1) has been applied in chemical heterogeneous catalysts, non-Newtonian fluids and also the theory of heat conduction in electrically conducting materials, see [101, 20] for detailed discussion. In this work, we are dealing with absorption zero order terms, that usually has a regularizing effect on the

solutions to (2.1.1), by starting from measure data for regularity results on the Lebesgue scale (in [44, 57]) and on the Marcinkiewicz one ([43]). We refer the reader to [106, 107] for another approach using results on elliptic and parabolic problems in the setting of Sobolev spaces. See also [11, 26, 75] for related topics.

In [44] the authors studied the regularizing effect of the term u^s on the solution to the following classical problem

$$\begin{cases} -\Delta u = f(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1.2}$$

when the term u^s is added in the left-hand side of (2.1.2), we obtain the following problem

$$\begin{cases} -\Delta u + u^s = f(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.1.3}$$

In [45] the authors study the following problem

$$\begin{cases} -\Delta u = \frac{f(x)}{u^\gamma} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.1.4}$$

Recently in [67] the authors study the regularity of the solution for the following problem

$$\begin{cases} -\Delta u + u^s = \frac{f(x)}{u^\gamma} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.1.5}$$

Definition 2.1.1 *A function $u \in W_0^{1,p}(\Omega)$ is a distributional solution to problem (2.1.1) in case $\alpha \leq 1$, $s \geq 1$ and $f \in L^r(\Omega)$ with $r \geq 1$ if*

$$\forall w \subset\subset \Omega \text{ exists } c_w > 0 \text{ s.t. } u \geq c_w \text{ a.e. in } w, \tag{2.1.6}$$

$$u^s \in L^1(\Omega),$$

and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi + \int_\Omega u^s\varphi = \int_\Omega \frac{f}{u^\alpha}\varphi \quad \forall \varphi \in \mathcal{C}_c^1(\Omega). \tag{2.1.7}$$

Our purpose is to establish the following result .

Theorem 2.1.1 *Let $\alpha \leq 1$, $s \geq 1$ and $0 \leq f \in L^r(\Omega)$ with $r \geq 1$. Then the problem $(2.1.1)$ has at least one distributional solution u in the sense of the Definition 2.1.1 . Moreover u belongs to $W_0^{1,p}(\Omega) \cap L^{s+1}(\Omega)$ if*

(i) $\alpha = 1$, $f \in L^1(\Omega)$ *or*

(ii) $\alpha < 1$, $f \in L^r(\Omega)$ *for some* $r > 1$ *and* $s \geq \frac{1-r\alpha}{r-1}$ *or*

(iii) $\alpha < 1$, $f \in L^{\frac{s+1}{s+\alpha}}(\Omega)$

while if

(iv) $\alpha < 1$, $f \in L^1(\Omega)$, *then* $u \in W_0^{1,\frac{p(s+\alpha)}{s+1}}(\Omega) \cap L^{s+\alpha}(\Omega)$.

The paper is organized as follows: Section 2.2 is devoted to describing the approximated problems and we prove some properties that we need in the proof of our main results. Finally, Section 2.3, we shall give the complete proof of Theorem 2.1.1.

2.2 Preliminary results

For a fixed $k > 0$, we define the truncation functions $T_k : \mathbb{R} \to \mathbb{R}$ and $G_k : \mathbb{R} \to \mathbb{R}$ as follows

$$T_k(s) := \max(-k; \min(s; k)) \text{ and } G_k(s) := (|s| - k)^+ \text{sign}(s).$$

We will also use the following functions

$$S_{\delta,k}(s) = 1 - V_{\delta,k}(s) \tag{2.2.1}$$

with

$$V_{\delta,k}(s) = \begin{cases} 1 & \text{if} \quad s \leq k, \\ \frac{k+\delta-s}{\delta} & \text{if } k < s < k+\delta, \\ 0 & \text{if } s \geq k+\delta, \end{cases}$$

we will denote with $\mathbb{R}^*$ the set $\mathbb{R}\backslash\{0\}$, with $\mathbb{R}^+$ the set $\{t \in \mathbb{R} \text{ s.t. } t > 0\}$, with r^* the Sobolev conjugate of $1 \leq r < N$, given by $\frac{Nr}{N-r}$, and with $r' = \frac{r}{r-1}$ the Hölder conjugate of $1 < r < \infty$ (if $r = 1$ we define $r' = \infty$, if $r = \infty$ we define $r' = 1$). Moreover, if no otherwise Specified, we will denote by c several positive constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data (for instance c can depend on Ω, α, s, p) but they will never depend on the indexes of the sequences we will introduce.

2.2.1 Approximating problems

Let us consider the following approximating problems,

$$\begin{cases} -\text{div}\left(|\nabla u_{n,k}|^{p-2}\nabla u_{n,k}\right) + T_k(|u_{n,k}|^{s-1}u_{n,k}) = \dfrac{f_n(x)}{(|u_{n,k}| + \frac{1}{n})^{\alpha}} & \text{in} \quad \Omega, \\ u_{n,k} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2.2}$$

where $n, k \in \mathbb{N}$, $0 \leq f_n(x) = T_n(f(x)) \in L^{\infty}(\Omega)$, $\alpha \leq 1$ and $s \geq 1$.

There exists $u_{n,k}$ weak solution to $(2.2.2)$, for each $n, k \in \mathbb{N}$ fixed (see [[94], Theorem 2]). Moreover $u_{n,k} \in L^{\infty}(\Omega)$ for all $n, k \in \mathbb{N}$, since if $m \geq 1$ is fixed, taking $G_m(u_{n,k}) \in W_0^{1,p}(\Omega)$ as test function in $(2.2.2)$ and using that $G_m(u_{n,k})$ and $T_k(|u_{n,k}|^{s-1}u_{n,k})$ have the same sign of $u_{n,k}$, we have taht

$$\int_{\Omega} |\nabla G_m(u_{n,k})|^p \leq \int_{\Omega} f_n G_m(u_{n,k}),$$

and so we can proceed as in [118] to end up with $u_{n,k} \in L^{\infty}(\Omega)$.

Moreover the previous L^{∞} estimate is independent from $k \in \mathbb{N}$.

Now by chossing $u_{n,k}$ as a test function in the weak formulation of $(2.2.2)$, we obtain

$$u_{n,k} \text{ is bounded in } W_0^{1,p}(\Omega) \text{ with respect to } k \text{ for } n \in \mathbb{N} \text{ fixed .}$$

Since $u_{n,k}$ is bounded in $L^{\infty}(\Omega)$ independently on k, for each $n \in \mathbb{N}$ fixed we choose k_n large enough in order to get the following scheme of approximation

$$\begin{cases} -\text{div}\left(|\nabla u_n|^{p-2}\nabla u_n\right) + |u_n|^{s-1}u_n = \dfrac{f_n(x)}{(|u_n| + \frac{1}{n})^{\alpha}} & \text{in} \quad \Omega, \\ u_n = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2.3}$$

where $u_n \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ is given by u_{n,k_n}.

As concerns the sign of u_n, by chosing $u_n^- := \min(u_n, 0) \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ as test function in $(2.2.3)$, we obtain

$$\int_{\Omega} |\nabla u_n^-|^p + \int_{\Omega} |u_n|^{s-1}(u_n^-)^2 = \int_{\Omega} \frac{f_n}{(|u_n| + \frac{1}{n})^{\alpha}} u_n^- \leq 0,$$

and so that $u_n \geq 0$ almost everywhere in Ω.

Now we prove some local positivity property that will guarantee that the limit of the approximations $(2.2.3)$ satisfies $(2.1.6)$.

Proposition 2.2.1 *For each $n \in \mathbb{N}$ fixed, the nonnegative $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ weak solution to $(2.2.3)$ is nondecreasing in $n \in \mathbb{N}$ and it results*

$$\forall w \subset\subset \Omega, \exists c_w > 0 \text{ (independent of } n \in \mathbb{N}) \text{ s.t. } u_n \geq c_w \text{ in } w \; \forall n \in \mathbb{N} \tag{2.2.4}$$

Proof 2.2.1

We can prove that the sequence u_n is nondecreasing in $n \in \mathbb{N}$ proceeding precisely as in [[45], Lemma 2.2], namely taking $(u_n - u_{n+1})^+ := max(u_n - u_{n+1}, 0) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function in the difference between the problem solved by u_n and the one solved by u_{n+1}, so we will omit the details. To prove $(2.2.4)$, we will instead use that

$$u_n \geq u_1 \quad \forall n \in \mathbb{N} \text{ a.e. in } \Omega, \tag{2.2.5}$$

and we will apply the strong maximum principle to $u_1 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, that solves

$$\begin{cases} -\text{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) + u_1^s = \dfrac{f_1(x)}{(u_1+1)^\alpha} \geq \dfrac{f_1(x)}{(\|u_1\|_{L^\infty(\Omega)}+1)^\alpha} \geq 0 & \text{in} \quad \Omega, \\ u_1 = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.2.6}$$

Indeed, since $u_1, \text{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) \in L^1_{loc}(\Omega)$, $u_1 \geq 0$ almost everywhere in Ω,

$$\text{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) \leq u_1^s$$

and

$$\int_0^{\cdot} (t^{s+1})^{-\frac{1}{2}} = \infty \Longleftrightarrow s \geq 1,$$

we can apply [[123],Theorem 1] and deduce that

$$\forall w \subset\subset \Omega, \exists c_w > 0 \text{ s.t. } u_1 \geq c_w \text{ in } w.$$

Then $(2.2.4)$ follows from $(2.2.5)$.

2.2.2 A priori estimates

Now we need some compactness results on the sequence of approximating solutions u_n, at least up to subsequences.

Proposition 2.2.2 *Let $n \in \mathbb{N}$ and $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a solution to (2.2.3) where $s \geq 1$.*

a) If one of the following holds

$$\begin{cases} \alpha = 1,\ f \in L^1(\Omega), \\ \alpha < 1,\ f \in L^r(\Omega) \text{ for some } r > 1 \text{ and } s \geq \frac{1-r\alpha}{r-1}, \\ \alpha < 1,\ f \in L^{\frac{s+1}{s+\alpha}}(\Omega), \end{cases}$$

then u_n is bounded in $W_0^{1,p}(\Omega) \cap L^{s+1}(\Omega)$.

b) If $\alpha < 1$ and $f \in L^1(\Omega)$ then u_n is bounded in $W_0^{1,\frac{p(s+\alpha)}{s+1}}(\Omega) \cap L^{s+\alpha}(\Omega)$.

Proof 2.2.2

a) The first case. Let us take $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function in $(2.2.3)$. We obtain

$$\int_\Omega |\nabla u_n|^p + \int_\Omega u_n^{s+1} \leq \int_\Omega f_n u_n^{1-\alpha}. \tag{2.2.7}$$

- *If $\alpha = 1$, we immediately find that u_n is bounded in $W_0^{1,p}(\Omega)$ and in $L^{s+1}(\Omega)$.*
- *If $\alpha < 1$, we apply Young's inequality with weights $(\epsilon, c(\epsilon))$ and exponents (r, r') on the right hand side of the previous, obtaining*

$$\int_\Omega |\nabla u_n|^p + \int_\Omega u_n^{s+1} \leq \frac{1}{c(\epsilon)} \int_\Omega f_n^r + \epsilon \int_\Omega u_n^{(1-\alpha)r'} \leq \frac{1}{c(\epsilon)} \int_\Omega f^r + \epsilon c \int_\Omega u_n^{s+1}.$$

If ϵ is small enough, we deduce the following estimate

$$\int_\Omega |\nabla u_n|^p + c(\Omega, \epsilon) \int_\Omega u_n^{s+1} dx \leq \frac{1}{c(\epsilon)} \int_\Omega f^r \leq c.$$

- *If $\alpha < 1$ and $f \in L^{\frac{s+1}{s+\alpha}}(\Omega)$, we apply Young's inequality with weights $(\epsilon, c(\epsilon))$ and exponents*

$$\Big(\frac{s+1}{s+\alpha}, \frac{s+1}{1-\alpha}\Big)$$

on the right hand side of $(2.2.7)$. Proceeding as before, we can easily prove the last assertion.

b) Here. by choosing $(u_n + \epsilon)^\alpha - \epsilon^\alpha \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function in $(2.2.3)$, where $0 < \epsilon < \frac{1}{n}$. one has

$$\alpha \int_\Omega |\nabla u_n|^p (u_n+\epsilon)^{\alpha-1} + \int_\Omega u_n^s ((u_n+\epsilon)^\alpha - \epsilon^\alpha) \leq \int_\Omega f_n.$$

Then, we can deduce that

$$\int_\Omega u_n^s ((u_n+\epsilon)^\alpha - \epsilon^\alpha) \leq \int_\Omega f,$$

and by letting $\epsilon \to 0$, we have

$$\int_\Omega u_n^{s+\alpha} \leq \int_\Omega f.$$

It follows that

$$\int_\Omega \frac{|\nabla u_n|^p}{(u_n+\epsilon)^{1-\alpha}} \leq c.$$

Now, if $q < p$, by thanking to Hölder's inequality with exponents $\frac{p}{q}$ and $\frac{p}{p-q}$, we obtain

$$\int_\Omega |\nabla u_n|^q = \int_\Omega \frac{|\nabla u_n|^q}{(u_n+\epsilon)^{(1-\alpha)\frac{q}{p}}} (u_n+\epsilon)^{(1-\alpha)\frac{q}{p}} \leq c\Big(\int_\Omega (u_n+\epsilon)^{\frac{(1-\alpha)q}{p-q}} \Big)^{1-\frac{q}{p}}$$

Now we choose q such that

$$\frac{(1-\alpha)q}{p-q} = s+\alpha.$$

It is not difficult to verify that

$$q = \frac{p(s+\alpha)}{(s+1)} \leq p,$$

which achieve the proof of this proposition.

2.3 Proof of the main results

2.3.1 Proof of Theorem 2.1.1

Proof 2.3.1 *Let u_n be a solution to* (2.2.2)*, then it follows from 2.2.2 that it is bounded in $W_0^{1,p}(\Omega)$ with respect to n. Hence there exists a function $u_p \in W_0^{1,p}(\Omega)$ such that u_n, up to subsequences, converges to u_p in $L^r(\Omega)$ for all $r < \frac{pN}{N-p}$ and weakly in $W_0^{1,p}(\Omega)$. proposition 2.2.2 also gives that $\dfrac{f_n(x)}{(|u_n| + \frac{1}{n})^\alpha}$ is bounded in $L^1_{loc}(\Omega)$ and clearly, $|u_n|^{s-1}u_n$ is bounded in $L^1(\Omega)$ with respect to n. Hence one can apply Theorem 2,1 of [46] which gives that ∇u_n converges to ∇u_p almost everywhere in Ω.*

Now we prove that u_p satisfies 2.1.7 by passing to the limit in n every term in the weak formulation of (2.2.2) *easily pass to the limit the first term in* (2.2.2) *with respect to n; hence we focus on the absorption term u^s, which we show to be equi-integrable.Indeed if we test* (2.2.2) *with $S_{\eta,k}(u_n)$(defined in* (2.2.1) *where $\eta, k > 0$ and we deduce*

$$\int_\Omega |\nabla u|^p S'_{\eta,k}(u_n) + \int_\Omega u_n^s S_{\eta,k}(u_n) \leq sup_{s\in[k,\infty)} \frac{1}{s^\alpha} \int_\Omega f_n S_{\eta,k}(u_n).$$

Which, observing that the first term on the left hand side is nonnegative and taking the limit with respect to $\eta \to 0$,implies

$$\int_{\{u_n \geq k\}} u_n^s S_{\eta,k}(u_n) \leq sup_{s\in[k,\infty)} \frac{1}{s^\alpha} \int_{\{u_n \geq k\}} f_n. \tag{2.3.1}$$

Which, since f_n converges to f in $L^r(\Omega), r \geq 1$ easily implies that u_n^s is equi-integrable and so it converges to u_p^s in $L^1(\Omega)$.This is sufficient to pass to the limit in the second term of the weak formulation of (2.2.2)*. For what concerns the right hand side, using (2.2.4), we find*

$$\Big|\frac{f_n\varphi}{(u_n + \frac{1}{n})^\alpha}\Big| \leq \Big|\frac{f\varphi}{c^\alpha_{supp\varphi}}\Big| \quad \forall \varphi \in \mathcal{C}^1_c(\Omega).$$

Then, thanks to Lebesgue Theorem, we can pass to the limit also in the right hand side of the distributional formulation of (2.2.3). This concludes the proof.

2.3.2 Some comments on the regularizing effect

Firstly, it is easy to verify that, if

$$\alpha < 1 \text{ and } s > \frac{N+p}{N-p} \tag{2.3.2}$$

then

$$\tfrac{s+1}{s+\alpha} < \Big(\frac{p^*}{1-\alpha}\Big)'$$

Since $f \in L^{\Big(\frac{p^*}{1-\alpha}\Big)'}(\Omega)$ is the weaker assumption on the datum in order to find a priori estimates in

$W_0^{1,p}(\Omega)$ for the sequence of approximating solutions to problem below:

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = \dfrac{f(x)}{u^{\alpha}} & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.3.3}$$

it follows that, if we add the term u^s, with s satisfying (2.3.2), in the left hand side of $(2.3.3),$ we find a priori estimates in $W_0^{1,p}(\Omega)$ for the sequence of approximating solutions also for less regular data.

Furthermore, if $f \in L^1(\Omega)$ and $\alpha < 1$, the Sobolev space in which the sequence of approximating solutions to (2.3.3) is bounded is given by $W_0^{1,\frac{N(\alpha+1)}{N-(1-\alpha)}}(\Omega)$ (see [56, 60, 68]).

It is easy to verify that, if

$$\alpha < 1 \text{ and } s > \frac{N+\alpha p}{N-p}, \tag{2.3.4}$$

then

$$\tfrac{N(\alpha+1)}{N-(1-\alpha)} < \tfrac{p(s+\alpha)}{s+1}.$$

So we have another regularizing effect of the lower order term u^s, with s such that (2.3.4) holds, on the a priori estimates for the approximating solutions.

Finally we recall that, if $f \in L^1(\Omega)$ and $s > \frac{N}{N-p}$, then the sequence of approximating solutions to the following problem:

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) + u^s = f(x) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.3.5}$$

is bounded in $W_0^{1,q}(\Omega)$ for all $q \in [1, \frac{ps}{(s+1)})$ (see [94]). Since

$$\frac{ps}{(s+1)} < \frac{p(s+\alpha)}{(s+1)} \Longleftrightarrow \alpha > 0,$$

we immediately obtain the, if we perturb the right hand side of (2.3.5) through the singular term $\frac{1}{u^{\alpha}}$ with $\alpha > 0$, we find a priori estimates on the sequence of approximating solutions.

Chapter 3

Semilinear elliptic problem with singular lower order terms

Abstract

The aim of this chapter is to discuss the existence of solution to a more general case associated to the following boundary value problem such as:

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) + u^s = \dfrac{f(x)}{u^\alpha} & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.0.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$ $(N > p)$, $0 < \alpha \leq 1$, $s \geq 1$ and f is a nonnegative function that belongs to some Lebesgue space.

3.1 Introduction and Assumptions

3.1.1 Introduction

In this chapter we establish the existence positive solutions to a more general case for the following elliptic problem

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) + u^s = \dfrac{f(x)}{u^\alpha} & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$ $(N > p)$, $0 < \alpha \leq 1$, $s \geq 1$ and f is just a nonnegative $L^1(\Omega)$ function. The presence of lower order terms u^s in elliptic problems of type (3.1.1) is quite important in applications; usually, a lower order term represents an absorption or a reaction, depending on its sign.

Problem (3.1.1) has been widely applied in many areas such as the contexts of chemical heterogeneous catalysts, non-Newtonian fluids and also the theory of heat conduction in electrically conducting materials, see [101, 20] for detailed discussion. In this work, we are dealing with absorption zero order terms, that usually have a regularizing effect on the solutions to (3.1.1), by starting from measure data for regularity results on the Lebesgue scale (in [44, 57]) and on the Marcinkiewicz one ([43]). We refer the reader to [106, 107] for another approach using results on elliptic and parabolic problems in the setting of Sobolev spaces. See also [11] for related topics.

In [44] the authors investigate the regularizing effect of the term us on the solution to the following classical problem

$$\begin{cases} \Delta u = f(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1.2}$$

when the term u^s is added in the left hand side of (3.1.2), namely when we consider the following

$$\begin{cases} \Delta u + u^s = f(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1.3}$$

In [45] the authors study the following problem

$$\begin{cases} \Delta u = \frac{f(x)}{u^\gamma} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1.4}$$

Recently in [67] the authors study the following problem when the addition of a singular sourcing term to a problem with only an absorption term and the addition of an absorption term to a problem with

only a singular sourcing term, will improve the regularity of the solution the following problem

$$\begin{cases} \Delta u + u^s = \frac{f(x)}{u^\gamma} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1.5}$$

3.1.2 Assumptions

The aim of this paper, is to study a more general case for problem (3.1.1), such as:

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) + \psi(u) = g(u)f(x) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1.6}$$

The absorption term $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is a nondecreasing and continuous function on $\mathbb{R}$ satisfying the assumptions listed below :

$$\psi(t)t \geq 0 \text{ for all } t \in \mathbb{R}, \tag{3.1.7}$$

$$\int_0 (\psi(t)t)^{-\frac{1}{2}} = \infty, \tag{3.1.8}$$

$$\psi(t) \geq t^s \text{ for all } t \geq 0 \text{ and for some } s \geq 1. \tag{3.1.9}$$

The singular souring term $g : \mathbb{R}^* \longrightarrow \mathbb{R}^+$ is a non increasing function on $\mathbb{R}^*$, continuous and bounded on $\mathbb{R}^*$ such that

$$\lim_{t \longrightarrow 0} g(t) = +\infty, \tag{3.1.10}$$

and such that

$$\text{there exist } c > 0, \delta^{'} > \delta > 0 \text{ and } \alpha, \eta \in (0,1] \text{ s.t. } \begin{cases} g(t) \leq \frac{c}{t^\eta} & \text{if} \quad t < \delta \\ g(t) \leq \frac{c}{t^\alpha} & \text{if} \quad t > \delta^{'}. \end{cases} \tag{3.1.11}$$

We take the following example for the function ψ :

$$\psi(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ t^s & \text{if} \quad 0 \leq t \leq 1 \\ t^{s+1} & \text{if} \quad t \geq 1, \end{cases}$$

while the function g is a general singular non increasing function with two different behaviors near zero and near infinite.

The plan of the paper is as follows : Section 3.2 is devoted to describe the approximated problems and we prove some properties. In section 3.3, we shall give the our main results.

3.2 Preminary results

Let us consider the approximating problem:

$$\begin{cases} -\text{div}\left(|\nabla u_{n,k}|^{p-2}\nabla u_{n,k}\right) + T_k(|u_{n,k}|^{s-1}u_{n,k}) = \dfrac{f_n(x)}{(|u_{n,k}| + \frac{1}{n})^{\alpha}} & \text{in} \quad \Omega, \\ u_{n,k} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.2.1}$$

where $n, k \in \mathbb{N}$, $0 \leq f_n(x) = T_n(f(x)) \in L^{\infty}(\Omega)$, $\alpha \leq 1$ and $s \geq 1$.

There exists $u_{n,k} \in L^{\infty}(\Omega)$ a weak solution(see [[94], Théorème 2]) to $(3.2.1)$, for each $n, k \in \mathbb{N}$ fixed.

Now we choose : $G_m(u_{n,k}) \in W_0^{1,p}(\Omega)$ with $m \geq 1$ is fixed , as test function in $(3.2.1)$ and since $G_m(u_{n,k})$ and $T_k(|u_{n,k}|^{s-1}u_{n,k})$ have the same sign of $u_{n,k}$, we obtain that

$$\int_{\Omega} |\nabla G_m(u_{n,k})|^p \leq \int_{\Omega} f_n G_m(u_{n,k}),$$

and by the same argument introduced in [118] we find that $u_{n,k} \in L^{\infty}(\Omega)$.then the L^{∞} estimate is independent from $k \in \mathbb{N}$.

Let us choose $u_{n,k}$ as a test function in the weak formulation of $(3.2.1)$, we can obtain :

$$u_{n,k} \text{ is bounded in } W_0^{1,p}(\Omega) \text{ for } n, k \in \mathbb{N} \text{ fixed}$$

Since $u_{n,k}$ is bounded in $L^{\infty}(\Omega)$ independently on k , for each $n \in \mathbb{N}$ fixed we can take k_n very large in order to obtain the following approximation

$$\begin{cases} -\text{div}\left(|\nabla u_n|^{p-2}\nabla u_n\right) + |u_n|^{s-1}u_n = \dfrac{f_n(x)}{(|u_n| + \frac{1}{n})^{\alpha}} & \text{in} \quad \Omega, \\ u_n = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.2.2}$$

where $u_n \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ is given by u_{n,k_n}.

Let us take $u_n^- := \min(u_n, 0) \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ as test function in $(1.2.1)$, we obtain

$$\int_{\Omega} |\nabla u_n^-|^p + \int_{\Omega} |u_n|^{s-1}(u_n^-)^2 = \int_{\Omega} \frac{f_n}{(|u_n| + \frac{1}{n})^{\alpha}} u_n^- \leq 0,$$

and so that $u_n \geq 0$ almost everywhere in Ω.

Now we prove the following proposition in order that the limit of the approximations $(3.3.3)$ satisfies $(5.3.1)$.

