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BICOMPLEX SEGAL BARGMANN TRANSFORMS AND ASSOCIATED FRACTINAL TRANSFORMS

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Contents

Résumé

Dans cette thèse, on a introduit l?espace de Bargmann bicomplex, et on a également discuté ses propriétés de base, y compris ses différentes décompositions idempotentes et l?existence du noyau reproduisant. Puis on a considéré et discuté certaines propriétés de la transformée de Fourier fractionnaire et de la transformée de Segal-Bargmann ainsi que leur inverse et la relation entre eux. Dans la deuxième partie, nous avons introduit deux types de transformées de Fourier-Wiegner que l?on peut voir comme des généralisations des transformées qu?on a définie auparavant. Et on a prouvé que leurs images sont une généralisation de l?espace de Bargmann bicomplexe. Il s?agit de l?espace des fonctions polyanalytiques bicomplexes. On a introduit également une nouvelle classe de polynômes orthogonaux de quatre indices de variable bicomplexe.

Abstract

in this thesis, we introduced the bicomplex Segal Bargmann space, we have also shown and discussed its basic properties, Including its different idempotent decompositions and the existence of the reproducing kernel, from this last one we built the transform $\mathcal{S}^{\nu}$ and the associated spaces $\mathcal{F}^{2,\frac{\nu}{2}}_{\mathbb{BC}}(\mathbb{C})$ and$\mathcal{F}^{2,\nu}_{i}(\mathbb{BC})$. As we consider and discuss some properties of the Fourier fractional transform and the segal Bargmann Transform as well their reverse and the relationship between them In a second article we have introduced two types of WEINER Fourier transform that we can see as generalizations of transforms that we defended before. And we realized their ranges as a generalization of the bicomplex segal bargmann space, it's about the space of bicomplex polyanalytic functions. We also provide a new class of four indices bivariate complex orthogonal polynomials of Hermit type

Introduction géneral

Les nombres bicomplexes $\mathbb{T}$ sont une généralisation spéciale des nombres complexes. autrement dit, se sont des nombres complexes à coefficients complexes, considérant telle que la solution de $x^2 = -1$ n'est pas unique. Depuis leur introduction par Corrado Segre en 1892 [32] plusieurs spécialités mathématiques ont été développées et étudiées,y compris l'analyse fonctionnelle bicomplexe [13,16,22,33,34], l'analyse différentielle bicomplexe [2,22,49] et la mécanique quantique bicomplexe[14,17,19,27,50]. Pour plus de détails sur la théorie des fonctions bicomplexes ainsi que leurs applications voir [2,10,11,22,25,29,30,49].

Pour compléter les études dans les travaux précités ci-dessus. L'objectif principal du ce projet est d'introduire et d'étudier en détail les propriétés de base de quelques transformations intégrales intéressantes dans le cadre de certains espaces d'Hilbert bicomplexes de dimension infinie résultant de l'analyse temps-fréquence. Spécifiquement, on s'est intéressé a la transformé de Segal- Bargmann bicomplexe (Chapitre 2), la transfromé Fourier-Wigner bicomplexe (Chapaitre 3) ainsi que la transformé de Fourier fractionnelle (Chapaitre 4).

Pour atteindre notre objectif, on commence par un chapitre introductif dans lequel on présente tout les concepts de base de l'analyse bicomplexe dont on a besoin, et on discute les différentes structures associées nécessaires.

Dans le deuxième chapitre , on considère l'espace de Bargmann bicomplexe $\mathcal{F}^{2,\nu}(\mathbb{T})$ et la transformée de Segal-Bargmann associée $\mathcal{B}_{\mathbb{T}}^{\sigma,\nu}$ dans le cadre des fonctions holomorphes au sens bicomplexe, l'analogue de l'espace de Segal Bargmann classique $\mathcal{F}^{2,\nu}(\mathbb{C})$ et aussi celui de la transformée de Segal-Bargmann $\mathcal{B}_{\mathbb{C}}^{\sigma,\gamma}$ pour les fonction holomorphes au sens complexe. Le premier résulta concernant $\mathcal{F}^{2,\nu}(\mathbb{T})$ est obtenu comme corollaire immédiat de l'observation clé (théorème2.2.1)). cela se décompose $\mathcal{F}^{2,\nu}(\mathbb{T})$ sous forme de somme des espaces $\mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$ en respectant la décomposition idempotente pour les fonctions holomorphes au sens complexe. Nous traitons ensuite des représentations intégrales des fonctions holomorphes au sens bicomplexes des carrés intégrables. Pour plus de précision le théorème 2.3.1 indique que $\mathcal{F}^{2,\nu}(\mathbb{T})$ est un espace de Hilbert bicomplexe à noyau reproduisant. D'autre part le théorème 2.3.5 nous donne une relation entre $\mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$ et le sous-espace spécial de $\mathcal{F}^{2,v}(\mathbb{T})$ constitué par les fonctions qui rend $\mathbb{C}_i := \mathbb{C} + j\{0\}$ invariant, à travers une transformation intégrale explicite et spéciale. Cette transformée est en outre une surjection de l'espace des fonctions des carrés intégrables sur le plan complexe $\mathbb{C}$ à valeurs bicomplexes vers l'espace de Segal Bargmann bicomplexe $\mathcal{F}^{2,\nu}(\mathbb{T})$. A propos de la transformée de Segal-Bargmann bicomplexe $\mathcal{B}_{\mathrm{T}}^{\sigma,\nu}$, on montre qu'il s'agit d'une transformation isométrique unitaire de l'espace des fonctions des carrés intégrables sur la droite réelle $\mathbb{R}$ à valeurs bicomplexes vers l'espace de Segal Bargmann bicomplexe $\mathcal{F}^{2,\nu}(\mathbb{T})$ (Theorem 2.4.2).). Nous donnons l'expression explicite de la sa transformée inverse $\left[\mathcal{B}_{\mathbb{T}}^{\sigma,\nu}\right]^{-1}$(Theorem 2.4.6).

Le chapitre 3 est une application immédiate des résultats obtenus dans le chapitre précédent.Ainsi, nous utilisons à la fois $\mathcal{B}_{\mathbb{T}}^{\sigma,\nu}$ et $\left[\mathcal{B}_{\mathbb{T}}^{\sigma,\nu}\right]^{-1}$ pour introduire une famille de transformées de Fourier fractionnelles bicomplexes (BFrFT) indexée par l'ensemble des nombres bicomplexes θ; $|\theta|, \theta \neq \pm 1, \pm ij$ de façon à récupérer la transformée classique i-Fourier ainsi j-Fourier. Les propriétés de base de BFrFT, telles que le théorème d'unicité, le théorème de Plancherel et que la formule d'inversion sont obtenus.

Dans le chapitre 4 on entame deux types d'analogues bicomplexes de la transformée de Fourier-Wigner classique (Fourier fenêtré) intensivement considérés dans l'analyse harmonique [51,67] et l'analyse temps-fréquence [39,47]. C'est une application bilinéaire $\mathcal{V}$: $(f,g) \longmapsto \mathcal{V}(f,g)$ sur $L^2_{\mathbb{C}}\left(\mathbb{R}^d\right) \times L^2_{\mathbb{C}}\left(\mathbb{R}^d\right)$ donnée par la fonction cross-Wigner [40,51,53,66, 67]

$$\mathcal{V}(f,g)(p,q) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle x-\frac{p}{2},q\rangle} f(x)\overline{g(x-p)}dx \tag{1}$$

pour tout $(p,q) \in \mathbf{R}^d \times \mathbb{R}^d$, où $\langle\rangle$, désigne le produit scalaire standard de $\mathbb{R}^d$. Pour une fenêtre fixée, elle est étroitement liée à la transformée de Gabor [44] ainsi a la transformation de Segal Bargmann [51,67]. Il se réduit à la distribution familière de Wigner quand $f = g$, [53,66]. La transformée $\mathcal{V}$ est un outil de base pour étudier la transformée de Weyl et à interpréter la mécanique quantique comme une forme dynamique statique non déterministe [48]. De plus il est très utile dans l'étude de la distribution de probabilité conjointe inexistante de l'impulsion positionnée dans un état donné [53].

Le but de ce chapitre est projeter cette transformation au cadre bicomplexe c'est-à-dire où $\mathbb{T} = \mathbb{C} + j\mathbb{C} = (\mathbb{R} \times \mathbb{R})e_+ + (\mathbb{R} \times \mathbb{R})e_-$est considéré à la place de l'espace de phase standard (temps-fréquence) $R \times \mathbb{R}$. Cela est dû à la décomposition idempotente des nombres bicomplexes. Bien que, cela peut être accomplir de plusieurs façons différentes mais on va s'intéresser à deux types de transformées de Fourier -Wigner principale qui sont très liés? groupe de Heisenberg bicomplexe $\mathbb{T} \times \mathbb{D}$ où $\mathbb{D}$ désigne l'ensemble des nombres hyperboliques. La représentation associée apparaît comme lastructure sous-jacente destransformées de Fourier-Wigner considérées. onétudie leurs propriétés de base comme lesidentités correspondantes de Moyal. oncaractérise aussi leurordre ce qui mène àunnouveau espaces de fonction bicomplexe polyanalytique et aussi on aintroduit une autre classe de polynômes orthogonaux d'Hermite dequatre indices qui forme une base orthonormée de $L^2_{\mathbb{T}}(\mathbb{T})$ l'espacede Hilbertinfini constitué par lesfonctionsdescarrés integrablles sur $\mathbb{T}$ à valeur bicomplexe

General introduction

The bicomplex $\mathbb{BC}$ are a special generalization of the complex numbers. Roughly speaking, they are complex numbers with complex coefficients. Since their introduction by Segre [32], different mathematical topics have been developed and investigated, including bicomplex functional analysis [13,16,22,33,34], bicomplex differentiability [2,22,49] and bicomplex quantum [14,17,19,27,50]. For a complete treatment on bicomplex function theory as well as their applications see, e.g., [2,10,11,22,25,29,30,49].

To complete the studies in the aforementioned works, the main purpose of the present project is to introduce and study in some details the basic properties of some interesting integral transforms in the framework of some infinite bicomplex Hilbert spaces, arising from time-frequency analysis. Mainly, we are concerned with the bicomplex analogues of Segal–Bargmann (Chapter 2), Fourier–Wigner transforms (Chapter 3), as well as the fractional Fourier transform (Chapter 4).

To this purpose, we begin by an introductory chapter in which we review briefly the basic concepts from bicomplex analysis, and discuss different needed associated structures.

In Chapter 2, we consider the bicomplex Bargmann space $\mathcal{F}^{2,\nu}(\mathbb{BC})$ and the associated bicomplex Segal–Bargmann transform $\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}$ in the context of the bicomplex holomorphic functions, the analogs of the classical ones $\mathcal{F}^{2,\gamma}(\mathbb{C})$ and $\mathcal{B}^{\sigma,\gamma}_{\mathbb{C}}$ for holomorphic functions. The first result concerning $\mathcal{F}^{2,\nu}(\mathbb{BC})$ is obtained as immediate corollary of the key observation (Theorem 2.2.1). It decomposes $\mathcal{F}^{2,\nu}(\mathbb{BC})$ in terms of the classical ones with respect to the idempotent decomposition of bicomplex holomorphic functions. We next deal with the integral representations of the L^2–bicomplex holomorphic functions. Namely, Theorem 2.3.1 expresses the fact that $\mathcal{F}^{2,\nu}(\mathbb{BC})$ is a reproducing kernel bicomplex Hilbert space. On the other hand, Theorem 2.3.5 connects $\mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$ to the special subspace of $\mathcal{F}^{2,\nu}(\mathbb{BC})$ leaving $\mathbb{C}_i := \mathbb{C} + j\{0\}$ invariant, through a special explicit integral transform. This transform is further a surjection from the space of L^2–holomorphic functions on the complex plane $\mathbb{C}$ with values in $\mathbb{BC}$ onto the bicomplex Bargmann space $\mathcal{F}^{2,\nu}(\mathbb{BC})$. Concerning the bicomplex Segal–Bargmann transform $\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}$, we show that it is a unitary isometric transform from the space of bicomplex-valued square integrable functions on the real line onto the bicomplex holomorphic Bargmann space (Theorem 2.4.2). We give the explicit expression of the inverse $[\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1}$ (Theorem 2.4.6).

Chapter 3 is an immediate application of the results obtained in the previous chapter. Thus, we use both $\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}$ and $[\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1}$ to introduce a two–parameter family of bicomplex fractional Fourier transforms (BFrFT) labeled by the set of bicomplex numbers θ; $|\theta| = 1$, $\theta \neq \pm 1, \pm ij$, so that one recovers the classical i-Fourier transform as well as its variant; the j-Fourier transform. The basic properties of BFrFT, such as the uniqueness theorem, the Plancherel theorem as well as the inversion formula are obtained.

Chapter 4 deals with the two bicomplex analogues of the classical (rescaled) Fourier–Wigner transform (windowed Fourier) intensively considered in harmonic analysis [51,67] and time–frequency analysis [39,47]. It is the well–defined bilinear mapping $\mathcal{V} : (f,g) \longmapsto \mathcal{V}(f,g)$ on $L^2_{\mathbb{C}}(\mathbb{R}^d) \times L^2_{\mathbb{C}}(\mathbb{R}^d)$ given by the cross–Wigner function [40,51,53,66,67]

$$\mathcal{V}(f,g)(p,q) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle x-\frac{p}{2},q\rangle} f(x)\overline{g(x-p)}dx \tag{2}$$

for every $(p,q) \in \mathbb{R}^d \times \mathbb{R}^d$, where $\langle , \rangle$ denotes the standard scaler product in $\mathbb{R}^d$. For fixed window state, it is closely related to Gabor's transform [44] as well as to the well–known Segal–Bargmann transform [51,67]. It reduces to the familiar Wigner distribution when $f = g$; see e.g [53,66]. The transform $\mathcal{V}$ is a basic tool in studying Weyl transform [53,67] and to interpreting quantum mechanics as a form of nondeterministic statical dynamics [48]. Moreover, it is very useful in the study of nonexisting joint probability distribution of positioned momentum in a given state [53].

The aim of this chapter is to extend this transform to the bicomplex setting, i.e. where $\mathbb{BC} = \mathbb{C} + j\mathbb{C} = (\mathbb{R} \times \mathbb{R})e_+ + (\mathbb{R} \times \mathbb{R})e_-$ is considered instead of the standard phase (time-frequency) space $\mathbb{R} \times \mathbb{R}$. This follows using the idempotent decomposition of bicomplex numbers. Although, this can be accomplished in a number of different ways, we shall confine our attention to two main natural bicomplex Fourier–Wigner transforms which are intimately connected to some bicomplex Heisenberg group $\mathbb{BC} \times \mathbb{D}$, where $\mathbb{D}$ denotes the set of hyperbolic numbers. The associated representation emerge as the underlying structure for the considered bicomplex Fourier–Wigner transforms. We investigate their basic properties such as the corresponding Moyal's identities (energy preservation principle). We also characterize their ranges leading to new bicomplex–polyanalytic functional spaces. We also provide a new class of four–indices bivariate complex orthogonal polynomials of Hermite type that form an orthonormal basis of $L^2_{\mathbb{BC}}(\mathbb{BC})$, the infinite Hilbert space of bicomplex–valued square integrable functions on bicomplex space.

The topics developed in this document are the author main contributions to the theory of bicomplex integral transforms, recently published in international indexed journals.

Chapter 1

BICOMPLEX NUMBERS AND ASSOCIATED STRUCTURES

This is a preliminary chapter in which we are concerned with a brief review and the introduction of some basic concepts and results related to bicomplex numbers, as well as to infinite bicomplex Hilbert space, including the Gaussian bicomplex Hilbert space $\mathcal{H}^{2,\nu}(\mathbb{BC})$, and bicomplex holomorphic functions. We also provide a brief description of next chapters.

1.1 The Bicomplex (or Tetra) numbers

The first section is devoted to recall the notation and the results concerning the theory of bicomplex holomorphic functions and bicomplex Hilbert spaces. For more details I suggest the reader to see the following references [26,27,49,50]. We start to recall some definitions and basic properties for bicomplex numbers.

1.1.1 Algebraic structure of bicomplex numbers

They are a special generalization of complex numbers, $z = x + iy; x, y \in \mathbb{R}, i^2 = -1$, by means of entities specified by four real numbers. In abstract algebra, a bicomplex number is a pair (z_1, z_2) of complex numbers constructed by the Cayley-Dickson process that defines the bicomplex conjugate $(z_1, -z_2)$.

More specifically, if we denote by $\mathbb{C}$ the set of complex numbers, and we let denote by $\mathbf{j}$ a pure imaginary unit independent of $\mathbf{i}$ but commuting with, i.e., $\mathbf{ij} = \mathbf{ji} = \mathbf{k}$ with $\mathbf{j}^2 = \mathbf{i}^2 = -1$, then

$$\mathbb{BC} := \{Z = z_1 + \mathbf{j}z_2; z_1, z_2 \in \mathbb{C}\}.$$

Moreover, a bicomplex number defined as $Z = z_1 + \mathbf{j}z_2$ admits several other forms of writing, or representations, which show different aspects of this number and which will help us to understand better the structure of the set $\mathbb{BC}$. First of all, if we write $z_1 = x_1 + \mathbf{i}y_1$, $z_2 = x_2 + \mathbf{i}y_2$ with real numbers x_1, y_1, x_2, y_2, then any bicomplex number can be written in the following different ways

$$\begin{aligned} Z &= (x_1 + \mathbf{i}y_1) + \mathbf{j}\,(x_2 + \mathbf{i}y_2) =: z_1 + \mathbf{j}z_2 \\ &= (x_1 + \mathbf{j}x_2) + \mathbf{i}\,(y_1 + \mathbf{j}y_2) =: \zeta_1 + \mathbf{i}\zeta_2 \\ &= (x_1 + \mathbf{k}y_2) + \mathbf{i}\,(y_1 - \mathbf{k}x_2) =: \mathfrak{z}_1 + \mathbf{i}\mathfrak{z}_2 \\ &= (x_1 + \mathbf{k}y_2) + \mathbf{j}\,(x_2 - \mathbf{k}y_1) =: \mathfrak{w}_1 + \mathbf{j}\mathfrak{w}_2 \\ &= (x_1 + \mathbf{i}y_1) + \mathbf{k}\,(y_2 - \mathbf{i}x_2) =: w_1 + \mathbf{k}w_2 \\ &= (x_1 + \mathbf{j}x_2) + \mathbf{k}\,(y_2 - \mathbf{j}y_1) =: \omega_1 + \mathbf{k}\omega_2 \\ &= x_1 + \mathbf{i}y_1 + \mathbf{j}x_2 + \mathbf{k}y_2. \end{aligned}$$

Notice for instance that addition and multiplication are defined in natural ways (term-by-term). Being indeed, we have

$$Z + W := (z_1 + w_1) + \mathbf{j}\,(z_2 + w_2) \tag{1.1}$$

and for two bicomplex numbers $Z = z_1 + \mathbf{j}z_2$ and $W = w_1 + \mathbf{j}w_2$ we have

$$Z \cdot W := (z_1 + \mathbf{j}z_2)\,(w_1 + \mathbf{j}w_2) = (z_1w_1 - z_2w_2) + \mathbf{j}\,(z_1w_2 + z_2w_1)\,. \tag{1.2}$$

Accordingly, it easy to see that the addition and multiplication are both commutative and associative. Moreover, the multiplication is distributive with respect to the addition operation. A specific subset of bicomplex numbers, is the set of hyperbolic (bireal) numbers

$$\mathbb{D} :== \{x + \mathbf{k}y \mid x, y \in \mathbb{R}\}.$$

defined with respect to given hyperbolic imaginary unit $\mathbf{k}$; $\mathbf{k}^2 = 1$.

