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Polymeromorphic complex Itô-Hermite and Zernike functions:a systematic study, spectral analysis and applications

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∵ In the name of God, most Gracious, most Merciful ∵

Polymeromorphic complex Itô–Hermite and Zernike functions: a systematic study, spectral analysis and applications

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Supervised by : **Allal Ghanmi**

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Dedication

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Abstract

The extensive development of numerical methods and the growing role of computer simulations have greatly increased interest in special functions, the reason why we are interested in considering new classes of such functions. Firstly, we provide a theoretical study of a new family of orthogonal functions on the punctured complex plane, extending the poly-analytic Itô–Hermite polynomials to the poly-meromorphic setting and solving the eigenvalue problems for some perturbed magnetic Laplacian modeling an Aharonov–Bohm effect. Additionally, we define a novel family of Hermite functions of real order β by means of a fractional Rodrigues formula involving the Caputo derivative. We discuss also and establish some of their properties as well as their representation in terms of the Kummer's function. Finally, we consider the so-called fractional Zernike functions defined on the punctured unit disc and generalizing the classical Zernike polynomials. Mainly, we show that they are orthogonal L^2-eigenfunctions for certain perturbed magnetic Laplacian. We establish their algebraic and analytic properties such as the description of their zeros, the differential equations, the recurrence and operational formulas they satisfy. Moreover, we discuss their regularity as poly-meromorphic functions and obtain some of their integral representations and generating functions, including a bilinear one of Hardy–Hille type. Furthermore, we prove that a truncated subclass defines a complete orthogonal system in the underlying Hilbert space giving rise to a specific Hilbertian orthogonal decomposition of a class of generalized Bergman spaces.

Keywords:

Poly-meromorphic Itô–Hermite functions; Generating functions; Perturbed magnetic Laplacian with Aharonov-Bohm effect; Fractional calculus; Fractional Zernike functions; β-modified poly-Bergman spaces.

Résumé

Le développement important des méthodes numériques et le rôle croissant des simulations informatiques ont considérablement accru l'intérêt pour les fonctions spéciales, raison pour laquelle nous nous intéressons à des nouvelles classes de fonctions spéciales. Dans ce travail, nous proposons premièrement une étude théorique d'une nouvelle famille de fonctions orthogonales sur le plan $\mathbb{C}^*$ étendant les polynômes poly-analytiques Itô–Hermite au cadre poly-méromorphe. Nous résolvons les problèmes aux valeurs propres du Laplacien magnétique perturbé par un vecteur potentiel singulier et modélisant l'effet d'Aharonov-Bohm. De plus, nous définissons une nouvelle famille de fonctions de type Hermite d'ordre réel β par une formule de Rodrigues fractionnaire et faisant appel à la dérivée de Caputo. Nous donnons également certaines de leurs propriétés ainsi que leur représentation en terme de la fonction hypergéométrique confluente. Nous considérons ensuite une famille de fonctions de Zernike fractionnaires sur le disque unité étoilé, généralisant les polynômes de Zernike classiques. Principalement, nous montrons qu'il s'agit de fonctions propres de carré intégrable, orthogonales pour certain Laplacien magnétique perturbé, et nous établissons leurs propriétés algébriques et analytiques comme la description de leurs zéros, les équations différentielles, les relations de récurrence et les formules opérationnelles qu'ils satisfont. De plus, nous discutons leur régularité en tant que fonctions poly-méromorphes et obtenons leurs représentations intégrales et fonctions génératrices, dont une bilinéaire de type Hardy–Hille. De plus, nous prouvons qu'une sous-classe tronquée définit un système orthogonal complet dans l'espace de Hilbert sous-jacent donnant lieu à une décomposition orthogonale hilbertienne spécifique pour une classe d'espaces de Bergman généralisés.

Mots clés:

Fonctions Itô-Hermite poly-méromorphes; Fonctions génératrices; Laplacien magnétique perturbé avec effet de Aharonov-Bohm; Calcul fractionnaire; Fonctions de Zernike fractionnaires; Espaces poly-Bergman β-modifié

Résumé de la thèse

Ce travail contient nos contributions à l'étude de trois nouvelles classes de fonctions orthogonales spéciales qui s'étendent dans différentes directions des polynômes orthogonaux classiques. Tout d'abord, nous fournissons une étude théorique d'une nouvelle famille de fonctions orthogonales sur $\mathbb{C}*$ résolvant les problèmes de valeurs propres pour un Laplacien magnétique perturbé par un potentiel vectoriel singulier avec un champ magnétique nul modélisant l'effet d'Aharonov–Bohm. Les fonctions sont définies par leur formule de type Rodrigues modifiée par β.

$$\psi_{n,m}^{\alpha,\beta}(z,\overline{z}) := (-1)^n z^{-\beta} e^{\alpha|z|^2} \frac{\partial^n}{\partial z^n}(z^{\beta+m} e^{-\alpha|z|^2}), \tag{0.0.1}$$

étendons les polynômes poly-analytiques d'Itô–Hermite au cadre poly-méromorphe. Nous dérivons principalement leurs différentes représentations opérationnelles et donnons leurs expressions explicites en termes de fonctions spéciales. Différentes fonctions génératrices et représentations intégrales sont obtenues. Ici, α et β sont des réels fixes donnés avec $\alpha > 0$, et n et m sont des entiers variables tels que $n = 0, 1, 2, \cdots$ et $m > -\beta - 1$. Cela conduit à une généralisation spéciale des polynômes d'Hermite complexes de Itô.

Deuxièmement, nous introduisons et étudions les fonctions d'Hermite dites fractionnaires, et leur extension au cadre fractionnaire d'un point de vue différent. Le point de départ de notre approche est la formule de Rodrigues pour les polynômes d'Hermite. Par conséquent, les principaux résultats de ce travail concernent les fonctions définies par les formules généralisées de type Rodrigues impliquant la dérivée fractionnaire de Liouville-Caputo ${}^cD^\beta$. En particulier, nous traitons

$${}_LH_{(\beta)}(x) = e^{x^2} {}^cD^\beta(e^{-x^2}), \tag{0.0.2}$$

pour $\beta \geq 0$, de sorte que pour l'ordre entier $\beta = n$, nous retrouvons les polynômes d'Hermite, à savoir ${}_LH_{(n)}(x) = (-1)^n H_n(x)$. Nous considérons également la classe associée à l'intégrale de Riemann Liouville ${}_LI^{-\beta}$ définie pour $\beta < 0$ par

$${}_LG_{(\beta)}(x) = e^{x^2} I^{-\beta}(e^{-x^2}). \tag{0.0.3}$$

Plus précisément, nous visons à étudier certaines de leurs propriétés utiles, y compris leurs trois termes et leurs formules de récurrence différentielle, et à explorer leurs différentes représentations hypergéométriques et intégrales.

Finalement, une généralisation spécifique des polynômes de Zernike sera le sujet principal de la discussion dans le présent document en insérant un ordre fractionnaire. Leur construction fait appel à la méthode de factorisation. Plus précisément, pour des nombres réels fixes $\rho, \kappa > -1$ et des entiers m et n tels que $m \geq 0$ et $n + \rho \geq 0$, nous traitons de la famille de fonctions

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) := (-1)^m z^{-\rho}(1-|z|^2)^{-\kappa} \frac{\partial^m}{\partial z^m} \left(z^{\rho+n}(1-|z|^2)^{\kappa+m}\right) \tag{0.0.4}$$

sur le disque unitaire $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. Nous considérons et fournissons une étude précise des fonctions de Zernike fractionnaires sur le disque unitaire $\mathbb{D}^*$, généralisant les polynômes de Zernike classiques et leurs fonctions de Zernike restreintes à β. Pour l'étude concrète des fonctions de Zernike fractionnaires $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})$ nous commençons par établir leurs expressions explicites, leurs différentes représentations hypergéométriques, leur expression en termes de polynômes de Jacobi ainsi que leur connexion aux polynômes de Zernike complexes. Ensuite, les ensembles de zéro de $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})$ sont décrits comme étant les cercles centrés dont les rayons sont les zéros des polynômes de Jacobi réels. L'orthogonalité dans l'espace de Hilbert $L^{2,\kappa}_{\rho}(\mathbb{D}) = L^2(\mathbb{D}, d\mu_{\kappa,\rho})$ est discutée et la norme carrée est explicitement calculée. L'appartenance à une classe spécifique de fonctions poly-méromorphes dans $\mathbb{D}$ est également considérée. De plus, nous étudions les formules opérationnelles qu'elles satisfont, y compris celles de type Burchnall. Nous discutons également de certaines relations de récurrence et des équations différentielles auxquelles elles obéissent. Certaines fonctions génératrices associées sont obtenues, telles qu'une fonction génératrice bilinéaire analogue à la fonction génératrice de Hardy–Hille pour les polynômes de Laguerre généralisés, qui peut être utilisée pour dériver une représentation intégrale spéciale pour $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})$. Une autre représentation intégrale de type Cauchy est obtenue sous la forme d'une intégrale sur le cercle unitaire.

Mots clés:

Fonctions Itô-Hermite poly-méromorphes; Fonctions génératrices; Laplacien magnétique perturbé avec effet de Aharonov-Bohm; Calcul fractionnaire; Fonctions de Zernike fractionnaires; Espaces poly-Bergman β-modifié.

Related publications

1. H. Dkhissi, A. Ghanmi, Polymeromorphic Itô–Hermite functions associated with a singular potential vector on the punctured complex plane. J. Math. Phys. 65 (2024), no. 6, Paper No. 063501, 16 pp.

2. H. Dkhissi, A. Ghanmi, S. Snoun, Fractional Zernike functions. J. Math. Anal. Appl. 532 (2024), no. 1, Paper No. 127923, 24 pp.

3. H. Dkhissi, A. Ghanmi, Fractional Hermite functions. Submitted.

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General introduction

State of the art and motivation

The theories of special functions and orthogonal polynomials have been the subject of intensive research programs over the years. Most of these studies have been directed towards the framework of the relevant theory within a unifying context, which from time to time has been identified with the theory of differential equations , the group theoretic point of view , umbral calculus and symbolic formalism, and so on. An important output of these studies has been the formalization of the theoretical foundations of the relevant properties in terms of the ordinary or partial differential equations. It also should be mentioned that the theory of special functions provides powerful tools for solving diverse mathematical problems, especially in Fourier and harmonic analysis, mathematical analysis and integral transforms [68, 84]. In fact, their well established properties and mathematical structure make them invaluable tools in various fields of research. Indeed, they are useful for solving the differential equations that model physical systems and problems related to quantum mechanics, statistical mechanics, signal processing, control theory, electrical circuits and mechanical vibrations[81]. They also appear in probability theory and statistics and used to enumerate combinatorial objects [10, 64]. Multi-variable special functions, including orthogonal polynomials, are natural solutions of the well-known equations of mathematical physics.

This work contains our contributions in studying three new classes of special orthogonal functions extending in different directions of the classical orthogonal polynomials we review below.

Real and complex Hermite polynomials and extensions. One of the broadest famillies of special functions, arising particularly in mathematical physics, consists of the Hermite polynomials $H_n(x)$, $n = 0, 1, 2, \cdots$, defined by the formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k}, \tag{0.0.5}$$

introduced for the first time in 1810 in **P.S Laplace**'s work on the use of definite integrals in calculating probabilities [67]. Next, they have been studied in detail by **P. Chebyshev** (1859) and were later named after **C. Hermite**, who wrote on the polynomials in 1864 [44], considering them as new. These Hermite polynomials form a fundamental class of orthogonal polynomials with weight $\rho(x) = e^{-x^2}$ on the interval $(-\infty, +\infty)$, and appear in the solutions of the Schrödinger equation for the quantum harmonic oscillator, one of the fundamental problems in quantum mechanics [41]. In fact, the real Hermite polynomials emerge naturally in the solution of the quantum harmonic oscillator problem, which is a fundamental concept in quantum mechanics. The wave functions $\psi_n(x)$ of the quantum harmonic oscillator

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right)\psi = E\psi$$

with corresponding energy levels given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

are expressed in terms of the Hermite polynomials in (0.0.5) through

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \quad n = 0, 1, 2, \ldots,$$

making them indispensable in this field.

On the other hand, such polynomials are systematically employed in signal processing and in combinatorics [10, 20]. Their orthogonal property and mathematical structure make them a valuable tool in analyzing and processing signals and in expanding the probability generating function of a generalized Poisson distribution [61]. Notice for instance that in 1961, **Bargmann** defined a unitary integral transform mapping the Hilbert space $L^2(\mathbb{R}^n)$ onto the Bargmann space (the Fock space), which consists of entire functions on phase space that are square integrable with respect to a Gaussian measure using a generating function of the Hermite polynomials on $\mathbb{C}^n$. Furthermore, he showed that the Bargmann space is a reproducing kernel Hilbert space, and therefore provided a mathematical framework for the study of quantum mechanics in phase space. The Bargmann transform and the Segal-Bargmann space are now standard tools in the study of quantum mechanics, quantum field theory and related areas.

On the other hand, Hermite polynomials have been generalized and extended to higher dimensions. Particularly, a two dimensional analog that is the tensor product of the real Hermite polynomials is given in [101] and it has been shown that Laguerre 2D-functions and Hermite 2D-functions are eigensolutions of a two-dimensional degenerate harmonic oscillator. In his paper of 1951 on multiple Wiener integral [56], **Itô** establishes a close relation between the real multiple Wiener integral and the Hermite polynomials of real variable. This motivates the introduction of the complex analog of the real Hermite polynomial within the framework of the complex multiple Wiener integral [55]. In fact, they are defined by their Rodrigues formula

$$H^{\alpha}_{m,n}(z,\overline{z}) = (-1)^{m+n} e^{\alpha|z|^2} \frac{\partial^{m+n}}{\partial \overline{z}^m \partial z^n} \left(e^{-\alpha|z|^2} \right), \ \alpha > 0. \tag{0.0.6}$$

Here $\partial/\partial z$ and $\partial/\partial \overline{z}$ denote the Wirtinger derivatives with respect to the variables z and $\overline{z}$, respectively. The polynomials in (0.0.6) associated with the gaussian measure is close to the so-called complex Hermite polynomials that are already introduced by Itô in [55] within the study of the complex multiple Wiener integrals. They have been extensively studied in connection with various branches of engineering sciences and mathematics and physics [32, 34, 35, 50, 53, 52, 55, 74, 91]. They form an orthogonal complete system in the Hilbert space $L^{2,\alpha}(\mathbb{C}) = L^2(\mathbb{C}, e^{-\alpha|z|^2} dxdy)$ (see [50, 32]). Moreover, they provide a concrete description of the spectral analysis of the Landau Hamiltonian [66]

$$\Delta_\alpha = \left(i\frac{\partial}{\partial x} - 2\alpha y \right)^2 + \left(i\frac{\partial}{\partial y} + 2\alpha x \right)^2; \ z = x + iy, \tag{0.0.7}$$

which is a model of Schrödinger operator with a constant magnetic field, and describes a non-relativistic particle confined to move on the complex plane in the presence of a constant homogeneous magnetic field. It is widely studied in the literature [8, 9, 36, 43, 48, 74, 77, 91] along with interesting applications to Feynman path integral (Feynman-Kac formula), oscillatory stochastic integral and theory of lattices electrons in uniform magnetic field [57, 73].

As already mentioned, their generalization in different directions has been investigated by many authors, including their extension within the framework of quantum calculus [99, 25, 99]. In fact, a specific generalized class of these polynomials is suggested by the magnetic Schrödinger operator in (0.0.7) when acting on mixed planar automorphic functions attached to a given equivariant pair [24]. This gave rise to

$$\mathfrak{G}^{\nu}_{m,n}(z,\overline{z}|\xi) = \frac{(-1)^{m+n}}{\nu^m} e^{\nu|z|^2 + \frac{\xi}{2}z} \frac{\partial^{m+n}}{\partial \overline{z}^m \partial z^n} \left(e^{-\nu|z|^2 - \frac{\xi}{2}z} \right) \tag{0.0.8}$$

and has been studied in sufficient detail in [32], see also [12] for a quite variant class. Some generating functions are obtained for these polynomials when $e^{-\frac{\xi}{2}z}$ is replaced by an arbitrary polynomial function [6] or a specific holomorphic function [32].

Complex Zernike (or disc) polynomials. This is another important class of special functions on the unit disc given by

$$\mathcal{Z}^{\gamma}_{m,n}(z,\overline{z}) := (-1)^{m+n}(1-|z|^2)^{-\gamma}\frac{\partial^{m+n}}{\partial z^m \partial \overline{z}^n}\left(1-|z|^2\right)^{\gamma+m+n} \tag{0.0.9}$$

$$= (-1)^m c^{\gamma}_{m,n}(1-|z|^2)^{-\gamma}\frac{\partial^m}{\partial z^m}\left(z^n\left(1-|z|^2\right)^{\gamma+m}\right), \tag{0.0.10}$$

for varying nonnegative integers m and n, and real $\gamma > -1$. They are named after **Fritz Zernike** (Noble prize in physics), and rapidly became popular within the optical community because of certain special properties that allow them to be applied in cases where the Seidel polynomials are not applicable. In Zernike's paper on the knife edge test and the phase contrast method [103], they have been defined as eigenfunctions of a rotational invariant second order partial differential equation, and have next been used in the Nijboer's works to develop the diffraction theory of optical aberrations. Since then, they have been extensively employed to express the propagation of a wavefront data in optical tests through imaging system [40, 51, 65] to represent the aberrations of optical systems (by atmospheric turbulence) [82, 98]. They are also used to study diffraction problems in the rotationally symmetric system with circular pupils [80, 104] and pattern recognition [62, 97]. More recently, they become a standard tool in describing the optical aberrations of the human eye. Moreover, they have been applied efficiently to characterize the shape of any portion of molecular surfaces and to evaluate the shape complementarity of protein-protein interfaces [76]. A systematic procedure for reconstructing this class of circle polynomials was presented in [13] by looking for those on the unit disc that are invariant by rotation of axes about the origin and satisfying the orthogonality relation. This approach was extended in [72] to construct two complete orthogonal sets of polynomials (invariant in form under rotation) using the Gram-Schmidt orthogonalization process. The generalized complex Zernike (or disc) polynomials follow this scheme and are defined as the orthogonal ones on the unit disc $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ with respect to the radial weight function $(1-|z|^2)^{\gamma}$ with finite values at the boundary. They are given by the Rodrigues type formulas (0.0.9) or (0.0.10), up to a multiplication constant. The provided definition agrees, up to a multiplicative constant, with the one provided by Koornwinder [64, (3.7)] as well as the one considered by Dunkl [23, P. 534] and Wünsche [100]. We need only to point out here that for some reasons, which will be clear in Chapter 3, we have interchanged the roles of m and n compared with [64] and [23]. This should not be confusing thanks to the symmetry relationship satisfied by these polynomials. The algebraic and analytic properties of $\mathcal{Z}^{\gamma}_{m,n}(z,\bar{z})$ have been discussed in many papers [5, 64, 100]. The corresponding Wiener and Paley type theorems have been obtained by Kanjin in [59]. Recently, they have shown to be useful in providing a concrete description of the spectral properties of different types of the Cauchy transform [27, 26].

Main question: Would happen when considering the case where such function is $z^{n+\beta}$, for an arbitrary fixed real β, replace $\dfrac{\partial^{m+n}}{\partial \overline{z}^m}\left(e^{-\nu|z|^2}\right)$ in (0.0.6) for the punctured complex plane and z^n in (0.0.10) for the punctured unit disc? What is the corresponding spectral realization as eigenfunctions of a like magnetic Laplacian? and what can be the physical meaning of the associated potential vector?

Main aims

In the present work, we mainly aim to answer the previous questions. Namely, this work is devoted to study some special extensions of the prescribed orthogonal polynomials, hoping that our study leads to new insights into the behavior of quantum mechanical systems and provide more accurate description of certain quantum phenomena. More precisely, we deal in Chapter 1 with the so-called poly-meromorphic Itô–Hermite functions, associated with a specific singular potential vector and modeling the Aharonov–Bohm effect on the punctured complex plane.

This is also the case of our introduced fractional Zernike functions that may lead to improve new methods for characterizing and correcting optical aberrations brought to our attention by one of the referees of [21].

On the other hand, the possibility of extending the physical equations within the framework of fractional calculus is one of the tasks of the present work. Thus, starting from the fact that the real Hermite polynomials in (0.0.5) are the natural solutions of the integer order differential equation

$$y'' - 2xy' + 2ny = 0,$$

a special fractional analog has been discussed in Chapter 2 and arose as solution of the fractional order differential equation

$$D^\beta D^\beta - 2x^\beta D^\beta y + 2\lambda y = 0$$

using Caputo fractional derivative D^β of order β.

It should be mentioned here that the fractional calculus, initiated by Leibniz and subsequently developed by Liouville and Riemann during the 19th century, remains an evolving domain and a valuable tool in understanding complex systems because of its ability to describe phenomena with fractional-order. It also finds applications in many fields including finance, control theory, physics and biology. For more details, one refers to [86]. Moreover, by incorporating fractional calculus into physics, researchers can better understand and characterize intricate phenomena, providing insights into the behavior of diverse physical systems and paving the way for advancements in understanding complex dynamics and transport processes. Applications of fractional calculus in image and signal processing have gained significant attention in recent years due to their ability to handle non-local and non-linear features of signals and images. In signal processing, fractional calculus techniques have proven valuable in the analysis and processing of non-stationary signals, such as biomedical signals and seismic data. This is also the case when dealing with the heat and mass transfer processes, image processing, electrical circuits, chemotherapy effects, electrical spectroscopy and the resolution and quality of nanoimages [93, 70, 89].

The next section describes briefly the obtained results.

Description of the content

In **Chapter 1**, we identify a special class of orthogonal and poly-meromorphic functions associated with the second order differential operator

$$\mathcal{D}_{\alpha,\beta} = \Delta_\alpha + S_{\alpha,\beta}.$$

This is essentially the Laplacian Δ_α in (0.0.7) perturbed by the first order differential operator $S_{\alpha,\beta}$ explicitly given by

$$S_{\alpha,\beta} = \frac{\beta}{|z|^2}\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}\right) - \beta\left(2\alpha - \frac{\beta}{|z|^2}\right) \tag{0.0.11}$$

and attached with the closed and singular potential vector

$$\widetilde{\theta}_\beta(z,\overline{z}) = -\frac{i\beta}{|z|^2}\left(\overline{z}dz - zd\overline{z}\right), \tag{0.0.12}$$

modeling the Aharonov–Bohm effect. Mainly, we aim to explore the role played by the injection of this singular potential in generating non-trivial orthogonal eigenstates within the factorization formalism. In fact, we provide an accurate study of the special functions

$$\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) := (-1)^n z^{-\beta} e^{\alpha|z|^2}\frac{\partial^n}{\partial z^n}\left(z^{\beta+m}e^{-\alpha|z|^2}\right), \tag{0.0.13}$$

referred to as poly-meromorphic (or β-modified complex) Itô-Hermite functions. Here, α and β are given fixed reals with $\alpha > 0$, and n and m are varying integers such that $n = 0, 1, 2, \cdots$ and $m > -\beta - 1$. This leads to a special generalization of the holomorphic Itô-Hermite polynomials in

(0.0.6). The polynomial case, shown in Section 1.3 to correspond to $m \geq n$ and $\beta \geq 0$, coincides with the polynomials $Z^{\beta}_{m,n}$ considered by Ismail and Zeng in [54, Section 3]. More precisely, we are concerned with certain basic algebraic, analytic and spectral properties of $\psi^{\alpha,\beta}_{n,m}(z,\overline{z})$ defined by their Rodrigues formula (0.0.13). In particular, the interrelation of $\psi^{\alpha,\beta}_{n,m}(z,\overline{z})$ with the special functions such as the Itô–Hermite polynomials and the confluent hypergeometric functions are considered in Section 1.1. Their spectral realization as eigenfunctions of the magnetic Laplacian $\mathcal{D}_{\alpha,\beta}$ as well as their regularity and their exact bi-order as poly-meromorphic functions on the complex plane are also studied in Sections 1.2 and 1.3, respectively. Their associated generating functions are obtained in Section 1.4, and next employed to discuss some of their applications such as their integral representations (Section 1.5) and special attached integral transforms (Section 1.6).

In **Chapter 2**, we introduce and study the so-called fractional Hermite functions, the extension of the fractional setting from a different point of view. The starting point in our approach is the Rodrigues formula for the real Hermite polynomials. Therefore, the chief results of this work concern the functions defined by the generalized Rodrigues type formulas involving the fractional Liouville-Caputo derivative ${}^cD^{\beta}$. Namely, we deal with

$$ {}_LH_{(\beta)}(x) = e^{x^2}{}^cD^{\beta}(e^{-x^2}), \tag{0.0.14}$$

for $\beta \geq 0$. The exact definitions of ${}^cD^{\beta}$ and of $I^{-\beta}$ above will be given in Chapter 2. For the integer order $\beta = n$, we recover the Hermite polynomials in (0.0.5), to wit ${}_LH_{(n)}(x) = (-1)^n H_n(x)$. We also consider the class associated with the Riemann-Liouville integral ${}_LI^{-\beta}$ defined, for $\beta < 0$, by

$$ {}_LG_{(\beta)}(x) = e^{x^2} I^{-\beta}(e^{-x^2}). \tag{0.0.15}$$

More precisely, we aim to investigate some of their useful properties including their three terms and differential recurrence formulas, and to explore their different hypergeometric and integral representations. Additionally, we demonstrate that these families can be defined using an extended explicit formula even if the fractional parameter β is non-integer. Moreover, the relation to the Bessel and Kummer's functions are obtained together with an integral representation generalizing the ordinary one

$$ H_n(x) = \frac{2^{n+1}}{\Gamma(1/2)} \int_0^{\infty} e^{x^2-t^2} t^n \cos\left(2xt - \frac{n\pi}{2}\right) dt. $$

We also show that ${}_LH_{(\beta)}(x)$ and ${}_LG_{(\beta)}$ are solutions of the fractional β-Hermite differential equation

$$ y''(x) - 2xy'(x) + 2\beta y = 0. \tag{0.0.16}$$

We anticipate that the acquired properties will provide advantages in modeling natural phenomena and advance the understanding of complex systems [71]. They may also be instrumental in representing the quantum mechanical harmonic oscillator through the application of fractional operator calculus. In fact, the Schrödinger equation $\psi'' + (2m/\hbar^2)(E - V(x))\psi = 0$, which reduces further to $\psi'' - x^2\psi = -(1+2\beta)\psi$, with $\beta = (m/2\hbar^2)^{1/2}E - (1/2)$, admits the fractional Hermite functions

$$e^{-x^2/2}T_{(\beta)} := \begin{cases} e^{-x^2/2}{}_LH_{(\beta)}, & \beta > 0 \\ e^{-x^2/2}{}_LG_{(\beta)}, & \beta < 0, \end{cases}$$

as a solution with the fraction $k = 1 + 2\beta$ as corresponding eigenvalue, which means that the function $e^{-x^2/2}T_{(\beta)}$ can be considered as a state of a particle of mass m in the potential $V(x)$, with fractional energy.

The content of Chapter 2 is outlined as follows. Needed preliminaries are reviewed and collected in Section 2.1. Section 2.2 is devoted to show that the suggested fractional Hermite functions can be expressed in terms of the hypergeometric functions. The main results are stated in Sections 2.3 and 2.4. In Section 2.5, we present numerical data to discern the distinctions between the proposed fractional order Hermite function and the ones of integer order. Additionally, we provide concluding remarks and discuss potential avenues for future research.

In **Chapter 3**, a specific generalization of the Zernike polynomials in (0.0.10) will be the main topic of the discussion by inserting a fractional order. Their construction made appeal to the factorization method. Namely, for fixed real numbers $\rho, \kappa > -1$ and varying integers m and n such that $m \geq 0$ and $n + \rho \geq 0$, we deal with the family of functions

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) := (-1)^m z^{-\rho}(1-|z|^2)^{-\kappa}\frac{\partial^m}{\partial z^m}\left(z^{\rho+n}(1-|z|^2)^{\kappa+m}\right) \tag{0.0.17}$$

on the punctured unit disc $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. It is worth noting that for an arbitrary nonnegative integer ρ, they reduce further to the Zernike polynomials in (0.0.10), since for every $\ell = 0, 1, \cdots$, we have

$$z^\ell \mathcal{Z}^{\kappa,\ell}_{m,n}(z,\overline{z}) = \frac{\mathcal{Z}^{\kappa}_{m,n+\ell}(z,\overline{z})}{(\kappa+m+1)_{n+\ell}}. \tag{0.0.18}$$

Otherwise, they are no longer polynomials. The main aim of the present work is to provide a systematic study of this class of functions. Notice for instance that their study for arbitrary ρ can be reduced to the subclass corresponding to $0 \leq \rho < 1$. More precisely, we have

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) = z^{-[\rho]}\mathcal{Z}^{\kappa,\widetilde{\rho}}_{m,n+[\rho]}(z,\overline{z}),$$

whenever $n+[\rho]\geq 0$. Here $[\rho]$ denotes the integer part of ρ and $0\leq\widetilde{\rho}=\rho-[\rho]<1$. This justifies somehow the following nomination of the functions $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})$ in (0.0.17), referred to as fractional Zernike functions, which can also be justified from being poly-meromorphic (see Theorem 3.2.10).