Proposition 3.2.1 *For each $n \in \mathbb{N}$ fixed, the nonnegative $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ weak solution to $(3.2.2)$ is nondecreasing in $n \in \mathbb{N}$ and it results*

$$\forall w \subset\subset \Omega, \exists c_w > 0 \text{ (independent of } n \in \mathbb{N}) \text{ s.t. } u_n \geq c_w \text{ in } w \; \forall n \in \mathbb{N} \tag{3.2.3}$$

3.2.1 proof of proposition 3.2.1

By the same argument as in [[45], Lemma 2.2],the sequence u_n is nondecreasing in $n \in \mathbb{N}$ we choose $(u_n - u_{n+1})^+ := \max(u_n - u_{n+1}, 0) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function in the difference between the problem solved by u_n and the one solved by u_{n+1}, .In order to prove $(3.2.3)$, we can use:

$$u_n \geq u_1 \forall n \in \mathbb{N} \text{ a.e. in } \Omega \tag{3.2.4}$$

and by applying the strong maximum principle to $u_1 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, that is a solution of

$$\begin{cases} -\text{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) + u_1^s = \dfrac{f_1(x)}{(u_1+1)^\alpha} \geq \dfrac{f_1(x)}{(\|u_1\|_{L^\infty(\Omega)}+1)^\alpha} \geq 0 & \text{in} \quad \Omega, \\ u_1 = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.2.5}$$

Since $u_1, \text{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) \in L^1_{loc}(\Omega)$, $u_1 \geq 0$ almost everywhere in Ω,

$$\text{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) \leq u_1^s$$

and

$$\int_0^{\cdot} (t^{s+1})^{-\frac{1}{2}} = \infty \Longleftrightarrow s \geq 1,$$

By applying [[123],Theorem 1] we can obtain

$$\forall w \subset\subset \Omega, \exists c_w > 0 \text{ s.t. } u_1 \geq c_w \text{ in } w.$$

Then $(3.2.3)$ follows from $(3.2.4)$.

3.2.2 A priori estimates

Now we need some thechnical results on the sequence of approximating solutions

Proposition 3.2.2 *Let $n \in \mathbb{N}$ and $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a solution to (3.2.2) where $s \geq 1$.*

i) If one of the following holds

$$\begin{cases} \alpha = 1,\ f \in L^1(\Omega), \\ \alpha < 1,\ f \in L^r(\Omega) \text{ for some } r > 1 \text{ and } s \geq \frac{1-r\alpha}{r-1}, \\ \alpha < 1,\ f \in L^{\frac{s+1}{s+\alpha}}(\Omega), \end{cases}$$

then u_n is bounded in $W_0^{1,p}(\Omega) \cap L^{s+1}(\Omega)$.

ii) If instead $\alpha < 1$ and $f \in L^1(\Omega)$ then u_n is bounded in $W_0^{1,\frac{p(s+\alpha)}{s+1}}(\Omega) \cap L^{s+\alpha}(\Omega)$.

3.2.3 proof of proposition 3.2.2

i) The first case. Let us take $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function in $(3.2.2)$. We obtain

$$\int_\Omega |\nabla u_n|^p + \int_\Omega u_n^{s+1} \leq \int_\Omega f_n u_n^{1-\alpha}. \tag{3.2.6}$$

- If $\alpha = 1$, we immediately find that u_n is bounded in $W_0^{1,p}(\Omega)$ and in $L^{s+1}(\Omega)$.
- If $\alpha < 1$, we apply Young's inequality with weights $(\epsilon, c(\epsilon))$ and exponents (r, r') on the right hand side of the previous, obtaining

$$\int_\Omega |\nabla u_n|^p + \int_\Omega u_n^{s+1} \leq \frac{1}{c(\epsilon)} \int_\Omega f_n^r + \epsilon \int_\Omega u_n^{(1-\alpha)r'} \leq \frac{1}{c(\epsilon)} \int_\Omega f^r + \epsilon c \int_\Omega u_n^{s+1}.$$

  If ϵ is small enough, we deduce the following estimate

$$\int_\Omega |\nabla u_n|^p + c(\Omega, \epsilon) \int_\Omega u_n^{s+1} dx \leq \frac{1}{c(\epsilon)} \int_\Omega f^r \leq c.$$

- If $\alpha < 1$ and $f \in L^{\frac{s+1}{s+\alpha}}(\Omega)$, we apply Young's inequality with weights $(\epsilon, c(\epsilon))$ and exponents

$$\Big(\frac{s+1}{s+\alpha}, \frac{s+1}{1-\alpha}\Big)$$

  on the right hand side of $(3.2.6)$. Proceeding as before, we can easily prove the last assertion.

ii) The second case. Let us take $(u_n+\epsilon)^\alpha-\epsilon^\alpha \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ as test function in $(3.2.2)$, where $0<\epsilon<\frac{1}{n}$. We obtain

$$\alpha\int_\Omega |\nabla u_n|^p(u_n+\epsilon)^{\alpha-1}+\int_\Omega u_n^s((u_n+\epsilon)^\alpha-\epsilon^\alpha)\le\int_\Omega f_n.$$

and so, in particular, we deduce that

$$\int_\Omega u_n^s((u_n+\epsilon)^\alpha-\epsilon^\alpha)\le\int_\Omega f,$$

that, letting $\epsilon\to 0$, implies

$$\int_\Omega u_n^{s+\alpha}\le\int_\Omega f.$$

Moreover we deduce

$$\int_\Omega \frac{|\nabla u_n|^p}{(u_n+\epsilon)^{1-\alpha}}\le c.$$

Now, if $q<p$, applying Hölder's inequality with exponents $\frac{p}{q}$ and $\frac{p}{p-q}$, we find

$$\int_\Omega |\nabla u_n|^q=\int_\Omega \frac{|\nabla u_n|^q}{(u_n+\epsilon)^{(1-\alpha)\frac{q}{p}}}(u_n+\epsilon)^{(1-\alpha)\frac{q}{p}}\le c\Big(\int_\Omega (u_n+\epsilon)^{\frac{(1-\alpha)q}{p-q}}\Big)^{1-\frac{q}{p}}$$

Finally we choose q such that

$$\frac{(1-\alpha)q}{p-q}=s+\alpha.$$

It is easy to verify that

$$q=\frac{p(s+\alpha)}{(s+1)}\le p,$$

and this gives us the result.

3.3 Main results

Let us start giving our definition of distributional solution to problem (3.1.6) :

Definition 3.3.1 *Let $\psi:\mathbb{R}\longrightarrow\mathbb{R}$ be a nondecreasing and continuous function satisfying (3.1.7), (3.1.8) and (3.1.9) and let $g:\mathbb{R}^*\longrightarrow\mathbb{R}^+$ be a non increasing function on $\mathbb{R}^*$, continuous and bounded on $\mathbb{R}^*$ satisfying (3.1.10) and (3.1.11). Under these assumptions a function $u\in W_0^{1,p}(\Omega)$ is a distributional*

solution to problem (3.1.6), with $f \in L^r(\Omega)$ and $r \geq 1$, if

$$\forall w \subset\subset \Omega \text{ exist } c_w > 0 \text{ s. t. } u \geq c_w \text{ a. e. in } w, \psi(u) \in L^1(\Omega) \tag{3.3.1}$$

and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx + \int_\Omega \psi(u)\varphi dx = \int_\Omega g(u) f \varphi dx \ \ \forall \varphi \in C_0^\infty(\Omega). \tag{3.3.2}$$

Remark 3.3.1 *The property (3.3.1) implies in particular that $g(u)f \in L^1_{loc}(\Omega)$, in the case with monotone singularity given by $g(u) = \frac{1}{u^\alpha}$ $(\alpha > 0)$. Consequently, the right hand side of (3.3.2) has a mining.*

Now our aim is to prove the following result.

Theorem 3.3.1 *Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be a nondecreasing and continuous function satisfying (3.1.7), (3.1.8) and (3.1.9) and let $g : \mathbb{R}^* \longrightarrow \mathbb{R}^+$ be a non increasing function on $\mathbb{R}^*$, continuous and bounded on $\mathbb{R}^*$, satisfying (3.1.10) and (3.1.11). If $f \in L^r(\Omega)$ $(r \geq 1)$, then there exists a distributional solution u to (3.1.6) then u belongs to $W_0^{1,p}(\Omega) \cap L^{s+1}(\Omega)$*

i) $\eta,\ \alpha = 1,\ f \in L^1(\Omega)$ *or*

ii) $\eta,\ \alpha < 1,\ f \in L^r(\Omega)$ *for some* $r > 1$ *and* $s \geq \frac{1-r\alpha}{r-1}$ *or*

iii) $\eta,\ \alpha < 1,\ f \in L^{\frac{s+1}{s+\alpha}}(\Omega)$*, while if*

iv) $\eta < \alpha < 1$ *and* $f \in L^1(\Omega)$,

then $u \in W_0^{1,\frac{p(s+\alpha)}{s+1}}(\Omega) \cap L^{s+\alpha}(\Omega)$.

3.3.1 proof of theorem 3.3.1

Let the following scheme of approximation for problem (3.1.6)

$$\begin{cases} -\text{div}\left(|\nabla u_{n,k}|^{p-2}\nabla u_{n,k}\right) + T_k(\psi(u_{n,k})) = g_n(u_{n,k}) f_n(x) & \text{in} \quad \Omega, \\ u_{n,k} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.3.3}$$

where $n,\ k \in \mathbb{N}$, $0 \leq f_n(x) = T_n(f(x)) \in L^\infty(\Omega)$ and $g_n(x) = T_n(g(x))$ there exists $u_{n,k} \in W_0^{1,p}(\Omega)$ weak solution to (3.3.3) for each $n,\ k \in \mathbb{N}$ fixed (see [94]). Now, we show that $u_{n,k} \in L^\infty(\Omega)$ for all $n,\ k \in \mathbb{N}$ and that this L^∞-estimate is independent on k and taking $u_{n,k}$ as a test function in the weak

formulation of the previous, we find once again that $u_{n,k}$ is bounded in $W_0^{1,p}(\Omega)$ with respect to k for $n \in \mathbb{N}$ fixed. Since $u_{n,k}$ is bounded in $L^\infty(\Omega)$ independently on k, as already done for problem (3.1.4) we choose k large enough to obtain the following scheme of approximation

$$\begin{cases} -\text{div}\left(|\nabla u_n|^{p-2}\nabla u_n\right) + \psi(u_n) = g_n(u_n)f_n(x) & \text{in} \quad \Omega, \\ u_n = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.3.4}$$

where $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is given by a certain $u_{n,k}$ for a k large enough.

Furthermore, analogously to what done in Proposition 3.2.1, we can to prove that the sequence u_n is nonnegative and nondecreasing in $n \in \mathbb{N}$.

In order to prove the local positivity of u_n, we will use, the strong maximum principle applied to $u_1 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ that solves

$$\begin{cases} -\text{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) + \psi(u_1) = g_1(u_1)f_1 \geq g_1(\|u_1\|_{L^\infty(\Omega)})f_1 & \text{in} \quad \Omega, \\ u_1 = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

thanks to (3.1.8) we apply ([123], Theorem 1) and deduce that

$$\forall w \subset\subset \Omega,\ \exists c_w > 0,\ s.t.\ u_1 \geq c_w\ a.e.\ \text{in}\ w.$$

i) In this case, we take u_n as a test function in the weak formulation of (3.3.4), we deduce that :

$$\begin{aligned} &\int_\Omega |\nabla u_n|^p dx + \int_\Omega u_n^{s+1} dx \leq \int_\Omega |\nabla u_n|^p + \int_\Omega \psi(u_n)u_n \\ &\leq \int_{\{u_n \leq \delta\}} f_n + \int_{\{\delta < u_n < \delta'\}} g_n(u_n)f_n u_n + \int_{\{u_n \geq \delta'\}} f_n \\ &\leq 2c\int_\Omega f + \delta^{'}|g|_{L^\infty(\delta,\delta')}\int_\Omega f. \end{aligned}$$

Then u_n is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and we have no problems in passing to the limit in the approximation (3.3.4), with $C_0^\infty(\Omega)$ test functions, we use Lebesgue Theorem for the singular sourcing term and monotone convergence Theorem for the absorption term.

ii) If $\eta,\ \alpha < 1$, let u_n as test function and we apply Young's inequality with weights $(\epsilon; c(\epsilon))$ and exponents $(r, r^{'})$ on the last term of the right hand side of the weak formulation, obtaining

$$\begin{aligned} &\int_\Omega |\nabla u_n|^p + \int_\Omega u_n^{s+1} \leq \int_\Omega |\nabla u_n|^p + \int_\Omega \psi(u_n)u_n \\ &\leq c\int_{\{u_n \leq \delta\}} f_n u_n^{1-\eta} + \delta^{'}\int_{\delta < u_n < \delta'} g_n(u_n)f_n u_n + c\int_{\{u_n \geq \delta'\}} f_n u_n^{1-\alpha} \\ &\leq c\delta^{1-\eta}\int_\Omega f + \delta^{'}|g|_{L^\infty(\delta,\delta')}\int_\Omega f + \frac{1}{c(\epsilon)}\int_\Omega f_n^r + c\epsilon\int_\Omega u_n^{(1-\alpha)r^{'}} \\ &\leq c\delta^{1-\eta}\int_\Omega f + \delta^{'}|g|_{L^\infty(\delta,\delta')}\int_\Omega f + \frac{1}{c(\epsilon)}\int_\Omega f^r + c\epsilon\int_\Omega u_n^{s+1}. \end{aligned}$$

Thus, if ϵ is small enough, we have

$$\int_\Omega |\nabla u_n|^p + c\int_\Omega u_n^{s+1} \leq c. \tag{3.3.5}$$

As in case **i)**, we can pass to the limit the approximation scheme (3.3.4) in order to prove the existence of a solution $u \in W_0^{1,p}(\Omega) \cap L^{s+1}(\Omega)$ to problem (3.1.6).

iii) In this case we proceed in a similarly to what done for the case **ii)**, we have

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^p + \int_\Omega u_n^{s+1} \leq \int_\Omega |\nabla u_n|^p + \int_\Omega \psi(u_n)u_n \\
&\leq c\int_{\{u_n\leq\delta\}} f_n u_n^{1-\eta} + \delta' \int_{\delta<u_n<\delta'} g_n(u_n)f_n + c\int_{\{u_n\geq\delta'\}} f_n u_n^{1-\alpha} \\
&\leq c\delta^{1-\eta}\int_\Omega f + \delta' \|g\|_{L^\infty(\delta,\delta')}\int_\Omega f + \frac{1}{c(\epsilon)}\int_\Omega f_n^{\frac{s+1}{s+\alpha}} + c\epsilon\int_\Omega u_n^{s+1}.
\end{aligned}$$

And so we deduce (3.3.5) once again and to obtain a solution $u \in W_0^{1,p}(\Omega) \cap L^{s+1}(\Omega)$ to (3.1.6)

iv) In this case we take $(u_n+\epsilon)^\alpha - \epsilon^\alpha \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ as test function in (3.3.4), where $0 < \epsilon < \frac{1}{n}$. We have

$$\begin{aligned}
&\alpha\int_\Omega |\nabla u_n|^p (u_n+\epsilon)^{\alpha-1} + \int_\Omega u_n^s((u_n+\epsilon)^\alpha - \epsilon^\alpha) \\
&\leq \alpha\int_\Omega |\nabla u_n|^p (u_n+\epsilon)^{\alpha-1} + \int_\Omega \psi(u_n)((u_n+\epsilon)^\alpha - \epsilon^\alpha) \\
&\leq c\int_{\{u_n\leq\delta\}} \frac{f_n(u_n+\epsilon)^\alpha}{u_n^\eta} + \int_{\delta<u_n<\delta'} g_n(u_n)f_n(u_n+\epsilon)^\alpha + c\int_{\{u_n\geq\delta'\}} \frac{f_n(u_n+\epsilon)^\alpha}{u_n^\alpha}
\end{aligned}$$

and so we have in particular

$$\begin{aligned}
&\int_\Omega u_n^s((u_n+\epsilon)^\alpha - \epsilon^\alpha) \\
&\leq c\int_{\{u_n\leq\delta\}} \frac{f_n(u_n+\epsilon)^\alpha}{u_n^\eta} + \int_{\delta<u_n<\delta'} g_n(u_n)f_n(u_n+\epsilon)^\alpha + c\int_{\{u_n\geq\delta'\}} \frac{f_n(u_n+\epsilon)^\alpha}{u_n^\alpha}
\end{aligned}$$

that, letting $\epsilon \longrightarrow 0$, implies

$$\int_\Omega u_n^{s+\alpha} \leq c\delta^{\alpha-\eta} + c.$$

Furthermore, we can reason as in proof case **ii)** of Proposition 3.2.2 obtaining that u_n is bounded in $W_0^{1,\frac{p(s+\alpha)}{s+1}}(\Omega)$ with respect to n. From here we conclude as in case **i)** of the current theorem. This concludes the proof of Theorem 3.3.1.

Part II

On some nonlinear elliptic problems in weighted Orlicz-Sobolev spaces

Chapter 4

On a nonlinear elliptic problems having large monotonocity with L^1-data in weighted Orlicz-Sobolev spaces

Abstract

In this chapter,We prove in weighted Orlicz-Sobolev spaces, the existence of entropy solution for a class of nonlinear elliptic equations of Leray-Lions type, with large monotonicity condition and right hand side $f \in L^1(\Omega)$

4.1 Introduction.

This chapter deals with existence of solutions to the following nonlinear Dirichlet problem

$$\begin{cases} A(u) = f(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.1.1}$$

where $A(u) = -\text{div}\ (\rho(x)a(x,u,\nabla u))$, Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$. $a(x,u,\nabla u) = (a_i(x,u,\nabla u))_{1\leq i\leq N}$, $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory functions (that is measurable with respect to x in Ω for every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ for almost every $x \in \Omega$) such that for all ξ, ξ' in $\mathbb{R}^N$,

$$|a_i(x,s,\xi)| \leq |\phi_i(x)| + K_i \overline{P}^{-1}(\rho^{-1}(x) M(c_2|s|)) + K_i \overline{M}^{-1} M(c_1 |\xi|), \tag{4.1.2}$$

$$\Big(a(x,s,\xi) - a(x,s,\xi^{'})\Big)(\xi - \xi^{'}) \geq 0, \tag{4.1.3}$$

$$a(x,s,\xi)\xi \geq M(\lambda_1|\xi|), \tag{4.1.4}$$

where c_1, c_2, λ_1 and K_i belongs to $\mathbb{R}_+$. M, P be two N-functions such that $P \prec\prec M$. Moreover $\overline{M}$, $\overline{P}$ be the complementary functions of M and P respectively, ρ be a weight function on Ω (that is, measurable and positive a.e. on Ω) and ϕ_0, $\phi_i \in E_{\overline{M}}(\Omega, \rho)$.

$$f \in L^1(\Omega), \tag{4.1.5}$$

It is well known that in the setting of weighted Orlicz–Sobolev spaces include many spaces as special spaces, such as Lebesgue spaces, weighted Lebesgue spaces and Orlicz spaces; see [104]. These spaces have many applications in various fields such as PDE, electrorheological fluids, and image restoration; see [69, 58, 90].

The feature of this paper, is to treat a class of problems for which the classical monotone operator methods (developed by Minty [100], Browder [55], Brézis [54] and Lions [91] in $W_0^{1,p}(\Omega)$ case) do not apply. The reason for this, is that $a(x,u,\nabla u)$ does not need to satisfy the strict monotonicity condition that is,

$$\Big(a(x,s,\xi) - a(x,s,\xi')\Big)(\xi - \xi') > 0, \text{ for all } \xi, \xi' \in \mathbb{R}^N, \ (\xi \neq \xi')$$

of a typical Leray–Lions operator but only a large monotonicity that is

$$\Big(a(x,s,\xi) - a(x,s,\xi')\Big)(\xi - \xi') \geq 0, \text{ for all } \xi, \xi' \in \mathbb{R}^N.$$

The aim of this paper is to demonstrate the existence of solutions for (4.1.1) under the weaker assumption large monotonicity condition, without using the almost everywhere convergence of the gradients of the approximate equations, since this is impossible to prove in our setting. The main tools of our proof are a version of Minty's Lemma (Now we use an idea of G. J. Minty [103]).

The chapter is organized as follows: We introduce some basic definitions and properties in the setting of weighted Orlicz–Sobolev spaces as well as an abstract theorem and we prepare some auxiliary results to prove our theorem, in the next section 6.2. In the final section 5.3, we state the main result and proofs.

4.2 Preliminary

This section present, some definitions and well-known about N-functions, weighted Orlicz-Sobolev spaces (standard references see [1] and [71].

4.2.1 N-function

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with

$$M(t) > 0 \text{ for } t > 0, \quad \tfrac{M(t)}{t} \to 0 \text{ as } t \to 0 \text{ and } \tfrac{M(t)}{t} \to \infty \text{ as } t \to \infty.$$

Equivalently M admits the representation :

$$M(t) = \int_0^t m(\tau) d\tau,$$

where $m : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing right continuous, with $m(0) = 0$, $m(t) > 0$ for $t > 0$ and $m(t) \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M defined by

$$\overline{M}(t) = \int_0^t \overline{m}(\tau) d\tau,$$

where $\overline{m} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{m}(t) = \sup\{s : m(s) \leq t\}$. It is well known that we can assume that m and $\overline{m}$ are continuous and strictly increasing. We will extend the N-functions into even function on all $\mathbb{R}^+$.

Clearly $\overline{\overline{M}} = M$ and has Young's inequality

$$st \leq M(t) + \overline{M}(s) \text{ for all } s, t \geq 0.$$

The N-function M is said to satisfy the Δ_2-condition every-where (resp. infinity) if there exist $k > 0$ (resp. $t_0 > 0$) such that, $M(2t) \leq kM(t)$ for all $t \geq 0$ (resp. $t \geq t_0$).

Let P and Q be two N-functions, the notation $P \prec\prec Q$ means that P grows essentially less rapidly than Q, that is to say for all $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to +\infty$. That is the case if and only if $\frac{Q^{-1}(t)}{P^{-1}(t)} \to 0$ as $t \to \infty$.

4.2.2 Orlicz-Sobolev space.

Let Ω be an open subset of $\mathbb{R}^N$ and M be an N-function.

The Orlicz classe $K_M(\Omega)$ (resp. the Orlicz spaces $L_M(\Omega)$) is the set of all (equivalence classes modulo equality a.e. in Ω of) real-valued measurable functions u defined in Ω and satisfying

$$\int_\Omega M(u(x))dx < \infty \Big(\text{resp. } \int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)dx < \infty \text{ for some } \lambda > 0\Big).$$

$L_M(\Omega)$ is a Banach space under the norm :

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)dx \le 1\right\}. \tag{4.2.1}$$

The closure in $L_M(\Omega)$ of the set of bounded measurable function with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$ (we have usual $E_M(\Omega) \subset K_M(\Omega) \subset L_M(\Omega)$).

The equality $L_M(\Omega) = E_M(\Omega)$ hold if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has a infinite measure or note .

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$ where $u \in L_M(\Omega)$ and $v \in L_{\overline{M}}(\Omega)$ and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$.

The space $L_M(\Omega)$ is reflexive if and only if M an $\overline{M}$ satisfy the Δ_2-condition for all t or for t large, according to whether Ω be infinite measure or note.

We return now to the Orlicz-Sobolev spaces $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all function u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. in $E_M(\Omega)$). It's Banach space under the norm :

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_{M,\Omega}. \tag{4.2.2}$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of $\prod L_M$ we have the weak topology $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ (resp. $W_0^1L_M(\Omega)$) is defined by the closure of $D(\Omega)$ in $W^1E_M(\Omega)$ (resp. $W^1L_M(\Omega)$) for the norm (4.2.2) (resp. for the topology $\sigma(\prod L_M, \prod E_{\overline{M}})$).

Definition 4.2.1 *The sequence u_n converges to u in $L_M(\Omega)$ for the modular convergence (denoted by $u_n \to u$ (mod) $L_M(\Omega)$) if $\int_\Omega M\Big(\frac{|u_n - u|}{\lambda}\Big)dx \to 0$ as $n \to \infty$ for some $\lambda > 0$.*

4.2.3 Weighted Orlicz–Sobolev space

Let Ω be a domain in $\mathbb{R}^N$, M be an N-function and $\rho(x)$ be a weight function on Ω, i.e. measurable positive a.e. on Ω such that :

$$\rho \in L^1(\Omega) \tag{4.2.3}$$

The weighted Orlicz classe $K_M(\Omega,\rho)$ (resp. the weighted Orlicz space $L_M(\Omega,\rho)$) is the set of all (equivalence classes modulo equality a.e. in Ω) of real-valued measurable functions u defined in Ω and satisfying $m_\rho(u,M)=\int_\Omega M(|u(x)|)\rho(x)dx<\infty$

$$\Big(\text{resp. } m_\rho\Big(\frac{u}{\lambda},M\Big)=\int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)\rho(x)dx<\infty \text{ for some } \lambda>0\Big).$$

$L_M(\Omega,\rho)$ is a Banach space under the norm :

$$\|u\|_{M,\rho}=\inf\Big\{\lambda>0; m_\rho\Big(\frac{u}{\lambda},M\Big)\le 1\Big\}. \tag{4.2.4}$$

The closure in $L_M(\Omega,\rho)$ of the set of bounded measurable function with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega,\rho)$ (we have usual $E_M(\Omega,\rho)\subset K_M(\Omega,\rho)\subset L_M(\Omega,\rho)$).

The equality $L_M(\Omega,\rho)=E_M(\Omega,\rho)$ hold if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has a infinite mesure or note.

Remmember that spaces $L_M(\Omega,\rho)$ and $L_{\overline{M}}(\Omega,\rho)$ are naturally associated by duality

$$\langle u,v\rangle=\int_\Omega u(x)v(x)\rho(x)dx,\ u\in L_M(\Omega,\rho), v\in L_{\overline{M}}(\Omega,\rho)$$

giving what's called the orlicz norm on $L_M(\Omega,\rho)$, which is

$$|||u|||_{M,\rho}=\sup\{\int_\Omega u(x)v(x)\rho(x)dx, m_\rho(v,\overline{M})\le 1\}$$

can be identify the dual of $E_M(\Omega,\rho)$ with $L_{\overline{M}}(\Omega,\rho)$ (see[34])
It is easy to prove (as in the case of non-weighted spaces) that

$$\|u\|_{M,\rho}\le|||u|||_{M,\rho}\le 2\|u\|_{M,\rho}$$

The space $L_M(\Omega,\rho)$ is reflexive if and only if M an $\overline{M}$ satisfy the Δ_2-condition for all t or for t large according to whether Ω be infinite measure or note.

