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Problèmes elliptiques et paraboliques fractionnaires avec termes singuliers

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Contents

Abstract

Lately, problems involving nonlocal operators have received considerable attention not only in the field of pure mathematical analysis but also in the real world of applications, as in thin obstacle problems, crystal dislocation, phase transition, optimization, finance, stratified materials, anomalous diffusion, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves.

In this thesis we are interested to study some non-local elliptic and parabolic problems.

A prototype of nonlocal operators is the fractional Laplacian operator, which can in fact be seen as the infinitesimal generator of the stable Lévy process. This is one of the main motivations behind the study of problems involving nonlocal operators.

Our main objective in this thesis is to extend some well-known results for the local case, to our non-local case. More precisely, we carry out an investigation of the existence and regularity of solutions to nonlocal problems elliptic and parabolic of fractional Laplacian type with a singular nonlinearity, which specifically are singular with respect to the unknown function. The content of the thesis is as follows. Chapter 1 is devoted to a functional background necessary to carry analysis of fractional Sobolev spaces. We also give some technical results necessary for the accomplishment of the work. In Chapter 2, we establish some existence and regularity results of solutions for a class of nonlocal equations involving the fractional Laplacian operator with singular nonlinearity and Radon measure data. In Chapter 3, We study a Lazer-Mckenna-type problem involving the fractional Laplacian and singular nonlinearity. We investigate the existence, regularity and uniqueness of solutions in light of the interplay between the nonlinearities and the summability of the datum.

Finally in Chapter 4, we study the existence of solutions for a parabolic problem involving the fractional Laplacian with singular nonlinearity.

Keywords :

Fractional Sobolev spaces; Fractional Laplacian; Nonlocal problems; Singular terms; Radon measures; Weak solutions; Energy solutions.

Résumé

Dernièrement, les problèmes impliquant des opérateurs non locaux ont reçu une attention considérable non seulement dans le domaine de l'analyse mathématique pure mais aussi dans le monde réel des applications, comme dans les problèmes d'obstacles minces, dislocation cristalline, transition de phase, optimisation, finance, matériaux stratifiés, diffusion anormale, flux quasi-géostrophiques, diffusion multiple, surfaces minimales, science des matériaux et vagues d'eau. Dans cette thèse, nous nous intéressons à l'étude de certains problèmes elliptiques et paraboliques non locaux.

Un prototype d'opérateurs non locaux est l'opérateur Laplacien fractionnaire, qui peut en fait être vu comme le générateur infinitésimal du processus de Lévy stable. C'est l'une des principales motivations de l'étude des problèmes impliquant des opérateurs non locaux.

Notre objectif principal dans cette thèse est d'étendre certains résultats bien connus pour le cas local, à notre cas non local. Plus précisément, nous menons une étude sur l'existence et la régularité de solutions à des problèmes non locaux elliptiques et paraboliques de type Laplacien fractionnaire avec une non-linéarité singulière, qui sont spécifiquement singuliers par rapport à la fonction inconnue. Le contenu de la thèse est le suivant. Le chapitre 1 est consacré à un arrière-plan fonctionnel nécessaire pour effectuer l'analyse des espaces fractionnaires de Sobolev. Nous donnons également quelques résultats techniques nécessaires à l'accomplissement des travaux. Dans le Chapitre 2, nous établissons des résultats d'existence et de régularité de solutions pour une classe d'équations non locales impliquant l'opérateur fractionnaire Laplacien à non-linéarité singulière et des données de mesure de Radon. Dans le Chapitre 3, nous étudions un problème de type Lazer-Mckenna impliquant le Laplacien fractionnaire et la non-linéarité singulière. Nous étudions l'existence, la régularité et l'unicité des solutions à la lumière de l'interaction entre les non-linéarités et la sommabilité de la donnée. Enfin au Chapitre 4, nous étudions l'existence de solutions pour un problème parabolique impliquant le Laplacien fractionnaire à non-linéarité singulière.

Mots clés :

Espace de Sobolev fractionnaires, Laplacien fractionnaire, Problèmes non locaux, Termes singuliers, Mesures de Radon, Solutions faibles, Solutions d'énergie.

General notations

Notation	Definition
Ω	: Open set of $\mathbb{R}^N,\ N \geq 1$,
dx	: Surface measurement on Ω
$x = (x_1, x_2, \cdots, x_N)$	: Generic point of $\mathbb{R}^N$
$a.e.$	: Almost everywhere
$\partial\Omega$	: Boundary of Ω,
$\mathcal{C}\Omega$	: Complement of the set Ω in $\mathbb{R}^N$,
$\omega \subset\subset \Omega$	: ω strongly included in $\Omega,$ i.e., $\overline{\omega}$ is compact and $\overline{\omega} \subset \Omega$,
$meas(E) = \|E\|$	: Lebesgue measure of E $\subset \mathbb{R}^N$,
$supp(f)$	: Support of a function f,
f^+	$:= \max(f, 0)$,
f^-	$:= -\min(f, 0)$,
T_k	: $T_k(s) = \max(-k, \min(s, k))$ Truncation function of level k,
p'	: The Hölder conjugate exponent of $p, p' = \dfrac{p}{p-1}$
p_s^*	: The fractional Sobolev critical exponent , $p_s^* = \dfrac{Np}{Np - ps}$
B_R	: An open ball of radius R centered at the origine,
∇u	: Gradient of the function u,
Δu	: Laplacian of u,
$\Delta_p u$	: p-Laplacian of u,
$\\| \cdot \\|_X$	: Norme in the space X,
X'	: Dual space of X,
$\rightharpoonup$	: Weak convergence,
$\mathcal{C}^\infty(\Omega)$	: Space of infinitely continuously differentiable functions on Ω,
$\mathcal{C}_0^\infty(\Omega)$	$:= \Big\{ f : \mathbb{R}^N \to \mathbb{R} / f \in \mathcal{C}^\infty(\mathbb{R}^N),\ supp(f) \text{ is compact and } supp(f) \subset \Omega \Big\}$,
$\mathcal{C}^{0,\beta}(\Omega)$	: Space of Hölder continuous fonctions on Ω,
$L^p(\Omega)$	$:= \{u : \Omega \to \mathbb{R},\ u \text{ is measurable and } \int_\Omega \|u\|^p < \infty\}$
$L^\infty(\Omega)$	$:= \{u : \Omega \to \mathbb{R},\ u \text{ is measurable and } \exists C > 0;\ \|u(x)\| \leq C\ a.e. \text{ in } \Omega, \}$

General notations

Notation		Definition
$(-\Delta)^s u$	:	Fractional Laplacian of u,
$(-\Delta)_p^s u$	:	Fractional p-Laplacian of u,
$W^{1,p}(\Omega), W_0^{1,p}(\Omega), H_0^1(\Omega), H_{loc}^1(\Omega)$	:	Sobolev spaces
$W^{s,p}(\Omega), W_0^{s,p}(\Omega), H_0^s(\Omega), X_0^s(\Omega)$	:	Fractional Sobolev spaces
$W_{loc}^{s,p}(\Omega), H_{loc}^s(\Omega)$	:	Local fractional Sobolev spaces

General introduction

This work is devoted to study, develop and deeply understand a set of problems, tools, or simply questions framed in the huge field of Partial Differential Equations (PDEs). In particular, we focus on the study of a class of elliptic and parabolic problems involving the fractional Laplacian operator with a singular nonlinearity, which specifically are singular with respect to the unknown function u, of the form $u^{-\gamma}$, $\gamma > 0$, therefore tends to infinity at the edge of the domain Ω. This singularity makes that the problems tackled present a certain number of difficulties, linked to the lack of regularity and therefore of the compactness of the solutions. Although these difficulties, the study of these problems have been an increasing interest. On one hand, the interest in such equations is motivated by their applications in the mathematical modeling of various real world processes, such as the thermo-conductivity [51], the boundary layer phenomena [32] and the theory non-Newtonian pseudoplastic fluids [64]. On the other hand, their theoretical study is very interesting from a purely mathematical point of view. This study was initiated in the pioneering works [38, 84] which constitutes the starting point of a wide literature about singular semilinear elliptic equations.

The classical Heat equation seems to describe in a satisfactory manner a wide variety of diffusive problems in Physics. However, the anomalous diffusion that follows non-Brownian scaling is leading to models governed by fractional Laplacian.

In the last few years, elliptic and parabolic equations involving nonlocal operators has attracted substantial attention. The interest brought to such equations is due to the emergence of this type of nonlocal operators in a wide range of phenomena – the crystal dislocation, thin obstacle problems, Physics, phase transitions, finance, stochastic control, quasi-geostrophic flows, materials science, water waves, anomalous diffusion to name a few (see e.g. [29, 30, 31, 39, 47, 75, 79, 80, 81, 83] and references therein). We also recall that the fractional Laplacian operator $(-\Delta)^s$ can be viewed as the infinitesimal generator of stable Lévy processes, see e.g. [13, 28, 86]. For an expository on fractional Laplacian, we refer the reader to [21, 28, 45] and the references therein.

For all these reasons, we are interested in studying some non-local elliptic and parabolic problems with term singular. The natural question concerns what changes between the local versions ($s = 1$) of the equations and their non-local equivalents. For a number of properties we find similar results described by $0 < s < 1$ but with some interesting variations. These variations are justified by the nonlocal aspect of the operator involved. We point out that a non-local operator is such that the value of the image of a function at a given point depends on other points rather not just a neighbourhood of the chosen point. In other words, if L^s is a nonlocal operator, being $u : \mathbb{R}^N \to \mathbb{R}$ a function, and fixing $x \in \mathbb{R}^N$, then the value of $L^s u(x)$ depends on the value of $u(y)$ in other points outside a neighbourhood of x. Contrary the more typical local operator, where the value of the image of a function at a certain point depends only on the value of the function close to this point.

Notice also that to have a well-defined Dirichlet problem in a non-local framework, it is not enough to prescribe the boundary condition at $\partial\Omega$. This is nothing but another consequence of the nonlocal nature of the operator, since in order to compute the value of $(-\Delta)^s u$ at any point in Ω, we need to know the value of u in the whole $\mathbb{R}^N$. In other words, the Dirichlet datum is given in $\mathbb{R}^N \backslash \Omega$ and not simply on $\partial\Omega$.

Our main objective in this thesis is to extend some well-known results for the local case to the non-local problems. More precisely, we carry out investigations on the existence and regularity of solutions to nonlocal problems of elliptic and parabolic type involving the fractional Laplacian operator with singular non-linearities, including specifically a singularity with respect to the unknown function.

Finally, we briefly summarize the organization of this thesis and the main results contained in every chapter. The thesis is conformed by four chapters.

In **Chapter 1**, we give some basic notations and necessary results that we will use in the accomplishment of the work. We start by presenting the fractional Sobolev spaces, the fractional Laplacian operator and their properties. Taking into account the algebraic character of our operator, we also present some algebraic inequalities that we will use regularly throughout this thesis.

Chapter 2 is devoted to investigate the existence and the regularity of solutions for a class of nonlocal equations involving the fractional Laplacian operator with singular nonlinearity and Radon measure data. In particular, we will study the following Dirichlet problem

$$\begin{cases} (-\Delta)^s u &= \dfrac{f(x)}{u^\gamma} + \mu \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \mathrm{R}^N \backslash \Omega, \end{cases} \tag{1}$$

where Ω is an open bounded subset in R^N, $N > 2s$, of class $\mathcal{C}^{0,1}$, $s \in (0,1)$, $\gamma > 0$, f is a non-negative function on Ω, μ is a non-negative bounded Radon measure on Ω and $(-\Delta)^s$ is the fractional Laplacian operator of order $2s$.
Our main objective is to extend the results in local case in [66] to our nonlocal case.

Our purpose in this chapter is to consider the problem (1) in the nonlocal framework and to prove the existence results of solutions to problem (1) with μ a bounded Radon measure and data $f \in L^1(\Omega)$. We use an approximation method that consists in analyzing the sequence of approximated problems truncating the datum f and the singular term $\dfrac{1}{u^\gamma}$ and approximating μ by smooth functions obtaining non singular problems with L^∞-data whose approximated solutions u_n can be obtained by a direct application of the Schauder fixed point theorem. We faced many difficulties in dealing with the nonlocal problem (1), but the main one is how to get estimations in appropriate fractional Sobolev spaces. The results contained in this chapter can be found in [89].

In **Chapter 3**, we consider the Lazer-Mckenna problem involving the fractional Laplacian and a singular nonlinearity. We investigate the existence, regularity and uniqueness of solutions in light of the interplay between the nonlinearities and the summability of the datum. More precisely, we will study the following nonlocal problem

$$\begin{cases} (-\Delta)^s u &= \dfrac{f(x)}{u^\gamma} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N > 2s$, of class $\mathcal{C}^{1,1}$, $s \in (0,1)$, $\gamma > 0$, $f \in L^m(\Omega)$, $m \geq 1$, is a non-negative function and $(-\Delta)^s$ is the fractional Laplacian operator. Our main goal in this chapter is to lead the investigations on the existence and regularity of positive solutions to (2) and establishing some missing results in [18, 34]. The case where $\gamma = 1$ is treated in [18, 34]. We study the case where $0 < \gamma < 1$ and $f \in L^m(\Omega)$ with $1 \leq m < \overline{m}$ which provides infinite energy solutions (see Theorem 3.2.1 bellow) and we prove the existence of finite energy solutions to problem (2) in the case $\gamma > 1$ under some suitable assumptions on the datum f. Further, to show the accuracy of our results we highlight the relationship with the Lazer-Mckenna condition. We also provide some regularity results for solutions as well as the uniqueness of finite energy solutions. At the end, we give an appendix which contains two auxiliary results necessary to the accomplishment of the work. Note that we obtain some results that extend to non-local problems those obtained in the local case in [14]. As regards non-local problems, some of our results with a more general data extend also those obtained in [10, 18].

This thesis ends with **Chapter 4** in which we analyze the existence of solutions for the following parabolic problem involving the fractional Laplacian with singular nonlinearity

$$\begin{cases} u_t + (-\Delta)^s u &= \dfrac{f(x,t)}{u^\gamma} \quad \text{in } \Omega_T := \Omega \times (0,T), \\ u &= 0 \quad \text{in } (\mathbb{R}^N \backslash \Omega) \times (0,T), \\ u(\cdot, 0) &= u_0(\cdot) \quad \text{in } \Omega, \end{cases} \tag{3}$$

where Ω is a bounded domain of class $\mathcal{C}^{0,1}$ in $\mathbb{R}^N$, $N > 2s$ with $s \in (0,1)$, $\gamma > 0$, $0 < T < +\infty$, $f \geq 0$, $f \in L^m(\Omega_T)$, $m \geq 1$, is a non-negative function on Ω_T and $u_0 \in L^\infty(\Omega)$ is a non-negative function on Ω which furthermore locally satisfies a positivity condition on Ω. The existence and regularity of the solutions of (3) are obtained under different assumptions on the summability of f and on γ. One of the main difficulties which arises in this problem is the proof of the positivity of the solutions inside the parabolic cylinder to make sense of the weak formulation of the solutions of the problem. In the proof of this property we only use the weak comparison principle. The content of this chapter is an extension to non-local problems of the results proved in [40] for the local case.

- Chapter 2 is the development of the published article [89] :

A. Youssfi and G. Ould Mohamed Mahmoud. On singular equations involving fractional Laplacian. Acta Math. Sci., 40B(5):1289-1315, 2020.

• Chapter 3 is the development of the published article [91] :

A. Youssfi and G. Ould Mohamed Mahmoud. Nonlocal semilinear elliptic problems with singular nonlinearity. *Calc. Var. Partial Differential Equations*, 60(153), 2021.

• Chapter 4 is the development of the article [90] :

A. Youssfi and G. Ould Mohamed Mahmoud. Fractional heat equation with singular terms. Submitted in 2021.

Chapter 1

Some preliminary tools and basic results

In this chapter, we present the functional setting and some auxiliary results that will play an important role throughout this thesis. We start by recalling the definition of fractional Sobolev spaces.

1.1 The fractional Sobolev spaces

In this section we provide some basic facts about fractional Sobolev spaces. We refer to [9, 45, 44, 21, 72] for more details. Let Ω be an open subset in R^N and let $\mathcal{C}\Omega := \mathrm{R}^N \backslash \Omega$. For any $0 < s < 1$ and for any $1 \leq q < +\infty$, the fractional Sobolev space $W^{s,q}(\Omega)$ is defined as the set of all functions (equivalence class) u in $L^q(\Omega)$ such that

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^q}{|x - y|^{N+qs}} dxdy < \infty.$$

$W^{s,q}(\Omega)$, also known as *Aronszajn*, *Gagliardo* or *Slobodeckij* spaces, is a Banach space when equipped with the natural norm

$$\|u\|_{W^{s,q}(\Omega)} = \left(\|u\|^q_{L^q(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^q}{|x - y|^{N+qs}} dxdy \right)^{\frac{1}{q}}. \tag{1.1}$$

It can be regarded as an intermediate space between $L^q(\Omega)$ and $W^{1,q}(\Omega)$. Recall that the space $W^{s,q}(\Omega)$ is reflexive for all $q > 1$ (see [58, Theorem 6.8.4]). We point out that if $0 < s \leq s' < 1$ then $W^{s',q}(\Omega)$ is continuously embedded in $W^{s,q}(\Omega)$ (see [45, Proposition 2.1]). Let us define

$W_0^{s,q}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{s,q}(\Omega)$ with respect to the norm defined in (1.1) where

$$\mathcal{C}_0^\infty(\Omega) = \Big\{ f : \mathbb{R}^N \to \mathbb{R} / f \in \mathcal{C}^\infty(\mathbb{R}^N), supp(f) \text{ is compact and } supp(f) \subset \Omega \Big\}.$$

Here and in the sequel $supp(f)$ stands for the support of the function f. $W_0^{s,q}(\Omega)$ is a Banach space under the norm $\|u\|_{W^{s,q}(\Omega)}$.
If Ω is bounded and is of class $\mathcal{C}^{0,1}$, we can give a fractional version of the Poincaré inequality in $W_0^{s,q}(\Omega)$, $1 \leq q < +\infty$, whose proof in the case where $q = 2$ can be found in [8]. For the convenience of the reader, we are giving the proof here.

Lemma 1.1.1. *(Fractional Poincaré-type inequality) Let Ω be a bounded open subset of $\mathbb{R}^N$ of class $\mathcal{C}^{0,1}$, $1 \leq q < +\infty$ and let $0 < s < 1$. Then there exists a constant $C(N, s, \Omega)$ such that for any $\varphi \in W_0^{s,q}(\Omega)$ one has*

$$\|\varphi\|^q_{L^q(\Omega)} \leq C(N, s, \Omega) \int_\Omega \int_\Omega \frac{|\varphi(x) - \varphi(y)|^q}{|x - y|^{N+qs}} dxdy.$$

Proof. Let $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Observe first that the above inequality holds if $\varphi = 0$. Assume that $\varphi \neq 0$ and set

$$\lambda(\Omega) = \inf_{\{\varphi \in \mathcal{C}_0^\infty(\Omega), \varphi \neq 0\}} \frac{\int_\Omega \int_\Omega \frac{|\varphi(x)-\varphi(y)|^q}{|x-y|^{N+qs}} dxdy}{\int_\Omega |\varphi(x)|^q dx}.$$

We shall prove that $\lambda(\Omega) > 0$. To do so, we argue by contradiction assuming that $\lambda(\Omega) = 0$. Thus, there exists a sequence $\{\varphi_n\}$ of $\mathcal{C}_0^\infty(\Omega)$ such that

$$\int_\Omega |\varphi_n(x)|^q dx = 1 \text{ and } \int_\Omega \int_\Omega \frac{|\varphi_n(x) - \varphi_n(y)|^q}{|x - y|^{N+qs}} dxdy \to 0 \text{ as } n \to \infty.$$

It follows that

$$\|\varphi_n\|_{W^{s,q}(\Omega)} \leq C.$$

By virtue of [45, Corollary 7.2], there exists a function f and a subsequence of $\{\varphi_n\}$, still indexed by n, such that

$$\varphi_n \to f \text{ in norm in } L^q(\Omega),$$
$$\varphi_n \to f \text{ a.e. in } \Omega.$$

Therefore,

$$\int_\Omega |f(x)|^q dx = 1 \text{ and } \frac{|\varphi_n(x) - \varphi_n(y)|^q}{|x - y|^{N+qs}} \to \frac{|f(x) - f(y)|^q}{|x - y|^{N+qs}} \text{ a.e in } \Omega \times \Omega.$$

Applying Fatou's lemma, we get

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^{N+qs}}dxdy \le \liminf_{n\to\infty}\int_\Omega\int_\Omega \frac{|\varphi_n(x)-\varphi_n(y)|^q}{|x-y|^{N+qs}}dxdy \to 0$$

and thus

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^{N+qs}}dxdy = 0. \tag{1.2}$$

Thus, we have $f \in W^{s,q}(\Omega)$. On the other hand, in view of (1.2) we can write

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|(\varphi_n(x)-f(x))-(\varphi_n(y)-f(y))|^q}{|x-y|^{N+qs}}dxdy\\
&\le 2^{q-1}\int_\Omega\int_\Omega \frac{|\varphi_n(x)-\varphi_n(y)|^q}{|x-y|^{N+qs}}dxdy\\
&+2^{q-1}\int_\Omega\int_\Omega \frac{|f(y)-f(x)|^q}{|x-y|^{N+qs}}dxdy\\
&= 2^{q-1}\int_\Omega\int_\Omega \frac{|\varphi_n(x)-\varphi_n(y)|^q}{|x-y|^{N+qs}}dxdy \to 0.
\end{aligned}$$

Hence, $\varphi_n \to f$ in $W^{s,q}(\Omega)$ and so $f \in W_0^{s,q}(\Omega)$. By (1.2), the function f has a constant value on Ω. The only possible value is $f \equiv 0$ which yields a contradiction with the fact that

$$\int_\Omega |f(x)|^q dx = 1.$$

So, we get

$$\|\varphi\|^q_{L^q(\Omega)} \le C(N,s,\Omega)\int_\Omega\int_\Omega \frac{|\varphi(x)-\varphi(y)|^q}{|x-y|^{N+qs}}dxdy,\ \forall\varphi\in\mathcal{C}_0^\infty(\Omega). \tag{1.3}$$

Now, for every $\varphi \in W_0^{s,q}(\Omega)$, there exists a sequence $\{\varphi_n\}$ of $\mathcal{C}_0^\infty(\Omega)$ functions such that

$$\varphi_n \to \varphi \text{ in norm in } W^{s,q}(\Omega).$$

Applying the inequality (1.3) for φ_n and passing to the limit, we conclude the result. □

Under the same assumptions of Lemma 1.1.1, the Banach space $W_0^{s,q}(\Omega)$ can be endowed with the norm

$$\|u\|_{W_0^{s,q}(\Omega)} = \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^q}{|x-y|^{N+qs}}dxdy\right)^{\frac{1}{q}}$$

which is equivalent to $\|u\|_{W^{s,q}(\Omega)}$. In the case where $q=2$, we note $W^{s,2}(\Omega) = H^s(\Omega)$ and $W_0^{s,2}(\Omega) = H_0^s(\Omega)$. Endowed with the inner product

$$\langle u,v\rangle_{H_0^s(\Omega)} = \int_\Omega\int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy.$$

$(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)})$ is a Hilbert space. Now, we define the following spaces

$$H^s_{loc}(\Omega) = \Big\{ u : \Omega \to \mathbb{R} : u \in L^2(K),\ \int_K \int_K \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dydx < \infty, \text{ for every compact } K \subset \Omega \Big\}$$

and

$$X_0^s(\Omega) = \Big\{ f \in H^s(\mathbb{R}^N) / f = 0 \text{ a.e. in } \mathcal{C}\Omega \Big\},$$

where from now on $\mathcal{C}\Omega := \mathbb{R}^N \setminus \Omega$ stands for the complementary of Ω in $\mathbb{R}^N$. Observe that if Ω has a continuous boundary, by [49, Theorem 6] (see also [53, Theorem 1.4.2.2]) we can infer that $X_0^s(\Omega) \subset H_0^s(\Omega)$. Indeed, if $f \in X_0^s(\Omega)$ then, by [49, Theorem 6] there exists a sequence $\{\rho_n\}_n$ that belongs to $\mathcal{C}_0^\infty(\Omega)$ satisfying

$$\|\rho_n - f\|_{H^s(\mathbb{R}^N)} \to 0 \text{ as } n \to +\infty$$

and in particular we obtain

$$\|\rho_n - f\|_{H^s(\Omega)} \to 0 \text{ as } n \to +\infty,$$

which yields $f \in H_0^s(\Omega)$. Under the same assumptions of Lemma 1.1.1, the following quantity

$$\|u\|_{X_0^s(\Omega)} = \Big(\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dydx \Big)^{\frac{1}{2}},$$

where $Q = \mathbb{R}^{2N} \backslash (\mathcal{C}\Omega \times \mathcal{C}\Omega)$, is a norm on $X_0^s(\Omega)$. It is well known that the pair $(X_0^s(\Omega), \|\cdot\|_{X_0^s(\Omega)})$ is a Hilbert space (see [76, Lemma 7]).

1.2 The fractional Laplacian operator

First, consider the Schwartz space $\mathcal{S}(\mathrm{R}^N)$ of rapidly decaying C^∞ functions in R^N, with the following semi-norm

$$\|\varphi\|_{\mathcal{S}(\mathrm{R}^N)} = \sup_{x \in \mathrm{R}^N} (1+|x|^k) \sum_{|\alpha| \le N} |D^\alpha \varphi(x)|, \qquad N = 1, 2, ...,$$

where $\varphi \in \mathcal{S}(\mathrm{R}^N)$. The fractional Laplacian operator $(-\Delta)^s$ of order $2s$, is defined as

$$(-\Delta)^s u = \alpha(N,s) P.V. \int_{\mathrm{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy,$$

where "P.V." stands for the integral in the principal value sense and $\alpha(N,s)$ is a positive renormalizing constant, depending only on N and s, given by

$$\alpha(N,s) = \frac{4^s\Gamma(\frac{N}{2}+s)}{\pi^{\frac{N}{2}}}\frac{s}{\Gamma(1-s)}$$

so that the identity

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u), \quad \xi \in \mathrm{R}^N, s \in (0,1) \text{ and } u \in \mathcal{S}(\mathrm{R}^N)$$

holds, where $\mathcal{F}u$ stands for the Fourier transform of u belonging to the Schwartz class $\mathcal{S}(\mathrm{R}^N)$ (cf. [60]). More details on the operator $(-\Delta)^s$ and the asymptotic behaviour of $\alpha(N,s)$ can be found in [45]. It is worth recalling that for any u and φ belonging to $H^s(\mathrm{R}^N)$, we have the following duality product

$$\int_{\mathrm{R}^N}(-\Delta)^s u\varphi dx = \frac{\alpha(N,s)}{2}\int_{\mathrm{R}^{2N}}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$

Thus, it can be seen that

$$(-\Delta)^s : H^s(\mathrm{R}^N) \to H^{-s}(\mathrm{R}^N)$$

is a continuous and symmetric operator defined on $H^s(\mathrm{R}^N)$.

In the particular case, if u and φ belong to $H^s(\mathrm{R}^N)$ with $u=\varphi=0$, on $\mathcal{C}\Omega$, we have

$$\int_{\mathrm{R}^N}(-\Delta)^s u\varphi dx = \frac{\alpha(N,s)}{2}\int_{Q}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx,$$

where $Q := \mathrm{R}^{2N}\backslash(\mathcal{C}\Omega\times\mathcal{C}\Omega)$.

For $N>2s$ we define the fractional Sobolev critical exponent $2^*_s = \dfrac{2N}{N-2s}$. The following result is a fractional version of the Sobolev inequality which provides a continuous embedding of $H^s_0(\Omega)$ in the critical Lebesgue space $L^{2^*_s}(\Omega)$. The proof can be found in [45, 72].

Theorem 1.2.1. *(Fractional Sobolev embedding)[45] Let $0<s<1$ be such that $N>2s$. Then, there exists a constant $S(N,s)$ depending only on N and s, such that for all $f\in\mathcal{C}^\infty_0(\mathrm{R}^N)$*

$$\|f\|^2_{L^{2^*_s}(\mathrm{R}^N)} \le S(N,s)\int_{R^N}\int_{R^N}\frac{|f(x)-f(y)|^2}{|x-y|^{N+2s}}dxdy.$$

Remark 1.2.1. *In particular, if Ω is an open bounded subset in R^N of class $\mathcal{C}^{0,1}$ with $N>2s$ and $0<s<1$ and $f\in\mathcal{C}^\infty_0(\Omega)$ we have*

$$\|f\|^2_{L^{2^*_s}(\Omega)} \le S(N,s,\Omega)\int_{\Omega}\int_{\Omega}\frac{|f(x)-f(y)|^2}{|x-y|^{N+2s}}dxdy.$$

Indeed, by [45, Theorem 5.4] we can write

$$\begin{aligned}\int_{R^N}\int_{R^N}\frac{|f(x)-f(y)|^2}{|x-y|^{N+2s}}dxdy &\leq \|f\|^2_{H^s(\mathrm{R}^N)} \leq C\|f\|^2_{H^s(\Omega)}\\ &= C\|f\|^2_{L^2(\Omega)} + C\int_\Omega\int_\Omega\frac{|f(x)-f(y)|^2}{|x-y|^{N+2s}}dxdy.\end{aligned}$$

The result follows then by Theorem 1.2.1 and Lemma 1.1.1.

1.3 Functional analysis and algebraic inequalities

In this section, we recall some well-known results can be founded for instance in [27, 50, 74, 92]. We start by present some estimates in the usual Marcinkiewicz space $\mathcal{M}^q(\Omega)$, $0 < q < \infty$, which consists of all measurable functions $u : \Omega \to \mathrm{R}$ such that there exists a constant $c = c(u) > 0$ satisfying

$$t^q meas(\{x : |u(x)| > t\}) \leq c,$$

for every $t > 0$. Here and in what follows, $meas(E)$ denotes the Lebesgue measure of a measurable subset E of Ω. It is worth recalling the following connection between Marcinkiewicz and Lebesgue spaces

$$L^q(\Omega) \hookrightarrow M^q(\Omega) \hookrightarrow L^{q-\varepsilon}(\Omega),$$

for every $1 < q < \infty$ and $0 < \varepsilon \leq q-1$ (see for instance [58]). We will also use the following truncation functions T_k and G_k, $k > 0$, defined for every $s \in \mathrm{R}$ by

$$T_k(s) = \max\{-k; \min\{k, s\}\} \text{ and } G_k(s) = s - T_k(s).$$

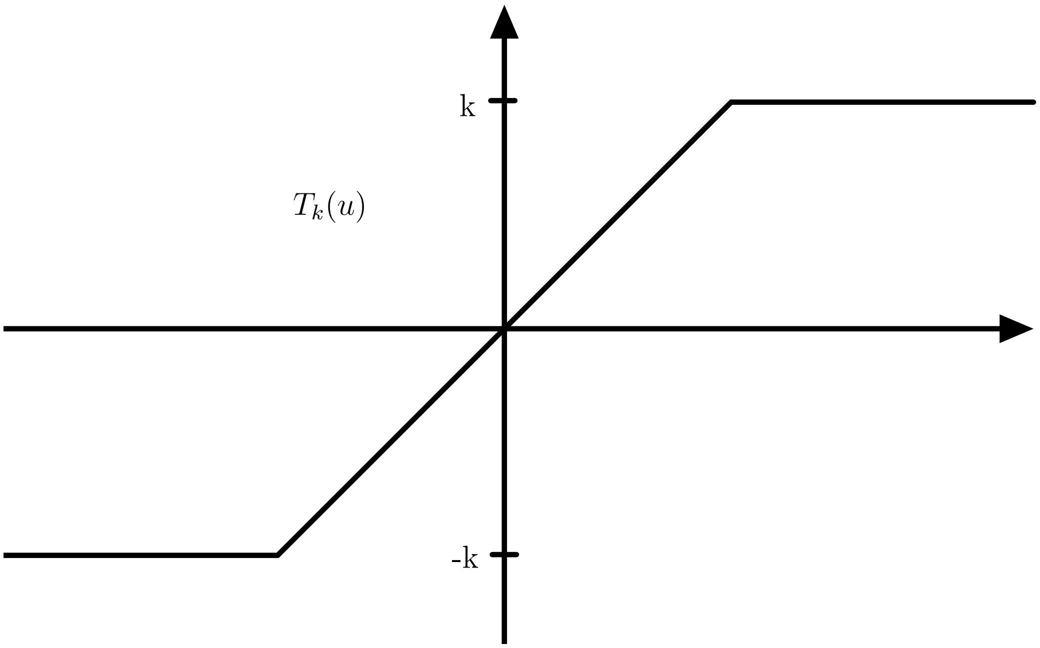

Figure 1.1: The truncation function $T_k(u)$

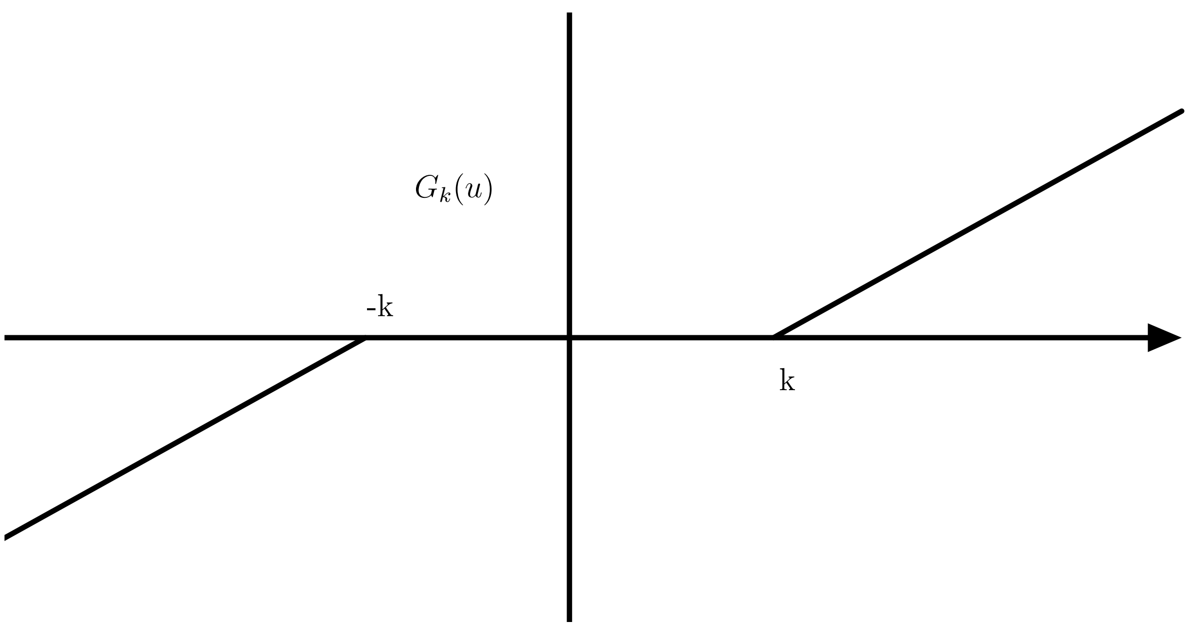

Figure 1.2: The function $G_k(u)$

We denote by $\mathcal{M}_b(\Omega)$ the space of all bounded Radon measures on Ω. The norm of a measure $\mu \in \mathcal{M}_b(\Omega)$ is given by $\|\mu\|_{\mathcal{M}_b(\Omega)} = \int_\Omega d|\mu|$.

Definition 1.3.1. *We say that the sequence of measurable functions $\{\mu_n\}$ is converging weakly* *to μ in the sense of the measures if*

$$\lim_{n\to\infty}\int_\Omega \varphi(x)\mu_n(x)dx = \int_\Omega \varphi d\mu, \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega).$$

Lemma 1.3.1. *(Dominated convergence theorem)[27, Theorem 4.2] Let $\{f_n\}$ be a sequence of functions in $L^1(\Omega)$ that satisfy*

- *$f_n(x) \to f(x)$, a.e. on Ω,*
- *there exists a function $g \in L^1(\Omega)$, such that for all n, $|f_n(x)| \leq |g(x)|$, a.e. on Ω.*

Then $f \in L^1(\Omega)$ and $\|f_n - f\|_{L^1(\Omega)} \to 0$.

Definition 1.3.2. *(Equi-integrability)[56, 4.12 Définition.] We say that $\{f_n\}$, a sequence of functions of $L^1(\Omega)$, is equi-integrable if :*

- *$\forall \varepsilon > 0$, there exist $A \subset \Omega$ of a finite Lebesque measure, such that for all $n \geq 1$*

$$\int_{\Omega\setminus A} |f_n| dx < \varepsilon$$

- *$\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall n \geq 1$ $\forall E \subset \Omega$, such that $|E| < \delta$, we have*

$$\int_E |f_n| dx < \varepsilon.$$

Theorem 1.3.1. *(Vitali's convergence theorem))[56, 92] Let $\{f_n\}$ be a sequence of functions of $L^1(\Omega)$ converges almost everywhere to a measurable function f. Then $\{f_n\}$ converges to f in $L^1(\Omega)$ if and only if $\{f_n\}$ is equi-integrable.*

Lemma 1.3.2. *(De La Vallé-Poussin)[70, Lemma 6.4] Let Ω be bounded. The sequence $\{f_n\}$ is sequentially weakly relatively compact in $L^1(\Omega)$ if and only if*

$$\sup_n \int_\Omega \Phi(|f_n|) dx < \infty,$$

for some continuous function $\Phi : [0, +\infty) \to \mathbb{R}$, with

$$\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty.$$

We also need the following technical algebraic inequalities that will play an important role throughout this thesis.

Lemma 1.3.3. *i)- Let $\alpha > 0$. For every x, $y \geq 0$ one has*

$$(x-y)(x^\alpha - y^\alpha) \geq \frac{4\alpha}{(\alpha+1)^2}(x^{\frac{\alpha+1}{2}} - y^{\frac{\alpha+1}{2}})^2.$$

ii)- Let $0 < \alpha \leq 1$. For every x, $y \geq 0$ with $x \neq y$ one has

$$\frac{x-y}{x^\alpha - y^\alpha} \leq \frac{1}{\alpha}(x^{1-\alpha} + y^{1-\alpha}).$$

iii)- Let $0 < \alpha \leq 1$, then for every x, $y \geq 0$ one has

$$|x^\alpha - y^\alpha| \leq |x-y|^\alpha.$$

iv)- Let $\alpha \geq 1$, then for every x, $y \geq 0$ one has

$$|x^\alpha - y^\alpha| \leq \alpha(x^{\alpha-1} + y^{\alpha-1})|x-y|.$$

v)- Let $\alpha \geq 1$, then for every x, $y \geq 0$ one has

$$|x+y|^{\alpha-1}|x-y| \leq C_\alpha |x^\alpha - y^\alpha|,$$

where C_α is a constant depending only on α.

Proof.

i)- It is proved in [7, Lemma 2.22].

ii)- If $x = 0$ or $y = 0$, the inequality trivially follows. Suppose now that

$x > y > 0$. We have

$$\exists\, \xi \in]y, x[\text{ such that } \quad x^\alpha - y^\alpha = (x-y)\alpha\xi^{\alpha-1}.$$

Then,

$$\frac{x-y}{x^\alpha - y^\alpha} = \frac{1}{\alpha}\xi^{1-\alpha} \leq \frac{1}{\alpha}(x^{1-\alpha} + y^{1-\alpha}).$$

By symmetry the desired result follows.

iii)- If $\alpha = 1$ the inequality is obvious. Assume $0 < \alpha < 1$. If $x = 0$ or $y = 0$ or $x = y$, the inequality is also trivial. Suppose that $x > y > 0$. Let us define the function $f(t) = (t-1)^\alpha - (t^\alpha - 1)$, for every $t > 1$. Observing that $f(t) > 0$, for every $t > 1$, we conclude the desired inequality by choosing $t = \dfrac{x}{y} > 1$.
By x/y symmetry we obtain the desired inequality.

iv)- If $\alpha = 1$ the inequality is obvious. Assume $\alpha > 1$. If $x = 0$ or $y = 0$ or $x = y$, the inequality is also trivial. Suppose that $x > y > 0$. We have

$$\exists\, \xi \in]y, x[\text{ such that } \quad x^\alpha - y^\alpha = \alpha \xi^{\alpha-1}(x-y).$$

Then,

$$|x^\alpha - y^\alpha| \leq \alpha(x^{\alpha-1} + y^{\alpha-1})|x-y|.$$

By symmetry the desired result follows.

v)- If $\alpha = 1$ the inequality is obvious. Assume $\alpha > 1$. If $x = 0$ or $y = 0$ or

$x = y$ the inequality is also trivial. Suppose that $x > y > 0$, we have

$$x^\alpha - y^\alpha \geq (x-y)\alpha y^{\alpha-1}.$$

Then

$$\alpha(x^{\alpha-1} + y^{\alpha-1}) \leq \frac{x^\alpha - y^\alpha}{x-y} + \alpha x^{\alpha-1} = \frac{x^\alpha - y^\alpha + \alpha x^\alpha - \alpha x^{\alpha-1} y}{x-y} \leq (\alpha+1)\frac{x^\alpha - y^\alpha}{x-y}.$$

It follows that

$$x^{\alpha-1} + y^{\alpha-1} \leq \frac{\alpha+1}{\alpha}\frac{x^\alpha - y^\alpha}{x-y}.$$

As $\alpha - 1 > 0$ we easily get $(x+y)^{\alpha-1} \leq (2^{\alpha-1}+1)(x^{\alpha-1} + y^{\alpha-1})$ which yield the inequality

$$|x+y|^{\alpha-1}|x-y| \leq (2^{\alpha-1}+1)\frac{\alpha+1}{\alpha}|x^\alpha - y^\alpha|.$$

By x/y symmetry we obtain the desired inequality.

Chapter 2

On singular equations involving fractional Laplacian[1]

In this chapter we study the existence and regularity of the solutions for a class of nonlocal equations involving the fractional Laplacian operator with singular nonlinearity and Radon measure data. We extend the results obtained in [66] to the same problem involving non-local fractional Laplacian operator.

