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## ملخص الأطروحة

تركز الأطروحة بشكل رئيسي على مسائل الممثلة ذات المستويين، وبالأخص النسخة العامة المعروفة بمسائل الممثلة ذات المستويين شبه المتجهية.

من ناحية، ندرس مسائل الممثلة متعددة الأهداف مع دالة هدف في المستوى الأدنى ذات قيمة متجهية. نسعى إلى استكشاف نهج التقييس الخاص بـ تشيرنز-كوبر للبرامج متعددة الأهداف واستخدام إعادة صياغة القيمة المثلى لتحويل المسألة القياسية الناتجة إلى مسألة ممثلة ذات مستوى واحد. أولاً، باستخدام حساب التفاضل والتكامل المعمم لـ بوريس موردوخوفيتش، نقترح تقديرًا جديدًا لدالة القيمة المثلى ونوفر الظروف الكافية لضمان خاصية ليبشيتز المحلية. بالإضافة إلى ذلك، نقوم بتطوير شروط المثالية الضرورية من الدرجة الأولى في السياقات السلسة وغير السلسة.

ومن ناحية أخرى، نتعامل مع حالة خاصة من مشكلة الممثلة ذات المستويين شبه المتجهية، حيث تكون مشكلة المستوى الأدنى هي مشكلة توازن ناش المعممة متعددة الأهداف. نطبق إعادة صياغة على مستوى واحد على هذا النوع من المشاكل. تتكون هذه الطريقة من تحويل المشكلة إلى مشكلة ممثلة أحادية المستوى باستخدام قيد الهدف الموزون وإعادة صياغة القيمة المثلى. هنا مرة أخرى، نستخدم حساب التفاضل والتكامل المعمم لـ بوريس موردوخوفيتش لاشتقاق شروط المثالية الضرورية من الدرجة الأولى، مفصلة بالكامل في بيئة سلسة.

وأخيرا، تناولنا مشكلة توازن ناش المعممة متعددة الأهداف، وهي مشكلة توازن ناش مع قيود تشمل مجموعات متعددة الأهداف. نحن نركز على دراسة شروط كاروش-كون-تاكر التقريبية لمشكلات توازن ناش المعممة متعددة الأهداف وتأثيرها على التقارب العام للخوارزميات. قمنا أولاً بتعريف شروط كاروش-كون-تاكر الضعيفة التقريبية القياسية لمشكلات توازن ناش المعممة متعددة الأهداف، وإظهار أن أي حل ضعيف الكفاءة يلبي هذه الشروط. نظرًا لأن شروط المثالية هذه مقيدة للغاية، فإننا نقترح مجموعة جديدة من شروط كاروش-كون-تاكر الضعيفة التقريبية، والتي تم تكييفها خصيصًا لمسائل توازن ناش المعممة متعددة الأهداف، وإظهار تقاربها مع نقاط كاروش-كون-تاكر الضعيفة، بشرط وجود استمرارية مخروطية الملكية راضية. على الرغم من أن شروط كاروش-كون-تاكر الضعيفة التقريبية الجديدة هذه لا تمثل شروطًا مثالية لمسائل توازن ناش العامة متعددة الأهداف، إلا أننا نوضح أنها صالحة للحالة الخاصة لنقطة توازن تباينية ضعيفة في مشكلة توازن ناش المعممة متعددة الأهداف والمحدبة المشتركة . أخيرًا، نقترح خوارزمية محسنة من نوع لاغرانج لتقدير تقريبي ضعيف لكاروش-كون-تاكر لمشكلة توازن ناش المعممة متعددة الأهداف وإظهار تقاربها العالمي.



***الكلمات المفتاحية:*** مشكلة ثنائية المستوى؛ الممثلة متعدد الأهداف؛ تقنية تشارنس-كوبر؛ وظيفة القيمة المثلى. الظروف المثالية؛ مشكلة توازن ناش المعممة. نهج القيد الموزون الموضوعي؛ عدم المساواة المتغيرة. ظروف كاروش-كون-تاكر التقريبية؛ مؤهلات القيد، خوارزمية لاغرانج.

# Abstract

The main focus of the thesis is on bilevel optimization problems, in particular the generalized version known as semivectorial bilevel optimization problems. On the one hand, we consider multi-objective optimization problems with a vector-valued lower-level objective function. We aim to explore the Charnes-Cooper scalarization approach for multi-objective programs and use the optimal value reformulation to convert the resulting scalar problem into a single-level optimization problem. First, using Boris Mordukhovich's extended differentiation calculus, we propose a new estimate of the optimal value function and provide sufficient conditions to guarantee its local Lipschitz property. In addition, we develop the necessary first-order optimality conditions in smooth and non-smooth contexts. On the other hand, we deal with a special case of a semivectorial bilevel optimization problem, the lower-level problem being a generalized multi-objective Nash equilibrium problem. We apply a single-level reformulation to this type of problem. This method consists of transforming the problem into a single-level optimization problem using the k-th weighted objective constraint and the optimal value reformulation. Here again, we use Boris Mordukhovich's generalized differentiation calculus to derive the necessary first-order optimality conditions, fully detailed in a smooth setting. Finally, we tackled the generalized multiobjective Nash equilibrium problem, which is a Nash equilibrium problem with constraints that include multiobjective sets. We focus on examining the approximate Karush-Kuhn-Tucker conditions for generalized multi-objective Nash equilibrium problems and their impact on the overall convergence of algorithms. We first define the standard approximate weak Karush-Kuhn-Tucker conditions for multiobjective generalized Nash equilibrium problems and show that any weakly efficient solution satisfies these conditions.

Since these optimality conditions are too restricted, we propose a new set of approximate weak Karush-Kuhn-Tucker conditions, specially adapted to multiobjective generalized Nash equilibrium problems, and demonstrate their convergence to weak Karush-Kuhn-Tucker points, provided that a conic continuity property is satisfied. Although these new approximate weak Karush-Kuhn-Tucker conditions do not serve as optimality conditions for general multiobjective generalized Nash equilibrium problems, we illustrate that they are valid for the special case of a weak variational equilibrium point in a joint convex multiobjective generalized Nash equilibrium problem. Lastly, we propose an improved Lagrangian-type algorithm for estimating an approximate weak Karush-Kuhn-Tucker of a generalized multi-objective Nash equilibrium problem and demonstrate its global convergence.

**Key Words :** Bilevel problem; multiobjective optimization; Charnes–Cooper technique; optimal value function; optimality conditions; generalized Nash equilibrium problem; kth-objective weighted-constraint approach; Variational inequalities; Approximate Karush–Kuhn–Tucker conditions; Constraint qualifications, Lagrangian-type algorithm.

# Résumé de la thèse

Cette thèse se concentre principalement sur les problèmes d'optimisation à deux niveaux, en particulier la version généralisée connue sous le nom de problèmes d'optimisation à deux niveaux semi-vectoriels. D'une part, nous étudions un problème d'optimisation multiobjectif à deux niveaux avec une fonction objective de niveau inférieur à valeur vectorielle. Nous revisitons la technique de scalarisation de Charnes-Cooper pour les programmes multi-objectifs et utilisons la reformulation de la valeur optimale pour transformer le problème scalaire en un problème d'optimisation à un seul niveau. En utilisant le calcul de différentiation généralisé développé par Boris Mordukhovich, nous fournissons une nouvelle estimation sous-différentielle de la fonction marginale et nous établissons des conditions suffisantes pour garantir la propriété de Lipschitz locale de la fonction de valeur optimale. En outre, nous construisons des conditions d'optimalité nécessaires du premier ordre dans le cadre lisse et non lisse. D'autre part, nous traitons un cas spécial de problèmes d'optimisation semi-vectoriels à deux niveaux, où le problème de niveau inférieur est un problème d'équilibre de Nash multiobjectif généralisé. Nous appliquons une reformulation à un seul niveau pour ce type de problème. Cette méthode consiste à transformer le problème en un problème d'optimisation à un seul niveau, en utilisant la contrainte du k-ième objectif pondéré et la reformulation de la fonction marginale. Nous utilisons une autre fois le calcul de différenciation généralisé de Boris Mordukhovich pour dériver les conditions d'optimalité nécessaires du premier ordre, entièrement détaillées dans le cadre lisse.

Finalement, nous traitons le problème d'équilibre de Nash généralisé multiobjectif qui est un problème d'équilibre de Nash avec des contraintes qui incluent des jeux multiobjectif. Nous nous concentrons sur

l'étude des conditions approximatives de Karush-Kuhn-Tucker pour les problèmes d'équilibre de Nash généralisés multiobjectif et leur impact sur la convergence globale des algorithmes. Pour cette raison, nous définissons d'abord les conditions Karush-Kuhn-Tucker faibles approximatives standards pour les problèmes d'équilibre de Nash généralisé multiobjectif et nous montrons que toute solution faiblement efficace satisfait à ces conditions. Étant donné que ces conditions d'optimalité sont trop restrictives, nous proposons un nouvel ensemble de conditions Karush-Kuhn-Tucker faibles approximatives, spécialement adaptées aux problèmes d'équilibre de Nash généralisé multiobjectif, et nous démontrons leur convergence vers les points Karush-Kuhn-Tucker faibles, à condition qu'une propriété de continuité conique soit satisfaite. Bien que ces nouvelles conditions Karush-Kuhn-Tucker faibles approximatives ne servent pas des conditions d'optimalité pour les problèmes d'équilibre de Nash généralisé multiobjectif généraux, nous illustrons qu'elles sont valables pour le cas particulier d'un point d'équilibre variationnel faible dans un problème d'équilibre de Nash généralisé multiobjectif convexe conjoint. Enfin, nous proposons un algorithme de type lagrangien amélioré pour estimer une condition Karush-Kuhn-Tucker faible approximative d'un problème d'équilibre de Nash généralisé multiobjectif et nous démontrons sa convergence globale.

**Mots clés :** Problème à deux niveaux ; optimisation multiobjectif ; technique de Charnes-Cooper ; fonction de valeur optimale ; conditions d'optimalité ; problème d'équilibre de Nash généralisé ; approche des contraintes pondérées du kème objectif ; inégalités variationnelles ; conditions approximatives de Karush-Kuhn-Tucker ; qualifications des contraintes ; algorithme de type lagrangien.

# Preface

**Published papers:**

1. Youness, E. Y., Lafhim, L.. Necessary optimality conditions for a semivectorial bilevel optimization problem via a revisited Charnes–Cooper scalarization. Optimization, 1–24 (2023). doi: 10.1080/02331934.2023.2249916

2. Youness, E. Y., Kalmoun, E. M., Lafhim, L.. Approximate weak optimality conditions in multiobjective generalized Nash equilibrium problems. Communications in Nonlinear Science and Numerical Simulation.(2024) 135, 108039. doi: 10.1016/j.cnsns.2024.108039

3. Youness, E. Y., Lafhim, L.. Optimization problems with nonconvex multiobjective generalized Nash equilibrium problem constraints. Communications in Combinatorics and Optimization. (2024). doi: 10.22049/cco.2024.29435.1993

**Submitted papers:**

4. El-Yahyaoui Youness, El Idrissi Rachid, Kalmoun El Mostafa, LAFHIM Lahoussine. Charnes-Cooper scalarization to non-smooth semivectorial bilevel optimization problems (2024).(Under review).

5. Rachid El Idrissi, Youness El-Yahyaoui El Mostafa Kalmoun and Lahoussine Lafhim. Optimality conditions for semivectorial bilevel optimization problems via convexificators (2024).(Under review).

**6.** Youness El-Yahyaoui , Mohsine Jennane, El Mostafa Kalmoun and Lahoussine Lafhim. Optimality conditions for nonsmooth semidefinite bilevel optimization problem via convexificators (2024).(Under review).

**Communications:**

**1.** Necessary optimality conditions for a semi-vectorial bilevel optimization problem via a revisited Charnes–Cooper scalarization. First National Conference on Pure Mathematics and Applications-Fez. May 22-23, 2023.

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# Abbreviations

| | |
|---|---|
| **CQ** : | Constraint Qualification |
| **KKT** : | Karush-Kuhn-Tucker |
| **AKKT** : | Approximate Karush-Kuhn-Tucker |
| **WKKT** : | Weak Karush-Kuhn-Tucker |
| **AWKKT** : | Approximate Weak Karush-Kuhn-Tucker |
| **MFCQ** : | Mangasarian-Fromowitz Constraint Qualification |
| **LICQ** : | Linear Independence Constraint Qualification. |
| **WMFCQ** : | Weak Mangasarian-Fromowitz Constraint Qualification. |
| **MPCC** : | Mathematical Program with Complementarity Constraints. |
| **MGNEP** : | Multiobjective Generalized Nash Equilibrium Problems |
| **SVBOP** : | Semivectorial Bilevel Optimization Problem |
| **BOP** : | Bilevel Optimization Problem |
| **VI** : | Variational Inequality. |
| **CCP** : | Cone Continuity Property |

# Notations

## Spaces and orthants

| | |
|---|---|
| $\mathbb{R}$ | the real numbers |
| $\mathbb{R}_+$ | the set of real numbers that are not negative |
| $\mathbb{R}^n$ | the n-dimensional Euclidean space |
| $\mathbb{R}^n_+$ | the nonnegative orthant in $\mathbb{R}^n$ |

## Sets

| | |
|---|---|
| $\mathbb{N}$ | the set of natural numbers |
| $\mathbb{B}$ | unit ball in $\mathbb{R}^n$ |
| $\mathbb{U}_\gamma$ | $\gamma$-closed ball |
| $\mathbb{B}_\gamma$ | $\gamma$-open ball |
| $\{0_n\}$ | origin of the space $\mathbb{R}^n$ |
| $\operatorname{co}\Omega$ | convex hull of the set $\Omega$ |
| $\operatorname{cl}\Omega$ | topological closure of the set $\Omega$ |
| $\operatorname{bd}\Omega$ | topological boundary of the set $\Omega$ |
| $\operatorname{int}\Omega$ | topological interior of $\Omega$ |

# Vectors

| | |
|---|---|
| $x \in \mathbb{R}^n$ | column vector in $\mathbb{R}^n$ |
| $x_i$ | i-th component of $x$ |
| $x \leq y$ | componentwise comparison $x_i \leq y_i,\ i = 1, \ldots, n$ |
| $x < y$ | componentwise comparison $x_i < y_i,\ i = 1, \ldots, n$ |
| $\|x\|$ | Euclidean norm of $x$ |

# Sequences

| | |
|---|---|
| $x^k$ | sequence in $\mathbb{R}^n$ |
| $x^k \to x$ | the sequence $x^k$ converges to $x$ |
| $x \overset{\phi}{\to} \bar{x}$ | means that $x \to \bar{x}$ with $\phi(x) \to \phi(\bar{x})$ |
| $x \overset{\Omega}{\to} \bar{x}$ | stands for $x \to \bar{x}$ with $x \in \Omega$ |

# Introduction

In this thesis, we focus our attention on the class of mathematical programs with multiobjective problems in the constraints. Bonnel and Morgan [11] referred to this type of problem as a "semivectorial bilevel optimization problem," which is given in the following form:

$$\min_{x,y} F(x,y) \ \text{ s.t. } x \in X,\ y \in \Phi(x), \tag{SVBOP}$$

where the upper-level feasible set is denoted by the nonempty closed set X, $F : \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{m}} \to \mathbb{R}$ is the objective function, and $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is the set-valued mapping expressing the solutions to the parametric multiobjective optimization problem represented as follows.

$$\min_{y} (f_1(x,y), \ldots, f_s(x,y)) \quad \text{s.t. } y \in Y(x), \tag{Mul}[x]$$

where $Y(x) \subseteq \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^s$ denote the feasible set and the multiobjective function of the lower-level problem, respectively.

In our case, in this doctoral research we define the meaning of 'minimization' in both (SVBOP) and (Mul[$x$]), on the basis of different notions of optimality and efficiency in semivectorial bilevel optimization problems. Notice that various concepts of efficiency allow the identification of valuable points in the feasible set when dealing with multiple objective functions. For example, in [10], Bonnel derived the necessary optimality conditions for the semivectorial bilevel optimization problem in the more general setting of Banach spaces, focusing the attention on both efficient and weakly efficient solutions for the parametric multiobjective optimization problem (Mul[$x$]). Indeed, the Bonnel's method incorporates

the weak or properly weak solution at the upper level of the underlying objective function. In this thesis, we consider the concepts of efficiency and weak efficiency.

In the case where $s = 1$, the (SVBOP) reduces to what is known as a standard bilevel optimization problem, which is used to model scenarios involving interdependent decision-making between two parts cooperating between them. Generally, the standard problem is usually considered as follows:

$$\min_{x,y} F(x,y) \ \text{ s.t. } x \in X,\ y \in S(x), \tag{BOP}$$

where the nonempty closed set X (resp., the function $F : \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{m}} \to \mathbb{R}$) denotes the upper-level feasible set (resp., objective function) while the multifunction $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ represents the solution mapping associated with the following parametric optimization problem

$$\min_{y} f(x,y) \ \text{ s.t. } y \in Y(x), \tag{P[$x$]}$$

where $Y(x) \subseteq \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ denote the feasible set and the objective function of the lower-level problem, respectively.

It is worth underlining here that this model is closely related to the optimistic approach in bilevel optimization. In the literature, the problems (BOP) and (P[$x$]) are actually referred to as upper-level and lower-level problems, respectively, so in this thesis we adopt this terminology even in the larger context we are addressing. Bilevel optimization is a commonly studied topic in the optimization community due to its multiple motivating applications in finance, economics, data science, natural sciences, and energy management. For an overview and the state of the art on bilevel optimization, we refer to the survey papers [22,23] as well as the monographs [8,20,62].

It is essential to notice that bilevel optimization problems are typically very irregular and nonconvex. On the other hand, there is no closed-form representation of the solution mapping $S$ associated with (P[$x$]), which makes the model (BOP) very difficult from both the theoretical and numerical aspects.

Let us now discuss the approaches involved in the known previous studies of the transformation of bilevel optimization problems. The mostly standard one involves transforming these problems into single-level (or ordinary) optimization problems. We would like to stress in this context that the major single-level representations of the bilevel optimization problem (BOP) are the optimal value function reformulation, initiated and developed by Outrata [44], while other optimizers, as in Chen and Florian [69], proceed rather by classical Karush-Kuhn-Tucker reformulation.

Regarding the semivectorial bilevel problem (SVBOP), this class of problems possesses a structure similar to standard bilevel programming problems, as stated in [8,20,62] for hierarchical optimization. The primary difference is that the lower-level (or follower's) problem (Mul[$x$]) has multiple objective functions, providing, henceforth, a much more general framework for the challenging vectorial problem under our consideration, which goes back to Bonnel and Morgan [11]. From another side, since the common notion of a minimizer does not apply to multiobjective systems, the lower-level decision-maker generally needs to compute the set of efficient or weakly efficient points for each fixed value of the upper-level (or leader's) variable. In the meantime we have to mention that it is well-known in the literature that scalarization techniques were once used to convert multiobjective optimization problems into scalar optimization problems. Therefore, when the lower-level problem (Mul[$x$]) has multiple objective functions, it seems natural to replace it with a scalarized version and consider the scalarization parameters as additional upper-level variables.

The conventional practice in semivectorial bilevel optimization research, particularly for problems with convex multiobjective data at the lower level, has involved weighted-sum scalarization techniques. Indeed, this approach converts the problem into a classical scalar bilevel program, which allows us to establish optimality conditions (see [29]). Another penalty method was developed in [73] in the case where the multiobjective lower-level problem is linear. This problem is also investigated in [33], when the upper-level objective function is vector-valued as well. In [10], Bonnel derived the necessary optimality conditions for the semivectorial bilevel optimization problem in very general Banach spaces.

Noting that the weighted-sum scalarization can only be employed in cases of convex problems. More recently, Lafhim investigated in [48] weak efficient solutions for nonconvex semivectorial bilevel programs using a different scalarization technique that involves minimizing one objective function and incorporating the rest of the weighted objectives as constraints. In particular, the author provided the necessary optimality conditions using the weak basic constraint qualification and Mordukhovich's generalized differentiation calculus. However, both scalarization techniques have limitations, particularly when dealing with efficient solutions. A significant drawback is the treatment of the scalarization parameter as an upper-level objective variable.

To address the aforementioned scenarios, this thesis focuses on the semivectorial bilevel program (SVBOP), aiming at solving efficiently the lower-level problem (Mul[$x$]). We proceed in two ways: we provide the existence of solutions, and we revisit the Charnes-Cooper scalarization technique, initially

introduced in the pioneering works [15,16] and recently applied to convex vector optimization problems in [17]. Using a revisited Charnes-Cooper technique, we are able to transform the multiobjective bilevel programming problem (Mul[$x$]) into a bilevel programming problem that incorporates a scalar lower-level problem. This innovative scalarization approach proves to be effective when we address significant classes of vectorial bilevel problems and permits us to overcome the limitations of previous scalarization methods.

Furthermore, we reformulate the bilevel optimization problem as a single-level optimization problem, ensuring global and local equivalence to the initial problem (SVBOP). In particular, this reformulation doesn't make recourse to any property of semi-compactness or inner-semicontinuity on the scalar lower-level solution set derived through the use of Charnes-Cooper. On the basis of this crucial key observation, in our case we establish the necessary optimality conditions for the vectorial problem subject to our treatment in this research.