1.1.2 Conjugation and moduli

The bicomplex numbers has many structures and properties common with the complex numbers. Nevertheless, there are more deep and even striking differences. Thus, unlike the quaternions, the bicomplex numbers form a commutative algebra over $\mathbb{C}$ and they are three types of conjugate, the $*$-conjugation, the $\dagger$-conjugation, and the $\widetilde{\ }$-conjugation, where each conjugation is an additive, involutive, and multiplicative operation on $\mathbb{BC}$. Thus, the complex conjugate of $Z = z_1 + \mathbf{j}z_2 \in \mathbb{BC}$ with respect to $\mathbf{j}$ is given by

$$Z^\dagger = z_1 - \mathbf{j}z_2,$$

$\widetilde{Z} = \overline{z_1} + \mathbf{j}\overline{z_2} and Z^* = \overline{z_1} - \mathbf{j}\overline{z_2}. where_1, \bar{z}_2$ are usual complex conjugates to $z_1, z_2 \in \mathbb{C}$. The other forms of complex conjugates are given by Accordingly, by following the same idea in defining the modulus of a complex number, which its square is obtained by multiplying a complex number by its conjugate one, we deal with three kinds possible moduli that arise in accordance with the formulas for their squares. The first one is defined by means of

$\dagger$-conjugation, to wit

$$|Z|_{\mathbf{i}}^2 := Z \cdot Z^{\dagger} = z_1^2 + z_2^2.$$

We also define

$$|Z|_{\mathbf{j}}^2 := Z \cdot \bar{Z} = \left(|z_1|^2 - |z_2|^2\right) + 2\,\mathrm{Re}\,(z_1\bar{z}_2)\,\mathbf{j}$$

and

$$|Z|_{\mathbf{k}}^2 := Z \cdot Z^* = \left(|z_1|^2 + |z_2|^2\right) - 2\,\mathrm{Im}\,(z_1\bar{z}_2)\,\mathbf{k}.$$

Unlike what happens in the complex case, such moduli are not $\mathbb{R}^+$-valued, nevertheless they preserve, fortunately, an important property related with the multiplication. By identifying $\mathbb{BC}$ to $\mathbb{C}^2$, we can consider the Euclidean norm $|Z|$ of a bicomplex number $Z = z_1 + \mathbf{j}z_2; z_1, z_2 \in \mathbb{C}$, to be

$$|Z| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2} = \sqrt{\mathrm{Re}\left(|Z|_{\mathbf{k}}^2\right)}.$$

It satisfies the identity

$$|Z \cdot W| \leq \sqrt{2}|Z| \cdot |W|. \tag{1.3}$$

1.1.3 Zero-Divisors and invertibility

According to the nullity of $ZZ^{\dagger} = z_1^2 + z_2^2$, we distinguish two interesting classes of bicomplex numbers. Indeed, $ZZ^{\dagger} \neq 0$ characterizes those that are invertible. While $ZZ^{\dagger} = 0$ is equivalent to $Z = \lambda(1 \pm ij)$ for certain complex number $\lambda \in \mathbb{C} = \mathbb{C}_i$ and characterizes those that are zero divisors in $\mathbb{BC}$.

1.1.4 Idempotent representation

In view of what precede, one considers the idempotent elements

$$e_+ = \frac{1+ij}{2} \quad \text{and} \quad e_- = \frac{1-ij}{2}$$

which satisfy the identities

$${e_+}^2 = e_+, \quad {e_-}^2 = e_-, \quad e_+ + e_- = 1, \quad e_+ - e_- = ij, \quad e_+e_- = 0.$$

The last identity shows in particular that $\mathbb{BC}$ is not a division algebra and that

$$e_+ = \frac{1}{2}(1, \mathbf{i}) \quad \text{and} \quad e_- = \frac{1}{2}(1, -\mathbf{i})$$

are orthogonal with respect to the Euclidean inner product in $\mathbb{C}^2$.

As crucial property in considering these special zero divisors is that any bicomplex number, $Z = z_1 + \mathbf{j}z_2 \in \mathbb{BC}$ can be rewritten in a unique way as

$$Z = (z_1 - \mathbf{i}z_2)e_+ + (z_1 + \mathbf{i}z_2)e_- = \alpha e_+ + \beta e_- \tag{1.4}$$

with $\alpha = z_1 - \mathbf{i}z_2, \beta = z_1 + \mathbf{i}z_2 \in \mathbb{C}$. Uniqueness of such decomposition is to be taken in the sense that if the bicomplex number Z has two idempotent representations $Z = \alpha e_+ + \beta e_- = \alpha' e_+ + \beta' e_-$ with complex coefficients $\alpha, \beta, \alpha', \beta' \in \mathbb{C}$, then $\alpha = \alpha'$ and $\beta = \beta'$. Moreover, it is immediate to check that the $Z^\dagger$-conjugate, the $\widetilde{Z}$-conjugate and the Z^*-conjugate read respectively

$$Z^\dagger = \beta e_+ + \alpha e_-, \quad \widetilde{Z} = \overline{\beta} e_+ + \overline{\alpha} e_- \quad \text{and} \quad Z^* = \overline{\alpha} e_+ + \overline{\beta} e_-.$$

The idempotent representation (1.4) is a central observation that simplifies considerably the computation with bicomplex numbers and reduces them to complex numbers. We conclude this section by noticing that, by viewing $\mathbb{BC}$ as $\mathbb{C}$-linear space, we can make the identification $\mathbb{BC} = \mathbb{C}^2$, by identifying the bases $\{1, \mathbf{j}\}$ and $\{e_+, e_-\}$, via $e_+ = \left(\frac{1}{2}, \frac{\mathbf{i}}{2}\right) \in \mathbb{C}^2$ and $e_- = \left(\frac{1}{2}, -\frac{\mathbf{i}}{2}\right) \in \mathbb{C}^2$. The transition formula is given by

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ \mathbf{i} & -\mathbf{i} \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{1.5}$$

The idempotent basis $\{e_+, e_-\}$ is orthogonal with respect to the Euclidean inner product

$$\langle (z_1, z_2), (w_1, w_2) \rangle_{\mathbb{C}^2} := z_1 \bar{w}_1 + z_2 \bar{w}_2$$

in $\mathbb{C}^2$. In fact, we have $\langle e_+, e_- \rangle_{\mathbb{C}^2} = 0$ and

$$\langle e_+, e_+ \rangle_{\mathbb{C}^2} = \langle e_-, e_- \rangle_{\mathbb{C}^2} = 1/2.$$

Thus $\{e_+, e_-\}$ is an orthogonal but not orthonormal basis for $\mathbb{C}^2$. Subsequently, for $Z = \alpha e_+ + \beta e_-$, we obtain

$$|Z| = \frac{1}{\sqrt{2}} \sqrt{|\alpha|^2 + |\beta|^2}. \tag{1.6}$$

1.2 $\mathbb{BC}$-Modules

The set of bicomplex numbers is a commutative ring. Just like vector spaces are defined over fields, modules are defined over rings [56]. A module M defined over the ring $\mathbb{BC}$ of bicomplex numbers is called a $\mathbb{BC}$ -module [57] [55]. Thus, if M is a $\mathbb{BC}$-module, then we consider the $\mathbb{C}$-vector spaces $V^+ = Me_+$and $V^- = Me_-$. Therefore, M can be seen as a $\mathbb{C}$-vector space by considering $M' = V^+ \oplus V^-$. In general, $V^+$and V^-bear no structural similarities. Any element $v^\pm \in V^\pm$ satisfies $v^\pm = v^\pm \mathbf{e}_\pm$.

1.2.1 Bicomplex norm and scalar product.

The norm of a vector is an important concept in vector space theory. Below, we generalize it to $\mathbb{BC}$-modules.

Definition 1.2.1. *A $\mathbb{BC}$-norm on a $\mathbb{BC}$-module M is a mapping $\| \cdot \| : M \longrightarrow \mathbb{R}$ such that*

(1) $\|\cdot\|$ *is a norm on the vector space* $V^+ \oplus V^-$,

(2) $\|\lambda\phi\| \leq \sqrt{2}|\lambda|\|\phi\|$ *for all* $\lambda \in \mathbb{BC}$ *and all* $\phi \in M$.

A $\mathbb{BC}$*-module with a* $\mathbb{BC}$*-norm is called a normed* $\mathbb{BC}$*-module.*

As in the theory of $\mathbb{C}$-vector spaces, a $\mathbb{BC}$-norm can always be induced from a $\mathbb{BC}$-scalar product. We define a bicomplex scalar product by

Definition 1.2.2. *Let* M *be a* $\mathbb{BC}$*-module. A inner product on* M *is a given functional* $\langle\cdot,\cdot\rangle : M \times M \longrightarrow \mathbb{BC}$ *satisfying*

1. $\langle\psi,\phi+\phi'\rangle = \langle\psi,\phi\rangle + \langle\psi,\phi'\rangle, \forall\psi,\phi,\phi' \in M$
2. $\langle\psi,\alpha\phi\rangle = \alpha\langle\psi,\phi\rangle, \forall\alpha \in \mathbb{BC}, \forall\psi,\phi \in M$
3. $\langle\psi,\phi\rangle = \langle\phi,\psi\rangle^*, \forall\psi,\phi \in M$
4. $\langle\psi,\psi\rangle = 0 \Leftrightarrow \psi = 0, \forall\psi \in M.$

Notice for instance that the assertion 3 implies that $\langle\psi,\psi\rangle \in \mathbb{D}$. Definition 1.2.1 seems to be very general. the natural restrictive situation requires the bicomplex scalar product $\langle\cdot,\cdot\rangle$ to be hyperbolic positive, that is,$\langle\psi,\psi\rangle \in \mathbb{D}^+$, for every $\psi \in M$. Moreover, from Definition 1.2.1, the projection $\langle\cdot,\cdot\rangle_{V^\pm}$ of $\langle\cdot,\cdot\rangle$ to each $V^\pm$ is a standard scalar product on $V^\pm$. More precisely, if ϕ,φ are in M identified to $V^+ \oplus V^-$, we have

$$\langle\phi,\varphi\rangle = \langle\phi^+,\varphi^+\rangle_{V^+} e_+ + \langle\phi^-,\varphi^-\rangle_{V^-} e_-, \tag{1.7}$$

where $\psi^+ := \psi e_+ \in V^+$ and $\psi^- := \psi e_- \in V^-$ for given $\psi \in M$. Notice that any $\mathbb{BC}$-scalar product on M is completely determined in this way (see [17, Theorem 2.6])

As motioned above, a $\mathbb{BC}$-norm is then induced from a $\mathbb{BC}$-scalar product by considering

$$\|\phi\|^2 = \frac{\langle\phi^+,\phi^+\rangle_{V^+} + \langle\phi^-,\phi^-\rangle_{V^-}}{2} = |\langle\phi,\phi\rangle|, \tag{1.8}$$

where $\phi = \phi^+ + \phi^-$ and the modulus $|\cdot|$ denotes the usual Euclidean norm of Z in $\mathbb{R}^4$ given by

$$|Z|^2 = |z_1|^2 + |z_2|^2 = \frac{|\alpha|^2 + |\beta|^2}{2} \tag{1.9}$$

for given $Z = z_1 + z_2 j = \alpha e_+ + \beta e_-; z_1, z_2, \alpha, \beta \in \mathbb{C}$. A trivial example of a $\mathbb{BC}$-inner product on $M = \mathbb{BC}$ is the following

$$\langle Z, W\rangle = ZW^* = \alpha\overline{\alpha'}e_+ + \beta\overline{\beta'}e_-, \tag{1.10}$$

where $Z = \alpha e_+ + \beta e_-$ and $W = \alpha' e_+ + \beta' e_-$ are the idempotent representations of Z and W in $\mathbb{BC}$, respectively.

The norm in (1.8) obeys a generalized Schwarz inequality ([17, Theorem 3.7]) bicomplex Hilbert space.

Theorem 1.2.3. *(Bicomplex Schwarz inequality). Let $\psi, \phi \in M$. Then*

$$|\langle \phi, \varphi \rangle| \leq \sqrt{2}\, \|\phi\|\, \|\varphi\| \, . \tag{1.11}$$

Proof. From the complex (in $\mathbb{C}(\mathbf{i})$) Schwarz inequality we have

$$|\langle \psi_\pm, \phi_\pm \rangle_{V^\pm}|^2 \leq ||\psi_\pm||^2_{V^\pm} \cdot ||\phi_\pm||^2_{V^\pm}, \quad \forall \psi_\pm, \phi_\pm \in V^\pm$$

Therefore, if $\psi, \phi \in M$, we obtain from (**??**) and (**??**)

$$\begin{aligned} |\langle \psi, \phi \rangle| &= |e_+ \langle \psi_+, \phi_+ \rangle_{V^+} + e_- \langle \psi_-, \phi_- \rangle_{V^-}| \\ &= \frac{1}{\sqrt{2}} \sqrt{|\langle \psi_+, \phi_+ \rangle_{V^+}|^2 + |\langle \psi_-, \phi_- \rangle_{V^-}|^2} \\ &\leq \frac{1}{\sqrt{2}} \sqrt{\ ||\psi\ _+||^2_{V^+} \cdot ||\phi\ _+||^2_{V^+} + ||\psi\ _-||^2_{V^-} \cdot ||\phi_-\ ||^2_{V^-}} \\ &\leq \sqrt{2} \|\psi\| \|\phi\|. \end{aligned}$$

■

1.2.2 Infinite bicomplex Hilbert space.

Definition 1.2.4. *Let M be a $\mathbb{BC}$-module and let $\langle \cdot, \cdot \rangle$ be a bicomplex scalar product defined on M. The space $\{M, \langle \cdot, \cdot \rangle\}$ is called a $\mathbb{BC}$-inner product space, or bicomplex pre-Hilbert space. When no confusion arises, $\{M, \langle \cdot, \cdot \rangle\}$ will simply be denoted by M.*

Theorem 1.2.5. *Let M be a bicomplex pre-Hilbert space. Then $(V^\pm, \langle \cdot, \cdot \rangle_{V^\pm})$ is a complex pre-Hilbert space*

Proof. If V^+ and V^- are complete, then $M' = V^+ \oplus V^-$ is a direct sum of two Hilbert spaces. It is easy to see that M' is also a Hilbert space, when the following natural scalar product is defined over the direct sum [58]

$$\langle \psi_+ \oplus \psi_-, \phi_+ \oplus \phi_- \rangle = \langle \psi_+, \phi_+ \rangle_{V^+} + \langle \psi_-, \phi_- \rangle_{V^-}.$$

From this scalar product, we can define a norm on the vector space M'

$$\|\phi\| := \frac{1}{\sqrt{2}} \sqrt{\langle \phi_+, \phi_+ \rangle_{V^+} + \langle \phi_-, \phi_- \rangle_{V^-}} = \frac{1}{\sqrt{2}} \sqrt{\,||\phi_+||^2_{V^+} + ||\phi||\,|^2_{V^-}}. \tag{1.12}$$

Here, we wrote $\|\phi_\pm\|_{V^\pm} = \sqrt{(\phi_\pm, \phi_\pm)_{V^\pm}}$, where $|| \cdot ||_{V^\pm}$ is the natural scalar-product-induced norm on $V^\pm$. The $1/\sqrt{2}$ factor in (1.12) is introduced so as to relate in a simple manner the norm with the bicomplex scalar product. Indeed, we have

$$\|\phi\| = \frac{1}{\sqrt{2}} \sqrt{\langle \phi_+, \phi_+ \rangle_{V^+} + \langle \phi_-, \phi_- \rangle_{V^-}} = |\sqrt{\langle \phi, \phi \rangle}|$$

which follows also from (1.7). Moreover, it is easy to check that $\| \cdot \|$ is a $\mathbb{BC}$ -norm on M

and that the $\mathbb{BC}$-module M is complete with respect to the metric $d(\phi, \psi) = \|\phi - \psi\|$ on M. Thus M is a complete $\mathbb{BC}$-module. ▬

The concept of infinite bicomplex Hilbert space was defined in [17]. It is a $\mathbb{BC}$-inner product space $(M, \langle\cdot,\cdot\rangle)$ which is complete with respect to the induced $\mathbb{BC}$-norm (1.8). The next result is interesting in itself and its proof is contained in Theorems 3.4, 3.5 and Corollary 3.6 in [17].

Theorem 1.2.6. *The $\mathbb{BC}$-module $(M, \langle\cdot,\cdot\rangle)$ is an infinite bicomplex Hilbert space if and only if $(V^{\pm}, \langle\cdot,\cdot\rangle_{V^{\pm}})$ are $\mathbb{C}$-Hilbert spaces.*

We conclude this subsection by recalling the bicomplex version of the classical Riesz' representation theorem [17, Theorem 3.7].

Theorem 1.2.7. *For every continuous $\mathbb{BC}$–valued linear functional f on a $\mathbb{BC}$-module M, there exists a unique $\phi_0 \in M$ such that for every $\phi \in M$ we have*

$$f(\phi) = \langle \phi, \phi_0 \rangle . \tag{1.13}$$

1.2.3 Basic example

In order to provide an interesting example of infinite bicomplex Hilbert space, we need to fix further notation. Let $L^{2,\nu}(X)$ be as in the end of the introductory section. We also consider the space $\mathcal{H}^{2,\nu}(\mathbb{BC})$ of all $\mathbb{BC}$–valued Borel measurable functions f on $\mathbb{BC}$ subject to $\|f\|_{\mathcal{H}^{2,\nu}(\mathbb{BC})} < +\infty$. The bicomplex norm $\|f\|_{\mathcal{H}^{2,\nu}(\mathbb{BC})}$ is the one induced from the bicomplex inner product

$$\langle f, g\rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} := c^{\nu}_{\mathcal{H}^{2,\nu}(\mathbb{BC})} \int_{\mathbb{BC}} \langle f(Z), g(Z)\rangle\, e^{-\nu|Z|^2} d\lambda(Z)$$

via (1.8), where $\langle\cdot,\cdot\rangle$, in the integrand, is the standard bicomplex inner product on $\mathbb{BC}$ defined by (1.10). In fact, for every $f = f_+e_+ + f_-e_-, g = g_+e_+ + jg_-e_-$, we have

$$\langle f, g\rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \langle f_+, g_+\rangle_{L^{2,\nu}(\mathbb{C}^2)}\, e_+ + \langle f_-, g_-\rangle_{L^{2,\nu}(\mathbb{C}^2}\, e_-,$$

where $f_{\pm}$ and $g_{\pm}$, are seen as $\mathbb{C}$–valued functions on $\mathbb{C}^2$ in the variables (z_1, z_2). Thus, we have

$$\|f\|^2_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \frac{1}{2}\left(\|f_+\|^2_{L^{2,\nu}(\mathbb{C}^2)} + \|f_-\|^2_{L^{2,\nu}(\mathbb{C}^2)}\right). \tag{1.14}$$

Subsequently, the following decomposition, with respect to the variables z_1 and z_2,

$$\mathcal{H}^{2,\nu}(\mathbb{BC}) = L^{2,\nu}(\mathbb{C}^2)e_+ + L^{2,\nu}(\mathbb{C}^2)e_-, \tag{1.15}$$

readily follows from (1.14). Another interesting decomposition of $\mathcal{H}^{2,\nu}(\mathbb{BC})$ with respect to the idempotent representation of bicomplex numbers is the following

Proposition 1.2.8. *We have*

$$\mathcal{H}^{2,\nu}(\mathbb{BC}) = L^{2,\frac{\nu}{2}}(\mathbb{C}^2)e_+ + L^{2,\frac{\nu}{2}}(\mathbb{C}^2)e_-. \tag{1.16}$$

More precisely, for every $f \in \mathcal{H}^{2,\nu}(\mathbb{BC})$ there exist some $\phi^\pm \in L^{2,\frac{\nu}{2}}(\mathbb{C}^2)$ such that

$$f(\alpha e_+ + \beta e_-) = \phi^+(\alpha,\beta)e_+ + \phi^-(\alpha,\beta)e_-$$

with

$$\|f\|^2_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \frac{1}{2}\left(\|\phi^+\|^2_{L^{2,\frac{\nu}{2}}(\mathbb{C}^2)} + \|\phi^-\|^2_{L^{2,\frac{\nu}{2}}(\mathbb{C}^2)}\right). \tag{1.17}$$

Moreover, the space $\mathcal{H}^{2,\nu}(\mathbb{BC})$ is an infinite bicomplex Hilbert space on $\mathbb{BC}$.

Proof. For any $f = f_1 + jf_2$ and $g = g_1 + jg_2 \in \mathcal{H}^{2,\nu}(\mathbb{BC})$, we consider the functions $\phi^\pm = f_1 \mp if_2$ and $\varphi^\pm = g_1 \mp ig_2$, seen as $\mathbb{C}$–valued functions on $\mathbb{C}^2$ in the variables (α,β), so that

$$f(\alpha e_+ + \beta e_-) = \phi^+(\alpha,\beta)e_+ + \phi^-(\alpha,\beta)e_-$$

and

$$g(\alpha e_+ + \beta e_-) = \varphi^+(\alpha,\beta)e_+ + \varphi^-(\alpha,\beta)e_-.$$

Thus, we have

$$\langle f,g\rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \left\langle \phi^+,\varphi^+\right\rangle_{L^{2,\frac{\nu}{2}}(\mathbb{C}^2)} e_+ + \left\langle \phi^-,\varphi^-\right\rangle_{L^{2,\frac{\nu}{2}}(\mathbb{C}^2)} e_-, \tag{1.18}$$

since the Lebesgue measure on $\mathbb{BC} \equiv \mathbb{R}^4$ reads

$$d\lambda(Z) = dx_1dy_1dx_2dy_2 = \frac{1}{4}d\lambda(\alpha)d\lambda(\beta).$$

Therefore, (1.14) reduces further to (1.17). This completes our check of (1.16). Finally, the result that $\mathcal{H}^{2,\nu}(\mathbb{BC})$ is an infinite bicomplex Hilbert space on $\mathbb{BC}$ readily follows making use of Theorem 1.2.6 thanks to (1.16) (or also (1.15)). ∎

Chapter 2

BICOMPLEX SEGAL–BARGMANN TRANSFORM

n this chapter, we consider and discuss some basic properties of the bicomplex analog of the classical Bargmann space. The explicit expression of the integral operator connecting the complex and bicomplex Bargmann spaces is also given. The corresponding bicomplex Segal–Bargmann transform is introduced and studied as well.