2.1 Introduction and main results

Lately, problems involving nonlocal operators and singular terms have recently received considerable attention in the literature. A good amount of investigations have focused on the existence and/or regularity of solutions to such problems governed by the fractional Laplacian with a singularity due to a negative power of the unknown or described by a potential, see for instance, [6, 7, 10, 19, 18, 34] and related papers.
A prototype of nonlocal operators is the fractional Laplacian operator of the form $(-\Delta)^s$, $0 < s < 1$, which is actually the infinitesimal generator of the radially symmetric and s-stable Lévy processes [13]. Fractional Laplacian operators naturally arise from a wide range of applications. They appear, for instance, in thin obstacle problems [39], crystal dislocation [47],

[1] A. Youssfi and G. Ould Mohamed Mahmoud. On singular equations involving fractional Laplacian. Acta Math. Sci., 40B(5):1289-1315, 2020

phase transition [83] and others.

In this chapter, we are interested in the existence and regularity of solutions to the following Dirichlet problem

$$\begin{cases} (-\Delta)^s u &= \dfrac{f(x)}{u^\gamma} + \mu \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \mathrm{R}^N \backslash \Omega, \end{cases} \tag{2.1}$$

where Ω is an open bounded subset in R^N, $N > 2s$, of class $\mathcal{C}^{0,1}$, $s \in (0,1)$, $\gamma > 0$, f is a non-negative function on Ω, μ is a non-negative bounded Radon measure on Ω and $(-\Delta)^s$ is the fractional Laplacian operator of order $2s$ defined by

$$(-\Delta)^s u = \alpha(N,s) P.V. \int_{\mathrm{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,$$

where "P.V." stands for the integral in the principal value sense.

The case $s = 1$ corresponds to the classical Laplacian operator. If further $\mu = 0$, an important result is due to Lazer-McKenna [61]. Under regularity assumptions on Ω and f, the authors present an obstruction to the existence of an energy solution. In fact, such a solution lying in $H_0^1(\Omega)$ should exists if and only if $\gamma < 3$ while it is not in $\mathcal{C}^1(\overline{\Omega})$ if $\gamma > 1$. As far as problem with L^1-data are concerned, the threshold 3 essentially due to the boundedness of the datum was sharpened in [88] while in [23] the existence of a distributional solution u is proved. In fact, it is proved in [23] that if $\gamma < 1$ and $f \in L^m(\Omega)$, $1 \leq m < \left(\dfrac{2^*}{1-\gamma}\right)'$, then $u \in W_0^{1,q}(\Omega)$ where $q = \dfrac{Nm(\gamma+1)}{N - m(1-\gamma)}$ while $u \in H_0^1(\Omega)$ if $f \in L^m(\Omega)$ with $m = \left(\dfrac{2^*}{1-\gamma}\right)'$. In the case where $f \in L^1(\Omega)$, if $\gamma = 1$ then $u \in H_0^1(\Omega)$; while $u \in H_{loc}^1(\Omega)$ if $\gamma > 1$. We note that in the latter case, the boundary datum is only assumed in a weaker sense than the usual one of traces, that is $u^{\frac{\gamma+1}{2}} \in H_0^1(\Omega)$. Let us point out here that solutions with infinite energy may exist if $\gamma > 1$ even for smooth data ([61]).

The nonhomogeneous case (i.e. $\mu \neq 0$) has been considered. In [66] the authors studied the existence of weak solutions for the problem

$$-\Delta u = \frac{f(x)}{u^\gamma} + \mu. \tag{2.2}$$

where $f \in L^1(\Omega)$ and μ is a bounded Radon measure. They prove the existence of a weak solution u of the problem (2.2) such that $u \in W_0^{1,q}(\Omega)$ for every $q < \dfrac{N}{N-1}$ when $\gamma \leq 1$

while if $\gamma > 1$, $u \in W^{1,q}_{loc}(\Omega)$ for every $q < \dfrac{N}{N-1}$ with the regularity $\Big(T_k(u)\Big)^{\frac{\gamma+1}{2}} \in H^1_0(\Omega)$, T_k being the truncation function at levels $\pm k$. Other related singular equations can be found for instance in [67, 57, 43, 38, 84].

Regarding nonlocal problems, the study of (2.2) with $\mu = 0$ was extended in [18, 34] where the Laplacian is substituted by the fractional Laplacian $(-\Delta)^s_p$, $0 < s < 1$ and $p > 1$. The authors obtain some existence and regularity results for the solutions depending on the summability of the datum f and γ (splitting in the cases $\gamma < 1$, $\gamma = 1$, $\gamma > 1$). Some fractional equations with measure data are studied in [71, 55, 12].

It is our purpose in this paper, to consider the problem (2.1) in the nonlocal framework and prove existence results of solutions to problem (2.1) with μ a bounded Radon measure and data $f \in L^1(\Omega)$. We use an approximation method that consists in analyzing the sequence of approximated problems truncating the datum f and the singular term $\dfrac{1}{u^\gamma}$ and approximating μ by smooth functions, obtaining non singular problems with L^∞-data whose approximated solutions u_n can be obtained by a direct application of the Schauder fixed point theorem. We faced many difficulties in dealing with the nonlocal problem (2.1), but the main one is how to get estimations in appropriate fractional Sobolev spaces.

Observe that in the local setting, if the approximated solutions are such that the sequence $\{\nabla u_n\}_n$ is uniformly bounded in the Marcinkiewicz space $\mathcal{M}^{\frac{N}{N-1}}(\Omega)$, then we conclude that the sequence $\{u_n\}_n$ is uniformly bounded in the Sobolev spaces $W^{1,q}_0(\Omega)$ for every $q < \dfrac{N}{N-1}$ (see [20]).

However, we underline here that given the fractional structure of the operator of the principal part, we can not retrieve the gradient of the approximate solutions and so appears the problem of getting a priori estimates in some fractional Sobolev spaces. To overcome this difficulty, we first prove the key result Lemma 2.3.1 (bellow) and use suitable test functions and algebraic inequalities that enable us to get appropriate a priori estimates in both cases $\gamma \leq 1$ and $\gamma > 1$. Taking into account that less regular data are involved, the classical notion of finite energy solution cannot be used. Instead, we shall consider the notion of weak solution whose meaning is defined as follows.

Definition 2.1.1. *Let $f \in L^1(\Omega)$ and let μ be a non-negative bounded Radon measure. By a* *weak solution of problem (2.1), we mean a measurable function u satisfying*

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0\ :\ u(x) \geq c_\omega > 0,\ \textit{in}\ \omega$$

and

$$\frac{\alpha(N,s)}{2}\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dxdy = \int_\Omega \frac{f\varphi}{u^\gamma}dx + \int_\Omega \varphi d\mu,$$

for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

We now state our main results. We give the existence and the regularity of weak solutions according to the values of $\gamma > 0$.

Theorem 2.1.1. *Let* Ω *be an open bounded subset in* R^N *of class* $\mathcal{C}^{0,1}$ *with* $N > 2s$ *and* $0 < s < 1$. *Let* $0 < \gamma \leq 1$ *and let* $f \in L^1(\Omega)$. *Then the problem (2.1) admits a weak solution* $u \in W_0^{s_1,q}(\Omega)$ *for every* $1 < q < \dfrac{N}{N-s}$ *and for every* $s_1 < s$.

Theorem 2.1.2. *Let* Ω *be an open bounded subset in* R^N *of class* $\mathcal{C}^{0,1}$ *with* $N > 2s$ *and* $0 < s < 1$. *Let* $\gamma > 1$ *and let* $f \in L^1(\Omega)$. *Then the problem* (2.1) *admits a weak solution* $u \in W_{loc}^{s_1,q}(\Omega)$ *for every* $1 < q < \dfrac{N}{N-s}$, *for all* $s_1 < s$. *Furthermore,* $T_k^{\frac{\gamma+1}{2}}(u) \in H_0^s(\Omega)$ *for every* $k > 0$.

We point out that the inclusion $W_0^{s_1,q}(\Omega) \subset W_0^{s_2,q}(\Omega)$ holds for any $s_2 < s_1$ (see [45]). Therefore, the range of s_1 in both Theorem 2.1.1 and Theorem 2.1.2 can be that of the set of the exponents s_1 close to s. Indeed, we can consider s_1 to be such that $\dfrac{s}{2-s} \leq s_1 < s$. So that when s tends to 1 one has also s_1 tends to 1^-. In addition, letting s tends to 1^- the operator $(-\Delta)^s$ is nothing but the standard Laplacian. So that the equation in (2.1) becomes

$$-\Delta u = \frac{f(x)}{u^\gamma} + \mu$$

and then the results in both Theorem 2.1.1 and Theorem 2.1.2 covers those obtained in [66].

2.2 Approximated problems : Existence and a comparison principle

Consider the sequence of approximate problems

$$\begin{cases} (-\Delta)^s u_n &= \dfrac{f_n}{(u_n+\frac{1}{n})^\gamma} + \mu_n \quad \text{in } \Omega, \\ u_n &> 0 \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \mathrm{R}^N\backslash\Omega, \end{cases} \tag{2.3}$$

where $f_n = T_n(f)$ is the truncation at level n of f and μ_n is a sequence of bounded non-negative smooth functions in $L^1(\Omega)$ converging weakly to μ in the sense of the measures.

We shall prove that for every fixed integer $n \in \mathrm{N}$, the problem (2.3) admits a unique weak solution u_n in the following sense :

$$\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dxdy = \int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx + \int_\Omega \mu_n\varphi dx,$$

for any $\varphi \in X_0^s(\Omega)$.

Lemma 2.2.1. *For each integer $n \in \mathrm{N}$, the problem (2.3) admits a non-negative weak solution $u_n \in H_0^s(\Omega) \cap L^\infty(\Omega)$.*

Proof. Let $n \in \mathrm{N}$ be fixed and let $v \in L^2(\Omega)$. We define the map

$$\begin{aligned} S : L^2(\Omega) &\to L^2(\Omega), \\ v &\mapsto S(v), \end{aligned}$$

where $w = S(v)$ is the weak solution to the following problem

$$\begin{cases} (-\Delta)^s w &= \dfrac{f_n}{(|v|+\frac{1}{n})^\gamma} + \mu_n \quad \text{in } \Omega, \\ \quad w &> 0 \quad \text{in } \Omega, \\ \quad w &= 0 \quad \text{on } \mathrm{R}^N\backslash\Omega. \end{cases} \tag{2.4}$$

The existence of w can be derived by classical minimization argument. Indeed, since $\dfrac{f_n}{(|v|+\frac{1}{n})^\gamma} + \mu_n \in L^\infty(\Omega)$, we already know (see [34, Lemma 2.1.]) that problem (2.4) has a unique weak solution $w \in X_0^s(\Omega)$, where

$$X_0^s(\Omega) = \Big\{\varphi \in H^s(\mathrm{R}^N) \text{ such that } \varphi = 0 \text{ a.e in } \mathrm{R}^N\backslash\Omega\Big\},$$

in the following sense

$$\frac{\alpha(N,s)}{2}\int_Q \frac{(w(x)-w(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dxdy = \int_\Omega \frac{f_n\varphi}{(|v|+\frac{1}{n})^\gamma}dx + \int_\Omega \mu_n\varphi dx,$$

for any $\varphi \in X_0^s(\Omega)$. Since Ω is regular enough, by [49, Theorem 6] the linear space $X_0^s(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H^s(\mathrm{R}^N)} = \left(\|u\|^2_{L^2(\mathrm{R}^N)} + \int_{\mathrm{R}^N}\int_{\mathrm{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}}.$$

Hence, by density arguments it follows that $X_0^s(\Omega) \subset H_0^s(\Omega)$. Thus, $w \in H_0^s(\Omega)$. As regards the uniqueness of w in $H_0^s(\Omega)$, we suppose there exist two solutions $w_1, w_2 \in H_0^s(\Omega)$ of (2.4). Summing up the both equations satisfied by w_1 and w_2 respectively, we get $(-\Delta)^s(w_1 - w_2) = 0$. Thus, taking $(w_1 - w_2)$ as a test function in this last equation and then integrating over R^N, we obtain

$$0 \leq \|w_1 - w_2\|^2_{H_0^s(\Omega)} \leq \int_Q \frac{|(w_1(x) - w_2(x)) - (w_1(y) - w_2(y))|^2}{|x-y|^{N+2s}} dxdy = 0.$$

So we get $w_1(x) = w_2(x)$, for almost every $x \in \Omega$. Since $w_1 = w_2 = 0$ on $\mathrm{R}^N \backslash \Omega$, we get $w_1(x) = w_2(x)$ for almost every $x \in \mathrm{R}^N$. Furthermore, by the comparison principle [19, Lemma 2.1] we get $w \geq 0$. Now, inserting w as a test function in (2.4) we obtain

$$\begin{aligned} \frac{\alpha(N,s)}{2} \int_Q \frac{(w(x) - w(y))^2}{|x-y|^{N+2s}} dydx &= \int_\Omega \frac{f_n w}{(|v| + \frac{1}{n})^\gamma} dx + \int_\Omega w \mu_n dx \\ &\leq n^{\gamma+1} \int_\Omega w dx + C(n) \int_\Omega w dx. \end{aligned}$$

By the Hölder inequality and the Sobolev embedding, we get

$$\|w\|_{H_0^s(\Omega)} \leq C'(n^{\gamma+1} + C(n)), \tag{2.5}$$

with C' and $C(n, s, N, \Omega)$ are independent of v, so that the ball of radius $C'(n^{\gamma+1} + C(n))$ is invariant under S in $H_0^s(\Omega)$.

Now, using the Schauder's fixed point theorem over S to prove the existence and uniqueness of solution of (2.3), we need to verify the continuity and compactness of S as an operator from $H_0^s(\Omega)$ to $H_0^s(\Omega)$.

First, we go to prove the continuity of S as an operator from $L^2(\Omega)$ to $L^2(\Omega)$. Let us consider a sequence v_k that converges to v in $L^2(\Omega)$, then up to a subsequence, we have

$$v_k \to v \text{ a.e. in } \Omega.$$

Denoting $w_k = S(v_k)$ and $w = S(v)$, we have

$$(-\Delta)^s w_k = \frac{f_n}{(|v_k| + \frac{1}{n})^\gamma} + \mu_n. \tag{2.6}$$

$$(-\Delta)^s w = \frac{f_n}{(|v| + \frac{1}{n})^\gamma} + \mu_n. \tag{2.7}$$

Taking $w_k(x)-w(x) \in X_0^s(\Omega)$ as test function in (2.6) and (2.7) respectively, then subtracting term at term the both resulting equations and using Hölder's inequality we arrive at

$$\begin{aligned}
&\frac{\alpha(N,s)}{2}\int_Q \frac{\Big(w_k(x)-w(x)-\big(w_k(y)-w(y)\big)\Big)^2}{|x-y|^{N+2s}}dydx \\
&= \int_\Omega \left(\frac{f_n}{(|v_k|+\frac{1}{n})^\gamma}-\frac{f_n}{(|v|+\frac{1}{n})^\gamma}\right)(w_k(x)-w(x))dx \\
&\le \|w_k-w\|_{L^{2_s^*}(\Omega)}\left(\int_\Omega \left(\frac{f_n}{(|v_k|+\frac{1}{n})^\gamma}-\frac{f_n}{(|v|+\frac{1}{n})^\gamma}\right)^{(2_s^*)'}dx\right)^{\frac{1}{(2_s^*)'}}.
\end{aligned}$$

Applying the Sobolev embedding and the Hölder's inequality with the exponents 2 and 2_s^*, we get

$$\|w_k-w\|_{L^2(\Omega)} \le \frac{2S(N,s)}{\alpha(N,s)}|\Omega|^{\frac{s}{N}}\left(\int_\Omega \left(\frac{f_n}{(|v_k|+\frac{1}{n})^\gamma}-\frac{f_n}{(|v|+\frac{1}{n})^\gamma}\right)^{(2_s^*)'}dx\right)^{\frac{1}{(2_s^*)'}}.$$

Since

$$\left|\frac{f_n}{(|v_k|+\frac{1}{n})^\gamma}-\frac{f_n}{(|v|+\frac{1}{n})^\gamma}\right|^{(2_s^*)'} \le 2^{(2_s^*)'}n^{(\gamma+1)(2_s^*)'} \tag{2.8}$$

and

$$\frac{f_n}{(|v_k|+\frac{1}{n})^\gamma}-\frac{f_n}{(|v|+\frac{1}{n})^\gamma} \to 0 \text{ a.e. in } \Omega,$$

then by the dominated convergence theorem we conclude that

$$\|w_k-w\|_{L^2(\Omega)} \to 0 \text{ as } k \to +\infty.$$

So, S is continuous from $L^2(\Omega)$ to $L^2(\Omega)$, it follows that S is continuous from $H_0^s(\Omega)$ to $H_0^s(\Omega)$. Now, we prove that S is compact from $H_0^s(\Omega)$ to $H_0^s(\Omega)$, let us consider a sequence $\{v_k\}_{k\in\mathrm{N}}$ such that $\|v_k\|_{H_0^s(\Omega)} \le C$, then by the compact embedding $H_0^s(\Omega)$ in $L^r(\Omega)$ for every $1 \le r < 2_s^*$ (see [45, Corollary 7.2]), we have

$$v_k \rightharpoonup v \text{ weakly in } H_0^s(\Omega),$$
$$v_k \to v \text{ in norm in } L^2(\Omega).$$

Denoting $w_k = S(v_k)$ and $w = S(v)$, by (2.5) we have

$$\|w_k\|_{H_0^s(\Omega)} \le C$$

where C is a constant not depending on k, then by the previous compact embedding and by the continuity of S on $L^2(\Omega)$ we get

$$S(v_k) = w_k \rightharpoonup \overline{w}, \text{ weakly in } H_0^s(\Omega),$$
$$S(v_k) = w_k \to S(v) = w, \text{ in norm in } L^2(\Omega).$$

So, by the uniqueness of the limit we have $\overline{w} = w$. In view of the previous, a similar argument, we have

$$(-\Delta)^s(w_k - w) = \frac{f_n}{(|v_k| + \frac{1}{n})^\gamma} - \frac{f_n}{(|v| + \frac{1}{n})^\gamma}.$$

Taking $w_k - w$ as a test function in the previous equation, using the Hölder's inequality and by (2.8), we obtain

$$\frac{\alpha(N,s)}{2}\|S(v_k) - S(v)\|^2_{H_0^s(\Omega)} \leq 2n^{\gamma+1}C(\Omega)\|S(v_k) - S(v)\|_{L^2(\Omega)}.$$

It follows that

$$\lim_{k\to+\infty} \|S(v_k) - S(v)\|_{H_0^s(\Omega)} = 0.$$

Hence, S is a compact operator from $H_0^s(\Omega)$ to $H_0^s(\Omega)$ and therefore by Schauder's fixed point theorem there exists $u_n \in H_0^s(\Omega)$ such that $u_n = S(u_n)$. This means that u_n is a weak solution to the problem

$$\begin{cases} (-\Delta)^s u_n &= \dfrac{f_n}{(u_n + \frac{1}{n})^\gamma} + \mu_n \quad \text{in } \Omega, \\ u_n &> 0 \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \mathbb{R}^N \backslash \Omega. \end{cases}$$

In addition, since the right hand side of belongs to $L^\infty(\Omega)$ by [63] we obtain $u_n \in L^\infty(\Omega)$. □

Lemma 2.2.2. *(A comparison principle.) The sequence* $\{u_n\}_{n\in\mathbb{N}}$ *is such that for every subset* $\omega \subset\subset \Omega$ *there exists a positive constant* c_ω*, independent on* n*, such that*

$$u_n(x) \geq c_\omega > 0, \quad \textit{for every } x \in \omega \textit{ and for every } n \in \mathbb{N}.$$

Proof. Consider the following problem

$$\begin{cases} (-\Delta)^s v_n &= \dfrac{f_n}{(v_n + \frac{1}{n})^\gamma} \quad \text{in } \Omega, \\ v_n &> 0 \quad \text{in } \Omega, \\ v_n &= 0 \quad \text{on } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{2.9}$$

In [18] the authors proved the existence of a weak solution v_n of (2.9) such that

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0\ :\ v_n(x) \geq c_\omega > 0,$$

for every $x \in \omega$ and for every $n \in \mathrm{N}$. Here the constant c_ω is independent on n. On the other hand, we have

$$(-\Delta)^s v_n = \frac{f_n}{(v_n + \frac{1}{n})^\gamma}$$

and

$$(-\Delta)^s u_n = \frac{f_n}{(u_n + \frac{1}{n})^\gamma} + \mu_n.$$

Then

$$(-\Delta)^s (v_n - u_n) = f_n \left(\frac{1}{(v_n + \frac{1}{n})^\gamma} - \frac{1}{(u_n + \frac{1}{n})^\gamma} \right) - \mu_n.$$

Hence

$$(-\Delta)^s (v_n - u_n) = f_n \left(\frac{(u_n + \frac{1}{n})^\gamma - (v_n + \frac{1}{n})^\gamma}{(v_n + \frac{1}{n})^\gamma (u_n + \frac{1}{n})^\gamma} \right) - \mu_n. \tag{2.10}$$

Since

$$\left((u_n + \frac{1}{n})^\gamma - (v_n + \frac{1}{n})^\gamma \right) (v_n - u_n)^+ \leq 0,$$

we obtain the following inequality

$$f_n \left(\frac{(u_n + \frac{1}{n})^\gamma - (v_n + \frac{1}{n})^\gamma}{(v_n + \frac{1}{n})^\gamma (u_n + \frac{1}{n})^\gamma} \right) (v_n - u_n)^+ - \mu_n (v_n - u_n)^+ \leq 0.$$

Now, taking $(v_n - u_n)^+$ as test function in (2.10) and then integrating over R^N, we get

$$\int_{\mathrm{R}^N} (-\Delta)^s (v_n - u_n)(v_n - u_n)^+ dx \leq 0.$$

Observe that for any function $g : \mathrm{R}^N \to \mathrm{R}$ the following inequality

$$(g(x) - g(y))(g^+(x) - g^+(y)) \geq (g^+(x) - g^+(y))^2$$

holds true for every x, $y \in \mathrm{R}^N$, where $g^+ = \max(g, 0)$. Therefore, we obtain

$$0 \leq \|(v_n - u_n)^+\|^2_{H^s_0(\Omega)} \leq 0$$

which implies that $u_n \geq v_n$ in Ω and so

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0\ :\ u_n(x) \geq c_\omega > 0$$

for every $x \in \omega$ and for every $n \in \mathrm{N}$. □

Remark 2.2.1. *Lemma 2.2.2 shows that the problem (2.3) has a unique solution. Indeed, if u_n and w_n are two solutions of problem (2.3), then as above taking $(u_n - w_n)^+$ as test function in the problem satisfied by $(u_n - w_n)$, we conclude that $u_n \leq w_n$ in Ω and again taking $(w_n - u_n)^+$ as a test function we get $w_n \leq u_n$ in Ω. Hence, follows $u_n = w_n$ in Ω.*

2.3 A priori estimates in fractional Sobolev spaces

In order to prove the existence of solutions for problem (2.1), we first need some a priori estimates on u_n. We start by proving the following lemma that we will use in both cases $\gamma \leq 1$ and $\gamma > 1$.

Lemma 2.3.1. *Let $v_n \in H_0^s(\Omega)$ be a sequence that satisfies the following assumptions*

1)- The sequence $\{v_n\}_n$ is uniformly bounded in $L^r(\Omega)$, for all $r < \dfrac{N}{N-2s}$.

2)- For any sufficient small $\theta \in (0,1)$

$$\int_\Omega \int_\Omega \frac{|w_n(x) - w_n(y)|}{|x-y|^{N+2s}} \frac{|w_n^\theta(x) - w_n^\theta(y)|}{w_n^\theta(x) w_n^\theta(y)} dydx \leq C,$$

where C is a constant not depending on n and $w_n = v_n + 1$. Then the sequence $\{v_n\}_n$ is uniformly bounded in the fractional Sobolev space $W_0^{s_1,q}(\Omega)$ for every $q < \dfrac{N}{N-s}$ and for all $s_1 < s$.

Proof. We shall prove that the sequence $\{v_n\}$ is uniformly bounded in the fractional Sobolev space $W_0^{s_1,q}(\Omega)$ for every $q < \dfrac{N}{N-s}$ and for all $s_1 < s$. That is there is a constant C not depending on n such that

$$\int_\Omega \int_\Omega \frac{|v_n(x) - v_n(y)|^q}{|x-y|^{N+qs_1}} dydx \leq C, \quad \text{for all } q < \frac{N}{N-s} \text{ and for all } s_1 < s. \tag{2.11}$$

To this aim, let $q < 2$ which will be chosen in a few lines. We can write

$$\int_\Omega \int_\Omega \frac{|v_n(x) - v_n(y)|^q}{|x-y|^{N+qs_1}} dydx = \int_\Omega \int_\Omega \frac{|w_n(x) - w_n(y)|^q}{|x-y|^{N+qs_1}} dydx$$

$$= \int_\Omega \int_{\{y\in\Omega : w_n(y) \neq w_n(x)\}} \frac{|w_n(x) - w_n(y)|^q}{|x-y|^{\frac{q}{2}N+qs}} \times \frac{(w_n^\theta(x) - w_n^\theta(y))}{(w_n(x) - w_n(y))(w_n^\theta(x) w_n^\theta(y))}$$

$$\times \frac{(w_n(x) - w_n(y))(w_n^\theta(x) w_n^\theta(y))}{(w_n^\theta(x) - w_n^\theta(y))|x-y|^{\frac{2-q}{2}N - q(s-s_1)}} dydx.$$

Pointing out that the quantity in the middle of the product inside the integral can be written as follows

$$\begin{aligned}\frac{(w_n^\theta(x)-w_n^\theta(y))}{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))} &= \left(\frac{(w_n^\theta(x)-w_n^\theta(y))}{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))}\right)^{\frac{q}{2}}\\ &\times\left(\frac{(w_n^\theta(x)-w_n^\theta(y))}{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))}\right)^{1-\frac{q}{2}}\end{aligned}$$

and using Hölder's inequality, we obtain

$$\begin{aligned}&\int_\Omega\int_\Omega \frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+qs_1}}dydx\\ &\leq \left[\int_\Omega\int_{\{y\in\Omega:w_n(y)\neq w_n(x)\}} \frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+2s}}\frac{|w_n^\theta(x)-w_n^\theta(y)|}{|w_n(x)-w_n(y)|(w_n^\theta(x)w_n^\theta(y))}dydx\right]^{\frac{q}{2}}\\ &\quad\times\left[\int_\Omega\int_{\{y\in\Omega:w_n(y)\neq w_n(x)\}}\left(\frac{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))}{(w_n^\theta(x)-w_n^\theta(y))}\right)^{\frac{2}{2-q}}\right.\\ &\quad\left.\times\frac{(w_n^\theta(x)-w_n^\theta(y))}{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))}\frac{dydx}{|x-y|^{N-\beta}}\right]^{\frac{2-q}{2}},\end{aligned}$$

where $\beta = \dfrac{2q(s-s_1)}{2-q} > 0$. Using Lemma 1.3.3, we get

$$\begin{aligned}&\int_\Omega\int_\Omega \frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+qs_1}}dydx\\ &\leq C^{\frac{q}{2}}\left(\int_\Omega\int_{\{y\in\Omega:w_n(y)\neq w_n(x)\}}\left(\frac{(w_n(x)-w_n(y))(w_n^\theta(x)w_n^\theta(y))}{[w_n^\theta(x)-w_n^\theta(y)]}\right)^{\frac{q}{2-q}}\frac{dydx}{|x-y|^{N-\beta}}\right)^{\frac{2-q}{2}}\\ &\leq \left(\frac{C}{\theta}\right)^{\frac{q}{2}}\left(\int_\Omega\int_\Omega\Big((w_n^{1-\theta}(x)+w_n^{1-\theta}(y))w_n^\theta(x)w_n^\theta(y)\Big)^{\frac{q}{2-q}}\frac{dydx}{|x-y|^{N-\beta}}\right)^{\frac{2-q}{2}}\\ &= \left(\frac{C}{\theta}\right)^{\frac{q}{2}}\left(\int_\Omega\int_\Omega\Big((w_n(x)w_n^\theta(y)+w_n(y)w_n^\theta(x))\Big)^{\frac{q}{2-q}}\frac{dydx}{|x-y|^{N-\beta}}\right)^{\frac{2-q}{2}}.\end{aligned}$$

Applying the Young inequality with the exponents $\dfrac{\theta+1}{\theta}$ and $\theta+1$, we have

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+qs_1}}dydx\\
&\le \Big(\frac{C}{\theta}\Big)^{\frac{q}{2}}\Bigg(\int_\Omega\int_\Omega\Big(w_n^{1+\theta}(x)+w_n^{1+\theta}(y)\Big)^{\frac{q}{2-q}}\frac{dydx}{|x-y|^{N-\beta}}\Bigg)^{\frac{2-q}{2}}\\
&\le 2^{\frac{2(q-1)}{2-q}}\Big(\frac{C}{\theta}\Big)^{\frac{q}{2}}\Bigg(\int_\Omega\int_\Omega\Big(w_n^{\frac{q(1+\theta)}{2-q}}(x)+w_n^{\frac{q(1+\theta)}{2-q}}(y)\Big)\frac{dydx}{|x-y|^{N-\beta}}\Bigg)^{\frac{2-q}{2}}\\
&\le 2^{\frac{2(q-1)}{2-q}}\Big(\frac{C}{\theta}\Big)^{\frac{q}{2}}\Bigg(\int_\Omega w_n^{\frac{q(1+\theta)}{2-q}}(x)\Big[\int_\Omega\frac{dy}{|x-y|^{N-\beta}}\Big]dx\Bigg)^{\frac{2-q}{2}}\\
&+2^{\frac{2(q-1)}{2-q}}\Big(\frac{C}{\theta}\Big)^{\frac{q}{2}}\Bigg(\int_\Omega w_n^{\frac{q(1+\theta)}{2-q}}(y)\Big[\int_\Omega\frac{dx}{|x-y|^{N-\beta}}\Big]dy\Bigg)^{\frac{2-q}{2}}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\int_\Omega\frac{dy}{|x-y|^{N-\beta}} &= \int_{\Omega\cap|x-y|>1}\frac{dy}{|x-y|^{N-\beta}}+\int_{\Omega\cap|x-y|\le1}\frac{dy}{|x-y|^{N-\beta}}\\
&\le|\Omega|+\int_{|z|\le1}\frac{dz}{|z|^{N-\beta}}=|\Omega|+\frac{|S^{N-1}|}{\beta}.
\end{aligned}\tag{2.12}$$

Here, $|S^{N-1}|$ stands for the Lebesgue measure of the unit sphere in R^N. By x/y symmetry, there exists a constant C, not depending on n, such that

$$\int_\Omega\int_\Omega \frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+qs_1}}dydx \le C\Bigg(\int_\Omega w_n^{\frac{q(1+\theta)}{2-q}}(y)dy\Bigg)^{\frac{2-q}{2}}.$$

Now we choose $\theta>0$ in order to get $\dfrac{q(1+\theta)}{2-q}<\dfrac{N}{N-2s}$. That is $\theta<\dfrac{2N-2q(N-s)}{q(N-2s)}$. To ensure the existence of θ we must have $2N-2q(N-s)>0$ which yields $q<\dfrac{N}{N-s}$. We then conclude that (2.11) is fulfilled and the sequence $\{v_n\}$ is uniformly bounded in $W_0^{s_1,q}(\Omega)$ for every $q<\dfrac{N}{N-s}$ and for all $s_1<s$. □

2.3.1 The case $\gamma\le1$

Lemma 2.3.2. *Let $u_n\in H_0^s(\Omega)$ be the solution of the problem (2.3). If $0<\gamma\le1$, then the sequence $\{u_n\}$ is uniformly bounded in $W_0^{s_1,q}(\Omega)$ for every $q<\dfrac{N}{N-s}$ and for all $s_1<s$.*

Proof. Let $k \geq 1$ be fixed. By Lemma 2.5.2 (in Appendix) the function $T_k(u_n)$ is an admissible test function in (2.3). Thus, inserting it in (2.3) we obtain

$$\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))(T_k(u_n(x))-T_k(u_n(y)))}{|x-y|^{N+2s}}dydx$$
$$= \int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}T_k(u_n)dx + \int_\Omega \mu_n T_k(u_n)dx.$$

By using Proposition 2.5.2 (in Appendix), we get

$$\frac{\alpha(N,s)}{2}\int_Q \frac{|T_k(u_n(x))-T_k(u_n(y))|^2}{|x-y|^{N+2s}}dydx \leq k^{1-\gamma}\|f\|_{L^1(\Omega)} + k\|\mu_n\|_{L^1(\Omega)} \leq Ck,$$

where $C = \|f\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}_b(\Omega)}$ is a constant not depending on n. Applying the Sobolev embedding theorem we get

$$\frac{1}{S}\left(\int_\Omega |T_k(u_n)(x)|^{2^*_s}dx\right)^{\frac{2}{2^*_s}} \leq Ck.$$

For the left hand side, observing that on the set $\{u_n \geq k\}$, we have $T_k(u_n) = k$, we get

$$\frac{1}{S}k^2(meas(\{u_n \geq k\}))^{\frac{2}{2^*_s}} \leq Ck,$$

which yields

$$meas(\{u_n \geq k\}) \leq \frac{C}{k^{\frac{N}{N-2s}}}. \tag{2.13}$$

Thus, the sequence $\{u_n\}$ is uniformly bounded in $M^{\frac{N}{N-2s}}(\Omega)$ and then so it is in $L^r(\Omega)$, for all $r < \dfrac{N}{N-2s}$. Let $s_1 \in (0,s)$ be fixed. For every $x \geq 0$ we define the function

$$\phi(x) = 1 - \frac{1}{(1+x)^\theta}, \text{ where } 0 < \theta \leq 1. \tag{2.14}$$

Observe that the function ϕ satisfies

$$\phi(x) \leq 1 \text{ and } \phi(x) \leq x^\gamma \text{ for any } 0 < \theta \leq \gamma \leq 1.$$

The function $\phi(u_n)$ is an admissible test function in (2.3). So that inserting it as a test function in (2.3) we obtain

$$\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))(\phi(u_n)(x)-\phi(u_n)(y))}{|x-y|^{N+2s}}dydx$$
$$= \int_\Omega \frac{f_n(x)\phi(u_n)}{(u_n+\frac{1}{n})^\gamma} + \int_\Omega \mu_n(x)\phi(u_n)dx$$
$$\leq \|f\|_{L^1(\Omega)} + \|\mu_n\|_{L^1(\Omega)} \leq C.$$

Being ϕ non-decreasing and $\Omega \times \Omega \subset Q$, the integral in the left-hand side can be treated as follows

$$\int_Q \frac{(u_n(x) - u_n(y))(\phi(u_n)(x) - \phi(u_n)(y))}{|x-y|^{N+2s}} dydx$$
$$\geq \int_\Omega \int_\Omega \frac{(u_n(x) - u_n(y))}{|x-y|^{N+2s}} \frac{(u_n(x)+1)^\theta - (u_n(y)+1)^\theta}{(u_n(x)+1)^\theta (u_n(y)+1)^\theta} dydx.$$

So that we obtain

$$\int_\Omega \int_\Omega \frac{(w_n(x) - w_n(y))}{|x-y|^{N+2s}} \frac{(w_n(x))^\theta - (w_n(y))^\theta}{(w_n(x))^\theta (w_n(y))^\theta} dydx \leq \frac{2C}{\alpha(N,s)},$$

where we have set $w_n = u_n + 1$. Therefore, by Lemma 2.3.1 with $0 < \theta \leq \gamma$ the sequence $\{u_n\}$ is uniformly bounded in $W_0^{s_1,q}(\Omega)$ for every $q < \dfrac{N}{N-s}$ and for all $s_1 < s$. □

2.3.2 The case $\gamma > 1$

Lemma 2.3.3. *Let $f \in L^1(\Omega)$ and let u_n be the solution of (2.3). For $k > 0$ and $\gamma > 1$ the sequence $\{T_k^{\frac{\gamma+1}{2}}(u_n)\}_n$ is uniformly bounded in $H_0^s(\Omega)$.*

Proof. Let us fix $k > 0$. Inserting $T_k^\gamma(u_n)$ as a test function in (2.3), we get

$$\frac{\alpha(N,s)}{2} \int_Q \frac{(u_n(x) - u_n(y))(T_k^\gamma(u_n(x)) - T_k^\gamma(u_n(y))}{|x-y|^{N+2s}} dydx$$
$$= \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\gamma} T_k^\gamma(u_n) dx + \int_\Omega \mu_n T_k^\gamma(u_n) dx$$
$$\leq \|f\|_{L^1(\Omega)} + k^\gamma \|\mu_n\|_{L^1(\Omega)} \leq C_1,$$

where $C_1 = \|f\|_{L^1(\Omega)} + k^\gamma \|\mu\|_{\mathcal{M}_b(\Omega)}$ is a constant not depending on n. By applying Proposition 2.5.2 (in Appendix) and Lemma 1.3.3, we have

$$\int_Q \frac{(u_n(x) - u_n(y))(T_k^\gamma(u_n(x)) - T_k^\gamma(u_n(y))}{|x-y|^{N+2s}} dydx$$
$$\geq \int_Q \frac{(T_k(u_n(x)) - T_k(u_n(y)))(T_k^\gamma(u_n(x)) - T_k^\gamma(u_n(y)))}{|x-y|^{N+2s}} dydx$$
$$\geq \frac{4\gamma}{(\gamma+1)^2} \int_Q \frac{|T_k^{\frac{\gamma+1}{2}}(u_n(x)) - T_k^{\frac{\gamma+1}{2}}(u_n(y))|^2}{|x-y|^{N+2s}} dydx.$$

Therefore, we obtain

$$\|T_k^{\frac{\gamma+1}{2}}(u_n)\|^2_{H_0^s(\Omega)} \leq \int_Q \frac{|T_k^{\frac{\gamma+1}{2}}(u_n(x)) - T_k^{\frac{\gamma+1}{2}}(u_n(y))|^2}{|x-y|^{N+2s}} dydx$$
$$\leq \frac{(\gamma+1)^2}{4\gamma} \frac{2}{\alpha(N,s)} C_1.$$

The proof is then achieved. □

Lemma 2.3.4. *Let u_n be the solution of the problem (2.3). If $\gamma > 1$, then the sequence $\{u_n\}$ is uniformly bounded in $W^{s_1,q}_{loc}(\Omega)$ for every $q < \dfrac{N}{N-s}$ and for all $s_1 < s$.*

Proof. For every $\omega \subset\subset \Omega$, for all $q < \dfrac{N}{N-s}$ and for all $s_1 < s$, we shall prove that there exists a constant $C = C(q, s_1, w)$, not depending on n, such that

$$\int_\omega\int_\omega \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+qs_1}}dydx \le C \text{ and } \int_\omega |u_n|^q dx \le C, \tag{2.15}$$

We begin by proving the left estimate in (2.15). Let $k_0 \ge 1$ be fixed. Let $q < 2$ and $s_1 < s$. Using the fact that $u_n = T_{k_0}(u_n) + G_{k_0}(u_n)$, we can write

$$\begin{aligned}
&\int_\omega\int_\omega \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+qs_1}}dydx \\
&= \int_\omega\int_\omega \frac{|T_{k_0}(u_n(x)) + G_{k_0}(u_n(x)) - T_{k_0}(u_n(y)) - G_{k_0}(u_n(y))|^q}{|x-y|^{N+qs_1}}dydx \\
&\le 2^{q-1}\int_\omega\int_\omega \frac{|T_{k_0}(u_n(x)) - T_{k_0}(u_n(y))|^q}{|x-y|^{N+qs_1}}dydx \\
&+2^{q-1}\int_\Omega\int_\Omega \frac{|G_{k_0}(u_n(x)) - G_{k_0}(u_n(y))|^q}{|x-y|^{N+qs_1}}dydx.
\end{aligned}$$

Applying the Hölder inequality, we get

$$\begin{aligned}
&\int_\omega\int_\omega \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+qs_1}}dydx \\
&\le 2^{q-1}\Big(\int_\omega\int_\omega \frac{|T_{k_0}(u_n(x)) - T_{k_0}(u_n(y))|^2}{|x-y|^{N+2s}}dydx\Big)^{\frac{q}{2}} \times \Big(\int_\Omega\int_\Omega \frac{dydx}{|x-y|^{N-\beta}}\Big)^{\frac{2-q}{2}} \\
&+2^{q-1}\int_\Omega\int_\Omega \frac{|G_{k_0}(u_n(x)) - G_{k_0}(u_n(y))|^q}{|x-y|^{N+qs_1}}dydx,
\end{aligned}$$

where $\beta = \dfrac{2q(s-s_1)}{2-q} > 0$. Thanks to (2.12), we have

$$2^{q-1}\Big(\int_\Omega\int_\Omega \frac{dydx}{|x-y|^{N-\beta}}\Big)^{\frac{2-q}{2}} \le C_3 := 2^{q-1}\Big(|\Omega| + \frac{|S^{N-1}|}{\beta}\Big)^{\frac{2-q}{2}}$$

which implies

$$\begin{aligned}
&\int_\omega\int_\omega \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+qs_1}}dydx \\
&\le C_3\Big(\int_\omega\int_\omega \frac{|T_{k_0}(u_n(x)) - T_{k_0}(u_n(y))|^2}{|x-y|^{N+2s}}dydx\Big)^{\frac{q}{2}} \\
&+2^{q-1}\int_\Omega\int_\Omega \frac{|G_{k_0}(u_n(x)) - G_{k_0}(u_n(y))|^q}{|x-y|^{N+qs_1}}dydx,
\end{aligned}$$