Another main concern of the thesis is the special case of the semivectorial bilevel optimization problem (SVBOP) when the lower level problem is the following multiobjective generalized Nash equilibrium problem composed by N players such that for any $\eta = 1, \ldots, N$, and $y^{-\eta}$, the $\eta$ th player chooses $y^{\eta}$ that solves

$$\begin{cases} \min\limits_{y^{\eta}} \; f^{\eta}\left(y^{\eta}, y^{-\eta}\right) = \left(f_1^{\eta}\left(y^{\eta}, y^{-\eta}\right), \ldots, f_{s_{\eta}}^{\eta}\left(y^{\eta}, y^{-\eta}\right)\right) \\ \text{s.t } y^{\eta} \in Y^{\eta}\left(y^{-\eta}\right), \end{cases} \qquad (P^{\eta}[y^{-\eta}])$$

by controlling his or her own strategy $y^{\eta} \in \mathbb{R}^{m_{\eta}}$ while taking into account the choices made by opponents, which are represented by the vector $y^{-\eta} \in \mathbb{R}^{m-m_{\eta}}$, where $m = m_1 + \ldots + m_N$ represents the total number of players' decision variables. Sometimes we use the notation $(y^{\eta}, y^{-\eta})$ instead of just $y$ to emphasize the variables of player within the vector $y$. The block components are not implied to be rearranged in any manner by this notation, and $y = (y^{\eta}, y^{-\eta})$ is always the case.

Furthermore, each player has an objective $f^{\eta} : \mathbb{R}^{m_{\eta}} \to \mathbb{R}^{s_{\eta}}$, called a utility function or payoff function, which depends on his own variables $y^{\eta}$ as well as the variables $y^{-\eta}$ of the other players. Moreover, each player's strategies belong to a set $Y^{\eta}\left(y^{-\eta}\right)$ that depends on the rival player's strategy. The components of this continuous function are denoted by $f_1^{\eta}, \ldots, f_{s_{\eta}}^{\eta} : \mathbb{R}^{m_{\eta}} \to \mathbb{R}$.

Our treatment of this problem begins with the case when $(P^{\eta}[y^{-\eta}])$ is solved up to weak efficiency. On one hand, the only existing result in the literature addressing mathematical programs with multiobjective generalized Nash equilibrium problems with constraints is [40]. In this work, the authors consider convex-

type assumptions. The resulting reformulation is a specific instance of the widely studied mathematical program with equilibrium constraints. We extend the existing results from [40] on generalized Nash equilibrium problems to nonconvex cases. Additionally, we provide a single-level reformulation for these problems and establish the necessary optimality conditions.

Furthermore, there has been a significant surge of interest in extending approximate versions of the KKT conditions (AKKT) to encompass generalized Nash equilibrium problems (GNEPs). Valuable contributions in this area include also the paper by Kanzow and Steck [45] and the research by Bueno, Haeser, and Rojas [13]. These studies explore optimality conditions and constraint qualifications by analyzing their interrelationships and implications for the global convergence of algorithms. It is important to note that although the AKKT conditions have been examined, they do not serve as optimality conditions for GNEPs. However, the authors proposed an augmented Lagrangian algorithm for GNEPs, demonstrating global convergence to a KKT point under a weak constraint qualification condition. Despite different and recurrent uses, the application of optimality conditions in the context of GNEPs with multiple objectives has not been investigated by means of these approximate optimality conditions.

In this thesis, we devote our efforts to the examination of the approximate Karush-Kuhn-Tucker (KKT) conditions for the multiobjective generalized Nash equilibrium problem that is solved up to weak efficiency and their impact on the global convergence of algorithms. In this way, more attention is devoted to estimating the weak Karush-Kuhn-Tucker (KKT) of the multiobjective generalized Nash equilibrium problem using an enhanced Lagrangian-type algorithm.

The remainder of the thesis is organized as follows:

The next chapter introduces fundamental concepts and results from variational analysis and generalized differentiation, which are extensively applied in this thesis. Additionally, this chapter provides a comprehensive overview of optimization problems involving geometric and operator constraints.

In Chapter 2, we investigate semivectorial bilevel optimization problems when the lower-level problem is solved up to efficiency. Our results in this part are based on the Charnes-Cooper scalarization approach applied to problems with continuously differentiable functions. However, many real-world applications of bilevel optimization involve non-smooth functions. Having this observation in mind, we aim at extending the previous works by proposing new necessary optimality conditions for semivectorial bilevel programs under weaker regularity assumptions involving the non-differentiability. Specifically, we revisit the Charnes-Cooper scalarization technique and present tailored optimality results for our problems

where all the appearing functions are only locally Lipschitz continuous. Through this generalization, our results provide a more flexible theoretical framework applicable to a broader class of non-smooth optimization models arising in practical semivectorial bilevel programming contexts.

In this regard, we first examine the graph-closedness of the associated efficiency mapping, and then we describe our new scalarization technique as well as the transformation process.

inherent in a single-level optimization problem. Furthermore, we establish a value function estimation in smooth and non-smooth cases and provide sufficient conditions to ensure its local Lipschitz property. Subsequently, suitable Abadie constraint qualifications are used to establish the necessary optimality conditions and then introduce a general partial calmness concept specifically tailored to the optimal value function reformulation of problem (SVBOP). Indeed, we use an appropriate weak Mangasarian-Fromowitz constraint qualification to establish the necessary optimality conditions. Moreover, we study the special case where the lower-level multiobjective problem is linear in the lower decision variable; afterward we end the chapter with concluding remarks and comments.

In Chapter 3, we examine a type of optimization problem where the lower-level problems are multiobjective generalized Nash equilibrium problems, which received only a restricted attention in the literature despite their numerous applications. In this regard, we present a single-level reformulation for these problems, emphasizing their equivalence in terms of global and local minimizers. Our approach converts the problem into a one-level optimization problem and then applies the kth-objective weighted constraint and optimal value reformulation. In this step, we make an appeal to the Mordukhovich generalized differentiation calculus to derive detailed first-order necessary optimality conditions in a smooth setting. To do that, we first introduce the notion of equilibrium for multiobjective extended Nash equilibrium problems and provide its scalarization formulation. Then, we present the associated mathematical programming problem and the corresponding surrogate standard optimization method. In this way, we examine the correspondence between global and local minimizers, so using the weak fundamental constraint qualification appropriate for mathematical optimization problems, we provide optimality conditions in terms of the limiting subdifferentials and the limiting normal cones. Then, by applying Mordukhovich's generalized differentiation calculus to the smooth scenario, fully specified first-order necessary optimality conditions are derived.

In Chapter 4, we analyze a multiobjective generalized Nash equilibrium problem (MGNEP), which is a Nash equilibrium problem with multiobjective games as constraints. In this research, we examine

the approximate Karush-Kuhn-Tucker (KKT) conditions for MGNEPs and explain how they affect the global convergence of algorithms. We begin by establishing the standard approximate weak Karush-Kuhn-Tucker (AWKKT) conditions for MGNEPs and show that any weakly efficient solution meets the standard approximate weak Karush-Kuhn-Tucker (AWKKT) condition. Given the strength of these optimality conditions, we provide a novel set of approximation weak KKT conditions designed particularly for MGNEPs, demonstrating their convergence to weak Karush-Kuhn-Tucker points when a cone-continuity condition is satisfied. It is essential to highlight that, while these recently established approximation weak Karush-Kuhn-Tucker conditions do not serve as optimality conditions for general MGNEPs, we show that they are valid for the specific case of a weak variational equilibrium point in a jointly convex multiobjective generalized Nash equilibrium problem (MGNEP). Finally, we present an updated Lagrangian-type algorithm for estimating an MGNEP's weak Karush-Kuhn-Tucker approximation and demonstrate its global convergence.

We end our thesis document with comments concluding our contribution and underlying some perspectives on the topic.

# Chapter 1

# Preliminaries

This chapter explores the general bilevel programming problem in greater depth, offering a cohesive framework for the two main reformulations—optimistic and pessimistic—as well as for the reformulations of the semivectorial bilevel optimization problem found in the literature. Additionally, we introduce the concept of the generalized Nash equilibrium problem and present fundamental concepts and results from variational analysis and generalized differentiation, which are extensively applied in the subsequent chapters.

## 1 Bilevel optimization problem

### 1.1 Definition of a bilevel optimization problem

This thesis focuses on the problem of hierarchical optimization

$$\text{"}\min_{x}\text{"}\ F(x,y), \quad \text{subject to } x \in X, y \in S(x). \tag{BPP}$$

In this context, the upper-level problem player (leader) aims to minimize its cost function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ with respect to the variable $x$, while considering the response $y$ of the lower-level problem player (follower). Here $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a set-valued mapping defined by

$$S(x) := \arg\min_{y}\{f(x,y), \quad y \in Y(x)\}, \tag{1.1}$$

the map $S$ describes the sets of optimal solutions for the following lower-level parametric optimization problem.

$$\min_{y} f(x,y), \quad \text{subject to } \; y \in Y(x), \tag{P[x]}$$

where $Y(x) \subseteq \mathbb{R}^m$ represents the feasible set of (P[x]) and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ denotes the multiobjective function of the lower-level problem.

The sets $X$ and $Y(x)$ are commonly referred to as the upper-level and lower-level feasible sets, respectively. To simplify the presentation of the main ideas, we focus on the case where the upper and lower-level constraint sets are explicitly defined as follows:

$$X := \{x \in \mathbb{R}^n : G(x) \leq 0\} \text{ and } Y(x) := \{y \in \mathbb{R}^m : g(x,y) \leq 0\},$$

with $G : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$.

## 1.2 Reformulation of the bilevel optimization problem

The bilevel problem is essentially the leader's problem. Quotation marks have been added in " $\min_{x}$ " $F(x,y)$ due to the unclear definition of the objective function value $F(x,y)$ from the leader's perspective, as the leader only controls $x$. This ambiguity arises if the set of optimal solutions to the lower-level problem is not a singleton [31]. Multiple optimal responses create a challenge for the leader, who cannot predict which solution the follower will choose. This uncertainty can significantly impact the leader's objective value and, consequently, their decision-making process. To address this complexity, many studies simply assume that the solution to the lower-level problem is unique. In other cases, a set-valued optimization problem and an optimistic or pessimistic approach are considered.

One approach to understanding the general bilevel program (BPP) is to examine it as the following set-valued optimization problem:

$$\min_{x \in X} F(x, S(x)) = \bigcup_{y \in S(x)} \{F(x,y)\}, \tag{1.2}$$

where minimization is examined according to some ordering cone. This technique is addressed in [21], and related advances are provided in [7,10,70] to establish the necessary optimality conditions for multiobjective bilevel programs.

To examine problem (BPP) using a scalar optimization objective, consider (1.2) as the following typical optimization problem:

$$\min_{x} \ \{F(x, S(x)), \quad x \in X\}, \tag{1.3}$$

given that $S(x)$ is single-valued for all $x \in X$; see [20] and its references for further information on algorithms and the necessary optimality conditions using this technique. It is worth noting that the primary difficulty with both techniques (1.2) and (1.3) is the implicit nature of the objective.

In the literature, there are two basic techniques to address the problem (BPP). On the one hand, we have the following optimistic formulation of the bilevel optimization problem :

$$\min_{x \in X} \varphi_0(x) = \min_{y} \ \{F(x, y), \quad y \in S(x)\}. \tag{1.4}$$

From an economic perspective, this reflects a scenario where the follower shares in the leader's profits, allowing for potential cooperation across different levels. Nevertheless, the leader may not always persuade the follower to make choices beneficial solely to the leader. Therefore, it becomes essential for the upper-level problem participant to mitigate the impact of unfavorable decisions made at the lower-level problem.

On the other hand, we have the following pessimistic formulation of the bilevel optimization problem:

$$\min_{x \in X} \varphi_p(x) = \max_{y} \ \{F(x, y), \quad y \in S(x)\}.$$

The former problem falls within the category referred to as minimax problems. Static minimax issues, which occur when the feasible set of the inner problem $S(x)$ remains static regardless of $x$, have attracted significant investigation in the literature; see works such as [32,62,68].

Simultaneously, it is widely acknowledged that when $S(x)$ represents varying solution sets of another optimization problem, the pessimistic formulation described earlier encounters numerous challenges, as discussed in [20] and the cited references.

The optimistic formulation (1.4) of the bilevel optimization problem is much easier to deal with, thus it has attracted the most attention in studies. However, in the following simpler model

$$\min_{x,y} F(x, y), \quad \text{subject to } x \in X, y \in S(x). \tag{BOP}$$

Problem (BOP) that we label here as the classical bilevel program has received the most research interest in the literature; it is commonly referred to as the "bilevel optimization problem" or "optimistic bilevel optimization problem." Shimizu et al. [62], Bard [8], and Dempe [20] have all written monographs on this topic.

One approach to investigating bilevel optimization problems (BOP) is to transform them into single-level (or ordinary) optimization problems. Initially, linear bilevel programming problems (where all the involved functions are affine) were typically transformed using linear programming duality or, equivalently, the Karush-Kuhn-Tucker (KKT) conditions for linear programming.

This transformation method has enabled the development of solution algorithms, as discussed by Candler and Townsley [67]. The resulting transformed problem is a special case of a mathematical program with equilibrium constraints (MPEC), also known as a mathematical program with complementarity constraints (MPCC), given by the following form:

$$\min_{x,y,u} \left\{ F(x,y) \mid \quad x \in X,\ \mathcal{L}(x,y,u) = 0,\ g(x,y) \leq 0,\ u \geq 0,\ u^{\top} g(x,y) = 0 \right\}, \tag{KKTR}$$

with the Lagrange function of the lower level problem is given by: $\mathcal{L}(x,y,u) = \nabla_y f(x,y) + \nabla_y g(x,y)^{\top} u$. This method is often referred to as the Karush-Kuhn-Tucker (KKT) transformation of the bilevel programming problem. This approach is also applicable to convex parametric lower-level problems that meet certain regularity assumptions.

Secend approach to investigate bilevel optimization problem (BOP), the so-called lower-level value function (LLVF) reformulation of problem (BPP) , introduced by Outrata [44] for the purpose of a numerical method is given by

$$\min_{x,y} F(x,y), \quad \text{subject to } x \in X, y \in Y(x),\ f(x,y) - \varphi(x) \leq 0, \tag{LLVR}$$

with $\varphi(x) = \inf_{y \in Y(x)} f(x,y)$.

Results on optimality conditions that take into consideration non-unique lower-level solution mappings have been traced back to Ye and Zhu [72] and Chen and Florian [69]. Both studies investigate lower-level value function (LLVR) reformulation. In the paper by Chen and Florian [69], the authors later switch to the classical Karush-Kuhn-Tucker reformulation. Therefore, the first KKT-type necessary optimality conditions for (BPP) using the (LLVR) reformulation are attributed to Ye and Zhu [72].

Since then, this reformulation has garnered significant interest, leading to many important contributions, as referenced in works such as [26,30,34,48,50,72]. Chen and Florian [69] are also credited with being among the first to highlight the failure of some well-known constraint qualifications (CQs) for a class of problem (BPP).

## 2 Semivectorial bilevel optimization problem

In the literature, there is a general case of a bilevel programming problem (BOP) called the semivectorial bilevel optimization problem, with the exception that the objective mapping of the lower-level problem is a vector function. Bonnel and Morgan [11] initially introduced this class of problems, which consider the following structure:

$$\min_{x,y} F(x,y) \ \text{ s.t. } x \in X,\ y \in S(x), \tag{SVBOP}$$

where the upper-level feasible set is denoted by the nonempty closed set X, $F : \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{m}} \to \mathbb{R}$ the objective function, and $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, the set of optimal solution to the multiobjective optimization problem represented below.

$$\min_{y} (f_1(x,y), \ldots, f_s(x,y)) \ \text{ s.t. } y \in Y(x), \tag{Mul}[x]$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^s$ the multiobjective function of the lower-level problem, respectively. Moreover, we recall the solution concept of multiobjective problems. To proceed, $Y(x) \subseteq \mathbb{R}^m$ should stand for the feasible set of (Mul[$x$]). A point $\overline{y}$ is an efficient (respectively weak efficient) solution for (Mul[$x$]) if there is no $y \in Y(x)$ such that $f(x,y) \leq f(x,\overline{y})$ and $f(x,y) \neq f(x,\overline{y})$. (respectively, $f(x,y) < f(x,\overline{y})$).

Common methods for solving a semivectorial bilevel optimization problem (SVBOP) are to convert it into a scalar (single-objective) problem using a set of parameters. This approach, also known as parameter-based scalarization, allows the scalarized problem to be solved using typical single-objective optimization methods and readily available methods.

The weighted sum approach converts the original problem (SVBOP) into a single-objective optimization problem by assigning weights to each objective function and minimizing the weighted sum of these functions. This scalarization technique transforms the multiobjective optimization problem (Mul[$x$])

into a standard one-level optimization problem.

$$\begin{cases} \min\limits_{y} \quad f(x,y,z) = \langle z, f(x,y)\rangle \\ \text{s.t.} \quad y \in Y(x), \end{cases}$$

where the new parameter vector $z$ is a nonnegative point on the unit sphere, i.e., $z$ belongs to

$$Z := \left\{ z \in \mathbb{R}^l : z \geq 0,\ \|z\| = 1 \right\}.$$

In contrast, Lafhim in [48] introduces a novel scalarization technique known as the weighted kth objective constraint problem. This approach transforms the multiobjective optimization problem (Mul[$x$]) into a standard one-level optimization problem. For this, let $e = (1, \ldots, 1) \in \mathbb{R}^s$ be the all-ones vector, and let

$$\mathbb{Z} := \left\{ z \in \mathbb{R}^s_+ \ , z > 0 \mid \sum_{k=1}^{s} z_k = z^T e = 1 \right\}.$$

To do this, fix $k$ and $z \in \mathbb{Z}$, and then consider the following scalarized problem:

$$\begin{cases} \min\limits_{y} \quad f_k(x,y) \\ \text{s.t.} \quad z_r\, f_r(x,y) \leq z_k\, f_k(x,y), \quad \forall r = 1, \ldots, s,\ r \neq k, \\ \qquad\quad y \in Y(x). \end{cases}$$

In recent years, many publications have made this problem the primary focus of their research, such as [10–12,25,47,48,50], where various types of necessary optimality conditions for finite-dimensional mathematical optimization problems for this type of problem.

Our aim in this thesis is to deal with the semivectorial optimistic bilevel optimization problem within the concept of efficient optimal solutions for the parametric multi-objective optimization problem. Specifically, we revisit the Charnes-Cooper scalarization technique and establish the necessary optimality conditions.

# 3 Generalized Nash equilibrium problem

Another type of problem studied in this thesis the generalized Nash equilibrium problem GNEP has $N$ players, with each $\eta$ controlling the variables $x^\eta \in \mathbb{R}^{n_\eta}$. We designate by $x$ the vector containing all of these choice variables $x := \left(x^1, \cdots, x^N\right)^\top$.

The vector $x$ has a dimension of $n := \sum_{\eta=1}^{N} n_\eta$. We denote by $x^{-\eta} \in \mathbb{R}^{n-n_\eta}$ the vector comprising all players' decision variables except those of player $\eta$. To highlight the $\eta$-th player's variables within $x$, we sometimes use the notation $(x^\eta, x^{-\eta})$ instead of $x$. It is important to note that this still represents the vector $x := \left(x^1, \cdots, x^\eta, \cdots, x^N\right)$ and does not imply that the block components of $x$ are reordered with $x^\eta$ as the first block.

Each player has an objective function $f^\eta : \mathbb{R}^n \to \mathbb{R}$, which is dependent on both his own variables $x^\eta$ and the variables $x^{-\eta}$ of all other players. The mapping $f^\eta$ is also known as the utility function of player $\eta$, payoff function, or loss function, depending on the application where the GNEP appears.

Furthermore, each player's strategy must belong to a set $Y^\eta\left(x^{-\eta}\right) \subseteq \mathbb{R}^{n_\eta}$ that depends on the rival players' strategies. This set is called the feasible set or strategy space of player $\eta$. The objective of player $\eta$, given the other players' strategies $x^{-\eta}$, is to choose a strategy $x^\eta$ that solves the minimization problem.

$$\min_{x^\eta} f^\eta\left(x^\eta, x^{-\eta}\right) \quad \text{subject to } x^\eta \in Y^\eta\left(x^{-\eta}\right). \tag{P$^\eta[x^{-\eta}]$}$$

For any $x^{-\eta}$, the solution set of problem (P$^\eta[x^{-\eta}]$) is denoted by $S^\eta\left(x^{-\eta}\right)$, $\eta = 1, \cdots, N$.

On the one hand, the only existing result in the literature that deals with mathematical programs with multi-objective generalized Nash equilibrium problems with constraints is [40]. In this result, the authors address this type of problem in the convex case; the transformed reformulation is a special case of the so-called mathematical program with equilibrium constraints, widely studied in the literature.

Our goal in this thesis is to tackle a special case of semivectoral bilevel optimization problems, with the lower-level problem being a multiobjective generalized Nash equilibrium problem. We generalize the existing results from [40] on the generalized Nash equilibrium problem for the nonconvex cases. Moreover, we provide a single-level reformulation for these types of problems and establish the necessary optimality conditions.

On the other hand, there has been a marked increase in interest in extending approximate versions of the Karush-Kuhn-Tucker (KKT) conditions to generalized Nash equilibrium problems (GNEPs). Notable contributions in this area include the work by Kanzow and Steck [45], as well as research by Bueno, Haeser, and Rojas [13]. These studies explore optimality conditions and constraint qualifications, examining their interrelationships and implications for the global convergence of algorithms.

In the final chapter of this thesis, we explore the approximate weak Karush-Kuhn-Tucker conditions for the multiobjective generalized Nash equilibrium problem MGNEP and their influence on overall algorithm convergence.

# 4 Mathematical tools

In this section, we introduce the fundamental tools of variational analysis that will be employed throughout the thesis. The primary generalized differentiation objects utilized are the subdifferentials, normal cones, and coderivatives as defined by Mordukhovich. Additionally, the relationships between these concepts and the corresponding Fréchet and Clarke nonsmooth tools will be frequently referenced, along with a general background on optimization problems. For a more comprehensive understanding of the topics briefly covered here, see the books [18,57,59].