From now on, we denote by $L^{2,\alpha}(X)$ the Hilbert spaces of all square integrable $\mathbb{C}$–valued functions on $X = \mathbb{R}, \mathbb{C}, \mathbb{C}^2$ with respect to the Gaussian measure

$$c^{\alpha}_{d_X} e^{-\alpha\|u\|^2_X} d\lambda_X(u).$$

The analog Hilbert spaces on $\mathbb{BC}$ or with values in $\mathbb{BC}$ are appropriately defined in Sections 4.2, 4.3 and 4.4. The normalization $c^{\alpha}_{d_X}$ varies from one Hilbert space to another and is taken such that

$$\int_X c^{\alpha}_{d_X} e^{-\alpha\|u\|^2_X} d\lambda_X(u) = 1.$$

The d_X in $c^{\alpha}_{d_X}$ can be interpreted as the complex dimension of X, so that $d_X = 0, 1, 2$ for $X = \mathbb{R}, \mathbb{C}, \mathbb{C}^2$ respectively. Thus, we have

$$c^{\alpha}_{\mathbb{BC}} = c^{\alpha}_2 = 4c^{\alpha/2}_2 = (c^{\alpha}_1)^2 = 4(c^{\alpha/2}_1)^2 = (c^{\alpha}_0)^4 = \left(\frac{\alpha}{\pi}\right)^2.$$

2.1 Bicomplex holomorphic functions

Following [49], a $\mathbb{BC}$–valued function $f = f_1 + jf_2$ on an open set $\Omega \subset \mathbb{BC}$ is said to be $\mathbb{BC}$-holomorphic at a point $Z_0 \in \mathbb{BC}$ if it admits a bicomplex derivative at Z_0, i.e., if the limit

$$\lim_{\substack{H \to 0 \\ H \notin \mathcal{NC}}} \frac{f(Z_0 + H) - f(Z_0)}{H}$$

exists and is finite, where $\mathcal{NC}$ denotes the null cone of bicomplex numbers defined by

$$\begin{aligned}\mathcal{NC} &= \{Z = z_1 + jz_2 \in \mathbb{BC};\, z_1^2 + z_2^2 = 0\} \\ &= \{Z = \alpha e_+ + \beta e_-;\, \alpha\beta = 0\}.\end{aligned}$$

This is equivalent to say that the $\mathbb{C}$–valued functions f_1 and f_2 are holomorphic in the variables (z_1, z_2) with $Z = z_1 + jz_2$ and satisfy the Cauchy-Riemann system

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \quad \text{and} \quad \frac{\partial f_1}{\partial z_2} = -\frac{\partial f_2}{\partial z_1},$$

which we can rewrite in matrix representation as

$$\begin{pmatrix} \partial_{z_1} & -\partial_{z_2} \\ \partial_{z_2} & \partial_{z_1} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here, the shorthand ∂_z is used to mean the differential operator $\partial/\partial z$. The following characterization of $\mathbb{BC}$-holomorphicity is given in [25] and shows that bicomplex holomorphic functions are once again solutions of a linear system of differential equations with constant coefficients. Namely, a given $f \in \mathcal{C}^1(\Omega)$ is $\mathbb{BC}$-holomorphic on Ω if and only if f satisfies the following three systems of differential equations

$$\frac{\partial f}{\partial Z^*} = \frac{\partial f}{\partial Z^\dagger} = \frac{\partial f}{\partial \widetilde{\overline{Z}}} = 0,$$

where

$$\frac{\partial}{\partial Z^*} = \frac{\partial}{\partial \overline{z_1}} + j\frac{\partial}{\partial \overline{z_2}}; \qquad \frac{\partial}{\partial Z^\dagger} = \frac{\partial}{\partial z_1} + j\frac{\partial}{\partial z_2}; \qquad \frac{\partial f}{\partial \widetilde{\overline{Z}}} = \frac{\partial}{\partial \overline{z_1}} - j\frac{\partial}{\partial \overline{z_2}}.$$

The above system is the foundation pillar for the theory of bicomplex holomorphic functions. Accordingly, any bicomplex holomorphic $\mathbb{BC}$–valued function f is of the form [49, Theorem 15.5]

$$f(Z) = f(\alpha e_+ + \beta e_-) = \phi^+(\alpha)e_+ + \phi^-(\beta)e_-, \tag{2.1}$$

where $\phi^\pm : \mathbb{C} \longrightarrow \mathbb{C}$ are holomorphic functions on $\mathbb{C}$. This is a key tool that we use in proving Theorem 2.2.1 below. We denote by $\mathcal{BH}ol(\mathbb{BC})$ the space of bicomplex holomorphic functions on $\mathbb{BC}$ taking their values in $\mathbb{BC}$.

2.2 $\mathbb{BC}$-Bargmann space

Recall first that the classical Bargmann space consists of all holomorphic functions on the complex plane subject to norm boundedness with respect to the Gaussian measure with given normalization constant c_1^γ,

$$\mathcal{F}^{2,\gamma}(\mathbb{C}) := Hol(\mathbb{C}) \cap L^2\left(\mathbb{C}; c_1^\gamma e^{-\gamma|z|^2} dxdy\right); \gamma > 0. \tag{2.2}$$

It is well–studied in the literature [8,31,64] and is a convenient setting for many problems in functional analysis, mathematical physics, and engineering. Basic references in these areas are e.g. [18,36,64,67] and the references therein.

The bicomplex counterpart of the classical Bargmann space in (2.2) is defined to be the subspace of the infinite Hilbert space $\mathcal{H}^{2,\nu}(\mathbb{BC})$ consisting of bicomplex holomorphic functions on $\mathbb{BC}$,

$$\mathcal{F}^{2,\nu}(\mathbb{BC}) = \mathcal{H}^{2,\nu}(\mathbb{BC}) \cap \mathcal{BH}ol(\mathbb{BC}).$$

The following result shows that $\mathcal{F}^{2,\nu}(\mathbb{BC})$ has a Hilbertian decomposition with respect to the (α, β)-idempotent variables.

Theorem 2.2.1. *We have*

$$\mathcal{F}^{2,\nu}(\mathbb{BC}) = \mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})e_+ + \mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})e_-. \tag{2.3}$$

Succinctly, every $f \in \mathcal{F}^{2,\nu}(\mathbb{BC})$ can be re-expressed as

$$f(\alpha e_+ + \beta e_-) = \phi^+(\alpha)e_+ + \phi^-(\beta)e_-.$$

The $\phi^\pm$ are $\mathbb{C}$–valued functions belonging to the classical Bargmann space $\mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$. Moreover, we have

$$\|f\|^2_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \frac{1}{2}\left(\|\phi^+\|^2_{L^{2,\frac{\nu}{2}}(\mathbb{C})} + \|\phi^-\|^2_{L^{2,\frac{\nu}{2}}(\mathbb{C})}\right). \tag{2.4}$$

Proof. By (1.16), every $\mathbb{BC}$–valued function f on $\mathbb{BC}$ is of the form $f(Z) = f(\alpha e_+ + \beta e_-) = \phi^+(\alpha,\beta)e_+ + \phi^-(\alpha,\beta)e_-$, where the $\mathbb{C}$–valued functions $\phi^\pm(\alpha,\beta)$ on $\mathbb{C}^2$ belong to $L^{2,\frac{\nu}{2}}(\mathbb{C}^2)$ and satisfy (1.17). Therefore, $\phi^\mp \in \mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C}^2)$. On the other hand, Theorem 15.5 in [49, p.87] shows that the function ϕ^+ (resp. ϕ^-) is a $\mathbb{C}$–valued holomorphic function on $\mathbb{C}^2$ that depends only in α (resp. β). Next, by Fubini's theorem and the fact $c_1^{\frac{\nu}{2}} \int_{\mathbb{C}} e^{-\frac{\nu}{2}|\xi|^2} d\lambda(\xi) = 1$, we obtain

$$\begin{aligned}
\|\phi^+\|^2_{L^{2,\frac{\nu}{2}}(\mathbb{C}^2)} &= c_2^{\frac{\nu}{2}} \int_{\mathbb{C}^2} |\phi^+(\alpha)|^2 e^{-\frac{\nu}{2}(|\alpha|^2+|\beta|^2)} d\lambda(\alpha) d\lambda(\beta) \\
&= c_2^{\frac{\nu}{2}} \left(\int_{\mathbb{C}} e^{-\frac{\nu}{2}|\beta|^2} d\lambda(\beta)\right)\left(\int_{\mathbb{C}} |\phi^+(\alpha)|^2 e^{-\frac{\nu}{2}|\alpha|^2} d\lambda(\alpha)\right) \\
&= \|\phi^+\|^2_{L^{2,\frac{\nu}{2}}(\mathbb{C})}.
\end{aligned}$$

Thus, the condition $\|\phi^+\|_{L^{2,\frac{\nu}{2}}(\mathbb{C}^2)} < +\infty$ implies that $\phi^+ \in \mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$. Similarly, we have $\|\phi^-\|^2_{L^{2,\frac{\nu}{2}}(\mathbb{C}^2)} = \|\phi^-\|^2_{L^{2,\frac{\nu}{2}}(\mathbb{C})}$ and therefore $\phi^- \in \mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$. Accordingly, (1.17) reduces further to (2.4). ∎

Remark 2.2.2. *This is a special way to create two models of Bargmann spaces to work with simultaneously. The first one is focused on e_+ and the other on e_-.*

Remark 2.2.3. *For bicomplex holomrphic functions* $f, g : \mathbb{BC} \longrightarrow \mathbb{BC}$ *with* $f = \phi^+ e_+ + \phi^- e_-$ *and* $g = \varphi^+ e_+ + \varphi^- e_-$*, the quantity in* (1.18) *reduces further to*

$$\langle f, g\rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \langle \phi^+, \varphi^+ \rangle_{L^{2,\frac{\nu}{2}}(\mathbb{C})} e_+ + \langle \phi^-, \varphi^- \rangle_{L^{2,\frac{\nu}{2}}(\mathbb{C})} e_-. \tag{2.5}$$

Corollary 2.2.4. *The space* $\mathcal{F}^{2,\nu}(\mathbb{BC})$ *is an infinite* $\mathbb{BC}$*-Hilbert space.*

Proof. This is an immediate consequence of Theorem 2.2.1 and Theorem 1.2.6. ▬

Corollary 2.2.5. *A function* f *belongs to* $\mathcal{F}^{2,\nu}(\mathbb{BC})$ *if and only if* f *can be expanded as* $f(Z) = \sum\limits_{n=0}^{\infty} A_n Z^n$ *for some bicomplex sequence* $(A_n)_n$ *satisfying the growth condition*

$$\sum_{n=0}^{\infty} \frac{2^n n!}{\nu^n} |A_n|^2 < +\infty. \tag{2.6}$$

Proof. In view of Theorem 2.2.1, the functions ϕ^+ and ϕ^- belong to $\mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$, and therefore they can be expanded in power series as $\phi^+(\alpha) = \sum\limits_{n=0}^{\infty} a_n^+ \alpha^n$ and $\phi^-(\beta) = \sum\limits_{n=0}^{\infty} a_n^- \beta^n$, where the complex sequences $(a_n^\pm)_n$ are subject to the growth conditions

$$\|\phi^\pm\|^2_{L^{2,\frac{\nu}{2}}(\mathbb{C})} = \sum_{n=0}^{\infty} \left(\frac{2}{\nu}\right)^n n!\, |a_n^\pm|^2 < \infty.$$

Subsequently, we get

$$f(Z) = \sum_{n=0}^{\infty} (a_n^+ e_+ + a_n^- e_-)(\alpha^n e_+ + \beta^n e_-) = \sum_{n=0}^{\infty} A_n Z^n,$$

where $A_n := a_n^+ e_+ + a_n^- e_- \in \mathbb{BC}$. Next, from (2.4), one obtains the growth condition (2.6). Indeed, we have

$$\|f\|^2_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{2}{\nu}\right)^n n! \left(|a_n|^2 + |b_n|^2\right) = \sum_{n=0}^{\infty} \left(\frac{2}{\nu}\right)^n n!\, |A_n|^2.$$

This completes the proof. ▬

2.3 Two integral representations of L^2-bicomplex holomorphic functions

The first result in this section is a consequence of Theorem 2.2.1.

Theorem 2.3.1. *The space* $\mathcal{F}^{2,\nu}(\mathbb{BC})$ *is a reproducing kernel infinite* $\mathbb{BC}$*-Hilbert space whose reproducing kernel is given by*

$$K^{\nu}_{\mathbb{BC}}(Z, W) = e^{\frac{\nu}{2} Z W^*}. \tag{2.7}$$

Succinctly, for every $f \in \mathcal{F}^{2,\nu}(\mathbb{BC})$*, we have*

$$f(Z) = c^{\nu}_{\mathcal{H}^{2,\nu}(\mathbb{BC})} \int_{\mathbb{BC}} e^{\frac{\nu}{2}ZW^*} f(W) e^{-\nu|W|^2} d\lambda(W). \tag{2.8}$$

Proof. The fact that $\mathcal{F}^{2,\nu}(\mathbb{BC})$ is a reproducing kernel infinite $\mathbb{BC}$-Hilbert is an immediate consequence of Theorem 2.2.1 and the $\mathbb{BC}$-Riesz′ representation theorem (1.13), since for every $f \in \mathcal{F}^{2,\nu}(\mathbb{BC})$, we have

$$|f(Z)| \leq \sqrt{2}|e^{\frac{\nu}{4}ZZ^*}| \, \|f\|_{\mathcal{F}^{2,\nu}(\mathbb{BC})} .$$

This can be handled making use of the generalized $\mathbb{BC}$-Schwarz inequality (1.11) and the expression (2.6) giving the norm $\|f\|_{\mathcal{F}^{2,\nu}(\mathbb{BC})}$. The explicit expression (2.7) of the reproducing kernel $K^{\nu}_{\mathbb{BC}}(Z,W)$ is obtained by only proving the reproducing property (2.8) thanks the uniqueness of the reproducing kernel. To this end, denote the right-hand side of (2.8) by $\mathcal{P}^{\nu} f(Z)$,

$$\mathcal{P}^{\nu} f(Z) = \langle f, K^{\nu}_{\mathbb{BC}}(\cdot, Z)\rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} ,$$

and notice that the restriction of $K^{\nu}_{\mathbb{BC}}$ in (2.7) to $\mathbb{C} \times \mathbb{C} = (\mathbb{C} + j\{0\}) \times (\mathbb{C} + j\{0\})$ reduces further to the reproducing function $K^{\frac{\nu}{2}}_{\mathbb{C}}$ of the Bargmann space $\mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$, so that

$$\phi(z) = \left\langle \phi, K^{\frac{\nu}{2}}_{\mathbb{C}}(\cdot, z)\right\rangle_{L^{2,\frac{\nu}{2}}(\mathbb{C})} = c_1^{\frac{\nu}{2}} \int_{\mathbb{C}} e^{\frac{\nu}{2}z\overline{\xi}} \phi(\xi) e^{-\frac{\nu}{2}|\xi|^2} d\lambda(\xi) \tag{2.9}$$

for every $\phi \in \mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$. Moreover, in virtue of Remark 2.2.3 as well as the facts $K^{\nu}_{\mathbb{BC}}(Z,W) = (K^{\nu}_{\mathbb{BC}}(W,Z))^*$ and

$$K^{\nu}_{\mathbb{BC}}(Z,W) = K^{\frac{\nu}{2}}_{\mathbb{C}}(\alpha, \xi)e_+ + K^{\frac{\nu}{2}}_{\mathbb{C}}(\beta, \zeta)e_-,$$

with $Z = \alpha e_+ + \beta e_-$ and $W = \xi e_+ + \zeta e_-$, we obtain

$$\begin{aligned} \mathcal{P}^{\nu} f(Z) &= \langle f, K^{\nu}_{\mathbb{BC}}(\cdot, Z)\rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} \\ &= \left\langle \phi^+, K^{\frac{\nu}{2}}_{\mathbb{C}}(\cdot, \xi)\right\rangle_{L^{2,\frac{\nu}{2}}(\mathbb{C})} e_+ + \left\langle \phi^-, K^{\frac{\nu}{2}}_{\mathbb{C}}(\cdot, \zeta)\right\rangle_{L^{2,\frac{\nu}{2}}(\mathbb{C})} e_-. \end{aligned}$$

Therefore, the desired result (2.8) follows making use of the reproducing property (2.9) for $\phi^{\pm}$ belonging to the classical Bargmann space $\mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$. ■

Remark 2.3.2. *The explicit expression of the corresponding reproducing kernel* $K^{\nu}_{\mathbb{BC}}$ *can also be obtained using the expansion formula* $K^{\nu}_{\mathbb{BC}}(Z,W) = \sum\limits_{n=0}^{\infty} \phi_n(Z)\phi_n^*(W)$*, where* ϕ_n *represents any orthonormal basis of* $\mathcal{H}^{2,\nu}(\mathbb{BC})$ *like the one provided in Lemma 2.3.3 below.*

Lemma 2.3.3. *The set of functions*

$$\phi_n(Z) := \left(\frac{\nu^n}{2^n n!}\right)^{\frac{1}{2}} Z^n \tag{2.10}$$

is a Schauder orthonormal basis of the infinite $\mathbb{BC}$*-Hilbert space* $\mathcal{F}^{2,\nu}(\mathbb{BC})$*.*

Proof. By means of Theorem 2.2.1 and Remark 2.2.3 combined with the fact $e_+ + e_- = 1$, it is clear that the monomials $Z^n = \alpha^n e_+ + \beta^n e_-$ form an orthogonal basis of $\mathcal{F}^{2,\nu}(\mathbb{BC})$, for the monomials $e_n(\alpha) = \alpha^n$ form an orthogonal basis of $\mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$. More precisely, we have

$$\langle E_n, E_m \rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \langle e_n, e_m \rangle_{L^{2,\frac{\nu}{2}}(\mathbb{C})} = \frac{2^n n!}{\nu^n} \delta_{n,m}.$$

For the density of the monomials Z^n in $\mathcal{F}^{2,\nu}(\mathbb{BC})$, let $f \in \mathcal{F}^{2,\nu}(\mathbb{BC})$ such that $\langle f, E_n \rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = 0$ for every nonnegative integer n. By expanding f in power series as $f(Z) = \sum\limits_{n=0}^{\infty} A_n Z^n$, we get

$$\langle f, E_n \rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \left\langle \sum_{m=0}^{\infty} A_m E_m, E_n \right\rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \frac{2^n n!}{\nu^n} A_n.$$

This implies that $A_n = 0$ for all n and consequently f is identically zero on $\mathbb{BC}$. ∎

We provide below another interesting integral representation of the elements of the bicomplex Bargmann space $\mathcal{F}^{2,\nu}(\mathbb{BC})$ by means of their restriction to $\mathbb{C} + j\{0\}$. Let $\mathcal{S}^\nu$ be the integral transform

$$\mathcal{S}^\nu(F)(Z) := c_1^{\frac{\nu}{2}} \int_{\mathbb{C}} F(\xi) e^{-\frac{\nu}{2} Z\overline{\xi} - \frac{\nu}{2}|\xi|^2} d\lambda(\xi). \tag{2.11}$$

It is well–defined on the classical Bargmann space $\mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$ of weight $\frac{\nu}{2}$, and connects it to the special subspace $\mathcal{F}_i^{2,\nu}(\mathbb{BC})$ consisting of $f \in \mathcal{F}^{2,\nu}(\mathbb{BC})$ leaving $\mathbb{C}_i := \mathbb{C} + j\{0\}$ invariant, i.e., $f(\mathbb{C} + j\{0\}) \subset \mathbb{C} + j\{0\}$. The space $\mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$, of L^2 holomorphic functions on the complex plane $\mathbb{C}$ with values in $\mathbb{BC}$, is then a particular subspace of $\mathcal{F}_{\mathbb{BC}}^{2,\frac{\nu}{2}}(\mathbb{C})$, $\mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C}) \subset \mathcal{F}_{\mathbb{BC}}^{2,\frac{\nu}{2}}(\mathbb{C})$. Thus, we prove the following

Theorem 2.3.4. *The transform $\mathcal{S}^\nu$ maps $\mathcal{F}_{\mathbb{BC}}^{2,\frac{\nu}{2}}(\mathbb{C})$ onto $\mathcal{F}^{2,\nu}(\mathbb{BC})$ and $\mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$ onto $\mathcal{F}_i^{2,\nu}(\mathbb{BC})$. Moreover, we have $\mathcal{S}^\nu(e_n)(Z) = Z^n$.*