So, it is sufficient to prove that $\{G_{k_0}(u_n)\}_n$ and $\{T_{k_0}(u_n)\}_n$ are uniformly bounded in $W_0^{s_1,q}(\Omega)$ and $H^s_{loc}(\Omega)$ respectively. We begin by proving that $G_{k_0}(u_n)$ is uniformly bounded in $W_0^{s_1,q}(\Omega)$ for all $q < \dfrac{N}{N-s}$ and for all $s_1 < s$. To do so, for $k > k_0$ we take $T_k(G_{k_0}(u_n))$ as a test function in (2.3) and use the fact that $G_{k_0}(u_n) = 0$ on $\{u_n \leq k_0\}$, we obtain

$$\begin{aligned}
&\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))[T_k(G_{k_0}(u_n(x))) - T_k(G_{k_0}(u_n(y)))]}{|x-y|^{N+2s}}dydx \\
&= \int_\Omega \frac{f_n T_k(G_{k_0}(u_n))}{(u_n+\frac{1}{n})^\gamma}dx + \int_\Omega \mu_n T_k(G_{k_0}(u_n))dx \\
&\leq k\int_{\{u_n>k_0\}} \frac{f}{(u_n+\frac{1}{n})^\gamma}dx + k\|\mu_n\|_{L^1(\Omega)} \leq C_1 k,
\end{aligned}$$

where $C_1 = k_0^{-\gamma}\|f\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}_b(\Omega)}$, is a constant not depending on n. Using the decomposition of u_n as $u_n = T_{k_0}(u_n) + G_{k_0}(u_n)$, we can write

$$\begin{aligned}
&\int_Q \frac{(u_n(x)-u_n(y))[T_k(G_{k_0}(u_n(x))) - T_k(G_{k_0}(u_n(y)))]}{|x-y|^{N+2s}}dydx \\
&= \int_Q \frac{(T_{k_0}(u_n(x))-T_{k_0}(u_n(y)))[T_k(G_{k_0}(u_n(x))) - T_k(G_{k_0}(u_n(y)))]}{|x-y|^{N+2s}}dydx \\
&+ \int_Q \frac{(G_{k_0}(u_n(x))-G_{k_0}(u_n(y)))[T_k(G_{k_0}(u_n(x))) - T_k(G_{k_0}(u_n(y)))]}{|x-y|^{N+2s}}dydx.
\end{aligned}$$

Let us observe that since T_{k_0} and $T_k(G_{k_0})$ are non-decreasing functions, we get

$$(T_{k_0}(u_n(x))-T_{k_0}(u_n(y)))[T_k(G_{k_0}(u_n(x))) - T_k(G_{k_0}(u_n(y)))] \geq 0 \text{ a.e. in Q}.$$

Hence, it follows

$$\begin{aligned}
&\int_Q \frac{(u_n(x)-u_n(y))[T_k(G_{k_0}(u_n(x))) - T_k(G_{k_0}(u_n(y)))]}{|x-y|^{N+2s}}dydx \\
&\geq \int_Q \frac{(G_{k_0}(u_n(x))-G_{k_0}(u_n(y)))[T_k(G_{k_0}(u_n(x))) - T_k(G_{k_0}(u_n(y)))]}{|x-y|^{N+2s}}dydx.
\end{aligned}$$

In the right-hand side of the above inequality, we decompose $G_{k_0}(u_n)$ as follows $G_{k_0}(u_n(x)) = G_k(G_{k_0}(u_n(x))) + T_k(G_{k_0}(u_n(x)))$ and we apply Proposition 2.5.2 (in Appendix) with $\alpha = 1$ obtaining

$$\int_\Omega\int_\Omega \frac{|T_k(G_{k_0}(u_n(x))) - T_k(G_{k_0}(u_n(y)))|^2}{|x-y|^{N+2s}}dydx \leq \frac{2kC_1}{\alpha(N,s)}.$$

Hence, using the fractional Sobolev inequality, we get again the inequality (2.13) for the function $G_{k_0}(u_n)$ that is

$$meas(\{G_{k_0}(u_n) \geq k\}) \leq Ck^{-\frac{N}{N-2s}}$$

which implies that $\{G_{k_0}(u_n)\}_n$ is uniformly bounded in $L^r(\Omega)$ for every $r < \dfrac{N}{N-2s}$.
Let ϕ be the function defined in (2.14). Observe that for every $0 < \theta < 1$ the function ϕ enjoys the following properties

$$\phi(x) \le x \text{ and } \phi(x) \le 1.$$

Inserting $\phi(G_{k_0}(u_n))$ as a test function in (2.3) we get

$$\begin{aligned}
&\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))\Big(\phi(G_{k_0}(u_n(x)))-\phi(G_{k_0}(u_n(y)))\Big)}{|x-y|^{N+2s}}dydx\\
&= \int_{\{u_n\ge k_0\}} \frac{f_n\phi(G_{k_0}(u_n))}{(u_n+\frac{1}{n})^\gamma}dx + \int_\Omega \mu_n(x)\phi(G_{k_0}(u_n))dx\\
&\le \int_{\{u_n\ge k_0\}} \frac{f_n G_{k_0}(u_n)}{(u_n+\frac{1}{n})^\gamma}dx + \|\mu_n\|_{L^1(\Omega)}\\
&\le \int_{\{u_n\ge k_0\}} \frac{|f|}{(u_n+\frac{1}{n})^{\gamma-1}}dx + \|\mu_n\|_{L^1(\Omega)}\\
&\le C_2 := k_0^{1-\gamma}\|f\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}_b(\Omega)}.
\end{aligned}$$

Then, writing the decomposition $u_n = T_{k_0}(u_n) + G_{k_0}(u_n)$ and using the fact that T_{k_0} and $\phi(G_{k_0})$ are non-decreasing functions, we obtain

$$\int_\Omega\int_\Omega \frac{(G_{k_0}(u_n)(x)-G_{k_0}(u_n)(y))\Big(\phi(G_{k_0}(u_n(x)))-\phi(G_{k_0}(u_n(y)))\Big)}{|x-y|^{N+2s}}dydx \le \frac{2C_2}{\alpha(N,s)}$$

which yields

$$\int_\Omega\int_\Omega \frac{(w_n(x)-w_n(y))}{|x-y|^{N+2s}}\frac{(w_n(x))^\theta-(w_n(y))^\theta}{(w_n(x))^\theta(w_n(y))^\theta}dydx \le C_3 := \frac{2C_2}{\alpha(N,s)},$$

where we have set $w_n = G_{k_0}(u_n)+1$. Thus, Lemma 2.3.1 ensures that the sequence $\{G_{k_0}(u_n)\}$ is uniformly bounded in $W_0^{s_1,q}(\Omega)$ for all $q < \dfrac{N}{N-s}$ and for all $s_1 < s$.
Now, we shall prove that $\{T_{k_0}(u_n)\}_n$ is uniformly bounded in $H^{s_1}_{loc}(\Omega)$. To do so, we insert $T^\gamma_{k_0}(u_n)$ as a test function in (2.3) obtaining

$$\begin{aligned}
&\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))\Big(T^\gamma_{k_0}(u_n(x))-T^\gamma_{k_0}(u_n(y))\Big)}{|x-y|^{N+2s}}dydx\\
&= \int_\Omega \frac{f_nT^\gamma_{k_0}(u_n)}{(u_n+\frac{1}{n})^\gamma}dx + \int_\Omega \mu_nT^\gamma_{k_0}(u_n)dx \le C_4 := \|f\|_{L^1(\Omega)} + k_0^\gamma\|\mu\|_{\mathcal{M}_b(\Omega)}.
\end{aligned}$$

By Lemma 1.3.3 (item v)) there exists a constant $C_\gamma > 0$, depending only on γ such that

$$\begin{aligned}
&\int_Q \frac{(u_n(x)-u_n(y))\Big(T_{k_0}^\gamma(u_n(x)) - T_{k_0}^\gamma(u_n(y))\Big)}{|x-y|^{N+2s}}dydx \\
&\geq \int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)||T_{k_0}^\gamma(u_n(x)) - T_{k_0}^\gamma(u_n(y))|}{|x-y|^{N+2s}}dydx \\
&\geq \frac{1}{C_\gamma}\int_\Omega\int_\Omega \frac{\Big|(u_n(x)-u_n(y))\Big(T_{k_0}(u_n(x)) - T_{k_0}(u_n(y))\Big)\Big|}{|x-y|^{N+2s}} \\
&\qquad\qquad \times (T_{k_0}(u_n(x)) + T_{k_0}(u_n(y)))^{\gamma-1}dydx.
\end{aligned}$$

Let now ω be a compact subset in Ω. By Proposition 2.5.2 (in Appendix) we can write

$$\begin{aligned}
&\int_Q \frac{(u_n(x)-u_n(y))[T_{k_0}^\gamma(u_n(x)) - T_{k_0}^\gamma(u_n(y))]}{|x-y|^{N+2s}}dydx \\
&\geq \frac{1}{C_\gamma}\int_\omega\int_\omega \frac{|T_{k_0}(u_n(x)) - T_{k_0}(u_n(y))|^2 (T_{k_0}(u_n(x)) + T_{k_0}(u_n(y)))^{\gamma-1}}{|x-y|^{N+2s}}dydx.
\end{aligned}$$

Pointing out that by Lemma 2.2.2 we have $T_{k_0}(u_n(x)) \geq \min(k_0, c_\omega)$ for every $x \in \omega$, we obtain

$$\begin{aligned}
&\int_Q \frac{(u_n(x)-u_n(y))[T_{k_0}^\gamma(u_n(x)) - T_{k_0}^\gamma(u_n(y))]}{|x-y|^{N+2s}}dydx \\
&\geq \frac{1}{C_\gamma}(2\min(k_0, c_\omega))^{\gamma-1}\int_\omega\int_\omega \frac{|T_{k_0}(u_n(x)) - T_{k_0}(u_n(y))|^2}{|x-y|^{N+2s}}dydx
\end{aligned}$$

which proves that $\{T_{k_0}(u_n)\}_n$ is uniformly bounded in $H^s_{loc}(\Omega)$.
We now prove the second estimate in (2.15). For $q < \dfrac{N}{N-s}$ and $s_1 < s$, writing

$$\begin{aligned}
\int_\omega |u_n|^q dx &\leq 2^{q-1}\int_\omega |T_{k_0}(u_n)|^q dx + 2^{q-1}\int_\omega |G_{k_0}(u_n)|^q dx \\
&\leq 2^{q-1}k_0^q|\omega| + 2^{q-1}\|G_{k_0}(u_n)\|^q_{L^q(\Omega)}
\end{aligned}$$

we conclude the result. In fact, for every $\gamma > 0$ the sequence $\{u_n\}$ is uniformly bounded in $L^q(\Omega)$ for all $1 \leq q < \dfrac{N}{N-2s}$. □

2.4 Proof of the main results

In this section, we show that in the both cases $\gamma \leq 1$ and $\gamma > 1$, the problem (2.1) has a weak solution obtained as the limit of approximate solutions $\{u_n\}_n$ of the problem (2.3).

2.4.1 The case $\gamma \le 1$

Proof. of Theorem 2.1.1. By virtue of Lemma 2.3.2 and the compact embedding of $W_0^{s_1,q}(\Omega)$ in $L^1(\Omega)$ (see [45, Corollary 7.2]), there exist a subsequence of $\{u_n\}_n$ still indexed by n and a measurable function $v \in W_0^{s_1,q}(\Omega)$ such that

$$u_n \rightharpoonup v \text{ weakly in } W_0^{s_1,q}(\Omega),$$
$$u_n \to v \text{ in norm in } L^1(\Omega),$$
$$u_n \to v \text{ a.e. in } \Omega.$$

Let u the function such that $u = v$ in Ω and $u = 0$ in $\mathrm{R}^N \setminus \Omega$. Thus, $u_n \to u$ a.e. in R^N which implies

$$\frac{|u_n(x) - u_n(y)|}{|x-y|^{N+2s}} \to \frac{|u(x) - u(y)|}{|x-y|^{N+2s}} \text{ a.e. in } Q.$$

Let $\rho > 0$ be a small enough real number that we will choose later. For any $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \int_\Omega \left[\frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}}\right]^{1+\rho} dydx$$
$$\le \int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y)|^{1+\rho}(\|D\varphi\|_{L^\infty(\Omega)}|x-y|)^{1+\rho}}{|x-y|^{N+(1+\rho)s_1}} \frac{dydx}{|x-y|^{\rho N+(1+\rho)(2s-s_1)}}$$
$$\le \|D\varphi\|_{L^\infty(\Omega)}^{1+\rho} \int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y)|^{1+\rho}|x-y|^{(1+\rho)(1+s_1-2s)-\rho N}}{|x-y|^{N+(1+\rho)s_1}} dydx.$$

We need that the term $|x-y|^{\rho N+(1+\rho)(2s-s_1)}$ vanishes from within the integral. To get this, it is sufficient to have $(1+\rho)(1+s_1-2s) - \rho N \ge 0$. To this aim, we consider s_1 to be very close of s. Precisely, we impose on s_1 the condition

$$\max(0, 1-3s) < s - s_1 < 1 - s.$$

We point out that with this range of values of s_1 and with the assumption $N > 2s$, we easily get

$$1 + s_1 - 2s > 0 \text{ and } N - 1 - s_1 + 2s > 0.$$

Thus, the fact that $(1+\rho)(1+s_1-2s) - \rho N \ge 0$ is equivalent to $0 < \rho \le \dfrac{1+s_1-2s}{N-1-s_1+2s}$.
Hence, we get

$$\int_\Omega \int_\Omega \left[\frac{|u_n(x) - u_n(y)||\varphi(x) - \varphi(y)|}{|x-y|^{N+2s}}\right]^{1+\rho} dydx$$
$$\le \|D\varphi\|_{L^\infty(\Omega)}^{1+\rho} diam(\Omega)^{(1+\rho)(1+s_1-2s)-\rho N} \int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y)|^{1+\rho}}{|x-y|^{N+(1+\rho)s_1}} dydx$$

where $diam(\Omega)$ stands for the diameter of Ω. Now we have to make a choice of ρ which enables us to use the uniform boundedness of $\{u_n\}_n$ in $W_0^{s_1,q}(\Omega)$ for every $q < \dfrac{N}{N-s}$. This is the case if $1+\rho < \dfrac{N}{N-s}$. Finally, we choose ρ to be such that

$$0 < \rho < \min\Big(\frac{s}{N-s}, \frac{1+s_1-2s}{N-1-s_1+2s}\Big)$$

Therefore, there is a constant $C>0$ not depending on n such that

$$\sup_n \int_\Omega\int_\Omega \Big[\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\Big]^{1+\rho} dydx \leq C.$$

Consequently by De La Vallée-Poussin and Dunford-Pettis theorems the sequence

$$\Big\{\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\Big\}$$

is equi-integrable in $L^1(\Omega\times\Omega)$. Now, taking $\varphi\in C_0^\infty(\Omega)$ as a test function in (2.3) we get

$$\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx = \int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx + \int_\Omega \varphi\mu_n dx. \tag{2.16}$$

We split the integral in left-hand side into three integrals as follows

$$\begin{aligned}\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx &= \int_\Omega\int_\Omega \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx \\ &+ \int_\Omega\int_{\mathcal{C}\Omega} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx \\ &+ \int_{\mathcal{C}\Omega}\int_\Omega \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx \\ &= I_1+I_2+I_3.\end{aligned} \tag{2.17}$$

By Vitali's lemma we have

$$\lim_{n\to\infty}\int_\Omega\int_\Omega \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx = \int_\Omega\int_\Omega \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$

For the second integral I_2 in (2.17), we start noticing that since $u_n(y)=\varphi(y)=0$ for every $y\in\mathcal{C}\Omega$ we can write

$$\Big|\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\Big| \leq \frac{|u_n(x)\varphi(x)|}{|x-y|^{N+2s}} \text{ for every } (x,y)\in\Omega\times\mathcal{C}\Omega.$$

Since $supp(\varphi)$ is a compact subset in Ω, we have

$$|x-y| \geq d_1 := dist(supp(\varphi),\partial\Omega) > 0 \text{ for every } (x,y)\in supp(\varphi)\times\mathcal{C}\Omega.$$

Therefore, an easy computation leads to

$$\int_{\mathcal{C}\Omega} \frac{dy}{|x-y|^{N+2s}} \leq \int_{d_1}^{+\infty} \frac{dz}{|z|^{N+2s}} \leq \frac{|S^{N-1}|}{2sd_1^{2s}}. \tag{2.18}$$

As a consequence of the convergence in norm of the sequence $\{u_n\}$ in $L^1(\Omega)$ there exist a subsequence of $\{u_n\}$ still indexed by n and a positive function g in $L^1(\Omega)$ such that

$$|u_n(x)| \leq g(x) \text{ a.e. in } \Omega,$$

which enables us to get

$$\frac{|(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}} \leq \frac{|g(x)\varphi(x)|}{|x-y|^{N+2s}} \text{ a.e. in } (x,y) \in \Omega \times \mathcal{C}\Omega.$$

We observe that by (2.18) the function $(x,y) \to \dfrac{|g(x)\varphi(x)|}{|x-y|^{N+2s}}$ belongs to $L^1(\Omega \times \mathcal{C}\Omega)$

$$\int_\Omega \int_{\mathcal{C}\Omega} \frac{|g(x)\varphi(x)|}{|x-y|^{N+2s}} = \int_{supp(\varphi)} \int_{\mathcal{C}\Omega} \frac{|g(x)\varphi(x)|}{|x-y|^{N+2s}} \leq \frac{|S^{N-1}| \|\varphi\|_{L^\infty(\Omega)} \|g\|_{L^1(\Omega)}}{2sd_1^{2s}}.$$

Thus, by the dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_\Omega \int_{\mathcal{C}\Omega} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dydx = \int_\Omega \int_{\mathcal{C}\Omega} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dydx.$$

For the third integral I_3 in (2.17), we can follow exactly the same lines as above using the x/y symmetry. We then conclude that

$$\lim_{n\to\infty} \int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dydx = \int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dydx,$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Now, for what concerns the right-hand side of (2.16), by virtue of lemma 3.3.2, for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$ with $Supp\, \varphi = \omega$, there exists a constant $c_\omega > 0$ not depending on n such that

$$0 \leq \left| \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} \right| \leq \frac{|f||\varphi|}{c_\omega^\gamma} \in L^1(\Omega)$$

obtaining by the dominated convergence theorem

$$\lim_{n\to\infty} \int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma} dx = \int_\Omega \frac{f\varphi}{u^\gamma} dx$$

and in the last term in (2.16), by the convergence of μ_n to μ we have

$$\lim_{n\to\infty} \int_\Omega \varphi(x)\mu_n(x)dx = \int_\Omega \varphi(x)d\mu.$$

Finally, passing to the limit as $n \to +\infty$, we obtain

$$\frac{\alpha(N,s)}{2} \int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dxdy = \int_\Omega \frac{f\varphi}{u^\gamma} dx + \int_\Omega \varphi d\mu,$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Therefore, u is a weak solution of (2.1). □

2.4.2 The case $\gamma > 1$

Proof. of Theorem 2.1.2. By virtue of Lemma 2.3.4, there exist a subsequence of $\{u_n\}_n$ still indexed by n and a measurable function $v \in W^{s_1,q}_{loc}(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup v \text{ in } W^{s_1,q}_{loc}(\Omega),\\ &u_n \to v \text{ in } L^1_{loc}(\Omega),\\ &u_n \to v \text{ a.e. in } \Omega. \end{aligned}$$

So that defining the function u by $u = v$ in Ω and $u = 0$ in $\mathrm{R}^N \setminus \Omega$, one has

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } W^{s_1,q}_{loc}(\Omega),\\ &u_n \to u \text{ in } L^1_{loc}(\Omega),\\ &u_n \to u \text{ a.e. in } \mathrm{R}^N,\\ &T_k^{\frac{\gamma+1}{2}}(u_n) \to T_k^{\frac{\gamma+1}{2}}(u) \text{ a.e. in } \Omega. \end{aligned}$$

Then for $\varphi \in \mathcal{C}^\infty_0(\Omega)$, we have

$$\frac{|(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}} \to \frac{|(u(x)-u(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}} \text{ a.e. in } Q.$$

Inserting $\varphi \in C^\infty_0(\Omega)$ as a test function in (2.3), we have

$$\frac{\alpha(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx = \int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx + \int_\Omega \varphi\mu_n dx. \qquad (2.19)$$

Let K be a compact subset of Ω such that $supp(\varphi) \subset K$ and $dist(supp(\varphi), \partial K) > 0$. The integral in the left-hand side of the previous equality can be splitted as

$$\begin{aligned} \int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx &= \int_{\mathrm{R}^N}\int_{\mathrm{R}^N} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\ &= \int_K\int_K \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\ &+ \int_K\int_{\mathcal{C}K} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\ &+ \int_{\mathcal{C}K}\int_K \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx. \end{aligned}$$

As in the proof of the Theorem 2.1.1, the same ideas allow to obtain

$$\lim_{n\to\infty}\int_K\int_K \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx = \int_K\int_K \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$

$$\lim_{n\to\infty}\int_K\int_{\mathcal{C}K}\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx=\int_K\int_{\mathcal{C}K}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$
$$\lim_{n\to\infty}\int_{\mathcal{C}K}\int_K\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx=\int_{\mathcal{C}K}\int_K\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$

We then conclude that

$$\lim_{n\to\infty}\int_Q\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx=\int_Q\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx,$$

for all $\varphi\in\mathcal{C}_0^\infty(\Omega)$. For what concerns the right-hand side of (2.19), it is exactly the same term in Theorem 2.1.1. Finally, passing to the limit as $n\to+\infty$, we obtain

$$\frac{\alpha(N,s)}{2}\int_Q\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx=\int_\Omega\frac{f\varphi}{u^\gamma}dx+\int_\Omega\varphi d\mu,$$

for all $\varphi\in\mathcal{C}_0^\infty(\Omega)$, So u is a weak solution to (2.1). Now, by virtue of lemma 2.3.3, and Fatou's lemma, we have

$$\int_\Omega\int_\Omega\frac{|T_k^{\frac{\gamma+1}{2}}(u(x))-T_k^{\frac{\gamma+1}{2}}(u(y))|^2}{|x-y|^{N+2s}}dxdy\le\liminf_{n\to+\infty}\int_\Omega\int_\Omega\frac{|T_k^{\frac{\gamma+1}{2}}(u_n(x))-T_k^{\frac{\gamma+1}{2}}(u_n(y))|^2}{|x-y|^{N+2s}}dxdy\le C.$$

It follows that $T_k^{\frac{\gamma+1}{2}}(u)\in H_0^s(\Omega)$, for every $k>0$. $\square$

2.5 Regularity of solutions

Now, we prove some regularities of the solution u of the problem (2.1).

Proposition 2.5.1. *Assume that μ is a Radon measure, $f\in L^1(\Omega)$ and $0<\gamma\le 1$. Then the solution u of the problem (2.1) obtained by approximation is such that*

$$u\in L^r(\Omega),\quad\forall\, r\in\Big(1,\frac{N}{N-2s}\Big).$$
$$|(-\Delta)^{\frac{s}{2}}u|\in L^r(\Omega),\quad\forall\, r\in\Big(1,\frac{N}{N-s}\Big).$$

Proof. We follow closely the lines in [63]. By (2.13) and Theorem 2.1.1, we can apply Fatou's Lemma, we conclude that $u\in L^r(\Omega)$, for every $1<r<\dfrac{N}{N-2s}$. Now, we will prove that $|(-\Delta)^{\frac{s}{2}}u_n|$ is bounded in the Marcinkiewicz space $M^{\frac{N}{N-s}}(\Omega)$. We fix $\beta>0$ and for any positive $k\ge 1$, we have

$$\begin{aligned}\{|(-\Delta)^{\frac{s}{2}}u_n|\ge\beta\} &= \{|(-\Delta)^{\frac{s}{2}}u_n|\ge\beta\ u_n<k\}\cup\{|(-\Delta)^{\frac{s}{2}}u_n|\ge\beta\ u_n\ge k\}\\ &\subset \{|(-\Delta)^{\frac{s}{2}}u_n|\ge\beta\ u_n<k\}\cup\{u_n\ge k\}.\end{aligned}$$

Then

$$meas(\{|(-\Delta)^{\frac{s}{2}}u_n| \geq \beta,\ u_n < k\}) \leq \frac{1}{\beta^2}\int_{\{u_n<k\}} |(-\Delta)^{\frac{s}{2}}u_n|^2 dx.$$

By using [63, Corollary 1] and Lemma 2.3.2, we get

$$\begin{aligned} meas(\{|(-\Delta)^{\frac{s}{2}}u_n| \geq \beta,\ u_n < k\}) &\leq \frac{1}{\beta^2}\int_{\{u_n<k\}} |(-\Delta)^{\frac{s}{2}}u_n|^2 dx \\ &\leq \frac{1}{\beta^2}\int_{\mathrm{R}^N} |(-\Delta)^{\frac{s}{2}}T_k(u_n)|^2 dx \\ &\leq \frac{C(N,s)}{\beta^2}\int_Q \frac{|T_k(u_n)(x)-T_k(u_n)(y)|^2}{|x-y|^{N+2s}}dxdy \leq C\frac{k}{\beta^2}. \end{aligned}$$

By using (2.13), we have

$$\begin{aligned} meas(\{|(-\Delta)^{\frac{s}{2}}u_n| \geq \beta\}) &\leq meas(\{|(-\Delta)^{\frac{s}{2}}u_n| \geq \beta,\ u_n < k\}) + meas(\{u_n \geq k\}) \\ &\leq C\frac{k}{\beta^2} + \frac{C}{k^{\frac{N}{N-2s}}}. \end{aligned}$$

Choosing $k = \beta^{\frac{N-2s}{N-s}}$, we get

$$meas(\{|(-\Delta)^{\frac{s}{2}}u_n| \geq \beta\}) \leq \frac{C}{\beta^{\frac{N}{N-s}}}.$$

This implies that $|(-\Delta)^{\frac{s}{2}}u_n|$ is bounded in the Marcinkiewicz space $M^{\frac{N}{N-s}}(\Omega)$. So, by the converges almost everywhere in the proof of Theorem 2.1.1, we can apply Fatou's Lemma, we conclude the result. □

Appendix

In this Appendix we give the functional and technical results we have used in the previous sections. We start with the following inequality whose proof in the cases where $\alpha = 1$ can be found [63]. Here we give a simple proof based on the monotony of the truncation functions.

Proposition 2.5.2. *Let $\alpha \geq 1$ and let $v : \mathrm{R}^N \to \mathrm{R}$ be a positive measurable function. Then for every $k > 0$ and for every $(x,y) \in \mathrm{R}^N \times \mathrm{R}^N$*

$$\Big(G_k(v(x)) - G_k(v(y))\Big)\Big(T_k(v(x))^\alpha - T_k(v(y))^\alpha\Big) \geq 0.$$

Proof. Let $x,\ y \in \mathrm{R}^N$ be arbitrary. Without loss of generality we can assume that $v(x) \geq v(y)$. Since the functions $s \mapsto T_k(s)$ and $s \mapsto G_k(s)$ are non-decreasing on R, we have

$$T_k(v(x))^\alpha \geq T_k(v(y))^\alpha \text{ and } G_k(v(x)) \geq G_k(v(y)).$$

Then

$$(G_k(v(x)) - G_k(v(y)))(T_k(v(x))^\alpha - T_k(v(y))^\alpha) \geq 0.$$

□

The next result, well known in classical Sobolev spaces, provides a necessary condition for a function to belong to the fractional Sobolev space $W_0^{s,p}(\Omega)$.

Lemma 2.5.1. *Let Ω be an open set in R^N of class $\mathcal{C}^{0,1}$ with bounded boundary, $1 \leq p < +\infty$ and let $0 < s < 1$. If $u \in W^{s,p}(\Omega)$ with supp(u) is a compact set in Ω, then $u \in W_0^{s,p}(\Omega)$.*

Proof. Let $u \in W^{s,p}(\Omega)$ be a function with $supp(u)$ be a compact subset included in Ω. Then there exists an open set ω such that

$$supp(u) \subset \omega \text{ and } \overline{\omega} \subset \Omega.$$

Then by [45, Corollary 5.5], there exists a sequence $\{u_n\}_n$ of functions $u_n \in \mathcal{C}_0^\infty(\mathrm{R}^N)$ such that

$$u_n \to u \text{ in norm in } W^{s,p}(\Omega).$$

Let $\varphi \in \mathcal{C}_0^\infty(\omega)$ be such that

$$\varphi = 1 \text{ on } Supp\, u \text{ and } 0 \leq \varphi \leq 1, \text{ a.e in } \omega.$$

It is clear that $\varphi u_n \in \mathcal{C}_0^\infty(\omega)$. Therefore, it sufficient to prove that

$$\varphi u_n \to u \text{ in } W^{s,p}(\Omega).$$

Using the fact that $\varphi u = u$ on Ω, we obtain

$$\int_\Omega |\varphi u_n - u|^p dx = \int_\Omega |\varphi u_n - \varphi u|^p dx \leq \int_\Omega |u_n - u|^p dx \to 0.$$

For the second part of the norm $\|\varphi u_n - u\|_{W^{s,p}(\Omega)}$, we can write it as follows

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{|(\varphi(x)u_n(x) - \varphi(y)u_n(y)) - (u(x) - u(y))|^p}{|x-y|^{N+ps}} dxdy \\
&= \int_\Omega \int_\Omega \left| \frac{\varphi(x)u_n(x) - \varphi(y)u_n(y)}{|x-y|^{\frac{N+ps}{p}}} - \frac{u(x) - u(y)}{|x-y|^{\frac{N+ps}{p}}} \right|^p dxdy \\
&= \int_\Omega \int_\Omega |F_n(x,y) - F(x,y)|^p dxdy,
\end{aligned}$$

where we have set

$$F_n(x,y) = \frac{\varphi(x)u_n(x) - \varphi(y)u_n(y)}{|x-y|^{\frac{N+ps}{p}}} \text{ and } F(x,y) = \frac{u(x)-u(y)}{|x-y|^{\frac{N+ps}{p}}}.$$

Thus, in order to prove that φu_n converges to u in $W^{s,p}(\Omega)$, it is sufficient to prove that up to a subsequence, $\{F_n(x,y)\}$ converges to $F(x,y)$ in norm in $L^p(\Omega\times\Omega)$. Since, up to a subsequence still indexed by n, u_n converges almost everywhere to u, we obtain

$$F_n(x,y) = \frac{\varphi(x)u_n(x) - \varphi(y)u_n(y)}{|x-y|^{\frac{N+ps}{p}}} \to \frac{u(x)-u(y)}{|x-y|^{\frac{N+ps}{p}}} = F(x,y) \text{ a.e in } \Omega\times\Omega.$$

The norm convergence of u_n to u in $W^{s,p}(\Omega)$, yields

$$\frac{u_n(x)-u_n(y)}{|x-y|^{\frac{N+ps}{p}}} \to \frac{u(x)-u(y)}{|x-y|^{\frac{N+ps}{p}}} \text{ in norm in } L^p(\Omega\times\Omega). \tag{2.20}$$

According to 2.20 and the norm convergence of $\{u_n\}$ in $L^p(\Omega)$, there exist a subsequence of $\{u_n\}$ still indexed by n and two positive functions g in $L^1(\Omega\times\Omega)$ and h in $L^1(\Omega)$ such that

$$\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} \le g(x,y) \text{ a.e in } \Omega\times\Omega$$

and

$$|u_n(x)|^p \le h(x) \text{ a.e in } \Omega.$$

So that writing

$$\begin{aligned} |F_n(x,y)|^p &= \frac{\Big|\varphi(x)u_n(x) - \varphi(x)u_n(y) + \varphi(x)u_n(y) - \varphi(y)u_n(y)\Big|^p}{|x-y|^{N+ps}} \\ &\le 2^{p-1}\frac{\Big|u_n(x)-u_n(y)\Big|^p}{|x-y|^{N+ps}} + 2^{p-1}\frac{|u_n(y)|^p|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}, \end{aligned}$$

we obtain

$$|F_n(x,y)|^p \le 2^{p-1}|g(x,y)| + 2^{p-1}\frac{|h(y)||\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}. \tag{2.21}$$

We need to prove that the function in the second term in the right-hand side in (2.21) belongs to $L^1(\Omega\times\Omega)$. To do so we can write

$$\begin{aligned} \int_\Omega\int_\Omega \frac{|h(y)||\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}dxdy &= \int_\Omega |h(y)|\Big[\int_{\Omega\cap|x-y|<1} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}dx\Big]dy \\ &+ \int_\Omega |h(y)|\Big[\int_{\Omega\cap|x-y|\ge 1} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}dx\Big]dy \end{aligned}$$

Since φ belongs at least to $\mathcal{C}_0^1(\Omega)$ and $0 \le \varphi \le 1$, a.e. in ω we have

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|h(y)||\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}dxdy &\le C_{lip}^p\int_\Omega |h(y)|\left[\int_{\Omega\cap|z|<1}\frac{dz}{|z|^{N+p(s-1)}}\right]dy\\ &+2^p\int_\Omega |h(y)|\left[\int_{\Omega\cap|z|\ge 1}\frac{dz}{|z|^{N+ps}}\right]dy\\ &\le 2\max\left(C_{lip}^p\frac{|\mathcal{S}^{N-1}|}{p(1-s)}, 2^p\frac{|\mathcal{S}^{N-1}|}{ps}\right)\int_\Omega |h(y)|dy < +\infty,\end{aligned}$$

where C_{lip} stands for the Lipschitz constant of φ and $|\mathcal{S}^{N-1}|$ stands for the Lebesgue measure of the surface area of the unit N-sphere $\mathcal{S}^{N-1}$ of $\mathcal{R}^N$. Applying the dominated convergence theorem, we conclude our claim and thus follows $u \in W_0^{s,p}(\Omega)$. □

Lemma 2.5.2. *Let Ω be an open set in R^N of class $\mathcal{C}^{0,1}$ with bounded boundary, $1 \le p < +\infty$ and let $0 < s < 1$. Let $\phi : \mathrm{R} \to \mathrm{R}$ be a uniformly Lipschitz function, with $\phi(0) = 0$. Then for every $u \in W_0^{s,p}(\Omega)$ one has $\phi(u) \in W_0^{s,p}(\Omega)$.*

Proof. Let us denote by K the Lipschitz constant of ϕ and let $u \in W_0^{s,p}(\Omega)$. There exists a sequence $\{u_n\}$ of $C_0^\infty(\Omega)$ functions which converges to u in norm in $W^{s,p}(\Omega)$. That is there exists $n_0 \in \mathrm{N}$ such that for all $n \in \mathrm{N}$ with $n \ge n_0$ one has

$$\|u_n - u\|_{W^{s,p}(\Omega)} < 1.$$

Defining $v_n = \phi(u_n)$, $G_n(x,y) = u_n(x) - u_n(y)$ and $G(x,y) = u(x) - u(y)$, we can write for every $n \ge n_0$

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}}dxdy &= \int_\Omega\int_\Omega \frac{|\phi(u_n)(x)-\phi(u_n)(y)|^p}{|x-y|^{N+ps}}dxdy\\ &\le K^p\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dxdy\\ &= K^p\int_\Omega\int_\Omega \frac{|G_n(x,y)|^p}{|x-y|^{N+ps}}dxdy\\ &\le 2^{p-1}K^p\int_\Omega\int_\Omega \frac{|G_n(x,y)-G(x,y)|^p}{|x-y|^{N+ps}}dxdy\\ &+2^{p-1}K^p\int_\Omega\int_\Omega \frac{|G(x,y)|^p}{|x-y|^{N+ps}}dxdy\\ &= 2^{p-1}K^p\|u_n-u\|_{W^{s,p}(\Omega)}^p + 2^{p-1}K^p\|u\|_{W^{s,p}(\Omega)}^p \le C_0\end{aligned}$$

and

$$\|v_n\|_{L^p(\Omega)} \leq K\|u_n\|_{L^p(\Omega)} \leq K\|u_n - u\|_{W^{s,p}(\Omega)} + K\|u\|_{W^{s,p}(\Omega)} \leq C_1,$$

C_0 and C_1 are constants not depending on n. Thus, $\{v_n\}$ is uniformly bounded in $W^{s,p}(\Omega)$. Since by $\phi(0) = 0$ the function v_n is compactly supported in Ω, so that by Lemma 2.5.1 we obtain $v_n \in W_0^{s,p}(\Omega)$. Now, we prove that

$$v_n \to \phi(u) \text{ in } W^{s,p}(\Omega).$$

Since the sequence $\{u_n\}$ converges to u in norm in $W^{s,p}(\Omega)$, then for a subsequence of $\{u_n\}$, still indexed by n, we have

$$u_n \to u \text{ a.e. in } \Omega.$$

Then, it follows

$$v_n = \phi(u_n) \to \phi(u) \text{ a.e. in } \Omega.$$

Furthermore,

$$\|v_n - \phi(u)\|_{L^p(\Omega)} = \|\phi(u_n) - \phi(u)\|_{L^p(\Omega)} \leq K\|u_n - u\|_{L^p(\Omega)} \to 0.$$

On the other hand we can write

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{|(v_n(x) - \phi(u)(x) - (v_n(y) - \phi(u)(y))|^p}{|x-y|^{N+ps}} dxdy \\
&= \int_\Omega \int_\Omega \left| \frac{v_n(x) - v_n(y)}{|x-y|^{\frac{N+ps}{p}}} - \frac{\phi(u(x)) - \phi(u(y)}{|x-y|^{\frac{N+ps}{p}}} \right|^p dxdy \\
&= \int_\Omega \int_\Omega |F_n(x,y) - F(x,y)|^p dxdy,
\end{aligned}$$

where we noted

$$F_n(x,y) = \frac{v_n(x) - v_n(y)}{|x-y|^{\frac{N+ps}{p}}} \text{ and } F(x,y) = \frac{\phi(u(x)) - \phi(u(y)}{|x-y|^{\frac{N+ps}{p}}}.$$

In order to show that v_n converges to $\phi(u)$ in $W^{s,p}(\Omega)$, it sufficient to prove that for a subsequence of $\{F_n(x,y)\}_{n\geq 1}$, still denoted by $\{F_n(x,y)\}_{n\geq 1}$, $\|F_n(x,y) - F(x,y)\|_{L^p(\Omega\times\Omega)} \to 0$. By the almost everywhere convergence of v_n to $\phi(u)$, we have

$$F_n(x,y) = \frac{v_n(x) - v_n(y)}{|x-y|^{\frac{N+ps}{p}}} \to \frac{\phi(u)(x) - \phi(u)(y)}{|x-y|^{\frac{N+ps}{p}}} = F(x,y), \quad a.e \text{ } in \text{ } \Omega \times \Omega.$$

Observe that the norm convergence of u_n to u in $W^{s,p}(\Omega)$ implies

$$\frac{u_n(x)-u_n(y)}{|x-y|^{\frac{N+ps}{p}}} \to \frac{u(x)-u(y)}{|x-y|^{\frac{N+ps}{p}}} \text{ in norm in } L^p(\Omega\times\Omega).$$

So that since

$$|F_n(x,y)| \le K\frac{|u_n(x)-u_n(y)|}{|x-y|^{\frac{N+ps}{p}}}$$

the sequence $\{|F_n(x,y)|^p\}_n$ is then equi-integrable. Applying Vitali's theorem we get $\|F_n(x,y) - F(x,y)\|_{L^p(\Omega\times\Omega)} \to 0$ which in turn implies $\|v_n - \phi(u)\|_{W^{s,p}(\Omega)}$ as $n \to +\infty$. Since the sequence $\{v_n\}$ belongs to the closed space $W_0^{s,p}(\Omega)$ forces the limit $\phi(u)$ to belong to $W_0^{s,p}(\Omega)$. □

Chapter 3

Nonlocal semilinear elliptic problems with singular nonlinearity[1]

In this chapter, we consider the Lazer-Mckenna problem involving the fractional Laplacian and singular nonlinearity. We investigate the existence, regularity and uniqueness of the solutions in light of the interplay between the nonlinearities and the summability of the datum. The study we are conducting extend some results obtained in [14] and [10, 18]. We also analyze the threshold 3 for integrable the data.

3.1 Introduction

The chapter deals with the existence, regularity and uniqueness of solutions for the following nonlocal problem

$$\begin{cases} (-\Delta)^s u &= \dfrac{f(x)}{u^\gamma} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N > 2s$, of class $\mathcal{C}^{1,1}$, $s \in (0,1)$, $\gamma > 0$, $f \in L^m(\Omega)$, $m \geq 1$, is a non-negative function and $(-\Delta)^s$ is the fractional Laplacian operator defined by

$$(-\Delta)^s u = a(N,s) P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,$$

[1] A. Youssfi and G. Ould Mohamed Mahmoud. Nonlocal semilinear elliptic problems with singular nonlinearity. *Calc. Var. Partial Differential Equations*, 60(153), 2021.

where "P.V." stands for the principal value and $a(N,s)$ is a positive renormalizing constant, depending only on N and s, given by

$$a(N,s) = \frac{4^s\Gamma(\frac{N}{2}+s)}{\pi^{\frac{N}{2}}}\frac{s}{\Gamma(1-s)}$$

to ensure that

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u), \quad \xi \in \mathbb{R}^N, s \in (0,1) \text{ and } u \in \mathcal{S}(\mathbb{R}^N),$$

where $\mathcal{F}u$ stands for the Fourier transform of u belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^N)$. More details on the operator $(-\Delta)^s$ and the asymptotic behaviour of $a(N,s)$ can be found in [45].

In the case of semilinear local problem corresponding to $s = 1$, the study of singular elliptic equations was initiated in the pioneering work [38] which constitutes the starting point of a wide literature about singular semilinear elliptic equations. Let us start recalling the important result of Lazer-McKenna [61]. Under regularity assumptions on Ω and if $0 < f \in \mathcal{C}^\alpha(\overline{\Omega})$, the authors obtained an optimal power related to the existence of finite energy solutions. In fact, a solution lying in $H_0^1(\Omega)$ should exists if and only if $\gamma < 3$ while it is not in $\mathcal{C}^1(\overline{\Omega})$ if $\gamma > 1$. The threshold 3 is analysed in [88] when the datum f is a positive L^1 function defined on Ω. In that paper [88], the authors provide an extension of the classical Lazer-McKenna obstruction. Existence and uniqueness results for (3.1) are obtained in [35] while in [26, 42] the authors showed that (3.1) has a solution u for every f in $L^1(\Omega)$ and for every $\gamma > 0$ and how the regularity of this solution u depends on the summability of f and on γ. In the case where the function f belongs to $L^m(\Omega)$ with $m \geq 1$, Boccardo and Orsina [23] proved the existence and regularity of a distributional solution $u \in W_0^{1,q}(\Omega)$ where $q = \dfrac{Nm(\gamma+1)}{N-m(1-\gamma)}$ if $0 < \gamma < 1$ and $f \in L^m(\Omega)$, $1 \leq m < \left(\dfrac{2^*}{1-\gamma}\right)'$, while $u \in H_0^1(\Omega)$ if $f \in L^m(\Omega)$ with $m = \left(\dfrac{2^*}{1-\gamma}\right)'$. In the case where $f \in L^1(\Omega)$, if $\gamma = 1$ then $u \in H_0^1(\Omega)$, while if $\gamma > 1$ then $u \in H^1_{loc}(\Omega)$ and $u^{\frac{\gamma+1}{2}} \in H_0^1(\Omega)$. In connection with the problem studied in [23], uniqueness of finite energy solutions was established in [22] where the main ingredient is the extension of the set of admissible test functions. We will use the same idea in this case of fractional Laplacian. In [14] the authors proved that if the non-negative function $f \in L^m(\Omega)$, $m > 1$, is strictly far away from zero on Ω (that is there exists a positive constant f_0 such that $f \geq f_0 > 0$ a.e. $x \in \Omega$)

then $u^\alpha \in H_0^1(\Omega)$ for every $\alpha \in \left(\frac{(m+1)(\gamma+1)}{4m}, \frac{\gamma+1}{2}\right]$ if $1 < \gamma < \frac{3m-1}{m+1}$. Some related existence and regularity results for local problems with singular nonlinearity involving reaction or absorption terms are proved in [37, 65, 66]. Let us also mention the contributions in [3, 33, 54, 59, 67, 69, 87] where related problems involving singular nonlinearities are considered. It is worth recalling here that singular local semilinear elliptic problems such as (3.1) arise in various contexts of chemical heterogeneous catalysts [15], non-Newtonian fluids [48] as well as heat conduction in electrically conducting materials (the term u^γ describes the resistivity of the material), see for instance [51, 64].