Contrary to the classical Zernike polynomials satisfying the symmetry relationships $\overline{\mathcal{Z}^{\kappa}_{m,n}(z,\overline{z})} = \mathcal{Z}^{\kappa}_{m,n}(\overline{z},z) = \mathcal{Z}^{\kappa}_{n,m}(z,\overline{z})$, which play a crucial rule in their study, this relation is no longer valid for the fractional Zernike functions $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})$ even when ρ is a positive integer. In fact, we have $\overline{\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})} = \mathcal{Z}^{\kappa,\rho}_{m,n}(\overline{z},z)$ only for arbitrary ρ and one gets from (0.0.18) the identity

$$\overline{\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})} = \frac{(\kappa+1)_m}{(\kappa+1)_{n+\rho}} z^{\rho}\overline{z}^{-\rho}\mathcal{Z}^{\kappa,\rho}_{n+\rho,m-\rho}(z,\overline{z}), \tag{0.0.19}$$

valid for ρ being a nonnegative integer. This reveals in particular that the analytic and spectral properties of the functions $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})$ can not be directly recovered from the Zernike polynomials, and the relevant properties may be completely different from the classical ones, essentially when ρ is non integer. Thus, a concrete description of their algebraic and analytic properties for fixed reals $\rho,\kappa>-1$ is desirable. For this purpose, we begin by considering the so-called β-restricted Zernike functions $\psi^{\gamma,\eta}_{m,n}$, which are shown to be a special class of polyanalytic excited states in the weighted Hilbert space $L^{2,\alpha}_{\beta}(\mathbb{D}) = L^2(\mathbb{D},d\mu_{\alpha,\beta})$ of all complex-valued functions that are square integrable with respect to the positive measure

$$d\mu_{\alpha,\beta}(z) := (1-|z|^2)^{\alpha}|z|^{2\beta}dxdy; \quad z=x+iy. \tag{0.0.20}$$

The main results concerning the functions $\psi^{\gamma,\eta}_{m,n}$ are summarized in Theorem 3.1.6. Namely, we prove that they form an orthogonal system of eigenfunctions in $L^{2,\alpha}_{\beta}(\mathbb{D})$ for a perturbed magnetic Laplacian, which is essentially the classical magnetic Schrödinger operator on the hyperbolic unit disc perturbed by a particular potential (with zero magnetic field) modeling the Aharonov–Bohm effect (see Remark 3.1.3). Moreover, the $L^{2,\alpha}_{\beta}$-eigenspace of the considered Laplacian associated with its lowest eigenvalue is shown to be the β-modified Bergman space $\mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D})$ on the punctured unit disc $\mathbb{D}^*$, recently introduced and studied in [37, 38]. The other $L^{2,\alpha}_{\beta}$-eigenspaces associated with the hyperbolic Landau levels for the considered Laplacian can be seen as the polyanalytic analogs of $\mathcal{A}^{2,\alpha}_{\beta}(\mathbb{D})$ (see Remark 3.1.7).

The motivation behind considering $\psi^{\gamma,\eta}_{m,n}$ is that they can be seen as the spectral side of fractional Zernike functions. For special values of γ and η they are closely connected to $\mathcal{Z}^{\kappa_m,\rho}_{m,n}$ by

$$\mathcal{Z}^{\kappa_m,\rho}_{m,n}(z,\overline{z}) = |z|^{2\eta}(1-|z|^2)^{\frac{\alpha+1-\kappa_m}{2}}\psi^{\gamma,\eta}_{m,n}(z,\overline{z}) \tag{0.0.21}$$

for $m, n \geq 0$ with $\rho = \beta - 2\eta$, and for κ depending on m and given by $\kappa = \kappa_m = \alpha - 2(\gamma + m) - 1$. However, the latter fact can not be employed to recover the global properties of the fractional Zernike functions $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})$. Only the local ones for every fixed m, n and ρ with the specific $\kappa = \kappa_m$ can be derived.

For the concrete study of $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})$ we begin by establishing their explicit expressions, their different hypergeometric representations, their expression in terms of the Jacobi polynomials and their connection to the complex Zernike polynomials in (0.0.10). Subsequently, the zero sets of $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})$ are described (Corollary 3.2.6) and shown to be the centered circles of radii being the zeros of the real Jacobi polynomials. The orthogonality in the Hilbert space $L^{2,\kappa}_{\rho}(\mathbb{D}) = L^2(\mathbb{D}, d\mu_{\kappa,\rho})$ is discussed and the square norm is explicitly computed. The membership to a specific class of poly-meromorphic functions in $\mathbb{D}$ is also considered (Theorem 3.2.10). Moreover, we investigate the operational formulas of such polynomials they satisfy including those of Burchnall type. Furthermore, we discuss some of its recurrence relations and the differential equations that it obeys (Theorems 3.2.12 and 3.2.15), and so on. Certain associated generating functions are obtained such as bilinear generating function analogous to the Hardy–Hille generating function for the generalized Laguerre polynomials, which can be employed to derive a special integral representation for $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})$. Another integral representation of Cauchy-type is obtained as an integral on the unit circle. In Theorem 3.2.25, we show that the truncated fractional Zernike functions

$$\Upsilon^{\kappa,\rho}_{m,s}(z,\overline{z}) := z^s |z|^{-s} \mathcal{Z}^{\kappa,\rho}_{m,m}(z,\overline{z}), \; s \in \mathbb{Z}, \; m = 0, 1, \cdots , \tag{0.0.22}$$

constitute an orthogonal basis of the Hilbert space $L^{2,\kappa}_{\rho}(\mathbb{D})$. Accordingly, we introduce a class of poly-meromorphic Bergman spaces leading to a complete microlocal orthogonal decomposition of the underlying Hilbert space $L^{2,\kappa}_{\rho}(\mathbb{D})$. The obtained results will contribute efficiently in the study of the associated isometric integral transforms (of Bargmann type) on the configuration space on the positive half real line.

The different sections of Chapter 3 are organized as follows. Section 3.1 deals with the spectral realization of the β-modified functions $\psi^{\gamma,\eta}_{m,n}$ by Schrödinger's factorization method. A proof that tha family $\psi^{\gamma,\eta}_{m,n}$ forms an orthogonal system of eigenfuctions in $L^{2,\alpha}_{\beta}(\mathbb{D})$ is also presented in this section. The basic properties of the fractional functions as described above are stated and proved in Section 3.2.

Chapter 1

Poly-meromorphic Itô–Hermite functions associated with a singular potential vector on the punctured complex plane

We provide a theoretical study of a new family of orthogonal functions on the punctured complex plane solving the eigenvalue problems for some magnetic Laplacian perturbed by a singular vector potential with zero magnetic field modeling the Aharonov–Bohm effect. The functions are defined by their β-modified Rodrigues type formula and extend the poly-analytic Itô–Hermite polynomials to the poly-meromorphic setting. Mainly, we derive their different operational representations and give their explicit expressions in terms of special functions. Different generating functions and integral representations are obtained.

1.1 Preliminary results

We begin by noticing that the functions $\psi^{\alpha,\beta}_{n,m}$ satisfy $\psi^{\alpha,\beta}_{m,n}(z,\overline{z}) = \overline{\psi^{\alpha,\beta}_{m,n}(\overline{z},z)}$ as well as the symmetry relation

$$\alpha^{m} z^{\beta} \psi^{\alpha,\beta}_{n+\beta,m-\beta}(z,\overline{z}) = \alpha^{n+\beta} \overline{z}^{\beta} \overline{\psi^{\alpha,\beta}_{m,n}(z,\overline{z})}, \tag{1.1.1}$$

valid for β being integer and for given non-negative integers m and n such that $n \geq max(0, -\beta)$. We have in addition

$$\overline{z}^\beta \psi^{1,\beta}_{m,m}(z,\overline{z}) = z^\beta \psi^{1,\beta}_{m+\beta,m-\beta}(z,\overline{z})$$

and

$$\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) = z^{-[\beta]} \psi^{\alpha,\tilde{\beta}}_{n,m+[\beta]}(z,\overline{z}),$$

valid for any real β such that $\beta + m > -1$, where $[\beta]$ denotes the integer part of β. For the particular case of $\beta = 1/2$ we have

$$\sqrt{\alpha|z|} 2^{2m+1} \psi^{\alpha,1/2}_{m,m}(z,\overline{z}) = H_{2m+1}(\sqrt{\alpha|z|}) = 2(-4)^m m! (\alpha|z|)^{1/2} L^{(1/2)}_m(\alpha|z|),$$

where $H_k(x)$ are the Hermite polynomials in (0.0.5) and $L^{(\alpha)}_k$ denotes the generalized Lageurre polynomials. For arbitrary non-negative integer β, the functions $\psi^{\alpha,\beta}_{m,n}(z,\overline{z})$ are closely connected to the complex Itô-Hermite polynomials in (0.0.6) on the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$,

$$z^\beta \psi^{\alpha,\beta}_{n,m}(z,\overline{z}) = \psi^{\alpha,0}_{n,m+\beta}(z,\overline{z}) = \alpha^{\frac{n-m-\beta}{2}} H_{m+\beta,n}(\sqrt{\alpha} z, \sqrt{\alpha}\overline{z}). \tag{1.1.2}$$

The interrelation with these polynomials for arbitrary β can be obtained by specifying f in the Burchnall's operational formula [34, Proposition 2.3]

$$(-1)^n e^{\alpha|z|^2} \frac{\partial^n}{\partial z^n} \left(z^m e^{-\alpha|z|^2} f \right) = \frac{n!}{\alpha^m} \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} H^\alpha_{m,n-k}(z,\overline{z}) \frac{\partial^k f}{\partial z^k}.$$

Thus, by setting $f(z) = z^\beta$ we obtain

$$\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) = \frac{n!}{\alpha^m} \sum_{k=0}^{n} \frac{(-1)^k \Gamma(\beta+1)}{k!(n-k)!\Gamma(\beta-k+1)} z^{-k} H^\alpha_{m,n-k}(z,\overline{z}).$$

For the explicit expression of $\psi^{\alpha,\beta}_{n,m}(z,\overline{z})$ (with $m > -\beta - 1$), we claim that

$$\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) = \sum_{k=0}^{n \wedge^* (m+\beta)} c^{\alpha,\beta}_{m,n,k} z^{m-k} \overline{z}^{n-k}, \tag{1.1.3}$$

where the starred minimum $m \wedge^* b$ is defined by the classical minimum $m \wedge^* b = \min(m,b)$ when b is an integer and by $m \wedge^* b = m$ otherwise. The involved constants $c^{\alpha,\beta}_{m,n,k}$ stand for

$$c^{\alpha,\beta}_{m,n,k} := \frac{(-1)^k n! \Gamma(\beta+m+1)}{k!(n-k)!\Gamma(\beta+m-k+1)} \alpha^{n-k}.$$

The expression in (1.1.3) can be handled by applying the Leibniz formula to (0.0.13) keeping in mind the fact that

$$z^{-(\beta+m-k)} \frac{\partial^k}{\partial z^k}(z^{\beta+m}) = \begin{cases} 0, & \beta = 0,1,2,\cdots, k > \beta+m, \\ \dfrac{\Gamma(\beta+m+1)}{\Gamma(\beta+m-k+1)} & \text{otherwise.} \end{cases}$$

Notice also that the monomials $z^q\overline{z}^p$ can be expressed in terms of the considered functions. In fact, by rewriting them as a derivation of the Gaussian function $e^{-\alpha|z|^2}$, we get

$$\begin{aligned}\alpha^p z^q\overline{z}^p &= (-1)^p z^q e^{\alpha|z|^2}\frac{\partial^p}{\partial z^p}(e^{-\alpha|z|^2})\\ &= (-1)^p z^q e^{\alpha|z|^2}\frac{\partial^p}{\partial z^p}(z^{-\beta-n+p-q}z^{\beta+n+q-p}e^{-\alpha|z|^2})\\ &= \sum_{n=0}^{p}\binom{p}{n}(-1)^p\frac{\partial^{p-n}}{\partial z^{p-n}}\left(z^{-\beta-n+p-q}\right)z^q e^{\alpha|z|^2}\frac{\partial^n}{\partial z^n}(z^{\beta+n+q-p}e^{-\alpha|z|^2}).\end{aligned}$$

The last equality holds making use of the Leibniz formula. Now, by means of the Rodrigues formulas (0.0.13) with $q-p\geq 0$, it follows

$$\alpha^p z^q\overline{z}^p = \sum_{n=0}^{p}\binom{p}{n}\frac{\Gamma(\beta+q)}{\Gamma(\beta+q+n-p)}\psi^{\alpha,\beta}_{n,n+q-p}(z,\overline{z}) \tag{1.1.4}$$

for every non-negative integers p and q with $p\leq q$.

The first few terms of $\psi^{\alpha,\beta}_{n,m}$ are given by $\psi^{\alpha,\beta}_{0,m}(z,\overline{z}) = z^m$ and $\psi^{\alpha,\beta}_{1,m}(z,\overline{z}) = z^{m-1}\left(\alpha z\overline{z}-(\beta+m)\right)$ when $n=0$ and $n=1$ respectively, while for $n=2$ we have

$$\psi^{\alpha,\beta}_{2,m}(z,\overline{z}) = z^{m-2}\left(\alpha^2 z^2\overline{z}^2 - 2\alpha(\beta+m)z\overline{z} + (\beta+m)(\beta+m-1)\right).$$

This reveals in particular that the $\psi^{\alpha,\beta}_{n,m}$ are no longer polynomials unless specifying β, m and n, which is the case of β being a non-positive integer. This becomes clear from their hypergeometric representation in terms of the hypergeometric functions defined by the series

$${}_pF_q\left(\begin{matrix}a_1,a_2,\cdots,a_p\\ c_1,c_2,\cdots,c_q\end{matrix}\middle| x\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(c_1)_n(c_2)_n\cdots(c_q)_n}\frac{x^n}{n!}$$

provided that $c_\ell \neq 0,-1,-2,\cdots$ for $\ell = 1,2,\cdots,q$. Indeed, from (1.1.3) and making use of the classical identities on Gamma function and Pochammer symbol, we get

$$\begin{aligned}\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) &= \alpha^n z^m\overline{z}^n\sum_{k=0}^{n\wedge^*(m+\beta)}(-n)_k(-\beta-m)_k\frac{(-\alpha z\overline{z})^{-k}}{k!}\\ &= \alpha^n z^m\overline{z}^n{}_2F_0\left(\begin{matrix}-n,-\beta-m\\ -\end{matrix}\middle| -\frac{1}{\alpha|z|^2}\right) \qquad (1.1.5)\\ &= \frac{(-1)^n\Gamma(\beta+m+1)}{\Gamma(\beta+m-n+1)}z^{m-n}{}_1F_1\left(\begin{matrix}-n\\ \beta+m-n+1\end{matrix}\middle|\alpha|z|^2\right), \qquad (1.1.6)\end{aligned}$$

provided that $\beta\in\mathbb{R}\setminus\mathbb{Z}$ or $\beta+m-n+1>0$ whenever β is integer. Otherwise, ie $\beta\in\mathbb{Z}$ and $\beta+m-n+1\leq 0$, the confluent hypergeometric function involved in (1.1.6) is undefined.

Subsequently, by means of (1.1.6) we obtain the following the identity expressing $\psi^{\alpha,\beta}_{n,m}$ in terms of the generalized Laguerre polynomials [69, p. 240]. In fact, we have

$$\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) = (-1)^n n! z^{m-n} L_n^{(\beta+m-n)}(\alpha|z|^2), \tag{1.1.7}$$

which holds only when $\beta + m - n + 1 > 0$. This can also be recovered starting from (0.0.13) using the derivation formula [69, p.241]. Accordingly, with the use of (1.1.7) and (1.1.2), it is straightforward to prove the orthogonality of $\psi^{\alpha,\beta}_{n,m}$ in the Hilbert space $L^{2,\alpha}_{\beta}(\mathbb{C}) := L^2(\mathbb{C}, d\mu_{\alpha,\beta})$ of square integrable functions with respect to the measure $d\mu_{\alpha,\beta}(z) = |z|^{2\beta} e^{-\alpha|z|^2} d\lambda(z)$, where $d\lambda(z) = dxdy$ denotes the Lebesgue measure on the complex plane with $z = x + iy$, $x, y \in \mathbb{R}$. More explicitly, we assert the following.

Proposition 1.1.1. *For $\beta + m - n > -1$ with $\beta \in \mathbb{R}$ or $\beta + m \geq 0$ when β integer we have*

$$\int_{\mathbb{C}} \psi^{\alpha,\beta}_{n,m}(z,\overline{z}) \overline{\psi^{\alpha,\beta}_{k,j}(z,\overline{z})} |z|^{2\beta} e^{-\alpha|z|^2} d\lambda(z) = \frac{\pi \alpha^n n!}{\alpha^{m+\beta+1}} \Gamma(\beta + m + 1) \delta_{m,j} \delta_{n,k}. \tag{1.1.8}$$

Proof. It is worth noting that for β being integer, the functions $\psi^{\alpha,\beta}_{n,m}$ are closely connected to the complex-Itô-Hermite polynomials through (1.1.2), which are known to be orthogonal with respected to the gaussian measure $e^{-|z|^2} d\lambda(z)$. For $\beta \in \mathbb{Z}$, we have

$$\begin{aligned}
\left\langle \psi^{\alpha,\beta}_{n,m}, \psi^{\alpha,\beta}_{k,j} \right\rangle_{\alpha,\beta} &= \left\langle z^\beta \psi_{n,m}, z^\beta \psi_{k,j} \right\rangle_{\alpha,0} \\
&= \int \alpha^{n-m+\beta} H_{m+\beta,n}(\sqrt{\alpha} z, \sqrt{\alpha} z) H_{j+\beta,k}(\sqrt{\alpha} z, \sqrt{\alpha} z) e^{-\alpha|z|^2 d\lambda(z)} \\
&= \alpha^{n-m-\beta-1} \left\langle H_{\beta+m,n}, H_{j+\beta,k} \right\rangle_{1,0} \\
&= \alpha^{n-m-\beta-1} (\beta + m)! n! \pi \delta_{m,j} \delta_{n,k}.
\end{aligned}$$

However, for the remaining case, we make use of the identity expressing $\psi^{\alpha,\beta}_{n,m}$ in terms of the generalized Laguerre polynomials to get

$$\begin{aligned}
\left\langle \psi^{\alpha,\beta}_{n,m}, \psi^{\alpha,\beta}_{k,j} \right\rangle_{\alpha,\beta} &= \int_{\mathbb{C}} \psi^{\alpha,\beta}_{n,m}(z,\overline{z}) \overline{\psi^{\alpha,\beta}_{k,j}(z,\overline{z})} |z|^{2\beta} e^{-\alpha|z|^2} d\lambda(z) \\
&= 2\pi (-1)^{n+k} n! k! \delta_{m-n,j-k} \int_0^\infty r^{m-n+j-k} L_n^{(\beta+m-n)}(\alpha r^2) L_k^{(\beta+j-k)}(\alpha r^2) r^{2\beta} e^{-\alpha r^2} r dr \\
&= \frac{(-1)^{n+k}}{\alpha^{m-n+\beta+1}} n! k! \pi \delta_{m,j} \delta_{n,k} \int_0^\infty t^{m-n+\beta+1} L_n^{(\beta+m-n)}(t) L_k^{(\beta+m-n)}(t) e^{-t} dt \\
&= \frac{\pi \alpha^n n!}{\alpha^{m+\beta+1}} \Gamma(\beta + m + 1) \delta_{m,j} \delta_{n,k}.
\end{aligned}$$

The last equality requires that $\beta + m - n > -1$.

□

In Section 1.2 below (Remark 1.2.3), we can prove that the orthogonality relation remains true in the case $m+\beta > -1$.

We conclude these preliminaries by noticing that starting from (0.0.13), it is clear that

$$z\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) = \psi^{\alpha,\beta-1}_{n,m+1}(z,\overline{z}).$$

Also, by rewriting ∂^{n+1} as $\partial^n\partial$, one obtains the recurrence formula

$$\alpha\overline{z}\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) = \psi^{\alpha,\beta}_{n+1,m}(z,\overline{z}) + (\beta+m)\psi^{\alpha,\beta}_{n,m-1}(z,\overline{z}). \tag{1.1.9}$$

Additional recurrence formulas can be derived from the different known ones for the generalized Laguerre polynomials. For example from those in [69, p. 241] one obtains

(1) $\psi^{\alpha,\beta-1}_{n,m+1}(z,\overline{z}) = \psi^{\alpha,\beta}_{n,m+1}(z,\overline{z}) + n\psi^{\alpha,\beta}_{n-1,m}(z,\overline{z})$

(2) $\psi^{\alpha,\beta}_{n+1,m+1}(z,\overline{z}) = \left(\alpha|z|^2 - [n+\beta+m+1]\right)\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) - n\left(\beta+m\right)\psi^{\alpha,\beta}_{n-1,m-1}(z,\overline{z})$

(3) $\alpha\overline{z}\psi^{\alpha,\beta}_{n,m+2}(z,\overline{z}) = \left(\alpha|z|^2 - n\right)\psi^{\alpha,\beta}_{n,m+1}(z,\overline{z}) - n\left(\beta+m+1\right)\psi^{\alpha,\beta}_{n-1,m}(z,\overline{z})$

(4) $\psi^{\alpha,\beta}_{n+1,m+1}(z,\overline{z}) = \left(\alpha|z|^2 - n - 1\right)\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) - z\left(\beta+m\right)\psi^{\alpha,\beta}_{n,m-1}(z,\overline{z})$

(5) $\left(\beta+m-n+\alpha|z|^2\right)\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) = z\left(\beta+m\right)\psi^{\alpha,\beta}_{n,m-1}(z,\overline{z}) + \alpha\overline{z}\psi^{\alpha,\beta}_{n,m+1}(z,\overline{z}).$

1.2 Spectral realization

The result below shows that the functions $\psi^{\alpha,\beta}_{m,n}$ are L^2-eigenfunctions of the perturbed magnetic Lapalcian defined by

$$\Delta_{\alpha,\beta} := -\frac{\partial^2}{\partial z\partial\overline{z}} + \left(\alpha - \frac{\beta}{|z|^2}\right)\overline{z}\frac{\partial}{\partial\overline{z}} \tag{1.2.1}$$

and the second order differential operator

$$\widetilde{\Delta}_{\alpha,\beta} := -\frac{\partial^2}{\partial z\partial\overline{z}} + \alpha z\frac{\partial}{\partial z} - \frac{\beta}{z}\frac{\partial}{\partial\overline{z}}. \tag{1.2.2}$$

Theorem 1.2.1. *The functions $\psi^{\alpha,\beta}_{n,m}$ form an orthogonal system in $L^{2,\alpha}_{\beta}(\mathbb{C})$ that solve the eigenvalue problems $\Delta_{\alpha,\beta} = \alpha n$ and $\widetilde{\Delta}_{\alpha,\beta} = \alpha m$.*

Proof. Notice first that the second order differential operator in (1.2.1) can be rewritten as

$$\Delta_{\alpha,\beta} = \nabla'_{z,\alpha,\beta}\nabla''_{z} = \nabla''_{z}\nabla'_{z,\alpha,\beta} - \alpha,$$

where $\nabla''_z = \partial/\partial\bar{z}$ and $\nabla'_{z,\alpha,\beta}$ are the first order differential operators given by

$$\nabla'_{z,\alpha,\beta} = -[\rho_{\alpha,\beta}(z)]^{-1}\frac{\partial}{\partial z}\left(\rho_{\alpha,\beta}(z)f(z)\right). \tag{1.2.3}$$

Here $\rho_{\alpha,\beta}(z) := |z|^{2\beta}e^{-\alpha|z|^2}$. Thus, using the commutation rule $\nabla''_z\nabla'_{z,\alpha,\beta} - \nabla'_{z,\alpha,\beta}\nabla''_z = \alpha Id$, one proceeds by induction to get the identity

$$\nabla''_z(\nabla'_{z,\alpha,\beta})^{n+1} = (\nabla'_{z,\alpha,\beta})^{n+1}\nabla''_z + \alpha(n+1)(\nabla'_{z,\alpha,\beta})^n. \tag{1.2.4}$$

Subsequently, one gets

$$\Delta_{\alpha,\beta}((\nabla'_{z,\alpha,\beta})^n(g)) = \left(\nabla''_z\nabla'_{z,\alpha,\beta} - \alpha\right)((\nabla'_{z,\alpha,\beta})^n(g)) = \alpha n((\nabla'_{z,\alpha,\beta})^n(g)),$$

for any $g \in \ker(\nabla''_z)$. Thus, by considering the case of the generic elements $g_m(z) = z^m$ with $m \in \mathbb{Z}$ for z in the punctured complex plane, one deduces that the function

$$\psi^{\alpha,\beta}_{n,m}(z,\bar{z}) := (\nabla'_{z,\alpha,\beta})^n(g_m) = (-1)^n[\rho_{\alpha,\beta}(z)]^{-1}\frac{\partial^n}{\partial z^n}\left(z^m\rho_{\alpha,\beta}(z)\right) \tag{1.2.5}$$

is an eigenfunction of $\Delta_{\alpha,\beta}$ with $n\alpha$ as a corresponding eigenvalue.

Now, to prove that $\widetilde{\Delta}_{\alpha,\beta}\psi^{\alpha,\beta}_{n,m} = \alpha m\psi^{\alpha,\beta}_{n,m}$, we make use of the partial raising operations

$$-\left(\frac{\partial}{\partial z} - \alpha\bar{z} + \frac{\beta}{z}\right)\psi^{\alpha,\beta}_{n,m}(z,\bar{z}) = \psi^{\alpha,\beta}_{n+1,m}(z,\bar{z}) \tag{1.2.6}$$

and

$$-\frac{1}{\alpha}\left(\frac{\partial}{\partial\bar{z}} - \alpha z\right)\psi^{\alpha,\beta}_{n,m}(z,\bar{z}) = \psi^{\alpha,\beta}_{n,m+1}(z,\bar{z}), \tag{1.2.7}$$

which are immediate by straightforward computation. On the other hand, from (1.1.9) and (1.2.6) one has

$$\left(\frac{\partial}{\partial z} + \frac{\beta}{z}\right)\psi^{\alpha,\beta}_{n,m}(z,\bar{z}) = (\beta+m)\psi^{\alpha,\beta}_{n,m-1}(z,\bar{z}). \tag{1.2.8}$$

Accordingly, by combining (1.2.7) and (1.2.8), it follows

$$\left(\frac{\partial}{\partial z} + \frac{\beta}{z}\right)\left(\frac{\partial}{\partial\bar{z}} - \alpha z\right)\psi^{\alpha,\beta}_{n,m}(z,\bar{z}) = -\alpha(\beta+m+1)\psi^{\alpha,\beta}_{n,m}(z,\bar{z}). \tag{1.2.9}$$

This completes the proof by observing that the operator $\widetilde{\Delta}_{\alpha,\beta}$ in (1.2.2) can be factorized, up to an additive constant, as

$$\widetilde{\Delta}_{\alpha,\beta} = -\left(\frac{\partial}{\partial z} + \frac{\beta}{z}\right)\left(\frac{\partial}{\partial\bar{z}} - \alpha z\right) - \alpha(\beta+1).$$

□

Remark 1.2.2. *Let $E = z\partial/\partial z$ be the Euler derivative operator and $\overline{E} = \overline{z}\partial/\partial\overline{z}$ be its complex conjugate. Then, the functions $\psi^{\alpha,\beta}_{n,m}$ satisfy*

$$(E - \overline{E})\psi^{\alpha,\beta}_{n,m} = (m-n)\psi^{\alpha,\beta}_{n,m},$$

which readily follows since $\widetilde{\Delta}_{\alpha,\beta} - \Delta_{\alpha,\beta} = \alpha(E - \overline{E})$. This is the analog for $\psi^{\alpha,\beta}_{n,m}$ at arbitrary integer m such that $m > -\beta - 1$ of the one obtained in [54].