Proof. Starting from the idempotent decomposition $f(Z) = \phi^+(\alpha)e_+ + \phi^-(\beta)e_-$ in $\mathcal{F}^{2,\nu}(\mathbb{BC})$ and using (2.9) for $\phi^\pm \in \mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$, we see that f admits the representation

$$\begin{aligned} f(Z) &= c_1^{\frac{\nu}{2}} \left(e_+ \int_{\mathbb{C}} e^{\frac{\nu}{2}\alpha\overline{\xi}} \phi^+(\xi) e^{-\frac{\nu}{2}|\xi|^2} d\lambda(\xi) + e_- \int_{\mathbb{C}} e^{\frac{\nu}{2}\beta\overline{\xi}} \phi^-(\xi) e^{-\frac{\nu}{2}|\xi|^2} d\lambda(\xi) \right) \\ &= c_1^{\frac{\nu}{2}} \int_{\mathbb{C}} \left(e^{\frac{\nu}{2}\alpha\overline{\xi}} e_+ + e^{\frac{\nu}{2}\beta\overline{\xi}} e_- \right) \left(\phi^+(\xi)e_+ + \phi^-(\xi)e_- \right) e^{-\frac{\nu}{2}|\xi|^2} d\lambda(\xi) \\ &= c_1^{\frac{\nu}{2}} \int_{\mathbb{C}+j\{0\}} e^{\frac{\nu}{2}Z\overline{\xi}} f|_{\mathbb{C}+j\{0\}} e^{-\frac{\nu}{2}|\xi|^2} d\lambda(\xi) \\ &= \mathcal{S}^\nu(f|_{\mathbb{C}+j\{0\}})(Z). \end{aligned}$$

The function $f|_{\mathbb{C}+j\{0\}}$ is clearly holomorphic on $\mathbb{C}$ but with coefficients in $\mathbb{BC}$. This proves that $\mathcal{S}^\nu : \mathcal{F}_{\mathbb{BC}}^{2,\frac{\nu}{2}}(\mathbb{C}) \longrightarrow \mathcal{F}^{2,\nu}(\mathbb{BC})$ and its restriction $\mathcal{S}^\nu|_{\mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})} : \mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C}) \longrightarrow \mathcal{F}_i^{2,\nu}(\mathbb{BC})$ are onto. A direct computation shows that the action of $\mathcal{S}^{\nu}$ on the monomials $e_n(\xi) = \xi^n$ is given by

$$\mathcal{S}^{\nu}(e_n)(Z) = c_1^{\frac{\nu}{2}} \int_{\mathbb{C}} \xi^n \left(e^{\frac{\nu}{2}\alpha\overline{\xi}} e_+ + e^{\frac{\nu}{2}\beta\overline{\xi}} e_- \right) e^{-\frac{\nu}{2}|\xi|^2} d\lambda(\xi) = Z^n,$$

since $\int_{\mathbb{C}} \xi^n e^{\gamma\alpha\overline{\xi}} e^{-\gamma|\xi|^2} d\lambda(\xi) = \left(\frac{\pi}{\gamma}\right) \alpha^n$. ▬

Remark 2.3.5. *We have*

$$\|\mathcal{S}^{\nu}(e_n)\|^2_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \|e_n\|^2_{L^{2,\frac{\nu}{2}}(\mathbb{C})}.$$

Remark 2.3.6. *The space $\mathcal{F}_i^{2,\nu}(\mathbb{BC})$ can be identified to the special phase subspace of $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ defined by*

$$\mathcal{A}^{2,\nu}(\mathbb{C}^2) := \left\{ f \in \mathcal{F}^{2,\nu}(\mathbb{C}^2);\ \left(\frac{\partial}{\partial z} + i\frac{\partial}{\partial w} \right) f = 0 \right\} \tag{2.12}$$

and characterized as the image of $L^{2,\sigma}(\mathbb{R})$ by the special one-to-one transform $\mathcal{B}^{2,\nu} \circ \mathcal{B}^{1,\nu}$ obtained as the composition operator of the 1*d and* 2*d Segal–Bargmann transforms [9, Theorem 2.2]. It can also be identified to the space of slice (left) regular functions on the quaternions leaving invariant the slice $\mathbb{C}_i \simeq \mathbb{C}$ in the quaternion $\mathbb{H}$ [9, Theorem 3.4].*

2.4 The bicomplex Segal–Bargmann transform

It is a known fact that the classical Bargmann space $\mathcal{F}^{2,\gamma}(\mathbb{C})$ is unitary isomorphic to the quantum mechanical configuration space $L^{2,\sigma}(\mathbb{R})$, of all $\mathbb{C}$–valued $e^{-\sigma x^2}dx$-square integrable functions on the real line, by considering the rescaled Segal–Bargmann transform ([31,36, 64,67])

$$\mathcal{B}_{\mathbb{C}}^{\sigma,\gamma}(\psi)(z) = c_0^{\sigma} \int_{\mathbb{R}} e^{-\sigma\left(x - \sqrt{\frac{\gamma}{2\sigma}}z\right)^2} \psi(x)dx. \tag{2.13}$$

Mathematical theory of Segal–Bargmann transform has interesting applications in many fields of mathematics and physics and is an essential tool in signal processing.

We propose in the present section a bicomplex analog of $\mathcal{B}_{\mathbb{C}}^{\sigma,\gamma}$ in (2.13) and study some of its basic properties. In fact, by Theorem 2.2.1, we can split any $f \in \mathcal{F}^{2,\nu}(\mathbb{BC})$ as

$$f(\alpha e_+ + \beta e_-) = \phi^+(\alpha)e_+ + \phi^-(\beta)e_-$$

for some $\phi^{\pm} \in \mathcal{F}^{2,\frac{\nu}{2}}(\mathbb{C})$. Then, by means of (2.13), there exist some $\varphi^+, \varphi^- \in L^{2,\sigma}(\mathbb{R})$ such that

$$\begin{aligned} f(Z) &= \mathcal{B}_{\mathbb{C}}^{\sigma,\frac{\nu}{2}}(\varphi^+)(\alpha)e_+ + \mathcal{B}_{\mathbb{C}}^{\sigma,\frac{\nu}{2}}(\varphi^-)(\beta)e_- && (2.14) \\ &= \int_{\mathbb{R}} \left(B_{\mathbb{C}}^{\sigma,\frac{\nu}{2}}(x;\alpha)e_+ + B_{\mathbb{C}}^{\sigma,\frac{\nu}{2}}(x;\beta)e_- \right) \left(\varphi^+(x)e_+ + \varphi^-(x)e_- \right) dx, \\ &= \int_{\mathbb{R}} B_{\mathbb{BC}}^{\sigma,\nu}(x;Z)\varphi(x)dx, && (2.15) \end{aligned}$$

where we have set $\varphi(x) := \varphi^+(x)e_+ + \varphi^-(x)e_-$. The function $\varphi : \mathbb{R} \longrightarrow \mathbb{BC}$ belongs to the bicomplex Hilbert space $L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$ consisting of all $e^{-\sigma x^2}dx$-square integrable $\mathbb{BC}$–valued functions whose component functions, with respect to the idempotent decomposition, belong to $L^{2,\sigma}(\mathbb{R})$. The $\mathbb{BC}$-norm in $L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$ is the one associated to the bicomplex inner product

$$\langle \varphi, \psi \rangle_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})} := c_0^\sigma \int_{\mathbb{R}} \varphi(x)\psi(x)^* e^{-\sigma x^2} dx. \tag{2.16}$$

The kernel function $B^{\sigma,\nu}_{\mathbb{BC}}(x;Z) := B^{\sigma,\frac{\nu}{2}}_{\mathbb{C}}(x;\alpha)e_+ + B^{\sigma,\frac{\nu}{2}}_{\mathbb{C}}(x;\beta)e_-$ is explicitly given by

$$B^{\sigma,\nu}{}_{\mathbb{BC}}(x;Z) = c_0^\sigma e^{-\sigma\left(x-\sqrt{\frac{\nu}{4\sigma}}Z\right)^2} \tag{2.17}$$

and can be seen as the natural extension of $B^{\sigma,\frac{\nu}{2}}(x;\xi)$ to the bicomplex (holomorphic) setting in the second variable, $Z = \alpha e_+ + \beta e_-$. The corresponding integral transform acts on $L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$ as defined by the left-hand side of (2.15), to wit

$$\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}(\varphi)(Z) := c_0^\sigma \int_{\mathbb{R}} e^{-\sigma\left(x-\sqrt{\frac{\nu}{4\sigma}}Z\right)^2} \varphi(x)dx, \tag{2.18}$$

provided that the integral exists.

Definition 2.4.1. *We call $\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}$ the bicomplex Segal–Bargmann transform.*

The action of $\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}$ on the rescaled real Hermite polynomials

$$H_n^\sigma(x) := (-1)^n e^{\sigma x^2} \frac{d^n}{dx^n}\left(e^{-\sigma x^2}\right) \tag{2.19}$$

is given by

$$\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}(H_n^\sigma)(\xi) = \frac{\|H_n^\sigma\|_{L^{2,\sigma}(\mathbb{R})}}{\|e_n\|_{L^{2,\frac{\nu}{2}}(\mathbb{C})}} Z^n = \frac{\|H_n^\sigma\|_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})}}{\|E_n^\nu\|_{L^{2,\nu}(\mathbb{BC})}} Z^n. \tag{2.20}$$

It follows by means of (2.14) combined with the facts that

$$\mathcal{B}^{\sigma,\frac{\nu}{2}}_{\mathbb{C}}(H_n^\sigma)(\xi) = \frac{\|H_n^\sigma\|_{L^{2,\sigma}(\mathbb{R})}}{\|e_n\|_{L^{2,\frac{\nu}{2}}(\mathbb{C})}} \xi^n \quad \text{and} \quad \|E_n\|_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \|e_n\|_{L^{2,\frac{\nu}{2}}(\mathbb{C})}.$$

Notice that we also have made use of $\|H_n^\sigma\|_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})} = \|H_n^\sigma\|_{L^{2,\sigma}(\mathbb{R})}$ for H_n^σ being real–valued. Therefore, the considered transform $\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}$ maps the orthonormal basis

$$\psi_n^\sigma(x) = \left(\frac{1}{2^n\sigma^n n!}\right)^{1/2} H_n^\sigma(x), \tag{2.21}$$

of the configuration space $L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$, to the monomials $\phi_n(Z)$ in (2.10), which form an orthonormal basis of $\mathcal{F}^{2,\nu}(\mathbb{BC})$. Using similar arguments, we can prove that

$$\|\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}\varphi\|^2_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \|\varphi\|^2_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})} \tag{2.22}$$

for every $\varphi \in L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$ such that $f = \mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}\varphi$. Indeed, the result readily follows by means of (2.4) and using $\|\mathcal{B}^{\sigma,\gamma}_{\mathbb{C}}(\varphi)\|^2_{L^{2,\gamma}(\mathbb{C})} = \|\varphi\|^2_{L^{2,\gamma}(\mathbb{R})}$.

The above discussion can be reformulated as follows

Theorem 2.4.2. *The bicomplex Segal–Bargmann transform $\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}$ in* (2.18) *realizes a unitary integral transform mapping isometrically the bicomplex Hilbert space $L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$ onto the bicomplex Bargmann space $\mathcal{F}^{2,\nu}(\mathbb{BC})$.*

Remark 2.4.3. *A variant of the bicomplex Segal–Bargmann transform can be defined on $L^{2,\sigma}_{\mathbb{R}}(\mathbb{D})$, the bicomplex Hilbert space of $\mathbb{BC}$-valued functions on the hyperbolic (bireal) numbers*

$$\mathbb{D} = \{U = xe_+ + ye_-; x, y \in \mathbb{R}\}$$

endowed with the bicomplex inner product

$$\langle \varphi, \psi \rangle_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{D})} := c^{\sigma}_1 \int_{\mathbb{D}} \varphi(U)\psi(U)^* e^{-\sigma\|U\|^2} d\lambda(U), \tag{2.23}$$

where $d\lambda(U) = dxdy$ is the Lebesgue measure on $\mathbb{D}$. In fact, we can consider

$$\widetilde{\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}}(\varphi)(Z) := c_1^{\frac{\nu}{2}} \int_{\mathbb{D}} e^{-\sigma\left(U - \frac{\nu}{4\sigma}Z\right)^2} \varphi(U) e^{-\sigma U^{\dagger 2}} d\lambda(U). \tag{2.24}$$

For every fixed Z, the restriction of the involved kernel function in the right–hand side of (2.18) *to the diagonal of the bireal numbers reduces further to the kernel function in the bicomplex Segal–Bargmann transform $\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}$.*

Remark 2.4.4. *It should also be noted that the kernel function $B^{\sigma,\nu}_{\mathbb{BC}}(x; Z)$ in* (2.17) *is closely connected to the generating function $G^{\sigma,\nu}(x; Z)$ of the Hermite polynomials $H^{\sigma}_n(x)$,*

$$B^{\sigma,\nu}_{\mathbb{BC}}(x; Z) = c^{\sigma}_0 e^{-\sigma x^2} G^{\sigma,\nu}(x; Z^*).$$

Indeed, for every given $(x; Z) \in \mathbb{R} \times \mathbb{BC}$, we have

$$G^{\sigma,\nu}(x; Z) := \sum_{n=0}^{\infty} \frac{H^{\sigma}_n(x) E_n(Z^*)}{\|H^{\sigma}_n\|_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})} \|E_n\|_{\mathcal{H}^{2,\nu}(\mathbb{BC})}} = e^{-\frac{\nu}{4}(Z^*)^2 + \sqrt{\sigma\nu}xZ^*}. \tag{2.25}$$

Remark 2.4.5. *By means of* (2.25)*, one can show that for every $Z, W \in \mathbb{BC}$, we have*

$$\begin{aligned}\langle G^{\sigma,\nu}(\cdot; Z^*), G^{\sigma,\nu}(\cdot; W) \rangle_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})} &= c^{\sigma}_0 \int_{\mathbb{R}} G^{\sigma,\nu}(x; Z^*) G^{\sigma,\nu}(x; W^*) e^{-\sigma x^2} dx \\ &= K^{\nu}_{\mathbb{BC}}(Z, W).\end{aligned}$$

Therefore, the function $G_Z^{\sigma,\nu} : x \mapsto G_Z^{\sigma,\nu}(x) := G^{\sigma,\nu}(x;Z)$*, for every fixed* $Z \in \mathbb{BC}$*, belongs to* $L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$*.*

The previous discussion shows that the transform $\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}$ admits an inverse mapping $\mathcal{F}^{2,\nu}(\mathbb{BC})$ onto $L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$. Basically, using the fact that

$$\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}(\psi_n^{\sigma})(Z) = \phi_n(Z), \tag{2.26}$$

where ψ_n^{σ} (resp. $\phi_n(Z)$) is the orthonormal basis of $L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$ (resp. $\mathcal{F}^{2,\nu}(\mathbb{BC})$) given by (2.21) (resp. (2.10)), we see that the inverse transform $[\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1}$ admits the expansion

$$\begin{aligned}
[\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1}(f)(x) &:= \sum_{n=0}^{\infty} c_n \left\| E_n \right\|_{\mathcal{H}^{2,\nu}(\mathbb{BC})} [\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1}(\phi_n)(x) \\
&= \sum_{n=0}^{\infty} \frac{\left\| E_n \right\|_{\mathcal{H}^{2,\nu}(\mathbb{BC})}}{\left\| H_n^{\sigma} \right\|_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})}} c_n H_n^{\sigma}(x)
\end{aligned}$$

for $f := \sum\limits_{n=0}^{\infty} c_n E_n \in \mathcal{F}^{2,\nu}(\mathbb{BC})$. Now, since $\langle E_n, E_n \rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} = \left\| E_n \right\|^2_{\mathcal{H}^{2,\nu}(\mathbb{BC})} \notin \mathcal{NC}$, we get

$$c_n = \langle f, E_n \rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} \left(\langle E_n, E_n \rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} \right)^{-1} = \frac{\langle f, E_n \rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})}}{\left\| E_n \right\|^2_{\mathcal{H}^{2,\nu}(\mathbb{BC})}},$$

so that

$$[\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1}(f)(x) = \sum_{n=0}^{\infty} \frac{\langle f, E_n \rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})}}{\left\| H_n^{\sigma} \right\|_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})} \left\| E_n \right\|_{\mathcal{H}^{2,\nu}(\mathbb{BC})}} H_n^{\sigma}(x). \tag{2.27}$$

This expansion leads to the integral representation

$$[\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1}(f)(x) = \langle f, (G^{\sigma,\nu}(x;\cdot))^* \rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} \tag{2.28}$$

that maps isometrically $\mathcal{F}^{2,\nu}(\mathbb{BC})$ onto $L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$. Thus, one can assert the following result.

Theorem 2.4.6. *The integral representation of the unitary transform* $[\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1} : \mathcal{F}^{2,\nu}(\mathbb{BC}) \longrightarrow L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$ *is given by*

$$[\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1}(f)(x) = c_T^{\nu} \int_{\mathbb{BC}} e^{-\nu|Z|^2 - \frac{\nu}{4}(Z^*)^2 + \sqrt{\sigma\nu} x Z^*} f(Z) d\lambda(Z). \tag{2.29}$$

In the next section, we will make use of the bicomplex Bargmann transform and its inverse to define a class of Fourier transforms generalizing the standard *i*– and *j*–Fourier transforms.

Chapter 3

BICOMPLEX FRACTIONAL FOURIER TRANSFORMS

The present chapter considers and discusses some basic properties of the bicomplex analog of the classical Bargmann space. The explicit expression of the integral operator connecting the complex and bicomplex Bargmann spaces is also given. The corresponding bicomplex Segal–Bargmann transform is introduced and studied as well. We make use of the explicit expression of the bicomplex Segal–Bargmann transform and its inverse to introduce a class of two–parameter bicomplex Fourier transforms (bicomplex fractional Fourier transform). This approach is convenient in exploring some useful properties of this bicomplex fractional Fourier transform, such as the inversion and Plancherel formulas.

3.1 Classical fractional Fourier transform

Fourier analysis is one of the oldest and special subjects in mathematical analysis that has great impact on different fields of mathematics, physics and engineery alike. The main tool in this theory is the Fourier transform on the real line

$$\mathcal{F}_f(w) = \hat{f}(w) := \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-i\omega t} dt; \quad w \in \mathbb{R}.$$

and enters in solving some differential equations. Moreover, it is essential for signal analysis and image processing. The basic properties of $\mathcal{F}$ are encoded in the kernel function $e^{i\omega t}$. Notice for instance that the Hermite functions are eigenfunctions of $\mathcal{F}$ and we have

$$\mathcal{F}h_m = i^m h_m.$$

The so-called fractional Fourier transform, is a special generalization of $\mathcal{F}$. Such transform was first appeared in 1920 by Wiener in his paper [**?**] discussing the extension of certain results of Hermann Weyl leading to Fourier developments of fractional order. Mainly, Wiener sets out to find a one–parameter family of unitary integral operators on $L^2(\mathbb{R})$,

$$\mathcal{K}_\theta \varphi(x) := \int_{-\infty}^{+\infty} K_\theta(x,y)\varphi(y)dy,$$

for which the n–th Hermite function $h_n(x) = H_n(x)e^{-x^2/2}$ is a eigenfunction with $e^{in\theta}$ as corresponding eigenvalue,

$$\mathcal{K}_\theta h_n(x) = e^{in\theta}h_n(x).$$

The explicit Wiener formula for the kernel function K_0 is a limiting case of the Mehler's formula [20] for the Hermite functions as showed earlier by Hörmander [**?**]. This transform was rediscovered later in quantum mechanics by Namias [**?**] (who was the first to attribute such concept),

$$\mathcal{F}_\alpha(f)(\omega) = \int_{-\infty}^{\infty} K_\alpha(t,w)f(t)\,dt,$$

with

$$K_\alpha(t,w) = \begin{cases} \sqrt{\frac{1-i\cot(\alpha)}{2\pi}}e^{-i\csc(\alpha)\omega t + i\cot(\alpha)(\frac{w^2}{2}+\frac{t^2}{2})} \; if \;\; \alpha \neq p\pi, \\ \delta(t-w) \; if \;\; \alpha = 2p\pi, \\ \delta(t+w) \; if \;\; \alpha = (2p-1)\pi, \end{cases}$$

where $\alpha \in \mathbb{R}$; p is an integer, and δ is the Dirac delta function.

Namias was able to generalize many results of classical Fourier transform to FrFT, based on the properties of the Hermite orthogonal polynomials. He derived a number of operational formulas which he used to solve several types of SchrÃ¶dinger equation. The fundamental mathematical foundation concerning the FrFT was developed later by McBride and Kerr in [**?**]. Applications of the FrFT are well–known in the context of signal processing [**? ? ?**] (signal restoration and noise removal), optics [**? ? ?**], and fractional differential equations [**?**]. For a new and brief introduction to the FrFT and its applications see [**?**]. A detailed overview of the theory of the fractional FT can be found in [**?**].