Let us now discuss the nonlocal problem (3.1). Recall first that a rich amount of research work has been done on nonlocal problems of either elliptic or parabolic types, we refer for instance to [5, 6, 7, 10, 62, 89]. Starting with the case $\gamma = 0$, the problem (3.1) with L^1-data was studied in [1, 34, 63] where a general fractional Laplacian operator including $(-\Delta)^s$ is involved, while for bounded Radon measure data it was investigated in [55, 71]. In the case where $\gamma > 0$, existence and regularity results of solutions to (3.1) were established in [10] when the datum f is a Hölder continuous function and behaviours basically as $\frac{1}{dist^\beta(x, \partial\Omega)}$ for some β such that $0 \leq \beta < 2s$. Existence and uniqueness results for positive solutions of the problem (3.1) have been also obtained in [18, 34]. It has been shown in [34] that (3.1) has a weak solution $u \in X_0^s(\Omega)$, when $0 < \gamma \leq 1$ and $f \in L^{\overline{m}}(\Omega)$ with $\overline{m} := \frac{2N}{N+2s+\gamma(N-2s)}$, while if $\gamma > 1$ and $f \in L^1(\Omega)$ then (3.1) has a weak solution $u \in H^s_{loc}(\Omega)$ with $u^{\frac{\gamma+1}{2}} \in X_0^s(\Omega)$. In the same spirit, the existence of positive solutions have been also established in [18] according to the range of $\gamma > 0$ and to the summability of f. Precisely, in that paper [18] it has been proven that if $\gamma \leq 1$ and $f \in L^{(2_s^*)'}(\Omega)$, $2_s^* := \frac{2N}{N-2s}$ and $(2_s^*)' := \frac{2N}{N+2s}$, then (3.1) has a solution $u \in X_0^s(\Omega) \cap L^{(\gamma+1)2_s^*}(\Omega)$, while if $\gamma > 1$ and $f \in L^1(\Omega)$ then (3.1) has a solution u such that $u^{\frac{\gamma+1}{2}} \in X_0^s(\Omega)$.

It is worth pointing out that the interest brought to the fractional Laplacian operator is due to the wide range of its applications, for instance in thin obstacle problems [39], in crystal dislocation [47] and in phase transition [83].

In the present paper, our aim is to lead investigations about the existence and regularity of positive solutions to (3.1) establishing some missing results in [18, 34]. The case where $\gamma = 1$ is treated in [18, 34]. We study the case where $0 < \gamma < 1$ and $f \in L^m(\Omega)$ with $1 \leq m < \overline{m}$

which provides infinite energy solutions (see Theorem 3.2.1 bellow) and we prove the existence of finite energy solutions to problem (3.1) in the case $\gamma > 1$ under some suitable assumptions on the datum f. Further, to show the accuracy of our results we highlight the relationship with the Lazer-Mckenna condition. We also provide some regularity results for solutions as well as the uniqueness of finite energy solutions.

3.2 Main results

We start with define the meaning we will give to the solution of the problem(3.1).

Definition 3.2.1. *Let $f \in L^1(\Omega)$ be a non-negative function. By a weak solution of the problem (3.1), we mean a measurable function u satisfying*

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0\ :\ u(x) \geq c_\omega > 0,\ \text{in } \omega \tag{3.2}$$

and

$$\frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx = \int_\Omega \frac{f\varphi}{u^\gamma}dx, \tag{3.3}$$

for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

Definition 3.2.2. *We say that $u \in X_0^s(\Omega)$ is a finite energy solution of (3.1) if it is a weak solution u of problem (3.1) which further satisfies (3.3) for every $\varphi \in X_0^s(\Omega)$.*

Remark 3.2.1. *By Lemma 3.4.4, if $u \in X_0^s(\Omega)$ is a weak solution of problem (3.1) (in the sense Definition 3.2.1), the u is a finite energy solution. In other words if $u \in X_0^s(\Omega)$ the two definitions 3.2.1 and 3.2.2 are equivalent.*

3.2.1 The case $0 < \gamma < 1$: Infinite energy solutions

We consider the problem (3.1) under the assumption $0 < \gamma < 1$. We recall that in this case it is proved in [34] that (3.1) has energy solutions when $f \in L^{\overline{m}}(\Omega)$, where $\overline{m}$ stands for the Hölder conjugate exponent of $\frac{2_s^*}{1-\gamma}$, that is $\overline{m} := \left(\frac{2_s^*}{1-\gamma}\right)' = \frac{2N}{N+2s+\gamma(N-2s)}$. It is in our purpose here to investigate the remaining range of summability of source terms corresponding to the data $f \in L^m(\Omega)$ with $1 \leq m < \overline{m}$. We show that the problem (3.1) has solutions lying in a fractional Sobolev space larger than $H_0^s(\Omega)$.

Theorem 3.2.1. *Let $0 < \gamma < 1$ and let $f \in L^m(\Omega)$, with $1 \le m < \overline{m}$. Then the problem (3.1) admits a weak solution $u \in W_0^{s_1,\overline{q}}(\Omega)$ for all $s_1 < s$ with $\overline{q} = \dfrac{Nm(1+\gamma)}{N - sm(1-\gamma)}$. Furthermore, $u \in L^\sigma(\Omega)$ where $\sigma = \dfrac{Nm(1+\gamma)}{N-2sm}$.*

Remark 3.2.2. *Note that $\overline{q} < 2$ since $m < \overline{m}$. Moreover, the exponent σ is well defined. Indeed, since $N > 2s$ we have*

$$4ms < m(N+2s) < m\Big(N + 2s + \gamma(N-2s)\Big).$$

As $m < \overline{m} := \dfrac{2N}{N+2s+\gamma(N-2s)}$, we get $4ms < 2N$.

Remark 3.2.3. *Observe that the inclusion $W_0^{s_1,q}(\Omega) \subset W_0^{s_2,q}(\Omega)$ holds for any $s_2 < s_1$ (see [45]). So we infer that it is sufficient to choose s_1 very close to s that is $\dfrac{s}{2-s} \le s_1 < s$ which implies that the results in Theorem 3.2.1 recovers those already obtained in [23, Theorem 5.6] when $s \to 1$.*

Remark 3.2.4. *Notice that if $\gamma = 0$ the problem (3.1) reduces to*

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.4}$$

In [63] the authors proved the existence of a unique weak solution u of the problem (3.4) such that

1. *If $f \in L^1(\Omega)$ then $u \in L^q(\Omega)$ for every $q < \dfrac{N}{N-2s}$.*
2. *If $f \in L^m(\Omega)$, with $1 < m < \dfrac{2N}{N+2s}$, then $u \in L^{\frac{Nm}{N-2sm}}(\Omega)$.*

We point out that when $1 < m < \overline{m}$ we have a kind of 'continuity' of the summability of the solution with respect to γ. If we let $\gamma \to 0$, the value of $\sigma = \dfrac{Nm(1+\gamma)}{N-2sm}$ tends to $\dfrac{Nm}{N-2sm}$ which is exactly the summability of solutions obtained in [63]. However, this 'continuity' fails to hold when $m = 1$ since $\sigma = \dfrac{N(1+\gamma)}{N-2s}$ tends to $\dfrac{N}{N-2s}$ but the solutions obtained in [63] belong to $L^q(\Omega)$ for every $q < \dfrac{N}{N-2s}$. In fact, the case where $\gamma = 0$ can not be considered, this is mainly due to the inequality (3.20) where we divide by γ.

3.2.2 The case $\gamma > 1$: Finite energy solutions

Let us recall that Lazer and McKenna [61] proved that the problem

$$\begin{cases} -\Delta u &= \dfrac{f(x)}{u^\gamma} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where the datum f is regular enough (say Hölder continuous) and bounded away from zero on Ω, admits a unique solution $u \in H_0^1(\Omega)$ if and only if $\gamma < 3$. In the case where f is a non-negative function such that $f \in L^m(\Omega)$ with $m > 1$ and strictly far away from zero on Ω, the authors [14] proved that if $1 < \gamma < \dfrac{3m-1}{m+1}$ then $u \in H_0^1(\Omega)$. As regards the case where the datum $f \in L^1(\Omega)$, the problem 3.5 has only a local solution $u \in H^1_{loc}(\Omega)$ which does not belong to $H_0^1(\Omega)$ (see [23, Theorem 4.2]). In the case of the fractional Laplacian operator, J.Giacomoni et al.[10] studied the following problem

$$\begin{cases} (-\Delta)^s u &= \dfrac{f(x)}{u^\gamma} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.6}$$

where f is a Hölder continuous function such that $f \simeq \dfrac{1}{dist^\beta(x, \partial\Omega)}$, with $0 \leq \beta < 2s$. They proved that if $\dfrac{\beta}{s} + \gamma > 1$ then the problem (3.6) admits a unique solution $u \in X_0^s(\Omega)$ if and only if $2\beta + \gamma(2s-1) < 2s+1$. This last inequality implies $\gamma(2s-1) < 2s+1$. So that letting s tends to 1^- one can find $\gamma < 3$ which is exactly the Lazer-Mckenna condition.

In this section, we investigate the existence of finite energy solutions for (3.1) when $\gamma > 1$ and $f \in L^m(\Omega)$, with $m \geq 1$. We impose some assumptions on the datum f and γ that provide solutions for (3.1) in $X_0^s(\Omega)$. The first result deals with data f strictly far away from zero.

Theorem 3.2.2. *Let $\gamma > 1$ and $s \in (0, 1)$. Assume that $f \in L^m(\Omega)$, $m > 1$, is such that there exists a positive constant f_0 satisfying $f(x) \geq f_0 > 0$ a.e. $x \in \Omega$. Then the problem (3.1) admits a weak solution $u \in H^s_{loc}(\Omega)$ such that $u^\alpha \in X_0^s(\Omega)$ for every $\alpha \in \Big(\max\Big(\frac{1}{2}, \frac{(\gamma+1)(2sm-m+1)}{4sm}\Big), \frac{\gamma+1}{2}\Big]$. In particular if γ satisfies*

$$(m(2s-1)+1)\gamma < m(2s+1) - 1, \tag{3.7}$$

then $u \in X_0^s(\Omega)$.

Remark 3.2.5. *Observe that from (3.7) we get* $\max\left(\frac{1}{2}, \frac{(\gamma+1)(2sm-m+1)}{4sm}\right) < 1 < \frac{\gamma+1}{2}$*, so that* $\alpha = 1$ *can be chosen to obtain* $u \in X_0^s(\Omega)$*. Furthermore, notice that for every* $m > 1$ *(3.7) reads as*

$$\gamma(2s-1) + \frac{\gamma}{m} < 2s + 1 - \frac{1}{m},$$

which implies $\gamma(2s-1) < 2s+1$ *and this is exactly the necessary and sufficient condition for the existence of the unique solution in* $X_0^s(\Omega)$ *obtained in [10, Theorem 1.2 ii)] when* $\beta = 0$*. We also observe that when* s *tends to* 1^-*, the condition (3.7) yields* $1 < \gamma < \frac{3m-1}{m+1}$ *and therefore Theorem 3.2.2 reduces to the same result stated in [14, Theorem 3]. Furthermore, letting* m *tends to* $+\infty$ *in the last inequality we get* $1 < \gamma < 3$*, which can be seen as an extension of the Lazer-Mckenna condition [61] for obtaining finite energy solutions to strictly positive* L^∞*-data.*

Remark 3.2.6. *In the local case corresponding to* $s = 1$*, it is known that the threshold* $\frac{3m-1}{m+1}$ *obtained in [14, Theorem 3] is not the optimal one. Using [88, Theorem 1], Oliva and Petitta [67] proved that the optimal threshold is* $3 - \frac{2}{m}$*. For the nonlocal problem (3.1), the situation is somehow different. Notice that for* $m > 1$ *if* $\frac{m-1}{2m} < s < 1$ *then* (3.7) *reads as*

$$\gamma < h(s) := \frac{m(2s+1)-1}{m(2s-1)+1}.$$

The optimality is lost since s *is varying, however we can obtain more information. Observe that the function* h *decreases from infinity to* $\frac{3m-1}{m+1}$ *as* $\frac{m-1}{2m} < s < 1$*. Setting* $\bar{s} := 1 - \frac{1}{2m}$*, one has* $\frac{m-1}{2m} < \bar{s} < 1$ *and* $h(\bar{s}) = 3 - \frac{2}{m}$*. Thus, for* $s < \bar{s}$ *we have* $h(\bar{s}) = 3 - \frac{2}{m} < h(s)$*. On the other hand, if* $0 < s \leq \frac{m-1}{2m}$ *then* (3.7) *is satisfied for every* $\gamma > 1$*. We conclude that the range of* γ *is wide than the one of the local case.*

We point out that we can avoid the hypothesis that the source term f is far from zero and we continue to obtain energy solutions. This is stated in the following theorem.

Theorem 3.2.3. *Let* $\gamma > 1$ *and* $s \in (0,1)$*. Suppose that* $f \in L^m(\Omega)$ *with* $m > 1$*. Then the problem (3.1) admits a weak solution* $u \in H^s_{loc}(\Omega)$ *such that* $u^\alpha \in X_0^s(\Omega)$ *for every* $\alpha \in \left(\max\left(\frac{1}{2}, \frac{sm(\gamma+1)-m+1}{2sm}\right), \frac{\gamma+1}{2}\right]$*. In particular, if* $1 < \gamma < 1 + \frac{m-1}{sm}$ *then* $u \in X_0^s(\Omega)$*.*

Here again, letting s tends to 1^- and m tends to $+\infty$ we obtain $1 < \gamma < 2$ which is a restriction of the Lazer-Mckenna condition to positive L^m-data, $m > 1$. Notice that the case where $m = 1$ can not be considered in the two last theorems, since the range of α will be empty. However, if we consider data $f \in L^1(\Omega)$ with compact support in Ω we can also obtain an energy solution. This is stated in the following theorem.

Theorem 3.2.4. *Let $\gamma > 1$ and $s \in (0,1)$. Suppose that $f \in L^1(\Omega)$ with compact support in Ω. Then the problem (3.1) admits a weak solution $u \in H^s_{loc}(\Omega)$ such that $u^\alpha \in X^s_0(\Omega)$ for every $\alpha \in \left(\frac{1}{2}, \frac{\gamma+1}{2}\right]$. In particular, $u \in X^s_0(\Omega)$.*

We point out that the Lazer-Mckenna condition vanishes when we deal with positive L^1-data having compact support.

3.2.3 Uniqueness of finite energy solutions

As mentioned in the introduction, the existence of weak solutions for the problem (3.1) lying $X^s_0(\Omega)$ has been proved in [34, Theorem 3.2] when $0 < \gamma \le 1$ and $f \in L^{\overline{m}}(\Omega)$. In the case where $\gamma > 1$, the existence of a weak solution $u \in X^s_0(\Omega)$ to the problem (3.1) is obtained in the previous theorems 3.2.2, 3.2.3 and 3.2.4. In the following theorem we prove the uniqueness of finite energy solutions to the problem (3.1).

Theorem 3.2.5. *Let $\gamma > 0$ and $s \in (0,1)$. Let $0 < f \in L^1(\Omega)$ be such that the problem (3.1) admits a finite energy solution $u \in X^s_0(\Omega)$ (in the sense of Definition 3.2.2). Then u is unique.*

3.3 Proof of main results

3.3.1 Approximated Problems

Consider the sequence of approximate problems

$$\begin{cases} (-\Delta)^s u_n &= \dfrac{f_n}{(u_n + \frac{1}{n})^\gamma} \quad \text{in } \Omega, \\ u_n &> 0 \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.8}$$

where $f_n = \min(f, n)$. The following results are proved in [18].

Lemma 3.3.1. *([18, Lemma 3.1]) For each integer $n \in \mathbb{N}$, the problem (3.8) admits a non-negative solution $u_n \in X_0^s(\Omega) \cap L^\infty(\Omega)$ in the sense*

$$\frac{a(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx = \int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx,$$

for every $\varphi \in X_0^s(\Omega)$.

Lemma 3.3.2. *([18, Lemma 3.2]) The sequence $\{u_n\}_{n\in\mathbb{N}}$ is an increasing and for every subset $\omega \subset\subset \Omega$, there exists a positive constant c_ω, not depending on n, such that*

$$u_n(x) \geq c_\omega > 0, \quad \textit{for every } x \in \omega \textit{ and for every } n \in \mathbb{N}.$$

Lemma 3.3.3. *Let $\gamma > 1$, $f \in L^1(\Omega)$ and let $u_n \in X_0^s(\Omega) \cap L^\infty(\Omega)$ be a solution of the problem (3.8). Then the sequence $\{u_n\}$ is uniformly bounded in $H^s_{loc}(\Omega)$.*

Proof. Taking u_n^γ a test function in (3.8), we obtain

$$\int_Q \frac{(u_n(x)-u_n(y))(u_n^\gamma(x)-u_n^\gamma(y))}{|x-y|^{N+2s}}dydx \leq \frac{2\|f\|_{L^1(\Omega)}}{a(N,s)}. \tag{3.9}$$

An application of the item $i)$ in Lemma 1.3.3 yields

$$\int_Q \frac{\left|u_n^{\frac{\gamma+1}{2}}(x)-u_n^{\frac{\gamma+1}{2}}(y)\right|^2}{|x-y|^{N+2s}}dydx \leq \frac{(\gamma+1)^2}{2\gamma a(N,s)}\|f\|_{L^1(\Omega)}.$$

Then by the Sobolev inequality (1.2.1) we get

$$\int_\Omega |u_n(x)|^{\frac{(\gamma+1)}{2}2_s^*}dx \leq \left(S(N,s)\frac{(\gamma+1)^2}{2\gamma a(N,s)}\right)^{\frac{N}{N-2s}}\|f\|_{L^1(\Omega)}^{\frac{N}{N-2s}}.$$

As $\dfrac{(\gamma+1)}{2}2_s^* > 2$, the sequence $\{u_n\}_n$ is uniformly bounded in $L^2(\Omega)$. On the other hand, let ω be a compact subset of Ω. Applying the item $v)$ in Lemma 1.3.3 (recall that $\gamma > 1$) and Lemma 3.3.2 in the left-hand side of the inequality (3.9), we obtain

$$\begin{aligned}
&\int_Q \frac{|u_n(x)-u_n(y)||u_n^\gamma(x)-u_n^\gamma(y)|}{|x-y|^{N+2s}}dydx \\
&\geq \frac{1}{C_\gamma}\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^2|u_n(x)+u_n(y)|^{\gamma-1}}{|x-y|^{N+2s}}dydx \\
&\geq \frac{1}{C_\gamma}\int_\omega\int_\omega \frac{|u_n(x)-u_n(y)|^2|u_n(x)+u_n(y)|^{\gamma-1}}{|x-y|^{N+2s}}dydx \\
&\geq \frac{1}{C_\gamma}(2c_\omega)^{\gamma-1}\int_\omega\int_\omega \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dydx.
\end{aligned}$$

This shows that $\{u_n\}_n$ is uniformly bounded in $H^s_{loc}(\Omega)$. □

Now, let $\phi \in X_0^s(\Omega) \cap L^\infty(\Omega)$ be the solution (see [63]) of the following problem

$$\begin{cases} (-\Delta)^s \phi &= 1 \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{3.10}$$

In order to prove Theorem 3.2.2, we shall prove the following comparison result for the approximate solutions u_n. In the proof of this comparison result, we use Lemma 2.7 and Lemma 2.9 of [73], which require that Ω is a bounded domain which satisfies the condition of the ball. Such a condition is equivalent (see [11, Lemma 2.2]) to say that Ω is a bounded domain of class $\mathcal{C}^{1,1}$.

Lemma 3.3.4. *(Comparison result) Let $\gamma > 1$, $\theta \in (1,2)$ and let u_n be a solution of the problem (3.8). Then there exists a positive constant T not depending on n such that*

$$u_n \geq \underline{u}_n := \left[T\phi^\theta + \frac{1}{n^{\frac{1+\gamma}{2}}}\right]^{\frac{2}{1+\gamma}} - \frac{1}{n}. \tag{3.11}$$

Proof. We shall prove that there exists a sub-solution $\underline{u}_n$ of the approximate problem (3.8), that is

$$\begin{cases} (-\Delta)^s \underline{u}_n &\leq \dfrac{f_n}{(\underline{u}_n + \frac{1}{n})^\gamma} \quad \text{in } \Omega, \\ \underline{u}_n &> 0 \quad \text{in } \Omega, \\ \underline{u}_n &= 0 \quad \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.12}$$

such that $u_n \geq \underline{u}_n$. Let $\underline{u}_n := \psi_n^{\frac{2}{1+\gamma}}(x) - \dfrac{1}{n}$, where we have set $\psi_n = T\phi^\theta + \dfrac{1}{n^{\frac{1+\gamma}{2}}}$ and $T > 0$ is a constant not depending on n and that will be chosen later. We will show that $\underline{u}_n$ satisfies (3.12). Applying the inequality (3.36) with $F(t) = t^{\frac{2}{1+\gamma}}$ yields

$$\begin{aligned} &(-\Delta)^s \underline{u}_n(x) = (-\Delta)^s \left(\psi_n^{\frac{2}{\gamma+1}} - \frac{1}{n}\right)(x) = (-\Delta)^s (F \circ \psi_n)(x) \\ &\leq F'(\psi_n(x))(-\Delta)^s \psi_n(x) - \frac{a(N,s)(\gamma+1)T^2}{2} F''(\psi_n(x)) \int_{\mathbb{R}^N} \frac{|\phi^\theta(x) - \phi^\theta(y)|^2}{|x-y|^{N+2s}} dy \\ &= \frac{2T}{1+\gamma} \psi_n^{\frac{1-\gamma}{1+\gamma}}(x)(-\Delta)^s(\phi^\theta(x)) + \frac{(\gamma-1)T^2}{(\gamma+1)\psi_n^{\frac{2\gamma}{1+\gamma}}(x)} a(N,s) \int_{\mathbb{R}^N} \frac{|\phi^\theta(x) - \phi^\theta(y)|^2}{|x-y|^{N+2s}} dy. \end{aligned}$$

Since $\theta > 1$, the function $g(t) = t^\theta$, $t > 0$, is convex so that one has the identity $g(t) - g(t') \leq g'(t)(t - t')$ which holds true for every t', t. Using the fact that ϕ solves (3.10), we get

$$(-\Delta)^s(\phi^\theta(x)) \leq \theta \phi^{\theta-1}(x)(-\Delta)^s(\phi(x)) = \theta\phi^{\theta-1}(x), \text{ for every } x \in \Omega.$$

Then, for every $x \in \Omega$ we get

$$(-\Delta)^s \underline{u}_n(x) \leq \frac{T}{\psi_n^{\frac{2\gamma}{1+\gamma}}(x)} \Big(\frac{2\theta}{1+\gamma} \psi_n(x) \phi^{\theta-1}(x) + \frac{(\gamma-1)T}{\gamma+1} a(N,s) \int_{\mathbb{R}^N} \frac{|\phi^\theta(x) - \phi^\theta(y)|^2}{|x-y|^{N+2s}} dy \Big). \tag{3.13}$$

On the other hand, let B_R be an open ball with radius $R > 0$ such that $\Omega \subset B_R$ and set $d_1 := dist(\partial\Omega, \partial B_R) > 0$. For every $x \in \Omega$, we can write

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|\phi^\theta(x) - \phi^\theta(y)|^2}{|x-y|^{N+2s}} dy &= \int_{B_R \setminus \Omega} \frac{|\phi^\theta(x) - \phi^\theta(y)|^2}{|x-y|^{N+2s}} dy + \int_{\mathbb{R}^N \setminus B_R} \frac{|\phi^\theta(x) - \phi^\theta(y)|^2}{|x-y|^{N+2s}} dy \\ &\quad + \int_{\Omega} \frac{|\phi^\theta(x) - \phi^\theta(y)|^2}{|x-y|^{N+2s}} dy \\ &= I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

We start by estimating the first integral I_1. Since Ω is a bounded domain of class $\mathcal{C}^{1,1}$, by [73, Lemma 2.7] there exists a positive constant C_1, depending only on Ω and s, such that $|\phi(x)| \leq C_1 \delta^s(x)$ for all $x \in \Omega$, where $\delta(x) := dist(x, \partial\Omega)$. Whence, we get

$$I_1(x) = \int_{B_R \setminus \Omega} \frac{|\phi^\theta(x)|^2}{|x-y|^{N+2s}} dy \leq C_1^{2\theta} \int_{B_R \setminus \Omega} \frac{|\delta^{s\theta}(x)|^2}{|x-y|^{N+2s}} dy.$$

Note that for $(x,y) \in \Omega \times B_R \setminus \Omega$, we have $\delta(x) \leq |x-y|$. Thus, we can write passing to the polar coordinates

$$\begin{aligned} I_1(x) &\leq C_1^{2\theta} \int_{B_R \setminus \Omega} \frac{dy}{|x-y|^{N-2s(\theta-1)}} dy \\ &\leq C_1^{2\theta} \int_{\{0 \leq |z| \leq 2R\}} \frac{dz}{|z|^{N-2s(\theta-1)}} \\ &= C_1^{2\theta} |S^{N-1}| \int_0^{2R} r^{2s(\theta-1)-1} dr = C_1', \end{aligned}$$

with $C_1' = \dfrac{(2R)^{2s(\theta-1)} C_1^{2\theta} |S^{N-1}|}{2s(\theta-1)}$, where from now on $|S^{N-1}|$ stands for the Lebesgue measure of the unit sphere in $\mathbb{R}^N$. For the second integral $I_2(x)$, noticing that

$$|x-y| \geq d_1 := dist(\partial\Omega, \partial B_R) > 0 \text{ for every } (x,y) \in \Omega \times (\mathbb{R}^N \setminus B_R),$$

we can estimate I_2 as follows

$$\begin{aligned} I_2(x) &= \int_{\mathbb{R}^N \setminus B_R} \frac{|\phi^\theta(x)|^2}{|x-y|^{N+2s}} dy \\ &\leq \|\phi\|_{L^\infty(\Omega)}^{2\theta} \int_{|z| \geq d_1} \frac{dz}{|z|^{N+2s}} dy \\ &= \|\phi\|_{L^\infty(\Omega)}^{2\theta} |S^{N-1}| \int_{d_1}^{+\infty} \frac{dr}{r^{2s+1}} = C_2', \end{aligned}$$

where $C_2' = \|\phi\|^{2\theta}_{L^\infty(\Omega)} \dfrac{|S^{N-1}|}{2sd_1^{2s}}$. We now turn to estimate $I_3(x)$. Combining $iii)$ et $iv)$ of Lemma 1.3.3, we obtain

$$|\phi^\theta(x) - \phi^\theta(y)|^2 \leq 2\theta^2|\phi(x) - \phi(y)|^{2\theta} + 8\theta^2\phi^{2(\theta-1)}(x)|\phi(x) - \phi(y)|^2. \tag{3.14}$$

By [73, Lemma 2.9] the function ϕ is $C^\beta(\Omega)$ for all $\beta \in (0, 2s)$. In particular and in what follows we make the choice $\beta \in (s, \min(1, s\theta))$. Furthermore, there exists a constant $C_3 > 0$, depending on Ω, s and β, such that for every $x \in \Omega$

$$|\phi(x) - \phi(y)| \leq C_3|x - y|^\beta \left(\frac{\delta(x)}{2}\right)^{s-\beta}, \tag{3.15}$$

for every $y \in B_{\frac{\delta(x)}{2}}(x)$, where $B_{\frac{\delta(x)}{2}}(x)$ stands for the open ball of radius $\dfrac{\delta(x)}{2}$ centered at x with $\delta(x) := dist(x, \partial\Omega)$. Now, using (3.14) we can write for every $x, y \in \Omega$

$$\begin{aligned} I_3(x) = \int_\Omega \frac{|\phi^\theta(x) - \phi^\theta(y)|^2}{|x - y|^{N+2s}}dy \ &\leq 2\theta^2 \int_\Omega \frac{|\phi(x) - \phi(y)|^{2\theta}}{|x - y|^{N+2s}}dy \\ &+8\theta^2 \int_\Omega \frac{\phi^{2(\theta-1)}(x)|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}}dy. \end{aligned}$$

Splitting the second integral on the right-hand side, we obtain

$$\begin{aligned} I_3(x) \ &\leq 2\theta^2 \int_\Omega \frac{|\phi(x) - \phi(y)|^{2\theta}}{|x - y|^{N+2s}}dy \\ &+8\theta^2 \int_{\{y\in\Omega:|x-y|\geq\frac{\delta(x)}{2}\}} \frac{\phi^{2(\theta-1)}(x)|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}}dy \\ &+8\theta^2 \int_{\{y\in\Omega:|x-y|<\frac{\delta(x)}{2}\}} \frac{\phi^{2(\theta-1)}(x)|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}}dy \\ &:= J_1(x) + J_2(x) + J_3(x). \end{aligned}$$

We shall estimate $J_1(x)$, $J_2(x)$ and $J_3(x)$. For $J_1(x)$, we note that by [73, Proposition 1.1] we have $\phi \in \mathcal{C}^s(\mathbb{R}^N)$. In addition, there exists a positive constant c_3 such that for every x, $y \in \mathbb{R}^N$, $|\phi(x) - \phi(y)| \leq c_3|x - y|^s$. Thus,

$$J_1(x) \leq 2\theta^2 c_3^{2\theta} \int_\Omega \frac{dy}{|x - y|^{N-2s(\theta-1)}}dy.$$

We calculate the integral using the change of variable $z = x - y$. We have

$$\begin{aligned}&\int_\Omega \frac{dy}{|x-y|^{N-2s(\theta-1)}}\\&= \int_{\Omega\cap|x-y|>1} \frac{dy}{|x-y|^{N-2s(\theta-1)}} + \int_{\Omega\cap|x-y|\le 1} \frac{dy}{|x-y|^{N-2s(\theta-1)}}\\&\le |\Omega| + \int_{|z|\le 1} \frac{dz}{|z|^{N-2s(\theta-1)}} = |\Omega| + \frac{|S^{N-1}|}{2s(\theta-1)}.\end{aligned} \tag{3.16}$$

Thus, we obtain

$$J_1(x) \le 2\theta^2 c_3^{2\theta}\Big(|\Omega| + \frac{|S^{N-1}|}{2s(\theta-1)}\Big).$$

For J_2 we use the fact that $\phi \in \mathcal{C}^s(\mathbb{R}^N)$ and $|\phi(x)| \le C_1\delta^s(x)$ for all $x \in \Omega$. By (3.16) we get

$$\begin{aligned}J_2(x) &\le 8\theta^2 c_3^2 \big(2^s C_1\big)^{2(\theta-1)} \int_\Omega \frac{dy}{|x-y|^{N-2s(\theta-1)}}\\&\le 8\theta^2 c_3^2 \big(2^s C_1\big)^{2(\theta-1)} \Big(|\Omega| + \frac{|S^{N-1}|}{2s(\theta-1)}\Big).\end{aligned}$$

While for $J_3(x)$ we use (3.15) and $|\phi(x)| \le C_1\delta^s(x)$ for all $x \in \Omega$. We arrive at

$$J_3(x) \le 8\theta^2\big(2^{\beta-s}C_1^{\theta-1}C_3\big)^2 \int_{\{y\in\Omega:|x-y|<\frac{\delta(x)}{2}\}} \frac{\delta^{2(s\theta-\beta)}(x)}{|x-y|^{N-2(\beta-s)}}dy.$$

The fact that $\beta \in (s, \min(1, s\theta))$ and that Ω is bounded, enables us to get

$$\begin{aligned}&J_3(x) \le\\&8\theta^2\big(2^{\beta-s}C_1^{\theta-1}C_3\big)^2\big(diam(\Omega)\big)^{2(s\theta-\beta)} \int_{\{y\in\Omega:|x-y|<\frac{\delta(x)}{2}\}} \frac{dy}{|x-y|^{N-2(\beta-s)}}\\&\le 4\theta^2\big(2^{\beta-s}C_1^{\theta-1}C_3\big)^2\big(diam(\Omega)\big)^{2s(\theta-1)}\frac{|S^{N-1}|}{\beta-s},\end{aligned}$$

where $diam(\Omega)$ stands for the diameter of Ω. Finally, there exists a constant $C_3' > 0$ depending on Ω, R, N, s, θ and β, such that

$$I_3(x) \le C_3'.$$

Let $T_0 = \min(1, f_0)$ and let us choose T small enough such that

$$\begin{aligned}0 &< T\Big[\frac{2\theta}{1+\gamma}\big(T\|\phi\|^\theta_{L^\infty(\Omega)} + 1)\big)\|\phi\|^{\theta-1}_{L^\infty(\Omega)} + \frac{3(\gamma-1)T}{\gamma+1}a(N,s)\max(C_1', C_2', C_3')\Big]\\&\le T_0.\end{aligned}$$

Going back to (3.13), we deduce that for every $x \in \Omega$

$$(-\Delta)^s \underline{u}_n(x) \le \frac{T_0}{\psi_n^{\frac{2\gamma}{1+\gamma}}(x)},$$

which yields

$$(-\Delta)^s \underline{u}_n(x) \leq \frac{f_n(x)}{(\underline{u}_n + \frac{1}{n})^\gamma}.$$

Thus, $\underline{u}_n$ is a sub-solution of (3.8). Now, we prove that $u_n(x) \geq \underline{u}_n(x)$ for every $x \in \Omega$. Assume by contradiction that there exists $\xi \in \Omega$ such that

$$u_n(\xi) < \underline{u}_n(\xi). \tag{3.17}$$

Then we have

$$\begin{aligned}(-\Delta)^s(u_n - \underline{u}_n)(\xi) &= (-\Delta)^s u_n(\xi) - (-\Delta)^s \underline{u}_n(\xi)\\ &\geq f_n(\xi)\left[\frac{1}{(u_n(\xi) + \frac{1}{n})^\gamma} - \frac{1}{(\underline{u}_n(\xi) + \frac{1}{n})^\gamma}\right] > 0.\end{aligned}$$

It follows from the weak maximum principle [81] that $(u_n - \underline{u}_n)(\xi) \geq 0$, which contradicts (3.17). Therefore, we have

$$u_n(x) + \frac{1}{n} \geq \psi_n^{\frac{2}{1+\gamma}}(x) = \left[T\phi^\theta(x) + \frac{1}{n^{\frac{1+\gamma}{2}}}\right]^{\frac{2}{1+\gamma}}.$$

□

3.3.2 The case $0 < \gamma < 1$: Proof of Theorem 3.2.1

In order to prove the existence of solutions for the problem (3.1), we first need to prove some a priori estimates on u_n.

3.3.2.1 A priori estimates

Lemma 3.3.5. *Let $f \geq 0$, $f \in L^m(\Omega)$, with $1 \leq m < \overline{m} := \dfrac{2N}{N + 2s + \gamma(N-2s)}$, and u_n be a solution of the problem (3.8). If $0 < \gamma < 1$, then $\{u_n\}$ is uniformly bounded in $W_0^{s_1,\overline{q}}(\Omega)$ for all $s_1 < s$, where $\overline{q} = \dfrac{Nm(1+\gamma)}{N - sm(1-\gamma)}$. Moreover, $\{u_n\}$ is uniformly bounded in $L^\sigma(\Omega)$, where $\sigma = \dfrac{Nm(1+\gamma)}{N - 2sm}$.*

Proof. Let $n \in \mathbb{N}$, $n \geq 1$, and let $\gamma \leq \theta < 1$ to be chosen later. Let $0 < \varepsilon < \dfrac{1}{n}$. By [63, Proposition 3.], the function $(u_n + \varepsilon)^\theta - \varepsilon^\theta$ is an admissible test function in (3.8). Taking it so, it yields

$$\int_\Omega\int_\Omega \frac{(u_n(x)-u_n(y))((u_n(x)+\varepsilon)^\theta-(u_n(y)+\varepsilon)^\theta)}{|x-y|^{N+2s}}dydx$$
$$\leq \frac{2}{a(N,s)}\int_\Omega f_n(u_n(x)+\varepsilon)^{\theta-\gamma}dx.$$

Passing to the limit as ε tends to 0, we obtain

$$\int_\Omega\int_\Omega \frac{(u_n(x)-u_n(y))(u_n^\theta(x)-u_n^\theta(y))}{|x-y|^{N+2s}}dydx \leq \frac{2}{a(N,s)}\int_\Omega f_n u_n(x)^{\theta-\gamma}dx. \tag{3.18}$$

By the item $i)$ of Lemma 1.3.3, we can minimize the term in the left-hand side of (3.18) as follows

$$\int_\Omega\int_\Omega \frac{\left|u_n^{\frac{\theta+1}{2}}(x)-u_n^{\frac{\theta+1}{2}}(y)\right|^2}{|x-y|^{N+2s}}dydx \leq \frac{(\theta+1)^2}{2a(N,s)\theta}\int_\Omega f_n u_n^{\theta-\gamma}dx.$$

Applying the fractional Sobolev inequality, we obtain

$$\int_\Omega |u_n(x)|^{\frac{N(\theta+1)}{N-2s}}dx \leq \left[\frac{S(N,s)(\theta+1)^2}{2a(N,s)\theta}\right]^{\frac{N}{N-2s}}\left[\int_\Omega f_n u_n^{\theta-\gamma}dx\right]^{\frac{N}{N-2s}}. \tag{3.19}$$

• If $m=1$, then the choice $\theta=\gamma$ gives

$$\int_\Omega |u_n(x)|^{\frac{N(\gamma+1)}{N-2s}}dx \leq \left[\frac{S(N,s)(\gamma+1)^2}{2a(N,s)\gamma}\right]^{\frac{N}{N-2s}}\|f\|_{L^1(\Omega)}^{\frac{N}{N-2s}}. \tag{3.20}$$

• While if $1<m<\overline{m}$ and $\gamma<\theta<1$, an application of Hölder's inequality in the right-hand side term of (3.19) with the exponents m and $m' := \dfrac{m}{m-1}$, gives

$$\begin{aligned}&\int_\Omega |u_n(x)|^{\frac{N(\theta+1)}{N-2s}}dx \leq \\ &\left[\frac{S(N,s)(\theta+1)^2}{2a(N,s)\theta}\right]^{\frac{N}{N-2s}}\|f\|_{L^m(\Omega)}^{\frac{N}{N-2s}}\left(\int_\Omega |u_n|^{(\theta-\gamma)m'}dx\right)^{\frac{N}{m'(N-2s)}}.\end{aligned} \tag{3.21}$$

We now choose θ to be such that $\dfrac{N(\theta+1)}{N-2s}=(\theta-\gamma)m'$, that is

$$\theta=\frac{N(m-1)+\gamma m(N-2s)}{N-2sm}.$$

Observe that the assumption $m<\overline{m}$ implies $\theta<1$ and since $\gamma>0$ we have $\gamma<\theta$. This choice of θ yields

$$\frac{N(\theta+1)}{N-2s}=\frac{Nm(1+\gamma)}{N-2sm}=\sigma.$$

Noticing that $\dfrac{N}{m'(N-2s)} < 1$ and using (3.21) we deduce the following inequality

$$\int_\Omega |u_n(x)|^{\frac{Nm(1+\gamma)}{N-2sm}} dx \le \left[\frac{S(N,s)(\theta+1)^2}{2a(N,s)\theta}\right]^{\frac{Nm}{N-2sm}} \|f\|_{L^m(\Omega)}^{\frac{Nm}{N-2sm}}. \tag{3.22}$$

Thus, from (3.20) and (3.22) we conclude that the sequence $\{u_n\}_n$ is uniformly bounded in $L^\sigma(\Omega)$ for $\sigma = \dfrac{Nm(1+\gamma)}{N-2sm}$ and $1 \le m < \overline{m}$.