Remark 1.2.3. *The orthogonality of $\psi^{\alpha,\beta}_{n,m}$ in $L^{2,\alpha}_{\beta}(\mathbb{C})$ can be reproved by observing that the operator $\nabla^{'}_{z,\alpha,\beta}$ is in fact the formal adjoint of $\nabla^{''}_{z}$ when acting on a densely domain of smooth functions in $L^{2,\alpha}_{\beta}(\mathbb{C})$. In fact, we get*

$$\begin{aligned}\left\langle \psi^{\alpha,\beta}_{n,m}, \psi^{\alpha,\beta}_{p,q} \right\rangle_{\alpha,\beta} &= \left\langle \nabla^{''}_{z}(\nabla^{'}_{z,\alpha,\beta})^{n}(g_m), (\nabla^{'}_{z,\alpha,\beta})^{p-1}(g_q) \right\rangle_{\alpha,\beta} \\ &= \left\langle (\nabla^{'}_{z,\alpha,\beta})^{n}\nabla^{''}_{z}(g_m) + n\alpha(\nabla^{'}_{z,\alpha,\beta})^{n-1}(g_m), (\nabla^{'}_{z,\alpha,\beta})^{p-1}(g_q) \right\rangle_{\alpha,\beta}\end{aligned}$$

by means of (1.2.4)*. without lost of generality we can assume that $n \leq p$. Mathematical induction combined with the fact $\nabla^{''}_{z}(g_m) = 0$ both infer*

$$\left\langle \psi^{\alpha,\beta}_{n,m}, \psi^{\alpha,\beta}_{p,q} \right\rangle_{\alpha,\beta} = \alpha n \left\langle \psi^{\alpha,\beta}_{n-1,m}, \psi^{\alpha,\beta}_{p-1,q} \right\rangle_{\alpha,\beta} = \alpha^{n} n! \left\langle g_m, \psi^{\alpha,\beta}_{p-n,q} \right\rangle_{\alpha,\beta}.$$

The case of $p = n$ leads to $\langle g_m, g_q \rangle_{\alpha,\beta} = \pi\alpha^{-(m+\beta+1)}\Gamma(\beta+m+1)\delta_{n,p}\delta_{m,q}$, since $\psi^{\alpha,\beta}_{0,m} = z^{q}$. However, for $p > n$ we get

$$\left\langle g_m, \psi^{\alpha,\beta}_{p-n,q} \right\rangle_{\alpha,\beta} = \left\langle g_m, (\nabla^{'}_{z,\alpha,\beta})^{p-n}(g_q) \right\rangle_{\alpha,\beta} = \left\langle (\nabla^{''}_{z})^{p-n}(g_m), g_q \right\rangle_{\alpha,\beta} = 0.$$

This completes the proof of

$$\left\langle \psi^{\alpha,\beta}_{n,m}, \psi^{\alpha,\beta}_{p,q} \right\rangle_{\alpha,\beta} = \frac{\pi\alpha^{n} n!}{\alpha^{m+\beta+1}}\Gamma(\beta+m+1)\delta_{n,p}\delta_{m,q}.$$

Below, for given reals α and β such that $\alpha > 0$ we prove that the considered functions are closely connected to the spectral analysis of a specific Schrödinger operator

$$\mathcal{D}_{\alpha,\beta} := \nabla^{*}_{\alpha,\beta}\nabla_{\alpha,\beta} = \frac{1}{4}(d + i\mathbf{ext}_{\theta_{\alpha,\beta}})^{*}(d + i\mathbf{ext}_{\theta_{\alpha,\beta}}),$$

where $\mathbf{ext}_{\theta_{\alpha,\beta}}(\omega) := \theta_{\alpha,\beta} \wedge \omega$ is the exterior multiplication by the real-valued singular vector potential (differential 1-form) given by

$$\theta_{\alpha,\beta} := -i(\partial - \overline{\partial})Log(\rho_{\alpha,\beta})$$

and $\nabla_{\alpha,\beta} = (d + i\mathbf{ext}_{\theta_{\alpha,\beta}})/2$ is the co-derivation operator acting on $\mathcal{C}_0^\infty(\mathbb{C})$, the space of smooth complex-valued functions on $\mathbb{C} \setminus \{0\}$ with compact support. The operator $\nabla_{\alpha,\beta}^*$ denotes its formal adjoint with respect to the Hermitian scalar product

$$\left(f, g\right)_0 = \int_{\mathbb{C}} f \wedge \star g, \tag{1.2.10}$$

where $\star$ is the Hodge star operator mapping the space of compactly supported smooth differential p-forms $\Omega_{p,c}^\infty(\mathbb{C} \setminus \{0\})$, $p = 0, 1, 2$, into $\Omega_{2-p,c}^\infty(\mathbb{C} \setminus \{0\})$. It is defined to satisfy $\star(f\omega) = \overline{f}(\star\omega)$ for scalar functions f, and $\star(dz \wedge d\bar{z}) = 2i$. This readily follows for the metric ds^2 being conformal to the Euclidean metric $ds^2(z) = dz \otimes d\bar{z}$.

The operator $\mathcal{D}_{\alpha,\beta} := \nabla_{\alpha,\beta}^* \nabla_{\alpha,\beta}$, with the operator domain being $\mathcal{C}_0^\infty(\mathbb{C})$, is symmetric and positive. Moreover, its deficiency indices are known to be $(2, 2)$ (see for example [29] or also [78, Proposition 5.7]). Therefore, it has self-adjoint extensions parametrized by (2×2)-unitary matrices [85]. Next, by considering the strong extensions (Cf. [46, p. 168 and p. 241] or also [31, 47, **?**, 79, 88]) of the differential operators d and ∇ initially defined on a domain of functions in $\mathcal{C}_0^\infty(\mathbb{C} \setminus \{0\})$, we can extend $\mathcal{D}_{\alpha,\beta}$ in the L^2-Hilbert space to the operator domain

$$D(\mathcal{D}_{\alpha,\beta}) = \{f, f \in \mathcal{C}^\infty(\mathbb{C} \setminus \{0\}), f \in L^2(\mathbb{C} \setminus \{0\}), \mathcal{D}_{\alpha,\beta} f \in L^2(\mathbb{C} \setminus \{0\})\}.$$

It should be mentioned here that all the possible hamiltonians, describing the non relativistic Aharonov-Bohm effect, have been constructed in [4] using the theory of self-adjoint extensions of von Neumann and Krein.

Lemma 1.2.4. *Keep $\mathcal{D}_{\alpha,\beta} := \nabla_{\alpha,\beta}^* \nabla_{\alpha,\beta}$, $\nabla_{\alpha,\beta}$ and $\theta_{\alpha,\beta}$ as above. Then, $\mathcal{D}_{\alpha,\beta}$ coincides with the second order differential operator $\widetilde{\mathcal{D}}_{\alpha,4\beta}$ in (**??**). More explicitly, we have*

$$\mathcal{D}_{\alpha,\beta} = -\left\{\frac{\partial^2}{\partial z \partial \bar{z}} + \left(\alpha - \frac{\beta}{|z|^2}\right)\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right)\right\} + \left(\alpha - \frac{\beta}{|z|^2}\right)^2 |z|^2. \tag{1.2.11}$$

Proof. Notice first that the differential 1-form $\theta_{\alpha,\beta}$ is explicitly given by $\theta_{\alpha,\beta} = ik_\beta^\alpha(z)\left(\bar{z}dz - zd\bar{z}\right)$ with $k_\beta^\alpha(z) := \alpha - \beta/|z|^2$. Straightforward computation making use of the well-known facts $d^* = - \star d \star$ and $(\mathbf{ext}_{\theta_{\alpha,\beta}})^* = \star(\mathbf{ext}_{\theta_{\alpha,\beta}})\star$ shows that for every smooth differential 1-form $\omega = Adz + Bd\bar{z}$ we have

$$d^*\left(Adz + Bd\bar{z}\right) = -2\left(\frac{\partial A}{\partial \bar{z}} + \frac{\partial B}{\partial z}\right) \tag{1.2.12}$$

and

$$(\mathbf{ext}_{\theta_{\alpha,\beta}})^*\left(Adz + Bd\bar{z}\right) = -2ik_\beta^\alpha\left(zA - \bar{z}B\right). \tag{1.2.13}$$

Therefore, by taking $\omega = df = \partial f dz + \overline{\partial} f d\bar{z}$ in (1.2.12) one recovers the explicit expression of the Hodge–de Rham operator

$$\nabla^*_{0,0}\nabla_{0,0} = \frac{1}{4}d^*d = -\frac{\partial^2}{\partial z\partial\bar{z}}. \tag{1.2.14}$$

Moreover, the explicit differential expression of the operators $d^*(\mathbf{ext}_{\theta_{\alpha,\beta}})$, $(\mathbf{ext}_{\theta_{\alpha,\beta}})^*d$ and $(\mathbf{ext}_{\theta_{\alpha,\beta}})^*(\mathbf{ext}_{\theta_{\alpha,\beta}})$ are given respectively by

$$d^*(\mathbf{ext}_{\theta_{\alpha,\beta}})f = 2ik^\alpha_\beta(E-\bar{E})f, \tag{1.2.15}$$

$$(\mathbf{ext}_{\theta_{\alpha,\beta}})^*df = -2ik^\alpha_\beta(E-\bar{E})f, \tag{1.2.16}$$

$$(\mathbf{ext}_{\theta_{\alpha,\beta}})^*(\mathbf{ext}_{\theta_{\alpha,\beta}})f = 4|zk^\alpha_\beta|^2 f. \tag{1.2.17}$$

Subsequently, by expanding $\nabla^*_{\alpha,\beta}\nabla_{\alpha,\beta}$ as

$$\nabla^*_{\alpha,\beta}\nabla_{\alpha,\beta} = \frac{1}{4}\left\{d^*d + i(d^*\mathbf{ext}_{\theta_{\alpha,\beta}} - (\mathbf{ext}_{\theta_{\alpha,\beta}})^*d) + (\mathbf{ext}_{\theta_{\alpha,\beta}})^*(\mathbf{ext}_{\theta_{\alpha,\beta}})\right\}, \tag{1.2.18}$$

and next making use of (1.2.14)–(1.2.17), we get its explicit expression given through $\mathcal{D}_{\alpha,\beta}$ in (3.1.2).

□

Remark 1.2.5. *The second order differential operator $\mathcal{D}_{\alpha,\beta}$ in* (3.1.2) *is a magnetic Laplacian with a constant homogeneous magnetic field of magnitude α applied perpendicularly on the complex plane. Indeed, we have*

$$d\theta_{\alpha,\beta} = d\theta_\alpha = 2i\partial\overline{\partial}(Log(\rho_{\alpha,\beta})) = 2i\alpha dz\wedge d\bar{z}.$$

Moreover, the operator $\mathcal{D}_{\alpha,\beta}$ is essentially the classical Landau Hamiltonian in (0.0.7) *perturbed by a first order differential operator associated with the potential* 1*-form $\widetilde{\theta}_\beta$ closed (with zero magnetic field), singular (at the origin) and modeling the Aharonov–Bohm effect.*

The spectral problem for a quantum system on the plane with a uniform magnetic field and an Aharonov-Bohm magnetic flux has already been treated in depth in the literature and has a rather long history. See for example [1, 4, 19, 49, 58, 75, 78, 94, 95] and the references therein. The following result shows that the introduced functions are closely connected to the spectral theory of $\mathcal{D}_{\alpha,\beta}$ in (3.1.2).

Theorem 1.2.6. *The functions $|z|^{2\beta}e^{-\alpha|z|^2}\psi^{2\alpha,2\beta}_{n,m}$ are L^2-eigenfunctions of the magnetic Laplacian $\mathcal{D}_{\alpha,\beta} = \nabla^*_{\alpha,\beta}\nabla_{\alpha,\beta}$ with $\alpha(2n+1)$ as associated eigenvalue.*

Proof. The proof is immediate using Theorem 1.2.1, Lemma 1.2.4 and observing that the operators $\Delta_{2\alpha}^{2\beta}$ in (1.2.1) and the magnetic Laplacian $\mathcal{D}_{\alpha,\beta}$ in (3.1.2) are unitary equivalent. More precisely, we have

$$\rho_{\alpha,\beta}\left(\Delta_{2\alpha}^{2\beta}+\alpha\right)\left((\rho_{\alpha,\beta})^{-1}f\right)=\mathcal{D}_{\alpha,\beta},$$

which readily follows since

$$\mathcal{D}_{\alpha,\beta}=B^{*_{-\alpha,-\beta}}\circ\nabla'_{z,\alpha,\beta}+\alpha=\nabla'_{z,\alpha,\beta}\circ B^{*_{-\alpha,-\beta}}-\alpha.$$

Here $\nabla'_{z,\alpha,\beta}$ is as in (1.2.3) and $B^{*_{\alpha,\beta}}$ is the differential operator given by

$$B^{*_{\alpha,\beta}}f=[\rho_{\alpha,\beta}(z)]^{-1}\frac{\partial}{\partial\overline{z}}\left(\rho_{\alpha,\beta}(z)f\right).$$

□

1.3 Analytical side (polymeromorphy)

In this section, we discuss the regularity of $\psi_{n,m}^{\alpha,\beta}(z,\overline{z})$ as polymeromorphic functions on the complex plane and we determinate the "bi-order" of its unique pole. Recall first from [11, p 199] that a n-meromorphic function (or polymeromorphic of order n) on an open set $U\subset\mathbb{C}$ is a complex-valued function for which there exist some meromorphic functions ψ_k; $k=0,1,\cdots,n-1$ on U such that

$$f(z,\overline{z})=\psi_0(z)+\overline{z}\psi_1(z)+\cdots+\overline{z}^{n-1}\psi_{n-1}(z). \quad (1.3.1)$$

They are called simply n-analytic (polyanalytic of order n) when the component functions are holomorphic in U, $\psi_k\in Hol(U)$. The latter ones can equivalently be defined as those satisfying the generalized Cauchy–Riemann equation $\partial^n/\partial\overline{z}^n=0$. In order to give the exact statement of the main result of this section, we need first to precise the notion of bi-order of a zero or a pole of a given polymeromorphic function on $\mathbb{C}$.

Definition 1.3.1. *For a given non-constant n-analytic function f on an open set $U\subseteq\mathbb{C}$, a point $z_0\in U$ is said to be a zero of bi-order (r,s), for given non-negative integers r,s with $0\leq r\leq n-1$ and $(r,s)\neq(0,0)$, if the following conditions are met*

(a) f can be rewritten as $f=\overline{(z-z_0)}^s g$ for certain $(n-s)$-analytic function

$$g=\sum_{k=0}^{n-s-1}\overline{(z-z_0)}^k\phi_k, \quad (1.3.2)$$

with $\phi_k\in Hol(U)$ and ϕ_0 is not identically zero on U.

(b) z_0 *is a zero of order r for the constant component function* ϕ_0 *in* (1.3.2).

The first condition (a) is to say that z_0 is a zero of order s for f seen as a polynomial in $\overline{z}$. Notice also that the suggested definition is equivalent to have

$$f(z,\overline{z}) = \overline{(z-z_0)}^s \left((z-z_0)^r \varphi_0 + \sum_{k=1}^{n-s-1} \left(\overline{z-z_0}\right)^k \phi_k \right) \tag{1.3.3}$$

for $\varphi_0, \phi_j \in Hol(U)$ with $\varphi(z_0) \neq 0$. Notice here that z_0 does not need to be a zero of the holomorphic components $\phi_k; k = 1, 2, \cdots, n-s-1$. However, for the particular case of z_0 being a common zero of ϕ_k the expression in (1.3.3) reduces to

$$f(z,\overline{z}) = (z-z_0)^r \overline{(z-z_0)}^s g(z,\overline{z}),$$

for certain non-vanishing polyanalytic function g. This makes z_0 a zero of f of bi-order (r,s).

Definition 1.3.2. *A point* $z_0 \in U$ *is said to be a pole of order* r $(r < 0)$ *for given n-polymeromorphic function* f *in* (1.3.1) *if* $(z-z_0)^{|r|} f$ *is a n-analytic function on* U *and* r *is the smallest negative integer satisfying this property.*

This is equivalent to z_0 being a pole for certain component meromorphic function ψ_j with

$$r = min\{Ord_p\left(z_0, \psi_j\right), j = 0, 1, \cdots, n-1\},$$

where $Ord_p\left(z_0, \psi_j\right)$ is exactly the multiplicity of z_0 if it is a pole of ψ_j and 0 otherwise.

Definition 1.3.3. *A pole is said to be of bi-order* (r,s)*, if in addition* (a) *is satisfied.*

Accordingly, we denote by bi-Ord$(z_0; f)$ the bi-order of a point z_0 when it is a zero or a pole of given n-polymeromorphic function f.

Theorem 1.3.4. *The functions* $\psi^{\alpha,\beta}_{n,m}(z,\overline{z})$ *are polymeromorphic on* $\mathbb{C}$*. The origin is either a zero or a pole of bi-order*

$$bi\text{-}Ord(0; \psi^{\alpha,\beta}_{n,m}) = \left(m - [n \wedge^* (\beta+m)], n - [n \wedge^* (\beta+m)]\right). \tag{1.3.4}$$

Proof. First of all, we point out that in view of (1.1.3), it is clear that the terms $\overline{z}^{n-k}$ are always regulars for every $k \leq n \wedge^* (m+\beta) \leq n$. The singularity of $\psi^{\alpha,\beta}_{n,m}(z,\overline{z})$ then lies in z^{m-k} for $k \leq n \wedge^* (m+\beta)$. In particular, the functions $\psi^{\alpha,\beta}_{n,m}(z,\overline{z})$ are polyanalytic (since they are polynomials in

z and $\overline{z}$) if and only if $m \geq n \wedge^* (m+\beta)$. The latter condition is equivalent to β be a non-positive integer or $m \geq n$. In this case, the expression of $\psi^{\alpha,\beta}_{n,m}(z,\overline{z})$ reduces further to

$$\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) = z^{m-[n\wedge^*(\beta+m)]}\overline{z}^{n-[n\wedge^*(\beta+m)]}R^{\alpha,\beta}_{n,m;n\wedge^*(\beta+m)}(z,\overline{z}),$$

where the involved $R^{\alpha,\beta}_{n,m;N}(z,\overline{z})$ are the radial polynomials given by

$$R^{\alpha,\beta}_{n,m;N}(z,\overline{z}) := \sum_{k=0}^{N} c^{\alpha,\beta}_{m,n,k}|z|^{2([n\wedge^*(\beta+m)]-k)}. \tag{1.3.5}$$

Subsequently, since $\beta+m+1 > n\wedge^*(\beta+m)$ and then $c^{\alpha,\beta}_{m,n,n\wedge^*(m+\beta)} \neq 0$, the origin is a zero of $\psi^{\alpha,\beta}_{n,m}(z,\overline{z})$ whenever $\min(m,n) > n\wedge^*(\beta+m)$. Its bi-order is then

$$\text{bi-Ord}(0;\psi^{\alpha,\beta}_{n,m}) = \begin{cases} (m-n,0), & \text{if} \quad \beta \in \mathbb{R}\setminus\mathbb{Z}^- \text{ or } m \geq n, \\ (-\beta, n-m-\beta), & \text{if} \quad \beta \in \mathbb{Z}^-, \beta \neq 0 \text{ and } n > \beta+m. \end{cases}$$

To achieve the proof, it remains sufficient to discuss the case of $m < n\wedge^*(m+\beta)$ (i.e. $n > m > -\beta-1$ and $\beta \notin \mathbb{Z}^-$). In this case we have

$$\begin{aligned} \psi^{\alpha,\beta}_{n,m}(z,\overline{z}) &= \sum_{k=0}^{m} c^{\alpha,\beta}_{m,n,k} z^{m-k}\overline{z}^{n-k} + \sum_{k=m+1}^{n\wedge^*(\beta+m)} c^{\alpha,\beta}_{m,n,k} z^{m-k}\overline{z}^{n-k} \\ &= \overline{z}^{n-m}R^{\alpha,\beta}_{n,m;m}(z,\overline{z}) + \overline{z}^{n-[n\wedge^*(\beta+m)]} \sum_{j=0}^{[n\wedge^*(\beta+m)]-m-1} c^{\alpha,\beta}_{m,n,m+1+j}\frac{\overline{z}^{[n\wedge^*(\beta+m)]-m-1-j}}{z^{j+1}} \\ &= \overline{z}^{n-m}R^{\alpha,\beta}_{n,m;m}(z,\overline{z}) + \frac{\overline{z}^{n-[n\wedge^*(\beta+m)]}}{z^{[n\wedge^*(\beta+m)]-m}} S^{\alpha,\beta}_{n,m;[n\wedge^*(\beta+m)]-m-1}(z,\overline{z}), \end{aligned}$$

where $R^{\alpha,\beta}_{n,m;m}(z,\overline{z})$ is as in (1.3.5) with $N=m$, and

$$S^{\alpha,\beta}_{n,m;N}(z,\overline{z}) = \sum_{j=0}^{[n\wedge^*(\beta+m)]-m-1} c^{\alpha,\beta}_{m,n,m+1+j}|\overline{z}|^{2([n\wedge^*(\beta+m)]-m-j-1)}.$$

It convenient to mention here that both $R^{\alpha,\beta}_{n,m;N}(z,\overline{z})$ and $S^{\alpha,\beta}_{n,m;N}(z,\overline{z})$ are polyanalytic radial polynomials on the whole complex plane for which the origin is not a zero (for again $c^{\alpha,\beta}_{m,n,n\wedge^*(\beta+m)} \neq 0$). This proves that the functions $\psi^{\alpha,\beta}_{n,m}$ are purely polymeromorphic functions with 0 as unique pole if and only if $\beta \notin \mathbb{Z}^-$ and $m < n$. The multiplicity of their unique pole is given by $\text{Ord}(0;\psi^{\alpha,\beta}_{n,m}) = m - [n\wedge^*(\beta+m)] < 0$ and then

$$\text{bi-Ord}(0;\psi^{\alpha,\beta}_{n,m}) = \begin{cases} (m-n,0) & \beta \in \mathbb{R}\setminus\mathbb{Z}, n > m, \\ (m-n,0) & \beta \in \mathbb{Z}^{+*}, \beta+m \geq n > m, \\ (-\beta, n-\beta-m) & \beta \in \mathbb{Z}^{+*}, n \geq \beta+m > m, \\ (-\beta, n-\beta-m) & \beta \in \mathbb{Z}^{-*}, n > m. \end{cases}$$

□

Remark 1.3.5. *The polynomial case, i.e., the restriction to the case of $m \geq n$ (with $\beta \geq 0$) leads to the class of polynomials $Z^{\beta}_{m,n}(z,w)$ introduced and studied by Ismail and Zeng [54, Section 3]. Some of the obtained results in the previous section generalize the one derived in [54, Section 3].*

1.4 Generating functions

Notice first that using the relation of $\psi^{\alpha,\beta}_{n,m}$ to the generalized Laguerre polynomials and the generating function for the latter ones [15, 69], one obtains

$$\frac{(-1)^n}{n!j!} z^{j+n}\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) = \sum_{k=0}^{n} \frac{(-1)^k z^k}{k!(n-k)!(j-(n-k))!}\psi^{\alpha,\beta}_{k,j}(z,\overline{z})$$

for all $j > max(-\beta, n)$. Moreover, by [69, p 242] with $\beta + k > -1$, we have

$$\sum_{n=0}^{+\infty} \frac{t^n \psi^{\alpha,\beta}_{n,n+k}(z,\overline{z})\psi^{\alpha,\beta}_{n,n+k}(w,\overline{w})}{n!(1+\beta+k)_n} = \frac{(wz)^k}{(1-t)^{\beta+k+1}} e^{\frac{\alpha t(|z|^2+|w|^2)}{(1-t)}} {}_0F_1\left(\begin{matrix} - \\ \beta+k+1 \end{matrix} \middle| \frac{|\alpha z w|^2}{t(1-t)^2} \right).$$

The next one is an analog of the standard one for the Itô–Hermite polynomials [34, p. 7], which appears as the special case when $\beta = 0$ and $\alpha = 1$. For its exact statement, we set $\mathbb{C}_\beta := \mathbb{C} \setminus \mathbb{R}_\beta$ and $\mathbb{C}_{\beta,z} := \mathbb{C} \setminus R_{\beta,z}$, where $\mathbb{R}_\beta$ and $R_{\beta,z}$ stand for

$$\mathbb{R}_\beta = \begin{cases} \mathbb{R}^- := \{x \in \mathbb{R}, x \leq 0\} & \beta \text{ non integer}, \\ \{0\} & -\beta = 0,1,2,\cdots, \\ \emptyset & \beta = 1,2,\cdots. \end{cases}$$

and

$$R_{\beta,z} := \begin{cases} \{x + i\Im(z), x \geq \Re(z)\} & \beta \text{ non integer}, \\ \{z\} & -\beta = 1,2,\cdots, \\ \emptyset & \beta = 0,1,2,\cdots. \end{cases}$$

Theorem 1.4.1. *Let $\beta > -1$. Then, for every $z \in \mathbb{C}_\beta$ and $v \in \mathbb{C}_{\beta,z}$, the functions $\psi^{\alpha,\beta}_{n,m}$ satisfy*

$$\sum_{m,n=0}^{+\infty} \frac{u^m v^n}{m!n!}\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) = \left(1 - \frac{v}{z}\right)^{\beta} e^{zu+\alpha v\overline{z}-uv}. \tag{1.4.1}$$

Proof. Starting from the left hand-side of (1.4.1), inserting (0.0.13) and next interchanging the sum in m and the n-th derivative, one obtains

$$\begin{aligned}\sum_{m,n=0}^{+\infty} \frac{u^m v^n}{m!n!}\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) &= z^{-\beta}e^{\alpha|z|^2} \sum_{n=0}^{+\infty} \frac{(-v)^n}{n!} \frac{d^n\left(\varphi_{\overline{z}}(x)\right)}{dx^n}|_{x=z} \\ &= z^{-\beta}e^{\alpha|z|^2}\varphi_{\overline{z}}(z-v),\end{aligned}$$

with $\varphi_{\bar{z}}(x) := z^{\beta}e^{-\alpha(\bar{z}+u)x}$. The last equality follows using the translation operator of the Taylor series of the involved function and gives rise to the right hand-side of (1.4.1). □

The following results are partial generating functions for $\psi^{\alpha,\beta}_{n,m}$ (with fixed n or m).

Proposition 1.4.2. *For $\beta > -1$ and for every $z \in \mathbb{C}_\beta$ and $v \in \mathbb{C}_{\beta,z}$, we have*

$$\sum_{n=0}^{+\infty} \psi^{\alpha,\beta}_{n,k}(z,\bar{z})\frac{v^n}{n!} = \frac{(z-v)^{k+\beta}}{z^{\beta}}e^{\alpha v\bar{z}} \tag{1.4.2}$$

as well as

$$\sum_{m=0}^{+\infty} \frac{(\gamma\bar{z})^m}{m!}\psi^{(\alpha,\beta)}_{n,m}(z,\bar{z}) = \frac{(-1)^n n!}{z^n}e^{\gamma|z|^2}L_n^{(\beta-n)}((\alpha-\gamma)|z|^2), \tag{1.4.3}$$

for $\beta - n > -1$ and $\alpha \geq \gamma$.

Proof. The first assertion can be handled starting from the Rodrigues formula (0.0.13) and next by expanding $e^{(z-v)u}$ in the second right hand-side of (1.4.1). Indeed, the identity (1.4.2) immediately follows from Theorem 1.4.1 by identifying the obtained series in u.

For (1.4.3), we have

$$\begin{aligned}\sum_{m=0}^{+\infty} \frac{(\gamma\bar{z})^m}{m!}\psi^{(\alpha,\beta)}_{n,m}(z,\bar{z}) &= \frac{(-1)^n}{z^{\beta}}e^{\alpha|z|^2}\frac{\partial^n}{\partial z^n}(z^{\beta}e^{(\gamma-\alpha)|z|^2})\\ &= \frac{(-1)^n n!}{z^n}e^{\gamma|z|^2}L_n^{(\beta-n)}((\alpha-\gamma)|z|^2).\end{aligned}$$

The latter expression, given in terms of the generalized Laguerre polynomials, is immediate by making use of the explicit expression of the generalized Laguerre polynomials [69, p.240]

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n}(-1)^m \binom{n+\alpha}{n-m}\frac{x^m}{m!}.$$

□

The exact statement of the next result, concerning a special generating function for the $\psi^{\alpha,\beta}_{n,m}(z,\bar{z})$, makes appeal to the lower incomplete Gamma function defined by

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt, \ \Re(s) > 0. \tag{1.4.4}$$

Its expansion series reads [69, p.337]

$$\gamma(s,x) = e^{-x}\sum_{k=0}^{\infty}\frac{x^{k+s}}{(s)_{k+1}}. \tag{1.4.5}$$

Theorem 1.4.3. *For every $\beta > 0$, $u, z \in \mathbb{C}_\beta$ and $v \in \mathbb{C}_{\beta,z}$ such that $|uv| < |uz|$, we have*

$$\sum_{m,n=0}^{+\infty} \frac{u^m v^n}{(\beta+1)_m n!} \psi^{\alpha,\beta}_{n,m}(z,\bar{z}) = \beta u^{-\beta} z^{-\beta} e^{u(z-v)+\alpha\bar{z}v} \gamma(\beta, u(z-v)). \tag{1.4.6}$$

Proof. The left hand-side in (1.4.6) can be expressed as

$$\sum_{m,n=0}^{+\infty} \frac{u^m v^n}{(\beta+1)_m n!} \psi^{\alpha,\beta}_{n,m}(z,\bar{z}) = \beta z^{-\beta} e^{\alpha|z|^2} \sum_{n=0}^{+\infty} \frac{(-v)^n}{n!} \frac{\partial^n}{\partial z^n} \left(u^{-\beta} e^{-\alpha|z|^2} \left[\sum_{m=0}^{+\infty} \frac{(zu)^{m+\beta}}{(\beta)_{m+1}} \right] \right).$$

This follows making use of (0.0.13) as well as the expansion (1.4.5). Moreover, we get

$$\begin{aligned}
\sum_{m,n=0}^{+\infty} \frac{u^m v^n}{(\beta+1)_m n!} \psi^{\alpha,\beta}_{n,m}(z,\bar{z}) &= \frac{\beta}{z^\beta u^\beta} e^{\alpha|z|^2} \sum_{n=0}^{+\infty} \frac{(-v)^n}{n!} \frac{\partial^n}{\partial z^n} \left(e^{z(u-\alpha\bar{z})} \gamma(\beta, zu) \right) \\
&= \frac{\beta}{z^\beta u^\beta} e^{zu} \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{(-v)^n (-\alpha\bar{z})^{n-k} (-u)^k}{k!(n-k)!} (1-\beta)_k \gamma(\beta-k, zu) \\
&= \frac{\beta}{(zu)^\beta} e^{zu+\alpha\bar{z}v} \sum_{k=0}^{+\infty} \frac{u^k v^k}{k!} (1-\beta)_k \gamma(\beta-k, zu).
\end{aligned}$$

The second equality follows using the Leibniz formula combined with the derivative formula given for the lower incomplete Gamma function in [15, p. 21]. Finally, by means of the series formula in [15, p. 460] we arrive at the expression

$$\sum_{m,n=0}^{+\infty} \frac{u^m v^n}{(\beta+1)_m n!} \psi^{\alpha,\beta}_{n,m}(z,\bar{z}) = \frac{\beta}{(zu)^\beta} \gamma(\beta, (z-u)v) e^{zu+\alpha\bar{z}v-uv},$$

valid for all $z, v \in \mathbb{C}$ such that $|v| < |z|$. □

Corollary 1.4.4. *For every complex numbers u, v and z such that $z \in \mathbb{C}_\beta$, $v \in \mathbb{C}_{\beta,z}$, $|z| > |v|$ and $\Re(u(z-v)) > 0$ with $\beta > 0$,*
we have

$$\sum_{m,n=0}^{+\infty} \frac{u^m v^n}{(\beta+1)_m n!} \psi^{\alpha,\beta}_{n,m}(z,\bar{z}) = \left(1 - \frac{v}{z}\right)^\beta e^{\alpha\bar{z}v} {}_1F_1 \left(\begin{matrix} 1 \\ \beta+1 \end{matrix} \middle| u(z-v) \right).$$

Proof. This can be handled by means of Theorem 1.4.3 and the hypergeometric representation of the lower incomplete Gamma function [69, p.337]

$$\gamma(s,x) = \frac{e^{-x}}{s} x^s {}_1F_1 \left(\begin{matrix} 1 \\ s+1 \end{matrix} \middle| x \right), \ \Re(x) > 0. \tag{1.4.7}$$

□

Remark 1.4.5. *The generating function in* (1.4.6) *can be rewritten in terms of the upper incomplete Gamma function [69, P. 337]*

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt, \tag{1.4.8}$$

since $\gamma(s,x) = \Gamma(s) - \Gamma(s,x)$. *Indeed, for* β *being a positive integer we have*

$$\sum_{m,n=0}^{+\infty} \psi^{\alpha,\beta}_{n,m}(z,\bar{z}) \frac{u^m v^n}{(\beta+1)_m n!} = \frac{\beta\left(\Gamma(\beta) - \Gamma(\beta, u(z-v))\right)}{(zu)^\beta} e^{\alpha\bar{z}v+u(z-v)}. \tag{1.4.9}$$

Remark 1.4.6. *As immediate consequence of Theorem 1.4.3, one can prove that the partial generating function in* (1.4.2) *remains valid for* (v,z) *in a special region of* $\mathbb{C}\times\mathbb{C}$.