The kernel can be expanded in terms of Hermite functions as

$$K_\alpha(t,w) = \sum_{n\geq 0}(-i)^{\frac{2\alpha n}{\pi}}\frac{h_n(t)h_n(w)}{||h_n||^2}.$$

Therefore, one can the Hermite functions are eigenfunctions of the fractional Fourier transformation whose corresponding eigenvalues are $e^{in\alpha}$; that is

$$\mathcal{F}_\alpha(h_n)(w) = e^{in\alpha}h_n(w).$$

It is also no difficult to see that $\mathcal{F}_\alpha$ satisfies the semi-Group property

$$\mathcal{F}_\alpha\mathcal{F}_\theta = \mathcal{F}_{\alpha+\theta}.$$

So that the inverse-FrFT with respect to angle α is the FrFT with angle $-\alpha$,

$$(\mathcal{F}_\alpha)^{-1} = \mathcal{F}_{-\alpha}.$$

While the Parseval's relation reads

$$\langle \mathcal{F}_\alpha(f), \mathcal{F}_\alpha(g)\rangle = \langle f, g\rangle$$

and hence the $\mathcal{F}_\alpha$ defines an isometric transformation on $L^2(\mathbb{R})$.

3.2 Bicomplex fraction Fourier transforms

The standard bicomplex Fourier transform is defined by

$$\mathcal{F}_{\mathbb{BC}}(\psi)(Z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixZ} \psi(x) dx, \quad Z = \alpha e_+ + \beta e_-.$$

Using the idempotent decomposition, the transform $\mathcal{F}_{\mathbb{BC}}$ can be seen as duplication over the two Fourier transforms with respect to complex frequencies α, β of given ψ on the real line (see for example [5,6,12]).

Now, for every fixed $\theta \in \{\theta \in S^1 e_+ + S^1 e_-;\ \theta \neq \pm 1, \pm ij\}$, we define the integral transform $\mathcal{F}^\sigma_\theta$ to be

$$\mathcal{F}^\sigma_\theta \psi(y) = \frac{c^\sigma_0}{\sqrt{1-\theta^2}} \int_{\mathbb{R}} \psi(x) e^{-\frac{\sigma}{1-\theta^2}(x-\theta y)^2} dx \tag{3.1}$$

for given bicomplex–valued function on the real line. It will be called bicomplex fractional Fourier transform (BFrFT). The following result shows in particular that the kernel function of $\mathcal{F}^\sigma_\theta$ is closely connected to the integral transform

$$[\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1} (\Gamma_\theta \mathcal{B}^{\sigma,\nu}_{\mathbb{BC}} \psi)(x),$$

where

$$\Gamma_\theta f(Z) := f(\theta Z).$$

We also establish its connection to the bicomplex version of Mehler formula for Hermite polynomials.

Theorem 3.2.1. *The transform $\mathcal{F}^\sigma_\theta$ is well–defined on the bicomplex Hilbert space $L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$. Moreover, for every $\psi \in L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$ and $x \in \mathbb{R}$, we have*

$$\mathcal{F}^\sigma_\theta \psi(x) = [\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1} (\Gamma_\theta \mathcal{B}^{\sigma,\nu}_{\mathbb{BC}} \psi)(x). \tag{3.2}$$

Proof. Starting from the integral representations of $\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}$ and $[\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1}$ given respectively by

$$\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}(\psi)(Z) = \langle \psi, G^{\sigma,\nu}(\cdot; Z)\rangle_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})}$$

and

$$[\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1}(f)(y) = \langle f, (G^{\sigma,\nu}(y;\cdot))^* \rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})},$$

we show that for every $\psi \in L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$ and $y \in \mathbb{R}$, we have

$$\begin{aligned}[\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1}(\Gamma_\theta \mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}\psi)(y) &= \left\langle \langle \psi(\cdot\cdot), G^{\sigma,\nu}(\cdot\cdot;\theta\cdot) \rangle_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})}, (G^{\sigma,\nu}(y;\cdot))^* \right\rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} \\ &= \left\langle \psi(\cdot\cdot), \langle G^{\sigma,\nu}(\cdot\cdot;\theta\cdot), G^{\sigma,\nu}(y;\cdot) \rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})} \right\rangle_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})}.\end{aligned}$$

The involved kernel function $F^\sigma_\theta(x,y) := \langle G^{\sigma,\nu}(x;\theta\cdot), G^{\sigma,\nu}(y;\cdot) \rangle_{\mathcal{H}^{2,\nu}(\mathbb{BC})}$ is independent of ν and can be computed explicitly using the explicit expression of $G^{\sigma,\nu}(x,Z)$ given through (2.25). Indeed, we obtain

$$\begin{aligned}F^\sigma_\theta(x,y) &= c^\nu_T \int_{\mathbb{BC}} e^{-\nu|Z|^2 - \frac{\nu}{4}Z^2 - \frac{\nu}{4}(\theta^* Z^*)^2 + \sqrt{\sigma\nu}yZ + \sqrt{\sigma\nu}x\theta^* Z^*} d\lambda(Z) \\ &= \frac{c^\nu_T}{4c_1^{\frac{\nu}{2}}} \left(I_{\alpha_\theta}(x,y)e_+ + I_{\beta_\theta}(x,y)e_- \right),\end{aligned}$$

where $\theta = \alpha_\theta e_+ + \beta_\theta e_-$ and $I_\xi(x,y)$ stands for

$$I_\xi(x,y) := \int_{\mathbb{C}} e^{-\frac{\nu}{2}|\zeta|^2 - \frac{\nu}{4}\zeta^2 - \frac{\nu}{4}\overline{\xi}^2\overline{\zeta}^2 + \sqrt{\sigma\nu}y\zeta + \sqrt{\sigma\nu}x\overline{\xi}\overline{\zeta}} d\lambda(\zeta)$$

for $x, y \in \mathbb{R}$ and $\xi \in \mathbb{C}$. Here, we recognize the integral formula

$$\int_{\mathbb{C}} e^{-\gamma|\zeta|^2 + a\zeta^2 + b\overline{\zeta}^2 + c\zeta + d\overline{\zeta}} d\lambda(\zeta) = \frac{\pi}{\sqrt{\gamma^2 - 4ab}} \exp\left(\frac{ad^2 + bc^2 + \gamma cd}{\gamma^2 - 4ab} \right)$$

valid for $|\Re(a+b)| < \gamma$, which implies in particular that $\gamma^2 - 4ab > 0$. This can be seen as a particular case of the Gaussian integral [67, p. 256]

$$\int_{\mathbb{R}^n} e^{-ayAy + by} dy = \left(\frac{\pi^n}{a^n \sqrt{\det A}} \right)^{1/2} e^{\frac{1}{4a} bA^{-1}b}, \tag{3.3}$$

where we have the limitation $a > 0$, $b \in \mathbb{C}^n$ and $A = (a_{mn})_{m,n} \in \mathbb{C}^{n\times n}$ is a symmetric n-matrix whose real part $\Re(A) = (\Re(a_{mn}))_{m,n}$ is positive definite. In our case, the limitation condition $|\Re(a+b)| < \gamma$ is equivalent to $|1 + \Re(\alpha_\theta| < \gamma$ and $|1 + \Re(\beta_\theta| < \gamma$ and therefore to $\theta \in \{\theta \in S^1 e_+ + S^1 e_-;\ \theta \neq \pm 1; \pm ij\}$. Under this assumption, we have

$$F^\sigma_\theta(x,y) = \frac{1}{c_1^{\frac{\nu}{2}} \sqrt{1 - (\theta^*)^2}} \exp\left(\frac{-\sigma(\theta^*)^2(x^2 + y^2) + 2\sigma\theta^* xy}{1 - (\theta^*)^2} \right).$$

This completes our check for (3.5) since $[\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1}(\Gamma_\theta \mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}\psi)(x)$ reduces further to

$$\frac{c^\sigma_0}{\sqrt{1-\theta^2}} e^{-\frac{\sigma\theta^2}{1-\theta^2}y^2} \int_{\mathbb{R}} \psi(y) e^{-\frac{\sigma}{1-\theta^2}x^2 + 2\frac{\sigma\theta}{1-\theta^2}xy} dx.$$

∎

Remark 3.2.2. *In our investigation, the condition* $|\theta| = 1$*, with* $\theta \neq \pm 1, \pm ij$*, is needed to deal with the bicomplex analog of the classical fractional Fourier transform. However, the kernel function* $T_\theta^\sigma(x,y)$ *in* (3.6)*, obtained by means of the bicomplex version of Mehler formula, can be used to define the integral transform*

$$\widetilde{\mathcal{F}_\theta^\sigma}(\varphi)(y) = \int_{\mathbb{R}} \varphi(x) T_\theta^\sigma(x,y) dx$$

for $y \in \mathbb{R}, \theta \in \mathbb{BC}$ *such that* $|\theta| < 1$ *and* $\varphi \in L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$*. In this case, the connection to the bicomplex Segal–Bargmann transform can be shown to be given by*

$$\widetilde{\mathcal{F}_\theta^\sigma} = [\mathcal{B}^{\sigma,\nu\gamma_\theta}_{\mathbb{BC}}]^{-1} \Gamma_\theta \mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}$$

for some positive real γ_θ *depending only on* θ*.*

Remark 3.2.3. *The authors of [12] define the Fourier transform for bicomplex holomorphic functions as the Cauchy-Kowalewski extension of the classical one on the real line (similarly as it has been done in the setting of Clifford analysis in* $\mathbb{R}^m$ *in [21]), namely for given bicomplex holomorphic function* $F(Z)$*, the bicomplex Fourier transform is given by*

$$\mathcal{F}_{\mathbb{BC}}(F)(Z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixZ} F|_{\mathbb{R}}(x) dx.$$

By proceeding in a similar way, we define the Cauchy-Kowalewski extension of the BFrFT for bicomplex holomorphic functions by considering

$$\mathcal{F}^{\sigma,\theta}_{\mathbb{BC}} F(Z) = \frac{c_0^\sigma}{\sqrt{1-\theta^2}} e^{-\frac{\sigma\theta^2}{1-\theta^2}Z^2} \int_{\mathbb{R}} F|_{\mathbb{R}}(x) e^{-\frac{\sigma}{1-\theta^2}x^2 + 2\frac{\sigma\theta}{1-\theta^2}xZ} dx.$$

The kernel function of the integral transform $\mathcal{F}^{\sigma,\theta}_{\mathbb{BC}}$ *is closely connected to the bicomplex version of the bilinear Mehler formula involving the product of the rescaled real Hermite polynomials* $H_n^\sigma(x)$ *in* (2.19) *and the bicomplex holomorphic Hermite polynomials* $H_n^\sigma(Z)$ *obtained from the first one by replacing* y *by the bicomplex* Z*, being indeed*

$$\sum_{n=0}^{\infty} \frac{\theta^n H_n^\sigma(Z) H_n^\sigma(y)}{2^n \sigma^n n!} = \frac{1}{\sqrt{1-\theta^2}} \exp\left(\frac{-\sigma\theta^2(Z^2+y^2) + 2\sigma\theta y Z}{1-\theta^2} \right). \tag{3.4}$$

Basic properties of this transform will be discussed in a forthcoming investigation.

3.3 Immediate consequences (Inversion and Plancherel formulas)

For the particular case $\theta = i$ (or also j, i.e. such that $\theta^2 = -1$), with $\sigma = 1$, the expression in (3.5) reduces further to the standard Fourier transform

$$\psi \longmapsto \left(y \longmapsto \frac{1}{\sqrt{2\pi}} e^{\frac{y^2}{2}} \int_{\mathbb{R}} e^{ixy} \psi(x) e^{\frac{-y^2}{2}} dx \right) \tag{3.5}$$

for every $\psi \in L^2_{\mathbb{BC}}(\mathbb{R}; e^{-x^2}dx)$ and $x \in \mathbb{R}$. Moreover, using the generating nature of $G^{\sigma,\nu}$ involving the functions ψ_n and ϕ_n given by (2.21) and (2.10), respectively, combined with the orthonormality of ψ_n in $\mathcal{H}^{2,\nu}(\mathbb{BC})$, we can rewrite the kernel $F^\sigma_\theta(x,y)$ as

$$F^\sigma_\theta(x,y) = \sum_{n=0}^{\infty} \frac{\theta^n H^\sigma_n(x) H^\sigma_n(y)}{2^n \sigma^n n!}.$$

The right-hand side is the bicomplex version of Mehler formula for the rescaled real Hermite polynomials

$$\sum_{m=0}^{\infty} \frac{\theta^n H^\sigma_n(x) H^\sigma_n(y)}{2^n \sigma^n n!} = \frac{1}{\sqrt{1-\theta^2}} \exp\left(\frac{-\sigma\theta^2(x^2+y^2) + 2\sigma\theta xy}{1-\theta^2} \right) \tag{3.6}$$

valid for every fixed $\theta \in \mathbb{BC}$ such that $|\theta| < 1$. It can be obtained easily form the classical one [4,20,24]. However, our approach shows that the expected formula is also valid for $\theta \in \{\theta \in S^1 e_+ + S^1 e_-; \theta \neq \pm 1; \pm ij\}$.

The uniqueness theorem as well as the Plancherel theorem and the inversion formula for the bicomplex fractional Fourier transform $\mathcal{F}^\sigma_\theta$ immediately follow in virtue of Theorem 3.2.1. Thus, we assert

Corollary 3.3.1. *If for given $\psi_1, \psi_2 \in L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$ we have $\mathcal{F}^\sigma_\theta \psi_1 = \mathcal{F}^\sigma_\theta \psi_2$, then $\psi_1 = \psi_2$.*

Corollary 3.3.2. *For every $\psi \in L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$, we have $\|\mathcal{F}^\sigma_\theta \psi\|_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})} = \|\psi\|_{L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})}$ and $\mathcal{F}^\sigma_\theta \psi_n = \theta^n \psi_n$.*

Corollary 3.3.3. *The inversion formula for $\mathcal{F}^\sigma_\theta$ is the FrFT with the parameter θ^*. More explicitly*

$$[\mathcal{F}^\sigma_\theta]^{-1}(\psi)(x) = \frac{c^\sigma_0}{\sqrt{1-(\theta^*)^2}} \int_{\mathbb{R}} \psi(y) e^{-\frac{\sigma}{1-(\theta^*)^2}(y-\theta^* x)^2} dy \tag{3.7}$$

for every $\psi \in L^{2,\sigma}_{\mathbb{BC}}(\mathbb{R})$.

Remark 3.3.4. *The inversion formula in* (3.7) *follows since*

$$[\mathcal{F}^\sigma_\theta]^{-1} = [\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}]^{-1} \Gamma_{\theta^*} \mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}$$

for every fixed bicomplex $|\theta| = 1$ with $\theta \neq \pm 1, \pm ij$. It can also be seen as an immediate consequence *of the semi-group property* $\mathcal{F}^{\sigma}_{\theta} \circ \mathcal{F}^{\sigma}_{\varrho} = \mathcal{F}^{\sigma}_{\theta\varrho}$ *which readily follows from Theorem 3.2.1 and the fact that* $\Gamma_{\theta_1} \circ \Gamma_{\theta_2} = \Gamma_{\theta_1\theta_2}$.

Chapter 4

BICOMPLEX FOURIER–WIGNER TRANSFORMS

We consider the 1-d and 2-d bicomplex analogues of the classical Fourier? Wigner transform. Their basic properties, including Moyal?s identity and characterization of their ranges giving rise to new bicomplex? polyanalytic functional spaces are discussed. Details concerning a special window function are developed explicitly. An orthogonal basis for the space of bicomplex?valued square integrable functions on the bicomplex numbers is constructed by means of a specific class of bicomplex Hermite functions.

Throughout the rest of the paper, the notation $\mathbb{C}_\tau$ (with $\tau^2 = -1$) will be used to mean the complex plane $\mathbb{C}_\tau := \{z_\nu := x + \tau y;\ x, y \in \mathbb{R}\}$. For $\tau = i$, the $\mathbb{C}_i$ is denoted simply $\mathbb{C}$.

4.1 Rescaled Fourier–Wigner transform.

We begin by reviewing the notion and the basic facts related to the rescaled Fourier–Wigner transform

$$\mathcal{V}^\sigma(f,g)(p,q) = \left(\frac{\sigma}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\sigma\left\langle x-\frac{p}{2},q\right\rangle} f(x)\overline{g(x-p)}dx. \tag{4.1}$$

Such transform can be rewritten in terms of the translation operator T_x and the modulation operator M^σ_ξ given respectively by $T_x g(t) := g(t-x)$ and $M^\sigma_\xi g(t) = e^{i\sigma\xi\cdot t}g(t)$. In fact, we have

$$\mathcal{V}^\sigma(f,g)(p,q) = \left(\frac{\sigma}{2\pi}\right)^{\frac{d}{2}} e^{-i\frac{\sigma}{2}\langle p,q\rangle} \left\langle f, M^\sigma_{-q}T_p g\right\rangle_{L^2_{\mathbb{C}}(\mathbb{R}^d)}, \tag{4.2}$$

where $L^2_{\mathbb{C}}(\mathbb{R}^d)$ is the space of $\mathbb{C}$–valued square integrable functions with respect to the Lebesgue measure dx on $\mathbb{R}^d$. The operators $M^\sigma_{-q}T_p$ involved in the right hand-side of (4.2) are fundamental in time frequency analysis. The underlying structure (when $\sigma = 2\pi$) can be described by the Schrödinger representation of the Heisenberg group $\mathbb{C}^d \times \mathbb{R}$ with

$$(z,t)\cdot(w,s) = \left(z + w, t + s + \frac{1}{2}\omega(z,w)\right)$$

as associated law group (see [47,51,67]). Here $\omega(z,w)=\Im(z\overline{w})$ is the canonical symplectic form on $\mathbb{C}^d$. The transform in (4.1) maps $L^2_{\mathbb{C}}(\mathbb{R}^d)\times L^2_{\mathbb{C}}(\mathbb{R}^d)$ into $L^2_{\mathbb{C}}(\mathbb{C}^d)$ (see e.g. [51,53]). An interesting result satisfied by $\mathcal{V}^\sigma$ is Moyal's formula

$$\langle \mathcal{V}^\sigma(f,g),\mathcal{V}^\sigma(\varphi,\psi)\rangle_{L^2_{\mathbb{C}}(\mathbb{C}^d)} \quad = \quad \langle f,\varphi\rangle_{L^2_{\mathbb{C}}(\mathbb{R}^d)}\,\langle \psi,g\rangle_{L^2_{\mathbb{C}}(\mathbb{R}^d)} \tag{4.3}$$

valid for all $f,g,\varphi,\psi\in L^2_{\mathbb{C}}(\mathbb{R}^d)$. It readily follows from the classical Moyal's formula for $\mathcal{V}$ ([41,51,53,67]) combined with the fact that $\mathcal{V}^\sigma(f,g)(p,q)=\sigma^{d/2}\mathcal{V}(f,g)(p,\sigma q)$. It interprets the fact that $\mathcal{V}^\sigma$ preserves the energy of a signal. Accordingly, it can be shown [41,53] that the Fourier–Wigner transform $\mathcal{V}^\sigma$ produces orthonormal bases for the Hilbert space $L^2_{\mathbb{C}}(\mathbb{C}^d)$ from the ones of $L^2_{\mathbb{C}}(\mathbb{R}^d)$. More precisely, if $\{\varphi_k, k\in\mathbb{N}\}$ is an orthonormal basis of $L^2_{\mathbb{C}}(\mathbb{R}^d)$, then $\{\varphi_{jk}=\mathcal{V}^\sigma(\varphi_j,\varphi_k);\ j,k=0,1,2,\cdots\}$ is an orthonormal basis of $L^2_{\mathbb{C}}(\mathbb{C}^d)$. This fact will be used, when dealing with the special bicomplex Fourier–Wigner transform discussed in Section 4, in order to obtain bicomplex four–indices orthogonal polynomials of Hermite type that are not tensor product of the Hermite polynomials on $\mathbb{R}$.

The next result is an analogue of Theorem 3.1 in [38] for the action of $\mathcal{V}^\sigma$ on the rescaled Hermite functions

$$h_n^\sigma(t)=(-1)^n e^{\frac{\sigma}{2}t^2}\frac{d^n}{dt^n}\left(e^{-\sigma t^2}\right)=\sqrt{\sigma}^n h_n(\sqrt{\sigma}t)$$

that form an orthogonal basis of $L^2_{\mathbb{C}}(\mathbb{R})$. It asserts that $\mathcal{V}^\sigma(h_m^\sigma,h_n^\sigma)$ is closely connected to the univariate polyanalytic Hermite function

$$h^\alpha_{m,n}(z,\overline{z}):=(-1)^{m+n}e^{\frac{\alpha}{2}|z|^2}\frac{\partial^{m+n}}{\partial\overline{z}^m\partial z^n}\left(e^{-\alpha|z|^2}\right),\quad \alpha>0. \tag{4.4}$$

We denote $h_{m,n}=h^1_{m,n}$.