Now, going back to the inequality (3.18) and following exactly the same lines as above, that is if $m = 1$ we choose $\theta = \gamma$ while if $1 \le m < \overline{m}$ we choose $\theta = \dfrac{N(m-1)+\gamma m(N-2s)}{N-2sm} < 1$. In both cases, applying the Hölder inequality we obtain

$$\int_\Omega\int_\Omega \frac{(u_n(x)-u_n(y))(u_n^\theta(x)-u_n^\theta(y))}{|x-y|^{N+2s}} dydx \le C, \tag{3.23}$$

where C is a positive constant not depending on n. Let $s_1 \in (0,s)$ be fixed and let $\overline{q} = \dfrac{Nm(1+\gamma)}{N-sm(1-\gamma)}$. We set $\theta = \dfrac{N(m-1)+\gamma m(N-2s)}{N-2sm}$ for $1 \le m < \overline{m}$ (we note that $\theta = \gamma$ if $m = 1$). We note that $\overline{q} \ge m(1+\gamma) > 1$ and the assumption $m < \overline{m}$ implies $\overline{q} < 2$. Thus, observe that $N + \overline{q}s_1$ can be splitted as follows

$$N + \overline{q}s_1 = \frac{\overline{q}}{2}N + \overline{q}s + \frac{2-\overline{q}}{2}N - \overline{q}(s-s_1).$$

Hence, setting $\tilde{\Omega} := \big\{y \in \Omega : u_n(y) \neq u_n(x)\big\}$ we can write

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}} dydx &= \int_\Omega\int_{\tilde{\Omega}} \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{\frac{\overline{q}}{2}N+\overline{q}s}} \times \frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))} \\ &\quad \times \frac{(u_n(x)-u_n(y))}{(u_n^\theta(x)-u_n^\theta(y))} \times \frac{dydx}{|x-y|^{\frac{2-\overline{q}}{2}N-\overline{q}(s-s_1)}}.\end{aligned}$$

Observe that the quantity in the middle of the product inside the integral can be written as follows

$$\frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))} = \left(\frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))}\right)^{\frac{\overline{q}}{2}} \times \left(\frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))}\right)^{\frac{2-\overline{q}}{2}},$$

we obtain

$$\begin{aligned}&\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}} dydx \\ &= \int_\Omega\int_{\{y\in\Omega: u_n(y)\neq u_n(x)\}} \left[\frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{\frac{\overline{q}}{2}N+\overline{q}s}} \times \left(\frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))}\right)^{\frac{\overline{q}}{2}}\right] \\ &\times \left[\left(\frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))}\right)^{\frac{2-\overline{q}}{2}} \times \frac{(u_n(x)-u_n(y))}{(u_n^\theta(x)-u_n^\theta(y))} \times \frac{1}{|x-y|^{\frac{2-\overline{q}}{2}N-\overline{q}(s-s_1)}}\right] dydx.\end{aligned}$$

Now using Hölder's inequality with the exponents $\frac{2}{\overline{q}}$ and $\frac{2}{2-\overline{q}}$, we obtain

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydx\\
&\leq \left[\int_\Omega\int_{\tilde{\Omega}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}} \times \frac{|u_n^\theta(x)-u_n^\theta(y)|}{|u_n(x)-u_n(y)|}dydx\right]^{\frac{\overline{q}}{2}}\\
&\times\left[\int_\Omega\int_{\tilde{\Omega}} \frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))} \times \left(\frac{(u_n(x)-u_n(y))}{(u_n^\theta(x)-u_n^\theta(y))}\right)^{\frac{2}{2-\overline{q}}} \times \frac{dydx}{|x-y|^{N-\beta}}\right]^{\frac{2-\overline{q}}{2}}\\
&= \left[\int_\Omega\int_{\tilde{\Omega}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}} \times \frac{|u_n^\theta(x)-u_n^\theta(y)|}{|u_n(x)-u_n(y)|}dydx\right]^{\frac{\overline{q}}{2}}\\
&\times\left[\int_\Omega\int_{\tilde{\Omega}} \left(\frac{(u_n(x)-u_n(y))}{(u_n^\theta(x)-u_n^\theta(y))}\right)^{\frac{2}{2-\overline{q}}} \times \frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))} \times \frac{dydx}{|x-y|^{N-\beta}}\right]^{\frac{2-\overline{q}}{2}},
\end{aligned} \tag{3.24}$$

where we have set $\beta = \frac{2\overline{q}(s-s_1)}{2-\overline{q}}$. Then,

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydx \leq\\
&\left(\int_\Omega\int_\Omega \frac{(u_n(x)-u_n(y))(u_n^\theta(x)-u_n^\theta(y))}{|x-y|^{N+2s}}dydx\right)^{\frac{\overline{q}}{2}}\\
&\times\left(\int_\Omega\int_{\tilde{\Omega}} \left(\frac{u_n(x)-u_n(y)}{u_n^\theta(x)-u_n^\theta(y)}\right)^{\frac{\overline{q}}{2-\overline{q}}} \times \frac{dydx}{|x-y|^{N-\beta}}\right)^{\frac{2-\overline{q}}{2}}.
\end{aligned}$$

Using the item $ii)$ of Lemma 1.3.3 and the inequality (3.23), we obtain

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydx \leq\\
&C_1\left(\int_\Omega\int_\Omega \left(u_n^{\frac{\overline{q}(1-\theta)}{2-\overline{q}}}(x) + u_n^{\frac{\overline{q}(1-\theta)}{2-\overline{q}}}(y)\right) \times \frac{dydx}{|x-y|^{N-\beta}}\right)^{\frac{2-\overline{q}}{2}},
\end{aligned}$$

where C_1 is a positive constant not depending on n. By x/y symmetry, there exists a constant C_2, not depending on n, such that

$$\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydx \leq C_2\left(\int_\Omega u_n^{\frac{\overline{q}(1-\theta)}{2-\overline{q}}}(x)\left[\int_\Omega \frac{dy}{|x-y|^{N-\beta}}\right]dx\right)^{\frac{2-\overline{q}}{2}}.$$

Observing that $\frac{\overline{q}(1-\theta)}{2-\overline{q}} = \sigma := \frac{Nm(1+\gamma)}{N-2s}$ and having in mind (3.16) we get

$$\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydx \leq C_3,$$

where C_3 is a positive constant not depending on n. Thus, $\{u_n\}$ is uniformly bounded in $W_0^{s_1,\overline{q}}(\Omega)$ for every $s_1 < s$. □

Remark 3.3.1. *Note that we can repeat the same lines as in the proof of Lemma 3.3.5 above with the exponent q instead of $\overline{q}$ in (3.24), with $1 \leq q \leq \overline{q}$. We obtain that $\{u_n\}$ is uniformly bounded in $W_0^{s_1,q}(\Omega)$ for all $1 \leq q \leq \overline{q}$ and for every $s_1 < s$ and $1 \leq m < \overline{m}$.*

3.3.2.2 Passage to the limit

Now, under the assumptions of Theorem 3.2.1, we are going to prove the existence of solution u to (3.1).

Proof. of Theorem 3.2.1.

From Lemma 3.3.5 and by the compact embedding of $W_0^{s_1,\overline{q}}(\Omega)$ into $L^1(\Omega)$ (see [45, Corollary 7.2] or [44, Theorem 4.54]), there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a measurable function $u \in W_0^{s_1,\overline{q}}(\Omega)$ such that

$$
\begin{aligned}
&u_n \rightharpoonup u \text{ weakly in } W_0^{s_1,\overline{q}}(\Omega),\\
&u_n \to u \text{ in norm in } L^1(\Omega),\\
&u_n \to u \text{ a.e. in } \mathbb{R}^N.
\end{aligned}
$$

Then

$$
\frac{u_n(x)-u_n(y)}{|x-y|^{N+2s}} \to \frac{u(x)-u(y)}{|x-y|^{N+2s}} \text{ a.e. in } Q.
$$

Let $\rho > 0$ be a small enough real number that we will choose later. For any $\varphi \in C_0^\infty(\Omega)$ we have

$$
\begin{aligned}
&\int_\Omega \int_\Omega \left[\frac{|(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}}\right]^{1+\rho} dydx\\
&\leq \int_\Omega \int_\Omega \frac{|u_n(x)-u_n(y)|^{1+\rho}(\|\nabla\varphi\|_{L^\infty(\Omega)}|x-y|)^{1+\rho}}{|x-y|^{N+(1+\rho)s_1}} \frac{dydx}{|x-y|^{\rho N+(1+\rho)(2s-s_1)}}\\
&\leq \|\nabla\varphi\|_{L^\infty(\Omega)}^{1+\rho} \int_\Omega \int_\Omega \frac{|u_n(x)-u_n(y)|^{1+\rho}|x-y|^{(1+\rho)(1+s_1-2s)-\rho N}}{|x-y|^{N+(1+\rho)s_1}} dydx.
\end{aligned}
$$

We now choose ρ to be such that $(1+\rho)(1+s_1-2s)-\rho N \geq 0$. To do so, we consider s_1 to be very close of s. Precisely, we impose on s_1 the condition

$$
\max(0, 1-3s) < s - s_1 < 1 - s.
$$

We point out that with this range of values of s_1 and with the assumption $N > 2s$, we obtain

$$
1 + s_1 - 2s > 0 \text{ and } N - 1 - s_1 + 2s > 0.
$$

Thus, the fact that $(1+\rho)(1+s_1-2s)-\rho N \geq 0$ is equivalent to $0<\rho \leq \dfrac{1+s_1-2s}{N-1-s_1+2s}$. Therefore, we have

$$\begin{aligned}&\int_\Omega\int_\Omega\left[\frac{|(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}}\right]^{1+\rho}dydx\\&\leq \|\nabla\varphi\|^{1+\rho}_{L^\infty(\Omega)}diam(\Omega)^{(1+\rho)(1+s_1-2s)-\rho N}\int_\Omega\int_\Omega\frac{|u_n(x)-u_n(y)|^{1+\rho}}{|x-y|^{N+(1+\rho)s_1}}dydx.\end{aligned}\tag{3.25}$$

Now we have to make a choice of ρ to prove that the right-hand integral in (3.25) is uniformly bounded. By Remark 3.3.1 we have the uniform boundedness of $\{u_n\}_n$ in $W^{s_1,q}_0(\Omega)$ for every $1 \leq q \leq \overline{q} = \dfrac{Nm(1+\gamma)}{N-sm(1-\gamma)}$. So it is sufficient to choose ρ such that $1+\rho \leq \overline{q} = \dfrac{Nm(1+\gamma)}{N-sm(1-\gamma)}$. Thus, the choice we need for ρ is the following

$$0<\rho\leq \min\Big(\frac{N(m-1)+m\gamma(N-s)+sm}{N-sm(1-\gamma)},\frac{1+s_1-2s}{N-1-s_1+2s}\Big).$$

Therefore, there is a constant $C>0$, not depending on n, such that

$$\sup_n\int_\Omega\int_\Omega\left[\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\right]^{1+\rho}dydx\leq C.$$

Finally, by De La Vallée Poussin and Dunford-Pettis theorems the sequence

$$\Big\{\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\Big\}$$

is equi-integrable in $L^1(\Omega\times\Omega)$. Now, inserting $\varphi\in C_0^\infty(\Omega)$ as a test function in (3.8) yields

$$\frac{a(N,s)}{2}\int_Q\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx=\int_\Omega\frac{f_n\varphi}{(u_n+\frac1n)^\gamma}dx.\tag{3.26}$$

We split the integral in the left-hand side of (3.26) into three integrals as follows

$$\begin{aligned}&\int_Q\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\&=\int_\Omega\int_\Omega\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\&+\int_\Omega\int_{\mathcal{C}\Omega}\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\&+\int_{\mathcal{C}\Omega}\int_\Omega\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\&=I_1+I_2+I_3.\end{aligned}\tag{3.27}$$

By Vitali's lemma we have

$$\lim_{n\to\infty}\int_\Omega\int_\Omega \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx$$
$$=\int_\Omega\int_\Omega \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$

For the second integral I_2 in (3.27), we start noticing that since $u_n(y)=\varphi(y)=0$ for every $y\in\mathcal{C}\Omega$ we can write

$$\left|\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\right| = \frac{|u_n(x)\varphi(x)|}{|x-y|^{N+2s}} \text{ for every } (x,y)\in\Omega\times\mathcal{C}\Omega.$$

As a consequence of the convergence in norm of the sequence $\{u_n\}$ in $L^1(\Omega)$ there exist a subsequence of $\{u_n\}$ still indexed by n and a positive function g in $L^1(\Omega)$ such that

$$|u_n(x)|\le g(x) \text{ a.e. in } \Omega,$$

which enables us to get

$$\frac{|(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}} \le \frac{|g(x)\varphi(x)|}{|x-y|^{N+2s}} \text{ a.e. in } (x,y)\in\Omega\times\mathcal{C}\Omega$$

and so we can write

$$\begin{aligned}\int_\Omega\int_{\mathcal{C}\Omega}\frac{|g(x)\varphi(x)|}{|x-y|^{N+2s}}dydx &= \int_{supp(\varphi)}\int_{\mathcal{C}\Omega}\frac{|g(x)\varphi(x)|}{|x-y|^{N+2s}}dydx\\ &\le \|\varphi\|_{L^\infty(\Omega)}\int_{supp(\varphi)}|g(x)|\left[\int_{\mathcal{C}\Omega}\frac{dy}{|x-y|^{N+2s}}\right]dx.\end{aligned}$$

Since $supp(\varphi)$ is a compact subset in Ω, we have

$$|x-y|\ge d_2 := dist(supp(\varphi),\partial\Omega)>0 \text{ for every } (x,y)\in supp(\varphi)\times\mathcal{C}\Omega.$$

Hence passing to the polar coordinates, an easy computation leads to

$$\int_{\mathcal{C}\Omega}\frac{dy}{|x-y|^{N+2s}} = \int_{\{z\in\mathbb{R}^N:|z|\ge d_2\}}\frac{dz}{|z|^{N+2s}} = \int_{d_2}^{+\infty}\int_{v=1}\frac{dvdr}{r^{2s+1}} = \frac{|S^{N-1}|}{2sd_2^{2s}}.$$

This shows that the function $(x,y)\to\dfrac{|g(x)\varphi(x)|}{|x-y|^{N+2s}}$ belongs to $L^1(\Omega\times\mathcal{C}\Omega)$. Therefore, by the Lebesgue dominated convergence theorem we obtain

$$\lim_{n\to\infty}\int_\Omega\int_{\mathcal{C}\Omega} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx$$
$$=\int_\Omega\int_{\mathcal{C}\Omega} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$

By x/y symmetry, the third integral I_3 in (3.27) can be treated in the similar way. Finally, we conclude that

$$\lim_{n\to\infty}\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx$$
$$=\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx,$$

for all $\varphi\in\mathcal{C}_0^\infty(\Omega)$. Now, for what concerns the right-hand side of (3.26), by virtue of Lemma 3.3.2, for any $\varphi\in\mathcal{C}_0^\infty(\Omega)$ with $supp(\varphi)=\omega$, there exists a constant $c_\omega>0$ not depending on n such that

$$0\le\left|\frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}\right|\le\frac{|f||\varphi|}{c_\omega^\gamma}\in L^1(\Omega).$$

So that by the Lebesgue dominated convergence theorem we get

$$\lim_{n\to\infty}\int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx=\int_\Omega\frac{f\varphi}{u^\gamma}dx.$$

Finally, passing to the limit in (3.26) as $n\to+\infty$ we obtain

$$\frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx=\int_\Omega\frac{f\varphi}{u^\gamma}dx,$$

for all $\varphi\in\mathcal{C}_0^\infty(\Omega)$. That is u is a weak solution of (3.1). Furthermore, from (3.20) and (3.22) we conclude by Fatou's lemma that $u\in L^\sigma(\Omega)$ with $\sigma=\dfrac{Nm(1+\gamma)}{N-2sm}$ and $1\le m<\overline{m}$. □

3.3.3 The case $\gamma>1$: Proof of Theorem 3.2.2

3.3.3.1 A priori estimates

Lemma 3.3.6. *Let $0<f_0\le f\in L^m(\Omega)$, $m>1$, where f_0 is a positive constant. Let $\gamma>1$, $s\in(0,1)$ and let u_n be a solution of the problem (3.8). Then the sequence $\{u_n^\alpha\}_n$ is uniformly bounded in $X_0^s(\Omega)$ for every $\alpha\in\Big(\max\Big(\frac{1}{2},\frac{(\gamma+1)(2sm-m+1)}{4sm}\Big),\frac{\gamma+1}{2}\Big]$. Furthermore, if γ satisfies*

$$\Big(m(2s-1)+1\Big)\gamma<m(2s+1)-1, \tag{3.28}$$

then $\{u_n\}_n$ is uniformly bounded in $X_0^s(\Omega)$.

Proof. We shall prove a priori estimates on u_n^α in $X_0^s(\Omega)$ for every α such that $\max\Big(\frac{1}{2},\frac{(\gamma+1)(2sm-m+1)}{4sm}\Big)<\alpha\le\frac{\gamma+1}{2}$. Let $n\ge1$ and let $0<\varepsilon<\dfrac{1}{n}$. For $\eta>0$, taking $(u_n+\varepsilon)^\eta-\varepsilon^\eta$ as a test function in (3.8), we obtain

$$\begin{aligned}&\frac{a(N,s)}{2}\int_Q\frac{(u_n(x)-u_n(y))((u_n(x)+\varepsilon)^\eta-(u_n(y)+\varepsilon)^\eta)}{|x-y|^{N+2s}}dydx\\&\le\int_\Omega\frac{f_n}{(u_n(x)+\frac{1}{n})^{\gamma-\eta}}dx.\end{aligned}$$

The passage to the limit in ε yields

$$\int_Q\frac{(u_n(x)-u_n(y))(u_n^\eta(x)-u_n^\eta(y))}{|x-y|^{N+2s}}dydx\le\frac{2}{a(N,s)}\int_\Omega\frac{f_n}{(u_n(x)+\frac{1}{n})^{\gamma-\eta}}dx.$$

An application of the item $i)$ in Lemma 1.3.3 and the Hölder inequality lead to

$$\begin{aligned}&\int_Q\frac{|u_n^{\frac{\eta+1}{2}}(x)-u_n^{\frac{\eta+1}{2}}(y)|^2}{|x-y|^{N+2s}}dydx\\&\le C(\eta,N,s)\|f\|_{L^m(\Omega)}\bigg(\int_\Omega\frac{dx}{\big(u_n(x)+\frac{1}{n}\big)^{(\gamma-\eta)m'}}\bigg)^{\frac{1}{m'}}.\end{aligned}$$

Let η be such that $0<\eta\le\gamma$. We can use (3.11) to get

$$\begin{aligned}&\int_Q\frac{|u_n^{\frac{\eta+1}{2}}(x)-u_n^{\frac{\eta+1}{2}}(y)|^2}{|x-y|^{N+2s}}dydx\\&\le C(\eta,N,s)\|f\|_{L^m(\Omega)}\Bigg(\int_\Omega\frac{dx}{\Big(T\phi^\theta(x)+\frac{1}{n^{\frac{1+\gamma}{2}}}\Big)^{\frac{2(\gamma-\eta)m'}{1+\gamma}}}\Bigg)^{\frac{1}{m'}}.\end{aligned}$$

From [19, Lemma 4.2] we know that there exists a positive constant $C>0$, depending only on Ω and s, such that for every $x\in\Omega$, $\phi(x)\ge C\delta^s(x)$, where $\delta(x):=dist(x,\partial\Omega)$. Using this, the above inequality reads as

$$\int_Q\frac{|u_n^{\frac{\eta+1}{2}}(x)-u_n^{\frac{\eta+1}{2}}(y)|^2}{|x-y|^{N+2s}}dydx\le C\|f\|_{L^m(\Omega)}\bigg[\int_\Omega\frac{dx}{\delta^{\frac{2s(\gamma-\eta)m'}{\gamma+1}\theta}(x)}\bigg]^{\frac{1}{m'}}.$$

Choosing $\alpha=\dfrac{\eta+1}{2}>\dfrac{1}{2}$, we must seek for the range of α that ensures the convergence of the integral in the right-hand side in the above inequality. If $\alpha=\dfrac{\gamma+1}{2}$ the integral obviously converges. If $\alpha<\dfrac{\gamma+1}{2}$ it is sufficient to have $\dfrac{2s(\gamma+1-2\alpha)m'}{\gamma+1}\theta<1$. If it is so, we get $\theta<\dfrac{\gamma+1}{2s(\gamma+1-2\alpha)m'}$. In order that $\theta\in(1,2)$ exists, it suffices to have $1<\dfrac{\gamma+1}{2s(\gamma+1-2\alpha)m'}$. This yields, $\dfrac{2sm-m+1}{4sm}(\gamma+1)<\alpha$. Finally, if $\max\Big(\dfrac{1}{2},\dfrac{(\gamma+1)(2sm-m+1)}{4sm}\Big)<\alpha\le$

$\frac{\gamma+1}{2}$ then the sequence $\{u_n^\alpha\}_n$ is uniformly bounded in $X_0^s(\Omega)$.

Furthermore, if the condition (3.28) holds then $\frac{(\gamma+1)(2sm-m+1)}{4sm} < 1$ and so we can chose $\alpha = 1$ obtaining the uniform boundedness of the sequence $\{u_n\}_n$ in $u \in X_0^s(\Omega)$. □

3.3.3.2 Passage to the limit

Proof. of Theorem 3.2.2.

By Lemma 3.3.6 the sequence $\{u_n^\alpha\}_n$ is uniformly bounded in $X_0^s(\Omega)$ and by the compact embedding in [45, Corollary 7.2] (see also [44, Theorem 4.54.]), there exists a subsequence of $\{u_n^\alpha\}_n$, still indexed by n, and a function $v_\alpha \in X_0^s(\Omega)$ such that $u_n^\alpha \to v_\alpha$ in $L^1(\Omega)$ and $u_n^\alpha \to v_\alpha$ a.e. in $\mathbb{R}^N$. In particular, the sequence $\{u_n\}$ is uniformly bounded in $L^{\frac{\gamma+1}{2}}(\Omega)$ and as $\frac{\gamma+1}{2} > 1$ it is also uniformly bounded in $L^1(\Omega)$. Thanks to Lemma 3.3.2, the sequence $\{u_n\}_n$ is increasing so that by Beppo-Levi's theorem the function $u(x) := \lim_{n\to\infty} u_n(x)$, for a.e. $x \in \Omega$, belongs to $L^1(\Omega)$. Since $u_n = 0$ on $\mathbb{R}^N \setminus \Omega$ we can extend u outside of Ω by setting $u = 0$ on $\mathbb{R}^N \setminus \Omega$ and then we obtain $u_n \to u$ a.e. in $\mathbb{R}^N$. By the uniqueness of the limit we get $v_\alpha = u^\alpha$ a.e. in $\mathbb{R}^N$. Therefore, $u^\alpha \in X_0^s(\Omega)$ for every $\max\left(\frac{1}{2}, \frac{(\gamma+1)(2sm-m+1)}{4sm}\right) < \alpha \leq \frac{\gamma+1}{2}$.

If the condition (3.28) holds, we can take $\alpha = 1$ obtaining $u \in X_0^s(\Omega)$.

Now, inserting $\varphi \in C_0^\infty(\Omega)$ as a test function in (3.8) we have

$$\frac{a(N,s)}{2} \int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dydx = \int_\Omega \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} dx. \tag{3.29}$$

The fact that $u_n \to u$ a.e. in $\mathbb{R}^N$ implies

$$\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \to \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \text{ a.e. in } \mathbb{R}^N \times \mathbb{R}^N.$$

By Lemma 3.3.3, the sequence $\{u_n\}_n$ is uniformly bounded in $H^s_{loc}(\Omega)$ and so we have

$$\frac{u_n(x)-u_n(y)}{|x-y|^{\frac{N+2s}{2}}} \rightharpoonup \frac{u(x)-u(y)}{|x-y|^{\frac{N+2s}{2}}} \text{ weakly in } L^2(K \times K) \tag{3.30}$$

for every $K \subset\subset \Omega$. Now we choose the compact K to be such that $supp(\varphi) \subset K$ and set $d_3 := dist(supp(\varphi), \partial K)) > 0$. Using the fact that $u_n(x) = u_n(y) = 0$ for every $(x,y) \in \mathcal{C}\Omega \times \mathcal{C}\Omega$ and $\varphi(x) = \varphi(y) = 0$ for every $(x,y) \in \mathcal{C}K \times \mathcal{C}K$, we can split the integral in the left-hand side of (3.29) as follows

$$\begin{aligned}
&\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx \\
&= \int_K\int_K \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx \\
&+ \int_K\int_{\mathcal{C}K} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx \\
&+ \int_{\mathcal{C}K}\int_K \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx. \\
&= I_n^1 + I_n^2 + I_n^3.
\end{aligned}$$

In order to pass to the limit as $n \to +\infty$ in I_n^1, observe that for all $\varphi \in \mathcal{C}_0^\infty(\Omega) \subset H^s(\Omega)$, we have

$$\frac{\varphi(x)-\varphi(y)}{|x-y|^{\frac{N+2s}{2}}} \in L^2(\Omega\times\Omega).$$

Then, by (3.30) we get

$$\lim_{n\to\infty} I_n^1 = \int_K\int_K \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$

For the integrals I_n^2 and I_n^3, we follow some ideas as in the the proof of Theorem 3.2.1 claiming that

$$\lim_{n\to\infty} I_n^2 = \int_K\int_{\mathcal{C}K} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx$$

and

$$\lim_{n\to\infty} I_n^3 = \int_{\mathcal{C}K}\int_K \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$

Indeed, let us start with the second integral I_n^2. For every $(x,y)\in K\times\mathcal{C}K$, using the fact that $\varphi(y)=0$ for every $y\in\mathcal{C}K$, we have

$$\begin{aligned}
\frac{|(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}} &\leq \frac{|u_n(x)\varphi(x)|}{|x-y|^{N+2s}} + \frac{|u_n(y)\varphi(x)|}{|x-y|^{N+2s}} \\
&= |G_n(x,y)| + |H_n(x,y)|.
\end{aligned} \tag{3.31}$$

We shall prove that the sequence $\{H_n(x,y)\}$ is uniformly bounded in $L^1(K\times\mathcal{C}K)$. Since $\varphi(x)=0$ on $K\backslash supp(\varphi)$ and $u_n(y)=0$ on $\mathcal{C}\Omega$, we obtain

$$\int_K\int_{\mathcal{C}K} |H_n(x,y)|dydx = \int_{supp(\varphi)}\int_{\Omega\backslash K} \frac{|u_n(y)\varphi(x)|}{|x-y|^{N+2s}}dydx.$$

Since for every $(x,y) \in supp(\varphi) \times \mathcal{C}K$, $|x-y| \geq d_3 := dist(supp(\varphi), \partial K) > 0$, we obtain the following estimation

$$\int_K \int_{\mathcal{C}K} |H_n(x,y)| dydx \leq \frac{\|\varphi\|_{L^\infty(\Omega)} |supp(\varphi)|}{d_3^{N+2s}} \|u_n\|_{L^1(\Omega)}.$$

As the sequence $\{u_n\}$ is increasing, then so is $\{H_n(x,y)\}$ and by Beppo-Levi's theorem and the fact that $u_n \to u$ a.e. in $\mathbb{R}^N$, we obtain

$$H_n(x,y) \to \frac{u(y)\varphi(x)}{|x-y|^{N+2s}} \text{ in } L^1(K \times \mathcal{C}K).$$

We deduce that there exist a subsequence of $\{u_n\}$, still indexed by n, and a positive function $h \in L^1(K \times \mathcal{C}K)$ such that

$$|H_n(x,y)| \leq h(x,y) \text{ a.e. in } K \times \mathcal{C}K. \tag{3.32}$$

As regards the sequence $\{G_n(x,y)\}$, we write

$$\begin{aligned}\int_K \int_{\mathcal{C}K} |G_n(x,y)| dydx &= \int_{supp(\varphi)} |u_n(x)\varphi(x)| \int_{\mathcal{C}K} \frac{dy}{|x-y|^{N+2s}} dx \\ &\leq \frac{|S^{N-1}| \|\varphi\|_{L^\infty(\Omega)} \|u_n\|_{L^1(\Omega)}}{d_3^{2s} 2s}.\end{aligned}$$

As above, the sequence $\{G_n(x,y)\}$ is increasing and by Beppo-Levi's theorem and the fact that $u_n \to u$ a.e. in $\mathbb{R}^N$, we obtain

$$G_n(x,y) \to \frac{u(x)\varphi(x)}{|x-y|^{N+2s}} \text{ in } L^1(K \times \mathcal{C}K).$$

Again we deduce that there exist a subsequence of $\{u_n\}$, still indexed by n, and a positive function $g \in L^1(K \times \mathcal{C}K)$ such that

$$|G_n(x,y)| \leq g(x,y) \text{ a.e. in } K \times \mathcal{C}K. \tag{3.33}$$

Combining (3.31), (3.32) and (3.33), we obtain

$$\frac{|(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))|}{|x-y|^{N+2s}} \leq g(x,y) + h(x,y) \in L^1(K \times \mathcal{C}K),$$

for every $(x,y) \in K \times \mathcal{C}K$. So that by Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to\infty} I_n^2 = \int_K \int_{\mathcal{C}K} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} dydx.$$

By x/y symmetry, one has

$$\lim_{n\to\infty} I_n^3 = \int_{\mathcal{C}K}\int_K \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$

Then, we conclude that

$$\lim_{n\to\infty}\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx$$
$$= \int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx,$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$. As regards the right-hand side of (3.29), we follow the same arguments as in Theorem 3.2.1 to obtain

$$\lim_{n\to\infty}\int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx = \int_\Omega \frac{f\varphi}{u^\gamma}dx.$$

Finally, the passage to the limit in (3.29), as $n \to +\infty$, shows that u is a weak solution of (3.1). □

3.3.4 The case $\gamma > 1$: Proof of Theorem 3.2.3

3.3.4.1 A priori estimates

Lemma 3.3.7. *Assume $\gamma > 1$. Let $s \in (0,1)$ and $f \in L^m(\Omega)$ with $m > 1$. Let u_n be a solution of the problem (3.8). Then the sequence $\{u_n^\alpha\}_n$ is uniformly bounded in $X_0^s(\Omega)$ for every $\alpha \in \left(\max\left(\frac{1}{2}, \frac{sm(\gamma+1)-m+1}{2sm}\right), \frac{\gamma+1}{2}\right]$. Furthermore, if γ satisfies*

$$1 < \gamma < 1 + \frac{m-1}{sm}, \tag{3.34}$$

then $\{u_n\}_n$ is uniformly bounded in $X_0^s(\Omega)$.

Proof. Before estimating the sequence $\{u_n^\alpha\}_n$ in $X_0^s(\Omega)$, we need to prove that

$$u_n(x) \geq C_0\delta^s(x), \text{ a.e. in } \Omega, \tag{3.35}$$

where $C_0 > 0$ is a constant not depending on n and $\delta(x) := dist(x, \partial\Omega)$. Observe that $0 \leq \dfrac{f_1}{(u_1+1)^\gamma} \in L^\infty(\Omega)$. Thus, applying [19, Lemma 4.2] we get

$$\frac{u_1(x)}{\delta^s(x)} \geq C\int_\Omega \frac{f_1(y)}{(u_1+1)^\gamma}\delta^s(y)dy \geq C\int_\Omega \frac{f_1(y)}{(\|u_1\|_{L^\infty(\Omega)}+1)^\gamma}\delta^s(y)dy$$
$$\geq C_0 := \frac{C\delta^s(\partial K, \partial\Omega)}{(\|u_1\|_{L^\infty(\Omega)}+1)^\gamma}\int_K f_1(y)dy$$

where K is an arbitrary compact in Ω. By Lemma 3.3.2, the sequence $\{u_n\}_n$ is increasing and therefore the inequality (3.35) is satisfied.

Now, we shall prove a priori estimates on u_n^α in $X_0^s(\Omega)$ for every α such that

$$\max\Big(\frac{1}{2}, \frac{sm(\gamma+1)-m+1}{2sm}\Big) < \alpha \le \frac{\gamma+1}{2}.$$

Let $n \ge 1$ and let $0 < \varepsilon < \dfrac{1}{n}$. For $\eta > 0$, taking $(u_n+\varepsilon)^\eta - \varepsilon^\eta$ as a test function in (3.8), we obtain

$$\frac{a(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))((u_n(x)+\varepsilon)^\eta - (u_n(y)+\varepsilon)^\eta)}{|x-y|^{N+2s}}dydx$$
$$\le \int_\Omega \frac{f_n}{(u_n(x)+\frac{1}{n})^{\gamma-\eta}}dx.$$

By Fatou's lemma we can pass to the limit in ε obtaining

$$\int_Q \frac{(u_n(x)-u_n(y))(u_n^\eta(x)-u_n^\eta(y))}{|x-y|^{N+2s}}dydx \le \frac{2}{a(N,s)}\int_\Omega \frac{f_n}{(u_n(x)+\frac{1}{n})^{\gamma-\eta}}dx.$$

Then, an application of the item $i)$ in Lemma 1.3.3 and the Hölder inequality respectively yield

$$\int_Q \frac{\Big|u_n^{\frac{\eta+1}{2}}(x) - u_n^{\frac{\eta+1}{2}}(y)\Big|^2}{|x-y|^{N+2s}}dydx \le C(\eta,N,s)\|f\|_{L^m(\Omega)}\Big(\int_\Omega \frac{dx}{u_n^{(\gamma-\eta)m'}(x)}\Big)^{\frac{1}{m'}}.$$

Let us choose $0 < \eta \le \gamma$. The inequality (3.35) implies

$$\int_Q \frac{\Big|u_n^{\frac{\eta+1}{2}}(x) - u_n^{\frac{\eta+1}{2}}(y)\Big|^2}{|x-y|^{N+2s}}dydx \le$$
$$C(\eta,N,s)C_0^{(\eta-\gamma)s}\|f\|_{L^m(\Omega)}\Big(\int_\Omega \frac{dx}{\delta^{(\gamma-\eta)sm'}(x)}\Big)^{\frac{1}{m'}}.$$

Now, choosing $\alpha = \dfrac{\eta+1}{2}$ one has $\dfrac{1}{2} < \alpha \le \dfrac{\gamma+1}{2}$ and then

$$\int_Q \frac{|u_n^\alpha(x)-u_n^\alpha(y)|^2}{|x-y|^{N+2s}}dydx \le$$
$$C(\eta,N,s)C_0^{(\eta-\gamma)s}\|f\|_{L^m(\Omega)}\Big(\int_\Omega \frac{dx}{\delta^{(\gamma-2\alpha+1)sm'}(x)}\Big)^{\frac{1}{m'}}.$$

Observe that the integral in the right-hand side of the above inequality converges if and only if $(\gamma-2\alpha+1)sm' < 1$, that is $\dfrac{sm(\gamma+1)-m+1}{2sm} < \alpha$. Therefore, the sequence $\{u_n^\alpha\}$ is uniformly bounded in $X_0^s(\Omega)$, for every $\alpha \in \Big(\max\Big(\dfrac{1}{2}, \dfrac{sm(\gamma+1)-m+1}{2sm}\Big), \dfrac{\gamma+1}{2}\Big]$.

In particular, if (3.34) holds then $\dfrac{sm(\gamma+1)-m+1}{2sm} < 1$ and so $\{u_n\}$ is uniformly bounded in $X_0^s(\Omega)$. □

3.3.4.2 Passage to the limit

Proof. of Theorem 3.2.3. We use similar arguments as in the proof of Theorem 3.2.2 obtaining that $u := \lim_{n\to\infty} u_n$ is a weak solution to (3.1) and $u^\alpha \in X_0^s(\Omega)$ for every
$\max\left(\frac{1}{2}, \frac{sm(\gamma+1)-m+1}{2sm}\right) < \alpha \leq \frac{\gamma+1}{2}$.
Furthermore, if (3.34) holds then $\frac{sm(\gamma+1)-m+1}{2sm} < 1$ and so $u \in X_0^s(\Omega)$. □

3.3.5 The case $\gamma > 1$: Proof of Theorem 3.2.4

Proof. of Theorem 3.2.4. Let $\gamma > 1$ and let u_n be a solution of (3.8). Let $0 < \varepsilon < \frac{1}{n}$, $n \geq 1$. For $\eta > 0$, taking $(u_n + \varepsilon)^\eta - \varepsilon^\eta$ as a test function in (3.8), we follow the same lines in the proof of Lemma (3.3.7). We obtain

$$\int_Q \frac{\left|u_n^{\frac{\eta+1}{2}}(x) - u_n^{\frac{\eta+1}{2}}(y)\right|^2}{|x-y|^{N+2s}} dydx \leq C(\eta, N, s) \int_{supp(f)} \frac{f}{u_n^{\gamma-\eta}} dx.$$

Now, let us choose $0 < \eta \leq \gamma$ and set $\alpha = \frac{\eta+1}{2}$, we get

$$\int_Q \frac{\left|u_n^\alpha(x) - u_n^\alpha(y)\right|^2}{|x-y|^{N+2s}} dydx \leq C(\eta, N, s) \int_{supp(f)} \frac{f}{u_n^{\gamma-(2\alpha-1)}} dx.$$

Applying Lemma 3.3.2, we obtain

$$\int_Q \frac{\left|u_n^\alpha(x) - u_n^\alpha(y)\right|^2}{|x-y|^{N+2s}} dydx \leq \frac{C(\eta, N, s)}{c_{supp(f)}^{\gamma-(2\alpha-1)}} \|f\|_{L^1(\Omega)}.$$

It follows that $\{u_n^\alpha\}$ is uniformly bounded in $X_0^s(\Omega)$ for every $\alpha \in \left(\frac{1}{2}, \frac{\gamma+1}{2}\right]$.
Arguing as above, it's easy to see that $u := \lim_{n\to\infty} u_n$ is a weak solution of (3.1) and $u^\alpha \in X_0^s(\Omega)$ for every $\alpha \in \left(\frac{1}{2}, \frac{\gamma+1}{2}\right]$. □

3.3.6 Uniqueness : Proof of Theorem 3.2.5

Proof. In order to prove the uniqueness of finite energy solutions, we assume that there exist two weak solutions u_1 and $u_2 \in X_0^s(\Omega)$ to (3.1). By Lemma 3.4.4 the weak solutions u_1 and u_2 both satisfy (3.38). By [63, Proposition 3] we have $(u_1 - u_2)^+ \in X_0^s(\Omega)$, hence $(u_1 - u_2)^+$

is an admissible test function in (3.38). Taking it so in the difference of formulations (3.38) solved by u_1 and u_2 we arrive at

$$\int_Q \frac{\Big((u_1(x)-u_2(x))-(u_1(y)-u_2(y))\Big)\Big((u_1-u_2)^+(x)-(u_1-u_2)^+(y)\Big)}{|x-y|^{N+2s}}dydx$$
$$= \frac{2}{a(N,s)}\int_\Omega f(x)\Big(\frac{1}{u_1^\gamma}-\frac{1}{u_2^\gamma}\Big)(u_1-u_2)^+(x)dx.$$

Observe that for any function $g:\mathbb{R}^N\to\mathbb{R}$ the following inequality

$$(g(x)-g(y))(g^+(x)-g^+(y)) \geq (g^+(x)-g^+(y))^2$$

holds true for every x, $y\in\mathbb{R}^N$. It follows that

$$\|(u_1-u_2)^+\|^2_{X_0^s(\Omega)} = 0,$$

which gives $u_2 \geq u_1$. By the u_1/u_2 symmetry we obtain $u_1 = u_2$. □

3.4 Some regularity results

We point out that if $f\in L^m(\Omega)$ with $m \geq \overline{m} := (\frac{2_s^*}{1-\gamma})' = \frac{2N}{N+2s+\gamma(N-2s)}$, then following the same lines as in the proof of [18, Lemma 3.4] we can prove that the sequence $\{u_n\}_n$ of non-negative solutions of the problem (3.8) is uniformly bounded in $X_0^s(\Omega)$. Furthermore, testing by a $\mathcal{C}_0^\infty(\Omega)$-function in (3.8) one can pass to the limit and obtain that $u := \lim_{n\to\infty} u_n$ is a weak solution for the problem (3.1) in the sense of Definition 3.2.1. In this section we give some further summability results of this weak solution u.