1.5 Integral representations

The aim below is to derive some integral representations for the considered polymeromorphic Itô–Hermite functions $\psi^{(\alpha,\beta)}_{n,m}$. The first one involves the Bessel function of order $\nu > -1$ of the first kind defined by [15, p. 675],

$$J_\nu(z) := \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu {}_0F_1\left(\nu+1; -\frac{z^2}{4}\right).$$

More specifically, we assert the following.

Proposition 1.5.1. *For fixed real* β *and integers* n, m *such that* $n = 0, 1, \cdots$ *and* $\beta + m - n > -1$, *we have*

$$\psi^{\alpha,\beta}_{n,m}(z,\bar{z}) = (-1)^n \frac{z^{m-n}e^{\alpha|z|^2}}{(\sqrt{\alpha}|z|)^{\beta+m-n}} \int_0^{+\infty} x^{n+m+\beta+1} J_{\beta+m-n}(2\sqrt{\alpha}|z|x)e^{-x^2}dt. \tag{1.5.1}$$

Proof. Making use of the close connection of $\psi^{\alpha,\beta}_{n,m}(z,\bar{z})$ to the Laguerre polynomials combined with their integral representation in terms of the Bessel function [69, p. 243]

$$L^{(\mu)}_n(x) = \frac{x^{-\mu/2}e^x}{n!}\int_0^{+\infty} e^{-t}t^{n+\frac{\mu}{2}} J_\mu(2\sqrt{tx})dt,$$

valid for $n = 0, 1, 2, \cdots$, and $n + \mu > -1$ with x being a real positive number, the expression (1.1.7) of $\psi^{\alpha,\beta}_{n,m}$ implies

$$\psi^{\alpha,\beta}_{n,m}(z,\bar{z}) = (-1)^n \frac{z^{m-n}e^{\alpha|z|^2}}{(\sqrt{\alpha}|z|)^{\beta+m-n}} \int_0^{+\infty} e^{-t}t^{\frac{n+m+\beta}{2}} J_{\beta+m-n}(2|z|\sqrt{\alpha t})dt, \tag{1.5.2}$$

for every integer m such that $\beta + m > -1$. Finally, the change of variable $t = x^2$ infers the expression in (1.5.1). □

The next integral representations are obtained by means of the generating functions (1.4.1) and (1.4.2).

Proposition 1.5.2. *The integral representation*

$$\psi^{(\alpha,\beta)}_{n,m}(z,\overline{z}) = \frac{1}{\pi^2 z^\beta}\int_{\mathbb{C}^2} u^m v^n \,(z-\overline{v})^\beta\, e^{-|u|^2-|v|^2+\alpha\overline{v}\overline{z}+\overline{u}z-\overline{u}\overline{v}} d\lambda(u,v) \tag{1.5.3}$$

holds for every $\beta > -1$ and $z \in \mathbb{C}_\beta$. Moreover, we have

$$\psi^{\alpha,\beta}_{n,k}(z,\overline{z})\frac{|v|^{2j}}{j!} = \frac{1}{z^\beta}\int_{\mathbb{C}} v^m (z-\overline{v})^{\beta+k} e^{-(v-\alpha\overline{z})\overline{v}} d\lambda(v). \tag{1.5.4}$$

Proof. Thanks to $\psi^{\alpha,\beta}_{m,n}(z,\overline{z}) = \overline{\psi^{\alpha,\beta}_{m,n}(\overline{z},z)}$, we can rewrite the generating function (1.4.1) in the following equivalent form

$$\sum_{m,n=0}^{+\infty} \frac{\overline{u}^m\overline{v}^n}{m!n!}\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) = \left(1-\frac{\overline{v}}{z}\right)^\beta e^{\overline{u}z+\alpha\overline{v}\overline{z}-\overline{u}\overline{v}}, \tag{1.5.5}$$

for any $z \in \mathbb{C}_\beta$ and v outside the zero-Lebesgue measure set $R_{\beta,z}$. Next, by multiplying the both sides by the monomials in u and v and integrating on the whole two-dimensional complex space endowed with the Gaussian measure, it follows

$$\int_{\mathbb{C}^2}\left(1-\frac{\overline{v}}{z}\right)^\beta u^m v^n e^{-|u|^2-|v|^2+\overline{u}z+\alpha\overline{v}\overline{z}-\overline{u}\overline{v}} d\lambda(u,v) = \pi^2\psi^{\alpha,\beta}_{n,m}(z,\overline{z}),$$

which leads to (1.5.3). Analogously, one gets (1.5.4) starting from (1.4.2). □

The generating function in Theorem 1.4.1 can be also employed to establish the following integral representation.

Proposition 1.5.3. *Let β be an integer such that $\beta + m \geq 0$. Then, for every $z \in \mathbb{C}_\beta$, we have*

$$\psi^{\alpha,\beta}_{n,m}(z,\overline{z}) = \frac{(-1)^{m+\beta}\alpha^{n+1}}{\pi z^\beta}\int_{\mathbb{C}} \xi^n\overline{\xi}^{m+\beta} e^{-\alpha(|\xi|^2-|z|^2+\xi z-\overline{\xi}\overline{z})} d\lambda(\xi). \tag{1.5.6}$$

Proof. Note that making use of the $2d$ fractional Fourier transform (1.2) introduced in [12] one obtains the following integral formula

$$e^{\alpha z w} = \frac{\alpha}{\pi}\int_{\mathbb{C}} e^{-\alpha|\xi|^2+\alpha(\xi z+\overline{\xi}w)} d\lambda(\xi) \tag{1.5.7}$$

for every complex numbers z, w and real $\alpha > 0$. It can also be viewed as a reproducing property for the reproducing kernel of the Segal–Bargmann space. Next, by rewriting the generating function in (1.4.2) in the following equivalent form

$$\sum_{n=0}^{+\infty} \psi^{\alpha,\beta}_{n,k-\beta}(z,\overline{z})\frac{v^n}{n!} = \frac{(-1)^k}{\alpha^k z^\beta} e^{\alpha|z|^2}\frac{\partial^k}{\partial\overline{z}^k}\left(e^{\alpha(v-z)\overline{z}}\right) \tag{1.5.8}$$

and making appeal to the formula (1.5.7), it follows

$$\sum_{n=0}^{+\infty} \psi^{\alpha,\beta}_{n,k-\beta}(z,\overline{z})\frac{v^n}{n!} = \frac{(-1)^k}{\alpha^k z^\beta}\frac{\alpha}{\pi}\int_{\mathbb{C}} (\alpha\overline{\xi})^k e^{-\alpha(|\xi|^2-|z|^2)+\alpha(\xi(v-z)+\overline{\xi}\overline{z})} d\lambda(\xi)$$
$$= \sum_{n=0}^{\infty} \frac{v^n}{n!}\left(\frac{(-1)^k}{z^\beta}\frac{\alpha}{\pi}\alpha^n \int_{\mathbb{C}} \xi^n\overline{\xi}^k e^{-\alpha(|\xi|^2-|z|^2+\xi z-\overline{\xi}\overline{z})} d\lambda(\xi)\right).$$

The result in (1.5.6) is then immediate by identification. □

Remark 1.5.4. *The identity* (1.5.6) *can be reproved starting from* (1.1.2) *and making use of the classical integral representation of the Itô–Hermite polynomials in [35].*

1.6 Applications

1.6.1 New integral formula for the generalized Laguerre polynomials

Using the obtained results one can derive new interesting integral formulas for the generalized Laguerre polynomials. Thus, we claim the following.

Theorem 1.6.1. *The integral identity*

$$L^{(\beta-n)}_n((\alpha-\gamma)|z|^2) = \frac{(-1)^n z^{n-\beta}}{n!\pi}\int_{\mathbb{C}} \overline{v}^n\,(z-v)^\beta\, e^{(\alpha-\gamma)v\overline{z}-|v|^2} d\lambda(v) \tag{1.6.1}$$

holds when $\beta > -1$, $\beta - n > -1$, $\alpha \geq \gamma$ *and* $z \in \mathbb{C}_\beta$.

Proof. From (1.4.1), one has

$$\sum_{m=0}^{+\infty} \frac{(\gamma\overline{z}))^m}{m!}\psi^{\alpha,\beta}_{k,m}(z,\overline{z}) = \frac{1}{\pi}\left\langle \sum_{m,n=0}^{+\infty}\frac{(\gamma\overline{z}))^m v^n}{m!n!}\psi^{\alpha,\beta}_{n,m}(z,\overline{z}), v^k\right\rangle_{L^2(\mathbb{C},e^{-|v|^2})}$$
$$= \frac{e^{\gamma|z|^2}}{\pi z^\beta}\int_{\mathbb{C}} \overline{v}^k\,(z-v)^\beta\, e^{(\alpha-\gamma)v\overline{z}} e^{-|v|^2} d\lambda(v).$$

Accordingly, the proof of (1.6.1) readily follows from (1.4.3). □

Remark 1.6.2. *As particular case we get*

$$L^{(\beta-n)}_n(z) = \frac{(-1)^n z^{n-\beta}}{n!\pi}\int_{\mathbb{C}} \overline{v}^n\,(z-v)^\beta\, e^{-v(\overline{v}-1)} d\lambda(v) \tag{1.6.2}$$

for any $z \in \mathbb{C}_\beta$ *and* $\beta - n > -1$ *by specifying* $\alpha = 0$ *and* $u = -1$. *Also, by taking* $\alpha = 1$ *and* $u = 0$, *we get*

$$L^{(\beta-n)}_n(|z|^2) = \frac{(-1)^n z^{n-\beta}}{n!\pi}\int_{\mathbb{C}} \overline{v}^n\,(z-v)^\beta\, e^{v\overline{z}} e^{-|v|^2} d\lambda(v), \quad \beta - n > -1. \tag{1.6.3}$$

1.6.2 Associated integral transforms

The orthogonality property of the functions $\psi^{\alpha,\beta}_{n,m}$ suggests the consideration of two special functional spaces $\mathcal{F}^{2,\alpha}_{\beta,n}(\mathbb{C})$ and $\widetilde{\mathcal{F}}^{2,\alpha}_{\beta,m}(\mathbb{C}_\beta)$ of polymeromorphic and anti-polymeromorphic functions on the punctured complex plane in $L^{2,\alpha}_{\beta}(\mathbb{C})$ for fixed non-negative integers m and n with $m+\beta>-1$. These spaces are spanned by $\psi^{\alpha,\beta}_{n,j}$, $j \geq [-\beta]$, and $\psi^{\alpha,\beta}_{k,m}$, $k=0,1,2,\cdots$, respectively. Moreover, they can be seen as the polymeromorphic analogs of the true polyanalytic (and anti-polyanalytic) Bargmann spaces [2, 96] defined as specific closed subspace in $\ker\left(\partial^{n+1}/\partial\overline{z}^{n+1}\right)\cap L^{2,\alpha}_{g}(\mathbb{C})$, and realized also as L^2-eigenspace $\mathcal{F}^{2,\alpha}_{n}(\mathbb{C})=\ker(\Delta_\alpha-\alpha n)$ associated with the n-th Landau level of the self-adjoint magnetic Laplacian

$$\Delta_\alpha=\Delta_{\alpha,0}=-\frac{\partial^2}{\partial z\partial\overline{z}}+\alpha\overline{z}\frac{\partial}{\partial\overline{z}}$$

acting on $L^{2,\alpha}_{g}(\mathbb{C})$ (see [8, 91]).

Next, we show that $\widetilde{\mathcal{F}}^{2,\alpha}_{\beta,m}(\mathbb{C}_\beta)=\overline{Span\{\psi^{\alpha,\beta}_{k,m},\, k=0,1,2,\cdots\}}^{L^{2,\alpha}_{\beta}(\mathbb{C})}$ can be realized as the image of the classical Segal–Bargmann space $\mathcal{F}^{2,\alpha}(\mathbb{C})=Hol(\mathbb{C})\cap L^{2,\alpha}_{g}(\mathbb{C})$ of holomorphic functions in the Hilbert space of the Gaussian functions, $L^{2,\alpha}_{g}(\mathbb{C}):=L^2(\mathbb{C},e^{-\alpha|\xi|^2}d\lambda)$, by means of the specific integral transform

$$\mathcal{B}^{\alpha,\beta}_{m}f(z):=\frac{\alpha}{\pi}\left(\frac{\alpha^{\beta+m}}{\Gamma(\beta+m+1)}\right)^{1/2}z^m\int_{\mathbb{C}}\left(1-\frac{\overline{w}}{z}\right)^{\beta+m}e^{-\alpha\overline{w}(w-\overline{z})}f(w)d\lambda(w),\qquad(1.6.4)$$

on $\mathbb{C}_\beta$, provided that the integral exists. Here m is a fixed integer such that $m>-\beta-1$ for given $\beta>-1$. In fact, the transform $\mathcal{B}^{\alpha,\beta}_{m}$ is well defined and maps the orthonormal basis $e^{\alpha}_{n}(z)=(\alpha^{n+1}/\pi n!)^{1/2}z^n$ of $\mathcal{F}^{2,\alpha}(\mathbb{C})$ to an orthonormal basis of $\widetilde{\mathcal{F}}^{2,\alpha}_{\beta,m}(\mathbb{C}_\beta)$. More precisely, we have

$$\mathcal{B}^{\alpha,\beta}_{m}(e_n)(z)=\left(\frac{\alpha^{\beta+m+1}}{\pi\alpha^n\Gamma(\beta+m+1)n!}\right)^{1/2}\psi^{\alpha,\beta}_{n,m}(z,\overline{z}).$$

This follows by observing that the integral kernel of the transform $\mathcal{B}^{\alpha,\beta}_{m}$ in (1.6.4) is the generating function in (1.4.2). Its inverse is given by

$$(\mathcal{B}^{\alpha,\beta}_{m})^{-1}(f)(w)=\frac{\alpha}{\pi}\left(\frac{\alpha^{\beta+m}}{\Gamma(\beta+m+1)}\right)^{1/2}\int_{\mathbb{C}}\frac{(\overline{z}-w)^{\beta+m}}{\overline{z}^{\beta}}e^{-\alpha(\overline{z}-w)z}f(z)d\lambda(z),\ w\in\mathbb{C}_\beta.\qquad(1.6.5)$$

It is worth noticing that for the particular case of $\beta=0$, the corresponding transform $\mathcal{B}^{\alpha,0}_{m}$ reduces further to the one considered in [12, Remark 2.13] mapping unitarily the Segal-Bargmann space to the true anti-polyanalytic Bargmann spaces $\widetilde{\mathcal{F}}^{2,\alpha}_{m}(\mathbb{C})$.

Similarly, associated with the kernel function on $\mathbb{C} \times \mathbb{C}$ given through the partial generating function in (1.4.3),

$$s_n^{(\alpha,\beta)}(u,z) := \frac{(-1)^n n!}{z^n} e^{uz} L_n^{(\beta-n)}(\alpha|z|^2 - uz), \quad \beta - n > -1, \tag{1.6.6}$$

we consider the integral transform

$$\mathcal{S}_n^{\alpha,\beta} f(z) := \frac{(-1)^n n!}{z^n} \int_{\mathbb{C}} L_n^{(\beta-n)}(\alpha|z|^2 - uz) e^{-u(\overline{u}-z)} f(u) d\lambda(u), \quad \beta - n > -1. \tag{1.6.7}$$

The image of $\mathcal{F}^{2,\alpha}(\mathbb{C})$ by $\mathcal{S}_n^{\alpha,\beta}$ for arbitrary $\beta > -1$ and $\beta - n > -1$ is the closed subspace of $L^{2,\alpha}_\beta(\mathbb{C})$ spanned by $\psi_{n,j}^{\alpha,\beta}$ for varying $j = 0, 1, 2, \cdots$,

$$\mathcal{S}_n^{\alpha,\beta}(\mathcal{F}^{2,\alpha}(\mathbb{C})) = \overline{Span\{\psi_{n,j}^{\alpha,\beta};\, j = 0, 1, 2, \cdots\}}^{L^{2,\alpha}_\beta(\mathbb{C})} =: \widehat{\mathcal{F}}^{2,\alpha}_{\beta,n}(\mathbb{C}_\beta).$$

For $\beta > 0$, it reduces to the restricted β-polymeromorphic space $\widehat{\mathcal{F}}^{2,\alpha}_{\beta,n}(\mathbb{C}_\beta)$ which is strictly contained in $\mathcal{F}^{2,\alpha}_{\beta,n}(\mathbb{C}) := \overline{Span\{\psi_{n,j}^{\alpha,\beta};\, j = [-\beta], [-\beta]+1, \cdots\}}^{L^{2,\alpha}_\beta(\mathbb{C})}$. For $-1 < \beta \leq 0$, this is exactly the polymeromorphic space $\widehat{\mathcal{F}}^{2,\alpha}_{\beta,n}(\mathbb{C}_\beta) = \mathcal{F}^{2,\alpha}_{\beta,n}(\mathbb{C})$.

Chapter 2

Fractional Hermite functions

Special functions of arbitrary order play a significant role in the theory of fractional differentiation and integration. The present work is devoted to study a new generalization of the Hermite polynomials of real order β obtained by a fractional Rodrigues formula based on the Caputo derivative. The proposed functions are solution of the known Hermite equation. Some properties of the fractional Hermite functions are derived. Also their representation in term of the Kummer's functions is given.

2.1 Notations

We fix and review from [45, 83] some needed notations and concepts in fractional calculus. Thus, for arbitrary fixed positive real number α and for a locally integrable function f on a given segment $[a, b]$ in the real line, the left-sided Riemann–Liouville integral of order α is defined by

$$ {}_L I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt. \tag{2.1.1} $$

Also, for given $f \in \mathcal{C}^{(n-1)}[a, b]$ such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$, we denote by ${}^c D^\alpha(f)$, $n - 1 < \alpha < n$, the left-handed Liouville-Caputo fractional derivative defined by

$$ {}^c D_a^\alpha(f)(x) = {}_L I_a^{n-\alpha} D^n(f) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt, \tag{2.1.2} $$

where $D = d/dx$. Here α is clearly assumed to be a noninteger positive real number. The n-th derivative $D^n(f)$ is then recovered from the Caputo derivative ${}^c D_a^\alpha(f)$ by letting $\alpha \to n$. From now on, we put $a = -\infty$, and the corresponding fractional integral in (2.1.1) and the Caputo fractional derivative in (2.1.2) will be denoted simply I^α and ${}^c D^\alpha$, respectively. Both can be considered as limits when $a \to -\infty$ of those in (2.1.1) and (2.1.2), respectively. By taking $f(t) = \varphi(t) := e^{-t^2}$ in (0.0.14) and (0.0.15), it becomes clear that the functions ${}_L H_{(\beta)}$ and ${}_L G_{(\beta)}$ can be defined by the integrals

$$ {}_L H_{(\beta)}(x) = \frac{e^{x^2}}{\Gamma(n-\beta)} \int_{-\infty}^x (x-t)^{n-\beta-1} (e^{-t^2})^{(n)} dt \tag{2.1.3} $$

with $n - 1 < \beta < n$, and

$$ {}_L G_{(\beta)}(x) = \frac{e^{x^2}}{\Gamma(-\beta)} \int_{-\infty}^x (x-t)^{-\beta-1} e^{-t^2} dt, \tag{2.1.4} $$

for $\beta < 0$. Accordingly, one gets

$$ {}_L H_{(\beta)}(-x) = (-1)^n \frac{e^{x^2}}{\Gamma(n-\beta)} \int_x^{+\infty} (t-x)^{n-\beta-1} (e^{-t^2})^{(n)} dt := (-1)^n {}_R H_{(\beta)}(x), $$

and

$$ {}_L G_{(\beta)}(-x) = \frac{e^{x^2}}{\Gamma(-\beta)} \int_x^{+\infty} (t-x)^{-\beta-1} e^{-t^2} dt := {}_R G_{(\beta)}(x). $$

This means that the suggested Hermite functions are related to the both right-handed Liouville-Caputo fractional derivative and the right-sided Riemann–Liouville integral at $a = +\infty$. It is also not difficult to see that the fractional Hermite function ${}_LH_{(\alpha)}$ can be expressed in terms of Riemann-Liouville derivative defined by ${}^{RL}D_a^{\alpha}f(t) := D^n \circ {}_LI_a^{n-\alpha}(f)$, since we have

$$\left({}^cD_a^{\alpha}f\right)(t) = \left({}^{RL}D_a^{\alpha}f\right)(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha} f^{(k)}(a)}{\Gamma(k-\alpha+1)}, \quad n-1<\alpha<n,$$

for any absolutely continuous function f [63, p. 91], then by specifying $f(t) := e^{-t^2}$, we get

$$\lim_{a\to-\infty} {}^cD_a^{\alpha}(f) = \lim_{a\to-\infty} {}^{RL}D_a^{\alpha}(f) = {}_LH_{(\alpha)}.$$

Thereafter, we use the unified notation $T_{(\beta)}$ to stand for

$$T_{(\beta)} := \begin{cases} {}_LH_{(\beta)}, & \beta>0 \quad (\text{with } n-1<\beta<n), \\ {}_LG_{(\beta)}, & \beta<0. \end{cases} \tag{2.1.5}$$

2.2 Hypergeometric representation for the fractional Hermite functions

In this section, we provide two equivalent explicit expressions of the considered fractional Hermite function in terms of the confluent hypergeometric functions ${}_1F_1$, and next discuss some special cases. For this purpose, let J_α stands for the exponential integrals

$$S_\beta(x) := \int_{-\infty}^{x} (x-t)^{-\beta-1} e^{x^2-t^2} dt \quad \text{and} \quad J_\alpha(x) := \int_0^{+\infty} u^{\alpha} e^{-u^2+2xu} du. \tag{2.2.1}$$

They are somehow equivalent. We first establish the following.

Lemma 2.2.1. *Fix the reals $\beta<0$ and $\beta \neq 0,-1,-2,\cdots$. We have*

$$S_\beta(x) = \Gamma(-\beta)\frac{\Gamma\left(1-\frac{\beta}{2}\right)}{\Gamma(1-\beta)}\, {}_1F_1\left(\frac{-\beta}{2},\frac{1}{2};x^2\right) + \Gamma\left(\frac{1-\beta}{2}\right) x\, {}_1F_1\left(\frac{1-\beta}{2},\frac{3}{2};x^2\right) \tag{2.2.2}$$

and

$$J_\beta(x) = \frac{1}{2}\Gamma\left(\frac{\beta+1}{2}\right) {}_1F_1\left(\frac{\beta+1}{2},\frac{1}{2};x^2\right) + x\Gamma\left(\frac{\beta+2}{2}\right) {}_1F_1\left(\frac{\beta+2}{2},\frac{3}{2};x^2\right). \tag{2.2.3}$$

Proof. Using the variable change $u = x-t$, we get

$$S_\beta(x) = \int_0^{+\infty} u^{-\beta-1} e^{-u^2+2ux} du = \int_0^{+\infty} u^{-\beta-1} e^{-u^2} \left(\sum_{k=0}^{+\infty} \frac{(2ux)^k}{k!}\right) du.$$

However, since

$$\int_0^{+\infty} u^{-\beta-1} e^{-u^2} \frac{(2ux)^k}{k!} du = \frac{2^{k-1}x^k}{k!}\Gamma\left(\frac{k-\beta}{2}\right),$$

and by splitting up the obtained series into the following two separated hypergeometric series

$$\sum_{k=0}^{+\infty}\frac{2^{k-1}x^k}{k!}\Gamma\left(\frac{k-\beta}{2}\right)=\frac{\Gamma\left(\frac{-\beta}{2}\right)}{2}\sum_{k=0}^{+\infty}\frac{x^{2k}}{k!}\frac{\left(\frac{-\beta}{2}\right)_k}{\left(\frac{1}{2}\right)_k}+x\Gamma\left(\frac{1-\beta}{2}\right)\sum_{k=0}^{+\infty}\frac{x^{2k}}{k!}\frac{\left(\frac{1-\beta}{2}\right)_k}{\left(\frac{3}{2}\right)_k},$$

we can interchange the integral and the summation, to obtain

$$S_\alpha(x)=\sum_{k=0}^{+\infty}\frac{2^{k-1}x^k}{k!}\Gamma\left(\frac{k-\beta}{2}\right),$$

which reduces further to the right hand side of (2.2.2). The proof of (2.2.3) can be handled in an analogical way making use of similar arguments as above. □

With the help of the above result, we can prove the following providing a unified hypergeometric representation for the fractional Hermite functions. Namely, we assert the following.