## 4.1 Subdifferentials

In this section, we present material from variational analysis that will be crucial to our thesis. For more details, see [18,57]. The Kuratowski-Painlevée upper limit of a set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at a point $\bar{x}$ is defined by

$$\limsup_{x\to\bar{x}} \Phi(x) := \{x^* \in \mathbb{R}^m : \ \exists x_k \to \bar{x},\ x_k^* \to x^* \text{ with } x_k^* \in \Phi(x_k) \text{ as } k \to \infty\}.$$

**Definition 1.1.** Let $\Omega$ be a nonempty subset of $\mathbb{R}^n$.

(i) Given $\bar{x} \in \Omega$ and $\epsilon \geq 0$, define the set of $\epsilon$-normals to $\Omega$ at $\bar{x}$ by

$$\widehat{N}_\epsilon(\bar{x};\Omega) := \left\{ x^* \in \mathbb{R}^n \mid \limsup_{x \overset{\Omega}{\to} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq \epsilon \right\}. \tag{1.5}$$

When $\epsilon = 0$, elements of (1.5) are called Fréchet normals and their collection, denoted by $N(\bar{x};\Omega)$, is the prenormal cone to $\Omega$ at $\bar{x}$. If $\bar{x} \notin \Omega$, we put $N_\epsilon(\bar{x};\Omega) := \emptyset$ for all $\epsilon \geq 0$.

(ii) Let $\bar{x} \in \Omega$. Then $x^* \in \mathbb{R}^n$ is a basic/limiting normal to $\Omega$ at $\bar{x}$ if there are sequences $\epsilon_k \downarrow 0$, $x_k \in \Omega$, $x_k \to \bar{x}$, and $x_k^* \to x^*$ such that $x_k^* \in N_{\epsilon_k}(x_k;\Omega)$ for all $k \in \mathbb{N}$. The collection of such normals

$$N(\bar{x};\Omega) := \limsup_{x \overset{\Omega}{\to} \bar{x}} \widehat{N}(x;\Omega), \tag{1.6}$$

is the (basic, limiting) normal cone to $\Omega$ at $\bar{x}$. Put $N(\bar{x};\Omega) := \emptyset$ for $\bar{x} \notin \Omega$.

where $x \overset{\Omega}{\to} \bar{x}$ stands for $x \to \bar{x}$ with $x \in \Omega$.

**Remark 1.2.** (1) The normal cone in (1.6) is a closed cone in $\mathbb{R}^n$.

(2) For convex set $\Omega$ of $\mathbb{R}^n$, the Fréchet and the limiting normals cone reduce to the normal cone of convex analysis.

The following example illustrates that the normal cone may be nonconvex in very simple settings.

**Example 1.3.** Consider the closed set

$$\Omega := \{(x, y) \in \mathbb{R}^2 \mid y \geq -|x|\}.$$

It is easy to see that

$$N((0,0);\Omega) = \{(v, v) \in \mathbb{R}^2 \mid v \leq 0\} \cup \{(v, -v) \in \mathbb{R}^2 \mid v \geq 0\},$$

which is a nonconvex subset of $\mathbb{R}^2$.

**Proposition 1.4.** *Let $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^m$ with $(\bar{x}_1, \bar{x}_2) \in \Omega_1 \times \Omega_2$. Then we have the product formula*

$$N((\bar{x}_1, \bar{x}_2); \Omega_1 \times \Omega_2) = N(\bar{x}_1; \Omega_1) \times N(\bar{x}_2; \Omega_2).$$

**Definition 1.5.** Let $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is assumed to be a lower semi-continuous function.

(1) For $\overline{x}$ in the domain of $\phi$, the Fréchet subdifferential of $\phi$ at $\bar{x}$ is

$$\widehat{\partial}\phi(\bar{x}) := \left\{ x^* \in \mathbb{R}^n : \liminf_{x \to \bar{x}} \frac{\phi(x) - \phi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

(2) The Mordukhovich (limiting) subdifferential of $\phi$ at $\bar{x}$ is given as follows:

$$\partial\phi(\bar{x}) := \limsup_{x \overset{\phi}{\to} \bar{x}} \widehat{\partial}\phi(x).$$

(3) The singular subdifferential of $\phi$ at $\overline{x}$ is defined by

$$\partial^\infty \phi(\overline{x}) := \left\{ x^* \in \mathbb{R}^n \ : \ \exists x_k \overset{\phi}{\to} \overline{x}, t_k \searrow 0, \text{ and } x_k^* \in \widehat{\partial}\phi(x_k) \, \forall k \text{ with } t_k x_k^* \to x^* \right\},$$

where $x \overset{\phi}{\to} \bar{x}$ means that $x \to \bar{x}$ with $\phi(x) \to \phi(\bar{x})$.

**Remark 1.6.** (1) $\partial\phi\,(\bar{x})$ is nonempty and compact for a local Lipschitz continuous function.

(2) The basic subdifferential $\partial\phi(\bar{x})$ reduces to the usual gradient $\{\nabla\phi(\bar{x})\}$ for smooth functions and to the subdifferential of convex analysis when $\phi$ is convex.

(3) The singular subdifferential $\partial^{\infty}\phi(\bar{x})$ reduces to $\{0\}$ for locally Lipschitzian functions.

**Definition 1.7.** The Clarke subdifferential of $\varphi$ at $\bar{x}$ is the convex hull of $\partial\varphi(\bar{x})$ can be defined by

$$\partial_C\phi\,(\overline{x}) := \text{co}\ \ \partial\phi\,(\overline{x})\,. \tag{1.7}$$

**Remark 1.8.** The following convex hull property, which is very important, thanks to this link between Mordukhovich and Clarke subdifferentials:

$$\text{co}\,\partial(-\phi)\,(\overline{x}) := -\,\text{co}\ \ \partial\phi\,(\overline{x})\,. \tag{1.8}$$

The concept of semismoothness can be applied to sets by means of the Euclidean distance function $d$ given by

$$d_{\Omega}(x) := \text{dist}(x;\Omega) = \inf_{u\in\Omega}\|x-u\|,\ \ \Omega\subseteq\mathbb{R}^n.$$

**Definition 1.9.** A set $\Omega\subseteq\mathbb{R}^n$ is called semismooth at $\overline{x}\in cl\ \Omega$ if for any sequence $x_k\longrightarrow\overline{x}$ with $x_k\in\Omega$ and $\|x_k-\overline{x}\|^{-1}(x_k-\overline{x})\longrightarrow d$ it holds that $\langle x_k^*,d\rangle\longrightarrow d$ for all selections $x_k^*\in\partial_C d_{\Omega}\,(x_k)$.

## 4.2 Coderivative and concepts of set-valued mappings

In this subsection, we examine the generalized differentiation of set-valued mappings or multifunctions $\Phi:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$, where the values $\Phi(x)\subset\mathbb{R}^m$ can be empty or singletons. When $\Phi(x)$ is a singleton for all $x$, we use the standard notation $h:\mathbb{R}^n\to\mathbb{R}^m$ for single-valued mappings. The graph of is $\Phi$ given by:

$$\text{gph}\,\Phi := \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m\mid y\in\Phi(x)\}\,.$$

and denote its domain by

$$\text{dom}\,\Phi := \{x\in\mathbb{R}^n\mid\Phi(x)\neq\emptyset\}\,.$$

**Definition 1.10.** Consider $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with $\operatorname{dom}\Phi \neq \emptyset$, and let $(\bar{x}, \bar{y}) \in \operatorname{gph}\Phi$. The coderivative for the set-valued mapping $\Phi$ at $(\bar{x}, \bar{y})$ is a mapping $D^*\Phi(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with the values

$$D^*\Phi(\bar{x}, \bar{y})(v) := \{u \in \mathbb{R}^n \mid (u, -v) \in N_{\operatorname{gph}\Phi}(\bar{x}, \bar{y})\} \text{ for } \quad v \in \mathbb{R}^m. \tag{1.9}$$

generated by the normal cone to the graph of $\Phi$ at $(\bar{x}, \bar{y})$.

In general, the coderivative may take nonconvex and also empty values. Let us illustrate this by direct calculations based on the definition.

**Example 1.11.** Consider first the function $h(x) := |x|$ on $\mathbb{R}$ and calculate its coderivative at $\bar{x} = 0$. Using the normal cone definition gives us

$$N((0,0); \operatorname{gph} h) = \{(x, y) \in \mathbb{R}^2 \mid y = |x| \text{ and } y \leq -|x|\}.$$

Thus the coderivative (1.9) of this function is calculated by

$$D^*h(0)(v) = \begin{cases} [-v, v] & \text{if } v \geq 0, \\ \{-v, v\} & \text{if } v < 0 \end{cases}$$

and has, in particular, nonconvex values when $v < 0$.

**Proposition 1.12.** *Let the mapping $h : \mathbb{R}^n \to \mathbb{R}^m$ be strictly differentiable at $\bar{x}$,. Then we have the representation*

$$D^*h(\bar{x})(y^*) = \left\{\nabla h(\bar{x})^\top y^*\right\} \quad \textit{for all } y^* \in \mathbb{R}^m.$$

**Theorem 1.13.** *Let $h : \mathbb{R}^n \to \mathbb{R}^m$ be continuous around $\bar{x}$. Then we have the inclusion*

$$\partial\langle v, h\rangle(\bar{x}) \subset D^*h(\bar{x})(v) \quad \textit{for all } v \in \mathbb{R}^m.$$

*If in addition $h$ is locally Lipschitzian around $\bar{x}$, then*

$$D^*h(\bar{x})(v) = \partial\langle v, h\rangle(\bar{x}) \quad \textit{for all } v \in \mathbb{R}^m.$$

**Definition 1.14.** Let $(\bar{x}, \bar{y}) \in \operatorname{gph}\Phi$. The set-valued mapping $\Phi$ is said to be

(1) inner semicompact at $\bar{x}$, $\Phi(\bar{x}) \neq \emptyset$, if and only if, for every sequence $x_k \longrightarrow \bar{x}$ with $\Phi(x_k) \neq \emptyset$ there is a sequence of $y_k \in \Phi(x_k)$ that contains a convergent subsequence as $k \longrightarrow +\infty$.

(2) inner semicontinuous at $(\bar{x}, \bar{y})$ if for any sequence $x_k \to \bar{x}$ there exists a sequence $y_k \in \Phi(x_k)$ which converges to $\bar{y}$ as $k \to \infty$.

(3) Lipschitz-like around $(\bar{x}, \bar{y})$ if there exists neighborhoods $U$ of $\bar{x}$, $V$ of $\bar{y}$, and a constant $\lambda > 0$ such that

$$d(y; \Phi(x)) \leq \lambda \|x - u\| \quad \forall x, u \in U \text{ and } \forall y \in \Phi(u) \cap V.$$

(4) calm at $(\bar{x}, \bar{y})$ if there are neighborhoods $U$ of $\bar{x}$, $V$ of $\bar{y}$, and a constant $\lambda > 0$ such that

$$d(y; \Phi(\bar{x})) \leq \lambda \|x - \bar{x}\| \quad \forall x \in U \text{ and } \forall y \in \Phi(x) \cap V.$$

(5) locally bounded around $\bar{x}$ if there exists a neighborhood $U$ of $\bar{x}$, and a bounded set $\mathbb{V} \subset \mathbb{R}^m$ such that $\Phi(x) \subset \mathbb{V}$ for all $x \in U$.

(6) has Aubin property at $(\overline{x}, \overline{y}) \in \text{gph } \Phi$ if there exist neighborhoods $U$ of $\overline{x}$, $V$ of $\overline{y}$, and a scalar $\tau > 0$ such that for all $x, x^* \in U$ and $y \in \Phi(x) \cap V$ there is $y^* \in \Phi(x^*)$ such that

$$\|y - y^*\| \leq \tau \|x - x^*\|.$$

In general, the concept of inner semicontinuity is a necessary condition for satisfying the Lipschitz property. The following coderivative provides a sufficient condition for the inner semicontinuity of $\Phi$ at $(\bar{x}, \bar{y})$ for closed graph mappings.

$$D^*\Phi(\bar{x}, \bar{y})(0) = \{0\}. \tag{1.10}$$

**Remark 1.15.** [25] The inner semicompactness of the set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ holds whenever $\Phi$ is uniformly bounded and has nonempty values around $\bar{x}$, i.e., there exists a neighborhood $U$ of $\bar{x}$ and a bounded set $V \subset \mathbb{R}^m$ such that $\Phi(x) \subset V$, for all $x \in U$.

## Chapter 2

# Charnes-Cooper scalarization to smooth and non-smooth semivectorial bilevel optimization problems

## 1 Introduction

Semivectorial bilevel optimization problems have a similar structure to standard bilevel programming problems based on hierarchical optimization. The key difference is that the objective function of the lower-level (or follower's) problem is a vector function. Bonnel and Morgan [11] originally introduced this class of problems, which take the following form:

$$\min_{x,y} F(x,y) \ \text{ s.t. } x \in X,\ y \in \Phi(x), \tag{SVBOP}$$

where the upper-level feasible set is denoted by the nonempty closed set X, $F : \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{m}} \to \mathbb{R}$ is the objective function, and $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is the set of efficient optimal solutions to the parametric multiobjective optimization problem represented below.

$$\min_{y} (f_1(x,y), \ldots, f_s(x,y)) \ \text{ s.t. } y \in Y(x), \tag{Mul[$x$]}$$

where $Y(x) \subseteq \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^s$ denote the feasible set and the multiobjective function

of the lower-level problem, respectively. When $s = 1$, we reduce to the classically optimistic bilevel optimization problem.

Except for [10,34], much of the existing literature has focused on the case when $\Phi$ is the weakly efficient optimal solution map of (Mul[$x$]). Earlier seminal works exploring this subject include [24,29, 33,48,63], providing optimality results and various types of solution algorithms.

Furthermore, the weighted-sum scalarization approach, which is commonly used in recent references to establish the necessary optimality conditions for related models, has numerous drawbacks when used to solve semivectorial nonconvex optimization problems. One notable problem is that the parameter employed in weighted-sum scalarization is treated as an upper-level objective variable. While this approach was originally designed to deal with cases involving weakly efficient solutions to lower-level problems, it fails to be effective for efficient solutions. Further, although weakly Pareto optimal sets are theoretically significant, they frequently lack utility in practice.

This chapter concentrates on the semivectorial bilevel program (SVBOP), which solves the lower-level problem (Mul[$x$]) efficiently. We revisit the Charnes-Cooper approach, which was first proposed in the seminal works [15,16] and recently applied to convex vector optimization problems in [17].

To proceed with the non-smooth analysis of problem (SVBOP), it is necessary to ensure that the efficiency map of a totally linear parametric multiobjective optimization problem has a closed graph. This criterion is essential for determining the existence of solutions to (SVBOP).

Using an improved Charnes-Cooper approach, we can then convert the multiobjective bilevel programming problem into a bilevel programming problem with a scalar lower-level problem. This novel scalarization approach effectively addresses significant classes of vectorial bilevel problems, overcoming the limitations of previous scalarization methods.

Moreover, we transform the bilevel optimization problem into a single-level optimization problem, ensuring global and local equivalence with the original problem. Remarkably, this reformulation is performed without requiring the scalar lower-level solution set obtained using the Charnes-Cooper technique to be either inner semicompact or inner semicontinuous. Building on this basis, we develop the necessary optimality conditions to address the problem under study.

The outline of this chapter is structured as follows: Section 1 first examines the graph-closedness of the associated efficiency mapping. Section 2 describes our new scalarization technique as well as the transformation process that results in a single-level optimization problem. Moving on to Section

3, we establish a value function estimation in the smooth and non-smooth cases and provide sufficient conditions to ensure its local Lipschitz property. Subsequently, in Section 4, suitable Abadie constraint qualifications are used to establish the necessary optimality conditions, and we introduce a general partial calmness concept specifically tailored to the optimal value function reformulation of problem (SVBOP). We use an appropriate weak Mangasarian-Fromowitz constraint qualification to establish the necessary optimality conditions. To enhance these conditions. Moreover, we study the special case where the lower-level multiobjective problem is linear in the lower decision variable. Finally, we draw our conclusions in Section 6.

# 2 Existence of optimal solutions

In [29], the authors demonstrate the existence of solutions for two-level scalar programming problems. In this chapter, however, we primarily focus on the two-level optimistic semivector optimization problem (SVBOP), where the set-valued mapping $\Phi = \Phi_{\text{eff}}$ represents the collection of efficient optimal solutions for the parametric multi-objective optimization problem (Mul$[x]$). Additionally, for the sake of clarity, we define the upper-level and lower-level feasible sets as follows throughout the chapter:

$$X := \{x \in \mathbb{R}^n : G_i(x) \leq 0,\ i \in I\} \text{ and } Y(x) := \{y \in \mathbb{R}^m : g_j(x,y) \leq 0,\ j \in J\},$$

with $I = \{1, \cdots, q\}$, $J = \{1, \cdots, p\}$, $G_i : \mathbb{R}^n \to \mathbb{R}$ and $g_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. We assume that all involved functions at both levels are Lipschitz continuous.

We also assume that the graph of $\Phi_{\text{eff}}$ is closed. This closedness assumption on the graph of the efficiency mapping is necessary to ensure the existence of solutions to (SVBOP). However, it is worth noting that when solving the lower-level problem (Mul$[x]$) efficiently, the graph of the corresponding efficiency map $\Phi_{\text{eff}}$ is generally not closed. To illustrate, consider the following lower-level problem:

$$\min_y \left(\lfloor y \rfloor; x^2 + \lfloor y + 2 \rfloor\right) \quad \text{s.t. } y \in [2,3],$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Here, the efficiency map is given by $\Phi_{\text{eff}}(x) = [2,3[$ for all $x \in \mathbb{R}$. Hence, gph $\Phi_{\text{eff}} = \mathbb{R} \times [2,3[$, which is not closed.

The non-closedness of gph $\Phi_{\text{eff}}$ poses challenges, for example in proving the existence of solutions to

(SVBOP). Considering the lower-level problem above, the upper-level problem

$$\min_{x,y} |x| - y + 3 \ \text{ s.t. } x \leq 0 \ \text{ s.t. } (x,y) \in \text{gph } \Phi_{\text{eff}}$$

clearly admits no solution.

In this section, we prove that for a completely linear multiobjective parametric optimization problem, the efficiency map has a closed graph. This closedness property ensures the existence of solutions to (SVBOP), as will be shown subsequently.

Recall that a fully linear multiobjective parametric optimization problem takes the following form:

$$\min_{y} Ay \ \text{ s.t. } Bx + Cy \leq e, \tag{L[x]}$$

where $A \in \mathbb{R}^{s\times m}$, $B \in \mathbb{R}^{p\times n}$, $C \in \mathbb{R}^{p\times m}$, and $e \in \mathbb{R}^p$. We represent every matrix in terms of its row vectors as $A = (a_k)_{1\leq k\leq s}$, $B = (b_j)_{1\leq j\leq p}$ and $C = (c_j)_{1\leq j\leq p}$. For this problem, the feasible set $Y(x)$, with $x \in \mathbb{R}^n$, is given by

$$Y(x) = \{y \in \mathbb{R}^m \ : \ Bx + Cy \leq e\}.$$

The following proposition shows that the efficiency map of a fully linear multiobjective parametric optimization problem possesses a closed graph, proven here without any scalarization.

**Proposition 2.1.** *The mapping* $\Phi_{\text{eff}}$ *associated to the problem* (L[x]) *possesses a closed graph.*

*Proof.* Suppose, on the contrary, that gph $\Phi_{\text{eff}}$ is not closed. Then, there exists a sequence $\{(x_t, y_t)\} \subseteq$ gph $\Phi_{\text{eff}}$ converging to $(x,y) \in$ gph $Y \setminus$ gph $\Phi_{\text{eff}}$. Hence, there is $\overline{y} \in Y(x)$ such that $a_k\overline{y} \leq a_k y$ for all $k = 1, \cdots, s$ and $a_k\overline{y} < a_k y$ for some $k = 1, \cdots, s$. Hence $\overline{y} - y \in \Gamma$, where

$$\Gamma := \{y \in \mathbb{R}^m \ : \ a_k y \leq 0 \ \text{ for all } k = 1, \cdots, s, \ \text{ and } a_k y < 0 \text{ for some } k = 1, \cdots, s\}.$$

Let $\mathcal{J} \subseteq J$ such that $\mathcal{J} = \mathcal{J}(t) := \{j \in J \ : \ b_j x_t + c_j y_t = e_j\}$. Then, $b_j x + c_j y = e_j$ and $c_j\overline{y} \leq c_j y$ for all $j \in \mathcal{J}$.

We claim there exists $j \in J \setminus \mathcal{J}$ with $c_j\overline{y} > c_j y$. Suppose to the contrary that $c_j(\overline{y} - y) \leq 0$ for all $j \in J \setminus \mathcal{J}$. Henceforth, for a given $z \in \Phi_{\text{eff}}(x)$, we have $(x, z + (\overline{y} - y)) \in \text{gph}Y$ since $(x,z) \in \text{gph}\Upsilon$. However, $z + (\overline{y} - y) - z = \overline{y} - y \in \Gamma$, so $a_k(z + (\overline{y} - y)) \leq a_k z$ for all $k$ and $a_k(z + (\overline{y} - y)) < a_k z$ for some $k$, contradicting $z$ being efficient for (L[x]). Hence there exists the claimed $j$.