We return now to the weighted Orlicz-Sobolev spaces $W^1L_M(\Omega,\rho)$ (resp. $W^1E_M(\Omega,\rho)$) is the space of all function u such that $u \in L_M(\Omega)$ (resp. $u \in E_M(\Omega)$) and its distributional derivatives up to order 1 lie in $L_M(\Omega,\rho)$ (resp. $E_M(\Omega,\rho)$). It's Banach space under the norm

$$\|u\|_{1,M,\rho} = \|u\|_M + \|\nabla u\|_{M,\rho}\,, \tag{4.2.5}$$

(where $\|u\|_M = \|u\|_{M,\Omega}$). Thus $W^1L_M(\Omega,\rho)$ and $W^1E_M(\Omega,\rho))$ can be identified with subspaces of $\prod L_{M,\rho} = L_M \times \prod L_M(\Omega,\rho)$ we have the weak topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$ and $\sigma(\prod L_{M,\rho}, \prod L_{\overline{M},\rho})$. The space $W_0^1E_M(\Omega,\rho)$ (resp. $W_0^1L_M(\Omega,\rho)$) is defined by the closure of $D(\Omega)$ in $W^1E_M(\Omega,\rho))$ (resp. $W^1L_M(\Omega,\rho)$) for the norm (4.2.5) (resp. for the topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$).

Definition 4.2.2 *The sequence u_n converges to u in $L_M(\Omega,\rho)$ for the modular convergence (denoted by $u_n \to u$ (mod) $L_M(\Omega,\rho)$) if $\int_\Omega M\Big(\frac{|u_n-u|}{\lambda}\Big)\rho(x)dx \to 0$ as $n \to \infty$ for some $\lambda > 0$.*

Definition 4.2.3 *The sequence u_n converges to u in $W^1L_M(\Omega,\rho)$ for the modular convergence (denoted by $u_n \to u$ (mod) $W^1L_M(\Omega,\rho)$) if for some $\lambda > 0$ $\int_\Omega M\Big(\frac{|u_n-u|}{\lambda}\Big)dx \to 0$ as $n \to \infty$ and $\int_\Omega M\Big(\frac{|D^\alpha(u_n-u)|}{\lambda}\Big)\rho(x)dx \to 0$ as $n \to \infty$ for $|\alpha| = 1$.*

Lemma 4.2.1 *[22] Let M be an N-function. If $u_n \in L_M(\Omega)$ converges a.e. to u and u_n bounded in $L_M(\Omega)$, then $u \in L_M(\Omega)$ and $u_n \to u$ for the topology $\sigma(L_M(\Omega), E_{\overline{M}}(\Omega))$.*

Lemma 4.2.2 *[22] If the sequence $u_n \in L_M(\Omega,\rho)$ converges to u a.e. and bounded in $L_M(\Omega,\rho)$, then $u \in L_M(\Omega,\rho)$ and $u_n \to u$ for the topology $\sigma(L_M(\Omega,\rho), E_{\overline{M}}(\Omega,\rho))$.*

4.2.4 Compactness results

Let Ω an open bounded locally-border lipschitzian in $\mathbb{R}^N$, ρ the weight function and M an N-function. Let the following integrability assumptions :

There exists a real $s > 0$ such that :

$$(M(t))^{\frac{s}{s+1}} \text{ be } N\text{-function and that } \rho^{-s} \in L^1(\Omega), \tag{4.2.6}$$

$$\int_1^{\infty} \frac{t}{M(t)^{1+\frac{s}{N(s+1)}}} dM(t) = \infty, \tag{4.2.7}$$

$$\lim_{t\to\infty} \frac{1}{M^{-1}(t)} \int_0^{t^{\frac{s+1}{s}}} \frac{M^{-1}(u)}{u^{1+\frac{s}{N(s+1)}}} du = 0. \tag{4.2.8}$$

Remark 4.2.1 *In the particular case where* $M(t) = \frac{|t|^p}{p}$ $(1 < p < \infty)$, *the first part of* (4.2.6) *is satisfied if* $s > \frac{1}{p-1}$.

Theorem 4.2.1 (See[2] Theorem 9-5) *Let* Ω *an open bounded subset of* $\mathbb{R}^N$ *with locally- lipschitzian and* M *an* N*-function. Suppose that assumptions* (4.2.6)–(4.2.8) *are satisfied. So we have the following compact injection :*

$$W^1 L_M(\Omega, \rho) \hookrightarrow\hookrightarrow E_M(\Omega).$$

We will also use the following technical lemmas.

4.2.5 Some technical lemmas

Lemma 4.2.3 *Let* $f_n, f \in L^1(\Omega)$ *such that* $f_n \geq 0$ *a.e. in* Ω, $f_n \to f$ *a.e. in* Ω *et* $\int_\Omega f_n(x)dx \to \int_\Omega f(x)dx$. *Then* $f_n \to f$ *stongly in* $L^1(\Omega)$.

Lemma 4.2.4 *Let* $F : \mathbb{R} \to \mathbb{R}$ *be uniformly Lipschitzian, with* $F(0) = 0$. *Let* $u \in W_0^1 L_M(\Omega, \rho)$. *Then* $F(u) \in W_0^1 L_M(\Omega, \rho)$. *Moreover, if the set* D *of discontinuity points of* F' *is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \quad in \{x \in \Omega; u(x) \notin D\}, \\ 0 & a.e. \quad in \{x \in \Omega; u(x) \in D\}. \end{cases}$$

4.2.6 proof of lemma 4.2.4

We suppose for the moment that F is also C^1, there exist a sequence $u_n \in D(\Omega)$ such that $u_n \to u$ (mod) $W^1 L_M(\Omega, \rho)$). Passing to subsequence, we can assume that

$$D^\alpha u_n \to D^\alpha u \quad \forall\, |\alpha| \leq 1 \text{ a.e. in } \Omega.$$

From the relation $|F(s)| \leq k\,|s|$, where k denote the Lipschitz constant for F, and $\frac{\partial}{\partial x_i}F(u_n) = F^{'}(u_n)\frac{\partial u_n}{\partial x_i}$, we deduce that $F(u_n)$ remains bounded in $W_0^1 L_M(\Omega,\rho)$. Thus going to to a further subsequence, we obtain

$$F(u_n) \to w \in W_0^1 L_M(\Omega,\rho) \text{ for } \sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho}),$$

and also by a local application of the compact imbedding theorem, $F(u_n) \to w$ a.e. in Ω. Consequently $w = F(u)$, and $F(u) \in W_0^1 L_M(\Omega,\rho)$. Finally, by the usual chain rule for weak derivatives,

$$\frac{\partial}{\partial x_i}F(u) = F^{'}(u)\frac{\partial u}{\partial x_i} \text{ a.e. in } \Omega. \tag{4.2.9}$$

For the general case. Taking convolution with the mollifiers, we get a sequence $F_n \in C^\infty(\mathbb{R})$ such that $F_n \to F$ uniformly on each compact, $F_n(0) = 0$ and $\left|F_n^{'}\right| \leq k$. For each n, $F_n \in W_0^1 L_M(\Omega,\rho)$, and we have (4.2.9) with F replaced by F_n. Finally (4.2.9) follows from the generalized chain rule for weak derivatives. The following lemmas follow from the previous lemma.

Lemma 4.2.5 *Let $u, v \in W_0^1 L_M(\Omega,\rho)$ and let $w = min(u,v)$. Then $w \in W_0^1 L_M(\Omega,\rho)$ and*

$$\frac{\partial w}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_i} \text{ a.e. in } \{x \in \Omega; u(x) \leq v(x)\}, \\ \frac{\partial v}{\partial x_i} \text{ a.e. in } \{x \in \Omega; u(x) > v(x)\}. \end{cases}$$

4.2.7 proof of lemma 4.2.5

Note that $\min(u,v) = u - (u-v)^+$ and apply Lemma 4.2.4 with $F(s) = s^+$.

We introduce the truncate operator. For a given constant $k > 0$, we define the function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} s \text{ if } |s| \leq k, \\ k\frac{s}{|s|} \text{ if } |s| > k. \end{cases}$$

Lemma 4.2.6 *Assume that* (4.2.6)–(4.2.8) *holds. Let $u \in W_0^1 L_M(\Omega,\rho)$, and let $T_k(u), k \in \mathbb{R}^+$, be the usual truncation then $T_k(u) \in W_0^1 L_M(\Omega,\rho)$. Moreover we have*

$$T_k(u) \to u \text{ strongly in } W_0^1 L_M(\Omega,\rho).$$

Lemma 4.2.7 *Let u_n be a sequence in $W_0^1 L_M(\Omega,\rho)$ such that $u_n \rightharpoonup u$ for the topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$. Then $T_k(u_n) \rightharpoonup T_k(u)$ for $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$.*

4.2.8 proof of lemma 4.2.7

Since $u_n \rightharpoonup u$ and $W_0^1 L_M(\Omega,\rho) \hookrightarrow\hookrightarrow E_M(\Omega)$, we have $u_n \to u$ strongly in $E_M(\Omega)$ and a.e. in Ω, then $T_k(u_n) \to T_k(u)$ a.e. in Ω. On the other hand, for some $\lambda > 0$,

$$\int_\Omega M\Big(\frac{|T_k(u_n)|}{\lambda}\Big)dx \leq \int_\Omega M\Big(\frac{|u_n|}{\lambda}\Big)dx$$

and

$$\int_\Omega M\Big(\frac{|\nabla T_k(u_n)|}{\lambda}\Big)\rho(x)dx = \int_\Omega M\Big(\frac{|T_k^{'}(u_n)|\,|\nabla u_n|}{\lambda}\Big)\rho(x)dx \leq \int_\Omega M\Big(\frac{|\nabla u_n|}{\lambda}\Big)\rho(x)dx$$

imply that

$$\|T_k(u_n)\|_{1,M,\rho} \leq \|u_n\|_{1,M,\rho}\,.$$

Then $(T_k(u))_n$ is bounded in $W_0^1 L_M(\Omega,\rho)$ hence by Lemma 4.2.2, we have $T_k(u_n) \rightharpoonup T_k(u)$ in $W_0^1 L_M(\Omega,\rho)$ for $\sigma(L_M(\Omega,\rho), E_{\overline{M}}(\Omega,\rho))$.

Lemma 4.2.8 *If the sequence $u_n \in E_M(\Omega,\rho)$ converges a.e. in Ω with $\rho \in L^1(\Omega)$, then it converges in norm in $E_M(\Omega,\rho)$ if and only if the norms are uniformly absolutely continuous, i.e. for each $\epsilon > 0$ there exist $\delta > 0$ such that $\|u_n\chi_E\|_{M,\rho} < \epsilon$, for all n and $E \subset \Omega$ with $|E| < \delta$.*

4.2.9 proof of lemma 4.2.8

by the same argument introduced in the proof of Lemma 11.2 in [89] we find $E_{n,m} = \{x \in \Omega : |u_n(x) - u_n(x)| > \alpha\}$, where $\alpha = M^{-1}\Big(\frac{\epsilon}{3\|\rho\|_1}\Big)$ and with $\delta > 0$ such that

$$\|u_n\chi_E\|_{M,\rho} < \frac{\epsilon}{3}.$$

We denote by $H(E_M(\Omega), r)$ the set of functions $u \in L_M(\Omega)$ whose distance to $E_M(\Omega)$ (with respect to the Orlicz norm) is strictly less than r and by $B_{L_M}(\Omega)(0,r)$ the ball in $L_M(\Omega)$ (with respect to the Orlicz norm) of radius r and center 0.

Lemma 4.2.9 *Let Ω be bounded subset of $\mathbb{R}^N$ with finite mesure.let M, R and Q be N-functions such that $Q << R$, and let f be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:*

$$|f(x,s)| \leq b(x) + k_1 R^{-1}\left(\rho^{-1}(x)Q(k_2\,|s|)\right), \tag{4.2.10}$$

where $0 \leq b(x) \in E_M(\Omega, \rho), \rho \in L^1(\Omega)$ *and* $k_1, k_2 \in \mathbb{R}^+$. *Then the Nemytskii operator*

$$N_f(u)(x) = f(x, u(x))$$

satisfies :

1. *Sends* $\left(H(E_Q(\Omega), \frac{1}{k_2})\right)^p$ *into* $L_R(\Omega, \rho)$ *and is continuous from* $\left(H(E_Q(\Omega), \frac{1}{k_2})\right)^p$ *to the norm topology of* $(L_Q(\Omega))^p$ *into* $L_R(\Omega, \rho)$ *to the modular convergence;*

2. *It's uniformly bounded on* $\left(B_{L_Q(\Omega)}(0, \frac{1}{k_2})\right)^p$;

3. *If* $b(x) \in E_{R_1}(\Omega, \rho)$ *with* $R_1 << R$, *then* N_f *is continuous to the norm topology of* $E_{R_1}(\Omega, \rho)$.

4.2.10 proof of lemma 4.2.9

(1) Let $u = (u_1, u_2, ..., u_p) \in \left(H(E_Q(\Omega), \frac{1}{k_2})\right)^p$. Since $d(u_i, E_Q(\Omega)) < \frac{1}{k_2}$ $(1 \leq i \leq p)$, we have $\int_\Omega Q(k_2 |u|)dx \leq 1$, (see Theorem 10.1 [89]). Let $\lambda \geq 2k_1$ such that $\frac{2b(x)}{\lambda} \in K_R(\Omega, \rho)$. By the growth condition (5.2.10) and the convexity of R, we get

$$\int_\Omega R\left(\frac{|f(x,s)|}{\lambda}\right)\rho(x)dx \leq \frac{1}{2}\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx + \frac{1}{2}\int_\Omega R\left(k_2 |u(x)|\right)dx < \infty.$$

On the other hand, suppose that

$$u_n \to u \in \left(H(E_Q(\Omega), \frac{1}{k_2})\right)^p,$$

and let $\alpha > 0$ such that $d(k_2 |u|, E_Q(\Omega)) < \alpha < 1$ and $d(k_2 |u|, E_Q(\Omega)) < 1 - \alpha < 1$.

We have $\frac{k_2}{\alpha} |u| \in K_Q(\Omega)$ and $\frac{k_2}{1-\alpha} |u| \in K_Q(\Omega)$ (see Theorem 10.1 [89]) and for $\lambda > 4k_1$ such that $\frac{4b(x)}{\lambda} \in K_R(\Omega, \rho)$, we get

$$\begin{aligned}
&\int_\Omega R\left(\frac{|f(x,u_n) - f(x,u)|}{\lambda}\right)\rho(x)dx \\
&\quad \leq \int_\Omega R\left(\frac{2b(x) + k_1 R^{-1}\left(\rho^{-1}(x)Q(k_2 |u_n|)\right) + k_1 R^{-1}\left(\rho^{-1}(x)Q(k_2 |u|)\right)}{\lambda}\right)\rho(x)dx \\
&\quad \leq \frac{1}{2}\int_\Omega R\left(\frac{4b(x)}{\lambda}\right)\rho(x)dx + \frac{1-\alpha}{4}\int_\Omega Q\left(\frac{k_2}{1-\alpha}|u_n - u|\right)\rho(x)dx \\
&\qquad + \frac{\alpha}{4}\int_\Omega Q\left(\frac{k_2}{\alpha}|u|\right)dx + \frac{1}{4}\int_\Omega Q\left(k_2 |u|\right)dx.
\end{aligned}$$

Since $Q\left(k_2\left|u\right|\right) \leq Q\left(\frac{k_2}{\alpha}\left|u\right|\right)$, the last inequality becomes

$$\int_\Omega R\left(\frac{|f(x,u_n)-f(x,u)|}{\lambda}\right)\rho(x)dx \leq \frac{1}{2}\int_\Omega R\left(\frac{4b(x)}{\lambda}\right)\rho(x)dx + \int_\Omega Q\left(\frac{k_2}{1-\alpha}\left|u_n-u\right|\right)dx + \int_\Omega Q\left(k_2\left|u\right|\right)dx,$$

which implies by using the Vitali's theorem

$$f(x,u_n) \to f(x,u) \text{ (mod) in } L_R(\Omega,\rho),$$

for a subsequence denoted again u_n (which holds for the whole sequence).

(2) Let now $u \in \left(B_{L_Q(\Omega)}(0,\frac{1}{k_2})\right)^p$ and let $\lambda \geq 2k$ such that $\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx \leq 1$. By the growth condition (5.2.10) and the convexity of R, we get

$$\int_\Omega R\left(\frac{|f(x,u)|}{\lambda}\right)\rho(x)dx \leq \frac{1}{2}\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx + \frac{1}{2}\int_\Omega Q\left(k_2\left|u(x)\right|\right)dx \leq 1,$$

which implies (2).

(3) Suppose that $b(x) \in E_{R_1}(\Omega,\rho)$ with $R_1 \prec\prec R$. As in (1), since $L_R(\Omega,\rho) \subset E_{R_1}(\Omega,\rho))$, we can show that $f(x,u) \in E_{R_1}(\Omega,\rho)$ for all $u \in \left(E_Q(\Omega),\frac{1}{k_2})\right)^p$. Suppose now that

$$u_n \to u \in \left(E_Q(\Omega),\frac{1}{k_2})\right)^p \text{ in } \left(L_Q(\Omega)\right)^p,$$

We shall show that

$$f(x,u_n) \to f(x,u) \text{(mod) in } E_{R_1}(\Omega,\rho).$$

Fix $\epsilon > 0$, we have as above

$$\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\epsilon}\right)\rho(x)dx \leq \frac{1}{4}\int_\Omega R_1\left(\frac{4b(x)}{\epsilon}\right)\rho(x)dx$$
$$+\frac{1}{4}\int_\Omega R_1\left(\frac{4k_1}{\epsilon}R^{-1}\left(\rho^{-1}(x)Q(k_2\left|u_n\right|)\right)\right)\rho(x)dx + \frac{1}{4}\int_\Omega R_1\left(\frac{4k_1}{\epsilon}R^{-1}\left(\rho^{-1}(x)Q(k_2\left|u\right|)\right)\right)\rho(x)dx.$$

Since $R_1 \prec\prec R$, there exists $K^{'}$ such that $R_1(\frac{4k_1}{\epsilon}t) \leq R(t)+K^{'}$ for all $t \geq 0$. Then, the last inequality can be written as the form

$$\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\epsilon}\right)\rho(x)dx \leq \frac{1}{4}\int_\Omega R_1\left(\frac{4b(x)}{\epsilon}\right)\rho(x)dx + \frac{1}{4}\int_\Omega Q(k_2\left|u_n\right|)dx$$
$$+\frac{1}{4}\int_\Omega Q(k_2\left|u\right|)dx + \frac{K^{'}}{2}\int_\Omega \rho(x)dx.$$

As in (1) by using the Vitali's theorem, we get

$$\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\epsilon}\right)\rho(x)dx \to 0 \text{ as } n \to \infty,$$

for a subsequence (which holds for the whole sequence). Since ϵ is arbitrary, we conclude.

4.3 Main results

Let Y be a closed subspace of $W^1L_M(\Omega,\rho)$ for $\sigma(\prod L_{M,\rho},\prod E_{\overline{M},\rho})$ and let

$$Y_0 = Y \cap W^1L_M(\Omega,\rho),$$

such that Y is the closure of Y_0 for $\sigma(\prod L_{M,\rho},\prod E_{\overline{M},\rho})$. In the next, we consider the complementary system (Y,Y_0,Z,Z_0) generated by Y i.e. Y_0^* can be identified to Z and Z_0^* can be identified to Y by the means $\langle .,. \rangle$. Let the mapping T (associated to the operator A) defined from $D(T) = \{u \in Y, a_0(x,u,\nabla u) \in L_{\overline{M}}(\Omega), a_i(x,u,\nabla u) \in L_{\overline{M}}(\Omega,\rho)\}$ into Z by the formula

$$a(u,v) = \int_\Omega a_0(x,u,\nabla u)v(x)dx + \sum_{1\le i\le N}\int_\Omega a_i(x,u,\nabla u)\frac{\partial v(x)}{\partial x_i}\rho(x)dx \quad \forall v \in Y_0.$$

We consider the complementary system

$$(Y,Y_0,Z,Z_0) = (W_0^1L_M(\Omega,\rho), W_0^1E_M(\Omega,\rho), W^{-1}E_{\overline{M}}(\Omega,\rho), W^{-1}L_{\overline{M}}(\Omega,\rho)).$$

Our main results are collected in the following theorem.

Theorem 4.3.1 *Under the assumptions (4.1.2)-(4.1.5),* (4.2.6)–(4.2.8) *and $\rho(x)$ be a weight function on Ω satisfy 4.2.3 there exist an entropy solution u of the problem* (4.1.1).

4.3.1 Main Lemma

Lemma 4.3.1 *Let u be a mesurable function such that $T_k(u)$ belongs to $W_0^1L_M(\Omega,\rho)$ for every $k>0$. Then*

$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx \le \int_\Omega f\,T_k(u-\phi)dx, \tag{4.3.1}$$

is equivalent to

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi)\,dx = \int_\Omega f\,T_k(u-\phi)dx, \tag{4.3.2}$$

for every $\phi \in W_0^1L_M(\Omega,\rho)$, and for every $k>0$.

4.3.2 Proof of lemma 4.3.1

In fact (4.3.2) implies (4.3.1)is easily proved adding and subtracting

$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx,$$

and then using assumption (4.1.3). Thus, it remains to prove that (4.3.1) implies (4.3.2). Let h and k be positive real numbers, let $\lambda \in]-1,1[$ and $\Psi \in W_0^1 L_M(\Omega,\rho) \cap L^\infty(\Omega)$.

choose, $\phi = T_h(u - \lambda T_k(u-\Psi)) \in W_0^1 L_M(\Omega,\rho) \cap L^\infty(\Omega)$ as test function in (4.3.1), we have :

$$I_{hk} \leq J_{hk}, \tag{4.3.3}$$

with

$$I_{hk} = \int_\Omega a(x,u,\nabla T_h(u - \lambda T_k(u-\Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u-\Psi)))\ dx,$$

and

$$J_{hk} = \int_\Omega f\ T_k(u - T_h(u - \lambda T_k(u-\Psi)))\ dx.$$

Put

$$A_{hk} = \{x \in \Omega, |u - T_h(u - \lambda T_k(u-\Psi))| \leq k\},$$

and

$$B_{hk} = \{x \in \Omega, |u - \lambda T_k(u-\Psi)| \leq h\}.$$

Then, we obtain

$$\begin{aligned} I_{hk} = &\int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u - \lambda T_k(u-\Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u-\Psi)))\ dx \\ &+ \int_{A_{kh}\cap B_{hk}^C} a(x,u,\nabla T_h(u - \lambda T_k(u-\Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u-\Psi)))\ dx \\ &+ \int_{A_{kh}^C} a(x,u,\nabla T_h(u - \lambda T_k(u-\Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u-\Psi)))\ dx. \end{aligned}$$

Since $\nabla T_k(u - T_h(u - \lambda T_k(u-\Psi)))$ is different to zero only on A_{kh}, we have

$$\int_{A_{kh}^C} a(x,u,\nabla T_h(u - \lambda T_k(u-\Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u-\Psi)))\ dx = 0. \tag{4.3.4}$$

Moreover, if $x \in B_{hk}^C$, we have $\nabla T_h(u - \lambda T_k(u-\Psi)) = 0$ and using (4.1.4), we deduce that,

$$\begin{aligned} &\int_{A_{kh}\cap B_{hk}^C} a(x,u,\nabla T_h(u - \lambda T_k(u-\Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u-\Psi)))\ dx \\ &= \int_{A_{kh}\cap B_{hk}^C} a(x,u,0)\nabla T_k(u - T_h(u - \lambda T_k(u-\Psi)))\ dx = 0. \end{aligned} \tag{4.3.5}$$

From (4.3.4) and (4.3.5), we obtain

$$I_{hk} = \int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u - \lambda T_k(u-\Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u-\Psi)))\ dx.$$

Letting $h \to +\infty, |\lambda| \leq 1$, we have

$$A_{kh} \to \{x, |\lambda||T_k(u-\Psi)| \leq h\} = \Omega, \tag{4.3.6}$$

$$B_{hk} \to \Omega \quad which\, implies \quad A_{kh} \cap B_{hk} \to \Omega. \tag{4.3.7}$$

Which and using Lebesgue theorem, we conclude that

$$\begin{aligned}\lim_{h\to+\infty} \int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx \\ = \lambda \int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx. \end{aligned}\tag{4.3.8}$$

thus implies that,

$$\lim_{h\to+\infty} I_{hk} = \lambda \int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx. \tag{4.3.9}$$

On the other hand, we have,

$$J_{hk} = \int_\Omega f\; T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx.$$

Then

$$\lim_{h\to+\infty} \int_\Omega f\; T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx = \lambda \int_\Omega f T_k(u-\Psi)dx, \tag{4.3.10}$$

i.e.,

$$\lim_{h\to+\infty} J_{hk} = \lambda \int_\Omega f T_k(u-\Psi)dx. \tag{4.3.11}$$

Together (4.3.9), (4.3.11) and passing to the limit in (4.3.3), we obtain,

$$\lambda \left(\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx \right) \leq \lambda \left(\int_\Omega f T_k(u-\Psi)dx \right)$$

for every $\Psi \in W_0^1 L_M(\Omega,\rho) \cap L^\infty(\Omega)$, and for every $k > 0$. Choosing $\lambda > 0$ dividing by λ, and then letting λ tend to zero, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\,dx \leq \int_\Omega f T_k(u-\Psi)dx. \tag{4.3.12}$$

for $\lambda < 0$, dividing by λ, and then letting λ tend to zero, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\,dx \geq \int_\Omega f T_k(u-\Psi)dx. \tag{4.3.13}$$

Combining (4.3.12) and (4.3.13), we conclude the following equality :

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\,dx = \int_\Omega f T_k(u-\Psi)dx. \tag{4.3.14}$$

This completes the proof of Lemma 4.3.1.

4.3.3 Proof of Theorem 4.3.1

Approximate problem and a priori estimate

For $n \in \mathbb{N}$, define $f_n := T_n(f)$. Let u_n be solution in $W_0^1 L_\varphi(\Omega)$ of the problem

$$\begin{cases} -\operatorname{div}(a(x, u_n, \nabla u_n)) = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3.15}$$

which exists thanks to [94]. Choosing $T_k(u_n)$ as test function in (4.3.15), we have

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n)\, dx = \int_\Omega f_n T_k(u_n) dx,$$

using $\nabla T_k(u_n) = \nabla u_{n\chi\{|u_n|\le k\}}$ and thanks to assumption (4.1.4), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n)\, dx \ge \int_\Omega \rho(x) M(\lambda_1 |\nabla T_k(u_n)|) dx,$$

then

$$\int_\Omega \rho(x) M(\lambda_1 |\nabla T_k(u_n)|) dx \le k \|f\|_{L^1(\Omega)}. \tag{4.3.16}$$

Then

$$\int_\Omega \rho(x) M(\lambda_1 |\nabla T_k(u_n)|) dx \le C_1 k, \tag{4.3.17}$$

where C_1 is a constant independently of n.