Proposition 4.1.1. *We have*

$$\mathcal{V}^\sigma(h_m^\sigma,h_n^\sigma)(p,q)=(-1)^n\frac{2^{m+n}}{\sqrt{2}}h^{\sigma/2}_{m,n}(z,\overline{z}). \tag{4.5}$$

Proof. A straightforward computation starting from the definition of $\mathcal{V}^\sigma$ and h_n^σ shows

$$\mathcal{V}^\sigma(h_m^\sigma,h_n^\sigma)(p,q)=\sqrt{\sigma}^{m+n}\mathcal{V}(h_m,h_n)(\sqrt{\sigma}p,\sqrt{\sigma}q).$$

Subsequently, by means of Theorem 3.1 in [38] combined with the fact that

$$h^\alpha_{m,n}(z,\overline{z}):=\sqrt{\alpha}^{m+n}h_{m,n}(\sqrt{\alpha}z,\sqrt{\alpha}\overline{z}),$$

we obtain

$$\begin{aligned}\mathcal{V}^{\sigma}(h^{\sigma}_{m}, h^{\sigma}_{n})(p,q) &= \sqrt{\sigma}^{m+n}(-1)^{n}\sqrt{2}^{m+n-1} h_{m,n}\left(\frac{\sqrt{\sigma}}{\sqrt{2}}z, \frac{\sqrt{\sigma}}{\sqrt{2}}\overline{z}\right)\\ &= (-1)^{n}\frac{2^{m+n}}{\sqrt{2}} h^{\sigma/2}_{m,n}(z,\overline{z}).\end{aligned}$$

∎

Remark 4.1.2. *The range of $L^2_{\mathbb{C}}(\mathbb{R})$ by the transform $\mathcal{V}^{\sigma}(\cdot, h^{\sigma}_{n})$ is the Hilbert space spanned by the complex Hermite functions $h^{\sigma/2}_{m,n}(z,\overline{z})$ for varying m, which is clearly connected to the so–called true–poly–Fock space $\mathcal{F}^{2,\sigma/2}_{n}(\mathbb{C})$ on $\mathbb{C}$ of level n in Vasilevski's terminology [60,78].*

In the sequel, we suggest some natural bicomplex analogues of the Fourier–Wigner transform with input functions belonging to the bicomplex Hilbert spaces $L^2_{\mathbb{BC}}(X)$ for $X = \mathbb{R}$ or $\mathbb{R}^2$, and output functions in $L^2_{\mathbb{BC}}(\mathbb{BC})$. We then investigate some of their basic properties, such as Moyal's identity. We identify their ranges and we establish the connection to some bicomplex transforms of Segal–Bargmann type, among others. The central idea in obtaining such bicomplex analogues is basically the idempotent decomposition of any $\varphi \in L^2_{\mathbb{BC}}(X)$ as $\varphi = \varphi^{-}e_{+} + \varphi^{-}e_{-}$ with $\varphi^{+}, \varphi^{-} \in L^2_{\mathbb{C}}(X)$.

4.2 Unidimensional bicomplex Fourier–Wigner transform.

For every given bicomplex number $Z = z_1 + jz_2$, where $z_\ell = x_\ell + iy_\ell$, we associate the hyperbolic numbers $X_e = x_1e_+ + x_2e_-$ and $Y_e = y_1e_+ + y_2e_-$, and consider the translation operator

$$T_{X_e}\varphi(t) := \varphi^{+}(t - x_1)e_{+} + \varphi^{-}(t - x_2)e_{-}$$

as well as the modified modulation operator

$$M^{\sigma,\nu,\mu}_{X_e,Y_e}\varphi(t) := e^{\sigma(\nu e_+ + \mu e_-)\left(t - \frac{X_e}{2}\right)Y_e}\varphi(t)$$

for given $\varphi = \varphi^{+}e_{+} + \varphi^{-}e_{-} \in L^2_{\mathbb{BC}}(\mathbb{R})$; $\varphi^{\pm} \in L^2_{\mathbb{C}}(\mathbb{R})$.

Definition 4.2.1. *We call unidimensional bicomplex Fourier–Wigner transform the integral transform $\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}$ on $L^2_{\mathbb{BC}}(\mathbb{R}) \times L^2_{\mathbb{BC}}(\mathbb{R})$ defined by*

$$\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(\varphi,\psi)(Z) \;:=\; \left(\frac{\sigma}{\pi}\right) e^{-\frac{\sigma}{4}\left((X_e^{\dagger})^2 + (Y_e^{\dagger})^2\right)} \int_{\mathbb{R}} \varphi(t)\left(M^{\sigma,\nu,\mu}_{X_e,Y_e}T_{X_e}\psi(t)\right)^{*} dt \qquad (4.6)$$

with $X_e^{\dagger} = x_2e_+ + x_1e_-$ and $Z^{*} = \overline{z}_1 - j\overline{z}_2$.

Remark 4.2.2. *The underlying representation emerged here, associated to translation operators T_{X_e} and the modulation operators $M^{\sigma,\nu,\mu}_{X_e,Y_e}$, yields a projective representation of two copies of the Heisenberg group $\mathbb{C} \times \mathbb{R}$. To this end, we introduce $z_{1_\nu} = x_1 + \nu y_1$ and $z_{2_\mu} = x_2 + \mu y_2$ with $z_\ell = x_\ell + iy_\ell$; $\ell = 1, 2$, and define $\Im_\nu(x + \nu y) := y$. In fact, we consider $\mathbb{BC} \times \mathbb{D}$, $\mathbb{D}$ being the set* *of hyperbolic numbers* $\mathbb{D} := \{U = ue_+ + u'e_-;\, u, u' \in \mathbb{R}\}$, *endowed with*

$$(Z,U)\cdot(W,V) = \left(Z+W, U+V+\frac{1}{2}\Omega(Z,W)\right),$$

where

$$\Omega(Z,W) = \Im_\nu(z_{1_\nu}\overline{w_{1_\nu}})e_+ + \Im_\mu(z_{2_\mu}\overline{w_{2_\mu}})e_- \in \mathbb{D}.$$

The corresponding representation on the bicomplex Hilbert space $L^2_{\mathbb{BC}}(\mathbb{R})$ *is defined by*

$$\rho^{\nu,\mu,\sigma}_{Z,U}\varphi(t) := e^{\sigma(\nu u e_+ + \mu u' e_-)} M^{\sigma,\nu,\mu}_{X_e,Y_e} T_{X_e}\varphi(t)$$

for given $\varphi = \varphi^+ e_+ + \varphi^- e_- \in L^2_{\mathbb{BC}}(\mathbb{R})$, *and satisfy*

$$\rho^{\nu,\mu,\sigma}_{Z,T} \circ \rho^{\nu,\mu,\sigma}_{W,S} = e^{\frac{\sigma}{2}\left(\Im_\nu(z_{1_\nu}\overline{w_{1_\nu}}e_+ + \mu\Im_\mu(z_{2_\mu}\overline{w_{2_\mu}}e_-\right)}\rho^{\nu,\mu,\sigma}_{Z+W,T+S}.$$

A variant representation associated to $\mathbb{BC}\times\mathbb{R}$ *may be introduced. It may be induced from the first one by taking* $u = u'$.

The following lemmas will play a crucial rule in establishing the main results of this section.

Lemma 4.2.3. *Let* $\varphi, \psi \in L^2_{\mathbb{BC}}(\mathbb{R})$. *Then, we have the splitting formula*

$$\begin{aligned} \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(\varphi,\psi)(Z) &= \left(\frac{2\sigma}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\sigma}{4}|z_{2_\mu}|^2}\mathcal{V}^{\sigma,\nu}(\varphi^+,\psi^+)(z_{1_\nu})e_+ \\ &\quad + \left(\frac{2\sigma}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\sigma}{4}|z_{1_\nu}|^2}\mathcal{V}^{\sigma,\mu}(\varphi^-,\psi^-)(z_{2_\mu})e_-. \end{aligned} \tag{4.7}$$

Moreover, $\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(\varphi,\psi) \in L^2_{\mathbb{BC}}(\mathbb{BC})$ *and we have Moyal identity*

$$\left\langle \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(\varphi_1,\psi_1), \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(\varphi_2,\psi_2)\right\rangle_{L^2_{\mathbb{BC}}(\mathbb{BC})} = \langle\varphi_1,\varphi_2\rangle_{L^2_{\mathbb{BC}}(\mathbb{R})}\langle\psi_1,\psi_2\rangle_{L^2_{\mathbb{BC}}(\mathbb{R})} \tag{4.8}$$

for every $\varphi_\ell, \psi_\ell \in L^2_{\mathbb{BC}}(\mathbb{R})$; $\ell = 1,2$.

Proof. The first assertion follows easily from (4.6) since

$$(X_e^\dagger)^2 + (Y_e^\dagger)^2 = |z_{2_\mu}|^2 e_+ + |z_{1_\nu}|^2 e_-,$$

$$M^{\sigma,\nu,\mu}_{X_e,Y_e}\varphi(t) = e^{\nu\sigma\left(t-\frac{x_1}{2}\right)y_1}\varphi^+(t)e_+ + e^{\mu\sigma\left(t-\frac{x_2}{2}\right)y_2}\varphi^-(t)e_-$$

and

$$T_{X_e}\psi(t) := T_{x_1}\psi^+(t)e_+ + T_{x_2}\psi^-(t)e_-.$$

In order to prove the second assertion, we notice first that $z_{1_\nu} \mapsto \mathcal{V}^{\sigma,\nu}(\varphi^+,\psi^+)(z_{1_\nu})$ is in

$L^2_{\mathbb{C}}(\mathbb{C}_\nu)$. Accordingly, the function

$$(z_{1_\nu}, z_{2_\mu}) \mapsto e^{-\frac{\sigma}{4}|z_{2_\mu}|^2}\mathcal{V}^{\sigma,\nu}(\varphi^+,\psi^+)(z_{1_\nu})$$

belongs to $L^2_{\mathbb{C}}(\mathbb{C}_\nu\times\mathbb{C}_\mu)$. The same observation holds true for $e^{-\frac{\sigma}{4}|z_{1_\nu}|^2}\mathcal{V}^{\sigma,\mu}(\varphi^-,\psi^-)(z_{2_\mu})$. This shows that $\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(\varphi,\psi)$ belongs to the Hilbert space

$$L^2_{\mathbb{C}}(\mathbb{C}_\nu)e_+ + L^2_{\mathbb{C}}(\mathbb{C}_\mu)e_- = L^2_{\mathbb{BC}}(\mathbb{C}_\nu\times\mathbb{C}_\mu) = 4L^2_{\mathbb{BC}}(\mathbb{BC}).$$

The multiplication factor 4 in the above formula is present to point out that $\|\psi\|^2_{L^2_{\mathbb{BC}}(\mathbb{C}_\nu\times\mathbb{C}_\mu)} = 4\,\|\widetilde{\psi}\|^2_{L^2_{\mathbb{BC}}(\mathbb{BC})}$ with $\psi(z_{1_\nu}, z_{2_\mu}) = \widetilde{\psi}(z_{1_\nu}e_+ + z_{2_\mu}e_-)$.

Now, by denoting the left–hand side of (4.8) by $M(\varphi_{1,2},\psi_{1,2})$ and making use of (4.7), we obtain

$$\begin{aligned}M(\varphi_{1,2},\psi_{1,2}) \;&=\; \left(\frac{\sigma}{2\pi}\right)\left(\int_{\mathbb{C}} e^{-\frac{\sigma}{2}|\xi|^2}d\lambda(\xi)\right)\Big(\langle\mathcal{V}^{\sigma,\nu}(\varphi_1^+,\psi_1^+),\mathcal{V}^{\sigma,\nu}(\varphi_2^+,\psi_2^+)\rangle_{L^2_{\mathbb{C}}(\mathbb{C}_\nu)}\,e_+ \\ &\qquad\qquad + \langle\mathcal{V}^{\sigma,\mu}(\varphi_1^-,\psi_1^-),\mathcal{V}^{\sigma,\mu}(\varphi_2^-,\psi_2^-)\rangle_{L^2_{\mathbb{C}}(\mathbb{C}_\mu)}\,e_-\Big).\end{aligned}$$

Consequently, from (4.3) we get

$$\begin{aligned}M(\varphi_{1,2},\psi_{1,2}) \;&=\; \langle\varphi_1^+,\varphi_2^+\rangle_{L^2_{\mathbb{C}}(\mathbb{R})}\,\langle\psi_1^+,\psi_2^+\rangle_{L^2_{\mathbb{C}}(\mathbb{R})}\,e_+ + \langle\varphi_1^-,\varphi_2^-\rangle_{L^2_{\mathbb{C}}(\mathbb{R})}\,\langle\psi_1^-,\psi_2^-\rangle_{L^2_{\mathbb{C}}(\mathbb{R})}\,e_- \\ &=\; \Big(\langle\varphi_1^+,\varphi_2^+\rangle_{L^2_{\mathbb{C}}(\mathbb{R})}\,e_+ + \langle\varphi_1^-,\varphi_2^-\rangle_{L^2_{\mathbb{C}}(\mathbb{R})}\,e_-\Big) \\ &\qquad\qquad \times\Big(\langle\psi_1^+,\psi_2^+\rangle_{L^2_{\mathbb{C}}(\mathbb{R})}\,e_+ + \langle\psi_1^-,\psi_2^-\rangle_{L^2_{\mathbb{C}}(\mathbb{R})}\,e_-\Big) \\ &=\; \langle\varphi_1,\varphi_2\rangle_{L^2_{\mathbb{BC}}(\mathbb{R})}\,\langle\psi_1,\psi_2\rangle_{L^2_{\mathbb{BC}}(\mathbb{R})}\,.\end{aligned}$$

This completes our check of (4.8) and hence the one of Lemma 4.2.3. ■

Another needed fact is the action of $\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}$ on the elementary functions

$$f^\sigma_{m,n}(t) := h^\sigma_m(t)e_+ + h^\sigma_n(t)e_-.$$

Namely, we assert

Lemma 4.2.4. *We have*

$$\begin{aligned}\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(f^\sigma_{m,n},f^\sigma_{r,s})(Z) \;&=\; \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}}(-1)^r 2^{m+r}e^{-\frac{\sigma}{4}|z_{2_\mu}|^2}h^{\sigma/2}_{m,r}(z_{1_\nu},\overline{z_{1_\nu}})e_+ \\ &\qquad + \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}}(-1)^s 2^{n+s}e^{-\frac{\sigma}{4}|z_{1_\nu}|^2}h^{\sigma/2}_{n,s}(z_{2_\mu},\overline{z_{2_\mu}})e_-.\end{aligned} \tag{4.9}$$

Proof. The result follows making use of (4.7) and Proposition 4.1.1. Indeed, we have

$$\begin{aligned}\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(f^{\sigma}_{m,n},f^{\sigma}_{r,s})(Z) &= \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(h^{\sigma}_{m}e_{+}+h^{\sigma}_{n}e_{-},h^{\sigma}_{r}e_{+}+h^{\sigma}_{s}e_{-})(Z)\\ &= \left(\frac{2\sigma}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\sigma}{4}\left(|z_{2_\mu}|^2e_{+}+|z_{1_\nu}|^2e_{-}\right)}\left(\mathcal{V}^{\sigma,\nu}(h^{\sigma}_{m},h^{\sigma}_{r})(z_{1_\nu})e_{+}+\mathcal{V}^{\sigma,\mu}(h^{\sigma}_{n},h^{\sigma}_{s})(z_{2_\mu})e_{-}\right)\\ &= \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\sigma}{4}\left(|z_{2_\mu}|^2e_{+}+|z_{1_\nu}|^2e_{-}\right)}\Big((-1)^{r}2^{m+r}h^{\sigma/2}_{m,r}(z_{1_\nu},\overline{z_{1_\nu}})e_{+}\\ &\qquad\qquad +(-1)^{s}2^{n+s}h^{\sigma/2}_{n,s}(z_{2_\mu},\overline{z_{2_\mu}})e_{-}\Big).\end{aligned}$$

$\blacksquare$

Below, we will discuss the basic properties of the transform $\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}$ for the special window function $\psi_0(t) := e^{-\frac{\sigma}{2}t^2}$. Therefore, we associate to each bicomplex number $Z = z_1 + jz_2 \in \mathbb{BC}$, its companion $Z^{e}_{\nu,\mu} = z_{1_\nu}e_{+} + z_{2_\mu}e_{-}$ and perform

$$\mathbb{BC}^{e}_{\nu,\mu} = \mathbb{C}_{\nu}e_{+} + \mathbb{C}_{\mu}e_{-} = \{Z^{e}_{\nu,\mu} = (x_1+\nu y_1)e_{+} + (x_2+\mu y_2)e_{-}, x_1,y_1,x_2,y_2 \in \mathbb{R}\}.$$

Therefore, any bicomplex–valued function $f(Z)$ on $\mathbb{BC}$ can be seen as a function on $\mathbb{BC}^{e}_{\nu,\mu}$. We define the following derivation operators are defined by

$$\begin{aligned}\frac{\partial}{\partial\overline{Z_{\nu,\mu}}^{e}} &= \frac{\partial}{\partial\bar{z}_{2_\mu}}e_{+} + \frac{\partial}{\partial\bar{z}_{1_\nu}}e_{-};\\ \frac{\partial}{\partial Z^{\dagger e}_{\nu,\mu}} &= \frac{\partial}{\partial z_{2_\mu}}e_{+} + \frac{\partial}{\partial z_{1_\nu}}e_{-};\\ \frac{\partial}{\partial Z^{*e}_{\nu,\mu}} &= \frac{\partial}{\partial\bar{z}_{1_\nu}}e_{+} + \frac{\partial}{\partial\bar{z}_{2_\mu}}e_{-}.\end{aligned}$$

Definition 4.2.5. *A bicomplex–valued function f on $\mathbb{BC}$ is said to be $\mathbb{BC}^{e}_{\nu,\mu}$–holomorphic if its companion $f^{e}(Z^{e}_{\nu,\mu}) := f(Z)$ is $\mathbb{BC}^{e}_{\nu,\mu}$–holomorphic in the sense that f^{e} satisfies the system of first order differential equations*

$$\frac{\partial f^{e}}{\partial\overline{Z_{\nu,\mu}}^{e}} = \frac{\partial f^{e}}{\partial Z^{\dagger e}_{\nu,\mu}} = \frac{\partial f^{e}}{\partial Z^{*e}_{\nu,\mu}} = 0. \tag{4.10}$$

This is clearly equivalent to rewrite f in the form

$$f(Z) = f^{e}(Z^{e}_{\nu,\mu}) = F(z_{1_\nu})e_{+} + G(z_{2_\mu})e_{-}$$

with $F \in \mathcal{H}ol(\mathbb{C}_{\nu})$ (resp $G \in \mathcal{H}ol(\mathbb{C}_{\mu})$) is a holomorphic function on $\mathbb{C}_{\nu}$ (resp. $\mathbb{C}_{\mu}$). A variant bicomplex Bargmann space of the one introduced in [46] is the following.

Definition 4.2.6. *We call companion $\mathbb{BC}^{e}_{\nu,\mu}$–Bargmann space, the Hilbert space $\mathcal{F}^{2,\sigma}(\mathbb{BC}^{e}_{\nu,\mu})$ of all bicomplex–valued $\mathbb{BC}^{e}_{\nu,\mu}$–holomorphic functions $f(Z) = f^{e}(Z^{e}_{\nu,\mu}) = F(z_{1_\nu})e_{+} + G(z_{2_\mu})e_{-}$ such* *that* $F \in L^{2,\sigma/2}_{\mathbb{C}}(\mathbb{C}_\nu)$ *and* $G \in L^{2,\sigma/2}_{\mathbb{C}}(\mathbb{C}_\mu)$. *Succinctly,*

$$\mathcal{F}^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu}) = \mathcal{F}^{2,\sigma/2}(\mathbb{C}_\nu)e_+ + \mathcal{F}^{2,\sigma/2}(\mathbb{C}_\mu)e_-,$$

where $\mathcal{F}^{2,\sigma/2}(\mathbb{C}_\tau)$ *denotes the classical complex Bargmann space of weight* $\sigma/2$ *on* $\mathbb{C}_\tau$.