Let $\mathcal{J}_1, \mathcal{J}_2 \subseteq J$ such that

$$\mathcal{J}_1 := \{j \in J \ : \ c_j(\overline{y} - y) \leq 0\} \text{ and } \mathcal{J}_2 := \{j \in J \ : \ c_j(\overline{y} - y) > 0\}.$$

Observe that $\mathcal{J} \subseteq \mathcal{J}_1$, $\mathcal{J}_2 \neq \emptyset$ and $\mathcal{J}_1 \cup \mathcal{J}_2 = J$. Fix $t \in \mathbb{N}$, and define

$$\rho_t := \min_{j \in \mathcal{J}_2} \frac{e_j - (b_j x_t + c_j y_t)}{c_j(\overline{y} - y)} = \frac{e_l - (b_l x_t + c_l y_t)}{c_l(\overline{y} - y)} > 0,$$

with $l \in \mathcal{J}_2$ and taking account the fact that $\mathcal{J} \nsubseteq \mathcal{J}_2$. Let

$$\tilde{y}_t := y_t + \rho_t(\overline{y} - y). \tag{2.1}$$

Since $\rho_t > 0$ and $\overline{y} \neq y$, then $\tilde{y}_t \neq y_t$. Two cases have to be considered.

- For $j \in \mathcal{J}_1$, we obtain

$$b_j x_t + c_j \tilde{y}_t = b_j x_t + c_j y_t + \rho_t c_j(\overline{y} - y) \leq b_j x_t + c_j y_t \leq e_j.$$

- For $j \in \mathcal{J}_2$, we obtain

$$\begin{aligned} b_j x_t + c_j \tilde{y}_t &= b_j x_t + c_j y_t + \rho_t c_j(\overline{y} - y) \\ &= b_j x_t + c_j y_t + \left(\rho_t \times \frac{c_j(\overline{y} - y)}{e_j - (b_j x_t + c_j y_t)} \times (e_j - (b_j x_t + c_j y_t))\right) \\ &\leq b_j x_t + c_j y_t + (1 \times (e_j - (b_j x_t + c_j y_t))) \\ &\leq e_j. \end{aligned}$$

Consequently, $(x_t, \tilde{y}_t) \in \text{gph } Y$. Moreover, we have from (2.1) that $a_k \tilde{y}_t \leq a_k y_t$ for all $k = 1, \cdots, s$ and $a_k \tilde{y}_t < a_k y_t$ for some $k = 1, \cdots, s$, which contradicts the fact that $y_t \in \Phi_{\text{eff}}(x_t)$. Therefore, gph $\Phi_{\text{eff}}$ is closed. □

# 3 Charnes-Cooper scalarization to semivectorial bilevel programs

In this section, the initial semivectorial bilevel problem (SVBOP) is transformed into a single optimistic bilevel programming problem. To this end, we revisit the Charnes-Cooper scalarization technique to

reformulate the lower-level problem (Mul[$x$]) into a standard one-level optimization problem. Taking advantage of this scalarization, we investigate the relationship between the solution set of the resulting problem and that of (SVBOP).

To begin, consider the following set

$$\Theta = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | x \in X \text{ and } y \in Y(x)\}.$$

Then, for $(x, y) \in \Theta$ the scalarization technique of the problem (Mul[$x$]) is given in the following form:

$$\begin{cases} \min\limits_{w} \sum\limits_{k=1}^{s} f_k(x, w) \\ f_k(x, w) - f_k(x, y) \leq 0, \ k = 1, \ldots, s \\ w \in Y(x). \end{cases} \tag{CCS[x,y]}$$

We refer to the problem (CCS[x,y]) as a revisited Charnes-Cooper scalarization problem.

To exploit the above scalarization we denote by $S(x, y)$ the solution set of (CCS[x,y]), and

$$\Xi(x, y) = \big\{w \in Y(x), \ f_k(x, w) - f_k(x, y) \leq 0, \ k = 1, \ldots, s\big\}, \tag{2.2}$$

the feasible set of problem (CCS[x,y]). When efficient solutions w.r.t $\mathbb{R}^s_+$ are considered for the lower-level problem (Mul[$x$]) we show the relation between (CCS[x,y]) and (Mul[$x$]).

**Proposition 2.2.** *Let $(x, y) \in \Theta$ and consider problems (Mul[$x$]) and (CCS[x,y]). Then, $y$ is an efficient solution of (Mul[$x$]); that is, $y \in \Phi_{\text{eff}}(x)$; if and only if $y$ is a solution of (CCS[x,y]).*

*Proof.* Assume that $y \in \Phi_{\text{eff}}(x)$. By contrary, suppose that $y$ is not a solution of (CCS[x,y]).

Hence, there exists $w_0 \in Y(x)$ such that:

$$f_k(x, w_0) \leq f_k(x, y), \quad \text{for all} \quad k \in \{1, \cdots, s\}, \tag{2.3}$$

and

$$\sum_{k=1}^{s} f_k(x, w_0) < \sum_{k=1}^{s} f_k(x, y). \tag{2.4}$$

From (2.4) there exists an index $j \in \{1, \cdots, s\}$ such that

$$f_j(x, w_0) < f_j(x, y),$$

In conclusion, we find $w_0 \in Y(x)$ such that $f_k(x, w_0) \leq f_k(x, y)$, for all $k \in \{1, \cdots, s\}$, and $f_j(x, w_0) < f_j(x, y)$ for some $j \in \{1, \cdots, s\}$, which is contrary to the fact that $y \in \Phi_{\text{eff}}(x)$.

Conversely, let $y$ be a solution of (CCS[x,y]) and assume that $y$ is not an efficient solution of (Mul[$x$]). Then, there exists $w_0 \in Y(x)$ such that:

$$f_k(x, w_0) \leq f_k(x, y), \quad \text{for all} \quad k \in \{1, \cdots, s\}. \tag{2.5}$$

$$\text{and} \quad f_j(x, w_0) < f_j(x, y), \quad \text{for some j.} \tag{2.6}$$

Since $w_0 \in Y(x)$ and (2.5) we have $w_0$ is a feasible solution of (CCS[x,y]). Now, by combining (2.5) and (2.6) we get a contradiction. □

We highlight that the proposition 2.2 is no longer true when weak efficient solutions for the lower-level problem are considered. To see these, we provide a simple example as follows.

**Example 2.3.** We consider the multiobjective optimization problem shown below:

$$\begin{cases} \min\limits_{y} \ ((x-y+1)^2+1, (x-y+2)^2+1, 3) \\ g_1(x,y) = y - 1 \leq 0, \\ g_2(x,y) = -y - 1 \leq 0. \end{cases} \tag{2.7}$$

where $X = \left\{x \in \mathbb{R} : x^2 - x \leq 0\right\} = [0, 1]$. Let $(x, y) \in \Theta$ with $y \in \Phi_{\text{eff}}(x)$, using the revisited Charnes-Cooper scalarization, the problem (2.7) is then equivalent to:

$$\begin{cases} \min\limits_{w} \ h(x,w) = (x-w+1)^2 + (x-w+2)^2 + 5 \\ (x-w+1)^2 - (x-y+1)^2 \leq 0, \\ (x-w+2)^2 - (x-y+2)^2 \leq 0, \\ -1 \leq w \leq 1. \end{cases} \tag{2.8}$$

Observe that for $\bar{x} = 0$, the point $\bar{y} = 0$ is a weak efficient solution of (2.7). However, $w = 1$ is a feasible set of (2.8) and $h(0, 1) = 6 < h(0, 0) = 10$, which means that $\bar{y} = 0$ is not a minimizer of the problem (2.8).

Now, exploiting the above arguments, the semivectorial optimization problem (SVBOP) can be replaced by the following standard bilevel optimization problem:

$$\begin{cases} \min\limits_{x,y} \ F(x,y) \\ \text{subject to} \quad x \in X, \ y \in S(x,y). \end{cases} \tag{2.9}$$

In the following result, we formalize the link between problems (2.9) and (SVBOP).

**Proposition 2.4.** *Consider problem (SVBOP)-(2.9). Then, any point $(\bar{x}, \bar{y})$ is a local (resp. global) optimal solution of the problem (SVBOP) if and only if $(\bar{x}, \bar{y})$ is a local (resp. global) optimal solution of the problem (2.9).*

*Proof.* Let $(\bar{x}, \bar{y})$ be a local optimal solution of the problem (SVBOP).

Suppose that $(\bar{x}, \bar{y})$ is not a solution to the problem (2.9), then there exists $(x^s, y^s)$ with $x^s \longrightarrow \bar{x}$, $y^s \longrightarrow \bar{y}$ and $x^s \in X$, $y^s \in S(x^s, y^s)$ such that

$$F(x^s, y^s) < F(\bar{x}, \bar{y}).$$

Since $x^s \in X$, and $y^s \in S(x^s, y^s)$ from Proposition 3.1 one has $y^s \in \Phi_{\text{eff}}(x^s)$.

Hence, from the above arguments, we have : $(x^s, y^s) \to (\bar{x}, \bar{y})$ with $x^s \in X, y^s \in \Phi_{\text{eff}}(x^s)$ such that

$$F(x^s, y^s) < F(\bar{x}, \bar{y}).$$

which contradicts the fact that $(\bar{x}, \bar{y})$ is a local optimal solution of the problem (SVBOP).

The proof of the reciprocal implication follows on the first part, since from Proposition 3.1 $\Phi_{\text{eff}}(x)$ and $S(x, y)$ play a symetrical role. □

The optimal value reformulation is a classical method to transform the latter problem into an optimization problem.

$$\begin{cases} \min\limits_{x,y} F(x, y) \\ x \in X, \quad y \in Y(x) \\ \sum\limits_{k=1}^{s} f_k(x, y) - \varphi(x, y) \leq 0, \end{cases} \tag{2.10}$$

here, $\varphi$ is the optimal value function associated with (CCS[x,y]) given by

$$\varphi(x, y) = \inf_{w \in \Xi(x,y)} \sum_{k=1}^{s} f_k(x, w), \tag{2.11}$$

and

$$\begin{aligned} S(x, y) &:= \arg\min_{w} \left\{ \sum_{k=1}^{s} f_k(x, w), w \in \Xi(x, y) \right\} \\ &= \left\{ w \in Y(x) : \sum_{k=1}^{s} f_k(x, w) \leq \varphi(x, y), \ f_k(x, w) \leq f_k(x, y), \ k = 1 \dots s \right\}. \end{aligned} \tag{2.12}$$

Note that the constraints $f_k(x,w) - f_k(x,y) \leqslant 0,\ k \in \{1 \cdots s\}$, do not appear in problem (2.10) since for $w = y$ they hold true.

**Remark 2.5.** When the lower-level problem is convex, other approaches such as the generalized equation or Karush-Kuhn-Tucker reformulations may be considered for the problem (2.9), for more information, see [27,28,71]. However, we will concentrate on reformulation (2.10) in order to develop the optimality conditions for the problem (SVBOP).

Before closing this section, some remarks on this new scalarization are in order. All efforts are made by investigating weak efficient solutions. In many practical multiobjective problems, efficient solutions are better than weak efficient solutions and more useful in numerical calculation. Unlike the previous works [11,25,47,48], we do not make use of either the closedness of the solution set of a scalarized problem or a more variable in the upper-level problem (for more details, see [11,25,47,48]).

# 4 Subdifferential estimate of Charnes-Cooper value function

## 4.1 Smooth case

In this subsection, we intend to derive a subdifferential estimate and establish the local Lipschitz continuity of $\varphi$. We start with the estimation of the coderivative of $\Xi$. To proceed, some definitions and notations are required. Namely, for $(\bar{x}, \bar{y}, \bar{w}) \in \text{gph } \Xi$,

the set-valued map $\Pi$ provided by

$$\begin{aligned} \Pi(v) \ := \ & \big\{ (x,y,w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m | \, (x,w) \in \text{gph } Y \text{ and} \\ & (f_1(x,w) - f_1(x,y), \cdots, f_s(x,w) - f_s(x,y)) \in v - \mathbb{R}^s_+ \big\}, \end{aligned} \tag{2.13}$$

the particular set of multipliers that play an important role in the remainder of the paper:

$$\Lambda(\overline{x}, \overline{w}, z) := \left\{ (\sigma, \beta) : \begin{array}{c} z + \displaystyle\sum_{k=1}^{s} \sigma_k \nabla_w f_k(\overline{x}, \overline{w}) + \nabla_w g(\overline{x}, \overline{w})^T \beta = 0 \\ \sigma \geq 0,\ \beta_j \geq 0,\ \beta_j g_j(\overline{x}, \overline{w}) = 0,\ j = 1, \cdots, p \end{array} \right\}. \tag{2.14}$$

Furthermore, to show that the optimal value function $\varphi$ is local Lipschitz continuous, we introduce the following assumptions associated with the lower-level problem.

$(\mathcal{H}_1)$ The mapping $\Pi$ is calm at $(0, \overline{x}, \overline{y}, \overline{w})$.

$(\mathcal{H}_2)$ The set-valued mapping $\Xi$ is locally bounded at $(\overline{x}, \overline{y})$ and $\Xi$ is Lipschitz-like at $(\overline{x}, \overline{y}, \overline{w})$.

$(\mathcal{H}_3)$ The lower level regularity condition holds at $(\overline{x}, \overline{y}, \overline{w})$

$$\left[\nabla_w g_0\left(\overline{x}, \overline{y}, \overline{w}\right)^T \eta = 0,\ \eta \geq 0,\ \eta^T g_0\left(\overline{x}, \overline{y}, \overline{w}\right) = 0\right] \implies \eta = 0.$$

where $g_0 : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^p$ is defined by $g_0\left(x, y, w\right) = g\left(x, w\right)$.

**Theorem 2.6.** *Consider the mapping $\Xi$ in (2.2) and let $(\overline{x}, \overline{y}, \overline{w}) \in gph\, \Xi$ such that $f_k\left(\overline{x}, \overline{w}\right) - f_k\left(\overline{x}, \overline{y}\right) = 0$ for all $k = 1, \cdots, s$. Assume the condition $(\mathcal{H}_1)$ and the lower-level regularity $(\mathcal{H}_3)$ be satisfied at $(\bar{x}, \bar{y}, \bar{w})$. Then, for all $z \in \mathbb{R}^m$ we obtain the coderivative estimate:*

$$D^*\Xi(\bar{x}, \bar{y}, \bar{w})\left(z\right) \ \subset \ \left( \bigcup_{(\sigma,\beta)\in\Lambda(\overline{x},\overline{w},z)} \begin{bmatrix} \nabla_x g\left(\overline{x}, \overline{w}\right)^T \beta \\ -\sum_{k=1}^{s} \sigma_k \nabla_y f_k(\bar{x}, \bar{y}) \end{bmatrix} \right).$$

*Proof.* Setting $\Delta = \{(x, y, w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m | g_0\left(x, y, w\right) \leq 0\}$,

$H\left(x, y, w\right) = (f_1(x, w) - f_1(x, y), \cdots, f_s(x, w) - f_s(x, y))$, and adding the calmness property $(\mathcal{H}_1)$, we have from [41, Theorem 4.1]

$$N_{\text{gph }\Xi}\left(\overline{x}, \overline{y}, \overline{w}\right) \ \subset \ \left( \bigcup_{\sigma \in N_{-\mathbb{R}^s_+}\left(H(\overline{x},\overline{y},\overline{w})\right)} D^*H\left(\overline{x}, \overline{y}, \overline{w}\right)\left(\sigma\right) + N_\Delta\left(\overline{x}, \overline{y}, \overline{w}\right) \right). \tag{2.15}$$

Because, for each $k = 1, \cdots s$, the constraint $f_k\left(\overline{x}, \overline{w}\right) - f_k\left(\overline{x}, \overline{y}\right) \leq 0$ is active at $(\overline{x}, \overline{y}, \overline{w})$, one has $H\left(\overline{x}, \overline{y}, \overline{w}\right) = 0$ and $N_{-\mathbb{R}^s_+}\left(H\left(\overline{x}, \overline{y}, \overline{w}\right)\right) = \mathbb{R}^s_+$. Based on these arguments, (2.15) becomes

$$N_{\text{gph }\Xi}\left(\overline{x}, \overline{y}, \overline{w}\right) \ \subset \ \left( \bigcup_{\sigma\in\mathbb{R}^s_+} \bigcup_{[\beta\geq 0, \beta^T g(\overline{x},\overline{w})=0]} \begin{bmatrix} 0 \\ -\sum_{k=1}^{s} \sigma_k \nabla_y f_k(\bar{x}, \bar{y}) \\ \sum_{k=1}^{s} \sigma_k \nabla_w f_k(\bar{x}, \bar{w}) \end{bmatrix} + \begin{bmatrix} \nabla_x g\left(\overline{x}, \overline{w}\right)^T \beta \\ 0 \\ \nabla_w g\left(\overline{x}, \overline{w}\right)^T \beta \end{bmatrix} \right) \tag{2.16}$$

while taking into account the definition of $g_0$ and the normal cone's expression

$$N_\Delta\left(\overline{x}, \overline{y}, \overline{w}\right) \subseteq \{\nabla g_0\left(\overline{x}, \overline{y}, \overline{w}\right)^T \beta | \beta \geq 0,\ \beta^T g_0\left(\overline{x}, \overline{y}, \overline{w}\right) = 0\}$$

which holds under the lower-level regularity $(\mathcal{H}_3)$ at $(\overline{x}, \overline{y}, \overline{w})$.

Now, fix $z \in \mathbb{R}^m$ and let $(a,b) \in D^*\Xi(\bar{x},\bar{y},\bar{w})(z)$. By definition, $(a,b,-z) \in N_{\text{gph }\Xi}(\overline{x},\overline{y},\overline{w})$. Then, there exist $\sigma \geq 0$ and $\beta \geq 0$ with $\beta^T g(\overline{x},\overline{w}) = 0$ such that

$$\begin{array}{rcl} a & = & \nabla_x g(\overline{x},\overline{w})^T \beta, \\ b & = & -\sum\limits_{k=1}^{s} \sigma_k \nabla_y f_k(\bar{x},\bar{y}) \\ 0 & = & z + \sum\limits_{k=1}^{s} \sigma_k \nabla_w f_k(\bar{x},\bar{w}) + \nabla_w g(\overline{x},\overline{w})^T \beta. \end{array} \tag{2.17}$$

Thus, for $(a,b) \in D^*\Xi(\bar{x},\bar{y},\bar{w})(z)$ there exist $(\sigma,\beta) \in \Lambda(\overline{x},\overline{w},z)$ such that

$$a = \nabla_x g(\overline{x},\overline{w})^T \beta \;\text{ and }\; b = -\sum_{k=1}^{s} \sigma_k \nabla_y f_k(\bar{x},\bar{y}).$$

Consequently,

$$D^*\Xi(\bar{x},\bar{y},\bar{w})(z) \;\subset\; \left( \bigcup_{(\sigma,\beta)\in\Lambda(\overline{x},\overline{w},z)} \begin{bmatrix} \nabla_x g(\overline{x},\overline{w})^T \beta \\ -\sum\limits_{k=1}^{s} \sigma_k \nabla_y f_k(\bar{x},\bar{y}) \end{bmatrix} \right),$$

which justifies the desired inclusion. □

In the following theorem, we derive the sensitivity result for $\varphi$ using the above coderivative estimate. Remembring the definition of the multiplier sets $\Lambda(\overline{x},\overline{w},z)$, we put

$$\Lambda\left(\overline{x},\overline{w},\sum_{k=1}^{s}\nabla_w f_k(\bar{x},\bar{w})\right) = \Lambda(\overline{x},\overline{w}).$$

**Theorem 2.7.** *Let $(\overline{x},\overline{y},\overline{w}) \in gph\ \Xi$ such that $f_k(\overline{x},\overline{w}) - f_k(\overline{x},\overline{y}) = 0$ for all $k = 1,\cdots,s$. Assume that $S$ is inner semicontinuous at $(\overline{x},\overline{y},\overline{w})$. In addition, let the condition $(\mathcal{H}_1)$ and the lower-level regularity $(\mathcal{H}_3)$ be satisfied at $(0,\bar{x},\bar{y},\bar{w})$ and $(\bar{x},\bar{y},\bar{w})$, respectively. Then, we obtain*

$$\partial\varphi(\bar{x},\bar{y}) \;\subset\; \begin{bmatrix} \sum\limits_{k=1}^{s} \nabla_x f_k(\bar{x},\bar{w}) \\ 0_{\mathbb{R}^m} \end{bmatrix} + \bigcup_{(\sigma,\beta)\in\Lambda(\overline{x},\overline{w})} \begin{bmatrix} \nabla_x g(\overline{x},\overline{w})^T \beta \\ -\sum\limits_{k=1}^{s} \sigma_k \nabla_y f_k(\bar{x},\bar{y}) \end{bmatrix}. \tag{2.18}$$

*If moreover $(\mathcal{H}_2)$ is satisfied at $(\bar{x},\bar{y},\bar{w})$, then $\varphi$ is Lipschitz continuous around $(\overline{x},\overline{y})$.*

*Proof.* Since $S$ is semicontinuity at $(\overline{x},\overline{y},\overline{w})$, we have from [57, Corollary 1.109] that

$$\partial\varphi(\bar{x},\bar{y}) \subset \sum_{k=1}^{s} \nabla_{x,y} f_k(\bar{x},\bar{w}) + D^*\Xi(\bar{x},\bar{y},\bar{w})\left(\sum_{k=1}^{s} \nabla_w f_k(\bar{x},\bar{w})\right).$$

As $S(x,y) \subset \Xi(x,y)$ for all $(x,y) \in \Theta$, the map $\Xi$ is also inner semicontinuous at $(\overline{x},\overline{y},\overline{w})$. Using the upper estimate of $\Xi$ from Theorem 2.6, we get

$$\partial\varphi(\bar{x},\bar{y}) \subset \begin{bmatrix} \sum_{k=1}^{s} \nabla_x f_k(\bar{x},\bar{w}) \\ 0_{\mathbb{R}^m} \end{bmatrix} + \bigcup_{(\sigma,\beta)\in\Lambda(\overline{x},\overline{w})} \begin{bmatrix} \nabla_x g(\overline{x},\overline{w})^T \beta \\ -\sum_{k=1}^{s} \sigma_k \nabla_y f_k(\bar{x},\bar{y}) \end{bmatrix}$$

while taking into account the satisfaction of both conditions $(\mathcal{H}_1)$ and the lower-level regularity $(\mathcal{H}_3)$ at $(0,\bar{x},\bar{y},\bar{w})$ and $(\bar{x},\bar{y},\bar{w})$, respectively. Thus, we have the claimed estimation (2.18). On the other hand, since $(\mathcal{H}_2)$ is satisfied at $(\bar{x},\bar{y},\bar{w})$ and $S$ is inner semicontinuous at $(\bar{x},\bar{y},\bar{w})$, from [51, Lemma 4.1], we get that $\varphi$ is Lipschitz continuous around $(\overline{x},\overline{y})$. □

**Remark 2.8.** It is important to note that there are several studies on sensitivity analysis for marginal functions in parametric programs. The linear independence constraint qualification (LICQ) and the Mangasarian-Fromovitz constraint qualification (MFCQ), two common constraint qualifications, were used in all of these publications to estimate the generalized subgradients of marginal functions.