Locally convergence of u_n in measure

Taking $\frac{1}{\lambda}|T_k(u_n)|$ in (4.3.15) and using (4.3.17), one has

$$\int_\Omega \rho(x) M(\lambda_1 \frac{|\nabla T_k(u_n)|}{\lambda}) dx \le \int_\Omega \rho(x) M(\lambda_1 |\nabla T_k(u_n)|) dx \le C_1 k. \tag{4.3.18}$$

Then we deduce by using (4.3.18), that

$$\begin{aligned} meas\{|u_n| > k\} &\le \frac{1}{\inf\limits_k M(\frac{k}{\lambda})} \int_{\{|u_n|>k\}} M(\frac{|u_n(x)|}{\lambda}) dx \\ &\le \frac{1}{\inf\limits_k M(\frac{k}{\lambda})} \int_\Omega M(\frac{1}{\lambda}|T_k(u_n)|) dx \\ &\le \frac{C_1 k}{\inf\limits_k M(\frac{k}{\lambda})} \quad \forall n, \quad \forall k \ge 0. \end{aligned} \tag{4.3.19}$$

For any $\beta > 0$, we have

$$meas\{|u_n - u_m| > \beta\} \leq meas\{|u_n| > k\} + meas\{|u_m| > k\} + meas\{|T_k(u_n) - T_k(u_m)| > \beta\},$$

and so that

$$meas\{|u_n - u_m| > \beta\} \leq \frac{2C_1 k}{\inf\limits_k M(\frac{k}{\lambda})} + meas\{|T_k(u_n) - T_k(u_m)| > \beta\}. \tag{4.3.20}$$

By using (4.3.17) and Poincaré inequality in weighted Orlicz-Sobolev spaces, we deduce that $(T_k(u_n))$ is bounded in $W_0^1 L_M(\Omega, \rho)$, and then there exists $\omega_k \in W_0^1 L_M(\Omega, \rho)$ such that $T_k(u_n) \rightharpoonup \omega_k$ weakly in $W_0^1 L_M(\Omega, \rho)$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$; strongly in $E_{\overline{M}}(\Omega, \rho)$ and a.e. in Ω.

Consequently, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω.

Let $\varepsilon > 0$, then by (4.3.20) and the fact that $\frac{2C_1 k}{\inf\limits_k M(\frac{k}{\lambda})} \to 0$ as $k \to +\infty$ there exists some $k = k(\varepsilon) > 0$ such that

$$meas\{|u_n - u_m| > \lambda\} < \varepsilon, \quad \text{for all } n, m \geq h_0(k(\varepsilon), \lambda).$$

This proves that u_n is a Cauchy sequence in measure, thus, u_n converges almost everywhere to some measurable function u. Finally, there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a function $u \in W_0^1 L_M(\Omega, \rho)$ such that

$$\begin{cases} u_n \rightharpoonup u \quad \text{weakly in } W_0^1 L_M(\Omega, \rho) \text{ for } \sigma(\Pi L_{M,\rho}, \Pi E_{\overline{M},\rho}) \\ u_n \longrightarrow u \quad \text{strongly in } E_M(\Omega, \rho) \text{ and a.e. in } \Omega. \end{cases} \tag{4.3.21}$$

An intermediate Inequality

In this step, we shall prove that for $\phi \in W_0^1 L_M(\Omega, \rho) \cap L^\infty(\Omega)$, we have

$$\int_\Omega a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)\, dx \leq \int_\Omega f_n T_k(u_n - \phi) dx. \tag{4.3.22}$$

We choose now $T_k(u_n - \phi)$ as test function in (4.3.15), with ϕ in $W_0^1 L_M(\Omega, \rho) \cap L^\infty(\Omega)$, we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \phi)\, dx = \int_\Omega f_n T_k(u_n - \phi) dx. \tag{4.3.23}$$

Adding and substracting the term $\int_\Omega a(x, u_n, \nabla\phi), \nabla T_k(u_n - \phi)\, dx$ i.e.,

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \phi)\, dx + \int_\Omega a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)\, dx \\ &- \int_\Omega a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)\, dx = \int_\Omega f_n T_k(u_n - \phi) dx. \end{aligned}$$

Thanks to assumption (4.1.3) and the definition of truncation function, we have

$$\int_\Omega (a(x,u_n,\nabla u_n)-a(x,u_n,\nabla\phi))\nabla T_k(u_n-\phi)\,dx \geq 0.$$

Combining (4.3.3) and (4.3.3), we obtain (4.3.22).

Passing to the limit

We shall prove that for $\phi \in W_0^1 L_M(\Omega,\rho)\cap L^\infty(\Omega)$, we have

$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx \leq \int_\Omega f_n T_k(u-\phi)dx.$$

Firstly, we claim that

$$\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx \to \int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx \text{ as } n\to+\infty.$$

Since $T_M(u_n)\rightharpoonup T_M(u)$ weakly in $W_0^1 L_M(\Omega,\rho)$, with $M=k+\|\phi\|_\infty$, then

$$T_k(u_n-\phi)\rightharpoonup T_k(u-\phi) \text{ in } W_0^1 L_M(\Omega,\rho), \tag{4.3.24}$$

which gives

$$\frac{\partial T_k}{\partial x_i}(u_n-\phi)\rightharpoonup \frac{\partial T_k}{\partial x_i}(u-\phi) \text{ weakly in } L_M(\Omega,\rho) \qquad \forall i=1,..,N. \tag{4.3.25}$$

Show that

$$a(x,T_M(u_n),\nabla\phi)\to a(x,T_M(u),\nabla\phi) \text{ strongly in } (L_{\overline{M}}(\Omega))^N.$$

Thanks to assumption (4.1.2), we obtain

$$|a_i(x,T_M(u_n),\nabla\phi)|\leq |\phi_i(x)|+K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|T_M(u_n)|))+K_i\overline{M}^{-1}M(c_1\,|\nabla\phi|),$$

with β and μ are positive constants. Since $T_M(u_n)\rightharpoonup T_M(u)$ weakly in $W_0^1 L_M(\Omega,\rho)$ and $W_0^1 L_M(\Omega,\rho)\hookrightarrow\hookrightarrow L_{\overline{M}}(\Omega,\rho)$, then $T_M(u_n)\rightharpoonup T_M(u)$ strongly in $L_M(\Omega,\rho)$ and a.e. in Ω, hence

$$|a(x,T_M(u_n),\nabla\phi)|\to|a(x,T_M(u),\nabla\phi)| \text{ a.e. in } \Omega.$$

and

$$|\phi_i(x)|+K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|T_M(u_n)|))+K_i\overline{M}^{-1}M(c_1\,|\nabla\phi|)\to$$

$$|\phi_i(x)| + K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|T_M(u)|)) + K_i\overline{M}^{-1}M(c_1\,|\nabla\phi|),$$

a.e. in Ω. Then, By Vitali's theorem, we deduce that

$$a(x, T_M(u_n), \nabla\phi) \to a(x, T_M(u), \nabla\phi) \text{ strongly in } (L_{\overline{M}}(\Omega, \rho))^N, \text{ as } n \to \infty. \tag{4.3.26}$$

Combining (4.3.25) and (4.3.26), we obtain

$$\int_\Omega a(x, u_n, \nabla\phi)\nabla T_k(u_n - \phi)\, dx \to \int_\Omega a(x, u, \nabla\phi)\nabla T_k(u - \phi)\, dx \text{ as } n \to +\infty. \tag{4.3.27}$$

Secondly, we show that

$$\int_\Omega f_n T_k(u_n - \phi)dx \to \int_\Omega f T_k(u - \phi)dx. \tag{4.3.28}$$

We have $f_n T_k(u_n - \phi) \to f T_k(u - \phi)$ a.e. in Ω and $|f T_k(u_n - \phi)| \leq k|f|$, then by using Vitali's theorem, we obtain (4.3.28). Thanks to (4.3.27) and (4.3.28) allow to pass to the limit in the inequality (4.3.22), so that $\forall\phi \in W_0^1 L_M(\Omega, \rho) \cap L^\infty(\Omega)$, we deduce

$$\int_\Omega a(x, u, \nabla\phi)\nabla T_k(u - \phi)\, dx \leq \int_\Omega f T_k(u - \phi)dx.$$

In view of main Lemma, we can deduce that u is an entropy solution of the problem (5.1.1). This completes the proof of Theorem 4.3.1.

Chapter 5

Existence of entropy solutions for nonlinear elliptic problem having large monotonocity in weighted Orlicz-Sobolev spaces

Abstract

We prove an existence result of entropy solution for a class of nonlinear elliptic problems of Leray-Lions type with large monotonicity condition in the framework of weighted Orlicz-Sobolev spaces and with right hand side $f \in L^1(\Omega)$.

5.1 Introduction

This paper deals with existence of solutions to the following nonlinear Dirichlet problem

$$\begin{cases} A(u) = f(x) - \operatorname{div} F(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5.1.1}$$

where $f \in L^1(\Omega), F \in (C^0(\mathbb{R}))^N$ and $A(u) = -\operatorname{div}(\rho(x)a(x,u,\nabla u))$, Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$.

Note that no growth hypothesis is assumed on the function F, which implies that the term $\operatorname{div} F(u)$ may be meaningless, even as a distribution. $a(x,u,\nabla u) = (a_i(x,u,\nabla u))_{1\leq i\leq N}$, $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory functions (that is measurable with respect to x in Ω for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and

continuous with respect to $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ for almost every $x \in \Omega$) such that for all $\xi,\ \xi'$ in $\mathbb{R}^N$, $(x,s) \in \Omega \times \mathbb{R}$

$$|a_i(x,s,\xi)| \leq |\phi_i(x)| + K_i \overline{P}^{-1}(\rho^{-1}(x) M(c_2|s|)) + K_i(\overline{M}^{-1} M(c_1\,|\xi|)), \tag{5.1.2}$$

$$\Big(a(x,s,\xi) - a(x,s,\xi')\Big)(\xi - \xi') \geq 0, \tag{5.1.3}$$

$$a(x,s,\xi)\xi \geq M(\lambda_1|\xi|), \tag{5.1.4}$$

where $c_1,\ c_2,\ \lambda_1, K_i > 0$. Let $M,\ P$ are two N-functions such that $P << M$. Moreover $\overline{M},\ \overline{P}$ are the complementary functions of M and P respectively, ρ is a weight function on Ω (that is, measurable and positive a.e. on Ω) and $\phi_0,\ \phi_i \in E_{\overline{M}}(\Omega,\rho)$, ($E_{\overline{M}}(\Omega,\rho)$ is introduced later.)

$$f \in L^1(\Omega), \tag{5.1.5}$$

and $F = (F_1, ..., F_N)$ satisfies

$$F \in (C^0(\mathbb{R}))^N. \tag{5.1.6}$$

The notion of entropy solution, used in [47], allows us to give a meaning to a possible solution of (5.1.1)

In fact Boccardo proved in [47], for $\rho(x) = 1$ and p such that $2 - 1/N < p < N$, the existence and regularity of an entropy solution u of problem (5.1.1) . For the case $\rho(x) = 1$, $\phi = 0$ and f is a bounded measure, Bénilan et al. proved in [48] the existence and uniqueness of entropy solutions,the same problem is treated using the notion of entropy solution introduced in [95] where $\rho(x) = 1$,$f \in L^1(\Omega)$, and $F \in L^{p'}(\Omega)^N$. We mention as a parallel development, the work of Lions and Murat [96] who consider similar problems in the context of the renormalized solutions introduced by Diperna and Lions [66] for the study of the Boltzmann equations. They can prove existence and uniqueness of renormalized solution,and afterwards by Dal Maso, Murat, Orsina and Prignet in [61] with general measure data.

In the general framework of weighted Orlicz-Sobolev spaces , the authors in [29] deal with the equation

$$-\operatorname{div}(\rho(x)a(x,u,\nabla u)) - \operatorname{div}\Phi(u) + g(x,u,\nabla u) = f$$

with $\Phi = 0$ and g is a nonlinear with the following (natural) growth condition:

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + M(|\xi|)\rho(x)),$$

which satisfies the sign condition

$$g(x,s,\xi)s \geq 0,$$

and the right-hand side $f \in W^{-1}E_M(\Omega,\rho)$,in the case where $f \in L^1(\Omega)$, $\phi = 0$ and $g = 0$ existence theorem have been proved by the authors in ([75]).

Several researches deals with the existence solutions of elliptic and parabolic problems under various assumptions and in different contexts (see [9, 8, 14, 18, 17, 19, 15, 13, 16, 12, 36, 37, 26, 27, 28, 49, 50, 51, 52, 53, 62, 70, 75, 78, 76, 109, 110, 111, 112, 113, 126, 127] for more details).

It is well known that the setting of weighted Orlicz–Sobolev spaces include many spaces as special spaces, such as Lebesgue spaces, weighted Lebesgue spaces and Orlicz spaces; see [104]. These spaces have many applications in various fields such as PDE, electrorheological fluids, and image restoration; see [69, 58, 90].

The feature of this paper, is to treat a class of problems for which the classical monotone operator methods (developed by Minty [100], Browder [55], Brézis [54] and Lions [91] in $W_0^{1,p}(\Omega)$ case) do not apply. The reason for this, is that $a(x,u,\nabla u)$ does not need to satisfy the strict monotonicity condition that is,

$$\Big(a(x,s,\xi) - a(x,s,\xi')\Big)(\xi - \xi') > 0, \text{ for all } \xi,\xi' \in \mathbb{R}^N, \ (\xi \neq \xi')$$

of a typical Leray–Lions operator but only a large monotonicity that is

$$\Big(a(x,s,\xi) - a(x,s,\xi')\Big)(\xi - \xi') \geq 0, \text{ for all } \xi,\xi' \in \mathbb{R}^N.$$

The aim of this paper is to prove the existence of solutions for (5.1.1) under the weaker assumption large monotonicity condition, without using the almost everywhere convergence of the gradients of the approximate equations, since this is impossible to prove in our setting. The main tools of our proof are a version of Minty's Lemma (here we use an idea of G. J. Minty [103]. We note that the techniques we used in the proof are different from those used in [35, 33, 72].

The paper is organized as follows: We introduce some basic definitions and properties in the setting of weighted Orlicz–Sobolev spaces as well as an abstract theorem and we prepare some auxiliary results to prove our theorem, in the next section 6.2. In the final section 5.3, we state the main result and proofs.

5.2 Preliminary

This section present some definitions and well-known about N-functions, weighted Orlicz-Sobolev spaces (standard references see [1] and [71].

5.2.1 N-function

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with

$$M(t) > 0 \text{ for } t > 0, \quad \tfrac{M(t)}{t} \to 0 \text{ as } t \to 0 \text{ and } \tfrac{M(t)}{t} \to \infty \text{ as } t \to \infty.$$

Equivalently M admits the representation :

$$M(t) = \int_0^t m(\tau) d\tau,$$

where $m : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing right continuous, with $m(0) = 0$, $m(t) > 0$ for $t > 0$ and $m(t) \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M defined by

$$\overline{M}(t) = \int_0^t \overline{m}(\tau) d\tau,$$

where $\overline{m} : \mathbb{R}^+ \to \mathbb{R}^+$ is given bay $\overline{m}(t) = \sup\{s : m(s) \leq t\}$. It is well known that we can assume that m and $\overline{m}$ are continuous and strictly increasing. We will extend the N-functions into even functions on all $\mathbb{R}^+$.

Clearly $\overline{\overline{M}} = M$ and has Young's inequality

$$st \leq M(t) + \overline{M}(s) \text{ for all } s, t \geq 0.$$

The N-function M is said to satisfy the Δ_2-condition every-where (resp. infinity) if there exist $k > 0$ (resp. $t_0 > 0$) such that, $M(2t) \leq kM(t)$ for all $t \geq 0$ (resp. $t \geq t_0$).

Let P and Q be two N-functions, the notation $P << Q$ means that P grows essentially less rapidly than Q, that is to say for all $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to +\infty$. That is the case if and only if $\frac{Q^{-1}(t)}{P^{-1}(t)} \to 0$ as $t \to \infty$.

5.2.2 Orlicz-Sobolev space

Let Ω be a open subset of $\mathbb{R}^N$ and M be an N-function.

The Orlicz classe $K_M(\Omega)$ (resp. the Orlicz spaces $L_M(\Omega)$) is the set of all (equivalence classes modulo equality a.e. in Ω of) real-valued measurable functions u defined in Ω and satisfying

$$\int_\Omega M(|u(x)|)dx < \infty \Big(\text{resp. } \int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)dx < \infty \text{ for some } \lambda > 0\Big).$$

$L_M(\Omega)$ it is a Banach space under the norm:

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)dx \le 1\right\}. \tag{5.2.1}$$

We denote by $B(\Omega)$ the set of bounded measurable functions with compact support in $\overline{\Omega}$ and by $E_M(\Omega)$ the closure of $B(\Omega)$ in $L_M(\Omega)$ (we have usual $E_M(\Omega) \subset K_M(\Omega) \subset L_M(\Omega)$).

The equality $L_M(\Omega) = E_M(\Omega)$ hold if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has a infinite measure or not .

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$ where $u \in L_M(\Omega)$ and $v \in L_{\overline{M}}(\Omega)$ and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$.

The space $L_M(\Omega)$ is reflexive if and only if M an $\overline{M}$ satisfy the Δ_2-condition for all t or for t large, according to whether Ω be infinite measure or note.

We return now to the Orlicz-Sobolev spaces $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all function u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. in $E_M(\Omega)$). It's Banach space under the norm :

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_{M,\Omega}. \tag{5.2.2}$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of $\prod L_M$ we have the weak topology $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ (resp. $W_0^1L_M(\Omega)$) is defined by the closure of $D(\Omega)$ in $W^1E_M(\Omega)$ (resp. $W^1L_M(\Omega)$) for the norm (5.2.2) (resp. for the topology $\sigma(\prod L_M, \prod E_{\overline{M}})$).

Definition 5.2.1 *The sequence u_n converges to u in $L_M(\Omega)$ for the modular convergence (denoted by $u_n \to u$ (mod) $L_M(\Omega)$) if $\int_\Omega M\Big(\frac{|u_n - u|}{\lambda}\Big)dx \to 0$ as $n \to \infty$ for some $\lambda > 0$.*

5.2.3 Weighted Orlicz–Sobolev space

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, M be an N-function and $\rho(x)$ be a weight function on Ω, i.e. measurable positive a.e. on Ω such that :

$$\rho \in L^1(\Omega). \tag{5.2.3}$$

The weighted Orlicz classe $K_M(\Omega,\rho)$ (resp. the weighted Orlicz space $L_M(\Omega,\rho)$) is the set of all (equivalence classes modulo equality a.e. in Ω) of real-valued measurable functions u defined in Ω and satisfying $m_\rho(u,M) = \displaystyle\int_\Omega M(|u(x)|)\rho(x)dx < \infty$

$$\Big(\text{resp. } m_\rho\Big(\frac{u}{\lambda},M\Big) = \int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)\rho(x)dx < \infty \text{ for some } \lambda > 0\Big).$$

$L_M(\Omega,\rho)$ is a Banach space under the norm :

$$\|u\|_{M,\rho} = \inf\Big\{\lambda > 0; m\Big(\frac{u}{\lambda},M\Big) \le 1\Big\}. \tag{5.2.4}$$

We denote by $B(\Omega)$ the set of bounded measurable functions with compact support in $\overline{\Omega}$ and by $E_M(\Omega,\rho)$ the closure of $B(\Omega)$ in $L_M(\Omega,\rho)$ (we have usual $E_M(\Omega,\rho) \subset K_M(\Omega,\rho) \subset L_M(\Omega,\rho)$).

The equality $L_M(\Omega,\rho) = E_M(\Omega,\rho)$ hold if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has a infinite measure or note.
The dual of $E_M(\Omega,\rho)$ can be identified with $L_{\overline{M}}(\Omega,\rho)$ by means of the pairing $\displaystyle\int_\Omega u(x)v(x)\rho(x)dx$ where $u \in L_M(\Omega,\rho)$ and $v \in L_{\overline{M}}(\Omega,\rho)$ and the dual norm on $L_{\overline{M}}(\Omega,\rho)$ is equivalent to $\|.\|_{\overline{M},\Omega}$ (see [34]).

The space $L_M(\Omega,\rho)$ is reflexive if and only if M an $\overline{M}$ satisfy the Δ_2-condition for all t or for t large according to whether Ω be infinite measure or note.

We return now to the weighted Orlicz-Sobolev spaces $W^1L_M(\Omega,\rho)$ (resp. $W^1E_M(\Omega,\rho)$) is the space of all function u such that $u \in L_M(\Omega)$ (resp. $u \in E_M(\Omega)$) and its distributional derivatives up to order 1 lie in $L_M(\Omega,\rho)$ (resp. $E_M(\Omega,\rho)$). It's Banach space under the norm

$$\|u\|_{1,M,\rho} = \|u\|_M + \|\nabla u\|_{M,\rho}, \tag{5.2.5}$$

(where $\|u\|_M = \|u\|_{M,\Omega}$). Thus $W^1L_M(\Omega,\rho)$ and $W^1E_M(\Omega,\rho)$) can be identified with subspaces of $\prod L_{M,\rho} = L_M \times \prod L_M(\Omega,\rho)$ we have the weak topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$ and $\sigma(\prod L_{M,\rho}, \prod L_{\overline{M},\rho})$.

The space $W_0^1 E_M(\Omega,\rho)$ (resp. $W_0^1 L_M(\Omega,\rho)$) is defined by the closure of $D(\Omega)$ in $W^1 E_M(\Omega,\rho)$) (resp. $W^1 L_M(\Omega,\rho)$) for the norm (5.2.5) (resp. for the topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$).

Definition 5.2.2 *The sequence u_n converges to u in $L_M(\Omega,\rho)$ for the modular convergence (denoted by $u_n \to u$ (mod) $L_M(\Omega,\rho)$) if* $\int_\Omega M\Big(\frac{|u_n-u|}{\lambda}\Big)\rho(x)dx \to 0$ *as $n\to\infty$ for some $\lambda>0$.*

Definition 5.2.3 *The sequence u_n converges to u in $W^1 L_M(\Omega,\rho)$ for the modular convergence (denoted by $u_n \to u$ (mod) $W^1 L_M(\Omega,\rho)$) if for some $\lambda>0$* $\int_\Omega M\Big(\frac{|u_n-u|}{\lambda}\Big)dx \to 0$ *as $n\to\infty$ and* $\int_\Omega M\Big(\frac{|D^\alpha(u_n-u)|}{\lambda}\Big)\rho(x)dx \to 0$ *as $n\to\infty$ for $|\alpha|=1$.*

Lemma 5.2.1 *[22] Let M be an N-function. If $u_n \in L_M(\Omega)$ converges a.e. to u and u_n bounded in $L_M(\Omega)$, then $u\in L_M(\Omega)$ and $u_n \to u$ for the topology $\sigma(L_M(\Omega), E_{\overline{M}}(\Omega))$.*

Lemma 5.2.2 *[22] If the sequence $u_n \in L_M(\Omega,\rho)$ converges to u a.e. is bounded in $L_M(\Omega,\rho)$, then $u\in L_M(\Omega,\rho)$ and $u_n\to u$ for the topology $\sigma(L_M(\Omega,\rho), E_{\overline{M}}(\Omega,\rho))$.*

5.2.4 Compactness results

Let Ω be a bounded open subset of $\mathbb{R}^N$ with locally Lipschitzian boundary, ρ the weight function and M an N-function. Let the following integrability assumptions :

There exists a real $s>0$ such that :

$$(M(t))^{\frac{s}{s+1}} \text{ is } N\text{-function and an } \rho^{-s}\in L^1(\Omega), \tag{5.2.6}$$

$$\int_1^\infty \frac{t}{M(t)^{1+\frac{s}{N(s+1)}}}dM(t)=\infty, \tag{5.2.7}$$

$$\lim_{t\to\infty}\frac{1}{M^{-1}(t)}\int_0^{t^{\frac{s+1}{s}}}\frac{M^{-1}(u)}{u^{1+\frac{s}{N(s+1)}}}du=0. \tag{5.2.8}$$

Remark 5.2.1 *In the particular case where $M(t)=\frac{|t|^p}{p}$ $(1<p<\infty)$, the first part of* (5.2.6) *is satisfied if $s>\frac{1}{p-1}$.*

Theorem 5.2.1 (See[2] Theorem 9-5) *Let Ω an open bounded locally- lipschitzian boundary in $\mathbb{R}^N$ and M an N-function. Suppose that assumptions* (5.2.6)–(5.2.8) *are satisfied. So we have the following compact injection :*

$$W^1 L_M(\Omega,\rho) \hookrightarrow\hookrightarrow E_M(\Omega).$$

Theorem 5.2.2 *(Weighted Poincaré inequality) Let Ω be a bounded open subset of $\mathbb{R}^N$ with locally Lipschitzian boundary, ρ a weight function, and M an N -function. If $u \in W_0^1 L_M(\Omega)$, then*

$$\|u\|_M \leq c\|\nabla u\|_{M,\rho}$$

where c is a positive constant, which implies that $\|\nabla u\|_{M,\rho}$ and $\|u\|_{1,M}$ are equivalent norms on $W_0^1 L_{M,\rho}$.