This functional space is trivially endowed with the bicomplex scaler product

$$\langle f_1, f_2\rangle_{\mathbb{BC}^e_{\nu,\mu}} = \langle F_1, F_2\rangle_{L^{2,\sigma/2}_{\mathbb{C}}(\mathbb{C}_\mu)} e_+ + \langle F_1, F_2\rangle_{L^{2,\sigma/2}_{\mathbb{C}}(\mathbb{C}_\mu)} e_- \tag{4.11}$$

for given $f_\ell(Z) = F(z_{1_\nu})e_+ + G(z_{2_\mu})e_-$. Moreover, it can be seen as subspace of $L^2_{\mathbb{BC}}(\mathbb{C}_\nu \times \mathbb{C}_\mu) = 4L^2_{\mathbb{BC}}(\mathbb{BC})$ by considering its range $\mathcal{M}^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu}) := M_{\sigma/2}(\mathcal{F}^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu}))$ by the multiplication operator

$$\begin{aligned} M_{\sigma/2}f(Z) &:= e^{-\frac{\sigma}{2}|Z^e_{\nu,\mu}|^2} f^e(Z^e_{\nu,\mu}) \\ &= e^{-\frac{\sigma}{4}\left(|z_{1_\nu}|^2+|z_{2_\mu}|^2\right)} \left(F(z_{1_\nu})e_+ + G(z_{2_\mu})e_-\right). \end{aligned}$$

In fact, for given $Y_\ell = M_{\sigma/2}f_\ell \in \mathcal{M}^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu})$; $\ell = 1, 2$, we have

$$\langle Y_1, Y_2\rangle_{L^2_{\mathbb{BC}}(\mathbb{BC})} = \int_{\mathbb{BC}} Y_1(Z)(Y_2(Z))^* d\lambda(Z) = \frac{2\pi}{2\sigma} \langle f_1, f_2\rangle_{\mathbb{BC}^e_{\nu,\mu}}.$$

The corresponding bicomplex norm is the one given through

$$\|Y\|^2_{\mathbb{BC}^e_{\nu,\mu}} := \frac{\pi}{4\sigma}\left(\|F\|^2_{L^{2,\sigma/2}_{\mathbb{C}}(\mathbb{C}_\nu)} + \|G\|^2_{L^{2,\sigma/2}_{\mathbb{C}}(\mathbb{C}_\mu)}\right). \tag{4.12}$$

Thus, we claim the following

Proposition 4.2.7. *The space* $\mathcal{M}^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu})$ *is a reproducing kernel bicomplex Hilbert space whose kernel function is given by*

$$K_\sigma(Z^e_{\nu,\mu}, W^e_{\nu,\mu}) = e^{-\frac{\sigma}{2}\left(|Z^e_{\nu,\mu}|^2+|W^e_{\nu,\mu}|^2+Z^e_{\nu,\mu}W^{*e}_{\nu,\mu}\right)}.$$

Moreover, we prove

Theorem 4.2.8. *The transform* $\mathcal{S}^{\sigma,\nu,\mu}_0(\varphi)$ *given by*

$$\mathcal{S}^{\sigma,\nu,\mu}_0(\varphi) := \left(\frac{\sigma}{\pi}\right)^{\frac{1}{4}} \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(\varphi, \psi_0)$$

defines an isometry from $L^2_{\mathbb{BC}}(\mathbb{R})$ *onto the Hilbert space* $\mathcal{M}^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu})$. *Moreover, the functions*

$$\varphi^{\sigma,\nu,\mu}_n(Z) = \left(\frac{\sigma}{\pi}\right)^{\frac{3}{4}} \sigma^n \left(Z^e_{\nu,\mu}\right)^n e^{-\frac{\sigma}{2}|Z^e_{\nu,\mu}|^2} \tag{4.13}$$

form an orthogonal basis of $\mathcal{M}^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu})$ *with norm given by*

$$\left\| \varphi_n^{\sigma,\nu,\mu} \right\|^2_{L^2_{\mathbb{BC}}(\mathbb{BC})} = \left(\frac{\pi}{\sigma}\right)^{\frac{1}{2}} 2^n \sigma^n n!. \tag{4.14}$$

Proof. Notice first that the window state is the Gaussian centred at the origin $\psi_0(t) := e^{-\frac{\sigma}{2}t^2} = f_{0,0}(t)$ and $h_n^\sigma = f^\sigma_{n,n}$ for $e_+ + e_- = 1$. Thus, from (4.9) and the fact

$$h^\alpha_{n,0}(\xi,\overline{\xi}) = \alpha^n \xi^n e^{-\frac{\alpha}{2}|\xi|^2},$$

we obtain

$$\begin{aligned}
\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(h_n^\sigma,\psi_0)(Z) &= \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(f^\sigma_{n,n},f_{0,0})(Z)\\
&= \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} 2^n \left(e^{-\frac{\sigma}{4}|z_{2_\mu}|^2} h^{\sigma/2}_{n,0}(z_{1_\nu},\overline{z_{1_\nu}})e_+ + e^{-\frac{\sigma}{4}|z_{1_\nu}|^2} h^{\sigma/2}_{n,0}(z_{2_\mu},\overline{z_{2_\mu}})e_- \right).\\
&= \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} \sigma^n \left(z^n_{1_\nu} e_+ + z^n_{2_\mu} e_- \right) e^{-\frac{\sigma}{4}\left(\left(|z_{2_\mu}|^2+|z_{1_\nu}|^2\right)e_+ + \left(|z_{1_\nu}|^2+|z_{2_\mu}|^2\right)\right)e_-}\\
&= \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} \sigma^n \left(z^n_{1_\nu} e_+ + z^n_{2_\mu} e_- \right) e^{-\frac{\sigma}{2}|Z^e_{\nu,\mu}|^2}.
\end{aligned} \tag{4.15}$$

These functions form an orthogonal system in the Hilbert space $L^2_{\mathbb{BC}}(\mathbb{BC})$ in virtue of Moyal identity (4.8) satisfied by $\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}$ and the orthogonality of h_n^σ in $L^2_{\mathbb{BC}}(\mathbb{R})$. Indeed, we have

$$\begin{aligned}
\left\langle \varphi_m^{\sigma,\nu,\mu}, \varphi_n^{\sigma,\nu,\mu} \right\rangle_{L^2_{\mathbb{BC}}(\mathbb{BC})} &= \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} \left\langle \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(h_m^\sigma,\psi_0), \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(h_n^\sigma,\psi_0) \right\rangle_{L^2_{\mathbb{BC}}(\mathbb{BC})}\\
&= \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} \langle h_m^\sigma, h_n^\sigma \rangle_{L^2_{\mathbb{BC}}(\mathbb{R})} \langle \psi_0,\psi_0 \rangle_{L^2_{\mathbb{BC}}(\mathbb{R})}\\
&= \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} \|\psi_0\|^2_{L^2_{\mathbb{C}}(\mathbb{R})} \|h_n^\sigma\|^2_{L^2_{\mathbb{C}}(\mathbb{R})} \delta_{m,n}\\
&= \|h_n^\sigma\|^2_{L^2_{\mathbb{C}}(\mathbb{R})} \delta_{m,n}.
\end{aligned}$$

This readily follows since $\|\psi_0\|^2_{L^2_{\mathbb{BC}}(\mathbb{R})} = \left(\frac{\pi}{\sigma}\right)^{1/2}$. Consequently, the obtained equality gives rise to (4.14) for $\|h_n^\sigma\|^2_{L^2_{\mathbb{BC}}(\mathbb{R})} = \left(\frac{\pi}{\sigma}\right)^{1/2} 2^n \sigma^n n!$. Identity (4.14) can also be handled by direct computation using the explicit expression of $\varphi_n^{\sigma,\nu,\mu}$. The previous result remains valid for any $\varphi \in L^2_{\mathbb{BC}}(\mathbb{R})$. Indeed, by applying Moyal identity (4.8), we get

$$\begin{aligned}
\left\| \mathcal{S}_0^{\sigma,\nu,\mu}(\varphi) \right\|^2_{L^2_{\mathbb{BC}}(\mathbb{BC})} &= \left(\frac{\sigma}{\pi}\right)^{1/2} \left| \left\langle \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(\varphi,\psi_0), \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(\varphi,\psi_0) \right\rangle_{L^2_{\mathbb{BC}}(\mathbb{BC})} \right|\\
&= \left(\frac{\sigma}{\pi}\right)^{1/2} \left| \langle \varphi,\varphi \rangle_{L^2_{\mathbb{BC}}(\mathbb{R})} \langle \psi_0,\psi_0 \rangle_{L^2_{\mathbb{BC}}(\mathbb{R})} \right|\\
&= \|\varphi\|^2_{L^2_{\mathbb{BC}}(\mathbb{R})}.
\end{aligned}$$

This shows in particular that $\mathcal{S}_0^{\sigma,\nu,\mu}(\varphi) \in \mathcal{M}^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu})$. One can conclude for the proof, by noting that the functions $\varphi_n^{\sigma,\nu,\mu}$ in (4.13) form a complete orthogonal system in $\mathcal{M}^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu})$ for the monomials $\left(Z^e_{\nu,\mu}\right)^n$ being an orthogonal basis of $L^{2,\sigma/2}_{\mathbb{BC}}(\mathbb{C}_\nu \times \mathbb{C}_\mu)$. Moreover, for any $\varphi(t) = \sum\limits_{n=0}^{\infty} c_n h_n^\sigma \in L^2_{\mathbb{BC}}(\mathbb{R})$, we have

$$\mathcal{S}_0^{\sigma,\nu,\mu}(\varphi) = \left(\frac{\sigma}{\pi}\right)^{1/4} \sum_{n=0}^{\infty} c_n \varphi_n^{\sigma,\nu,\mu}$$

which follows by means of (4.15) and the continuity of the linear mapping $\mathcal{S}_0^{\sigma,\nu,\mu}$. ▬

Corollary 4.2.9. *The transform $\mathcal{S}_0^{\sigma,\nu,\mu}$ is closely connected to the bicomplex Segal–Bargmann transform $\mathcal{B}^{\sigma,\nu}_{\mathbb{BC}}$ introduced in [46]. More precisely, we have*

$$\mathcal{S}_0^{\sigma,i,i}(\varphi)(Z) = \left(\frac{\sigma}{\pi}\right) e^{-\frac{\sigma}{2}|Z^e_{\nu,\mu}|^2} \int_{\mathbb{R}} e^{-\sigma\left(t-\frac{Z^e_{\nu,\mu}}{2}\right)^2} e^{\frac{\sigma}{2}t^2} \varphi(t)dt. \tag{4.16}$$

Proof. Identity (4.16) follows by a tedious but straightforward computation. Indeed, we obtain

$$\mathcal{S}_0^{\sigma,\nu,\mu}(\varphi)(Z) \quad = \quad \left(\frac{\sigma}{\pi}\right) e^{-\frac{\sigma}{2}|Z^e_{\nu,\mu}|^2} \int_{\mathbb{R}} e^{-\sigma\left(t-\frac{Z^e_{\nu,\mu}}{2}\right)^2} e^{\frac{\sigma}{2}t^2} \varphi(t)dt,$$

so that for $\nu = \mu = i$, we recover the bicomplex Segal–Bargmann transform introduced in [46, Eq. (5.6)] from $L^2_{\mathbb{BC}}(\mathbb{R})$ onto the bicomplex Bargmann space $\mathcal{F}^{2,\sigma}(\mathbb{BC}^e_{i,i})$. ▬

The last result of this section identifies the range $\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(L^2_{\mathbb{BC}}(\mathbb{R}) \times L^2_{\mathbb{BC}}(\mathbb{R}))$ as special bicomplex–analytic closed subspace of $L^2_{\mathbb{BC}}(\mathbb{BC})$. To do this, we introduce the following.

Definition 4.2.10. *We call bicomplex $(1^-, 1^\dagger, n^*)$–$\mathbb{BC}^e_{\nu,\mu}$–companion Bargmann space of first kind, the Hilbert space $\mathcal{F}_n^{2,\sigma,\nu,\mu}(\mathbb{BC}^e_{\nu,\mu})$ of all bicomplex–valued functions $f(Z) = f^e(Z^e_{\nu,\mu}) = F(z_{1_\nu})e_+ + G(z_{2_\mu})e_-$ satisfying the system*

$$\frac{\partial f^e}{\partial \overline{(Z^e_{\nu,\mu})}} = \frac{\partial f^e}{\partial (Z^e_{\nu,\mu})^\dagger} = \frac{\partial^{n+1} f^e}{\partial (Z^{*e}_{\nu,\mu})^{n+1}} = 0, \tag{4.17}$$

and $\|F\|^2_{L^{2,\sigma/2}_{\mathbb{C}}(\mathbb{C}_\nu)}$ and $\|G\|^2_{L^{2,\sigma/2}_{\mathbb{C}}(\mathbb{C}_\mu)}$ are finite.

Hence, we claim the following (we omit the proof for its similarity to one provided above in the case $n = 0$).

Lemma 4.2.11. *The spaces $\mathcal{M}_n^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu}) := M_{\sigma/2}(\mathcal{F}_n^{2,\sigma,\nu,\mu}(\mathbb{BC}^e_{\nu,\mu}))$, for varying n, are closed subspaces of $L^2_{\mathbb{BC}}(\mathbb{BC})$ and we have*

$$\mathcal{M}_n^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu}) = e^{-\frac{\sigma}{2}|Z^e_{\nu,\mu}|^2} \left(\mathcal{F}_n^{2,\sigma/2}(\mathbb{C}_\nu)e_+ + \mathcal{F}_n^{2,\sigma/2}(\mathbb{C}_\mu)e_-\right). \tag{4.18}$$

Moreover, they are pairwise orthogonal in $L^2_{\mathbb{BC}}(\mathbb{BC})$ *in the sense that* $\langle \varphi, \psi \rangle_{L^2_{\mathbb{BC}}(\mathbb{BC})} = 0$ *for every* $\varphi \in \mathcal{M}^{2,\sigma}_n(\mathbb{BC}^e_{\nu,\mu})$ *and* $\psi \in \mathcal{M}^{2,\sigma}_m(\mathbb{BC}^e_{\nu,\mu})$ *with* $n \neq m$.

The decomposition (4.18) must be understood in the sense that every bicomplex–valued function $f \in \mathcal{F}^{2,\sigma,\nu,\mu}_n(\mathbb{BC}^e_{\nu,\mu})$ is of the form

$$f(Z) = F(z_{1_\nu})e_+ + G(z_{2_\mu})e_-$$

with F and G are certain $\mathbb{C}$–valued functions belonging to $\mathcal{F}^{2,\sigma}_n(\mathbb{C}_\nu)$ and $\mathcal{F}^{2,\sigma}_n(\mathbb{C}_\mu)$, respectively. We endow $\mathcal{F}^{2,\sigma,\nu,\mu}_n(\mathbb{BC}^e_{\nu,\mu})$ with the bicomplex scalar product $\langle \cdot, \cdot \rangle_{\mathbb{BC}^e_{\nu,\mu}}$ in (4.11). The associated bicomplex norm is given by (4.12).

Proposition 4.2.12. *The functions* $\psi^{\sigma,\nu,\mu}_{m,n}(Z^e_{\nu,\mu}, Z^{*e}_{\nu,\mu}) = e^{-\frac{\sigma}{2}|Z^e_{\nu,\mu}|^2} H^{\sigma}_{m,n}(Z^e_{\nu,\mu}, Z^{*e}_{\nu,\mu})$, *where*

$$h^{\sigma}_{m,n}(Z^e_{\nu,\mu}, Z^{*e}_{\nu,\mu}) := (-1)^{m+n} e^{\frac{\sigma}{4} Z^e_{\nu,\mu} Z^{*e}_{\nu,\mu}} \frac{\partial^{m+n}}{\partial (Z^{*e}_{\nu,\mu})^m \partial (Z^e_{\nu,\mu})^n} \left(e^{-\frac{\sigma}{2} Z^e_{\nu,\mu} Z^{*e}_{\nu,\mu}} \right) \qquad (4.19)$$

for varying m*, form an orthogonal basis of the infinite* $\mathbb{BC}^e_{\nu,\mu}$*-Hilbert space* $\mathcal{M}^{2,\sigma}_n(\mathbb{BC}^e_{\nu,\mu})$.

Proof. The proof is similar to one provided for $n = 0$, but here we make use of the fact that the complex Hermite functions $h^{\sigma/2}_{m,n}(\xi)$ is an orthogonal basis of $L^{2,\sigma/2}_{\mathbb{C}}(\mathbb{C}_\tau)$ and that

$$H^{\sigma/2}_{m,n}(z_{1_\nu})e_+ + H^{\sigma/2}_{m,n}(z_{2_\mu})e_- = H^{\sigma}_{m,n}(Z^e_{\nu,\mu}, Z^{*e}_{\nu,\mu}).$$

▬

Theorem 4.2.13. *The transform*

$$\mathcal{S}^{\sigma,\nu,\mu}_n f := \frac{\left(\frac{\sigma}{\pi}\right)^{1/4}}{\sqrt{2^n \sigma^n n!}} \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(f, h^{\sigma}_n)$$

corresponding to the window function h^{σ}_n *defines an isometry from* $L^2_{\mathbb{BC}}(\mathbb{R})$ *onto the Hilbert space* $\mathcal{M}^{2,\sigma}_n(\mathbb{BC}^e_{\nu,\mu})$.

Proof. The proof can be handled in a similar way as for Theorem 4.2.8 (where $n = 0$) with $h^{\sigma}_0 = \psi_0$. Let just mention that the expression of the functions $\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(h^{\sigma}_m, h^{\sigma}_n)$ is a particular case of (4.9). Hence,

$$\begin{aligned} \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}(h^{\sigma}_m, h^{\sigma}_n)(Z) &= \mathcal{V}^{\sigma,\nu}(f^{\sigma}_{m,m}, f^{\sigma}_{n,n})(Z) \\ &= (-1)^n 2^{m+n} \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\sigma}{2}|Z^e_{\nu,\mu}|^2} \left(h^{\sigma/2}_{m,n}(z_{1_\nu})e_+ + h^{\sigma/2}_{m,n}(z_{2_\mu})e_- \right), \\ &= (-1)^n 2^{m+n} \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} \psi^{\sigma,\nu,\mu}_{m,n}(Z^e_{\nu,\mu}, Z^{*e}_{\nu,\mu}), \qquad (4.20) \end{aligned}$$

where $\psi^{\sigma,\nu,\mu}_{m,n}$ are as in Proposition 4.2.12. The range of $L^2_{\mathbb{BC}}(\mathbb{R})$ by $\mathcal{S}^{\sigma,\nu,\mu}_n$ is then spanned by the bicomplex Hermite functions $\psi^{\sigma,\nu,\mu}_{m,n}$ for varying m (n fixed). Thus, one can conclude making use of Proposition 4.2.12 and Moyal identity (4.8). ▬

Theorem 4.2.14. *The transform* $\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}$ *defines an isometry from* $L^2_{\mathbb{BC}}(\mathbb{R}) \times L^2_{\mathbb{BC}}(\mathbb{R})$ *onto the Hilbert space*

$$\mathcal{G}^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu}) := \bigoplus_{n=0}^{+\infty} \mathcal{M}^{2,\sigma}_n(\mathbb{BC}^e_{\nu,\mu}).$$

Proof. By Lemma 4.2.11, it is clear that the bicomplex Hermite functions $\psi^{\sigma,\nu,\mu}_{m,n}$ for varying m and n form an orthogonal basis of the range of $L^2_{\mathbb{BC}}(\mathbb{R}) \times L^2_{\mathbb{BC}}(\mathbb{R})$ by $\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R},\mathbb{BC}}$. ∎

Remark 4.2.15. *The space* $\mathcal{G}^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu})$ *is strictely contained in* $L^2_{\mathbb{BC}}(\mathbb{BC})$ *since the functions*

$$\varphi^{\sigma,\nu,\mu}_{m,n}(Z) = \left(\sigma^m z^m_{2_\mu} e_+ + \sigma^n z^n_{1_\nu} e_-\right) e^{-\frac{\sigma}{2}|Z^e_{\nu,\mu}|^2}$$

belong to $L^2_{\mathbb{BC}}(\mathbb{BC})$ *whenever* $m \neq n$*, but do not belongs to* $\mathcal{G}^{2,\sigma}(\mathbb{BC}^e_{\nu,\mu})$.

4.3 Bidimensional bicomplex Fourier–Wigner transform.

In this section, we consider a natural extention to the bicomplex Hilbert space $L^2_{\mathbb{BC}}(\mathbb{R}^2)$ of the operators defined on $L^2_{\mathbb{C}}(\mathbb{R}^2)$ by

$$M^{\nu,\sigma}_{X,Y} g(U) = e^{\nu\sigma\langle U-\frac{X}{2},Y\rangle} g(U) \quad \text{and} \quad T_X g(U) := g(U-X)$$

where $X, Y \in \mathbb{R}^2$. Namely, we define

$$\widetilde{M^{\sigma,\nu,\mu}_{X,Y}}\varphi = M^{\sigma,\nu}_{X,Y}\varphi^+ e_+ + M^{\sigma,\mu}_{X,Y}\varphi^- e_-$$

and

$$\widetilde{T_X}\psi(U) := \psi^+(U-X)e_+ + \psi^-(U-X)e_-$$

for given $\varphi = \varphi^+ e_+ + \varphi^- e_-$ and $\psi = \psi^+ e_+ + \psi^- e_-$ in $L^2_{\mathbb{BC}}(\mathbb{R}^2)$ with $\varphi^+, \varphi^-, \psi^+, \psi^- \in L^2_{\mathbb{C}}(\mathbb{R}^2)$.