However, in the present case, we take advantage of the set-valued map Π's calmness property. The fundamental reason for this is that MFCQ is stronger than the partial calmness $(\mathcal{H}_1)$, see [51] and [57, Corollary 4.39]. More precisely, MFCQ is only sufficient but not necessary for the presence of the Lipschitz-like property. The latter property is stronger than the calmness property, which is automatically satisfied if, e.g., the functions $f$ and $g$ are fully linear [30].

Another essential element of this is that MFCQ might not hold, particularly when $f_k(x,w) = \varphi_k(x)$ for some $k = 1,\cdots,s$ with $\varphi_k(x) = \inf_{y\in Y(x)} f_k(x,y)$.

**Remark 2.9.** It could have been stated that $(\mathcal{H}_2)$ is frequently utilized in the literature for ensuring local Lipschitz continuity of marginal functions; see, e.g., [55, Theorem 5.3 (i)]. In addition, in [51], the authors present additional criteria for establishing local Lipschitz continuity of marginal functions in nonlinear parametric optimization problems with a feasible region represented by Ξ. More detail on these and related advancements may be found in the aforementioned articles, notably in Lemma 4.1 and Lemma 4.2.

## 4.2 Non-smooth case

This subsection investigates the local sensitivity and stability properties of the optimal value function $\varphi$ defined in (2.11). We derive an upper estimate of the Mordukhovich subdifferential of $\varphi$ at a reference point, providing insight into its point-based sensitivity. Additionally, various conditions are established for the local Lipschitz continuity of $\varphi$ around the reference point.

To facilitate the analysis, some key definitions and notation are introduced. Let $(x, y, w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$, and set:

$$\begin{aligned} f_{0k}(x,y,w) &:= f_k(x,w), \text{ for all } k = 1, \cdots, s; \\ f_0(x,y,w) &:= \sum_{k=1}^{s} f_{0k}(x,y,w); \\ g_0(x,y,w) &:= g(x,w). \end{aligned}$$

We impose the calmness property as a significant constraint qualification at the reference points of the following set-valued mappings:

$$\Psi_{Y_0}(\mu) := \{(x,y,w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \ : \ g_0(x,y,w) \in \mu - \mathbb{R}^p_+\}, \quad \text{for all } \mu \in \mathbb{R}^p$$

$$\Psi_{\Xi}(\kappa) := \left\{(x,y,w) \in \text{gph } Y_0 \ : \ H(x,y,w) \in \kappa - \mathbb{R}^s_+\right\}, \quad \text{for all } \kappa \in \mathbb{R}^s,$$

where $Y_0 : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is the set-valued mapping given by $Y_0(x,y) = Y(x)$ and $H(x,y,w) = \Big(f_1(x,w) - f_1(x,y), \cdots, f_s(x,w) - f_s(x,y)\Big)$.

**Theorem 2.10.** *Let $(\overline{x}, \overline{y})$ be a feasible point of problem* (SVBOP). *Assume that $\Xi$ possesses the Aubin property at $(\overline{x}, \overline{y}, w)$ for all $w \in S(\overline{x}, \overline{y})$. Moreover, suppose that the set valued-mapping $S$ is inner semi-compact at $(\overline{x}, \overline{y})$ and that the set-valued mapping $\Psi_{\Xi}$ (respectively, $\Psi_{Y_0}$) is calm at $(0_{\mathbb{R}^s}, \overline{x}, \overline{y}, w)$ (respectively, at $(0_{\mathbb{R}^p}, \overline{x}, \overline{y}, w)$) for all $w \in S(\overline{x}, \overline{y})$. Then, the optimal value function $\upsilon$ is Lipschitz continuous near $(\overline{x}, \overline{y})$, and we have:*

$$\begin{aligned} \partial\varphi(\overline{x}, \overline{y}) \subseteq \bigcup_{w \in S(\overline{x},\overline{y})} \bigcup_{\delta \geq 0} \bigcup_{\vartheta \geq 0} \Bigg\{ & (\overline{x}^*, \overline{y}^*) \ : \ (\overline{x}^*, \overline{y}^*, 0_{\mathbb{R}^m}) \in \sum_{k=1}^{s} \partial f_k(\overline{x}, w) \\ & + \sum_{k=1}^{s} \delta_k \left(\partial f_k(\overline{x}, w) - conv\ \partial f_k(\overline{x}, \overline{y})\right) + \sum_{j=1}^{p} \vartheta_j \partial g_j(\overline{x}, w) \Bigg\}. \end{aligned}$$

*Proof.* Due to the inner semicompactness of $S$ at $(\overline{x},\overline{y})$ and the fact that $f_{0k}$, $k=1,\cdots,s$, is Lipschitz continuous around $(\overline{x},\overline{y},w)$ for all $w\in S(\overline{x},\overline{y})$, we have from [58, Theorem 7(ii)] that

$$\begin{aligned}\partial^{\infty}\varphi(\overline{x},\overline{y}) &\subseteq \bigcup_{w\in S(\overline{x},\overline{y})} D^{*}\Xi(\overline{x},\overline{y},w)(0), \text{ and}\\ \partial\varphi(\overline{x},\overline{y}) &\subseteq \bigcup_{w\in S(\overline{x},\overline{y})} \Big\{(\eta,0)^{\top}+D^{*}\Xi(\overline{x},\overline{y},w)(\sigma),\ (\eta,0,\sigma)\in\partial f_0(\overline{x},\overline{y},w)\Big\}.\end{aligned}$$

By the Aubin property of $\Xi$ at $(\overline{x},\overline{y},w)$ for all $w\in S(\overline{x},\overline{y})$, we have the inclusion $D^{*}\Xi(\overline{x},\overline{y},w)(0)\subseteq\{0\}$ for all $w\in S(\overline{x},\overline{y})$. Thus, $\partial^{\infty}\varphi(\overline{x},\overline{y})=\{0\}$. Consequently, the optimal value function $\varphi$ is Lipshitz near $(\overline{x},\overline{y})$.

Furthermore, for any $(\overline{x}^{*},\overline{y}^{*})\in\partial\varphi(\overline{x},\overline{y})$, there exists $\overline{w}\in S(\overline{x},\overline{y})$ such that for some $(\eta,0,\sigma)\in\partial f_0(\overline{x},\overline{y},\overline{w})$ one has

$$\begin{pmatrix}\overline{x}^{*}\\ \overline{y}^{*}\end{pmatrix}\in\begin{pmatrix}\eta\\ 0\end{pmatrix}+D^{*}\Xi(\overline{x},\overline{y},\overline{w})(\sigma).$$

Exploiting the coderivative definition together with the subdifferential sum rule in the last inclusion, we get

$$(\overline{x}^{*}-\eta,\overline{y}^{*},-\sigma)\in N\Big((\overline{x},\overline{y},\overline{w}),\operatorname{gph}\Xi\Big).$$

Combining this with the calmness of $\Psi_{\Xi}$ while observing that $\Psi_{\Xi}(0)=\operatorname{gph}\Xi$, we conclude from [41, Theorem 4.1] that there exists $\delta\in N\left(H(\overline{x},\overline{y},\overline{w}),-\mathbb{R}^{s}_{+}\right)$ such that

$$(\overline{x}^{*}-\eta,\overline{y}^{*},-\sigma)\in\partial\langle\delta,H\rangle(\overline{x},\overline{y},\overline{w})+N\Big((\overline{x},\overline{y},\overline{w}),\operatorname{gph}Y_0\Big).$$

Invoking the structure of the functions $f_{0k}$ and utilizing the fact that $N\left(H(\overline{x},\overline{y},\overline{w}),-\mathbb{R}^{s}_{+}\right)\subseteq\mathbb{R}^{s}_{+}$, it follows that there exists $(\eta_k,0,\sigma_k)\in\partial f_{0k}(\overline{x},\overline{y},\overline{w})$, with $k=1,\cdots,s$, such that $(\eta,0,\sigma)=\sum_{k=1}^{s}(\eta_k,0,\sigma_k)$. Additionally, there exist positive scalars $\delta_k$, with $k=1,\cdots,s$ such that

$$(\overline{x}^{*}-\eta,\overline{y}^{*},-\sigma)\in\sum_{k=1}^{s}\delta_k\left(\partial f_k(\overline{x},\overline{w})-\operatorname{conv}\partial f_k(\overline{x},\overline{y})\right)+N\Big((\overline{x},\overline{y},\overline{w}),\operatorname{gph}\Upsilon_0\Big).\tag{2.19}$$

This follows by applying the calculus rules for generalized derivatives along with the convexity property stated in (1.8).

Let us now compute the normal cone $N\Big((\overline{x},\overline{y},\overline{w}),\operatorname{gph}Y_0\Big)$. Taking into consideration that $\operatorname{gph}\Upsilon_0=\Psi_{Y_0}(0)$ and that $\Psi_{Y_0}$ is calm at $(0_{\mathbb{R}^p},\overline{x},\overline{y},\overline{w})$, we derive from [41, Theorem 4.1] that for any $(a,b,c)\in$

$N\Big((\overline{x},\overline{y},\overline{w}),\text{gph } Y_0\Big)$ there exits $\vartheta \in N\Big(g_0(\overline{x},\overline{y},\overline{w}),-\mathbb{R}^p_+\Big) \subseteq \mathbb{R}^p_+$ fulfilling

$$(a,b,c) \in \sum_{j=1}^{p} \vartheta_j \partial g_j(\overline{x},\overline{w}). \tag{2.20}$$

Hence, (2.19) and (2.20) yield

$$(\overline{x}^* - \eta, \overline{y}^*, -\sigma) \in \sum_{k=1}^{s} \delta_k \left(\partial f_k(\overline{x},\overline{w}) - \text{conv } \partial f_k(\overline{x},\overline{y})\right) + \sum_{j=1}^{p} \vartheta_j \partial g_j(\overline{x},\overline{w}).$$

This proves the claimed estimate for $\partial\varphi(\overline{x},\overline{y})$. □

**Remark 2.11.**

1. Let $j \in J$, one can observe that $N(g_{0j}(\overline{x},\overline{y},\overline{w}),-\mathbb{R}_+) = \mathbb{R}_+$ when $g_{0j}(\overline{x},\overline{y},\overline{w}) = 0$; and $N(g_{0j}(\overline{x},\overline{y},\overline{w}),-\mathbb{R}_+) = \{0\}$ when $g_{0j}(\overline{x},\overline{y},\overline{w}) < 0$. Hence, we deduce from the proof of Theorem 2.10 that $\vartheta_j \geq 0$ and $\vartheta_j g_{0j}(\overline{x},\overline{y},\overline{w}) = 0$ for all $j \in J$.

2. It bears noting that the Aubin property imposed on the set-valued mapping $\Xi$ is a sufficient but not necessary condition. As an alternative, the following constraint qualification could be imposed for each $w \in S(\overline{x},\overline{y})$:

$$\left[\begin{array}{l} (x^*,y^*,0) \in \partial\langle\delta_g,g_0\rangle(\overline{x},\overline{y},w) + \partial\langle\delta_H,H\rangle(\overline{x},\overline{y},w) \\ \\ \text{with } \delta_g \in N(g_0(\overline{x},\overline{y},w),-\mathbb{R}^p_+) \text{ and } \delta_H \in \mathbb{R}^s_+ \end{array}\right] \Rightarrow (x^*,y^*) = (0,0).$$

   This point-based constraint qualification establishes a weaker assumption that could still ensure the conclusions of Theorem 2.10 are valid, namely the claimed subdifferential estimate and Lipschitz continuity of $\varphi$.

3. In fact, the Aubin property appears to be strong, making it difficult to satisfy; however, many studies in the literature have a large number of properties and sufficient conditions to ensure its validity, see [41,42,57].

In Theorem 2.10, it is certainly important to simplify further the upper estimate of the subdifferential for the value function $\varphi$. One way to achieve this is by applying the inner semicontinuity property of the solution map $S$ rather than its inner semicompactness.

**Theorem 2.12.** *Let $(\overline{x},\overline{y})$ be a feasible point of problem* (SVBOP). *Assume that $\Xi$ possesses the Aubin property at $(\overline{x},\overline{y},\overline{w})$ while $S$ is inner semicontinuous at this point. Suppose that the set-valued mapping $\Psi_\Xi$ (respectively, $\Psi_{Y_0}$) is calm at $(0_{\mathbb{R}^s},\overline{x},\overline{y},\overline{w})$ (respectively, at $(0_{\mathbb{R}^p},\overline{x},\overline{y},\overline{w})$). Then, the optimal value function $\varphi$ is Lipschitz continuous near $(\overline{x},\overline{y})$, and we have:*

$$\partial\varphi(\overline{x},\overline{y}) \subseteq \bigcup_{\delta\geq 0}\bigcup_{\vartheta\geq 0}\Big\{(\overline{x}^*,\overline{y}^*) : (\overline{x}^*,\overline{y}^*,0_{\mathbb{R}^m})\in\sum_{k=1}^{s}\partial f_k(\overline{x},\overline{w}) + \sum_{k=1}^{s}\delta_k\left(\partial f_k(\overline{x},\overline{w}) - conv\,\partial f_k(\overline{x},\overline{y})\right)+\sum_{j=1}^{p}\vartheta_j\partial g_j(\overline{x},\overline{w})\Big\}.$$

*Proof.* We have that $S$ is inner semicontinuous at $(\overline{x},\overline{y},\overline{w})$. As stated previously, [58, Theorem 7(i)] ensures the optimal value function $\upsilon$ is Lipschitz near $(\overline{x},\overline{y})$, owing to the validity of the Aubin property of $\Xi$ at $(\overline{x},\overline{y},\overline{w})$. Again from [58, Theorem 7(i)], by the inner semicontinuity of $S$ at $(\overline{x},\overline{y},\overline{w})$ and the local Lipschitz continuity of $f_{0k}$, $k=1,\ldots,s$, at this point, for any $(\overline{x},\overline{y})\in\partial\upsilon(\overline{x},\overline{y})$ there exists $(\eta,0,\sigma)\in\partial f_0(\overline{x},\overline{y},\overline{w})$ such that

$$\begin{pmatrix}\overline{x}^*\\ \overline{y}^*\end{pmatrix}\in\begin{pmatrix}\eta\\ 0\end{pmatrix}+D^*\Xi(\overline{x},\overline{y},\overline{w})(\sigma).$$

As shown in the proof of Theorem 2.10, there exist $(\eta_k,0,\sigma_k)\in\partial f_{0k}(\overline{x},\overline{y},\overline{w})$, $k=1,\ldots,s$ such that $(\eta,0,\sigma)=\sum_{k=1}^{s}(\eta_k,0,\sigma_k)$, $\delta\in N((H(\overline{x},\overline{y},\overline{w})),-\mathbb{R}+^s)\subseteq\mathbb{R}^s+$ and $\vartheta\in N((g_0(\overline{x},\overline{y},\overline{w})),-\mathbb{R}^p+)\subseteq\mathbb{R}^p+$ fulfilling

$$(\overline{x}^*-\eta,\overline{y}^*,-\sigma)\in\sum_{k=1}^{s}\delta_k\left(\partial f_k(\overline{x},\overline{w})-\text{conv}\,\partial f_k(\overline{x},\overline{y})\right)+\sum_{j=1}^{p}\vartheta_j\partial g_j(\overline{x},\overline{w}).$$

In conclusion, this establishes the upper estimate of $\partial\varphi(\overline{x},\overline{y})$ that the theorem claims. □

**Remark 2.13.** It should be noted that the Aubin property of the set-valued mapping $\Xi$ is not required for the upper estimates of the fundamental subdifferential of $\upsilon$ given in Theorems 2.10 and 2.12. The Aubin property is exclusively needed to obtain the Lipschitz continuity of the already defined lower-level value function.

**Remark 2.14.** Assume the functions involved in (SVBOP) are continuously differentiable around the reference point.

(*a*) Under the inner semicontinuity of $S$, the upper estimate of the optimal value function in Theorem 2.12 reduces to that achieved in [34, Theorem 4.2].

($b$) Under the inner semicompactness of $S$, the optimal value function estimation in Theorem 2.10 reduces to the following estimation:

$$\partial\varphi(\overline{x},\overline{y}) \subseteq \bigcup_{w\in S(\overline{x},\overline{y})}\bigcup_{\delta\geq 0}\bigcup_{\vartheta\geq 0}\Big\{(\overline{x}^*,\overline{y}^*) : (\overline{x}^*,\overline{y}^*,0_{\mathbb{R}^m})\in\sum_{k=1}^{s}\nabla f_k(\overline{x},w) + \sum_{k=1}^{s}\delta_k\left(\nabla f_k(\overline{x},w)-\nabla f_k(\overline{x},\overline{y})\right)+\sum_{j=1}^{p}\vartheta_j\nabla g_j(\overline{x},w)\Big\}.$$

This estimation extends the one stated in [34, Theorem 4.2], and provides new necessary optimality conditions in smooth settings.

# 5 Necessary Optimality Conditions

## 5.1 N.O.C. via Abadie Constraint qualification

In this subsection, we introduce a weak form of the well-known Abadie constraint qualification before deriving Karush-Kuhn-Tucker conditions for the optimal value reformulation (2.10) of problem (SVBOP). To continue, let $(\overline{x},\overline{y})\in\Theta$, and denote by:

$$I^g(\bar{x},\bar{y}) := \big\{j\in\{1\cdots p\} : \quad g_j(\bar{x},\bar{y})=0\big\} \text{ and}$$
$$I^G(\bar{x}) := \big\{i\in\{1\cdots q\} : \quad G_i(\bar{x})=0\big\}.$$

the index sets of the upper and lower problems, respectively.

$$\Omega=\left\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m | x\in X,\ y\in Y(x),\ \text{and}\ \sum_{k=1}^{s}f_k(x,y)-\varphi(x,y)\leq 0\right\} \tag{2.21}$$

the feasible set of the problem (2.10).

Based on the previous section and assuming that $(\mathcal{H}_1)$ $(\mathcal{H}_2)$, and $(\mathcal{H}_3)$ hold, we define the linearized cone using the Clarke directional derivative. That is to say:

$$\mathcal{L}(\Omega,(\bar{x},\bar{y}))=\left\{d=(d_1,d_2)\ \middle|\ \begin{array}{rl}\nabla G_i(\bar{x}).d_1 & \leq 0,\quad i\in I^G(\bar{x})\\ \nabla g_j(\bar{x},\bar{y}).d & \leq 0,\quad j\in I^g(\bar{x},\bar{y})\\ \sum_{k=1}^{s}\nabla f_k(\bar{x},\bar{y}).d-\varphi^\circ(\bar{x},\bar{y}).d & \leq 0\end{array}\right\}, \tag{2.22}$$

where the Clarke directional derivative for a given function $C:\mathbb{R}^n\to\mathbb{R}$ is defined by:

$$C^\circ(\bar{x},d)=\limsup_{\lambda\to 0^+,y\to\bar{x}}\frac{C(y+\lambda d)-C(y)}{\lambda},\ d\in\mathbb{R}^n,\ \lambda\in\mathbb{R}.$$

We consider the following sets:

$$\mathcal{E} = \bigcup_{j\in I^g(\bar{x},\bar{y})} \{\nabla g_j(\bar{x},\bar{y})\},\ \mathcal{F} = \bigcup_{i\in I^G(\bar{x})} \{\nabla G_i(\bar{x})\}\times\{0\} \quad \text{and}$$
$$\mathcal{G} = \left\{\sum_{k=1}^{s} \nabla f_k(\bar{x},\bar{y})\right\} - \partial_C\varphi(\bar{x},\bar{y}) \qquad .$$

**Definition 2.15.** Let $(\overline{x},\overline{y})$ represent a feasible solution to the problem (2.10). Problem (2.10) is said to satisfy the Abadie type regularity condition at $(\overline{x},\overline{y})$ if

$$\mathcal{L}\left(\Omega,(\bar{x},\bar{y})\right) = T(\Omega,(\overline{x},\overline{y}),$$

where $T(\Omega,(\overline{x},\overline{y})$ is the tangent Bouligand cone to $\Omega$ at $(\bar{x},\bar{y})\in\Omega$ defined by

$$T(\Omega,\bar{x},\bar{y}) := \left\{d\in\mathbb{R}^{n+m} \mid \exists t_k\downarrow 0, d_k\to d : (\bar{x},\bar{y}) + t_k d_k\in\Omega,\ \forall k\in\mathbb{N}\right\}.$$

Now, applying Abadie constraint qualification, we are ready to state our necessary optimality conditions.