Proof 5.2.1 *Under the assumptions* (5.2.6)-(5.2.8)*, the Sobolev conjugate N -function M_s^* of M_s is well defined by*

$$M_s^{*-1} = \int_0^s \frac{M^{-1}(t)}{t^{1+\frac{1}{N}}} dt$$

and we have $W_0^1 L_{M_s} \subset L_{M_s^*}$. And since $M \ll M_s^*$, we have $L_{M_s^*} \subset L_M$. Hence

$$\|u\|_M \leq c_1\|u\|_{M_s^*} \leq c_2\|u\|_{1,M_s},$$

where c_1 and c_2 are two positive constants. Then, by using the Poincaré inequality in the non-weighted Orlicz-Sobolev space, there exists a positive constant c' such that

$$\|u\|_{1,M_s} \leq c'\|\nabla u\|_{M_s}.$$

We will show that

$$\|\nabla u\|_{M_s} \leq c\|\nabla u\|_{M,\rho}.$$

For that we have

$$\begin{aligned}
\|v\|_{M_s} \leq \int_\Omega M_s(v(x))dx + 1 &= \int_\Omega M_s(v(x))\frac{1}{\rho(x)}\rho(x)dx + 1 \\
&\leq \int_\Omega S\left(M_s(v(x))\right)\rho(x)dx + \int_\Omega \frac{1}{\rho(x)}\rho(x)dx + 2 \\
&= \int_\Omega M(v(x))\rho(x)dx + \int_\Omega \rho^{-s}(x)dx + 1,
\end{aligned}$$

which implies that

$$\|v\|_{M_s} \leq c\|v\|_{M,\rho},$$

for some positive constant c. In fact, if this is not true, then there exists a sequence v_n such that $\|v_n\|_{M_s} \to \infty$ and for n large, $\|v_n\|_{M,\rho} \leq 1$. Hence, for n sufficiently large we get

$$\int_\Omega M\left(v_n(x)\right)\rho(x)dx \leq \|v_n\|_{M,\rho} \leq 1.$$

Then

$$\begin{aligned}\|v_n\|_{M_s} &\leq \int_\Omega M(v(x))\rho(x)dx + \int_\Omega \rho^{-s}(x)dx + 1 \\ &\leq \|v_n\|_{M,\rho} + \int_\Omega \rho^{-s}(x)dx + 1,\end{aligned}$$

which is a contradiction, since the left hand-side tends to infinity while the right hand-side is bounded. Finally, by taking $v = \nabla u$, we conclude the result.

We will also use the following technical lemmas.

5.2.5 Some technical lemmas

Lemma 5.2.3 *[87]Let $f_n, f \in L^1(\Omega)$ such that $f_n \geq 0$ a.e. in Ω, $f_n \to f$ a.e. in Ω et $\int_\Omega f_n(x)dx \to \int_\Omega f(x)dx$. Then $f_n \to f$ stongly in $L^1(\Omega)$.*

Lemma 5.2.4 *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W_0^1 L_M(\Omega, \rho)$. Then $F(u) \in W_0^1 L_M(\Omega, \rho)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \quad in \left\{x \in \Omega; u(x) \notin D\right\}, \\ 0 & a.e. \quad in \left\{x \in \Omega; u(x) \in D\right\}. \end{cases}$$

Proof 5.2.2 *We suppose for the moment that F is also C^1, there exist a sequence $u_n \in D(\Omega)$ such that $u_n \to u$ (mod) $W^1 L_M(\Omega, \rho)$). Passing to subsequence, we can assume that*

$$D^\alpha u_n \to D^\alpha u \quad \forall\, |\alpha| \leq 1 \text{ a.e. in } \Omega.$$

From the relation $|F(s)| \leq k\,|s|$, where k denote the Lipschitz constant for F, and $\frac{\partial}{\partial x_i} F(u_n) = F'(u_n)\frac{\partial u_n}{\partial x_i}$, we deduce that $F(u_n)$ remains bounded in $W_0^1 L_M(\Omega, \rho)$. Thus going to a further subsequence, we obtain

$$F(u_n) \to w \in W_0^1 L_M(\Omega, \rho) \text{ for } \sigma(\textstyle\prod L_M, \prod E_{\overline{M}}),$$

and also by a local application of the compact imbedding theorem, $F(u_n) \to w$ a.e. in Ω. Consequently $w = F(u)$, and $F(u) \in W_0^1 L_M(\Omega, \rho)$. Finally, by the usual chain rule for weak derivatives,

$$\frac{\partial}{\partial x_i} F(u) = F'(u)\frac{\partial u}{\partial x_i} \text{ a.e. in } \Omega. \tag{5.2.9}$$

For the general case. Taking convolution with the mollifiers, we get a sequence $F_n \in C^\infty(\mathbb{R})$ such that $F_n \to F$ uniformly on each compact, $F_n(0) = 0$ and $\left|F_n'\right| \leq k$. For each n, $F_n \in W_0^1 L_M(\Omega, \rho)$, and we have (5.2.9) *with F replaced by F_n. Finally* (5.2.9) *follows from the generalized chain rule for weak derivatives.*

The following lemmas follow from the previous lemma.

Lemma 5.2.5 *Let $u, v \in W_0^1 L_M(\Omega, \rho)$ and let $w = min(u, v)$. Then $w \in W_0^1 L_M(\Omega, \rho)$ and*

$$\frac{\partial w}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_i} \text{ a.e. in } \{x \in \Omega; u(x) \leq v(x)\}, \\ \frac{\partial v}{\partial x_i} \text{ a.e. in } \{x \in \Omega; u(x) > v(x)\}. \end{cases}$$

Proof 5.2.3 *Note that $min(u, v) = u - (u - v)^+$ and apply Lemma 5.2.4 with $F(s) = s^+$.*

We introduce the truncate operator. For a given constant $k > 0$, we define the function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} s \text{ if } |s| \leq k, \\ k\frac{s}{|s|} \text{ if } |s| > k. \end{cases}$$

Lemma 5.2.6 *Assume that* (5.2.6)–(5.2.8) *holds. Let $u \in W_0^1 L_M(\Omega, \rho)$, and let $T_k(u), k \in \mathbb{R}^+$, be the usual truncation then $T_k(u) \in W_0^1 L_M(\Omega, \rho)$. Moreover we have*

$$T_k(u) \to u \text{ strongly in } W_0^1 L_M(\Omega, \rho).$$

Lemma 5.2.7 *Let u_n be a sequence of $W_0^1 L_M(\Omega, \rho)$ such that $u_n \rightharpoonup u$ weakly in $W_0^1 L_M(\Omega, \rho)$ for the topology $\sigma(L_M(\Omega, \rho), E_{\overline{M}}(\Omega, \rho))$. Then $T_k(u_n) \rightharpoonup T_k(u)$.*

Proof 5.2.4 *Since $u_n \rightharpoonup u$ and $W_0^1 L_M(\Omega, \rho) \hookrightarrow\hookrightarrow E_M(\Omega)$, we have $u_n \to u$ strongly in $E_M(\Omega)$ and a.e. in Ω, then $T_k(u_n) \to T_k(u)$ a.e. in Ω. On the other hand, for some $\lambda > 0$,*

$$\int_\Omega M\Big(\frac{|T_k(u_n)|}{\lambda}\Big)dx \leq \int_\Omega M\Big(\frac{|u_n|}{\lambda}\Big)dx$$

and

$$\int_\Omega M\Big(\frac{|\nabla T_k(u_n)|}{\lambda}\Big)\rho(x)dx = \int_\Omega M\Big(\frac{|T_k'(u_n)|\,|\nabla u_n|}{\lambda}\Big)\rho(x)dx \leq \int_\Omega M\Big(\frac{|\nabla u_n|}{\lambda}\Big)\rho(x)dx$$

imply that

$$\|T_k(u_n)\|_{1,M,\rho} \leq \|u_n\|_{1,M,\rho}\,.$$

Then $(T_k(u))_n$ *is bounded in* $W_0^1 L_M(\Omega,\rho)$ *hence by Lemma 5.2.2, we have* $T_k(u_n) \rightharpoonup T_k(u)$ *in* $W_0^1 L_M(\Omega,\rho)$ *for* $\sigma(L_M(\Omega,\rho), E_{\overline{M}}(\Omega,\rho))$.

Lemma 5.2.8 *If the sequence* $u_n \in E_M(\Omega,\rho)$ *converges a.e. in* Ω *with* $\rho \in L^1(\Omega)$*, then it converges in norm in* $E_M(\Omega,\rho)$ *if and only if the norms are uniformly absolutely continuous, i.e. for each* $\epsilon > 0$ *there exist* $\delta > 0$ *such that* $\|u_n\chi_E\|_{M,\rho} < \epsilon$*, for all* n *and* $E \subset \Omega$ *with* $|E| < \delta$.

Proof 5.2.5 *by the same argument introduced in the proof of Lemma 11.2 in [89] we find* $E_{n,m} = \{x \in \Omega : |u_n(x) - u_n(x)| > \alpha\}$*, where* $\alpha = M^{-1}\Big(\frac{\epsilon}{3\|\rho\|_1}\Big)$ *and with* $\delta > 0$ *such that*

$$\|u_n\chi_E\|_{M,\rho} < \frac{\epsilon}{3}.$$

We denote by $H(E_M(\Omega), r)$ *the set of functions* $u \in L_M(\Omega)$ *whose distance to* $E_M(\Omega)$ *(with respect to the Orlicz norm) is strictly less than* r *and by* $B_{L_M}(\Omega)(0,r)$ *the ball in* $L_M(\Omega)$ *(with respect to the Orlicz norm) of radius* r *and center* 0.

Lemma 5.2.9 *Let* Ω *be bounded subset of* $\mathbb{R}^N$ *with finite measure.let* M, R *and* Q *be N-functions such that* $Q << R$*, and let* f *be a Carathéodory function such that for a.e.* $x \in \Omega$ *and all* $s \in \mathbb{R}$:

$$|f(x,s)| \leq b(x) + k_1 R^{-1}\left(\rho^{-1}(x) Q(k_2\,|s|)\right), \tag{5.2.10}$$

where $0 \leq b(x) \in E_M(\Omega,\rho)$, $\rho \in L^1(\Omega)$ *and* $k_1, k_2 \in \mathbb{R}^+$*. Then the Nemytskii operator*

$$N_f(u)(x) = f(x, u(x))$$

satisfies :

1. *Sends* $\Big(H(E_Q(\Omega), \frac{1}{k_2})\Big)^p$ *into* $L_R(\Omega,\rho)$ *and is continuous from* $\Big(H(E_Q(\Omega), \frac{1}{k_2})\Big)^p$ *to the norm topology of* $(L_Q(\Omega))^p$ *into* $L_R(\Omega,\rho)$ *to the modular convergence;*
2. *It's uniformly bounded on* $\Big(B_{L_Q(\Omega)}(0, \frac{1}{k_2})\Big)^p$;
3. *If* $b(x) \in E_{R_1}(\Omega,\rho)$ *with* $R_1 << R$*, then* N_f *is continuous to the norm topology of* $E_{R_1}(\Omega,\rho)$.

Proof 5.2.6 (1) *Let* $u=(u_1,u_2,...,u_p)\in\left(H(E_Q(\Omega),\frac{1}{k_2})\right)^p$. *Since* $d(u_i,E_Q(\Omega))<\frac{1}{k_2}$ $(1\leq i\leq p)$, *we have* $\int_\Omega Q(k_2|u|)dx\leq 1$, *(see Theorem 10.1 [89]). Let* $\lambda\geq 2k_1$ *such that* $\frac{2b(x)}{\lambda}\in K_R(\Omega,\rho)$. *By the growth condition* (5.2.10) *and the convexity of* R, *we get*

$$\int_\Omega R\left(\frac{|f(x,s)|}{\lambda}\right)\rho(x)dx\leq\frac{1}{2}\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx+\frac{1}{2}\int_\Omega R\left(k_2|u(x)|\right)dx<\infty.$$

On the other hand, suppose that

$$u_n\to u\in\left(H(E_Q(\Omega),\frac{1}{k_2})\right)^p,$$

and let $\alpha>0$ *such that* $d(k_2|u|,E_Q(\Omega))<\alpha<1$ *and* $d(k_2|u|,E_Q(\Omega))<1-\alpha<1$.

We have $\frac{k_2}{\alpha}|u|\in K_Q(\Omega)$ *and* $\frac{k_2}{1-\alpha}|u|\in K_Q(\Omega)$ *(see Theorem 10.1 [89]) and for* $\lambda>4k_1$ *such that* $\frac{4b(x)}{\lambda}\in K_R(\Omega,\rho)$, *we get*

$$\begin{aligned}&\int_\Omega R\left(\frac{|f(x,u_n)-f(x,u)|}{\lambda}\right)\rho(x)dx\\&\quad\leq\int_\Omega R\left(\frac{2b(x)+k_1R^{-1}\left(\rho^{-1}(x)Q(k_2|u_n|)\right)+k_1R^{-1}\left(\rho^{-1}(x)Q(k_2|u|)\right)}{\lambda}\right)\rho(x)dx\\&\quad\leq\frac{1}{2}\int_\Omega R\left(\frac{4b(x)}{\lambda}\right)dx+\frac{1-\alpha}{4}\int_\Omega Q\left(\frac{k_2}{1-\alpha}|u_n-u|\right)dx\\&\qquad+\frac{\alpha}{4}\int_\Omega Q\left(\frac{k_2}{\alpha}|u|\right)dx+\frac{1}{4}\int_\Omega Q\left(k_2|u|\right)dx.\end{aligned}$$

Since $Q\left(k_2|u|\right)\leq Q\left(\frac{k_2}{\alpha}|u|\right)$, *the last inequality becomes*

$$\begin{aligned}&\int_\Omega R\left(\frac{|f(x,u_n)-f(x,u)|}{\lambda}\right)\rho(x)dx\\&\qquad\leq\frac{1}{2}\int_\Omega R\left(\frac{4b(x)}{\lambda}\right)\rho(x)dx+\int_\Omega Q\left(\frac{k_2}{1-\alpha}|u_n-u|\right)dx+\int_\Omega Q\left(k_2|u|\right)dx,\end{aligned}$$

which implies by using the Vitali's theorem

$$f(x,u_n)\to f(x,u)\ \textit{(mod) in}\ L_R(\Omega,\rho),$$

for a subsequence denoted again u_n *(which holds for the whole sequence).*

(2) *Let now* $u\in\left(B_{L_Q(\Omega)}(0,\frac{1}{k_2})\right)^p$ *and let* $\lambda\geq 2k$ *such that* $\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx\leq 1$.
By the growth condition (5.2.10) *and the convexity of* R, *we get*

$$\int_\Omega R\left(\frac{|f(x,u)|}{\lambda}\right)\rho(x)dx\leq\frac{1}{2}\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx+\frac{1}{2}\int_\Omega Q\left(k_2|u(x)|\right)dx\leq 1,$$

which implies (2).

(3) *Suppose that* $b(x) \in E_{R_1}(\Omega,\rho)$ *with* $R_1 \prec\prec R$. *As in* (1), *since* $L_R(\Omega,\rho) \subset E_{R_1}(\Omega,\rho))$, *we can show that* $f(x,u) \in E_{R_1}(\Omega,\rho)$ *for all* $u \in \left(E_Q(\Omega), \frac{1}{k_2})\right)^p$. *Suppose now that*

$$u_n \to u \in \left(E_Q(\Omega), \frac{1}{k_2})\right)^p \text{ in } (L_Q(\Omega))^p,$$

We shall show that

$$f(x,u_n) \to f(x,u) \textit{(mod) in } E_{R_1}(\Omega,\rho).$$

Fix $\epsilon > 0$, *we have as above*

$$\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\epsilon}\right)\rho(x)dx \le \frac{1}{4}\int_\Omega R_1\left(\frac{4b(x)}{\epsilon}\right)\rho(x)dx$$
$$+\frac{1}{4}\int_\Omega R_1\left(\frac{4k_1}{\epsilon}R^{-1}\left(\rho^{-1}(x)Q(k_2\,|u_n|)\right)\right)\rho(x)dx + \frac{1}{4}\int_\Omega R_1\left(\frac{4k_1}{\epsilon}R^{-1}\left(\rho^{-1}(x)Q(k_2\,|u|)\right)\right)\rho(x)dx.$$

Since $R_1 \prec\prec R$, *there exists* $K^{'}$ *such that* $R_1(\frac{4k_1}{\epsilon}t) \le R(t)+K^{'}$ *for all* $t \ge 0$. *Then, the last inequality can be written as the form*

$$\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\epsilon}\right)\rho(x)dx \le \frac{1}{4}\int_\Omega R_1\left(\frac{4b(x)}{\epsilon}\right)\rho(x)dx + \frac{1}{4}\int_\Omega Q(k_2\,|u_n|)dx$$
$$+\frac{1}{4}\int_\Omega Q(k_2\,|u|)dx + \frac{K^{'}}{2}\int_\Omega \rho(x)dx.$$

As in (1) *by using the Vitali's theorem, we get*

$$\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\epsilon}\right)\rho(x)dx \to 0 \textit{ as } n \to \infty,$$

for a subsequence (which holds for the whole sequence). Since ϵ *is arbitrary, we conclude* (3).

Definition 5.2.4 *[1] (**Segment property**) : A domain* Ω *is said to satisfy the segment property, if there exist a finite open covering* $\{\theta\}_{i=1}^k$ *of* $\bar{\Omega}$ *and a corresponding nonzero vectors* $z_i \in R^N$ *such that* $\left(\bar{\Omega}\cap\theta_i\right)+tz_i \subset \Omega$ *for all* $t \in (0,1)$ *and* $i = 1,\ldots,k$.

Lemma 5.2.10 *[84] Suppose that* Ω *satisfies the segment property and let* $u \in W_0^1 L_M(\Omega,\rho)$. *Then, there exists a sequence* $u_n \in D(\Omega)$ *such that*

$$u_n \to u \textit{ for modular convergence in } W_0^1 L_M(\Omega,\rho).$$

Furthermore, if $u \in W_0^1 L_M(\Omega,\rho) \cap L^\infty(\Omega)$ *then* $\|u_n\|_\infty \le (N+1)\|u\|_\infty$.

Lemma 5.2.11 *Let Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property. If $u \in (W_0^1 L_{M,\rho}(\Omega))^N$ then $\int_\Omega \operatorname{div}(u)dx = 0$.*

Proof 5.2.7 Fix *a vector $u = \left(u^1, \ldots, u^N\right) \in \left(W_0^1 L_{M,\rho}(\Omega)\right)^N$. Since $W_0^1 L_{M,\rho}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^1 L_{M,\rho}(\Omega)$, then each term u^i can be approximated by a suitable sequence $u_k^i \in D(\Omega)$ such that u_k^i converges to u^i in $W_0^1 L_{M,\rho}(\Omega)$. Moreover, due to the fact that $u_k^i \in C_0^\infty(\Omega)$, then the Green formula gives*

$$\int_\Omega \frac{\partial u_k^i}{\partial x_i} = \int_{\partial\Omega} u_k^i \vec{n} ds = 0. \tag{5.2.11}$$

On the other hand, $\frac{\partial u_k^i}{\partial x_i} \to \frac{\partial u^i}{\partial x_i}$ in $L_{M,\rho}(\Omega)$. Thus $\frac{\partial u_k^i}{\partial x_i} \to \frac{\partial u^i}{\partial x_i}$ in $L^1(\Omega)$, which gives in view of (5.2.11) *that*

$$\int_\Omega \operatorname{div}(u)dx = 0.$$

5.3 Main results

Let Y be a closed subspace of $W^1 L_M(\Omega,\rho)$ for $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$ and let

$$Y_0 = Y \cap W^1 L_M(\Omega,\rho),$$

such that Y is the closure of Y_0 for $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$. In the next, we consider the complementary system (Y, Y_0, Z, Z_0) generated by Y i.e. Y_0^* can be identified to Z and Z_0^* can be identified to Y by the means $\langle ., . \rangle$. Let the mapping T (associated to the operator A) defined from $D(T) = \{u \in Y, a_0(x,u,\nabla u) \in L_{\overline{M}}(\Omega), a_i(x,u,\nabla u) \in L_{\overline{M}}(\Omega,\rho)\}$ into Z by the formula

$$a(u,v) = \int_\Omega a_0(x,u,\nabla u)v(x)dx + \sum_{1\le i\le N} \int_\Omega a_i(x,u,\nabla u)\frac{\partial v(x)}{\partial x_i}\rho(x)dx \quad \forall v \in Y_0.$$

We consider the complementary system

$$(Y, Y_0, Z, Z_0) = (W_0^1 L_M(\Omega,\rho), W_0^1 E_M(\Omega,\rho), W^{-1} E_{\overline{M}}(\Omega,\rho), W^{-1} L_{\overline{M}}(\Omega,\rho)).$$

As in [47], we define the entropy solution of our problem.

Definition 5.3.1 *An entropy solution of the problem* (5.1.1) *is a measurable function u such that* $T_k(u) \in W_0^1 L_M(\Omega,\rho)$ *for every* $k>0$ *and such that*

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi)\rho(x)dx \le \int_\Omega f T_k(u-\phi)dx + \int_\Omega F(u)\nabla T_k(u-\phi)$$

for every $\phi \in W_0^1 E_M(\Omega,\rho)\cap L^\infty(\Omega)$.

Our main results are collected in the following theorem.

Theorem 5.3.1 *Under the assumptions (5.1.2)-(5.1.6),*(5.2.6)–(5.2.8) *and* $\rho(x)$ *be a weight function on* Ω *satisfy* (5.2.3) *there exist an entropy solution* u *of the problem* (5.1.1).

5.3.1 Main Lemma

Lemma 5.3.1 *Let* u *be a mesurable function such that* $T_k(u)$ *belongs to* $W_0^1 L_M(\Omega,\rho)$ *for every* $k>0$. *Then*

$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx \le \int_\Omega f\,T_k(u-\phi)dx + \int_\Omega F(u)\nabla T_k(u-\phi) \tag{5.3.1}$$

is equivalent to

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi)\,dx = \int_\Omega f\,T_k(u-\phi)dx + \int_\Omega F(u)\nabla T_k(u-\phi) \tag{5.3.2}$$

for every $\phi \in W_0^1 L_M(\Omega,\rho)\cap L^\infty(\Omega)$, *and for every* $k>0$.

5.3.2 Proof of lemma 5.3.1

In fact (5.3.2) implies (5.3.1) is easily proved adding and subtracting

$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx,$$

and then using assumption (5.1.3). Thus, it remains to prove that (5.3.1) implies (5.3.2). Let h and k be positive real numbers, let $\lambda \in\,]-1,1[$ and $\Psi \in W_0^1 L_M(\Omega,\rho)\cap L^\infty(\Omega)$.

choose, $\phi = T_h(u-\lambda T_k(u-\Psi)) \in W_0^1 L_M(\Omega,\rho)\cap L^\infty(\Omega)$ as test function in (5.3.1), we have:

$$I_{hk} \le J_{hk}, \tag{5.3.3}$$

with

$$I_{hk} = \int_\Omega a(x,u,\nabla T_h(u - \lambda T_k(u - \Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u - \Psi))) \; dx,$$

and

$$J_{hk} = \int_\Omega f\, T_k(u - T_h(u - \lambda T_k(u - \Psi)))\, dx + \int_\Omega F(u)\nabla T_k(u - T_h(u - \lambda T_k(u - \Psi))).$$

Put

$$A_{hk} = \{x \in \Omega, |u - T_h(u - \lambda T_k(u - \Psi))| \leq k\},$$

and

$$B_{hk} = \{x \in \Omega, |u - \lambda T_k(u - \Psi)| \leq h\}.$$

Then, we obtain

$$\begin{aligned} I_{hk} = &\int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u - \lambda T_k(u - \Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u - \Psi)))\, dx \\ &+ \int_{A_{kh}\cap B_{hk}^C} a(x,u,\nabla T_h(u - \lambda T_k(u - \Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u - \Psi)))\, dx \\ &+ \int_{A_{kh}^C} a(x,u,\nabla T_h(u - \lambda T_k(u - \Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u - \Psi)))\, dx. \end{aligned}$$

Since $\nabla T_k(u - T_h(u - \lambda T_k(u - \Psi)))$ is different to zero only on A_{kh}, we have

$$\int_{A_{kh}^C} a(x,u,\nabla T_h(u - \lambda T_k(u - \Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u - \Psi)))\, dx = 0. \tag{5.3.4}$$

Moreover, if $x \in B_{hk}^C$, we have $\nabla T_h(u - \lambda T_k(u - \Psi)) = 0$ and using (5.1.4), we deduce that,

$$\begin{aligned} &\int_{A_{kh}\cap B_{hk}^C} a(x,u,\nabla T_h(u - \lambda T_k(u - \Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u - \Psi)))\, dx \\ &\quad = \int_{A_{kh}\cap B_{hk}^C} a(x,u,0)\nabla T_k(u - T_h(u - \lambda T_k(u - \Psi)))\, dx = 0. \end{aligned} \tag{5.3.5}$$

From (5.3.4) and (5.3.5), we obtain

$$I_{hk} = \int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u - \lambda T_k(u - \Psi)))\nabla T_k(u - T_h(u - \lambda T_k(u - \Psi)))\, dx.$$

Letting $h \to +\infty, |\lambda| \leq 1$, we have

$$A_{kh} \to \{x, |\lambda||T_k(u - \Psi)| \leq h\} = \Omega, \tag{5.3.6}$$

$$B_{hk} \to \Omega \quad which\, implies \quad A_{kh} \cap B_{hk} \to \Omega. \tag{5.3.7}$$

Which and using Lebesgue theorem, we conclude that

$$\lim_{h\to+\infty}\int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx$$
$$=\lambda\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx. \tag{5.3.8}$$

thus implies that,

$$\lim_{h\to+\infty} I_{hk}=\lambda\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx. \tag{5.3.9}$$

On the other hand, we have,

$$J_{hk}=\int_\Omega f\,T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx+\int_\Omega F(u)\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx.$$

Then

$$\lim_{h\to+\infty}\int_\Omega f\,T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx \tag{5.3.10}$$
$$+\int_\Omega F(u)\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx \tag{5.3.11}$$
$$=\lambda(\int_\Omega fT_k(u-\Psi)dx+\int_\Omega F(u)\nabla T_k(u-\Psi)\,dx), \tag{5.3.12}$$

i.e.,

$$\lim_{h\to+\infty} J_{hk}=\lambda(\int_\Omega fT_k(u-\Psi)dx+\int_\Omega F(u)\nabla T_k(u-\Psi)\,dx),. \tag{5.3.13}$$

Together (5.3.9), (5.3.13) and passing to the limit in (5.3.3), we obtain,

$$\lambda\left(\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx\right)\le\lambda\left(\int_\Omega fT_k(u-\Psi)dx+\int_\Omega F(u)\nabla T_k(u-\Psi)\,dx\right)$$

for every $\Psi\in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$, and for every $k>0$. Choosing $\lambda>0$ dividing by λ, and then letting λ tend to zero, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\,dx\le\int_\Omega fT_k(u-\Psi)dx+\int_\Omega F(u)\nabla T_k(u-\Psi)\,dx. \tag{5.3.14}$$

for $\lambda<0$, dividing by λ, and then letting λ tend to zero, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\,dx\ge\int_\Omega fT_k(u-\Psi)dx+\int_\Omega F(u)\nabla T_k(u-\Psi)\,dx. \tag{5.3.15}$$

Combining (5.3.14) and (5.3.15), we conclude the following equality :

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\,dx=\int_\Omega fT_k(u-\Psi)dx+\int_\Omega F(u)\nabla T_k(u-\Psi)\,dx. \tag{5.3.16}$$

This completes the proof of Lemma 5.3.1.