Theorem 2.2.2. *For any noninteger real β, the fractional Hermite function $T_{(\beta)}$ in* (2.1.5) *can be expressed as*

$$T_{(\beta)}(x)=\frac{\Gamma\left(\frac{2-\beta}{2}\right)}{\Gamma(1-\beta)}{}_1F_1\left(-\frac{\beta}{2},\frac{1}{2};x^2\right)+x\frac{\Gamma\left(\frac{1-\beta}{2}\right)}{\Gamma(-\beta)}{}_1F_1\left(\frac{1-\beta}{2},\frac{3}{2};x^2\right). \qquad (2.2.4)$$

Proof. The case $\beta<0$ follows from (2.2.2) in Lemma 2.2.1. The main idea in proving (2.2.4) when $\beta>0$ is to rewrite ${}_LH_{(\beta)}$ as

$${}_LH_{(\beta)}(x)=(-1)^n\frac{1}{\Gamma(n-\beta)}\int_0^{+\infty}u^{n-\beta-1}H_n(x-u)e^{-u^2+2xu}du, \qquad (2.2.5)$$

and next use the explicit expression of the classical real Hermite polynomials to re-express ${}_LH_{(\beta)}$ in terms of the exponential integral J_α in (2.2.1). Namely, we obtain

$${}_LH_{(\beta)}(x)=(-1)^n\frac{n!}{\Gamma(n-\beta)}\sum_{m=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{n-2m}\frac{(-1)^{m+n-s}2^{n-2m}}{m!(n-2m)!}\binom{s}{n-2m}x^sJ_{(2(n-m)-\beta-s-1)}(x).$$

Now, from (2.2.3) combined with the following recurrence relations for the Kummer's functions [69, § 6.2., p. 267]

$$(c-1){}_1F_1(a;c-1;z)+(a+1-c){}_1F_1(a;c;z)-a{}_1F_1(a+1;c;z)=0 \qquad (2.2.6)$$

and

$$c\,{}_1F_1(a;c;z)-c\,{}_1F_1(a-1;c;z)-z\,{}_1F_1(a;c+1;z)=0, \qquad (2.2.7)$$

one reduces the expression of the Hermite functions ${}_LH_{(\beta)}$ in terms of the special functions ${}_1F_1(-\beta/2,1/2;x^2)$ and $x{}_1F_1((1-\beta)/2,3/2;x^2)$. □

Subsequently, for specific values of β, we get special linear combinations of the modified Bessel function of the first kind given by its series expansion [69, p. 66]

$$I_v(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+v+1)k!}\left(\frac{z}{2}\right)^{2k+v}.$$

Corollary 2.2.3. *For the special values $\beta = 1/2$ and $\beta = -1/2$, we have respectively*

$$_LH_{(\frac{1}{2})}(x) = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\sqrt{2}\Gamma\left(\frac{1}{2}\right)} e^{x^2/2} x|x|^{\frac{1}{2}}\left(I_{\frac{-3}{4}}\left(\frac{x^2}{2}\right) + I_{\frac{3}{4}}\left(\frac{x^2}{2}\right) - I_{\frac{-1}{4}}\left(\frac{x^2}{2}\right) - I_{\frac{1}{4}}\left(\frac{x^2}{2}\right)\right), \tag{2.2.8}$$

and

$$_LG_{(\frac{-1}{2})}(x) = \sqrt{2}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{5}{4})}{\Gamma(\frac{1}{2})} e^{x^2/2}|x|^{\frac{1}{2}}\left(I_{\frac{-1}{4}}\left(\frac{x^2}{2}\right) + I_{\frac{1}{4}}\left(\frac{x^2}{2}\right)\right). \tag{2.2.9}$$

Proof. For the case of $\beta = 1/2$, the obtained hypergeometric representation of $_LH_{(\frac{1}{2})}$ in Theorem 2.2.2 reduces further to (2.2.8) making use of the identity [15, p. 622, Eq. (5)] asserting

$$_1F_1(a; 2a+1; x) = 2^{2a-1}\Gamma\left(a+\frac{1}{2}\right) z^{\frac{1}{2}-a} e^{\frac{x}{2}}\left(I_{a-\frac{1}{2}}\left(\frac{x}{2}\right) - I_{a+\frac{1}{2}}\left(\frac{x}{2}\right)\right),\ x > 0.$$

However, the expression (2.2.9) corresponding to $\beta = -1/2$ follows by means of the fact

$$_1F_1(a; 2a; z) = e^{z/2} {}_0F_1\left(; a+\frac{1}{2}; \frac{z^2}{16}\right)$$

combined with

$$_1F_1(a; 2a; z) = 2^{2a-1}\Gamma\left(a+\frac{1}{2}\right) z^{\frac{1}{2}-a} e^{z/2} I_{a-\frac{1}{2}}\left(\frac{z}{2}\right).$$

□

The next result provides another generalized and equivalent expression for the fractional Hermite functions.

Corollary 2.2.4. *For any real $\beta \neq 0, -1, -2, \cdots$, we have*

$$_LH_{(\beta)}(x) = 2\frac{\Gamma(\beta)}{\Gamma(\frac{\beta}{2})}\cos\left(\frac{\pi\beta}{2}\right) {}_1F_1\left(-\frac{\beta}{2}, \frac{1}{2}; x^2\right) - \beta\frac{2^\beta}{\sqrt{\pi}}\Gamma\left(\frac{\beta}{2}\right)\sin\left(\frac{\pi\beta}{2}\right) x\, {}_1F_1\left(\frac{1-\beta}{2}, \frac{3}{2}; x^2\right).$$

Proof. The proof can be handled by rewriting the involved constants in $_LH_{(\beta)}$ giving by (2.2.4) as

$$a^0_\beta := \frac{\Gamma(1-\frac{\beta}{2})}{\Gamma(1-\beta)} = 2\frac{\Gamma(\beta)}{\Gamma(\frac{\beta}{2})}\cos\left(\frac{\pi\beta}{2}\right) \quad \text{and} \quad a^1_\beta := \frac{\Gamma(\frac{1-\beta}{2})}{\Gamma(-\beta)} = -\frac{2^\beta\beta}{\sqrt{\pi}}\Gamma\left(\frac{\beta}{2}\right)\sin\left(\frac{\pi\beta}{2}\right),$$

which readily follows using the following well known identities for the Gamma function [69, p. 2-3]

$$\Gamma(x)\Gamma(-x) = -\frac{\pi}{x\sin(\pi x)} \quad \text{and} \quad \Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{2\pi}2^{\frac{1}{2}-2x}\Gamma(2x).$$

□

Remark 2.2.5. *By specifying $\beta = n$ in the Rodrigues formula of ${}_LH_{(\beta)}$, we easily recover the Hermite polynomials, $H_{(n)} = (-1)^n H_n$. This can also be recovered from the established corollary. In fact, by evaluating at $2n$, the term $\sin(\pi n)$ vanishes and a^0_{2n} reduces further to $a^0_{2n} = (-1)^n(2n)!/n!$. Therefore, one gets*

$$ {}_LH_{(2n)}(x) = (-1)^n \frac{(2n)!}{n!} \, {}_1F_1\left(-n, \frac{1}{2}; x^2\right) = H_{2n}(x). $$

In a similar way we have $a^0_{2n+1} = 0$ when $\beta = 2n+1$, and the remaining coefficient leads to

$$ {}_LH_{(2n+1)}(x) = 2(-1)^{n+1} \frac{(2n+1)!}{n!} x \, {}_1F_1\left(-n, \frac{3}{2}; x^2\right) = -H_{2n+1}(x). $$

As immediate consequence of Theorem 2.2.2, one provides an integral formula for the product of gamma function which for our knowledge seems to be a new result.

Corollary 2.2.6. *For all $\beta \in]n-1, n[$, and n a positive integer, we have*

$$ \Gamma(n-\beta)\Gamma\left(1 - \frac{\beta}{2}\right) = \Gamma(1-\beta) \int_0^{+\infty} u^{n-\beta-1} H_n(u) e^{-u^2} du. $$

Proof. The proof follows from the integral formula (2.2.5) for the case $x = 0$. □

2.3 Fractional Hermite functions as solutions of the β-Hermite equation

In this section, we show that the considered fractional Hermite functions are solutions of the β-Hermite equation in (0.0.16) with fractional order. To this end, we first derive an explicit expansion series for the functions $T_{(\beta)}$.

Lemma 2.3.1. *For every noninteger real β, the functions $T_{(\beta)}$ are analytic and are explicitly given by the absolutely convergent series*

$$ T_{(\beta)}(x) = \frac{1}{2\Gamma(-\beta)} \sum_{j=0}^{+\infty} 2^j \Gamma\left(\frac{j-\beta}{2}\right) \frac{x^j}{j!}. $$

Proof. The proof follows by direct computation starting from Theorem 2.2.2. Indeed, we have

$$\begin{aligned}
T_{(\beta)}(x) &= \frac{\Gamma\left(\frac{2-\beta}{2}\right)}{\Gamma(1-\beta)} {}_1F_1\left(-\frac{\beta}{2}, \frac{1}{2}; x^2\right) + x\frac{\Gamma\left(\frac{1-\beta}{2}\right)}{\Gamma(-\beta)} {}_1F_1\left(\frac{1-\beta}{2}, \frac{3}{2}; x^2\right) \\
&= \frac{\Gamma\left(\frac{2-\beta}{2}\right)}{\Gamma(1-\beta)} \sum_{k=0}^{+\infty} \frac{\left(-\frac{\beta}{2}\right)_k}{\left(\frac{1}{2}\right)_k k!} x^{2k} + \frac{\Gamma\left(\frac{1-\beta}{2}\right)}{\Gamma(-\beta)} \sum_{k=0}^{+\infty} \frac{\left(\frac{1-\beta}{2}\right)_k}{\left(\frac{3}{2}\right)_k k!} x^{2k+1} \\
&= \frac{1}{\Gamma(-\beta)} \sum_{k=0}^{+\infty} \frac{\Gamma\left(\frac{2k-\beta}{2}\right)}{2\left(\frac{1}{2}\right)_k k!} x^{2k} + \frac{1}{\Gamma(-\beta)} \sum_{k=0}^{+\infty} \frac{\Gamma\left(\frac{1+2k-\beta}{2}\right)}{\left(\frac{3}{2}\right)_k k!} x^{2k+1} \\
&= \frac{1}{\Gamma(-\beta)} \sum_{k=0}^{+\infty} \frac{\Gamma\left(\frac{2k-\beta}{2}\right)}{2(2k)!} (2x)^{2k} + \frac{1}{\Gamma(-\beta)} \sum_{k=0}^{+\infty} \frac{\Gamma\left(\frac{1+2k-\beta}{2}\right)}{2(2k+1)!} (2x)^{2k+1} \\
&= \frac{1}{2\Gamma(-\beta)} \sum_{j=0}^{+\infty} \frac{\Gamma\left(\frac{j-\beta}{2}\right)}{j!} (2x)^j.
\end{aligned}$$

The absolute convergence of the occurred series can be proved using the ratio test. In fact, one shows that

$$\frac{\Gamma(\frac{j+1-\beta}{2})j!}{(j+1)!\Gamma(\frac{j-\beta}{2})} \backsim \frac{1}{\sqrt{2j}}$$

vanishes for j being large enough. □

For $\beta = n$ being an integer, it is known that the Hermite polynomial H_n is solution of the classical Hermite equation in [**?**, p. 608], to wit $y'' - 2xy' + 2ny = 0$. However, for arbitrary β, we establish the following.

Theorem 2.3.2. *The fractional Hermite functions $T_{(\beta)}$ are solutions of the generalized β-Hermite equation*

$$y'' - 2xy' + 2\beta y = 0. \tag{2.3.1}$$

Proof. By the expressions defining $T_{(\beta)}$, and the derivatives $2xT'_{(\beta)}$ and $T''_{(\beta)}$ obtained by means of Lemma 2.3.1, it is clear

$$T''_{(\beta)}(x) - 2xT'_{(\beta)}(x) + 2\beta T_{(\beta)}(x) = \frac{1}{2\Gamma(-\beta)} \sum_{j=0}^{+\infty} \frac{2^{j+1}}{j!} s_{\beta,j} x^j$$

for any $\beta \neq 0, 1, 2, \cdots$, where $s_{\beta,j}$ stands for

$$s_{\beta,j} := 2\Gamma\left(\frac{j+2-\beta}{2}\right) - j\Gamma\left(\frac{j-\beta}{2}\right) + \beta\Gamma\left(\frac{j-\beta}{2}\right).$$

for any non-negative integer j. Moreover, we have $s_{\beta,j} = 0$. Thus, it becomes clear that the fractional Hermite functions $T_{(\beta)}$ are solutions of (2.3.1). This completes the proof since for $\beta = n$ being a nonnegative integer, the function $T_{(n)}$ is exactly the Hermite polynomial $(-1)^n H_n$ which is well known to satisfy (2.3.1). □

Proposition 2.3.3. *The fractional Hermite functions $T_{(\beta)}$ satisfy the differential-recurrence formula $T'_{(\beta)} = -2\beta\, T_{(\beta-1)}$ as well as the recurrence relation $T_{(\beta+1)} + 2x\, T_{(\beta)} + 2\beta\, T_{(\beta-1)} = 0$.*

Proof. Follows from Lemma 2.3.1 by straightforward computation. □

Remark 2.3.4. *In view of Remark 2.2.5, the previous result is a generalization of both the classical recurrence formula $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$, $n \geq 2$, and the derivative formula $H'_n(x) = 2nH_{n-1}(x)$ for the Hermite polynomials.*

2.4 Integral representations and generating functions

We begin by proving the following.

Theorem 2.4.1. *We have the integral representations*

$$ {}_L H_{(\beta)}(x) = \frac{2^{\beta+1}}{\sqrt{\pi}} \int_0^{+\infty} e^{x^2-u^2} u^\beta \cos\left(2xu + \frac{\pi\beta}{2}\right) du, \quad \beta \geq 0, \tag{2.4.1}$$

and

$$ {}_L G_{(\beta)}(x) = \frac{1}{\Gamma(-\beta)} \int_0^{+\infty} e^{2xu-u^2} u^{-(\beta+1)} du, \quad \beta < 0. \tag{2.4.2}$$

Proof. From the integral formula in [69, p. 275], and the first transformation formula giving in [69, p. 267], we get

$$ {}_1F_1\left(\frac{-\beta}{2}, \frac{1}{2}; x^2\right) = e^{x^2} {}_1F_1\left(\frac{1+\beta}{2}, \frac{1}{2}; -x^2\right) = \frac{e^{x^2}}{\Gamma(\frac{1+\beta}{2})} \int_0^{+\infty} e^{-t} t^{\frac{\beta-1}{2}} {}_0F_1\left(\frac{1}{2}; -tx^2\right) dt,$$

and

$$ {}_1F_1\left(\frac{1-\beta}{2}, \frac{3}{2}; x^2\right) = e^{x^2} {}_1F_1\left(1+\frac{\beta}{2}, \frac{3}{2}; -x^2\right) = \frac{e^{x^2}}{\Gamma(\frac{2+\beta}{2})} \int_0^{+\infty} e^{-t} t^{\frac{\beta}{2}} {}_0F_1\left(\frac{3}{2}; -tx^2\right) dt$$

for $\beta \geq 0$. However, making use of the facts ${}_0F_1\left(1/2; -z\right) = \cos\left(2\sqrt{z}\right)$ and $2\sqrt{z}{}_0F_1(3/2; -z) = \sin(2\sqrt{z})$, it follows

$$\begin{aligned} {}_L H_{(\beta)}(x) = {} & \frac{2^\beta}{\sqrt{\pi}} \cos\left(\frac{\pi\beta}{2}\right) \int_0^{+\infty} e^{x^2-t} t^{\frac{\beta-1}{2}} \cos(2|x|\sqrt{t}) dt \\ & - sign(x) \frac{2^\beta}{\sqrt{\pi}} \sin\left(\frac{\pi\beta}{2}\right) \int_0^{+\infty} e^{x^2-t} t^{\frac{\beta-1}{2}} \sin(2|x|\sqrt{t}) dt. \end{aligned}$$

The variable change $u = \sqrt{t}$ completes the proof of (2.4.1). The second integral (2.4.2) can be deduced from the definition given through (2.1.4) making use of the variable change $u = x - t$.

□

The main results of this section are fractional generalizations of the generating function for Hermite polynomials [7, p. 817]

$$e^{-z^2+2zx} = \sum_{n=0}^{\infty} H_n(x)\frac{z^n}{n!}. \tag{2.4.3}$$

Theorem 2.4.2. *For positive noninteger real β, let n be the integer satisfying $n-1<\beta<n$. Then, for given reals x and z we have*

$$\sum_{k+n=0}^{\infty} \frac{{}_LH_{(\beta+k+1)}(x)}{(n+k)!}z^{n+k} = \frac{e^{-z^2-2xz}}{\Gamma(n-\beta)}\left\{\Gamma\left(\frac{n-\beta+1}{2}\right){}_1F_1\left(\frac{n-\beta-1}{2},\frac{1}{2};(x+z)^2\right)\right.$$
$$\left. + \left(\frac{n-\beta-1}{2}\right)\Gamma\left(\frac{n-\beta}{2}\right)(x+z){}_1F_1\left(\frac{n-\beta}{2},\frac{3}{2};(x+z)^2\right)\right\}.$$

Proof. The integral expression in (2.1.3) defining ${}_LH_{(\beta)}$ and the ordinary generating function (2.4.3) show that the series

$$S(x,z) = \sum_{k+n=0}^{\infty} \frac{{}_LH_{(\beta+k+1)}(x)}{(n+k)!}z^{n+k}$$

can be re-expressed in terms of the quantities J_α in (2.2.1) as

$$\begin{aligned} S(x,z) &= -2\frac{e^{-z^2-2xz}}{\Gamma(n-\beta)}\int_0^{+\infty}(z+x-u)(u)^{n-\beta-1}e^{-u^2+2u(z+x)}du \\ &= -2\frac{e^{-z^2-2xz}}{\Gamma(n-\beta)}\left((z+x)J_{n-\beta-1}(z+x) - J_{n-\beta}(z+x)\right). \end{aligned}$$

The last equality follows from (2.2.3). The assertion of the theorem then holds by means of (2.2.6) and (2.2.7). □

Remark 2.4.3. *It is worth noting that when β is an integer we recover the generating function in [69, p. 252]. The particular case of $\beta = 0$ (and then $n = 1$ and $\beta_n = 0$) reduces further to the generating function in [7, p. 817] given by* (2.4.3).

Theorem 2.4.4. *We have the following*

$$\sum_{n\geq 1}^{\infty} {}_LG_{(-n)}(x)z^{n-1} = \frac{\sqrt{\pi}}{2}\left(1+\mathrm{Erf}(x+\frac{z}{2})\right)e^{(x+\frac{z}{2})^2}, \tag{2.4.4}$$

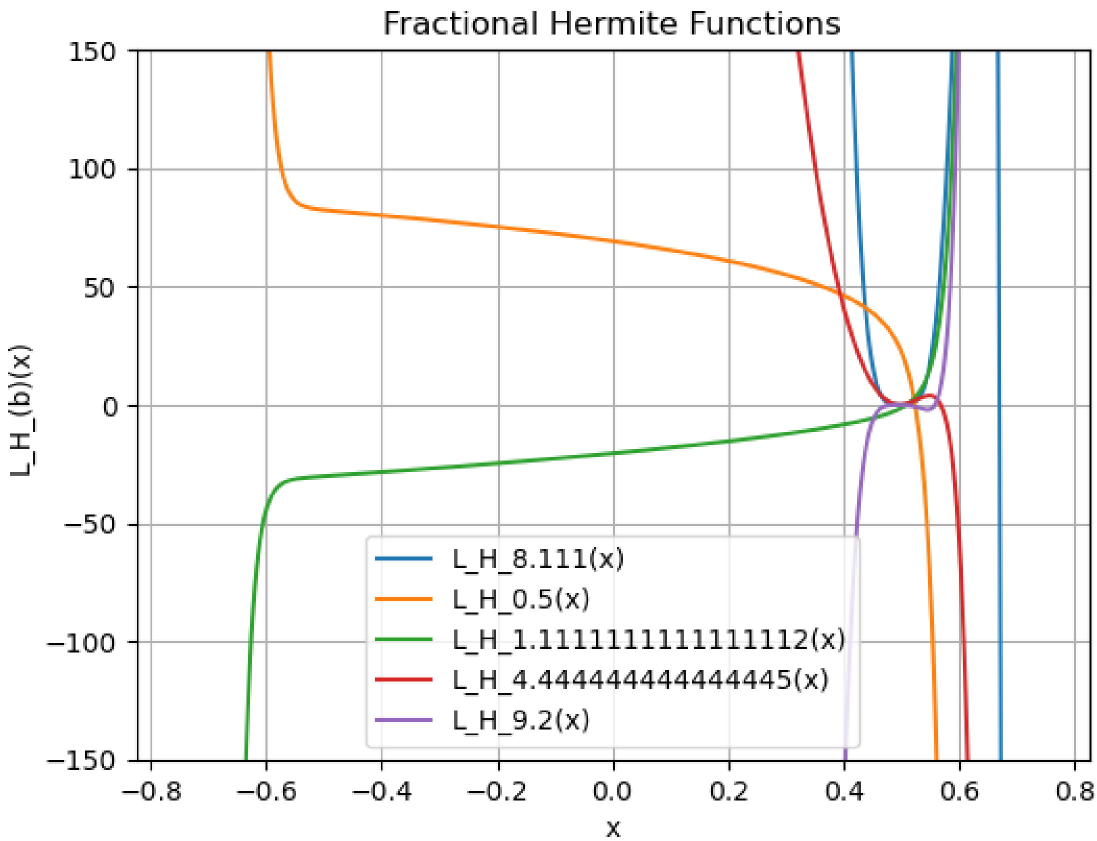
where $\mathrm{Erf}(x)$ *is the error function [69, p. 349] defined by*

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

Proof. It follows from formula (2.2.3) combined with the obtained integral representation (2.4.2). □

2.5 Discussion

A notable point is that well-known formulas used to derive orthogonal Hermite polynomials can be extended for the generalization of fractional derivatives. An illustration of this is found in the integral representation, differential equation and recurrence formulas of Hermite polynomials.



The explicit expression of fractional Hermite functions assists in plotting their truncated forms. Nonetheless, considering the large function values, the plot is confined within the specified interval $[-0.8, 0.8]$.

Nonetheless, the suggested functions lack the polynomial characteristic, presenting difficulties in expressing them in relation to other special functions, or in studying their orthogonality. In this setting, it is worth noting that the Hermite polynomials satisfies the orthogonality relation over the real line with respect to the gaussian weight $w(x) = e^{-x^2}$, which explicitly reads

$$\int_{-\infty}^{+\infty} H_m(x)H_n(x)e^{-x^2}dx = 2^n n!\sqrt{\pi}\delta_{m,n}.$$

However, the orthogonality of functions of fractional order is still an open problem and needs further investigation, including the one for our introduced fractional Hermite functions $T_{(\beta)}$ (a problem under consideration). Notice for instance that the orthogonality relation is not valid with the weight function e^{-x^2}. In fact, we have

$$\begin{aligned}\left\langle T_{(\beta)}, T_{(\alpha)} \right\rangle_g &:= \int_{-\infty}^{\infty} e^{-x^2} T_{(\beta)}(x)T_{(\alpha)}(x)dx \\ &= [Z_{(\beta)}Z'_{(\alpha)} - Z_{(\alpha)}Z'_{(\beta)}]_{-\infty}^{+\infty} \\ &= [2\beta Z_{(\beta-1)}Z_{(\alpha)} - 2\alpha Z_{(\alpha-1)}Z_{(\beta)}]_{-\infty}^{+\infty},\end{aligned}$$

for given reals β and α (not both positive integers), where we have set $Z_{(\beta)} = e^{\frac{-x^2}{2}} T_{(\beta)}$. Therefore, $\left\langle T_{(\beta)}, T_{(\alpha)} \right\rangle_g = +\infty$. Indeed, by means of [69, p.289] it follows that

$$T_{(\beta)} = \frac{2\Gamma(\frac{3}{2})}{\Gamma(-\beta)}|x|^{-1-\beta}e^{x^2}\left[1 + O\left(\frac{1}{x^2}\right)\right],$$

for x large enough ($x \to +\infty$), and

$$2\beta Z_{(\beta-1)}Z_{(\alpha)} - 2\alpha Z_{(\alpha-1)}Z_{(\beta)} = \frac{8\left(\Gamma(\frac{3}{2})\right)^2}{\Gamma(-\beta)\Gamma(-\alpha)}|x|^{-1-\alpha-\beta}e^{x^2}\left[O\left(\frac{1}{x^2}\right) + O\left(\frac{1}{x^4}\right)\right].$$

Moreover, $x \to -\infty$, we have

$$T_{(\beta)} = \frac{\Gamma(\frac{3}{2})}{\Gamma(-\beta)}|x|^{-1-\beta}e^{x^2}O\left(\frac{1}{x^2}\right),$$

and then

$$2\beta Z_{(\beta-1)}Z_{(\alpha)} - 2\alpha Z_{(\alpha-1)}Z_{(\beta)} = \frac{2\left(\Gamma(\frac{3}{2})\right)^2}{\Gamma(-\beta)\Gamma(-\alpha)}|x|^{-1-\alpha-\beta}e^{x^2}O\left(\frac{1}{x^4}\right).$$

Chapter 3

Fractional Zernike functions

We consider and provide an accurate study for the fractional Zernike functions on the punctured unit disc, generalizing the classical Zernike polynomials and their associated β-restricted Zernike functions. Mainly, we give the spectral realization of the latter ones and show that they are orthogonal L^2-eigenfunctions for certain perturbed magnetic (hyperbolic) Laplacian. The algebraic and analytic properties for the fractional Zernike functions to be established include the connection to special functions, their zeros, their orthogonality property, as well as the differential equations, recurrence and operational formulas they satisfy. Integral representations are also obtained. Their regularity as poly-meromorphic functions is discussed and their generating functions including a bilinear one of Hardy–Hille type are derived. Moreover, we prove that a truncated subclass defines a complete orthogonal system in the underlying Hilbert space giving rise to a specific Hilbertian orthogonal decomposition in terms of a class of generalized Bergman spaces.

3.1 The β-restricted Zernike functions (spectral realization)

In this section we are concerned with the functions

$$\psi^{\gamma,\eta}_{m,n}(z,\overline{z}) = (-1)^m z^{\eta-\beta}\overline{z}^{-\eta}(1-|z|^2)^{\gamma-\alpha+m}\frac{\partial^m}{\partial z^m}\left(z^{n+\beta-2\eta}(1-|z|^2)^{\alpha-2\gamma-m-1}\right), \tag{3.1.1}$$

for given reals $\alpha, \beta, \gamma, \eta$. They referred to as the β-restricted Zernike functions (justified by Remark 3.1.7 below). We aim to derive their basic properties and show that they form an orthogonal system of $L^{2,\alpha}_{\beta}$-eigenfunctions for a perturbed magnetic Laplacian of the form

$$\Delta^{c,d}_{a,b} = \Delta_{hyp} + (1-|z|^2)\left(H^b_a(z)E - H^d_c(z)\overline{E}\right) + H^b_a(z)H^d_c(z)|z|^2 \tag{3.1.2}$$

acting on the weighted Hilbert space $L^{2,\alpha}_{\beta}(\mathbb{D})$. Above a, b, c and d are given real numbers, $\Delta_{hyp} = -(1-|z|^2)^2\partial^2/\partial z\partial\bar{z}$ is the Laplace–Beltrami operator on the hyperbolic disc, $\overline{E} = \bar{z}\partial/\partial\bar{z}$ denotes the complex conjugate of the complex Euler operator $E := z\partial/\partial z$ and

$$H^b_a(z) := a + b - \frac{b}{|z|^2}. \tag{3.1.3}$$

It is worth noting that for particular values of a, b, c, d one recovers the magnetic Schrödinger operator on the hyperbolic unit disc, representing the Hamiltonian of a charged particle in motion under an external uniform magnetic field [18, 14, 36, 39].

To this end, we have to factorize the considered Laplacian in terms of some first order differential operators (leading in particular to their Rodrigues type formula). Thus, if we set $h_{\alpha,\beta}(z) = h(z)^\alpha |z|^{2\beta}$ with $h(z) = 1-|z|^2$, we can consider the first order differential operator

$$A_{\gamma,\eta} f(z) := h_{1-\gamma,-\eta}(z)\frac{\partial}{\partial\bar{z}}(h_{\gamma,\eta} f)(z)$$

for given fixed reals γ and η. Its explicit expression is given by

$$A_{\gamma,\eta} f(z) = \left\{(1-|z|^2)\frac{\partial}{\partial\bar{z}} - H^\eta_\gamma(z)\right\} f(z). \tag{3.1.4}$$

The corresponding null space is closely connected to the set $\mathrm{Hol}(\mathbb{D}^*)$ of holomorphic functions on the punctured unit disc. Namely, we have $\ker(A_{\gamma,\eta}) = h_{-\gamma,-\eta}\mathrm{Hol}(\mathbb{D}^*)$. Moreover, the formal adjoint operator $A^{*_{\alpha,\beta}}_{\gamma,\eta}$ of $A_{\gamma,\eta}$ with respect to the inner scalar product

$$\langle f, g\rangle_{\alpha,\beta} := \int_{\mathbb{D}} f(z)\overline{g(z)} d\mu_{\alpha,\beta}(z) \tag{3.1.5}$$

in $L^{2,\alpha}_{\beta}(\mathbb{D})$ is given by

$$A^{*_{\alpha,\beta}}_{\gamma,\eta} f(z) := -h_{\gamma-\alpha,\eta-\beta}(z)\frac{\partial}{\partial z}(h_{\alpha-\gamma+1,\beta-\eta} f)(z) \tag{3.1.6}$$

in account of the conventional calculation. Accordingly, we perform

$$\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta} = A_{\gamma,\eta} A^{*_{\alpha,\beta}}_{\gamma,\eta} \quad \textit{and} \quad \mathcal{L}^{\alpha,\beta,-}_{\gamma,\eta} = A^{*_{\alpha,\beta}}_{\gamma,\eta} A_{\gamma,\eta}. \tag{3.1.7}$$

Straightforward computation leads to the explicit expression of these second order differential operators in terms of $\Delta^{c,d}_{a,b}$ in (3.1.2) (we omit the proof).

Lemma 3.1.1. *The expression of $\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta}$ in the z-coordinate is given by*

$$\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta} = \Delta^{\gamma-\alpha-1,\eta-\beta}_{\gamma+1,\eta} + (\alpha-\gamma+1).$$

Moreover, the operators $\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta}$ and $\mathcal{L}^{\alpha,\beta,-}_{\gamma,\eta}$ satisfy

$$\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta} = \mathcal{L}^{\alpha,\beta,-}_{\gamma+1,\eta} + (\alpha-2\gamma). \tag{3.1.8}$$

Remark 3.1.2. *For $\alpha = -2$ and $\beta = 0$ we have $\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta} = \mathcal{L}^{\alpha,\beta,-}_{\gamma+1,\eta} - 2(\gamma+1)$. Also $H^{\beta-\eta}_{\alpha-\gamma+1} = -H^{\eta}_{\gamma+1}$ so that the Laplacian $\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta}$ reduces further to*

$$\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta} = -h\left\{h\frac{\partial^2}{\partial z\partial\bar{z}} - H^{\eta}_{\gamma+1}(z)\left(E-\overline{E}\right)\right\} + (H^{\eta}_{\gamma+1}(z))^2|z|^2 - (\gamma+1). \tag{3.1.9}$$

For the particular cases of γ,η we recover the Landau-like Hamiltonian on $\mathbb{D}$ (see e.g. [14, 36, 39]).