**Theorem 2.16.** *Consider (2.10) in which all objective functions $F$ and $f_k$'s and constraint functions $g_j$'s, and $G_i$'s are strictly differentiable. Let $(\bar{x},\bar{y})$ be a local minimizer of (2.10). Assume that the convex cone $\Xi$ generated by $\mathcal{E}\cup\mathcal{F}\cup\mathcal{G}$ is closed and the Abadie type regularity condition is satisfied at $(\bar{x},\bar{y})$. Then, there exist scalars $\gamma_i\geq 0$, $i\in I^G(\overline{x})$, $\zeta_j\geq 0$, $j\in J^g(\overline{x},\overline{y})$ and $\tau\geq 0$ such that*

$$\begin{aligned} 0 \ \in\ & \nabla F(\bar{x},\bar{y}) + \sum_{j\in I^g(\bar{x},\bar{y})} \zeta_j\nabla g_j(\bar{x},\bar{y})\} \ + \sum_{i\in I^G(\bar{x})} \gamma_i\nabla G_i(\bar{x}) \\ & + \tau\sum_{k=1}^{s} \nabla f_k(\bar{x},\bar{y}) - \mathit{cone}\ \left(\partial\varphi(\bar{x},\bar{y})\right). \end{aligned}$$

*Proof.* Let $(\bar{x},\bar{y})$ be a local minimizer of (2.10). Then, from the classical necessary optimiality conditions, we have

$$\langle\nabla F\left(\bar{x},\bar{y}\right),d\rangle\geq 0,\ \text{for all } d\in T(\Omega,(\bar{x},\bar{y})). \tag{2.23}$$

Applying the nonsmooth Abadie constraint qualification, which is satisfied at $(\bar{x},\bar{y})$, relation (2.23) becomes,

$$\langle\nabla F\left(\bar{x},\bar{y}\right),d\rangle\geq 0,\ \text{for all } d\in\mathcal{L}(\Omega,(\bar{x},\bar{y})).$$

Remembering that $\Xi = \mathit{cone}(\mathcal{E}\cup\mathcal{F}\cup\mathcal{G})$, one deduces that

$$\langle\nabla F\left((\bar{x},\bar{y})\right),d\rangle\geq 0,\ \text{when}\ \max_{\nu\in\Xi}\nu^\top d\leq 0,$$

Consequently, the function $d \to \langle \nabla F(\bar{x},\bar{y}), d\rangle + \delta_{\Xi^0}(d)$ reaches its minimum at 0, where $\delta_{\Xi^0}$ is the indicator function of set $\Xi^0$. Using calculus rule, we get $0 \in \nabla F(\bar{x},\bar{y}) + \partial\delta_{\Xi^0}(0)$. Besides, we have that $\partial\delta_{\Xi^0}(0) = \Xi^{00} = \text{cl }\Xi$, thus,

$$0 \in \nabla F(\bar{x},\bar{y}) + \text{cl }\Xi.$$

According to the closedness of $\Xi$ and its expression, one obtains

$$\begin{aligned} 0 \;\in\; & \nabla F(\bar{x},\bar{y}) + \text{cone}\left( \bigcup_{j\in I^g(\bar{x},\bar{y})} \{\nabla g_j(\bar{x},\bar{y})\} \;\cup \bigcup_{i\in I^G(\bar{x})} \{\nabla G_i(\bar{x})\}\times\{0\} \right. \\ & \left. \cup \left\{ \sum_{k=1}^{s} \nabla f_k(\bar{x},\bar{y}) \right\} - \partial_C\varphi(\bar{x},\bar{y}) \right). \end{aligned}$$

Consequently, there exist scalars $\gamma_i \geq 0$, $i \in I^G(\overline{x})$, $\zeta_j \geq 0$, $j \in I^g(\overline{x},\overline{y})$ and $\tau \geq 0$ such that

$$\begin{aligned} 0 \;\in\; & \nabla F(\bar{x},\bar{y}) + \sum_{j\in I^g(\bar{x},\bar{y})} \zeta_j \nabla g_j(\bar{x},\bar{y})\} \;+ \sum_{i\in I^G(\bar{x})} \gamma_i \nabla G_i(\bar{x})\times\{0\} \\ & + \tau \left( \sum_{k=1}^{s} \nabla f_k(\bar{x},\bar{y}) - \text{cone }\left(\partial_C\varphi(\bar{x},\bar{y})\right) \right). \end{aligned}$$

Applying the conic hull property of a subset $B$ of $\mathbb{R}^n$, $\text{cone}\,(co\,(B)) \subset \text{cone}\,(B)$ to $\partial_C\varphi(\bar{x},\bar{y}) = \text{co }\partial\varphi(\bar{x},\bar{y})$, we get

$$\begin{aligned} 0 \;\in\; & \nabla F(\bar{x},\bar{y}) + \sum_{j\in I^g(\bar{x},\bar{y})} \zeta_j \nabla g_j(\bar{x},\bar{y})\} \;+ \sum_{i\in I^G(\bar{x})} \gamma_i \nabla G_i(\bar{x}) \\ & + \tau \sum_{k=1}^{s} \nabla f_k(\bar{x},\bar{y}) - \text{cone }\left(\partial\varphi(\bar{x},\bar{y})\right). \end{aligned}$$

This completes the poof. □

**Example 2.17.** Consider the semivectorial optimistic bilevel optimization problem below:

$$\min_{x,y} F(x,y) = x^2 + y^2 : \quad G(x) = -x \leq 0, \quad y \in \Phi_{\text{eff}}(x). \tag{2.24}$$

The solution set of the following multiobjective optimization problem is represented by $\Phi_{\text{eff}}\,(x)$.

$$\begin{cases} \min\limits_{y} \left(x^2 + (y+1)^2 \;,\; xy + x\right) \\ g_1(x,y) = -y \leq 0. \end{cases} \tag{2.25}$$

where $X = \{x \in \mathbb{R} : -x \leq 0\} = \mathbb{R}^+$. Let $(x, y) \in \Theta = \mathbb{R}^+ \times \mathbb{R}^+$ with $y \in \Phi_{\text{eff}}(x)$, using the revisited Charnes-Cooper scalarization, problem (2.51) is then equivalent to

$$\begin{cases} \min\limits_{z} (z+1)^2 + x^2 + xz + x \\ (z+1)^2 - (y+1)^2 \leq 0, \\ xz - yz \leq 0, \\ z \geq 0. \end{cases} \tag{2.26}$$

The optimal value function of the problem (2.52) is given by the formula:

$$\varphi(x, y) = x^2 + x + 1.$$

The optimal value reformulation of the problem (2.50) is given by:

$$\begin{cases} \min\limits_{x,y} F(x, y) = x^2 + y^2 \\ (y+1)^2 + yx - 1 \leq 0, \\ y \geq 0,\ x \geq 0. \end{cases} \tag{2.27}$$

The feasible set of problem (2.53), the linearized cone, and the Bouligand cone tangent to $\Omega$ at $(\bar{x}, \bar{y}) = (0, 0) \in \Omega$ are given by:

$$\Omega = \mathbb{R}^+ \times \{0\}, \quad \mathcal{L}(\Omega, (0,0)) = \mathbb{R}^+ \times \{0\}, \text{ and } T(\Omega, (0,0)) = \mathbb{R}^+ \times \{0\}.$$

Observe first that $\bar{y} = 0 \in \Phi_{\text{eff}}(0)$ and $(\bar{x}, \bar{y}) = (0, 0)$ is a solution of the problem (2.53). It is easy to see that the convex cone $\Xi$ generated by $\{(-1, 0)\} \cup \{(0, -1)\} \cup \{(0, 2)\}$ equal $\mathbb{R}^- \times ]-\infty, 2]$ is closed, and the Abadie type regularity condition is satisfied at $(\bar{x}, \bar{y}) = (0, 0)$. Then, for $\zeta_1 = 1$,$\gamma = 1$, and $\tau = \dfrac{1}{2}$ we get

$$0 \in \nabla F(0,0) + \ \zeta_1 \nabla g_1(0,0) + \gamma \nabla G(0) + \tau(\nabla f_1(0,0) + \nabla f_2(0,0)) - \text{cone } (\partial\varphi(0,0)).$$

**Remark 2.18.** The additional condition, the closedness of $\Xi = \text{cone } (\mathcal{E} \cup \mathcal{F} \cup \mathcal{G})$, has also been used by several authors to detect the M-stationarity conditions for nonsmooth mathematical programs with equilibrium constraints (see [5] and [49]). Without it, the necessary optimality condition is supposed to be

$$\begin{aligned} 0 \ \in \ & \nabla F(\bar{x}, \bar{y}) + \text{cl cone} \Big( \bigcup_{j \in I^g(\bar{x}, \bar{y})} \{\nabla g_j(\bar{x}, \bar{y})\} \ \cup \ \bigcup_{i \in I^G(\bar{x})} \{\nabla G_i(\bar{x})\} \\ & \cup \Big\{ \sum_{k=1}^{s} \nabla f_k(\bar{x}, \bar{y}) \Big\} - \partial_C \varphi(\bar{x}, \bar{y}) \Big). \end{aligned}$$

There are various sufficient conditions for the validity of the closedness of $\Xi$. We mention the work by Hiriart-Urruty [43, Proposition 1.4.7] in the framework where $\Xi$ is compact set such that $0 \notin \operatorname{conv} \Xi$. Besides, Rockafellar in [60, Corollary 19.7.1] requires conv $\Xi$ to be a polyhedral set containing the origin as a sufficient condition ensuring the closedness of $\Xi$.

**Remark 2.19.** Because Abadie constraint qualification is not always verified due to the lack of closure of the set $\Xi$ and the unbounded cone multiplier, we will introduce other necessary optimality conditions in the following subsection to address this problem.

## 5.2 N.O.C. via partial calmness

In this subsection, we apply a partial exact penalization idea to the lower-level value refomulation problem (2.10), which is popular in bilevel programming (see [72]). To continue, we set

$$\psi(x,y) = \sum_{k=1}^{s} f_k(x,y) - \varphi(x,y). \tag{2.28}$$

**Definition 2.20.** [72] Let $(\overline{x},\overline{y})$ be a local solution of (2.10). The bilevel programming problem (2.10) is said to be partially calm at $(\overline{x},\overline{y})$, if and only if there exist $\nu > 0$ and a neighborhood $U$ of $(\overline{x},\overline{y},0) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ such that

$$\begin{aligned} &F(x,y) + \nu u \geq 0, \\ &\forall (x,y,u) \in U, \quad x \in X, \quad y \in Y(x), \quad \psi(x,y) + u = 0. \end{aligned}$$

**Proposition 2.21.** *[72] Let $(\overline{x},\overline{y})$ be a local solution of* (2.10)*. This problem is partially calm at $(\overline{x},\overline{y})$ if and only if there exists $\nu > 0$ such that $(\overline{x},\overline{y})$ is a local optimal solution of the problem*

$$\begin{cases} \min\limits_{x,y} F(x,y) + \nu\psi(x,y), \\ x \in X,\ y \in Y(x). \end{cases} \tag{2.29}$$

It is well worth mentioning that Abadie is the weakly nonsmooth constraint qualification. However, this constraint qualification may not exist. This drawback can be removed by using partial calmness. The following example shows that this latter may hold even when the Abadie constraint qualification is not checked.

**Example 2.22.** Consider the semivectorial optimistic bilevel optimization problem shown below:

$$\min_{x,y} F(x,y) = x + y : \quad x \in X, \quad y \in \Phi_{\text{eff}}(x), \tag{2.30}$$

with the $\Phi_{\text{eff}}(x)$ is the solution set of the following multiobjective optimization problem

$$\begin{cases} \min\limits_{y} \ ((x-1)^2 + (y+1)^2 \ , \ (x+1)^2 + (y+1)^2) \\ g_1(x,y) = -y - 1 \leq 0, \end{cases} \tag{2.31}$$

where $X = [-1, +\infty[$. Let $(x,y) \in \Theta = [-1, +\infty[\times[-1, +\infty[$ with $y \in \Phi_{\text{eff}}(x)$, using the revisited Charnes-Cooper scalarization, the problem (2.31) is equivalent to

$$\begin{cases} \min\limits_{w} (x-1)^2 + (x+1)^2 + 2(w+1)^2 \\ (w+1)^2 - (y+1)^2 \leq 0, \\ (w+1)^2 - (y+1)^2 \leq 0, \\ w \geq -1. \end{cases} \tag{2.32}$$

The following formula gives the optimal value function of the problem (2.32):

$$\varphi(x,y) = (x-1)^2 + (x+1)^2.$$

Using the optimal value reformulation, we transform (2.30) into the following optimization problem:

$$\begin{cases} \min\limits_{x,y} F(x,y) = x + y \\ 2(y+1)^2 \leq 0, \ y \geq -1, \ x \geq -1. \end{cases} \tag{2.33}$$

The feasible set of problem (2.33), the linearized cone, and the Bouligand cone tangent to $\Omega$ at a point $(-1,-1) \in \Omega$ are given by:

$$\Omega = [-1, +\infty[\times\{-1\}, \quad \mathcal{L}(\Omega, (-1,-1)) = \mathbb{R}^+ \times \mathbb{R}^+, \text{ and } T(\Omega, (-1,-1)) = \mathbb{R}^+ \times \{0\}.$$

Observe first that $\bar{y} = -1 \in \Phi_{\text{eff}}(-1)$ and $(\bar{x}, \bar{y}) = (-1,-1)$ is a local solution optimal solution of the problem (2.33). On the other hand, for all $(x,y) \in \Omega$, and $\nu = 1$, consider the penalized problem

$$\min_{x,y} x + y + 2(y+1)^2 : \ -1 \leq y, \ -1 \leq x. \tag{2.34}$$

where

$$\psi(x,y) = 2(y+1)^2.$$

The point $(\bar{x}, \bar{y}) = (-1, -1)$ is a local solution of the problem (2.34). Hence, at the optimal solution $(-1, -1)$, the problem (2.33) is partially calm, but the smooth Abadie constraint qualification fails at the point $(-1, -1)$.

Following that, we derive a necessary optimality condition for this problem from the feasible set using partial penalization, which eliminates the value function constraint responsible for the failure of constraint qualifications. To continue, we will rewrite problem (2.10) as follows:

$$\min_{x,y} F(x, y), \quad (x, y) \in \Theta, \ \psi(x, y) \leq 0. \tag{2.35}$$

We also require additional constraint qualifications. The following is the smooth counterpart of the lower- and upper-level regularity conditions:

The upper-level regularity of (SVBOP) at $\bar{x} \in X$ is given by

$$\left.\begin{array}{r} \displaystyle\sum_{i=1}^{q} \alpha_i \nabla G_i(\bar{x}) = 0, \\ \alpha_i \geqslant 0, \ \alpha_i G_i(\bar{x}) = 0, \ i = 1, \cdots, q \end{array}\right\} \implies \alpha_i = 0, \ i = 1, \cdots, q. \tag{$\mathcal{H}_4$}$$

The lower-level regularity of (Mul[$x$]) at $(\bar{x}, \bar{y})$ is given by

$$\left.\begin{array}{r} \displaystyle\sum_{j=1}^{p} \gamma_j \nabla_y g_j(\bar{x}, \bar{y}) = 0 \\ \gamma_j \geq 0, \quad \gamma_j g_j(\bar{x}, \bar{y})) = 0, \quad j \in J, \end{array}\right\} \implies \gamma_j = 0, \ j = 1, \cdots, p \tag{$\mathcal{H}_5$}$$

The necessary optimality conditions in the case $k \geqslant 2$ are demonstrated in the following theorem.

**Theorem 2.23.** *Consider (2.10) in which all objective functions $F$ and $f_k$'s and constraint functions $g_j$'s, and $G_i$'s are strictly differentiable. Let $(\overline{x}, \overline{y})$ be a local minimizer of* (2.10)*. Suppose that* (2.10) *is partially calm at $(\overline{x}, \overline{y})$. Assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, $(\mathcal{H}_4)$ and $(\mathcal{H}_5)$ are satisfied at $(0, \overline{x}, \overline{y}, \overline{y})$, $(\overline{x}, \overline{y}, \overline{y})$, $(\overline{x}, \overline{y}, \overline{y})$, $\overline{x}$ and $(\overline{x}, \overline{y})$, respectively. Moreover, suppose that $S$ is inner semicontinuous at $(\overline{x}, \overline{y}, \overline{y})$. Then,*

*there exists* $\nu > 0$, $\alpha \in \mathbb{R}^p_+$, $\gamma \in \mathbb{R}^q_+$, $\sigma^l_k$, $k \in S$, $\beta^l_j$, $j \in J$, *and* $\epsilon^l$, *with* $l = 1, \cdots n+m+1$, *such that*

$$\nabla_x F(\overline{x},\overline{y}) + \sum_{i=1}^{q} \alpha_i \nabla G_i(\bar{x}) + \sum_{j=1}^{p} \gamma_j \nabla_x g_j(\bar{x},\bar{y}) - \nu \sum_{l=1}^{n+m+1} \epsilon^l \sum_{j=1}^{p} \beta^l_j \ \nabla_x g_j(\overline{x},\overline{y}) = 0,$$
$$\nabla_y F(\overline{x},\overline{y}) + \sum_{j=1}^{p} \gamma_j \nabla_y g_j(\bar{x},\bar{y}) + \nu \left( \sum_{k=1}^{s} \nabla_y f_k(\bar{x},\bar{y}) + \sum_{l=1}^{n+m+1} \epsilon^l \sum_{k=1}^{s} \sigma^l_k \nabla_y f_k(\overline{x},\overline{y}) \right) = 0,$$

$$\forall\ l = 1, \cdots, n+m+1,\ \sum_{k=1}^{s} \nabla_y f_k(\overline{x},\overline{y}) + \sum_{k=1}^{s} \sigma^l_k \nabla_y f_k(\overline{x},\overline{y}) + \nabla_y g(\overline{x},\overline{y})^T \beta^l = 0,$$

$$\forall\ j = 1, \cdots, p,\ \forall\ l = 1, \ldots, n+m+1,\ \sigma^l \geq 0,\ \beta^l_j \geq 0,\ \beta^l_j g_j(\overline{x},\overline{y}) = 0,$$

$$\forall\ i = 1, \cdots, q,\ \alpha_i \geq 0,\ \alpha_i G_i(\bar{x}) = 0,$$

$$\forall\ j = 1, \cdots, p,\ \gamma_j \geq 0,\ \gamma_j g_j(\bar{x},\bar{y}) = 0,$$

$$\forall\ l = 1, \cdots, n+m+1,\ \epsilon^l \geq 0,\ \sum_{l=1}^{n+m+1} \epsilon^l = 1.$$

*Proof.* Let $(\bar{x},\bar{y})$ be a local solution of (2.10). First, note that the validity of $(\mathcal{H}_2)$ at $(\bar{x},\bar{y},\bar{y})$ ensures from Theorem 2.7 that $\varphi$ is locally Lipschitz continuous at $(\overline{x},\overline{y})$.

Secondly, since problem (2.10) is partially calm at $(\overline{x},\overline{y})$, from Proposition 2.21 there exists $\nu \geq 0$ such that $(\bar{x},\bar{y})$ is a local solution of (2.29). By ([57],Proposition 5.3 ), since $F$ and $\Phi$ are locally Lipschitz continuous at $(\overline{x},\overline{y})$, we have

$$0 \in \nabla F(\bar{x},\bar{z}) + \nu\ \partial\psi(\bar{x},\bar{y}) + N_\Theta(\bar{x},\bar{y}). \tag{2.36}$$

Let $\chi(x,y) = [G_i(x),\ i = 1, \cdots, q,\ g_j(x,y), j = 1, \cdots, p]^\top$. Starting with $N_\Theta(\bar{x},\bar{y})$, let us note that $\Theta$ can be rewritten as $\Theta = \{(x,y) : \chi(x,y) \leq 0\}$.

Now, the fulfillment of both the upper-level $(\mathcal{H}_4)$ and lower-level $(\mathcal{H}_5)$ regularity conditions, respectively at $\overline{x}$ and $(\overline{x},\overline{y})$, implies the satisfaction of the following implication.

$$\left.\begin{array}{r} \langle u, \nabla\chi(\bar{x},\bar{y})\rangle = 0 \\ u \in \mathbb{R}^{p+q}_+, \{u, \chi(\bar{x},\bar{y})\rangle = 0 \end{array}\right\} \Longrightarrow [u = 0].$$

Hence, from [59, Theorem 6.14], we obtain

$$N_\Theta(\bar{x},\bar{y}) \subseteq \left\{ \begin{array}{l} \left( \begin{array}{c} \sum_{i=1}^{q} \alpha_i \nabla G_i(\bar{x}) + \sum_{j=1}^{p} \gamma_j \nabla_x g_j(\bar{x},\bar{y}) \\ \sum_{j=1}^{p} \gamma_j \nabla_y g_j(\bar{x},\bar{y}) \end{array} \right) : \\ \alpha_i \geq 0, \quad \alpha_i G_i(\bar{x}) = 0, \quad i = 1,\cdots,q, \\ \gamma_j \geq 0, \quad \gamma_j g_j(\bar{x},\bar{y}) = 0, \quad j = 1,\cdots,p. \end{array} \right\} \tag{2.37}$$

From the structure of $\Phi$, the basic subdifferential of $\Phi$ is given by

$$\partial\psi(\bar{x},\bar{y}) = \partial\Big(\sum_{k=1}^{s} f_k(\bar{x},\bar{y}) - \varphi(\bar{x},\bar{y})\Big).$$

Then

$$\partial\psi(\bar{x},\bar{y}) \subseteq \sum_{k=1}^{s} \nabla f_k(\bar{x},\bar{y}) + \partial(-\varphi)(\bar{x},\bar{y}).$$

Since

$$\partial(-\varphi)(\bar{x},\bar{y}) \subseteq -\mathrm{co}\left(\partial\varphi(\bar{x},\bar{y})\right),$$

one gets

$$\partial\psi(\bar{x},\bar{y}) \subseteq \sum_{k=1}^{s} \nabla f_k(\bar{x},\bar{y}) - \mathrm{co}\left(\partial\varphi(\bar{x},\bar{y})\right). \tag{2.38}$$

Besides, the solution set-valued mapping $\Psi$ is inner semicontinuous at $(\bar{x},\bar{y},\bar{y})$, from Theorem 2.7 we have

$$\partial\varphi(\bar{x},\bar{y}) \subset \left[ \begin{array}{c} \sum_{k=1}^{s} \nabla_x f_k(\bar{x},\bar{y}) \\ 0_{\mathbb{R}^m} \end{array} \right] + \bigcup_{(\sigma,\beta)\in\Lambda(\overline{x},\overline{y})} \left[ \begin{array}{c} \nabla_x g(\overline{x},\overline{y})^\top \beta \\ -\sum_{k=1}^{s} \sigma_k \nabla_y f_k(\bar{x},\bar{y}) \end{array} \right] \tag{2.39}$$

while taking into account the validity of $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ at $(0,\overline{x},\overline{y},\overline{y})$ and $(\overline{x},\overline{y},\overline{y})$, respectively.