5.3.3 Proof of Theorem 5.3.1

Approximate problem and a priori estimate

For $n \in \mathbb{N}$, define $f_n := T_n(f), F_n = F(T_n)$. Let u_n be solution in $W_0^1 L_\varphi(\Omega)$ of the problem

$$\begin{cases} -\operatorname{div}(a(x, u_n, \nabla u_n)) = f_n - \operatorname{div} F_n(u_n) & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.3.17}$$

which exists thanks to [94]. Choosing $T_k(u_n)$ as test function in (5.3.17), we have

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n)\, dx = \int_\Omega f_n T_k(u_n)\, dx + \int_\Omega F_n(u_n)\nabla T_k(u_n)\, dx,$$

We claim that:

$$\int_\Omega F_n(u_n)\nabla T_k(u_n)\, dx = 0, \tag{5.3.18}$$

using $\nabla T_k(u_n) = \nabla u_{n\chi\{|u_n|\le k\}}$, define $\Theta(t) = F_n(t)\chi\{t| \le k\}$, and $\tilde{\Theta}(t) = \displaystyle\int_0^t \Theta(\tau)\, d\tau$,

We have by Lemma 6.3.9 $\tilde{\Theta}(u_n) \in (W_0^1 L_\varphi(\Omega))^N$

$$\int_\Omega F_n(u_n)\nabla T_k(u_n)\, dx = \int_\Omega F_n(u_n)\chi\{|u_n| \le k\}\nabla u_n\, dx$$

$$= \int_\Omega \Theta(u_n)\nabla u_n\, dx = \int_\Omega \operatorname{div}(\tilde{\Theta}(u_n))\, dx = 0 \tag{5.3.19}$$

(by 6.3.9) which proves the claim.

Now thanks to assumption (5.1.4), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n)\, dx \ge \int_\Omega \rho(x) M(\lambda_1 |\nabla T_k(u_n)|) dx,$$

then

$$\int_\Omega \rho(x) M(\lambda_1 |\nabla T_k(u_n)|) dx \le k \|f\|_{L^1(\Omega)}. \tag{5.3.20}$$

Then

$$\int_\Omega \rho(x) M(\lambda_1 |\nabla T_k(u_n)|) dx \le C_1 k, \tag{5.3.21}$$

where C_1 is a constant independently of n.

Locally convergence of u_n in measure

Taking $\lambda|T_k(u_n)|$ in (5.3.17) and using (5.3.21), one has

$$\int_\Omega \rho(x)M(\lambda_1\frac{|\nabla T_k(u_n)|}{\lambda})dx \leq \int_\Omega \rho(x)M(\lambda_1|\nabla T_k(u_n)|)dx \leq C_1k. \tag{5.3.22}$$

Then we deduce by using (5.3.22), that

$$\begin{aligned} meas\{|u_n| > k\} &\leq \frac{1}{\inf\limits_k M(\frac{k}{\lambda})}\int_{\{|u_n|>k\}} M(\frac{|u_n(x)|}{\lambda})dx \\ &\leq \frac{1}{\inf\limits_k M(\frac{k}{\lambda})}\int_\Omega M(\frac{1}{\lambda}|T_k(u_n)|)dx \\ &\leq \frac{C_1k}{\inf\limits_k M(\frac{k}{\lambda})} \quad \forall n, \quad \forall k \geq 0. \end{aligned} \tag{5.3.23}$$

For any $\beta > 0$, we have

$$meas\{|u_n - u_m| > \beta\} \leq meas\{|u_n| > k\} + meas\{|u_m| > k\} + meas\{|T_k(u_n) - T_k(u_m)| > \beta\},$$

and so that

$$meas\{|u_n - u_m| > \beta\} \leq \frac{2C_1k}{\inf\limits_{x\in\Omega} M(\frac{k}{\lambda})} + meas\{|T_k(u_n) - T_k(u_m)| > \beta\}. \tag{5.3.24}$$

By using (5.3.21) and Poincaré inequality in weighted Orlicz-Sobolev spaces (Theorem5.2.2), we deduce that $(T_k(u_n))$ is bounded in $W_0^1L_M(\Omega,\rho)$, and then there exists $\omega_k \in W_0^1L_M(\Omega,\rho)$ such that $T_k(u_n) \rightharpoonup \omega_k$ weakly in $W_0^1L_M(\Omega,\rho)$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$; strongly in $E_{\overline{M}}(\Omega,\rho)$ and a.e. in Ω.
Consequently, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω.
Let $\varepsilon > 0$, then by (5.3.24) and the fact that $\frac{2C_1k}{\inf\limits_{x\in\Omega} M(\frac{k}{\lambda})} \to 0$ as $k \to +\infty$ there exists some $k = k(\varepsilon) > 0$ such that

$$meas\{|u_n - u_m| > \lambda\} < \varepsilon, \quad \text{for all } n, m \geq h_0(k(\varepsilon), \lambda).$$

This proves that u_n is a Cauchy sequence in measure, thus, u_n converges almost everywhere to some measurable function u. Finally, there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a function $u \in W_0^1L_M(\Omega,\rho)$ such that

$$\begin{cases} u_n \rightharpoonup u \quad \text{weakly in } W_0^1L_M(\Omega,\rho) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}) \\ u_n \longrightarrow u \quad \text{strongly in } E_M(\Omega,\rho) \text{ and a.e. in } \Omega. \end{cases} \tag{5.3.25}$$

An intermediate Inequality

In this step, we shall prove that for $\phi \in W_0^1 L_M(\Omega,\rho) \cap L^\infty(\Omega)$, we have

$$\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx \le \int_\Omega f_n T_k(u_n-\phi)\,dx + \int_\Omega F_n \nabla T_k(u_n-\phi)\,dx. \tag{5.3.26}$$

We choose now $T_k(u_n-\phi)$ as test function in (5.3.17), with ϕ in $W_0^1 L_M(\Omega,\rho)\cap L^\infty(\Omega)$, we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\phi)\,dx = \int_\Omega f_n T_k(u_n-\phi)\,dx + \int_\Omega F_n \nabla T_k(u_n-\phi)\,dx. \tag{5.3.27}$$

Adding and substracting the term $\int_\Omega a(x,u_n,\nabla\phi),\nabla T_k(u_n-\phi)\,dx$ i.e.,

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\phi)\,dx + \int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx \tag{5.3.28}$$

$$-\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx = \int_\Omega f_n T_k(u_n-\phi)dx + \int_\Omega F_n\nabla T_k(u_n-\phi)\,dx$$

.

Thanks to assumption (5.1.3) and the definition of truncation function, we have

$$\int_\Omega (a(x,u_n,\nabla u_n)-a(x,u_n,\nabla\phi))\nabla T_k(u_n-\phi)\,dx \ge 0. \tag{5.3.29}$$

Combining (5.3.28) and (5.3.29), we obtain (5.3.26).

Passing to the limit

We shall prove that for $\phi \in W_0^1 L_M(\Omega,\rho)\cap L^\infty(\Omega)$, we have

$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx \le \int_\Omega f T_k(u-\phi)dx + \int_\Omega F\nabla T_k(u-\phi)\,dx.$$

Firstly, we claim that

$$\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx \to \int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx \text{ as } n\to+\infty.$$

Since $T_M(u_n) \rightharpoonup T_M(u)$ weakly in $W_0^1 L_M(\Omega,\rho)$, with $M = k+\|\phi\|_\infty$, then

$$T_k(u_n-\phi) \rightharpoonup T_k(u-\phi) \text{ in } W_0^1 L_M(\Omega,\rho), \tag{5.3.30}$$

which gives

$$\frac{\partial T_k}{\partial x_i}(u_n - \phi) \rightharpoonup \frac{\partial T_k}{\partial x_i}(u - \phi) \text{ weakly in } L_M(\Omega, \rho) \qquad \forall i = 1, .., N. \tag{5.3.31}$$

Show that

$$a(x, T_M(u_n), \nabla\phi) \to a(x, T_M(u), \nabla\phi) \text{ strongly in } (L_{\overline{M}}(\Omega))^N.$$

Thanks to assumption (5.1.2), we obtain

$$|a_i(x, T_M(u_n), \nabla\phi)| \leq |\phi_i(x)| + K_i \overline{P}^{-1}(\rho^{-1}(x) M(c_2|T_M(u_n)|)) + K_i \overline{M}^{-1} M(c_1 |\nabla\phi|),$$

with β and μ are positive constants. Since $T_M(u_n) \rightharpoonup T_M(u)$ weakly in $W_0^1 L_M(\Omega, \rho)$ and $W_0^1 L_M(\Omega, \rho) \hookrightarrow\hookrightarrow L_{\overline{M}}(\Omega, \rho)$, then $T_M(u_n) \rightharpoonup T_M(u)$ strongly in $L_M(\Omega, \rho)$ and a.e. in Ω, hence

$$|a(x, T_M(u_n), \nabla\phi)| \to |a(x, T_M(u), \nabla\phi)| \text{ a.e. in } \Omega.$$

and

$$|\phi_i(x)| + K_i \overline{P}^{-1}(\rho^{-1}(x) M(c_2|T_M(u_n)|)) + K_i \overline{M}^{-1} M(c_1 |\nabla\phi|) \to$$

$$|\phi_i(x)| + K_i \overline{P}^{-1}(\rho^{-1}(x) M(c_2|T_M(u)|)) + K_i \overline{M}^{-1} M(c_1 |\nabla\phi|),$$

a.e. in Ω. Then, By Vitali's theorem, we deduce that

$$a(x, T_M(u_n), \nabla\phi) \to a(x, T_M(u), \nabla\phi) \text{ strongly in } (L_{\overline{M}}(\Omega, \rho))^N, \text{ as } n \to \infty. \tag{5.3.32}$$

Combining (5.3.31) and (5.3.32), we obtain

$$\int_\Omega a(x, u_n, \nabla\phi) \nabla T_k(u_n - \phi)\, dx \to \int_\Omega a(x, u, \nabla\phi) \nabla T_k(u - \phi)\, dx \text{ as } n \to +\infty. \tag{5.3.33}$$

Secondly, we show that

$$\int_\Omega f_n T_k(u_n - \phi) dx \to \int_\Omega f T_k(u - \phi) dx. \tag{5.3.34}$$

and

$$\int_\Omega F_n \nabla T_k(u_n - \phi)\, dx \to \int_\Omega F \nabla T_k(u - \phi)\, dx \tag{5.3.35}$$

We have $f_n T_k(u_n - \phi) \to f T_k(u - \phi)$ a.e. in Ω and $|f T_k(u_n - \phi)| \leq k|f|$, and

$F_n \nabla T_k(u_n - \phi) \to F \nabla T_k(u - \phi)$ a.e. in Ω, and $|F \nabla T_k(u_n - \phi)| \leq k|F|$ then by using Vitali's theorem,

we obtain (5.3.34)and (5.3.35) . Thanks to (5.3.33), (5.3.34)and (5.3.35) allow to pass to the limit in the inequality (5.3.26), so that $\forall \phi \in W_0^1 L_M(\Omega, \rho) \cap L^\infty(\Omega)$, we deduce

$$\int_\Omega a(x, u, \nabla\phi)\nabla T_k(u - \phi)\, dx \leq \int_\Omega f T_k(u - \phi)dx + \int_\Omega F\nabla T_k(u - \phi)dx.$$

In view of main Lemma, we can deduce that u is an entropy solution of the problem (5.1.1). This completes the proof of Theorem 5.3.1.

Part III

Nonlinear elliptic problem in Musielak-Orlicz-Sobolev spaces

Chapter 6

Existence of renomalized solution for nonlinear elliptic boundary value problem without Δ_2 -condition

Abstract

In this paper we will prove in Musielak-Orlicz spaces, the existence of renomalized solution for nonlinear elliptic equations of Leray-Lions type, in the case where the Musielak-Orlicz function φ doesn't satisfy the Δ_2 condition while the right hand side f belongs to $W^{-1}E_{\psi}(\Omega)$.

6.1 Introduction and Basic Assumptions

This work deals with existence of solutions for strongly nonlinear boundary value problem whose model is:

$$\begin{cases} A(u) - \operatorname{div}\Phi(u) + g(x,u,\nabla u) = f \quad in \quad \Omega \\ \qquad\qquad\qquad\qquad\qquad\qquad u \equiv 0, \quad on \quad \partial\Omega \end{cases} \tag{6.1.1}$$

where Ω be a bounded domain of $\mathbb{R}^N, N \geq 2$, $A(u) = -\operatorname{div} a(x,u,\nabla u)$ be a Leray-Lions operator defined from the space $W_0^1 L_{\varphi}(\Omega)$ into its dual $W^{-1}L_{\psi}(\Omega)$, and $\Phi \in \mathrm{C}^0\left(\mathrm{R}, \mathrm{R}^N\right)$. where a is a function

satisfying the following conditions :

$$a(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}^N \text{ is a Carathéodory function.} \tag{6.1.2}$$

There exist two Musielak-Orlicz functions φ and γ such that $\gamma\prec\prec\varphi$, a positive function $d(\cdot)\in E_\psi(\Omega)$ and positive constants k_1,k_2 and k_3 such that for a.e. $x\in\Omega$ and for all $s\in\mathbb{R},\xi\in\mathbb{R}^N$

$$|a(x,s,\xi)|\leq k_1\left(d(x)+\psi_x^{-1}\gamma(x,k_2|s|)\right)+\psi_x^{-1}\varphi(x,k_3|\xi|); \tag{6.1.3}$$

$$\left(a(x,s,\xi)-a\left(x,s,\xi'\right)\right)\left(\xi-\xi'\right)>0; \tag{6.1.4}$$

$$a(x,s,\xi).\xi\geq\alpha\varphi(x,|\xi|). \tag{6.1.5}$$

Furthermore, let $g(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ be a Carathéodory function such that for a.e. $x\in\Omega$ and for all $s\in\mathbb{R},\xi\in\mathbb{R}^N$, satisfying the following conditions

$$|g(x,s,\xi)|\leq c(x)+b(|s|)\varphi(x,|\xi|); \tag{6.1.6}$$

$$g(x,s,\xi)s\geq 0; \tag{6.1.7}$$

where $b:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a continuous positive function which belongs to $L^1\left(\mathbb{R}^+\right)$ and $c(\cdot)\in L^1(\Omega)$ The right-hand side of (6.1.1) and $\Phi:\mathbb{R}\to\mathbb{R}^N$ are assumed to satisfy

$$f\in W^{-1}E_\psi(\Omega); \tag{6.1.8}$$

$$\Phi\in\mathcal{C}^0\left(\mathbb{R},\mathbb{R}^N\right). \tag{6.1.9}$$

Note that no growth hypothesis is assumed on the function Φ, which implies that the term $-\text{div }\Phi(u)$ may be meaningless, even as a distribution.

Several researches deals with the existence solutions of elliptic and parabolic problems under various assumptions and in different contexts (see [9, 12, 13, 26, 31, 32, 29, 75, 65, 66, 76, 14, 15, 16, 17, 18, 19, 36, 37, 27, 28, 126, 127, 108, 114] for more details), indeed we can't recite all examples; we will just choose some of them, So we mention that:

the problem (6.1.1) was treated by Boccardo (see [47]) in the case $g\equiv 0$ and for p such that $2-1/N<p<N$ where he proved the existence and regularity of an entropy solution u that is $u\in W_0^{1,q}(\Omega),\quad q<$

$\tilde{p} = \frac{(p-1)N}{N-1}$, $T_k(u) \in W_0^{1,p}(\Omega), \quad \forall k > 0$. The same problem have been studied by Diperna and lions in [66] where they introduced the idea of renormalized solutions.

In the framework of variable exponent Sobolev spaces in [38] have proved the existence result of solutions for the problem 6.1.1 without sign condition involving nonstandard growth.

In the setting of Musielak spaces and in variational case, the existence of a weak solution for the problem (6.1.1) was treated by M. L. Ahmed Oubeid, A. Benkirane and M. Sidi El Vally in [5] where $\operatorname{div} \Phi \equiv 0$.

Our purpose in this paper is to show the existence of renormalized solutions for problem (6.1.1) in Musielak Orlicz spaces in the case where the Musielak-Orlicz function φ doesn't satisfy the Δ_2 condition,while the right-hand side belongs to $W^{-1}E_{\psi}(\Omega)$, $\Phi \in \mathcal{C}^0\left(\mathbb{R}, \mathbb{R}^N\right)$. and a nonlinearity $g(x, s, \xi)$ having natural growth with respect to the gradient.

The paper is organized as follows: In section 6.2 , we give some preliminaries and background. Section 6.3 is devoted to some technical lemmas which can be used to our result. In the final section 6.4, we state our main result and give the prove of an existence solution .

6.2 Some Preliminaries and Background

Here we give some definitions and properties that concern Musielak-Orlicz spaces (see [104]).

Let Ω be an open subset of $\mathbb{R}^n$, a Musielak-Orlicz function φ is a real-valued function defined in $\Omega \times \mathbb{R}_+$ such that

a): $\varphi(x, t)$ is an N-function i.e. convex, nondecreasing, continuous, $\varphi(x, 0) = 0$, $\varphi(x, t) > 0$ for all $t > 0$ and

$$\lim_{t \to 0} \sup_{x \in \Omega} \frac{\varphi(x, t)}{t} = 0, \qquad \lim_{t \to \infty} \inf_{x \in \Omega} \frac{\varphi(x, t)}{t} = 0$$

b): $\varphi(x, t)$ is a measurable function for all $t \geq 0$.

Now, let $\varphi_x(t) = \varphi(x, t)$ and let φ_x^{-1} be the non-negative reciprocal function with respect to t, i.e the

function that satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi\left(x, \varphi_x^{-1}(t)\right) = t$$

The Musielak-orlicz function φ is said to satisfy the Δ_2 -condition if for some $k > 0$, and a non negative function h, integrable in Ω, we have

$$\varphi(x,2t) \leq k\varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \tag{6.2.1}$$

When (6.2.1) holds only for $t \geq t_0 > 0$, then φ is said to satisfy the Δ_2 -condition near infinity. Let φ and γ be two Musielak-orlicz functions, we say that φ dominate γ and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x \in \Omega$

$\gamma(x,t) \leq \varphi(x,ct)$ for all $t \geq t_0$, (resp. for all $t \geq 0$ i.e. $t_0 = 0$) We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma \prec\prec \varphi$ if for every positive constant c we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0, \quad \left(\text{ resp. } \lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0\right)$$

"Remark 2.1": (see [7]) If $\gamma \prec \varphi$ near infinity such that γ is locally integrable on Ω, then $\forall c > 0$ there exists a nonnegative integrable function h such that

$$\gamma(x,t) \leq \varphi(x,ct) + h(x), \text{. for all } t \geq 0 \text{ and for a. e. } x \in \Omega.$$

For a Musielak-Orlicz function φ and a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx$$

The set $K_\varphi(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \text{ measurable } / \rho_{\varphi,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable } / \rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < \infty, \text{ for some } \lambda > 0\right\}$$

For a Musielak-Orlicz function φ we put: $\psi(x,s) = \sup_{t>0}\{st - \varphi(x,t)\}$, ψ is the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sens of Young with respect to the variable s In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda > 0/ \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}$$

which is called the Luxemburg norm and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)v(x)|dx$$

where ψ is the Musielak Orlicz function complementary to φ. These two norms are equivalent (see [104])

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_\varphi(\Omega)$, It is a separable space (see [104], Theorem 7.10) .

We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall |\alpha| \leq m, D^\alpha u \in L_\varphi(\Omega)\}$$

and

$$W^m E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) : \forall |\alpha| \leq m, D^\alpha u \in E_\varphi(\Omega)\}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + \ldots + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.

Let

$$\bar{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \rho_{\varphi,\Omega}\left(D^\alpha u\right) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\left\{\lambda > 0 : \bar{\rho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

for $u \in W^m L_\varphi(\Omega)$. These functionals are a convex modular and a norm on $W^m L_M(\Omega)$, respectively, and the pair $\left(W^m L_\varphi(\Omega), |||^m_{\varphi,\Omega}\right)$ is a Banach space if φ satisfies the following condition (see[104]):

$$\text{there exist a constant } c_0 > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c_0 \tag{6.2.2}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closed.

The space $W_0^m L_\varphi(\Omega)$ is defined as the $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. and the space $W_0^m E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W_0^m L_\varphi(\Omega)$ be the $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$ The following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \right\}$$

and

$$W^{-m} E_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \right\}$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n \to \infty} \bar{\rho}_{\varphi,\Omega} \left(\frac{u_n - u}{k} \right) = 0$$

For φ and her complementary function ψ, the following inequality is called the Young inequality (see[104]):

$$ts \leq \varphi(x,t) + \psi(x,s), \quad \forall t, s \geq 0, x \in \Omega \tag{6.2.3}$$

This inequality implies that

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1 \tag{6.2.4}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} > 1 \tag{6.2.5}$$

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} \leq 1 \tag{6.2.6}$$

For two complementary Musielak Orlicz functions φ and ψ, let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, then we have the Holder inequality (see[104]):

$$\left| \int_\Omega u(x) v(x) dx \right| \leq \|u\|_{\varphi,\Omega} \| |v| \|_{\psi,\Omega} \tag{6.2.7}$$

We will use the following technical lemmas.

6.3 Some Technical Lemmas

Lemma 6.3.1 *[31]*

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:

i)There exist a constant $c > 0$ such that inf $_{x\in\Omega}\varphi(x, 1) \geq c$.

ii) There exist a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$ we have

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{\log\left(\frac{1}{|x-y|}\right)}\right)}, \quad \forall t \geq 1 \tag{6.3.1}$$

iii)

$$\text{If } D \subset \Omega \text{ is a bounded measurable set, then } \int_D \varphi(x, 1)dx < \infty \tag{6.3.2}$$

iv) There exist a constant $C > 0$ such that $\psi(x, 1) \leq C$ a.e in Ω.

Under this assumptions, $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\bar{\Omega})$ is dense in $W^1 L_\varphi(\Omega)$ the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L_\psi(\Omega)$ on an element u of $W_0^1 L_\varphi(\Omega)$ is well defined. It will be denoted by $< S, u >$.

Lemma 6.3.2 *([113]) Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak-Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$ Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{a.e } in\{x \in \Omega : u(x) \in D\} \\ 0 & \text{a.e in } \{x \in \Omega : u(x) \notin D\} \end{cases}$$

Lemma 6.3.3 *[6] (Poincare's inequality). Let φ a Musielak Orlicz function which satisfies the assumptions of lemma 6.3.1, suppose that $\varphi(x, t)$ decreases with respect of one of coordinate of x Then, that exists a constant $c > 0$ depends only of Ω such that*

$$\int_\Omega \varphi(x, |u(x)|)dx \leq \int_\Omega \varphi(x, c|\nabla u(x)|)dx, \quad \forall u \in W_0^1 L_\varphi(\Omega)$$

Lemma 6.3.4 *[31] Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then, there exists a sequence $(u_n) \subset \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_\varphi(\Omega)$$

Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.

Lemma 6.3.5 *Let $(f_n), f \in L^1(\Omega)$ such that*
i) $f_n \geq 0$ a.e in Ω
ii) $f_n \longrightarrow f$ a.e in Ω
iii) $\int_\Omega f_n(x)dx \longrightarrow \int_\Omega f(x)dx$
then $f_n \longrightarrow f$ strongly in $L^1(\Omega)$

Lemma 6.3.6 *[32] If a sequence $g_n \in L_\varphi(\Omega)$ converges in measure to a measurable function g and if g_n remains bounded in $L_\varphi(\Omega)$, then $g \in L_\varphi(\Omega)$ and $g_n \rightharpoonup g$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$*

Lemma 6.3.7 *(Jensen inequality). [115] Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ a convex function and $g : \Omega \longrightarrow \mathbb{R}$ is function measurable, then*

$$\varphi\left(\int_\Omega g d\mu\right) \leq \int_\Omega \varphi \circ g d\mu.$$

Lemma 6.3.8 *(The Nemytskii Operator)[7]. Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak Orlicz functions. Let $f : \Omega \times \mathbb{R}^p \longrightarrow \mathbb{R}^q$ be a Carathodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:*

$$|f(x,s)| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|)$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$ Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is continuous from

$$\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right)^p = \prod \left\{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\right\}$$

into $(L_\psi(\Omega))^q$ for the modular convergence.
Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right)^p$ to $(E_\gamma(\Omega))^q$

Lemma 6.3.9 *Let Ω be a bounded open subset of R^N with the segment property. If $u \in (W_0^1 L_\varphi(\Omega))^N$ then $\int_\Omega$ div $u\, dx = 0$.*

Proof of lemma 6.3.9: The proof of this lemma is based on [[73], Lemma 3.2]

6.4 Main Result

We consider the following boundary value problem

$$(\mathcal{P})\begin{cases} A(u)-\operatorname{div}\Phi(u)+g(.,u,\nabla u)=f\in W^{-1}E_{\psi}(\Omega), & \text{in } \Omega \\ u\equiv 0, & \text{on } \partial\Omega \end{cases}$$

Let us define

$$\mathcal{T}_0^{1,\varphi}(\Omega)=\left\{u \text{ measurable} \quad \text{such that} \quad T_k(u)\in W_0^1L_{\varphi}(\Omega),\forall k>0\right\}.$$

As in [39] we define the following notion of renormalized solution, which gives a meaning to a possible solution of $(\mathcal{P})$

Definition 6.4.1 *Let φ be a Musielak function satisfying the conditions of Lemma 6.3.1 and lemma 6.3.3. Suppose that assumptions* (6.1.2) − (6.1.8) *and* (6.1.9) *hold true. A function u is a renormalized solution of the problem $(\mathcal{P})$ if*

$$\begin{cases} u\in \mathcal{T}_0^{1,\varphi}(\Omega), g(.,u,\nabla u)\in L^1(\Omega), g(.,u,\nabla u)u\in L^1(\Omega) \\ \displaystyle\int_{\Omega} a(x,u,\nabla u)h(u)\nabla v dx+\int_{\Omega} a(x,u,\nabla u)h'(u)\nabla u v dx+\int_{\Omega}\Phi(u)h(u)\nabla v dx+\int_{\Omega}\Phi(u)h'(u)\nabla u v dx \\ \qquad\qquad +\displaystyle\int_{\Omega} g(x,u,\nabla u)h(u)v dx=\int_{\Omega} fh(u)v dx \end{cases} \tag{6.4.1}$$

for all $h\in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support in $\mathbb{R}$, and for all $v\in W_0^1L_{\varphi}(\Omega)\cap L^{\infty}(\Omega)$.