Remark 3.1.3. *The considered operators $\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta}$ and $\mathcal{L}^{\alpha,\beta,-}_{\gamma,\eta}$ can be realized geometrically as magnetic Schrödinger operators associated with a singular real differential 1-form (vector potential) $\theta_{\alpha,\beta} = \theta_\alpha + \widetilde{\theta_\beta}$ with $d\widetilde{\theta_\beta} = 0$ and $d\theta_\alpha$ is the Khäler two form on the hyperbolic unit disc up to a multiplicative constant. More precisely, we have*

$$\theta_{\alpha,\beta}(z) = \frac{i\alpha\left(\bar{z}dz - zd\bar{z}\right)}{1-|z|^2} - i\beta\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right). \tag{3.1.10}$$

Now, by means of the identity (3.1.8) we can establish the following (we omit the proof).

Lemma 3.1.4. *The following commutation rules hold trues*

$$(i) \qquad \mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta}A_{\gamma,\eta} = A_{\gamma,\eta}\mathcal{L}^{\alpha,\beta,-}_{\gamma,\eta} \quad \text{and} \quad A^{*_{\alpha,\beta}}_{\gamma,\eta}\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta} = \mathcal{L}^{\alpha,\beta,-}_{\gamma,\eta}A^{*_{\alpha,\beta}}_{\gamma,\eta}.$$

$$(ii) \qquad \mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta}A^{*_{\alpha,\beta}}_{\gamma+1,\eta} = A^{*_{\alpha,\beta}}_{\gamma+1,\eta}\left(\mathcal{L}^{\alpha,\beta,+}_{\gamma+1,\eta} + (\alpha-2\gamma)\right).$$

$$(iii) \qquad A_{\gamma+1,\eta}\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta} = \left(\mathcal{L}^{\alpha,\beta,+}_{\gamma+1,\eta} + (\alpha-2\gamma)\right)A_{\gamma+1,\eta}.$$

$$(iv) \qquad \mathcal{L}^{\alpha,\beta,-}_{\gamma+1,\eta}A_{\gamma,\eta} = A_{\gamma,\eta}\left(\mathcal{L}^{\alpha,\beta,-}_{\gamma,\eta} - (\alpha-2\gamma)\right).$$

$$(v) \qquad A^{*_{\alpha,\beta}}_{\gamma,\eta}\mathcal{L}^{\alpha,\beta,-}_{\gamma+1,\eta} = \left(\mathcal{L}^{\alpha,\beta,-}_{\gamma,\eta} - (\alpha-2\gamma)\right)A^{*_{\alpha,\beta}}_{\gamma,\eta}.$$

Lemma 3.1.4 is efficient in analyzing and studying the family of functions $\psi^{\gamma,\eta}_{m,n}$ in (3.1.1). In fact, we show that they can be obtained by successive application of $A^{*}_{\gamma+j,\eta}$; $j = 1,2,\cdots,m$, to the ground state functions. Namely, we consider the differential operator

$$A^{*,m}_{\gamma,\eta}(f) := A^{*_{\alpha,\beta}}_{\gamma+1,\eta}\circ A^{*_{\alpha,\beta}}_{\gamma+2,\eta}\circ\cdots\circ A^{*_{\alpha,\beta}}_{\gamma+m,\eta}(f).$$

Then, we claim the following.

Lemma 3.1.5. *The closed expression of Rodrigues type for $A^{*,m}_{\gamma,\eta}$ is given by*

$$A^{*,m}_{\gamma,\eta}(f) = (-1)^m h_{\gamma-\alpha+m,\eta-\beta}\frac{\partial^m}{\partial z^m}(h_{\alpha-\gamma,\beta-\eta}f). \tag{3.1.11}$$

Proof. Starting from the definition of $A^{*,m}_{\gamma,\eta}$ one gets

$$A^{*,m}_{\gamma,\eta}(f) = (-1)^m h_{\gamma-\alpha-1,\eta-\beta}\left(h^2\frac{\partial}{\partial z}\right)^m(h_{\alpha-\gamma-m+1,\beta-\eta}f).$$

Then (3.1.11) readily follows thanks to the fact $(h^2\partial)^m(f) = h^{m+1}\partial^m(h^{m-1}f)$ in [33]. □

The main result in this section is the following.

Theorem 3.1.6. *Fix γ such that $\alpha > 2\gamma + 1$. Then, for integers m, n such that $n > 2\eta - \beta - 1$ and $0 \leq m < (\alpha - 1 - 2\gamma)/2$, the following assertions hold.*

(i) The function $\psi^{\gamma,\eta}_{m,n}$ is a $L^{2,\alpha}_{\beta}$-eigenfunction of $\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta}$ with $E^{\gamma,\alpha}_m = (m+1)(\alpha - 2\gamma - m)$ as corresponding eigenvalue.

(ii) The functions $\psi^{\gamma,\eta}_{m,n}$ form an orthogonal system in the Hilbert space $L^{2,\alpha}_{\beta}(\mathbb{D})$ and their square norm (induced from (3.1.5)*) is given by*

$$\|\psi^{\gamma,\eta}_{m,n}\|^2_{\alpha,\beta} = \frac{\pi m!}{(\alpha - 2(\gamma+m) - 1)} \frac{\Gamma(\alpha - 2\gamma - m)\Gamma(n + \beta - 2\eta + 1)}{\Gamma(n + \alpha + \beta - 2(\gamma + \eta + m))}. \tag{3.1.12}$$

Here Γ is the Gamma Euler function.

Proof. In virtue of the algebraic identity (iii) in Lemma 3.1.4 and the identity (3.1.8) we can proceed by mathematical induction to get

$$\begin{aligned}
\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta} A^{*,m}_{\gamma,\eta} &= A^{*,m}_{\gamma,\eta} \mathcal{L}^{\alpha,\beta,+}_{\gamma+m,\eta} + \sum_{j=0}^{m-1} (\alpha - 2(\gamma + j)) A^{*,m}_{\gamma,\eta} \\
&= A^{*,m}_{\gamma,\eta} \left(\mathcal{L}^{\alpha,\beta,-}_{\gamma+m+1,\eta} + (\alpha - 2(\gamma+m)) \right) + \sum_{j=0}^{m-1} (\alpha - 2(\gamma + j)) A^{*,m}_{\gamma,\eta} \\
&= A^{*,m}_{\gamma,\eta} \mathcal{L}^{\alpha,\beta,-}_{\gamma+m+1,\eta} + \sum_{j=0}^{m} (\alpha - 2(\gamma + j)) A^{*,m}_{\gamma,\eta} \\
&= A^{*,m}_{\gamma,\eta} \mathcal{L}^{\alpha,\beta,-}_{\gamma+m+1,\eta} + (m+1)(\alpha - 2\gamma - m) A^{*,m}_{\gamma,\eta}.
\end{aligned}$$

Accordingly, it becomes clear that the functions $A^{*,m}_{\gamma,\eta}(\varphi^{\gamma,\eta}_m)$ are eigenfunctions of $\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta}$ whenever $\varphi^{\gamma,\eta}_m$ belongs to the null space of $A_{\gamma+m+1,\eta}$,

$$\ker(A_{\gamma+m+1,\eta}) = \{f : \mathbb{D}^* \longrightarrow \mathbb{C};\ A_{\gamma+m+1,\eta} f = 0\} \subseteq \ker\left(\mathcal{L}^{-}_{\gamma+m+1,\eta}\right).$$

This is the case when considering

$$\varphi^{\gamma,\eta}_m(z) = \varphi^{\gamma,\eta}_{m,n}(z) := z^n (1 - |z|^2)^{-(\gamma+m+1)} |z|^{-2\eta};\ \ n \in \mathbb{Z}. \tag{3.1.13}$$

More precisely, the functions $A^{*,m}_{\gamma,\eta}(\varphi^{\gamma,\eta}_m) = A^{*,m}_{\gamma,\eta}(z^n h_{-(\gamma+m+1),-\eta})$ are given by

$$\begin{aligned}
A^{*,m}_{\gamma,\eta}(z^n h_{-(\gamma+m+1),-\eta}) &= (-1)^m h_{\gamma-\alpha+m,\eta-\beta} \frac{\partial^m}{\partial z^m} (z^n h_{\alpha-2\gamma-m-1,\beta-2\eta}) \qquad (3.1.14) \\
&= (-1)^m (1 - |z|^2)^{\gamma-\alpha+m} |z|^{2(\eta-\beta)} \frac{\partial^m}{\partial z^m} \left(z^n |z|^{2(\beta-2\eta)} (1 - |z|^2)^{\alpha-2(\gamma+m)+m-1} \right)
\end{aligned}$$

thanks to Lemma 3.1.5. The latter formula reduces further to the expression of the β-restricted Zernike functions in (3.1.1). Moreover, they satisfy

$$\mathcal{L}^{\alpha,\beta,+}_{\gamma,\eta}\left(\psi^{\gamma,\eta}_{m,n}\right) = (m+1)(\alpha-2\gamma-m)\psi^{\gamma,n}_{m,n} = E^{\gamma,\alpha}_{m}\psi^{\gamma,\eta}_{m,n}. \tag{3.1.15}$$

Now, for their orthogonality in $L^{2,\alpha}_{\beta}(\mathbb{D})$ one can use their explicit expressions in terms of certain special functions (see for example Remark 3.2.7 below). However, we present below another proof using the factorization method. To this purpose, notice first that $A^{*,m}_{\gamma,\eta} = A^{*_{\alpha,\beta}}_{\gamma+1,\eta}\circ A^{*,m-1}_{\gamma+1,\eta}$ and that $\varphi^{\gamma,\eta}_{m,n} = \varphi^{\gamma+1,\eta}_{m-1,n}$. It follows

$$\psi^{\gamma,\eta}_{m,n} = A^{*,m}_{\gamma,\eta}(\varphi^{\gamma,\eta}_{m,n}) = A^{*_{\alpha,\beta}}_{\gamma+1,\eta}\circ A^{*,m-1}_{\gamma+1,\eta}(\varphi^{\gamma,\eta}_{m,n}) = A^{*_{\alpha,\beta}}_{\gamma+1,\eta}(\psi^{\gamma+1,\eta}_{m-1,n}).$$

Accordingly, making use of (3.1.15) we obtain

$$\left\langle \psi^{\gamma,\eta}_{m,n}, \psi^{\gamma,\eta}_{j,k}\right\rangle = \left\langle \mathcal{L}^{\alpha,\beta,+}_{\gamma+1,\eta}(\psi^{\gamma+1,\eta}_{m-1,n}), \psi^{\gamma+1,\eta}_{j-1,k}\right\rangle = E^{\gamma+1,\alpha}_{m-1}\left\langle \psi^{\gamma+1,\eta}_{m-1,n}, \psi^{\gamma+1,\eta}_{j-1,k}\right\rangle.$$

More generally, by induction we arrive at

$$\left\langle \psi^{\gamma,\eta}_{m,n}, \psi^{\gamma,\eta}_{j,k}\right\rangle = \prod_{\ell=1}^{s} E^{\gamma+\ell,\alpha}_{m-\ell}\left\langle \psi^{\gamma+s,\eta}_{m-s,n}, \psi^{\gamma+s,\eta}_{j-s,k}\right\rangle;\ 1\leq s\leq m.$$

Therefore, without lost of generality we can assume that $m\leq j$ and take $s=m$ to get

$$\begin{aligned}\left\langle \psi^{\gamma,\eta}_{m,n}, \psi^{\gamma,\eta}_{j,k}\right\rangle &= \prod_{\ell=1}^{m} E^{\gamma+\ell,\alpha}_{m-\ell}\left\langle \psi^{\gamma+m,\eta}_{0,n}, \psi^{\gamma+m,\eta}_{j-m,k}\right\rangle\\ &= \prod_{\ell=1}^{m} E^{\gamma+\ell,\alpha}_{m-\ell}\left\langle \psi^{\gamma+m,\eta}_{0,n}, A^{*}_{\gamma+m+1,\eta}\circ A^{*}_{\gamma+m+2,\eta}\circ\cdots\circ A^{*}_{\gamma+j,\eta}(\varphi^{\gamma+m,\eta}_{j-m,k})\right\rangle\\ &= \prod_{\ell=1}^{m} E^{\gamma+\ell,\alpha}_{m-\ell}\left\langle A_{\gamma+j,\eta}\circ\cdots\circ A_{\gamma+m+1,\eta}(\varphi^{\gamma+m,\eta}_{0,n}), \varphi^{\gamma+m,\eta}_{j-m,k}\right\rangle.\end{aligned}$$

The last identity holds by observing that $\psi^{\gamma+s,\eta}_{0,n} = \varphi^{\gamma+s,\eta}_{0,n}$, which readily follows from (3.1.11) or (3.1.14). Next, since $\varphi^{\gamma+m,\eta}_{0,n}$ belongs to $\ker(A_{\gamma+m+1,\eta})$ and then $A_{\gamma+j,\eta}\circ\cdots\circ A_{\gamma+m+2,\eta}\circ A_{\gamma+m+1,\eta}(\varphi^{\gamma+m,\eta}_{0,n})$ vanishes whenever $m<j$, we obtain

$$\left\langle \psi^{\gamma,\eta}_{m,n}, \psi^{\gamma,\eta}_{j,k}\right\rangle = \left(\prod_{\ell=1}^{m} E^{\gamma+\ell,\alpha}_{m-\ell}\right)\left\langle \varphi^{\gamma+m,\eta}_{0,n}, \varphi^{\gamma+m,\eta}_{0,k}\right\rangle\delta_{m,j}.$$

To the computation of the quantity $\left\langle \varphi^{\gamma+m,\eta}_{0,n}, \varphi^{\gamma+m,\eta}_{0,k}\right\rangle$ we make use of (3.1.13) giving the explicit expression of $\varphi^{\gamma,\eta}_{m,n}$. This yields

$$\begin{aligned}\left\langle \varphi^{\gamma+m,\eta}_{0,n}, \varphi^{\gamma+m,\eta}_{0,k}\right\rangle &= \int_{\mathbb{D}}(1-|z|^2)^{\alpha-2(\gamma+m+1)}|z|^{2(\beta-2\eta)}z^n\overline{z}^k d\lambda(z)\\ &= \pi\left(\int_0^1(1-t)^{\alpha-2(\gamma+m+1)}t^{n+\beta-2\eta}dt\right)\delta_{n,k}\\ &= \pi B(n+\beta-2\eta+1,\alpha-2(\gamma+m)-1)\delta_{n,k},\end{aligned}$$

where $B(a,b)$ denotes the classical beta function. The validity of the previous formula requires that $n > 2\eta - \beta - 1$ and $\alpha - 2\gamma - 1 > 2m$ with $\alpha - 2\gamma - 1 > 0$. Finally, since

$$\prod_{\ell=1}^{m} E_{m-\ell}^{\gamma+\ell,\alpha} = m!(\alpha - 2(\gamma+m))_m = m!\frac{\Gamma(\alpha - 2\gamma - m)}{\Gamma(\alpha - 2(\gamma+m))},$$

we arrive at

$$\left\langle \psi_{m,n}^{\gamma,\eta}, \psi_{j,k}^{\gamma,\eta} \right\rangle = \frac{\pi m!}{(\alpha - 2(\gamma+m) - 1)} \frac{\Gamma(\alpha - 2\gamma - m)\Gamma(n+\beta-2\eta+1)}{\Gamma(n+\alpha+\beta-2(\gamma+\eta+m))} \delta_{m,j}\delta_{n,k}.$$

This completes the proof. □

Remark 3.1.7. *The functions in* (3.1.1) *corresponding to* $\gamma = -1$, $\eta = 0$ *and* $m = 0$ *reduce further to* $\psi_{0,n}^{-1,0}(z,\overline{z}) = z^n$, *for varying integer* $n > -(\beta+1)$, *whose square norm in* $L_{\beta}^{2,\alpha}(\mathbb{D})$ *is given by*

$$\left\| \psi_{0,n}^{-1,0} \right\|_{\alpha,\beta}^2 = \pi \frac{\Gamma(\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)}.$$

They form an orthogonal basis of the β*-modified Bergman space* $\mathcal{A}_{\beta}^{2,\alpha}(\mathbb{D})$ *defined as the closed subspace in* $L_{\beta}^{2,\alpha}(\mathbb{D})$ *formed by the holomorphic functions on the punctured disc* $\mathbb{D}^*$ *(see [37, 38] for details). In other words, the* β*-modified Bergman space is the* L^2*-eigenspace of our magnetic Laplacian* $\mathcal{L}_{\gamma+1,\eta}^{\alpha,\beta,+}$ *associated with its lowest Landau level. For the particular case of* $\beta = 0$ *we recover the classical Bergman space on the unit disc with respect to the weight function being of the generalized Gegenbauer form* $(1-|z|^2)^{\alpha}$.

Remark 3.1.8. *The functions* $\psi_{m,n}^{\gamma,\eta}$ *do not form a complete system in* $L_{\beta}^{2,\alpha}(\mathbb{D})$. *However, for fixed m such that* $0 \leq m < (\alpha - 1 - 2\gamma)/2$ *and varying integer* $n \geq 2\eta - \beta$ *they span a specific closed subspace* $\mathcal{A}_{\beta,m}^{2,\alpha}(\mathbb{D})$ *in* $L_{\beta}^{2,\alpha}(\mathbb{D})$. *This gives rise to what can be called the m-th generalized (or also poly-meromophic)* β*-modified Bergman space on* $\mathbb{D}^*$ *and can be seen as the polyanalytic analog of the* β*-modified Bergman space. Its reproducing kernel is given in Remark 3.2.20 below.*

3.2 Fractional Zernike functions

In this section we provide an accurate theoretical study for the fractional Zernike functions in (0.0.17). We discuss their connection with some special functions, zeros, orthogonality in $L_{\rho}^{2,\kappa}(\mathbb{D})$, regularity, differential equations, recurrence and operational formulas. Some results concerning the generating functions, the integral representations and completeness are also obtained.

3.2.1 Connection to special functions and explicit expression

We begin by establishing the explicit expression of the fractional Zernike functions $\mathcal{Z}^{\kappa,\rho}_{m,n}$ in terms of the classical Zernike polynomials. Thus, for given real b and nonnegative integer m we define the infected minimum $m \wedge^* b$ to be

$$m \wedge^* b = \begin{cases} \min(m,b), & b = 0,1,2,\cdots \\ m, & b \in \mathbb{R},\, b \neq 0,1,2,\cdots . \end{cases}$$

Proposition 3.2.1. *For every $\rho > -1$ we have*

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\bar{z}) = \frac{m!\Gamma(\rho+1)}{(\kappa+m+1)_n} \sum_{j=0}^{m\wedge^*\rho} \frac{(-1)^j}{j!(m-j)!\Gamma(\rho-j+1)} \frac{\left(1-|z|^2\right)^j}{z^j} \mathcal{Z}^{\kappa+j}_{m-j,n}(z,\bar{z}). \tag{3.2.1}$$

Proof. Using the facts (3.2.4) and

$$z^n(1-|z|^2)^{\kappa+m} = \frac{(-1)^n}{(\kappa+m+1)_n} \frac{\partial^n}{\partial \bar{z}^n} \left((1-|z|^2)^{\kappa+m+n}\right),$$

we get

$$\begin{aligned} \mathcal{Z}^{\kappa,\rho}_{m,n}(z,\bar{z}) &= \frac{(-1)^{m+n}}{(\kappa+m+1)_n} z^{-\rho}(1-|z|^2)^{-\kappa} \frac{\partial^m}{\partial z^m} \left(z^\rho \frac{\partial^n}{\partial \bar{z}^n} \left((1-|z|^2)^{\kappa+m+n}\right)\right) \\ &= \frac{(-1)^{m+n} m!}{(\kappa+m+1)_n} z^{-\rho}(1-|z|^2)^{-\kappa} \sum_{j=0}^{m} \frac{(-\rho)_j}{j!(m-j)!} z^{\rho-j} \frac{\partial^{m-j+n}}{\partial z^{m-j}\partial \bar{z}^n} \left((1-|z|^2)^{\kappa+j+m-j+n}\right), \end{aligned}$$

which can be rewritten as (3.2.1). □

Remark 3.2.2. *For $\rho = 0$ we recover the Zernike polynomials $\mathcal{Z}^{\kappa}_{m,n}(z,\bar{z})$ up to the multiplicative constant $1/(\kappa+m+1)_n$, while when $\rho = 1$ we get*

$$\mathcal{Z}^{\kappa}_{m,n+1}(z,\bar{z}) = (\kappa+m+n+1)\left(z\mathcal{Z}^{\kappa}_{m,n}(z,\bar{z}) + m\left(1-|z|^2\right)\mathcal{Z}^{\kappa+1}_{m-1,n}(z,\bar{z})\right),$$

which is exactly the three terms recurrence formula for the Zernike polynomials [5, p. 403, Eq. (5.1)]. This follows since that $(-\rho)_j = 0$ for $j \geq \rho+1$ whenever $\rho = 0,1,2,\cdots$. More generally, from (0.0.18) *with ρ being a nonnegative integer we obtain new recurrence formula for the classical complex Zernike polynomials*

$$\mathcal{Z}^{\kappa}_{m,n+\rho}(z,\bar{z}) = m!\Gamma(\rho+1)(\kappa+m+n+1)_\rho \sum_{j=0}^{m\wedge\rho} \frac{(-1)^j z^{\rho-j}\left(1-|z|^2\right)^j}{j!(m-j)!\Gamma(\rho-j+1)} \mathcal{Z}^{\kappa+j}_{m-j,n}(z,\bar{z}). \tag{3.2.2}$$

The explicit expression of the few first terms of $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\bar{z})$ can be computed easily from the Rodrigues formula (0.0.17) or also using (3.2.1). Thus, those corresponding to $m = 0$ reduce to the monomials, $\mathcal{Z}^{\kappa,\rho}_{0,n}(z,\bar{z}) = z^n$. For $m = 1$ and $m = 2$ we get respectively

$$\mathcal{Z}^{\kappa,\rho}_{1,n}(z,\bar{z}) = (\kappa+n_\rho+1)\bar{z}z^n - n_\rho z^{n-1}$$

and

$$\mathcal{Z}^{\kappa,\rho}_{2,n}(z,\overline{z}) = (\kappa+n_\rho+1)(\kappa+n_\rho+2)\overline{z}^2z^n - 2n_\rho(\kappa+n_\rho+1)\overline{z}z^{n-1} + n_\rho(n_\rho-1)z^{n-2},$$

where we have set $n_\rho = n+\rho$. A general formula for the explicit expression of $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})$ is given by the following assertion.

Proposition 3.2.3. *We have*

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) = \sum_{j=0}^{m\wedge^*(n+\rho)} \frac{(-1)^j m!\Gamma(n+\rho+1)\Gamma(\kappa+m+1)}{j!(m-j)!\Gamma(n+\rho-j+1)\Gamma(\kappa+j+1)} \left(1-|z|^2\right)^j z^{n-j}\overline{z}^{m-j}. \tag{3.2.3}$$

Proof. Using the known fact that $(x)^{\underline{n}} = \Gamma(x+1)/\Gamma(x-n+1)$ for the decreasing factorial defined by $(x)^{\underline{n}} = x(x-1)\cdots(x-n+1)$, we obtain

$$\frac{\partial^m}{\partial z^m}(1-xz)^a = (-1)^m\frac{\Gamma(a+1)}{\Gamma(a+1-m)}x^m(1-xz)^{a-m}$$

and

$$\frac{\partial^m}{\partial z^m}(z^a) = (-a)_m z^{a-m} = \varepsilon^*_{a,m}\frac{\Gamma(a+1)}{\Gamma(a+1-m)}z^{a-m}, \tag{3.2.4}$$

where for the nonnegative integer m we have set

$$\varepsilon^*_{a,m} = \begin{cases} 1, & a\geq m;\ a=0,1,\cdots \\ 0, & a<m;\ a=0,1,\cdots \\ 1, & a\in\mathbb{R},\ a\neq 0,1,\cdots. \end{cases}$$

Thus, applying the Leibnitz formula for high order derivation of a product yields

$$\begin{aligned}\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) &= (-1)^m z^{-\rho}(1-|z|^2)^{-\kappa}\frac{\partial^m}{\partial z^m}\left(z^{n+\rho}(1-|z|^2)^{\kappa+m}\right) \\ &= \sum_{j=0}^{m}\varepsilon^*_{n+\rho,j}\frac{(-1)^j m!\Gamma(n+\rho+1)\Gamma(\kappa+m+1)}{j!(m-j)!\Gamma(n+\rho-j+1)\Gamma(\kappa+j+1)}\overline{z}^{m-j}z^{n-j}\left(1-|z|^2\right)^j.\end{aligned}$$

This gives rise to (3.2.3). □

Below, we present different hypergeometric representations of $\mathcal{Z}^{\kappa,\rho}_{m,n}$ in terms of the Gauss hypergeometric function defined on the open unit disc by the power series

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle| z\right) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

provided that $c\neq 0,-1,-2,\cdots$.

Proposition 3.2.4. *The functions* $\mathcal{Z}_{m,n}^{\kappa,\rho}(z,\overline{z})$ *are given in terms of the* ${}_2F_1$ *function by*

$$\mathcal{Z}_{m,n}^{\kappa,\rho}(z,\overline{z}) = (\kappa+1)_m z^n \overline{z}^m {}_2F_1\left(\begin{array}{c} -m, -n-\rho \\ \kappa+1 \end{array} \middle| 1 - \frac{1}{|z|^2} \right). \tag{3.2.5}$$

Proof. By means of $(a)^{\underline{n}} = (-1)^n(-a)_n = \Gamma(a+1)/\Gamma(a-n+1)$ combined with $(-1)^j(-m)_j(m-j)! = m!$, we can rewrite (3.2.3) as

$$\begin{aligned} \mathcal{Z}_{m,n}^{\kappa,\rho}(z,\overline{z}) &= (\kappa+1)_m z^n \overline{z}^m \sum_{j=0}^{m\wedge^*(n+\rho)} \frac{(-m)_j(-n-\rho)_j}{(\kappa+1)_j j!} \left(1 - \frac{1}{|z|^2}\right)^j \\ &= (\kappa+1)_m z^n \overline{z}^m {}_2F_1\left(\begin{array}{c} -m, -n-\rho \\ \kappa+1 \end{array} \middle| 1 - \frac{1}{|z|^2} \right). \end{aligned}$$

□

There are several equivalent expressions for $\mathcal{Z}_{m,n}^{\kappa,\rho}$ in terms of the Gauss hypergeometric functions which follow from the well-known linear transformations for ${}_2F_1$. Thus, from the second and the third ones in [69, § 2.4., p. 47], it follows

$$\mathcal{Z}_{m,n}^{\kappa,\rho}(z,\overline{z}) = (\kappa+1)_m z^{n-m} {}_2F_1\left(\begin{array}{c} -m, n+\kappa+\rho+1 \\ \kappa+1 \end{array} \middle| 1 - |z|^2 \right) \tag{3.2.6}$$

and

$$\mathcal{Z}_{m,n}^{\kappa,\rho}(z,\overline{z}) = (\kappa+1)_m z^{-\rho} \overline{z}^{m-n-\rho} {}_2F_1\left(\begin{array}{c} -n-\rho, \kappa+m+1 \\ \kappa+1 \end{array} \middle| 1 - |z|^2 \right). \tag{3.2.7}$$

However, starting from (3.2.5) and applying the linear transformation [69, § 2.4.1, p. 47], one obtains

$$\mathcal{Z}_{m,n}^{\kappa,\rho}(z,\overline{z}) = (\kappa+\rho+n+1)_m z^n \overline{z}^m {}_2F_1\left(\begin{array}{c} -m, -n-\rho \\ -\kappa-\rho-n-m \end{array} \middle| \frac{1}{|z|^2} \right). \tag{3.2.8}$$

The same transformation applied to (3.2.6) gives rise to

$$\mathcal{Z}_{m,n}^{\kappa,\rho}(z,\overline{z}) = (-n-\rho)_m z^{n-m} {}_2F_1\left(\begin{array}{c} -m, \kappa+n+\rho+1 \\ n+\rho-m+1 \end{array} \middle| |z|^2 \right). \tag{3.2.9}$$

The latter one remains valid for ρ being integer and $m \leq n+\rho$.

The next result is concerned with the expression of $\mathcal{Z}_{m,n}^{\kappa,\rho}(z,\overline{z})$ in terms of the the real Jacobi polynomials.