Lemma 3.4.1. *Suppose that $0<\gamma<1$. Let u be the weak solution of (3.1) corresponding to $f\in L^m(\Omega)$ with $m \geq (\frac{2_s^*}{1-\gamma})' = \frac{2N}{N+2s+\gamma(N-2s)}$. If $(\frac{2_s^*}{1-\gamma})' \leq m < \frac{N}{2s}$, then $u\in L^\sigma(\Omega)$ where $\sigma = \frac{Nm(\gamma+1)}{N-2sm}$.*

Proof. Let $u_n\in X_0^s(\Omega)\cap L^\infty(\Omega)$ be a solution of the problem (3.8). Inserting u_n^θ, $\theta>1$, as a test function in (3.8) we get

$$\int_Q \frac{(u_n(x)-u_n(y))(u_n^\theta(x)-u_n^\theta(y))}{|x-y|^{N+2s}}dydx \leq \frac{2}{a(N,s)}\int_\Omega f_n u_n^{\theta-\gamma}(x)dx.$$

Applying the item $i)$ in Lemma 1.3.3 in the right-hand side and Hölder's inequality in the left hand-side, we get

$$\int_Q \frac{|u_n(x)^{\frac{\theta+1}{2}} - u_n(y)^{\frac{\theta+1}{2}}|^2}{|x-y|^{N+2s}} dydx \leq C_1 \|f\|_{L^m(\Omega)} \Big(\int_\Omega u_n^{(\theta-\gamma)m'}(x)dx \Big)^{\frac{1}{m'}}.$$

where $C_1 = \dfrac{(\theta+1)^2}{2\theta a(N,s)}$. Applying fractional Sobolev's inequality, we obtain

$$\int_\Omega |u_n(x)|^{\frac{N(\theta+1)}{N-2s}} dx \leq C_2 \|f\|_{L^m(\Omega)}^{\frac{N}{N-2s}} \Big(\int_\Omega u_n^{(\theta-\gamma)m'}(x)dx \Big)^{\frac{N}{m'(N-2s)}},$$

with $C_2 = (S(N,s)C_1)^{\frac{N}{N-2s}}$. Now we choose $\theta > 1$ in order to get $\dfrac{N(\theta+1)}{N-2s} = (\theta-\gamma)m'$, that is

$$\theta = \frac{N(m-1) + \gamma m(N-2s)}{N-2sm}.$$

Observe that $\theta > 1$ and

$$\frac{N(\theta+1)}{N-2s} = \frac{Nm(\gamma+1)}{N-2sm}.$$

In addition the assumption $m < \dfrac{N}{2s}$ implies $\dfrac{N}{m'(N-2s)} < 1$. Then it follows

$$\int_\Omega |u_n(x)|^{\frac{Nm(1+\gamma)}{N-2sm}} dx \leq C_2^{\frac{m(N-2s)}{N-2sm}} \|f\|_{L^m(\Omega)}^{\frac{Nm}{N-2sm}}.$$

By Fatou's Lemma, we obtain $u \in L^\sigma(\Omega)$ with $\sigma = \dfrac{Nm(\gamma+1)}{N-2sm}$. □

Remark 3.4.1. *In the particular case where $m = (2_s^*)'$, we obtain $u \in L^{(1+\gamma)2_s^*}(\Omega)$ which is exactly the result stated in [18, Proposition 3.8]. While if $s = 1$ the exponent of summability $\sigma = \dfrac{Nm(\gamma+1)}{N-2sm}$ coincides with the one given [23, Lemma 5.5] in the local case.*

Lemma 3.4.2. *(Limit case : Exponential summability) Assume that $\gamma > 0$. Let $f \in L^{\frac{N}{2s}}(\Omega)$ and let u be the weak solution of the problem (3.1) given by Theorem 3.2.3 if $\gamma > 1$ or given by [34, Theorem 3.2.] if $0 < \gamma \leq 1$. Then there exists $\lambda > 0$ such that $e^{\lambda \frac{N(1+\gamma)}{N-2s} u} \in L^1(\Omega)$.*

Proof. Let us start with the case $\gamma > 1$. For $\lambda > 0$, we consider the locally Lipschitz function $t \to \psi(t) = (e^{\lambda t} - 1)^{\frac{\gamma+1}{2}}$. Let $u_n \in X_0^s(\Omega) \cap L^\infty(\Omega)$ be a non-negative solution of the problem (3.8). Since $\psi(0) = 0$ and we can take $\psi'(u_n)\psi(u_n)$ as a test function in (3.8). As $\gamma > 1$, the function ψ is convex so that according with [63, Proposition 4.] we arrive at

$$\begin{aligned}&\frac{a(N,s)}{2}\int_Q \frac{|\psi(u_n)(x)-\psi(u_n)(y)|^2}{|x-y|^{N+2s}}dydx\\ \leq & \int_\Omega \psi'(u_n)\psi(u_n)(-\Delta)^s u_n(x)dx\\ = & \int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\psi'(u_n)\psi(u_n)dx.\end{aligned}$$

Using the Sobolev inequality, we obtain

$$\|\psi(u_n)\|^2_{L^{2^*_s}(\Omega)} \leq \frac{2S(N,s)}{a(N,s)}\int_\Omega \frac{f}{u_n^\gamma}\psi'(u_n)\psi(u_n)dx.$$

Using the elementary inequality $\dfrac{e^a-1}{a}\leq e^a$ for every $a>0$, we get

$$\frac{\psi'(u_n)\psi(u_n)}{u_n^\gamma}\leq \frac{\gamma+1}{2}\lambda^{\gamma+1}e^{\lambda(\gamma+1)u_n}\leq C(\gamma)\lambda^{\gamma+1}\psi^2(u_n)+C(\lambda,\gamma),$$

where we have set $C(\gamma)=2^\gamma\dfrac{\gamma+1}{2}$ and $C(\lambda,\gamma)=\lambda^{\gamma+1}C(\gamma)$. Then, using Hölder's inequality we obtain

$$\begin{aligned}\|\psi(u_n)\|^2_{L^{2^*_s}(\Omega)} &\leq \frac{2S(N,s)C(\gamma)\lambda^{\gamma+1}}{a(N,s)}\int_\Omega f\psi^2(u_n)+C(\lambda,\gamma,\Omega)\|f\|_{L^{\frac{N}{2s}}(\Omega)}\\ &\leq \frac{2S(N,s)C(\gamma)\lambda^{\gamma+1}}{a(N,s)}\|f\|_{L^{\frac{N}{2s}}(\Omega)}\|\psi(u_n)\|^2_{L^{2^*_s}(\Omega)}\\ &+C(\lambda,\gamma,\Omega)\|f\|_{L^{\frac{N}{2s}}(\Omega)}.\end{aligned}$$

Choosing $\lambda>0$ to be such that $\dfrac{2S(N,s)C(\gamma)\|f\|_{L^{\frac{N}{2s}}(\Omega)}\lambda^{\gamma+1}}{a(N,s)}<1$, we deduce that

$$\int_\Omega e^{\lambda\frac{N(1+\gamma)}{N-2s}u_n}dx\leq C,$$

where C is a constant not depending on n. Applying Fatou's lemma, we conclude the result.

We turn now to the case $\gamma\leq 1$. We consider the convex and locally Lipschitz function $t\to\psi(t)=e^{\frac{\gamma+1}{2}\lambda t}-1$ and we insert $\psi'(u_n)\psi(u_n)$ as a test function in (3.8). Again by [63, Proposition 4.] and the Sobolev inequality we obtain

$$\|\psi(u_n)\|^2_{L^{2^*_s}(\Omega)} \leq \frac{2S(N,s)}{a(N,s)}\int_\Omega \frac{f}{u_n^\gamma}\psi'(u_n)\psi(u_n)dx.$$

Since $0 < \dfrac{\gamma+1}{2} \le 1$, we can apply the inequality in the item *iii*) in Lemma 1.3.3 obtaining

$$\frac{\psi'(u_n)\psi(u_n)}{u_n^\gamma} \le \frac{\gamma+1}{2}\lambda \frac{e^{\frac{\gamma+1}{2}\lambda u_n}\left(e^{\lambda u_n}-1\right)^{\frac{\gamma+1}{2}}}{u_n^\gamma}.$$

Noticing that $u_n^{\frac{\gamma+1}{2}} \le u_n^\gamma$ on the subset $\{u_n \le 1\} := \{x \in \Omega : u_n(x) \le 1\}$, we can write

$$\begin{aligned}\int_\Omega \frac{f}{u_n^\gamma}\psi'(u_n)\psi(u_n)dx &\le \frac{\gamma+1}{2}\lambda \int_{\{u_n\le 1\}} \frac{fe^{\frac{\gamma+1}{2}\lambda u_n}\left(e^{\lambda u_n}-1\right)^{\frac{\gamma+1}{2}}}{u_n^{\frac{\gamma+1}{2}}}dx \\ &+\frac{\gamma+1}{2}\lambda \int_{\{u_n>1\}} fe^{\frac{\gamma+1}{2}\lambda u_n}\left(e^{\lambda u_n}-1\right)^{\frac{\gamma+1}{2}}dx.\end{aligned}$$

Using the elementary inequality $\dfrac{e^a-1}{a} \le e^a$, which holds for every $a > 0$, in the first integral in the right-hand side of the previous inequality, we obtain

$$\begin{aligned}\int_\Omega \frac{f}{u_n^\gamma}\psi'(u_n)\psi(u_n)dx &\le \frac{\gamma+1}{2}\lambda^{\frac{\gamma+3}{2}} \int_{\{u_n\le 1\}} fe^{(\gamma+1)\lambda u_n} \\ &+\frac{\gamma+1}{2}\lambda \int_{\{u_n>1\}} fe^{(\gamma+1)\lambda u_n}dx \\ &\le \frac{\gamma+1}{2}\lambda^{\frac{\gamma+3}{2}} e^{(\gamma+1)\lambda} \int_\Omega fdx \\ &+\frac{\gamma+1}{2}\lambda \int_\Omega f(\psi(u_n)+1)^2dx.\end{aligned}$$

Using the fact that $(\psi(u_n)+1)^2 \le 2(\psi(u_n)^2+1)$, we get

$$\begin{aligned}\int_\Omega \frac{f}{u_n^\gamma}\psi'(u_n)\psi(u_n)dx &\le \frac{\gamma+1}{2}\lambda^{\frac{\gamma+3}{2}} e^{(\gamma+1)\lambda} \int_\Omega fdx \\ &+(\gamma+1)\lambda \int_\Omega f(\psi^2(u_n)+1)dx \\ &\le \left(\frac{\gamma+1}{2}\lambda^{\frac{\gamma+3}{2}} e^{(\gamma+1)\lambda} + (\gamma+1)\lambda\right) \int_\Omega fdx \\ &+(\gamma+1)\lambda \int_\Omega f\psi^2(u_n)dx.\end{aligned}$$

An application of Hölder's inequality with the exponents $\dfrac{N}{N-2s}$ and $\dfrac{N}{2s}$ gives

$$\begin{aligned}\|\psi(u_n)\|^2_{L^{2^*_s}(\Omega)} &\le \frac{S(N,s)}{a(N,s)}(\gamma+1)\left(\lambda^{\frac{\gamma+3}{2}} e^{(\gamma+1)\lambda} + 2\lambda\right)|\Omega|^{\frac{N-2s}{N}}\|f\|_{L^{\frac{N}{2s}}(\Omega)} \\ &+\frac{2S(N,s)(\gamma+1)}{a(N,s)}\lambda\|f\|_{L^{\frac{N}{2s}}(\Omega)}\|\psi(u_n)\|^2_{L^{2^*_s}(\Omega)}.\end{aligned}$$

Therefore, choosing $\lambda > 0$ such that $\lambda < \dfrac{a(N,s)}{2S(N,s)(\gamma+1)\|f\|_{L^{\frac{N}{2s}}(\Omega)}}$ we obtain

$$\int_\Omega e^{\lambda\frac{N(1+\gamma)}{N-2s}u_n}dx \le C,$$

where C is a constant not depending on n, and by Fatou's lemma we conclude the result. □

Remark 3.4.2. *Recall that the inequality $e^x \ge \dfrac{x^k}{k!}$ holds for every $x > 0$ and $k \in \mathbb{N}$. Thus, we conclude that $u \in L^r(\Omega)$ for every $r < \infty$.*

Appendix

We start by proving the following lemma which we have used in the proof of Lemma 3.3.4.

Lemma 3.4.3. *Let $F(x) = x^r$, $0 < r < 1$, for every $x > 0$. Then for every function $v : \mathbb{R}^N \to]0,+\infty[$ that satisfies*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}dydx < \infty,$$

we have

$$\begin{aligned}&(-\Delta)^s(F\circ v)(x) \le \\ &F'(v(x))(-\Delta)^s v(x) - \frac{F''(v(x))}{r}a(N,s)\int_{\mathbb{R}^N}\frac{\big(v(x)-v(y)\big)^2}{|x-y|^{N+2s}}dy.\end{aligned} \tag{3.36}$$

Proof. Following [36, Lemma 2.3.], we can use Taylor's formula obtaining for every $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$

$$F(v(y)) - F(v(x)) = F'(v(x))(v(y)-v(x)) + R(F), \tag{3.37}$$

where

$$\begin{aligned}R(F) &= \int_{v(x)}^{v(y)}(v(y)-t)F''(t)dt \\ &= (v(y)-v(x))^2\int_0^1(1-s)F''(v(x)+s(v(y)-v(x)))ds.\end{aligned}$$

On the other hand, since the function F'' is increasing we have

$$\begin{aligned}(1-s)v(x) &\le v(x)+s(v(y)-v(x)) \\ &\Rightarrow F''((1-s)v(x)) \le F''(v(x)+s(v(y)-v(x))).\end{aligned}$$

Hence, it follows

$$\begin{aligned}-R(F) &\leq -(v(y)-v(x))^2\int_0^1 (1-s)F''((1-s)v(x))ds\\ &= -(v(y)-v(x))^2 F''(v(x))\int_0^1 (1-s)^{r-1}ds.\end{aligned}$$

Then, from (3.37) we obtain

$$F(v(x))-F(v(y)) \leq F'(v(x))(v(x)-v(y)) - \frac{F''(v(x))}{r}(v(y)-v(x))^2.$$

Dividing both sides of this inequality by $|x-y|^{N+2s}$ and then integrating with respect to the variable y we arrive at

$$\begin{aligned}a(N,s)P.V.\int_{\mathbb{R}^N}\frac{F(v(x))-F(v(y))}{|x-y|^{N+2s}}dy &\leq F'(v(x))a(N,s)P.V.\int_{\mathbb{R}^N}\frac{(v(x)-v(y))}{|x-y|^{N+2s}}dy\\ &\quad -\frac{F''(v(x))}{r}a(N,s)P.V.\int_{\mathbb{R}^N}\frac{(v(y)-v(x))^2}{|x-y|^{N+2s}}dy,\end{aligned}$$

which proves (3.36). □

In the following result we extend the space of admissible test functions in (3.3).

Lemma 3.4.4. *Let $u \in X_0^s(\Omega)$ be a solution of the problem (3.1) taken in the sense of Definition 3.2.1 with $f \in L^1(\Omega)$. Then for every $\phi \in X_0^s(\Omega)$ we get $\frac{f\phi}{u^\gamma} \in L^1(\Omega)$ and*

$$\frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}dydx = \int_\Omega \frac{f\phi}{u^\gamma}dx. \tag{3.38}$$

Proof. Take an arbitrary $\phi \in X_0^s(\Omega)$. By [49, Theorem 6] there exists a sequence $\{\varphi_n\}_n \subset \mathcal{C}_0^\infty(\Omega)$ such that $\varphi_n \to \phi$ in norm in $H^s(\mathbb{R}^N)$. Writing (3.3) with $\varphi_n \in \mathcal{C}_0^\infty(\Omega)$ we obtain

$$\frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))(\varphi_n(x)-\varphi_n(y))}{|x-y|^{N+2s}}dydx = \int_\Omega \frac{f\varphi_n}{u^\gamma}dx, \tag{3.39}$$

in which we shall pass to the limit as n tends to $+\infty$. Starting with the left-hand side of (3.39), we consider the following two functions

$$F_n(x,y) = \frac{(\varphi_n(x)-\varphi_n(y))}{|x-y|^{\frac{N+2s}{2}}} \text{ and } F(x,y) = \frac{(\phi(x)-\phi(y))}{|x-y|^{\frac{N+2s}{2}}}.$$

Notice that the convergence $\varphi_n \to \phi$ in norm in $H^s(\mathbb{R}^N)$ implies that the sequence $\{F_n(x,y)\}_n$ converges to $F(x,y)$ in $L^2(\mathbb{R}^{2N})$ and, up to a subsequence if necessary, we can assume that

$\{F_n(x,y)\}_n$ converges almost everywhere in $\mathbb{R}^{2N}$. As $u \in X_0^s(\Omega)$ we have $\dfrac{(u(x)-u(y))}{|x-y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^{2N})$ implying

$$\lim_{n\to\infty}\int_Q \frac{(u(x)-u(y))(\varphi_n(x)-\varphi_n(y))}{|x-y|^{N+2s}}dydx$$
$$= \int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}dydx.$$

For the term in the right-hand side of (3.39), we first note that thanks to [63, Proposition 3.] the two functions $(\varphi_n-\varphi_k)^+$ and $(\varphi_n-\varphi_k)^-$ are both admissible test functions in (3.3). Taking them so we obtain

$$\int_\Omega \frac{f}{u^\gamma}(\varphi_n-\varphi_k)^+(x)dx$$
$$= \frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))\big((\varphi_n-\varphi_k)^+(x)-(\varphi_n-\varphi_k)^+(y)\big)}{|x-y|^{N+2s}}dydx$$

and

$$\int_\Omega \frac{f}{u^\gamma}(\varphi_n-\varphi_k)^-(x)dx$$
$$= \frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))\big((\varphi_n-\varphi_k)^-(x)-(\varphi_n-\varphi_k)^-(y)\big)}{|x-y|^{N+2s}}dydx.$$

Then, summing up both the two equalities we have

$$\int_\Omega \frac{f}{u^\gamma}\Big|\varphi_n-\varphi_k\Big|dx$$
$$= \frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))\Big(|\varphi_n(x)-\varphi_k(x)|-|\varphi_n(y)-\varphi_k(y)|\Big)}{|x-y|^{N+2s}}dydx$$
$$\leq \frac{a(N,s)}{2}\int_Q \frac{|u(x)-u(y)|\Big|(\varphi_n(x)-\varphi_k(x))-(\varphi_n(y)-\varphi_k(y))\Big|}{|x-y|^{N+2s}}dydx$$

and then the Hölder inequality implies

$$\int_\Omega \left|\frac{f\varphi_n}{u^\gamma}-\frac{f\varphi_k}{u^\gamma}\right|dx \leq \frac{a(N,s)}{2}\|u\|_{X_0^s(\Omega)}\|\varphi_n-\varphi_k\|_{X_0^s(\Omega)}.$$

Thus, we deduce that $\left\{\dfrac{f\varphi_n}{u^\gamma}\right\}_n$ is a Cauchy sequence in $L^1(\Omega)$. Since φ_n converges to φ a.e. in Ω, the sequence $\left\{\dfrac{f\varphi_n}{u^\gamma}\right\}_n$ converges to $\dfrac{f\phi}{u^\gamma} \in L^1(\Omega)$ in norm in $L^1(\Omega)$. So that the passage to the limit as n tends to infinity in (3.39) yields

$$\frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}dydx = \int_\Omega \frac{f\phi}{u^\gamma}dx,$$

for every $\phi \in X_0^s(\Omega)$. □

Chapter 4

Fractional heat equation with singular terms[1]

In this chapter we consider the nonlocal heat equation involving singular terms. Our aim in this chapter is to analyze the existence of solutions for this problem. In light of the interplay between the summability of the data and the nonlinearity some results are proven. Some of them extend those obtained in [40] for the local case.

4.1 Introduction

We are interested in the existence and regularity of solutions of the following initial-boundary value problem

$$\begin{cases} u_t + (-\Delta)^s u &= \dfrac{f(x,t)}{u^\gamma} \quad \text{in } \Omega_T := \Omega \times (0,T), \\ u &= 0 \quad \text{in } (\mathbb{R}^N \backslash \Omega) \times (0,T), \\ u(\cdot,0) &= u_0(\cdot) \quad \text{in } \Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain of class $\mathcal{C}^{0,1}$ in $\mathbb{R}^N$, $N > 2s$ with $s \in (0,1)$, $\gamma > 0$, $f \geq 0$, $f \in L^m(\Omega_T)$, $m \geq 1$, is a non-negative function on Ω_T, $u_0 \in L^\infty(\Omega)$ is a non-negative function on Ω which further satisfies

$$\forall \omega \subset\subset \Omega \quad \exists d_\omega > 0, \text{ such that } u_0 \geq d_\omega. \tag{4.2}$$

The operator $(-\Delta)^s$ is the fractional Laplacian operator defined by

$$(-\Delta)^s u(x,t) = P.V. \int_{\mathbb{R}^N} \frac{u(x,t) - u(y,t)}{|x-y|^{N+2s}} dy,$$

[1]Submitted in 2021

where "P.V." stands for the integral in the principal value sense.

The classical Heat equation seems to describe in a satisfactory manner a wide variety of diffusive problems in Physics. However, the anomalous diffusion that follows non-Brownian scaling is leading to models governed by fractional Laplacian. In the last few years, elliptic and parabolic equations involving nonlocal operators has attracted substantial attention. The interest brought to such equations is due to the emergence of this type of nonlocal operators in a wide range of phenomena – the crystal dislocation, thin obstacle problems, Physics, phase transitions, finance, stochastic control, quasi-geostrophic flows, anomalous diffusion to name a few (see e.g. [30, 39, 47, 75, 81, 83] and references therein). We also recall that the fractional Laplacian operator $(-\Delta)^s$ can be viewed as the infinitesimal generator of stable Lévy processes, see e.g. [13, 28, 86]. For an expository on fractional Laplacian, we refer the reader to [21, 28, 45] and the references therein.

As a prelude to study the initial-boundary value problem (4.1), it is worth recalling some latest important known results about this kind of parabolic problems (local/nonlocal) with singular terms. Let us star discussing the local case (i.e. $s = 1$). The initial-boundary value problem corresponding to (4.1) was studied in [40, 68]. The authors in [40] have considered the following problem

$$\begin{cases} u_t - \Delta_p u &= \dfrac{f(x,t)}{u^\gamma}, \quad \text{in } \Omega_T = \Omega \times (0,T), \\ \qquad\quad u &= 0 \qquad\qquad \text{on } \partial\Omega \times (0,T), \\ \quad u(\cdot,0) &= u_0(\cdot) \qquad\quad \text{in } \Omega, \end{cases} \tag{4.3}$$

where $p \geq 2$, $0 \leq f \in L^m(\Omega_T)$ with $m \geq 1$. Assuming that 4.2 is fulfilled, they proved the existence of a solution u of problem (4.3) such that

- If $0 < \gamma \leq 1$ and $f \in L^{m_0}(\Omega_T)$, with $m_0 = \dfrac{p(N+2)}{p(N+2) - N(1-\gamma)}$, then $u \in L^p(0,T; W_0^{1,p}(\Omega))$.
- If $0 < \gamma < 1$ and $f \in L^m(\Omega_T)$ with $1 \leq m < \dfrac{p(N+2)}{p(N+2) - N(1-\gamma)}$, then

$$u \in L^{q_0}(0,T; W_0^{1,q_0}(\Omega)) \text{ with } q_0 = \frac{m[N(p+\gamma-1) + p(\gamma+1)]}{N+2-m(1-\gamma)},$$

- If $\gamma > 1$ and $f \in L^1(\Omega_T)$ then $u \in L^p(0,T; W_{loc}^{1,p}(\Omega))$ and $u^{\frac{p+\gamma-1}{p}} \in L^p(0,T; W_0^{1,p}(\Omega))$.

Some other related local parabolic equations with singular terms are studied in [17, 25, 41].

Let us now discuss the nonlocal problem (4.1). In [2, 85] the authors studied (4.1) with a general fractional Laplacian operator including $(-\Delta)^s$ in the case where $\gamma = 0$ and $(f, u_0) \in (L^1(\Omega_T) \times L^1(\Omega))$. In [2] the authors proved the existence of a weak solution and the existence of a non-negative entropy one, while in [85] the authors proved the existence and uniqueness of renormalized solutions.

Regarding parabolic problems involving the fractional Laplacian with singular terms, some few variants have been investigated. We refer to [52] where the authors considered the following problem

$$\begin{cases} u_t + (-\Delta)^s u &= \dfrac{1}{u^\gamma} + f(x,t), \quad u > 0 \quad \text{in } \Omega \times (0,T), \\ u &= 0 \qquad\qquad\qquad\qquad\quad \text{on } (\mathbb{R}^N \backslash \Omega) \times (0,T), \\ u(\cdot, 0) &= u_0(\cdot) \qquad\qquad\qquad\quad\ \text{in } \mathbb{R}^N. \end{cases} \tag{4.4}$$

Under some suitable assumptions on the data f and u_0 and using the semi-discretization in time coupled with the implicit Euler method, they proved the existence and uniqueness of a weak solution to the problem (4.4). Lately, in [4] the existence and nonexistence of positive solutions were obtained for the problem (4.1) with the singular term $\lambda \dfrac{u^p}{\delta^{2s}(x)}$, $\delta(x) := dist(x, \partial\Omega)$ λ, $p > 0$, instead of $\dfrac{f}{u^\gamma}$. It is worth recalling that the case nonlocal elliptic counterpart of (4.1) have been studied in [10, 18, 34].

In this paper, we will be concerned with the nonlocal problem (4.1) that involves the fractional Laplacian. Our aim is to prove the existence and regularity of the positive weak solutions of the problem (4.1). Two remarks concerning the difficulties in dealing with problem (4.1) are in order. First of all, observe that since $\gamma > 0$ the second term is singular so that classical existence results can not be applied even if f is smooth enough. To overcome this problem, we will consider approximate problems in which we 'cut' by means of truncatures the singularity in order to get smooth source terms. This makes it possible to use Schauder's fixed point theorem obtaining approximate solutions. On the other hand, the other important difficulty that arises in problem (4.1) is the proof of the strict positivity of the solution in the interior of the parabolic cylinder. This is the reason why we impose the hypothesis (4.2) to make sense of the notion of weak solution of the problem (4.1). Thus, since we will construct weak solutions as limit of approximate solutions we need to prove an analogous inequality to (4.2) for the approximate solutions (Lemma 3.3.2). Let us recall that in the classical case (s=1), the positivity of the approximate solutions inside the parabolic cylinder has been obtained in [24] by comparing the approximate solutions to those of a suitable homogeneous problem which are Hölder continuous on every $w \times [0,T)$, $w \subset\subset \Omega$, and on which a classical form of Harnack's inequality ([16, Theorem 5]) is then applied. Inspired by [46, Theorem 1.1] (page 93), the authors [40] proved the intrinsic form of the Harnack inequality. Combined with the idea in [24], they obtain the positivity of the approximate solutions.

Coming to our non-local problem (4.1), we follow the idea of [24] to show the positivity of the approximate solutions inside the parabolic cylinder without using neither the Hölder continuity for solutions of auxiliary homogeneous problems nor any form of Harnack inequality as it was the case in [24] and [40].

The idea we use consists in involving the smallest eigenvalue of the fractional Laplacian (see e.g. [77, Proposition 4] and [78, Proposition 9]) together with its associate eigenfunction to build an homogeneous problem with suitable initial datum whose solution (which is by construction locally bounded from below) is comparable with the approximate solutions of the problem by means of the weak comparison principle.

4.2 Functional setting and main results

As in the classical case, we define the corresponding parabolic spaces as the following :

$$L^q(0,T;W_0^{s,q}(\Omega)) = \Big\{u \in L^q(\Omega\times(0,T)),\ \|u\|_{L^q(0,T;W_0^{s,q}(\Omega))} < \infty\Big\},$$

$$L^2(0,T;X_0^s(\Omega)) = \Big\{u \in L^2(\mathbb{R}^N\times(0,T)),\ \|u\|_{L^2(0,T;X_0^s(\Omega))} < \infty\Big\},$$

where

$$\|u\|_{L^q(0,T;W_0^{s,q}(\Omega))} = \left(\int_0^T\int_\Omega\int_\Omega \frac{|u(x,t)-u(y,t)|^q}{|x-y|^{N+qs}}dxdydt\right)^{\frac{1}{q}},$$

$$\|u\|_{L^2(0,T;X_0^s(\Omega))} = \left(\int_0^T\int_Q \frac{|u(x,t)-u(y,t)|^2}{|x-y|^{N+2s}}dxdydt\right)^{\frac{1}{2}},$$

with its dual spaces $L^{q'}(0,T;W^{-s,q'}(\Omega))$ and $L^2(0,T;X^{-s}(\Omega))$ respectively. We also define

$$\begin{aligned} L^2(0,T;H^s_{loc}(\Omega)) \ &= \Big\{u \in L^2(K\times(0,T)), \text{s.t.} \int_0^T\int_K\int_K \frac{|u(x,t)-u(y,t)|^2}{|x-y|^{N+2s}}dxdydt < \infty, \\ &\qquad \text{for every compact } K\subset\Omega\Big\}. \end{aligned}$$

We shall consider the notion of weak solution whose meaning is defined as follows.

Definition 4.2.1. *Let $f \in L^1(\Omega_T)$. By a weak solution of problem (4.1), we mean a measurable function $u \in \mathcal{C}([0,T], L^1_{loc}(\Omega))$ satisfying*

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0\ :\ u(x,t) \geq c_\omega > 0,\ \text{in } \omega \times [0,T)$$

and

$$- \int_{\Omega_T} u\varphi_t dxdt - \int_\Omega u_0(x)\varphi(x,0)dx + \frac{1}{2}\int_{Q_T} \frac{(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} dxdydt = \int_{\Omega_T} \frac{f\varphi}{u^\gamma} dxdt,$$

for any $\varphi \in \mathcal{C}^\infty_0(\Omega \times [0,T))$. Here, $Q_T = Q \times (0,T)$ and $Q := \mathbb{R}^{2N} \backslash (\mathcal{C}\Omega \times \mathcal{C}\Omega)$.

We now state our main results. We give the existence and the regularity of weak solutions according to the values of $\gamma > 0$ and the summability of the datum f.

Theorem 4.2.1. *Let $\gamma = 1$. Assume that $(f, u_0) \in L^1(\Omega_T) \times L^\infty(\Omega)$, with $f \geq 0$ and the condition (4.2) is fulfilled. Then the problem (4.1) admits at least one weak solution $u \in L^2(0,T; X^s_0(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$.*

Theorem 4.2.2. *Let $0 < \gamma < 1$. Assume that $u_0 \in L^\infty(\Omega)$ satisfies the condition (4.2) and $f \geq 0$ is such that*

i) $f \in L^{\frac{2}{\gamma+1}}\left(0,T; L^{\left(\frac{2^*_s}{1-\gamma}\right)'}(\Omega)\right)$,

or

ii) $f \in L^{\overline{m}}(\Omega_T)$ *with*

$$\overline{m} := \frac{2(N+2s)}{2(N+2s) - N(1-\gamma)}.$$

Then the problem (4.1) has at least one weak solution $u \in L^2(0,T; X^s_0(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$.

Remark 4.2.1. *We point out that since $\gamma < 1$ we have*

$$\left(\frac{2^*_s}{1-\gamma}\right)' = \frac{2N}{2N - (1-\gamma)(N-2s)} < \overline{m} < \frac{2}{\gamma+1}.$$

Then the two spaces $L^{\frac{2}{\gamma+1}}\left(0,T; L^{\left(\frac{2^*_s}{1-\gamma}\right)'}(\Omega)\right)$ and $L^{\overline{m}}(\Omega_T)$ cannot be compared. The case $\gamma = 1$ cannot be considered since in (4.16) we use the Hölder inequality with the exponent $\dfrac{2}{\gamma+1}$ and

$\dfrac{2}{1-\gamma}$. *Note that the range of the values of* s *is the same for both* $\gamma = 1$ *and* $\gamma < 1$. *Therefore, there is, in some sense, "continuity" of the summability of the solution with respect to* γ. *If* γ *tends to 1, then* $\overline{m}$ *tends to* 1 *and* $\dfrac{2}{\gamma+1}$ *and* $\left(\dfrac{2_s^*}{1-\gamma}\right)' = \dfrac{2N}{2N-(1-\gamma)(N-2s)}$ *tend at the same time to* 1, *so that* f *will belong to* $L^1(\Omega_T)$.

If $m < \overline{m}$, we no longer find solutions in $L^2(0,T;X_0^s(\Omega))$ but in a larger space depending on m and $s_1 < s$.

Theorem 4.2.3. *Let* $0 < \gamma < 1$. *Assume that* $0 \le f \in L^m(\Omega_T)$, *with* $1 \le m < \overline{m}$, *and that* $u_0 \in L^\infty(\Omega)$ *satisfies the condition (4.2). Then the problem (4.1) admits at least one weak solution* $u \in L^{\overline{q}}(0,T;W_0^{s_1,\overline{q}}(\Omega)) \cap L^\infty(0,T;L^{1+\gamma}(\Omega))$, *for every* $s_1 < s$ *with*

$$\overline{q} = \frac{m(\gamma+1)(N+2s)}{N+2s-sm(1-\gamma)}.$$

Moreover $u \in L^\sigma(\Omega_T)$, *where*

$$\sigma = \frac{m(\gamma+1)(N+2s)}{N-2s(m-1)}.$$

Remark 4.2.2. *We can easily check that* $\overline{q} \ge m(\gamma+1) > 1$ *and* $\sigma \ge m(\gamma+1) > 1$. *Observe that* $m < \overline{m}$ *is equivalent to* $\overline{q} < 2$ *which implies* $L^2(0,T;X_0^s(\Omega)) \subset L^{\overline{q}}(0,T;W_0^{s_1,\overline{q}}(\Omega))$. *The condition* $\overline{q} < 2$ *is needed in (4.27). Thus, in Theorem 4.2.3 the case* $\gamma = 1$ *is not allowed since it yields* $\overline{q} = 2m \ge 2$ *which contradicts* $\overline{q} < 2$. *Furthermore, the case* $\gamma = 0$ *is also not allowed, since the choice of the test function depends on* γ *by which we divide in (2.8).*
If $m = 1$, *the "continuity" with respect to* γ *is broken down since* $\overline{q}$ *tends to* 2 *when* γ *tends to* 1 *and the solution belongs to the larger space* $L^2(0,T;H_0^{s_1}(\Omega))$ *than* $L^2(0,T;X_0^s(\Omega))$.

Remark 4.2.3. *The case* $\gamma = 0$ *corresponding to the problem*

$$u_t + (-\Delta)^s u = f. \tag{4.5}$$

was studied in [63]. The authors proved the existence of a unique weak solution u *of the problem (4.5) satisfying*

1. *if* $f \in L^1(\Omega_T)$ *then* $u \in L^q(\Omega_T)$ *for every* $q < \dfrac{N+2s}{N}$ *(see [63, Theorem 28]),*
2. *if* $f \in L^m(\Omega_T)$, *with* $1 < m < \dfrac{2N}{N+2s}$, *then* $u \in L^{\frac{Nm}{N-2sm}}(\Omega_T)$ *(see [63, Theorem 24]).*

Theorem 4.2.3 provides a solution $u \in L^{\sigma}(\Omega_T)$ with $\sigma = \dfrac{m(\gamma+1)(N+2s)}{N-2s(m-1)}$. if γ tends to zero, the value of σ tends to $\dfrac{m(N+2s)}{N-2s(m-1)} < \dfrac{Nm}{N-2sm}$. Thus, even if $m>1$ the solution may not belong to $L^{\frac{Nm}{N-2sm}}(\Omega_T)$, while if $m=1$ the value of σ tends to $\dfrac{N+2s}{N}$, this fact is in contrast with the result in [63, Theorem 28] since if $f \in L^1(\Omega_T)$ the solution of $u_t + (-\Delta)^s u = f$ does not belong to $L^{\frac{N+2s}{N}}(\Omega_T)$ but to $L^q(\Omega_T)$ for every $q < \dfrac{N+2s}{N}$. This explains the fact that $\gamma > 0$ in Theorem 4.2.3.

We deal now with the case $\gamma > 1$. Here again we can not find solutions in $L^2(0,T;X_0^s(\Omega))$, if we look for $L^2(0,T;H^s(\Omega))$ estimates, we can only get them in $L^2(0,T;H^s_{loc}(\Omega))$.

Theorem 4.2.4. *Let $\gamma > 1$. Assume that $0 \le f \in L^1(\Omega_T)$ and that $u_0 \in L^\infty(\Omega)$ satisfies the condition (4.2). Then the problem (4.1) admits at least one weak solution $u \in L^2(0,T;H^s_{loc}(\Omega)) \cap L^\infty(0,T;L^{1+\gamma}(\Omega))$, such that $u^{\frac{\gamma+1}{2}} \in L^2(0,T;X_0^s(\Omega))$.*

Remark 4.2.4. *Since Ω is bounded, the conclusion in Theorem 4.2.4 remains true for every $f \in L^m(\Omega_T)$, with $m \ge 1$.*

To give a meaning to the initial condition in the problem (4.1), we shall prove that the weak solutions obtained in the previous results are continuous in time.

Proposition 4.2.1. *Let $\gamma > 0$ and suppose that $f \in L^1(\Omega_T)$. Let u be the weak solution of the problem (4.1) given by Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.3 and Theorem 4.2.4. Then $u \in C([0,T];L^1_{loc}(\Omega))$.*

Notations In the sequel, for any open subset ω, the notation $\omega \subset\subset \Omega$ means that $\overline{\omega} \subset \Omega$ and $\overline{\omega}$ is compact. We denote by $\chi_{(0,\tau)}$ the characteristic function of $(0,\tau)$ in $(0,T]$ and $\Omega_\tau := \Omega \times (0,\tau)$. For any measurable subset E of Ω, $|E|$ stands for the Lebesgue measure of E.

4.3 Approximated Problems : Existence and Positivity

Consider the sequence of approximate problems

$$\begin{cases} (u_n)_t + (-\Delta)^s u_n &= \dfrac{f_n(x,t)}{(u_n+\frac{1}{n})^\gamma} & \text{in } \Omega_T = \Omega\times(0,T),\\ u_n &= 0 & \text{on } (\mathbb{R}^N\backslash\Omega)\times(0,T),\\ u_n(\cdot,0) &= u_0(\cdot) & \text{in } \Omega, \end{cases} \tag{4.6}$$

where $f_n = \min(f,n)$. We shall prove that for every fixed integer $n\in\mathbb{N}$, the problem (4.6) admits a non-negative solution u_n.

Lemma 4.3.1. *For each integer $n\in\mathbb{N}$, the problem (4.6) admits a non-negative solution $u_n\in L^2(0,T;X_0^s(\Omega))\cap L^\infty(\Omega_T)$ with $(u_n)_t\in L^2(0,T;X^{-s}(\Omega))$ satisfying for every $\varphi\in L^2(0,T;X_0^s(\Omega))$*

$$\begin{aligned}&\int_0^T\int_\Omega u_n)t\varphi dxdt + \frac{1}{2}\int_0^T\int_Q \frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt\\ &= \int_0^T\int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dxdt.\end{aligned} \tag{4.7}$$

Proof. Let $n\in\mathbb{N}$ be fixed and let $v\in L^2(\Omega_T)$. We define the map

$$\begin{aligned} S: L^2(\Omega_T) &\to L^2(\Omega_T),\\ v &\mapsto S(v),\end{aligned}$$

where $w=S(v)$ is the unique solution to the following problem

$$\begin{cases} w_t + (-\Delta)^s w &= \dfrac{f_n(x,t)}{(|v|+\frac{1}{n})^\gamma} & \text{in } \Omega\times(0,T),\\ w &= 0 & \text{in } (\mathbb{R}^N\backslash\Omega)\times(0,T),\\ w(\cdot,0) &= u_0(\cdot) & \text{on } \Omega. \end{cases}$$

Since $0\le \dfrac{f_n}{(|v|+\frac{1}{n})^\gamma}\in L^\infty(\Omega_T)\subset L^2(0,T;X^{-s}(\Omega))$ and $0\le w(\cdot,0)=u_0(\cdot)\in L^\infty(\Omega)$, then by [63, Theorem 26.], we have the existence and uniqueness of $0\le w\in L^2(0,T;X_0^s(\Omega))\cap C([0,T];L^2(\Omega))$ with $w_t\in L^2(0,T;X^{-s}(\Omega))$, that is w satisfies for every $\varphi\in L^2(0,T;X_0^s(\Omega))$

$$\begin{aligned}&\int_0^T\int_\Omega w_t\varphi dxdt + \frac{1}{2}\int_0^T\int_Q \frac{(w(x,t)-w(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt\\ &= \int_0^T\int_\Omega \frac{f_n\varphi}{(|v|+\frac{1}{n})^\gamma}dxdt.\end{aligned} \tag{4.8}$$

Furthermore by [63, Corollary 3.] we have $w \in L^\infty(\Omega_T)$. Testing in (4.8) by the function w, we get

$$\int_0^T\int_\Omega w_t w dxdt + \frac{1}{2}\int_0^T\int_Q \frac{(w(x,t)-w(y,t))^2}{|x-y|^{N+2s}}dydxdt = \int_0^T\int_\Omega \frac{f_n w}{(|v|+\frac{1}{n})^\gamma}dxdt.$$

Then

$$\int_\Omega w^2(x,T)dx + \int_0^T\int_Q \frac{(w(x,t)-w(y,t))^2}{|x-y|^{N+2s}}dydxdt \le 2n^{1+\gamma}\int_0^T\int_\Omega wdxdt + \int_\Omega u_0^2(x)dx.$$

Dropping the positive part and applying the Hölder inequality and then Lemma 1.1.1 with $q=2$, we obtain

$$\begin{aligned}\int_0^T \|w\|^2_{X_0^s(\Omega)}dt &\le (T|\Omega|)^{\frac{1}{2}}2n^{1+\gamma}\Big(\int_0^T \|w\|^2_{L^2(\Omega)}dt\Big)^{\frac{1}{2}} + \|u_0\|^2_{L^\infty(\Omega)}|\Omega| \\ &\le \Big(T|\Omega|C(N,s,\Omega)\Big)^{\frac{1}{2}}2n^{1+\gamma}\Big(\int_0^T \|w\|^2_{X_0^s(\Omega)}dt\Big)^{\frac{1}{2}} + \|u_0\|^2_{L^\infty(\Omega)}|\Omega|.\end{aligned}$$

If follows that

$$\|w\|^2_{L^2(0,T;X_0^s(\Omega))} = \int_0^T \|w\|^2_{X_0^s(\Omega)}dt \le C_1 := C_1(n,s,N,\gamma,T,\Omega,u_0), \tag{4.9}$$

where the constant C_1 is not depending on v. Thus, the ball of radius C_1 is invariant under the map S in $L^2(0,T;X_0^s(\Omega))$. Now we shall prove that the map S is continuous and compact from $L^2(0,T;X_0^s(\Omega))$ to itself. First, we start proving the continuity of S as an operator from $L^2(\Omega_T)$ to $L^2(\Omega_T)$. Let $\{v_k\}$ be a sequence such that $\|v_k - v\|_{L^2(\Omega_T)} \to 0$. Then up to a subsequence still indexed by k, the sequence v_k converges almost everywhere to v in Ω_T. Denoting $w_k = S(v_k)$ and $w = S(v)$, we can write

$$\begin{cases} (w_k-w)_t + (-\Delta)^s(w_k-w) = \dfrac{f_n(x,t)}{(|v_k|+\frac{1}{n})^\gamma} - \dfrac{f_n(x,t)}{(|v|+\frac{1}{n})^\gamma} & \text{in } \Omega\times(0,T), \\ (w_k-w)(x,t) = 0 & \text{in } (\mathbb{R}^N\setminus\Omega)\times(0,T), \\ (w_k-w)(x,0) = 0 & \text{on } \Omega. \end{cases} \tag{4.10}$$

Taking $w_k(x)-w(x)$ as a test function in (4.10) and using Hölder's inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega (w_k(x,T)-w(x,T))^2dx + \frac{1}{2}\int_0^T\int_Q \frac{|(w_k(x,t)-w(x,t))-(w_k(y,t)-w(y,t))|^2}{|x-y|^{N+2s}}dydxdt \\ &= \int_{\Omega_T}\Big(\frac{f_n(x,t)}{(|v_k|+\frac{1}{n})^\gamma} - \frac{f_n(x,t)}{(|v|+\frac{1}{n})^\gamma}\Big)(w_k-w)dxdt \\ &\le \|w_k-w\|_{L^2(\Omega_T)}\Big(\int_\Omega\Big(\frac{f_n(x,t)}{(|v_k|+\frac{1}{n})^\gamma} - \frac{f_n(x,t)}{(|v|+\frac{1}{n})^\gamma}\Big)^2dxdt\Big)^{\frac{1}{2}}.\end{aligned}$$

Dropping the positive term and applying Lemma (1.1.1) with $q = 2$, we obtain

$$\|w_k - w\|_{L^2(\Omega_T)} \leq 2C(N, s, \Omega)\left(\int_{\Omega_T}\left(\frac{f_n(x,t)}{(|v_k| + \frac{1}{n})^\gamma} - \frac{f_n(x,t)}{(|v| + \frac{1}{n})^\gamma}\right)^2 dxdt\right)^{\frac{1}{2}}.$$