The weaker problem (6.4.1) is obtained by using the test function $h(u)v$ where $h\in W^{1,\infty}(\mathbb{R})$. and $v\in W_0^1L_{\varphi}(\Omega)\cap L^{\infty}(\Omega)$ in $(\mathcal{P})$.

Remark 6.4.1 *Let us note that in* (6.4.1) *every term is meaningful in the distributional sense.*

Theorem 6.4.1 *Let φ be a Musielak function satisfying the conditions of Lemma 6.3.1 and lemma 6.3.3. Suppose that assumptions* (6.1.2) − (6.1.8) *and* (6.1.9) *hold true. there exists at least a renormalized solution u in the sense of definition 6.4.1 of problem $(\mathcal{P})$.*

Let us introduce the truncate operator. For a given constant $k>0$, we define the function $T_k:\mathbb{R}\to\mathbb{R}$ as

$$T_k(s)=\begin{cases} s \text{ if } |s|\le k, \\ k\frac{s}{|s|} \text{ if } |s|>k. \end{cases}$$

6.4.1 Proof of Theorem 6.4.1

Approximate problem and a priori estimate

We use an idea contained in [114] (Theorem 1.1), based on the approximation of the original problem and a priori estimate. For $n \in \mathbb{N}$, let $(f_n)_n$ be a sequence in $W^{-1}E_\psi(\Omega) \cap L^1(\Omega)$ such that $f_n \longrightarrow f$ in $L^1(\Omega)$ with $\|f_n\|_1 \leq \|f\|_1, \phi_n(s) = \phi(T_n(s))$ and $g_n(x,s,\xi) = T_n(g(x,s,\xi))$. The following approximate problem

$$(P_n)\begin{cases} -\operatorname{div}(a(\cdot,u_n,\nabla u_n)) + g_n(\cdot,u_n,\nabla u_n) = f_n + \operatorname{div}(\Phi_n(u_n)) & \text{in } D'(\Omega) \\ \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution u_n in $W_0^1 L_\varphi(\Omega)$.

Now Choosing u_n as a function test in problem (P_n), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n dx + \int_\Omega \Phi_n(u_n)\cdot\nabla u_n dx + \int_\Omega g_n(x,u_n,\nabla u_n)\,u_n dx = \langle f, u_n\rangle \tag{6.4.2}$$

By posing

$$\widetilde{\Phi}_n(t) = \int_0^t \Phi_n(\tau)d\tau$$

we obtain

$$\bar{\Phi}_n(0) = 0.$$

As each component of $\bar{\Phi}_n$ is uniformly Lipschitizian, and according to [[85], Lemma 2], it follows that the function $\bar{\Phi}_n(u_n)$ belongs to $(W_0^1 L_\varphi(\Omega))^N$.

therefore by using Lemma 6.3.9

$$\int_\Omega \Phi_n(u_n)\cdot\nabla u_n dx = \int_\Omega \operatorname{div}\left(\widetilde{\Phi}_n(u_n)\right)dx = 0$$

According to (6.1.7) and using Young's inequality, we have

$$\left|\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n dx\right| \leq C_1 + \frac{\alpha}{2}\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx. \tag{6.4.3}$$

which together with (6.1.5) gives

$$\int_{\Omega} \varphi\left(x, \left|\nabla T_k\left(u_n\right)\right|\right) dx \leq C_2 \tag{6.4.4}$$

Poincare inequality (see Lemma6.3.3) implies that

$$\int_{\Omega} \varphi\left(x, \frac{\left|T_k\left(u_n\right)\right|}{c}\right) dx \leq \int_{\Omega} \varphi\left(x, \left|\nabla T_k\left(u_n\right)\right|\right) dx \leq c_2 k \tag{6.4.5}$$

On the other hand we have

$$\int_{\Omega} g_n\left(x, u_n, \nabla u_n\right) u_n dx \leq C_3 \tag{6.4.6}$$

so it follows that $\left(T_k\left(u_n\right)\right)_n$ and $\left(\nabla T_k\left(u_n\right)\right)_n$ are bounded in $L_\varphi(\Omega)$, Thus

$$\left(T_k\left(u_n\right)\right)_n is \text{ bounded in } W_0^1 L_\varphi(\Omega),$$

there exists some $v_k \in W_0^1 L_\varphi(\Omega)$ such that

$$\begin{cases} T_k\left(u_n\right) \rightharpoonup v_k & \text{weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma\left(\Pi L_\varphi, \Pi E_\psi\right) \\ T_k\left(u_n\right) \longrightarrow v_k & \text{strongly in } E_\psi(\Omega). \end{cases} \tag{6.4.7}$$

Now one suppose that exists a function φ satisfies $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$ and $\varphi(t) \leq \operatorname{ess\,inf}_{x\in\Omega} \varphi(x,t)$ Let $k > 0$ large enough, by using (6.4.5) we have

$$\begin{aligned} \varphi(k) \operatorname{meas}\left\{\left|u_n\right| > k\right\} &= \int_{\left\{\left|u_n\right|>k\right\}} \varphi\left(\left|T_k\left(u_n\right)\right|\right) dx \\ &\leq \int_{\left\{\left|u_n\right|>k\right\}} \varphi\left(x, \left|T_k\left(u_n\right)\right|\right) dx \leq \int_{\Omega} \varphi\left(x, \left|T_k\left(u_n\right)\right|\right) dx \\ &\leq c_3 k \end{aligned}$$

Hence

$$\operatorname{meas}\left\{\left|u_n\right| > k\right\} \leq \frac{c_3 k}{\varphi(k)} \longrightarrow 0 \text{ as } k \longrightarrow \infty$$

For every $\lambda > 0$, we have

$$\begin{aligned} \operatorname{meas}\left\{\left|u_n - u_m\right| > \lambda\right\} \leq & \operatorname{meas}\left\{\left|u_n\right| > k\right\} + \operatorname{meas}\left\{\left|u_m\right| > k\right\} \\ & + \operatorname{meas}\left\{\left|T_k\left(u_n\right) - T_k\left(u_m\right)\right| > \lambda\right\} \end{aligned} \tag{6.4.8}$$

then, by using (6.4.5) one suppose that $\left(T_k\left(u_n\right)\right)_n$ is a Cauchy sequence in measure in Ω, Let $\varepsilon > 0$, then by (6.4.8) there exists some $k = k(\varepsilon) > 0$ such that

$$\operatorname{meas}\left\{\left|u_n - u_m\right| > \lambda\right\} < \varepsilon, \quad \text{for all } n, m \geq h_0(k(\varepsilon), \lambda)$$

which means that $(u_n)_n$ is a Cauchy sequence in measure in Ω, thus converge almost every where to u.

Consequently

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma\left(\Pi L_\varphi, \Pi E_\psi\right) \\ u_n \longrightarrow u \text{ strongly in } E_\psi(\Omega). \end{cases} \tag{6.4.9}$$

In this step we shall show the boundedness of $\left(a\left(\cdot, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\right)_n$ in $\left(L_\psi(\Omega)\right)^N$

Let $\vartheta \in E_\varphi(\Omega)^N$ such that $\|\vartheta\|_{\varphi,\Omega} \leq 1$, the hypothesis (6.1.4) gives We have

$$\int_\Omega \left[a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) - a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right]\left[\nabla T_k\left(u_n\right) - \frac{\vartheta}{k_3}\right] dx > 0$$

This implies that

$$\begin{aligned} &\int_\Omega \frac{1}{k_3} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \vartheta dx \\ &\leq \int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) dx \\ &\quad - \int_\Omega a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\left(\nabla T_k\left(u_n\right) - \frac{\vartheta}{k_3}\right) dx \\ &\leq c_2 k - \int_\Omega a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right) \nabla T_k\left(u_n\right) dx \\ &\quad + \frac{1}{k_3} \int_\Omega a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right) \vartheta dx \end{aligned}$$

By using Young's inequality in the last two terms of the last side and (6.4.5) we get

$$\begin{aligned} &\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \vartheta dx \leq c_2 k k_3 \\ &\quad + 3k_1\left(1 + k_3\right) \int_\Omega \psi\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right|}{3k_1}\right) dx \\ &\quad + 3k_1 k_3 \int_\Omega \varphi\left(x, \left|\nabla T_k\left(u_n\right)\right|\right) dx + 3k_1 \int_\Omega \varphi(x, |\vartheta|) dx \\ &\leq c_2 k k_3 + 3k_1 k_3 c_2 k + 3k_1 \\ &\quad + 3k_1\left(1 + k_3\right) \int_\Omega \psi\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right|}{3k_1}\right) dx \end{aligned}$$

Now, by using (6.1.3) and the convexity of ψ we get

$$\psi\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right|}{3k_1}\right) \leq \frac{1}{3}\left(\psi(x, d(x)) + \gamma\left(x, k_2\left|T_k\left(u_n\right)\right|\right) + \varphi(x, |\vartheta|)\right).$$

Thanks to **"Remark 2.1"** there exists $h \in L^1(\Omega)$ such that

$$\gamma\left(x, k_2\left|T_k\left(u_n\right)\right|\right) \leq \gamma\left(x, k_2 k\right) \leq \varphi(x, 1) + h(x);$$

then by integrating over Ω we deduce that

$$\int_\Omega \psi\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{v}{k_3}\right)\right|}{3k_1}\right) dx \leq \frac{1}{3}\left(\int_\Omega \psi(x, c(x))dx + \int_\Omega h(x)dx + \int_\Omega \varphi(x, 1)dx + \int_\Omega \varphi(x, |\vartheta|)dx\right) \leq c_k,$$

where c_k is a constant depending on k. So,

$$\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\vartheta dx \leq c_k, \quad \forall \vartheta \in \left(E_\varphi(\Omega)\right)^N \quad \text{with } \|\vartheta\|_{\varphi,\Omega} = 1$$

and thus $\left\|a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\right\|_{\psi,\Omega} \leq c_k$, which implies that,

$$\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\right)_n \text{ is bounded in } L_\psi(\Omega)^N. \tag{6.4.10}$$

Let us show that :

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{(m\leq|u_n|\leq m+1|} a\left(x, u_n, \nabla u_n\right)\cdot\nabla u_n dx = 0$$

Defining

$$\theta_m(r) = T_{m+1}(r) - T_m(r) \qquad \text{For any } m \geq 1,$$

in view of [[85], Lemma2] one get $\theta_m\left(u_n\right) \in W_0^1 L_\varphi(\Omega)$.

Now let us taking $\theta_m\left(u_n\right)$ as a test function in (P_n) we have

$$\int_\Omega a\left(x, u_n, \nabla u_n\right)\nabla\theta_m\left(u_n\right)dx + \int_\Omega \Phi_n\left(u_n\right)\nabla\theta_m\left(u_n\right)dx$$

$$+\int_{\Omega} g_n\left(x,u_n,\nabla u_n\right)\theta_m\left(u_n\right)dx=\int_{\Omega} f_n\theta_m\left(u_n\right)dx$$

Consider,

$$\begin{aligned}\phi(t)&=\Phi_n(t)\chi_{\{s\in\mathbb{R},m\leq|s|\leq m+1\}}(t)\\ \tilde{\phi}(t)&=\int_0^t\phi(\tau)d\tau\end{aligned}$$

hence $\tilde{\phi}\left(u_n\right)\in\left(W_0^1L_\varphi(\Omega)\right)^N$ (by Lemma6.3.2). We obtain, by Lemma 6.3.9,

$$\begin{aligned}\int_{\Omega}\Phi_n\left(u_n\right)\nabla\theta_m\left(u_n\right)dx&=\int_{\Omega}\Phi_n\left(u_n\right)\chi_{\{s\in\mathbb{R},m\leq|s|\leq m+1\}}\left(u_n\right)\nabla u_ndx\\&=\int_{\Omega}\phi\left(u_n\right)\nabla u_ndx=\int_{\Omega}\operatorname{div}\left(\tilde{\phi}\left(u_n\right)\right)dx=0\end{aligned}$$

Using the sign condition (6.1.7) we have $g_n\left(x,u_n,\nabla u_n\right)\theta_m\left(u_n\right)\geq 0$ a.e. in Ω, and knowing that $\nabla\theta_m\left(u_n\right)=\nabla u_n\chi_{\{m\leq|u_n|\leq m+1\}}$ a.e. in Ω, we get

$$\int_{\{m\leq|u_n|\leq m+1\}} a\left(x,u_n,\nabla u_n\right)\nabla u_ndx\leq\left\langle f,\theta_m\left(u_n\right)\right\rangle.$$

It is not difficult to see that

$$\left\|\nabla\theta_m\left(u_n\right)\right\|_{\varphi,\Omega}\leq\left\|\nabla u_n\right\|_{\varphi,\Omega}.$$

then in view of (6.4.4) and (6.4.9) it follows that

$$\theta_m\left(u_n\right)\rightharpoonup\theta_m(u)\quad\text{ weakly in }W_0^1L_\varphi(\Omega)\quad\text{for }\sigma\left(\Pi L_\varphi(\Omega),\Pi E_\varphi(\Omega)\right)$$

Therefore, we get

$$\lim_{n\to\infty}\int_{\{m\leq|u_u|\leq m+1\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_ndx\leq\left\langle f,\theta_m(u)\right\rangle$$

as $\theta_m(u)\rightharpoonup 0$ weakly in $W_0^1L_\varphi(\Omega,)$ for $\sigma\left(\Pi L_\varphi(\Omega),\Pi E_\varphi(\Omega)\right)$ one obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{(m\leq||n_n|\leq m+1|} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_ndx\leq\lim_{m\to\infty}\left\langle f,\theta_m(u)\right\rangle=0$$

By (6.1.5), we get

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{(m\leq|u_k|\leq m+1|} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_ndx=0\tag{6.4.11}$$

In this subsubsection we pose $\phi(s) = se^{\lambda s^2}$ where $\lambda = \left(\frac{b(k)}{2\alpha}\right)^2$. it is easy to get,

$$\text{for all } s \in \mathbb{R}, \qquad \phi'(s) - \frac{b(k)}{\alpha}|\phi(s)| \geq \frac{1}{2} \tag{6.4.12}$$

For $m \geq k$, definning

$$\psi_m(s) = \begin{cases} 1 & \text{if } |s| \leq m \\ m+1-|s| & \text{if } m \leq |s| \leq m+1 \\ 0 & \text{if } |s| \geq m+1 \end{cases}$$

Let $\{v_j\}_j \subset D(\Omega)$ be a sequence such that $v_j \to u$ in $W_0^1 L_\varphi(\Omega)$ for the modular convergence and a e. in Ω. And let us define the functions

$$\theta_n^j = T_k(u_n) - T_k(v_j), \theta^j = T_k(u) - T_k(v_j) \text{ and } z_{n,m}^j = \phi\left(\theta_n^j\right)\psi_m(u_n).$$

Using $z_{n,m}^j \in W_0^1 L_\varphi(\Omega)$ as a test function in (P_n) we get

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla z_{n,m}^j dx + \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \cdot \nabla u_n \psi_m'(u_n) \phi(T_k(u_n) - T_k(v_j)) dx \tag{6.4.13}$$

$$+ \int_\Omega \Phi_n(u_n) \cdot \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx + \int_\Omega g_n(x, u_n, \nabla u_n) z_{n,m}^j dx = \int_\Omega f z_{n,m}^j dx$$

From now on, we denote by $\epsilon_i(n,j), i = 0, 1, 2, \ldots$, various sequences of real numbers which tend to zero as n and $j \to \infty$, i.e.,

$$\lim_{j \to +\infty} \lim_{n \to +\infty} \epsilon_i(n,j) = 0$$

by using (6.4.7) one has $z_{n,m}^j \to \phi(\theta^j)\psi_m(u)$ weakly in $L^\infty(\Omega)$ for $\sigma^*(L^\infty, L^1)$ as $n \to \infty$ which give

$$\lim_{n \to \infty} \int_\Omega f z_{n,m}^j dx = \int_\Omega f \phi\left(\theta^j\right) \psi_m(u) dx$$

and $\phi(\theta^j) \to 0$ weakly in $L^\infty(\Omega)$ for $\sigma(L^\infty, L^1)$ as $j \to \infty$, we have

$$\lim_{j \to \infty} \int_\Omega f \phi\left(\theta^j\right) \psi_m(u) dx = 0$$

Therefore, by denoting

$$\int_\Omega f z_{n,m}^j dx = \epsilon_0(n,j),$$

the divergence lemma implies that

$$\int_{\{m\leq|u_n|\leq m+1\}} \Phi_n(u_n)\cdot\nabla u_n\psi_m'(u_n)\,\phi\left(T_k(u_n)-T_k(v_j)\right)dx=0.$$

The third term in the left-hand side of (6.4.13) can be written as follows

$$\begin{aligned}&\int_\Omega \Phi_n(u_n)\cdot\nabla\phi\left(T_k(u_n)-T_k(v_j)\right)\psi_m(u_n)\,dx\\&\qquad=\int_\Omega \Phi_n(u_n)\cdot\nabla T_k(u_n)\,\phi'\left(\theta_n^j\right)\psi_m(u_n)\,dx-\int_\Omega \Phi_n(u_n)\cdot\nabla T_k(v_j)\,\phi'\left(\theta_n^j\right)\psi_m(u_n)\,dx\end{aligned}$$

Applying the divergence lemma we have,

$$\int_\Omega \Phi_n(u_n)\cdot\nabla T_k(u_n)\,\phi'\left(\theta_n^j\right)\psi_m(u_n)\,dx=0.$$

By (6.4.7) one obtain

$$\Phi_n(u_n)\,\phi'\left(\theta_n^j\right)\psi_m(u_n)\to\Phi(u)\phi'\left(\theta^j\right)\psi_m(u)\text{ a.e. in }\Omega\quad as\quad n\to+\infty$$

now, we can verify that

$$\left\|\Phi_n(u_n)\,\phi'\left(\theta_n^j\right)\psi_m(u_n)\right\|_{\varphi,\Omega}\leq\psi\left(x,c_m\phi'(2k)\right)|\Omega|+1$$

with $c_m=\max_{|t|\leq m+1}\Phi(t)$.

Thanks to [[89], Theorem 14.6], we have

$$\lim_{n\to+\infty}\int_\Omega \Phi_n(u_n)\cdot\nabla T_k(v_j)\,\phi'\left(\theta_n^j\right)\psi_m(u_n)\,dx=\int_\Omega \Phi(u)\cdot\nabla T_k(v_j)\,\phi'\left(\theta^j\right)\psi_m(u)dx$$

Using the modular convergence of the sequence $\{v_j\}_j$, it follows that

$$\lim_{j\to+\infty}\lim_{n\to+\infty}\int_\Omega \Phi_n(u_n)\cdot\nabla T_k(v_j)\,\phi'\left(\theta_n^j\right)\psi_m(u_n)\,dx=\int_\Omega \Phi(u)\cdot\nabla T_k(u)\psi_m(u)dx$$

Then, thanks to Lemma 6.3.9 we obtain

$$\int_\Omega \Phi(u)\cdot\nabla T_k(u)\psi_m(u)dx=0$$

Therefore,we denote

$$\int_\Omega \Phi_n(u_n)\cdot\nabla\phi\left(T_k(u_n)-T_k(v_j)\right)\psi_m(u_n)\,dx=\epsilon_1(n,j).$$

since $g_n(x,u_n,\nabla u_n)\,z_{nm}^j\geq 0$ on the set $\{|u_n|>k\}$ and $\psi_m(u_n)=1$ on the set $\{|u_n|\leq k\}$, by according to 6.4.13 we get

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla z_{nm}^j dx+\int_{\{|u_n|\leq k|}} g_n(x,u_n,\nabla u_n)\,\phi\left(\theta_n^j\right)dx\leq\epsilon_2(n,j)\tag{6.4.14}$$

For the first term of the left-hand side of (6.4.14) we can write

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla z_{n,m}^j dx=\int_\Omega a(x,u_n,\nabla u_n)\cdot\left(\nabla T_k(u_n)-\nabla T_k(v_j)\right)\phi'\left(\theta_n^j\right)\psi_m(u_n)\,dx$$

$$+\int_{\Omega} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n\phi\left(\theta_n^j\right)\psi_m'\left(u_n\right)dx$$

$$=\int_{\Omega} a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\right)\phi'\left(\theta_n^j\right)dx$$

$$-\int_{||u_n|>k|} a\left(x,u_n,\nabla u_n\right)\cdot\nabla T_k\left(v_j\right)\phi'\left(\theta_n^j\right)\psi_m\left(u_n\right)dx+\int_{\Omega} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n\phi\left(\theta_n^j\right)\psi_m'\left(u_n\right)dx$$

therefore

$$\int_{\Omega} a\left(x,u_n,\nabla u_n\right)\cdot\nabla z_{nm}^j dx$$

$$=\int_{\Omega}\left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\phi'\left(\theta_n^j\right)dx$$

$$+\int_{\Omega} a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j'\right)\phi'\left(\theta_n^j\right)dx$$

$$-\int_{\Omega\setminus\Omega_j^*} a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(v_j\right)\phi'\left(\theta_n^j\right)dx$$

$$-\int_{[|u_u|>k]} a\left(x,u_n,\nabla u_n\right)\cdot\nabla T_k\left(v_j\right)\phi'\left(\theta_n^j\right)\psi_m\left(u_n\right)dx+\int_{\Omega} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n\phi\left(\theta_n^j\right)\psi_m'\left(u_n\right)dx \tag{6.4.15}$$

let us define x_j^s, $s>0$, and the characteristic function of the subset $\Omega_j^s=\{x\in\Omega:|\nabla T_k\left(v_j\right)|\leq s\}$.

By fixing m and s, we will pass to the limit in n and in j in the second, third, fourth and fifth term on the right hand side of (6.4.15) .

For the second term, we have

$$\int_{\Omega} a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\phi'\left(\theta_n^j\right)dx$$

$$\rightarrow\int_{\Omega}(a\left(x,T_k(u),\nabla T_k\left(v_j\right)\chi_j^s\right)\cdot\left(\nabla T_k(u)-\nabla T_k\left(v_j\right)\chi_j^s\right)\phi'\left(\theta^j\right))dx\quad\text{ as } n\rightarrow+\infty$$

thinks to 6.3.8, one has

$$a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)x_j^s\right)\phi'\left(\theta_n^j\right)\rightarrow a\left(x,T_k(u),\nabla T_k\left(v_j\right)\chi_j^s\right)\phi'\left(\theta^j\right)\text{ strongly in }\left(E_{\varphi}(\Omega)\right)^N\text{ as n }\rightarrow$$

$$\infty$$

and by (6.4.4)

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \quad \text{weakly in } (L_\varphi(\Omega))^N$$

Let us define χ^s the characteristic function of the subset

$$\Omega^s = \{x \in \Omega : |\nabla T_k(u)| \leq s\}$$

As $\nabla T_k(v_j)\chi_j^s \to \nabla T_k(u)\chi^s$ strongly in $(E_\varphi(\Omega))^N$ as $j \to \infty$, we get

$$\int_\Omega a\left(x, T_k(u), \nabla T_k(v_j)\chi_j^s\right) \cdot \left(\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s\right) \phi'\left(\theta^j\right) dx \to 0 \quad \text{as } j \to \infty$$

thus,

$$\int_\Omega a\left(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s\right) \cdot \left(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right) \phi'\left(\theta_n^j\right))dx = \epsilon_3(n,j) \tag{6.4.16}$$

For third term estimation of (6.4.15) . It's it is clear that by (6.1.5) one can verify that $a(x,s,0) = 0$ for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$.