Proposition 3.2.5. *For $\rho = 0, 1, \cdots$, we have*

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) = \frac{(\kappa+1)_{m\vee(n+\rho)}(m\wedge(n+\rho))!}{(\kappa+1)_{n+\rho}} \frac{z^n\overline{z}^m}{|z|^{2(m\wedge(n+\rho))}} P^{(\kappa,|m-n-\rho|)}_{m\wedge(n+\rho)}(2|z|^2-1), \tag{3.2.10}$$

while when $\rho > -1$ is non-integer or $m = m \wedge (n+\rho)$ we have

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) = m! z^{n-m} P^{(\kappa,n-m+\rho)}_m(2|z|^2-1). \tag{3.2.11}$$

Proof. Consider first the case of $\rho = 0, 1, \cdots$. Starting from (3.2.6) and the assumption $m \leq n+\rho$ one can make use of the facts $(-n-\rho)_m = (-1)^m(n+\rho-m+1)_m$, $P^{(a,b)}_m(x) = (-1)^m P^{(b,a)}_m(-x)$ and

$$_2F_1\left(\begin{matrix} -m, m+a+b+1 \\ a+1 \end{matrix} \middle| x \right) = \frac{m!}{(a+1)_m} P^{(a,b)}_m(1-2x) \tag{3.2.12}$$

to get

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) = m! z^{n-m} P^{(\kappa,n+\rho-m)}_m(2|z|^2-1). \tag{3.2.13}$$

The result corresponding to the case $m \geq n+\rho$ is immediate from the previous one using the like-symmetry relationship (0.0.19) (it is also immediate from (3.2.7)). In fact, we have

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) = \frac{(\kappa+1)_m(n+\rho)!}{(\kappa+1)_{n+\rho}} z^{-\rho}\overline{z}^{m-n-\rho} P^{(\kappa,m-n-\rho)}_{n+\rho}(2|z|^2-1). \tag{3.2.14}$$

To conclude one observes that both expressions can rewritten in the unified form

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) = \frac{(\kappa+1)_{m\vee(n+\rho)}(m\wedge(n+\rho))!}{(\kappa+1)_{n+\rho}} |z|^{|m-n-\rho|-\rho} e^{i(n-m)\arg z} P^{(\kappa,|m-n-\rho|)}_{m\wedge(n+\rho)}(2|z|^2-1),$$

which clearly leads to (3.2.10). This can also be deduced from the functions $z^\rho \mathcal{Z}^{\kappa,\rho}_{m,n}$ being the classical Zernike functions in (0.0.10) up to multiplicative constant and next making appeal to the one in [33].

Finally, the formula (3.2.11) is exactly (3.2.10) when ρ integer and $m \leq n+\rho$. It can be obtained by means of (3.2.6) and (3.2.12) valid whenever $m = m \wedge^* (n+\rho)$, which corresponds to $\rho > -1$ being non-integer or also to $m = m \wedge (n+\rho)$. A direct proof can be handled starting from the derivation formula [15, p. 1, 1.1.2 Eq. 2] that we can rewrite as

$$\frac{d^m}{dz^m}\left(z^a(1-xz)^n\right) = m! z^{a-m}(1-xz)^{b-m} P^{(a-m,b-m)}_m(1-2xz). \tag{3.2.15}$$

Therefore, with the specification $a = n+\rho$, $b = \kappa+m$ and $x = \overline{z}$, the expression of $\mathcal{Z}^{\kappa,\rho}_{m,n}$ in (0.0.17) becomes (3.2.11). □

The next result concerning the zeros of $\mathcal{Z}^{\kappa,\rho}_{m,n}$ is an immediate consequence of Proposition 3.2.5.

Corollary 3.2.6. *The point* 0 *is a zero of* $\mathcal{Z}^{\kappa,\rho}_{m,n}$ *when* $\rho = 0,1,2,\cdots$ *if and only if* $n > m$ *or* $m > n$ *with* $\rho = 0$*. However, when* ρ *is non-integer, it is a zero of* $\mathcal{Z}^{\kappa,\rho}_{m,n}$ *only when* $n > m$*. The other zeros are the circles centered at the origin with radii* $(1 + r^{\kappa,\rho}_{m,n})/2$*, where* $r^{\kappa,\rho}_{m,n}$ *are the zeros, located at the segment* $(0,1)$*, of the real Jacobi polynomials* $P_m^{(\kappa,n-m+\rho)}(x)$ *when* ρ *is non-integer and* $P^{(\kappa,|n-m+\rho|)}_{m\wedge(n+\rho)}(x)$ *when* ρ *is a non-negative integer.*

Remark 3.2.7. *According to Proposition 3.2.5, the expression of the* β*-restricted Zernike functions* $\psi^{\gamma,\eta}_{m,n}$ *in terms of the Jacobi polynomials reads*

$$\psi^{\gamma,\eta}_{m,n}(z,\overline{z}) = (-1)^m m! \frac{z^{n-m}}{|z|^{2\eta}} (1-|z|^2)^{\frac{\kappa-\alpha-1}{2}} P_m^{(n-m+\rho,\kappa)}(1-2|z|^2), \tag{3.2.16}$$

where $\rho = \beta - 2\eta > -1$ *is non-integer and* $\kappa = \alpha - 2(\gamma + m) - 1$*.*

We conclude this subsection by discussing the orthogonality of the considered functions.

Corollary 3.2.8. *The functions* $\mathcal{Z}^{\kappa,\rho}_{m,n}$ *form an orthogonal system in* $L^{2,\kappa}_{\rho}(\mathbb{D})$ *with square norm given by*

$$\|\mathcal{Z}^{\kappa,\rho}_{m,n}\|^2_{L^{2,\kappa}_{\rho}(\mathbb{D})} = \frac{\pi m!(n+\rho)!\Gamma(m+\kappa+1)}{(m+n+\rho+\kappa+1)\Gamma(n+\rho+\kappa+1)} =: \frac{1}{\gamma^{\kappa,\rho}_{m,n}}. \tag{3.2.17}$$

Proof. We provide explicit computation only when ρ being integer. For the case of ρ non-integer one can proceeds as for $m = m \wedge (n+\rho)$ and ρ is integer. Thus, let ρ a fixed integer and set

$$d^{\rho,\kappa}_{m,n} := \frac{(\kappa+1)_{m\vee(n+\rho)}(m\wedge(n+\rho))!}{(\kappa+1)_{n+\rho}}.$$

Then, from Proposition 3.2.5 and the use of the polar coordinates $z = \sqrt{t}e^{i\theta}; 0 \le t < 1, 0 \le \theta < 2\pi$, we get

$$\begin{aligned}
I^{\rho,\kappa}_{m,n,j,k} &:= \int_D \mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})\overline{\mathcal{Z}^{\kappa,\rho}_{j,k}(z,\overline{z})}|z|^{2\rho}(1-|z|^2)^{\kappa} d\Lambda(z) \\
&= \pi d^{\rho,\kappa}_{m,n} d^{\rho,\kappa}_{j,k} \left(\int_0^1 t^{|m-n-\rho|}(1-t)^{\kappa} P^{(\kappa,|m-n-\rho|)}_{m\wedge(n+\rho)}(2t-1) P^{(\kappa,|m-n-\rho|)}_{j\wedge(k+\rho)}(2t-1) dt \right) \delta_{n-m,k-j} \\
&= \frac{\pi d^{\rho,\kappa}_{m,n} d^{\rho,\kappa}_{j,k}}{2^{|m-n-\rho|+\kappa+1}} \left(\int_{-1}^1 (1+x)^{|m-n-\rho|}(1-x)^{\kappa} P^{(\kappa,|m-n-\rho|)}_{m\wedge(n+\rho)}(x) P^{(\kappa,|m-n-\rho|)}_{j\wedge(k+\rho)}(x) dx \right) \delta_{n-m,k-j}.
\end{aligned}$$

Now, by the orthogonal property for the classical Jacobi polynomials [69, p.212], it follows

$$\begin{aligned}
I^{\rho,\kappa}_{m,n,j,k} &= \frac{\pi d^{\rho,\kappa}_{m,n} d^{\rho,\kappa}_{j,k}}{2^{|m-n-\rho|+\kappa+1}} \left\| P^{(\kappa,|m-n-\rho|)}_{m\wedge(n+\rho)} \right\|^2 \delta_{n-m,k-j}\delta_{m\wedge(n+\rho),j\wedge(k+\rho)} \\
&= \frac{\pi m!(n+\rho)!\Gamma(m+\kappa+1)}{(m+n+\rho+\kappa+1)\Gamma(n+\rho+\kappa+1)} \delta_{m,j}\delta_{n,k}.
\end{aligned}$$

This proves the orthogonality of $\mathcal{Z}^{\kappa,\rho}_{m,n}$ in the Hilbert space $L^{2,\kappa}_{\rho}(\mathbb{D})$. □

3.2.2 Poly-meromorphy

In analogy with the definition of the polyanalytic functions defined as those satisfying the Cauchy–Riemann equation $\partial^n/\partial\overline{z}^n = 0$ one has to suggest the following or fthe poly-meromorpy [11, p. 199].

Definition 3.2.9. *A complex-valued function f on an open set U in the complex plane is said to be poly-meromorphic of order n (of first kind) if there exist certain meromorphic functions ψ_k; $k = 0, 1, \cdots, n-1$ on U such that*

$$f(z) = \psi_0(z) + \overline{z}\psi_1(z) + \cdots + \overline{z}^{n-1}\psi_{n-1}(z).$$

The main result in this subsection discusses the regularity of the considered fractional Zernike functions.

Theorem 3.2.10. *The fractional Zernike functions $\mathcal{Z}^{\kappa,\rho}_{m,n}$ are polynomials in z and $\overline{z}$ if and only if $\rho = 0$ or $m \leq n$. Alternatively, they are poly-meromorphic functions of order m with 0 as unique pole. Its order of multiplicity is given by $Ord^{\kappa,\rho}_{m,n} = \rho$ for $m > n+\rho$ when $\rho = 1, 2, \cdots$, and by $Ord^{\kappa,\rho}_{m,n} = m - n$ for $m > n$ when $\rho > -1$ is non-integer, or when $n < m \leq n+\rho$ with $\rho = 1, 2, \cdots$.*

Proof. Set

$$c^{\kappa,\rho}_{m,n,j} := \frac{(-1)^j m!\Gamma(n+\rho+1)\Gamma(\kappa+m+1)}{j!(m-j)!\Gamma(n+\rho-j+1)\Gamma(\kappa+j+1)},$$

and for $p < q \leq m \wedge^* (n+\rho)$ consider the quantities

$$S^{\kappa,\rho,m,n}_{q,p} := R^{\kappa,\rho,m,n}_{q} - X^{p-q} R^{\kappa,\rho,m,n}_{p},$$

where $R^{\kappa,\rho,m,n}_{p}$ is the polynomial of degree less or equal to p given by

$$R^{\kappa,\rho,m,n}_{p}(X) = \sum_{k=0}^{p} \left(\sum_{j=0}^{p-k} (-1)^j \frac{(j+k)!}{j!k!} c^{\kappa,\rho}_{m,n,j+k} \right) X^{p-k}. \tag{3.2.18}$$

Notice for instance that its constant coefficient is given by $S^{\kappa,\rho,m,n}_{q,p}(0) = R^{\kappa,\rho,m,n}_{q}(0) = c^{\kappa,\rho}_{m,n,q}$. Thus, starting from (3.2.3) we can rewrite $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})$ as

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) = z^{n-[m\wedge^*(n+\rho)]}\overline{z}^{m-[m\wedge^*(n+\rho)]} R^{\kappa,\rho,m,n}_{m\wedge^*(n+\rho)}(|z|^2). \tag{3.2.19}$$

But, since $c^{\kappa,\rho}_{m,n,p} \neq 0$, it becomes clear from (3.2.19) that the functions $\mathcal{Z}^{\kappa,\rho}_{m,n}$ are polynomials if and only if $\rho = 0$ or $m \leq n$ independently of $\rho > -1$ being integer or not. This assertion is also immediate from Proposition 3.2.5. Next, using the fact that $R_q^{\kappa,\rho,m,n} = X^{q-p}R_p^{\kappa,\rho,m,n} + S_{q,p}^{\kappa,\rho,m,n}$ we obtain

$$\mathcal{Z}_{m,n}^{\kappa,\rho}(z,\overline{z}) = \overline{z}^{m-n}R_n^{\kappa,\rho,m,n}(|z|^2) + z^{n-[m\wedge^*(n+\rho)]}\overline{z}^{m-[m\wedge^*(n+\rho)]}S_{m\wedge^*(n+\rho),n}^{\kappa,\rho,m,n}(|z|^2). \tag{3.2.20}$$

A meticulous study of the different possible cases of m compared to n and $n+\rho$ for given $\rho > -1$ leads to

$$\mathcal{Z}_{m,n}^{\kappa,\rho}(z,\overline{z}) = \begin{cases} z^{n-m}R_m^{\kappa,\rho,m,n}(|z|^2) & \text{if } m \leq n;\ \rho > -1 \\ \overline{z}^{m-n}R_n^{\kappa,\rho,m,n}(|z|^2) & \text{if } m > n;\ \rho = 0 \\ \overline{z}^{m-n}R_n^{\kappa,\rho,m}(|z|^2) + \dfrac{1}{z^{m-n}}S_m^{\kappa,\rho,m,n}(|z|^2) & \text{if } m > n;\ \rho \text{ non-integer} \\ & \text{or } n < m \leq n+\rho;\ \rho = 1,2,\cdots \\ \overline{z}^{m-n}R_n^{\kappa,\rho,m,n}(|z|^2) + \dfrac{\overline{z}^{m-(n+\rho)}}{z^\rho}S_{n+\rho}^{\kappa,\rho,m,n}(|z|^2) & \text{if } m > n+\rho;\ \rho = 1,2,\cdots. \end{cases}$$

Moreover, this reveals that the regular part of $\mathcal{Z}_{m,n}^{\kappa,\rho}$ is always a polyanalytic function of order m and anti-polyanalytic of order n. It is given by

$$R_{m,n}^{\kappa,\rho}(|z|^2) = \begin{cases} z^{n-m}R_m^{\kappa,\rho,m,n}(|z|^2), & m \leq n \\ \overline{z}^{m-n}R_n^{\kappa,\rho,m,n}(|z|^2), & m \geq n \end{cases} = z^{n-m\wedge n}\overline{z}^{m-m\wedge n}R_{m\wedge n}^{\kappa,\rho,m,n}(|z|^2). \tag{3.2.21}$$

However, in general $\mathcal{Z}_{m,n}^{\kappa,\rho}$ are poly-meremorphic with 0 as the unique pole for $\rho \neq 0$ and $m > n$. Thus, for $m > n+\rho$ with $\rho = 1,2,\cdots$, the singular part is clearly $z^{-\rho}\overline{z}^{m-(n+\rho)}S_{n+\rho}^{\kappa,\rho,m,n}(|z|^2)$. It reduces to $z^{n-m}S_m^{\kappa,\rho,m,n}(|z|^2)$ whenever $n < m$ and $\rho > -1$ non-integer or $n < m \leq n+\rho$ when $\rho = 1,2,\cdots$. Therefore, it becomes clear that the multiplicity of the singularity is given by

$$Ord_{m,n}^{\kappa,\rho} = \begin{cases} \rho & \text{if } m > n+\rho;\ \rho = 1,2,\cdots \\ m-n & \text{if } n < m;\ \rho \text{ non-integer} \\ & \text{or } n < m \leq n+\rho;\ \rho = 1,2,\cdots. \end{cases}$$

This completes the proof. □

Remark 3.2.11. *The obtained result can justifies somehow the appellation of $\mathcal{Z}_{m,n}^{\kappa_m,\rho}$ by fractional Zernike functions.*

3.2.3 Differential equations

In this subsection we are concerned with some second order differential equations satisfied by the fractional Zernike functions.

Theorem 3.2.12. *Let $m_\rho = m+\rho$. Then, the function $\mathcal{Z}^{\kappa,\rho}_{m,n}$ is solution of the differential equation*

$$z^2(1-|z|^2)\frac{\partial^2}{\partial z^2} + \left((m_\rho - n+1) - (\kappa + m_\rho - n+2)|z|^2\right) z\frac{\partial}{\partial z} + n(\kappa+m_\rho+1)|z|^2 = \rho(n-m).$$

Proof. Consider the first order differential operator

$$\nabla^{\kappa,\rho}(f)(z) = -\left((1-|z|^2)\frac{\partial}{\partial z} - \mathcal{Z}^{\kappa,\rho}_{1,0}(z,\bar{z})\right)(f)(z). \tag{3.2.22}$$

Also, for varying $j = 1,2,\cdots,m$ we set

$$\nabla^{\kappa,\rho}_j(f) := -z^{-\rho}(1-|z|^2)^{-\kappa-j+1}\frac{\partial}{\partial z}\left(z^\rho(1-|z|^2)^{\kappa+j}f\right), \tag{3.2.23}$$

so that $\nabla^{\kappa,\rho}(f)(z) = \nabla^{\kappa,\rho}_1(f)(z)$. Successive application of $\nabla^{\kappa,\rho}_j$ leads to the operator $\widetilde{\nabla}^{\kappa,\rho}_m := \nabla^{\kappa,\rho}_1 \circ \nabla^{\kappa,\rho}_2 \circ \cdots \circ \nabla^{\kappa,\rho}_m$ satisfying $\widetilde{\nabla}^{\kappa,\rho}_{m+1} = \nabla^{\kappa,\rho}_1 \circ \widetilde{\nabla}^{\kappa+1,\rho}_m$ since $\nabla^{\kappa,\rho}_{j+1} = \nabla^{\kappa+1,\rho}_j$. It is explicitly given by

$$\widetilde{\nabla}^{\kappa,\rho}_m(f)(z) = (-1)^m z^{-\rho}(1-|z|^2)^{-\kappa}\frac{\partial^m}{\partial z^m}\left(z^\rho(1-|z|^2)^{\kappa+m}f\right). \tag{3.2.24}$$

Thus, in view of (0.0.17) it is clear that $\widetilde{\nabla}^{\kappa,\rho}_m(e_n)(z) = \mathcal{Z}^{\kappa,\rho}_{m,n}(z,\bar{z})$ for $e_n(z) := z^n$, and therefore

$$\nabla^{\kappa,\rho}(\mathcal{Z}^{\kappa+1,\rho}_{m,n}(z,\bar{z})) = \nabla^{\kappa,\rho}_1\left(\nabla^{\kappa+1,\rho}_1 \circ \nabla^{\kappa+1,\rho}_2 \circ \cdots \circ \nabla^{\kappa+1,\rho}_m\right)(e_n) = \mathcal{Z}^{\kappa,\rho}_{m+1,n}(z,\bar{z}). \tag{3.2.25}$$

Now associated with the Euler differential operator $E_z = z\partial/\partial z$ and $c^{\kappa,\rho}_{m,n} = m(n+\rho+\kappa+1)$, we define the first order differential operator

$$D^{\kappa,\rho}_{m,n} = \frac{1}{c^{\kappa,\rho}_{m,n}\bar{z}}\left(E_z - (n-m)\right) = \frac{1}{c^{\kappa,\rho}_{m,n}}\left(\frac{z}{\bar{z}}\frac{\partial}{\partial z} - \frac{(n-m)}{\bar{z}}\right).$$

Hence making use of (3.2.11) combined with the differentiation formula of Jacobi polynomials in [69, p. 213] we obtain the identity

$$D^{\kappa,\rho}_{m,n}(\mathcal{Z}^{\kappa,\rho}_{m,n}) = \mathcal{Z}^{\kappa+1,\rho}_{m-1,n}. \tag{3.2.26}$$

Therefore, from (3.2.25) and (3.2.26) it is immediate that $\nabla^{\kappa,\rho} \circ D^{\kappa,\rho}_{m,n}(\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\bar{z})) = \mathcal{Z}^{\kappa,\rho}_{m,n}(z,\bar{z})$. A direct computation shows that $-c^{\kappa,\rho}_{m,n}\bar{z}\nabla^{\kappa,\rho} \circ D^{\kappa,\rho}_{m,n}$ is given by

$$z(1-|z|^2)\frac{\partial^2}{\partial z^2} + \left([m_\rho - n+1] - [\kappa+m_\rho-n+2]|z|^2\right)\frac{\partial}{\partial z} + (m-n)\left(\frac{\rho}{z} - [\kappa+\rho+1]\bar{z}\right).$$

This shows that the fractional Zernike function $\mathcal{Z}^{\kappa,\rho}_{m,n}$ satisfy the desired differential equation. □

Remark 3.2.13. *In view of* (3.2.25) *and* (3.2.26) *the considered operators $\nabla^{\kappa,\rho}$ and $D^{\kappa,\rho}_{m,n}$ appear as creation and annihilation operators for the fractional Zernike functions.*

Remark 3.2.14. *Let $(P_j f)(z) = z^j f(z)$. Then, the commutation relation $P_j \circ \nabla_m^{\kappa,\rho} \circ P_j^{-1} = \nabla_m^{\kappa,\rho-j}$ holds for all $z \in \mathbb{D}^*$. This follows by observing that from* (0.0.17)*, we have*

$$z^j \mathcal{Z}_{m,n}^{\kappa,\rho}(z,\overline{z}) = \mathcal{Z}_{m,n+j}^{\kappa,\rho-j}(z,\overline{z}). \tag{3.2.27}$$

The following can be proved using the close connection of $\mathcal{Z}_{m,n}^{\kappa_m,\rho}$ to the β-restricted Zernike functions studied in Section 2.

Theorem 3.2.15. *For given fixed nonnegative integer m and reals α and γ such that $\kappa_m = \alpha - 2(\gamma+m) - 1$, the fractional Zernike functions $\mathcal{Z}_{m,n}^{\kappa_m,\rho}$ for varying n are eigenfunctions of*

$$-(1-|z|^2)^2\partial\overline{\partial} - (1-|z|^2)\left(mE - H^{\rho}_{\kappa_m+m+1}(z)\overline{E}\right) + mH^{\rho}_{\kappa_m+m+1}(z)|z|^2, \tag{3.2.28}$$

with $m(\kappa_m + m + 1)$ as corresponding eigenvalue.

Proof. For the proof observe that the fractional Zernike functions $\mathcal{Z}_{m,n}^{\kappa_m,\rho}$ are closely connected to the β-restricted Zernike functions $\psi_{m,n}^{\gamma,\eta}(z,\overline{z})$ by (3.1.1) for every fixed nonnegative integer m. The latter ones are eigenfunctions of the hamiltonian $\mathcal{L}_{\gamma,\eta}^{\alpha,\beta,+}$ in (3.1.7) with $E_m^{\gamma,\alpha} = (m+1)(\alpha - 2\gamma - m)$ as corresponding eigenvalue (see (i) in Theorem 3.1.6). The key observation to conclude is that the operators $\mathcal{L}_{\gamma,\eta}^{\alpha,\beta,+}$ and $\mathcal{L}_{\gamma+a,\eta+b}^{\alpha+2a,\beta+2b,+}$ are unitary equivalent for arbitrary reals a and b. More precisely, since $A_{\gamma,\eta}^{*_{\alpha,\beta}}(h_{a,b}f) = h_{a,b}A_{\gamma+a,\eta+b}^{*_{\alpha+2a,\beta+2b}}(f)$ and $A_{\gamma,\eta}(h_{a,b}f) = h_{a,b}A_{\gamma+a,\eta+b}(f)$, one obtains

$$\mathcal{L}_{\gamma,\eta}^{\alpha,\beta,+}(h_{a,b}f) = A_{\gamma,\eta}A_{\gamma,\eta}^{*_{\alpha,\beta}}(h_{a,b}f) = h_{a,b}A_{\gamma+a,\eta+b}A_{\gamma+a,\eta+b}^{*_{\alpha+2a,\beta+2b}}(f) = h_{a,b}\mathcal{L}_{\gamma+a,\eta+b}^{\alpha+2a,\beta+2b,+}(f).$$

Subsequently, for $\kappa_m = \alpha - 2(\gamma+m) - 1$, $\rho = \beta - 2\eta$, $b = -\eta$ and $a = (\kappa_m - \alpha - 1)/2 = -(\gamma + m + 1)$, the fractional Zernike function $\mathcal{Z}_{m,n}^{\kappa_m,\rho}$ satisfies

$$\mathcal{L}_{-m-1,0}^{\kappa_m-1,\rho,+}\mathcal{Z}_{m,n}^{\kappa_m,\rho} = E_m^{\gamma,\alpha}\mathcal{Z}_{m,n}^{\kappa_m,\rho}.$$

However, from Lemma 3.1.1, it is clear that the second order partial differential equation in (3.2.28) is exactly $\mathcal{L}_{-m-1,0}^{\kappa_m-1,\rho,+} - (\kappa_m + m + 1)$. □

Corollary 3.2.16. *The Zernike polynomials $\mathcal{Z}_{m,n}^{-1}$, corresponding to the limit case of $\kappa_m = -1$ and fixed m, are harmonic functions for the Laplacian*

$$\left\{(1-|z|^2)\partial\overline{\partial} + m\left(E - \overline{E}\right) - m^2\right\}\mathcal{Z}_{m,n}^{-1}.$$

Proof. This readily follows by specifying $\rho = 0$ in Theorem 3.2.15 and choosing α and γ such that $m = (\alpha/2) - \gamma$. Indeed, in this case we have $\kappa_m + m + 1 = m$ and the left hand side of (3.2.28) reduces further to the Landau Hamiltonian $(1-|z|^2)\left\{(1-|z|^2)\partial\overline{\partial} + m\left(E - \overline{E}\right)\right\} + m^2|z|^2$ with quantized constant magnetic field of magnitude m. □

3.2.4 Recurrence and operational formulas

From the three terms recurrence formula in [69, p. 213] for the Jacobi polynomials one can deduces

$$A_{m,b}z^2\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\bar{z}) + B_{m,b}z\mathcal{Z}^{\kappa,\rho-1}_{m-1,n}(z,\bar{z}) + C_{m,b}\mathcal{Z}^{\kappa,\rho-2}_{m-2,n}(z,\bar{z}) = 0,$$

valid for all $m = 2, 3, 4, \cdots$, where we have set $A_{m,b} := (b-m)(b+2)$, $B_{m,b} := b(b-1)(b-\kappa-1)$ and $C_{m,b} := b(m-1)(\kappa+m-1)(b-\kappa-m)$ with $b = \kappa+n+m+\rho$. However, starting from the Rodriguez formula for the fractional Zernike functions and rewriting it in the form

$$\mathcal{Z}^{\kappa,\rho}_{m,n} = (-1)^m z^{-\rho}(1-|z|^2)^{-\kappa}\partial_z^{m-1}\left(\partial_z(z^{n+\rho}(1-|z|^2)^{\kappa+m})\right),$$

one derives the recurrence formula

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\bar{z}) = (\kappa+m)\bar{z}\mathcal{Z}^{\kappa,\rho}_{m-1,n}(z,\bar{z}) - (n+\rho)(1-|z|^2)\mathcal{Z}^{\kappa+1,\rho}_{m-1,n-1}(z,\bar{z}). \tag{3.2.29}$$

But, by means of (3.2.27) we can rewrite the recurrence formula (3.2.29) as

$$z\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\bar{z}) = (\kappa+m)\bar{z}\mathcal{Z}^{\kappa,\rho-1}_{m-1,n+1}(z,\bar{z}) - (n+\rho)(1-|z|^2)\mathcal{Z}^{\kappa+1,\rho-1}_{m-1,n}(z,\bar{z}). \tag{3.2.30}$$

Moreover, we can prove the following

$$\mathcal{Z}^{\kappa-1,\rho-1}_{m+1,n+1} = [\kappa|z|^2 + (m-n-\rho)(1-|z|^2)]\mathcal{Z}^{\kappa,\rho}_{m,n} - m(n+\rho+\kappa+1)\bar{z}(1-|z|^2)\mathcal{Z}^{\kappa+1,\rho}_{m-1,n}. \tag{3.2.31}$$

Indeed, this follows from the use (3.2.26), leading to

$$z\frac{\partial}{\partial_z}(\mathcal{Z}^{\kappa,\rho}_{m,n}) - (n-m)\mathcal{Z}^{\kappa,\rho}_{m,n} = \bar{z}m(n+\rho+\kappa+1)\mathcal{Z}^{\kappa+1,\rho}_{m-1,n},$$

combined with the derivation formula

$$(1-|z|^2)\frac{\partial}{\partial_z}(\mathcal{Z}^{\kappa,\rho}_{m,n}) = -\rho(1-|z|^2)\mathcal{Z}^{\kappa,\rho+1}_{m,n-1} + \kappa\bar{z}\mathcal{Z}^{\kappa,\rho}_{m,n} - \mathcal{Z}^{\kappa-1,\rho}_{m+1,n}.$$

The latter one follows from the Rodrigues Formula (0.0.17). In the sequel, we obtain non-trivial recurrence formulas of Nielsen type for the fractional Zernike functions. This follows as specific cases of the so-called Burchnall representation type formulas for such functions. For the exact statement, we let $\mathbf{a}^{\kappa,\rho,m,n}_{q,j,\ell,k}$ and $\mathbf{b}^{\kappa,\rho}_{m,n,j,k}$ respectively, be the constants given by

$$\mathbf{a}^{\kappa,\rho,m,n}_{q,j,\ell,k} := \varepsilon^*_{\rho,m-j}\frac{(-1)^{m+j+\ell}m!q!\Gamma(\rho+1)\Gamma(\kappa+m+1)}{\ell!(m-j)!(j-\ell)!(q-\ell)!\Gamma(\rho-m+j+1)\Gamma(\kappa+m+n+1)}, \tag{3.2.32}$$

and

$$\mathbf{b}^{\kappa,\rho,m,n}_{j,k} := \frac{(-1)^{j+k}m!n!\Gamma(\kappa+m+n+1)}{j!k!(m-j)!(n-k)!\Gamma(\kappa+m+k+1)}. \tag{3.2.33}$$