Definition 4.3.1. *We call bidimensional bicomplex Fourier–Wigner transform the one associated to the "bicomplex time–frequency shift" operator* $\widetilde{M^{\sigma,\nu,\mu}_{X,-Y}}\widetilde{T_X}$ *on* $L^2_{\mathbb{BC}}(\mathbb{R}^2)$ *and given explicitely by*

$$\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(\varphi,\psi)(Z) := \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}^2} e^{\sigma(\nu e_+ + \mu e_-)\langle U-\frac{X}{2},Y\rangle_{\mathbb{R}^2}} \varphi(U)\psi^*(U-X)d\lambda(U) \tag{4.21}$$

with $Z = z_1 + jz_2 \in \mathbb{BC}$*,* $z_\ell = x_\ell + iy_\ell$*,* $X = (x_1, x_2)$ *and* $Y = (y_1, y_2)$.

By proceeding in a similar way as in the previous section, we can prove the following (we omit the proof).

Lemma 4.3.2. *We have*

$$\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(\varphi,\psi)(Z) = \mathcal{V}^{\sigma,\nu}_{\mathbb{R}^2,\mathbb{C}^2}(\varphi^+,\psi^+)(X,Y)e_+ + \mathcal{V}^{\sigma,\mu}_{\mathbb{R}^2,\mathbb{C}^2}(\varphi^-,\psi^-)(X,Y)e_- \tag{4.22}$$

as well as

$$\left\langle \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(\varphi_1,\psi_1), \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(\varphi_2,\psi_2)\right\rangle_{L^2_{\mathbb{BC}}(\mathbb{BC})} = \langle \varphi_1,\varphi_2\rangle_{L^2_{\mathbb{BC}}(\mathbb{C})} \langle \psi_1,\psi_2\rangle_{L^2_{\mathbb{BC}}(\mathbb{C})}. \tag{4.23}$$

Proposition 4.3.3. *The bicomplex Fourier–Wigner transform $\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}$ defines a surjection from $L^2_{\mathbb{BC}}(\mathbb{R}^2) \times L^2_{\mathbb{BC}}(\mathbb{R}^2)$ onto $L^2_{\mathbb{BC}}(\mathbb{BC})$.*

Proof. Let $F \in L^2_{\mathbb{BC}}(\mathbb{BC})$. Then, we can rewrite F as $F = F^+e_+ + F^-e_-$ for certain $F^\pm \in L^2_{\mathbb{C}}(\mathbb{C}^2)$. By the surjectivity of $\mathcal{V}^{\sigma,\nu}_{\mathbb{R}^2,\mathbb{C}^2}$ and $\mathcal{V}^{\sigma,\mu}_{\mathbb{R}^2,\mathbb{C}^2}$ from $L^2_{\mathbb{C}}(\mathbb{R}^2) \times L^2_{\mathbb{C}}(\mathbb{R}^2)$ onto $L^2_{\mathbb{C}}(\mathbb{C}^2)$, we can exhibit $\varphi^\pm, \psi^\pm \in L^2_{\mathbb{C}}(\mathbb{R}^2)$ such that

$$F^+(Z) = \mathcal{V}^{\sigma,\nu}_{\mathbb{R}^2,\mathbb{C}^2}(\varphi^+,\psi^+)(X,Y)$$

and

$$F^-(Z) = \mathcal{V}^{\sigma,\mu}_{\mathbb{R}^2,\mathbb{C}^2}(\varphi^-,\psi^-)(X,Y).$$

Accordingly,

$$F(Z) = \mathcal{V}^{\sigma,\nu}_{\mathbb{R}^2,\mathbb{C}^2}(\varphi^+,\psi^+)(X,Y)e_+ + \mathcal{V}^{\sigma,\nu}_{\mathbb{R}^2,\mathbb{C}^2}(\varphi^-,\psi^-)(X,Y)e_-.$$

In virtue of (4.22) and setting $\varphi := \varphi^+e_+ + \varphi^-e_-$ and $\psi := \psi^+e_+ + \psi^-e_-$, we get

$$F(Z) = \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(\varphi^+e_+ + \varphi^-e_-, \psi^+e_+ + \psi^-e_-)(Z) = \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(\varphi,\psi)(Z).$$

Notice finally that $\varphi,\psi \in L^2_{\mathbb{BC}}(\mathbb{R}^2)$ since $\varphi^\pm,\psi^\pm \in L^2_{\mathbb{C}}(\mathbb{R}^2)$. ▬

In the sequel, we provide a nontrivial basis for the bicomplex Hilbert space $L^2_{\mathbb{BC}}(\mathbb{BC})$. In fact, Moyal identity (4.23) is an effective tool for constructing orthogonal bases for $L^2_{\mathbb{BC}}(\mathbb{BC})$ from those of $L^2_{\mathbb{BC}}(\mathbb{C})$. Namely, we assert

Proposition 4.3.4. *Let $(\phi_n)_n$ be a system in $L^2_{\mathbb{BC}}(\mathbb{R}^2)$ such that $\phi_n = \phi_n^+e_+ + \phi_n^-e_-$ with $\phi_n^+, \phi_n^- \in L^2_{\mathbb{C}}(\mathbb{C})$. If $(\phi_n^+)_n$ and $(\phi_n^-)_n$ are orthonormal bases of $L^2_{\mathbb{C}}(\mathbb{C})$, then the family of functions*

$$\phi_{m,n} := \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(\phi_m,\phi_n);\ m,n = 0,1,2,\cdots,$$

is an orthonormal basis of $L^2_{\mathbb{BC}}(\mathbb{BC})$.

Proof. Under the assumption that $(\phi_n^+)_n$ and $(\phi_n^-)_n$ are orthogonal in $L^2_{\mathbb{C}}(\mathbb{C})$, i.e.,

$$\langle \phi_n^+, \phi_{'}^+\rangle_{L^2_{\mathbb{C}}(\mathbb{C})} = \langle \phi_n^-, \phi_{n'}^-\rangle_{L^2_{\mathbb{C}}(\mathbb{C})} = 0; \quad n \neq n',$$

it follows

$$\langle \phi_n, \phi_{n'}\rangle_{L^2_{\mathbb{BC}}(\mathbb{C})} = \langle \phi_n^+, \phi_{'}^+\rangle_{L^2_{\mathbb{C}}(\mathbb{C})} e_+ + \langle \phi_n^-, \phi_{n'}^-\rangle_{L^2_{\mathbb{C}}(\mathbb{C})} e_- = 0$$

for $n \neq n'$, and therefore $(\phi_n)_n$ is orthogonal in $L^2_{\mathbb{BC}}(\mathbb{C})$. Thus

$$\langle \phi_m, \phi_{m'}\rangle_{L^2_{\mathbb{BC}}(\mathbb{C})} \langle \phi_n, \phi_{n'}\rangle_{L^2_{\mathbb{BC}}(\mathbb{C})} = 0;\ \text{for } (m,n) \neq (m',n').$$

Subsequently, the family $(\mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(\phi_m,\phi_n))_n$ is orthogonal in $L^2_{\mathbb{BC}}(\mathbb{BC})$ by means of (4.23). Moreover, the corresponding bicomplex norm is given by

$$\begin{aligned} \|\phi_{m,n}\|^2_{L^2_{\mathbb{BC}}(\mathbb{BC})} &= \left| \left\langle \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(\phi_m,\phi_n), \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(\phi_m,\phi_n) \right\rangle_{L^2_{\mathbb{BC}}(\mathbb{BC})} \right| \\ &= \left| \langle \phi_m,\phi_m\rangle_{L^2_{\mathbb{BC}}(\mathbb{C})} \langle \phi_n,\phi_n\rangle_{L^2_{\mathbb{BC}}(\mathbb{C})} \right| \\ &= \left| \langle \phi^+_m,\phi^+_n\rangle_{L^2_{\mathbb{C}}(\mathbb{C})} \langle \phi^+_m,\phi^+_n\rangle_{L^2_{\mathbb{C}}(\mathbb{C})} e_+ + \langle \phi^-_m,\phi^-_n\rangle_{L^2_{\mathbb{C}}(\mathbb{C})} \langle \phi^-_m,\phi^-_n\rangle_{L^2_{\mathbb{C}}(\mathbb{C})} \right| e_- \\ &= \frac{1}{2}\left(\|\phi^+_m\|^2_{L^2_{\mathbb{C}}(\mathbb{C})} \|\phi^+_n\|^2_{L^2_{\mathbb{C}}(\mathbb{C})} + \|\phi^-_m\|^2_{L^2_{\mathbb{C}}(\mathbb{C})} \|\phi^-_n\|^2_{L^2_{\mathbb{C}}(\mathbb{C})} \right). \end{aligned}$$

so that $\|\phi_{m,n}\|^2_{L^2_{\mathbb{BC}}(\mathbb{BC})} = 1$ for $(\phi^+_n)_n$ and $(\phi^-_n)_n$ being orthonormal in $L^2_{\mathbb{C}}(\mathbb{C})$. The fact that $(\phi_{m,n})_{m,n}$ is a basis of $L^2_{\mathbb{BC}}(\mathbb{BC}) = L^2_{\mathbb{C}}(\mathbb{C}^2)e_+ + L^2_{\mathbb{C}}(\mathbb{C}^?2)e_-$ follows easily since this is equivalent to $(\mathcal{V}^{\sigma,\nu}_{\mathbb{R}^2,\mathbb{C}^2}(\phi^+_m,\phi^+_n))_{m,n}$ and $(\mathcal{V}^{\sigma,\mu}_{\mathbb{R}^2,\mathbb{C}^2}(\phi^-_m,\phi^-_n))_{m,n}$ be bases of $L^2_{\mathbb{C}}(\mathbb{C}^2)$ in view of the idempotent decomposition (4.22). This holds true since $(\phi^+_n)_n$ and $(\phi^-_n)_n$ are bases of $L^2_{\mathbb{C}}(\mathbb{C})$ and $\mathcal{V}^{\sigma,\tau}_{\mathbb{R}^2,\mathbb{C}^2}$ is the standard Fourier–Wigner transform. This completes the proof. ■

Corollary 4.3.5. *The functions*

$$h^{\sigma}_{m,n,m',n'} := \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(h^{\sigma}_{m,n}, h^{\sigma}_{m',n'})$$

for varying $m,n,m',n' = 0,1,2,\cdots$, and any fixed $\sigma > 0$, form an orthogonal basis of $L^2_{\mathbb{BC}}(\mathbb{BC})$.

Proof. This is an immediate consequence of Proposition 4.3.4 since the univariate complex Hermite functions $h^{\sigma}_{m,n}(\xi,\overline{\xi}) = h^{\sigma}_{m,n}(\xi,\overline{\xi})e_+ + h^{\sigma}_{m,n}(\xi,\overline{\xi})e_-$ is an orthogonal basis of $L^2_{\mathbb{BC}}(\mathbb{C})$. ■

Remark 4.3.6. *The polynomials associated to $h^{\sigma}_{m,n,m',n'}$ form a new class of univariate bicomplex polynomials of Hermite type which are not a tensor product of four one–dimensional copies of the classical Hermite functions h^{σ}_n, nor a tensor product of two copies of the complex Hermite functions $h^{\sigma}_{m,n}$. They can be considered as the bicomplex version of the bivariate polyanalytic Hermite polynomials recently introduced and studied in [45].*

Remark 4.3.7. *For the special window function $\psi_0(U) := h^{\sigma}_0(u)h^{\sigma}_0(v) = h^{\sigma}_{0,0}(U_\tau)$ with $U_\tau = u + \tau v$ and $U^2 = u^2 + v^2$ for $U = (u,v)$, the transfrom $\varphi \longmapsto \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(\varphi,\psi_0)$ on $L^2_{\mathbb{BC}}(\mathbb{R}^2)$ is closely connected to the bidimensional Segal–Bargmann transform. Indeed, we have*

$$\begin{aligned} \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(\varphi,\psi_0)(Z) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} e^{\frac{\sigma}{2}(\nu e_+ + \mu e_-)(2UY - XY) - \frac{\sigma}{2}(U-X)^2} \varphi(U)d\lambda(U) \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{\sigma}{4}\left(X^2+Y^2\right)} e^{\frac{\sigma}{4}[X+(\nu e_+ + \mu e_-)Y]^2} \int_{\mathbb{R}^2} e^{-\frac{\sigma}{2}(U-[X+(\nu e_+ + \mu e_-)Y])^2} \varphi(U)d\lambda(U) \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{\sigma}{4}|S_{\nu e_+ + \mu e_-}|^2} e^{\frac{\sigma}{4}(S_{\nu e_+ + \mu e_-})^2} \int_{\mathbb{R}^2} e^{-\frac{\sigma}{2}\left(U - S_{\nu e_+ + \mu e_-}\right)^2} \varphi(U)d\lambda(U), \end{aligned}$$

where $S_{\nu e_+ + \mu e_-} = (z,w) = X + (\nu e_+ + \mu e_-)Y \in \mathbb{C}^2_{\nu e_+ + \mu e_-}$ with $X = (x_1,x_2)$, $Y = (y_1,y_2)$, $Z = z_1 + jz_2 \in \mathbb{BC}$ and $z_\ell = x_\ell + iy_\ell$; $\ell = 1,2$.

Proposition 4.3.8. *we can choose the nth Hermite function*

$$h_n(t) = c_n e^{\pi t^2} \left(\frac{d}{dt}\right)^n \left(e^{-2\pi t^2}\right),$$

where c_n is chosen so that $\|h_n\|_2 = 1$, as a special window in 4.6 , and find an important and useful relation between Unidimensional bicomplex Fourier–Wigner transforms with Hermite functions and the bi- polyanalytic Bargmann transforms of general order n^*:

$$\left(B_{\mathbb{T}}^{n,\sigma,\nu} f\right) = e^{\frac{\nu}{2}[ZZ^* + i(x_1 y_1 - x_2 y_2) + j(x_1 x_2 - y_1 y_2)]} \mathcal{V}_{\mathbb{R},\mathbb{BC}}^{\sigma,\nu,\mu}(f, h_n)(Z)$$

where $Z = x_1 + iy_1 + jx_2 + ijy_2$

Proof. Let $f \in L_{\mathbb{T}}^{2,\sigma}(\mathbb{R})$, so f is of the form $f(x) = f^+(x)e^+ + f^-(x)e^-$ ∎

for all real x, where $f^{\mp} \in \mathbb{C}^{2,\sigma}(\mathbb{R})$. Moreover, we have

$$\begin{aligned}\left(B_{\mathbb{T}}^{n,a,\nu} f\right)(z) &= \left(B_{\mathbb{C}}^{n,\sigma,\frac{5}{3}} f^+\right)(\alpha)e^+ + \left(B_{\mathbb{C}}^{n,\sigma,\frac{r}{4}} f^-\right)(\beta)e^- \\ &= e^{i\pi u_1 u_2 + \pi\frac{|\alpha|^2}{2}} \mathcal{V}_{h_n}^{\sigma} f(\alpha)e^+ + e^{i\pi v_1 v_2 + \pi\frac{|\alpha|^2}{2}} \mathcal{V}_{h_n}^{\sigma} f(\beta)e^-\end{aligned}$$

where $\alpha = u_1 - u_2$, $\beta = v_1 - iv_2$ and $Z = \alpha e^+ + \beta e^-$, based on simplification and factorisation, we obtain

$$\begin{aligned}\left(B_{\mathbb{T}}^{n,a,\nu} f\right)(z) &= e^{\frac{\nu}{2}\left(ZZ^* + iu_1 v_1 e^+ + iu_2 v_2 e^-\right)} \left(\mathcal{V}^{\sigma}\left(f^+, h_n\right)(\alpha)\, e_+ + \mathcal{V}^{\sigma}\left(f^-, h_n\right)(\alpha)\, e_-\right) \\ &= e^{\frac{\nu}{2}[ZZ^* + i(x_1 y_1 - x_2 y_2) + j(x_1 x_2 - y_1 y_2)]} \mathcal{V}_{\mathbb{R},\mathbb{BC}}^{\sigma,\nu,\mu}(f, h_n)(Z)\end{aligned}$$

We have considered two bicomplex analogues of the classical (rescaled) Fourier–Wigner transform. This follows using the idempotent decomposition of bicomplex numbers. The standard phase (or time–frequency) space $\mathbb{R} \times \mathbb{R}$ is replaced here by the bicomplex $(\mathbb{R} \times \mathbb{R})e_+ + (\mathbb{R} \times \mathbb{R})e_-$. The concrete description of analytic properties of these transforms are obtained and gives rise to special generalization of the bicomplex Bargmann space studied in [46]. One of the advantages of this setting is to work simultaneously with two models of the polyanalytic Bargmann space $\mathcal{F}_n^{2,\sigma}(\mathbb{C}_\tau)$, the first one is focused on e_+ and the other on e_-. This is the case of the first transform and the obtained functional spaces are particular subclasses of the so–called $(1^-, 1^\dagger, n^*)$–$\mathbb{BC}$–polyanalytic functions of first kind. More generally, a bicomplex–valued function f on $\mathbb{BC}$ is said to be $(k^-, m^\dagger, n^*)$–$\mathbb{BC}$–polyanalytic if it satisfies the system of first order differential equations

$$\frac{\partial^{k+1} f}{\partial \overline{Z}^{k+1}} = \frac{\partial^{m+1} f}{\partial (Z^\dagger)^{m+1}} = \frac{\partial^{n+1} f}{\partial (Z^*)^{n+1}} = 0. \tag{4.24}$$

These spaces (and others) will be the subject of a forthcoming paper.

As signaled in Section 3, the range of the first transform is strictly contained in $L^2_{\mathbb{BC}}(\mathbb{BC})$. This is not the case for the second transform studied in Section 4. In fact, we obtain a Hilbertian orthogonal decomposition of $L^2_{\mathbb{BC}}(\mathbb{BC})$,

$$L^2_{\mathbb{BC}}(\mathbb{BC}) := \bigoplus_{m,n=0}^{+\infty} \mathcal{M}^{2,\sigma}_{m,n}(\mathbb{BC}^e_{\nu,\mu}),$$

in terms of the ranges $\mathcal{M}^{2,\sigma}_{m,n}(\mathbb{BC}^e_{\nu,\mu}) := \mathcal{S}^{\sigma,\nu,\mu}_{m,n}(L^2_{\mathbb{BC}}(\mathbb{R}^2))$ of $L^2_{\mathbb{BC}}(\mathbb{R}^2)$ by the transforms $\mathcal{S}^{\sigma,\nu,\mu}_{m,n} = \mathcal{V}^{\sigma,\nu,\mu}_{\mathbb{R}^2,\mathbb{BC}}(\cdot, h^{\sigma}_{m,n})$ (this is contained in Corollary 4.3.5). It is of interest to provide a concrete description of the functions in Corollary 4.3.5. This will be treated in some detail in a forthcoming paper from a different point of view.

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Résume

Dans cette thèse, on a introduit l'espace de Bargmann bicomplex, et on a également discuté ses propriétés de base, y compris ses différentes décompositions idempotentes et l'existence du noyau reproduisant. Puis on a considéré et discuté certaines propriétés de la transformée de Fourier fractionnaire et de la transformée de Segal-Bargmann ainsi que leur inverse et la relation entre eux. Dans la deuxième partie, nous avons introduit deux types de transformées de Fourier-Wiegner que l'on peut voir comme des généralisations des transformées qu'on a définie auparavant. Et on a prouvé que leurs images sont une généralisation de l'espace de Bargmann bicomplexe. Il s'agit de l'espace des fonctions polyanalytiques bicomplexes. On a introduit également une nouvelle classe de polynômes orthogonaux de quatre indices de variable bicomplexe.

Mots-clefs: Transformées intégrales ; Bicomplexe ; Fonctions génératrices; Fonctions polyanalytique; Transformées de Segal-Bargmann ; Espaces de Hilbert bicomplexe , Transformée de Fourier-Weigner bicomplex , Transformée de Fourier fractionnelle bicomplex , Espace de Bargmann bicomplex.

Abstract

In this thesis, we introduced the bicomplex Segal Bargmann space, we have also shown and discussed its basic properties, including its different idempotent decompositions and the existence of the reproducing kernel, from this last one. As we consider and discuss some properties of the Fourier fractional transform and the Segal Bargmann Transform as well their reverse and the relationship between them. In a second article, we have introduced two types of Weiner Fourier transform that we can see as generalizations of transforms that we defended before. In addition, we realized their ranges as a generalization of the bicomplex Segal Bargmann space; it is about the space of bicomplex polyanalytic functions. We also provide a new class of four indices bivariate complex orthogonal polynomials of Hermit type.

Key Words: Integral transforms; Bicomplex; Generating functions; Polyanalytical functions; Segal-Bargmann transforms; Bicomplex Hilbert spaces , Bicomplex Fourier-Weigner transform , Bicomplex fractional Fourier transform , Bicomplex Bargmann space.