Since

$$\left|\frac{f_n(x,t)}{(|v_k| + \frac{1}{n})^\gamma} - \frac{f_n(x,t)}{(|v| + \frac{1}{n})^\gamma}\right|^2 \leq 2^2 n^{2(\gamma+1)}$$

and

$$\frac{f_n(x,t)}{(|v_k| + \frac{1}{n})^\gamma} - \frac{f_n(x,t)}{(|v| + \frac{1}{n})^\gamma} \to 0 \text{ a.e. in } \Omega_T \text{ as } k \to +\infty,$$

by the dominated convergence theorem, we get

$$\|w_k - w\|_{L^2(\Omega_T)} \to 0 \text{ as } k \to +\infty.$$

Hence, S is continuous from $L^2(\Omega_T)$ to $L^2(\Omega_T)$. Now, we prove the compactness of $S : L^2(\Omega_T) \to L^2(\Omega_T)$. Let us $\{v_k\}$ be a sequence such that $\|v_k\|_{L^2(\Omega_T)} \leq C$, where C is a positive constant independent on k and set $w_k = S(v_k)$. We shall prove that w_k has a subsequence $\{w_k\}$ that converges in $L^2(\Omega_T)$. As w_k is the solution of the problem

$$\begin{cases} (w_k)_t + (-\Delta)^s w_k &= \dfrac{f_n(x,t)}{(|v_k| + \frac{1}{n})^\gamma} \quad \text{in } \Omega \times (0,T), \\ w_k &= 0 \quad \text{in } (\mathbb{R}^N \backslash \Omega) \times (0,T), \\ w_k(\cdot, 0) &= u_0(\cdot) \quad \text{on } \Omega, \end{cases} \tag{4.11}$$

we can take it as a test function in (4.11) and using (4.9), we obtain

$$\|w_k\|^2_{L^2(0,T;X_0^s(\Omega))} = \int_0^T \|w_k\|^2_{X_0^s(\Omega)} dt \leq C_1,$$

where C_1 is a constant not depending on v_k. This implies that $\{w_k\}_k$ is uniformly bounded with respect to k in $L^2(0,T;X_0^s(\Omega))$ and therefore $\{(-\Delta)^s w_k\}$ is uniformly bounded with respect to k in $L^2(0,T;X^{-s}(\Omega))$. From the equation (4.11), it follows that $\{(w_k)_t\}$ is uniformly bounded in $L^2(0,T;X^{-s}(\Omega))$. Then by the compact embedding $L^2(0,T;X_0^s(\Omega))$ in $L^2(\Omega_T)$ (see [82, Corollary 4]), there exist a subsequence of $\{w_k\}_k$, still indexed by k, and a measurable function w such that w_k strongly converge to w in $L^2(\Omega_T)$. Hence, S is a compact operator from $L^2(\Omega_T)$ to $L^2(\Omega_T)$ and therefore by Schauder's fixed point theorem there exists a non-negative function $u_n \in L^2(0,T;X_0^s(\Omega)) \cap L^\infty(\Omega_T)$ and $(u_n)_t \in L^2(0,T;X^{-s}(\Omega))$ such that $u_n = S(u_n)$. This ends the proof. □

Lemma 4.3.2. *Let u_n be a solution of (4.6). Then the sequence $\{u_n\}_{n\in\mathbb{N}}$ is increasing and for every subset $\omega \subset\subset \Omega$ there exists a positive constant c_ω, not depending on n, such that*

$$u_n(x,t) \geq c_\omega > 0, \quad \textit{for every } (x,t) \in \omega \times [0,T), \ \forall n \in \mathbb{N}. \tag{4.12}$$

Proof. We first prove that the sequence $\{u_n\}_n$ is increasing. Let u_n and u_{n+1} be two solutions to the following problems respectively

$$\begin{cases} (u_n)_t + (-\Delta)^s u_n &= \dfrac{f_n(x,t)}{(u_n + \frac{1}{n})^\gamma} \quad \text{in } \Omega_T = \Omega \times (0,T),\\ u_n &= 0 \quad \text{on } (\mathbb{R}^N \backslash \Omega) \times (0,T),\\ u_n(\cdot,0) &= u_0(\cdot) \quad \text{in } \Omega \end{cases}$$

and

$$\begin{cases} (u_{n+1})_t + (-\Delta)^s u_{n+1} &= \dfrac{f_{n+1}(x,t)}{(u_{n+1} + \frac{1}{n+1})^\gamma} \quad \text{in } \Omega_T = \Omega \times (0,T),\\ u_{n+1} &= 0 \quad \text{on } (\mathbb{R}^N \backslash \Omega) \times (0,T),\\ u_{n+1}(\cdot,0) &= u_0(\cdot) \quad \text{in } \Omega. \end{cases}$$

Subtracting the two equations, taking $(u_n - u_{n+1})^+$ as a test function and using the following inequality

$$(g(x,t) - g(y,t))(g^+(x,t) - g^+(y,t)) \geq (g^+(x,t) - g^+(y,t))^2$$

which holds true for every x, $y \in \mathbb{R}^N$, $t \in \mathbb{R}$, we arrive at

$$\begin{aligned} &\int_{\Omega_T} (u_n - u_{n+1})_t (u_n - u_{n+1})^+ dxdt + \frac{1}{2}\int_{Q_T} \frac{|(u_n - u_{n+1})^+(x,t) - (u_n - u_{n+1})^+(y,t))|^2}{|x-y|^{N+2s}} dydxdt \\ &\leq \int_{\Omega_T} \left(\frac{f_n(x,t)}{(u_n + \frac{1}{n})^\gamma} - \frac{f_{n+1}(x,t)}{(u_{n+1} + \frac{1}{n+1})^\gamma} \right) (u_n - u_{n+1})^+ dxdt \\ &\leq \int_{\Omega_T} f_{n+1} \left(\frac{(u_{n+1} + \frac{1}{n+1})^\gamma - (u_n + \frac{1}{n})^\gamma}{(u_n + \frac{1}{n})^\gamma (u_{n+1} + \frac{1}{n+1})^\gamma} \right) (u_n - u_{n+1})^+ dxdt \leq 0. \end{aligned}$$

Pointing out that $(u_n - u_{n+1})^+(x,0) = 0$, we get

$$\begin{aligned} \int_{\Omega_T} (u_n - u_{n+1})_t (u_n - u_{n+1})^+ dxdt &= \frac{1}{2}\int_0^T \int_\Omega \frac{d}{dt}\Big((u_n - u_{n+1})^+\Big)^2 dxdt \\ &= \frac{1}{2}\int_\Omega \Big((u_n - u_{n+1})^+(x,T)\Big)^2 dx \geq 0. \end{aligned}$$

Therefore,

$$0 \leq \|(u_n - u_{n+1})^+\|^2_{L^2(0,T;X_0^s(\Omega))} = \int_{Q_T} \frac{|(u_n - u_{n+1})^+(x,t) - (u_n - u_{n+1})^+(y,t))|^2}{|x-y|^{N+2s}} dydxdt \leq 0,$$

which implies that the sequence $\{u_n\}_n$ is increasing with respect to n.

We now turn to prove the inequality (4.12). Starting as in [24], we need in a first time to prove the for every open ball $B_r(x_0)$ included in Ω there exists a positive constant $C(x_0, r)$ depending on x_0 and r but not on n such that

$$u_n(x,t) \geq C(x_0, r), \quad \forall (x,t) \in B_{r/2}(x_0) \times [0,T), \quad \forall n \in \mathbb{N}. \tag{4.13}$$

Consider the compact subset $K := \overline{B_{3r/4}(x_0)}$. By the initial condition (4.2) on u_0 there is a constant d_K such that

$$u_0(x) \geq d_K, \text{ for all } x \in K.$$

This implies

$$u_n(x,0) = u_0(x) \geq d_K, \quad \forall (x,t) \in K \times \{0\}. \tag{4.14}$$

Let v be the non-negative solution (see [63, Theorem 26]) of the following problem

$$\begin{cases} v_t + (-\Delta)^s v &= 0 \quad \text{in } \Omega_T = \Omega \times (0,T), \\ v(x,t) &= 0 \quad \text{on } (\mathbb{R}^N \backslash \Omega) \times (0,T), \\ v(x,0) &= v_0 \quad \text{in } \Omega, \end{cases}$$

where

$$v_0(x) = \begin{cases} d_K & \text{if } x \in K, \\ 0 & \text{if } x \in \Omega \backslash K. \end{cases}$$

By the weak comparison principle, (see [6, Lemma 2.2] or [8, Lemma 2.9]), we obtain

$$u_n(x,t) \geq v(x,t), \quad \text{for all } (x,t) \in \mathbb{R}^N \times (0,T).$$

Let $\Omega' := B_{3r/4}(x_0)$ be the open ball and define the function

$$g(x,t) = \begin{cases} v(x,t) & \text{if } (x,t) \in (\Omega \backslash \Omega') \times (0,T), \\ 0 & \text{if } (x,t) \in (\mathbb{R}^N \backslash \Omega) \times (0,T). \end{cases}$$

It's easy to see that v is the solution of the problem

$$\begin{cases} v_t + (-\Delta)^s v &= 0 \quad \text{in } \Omega' \times (0,T), \\ v(x,t) &= g \quad \text{on } (\mathbb{R}^N \backslash \Omega') \times (0,T), \\ v(x,0) &= v_0 \quad \text{in } \Omega'. \end{cases}$$

On the other hand, let w be the solution (see [63, Theorem 26]) of the following problem

$$\begin{cases} w_t + (-\Delta)^s w &= 0 \quad \text{in } \Omega' \times (0,T), \\ w(x,t) &= 0 \quad \text{on } (\mathbb{R}^N \backslash \Omega') \times (0,T), \\ w(x,0) &= v_0 \quad \text{in } \Omega'. \end{cases}$$

Using the fact that $g(x,t) \geq 0$ on $(\mathbb{R}^N \backslash \Omega') \times (0,T)$ and again by the weak comparison principle, (see [6, Lemma 2.2] or [8, Lemma 2.9]), we get

$$v(x,t) \geq w(x,t), \quad \text{for all } (x,t) \in \mathbb{R}^N \times (0,T).$$

Now, let $\phi_1 \in X_0^s(\Omega') \cap L^\infty(\Omega')$ be the (eigenfunction) solution (see e.g. [77, Proposition 4] and [78, Proposition 9]) of the following eigenvalue problem

$$\begin{cases} (-\Delta)^s \phi &= \lambda \phi \quad \text{in } \Omega', \\ \phi &= 0 \quad \text{on } \mathbb{R}^N \backslash \Omega'. \end{cases}$$

corresponding to the smallest eigenvalue λ_1. So that defining $z(x,t) = \dfrac{d_K}{\|\phi_1\|_{L^\infty(\Omega')}} e^{-\lambda_1 t} \phi_1(x)$, we observe that the function z is a solution to the following problem

$$\begin{cases} z_t + (-\Delta)^s z &= 0 \quad \text{in } \Omega' \times (0,T), \\ z(x,t) &= 0 \quad \text{on } (\mathbb{R}^N \backslash \Omega') \times (0,T), \\ z(x,0) &= \dfrac{d_K}{\|\phi_1\|_{L^\infty(\Omega')}} \phi_1(x) \quad \text{in } \Omega'. \end{cases}$$

Since $z(x,0) \leq w(x,0)$ on Ω', applying again the weak comparison principle (see [6, Lemma 2.2] or [8, Lemma 2.9]), we obtain

$$w(x,t) \geq z(x,t), \quad \text{for all } (x,t) \in \mathbb{R}^N \times (0,T).$$

Which in particular implies

$$w(x,t) \geq \frac{d_K}{\|\phi_1\|_{L^\infty(\Omega')}} e^{-\lambda_1 t} \phi_1(x), \quad \forall (x,t) \in \Omega' \times (0,T).$$

From [19, Lemma 4.2], we know that there exists a positive constant $C_1 > 0$, depending on Ω', such that for every $x \in \Omega'$, one has $\phi_1(x) \geq C_1 \big(\delta'(x)\big)^s$, where $\delta'(x) = dist(x, \partial\Omega')$. Therefore, we have

$$w(x,t) \geq C_1 \frac{d_K}{\|\phi_1\|_{L^\infty(\Omega')}} e^{-\lambda_1 T} \big(\delta'(x)\big)^s, \quad \text{for all } (x,t) \in \Omega' \times (0,T).$$

Taking into the account the above comparisons, we obtain

$$u_n(x,t) \geq C_1 \frac{d_K}{\|\phi_1\|_{L^\infty(\Omega')}} e^{-\lambda_1 T}\left(\frac{r}{4}\right)^s, \qquad \text{for all } (x,t) \in B_{r/2}(x_0) \times (0,T).$$

Now, having in mind (4.14) for $t = 0$ we conclude that for every $(x,t) \in B_{r/2}(x_0) \times [0,T)$

$$u_n(x,t) \geq C(x_0,r) := \min\left(\frac{d_K C_1}{\|\phi_1\|_{L^\infty(\Omega')}} e^{-\lambda_1 T}\big(\frac{r}{4}\big)^s, d_K\right).$$

Let now $\omega \subset\subset \Omega$ be an arbitrary open set. We know that $\overline{\omega}$ is covered by a finite number m of open balls $B_{r_1/2}(x_1)$, $B_{r_2/2}(x_2)$, $\cdots$, $B_{r_m/2}(x_m)$, that is

$$\overline{\omega} \subset \bigcup_{j=1}^{m} B_{r_j/2}(x_j).$$

Therefore, applying (4.13) to every ball $B_{r_j/2}(x_j)$ and choosing

$$c_\omega = \min_{1\leq j\leq m}\Big(C(x_j, r_j/2)\Big).$$

We easily conclude (4.12). □

4.4 Proof of main results

4.4.1 Proof of Theorems 4.2.1 and 4.2.2

In order to prove the existence of solutions for the problem (4.1), we first need some a priori estimates on u_n.

4.4.1.1 A priori estimates in fractional Sobolev spaces

Lemma 4.4.1. *Let $\gamma = 1$ and $(f, u_0) \in (L^1(\Omega_T) \times L^\infty(\Omega))$. Let u_n be a solution of (4.6), then the sequence $\{u_n\}$ is uniformly bounded in $L^2(0,T; X_0^s(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$.*

Proof. Using $u_n(x,t)\chi_{(0,\tau)}(t)$, $\tau \in (0,T]$, as a test function in (4.7), we get

$$\int_0^\tau \int_\Omega (u_n)_t u_n dxdt + \frac{1}{2}\int_0^\tau \int_Q \frac{(u_n(x,t) - u_n(y,t))^2}{|x-y|^{N+2s}} dydxdt \leq \int_{\Omega_T} \frac{f(x,t)u_n}{(u_n + \frac{1}{n})} dxdt.$$

Then we have

$$\int_\Omega u_n^2(x,\tau)dx + \int_0^\tau \int_Q \frac{(u_n(x,t) - u_n(y,t))^2}{|x-y|^{N+2s}} dydxdt \leq 2\|f\|_{L^1(\Omega_T)} + \|u_0\|^2_{L^\infty(\Omega)}|\Omega|.$$

Taking the supremum of the left term on $[0,T]$, we get

$$\sup_{0\le\tau\le T}\int_\Omega u_n^2(x,\tau)dx+\int_0^T\int_Q\frac{(u_n(x,t)-u_n(y,t))^2}{|x-y|^{N+2s}}dydxdt\le 2\|f\|_{L^1(\Omega_T)}+\|u_0\|^2_{L^\infty(\Omega)}|\Omega|,$$

which implies that the sequence $\{u_n\}_n$ is uniformly bounded in $L^2(0,T;X_0^s(\Omega))\cap L^\infty(0,T;L^2(\Omega))$. □

Lemma 4.4.2. *Let $0<\gamma<1$ and $u_0\in L^\infty(\Omega)$. Let u_n be a solution of (4.6). Then,*

i) if $f\in L^{\frac{2}{\gamma+1}}(0,T;L^{\left(\frac{2_s^*}{1-\gamma}\right)'}(\Omega))$,

or

ii) if $f\in L^{\overline{m}}(\Omega_T)$, with $\overline{m}=\dfrac{2(N+2s)}{2(N+2s)-N(1-\gamma)}$,

the sequence $\{u_n\}$ is uniformly bounded in $L^2(0,T;X_0^s(\Omega))\cap L^\infty(0,T;L^2(\Omega))$.

Proof. Taking $u_n(x,t)\chi_{(0,\tau)}(t)$ as a test function in (4.7), by similar arguments in the previous lemma, we obtain

$$\sup_{0\le\tau\le T}\int_\Omega u_n^2(x,\tau)dx+\int_0^T\int_Q\frac{(u_n(x,t)-u_n(y,t))^2}{|x-y|^{N+2s}}dydxdt\le 2\int_{\Omega_T}fu_n^{1-\gamma}dxdt+\|u_0\|^2_{L^\infty(\Omega)}|\Omega|. \tag{4.15}$$

$i)$ Since $f\in L^{\frac{2}{\gamma+1}}(0,T;L^{\left(\frac{2_s^*}{1-\gamma}\right)'}(\Omega))$, we apply the Hölder inequality twice obtaining

$$\begin{aligned}\int_{\Omega_T}fu_n^{1-\gamma}dxdt &\le \int_0^T\left(\int_\Omega|f(x,t)|^{\left(\frac{2_s^*}{1-\gamma}\right)'}dx\right)^{\frac{1}{\left(\frac{2_s^*}{1-\gamma}\right)'}}\left(\int_\Omega|u_n(x,t)|^{2_s^*}dx\right)^{\frac{1-\gamma}{2_s^*}}dt\\ &=\int_0^T\|f\|_{L^{\left(\frac{2_s^*}{1-\gamma}\right)'}(\Omega))}\|u_n\|^{1-\gamma}_{L^{2_s^*}(\Omega))}dt \qquad (4.16)\\ &\le\left(\int_0^T\|f\|^{\frac{2}{1+\gamma}}_{L^{\left(\frac{2_s^*}{1-\gamma}\right)'}(\Omega)}dt\right)^{\frac{1+\gamma}{2}}\left(\int_0^T\|u_n\|^2_{L^{2_s^*}(\Omega)}dt\right)^{\frac{1-\gamma}{2}}.\end{aligned}$$

An application of the Sobolev embedding in the last term on the right-hand side in (4.16) yields

$$\int_{\Omega_T}fu_n^{1-\gamma}dxdt\le\big(S(N,s)\big)^{\frac{1-\gamma}{2}}\|f\|_{L^{\frac{2}{\gamma+1}}(0,T;L^{\left(\frac{2_s^*}{1-\gamma}\right)'}(\Omega))}\left[\int_0^T\int_Q\frac{(u_n(x,t)-u_n(y,t))^2}{|x-y|^{N+2s}}dydxdt\right]^{\frac{1-\gamma}{2}}.$$

Using Young's inequality we deduce from (4.15)

$$\sup_{0\le\tau\le T}\int_\Omega u_n^2(x,\tau)dx+\int_0^T\int_Q\frac{(u_n(x,t)-u_n(y,t))^2}{|x-y|^{N+2s}}dydxdt\le C,$$

where C is a positive constant which does not depend on n.

ii) As $f \in L^{\overline{m}}(\Omega_T)$, applying the Hölder inequality in the first term on the right hand-side in (4.15), we get

$$\begin{aligned} &\sup_{0\le\tau\le T}\int_\Omega u_n^2(x,\tau)dx+\int_0^T\int_Q \frac{|u_n(x,t)-u_n(y,t)|^2}{|x-y|^{N+2s}}dydxdt \\ &\le 2\|f\|_{L^{\overline{m}}(\Omega_T)}\Big[\int_{\Omega_T} u_n^{(1-\gamma)\overline{m}'}dxdt\Big]^{\frac{1}{\overline{m}'}}+\|u_0\|^2_{L^\infty(\Omega)}|\Omega|. \end{aligned} \tag{4.17}$$

On other hand, by the Hölder inequality with the exponents $\dfrac{N}{N-2s}$ and $\dfrac{N}{2s}$ and applying the Sobolev embedding, we can write

$$\begin{aligned} \int_{\Omega_T}|u_n|^{\frac{2(N+2s)}{N}}dxdt &= \int_{\Omega_T}|u_n|^2|u_n|^{\frac{4s}{N}}dxdt \\ &\le \int_0^T\Big[\int_\Omega |u_n(x,t)|^{\frac{2N}{N-2s}}dx\Big]^{\frac{N-2s}{N}}\Big[\int_\Omega|u_n(x,t)|^2dx\Big]^{\frac{2s}{N}}dt \\ &= \int_0^T\Big[\int_\Omega|u_n(x,t)|^2dx\Big]^{\frac{2s}{N}}\|u_n\|^2_{L^{2_s^*}(\Omega))}dt \\ &\le S(N,s)\Big[\sup_{0\le t\le T}\int_\Omega|u_n(x,t)|^2dx\Big]^{\frac{2s}{N}}\int_0^T\int_Q\frac{(u_n(x,t)-u_n(y,t))^2}{|x-y|^{N+2s}}dydxdt. \end{aligned} \tag{4.18}$$

So that by (4.17) we get

$$\begin{aligned} \int_{\Omega_T}|u_n|^{\frac{2(N+2s)}{N}} &\le S(N,s)\Big(2\|f\|_{L^{\overline{m}}(\Omega_T)}\Big(\int_{\Omega_T}u_n^{(1-\gamma)\overline{m}'}dxdt\Big)^{\frac{1}{\overline{m}'}}+\|u_0\|^2_{L^\infty(\Omega)}|\Omega|\Big)^{\frac{N+2s}{N}} \\ &\le S(N,s)2^{\frac{2s}{N}}\Big(\Big(2\|f\|_{L^{\overline{m}}(\Omega_T)}\Big)^{\frac{N+2s}{N}}\Big(\int_{\Omega_T}u_n^{(1-\gamma)\overline{m}'}\Big)^{\frac{N+2s}{N\overline{m}'}}+\Big(\|u_0\|^2_{L^\infty(\Omega)}|\Omega|\Big)^{\frac{N+2s}{N}}\Big). \end{aligned}$$

Pointing out that $(1-\gamma)\overline{m}' = \dfrac{2(N+2s)}{N}$ and $\dfrac{N+2s}{N\overline{m}'} = \dfrac{1-\gamma}{2} < 1$, we use the Young inequality to obtain

$$\int_{\Omega_T}|u_n|^{\frac{2(N+2s)}{N}}dxdt \le C,$$

where C is a positive constant not depending on n. Therefore, by (4.17) we conclude that $\{u_n\}$ is uniformly bounded in $L^2(0,T;X_0^s(\Omega))\cap L^\infty(0,T;L^2(\Omega))$. □

4.4.1.2 Passing to the limit

Proof. of Theorem 4.2.2 and Theorem 4.2.1.

Since by Lemma 4.4.2 and Lemma 4.4.1, the sequence $\{u_n\}$ is uniformly bounded in the reflexive space $L^2(0,T;X_0^s(\Omega))$, there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a measurable function $u \in L^2(0,T;X_0^s(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^2(0,T;X_0^s(\Omega))$. On other hand, by the Poincaré inequality (Lemma 1.1.1) the increasing sequence $\{u_n\}$ is uniformly bounded in $L^2(\Omega_T) \subset L^1(\Omega_T)$, so that by Beppo-Levi's theorem u_n converges to a function v in norm in $L^1(\Omega_T)$ and (for a subsequence if necessary) a.e. in $\Omega \times (0,T)$. As $L^2(0,T;X_0^s(\Omega)) \subset L^2(\Omega_T)$ we have the identification $u = v$ a.e. in $\Omega \times (0,T)$. In addition, since $u_n = u = 0$ on $\mathcal{C}\Omega \times (0,T)$, we obtain $u_n \to u$ for a.e. $(x,t) \in \mathbb{R}^N \times (0,T)$. Hence follows

$$\frac{u_n(x,t)-u_n(y,t)}{|x-y|^{\frac{N+2s}{2}}} \to \frac{u(x,t)-u(y,t)}{|x-y|^{\frac{N+2s}{2}}} \text{ a.e. in } Q \times (0,T).$$

Testing by an arbitrary function $\varphi \in C_0^\infty(\Omega \times [0,T))$ in (4.7) we get

$$\begin{aligned} &-\int_{\Omega_T} u_n \varphi_t dxdt - \int_\Omega u_n(x,0)\varphi(x,0)dx + \frac{1}{2}\int_{Q_T} \frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt \\ &= \int_{\Omega_T} \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dxdt. \end{aligned} \tag{4.19}$$

It is clear that

$$\lim_{n\to\infty}\int_{\Omega_T} u_n\varphi_t dxdt = \int_{\Omega_T} u\varphi_t dxdt$$

and

$$\lim_{n\to\infty}\int_\Omega u_n(x,0)\varphi(x,0)dx = \int_\Omega u_0(x)\varphi(x,0)dx.$$

Define

$$F_n(x,y,t) = \frac{u_n(x,t)-u_n(y,t)}{|x-y|^{\frac{N+2s}{2}}} \text{ and } F(x,y,t) = \frac{u(x,t)-u(y,t)}{|x-y|^{\frac{N+2s}{2}}}.$$

Since $\{u_n\}$ is uniformly bounded in $L^2(0,T;X_0^s(\Omega))$, then so is $\{F_n(x,y,t)\}_n$ in $L^2(Q\times(0,T))$ which implies that for a subsequence $F_n \rightharpoonup F$ in $L^2(Q\times(0,T))$. On the other hand, observe that for every $\varphi \in C_0^\infty(\Omega\times[0,T)) \subset L^2(0,T;X_0^s(\Omega))$, we have

$$\frac{\varphi(x,t)-\varphi(y,t)}{|x-y|^{\frac{N+2s}{2}}} \in L^2(Q\times(0,T)).$$

Thus, we have

$$\begin{aligned} \lim_{n\to\infty}\int_{Q_T} &\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt = \\ &\int_{Q_T}\frac{(u(x)-u(y))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt, \end{aligned}$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega \times [0,T))$. As regards the last term on the right-hand side in (4.19) we use Lemma 4.3.2. Whence, for any $\varphi \in \mathcal{C}_0^\infty(\Omega \times [0,T))$ with $supp(\varphi) \subset \omega \times [0,T)$, there exists a constant $c_\omega > 0$, not depending on n, such that

$$0 \leq \left| \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} \right| \leq \frac{\|\varphi\|_{L^\infty(\Omega_T)} |f|}{c_\omega^\gamma} \in L^1(\Omega_T),$$

so that by the dominated convergence theorem we obtain

$$\lim_{n\to\infty} \int_{\Omega_T} \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} dxdt = \int_{\Omega_T} \frac{f\varphi}{u^\gamma} dxdt.$$

Finally, passing to the limit as $n \to +\infty$ we get

$$-\int_{\Omega_T} u\varphi_t dxdt - \int_\Omega u_0(x)\varphi(x,0)dx + \frac{1}{2}\int_{Q_T} \frac{(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} dxdydt$$
$$= \int_{\Omega_T} \frac{f\varphi}{u^\gamma} dxdt,$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega \times [0,T)$. So u is a weak solution of (4.1). □

4.4.2 Proof of Theorem 4.2.3

4.4.2.1 A priori estimates in fractional Sobolev spaces

Lemma 4.4.3. *Assume that $0 < \gamma < 1$ and $(f, u_0) \in (L^m(\Omega_T) \times L^\infty(\Omega))$ with $1 \leq m < \overline{m} := \dfrac{2(N+2s)}{2(N+2s) - N(1-\gamma)}$. Let u_n be a solution of (4.6). Then the sequence $\{u_n\}$ is uniformly bounded in $L^{\overline{q}}(0,T; W_0^{s_1,\overline{q}}(\Omega)) \cap L^\infty(0,T; L^{1+\gamma}(\Omega))$ for every $s_1 < s$ with*

$$\overline{q} = \frac{m(\gamma+1)(N+2s)}{N+2s-sm(1-\gamma)}.$$

Moreover, $\{u_n\}_n$ is uniformly bounded in $L^\sigma(\Omega_T)$, with $\sigma = \dfrac{m(\gamma+1)(N+2s)}{N-2s(m-1)}$.

Proof. Let $\gamma \leq \theta < 1$, $0 < \varepsilon < \dfrac{1}{n}$ and let $\tau \in (0,T]$. Taking $\Big((u_n(x,t)+\varepsilon)^\theta - \varepsilon^\theta\Big)\chi_{(0,\tau)}(t)$ as a test function in (4.7), we have

$$\int_0^\tau \int_\Omega (u_n)_t[(u_n+\varepsilon)^\theta - \varepsilon^\theta] dxdt$$
$$+\frac{1}{2}\int_0^\tau \int_Q \frac{(u_n(x,t)-u_n(y,t))((u_n(x,t)+\varepsilon)^\theta - (u_n(y,t)+\varepsilon)^\theta)}{|x-y|^{N+2s}} dydxdt$$
$$\leq \int_{\Omega_T} f(x,t)u_n(u_n(x,t)+\varepsilon)^{\theta-\gamma} dxdt.$$

Letting ε tends to zero, we get

$$\frac{2}{\theta+1}\int_{\Omega}|u_n(x,\tau)|^{\theta+1}dx \quad +\int_0^{\tau}\int_Q \frac{(u_n(x,t)-u_n(y,t))(u_n^{\theta}(x,t)-u_n^{\theta}(y,t))}{|x-y|^{N+2s}}dydxdt$$
$$\leq 2\int_{\Omega_T} f_n u_n^{\theta-\gamma}dxdt+\frac{2}{\theta+1}\int_{\Omega}u_0^{\theta+1}(x)dx.$$

As this inequality holds true for every $\tau\in[0,T]$, we can pass to the supremum on the left-hand side obtaining

$$\frac{2}{\theta+1}\sup_{0\leq\tau\leq T}\int_{\Omega}|u_n(x,\tau)|^{\theta+1}dx \quad +\int_0^{T}\int_Q \frac{(u_n(x,t)-u_n(y,t))(u_n^{\theta}(x,t)-u_n^{\theta}(y,t))}{|x-y|^{N+2s}}dydxdt$$
$$\leq 2\int_{\Omega_T} f u_n^{\theta-\gamma}dxdt+\frac{2|\Omega|}{\theta+1}\|u_0\|^{\theta+1}_{L^{\infty}(\Omega)}. \tag{4.20}$$

Then an application of Lemma 1.3.3 yields

$$\begin{aligned}\frac{\theta+1}{2\theta}\sup_{0\leq\tau\leq T}\int_{\Omega}|u_n(x,\tau)|^{\theta+1}dx \quad &+\int_0^{T}\int_Q \frac{\left|u_n^{\frac{\theta+1}{2}}(x,t)-u_n^{\frac{\theta+1}{2}}(y,t)\right|^2}{|x-y|^{N+2s}}dydxdt\\
&\leq \frac{(\theta+1)^2}{2\theta}\int_{\Omega_T} f u_n^{\theta-\gamma}dxdt+\frac{(\theta+1)|\Omega|}{2\theta}\|u_0\|^{\theta+1}_{L^{\infty}(\Omega)}\\
&\leq \frac{2}{\theta}\int_{\Omega_T} f u_n^{\theta-\gamma}dxdt+\frac{|\Omega|}{\theta}\|u_0\|^{\theta+1}_{L^{\infty}(\Omega)}\\
.\qquad &\leq \frac{\max(|\Omega|,2)}{\theta}\left(\int_{\Omega_T} f u_n^{\theta-\gamma}dxdt+\|u_0\|^{\theta+1}_{L^{\infty}(\Omega)}\right).\end{aligned} \tag{4.21}$$

In the previous inequality, the term $\dfrac{\theta+1}{2\theta}>1$, can be dropped. On one hand, an application of the Hölder inequality with $\dfrac{N}{N-2s}$ and $\dfrac{N}{2s}$ gives

$$\begin{aligned}\int_{\Omega_T}|u_n|^{\frac{(\theta+1)(N+2s)}{N}}dxdt &= \int_{\Omega_T}|u_n|^{2\frac{\theta+1}{2}}|u_n|^{\frac{4s}{N}\frac{\theta+1}{2}}dxdt\\
&\leq \int_0^T\left(\int_{\Omega}|u_n(x,t)|^{2_s^*\frac{\theta+1}{2}}dx\right)^{\frac{N-2s}{N}}\left[\int_{\Omega}|u_n(x,t)|^{\theta+1}dx\right]^{\frac{2s}{N}}dt\\
&= \int_0^T\|u_n^{\frac{\theta+1}{2}}\|^2_{L^{2_s^*}}\left[\int_{\Omega}|u_n(x,t)|^{\theta+1}dx\right]^{\frac{2s}{N}}dt.\end{aligned}$$

Taking the supremum on $[0,T]$ and applying the Sobolev embedding, we obtain

$$\int_{\Omega_T}|u_n|^{\frac{(\theta+1)(N+2s)}{N}}\leq S(N,s)\left(\sup_{0\leq t\leq T}\int_{\Omega}|u_n(x,t)|^{\theta+1}dx\right)^{\frac{2s}{N}}\int_{Q_T}\frac{\left|u_n^{\frac{\theta+1}{2}}(x,t)-u_n^{\frac{\theta+1}{2}}(y,t)\right|^2}{|x-y|^{N+2s}}dydxdt.$$

By (4.21) we have

$$\begin{aligned}\int_{\Omega_T} |u_n|^{\frac{(\theta+1)(N+2s)}{N}} dxdt &\le S(N,s)\left(\frac{\max(|\Omega|,2)}{\theta}\right)^{\frac{N+2s}{N}} \left(\int_{\Omega_T} f u_n^{\theta-\gamma} dxdt + \|u_0\|_{L^\infty(\Omega)}^{\theta+1}\right)^{\frac{N+2s}{N}} \\ &\le S(N,s)2^{\frac{2s}{N}}\left(\frac{\max(|\Omega|,2)}{\theta}\right)^{\frac{N+2s}{N}} \left(\left(\int_{\Omega_T} f u_n^{\theta-\gamma} dxdt\right)^{\frac{N+2s}{N}} + \|u_0\|_{L^\infty(\Omega)}^{\frac{(\theta+1)N+2s}{N}}\right).\end{aligned} \tag{4.22}$$

If $m = 1$ then we take $\theta = \gamma$ in the previous estimation. So we obtain

$$\int_{\Omega_T} |u_n|^{\frac{(\gamma+1)(N+2s)}{N}} dxdt \le S(N,s)2^{\frac{2s}{N}}\left(\frac{\max(|\Omega|,2)}{\gamma}\right)^{\frac{N+2s}{N}} \left(\|f\|_{L^1(\Omega_T)}^{\frac{N+2s}{N}} + \|u_0\|_{L^\infty(\Omega)}^{\frac{(\gamma+1)(N+2s)}{N}}\right). \tag{4.23}$$

On the other hand, by (4.21) we easily have

$$\|u_n\|_{L^\infty(0,T;L^{\gamma+1}(\Omega))} \le \frac{2\max(|\Omega|,2)}{\gamma+1}\left[\|f\|_{L^1(\Omega_T)} + \|u_0\|_{L^\infty(\Omega)}^{\gamma+1}\right]. \tag{4.24}$$

We now consider the case $m > 1$. Let $\gamma < \theta < 1$. Using (4.22) and applying the Hölder inequality, we obtain

$$\int_{\Omega_T} u_n^{\frac{(\theta+1)(N+2s)}{N}} dxdt \le C_1\|f\|_{L^m(\Omega_T)}^{\frac{N+2s}{N}}\left[\int_{\Omega_T} u_n^{m'(\theta-\gamma)} dxdt\right]^{\frac{N+2s}{Nm'}} + C_1\|u_0\|_{L^\infty(\Omega)}^{\frac{(\theta+1)(N+2s)}{N}},$$

where $C_1 = S(N,s)2^{\frac{2s}{N}}\left(\frac{\max(|\Omega|,2)}{\theta}\right)^{\frac{N+2s}{N}}$. We can choose $\gamma < \theta < 1$ to be such that

$$\frac{(\theta+1)(N+2s)}{N} = m'(\theta-\gamma).$$

That is

$$\theta = \frac{(N+2s)(m-1) + Nm\gamma}{N - 2s(m-1)}.$$

Notice that the condition $\theta < 1$ is equivalent to $m < \overline{m}$; while $\gamma < \theta$ is always fulfilled. Since $\frac{N+2s}{Nm'} < 1$, applying Young's inequality with $\varepsilon > 0$, we get

$$\int_{\Omega_T} u_n^{\frac{(\theta+1)(N+2s)}{N}} dxdt \le C_1\|f\|_{L^m(\Omega_T)}^{\frac{N+2s}{N}}\left(\varepsilon\int_{\Omega_T} u_n^{\frac{(\theta+1)(N+2s)}{N}} dxdt + C(\varepsilon)\right) + C_1\|u_0\|_{L^\infty(\Omega)}^{\frac{(\theta+1)(N+2s)}{N}}.$$

Choosing ε small enough such that $\varepsilon C_1\|f\|_{L^m(\Omega_T)}^{\frac{N+2s}{N}} < 1$ and using the fact that

$$\sigma := \frac{m(\gamma+1)(N+2s)}{N-2s(m-1)} = \frac{(\theta+1)(N+2s)}{N},$$

we get

$$\int_{\Omega_T} |u_n|^\sigma dxdt \le C, \tag{4.25}$$

where C is a positive constant which does not depend on n.

It remains to prove that $\{u_n\}$ is uniformly bounded in $L^\infty(0,T;L^{\gamma+1}(\Omega))$. By (4.21) we have

$$\begin{aligned}\sup_{0\le\tau\le T}\int_\Omega u_n^{\theta+1}(x,\tau)dx &\le \frac{\max(|\Omega|,2)}{\theta}\left(\int_{\Omega_T} f u_n^{\theta-\gamma}dxdt + \|u_0\|^{\theta+1}_{L^\infty(\Omega)}\right)\\ &\le \frac{\max(|\Omega|,2)}{\theta}\left(\|f\|_{L^m(\Omega_T)}\|u_n\|^{\frac{\sigma}{m'}}_{L^\sigma(\Omega_T)} + \|u_0\|^{\theta+1}_{L^\infty(\Omega)}\right).\end{aligned}$$

Since $\gamma<\theta$ and by (4.25) we conclude that $\{u_n\}$ is uniformly bounded in $L^\infty(0,T;L^{\gamma+1}(\Omega))$. Finally, we conclude that in both cases, that is $1\le m<\overline{m}$, the sequence $\{u_n\}$ is uniformly bounded in $L^\sigma(\Omega_T)$, $\sigma := \dfrac{m(\gamma+1)(N+2s)}{N-2s(m-1)}$ and in $L^\infty(0,T;L^{\gamma+1}(\Omega))$. Thus, by (4.20), we have

$$\int_0^T\int_Q \frac{(u_n(x,t)-u_n(y,t))(u_n^\theta(x,t)-u_n^\theta(y,t))}{|x-y|^{N+2s}}dydxdt \le C, \tag{4.26}$$

where C is a positive constant which does not depend on n.

Now, we shall prove that the sequence $\{u_n\}$ is uniformly bounded in a suitable fractional Sobolev space. Let $s_1\in(0,s)$, be fixed and let $1<\overline{q}<2$ that will be chosen later. Applying Hölder's inequality, we get

$$\begin{aligned}&\int_0^T\int_\Omega\int_\Omega \frac{|u_n(x,t)-u_n(y,t)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydxdt\\ &=\int_0^T\int_\Omega\int_{\{y\in\Omega:u_n(y,t)\ne u_n(x,t)\}} \frac{|u_n(x,t)-u_n(y,t)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}} \times \frac{u_n^\theta(x,t)-u_n^\theta(y,t)}{u_n(x,t)-u_n(y,t)} \times \frac{u_n(x,t)-u_n(y,t)}{u_n^\theta(x,t)-u_n^\theta(y,t)}\\ &\le\left(\int_0^T\int_\Omega\int_{\{y\in\Omega:u_n(y,t)\ne u_n(x,t)\}} \frac{|u_n(x,t)-u_n(y,t)|^2}{|x-y|^{N+2s}} \times \frac{u_n^\theta(x,t)-u_n^\theta(y,t)}{u_n(x,t)-u_n(y,t)}dydxdt\right)^{\frac{\overline{q}}{2}}\\ &\times\left(\int_0^T\int_\Omega\int_{\{y\in\Omega:u_n(y,t)\ne u_n(x,t)\}} \left(\frac{u_n(x,t)-u_n(y,t)}{u_n^\theta(x,t)-u_n^\theta(y,t)}\right)^{\frac{2}{2-\overline{q}}} \times \frac{u_n^\theta(x,t)-u_n^\theta(y,t)}{u_n(x,t)-u_n(y,t)} \times \frac{dydxdt}{|x-y|^{N-\beta}}\right)^{\frac{2-\overline{q}}{2}},\end{aligned} \tag{4.27}$$

where $\beta = \dfrac{2\overline{q}(s-s_1)}{2-\overline{q}}$. Then

$$\begin{aligned}&\int_0^T\int_\Omega\int_\Omega \frac{|u_n(x,t)-u_n(y,t)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydxdt\\ &\le \left(\int_0^T\int_\Omega\int_\Omega \frac{(u_n(x,t)-u_n(y,t))(u_n^\theta(x,t)-u_n^\theta(y,t))}{|x-y|^{N+2s}}dydxdt\right)^{\frac{\overline{q}}{2}}\\ &\times\left(\int_0^T\int_\Omega\int_{\{y\in\Omega:u_n(y,t)\neq u_n(x,t)\}}\left(\frac{u_n(x,t)-u_n(y,t)}{u_n^\theta(x,t)-u_n^\theta(y,t)}\right)^{\frac{\overline{q}}{2-\overline{q}}}\times\frac{dydxdt}{|x-y|^{N-\beta}}\right)^{\frac{2-\overline{q}}{2}}.\end{aligned}$$

By the inequality (4.26) and Lemma 1.3.3, we get

$$\int_0^T\int_\Omega\int_\Omega \frac{|u_n(x,t)-u_n(y,t)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydxdt \le C_1\left(\int_0^T\int_\Omega\int_\Omega \frac{u_n^{\frac{\overline{q}(1-\theta)}{2-\overline{q}}}(x,t)+u_n^{\frac{\overline{q}(1-\theta)}{2-\overline{q}}}(y,t)}{|x-y|^{N-\beta}}dydxdt\right)^{\frac{2-\overline{q}}{2}}.$$

By x/y symmetry, there exists a constant C_2, not depending on n, such that

$$\int_0^T\int_\Omega\int_\Omega \frac{|u_n(x,t)-u_n(y,t)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydxdt \le C_2\left(\int_0^T\int_\Omega u_n^{\frac{\overline{q}(1-\theta)}{2-\overline{q}}}(x,t)\left(\int_\Omega\frac{dy}{|x-y|^{N-\beta}}\right)dxdt\right)^{\frac{2-\overline{q}}{2}}.$$

Since Ω is bounded, then there exists a constant $R>0$, such that $\Omega\subset B_R$, where B_R, is the ball of radius R, so, an easy computation leads to

$$\int_\Omega\frac{dy}{|x-y|^{N-\beta}} \le \int_0^R\frac{dz}{|z|^{N-\beta}} \le \frac{|S^{N-1}|}{\beta}R^\beta,$$

where $|S^{N-1}|$ stands for the Lebesgue measure of the unit sphere in $\mathbb{R}^N$. We now choose $\overline{q}$ to be such that

$$\frac{\overline{q}(1-\theta)}{2-\overline{q}} = \sigma := \frac{m(\gamma+1)(N+2s)}{N-2s(m-1)},$$

that is

$$\overline{q} = \frac{m(\gamma+1)(N+2s)}{N+2s-sm(1-\gamma)}.$$

We point out that $\overline{q}<2$ is equivalent to $m<\overline{m}$; while $\overline{q}>1$ is always fulfilled. Then we get

$$\int_0^T\int_\Omega\int_\Omega \frac{|u_n(x,t)-u_n(y,t)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydxdt \le C_2\left(\frac{|S^{N-1}|}{\beta}R^\beta\int_{\Omega_T}u_n^\sigma(x,t)dxdt\right)^{\frac{2-\overline{q}}{2}} \le C_3,$$

where C_3 is a positive constant which does not depend on n. Thus, $\{u_n\}$ is uniformly bounded in $L^{\overline{q}}(0,T;W_0^{s_1,\overline{q}}(\Omega))$ for every $s_1<s$. □

Remark 4.4.1. *Notice that by (4.25) the sequence $\{u_n\}$ is uniformly bounded in $L^r(\Omega)$ for every $1\le r\le\sigma$, then as is the same lines of the proof of the previous Lemma 4.4.3 with the exponent q instead of $\overline{q}$ in 4.27, we can obtain that $\{u_n\}$ is uniformly bounded in $L^q(0,T;W_0^{s_1,q}(\Omega))$ for all $1<q\le\overline{q}$ and for every $s_1<s$ and $1\le m<\overline{m}$.*

4.4.2.2 Passing to the limit

Proof. of Theorem 4.2.3.