For any $(a,b) \in \mathrm{co}\left(\partial\varphi(\bar{x},\bar{y})\right)$. Carathéodory's theorem gives us from (2.39) $\sigma_k^l$, $k \in S$, $\beta_j^l$, $j \in J$ and $\epsilon^l$, with $l = 1,\cdots n+m+1$, such that

$$\sum_{l=1}^{n+m+1} \epsilon^l = 1, \quad \text{and} \quad \left(\sigma^l,\beta^l\right) \in \Lambda(\overline{x},\overline{y}) \tag{2.40}$$

$$\begin{array}{rcl} a & = & \displaystyle\sum_{k=1}^{s} \nabla_x f_k(\bar{x},\bar{y}) + \sum_{l=1}^{n+m+1} \epsilon^l \sum_{j=1}^{p} \beta_j^l \ \nabla_x g_j\left(\overline{x},\overline{y}\right) \\ b & = & -\displaystyle\sum_{l=1}^{n+m+1} \epsilon^l \sum_{k=1}^{s} \sigma_k^l \nabla_y f_k\left(\overline{x},\overline{y}\right) \end{array} \tag{2.41}$$

Combining (2.48),(2.48)-(2.49) with (2.40)-(2.41), while exploiting the definition of $\Lambda\left(\overline{x},\overline{y}\right)$ we obtain the result. □

**Example 2.24.** Consider the following semivectorial optimistic bilevel optimization problem:

$$\min_{x,y} \ F(x,y) = x^2 + y^2 + xy: \ G_1(x) = x^2 - x \leq 0, \ y \in \Phi_{\text{eff}}(x), \tag{2.42}$$

here, $\Phi_{\text{eff}}\left(x\right)$ is the solution set of the following multiobjective optimization problem:

$$\min_{y} \ \left(x+y \ , \ (x-1)^2 + y\right): \ g_1(x,y) = -y \ \leq 0, \ g_2(x,y) = y - 1 \ \leq 0. \tag{2.43}$$

where $X = \left\{x \in \mathbb{R} : x^2 - x \leq 0\right\} = [0,1]$, and $Y\left(x\right) = [0,1]$. Let $(x,y) \in \Theta = [0,1] \times [0,1]$ with $y \in \Phi_{\text{eff}}\left(x\right)$, applying the revisited Charnes-Cooper scalarization, problem (2.43) is then equivalent to

$$\begin{cases} \min\limits_{w} \ (x-1)^2 + 2w + x \\ 0 \leq w \leq 1, \ w \leq y, \ w \leq y. \end{cases} \tag{2.44}$$

The feasible set of problem (2.44) is given by

$$\Xi\left(x,y\right) = \left\{w \in [0,1], \ w \leq y, \ w \leq y\right\} = [0,y]. \tag{2.45}$$

The following formula gives the optimal value function of the problem (2.44):

$$\varphi(x,y) = (x-1)^2 + x.$$

Applying the optimal value reformulation, we transform (2.42) into the following optimization problem

$$\begin{cases} \min\limits_{x,y} \ F(x,y) = x^2 + y^2 + xy \\ 0 \leq y \leq 1, \\ 0 \leq x \leq 1, \\ 2y \leq 0. \end{cases} \tag{2.46}$$

We remark that $\overline{y} = 0 \in \Phi_{\text{eff}}\left(0\right)$ and $(\bar{x},\bar{y}) = (0,0)$ is a local solution for the problem (2.46). For $\nu = 1 \geqslant 0$ we obtain

$$\min_{x,y} \ x^2 + y^2 + xy + 2y: \ 0 \leq y \leq 1, \ 0 \leq x \leq 1. \tag{2.47}$$

with

$$\psi(x,y)=2y.$$

Then $(\bar{x},\bar{y})=(0,0)$ is a local minimizer of (2.47). Hence (2.46) is partially calm at $(0,0)$.

Besides, looking at the structure of gph $Y$, $\Xi$ and $S$, on can easily check that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ hold together with the inner semicontinuity of the set valued-map $S$. Moreover, the conditions $(\mathcal{H}_3)$-$(\mathcal{H}_4)$ hold true.

Then, there exists $\nu=1$, $\alpha=0$, $\gamma_1=3$, $\gamma_2=0$, $\sigma_1^l=\dfrac{1}{2}$, $\sigma_2^l=\dfrac{1}{2}$, and for $l=1,\cdots 3$, $\beta_1^l=\beta_2^l=3$, and $\epsilon^l=\dfrac{1}{3}$ such that

$$\nabla_x F(0,0)+\alpha\nabla G_1(0)+\sum_{j=1}^{2}\gamma_j\nabla_x g_j(0,0)-\nu\sum_{l=1}^{3}\epsilon^l\sum_{j=1}^{2}\beta_j^l\,\nabla_x g_j(0,0)=0,$$
$$\nabla_y F(0,0)+\sum_{j=1}^{2}\gamma_j\nabla_y g_j(0,0)+\nu\left(\sum_{k=1}^{2}\nabla_y f_k(0,0)+\sum_{l=1}^{3}\epsilon^l\sum_{k=1}^{2}\sigma_k^l\nabla_y f_k(0,0)\right)=0,$$

$$\forall\ l=1,\cdots,3,\ \sum_{k=1}^{2}\nabla_y f_k(0,0)+\sum_{k=1}^{2}\sigma_k^l\nabla_y f_k(0,0)+\nabla_y g(0,0)^T\beta^l=0,$$

$$\forall\ j=1,2,\ \forall\ l=1,\ldots,3,\ \sigma^l\geq 0,\ \beta_j^l\geq 0,\ \beta_j^l g_j(0,0)=0,$$

$$\forall\ i=1,\ \alpha_1\geq 0,\ \alpha_1 G_1(0)=0,\ \ \forall\ j=1,2,\ \gamma_j\geq 0,\ \gamma_j g_j(0,0)=0,$$

$$\forall\ l=1,\cdots,3,\ \epsilon^l\geq 0,\ \sum_{l=1}^{3}\epsilon^l=1.$$

We also require additional constraint qualifications. The following is the non-smooth counterpart of the lower- and upper-level regularity conditions:

- The upper-level regularity of (SVBOP) at $\overline{x}$ is given by

$$\left.\begin{array}{r} 0\in\sum_{i=1}^{q}\zeta_i\partial G_i(\overline{x}) \\ \zeta_i\geq 0,\ \zeta_i G_i(\overline{x})=0,\ i\in I \end{array}\right\}\Longrightarrow\ \zeta_i=0,\ i=1,\cdots,q. \qquad \text{(ULR)}$$

- The lower-level regularity of (Mul[$x$]) at $(\overline{x},\overline{y})$ is given by

$$\left.\begin{array}{r} 0\in\sum\limits_{j=1}^{p}\gamma_j\partial_y g_j\left(\overline{x},\overline{y}\right)\\ \gamma_j\geq 0,\ \gamma_j g_j\left(\overline{x},\overline{y}\right)=0,\quad j\in J, \end{array}\right\}\implies \gamma_j=0,\ j=1,\cdots,p. \tag{LLR}$$

The next result gives us the normal cone formula for $\Theta$ at each point $(\overline{x},\overline{y})\in\Theta$.

**Lemma 2.25.** *Let $(\overline{x},\overline{y})\in\Theta$. Assume the upper-level and lower-level regularities hold at $\overline{x}$ and $(\overline{x},\overline{y})$, respectively. Then*

$$N\left(\left(\overline{x},\overline{y}\right),\Theta\right)\subseteq$$
$$\left\{\begin{pmatrix}\sum\limits_{i=1}^{q}\zeta_i\partial G_i\left(\overline{x}\right)+\sum\limits_{j=1}^{p}\gamma_j\partial_x g_j\left(\overline{x},\overline{y}\right)\\ \sum\limits_{j=1}^{p}\gamma_j\partial_y g_j\left(\overline{x},\overline{y}\right)\end{pmatrix}\ :\ \begin{array}{l}\zeta_i\geq 0,\quad \zeta_i G_i\left(\overline{x}\right)=0, i\in I\\ \gamma_j\geq 0,\quad \gamma_j g_j\left(\overline{x},\overline{y}\right)=0, j\in J.\end{array}\right\}$$

*Proof.* Setting $\chi(x,y)=[G_i(x),\ i=1,\cdots,q,\ g_j(x,y), j=1,\cdots,p]^\top$, the set $\Theta$ can be rewritten as $\Theta=\{(x,y):\chi(x,y)\leq 0\}$. Then, fulfillment of the upper-level and lower-level regularities at $\overline{x}$ and $(\overline{x},\overline{y})$ respectively implies

$$\left.\begin{array}{r} 0\in\sum\limits_{i=1}^{q}\zeta_i\partial\chi_i\left(\overline{x},\overline{y}\right)+\sum\limits_{j=1}^{p}\gamma_i\partial\chi_{q+j}\left(\overline{x},\overline{y}\right)\\ \zeta_i\geq 0,\ \zeta_i\chi_i\left(\overline{x},\overline{y}\right)=0,\ i\in I\\ \gamma_j\geq 0,\ \gamma_j\chi_{q+j}\left(\overline{x},\overline{y}\right)=0,\ j\in J \end{array}\right\}\implies \zeta_1,\cdots,\zeta_q,\gamma_{q+1},\cdots,\gamma_{q+p}=0.$$

Consequently, from [59, Theorem 6.14], we obtain the desired result. □

**Theorem 2.26.** *Let $(\overline{x},\overline{y})$ be a local optimal solution of problem (2.10). Assume that the problem (2.10) is partially calm at $(\overline{x},\overline{y})$, and that the upper-level and lower-level regularities are satisfied at $\overline{x}$ and $(\overline{x},\overline{y})$, respectively. Then the following statements are valid:*

*(a) Based on the assumptions of Theorem 2.10, there exist $\nu>0$, $\xi\in\mathbb{R}^{n+m+1}_+$, $w_r\in S\left(\overline{x},\overline{y}\right)$, $\delta_r\in\mathbb{R}^s_+$, $\vartheta_r\in\mathbb{R}^p_+$, $v^k_r\in\mathbb{R}^{n+m+1}_+$, $(\zeta,\gamma)\in\mathbb{R}^q\times\mathbb{R}^p$, and*

$$\begin{array}{lll}\left(x^*_F,y^*_F\right)\in\partial F\left(\overline{x},\overline{y}\right), & x^*_{G_i}\in\partial G_i\left(\overline{x}\right), & \left(x^*_{f_k},y^*_{f_k}\right)\in\partial f_k\left(\overline{x},\overline{y}\right),\\ \left(x^*_{g_j},y^*_{g_j}\right)\in\partial g\left(\overline{x},\overline{y}\right), & \left(x^{*r}_{f_k},w^{*r}_{f_k}\right)\in\partial f_k\left(\overline{x},w_r\right), & \left(\tilde{x}^r_{f_k},\tilde{w}^r_{f_k}\right)\in\partial f_k\left(\overline{x},w_r\right),\\ \left(\hat{x}^{r,l}_{f_k},\hat{y}^{r,l}_{f_k}\right)\in\partial f_k\left(\overline{x},\overline{y}\right), & \left(\underline{x}^r_{g_j},\underline{w}^r_{g_j}\right)\in\partial g_j\left(\overline{x},w_r\right), & \end{array}$$

*with* $i \in I$, $j \in J$, $k = 1, \cdots, s$, $r = 1, \cdots, n+m+1$, $l = 1, \cdots, n+m+1$, $\sum_{r=1}^{n+m+1} \xi_r = 1$ *and* $\sum_{l=1}^{n+m+1} v_{r,l}^k = 1$, $r = 1, \cdots, n+m+1$, $k = 1, \cdots, s$, *such that:*

$$
\begin{aligned}
0 &= x_F^* + \sum_{i=1}^{q} \zeta_i x_{G_i}^* + \nu \sum_{k=1}^{s} x_{f_k}^* + \sum_{j=1}^{p} \gamma_j x_{g_j}^* - \nu \sum_{r=1}^{n+m+1} \xi_r \Bigg( \sum_{k=1}^{s} x_{f_k}^{*r} \\
&\quad + \sum_{k=1}^{s} \delta_{r,k} \left( \tilde{x}_{f_k}^r - \sum_{l=1}^{n+m+1} v_{r,l}^k \hat{x}_{f_k}^{r,l} \right) + \sum_{j=1}^{p} \vartheta_{r,j} \underline{x}_{g_j}^r \Bigg), \\
0 &= y_F^* + \nu \sum_{k=1}^{s} y_{f_k}^* + \sum_{j=1}^{p} \gamma_j y_{g_j}^* + \nu \sum_{r=1}^{n+m+1} \xi_r \sum_{k=1}^{s} \delta_{r,k} \sum_{l=1}^{n+m+1} v_{r,l}^k \hat{y}_{f_k}^{r,l}, \\
0 &= \sum_{k=1}^{s} w_{f_k}^{*r} + \sum_{k=1}^{s} \delta_{r,k} \tilde{w}_{f_k}^r + \sum_{j=1}^{p} \vartheta_{r,j} \underline{w}_{g_j}^r, \quad \forall r = 1, \cdots, n+m+1, \\
&\zeta_i \geqslant 0,\ \zeta_i G_i(\overline{x}) = 0, \quad \forall i = 1, \ldots, q, \\
&\gamma_j \geqslant 0,\ \gamma_j g_j(\overline{x}, \overline{y}) = 0, \quad \forall j = 1, \ldots, p \\
&\vartheta_{r,j} \geqslant 0,\ \vartheta_{r,j} g_j(\overline{x}, w_r) = 0, \quad \forall j = 1, \ldots, p, \quad \forall r = 1, \cdots, n+m+1.
\end{aligned}
$$

(*b*) *Based on the assumptions of Theorem 2.12, there exist* $\nu > 0$, $\xi \in \mathbb{R}_+^{n+m+1}$, $\delta_r \in \mathbb{R}_+^s$, $\vartheta_r \in \mathbb{R}_+^p$, $v_r^k \in \mathbb{R}_+^{n+m+1}$, $(\zeta, \gamma) \in \mathbb{R}^q \times \mathbb{R}^p$, *and*

$$
\begin{aligned}
&(x_F^*, y_F^*) \in \partial F(\overline{x}, \overline{y}), && x_{G_i}^* \in \partial G_i(\overline{x}), && \left(x_{f_k}^*, y_{f_k}^*\right) \in \partial f_k(\overline{x}, \overline{y}), \\
&\left(x_{g_j}^*, y_{g_j}^*\right) \in \partial g(\overline{x}, \overline{y}), && \left(x_{f_k}^{*r}, y_{f_k}^{*r}\right) \in \partial f_k(\overline{x}, \overline{y}), && \left(\tilde{x}_{f_k}^r, \tilde{y}_{f_k}^r\right) \in \partial f_k(\overline{x}, \overline{y}), \\
&\left(\hat{x}_{f_k}^{r,l}, \hat{y}_{f_k}^{r,l}\right) \in \partial f_k(\overline{x}, \overline{y}), && \left(\underline{x}_{g_j}^r, \underline{y}_{g_j}^r\right) \in \partial g_j(\overline{x}, \overline{y}),
\end{aligned}
$$

*with* $i \in I$, $j \in J$, $k = 1, \cdots, s$, $r = 1, \cdots, n+m+1$, $l = 1, \cdots, n+m+1$, $\sum_{r=1}^{n+m+1} \xi_r = 1$ *and*

$\sum_{l=1}^{n+m+1} v_{r,l}^k = 1$, $r = 1, \cdots, n+m+1$, $k = 1, \cdots, s$, *such that:*

$$
\begin{aligned}
0 &= x_F^* + \sum_{i=1}^{q} \zeta_i x_{G_i}^* + \nu \sum_{k=1}^{s} x_{f_k}^* + \sum_{j=1}^{p} \gamma_j x_{g_j}^* - \nu \sum_{r=1}^{n+m+1} \xi_r \Bigg( \sum_{k=1}^{s} x_{f_k}^{*r} \\
&\quad + \sum_{k=1}^{s} \delta_{r,k} \left( \tilde{x}_{f_k}^r - \sum_{l=1}^{n+m+1} v_{r,l}^k \hat{x}_{f_k}^{r,l} \right) + \sum_{j=1}^{p} \vartheta_{r,j} \underline{x}_{g_j}^r \Bigg), \\
0 &= y_F^* + \nu \sum_{k=1}^{s} y_{f_k}^* + \sum_{j=1}^{p} \gamma_j y_{g_j}^* + \nu \sum_{r=1}^{n+m+1} \xi_r \sum_{k=1}^{s} \delta_{r,k} \sum_{l=1}^{n+m+1} v_{r,l}^k \hat{y}_{f_k}^{r,l}, \\
0 &= \sum_{k=1}^{s} y_{f_k}^{*r} + \sum_{k=1}^{s} \delta_{r,k} \tilde{y}_{f_k}^r + \sum_{j=1}^{p} \vartheta_{r,j} \underline{y}_{g_j}^r, \quad \forall r = 1, \cdots, n+m+1, \\
&\zeta_i \geqslant 0, \ \zeta_i G_i(\overline{x}) = 0, \quad \forall i = 1, \ldots, q, \\
&\gamma_j \geqslant 0, \ \gamma_j g_j(\overline{x}, \overline{y}) = 0, \quad \forall j = 1, \ldots, p, \\
&\vartheta_{r,j} \geqslant 0, \ \vartheta_{r,j} g_j(\overline{x}, \overline{y}) = 0, \quad \forall j = 1, \ldots, p.
\end{aligned}
$$

*Proof.* Let $(\overline{x}, \overline{y})$ be a local optimal solution of (2.10). By either assumptions of Theorem 2.10 or Theorem 2.12, the lower-level optimal value function $\upsilon$ is locally Lipscitz at this point.

Since problem (2.10) is partially calm at $(\overline{x}, \overline{y})$, we deduce from Proposition 2.21 that there exists a constant $\nu > 0$ such that $(\overline{x}, \overline{y})$ is a local optimal solution of the penalized problem (2.29).

Now, taking into account the local Lipschitz continuity of $F$ and $\psi$ at $(\overline{x}, \overline{y})$, we have from [57, Proposition 5.3]

$$
0 \in \partial F(\overline{x}, \overline{y}) + \nu \, \partial \psi(\overline{x}, \overline{y}) + N\left((\overline{x}, \overline{y}), \Theta\right). \tag{2.48}
$$

From the structure of $\psi$, the basic subdifferential is given by

$$
\partial \psi(\overline{x}, \overline{y}) = \partial \Bigg( \sum_{k=1}^{s} f_k(\overline{x}, \overline{y}) - \varphi(\overline{x}, \overline{y}) \Bigg).
$$

Therefore,

$$
\partial \psi(\overline{x}, \overline{y}) \subseteq \sum_{k=1}^{s} \partial f_k(\overline{x}, \overline{y}) + \partial(-\varphi)(\overline{x}, \overline{y}).
$$

Since $\partial(-\varphi)(\overline{x}, \overline{y}) \subseteq -\mathrm{co}\left(\partial \varphi(\overline{x}, \overline{y})\right)$, one gets

$$
\partial \psi(\overline{x}, \overline{y}) \subseteq \sum_{k=1}^{s} \partial f_k(\overline{x}, \overline{y}) - \mathrm{co}\left(\partial \varphi(\overline{x}, \overline{y})\right). \tag{2.49}
$$

By combining inclusions (2.48), (2.49) with Lemma 2.25, Theorems 2.10 and 2.12, and Remark 2.11, we will obtain results $(a)$ and $(b)$ by utilizing the Carathédory theorem to compute an element of conv $\partial\varphi\left(\overline{x},\overline{y}\right)$. □

**Example 2.27.** In this example, we apply the necessary optimality conditions derived in Theorem 2.26 to verify they are satisfied at the given candidate optimal solution.

Let us first define the problem data for (SVBOP). For $(x,y)\in\mathbb{R}\times\mathbb{R}$, set

$$\begin{aligned} &F(x,y)=|x|+(y+2)^2, \quad G(x)=-x,\\ &f_1(x,y)=x+(y+2)^2, \quad f_2(x,y)=x^2+y+2, \quad g(x,y)=|y|-2. \end{aligned}$$

Then, we have $\Theta=\mathbb{R}^+\times[-2,2]$, and $\Xi(x,y)=[-2,y]$ for all $(x,y)\in\Theta$. Moreover, we have

$$S(x,y)=\{y\}, \quad \varphi(x,y)=x+x^2.$$

The point $(\overline{x},\overline{y}):=(0,-2)$ is an optimal solution of the considered problem. It is then shown that:

- The functions $F$, $g$, $f_1$, and $f_2$ are locally Lipschitz continuous.
- The open Cartesian cube $U:=(-1,1)\times(-3,-1)\times(-1,1)$ is a neighborhood of $(\overline{x},\overline{y},0)$. For $(x,y,u)\in U$ such that $x\in X$, $y\in Y(x)$, and $\psi(x,y)+u=0$, we have

  $$F(x,y)-F\left(\overline{x},\overline{y}\right)+\nu|u|=F(x,y)+\nu|u|\geq 0.$$

  Therefore, problem (2.10) is partially calm at $(\overline{x},\overline{y})$.
- The set-valued map $\Xi$ possesses the Aubin property at $(\overline{x},\overline{y},w)$ with $w\in S\left(\overline{x},\overline{y}\right)$.
- By construction, the set-valued mapping $S$ is inner semi-compact and inner semicontinuous at $(\overline{x},\overline{y})$.
- For $\mu\in\mathbb{R}$, we have

  $$\Psi_{Y_0}\left(\mu\right)=\left\{(x,y,w)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\ :\ |w|-2\leq\mu\right\}.$$

  Hence, the set-valued mapping $\Psi_{Y_0}$ is calm at $(0,\overline{x},\overline{y},w)$ with $w\in S\left(\overline{x},\overline{y}\right)$.