Thus, from (6.4.10) we have that

$$\left(a\left(x, T_k(u_n), \nabla T_k(u_n)\right)\right)_n \text{ is bounded in } (L_\varphi(\Omega))^N \text{ for all } k \geq 0.$$

Therefore, there exist a subsequence still indexed by n and a function l_k in $(L_\varphi(\Omega))^N$ such that

$$a\left(x, T_k(u_n), \nabla T_k(u_n)\right) \rightharpoonup l_k \text{ weakly in } (L_\varphi(\Omega))^N \text{ for } \sigma\left(\Pi L_\psi, \Pi E_\varphi\right).$$

Then, by using the fact that $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_j^s} \in (E_\varphi(\Omega))^N$, we get

$$\int_{\Omega|\Omega_j^s} a\left(x, T_k(u_n), \nabla T_k(u_n)\right) \cdot \nabla T_k(v_j) \phi'\left(\theta_n^j\right) dx \to \int_{\Omega\setminus\Omega_j^s} l_k \cdot \nabla T_k(v_j) \phi'\left(\theta^j\right) dx \quad \text{as } n \to \infty.$$

The modular convergence of $\{v_j\}$ give

$$-\int_{\Omega\setminus\Omega_j} l_k \cdot \nabla T_k(v_j) \phi'\left(\theta^j\right) dx \to -\int_{\Omega|\Omega^s} l_k \cdot \nabla T_k(u) dx \quad \text{as } j \to \infty$$

Consequently

$$-\int_{\Omega\setminus\Omega_j^{*=s}} a\left(x, T_k(u_n), \nabla T_k(u_n)\right) \cdot \nabla T_k(v_j) \phi'\left(\theta_n^j\right) dx = -\int_{\Omega\setminus\Omega^s} l_k \cdot \nabla T_k(u) dx + \epsilon_4(n,j) \tag{6.4.17}$$

For the fourth term, we remark that $\psi_m(u_n) = 0$ on the subset $\{|u_n| \geq m+1\}$, then we obtain

$$-\int_{\{|u_n|>k\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla T_k\left(v_j\right)\phi'\left(\theta_n^j\right)\psi_m\left(u_n\right)dx$$

$$=-\int_{\{|u_n|>k\}} a\left(x,T_{m+1}\left(u_n\right),\nabla T_{m+1}\left(u_n\right)\right)\cdot\nabla T_k\left(v_j\right)\phi'\left(\theta_n^j\right)\psi_m\left(u_n\right)dx$$

By using the same procedure as above we have

$$-\int_{\{|u_n|>k\}} a\left(x,T_{m+1}\left(u_n\right),\nabla T_{m+1}\left(u_n\right)\right)\cdot\nabla T_k\left(v_j\right)\phi'\left(\theta_n^j\right)\psi_m\left(u_n\right)dx$$

$$=-\int_{\{|u|>k\}} l_{m+1}\cdot\nabla T_k(u)\psi_m(u)dx+\epsilon_5(n,j)$$

By observing that $\nabla T_k(u)=0$ on the subset $\{|u|>k\}$, we can write

$$-\int_{\{|u_n|>k\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla T_k\left(v_j\right)\phi'\left(\theta_n^j\right)\psi_m\left(u_n\right)dx=\epsilon_5(n,j) \tag{6.4.18}$$

For the last term of (6.4.15) we obtain

$$\begin{aligned}\left|\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n\phi\left(\theta_n^j\right)\psi_m'\left(u_n\right)dx\right| &= \left|\int_{\{m\le|u_k|\le m+1\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n\phi\left(\theta_n^j\right)\psi_m'\left(u_n\right)dx\right| \\ &\le \phi(2k)\int_{\{m\le|u_k|\le m+1\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n dx\end{aligned}$$

By taking $T_1\left(u_n-T_m\left(u_n\right)\right)\in W_0^1L_\varphi(\Omega)$ as test in (P_n) one has

$$\int_{\{m\le|u_k|\le m+1\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n dx+\int_{\{m\le|u_n|\le m+1\}}\Phi_n\left(u_n\right)\cdot\nabla u_n dx$$

$$+\int_{\{|u_x|\ge m\}} g_n\left(x,u_n,\nabla u_n\right)T_1\left(u_n-T_m\left(u_n\right)\right)dx=\langle f,T_1\left(u_n-T_m\left(u_n\right)\right)\rangle.$$

by according to Lemma 6.3.9, we get

$$\int_{\{m\le|u_k|\le m+1\}}\Phi_n\left(u_n\right)\cdot\nabla u_n dx=0$$

Since $g_n\left(x,u_n,\nabla u_n\right)T_1\left(u_n-T_m\left(u_n\right)\right)\ge 0$ on the subset $\{|u_n|\ge m\}$, we have

$$\int_{\{m\le|u_k|\le m+1\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n dx\le\langle f,T_1\left(u_n-T_m\left(u_n\right)\right)\rangle$$

By observing f as $f = -\operatorname{div} F$, where $F \in (E_\varphi(\Omega))^N$, and applying Young's inequality, we get

$$\int_{\{m\leq|u_k|\leq m+1\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n dx \leq \alpha \quad \int_{\{m\leq|u_n|\leq m+1\}} \psi\left(x,\frac{2}{\alpha}|F|\right)dx$$

which implies that

$$\left|\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n \phi\left(\theta_n^j\right)\psi_m'\left(u_n\right)dx\right| \leq \alpha\phi(2k) \quad \int_{\{m\leq|u_k|\leq m+1\}} \psi\left(x,\frac{2}{\alpha}|F|\right)dx \tag{6.4.19}$$

thinks to (6.4.15), (6.4.17), (6.4.18) and 6.4.19 we get

$$\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\nabla z_{n,m}^j dx \geq \tag{6.4.20}$$

$$\int_\Omega \left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)x_j^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)x_j^s\right)\phi'\left(\theta_n^j\right)dx$$

$$-\alpha\phi(2k) \quad \int_{\{m\leq|u_k|\leq m+1\}} \psi\left(x,\frac{2}{\alpha}|F|\right)dx - \int_{\Omega\setminus\Omega^s} l_k\cdot\nabla T_k(u)dx + e_6(n,j) \tag{6.4.21}$$

Now, we turn to the second term on the left-hand side of (6.4.15) and by using the hypothesis (6.1.6) one has

$$\begin{aligned}\left|\int_{\{|u_n|\leq k\}} g_n\left(x,u_n,\nabla u_n\right)\phi\left(\theta_n^j\right)dx\right| &= \left|\int_{\{|u_x|\leq k\}} g_n\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\phi\left(\theta_n^j\right)dx\right| \\ &\leq b(k)\int_\Omega \varphi\left(x,|\nabla T_k\left(u_n\right)|\right)\left|\phi\left(\theta_n^j\right)\right|dx + b(k)\int_\Omega c(x)\left|\phi\left(\theta_n^j\right)\right|dx \\ &\leq \frac{b(k)}{\alpha}\int_\Omega a_n\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(u_n\right)\left|\phi\left(\theta_n^j\right)\right|dx + \epsilon_7(n,j)\end{aligned}$$

Therefore,

$$\left|\int_{|u_k|\leq k\}} g_n\left(x,u_n,\nabla u_n\right)\phi\left(\theta_n^j\right)dx\right| \leq$$

$$\frac{b(k)}{\alpha}\int_\Omega \left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\left|\phi\left(\theta_n^j\right)\right|dx$$

$$+\frac{b(k)}{\alpha}\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\left|\phi\left(\theta_n^j\right)\right|dx$$

$$+\frac{b(k)}{\alpha}\int_\Omega a_n\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(v_j\right)\chi_j^s\left|\phi\left(\theta_n^j\right)\right|dx + \epsilon_7(n,j) \tag{6.4.22}$$

Using the same procedure as above we get

$$\frac{b(k)}{\alpha}\int_{\Omega} a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\left|\phi\left(\theta_n^j\right)\right|dx=\epsilon_8(n,j)$$

and

$$\frac{b(k)}{\alpha}\int_{\Omega} a_n\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(v_j\right)\chi_j^s\left|\phi\left(\theta_n^j\right)\right|dx=\epsilon_9(n,j).$$

thus, we obtain

$$\left|\int_{\{|u_n|\le k\}} g_n\left(x,u_n,\nabla u_n\right)\phi\left(\theta_n^j\right)dx\right|\le\frac{b(k)}{\alpha}\int\left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)$$
$$\times\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\left|\phi\left(\theta_n^j\right)\right|dx+\epsilon_{10}(n,j) \tag{6.4.23}$$

By combining (6.4.14),(6.4.20) and (6.4.23) we have

$$\int_{\Omega}\left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)$$
$$\times\left(\phi'\left(\theta_n^j\right)-\frac{b(k)}{\alpha}\left|\phi\left(\theta_n^j\right)\right|\right)dx$$

$$\le\int_{\Omega\setminus\Omega^s} l_k\cdot\nabla T_k(u)dx+\alpha\phi(2k)\psi\left(x,\frac{2}{\alpha}|F|\right)dx+\epsilon_{11}(n,j)$$

thinks to (6.4.12), we get

$$\int_{\Omega}\left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\rho(x)dx$$
$$\le 2\int_{\Omega\setminus\Omega^s} l_k\cdot\nabla T_k(u)dx+2\alpha\phi(2k)\int_{\{m\le|u_n|\le m+1\}}\psi\left(x,\frac{2}{\alpha}|F|\right)dx+\epsilon_{11}(n,j) \tag{6.4.24}$$

On the other hand, we have

$$\int_{\Omega}\left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k(u)\chi^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k(u)\chi^s\right)dx$$

$$=\int_{\Omega}\left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)dx$$

$$+\int_{\Omega} a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k(u)\chi^s\right)dx$$

$$-\int_{\Omega} a\left(x,T_k\left(u_n\right),\nabla T_k(u)\chi^s\right)\cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k(u)\chi^s\right)dx$$

$$+\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right) \cdot \left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right) dx$$

We will passe to the limit in n and then in j in the last three terms of the right-hand side of the above equality.

using the same procedure as is done in (6.4.15) and (6.4.22), we get

$$\begin{aligned}
&\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \left(\nabla T_k\left(v_j\right) \chi_j^s-\nabla T_k(u) \chi^s\right) dx=\epsilon_{12}(n, j) \\
&\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k(u) \chi^s\right) \cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k(u) \chi^s\right) dx=\epsilon_{13}(n, j) \\
&\int_{\Omega} a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right) \cdot \left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right) dx=\epsilon_{14}(n, j)
\end{aligned} \tag{6.4.25}$$

Therefore,

$$\begin{aligned}
&\int_{\Omega}\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k(u) \chi^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k(u) \chi^s\right) dx \\
&=\int_{\Omega}\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right) dx+\epsilon_{15}(n, j)
\end{aligned} \tag{6.4.26}$$

Let $r \leq s$. Thinks to (6.1.4) , (6.4.24) and (6.4.26) we have

$$\begin{aligned}
0 & \leq \int_{\Omega^r}\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k(u)\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k(u)\right) dx \\
& \leq \int_{\Omega^s}\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k(u)\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k(u)\right) dx \\
& =\int_{\Omega^s}\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k(u) \chi^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k(u) \chi^s\right) dx \\
& \leq \int_{\Omega}\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k(u) \chi^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k(u) \chi^s\right) dx \\
& =\int_{\Omega}\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right) dx+\epsilon_{15}(n, j)
\end{aligned}$$

by passing to the limit in n and then in j one has,

$$\begin{aligned}
0 & \leq \limsup_{n \rightarrow \infty} \int\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k(u)\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k(u)\right) dx \\
& \leq 2 \int_{\Omega \backslash \Omega^s} l_k \cdot \nabla T_k(u) dx+2 \alpha \phi(2 k) \int_{\{m \leq|u| \leq m+1\}} \psi\left(x, \frac{2}{\alpha}|F|\right) dx .
\end{aligned}$$

Let $s \rightarrow+\infty$ and $m \rightarrow+\infty$, using the fact that $l_k \cdot \nabla T_k(u) \in L^1(\Omega),|F| \in\left(E_\varphi(\Omega)\right)^N,|\Omega \backslash \Omega^s| \rightarrow 0$ and $|\{m \leq|u| \leq m+1\}| \rightarrow 0$, we obtain

$$\int_{\Omega^r}\left(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k(u)\right)\right) \cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k(u)\right) dx \rightarrow 0 \quad \text { as } n \rightarrow \infty$$

Thinks to [83] there exists a subsequence of $\{u_n\}$ still indexed by n such that

$$\nabla u_n \to \nabla u \quad \text{a.e.in } \Omega \tag{6.4.27}$$

Thus, by taking account that (6.4.7),(6.4.9) and (6.4.10) we can apply [[89], Theorem 14.6] to obtain $a(x,u,\nabla u) \in (L_\varphi(\Omega))^N$ and

$$a\left(x,u_n,\nabla u_n\right) \rightharpoonup a(x,u,\nabla u) \text{ weakly in } \left(L_\varphi(\Omega)\right)^N \text{for } \sigma\left(\Pi L_\varphi(\Omega),\Pi E_\psi(\Omega)\right). \tag{6.4.28}$$

Now,we shall prove that

$$\text{for every } k>0, \quad T_k\left(u_n\right) \to T_k(u) \quad \text{ in } W_0^1 L_\varphi(\Omega) \text{ for the modular convegence}$$

.

From inequality (6.4.24), we obtain

$$\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(u_n\right)dx \le \int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(v_j\right)\chi_j^s dx$$

$$+\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)x_j^s\right)\cdot\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)dx$$

$$+2\alpha\phi(2k) \quad \int_{\{m\le|u_k|\le m+1\}}\psi\left(x,\frac{2}{\alpha}|F|\right)dx$$

$$+2\int_{\Omega|\Omega^s} a\left(x,T_k(u),\nabla T_k(u)\right)\cdot\nabla T_k(u)dx+\epsilon_{11}(n,j).$$

thinks to (6.4.25), we obtain

$$\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(u_n\right)dx \le$$

$$\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(v_j\right)\chi_j^s dx+2\alpha\phi(2k) \quad \int_{\{m\le|u_n|\le m+1\}}\psi\left(x,\frac{2}{\alpha}|F|\right)dx$$

$$+2\int_{\Omega|\Omega^s} a\left(x,T_k(u),\nabla T_k(u)\right)\cdot\nabla T_k(u)dx+\epsilon_{17}(n,j).$$

the passage to the limitto the limit in n on both sides of this inequality and using (6.4.28) implies that

$$\limsup_{n\to\infty}\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(u_n\right)dx\leq$$

$$\int_\Omega a\left(x,T_k(u),\nabla T_k(u)\right)\cdot\nabla T_k\left(v_j\right)\chi_j^s dx+2\alpha\phi(2k)\quad\int_{\{m\leq|u|\leq m+1\}}\psi\left(x,\frac{2}{\alpha}|F|\right)dx$$

$$+2\int_{\Omega\setminus\Omega^s}a\left(x,T_k(u),\nabla T_k(u)\right)\cdot\nabla T_k(u)dx.$$

and by passing to the limit in j we obtain

$$\limsup_{n\to\infty}\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(u_n\right)dx$$

$$\leq\int_\Omega a\left(x,T_k(u),\nabla T_k(u)\right)\cdot\nabla T_k(u)\chi^s dx+2\alpha\phi(2k)\quad\int_{\{m\leq|u|\leq m+1\}}\psi\left(x,\frac{2}{\alpha}|F|\right)dx$$

$$+2\int_{\Omega\setminus\Omega^s}a\left(x,T_k(u),\nabla T_k(u)\right)\cdot\nabla T_k(u)dx.$$

Let s and $m\to\infty$, we get

$$\limsup_{n\to\infty}\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(u_n\right)dx\leq\int_\Omega a\left(x,T_k(u),\nabla T_k(u)\right)\cdot\nabla T_k(u)dx$$

Now, thinks to (6.1.5),(6.4.4),(6.4.27) and applying Fatou's lemma, we get

$$\int_\Omega a\left(x,T_k(u),\nabla T_k(u)\right)\cdot\nabla T_k(u)dx\leq\liminf_{n\to\infty}\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(u_n\right)dx$$

thus,

$$\lim_{n\to\infty}\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(u_n\right)dx=\int_\Omega a\left(x,T_k(u),\nabla T_k(u)\right)\cdot\nabla T_k(u)dx$$

In view of Lemma 6.3.5, we deduce that for every $k>0$

$$a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(u_n\right)\to a\left(x,T_k(u),\nabla T_k(u)\right)\cdot\nabla T_k(u)\quad\text{ strongly in } L^1(\Omega)\qquad(6.4.29)$$

by vertu of hypothesis (6.1.5) and using the convexity of φ we get

$$\varphi\left(x,\tfrac{|\nabla T_k(u_n)-\nabla T_k(u)|}{2}\right)\leq\tfrac{1}{2\alpha}a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\cdot\nabla T_k\left(u_n\right)+\tfrac{1}{2\alpha}a\left(x,T_k(u),\nabla T_k(u)\right)\cdot\nabla T_k(u)$$

By applying Vitali's theorem we obtain

$$\lim_{|\varepsilon|\to0}\sup_n\int_E\varphi\left(x,\frac{|\nabla T_k\left(u_n\right)-\nabla T_k(u)|}{2}\right)dx=0$$

Consequently, for every $k>0$

$$T_k\left(u_n\right)\to T_k(u)\quad\text{ in } W_0^1L_\varphi(\Omega).$$

for the modular convegence.

We shall show that

$$g_n\left(x,u_n,\nabla u_n\right)\to g(x,u,\nabla u)\quad in\quad L^1(\Omega) \tag{6.4.30}$$

From (6.4.9) and (6.4.27) we have

$$g_n\left(x,u_n,\nabla u_n\right)\to g(x,u,\nabla u)\quad \text{a.e. in } \Omega \tag{6.4.31}$$

Let E be a measurable subset of Ω and let $m>0$. by taking account of (6.1.5) and (6.1.6) we obtain

$$\int_E\left|g_n\left(x,u_n,\nabla u_n\right)\right|dx=\int_{E\cap[|u_k|\le m\}}\left|g_n\left(x,u_n,\nabla u_n\right)\right|dx+\int_{E\cap\{|u_n|>m\}}\left|g_n\left(x,u_n,\nabla u_n\right)\right|dx$$

$$\le b(m)\int_E c(x)dx+b(m)\int_E a\left(x,T_m\left(u_n\right),\nabla T_m\left(u_n\right)\right)\cdot\nabla T_m\left(u_n\right)dx+\frac{1}{m}\int_\Omega g_n\left(x,u_n,\nabla u_n\right)u_n dx$$

By (6.1.7) and (6.4.6) it follows that

$$\lim_{m\to\infty}\frac{1}{m}\int_\Omega g_n\left(x,u_n,\nabla u_n\right)u_n dx=0$$

By using (6.4.29) the sequence

$$\left\{a\left(x,T_m\left(u_n\right),\nabla T_m\left(u_n\right)\right)\cdot\nabla T_m\left(u_n\right)\right\}_n \text{ is equi-integrable ,}$$

Consequently

$$\lim_{|E|\to 0}\sup_n\int_E a\left(x,T_m\left(u_n\right),\nabla T_m\left(u_n\right)\right)\cdot\nabla T_m\left(u_n\right)dx=0$$

This proves that $g_n\left(x,u_n,\nabla u_n\right)$ is equi-integrable.

Therefore, Vitali's theorem allows us to get

$$g(x,u,\nabla u)\in L^1(\Omega),$$

and

$$g_n\left(x,u_n,\nabla u_n\right)\to g(x,u,\nabla u)\quad \text{strongly in } L^1(\Omega). \tag{6.4.32}$$

In this subsubsection we prove that

$$\lim_{m\to\infty}\int_{\{m\leq|u|\leq m+1\}} a(x,u,\nabla u)\cdot\nabla u dx=0 \tag{6.4.33}$$

for any $m\geq 0$ we have

$$\int_{\{m\leq|u_n|\leq m+1\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n dx=\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\left(\nabla T_{m+1}\left(u_n\right)-\nabla T_m\left(u_n\right)\right)dx$$

$$=\int_\Omega a\left(x,T_{m+1}\left(u_n\right),\nabla T_{m+1}\left(u_n\right)\right)\cdot\nabla T_{m+1}\left(u_n\right)dx-\int_\Omega a\left(x,T_m\left(u_n\right),\nabla T_m\left(u_n\right)\right)\cdot\nabla T_m\left(u_n\right)dx$$

Thinks to (6.4.29) and passing to the limit as $n\to+\infty$ for fixed $m\geq 0$

$$\lim_{n\to\infty}\int_{\{m\leq|u_n|\leq m+1\}} a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n dx$$

$$=\int_\Omega a\left(x,T_{m+1}(u),\nabla T_{m+1}(u)\right)\cdot\nabla T_{m+1}(u)dx-\int_\Omega a\left(x,T_m(u),\nabla T_m(u)\right)\nabla T_m(u)dx$$

$$=\int_\Omega a(x,u,\nabla u)\cdot\left(\nabla T_{m+1}(u)-\nabla T_m(u)\right)dx$$

$$=\int_{\{m\leq|u|\leq m+1\}} a(x,u,\nabla u)\cdot\nabla u dx.$$

according to (6.4.11), we can pass to the limit as $m\to+\infty$ in order to have (6.4.33).

Finally, in this step thanks to (6.4.29) and Lemma 6.3.5, one has

$$a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n\to a(x,u,\nabla u)\cdot\nabla u\quad\text{ strongly in } L^1(\Omega) \tag{6.4.34}$$

Let $h\in\mathbb{C}_c^1(\mathbb{R})$ and $\varrho\in\mathcal{D}(\Omega)$. we choose $h\left(u_n\right)\varrho$ as a test function in (P_n) we obtain

$$\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n h'\left(u_n\right)\varrho dx+\int_\Omega \rho(x)a\left(x,u_n,\nabla u_n\right)\cdot\nabla\varrho h\left(u_n\right)dx$$

$$+\int_{\Omega}\Phi_n\left(u_n\right)\cdot\nabla\left(h\left(u_n\right)\varrho\right)dx+\int_{\Omega}g_n\left(x,u_n,\nabla u_n\right)h\left(u_n\right)\varrho dx=\left\langle f,h\left(u_n\right)\varrho\right\rangle \tag{6.4.35}$$

Now, we can pass to the limit as $n\to+\infty$ in each term of equality (6.4.35). since h and h' have a compact support on $\mathbb{R}$, there exists a real number $v>0$, such that supp $h\subset[-v,v]$ and supp $h'\subset[-v,v]$. For $n>v$, we can have

$$\Phi_n(t)h(t)=\Phi\left(T_v(t)\right)h(t)\quad\text{ and }\quad\Phi_n(t)h'(t)=\Phi\left(T_v(t)\right)h'(t)$$

Moreover,

$$\text{the functions }\Phi h\text{ and }\Phi h'\text{ belong to }\left(\mathcal{C}^0(\mathbb{R})\cap L^{\infty}(\mathbb{R})\right)^N.$$

Now we can see that the sequence $\left\{h\left(u_n\right)\varrho\right\}_n$ is bounded in $W_0^1L_{\varphi}(\Omega)$.
Indeed, let $c^{'}>0$ be a positive constant such that $\left\|h\left(u_n\right)\nabla\varrho\right\|_{\infty}\leq c^{'}$ and $\left\|h'\left(u_n\right)\varrho\right\|_{\infty}\leq c^{'}$.
Thinks to (6.3.2) we obtain

$$\begin{aligned}\int_{\Omega}\varphi\left(x,\frac{\left|\nabla\left(h\left(u_n\right)\varrho\right)\right|}{2c'}\right)dx&\leq\int_{\Omega}\varphi\left(x,\frac{\left|h\left(u_n\right)\nabla\varrho\right|+\left|h'\left(u_n\right)\varrho\|\nabla u_n\right|}{2c'}\right)dx\\&\leq\frac{1}{2}\int_{\Omega}\varphi(x,1)dx+\frac{1}{2}\int_{\Omega}\varphi\left(x,\left|\nabla T_M\left(u_n\right)\right|\right)dx\\&\leq c\end{aligned}$$

Which jointly with (6.4.9) it follows that

$$h\left(u_n\right)\varrho\to h(u)\varrho\text{ weakly in }W_0^1L_{\varphi}(\Omega)\text{ for for }\sigma\left(\Pi L_{\varphi},\Pi E_{\psi}\right). \tag{6.4.36}$$

Which give

$$\left\langle f,h\left(u_n\right)\varphi\right\rangle\to\left\langle f,h(u)\varphi\right\rangle.$$

Let E be a measurable subset of Ω. we pose $c_v=\max_{|t|\leq v}\Phi(t)$. And denoting by $\|v|_{\varphi,\Omega}$ the Orlicz norm of a function $v\in L_{\varphi}(\Omega)$. We thinking to the strengthened $H\ddot{o}lder$ inequality with both Orlicz and Luxemburg norms, we have

$$\begin{aligned}\left\|\Phi\left(T_v\left(u_n\right)\right)\chi_E\right\|_{\psi,\Omega}&=\sup_{\|v\|_{\varphi,\Omega}\leq1}\left|\int_E\Phi\left(T_v\left(u_n\right)\right)vdx\right|\\&\leq c_v\sup_{\|v\|_{\varphi,\Omega}\leq1}\left\|\chi_E\right\|_{\psi,\Omega}\left\|v\right\|_{\varphi,\Omega}\\&\leq c_v|E|\varphi^{-1}\left(x,\frac{1}{|E|}\right)\end{aligned}$$

Consequently,

$$\lim_{|E|\to 0}\sup_{n}\left\|\Phi\left(T_v\left(u_n\right)\right)\chi_E\right\|_{(\psi,\Omega)}=0$$

Then, in view of (6.4.9) and by applying [[89], Lemma 11.2], we get

$$\Phi\left(T_v\left(u_n\right)\right)\to\Phi\left(T_v(u)\right)\quad\text{ strongly in }\ \left(E_\psi(\Omega)\right)^N$$

which together with (6.4.36) allow us to pass to the limit in the third term of (6.4.35) to obtain

$$\int_\Omega\Phi\left(T_v\left(u_n\right)\right)\cdot\nabla\left(h\left(u_n\right)\varrho\right)dx\to\int_\Omega\Phi\left(T_v(u)\right)\cdot\nabla(h(u)\varrho)dx$$

Observing that

$$\left|a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n h'\left(u_n\right)\varrho\right|\leq c^{'}a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n$$

Therefore, thinks to (6.4.34) and applying Vitali's theorem, we get

$$\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\nabla u_n h'\left(u_n\right)\varrho dx\to\int_\Omega a(x,u,\nabla u)\cdot\nabla u h'(u)\varrho dx$$

Concerning the second term of (6.4.35) using the same procedure as above we obtain

$$h\left(u_n\right)\nabla\varrho\to h(u)\nabla\varrho\quad\text{ strongly in }\ \left(E_\varphi(\Omega)\right)^N$$

which jointly with (6.4.28) implies that

$$\int_\Omega a\left(x,u_n,\nabla u_n\right)\cdot\nabla\varrho h\left(u_n\right)dx\to\int_\Omega a(x,u,\nabla u)\cdot\nabla\varrho h(u)dx$$

Remark that $h\left(u_n\right)\varrho\to h(u)\varrho$ weakly in $L^\infty(\Omega)$ for $\sigma^*\left(L^\infty,L^1\right)$ with (6.4.30) we can pass to the limit in the Fourth term of (6.4.35) in order to have

$$\int_\Omega g_n\left(x,u_n,\nabla u_n\right)h\left(u_n\right)\varrho dx\to\int_\Omega g(x,u,\nabla u)h(u)\varrho dx$$

Finally, we can pass to the limit in each term of (6.4.35) so as to obtain

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\cdot\left[\nabla\varphi h(u)+h'(u)\varrho\nabla u\right]dx+\int\Phi(u)h'(u)\varrho\cdot\nabla u dx\\+\int_\Omega\Phi(u)h(u)\cdot\nabla\varrho dx+\int_\Omega g(x,u,\nabla u)h(u)\varrho dx=\langle f,h(u)\varrho\rangle\end{aligned}$$

for all $h\in\mathcal{C}_c^1(\mathbb{R})$ and for all $\varrho\in\mathcal{D}(\Omega)$. Thus, as well (6.1.7),(6.4.6) and (6.4.31) we apply Fatou's lemma to get $g(x,u,\nabla u)u\in L^1(\Omega)$.

Consequently, thinks to (6.4.9),(6.4.28),(6.4.32),(6.4.33), the function u is a renormalized solution of problem $(\mathcal{P})$.

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