By virtue of Lemma 4.4.3 the sequence $\{u_n\}$ is uniformly bounded in the reflexive space $L^{\overline{q}}(0,T;W_0^{s_1,\overline{q}}(\Omega))$, then there exist a subsequence of $\{u_n\}_n$ still indexed by n and a measurable function $u \in L^{\overline{q}}(0,T;W_0^{s_1,\overline{q}}(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^{\overline{q}}(0,T;W_0^{s_1,\overline{q}}(\Omega))$. Let's reason as above, the sequence $\{u_n\}$ is increasing and uniformly bounded in $L^1(\Omega_T)$, so that by Beppo-Levi's theorem $u_n \to u$ in norm in $L^1(\Omega_T)$ and a.e. in $\Omega\times(0,T)$ and since $u_n = 0$ on $\mathcal{C}\Omega\times(0,T)$, extending u by zero outside Ω, we get $u_n \to u$ for a.e $(x,t) \in \mathbb{R}^N \times (0,T)$. Let now $\rho > 0$ be a small enough real number that we will choose later. For any $\varphi \in C_0^\infty(\Omega\times[0,T))$ we have

$$
\begin{aligned}
&\int_0^T\int_\Omega\int_\Omega \left[\frac{|(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))|}{|x-y|^{N+2s}}\right]^{1+\rho} dydxdt\\
&\le \int_0^T\int_\Omega\int_\Omega \frac{|u_n(x,t)-u_n(y,t)|^{1+\rho}(\|D\varphi\|_{L^\infty(\Omega_T)}|x-y|)^{1+\rho}}{|x-y|^{N+(1+\rho)s_1}}\frac{dydxdt}{|x-y|^{\rho N+(1+\rho)(2s-s_1)}}\\
&\le \|D\varphi\|^{1+\rho}_{L^\infty(\Omega_T)}\int_0^T\int_\Omega\int_\Omega \frac{|u_n(x,t)-u_n(y,t)|^{1+\rho}|x-y|^{(1+\rho)(1+s_1-2s)-\rho N}}{|x-y|^{N+(1+\rho)s_1}}dydx.
\end{aligned}
$$

We need an adequate choice of ρ to assure that $(1+\rho)(1+s_1-2s)-\rho N \ge 0$. To this aim, we consider s_1 to be very close to s. Precisely, we impose on s_1 the condition

$$\max(0,1-3s) < s-s_1 < 1-s.$$

We point out that with this range of values of s_1 and with the assumption $N > 2s$, we obtain

$$1+s_1-2s > 0 \text{ and } N-1-s_1+2s > 0.$$

Thus, the fact that $(1+\rho)(1+s_1-2s)-\rho N \ge 0$ is equivalent to $0 < \rho \le \dfrac{1+s_1-2s}{N-1-s_1+2s}$. Hence, we get

$$
\begin{aligned}
&\int_0^T\int_\Omega\int_\Omega \left[\frac{|(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))|}{|x-y|^{N+2s}}\right]^{1+\rho} dydxdt\\
&\le \|D\varphi\|^{1+\rho}_{L^\infty(\Omega_T)} diam(\Omega)^{(1+\rho)(1+s_1-2s)-\rho N}\int_\Omega\int_\Omega \frac{|u_n(x,t)-u_n(y,t)|^{1+\rho}}{|x-y|^{N+(1+\rho)s_1}}dydxdt,
\end{aligned}
\tag{4.28}
$$

where $diam(\Omega)$ stands for the diameter of Ω. Now we make the adequate choice of ρ to prove that the right-hand integral in (4.28) is uniformly bounded. By Remark 4.4.1 we have the uniform boundedness of $\{u_n\}_n$ in $L^{\overline{q}}(0,T;W_0^{s_1,\overline{q}})$ for every $1<q\leq \overline{q}=\dfrac{m(\gamma+1)(N+2s)}{N+2s-sm(1-\gamma)}$. So it is sufficient to choose ρ such that $1+\rho\leq \overline{q}=\dfrac{m(\gamma+1)(N+2s)}{N+2s-sm(1-\gamma)}$. Finally, we choose ρ to be such that

$$0<\rho\leq \min\Big(\frac{(N+2s)(m-1)+m\gamma(N+s)+sm}{N+2s-sm(1-\gamma)},\frac{1+s_1-2s}{N-1-s_1+2s}\Big).$$

Therefore, there is a constant $C>0$ which does not depend on n such that

$$\sup_n\int_0^T\int_\Omega\int_\Omega\Big[\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}\Big]^{1+\rho}dydxdt\leq C.$$

Therefore, by De La Vallée-Poussin and Dunford-Pettis theorems the sequence

$$\Big\{\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}\Big\}$$

is equi-integrable in $L^1(\Omega\times\Omega\times(0,T))$. Now, taking $\varphi\in C_0^\infty(\Omega\times[0,T))$ as a test function in (4.7) we get

$$\begin{aligned}&-\int_{\Omega_T}u_n\varphi_t dxdt-\int_\Omega u_n(x,0)\varphi(x,0)dx+\frac{1}{2}\int_{Q_T}\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt\\&=\int_{\Omega_T}\frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dxdt.\end{aligned}\tag{4.29}$$

For the third integral on the left-hand side of (4.29), we split it into three integrals as follows

$$\begin{aligned}&\int_0^T\int_Q\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt\\&=\int_0^T\int_\Omega\int_\Omega\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt\\&+\int_0^T\int_\Omega\int_{\mathcal{C}\Omega}\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt\\&+\int_0^T\int_{\mathcal{C}\Omega}\int_\Omega\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt\\&=I_1+I_2+I_3.\end{aligned}\tag{4.30}$$

The almost everywhere convergence of $\{u_n\}$ to u allows us to get for every $\varphi\in C_0^\infty(\Omega\times[0,T))$

$$\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}\to\frac{(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}\ \text{a.e. in } Q\times(0,T).$$

Then by Vitali's lemma we have

$$\lim_{n\to\infty}\int_0^T\int_\Omega\int_\Omega \frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt$$
$$=\int_0^T\int_\Omega\int_\Omega \frac{(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt.$$

For the second integral I_2 in (4.30), we start noticing that since $u_n(y,t)=\varphi(y,t)=0$ for every $(y,t)\in \mathcal{C}\Omega\times(0,T)$ we can write

$$\Big|\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}\Big| = \frac{|u_n(x,t)\varphi(x,t)|}{|x-y|^{N+2s}} = |G_n(x,y,t)| \text{ in } (x,y,t)\in\Omega\times\mathcal{C}\Omega\times(0,T).$$

We need to prove that the sequence $\{G_n(x,y,t)\}_n$ is uniformly bounded in $L^1(\Omega\times\mathcal{C}\Omega\times(0,T))$. Since $supp(\varphi)$ is a compact subset in Ω, we have

$$|x-y|\geq d_1 := dist(supp(\varphi),\partial\Omega)>0 \text{ for every } (x,y)\in supp(\varphi)\times\mathcal{C}\Omega.$$

Therefore, an easy computation leads to

$$\begin{aligned}\int_0^T\int_\Omega\int_{\mathcal{C}\Omega}\frac{|u_n(x,t)\varphi(x,t)|}{|x-y|^{N+2s}}dydxdt &= \int_0^T\int_{supp(\varphi)}|u_n(x,t)\varphi(x,t)|\Big(\int_{\mathcal{C}\Omega}\frac{dy}{|x-y|^{N+2s}}\Big)dxdt\\ &\leq \frac{|S^{N-1}|\|\varphi\|_{L^\infty(\Omega_T)}\|u_n\|_{L^1(\Omega_T)}}{2sd_1^{2s}},\end{aligned}$$

where $|S^{N-1}|$ stands for the Lebesgue measure of the unit sphere in $\mathbb{R}^N$. On the other hand, since $\{u_n\}$ is increasing then so is $\{G_n(x,y,t)\}_n$. Hence Beppo-Levi's theorem and the almost everywhere converge of u_n to u yield

$$G_n(x,y,t)\to\frac{u(x,t)\varphi(x,t)}{|x-y|^{N+2s}} \text{ in } L^1(\Omega\times\mathcal{C}\Omega\times(0,T)).$$

Which implies

$$\lim_{n\to\infty}\int_0^T\int_\Omega\int_{\mathcal{C}\Omega} \frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt$$
$$=\int_0^T\int_\Omega\int_{\mathcal{C}\Omega} \frac{(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt.$$

As regards the third integral I_3 in (4.30), we can follow exactly the same lines as above using the x/y symmetry. We then conclude that

$$\lim_{n\to\infty}\int_{Q_T} \frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt$$
$$=\int_{Q_T} \frac{(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt,$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega \times [0,T))$. Now for what concerns the first, the second integrals and the right-hand side of (4.29), we follow the same arguments as in the proof of Theorem 4.2.2. Finally, passing to the limit as $n \to +\infty$, we get

$$-\int_{\Omega_T} u\varphi_t dxdt - \int_\Omega u_0(x)\varphi(x,0)dx + \frac{1}{2}\int_{Q_T} \frac{(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dxdydt = \int_{\Omega_T} \frac{f\varphi}{u^\gamma}dxdt,$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega \times [0,T))$. This shows that u is a weak solution of (4.1). □

4.4.3 Proof of Theorem 4.2.4

4.4.3.1 A priori estimates in fractional Sobolev spaces

Lemma 4.4.4. *Suppose that $\gamma > 1$ and $(f, u_0) \in L^1(\Omega_T) \times L^\infty(\Omega)$. Let u_n be a solution of (4.6). Then the sequence $\{u_n\}$ is uniformly bounded in $L^2(0,T; H^s_{loc}(\Omega)) \cap L^\infty(0,T; L^{\gamma+1}(\Omega))$. Moreover $\{u_n^{\frac{\gamma+1}{2}}\}_n$ is uniformly bounded in $L^2(0,T; X_0^s(\Omega))$.*

Proof. Taking $u_n^\gamma(x,t)\chi_{(0,\tau)}(t)$, $\tau \in (0,T]$, as a test function in (4.7) and then the supremum on $[0,T]$ in the left-hand side, we get

$$\frac{2}{\gamma+1}\sup_{0\le\tau\le T}\int_\Omega u_n^{\gamma+1}(x,\tau)dx + \int_0^T\int_Q \frac{(u_n(x,t)-u_n(y,t))(u_n^\gamma(x,t)-u_n^\gamma(y,t))}{|x-y|^{N+2s}}dydxdt \le 2\|f\|_{L^1(\Omega_T)} + \frac{2|\Omega|}{\gamma+1}\|u_0\|^{\gamma+1}_{L^\infty(\Omega)}. \tag{4.31}$$

Using the item i) of Lemma 1.3.3, we get

$$\sup_{0\le\tau\le T}\int_\Omega u_n^{\gamma+1}(x,\tau)dx + \frac{2\gamma}{\gamma+1}\int_{Q_T} \frac{|u_n^{\frac{\gamma+1}{2}}(x,t)-u_n^{\frac{\gamma+1}{2}}(y,t)|^2}{|x-y|^{N+2s}}dydxdt \le (\gamma+1)\|f\|_{L^1(\Omega_T)} + |\Omega|\|u_0\|^{\gamma+1}_{L^\infty(\Omega)}. \tag{4.32}$$

It follows that $\{u_n\}$ and $\{u_n^{\frac{\gamma+1}{2}}\}_n$ are uniformly bounded in $L^\infty(0,T; L^{\gamma+1}(\Omega))$ and $L^2(0,T; X_0^s(\Omega))$ respectively.

We now prove that $\{u_n\}$ is uniformly bounded in $L^2(0,T; H^s_{loc}(\Omega))$. Since $\{u_n\}_n$ is uniformly bounded in $L^\infty(0,T; L^{\gamma+1}(\Omega)) \subset L^2(0,T; L^{\gamma+1}(\Omega))$ and since $\gamma > 1$, we conclude that $\{u_n\}_n$ is uniformly bounded in $L^2(0,T; L^2(\Omega)) = L^2(\Omega_T)$ and in particular in $L^2(K \times (0,T))$, for every compact $K \subset \Omega$. On the other hand, by (4.31) we can write

$$\int_0^T \int_K \int_K \frac{(u_n(x,t)-u_n(y,t)))(u_n^\gamma(x,t)-u_n^\gamma(y,t)))}{|x-y|^{N+2s}} dydxdt \leq 2\|f\|_{L^1(\Omega_T)} + \frac{2|\Omega|}{\gamma+1}\|u_0\|_{L^\infty(\Omega)}^{\gamma+1},$$

for every compact $K \subset \Omega$. Applying the item v) of Lemma 1.3.3, we have

$$\int_0^T \int_K \int_K \frac{|u_n(x,t)-u_n(y,t))|^2|u_n(x,t)+u_n(y,t)|^{\gamma-1}}{|x-y|^{N+2s}} \leq 2C_\gamma\Big(\|f\|_{L^1(\Omega_T)} + |\Omega|\|u_0\|_{L^\infty(\Omega)}^{\gamma+1}\Big).$$

Using Lemma 4.3.2, we get

$$\int_0^T \int_K \int_K \frac{|u_n(x,t)-u_n(y,t))|^2}{|x-y|^{N+2s}} dydxdt \leq \frac{2^{2-\gamma}C_\gamma}{c_K^{\gamma-1}}\Big(\|f\|_{L^1(\Omega_T)} + |\Omega|\|u_0\|_{L^\infty(\Omega)}^{\gamma+1}\Big). \tag{4.33}$$

The lemma is then proved. □

4.4.3.2 Passing to the limit

Proof. of Theorem 4.2.4.

By virtue of Lemma 4.4.4, we have $\{u_n^{\frac{\gamma+1}{2}}\}$ is uniformly bounded in $L^2(0,T;X_0^s(\Omega)) \subset L^2(\Omega_T)$, this implies that the increasing sequence $\{u_n\}$ is uniformly bounded in $L^1(\Omega_T)$. Then, by Beppo Levi's theorem there exists a measurable function $u \in L^1(\Omega_T)$ such that $u_n \to u$ a.e. in Ω_T. Since $u_n = 0$ on $(\mathbb{R}^N\backslash\Omega)\times(0,T)$, extending u by zero outside of Ω we conclude that $u_n \to u$ a.e. in $\mathbb{R}^N \times (0,T)$ with $u=0$ on $(\mathbb{R}^N\backslash\Omega)\times(0,T)$.

Using Fatou's lemma in the two estimates (4.33) and (4.32), we obtain $u \in L^2(0,T;H^s_{loc}(\Omega)))\cap L^\infty(0,T;L^{\gamma+1}(\Omega))$ and $u^{\frac{\gamma+1}{2}} \in L^2(0,T;X_0^s(\Omega))$.

Testing by an arbitrary function $\varphi \in C_0^\infty(\Omega\times[0,T))$ in (4.7) we get

$$\begin{aligned} -\int_{\Omega_T} u_n\varphi_t dxdt \quad -\int_\Omega u_n(x,0)\varphi(x,0) + \frac{1}{2}\int_{Q_T}\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt \\ = \int_{\Omega_T}\frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dxdt. \end{aligned} \tag{4.34}$$

Let K be a compact subset of Ω such that $supp(\varphi) \subset K$ and $dist(\partial K, \partial\Omega) > 0$. As a consequence of $u_n(x,t) = u_n(y,t) = 0$ for every $(x,y,t) \in \mathcal{C}\Omega\times\mathcal{C}\Omega\times(0,T)$ and $\varphi(x,t) = \varphi(y,t) = 0$ for every $(x,y,t) \in \mathcal{C}K\times\mathcal{C}K\times(0,T)$, we can split the third integral on the left-hand side of (4.34) into three integrals as follows

$$\begin{aligned}
&\int_0^T \int_Q \frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} dydxdt \\
&= \int_0^T \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} dydxdt \\
&= \int_0^T \int_K \int_K \frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} dydxdt \\
&+ \int_0^T \int_K \int_{\mathcal{C}K} \frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} dydxdt \\
&+ \int_0^T \int_{\mathcal{C}K} \int_K \frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} dydxdt \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{4.35}$$

The almost everywhere convergence of $\{u_n\}$ to u allows us to get for every $\varphi \in C_0^\infty(\Omega \times [0,T))$

$$\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} \to \frac{(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} \text{ a.e. in } Q\times(0,T).$$

We follow the same ideas as in the proof of Theorem 4.2.2 for the first integral I_1, obtaining

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \int_K \int_K \frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} dydxdt \\
&\quad = \int_0^T \int_K \int_K \frac{(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} dydxdt.
\end{aligned}$$

For the integrals I_2 and I_3 in (4.35), we follow the same ideas as in the proof of Theorem 4.2.3, we get

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \int_K \int_{\mathcal{C}K} \frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} dydxdt \\
&\quad = \int_0^T \int_K \int_{\mathcal{C}K} \frac{(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} dydxdt,
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \int_{\mathcal{C}K} \int_K \frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} dydxdt \\
&\quad = \int_0^T \int_{\mathcal{C}K} \int_K \frac{(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}} dydxdt
\end{aligned}$$

which yields

$$\lim_{n\to\infty}\int_{Q_T}\frac{(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt$$
$$=\int_{Q_T}\frac{(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dydxdt.$$

For what concerns the first, the second integrals and the right-hand side of (4.34), we follow the same arguments as in the proof of Theorem 4.2.2. Finally, passing to the limit as $n\to+\infty$, we obtain

$$-\int_{\Omega_T}u\varphi_t dxdt-\int_{\Omega}u_0(x)\varphi(x,0)dx+\frac{1}{2}\int_{Q_T}\frac{(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))}{|x-y|^{N+2s}}dxdydt$$
$$=\int_{\Omega_T}\frac{f\varphi}{u^\gamma}dxdt,$$

for all $\varphi\in\mathcal{C}_0^\infty(\Omega\times[0,T))$. Therefore, u is a weak solution of (4.1). □

4.4.4 Proof of Proposition 4.2.1

Proof. of Proposition 4.2.1.

Let $\varphi\in C_0^\infty(\Omega)$ and let Ω' be an open subset of Ω $supp(\varphi)\subset\Omega'\subset\subset\Omega$. Let now $\eta\in X_0^s(\Omega')\cap L^\infty(\Omega')$ be the unique positive solution (see [63, Theorem 12 and Theorem 13] of the following problem

$$\begin{cases}(-\Delta)^s\eta &=1 \quad \text{in } \Omega',\\ \eta &=0 \quad \text{in } \mathbb{R}^N\backslash\Omega'.\end{cases}\tag{4.36}$$

Following the lines in [4], we shall show that $\{u_n\eta\}_n$ is a Cauchy sequence in $\mathcal{C}([0,T],L^1(\Omega))$. Indeed, let us fix $n>m>1$. It is easy to see that the function $w_{n,m}:=u_n-u_m\geq 0$ solves the problem

$$\begin{cases}(w_{n,m})_t+(-\Delta)^s w_{n,m} &=\dfrac{f_n(x,t)}{(u_n+\frac{1}{n})^\gamma}-\dfrac{f_m(x,t)}{(u_m+\frac{1}{m})^\gamma} \quad \text{in } \Omega_T=\Omega\times(0,T),\\ w_{n,m} &=0 \quad \text{in } (\mathbb{R}^N\backslash\Omega)\times(0,T),\\ w_{n,m}(\cdot,0) &=0 \quad \text{in } \Omega,\end{cases}$$

Using $\eta(x)\chi_{[0,\tau]}(t)\in L^2(0,T;X_0^s(\Omega))$, $\tau\in(0,T]$, as a test function in the formulation of solution (4.7) of the problem (4.6) solved by u_n and u_m respectively and then subtracting the two equations we obtain

$$\int_{\Omega_\tau}(w_{n,m})_t\eta(x)dxdt+\int_{\Omega_\tau}(-\Delta)^s w_{n,m}(x,t)\eta(x)dx\leq\int_{\Omega_T}\left|\frac{f_n(x,t)}{(u_n+\frac{1}{n})^\gamma}-\frac{f_m(x,t)}{(u_m+\frac{1}{m})^\gamma}\right|\eta(x)dxdt.$$

Thus, the symmetry of $(-\Delta)^s$ and (4.36) yield

$$\int_{\Omega_\tau}(w_{n,m})_t\eta(x)dxdt+\int_{\Omega_\tau}w_{n,m}(x,t)dxdt\le\int_{\Omega_T}\left|\frac{f_n(x,t)}{(u_n+\frac{1}{n})^\gamma}-\frac{f_m(x,t)}{(u_m+\frac{1}{m})^\gamma}\right|\eta(x)dxdt.$$

Dropping the non negative term, it follows that

$$\int_\Omega|u_n-u_m|(x,\tau)\eta(x)dx\le\int_{\Omega_T}\left|\frac{f_n(x,t)}{(u_n+\frac{1}{n})^\gamma}-\frac{f_m(x,t)}{(u_m+\frac{1}{m})^\gamma}\right|\eta(x)dxdt.$$

Since $supp(\eta)\subset\overline{\Omega'}$, by Lemma 4.3.2 we have

$$\left|\frac{f_n(x,t)\eta(x)}{(u_n+\frac{1}{n})^\gamma}\right|\le\left|\frac{\|\eta\|_{L^\infty(\Omega)}f}{c^\gamma_{\Omega'}}\right|\in L^1(\Omega_T).$$

Using the fact that $u_n\to u$ a.e. in Ω, we conclude by the Lebesgue dominated convergence theorem that the sequence $\left\{\dfrac{f_n\eta}{(u_n+\frac{1}{n})^\gamma}\right\}_n$ converges to $\dfrac{f\eta}{u^\gamma}$ in $L^1(\Omega_T)$ and so

$$\sup_{\tau\in[0,T]}\int_\Omega|u_n-u_m|(x,\tau)\eta(x)dx\to0,\ \text{as}\ n,m\to+\infty,$$

Thus, $\{u_n\eta\}_n$ is a Cauchy sequence in $C([0,T];L^1(\Omega))$. Let $\delta'(x):=dist(x,\partial\Omega')$. Going back to [19, Lemma 4.2] we conclude that

$$\eta(x)\ge c(\Omega')(\delta')^s(x),\quad\forall x\in\Omega'.$$

Particularly, for every $x\in supp(\varphi)\subset\Omega'$ one has

$$\eta(x)\ge c(\varphi,s):=c(\Omega')\Big(dist(supp(\varphi),\partial\Omega')\Big)^s.$$

So that, we obtain

$$\begin{aligned}\sup_{\tau\in[0,T]}\int_\Omega|u_n-u_m|(x,\tau)|\varphi|dx&=\sup_{\tau\in[0,T]}\int_{supp(\phi)}|u_n-u_m|(x,\tau)\eta(x)\frac{|\varphi|}{\eta}dx\\&\le\frac{\|\varphi\|_\infty}{c(\varphi,s)}\sup_{\tau\in[0,T]}\int_\Omega|u_n-u_m|(x,\tau)\eta(x)dx.\end{aligned}$$

Therefore, for every $\phi\in C_0^\infty(\Omega)$

$$\sup_{\tau\in[0,T]}\int_\Omega|u_n-u_m|(x,\tau)|\varphi|dx\to0,\ \text{as}\ n,m\to+\infty.$$

This shows that $\{u_n\phi\}_n$ is a Cauchy sequence in $C([0,T];L^1(\Omega))$ for every $\phi\in C_0^\infty(\Omega)$ and so $u\in C([0,T];L^1_{loc}(\Omega))$. Moreover

$$\int_\Omega u(x,t)\varphi(x)dx\to\int_\Omega u_0(x)\varphi(x)dx\ \text{as}\ t\to0.$$

□

Conclusion and Outlook

Conclusion

In this thesis, we have provided existence, regularity and uniqueness of solutions for nonlocal elliptic and parabolic problems of fractional Laplacian type with a singular nonlinearity, which specifically are singular with respect to the unknown function.

We have succeeded to extend some well-known results for the local case, to the non-local one. Despite the difficulties encountered because of the difference between the local case and the non-local case we needed to develop the methods used in the local case and using some algebraic inequalities.

We have also improved some results in the non-local case by bringing in more general data.

Outlook

We give some perspectives and open questions encountered during the accomplishment of this work in the same order as the chapters of this thesis.

In Chapter 2, we have established some existence and regularity results of solutions for a non-local elliptic problem involving the fractional Laplacian operator with singular nonlinearity and Radon measure data. It is interesting to generalize the results to the more general case of the fractional p-Laplacian operator.

In Chapter 3 we mentioned in Remark 3.2.6 that the threshold in our result extends the one obtained in the local case in [14]. However, unlike what is proved in [67], we have observed that this threshold is not optimal. The open question that arises here is whether we are able to reach the optimality of this threshold. We also hope to perform some results with measure data.

As regards Chapter 4, we have studied the existence of solutions for a parabolic problem involving the fractional Laplacian with singular nonlinearity. It is interesting to extend the results to more general data and general operators as for instance the fractional p-Laplacian operator.

Bibliography

[1] B. Abdellaoui, A. Attar, and R. Bentifour. On the fractional p-Laplacian equations with weight and general datum. *Adv. Nonlinear Anal.*, 8(1):144–174, 2019.

[2] B. Abdellaoui, A. Attar, R. Bentifour, and I. Peral. On fractional p-Laplacian parabolic problem with general data. *Ann. Mat. Pura Appl. (4)*, 197(2):329–356, 2018.

[3] B. Abdellaoui, K. Biroud, J. Davila, and F. Mahmoudi. Nonlinear elliptic problem related to the Hardy inequality with singular term at the boundary. *Commun. Contemp. Math.*, 17(3):1450033, 28, 2015.

[4] B. Abdellaoui, K. Biroud, and E.-H Laamri. Existence and nonexistence of positive solutions to a fractional parabolic problem with singular weight at the boundary. *To appear in J. Evol. Equ.*, 2020.

[5] B. Abdellaoui, K. Biroud, and A. Primo. Nonlinear fractional elliptic problem with singular term at the boundary. *Complex Var. Elliptic Equ.*, 64(6):909–932, 2019.

[6] B. Abdellaoui, A. Boucherif, and T.M. Touaoula. Fractional parabolic problems with a nonlocal initial condition. *Moroccan J. of Pure and Appl. Anal.*, 3(1):116–132, 2017.

[7] B. Abdellaoui, M. Medina, I. Peral, and A. Primo. The effect of the Hardy potential in some Calderón-Zygmund properties for the fractional Laplacian. *J. Differential Equations*, 260(11):8160–8206, 2016.

[8] B. Abdellaoui, M. Medina, I. Peral, and A. Primo. Optimal results for the fractional heat equation involving the Hardy potential. *Nonlinear Anal.*, 140:166–207, 2016.

[9] R. A. Adams. *Sobolev spaces*. Academic Press, New York-London, 1975.

[10] A. Adimurthi, J. Giacomoni, and S. Santra. Positive solutions to a fractional equation with singular nonlinearity. *J. Differential Equations*, 265(4):1191–1226, 2018.

[11] H. Aikawa, T. Kilpeläinen, N. Shanmugalingam, and X. Zhong. Boundary Harnack principle for p-harmonic functions in smooth Euclidean domains. *Potential Anal.*, 26(3):281–301, 2007.

[12] N. Alibaud, B. Andreianov, and M. Bendahmane. Renormalized solutions of the fractional Laplace equation. *C. R. Math. Acad. Sci. Paris*, 348(13-14):759–762, 2010.

[13] D. Applebaum. *Lévy processes and stochastic calculus*, volume 116. Cambridge Univ. Press, Cambridge, 2009. Second edition. Cambridge Studies in Advanced Mathematics.

[14] D. Arcoya and L. Moreno-Mérida. Multiplicity of solutions for a Dirichlet problem with a strongly singular nonlinearity. *Nonlinear Anal.*, 95:281–291, 2014.

[15] R. Aris. *The mathematical theory of diffusion and reaction in permeable catalysts*, volume I and II. Oxford University Press, New York, (1975).

[16] D.G. Aronson and J. Serrin. Local behavior of solutions of quasilinear parabolic equations. *Arch. Rational Mech. Anal.*, 25:81–122, 1967.

[17] M. Badra, K. Bal, and J. Giacomoni. A singular parabolic equation: existence, stabilization. *J. Differential Equations*, 252(9):5042–5075, 2012.

[18] B. Barrios, I. De Bonis, M. Medina, and I. Peral. Semilinear problems for the fractional laplacian with a singular nonlinearity. *Open Math.*, 13:390–407, 2015.

[19] B. Barrios, M. Medina, and I. Peral. Some remarks on the solvability of non-local elliptic problems with the Hardy potential. *Commun. Contemp. Math.*, 16(4):1350046, 29, 2014.

[20] P. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre, and J.L. Vasquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Annali Scuola Norm. Sup. Pisa*, 22(4):241–273, 1995.

[21] G.M. Bisci, V.D. Radulescu, and R. Servadei. *Variational methods for nonlocal fractional problems*, volume 162 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2016.

[22] L. Boccardo and J. Casado-Díaz. Some properties of solutions of some semilinear elliptic singular problems and applications to the G-convergence. *Asymptot. Anal.*, 86(1):1–15, 2014.

[23] L. Boccardo and L. Orsina. Semilinear elliptic equations with singular nonlinearities. *Calc. Var. Partial Differential Equations*, 37(3-4):363–380, 2010.

[24] L. Boccardo, L. Orsina, and M.M. Porzio. Existence results for quasilinear elliptic and parabolic problems with quadratic gradient terms and sources. *Adv. Calc. Var.*, 4(4):397–419, 2011.

[25] B. Bougherara and J. Giacomoni. Existence of mild solutions for a singular parabolic equation and stabilization. *Adv. Nonlinear Anal.*, 4(2):123–134, 2015.

[26] B. Brandolini, F. Chiacchio, and C. Trombetti. Symmetrization for singular semilinear elliptic equations. *Ann. Mat. Pura Appl. (4)*, 193(2):389–404, 2014.

[27] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[28] C. Bucur and E. Valdinoci. *Nonlocal diffusion and applications*, volume 20 of *Lecture Notes of the Unione Matematica Italiana*. Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016.

[29] L. Caffarelli. Non-local diffusions, drifts and games. In *Nonlinear partial differential equations*, volume 7 of *Abel Symp.*, pages 37–52. Springer, Heidelberg, 2012.

[30] L. Caffarelli and L. Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62(5):597–638, 2009.

[31] L.A. Caffarelli, S. Salsa, and L. Silvestre. Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math.*, 171(2):425–461, 2008.

[32] A. Callegari and A. Nachman. Some singular, nonlinear differential equations arising in boundary layer theory. *J. Math. Anal. Appl.*, 64(1):96–105, 1978.

[33] A. Canino and M. Degiovanni. A variational approach to a class of singular semilinear elliptic equations. *J. Convex Anal.*, 11(1):147–162, 2004.

[34] A. Canino, L. Montoro, B. Sciunzi, and M. Squassina. Nonlocal problems with singular nonlinearity. *Bull. Sci. Math.*, 141(3):223–250, 2017.

[35] A. Canino, B. Sciunzi, and A. Trombetta. Existence and uniqueness for p-Laplace equations involving singular nonlinearities. *NoDEA Nonlinear Differential Equations Appl.*, 23(2):Art. 8, 18, 2016.

[36] H. Chen and L. Véron. Semilinear fractional elliptic equations involving measures. *J. Differential Equations*, 257(5):1457–1486, 2014.

[37] G.M. Coclite and M.M. Coclite. On the summability of weak solutions for a singular Dirichlet problem in bounded domains. *Adv. Differential Equations*, 19(5-6):585–612, 2014.

[38] M.G. Crandall, P.H. Rabinowitz, and L. Tartar. On a Dirichlet problem with a singular nonlinearity. *Comm. Partial Differential Equations*, 2(2):193–222, (1977).

[39] D. Danielli and S. Salsa. Obstacle problems involving the fractional Laplacian. In *Recent developments in nonlocal theory*, pages 81–164. De Gruyter, Berlin, 2018.

[40] I. De Bonis and L. M. De Cave. Degenerate parabolic equations with singular lower order terms. *Differential Integral Equations*, 27(9-10):949–976, 2014.

[41] I. De Bonis and D. Giachetti. Singular parabolic problems with possibly changing sign data. *Discrete Contin. Dyn. Syst. Ser. B*, 19(7):2047–2064, 2014.

[42] L.M. De Cave. Nonlinear elliptic equations with singular nonlinearities. *Asymptot. Anal.*, 84(3-4):181–195, 2013.

[43] L.M. De Cave and F. Oliva. Elliptic equations with general singular lower order term and measure data. *Nonlinear Anal.*, 128:391–411, 2015.

[44] F. Demengel and G. Demengel. *Functional spaces for the theory of elliptic partial differential equations*. Universitext. Springer, London; EDP Sciences, Les Ulis, 2012.

[45] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[46] E. DiBenedetto, U. Gianazza, and V. Vespri. *Harnack's inequality for degenerate and singular parabolic equations*. Springer Monographs in Mathematics. Springer, New York, 2012.

[47] S. Dipierro, A. Figalli, and E. Valdinoci. Strongly nonlocal dislocation dynamics in crystals. *Comm. Partial Differential Equations*, 39(12):2351–2387, 2014.

[48] H.V. Emden and A.W. Shaker. Multigrid methods for a semilinear pde in the theory of pseudoplastic fluids. *Calhoun*, pages 231–242, (1993).

[49] A. Fiscella, R. Servadei, and E. Valdinoci. Density properties for fractional Sobolev spaces. *Ann. Acad. Sci. Fenn. Math.*, 40(1):235–253, 2015.

[50] G.B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1984. Modern techniques and their applications, A Wiley-Interscience Publication.

[51] W. Fulks and J.S. Maybee. A singular non-linear equation. *Osaka Math. J.*, 12:1–19, (1960).

[52] J. Giacomoni, T. Mukherjee, and K. Sreenadh. Existence and stabilization results for a singular parabolic equation involving the fractional Laplacian. *Discrete Contin. Dyn. Syst. Ser. S*, 12(2):311–337, 2019.

[53] P. Grisvard. *Elliptic problems in nonsmooth domains.* Pitman, Boston, 1985.

[54] C. Gui and F.H. Lin. Regularity of an elliptic problem with a singular nonlinearity. *Proc. Roy. Soc. Edinburgh Sect. A*, 123(6):1021–1029, 1993.

[55] K.H. Karlsen, F. Petitta, and S. Ulusoy. A duality approach to the fractional Laplacian with measure data. *Publ. Mat.*, 55(1):151–161, 2011.

[56] O. Kavian. *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, volume 13 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Paris, 1993.

[57] T. Klimsiak. Reduced measures for semilinear elliptic equations involving Dirichlet operators. *Calc. Var. Partial Differential Equations*, 55(4):Art. 78, 27, 2016.

[58] A. Kufner, O. John, and S. Fučík. *Function spaces.* Noordhoff International Publishing, Leyden; Academia, Prague, 1977. Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis.

[59] A.V. Lair and A.W. Shaker. Classical and weak solutions of a singular semilinear elliptic problem. *J. Math. Anal. Appl.*, 211(2):371–385, 1997.

[60] N. Landkof. *Foundations of modern potential theory, in: Die Grundlehren der Mathematischen Wissenschaften*, volume Band 180. Springer-Verlag, New York, Heidelberg, 1972.

[61] A.C. Lazer and P.J. McKenna. On a singular nonlinear elliptic boundary-value problem. *Proc. Amer. Math. Soc.*, 111(3):721–730, 1991.

[62] A.R. Leggat and S.E. Miri. Anisotropic problem with singular nonlinearity. *Complex Var. Elliptic Equ.*, 61(4):496–509, (2016).

[63] T. Leonori, I. Peral, A. Primo, and F. Soria. Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations. *Discrete Contin. Dyn. Syst.*, 35(12):6031–6068, 2015.

[64] A. Nachman and A. Callegari. A nonlinear singular boundary value problem in the theory of pseudoplastic fluids. *SIAM J. Appl. Math.*, 38(2):275–281, (1980).

[65] F. Oliva. Regularizing effect of absorption terms in singular problems. *J. Math. Anal. Appl.*, 472(1):1136–1166, 2019.

[66] F. Oliva and F. Petitta. On singular elliptic equations with measure sources. *ESAIM Control Optim. Calc. Var.*, 22(1):289–308, 2016.

[67] F. Oliva and F. Petitta. Finite and infinite energy solutions of singular elliptic problems: existence and uniqueness. *J. Differential Equations*, 264(1):311–340, 2018.

[68] F. Oliva and F. Petitta. A nonlinear parabolic problem with singular terms and nonregular data. *Nonlinear Anal.*, 194:111472, 13, 2020.

[69] L. Orsina and F. Petitta. A Lazer-McKenna type problem with measures. *Differential Integral Equations*, 29(1-2):19–36, 2016.

[70] P. Pedregal. *Parametrized measures and variational principles*, volume 30 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 1997.

[71] F. Petitta. Some remarks on the duality method for integro-differential equations with measure data. *Adv. Nonlinear Stud.*, 16(1):115–124, 2016.

[72] A.C. Ponce. *Elliptic PDEs, measures and capacities*, volume 23 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2016.

[73] X. Ros-Oton and J. Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl. (9)*, 101(3):275–302, 2014.

[74] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.

[75] R. Servadei. The Yamabe equation in a non-local setting. *Adv. Nonlinear Anal.*, 2(3):235–270, 2013.

[76] R. Servadei and E. Valdinoci. Mountain pass solutions for non-local elliptic operators. *J. Math. Anal. Appl.*, 389(2):887–898, 2012.

[77] R. Servadei and E. Valdinoci. A Brezis-Nirenberg result for non-local critical equations in low dimension. *Commun. Pure Appl. Anal.*, 12(6):2445–2464, 2013.

[78] R. Servadei and E. Valdinoci. Variational methods for non-local operators of elliptic type. *Discrete Contin. Dyn. Syst.*, 33(5):2105–2137, 2013.

[79] R. Servadei and E. Valdinoci. The Brezis-Nirenberg result for the fractional Laplacian. *Trans. Amer. Math. Soc.*, 367(1):67–102, 2015.

[80] R. Servadei and E. Valdinoci. Fractional Laplacian equations with critical Sobolev exponent. *Rev. Mat. Complut.*, 28(3):655–676, 2015.

[81] L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.*, 60(1):67–112, 2007.

[82] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[83] Y. Sire and E. Valdinoci. Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result. *J. Funct. Anal.*, 256(6):1842–1864, 2009.

[84] C.A. Stuart. Existence and approximation of solutions of non-linear elliptic equations. *Math. Z.*, 147(1):53–63, 1976.

[85] K. Teng, C. Zhang, and S. Zhou. Renormalized and entropy solutions for the fractional p-Laplacian evolution equations. *J. Evol. Equ.*, 19(2):559–584, 2019.

[86] J.L. Vázquez. The mathematical theories of diffusion: nonlinear and fractional diffusion. In *Nonlocal and nonlinear diffusions and interactions: new methods and directions*, volume 2186 of *Lecture Notes in Math.*, pages 205–278. Springer, Cham, 2017.

[87] M. Yao and J. Zhao. Positive solution of a singular non-linear elliptic boundary value problem. *Appl. Math. Comput.*, 148(3):773–782, (2004).

[88] S. Yijing and Z. Duanzhi. The role of the power 3 for elliptic equations with negative exponents. *Calc. Var. Partial Differential Equations*, 49(3-4):909–922, 2014.

[89] A. Youssfi and G. Ould Mohamed Mahmoud. On singular equations involving fractional Laplacian. *Acta Math. Sci.*, 40B(5):1289–1315, 2020.

[90] A. Youssfi and G. Ould Mohamed Mahmoud. Fractional heat equation with singular terms. *Submitted*, 2021.

[91] A. Youssfi and G. Ould Mohamed Mahmoud. Nonlocal semilinear elliptic problems with singular nonlinearity. *Calc. Var. Partial Differential Equations*, 60(153), 2021.

[92] W.P. Ziemer. *Modern real analysis*, volume 278 of *Graduate Texts in Mathematics*. Springer, Cham, second edition, 2017. With contributions by Monica Torres.