Proposition 3.2.17. *Let κ, ρ, m and n be as above. Let p be a nonnegative integer and u a real such that $u \geq \max(-\kappa, -1)$. Then, we have*

$$\mathcal{Z}^{\kappa,\rho}_{m,n+q}(z,\overline{z}) = z^q \sum_{j=0}^{m} \sum_{\ell=0}^{j\wedge q} \mathbf{a}^{\kappa,\rho,m,n}_{q,j,\ell,k} \left(\frac{1-|z|^2}{z}\right)^{m+\ell-j} \mathcal{Z}^{\kappa+m+\ell-j}_{j-\ell,n}(z,\overline{z}), \tag{3.2.34}$$

$$\mathcal{Z}^{\kappa,\rho}_{m,n+q}(z,\overline{z}) = \frac{m!q!\Gamma(\kappa+m+1)}{\Gamma(\kappa+m+n+1)} z^q \sum_{j=0}^{m\wedge q} \sum_{k=0}^{n} \frac{(-1)^j}{j!(m-j)!(q-j)!} \left(\frac{1-|z|^2}{z}\right)^{j} \mathcal{Z}^{\kappa+j,\rho}_{m-j,n}(z,\overline{z}) \tag{3.2.35}$$

and

$$\mathcal{Z}^{\kappa+u,\rho}_{m,n}(z,\overline{z}) = \frac{\Gamma(\kappa+u+m+1)}{\Gamma(\kappa+u+m+n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} (-1)^{j+k} \mathbf{b}^{\kappa,\rho,m,n}_{j,k} \mathcal{Z}^{u-j-k}_{j,k}(z,\overline{z}). \tag{3.2.36}$$

Proof. Consider the operator

$$B^{\kappa,\rho}_{m,n}(f) = \frac{\partial^m}{\partial z^m}\left(z^\rho \frac{\partial^n}{\partial \overline{z}^n} \left((1-|z|^2)^{\kappa+m+n} f \right) \right), \tag{3.2.37}$$

for every sufficiently differentiable function f. Making use of the Leibnitz formula applied from outside to inside we arrive at the Burchnall type formula

$$\begin{aligned} B^{\kappa,\rho}_{m,n}(f) &= \sum_{j=0}^{m} \binom{m}{j} \frac{\partial^{m-j}}{\partial z^{m-j}}(z^\rho) \frac{\partial^j}{\partial z^j} \left(\frac{\partial^n}{\partial \overline{z}^n} [(1-|z|^2)^{\kappa+m+n} f] \right) \\ &= z^\rho (1-|z|^2)^{\kappa+m} \sum_{j=0}^{m} \sum_{\ell=0}^{j} \sum_{k=0}^{n} \mathbf{a}^{\kappa,\rho}_{m,n,j,\ell,k} z^{j-m} (1-|z|^2)^{k+\ell-j} \mathcal{Z}^{\kappa+m+k+\ell-j}_{j-\ell,n-k}(z,\overline{z}) \frac{\partial^{\ell+k}}{\partial z^\ell \partial \overline{z}^k}(f), \end{aligned} \tag{3.2.38}$$

where the involved constant is given by

$$\mathbf{a}^{\kappa,\rho}_{m,n,j,\ell,k} := (-1)^{m+n+k} \frac{n!(q-\ell)!\Gamma(\kappa+m+n+1)}{q!k!(n-k)!\Gamma(\kappa+m+1)} \mathbf{a}^{\kappa,\rho,m,n}_{q,j,\ell}.$$

Similarly, we get (by Leibnitz formula from inside to outside)

$$\begin{aligned} B^{\kappa,\rho}_{m,n}(f) &= \frac{\partial^m}{\partial z^m} \left(z^\rho \left[\sum_{k=0}^{n} \binom{n}{k} \frac{\partial^{n-k}}{\partial \overline{z}^{n-k}} \left((1-|z|^2)^{\kappa+m+n} \right) \frac{\partial^k}{\partial \overline{z}^k}(f) \right] \right) \\ &= (-1)^{m+n} z^\rho \sum_{j=0}^{m} \sum_{k=0}^{n} \mathbf{b}^{\kappa,\rho}_{m,n,j,k} (1-|z|^2)^{\kappa+j+k} \mathcal{Z}^{\kappa+j+k,\rho}_{m-j,n-k}(z,\overline{z}) \frac{\partial^{j+k}}{\partial z^j \partial \overline{z}^k}(f). \end{aligned} \tag{3.2.39}$$

Therefore, since the action of $B^{\kappa,\rho}_{m,n}$ for the specific case of $f = e_q = z^q$ reduces to

$$B^{\kappa,\rho}_{m,n}(z^q) = (-1)^{m+n} (\kappa+m+1)_n z^\rho (1-|z|^2)^\kappa \mathcal{Z}^{\kappa,\rho}_{m,n+q}(z,\overline{z}),$$

we obtain (3.2.34) (resp. (3.2.35)) from (3.2.38) (resp. (3.2.39)). The identity (3.2.36) follows from (3.2.39) by considering the particular case of $f(z) = (1-|z|^2)^u$ and observing that $B^{\kappa,\rho}_{m,n}((1-|z|^2)^u g) = B^{\kappa+u,\rho}_{m,n}(g)$. ☐

Remark 3.2.18. *The Burchnall representation in* (3.2.39) *for f being an holomorphic function simply reads*

$$B^{\kappa,\rho}_{m,n}(f) = (-1)^{m+n} z^{\rho} h^{\kappa} \sum_{j=0}^{m} \mathbf{b}^{\kappa,\rho}_{m,n,j,0} (1-|z|^2)^j \mathcal{Z}^{\kappa+j,\rho}_{m-j,n}(z,\overline{z}) \frac{\partial^j (f)}{\partial z^j}. \tag{3.2.40}$$

Analog representation for arbitrary f (not necessary holomorphic) can be developed using the differential operator

$$A^{\rho,\kappa}_{m,n}(f) = \frac{\partial^m}{\partial z^m}\left(z^{n+\rho}(1-|z|^2)^{\kappa+m} f\right). \tag{3.2.41}$$

More precisely, one obtains

$$A^{\rho,\kappa}_{m,n}(f) = (-1)^m m! z^{\rho} h^{\kappa} \sum_{j=0}^{m} \frac{(-1)^j (1-|z|^2)^j}{j!(m-j)!} \mathcal{Z}^{\kappa+j,\rho}_{m-j,n}(z,\overline{z}) \frac{\partial^j f}{\partial z^j}. \tag{3.2.42}$$

3.2.5 Generating and bilinear generating functions

The aim here is to obtain some generating and bilinear generating functions for the fractional Zernike functions. First, it is worth noting that from Proposition 3.2.5 and making use of the generating function for the Jacobi polynomials in [69, p. 213] we obtain

$$\sum_{m=0}^{+\infty} \frac{u^m}{m!} \mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) = 2^n z^{2n+1-m+\rho+\kappa} \frac{(z-u+R(u,z))^{m-n-\rho}(z+u+R(u,z))^{-\kappa}}{R(u,z)}.$$

Here $R(u,z) = 1$ for $u = 0$ and $R(u,z) = (z^2 + 2uz(1-2|z|^2) + z^2(1-2|z|^2)^2)^{1/2}$ when $u \neq 0$.

The next result gives the expression of a special bilinear generating functions as derivative of the confluent and Gauss hypergeometric functions by means of the partial differential operator

$$R^{\kappa,\rho}_m f(z) = \frac{1}{(z\overline{w})^{\rho}(1-|z|^2)^{\kappa}(1-|w|^2)^{\kappa}} \frac{\partial^{2m}}{\partial z^m \partial \overline{w}^m}\left((z\overline{w})^{\rho}(1-|z|^2)^{\kappa+m}(1-|w|^2)^{\kappa+m} f\right)(z)$$

for sufficiently differential function f.

Proposition 3.2.19. *We have*

$$\sum_{n=0}^{+\infty} \frac{(a)_n}{n!(c)_n} \mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) \overline{\mathcal{Z}^{\kappa,\rho}_{m,n}(w,\overline{w})} = R^{\kappa,\rho}_m \left({}_1F_1 \left(\begin{matrix} a \\ c \end{matrix} \middle| z\overline{w} \right)\right) \tag{3.2.43}$$

and

$$\sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{n!(c)_n} \mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) \overline{\mathcal{Z}^{\kappa,\rho}_{m,n}(w,\overline{w})} = R^{\kappa,\rho}_m \left({}_2F_1 \left(\begin{matrix} a,b \\ c \end{matrix} \middle| z\overline{w} \right)\right). \tag{3.2.44}$$

Proof. This readily follows by means of the Rodrigues' formula for $\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})$. Indeed, we can rewrite the left-hand side in (3.2.43) as

$$\frac{1}{(z\overline{w})^{\rho}(1-|z|^2)^{\kappa}(1-|w|^2)^{\kappa}} \frac{\partial^{2m}}{\partial z^m \partial \overline{w}^m}\left((z\overline{w})^{\rho}[(1-|z|^2)(1-|w|^2)]^{\kappa+m} \sum_{n=0}^{+\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}\right),$$

which reduces further to (3.2.43). The formula (3.2.44) follows in a similar way. □

Remark 3.2.20. *For the special values of* $a = 1$, $b = \kappa + \rho + 1$ *and* $c = \rho + 1$ *with* $\rho = \beta - 2\eta$ *and* $\kappa = \kappa_m = \alpha - 2(\gamma + m) - 1$, *the quantity* $n!(c)_n/(a)_n(b)_n$ *reduces to be the square norm of* $\psi^{\gamma,\eta}_{m,n}$ *in* (3.1.12) *up to a multiplicative constant* $d^{\kappa,\rho}_m$ *independent of* n. *Thus, formula in* (3.2.44) *leads to the reproducing kernel*

$$K^{\alpha,\beta}_{\gamma,\eta,m}(z,w) = \sum_{n=0}^{+\infty} \frac{\psi^{\gamma,\eta}_{m,n}(z,\overline{z})\overline{\psi^{\gamma,\eta}_{m,n}(w,\overline{w})}}{\left\|\psi^{\gamma,\eta}_{m,n}\right\|^2_{\alpha,\beta}}$$

of the m-th generalized β-modified Bergman space introduced in Remark 3.1.8. In fact, we have

$$K^{\alpha,\beta}_{\gamma,\eta,m}(z,w) = d^{\kappa,\rho}_m \frac{[(1-|z|^2)(1-|w|^2)]^{\gamma+m}}{|zw|^{2\eta}} R^{\kappa,\rho}_m\left({}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix} \middle| z\overline{w} \right)\right).$$

A closed formula for $K^{\alpha,\beta}_{\gamma,\eta,m}(z,w)$ *needs further investigation.*

Below, we prove another bilinear generating function for the fractional Zernike function that looks like the Hardy–Hille formula for the generalized Laguerre polynomials. Thus, we deal with

$$G^{\kappa,\rho}_n(z,w|t) := \sum_{m=0}^{+\infty} \frac{t^m \mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z})\mathcal{Z}^{\kappa,\rho}_{m,n}(\overline{w},w)}{m!(\kappa+1)_m}. \tag{3.2.45}$$

Proposition 3.2.21. *For every sufficiently small* t, *the closed expression of* $G^{\kappa,\rho}_n(z,w|t)$ *is given by*

$$G^{\kappa,\rho}_n(z,w|t) = \frac{(z+tw)^{n+\rho}(\overline{w}+t\overline{z})^{n+\rho}}{(z\overline{w})^{\rho}(1+t\overline{z}w)^{2(n+\rho)+\kappa+1}} {}_2F_1\left(\begin{matrix} -n-\rho, -n-\rho \\ \kappa+1 \end{matrix} \middle| -\frac{t(1-|z|^2)(1-|w|^2)}{(z+tw)(\overline{w}+t\overline{z})} \right).$$

Proof. Using the hypergeometric representation (Proposition 3.2.5) we can rewrite $G^{\kappa,\rho}_n(z,w|t)$ as

$$\begin{aligned} G^{\kappa,\rho}_n(z,w|t) &= (z\overline{w})^n \sum_{m=0}^{+\infty} \frac{(\kappa+1)_m}{m!}(t\overline{z}w)^m \\ &\quad\times {}_2F_1\left(\begin{matrix} -m, -n-\rho \\ \kappa+1 \end{matrix} \middle| 1-\frac{1}{|z|^2} \right) {}_2F_1\left(\begin{matrix} -m, -n-\rho \\ \kappa+1 \end{matrix} \middle| 1-\frac{1}{|w|^2} \right). \end{aligned}$$

Thus, one concludes for the result in Proposition 3.2.21 by making use of the Meixner bilinear relation in [28, Eq. (12), p. 85]. □

3.2.6 Integral representations

By means of the classical integral representations for the Gauss hypergeometric functions in the right hand side of (3.2.6) and (3.2.9) or for the Jacobi polynomials in (3.2.11), we can derive different integral representations for $\mathcal{Z}^{\rho,\kappa}_{m,n}(z,\bar{z})$. However, we give below some non-trivial ones. The first one is based on the Cauchy integral formula for holomorphic functions as in [60].

Theorem 3.2.22. *The fractional Zernike function $\mathcal{Z}^{\kappa,\rho}_{m,n}$ admits the following integral representation*

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) = \frac{(-1)^m m!}{2\pi i} z^{-\rho}(1-|z|^2)^{-\kappa} \oint_{|t|=1} t^{n+m+\rho+\kappa} \frac{(\overline{t}-\overline{z})^{\kappa+m}}{(t-z)^{m+1}} dt. \tag{3.2.46}$$

Proof. Make use of the ordinary binomial expansion with the factorial function [84, p. 56],

$$(1-\xi)^{-a} = \sum_{j=0}^{+\infty} (a)_j \frac{\xi^j}{j!},$$

to expand the factor $(1-|z|^2)^j$ in the explicit expression of $\mathcal{Z}^{\kappa_m,\rho}_{m,n}$ given by (3.2.3). Also, we need to recall that

$$\frac{\partial^m}{\partial z^m}(z^{j+n+\rho}) = \frac{m!}{2\pi i} \oint_{|t|=1} \frac{t^{j+n+\rho}}{(t-z)^{m+1}} dt,$$

which follows from the Cauchy integral formula applied to the function $\varphi_z(t) = t^{m+j+\rho}/(t-z)$. Thus, we obtain

$$\begin{aligned}
\mathcal{Z}^{\kappa,\rho}_{m,n}(z,\overline{z}) &= (-1)^m z^{-\rho}(1-|z|^2)^{-\kappa} \sum_{j=0}^{+\infty} \frac{(-\kappa-m)_j}{j!} \overline{z}^j \frac{\partial^m}{\partial z^m}\left(z^{n+\rho+j}\right) \\
&= \frac{(-1)^m m!}{2\pi i} z^{-\rho}(1-|z|^2)^{-\kappa} \oint_{|t|=1} \frac{t^{n+\rho}}{(t-z)^{m+1}} \left(\sum_{j=0}^{+\infty}(-\kappa-m)_j \frac{(t\overline{z})^j}{j!}\right) dt \\
&= \frac{(-1)^m m!}{2\pi i} z^{-\rho}(1-|z|^2)^{-\kappa} \oint_{|t|=1} \frac{t^{n+\rho}(1-t\overline{z})^{\kappa+m}}{(t-z)^{m+1}} dt \\
&= \frac{(-1)^m m!}{2\pi i} z^{-\rho}(1-|z|^2)^{-\kappa} \oint_{|t|=1} t^{n+m+\rho+\kappa} \frac{(\overline{t}-\overline{z})^{\kappa+m}}{(t-z)^{m+1}} dt.
\end{aligned}$$

This proves (3.2.46). □

The next integral representation for the fractional Zernike functions appears as corollary of the bilinear generating function in Proposition 3.2.21.

Proposition 3.2.23. *Let $\gamma^{\kappa,\rho}_{m,n}$ be as in* (3.2.17). *The fractional Zernike functions have the integral representation*

$$\begin{aligned}
\mathcal{Z}^{\kappa,\rho}_{m,n}(w,\overline{w}) = \frac{m!(\kappa+1)_m \gamma^{\kappa,\rho}_{m,n}}{w^\rho t^m} \int_D \Xi^{\kappa,\rho}_{m,n}(z,w|t) {}_2F_1\left(\begin{matrix} -m, n+\kappa+\rho+1 \\ \kappa+1 \end{matrix} \middle| 1-|z|^2\right) \\
\times {}_2F_1\left(\begin{matrix} -n-\rho, -n-\rho \\ \kappa+1 \end{matrix} \middle| -\frac{t(1-|z|^2)(1-|w|^2)}{(w+tz)(\overline{z}+t\overline{w})}\right) d\lambda(z),
\end{aligned} \tag{3.2.47}$$

where

$$\Xi^{\kappa,\rho}_{m,n}(z,w|t) := \frac{\overline{z}^{m-n-\rho}(w+tz)^{n+\rho}(\overline{z}+t\overline{w})^{n+\rho}}{(1+tz\overline{w})^{2(n+\rho)+\kappa+1}}(1-|z|^2)^\kappa.$$

Proof. Starting from the expansion (3.2.45) one gets

$$\mathcal{Z}^{\kappa,\rho}_{m,n}(w,\overline{w}) = \frac{m!(\kappa+1)_m \gamma^{\kappa,\rho}_{m,n}}{t^m}\overline{\langle G^{\kappa,\rho}_n(\cdot,w|t), \mathcal{Z}^{\kappa,\rho}_{m,n}\rangle_{L^{2,\kappa}_\rho(\mathbb{D})}}.$$

Next, by means of the explicit expression of the kernel function $G^{\kappa,\rho}_n(z,w|t)$ given in Proposition 3.2.21 combined with the hypergeometric representation in (3.2.6), one derives the integral representation (3.2.47). □

Remark 3.2.24. *The integral in the right hand side of* (3.2.47) *is a rigid integral on complex domain* $\mathbb{D}$ *in the sense that it is nontrivial and can not be reduced to classical integral on real domains.*

3.2.7 Completeness

Here we discuss the completeness of the fractional Zernike functions in $L^{2,\kappa}_\rho(\mathbb{D})$. Notice for instance that for $\rho = 0,1,2,\cdots$, the functions $z^\rho \mathcal{Z}^{\kappa,\rho}_{m,n}$ for varying $m, n+\rho = 0,1,2,\cdots$, constitute an orthogonal basis of $L^{2,\kappa}_\rho(\mathbb{D})$ since they are closely connected to the complex Zernike polynomials $\mathcal{Z}^{\kappa}_{m,n+\rho}$ by (0.0.18). The latter ones are known to form an orthogonal basis for the Hilbert space $L^{2,\kappa}_0(\mathbb{D})$ (see e.g., [23, 59]). A direct proof starts from the observation that each $(\mathcal{Z}^{\kappa,\ell}_{m,n})_{m,n}$ is a polynomial of exact degrees m in $\overline{z}$ and n in z, so that any $z^n\overline{z}^m$ can be rewritten as

$$\overline{z}^m z^n = \sum_{j=0}^{m}\sum_{k=0}^{n} a^{m,n}_{j,k}\, \mathcal{Z}^{\kappa,\ell}_{j,k}(z,\overline{z}).$$

The coefficients $a^{m,n}_{j,k}$ are explicit and can be computed by the formula

$$a^{m,n}_{j,k} = \gamma^{\kappa,\rho}_{m,n}\int_{\mathbb{D}} \overline{z}^m z^n \overline{\mathcal{Z}^{\kappa,\ell}_{j,k}}(z,\overline{z})|z|^{2\rho}(1-|z|^2)^\kappa d\lambda(z),$$

where $\gamma^{\kappa,\rho}_{m,n}$ is as in (3.2.17). In the sequel, we consider only the case of ρ is non-integer $\rho > -1$.

Theorem 3.2.25. *The functions* $\Upsilon^{\kappa,\rho}_{m,s} := \overline{z}^{-s/2}\mathcal{Z}^{\kappa,\rho-s/2}_{m,m+s/2}$, *for varying nonnegative integer m and varying integer s, form an orthogonal complete system in* $L^{2,\kappa}_\rho(\mathbb{D})$.

Proof. The orthogonality in $L^{2,\kappa}_\rho(\mathbb{D})$ is quite trivial for $\Upsilon^{\kappa,\rho}_{m,s}$ being a direct product of an orthogonal system in radial variable and an orthogonal system in the angular variable. This also follows by means of the Jacobi polynomials $P^{(\kappa,\rho)}_m$ in the Hilbert $L^2_{\kappa,\rho}$ of square integrable functions on $[-1,1[$ with respect to the measure $(1-x)^\kappa(1+x)^\rho dx$, since

$$\Upsilon^{\kappa,\rho}_{m,s}(z,\overline{z}) = \frac{z^s}{|z|^s}\mathcal{Z}^{\kappa,\rho}_{m,m}(z,\overline{z}) = m! e^{is\theta}P^{(\kappa,\rho)}_m(x),$$

for $z=\sqrt{(1+x)/2}\,e^{i\theta}$ with $x\in[-1,1[$ and $\theta\in[0,2\pi[$. More exactly, we have

$$\int_D Y^{\kappa,\rho}_{m,s}(z,\overline{z})\overline{Y^{\kappa,\rho}_{n,r}(z,\overline{z})}|z|^{2\rho}(1-|z|^2)^{\kappa}d\lambda(z)=\frac{m!n!\pi}{2^{\kappa+\rho+1}}\left\|P^{(\kappa,\rho)}_m\right\|^2_{L^2_{\kappa,\rho}}\delta_{m,n}\delta_{s,r}.$$

For their completeness, let $f\in F^{\perp}$, the orthogonal of $F=Span\{Y^{\kappa,\rho}_{m,s};m=0,1,2,\cdots,s\in\mathbb{Z}\}$ in $L^{2,\kappa}_{\rho}(\mathbb{D})$. Thus, by Proposition 3.2.5, the assumption that $f\in F^{\perp}$ becomes equivalent to

$$\langle f,Y^{\kappa,\rho}_{m,s}\rangle_{L^{2,\kappa}_{\rho}}=\frac{m!}{2^{\kappa+\rho+2}}\int_{-1}^{1}(1-x)^{\kappa/2}(1+x)^{\rho/2}P^{(\kappa,\rho)}_m(x)\hat{f}^{\kappa,\rho}_s(x)dx=0,$$

for every integer s and $m=0,1,2,\cdots$. The involved function is defined by

$$\hat{f}^{\kappa,\rho}_s(x):=(1-x)^{\kappa/2}(1+x)^{\rho/2}\hat{f}_s(x),$$

where $\hat{f}_s(x)$ denotes the s-th Fourier coefficient of the function $f_x:=\theta\longmapsto f(\sqrt{(1+x)/2}\,e^{i\theta})$ for every fixed $x\in[-1,1[$. Clearly $\hat{f}^{\kappa,\rho}_s$ belongs to $L^2([-1,1[;dt)$ since by means of the Cauchy-Schwartz inequality and the Fubini's theorem one gets

$$\int_{-1}^{1}|\hat{f}^{\kappa,\rho}_s(x)|^2dx\leq 2\pi\int_{-1}^{1}(1-x)^{\kappa}(1+x)^{\rho}\left(\int_0^{2\pi}\left|f\left(\sqrt{\frac{1+x}{2}}e^{i\theta}\right)\right|^2d\theta\right)dx=2^{\kappa+\rho+3}\pi\|f\|^2_{L^{2,\kappa}_{\rho}}.$$

Therefore, $\hat{f}^{\kappa,\rho}_s=0$ a.e on $[-1,1[$ for every $s\in\mathbb{Z}$ for the functions $(1-x)^{\kappa/2}(1+x)^{\rho/2}P^{(\kappa,\rho)}_m$, $m=0,1,2,\cdots$, being an orthogonal basis of $L^2([-1,1[,dt)$. This implies in particular that the Fourier transform of $f_x\in L^2([0,2\pi[,d\theta)$ satisfies $\mathcal{F}(f_x)(\ell)=\hat{f}_{-s}(x)=0$ for every $s\in\mathbb{Z}$ and every fixed $x\in[-1,1[\backslash N$, where we have set $N:=\cup_s\{x\in[-1,1[;\hat{f}_s(x)\neq 0\}$. This proves that the function $f_x=0$ a.e. on $[0,2\pi[$ for almost every $x\in[-1,1[$. Therefore, f is a vanishing function almost everywhere on D. This completes the proof. □

Corollary 3.2.26. *The Hilbert spaces* $A^{\kappa,\rho}_s(\mathbb{D}):=\overline{Span\{\overline{z}^{-s/2}\mathcal{Z}^{\kappa,\rho-s/2}_{m,m+s/2};m=0,1,2,\cdots\}}^{L^{2,\kappa}_{\rho}}$; $s\in\mathbb{Z}$ *defines a Hilbertian orthogonal decomposition of* $L^{2,\kappa}_{\rho}(\mathbb{D})$. *Namely, we have*

$$L^{2,\kappa}_{\rho}(\mathbb{D})=\bigoplus_{s\in\mathbb{Z}}A^{\kappa,\rho}_s(\mathbb{D}).$$

Definition 3.2.27. *The closed subspace* $A^{\kappa,\rho}_s(\mathbb{D})$ *are called generalized (poly-meromorphic) Bergman spaces of second kind.*

FUTURE WORK

The fractional side as well as the associated functional spaces of Segal–Bargmann type will be introduced and studied in details in a forthcoming research paper.

3.3 Concluding remaks

The set of the classical real Zernike polynomials has found numerous applications in a variety of fields as explained in the introductory section, including optical engineering and geometrical optics. In particular, they have been applied in visual optics and for fitting visual surfaces as corneal and eye surfaces [16, 90, 92, 102]. For example, recent discussions address the precision and accuracy of these polynomials when applied to surfaces such as the human cornea. In this context, their accuracy of different orders when fitting several types of theoretical corneal and wave-front surface data has been investigated in [16]. The introduced fractional Zernike functions appear as special generalizations of these polynomials and contain additional parameters that may encode many information. Accordingly, we think that the present theoretic study will serve as a valuable resource for experts aiming to illustrate the usefulness of this new class in various optical disciplines. Moreover, this study may shed new light on some new and potential applications for generating representations of corneal surfaces, optical testing and ophthalmic optics.

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Résumé

Le développement important des méthodes numériques et le rôle croissant des simulations informatiques ont considérablement accru l'intérêt pour les fonctions spéciales, raison pour laquelle nous nous intéressons à des nouvelles classes de fonctions spéciales. Dans ce travail, nous proposons premièrement une étude théorique d'une nouvelle famille de fonctions orthogonales sur le plan C* étendant les polynômes poly- analytiques Itô-Hermite au cadre poly-méromorphe. Nous résolvons les problèmes aux valeurs propres du Laplacien magnétique perturbé par un vecteur potentiel singulier et modélisant l'effet d'Aharonov-Bohm. De plus, nous définissons une nouvelle famille de fonctions de type Hermite d'ordre réel β par une formule de Rodrigues fractionnaire et faisant appel à la dérivée de Caputo. Nous donnons également certaines de leurs propriétés ainsi que leur représentation en termes de la fonction hypergéométrique confluente. Nous considérons ensuite une famille de fonctions de Zernike fractionnaires sur le disque unité étoilé, généralisant les polynômes de Zernike classiques. Principalement, nous montrons qu'il s'agit de fonctions propres de carré intégrable, orthogonales pour certain Laplacien magnétique perturbé, et nous établissons leurs propriétés algébriques et analytiques comme la description de leurs zéros, les équations différentielles, les relations de récurrence et les formules opérationnelles qu'ils satisfont. De plus, nous discutons leur régularité en tant que fonctions poly-méromorphes et obtenons leurs représentations intégrales et fonctions génératrices, dont une bilinéaire de type Hardy-Hille. De plus, nous prouvons qu'une sous-classe tronquée définit un système orthogonal complet dans l'espace de Hilbert sous-jacent donnant lieu à une décomposition orthogonale hilbertienne spécifique pour une classe d'espaces de Bergman généralisés.

Mots-clefs : Fonctions d'Hermite polyméromorphes ; Fonctions génératrices ; Laplacien magnétique perturbé ; Calcul fractionnaire ; Fonctions de Zernike fractionnaires.

Abstract

The extensive development of numerical methods and the growing role of computer simulations have greatly increased interest in special functions, the reason why we are interested in considering new classes of such functions. Firstly, we provide a theoretical study of a new family of orthogonal functions on the punctured complex plane, extending the poly-analytic It\^o--Hermite polynomials to the poly-meromorphic setting and solving the eigenvalue problems for some perturbed magnetic Laplacian modeling an Aharonov-- Bohm effect. Additionally, we define a novel family of Hermite functions of real order β by means of a fractional Rodrigues formula involving the Caputo derivative. We discuss also and establish some of their properties as well as their representation in terms of the Kummer's function. Finally, we consider the so-called fractional Zernike functions defined on the punctured unit disc and generalizing the classical Zernike polynomials. Mainly, we show that they are orthogonal L^2-eigenfunctions for certain perturbed magnetic Laplacian. We establish their algebraic and analytic properties such as the description of their zeros, the differential equations, the recurrence and operational formulas they satisfy. Moreover, we discuss their regularity as poly-meromorphic functions and obtain some of their integral representations and generating functions, including a bilinear one of Hardy--Hille type. Furthermore, we prove that a truncated subclass defines a complete orthogonal system in the underlying Hilbert space giving rise to a specific Hilbertian orthogonal decomposition of a class of generalized Bergman spaces.

Keywords : Poly-meromorphic Ito–Hermite functions; Generating functions; Perturbed magnetic Laplacian; Fractional calculus; Fractional Zernike functions.