- For $\kappa \in \mathbb{R}$, we have

$$\Psi_{\Xi}(\kappa) = \{(x, y, w) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ : \ |w| - 2 \leq 0, \ w - y \leq \kappa\}.$$

  Hence, the set-valued mapping $\Psi_{\Xi}$ is calm at $(0, \overline{x}, \overline{y}, w)$ with $w \in S(\overline{x}, \overline{y})$.

- The upper-level and lower-level regularities are satisfied at $\overline{x}$ and $(\overline{x}, \overline{y})$ respectively.

By explicitly constructing the multipliers and constants, the optimality conditions (a) and (b) in Theorem 2.26 are shown to hold at the optimal solution $(\overline{x}, \overline{y}) = (0, -2)$.

The multipliers and constants are constructed as follows: $(x_F^*, y_F^*) := (1, 0)$, $x_G^* := -1$, $\left(x_{f_1}^*, y_{f_1}^*\right) = \left(x_{f_1}^{*r}, w_{f_1}^{*r}\right) = \left(\tilde{x}_{f_1}^r, \tilde{w}_{f_1}^r\right) = \left(\hat{x}_{f_1}^{r,l}, \hat{y}_{f_1}^{r,l}\right) := (1, 0)$, $\left(x_{f_2}^*, y_{f_2}^*\right) = \left(x_{f_2}^{*r}, w_{f_2}^{*r}\right) = \left(\tilde{x}_{f_2}^r, \tilde{w}_{f_2}^r\right) = \left(\hat{x}_{f_2}^{r,l}, \hat{y}_{f_2}^{r,l}\right) := (0, 1)$, $\left(x_g^*, y_g^*\right) = \left(\underline{x}_g^r, \underline{w}_g^r\right) := (0, -1)$, $\nu := 1$, $\xi = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)^\top$, $w_r := -2 \in S(\overline{x}, \overline{y})$, $\delta_{r,1} := 3$, $\delta_{r,2} := 1$, $\vartheta_r := 2$, $v_r^k := \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)^\top$, $\zeta := 1$, and $\gamma := 2$, $r = 1, 2, 3$, $l = 1, 2, 3$, $k = 1, 2$.

## 5.3 N.O.C via weak Mangasarian-Fromowitz constraint qualification

As stated in [25], the weak basic constraint qualification (WMFCQ) has been demonstrated to work, particularly for simple bilevel programming problems. In this section, we determine necessary optimality conditions for the auxiliary optimistic bilevel programming problem (2.10) using the lower-level value function (2.11) and sensitivity results from Section 4.2.

First, we define

$$\psi(x, y) = \sum_{k=1}^{s} f_k(x, y) - \varphi(x, y),$$
$$\Sigma = \{(x, y) \in \Theta : \ \psi(x, y) \leq 0\}.$$

We also require additional constraint qualifications to establish necessary optimality conditions. The following conditions are nonsmooth counterparts of regularity assumptions at the lower and upper levels:

- We say that the weak Mangasarian-Fromowitz constraint qualification is satisfied at $(\overline{x}, \overline{y})$ if

$$\partial\psi(\overline{x}, \overline{y}) \cap -\text{bd } N_{\Theta}(\overline{x}, \overline{y}) = \emptyset. \tag{WMFCQ}$$

Subsequently, we are going to provide one of the primary results of this study, which gives the necessary optimality conditions for the auxiliary problem (2.10). We initially focus on the situation where the weak Mangasarian-Fromowitz constraint qualification holds.

**Theorem 2.28.** *Let $(\overline{x},\overline{y})$ be a local optimal solution of problem (2.10). Assume $\Theta$ is regular and semismooth at $(\overline{x},\overline{y})$ and (WMFCQ) holds at this point. Moreover, suppose that the upper-level and lower-level regularities are satisfied at $\overline{x}$ and $(\overline{x},\overline{y})$, respectively. Then the following statements are valid:*

*(a) Under the hypotheses of Theorem 2.10, there exist $\nu \geqslant 0$, $\xi \in \mathbb{R}^{n+m+1}_+$, $w_r \in S(\overline{x},\overline{y})$, $\delta_r \in \mathbb{R}^s_+$, $\vartheta_r \in \mathbb{R}^p_+$, $v^k_r \in \mathbb{R}^{n+m+1}_+$, $(\zeta,\gamma) \in \mathbb{R}^q \times \mathbb{R}^p$, and*

$$
\begin{array}{lll}
(x^*_F, y^*_F) \in \partial F(\overline{x},\overline{y}), & x^*_{G_i} \in \partial G_i(\overline{x}), & \left(x^*_{f_k}, y^*_{f_k}\right) \in \partial f_k(\overline{x},\overline{y}), \\
\left(x^*_{g_j}, y^*_{g_j}\right) \in \partial g(\overline{x},\overline{y}), & \left(x^{*r}_{f_k}, w^{*r}_{f_k}\right) \in \partial f_k(\overline{x},w_r), & \left(\tilde{x}^r_{f_k}, \tilde{w}^r_{f_k}\right) \in \partial f_k(\overline{x},w_r), \\
\left(\hat{x}^{r,l}_{f_k}, \hat{y}^{r,l}_{f_k}\right) \in \partial f_k(\overline{x},\overline{y}), & \left(\underline{x}^r_{g_j}, \underline{w}^r_{g_j}\right) \in \partial g_j(\overline{x},w_r), &
\end{array}
$$

*with $i \in I$, $j \in J$, $k = 1,\cdots,s$, $r = 1,\cdots,n+m+1$, $l = 1,\cdots,n+m+1$, $\sum\limits_{r=1}^{n+m+1} \xi_r = 1$ and $\sum_{l=1}^{n+m+1} v^k_{r,l} = 1$, $r = 1,\cdots,n+m+1$, $k = 1,\cdots,s$, such that:*

$$
\begin{aligned}
0 &= x^*_F + \sum_{i=1}^{q} \zeta_i x^*_{G_i} + \nu \sum_{k=1}^{s} x^*_{f_k} + \sum_{j=1}^{p} \gamma_j x^*_{g_j} - \nu \sum_{r=1}^{n+m+1} \xi_r \Bigg( \sum_{k=1}^{s} x^{*r}_{f_k} \\
&\quad + \sum_{k=1}^{s} \delta_{r,k} \left( \tilde{x}^r_{f_k} - \sum_{l=1}^{n+m+1} v^k_{r,l} \hat{x}^{r,l}_{f_k} \right) + \sum_{j=1}^{p} \vartheta_{r,j} \underline{x}^r_{g_j} \Bigg), \\
0 &= y^*_F + \nu \sum_{k=1}^{s} y^*_{f_k} + \sum_{j=1}^{p} \gamma_j y^*_{g_j} + \nu \sum_{r=1}^{n+m+1} \xi_r \sum_{k=1}^{s} \delta_{r,k} \sum_{l=1}^{n+m+1} v^k_{r,l} \hat{y}^{r,l}_{f_k}, \\
0 &= \sum_{k=1}^{s} w^{*r}_{f_k} + \sum_{k=1}^{s} \delta_{r,k} \tilde{w}^r_{f_k} + \sum_{j=1}^{p} \vartheta_{r,j} \underline{w}^r_{g_j}, \quad \forall r = 1,\cdots,n+m+1,
\end{aligned}
$$

$$
\begin{aligned}
&\zeta_i \geqslant 0,\ \zeta_i G_i(\overline{x}) = 0, \quad \forall i = 1,\ldots,q, \\
&\gamma_j \geqslant 0,\ \gamma_j g_j(\overline{x},\overline{y}) = 0, \quad \forall j = 1,\ldots,p \\
&\vartheta_{r,j} \geqslant 0,\ \vartheta_{r,j} g_j(\overline{x},w_r) = 0, \quad \forall j = 1,\ldots,p, \quad \forall r = 1,\cdots,n+m+1.
\end{aligned}
$$

*(b) Under the hypotheses of Theorem 2.12, there exist $\nu \geqslant 0$, $\xi \in \mathbb{R}^{n+m+1}_+$, $\delta_r \in \mathbb{R}^s_+$, $\vartheta_r \in \mathbb{R}^p_+$, $v^k_r \in$*

$\mathbb{R}^{n+m+1}_+$, $(\zeta,\gamma)\in\mathbb{R}^q\times\mathbb{R}^p$, *and*

$$
\begin{array}{lll}
(x_F^*,y_F^*)\in\partial F(\overline{x},\overline{y}), & x_{G_i}^*\in\partial G_i(\overline{x}), & \left(x_{f_k}^*,y_{f_k}^*\right)\in\partial f_k(\overline{x},\overline{y}),\\
\left(x_{g_j}^*,y_{g_j}^*\right)\in\partial g(\overline{x},\overline{y}), & \left(x_{f_k}^{*r},y_{f_k}^{*r}\right)\in\partial f_k(\overline{x},\overline{y}), & \left(\tilde{x}_{f_k}^r,\tilde{y}_{f_k}^r\right)\in\partial f_k(\overline{x},\overline{y}),\\
\left(\hat{x}_{f_k}^{r,l},\hat{y}_{f_k}^{r,l}\right)\in\partial f_k(\overline{x},\overline{y}), & \left(\underline{x}_{g_j}^r,\underline{y}_{g_j}^r\right)\in\partial g_j(\overline{x},\overline{y}), &
\end{array}
$$

*with* $i\in I$, $j\in J$, $k=1,\cdots,s$, $r=1,\cdots,n+m+1$, $l=1,\cdots,n+m+1$, $\sum\limits_{r=1}^{n+m+1}\xi_r=1$ *and* $\sum_{l=1}^{n+m+1}v_{r,l}^k=1$, $r=1,\cdots,n+m+1$, $k=1,\cdots,s$, *such that:*

$$
\begin{aligned}
0&=x_F^*+\sum_{i=1}^{q}\zeta_i x_{G_i}^*+\nu\sum_{k=1}^{s}x_{f_k}^*+\sum_{j=1}^{p}\gamma_j x_{g_j}^*-\nu\sum_{r=1}^{n+m+1}\xi_r\Bigg(\sum_{k=1}^{s}x_{f_k}^{*r}\\
&\quad+\sum_{k=1}^{s}\delta_{r,k}\left(\tilde{x}_{f_k}^r-\sum_{l=1}^{n+m+1}v_{r,l}^k\hat{x}_{f_k}^{r,l}\right)+\sum_{j=1}^{p}\vartheta_{r,j}\underline{x}_{g_j}^r\Bigg),\\
0&=y_F^*+\nu\sum_{k=1}^{s}y_{f_k}^*+\sum_{j=1}^{p}\gamma_j y_{g_j}^*+\nu\sum_{r=1}^{n+m+1}\xi_r\sum_{k=1}^{s}\delta_{r,k}\sum_{l=1}^{n+m+1}v_{r,l}^k\hat{y}_{f_k}^{r,l},\\
0&=\sum_{k=1}^{s}y_{f_k}^{*r}+\sum_{k=1}^{s}\delta_{r,k}\tilde{y}_{f_k}^r+\sum_{j=1}^{p}\vartheta_{r,j}\underline{y}_{g_j}^r,\quad\forall r=1,\cdots,n+m+1,\\
&\zeta_i\geqslant 0,\ \zeta_i G_i(\overline{x})=0,\quad\forall i=1,\ldots,q,\\
&\gamma_j\geqslant 0,\ \gamma_j g_j(\overline{x},\overline{y})=0,\quad\forall j=1,\ldots,p,\\
&\vartheta_{r,j}\geqslant 0,\ \vartheta_{r,j}g_j(\overline{x},\overline{y})=0,\quad\forall j=1,\ldots,p.
\end{aligned}
$$

*Proof.* Let $(\overline{x},\overline{y})$ be a local optimal solution of (2.10). Regarding the hypotheses of $(a)$ or $(b)$ the lower-level value function $\upsilon$ is Lipschitz continuous near $(\overline{x},\overline{y})$, as shown in Theorems 2.10 and 2.12.

Considering the closedness of the set $\Sigma$ and Lipschitz property of all functions, we obtain by applying the necessary optimality conditions from [57, Proposition 5.3] to problem (2.10) that:

$$0\in\partial F(\overline{x},\overline{y})+N((\overline{x},\overline{y}),\Sigma).$$

Then, applying [25, Lemma 3.3] together with (WMFCQ), there exists $\nu\geq 0$ such that:

$$0\in\partial F(\overline{x},\overline{y})+\nu\sum_{k=1}^{s}\partial f_k(\overline{x},\overline{y})+\nu\partial(-\varphi)(\overline{x},\overline{y})+N((\overline{x},\overline{y}),\Theta)$$

while considering the regularity and semi-smoothness of $\Theta$.

Using the convex hull properties, conv $\partial(-\varphi)(\overline{x},\overline{y}) = -\text{conv}\ \partial\varphi(\overline{x},\overline{y})$, we get

$$0 \in \partial F(\overline{x},\overline{y}) + \nu\sum_{k=1}^{s}\partial f_k(\overline{x},\overline{y}) - \nu\ \text{conv}\ \partial\varphi(\overline{x},\overline{y}) + N((\overline{x},\overline{y}),\Theta).$$

Observing this expression along with Lemma 2.25 and the upper estimate for $\upsilon$ from Theorems 2.10 and 2.12, and Remark 2.11, we can deduce assertions $(a)$ and $(b)$ of the theorem, Specifically, we apply Carathéodory's theorem to compute an element of conv $\partial\varphi(\overline{x},\overline{y})$. □

**Example 2.29.** Consider the following semivectorial optimistic bilevel optimization problem:

$$\min_{x,y} F(x,y) = |x| + x + y^2 \ \text{ s.t. } G(x) = x^2 - 2x \leq 0, \quad y \in \Phi_{\text{eff}}(x), \tag{2.50}$$

where $\Phi_{\text{eff}}(x)$ represents the solution set of the lower-level multiobjective optimization problem:

$$\min_{y} \left(x^2 + xy + x\ ,\ y^2 + xy + y\right) \ \text{ s.t. } g(x,y) = -y + x \leq 0. \tag{2.51}$$

For this example, we have:

$$X = [0,2], \quad Y(x) = [x,+\infty[, \quad \text{and} \quad \Theta = \left\{(x,y) \in \mathbb{R}^2 :\ (x,y) \in [0,2] \times [x,+\infty[\right\}.$$

Using the Charnes-Cooper scalarization technique, we can conclude that the lower-level multiobjective problem (2.51) is equivalent to the following single-objective program:

$$\begin{cases} \min\limits_{w}\ w^2 + w + x^2 + 2xw + x \\ (w-y)(w+y+x+1) \leq 0, \\ xw - xy \leq 0, \\ x \leq w. \end{cases} \tag{2.52}$$

Let $(x,y) \in \Theta$, we have $\Xi(x,y) = [x,y]$. The optimal value function of problem (2.52) is given by:

$$\varphi(x,y) = 4x^2 + 2x.$$

Therefore, the optimal value reformulation of the original problem (2.50) is given by:

$$\begin{cases} \min\limits_{x,y}\ F(x,y) = |x| + x + y^2 \\ y^2 - 3x^2 + 2xy + y - x \leq 0, \\ x \leq y, \\ 0 \leq x \leq 2. \end{cases} \tag{2.53}$$

Observe first that $0 \in \Phi_{\text{eff}}(0)$, then $(\overline{x}, \overline{y}) := (0,0)$ is an optimal solution of the problem (2.53). The normal cone to $\Theta$ at $(\overline{x}, \overline{y})$, and the partial of $\psi$ at $(\overline{x}, \overline{y})$ are given by:

$$N((\overline{x}, \overline{y}), \Theta) = \mathbb{R} \times \mathbb{R}^{-}, \quad \partial\psi(\overline{x}, \overline{y}) = \{(-1, 1)\}.$$

Then, $\partial\psi(\overline{x}, \overline{y}) \cap -\text{bd } N((\overline{x}, \overline{y}), \Theta) = \emptyset$.

- The constraint qualification (WMFQC) is satisfied at $(\overline{x}, \overline{y})$.
- The set-valued mapping $S$ given by $S(x,y) = \{x\}$, for all $(x,y) \in \Theta$, is inner semi-compact and inner semicontinuous at $(\overline{x}, \overline{y})$.
- For $\mu \in \mathbb{R}$, $\Psi_{\Upsilon_0}(\mu) = \{(x,y,w) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ : \ -w + x \leq \mu\}$. Hence, the set-valued mapping $\Psi_{\Upsilon_0}$ is calm at $(0, \overline{x}, \overline{y}, w)$ for $w \in S(\overline{x}, \overline{y})$.
- For $\kappa \in \mathbb{R}$, $\Psi_{\Xi}(\kappa) = \{(x,y,w) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ : \ -w + x \leq 0, \ xw - xy \leq \kappa\}$. Hence, the set-valued mapping $\Psi_{\Xi}$ is calm at $(0, \overline{x}, \overline{y}, w)$ for $w \in S(\overline{x}, \overline{y})$.
- The upper-level and lower-level regularities are satisfied at $\overline{x}$ and $(\overline{x}, \overline{y})$, respectively.

Finally, for $(x_F^*, y_F^*) := (2,0)$, $x_G^* := -2$, $\left(x_{f_1}^*, y_{f_1}^*\right) = \left(x_{f_1}^{*r}, w_{f_1}^{*r}\right) = \left(\tilde{x}_{f_1}^r, \tilde{w}_{f_1}^r\right) = \left(\hat{x}_{f_1}^{r,l}, \hat{y}_{f_1}^{r,l}\right) := (1,0)$, $\left(x_{f_2}^*, y_{f_2}^*\right) = \left(x_{f_2}^{*r}, w_{f_2}^{*r}\right) = \left(\tilde{x}_{f_2}^r, \tilde{w}_{f_2}^r\right) = \left(\hat{x}_{f_2}^{r,l}, \hat{y}_{f_2}^{r,l}\right) := (0,1)$, $\left(x_g^*, y_g^*\right) = \left(\underline{x}_g^r, \underline{w}_g^r\right) := (1,-1)$, $\nu := 1$, $\xi = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)^{\top}$, $w_r := 0 \in S(\overline{x}, \overline{y})$, $\delta_{r,1} := 3$, $\delta_{r,2} := 1$, $\vartheta_r := 2$, $v_r^k := \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)^{\top}$, $\zeta := 1$, and $\gamma := 2$, $r = 1,2,3$, $l = 1,2,3$, $k = 1,2$, the optimality conditions $(a)$ and $(b)$ stated in Theorem 2.28 are satisfied.

**Remark 2.30.** Obviously, the necessary optimality conditions obtained in Theorems 2.28 and 2.26 are almost the same; the only difference between them is that the scalar $\nu$ is just a nonnegative scalar in Theorem 2.28, and in Theorem 2.26 it is a positive scalar.

To conclude this section, we demonstrate a case where the partial calmness property holds for problem (2.10) at every local optimum. Specifically, consider the semi-vectorial optimization problem (SVBOP) with the parametric linear multiobjective problem (L[x]). The scalarized formulation is the linear program:

$$\min_{w} \sum_{k=1}^{s} a_k w \text{ s.t. } a_k w - a_k y \leq 0, \ k = 1, \cdots, s, \quad b_j x + c_j y \leq e_j, \ j = 1, \cdots, p. \tag{2.54}$$

The feasible set and the solution set of problem (2.54), respectively, are given by:

$$\Xi(x,y) = \{w \in \mathbb{R}^m \ : \ a_k w - a_k y \leq 0, \ k = 1, \cdots, s; \ \ b_j x + c_j y \leq e_j, \ j = 1, \cdots, p\},$$

$$S(x,y) = \arg\min_{w} \left\{ \sum_{k=1}^{s} a_k w \ : \ w \in \Xi(x,y) \right\}.$$

The optimal value function $\upsilon : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ associated with (2.54) is given by:

$$\varphi(x,y) = \inf_{w \in \Xi(x,y)} \ \sum_{k=1}^{s} a_k w, \ \ \text{for all } \ (x,y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Then, problem (SVBOP) is equivalent to the following single-level optimization problem:

$$\begin{cases} \min\limits_{x,y} F(x,y) \\ G_i(x) \leq 0, \ i = 1 \cdots, q \\ b_j x + c_j y \leq e_j, \ j = 1, \cdots, p \\ \sum\limits_{k=1}^{s} a_k y - \varphi(x,y) \leq 0. \end{cases} \tag{2.55}$$

**Theorem 2.31.** *Let $(\overline{x}, \overline{y})$ be a local optimal solution of problem (2.55). Assume that the upper-level and lower-level regularities are satisfied at $\overline{x}$ and $(\overline{x}, \overline{y})$, respectively. Then the following statements are valid:*

*(a) Under the hypotheses of Theorem 2.10, there exist $\nu > 0$, $\xi \in \mathbb{R}^{n+m+1}$, $w_r \in S(\overline{x}, \overline{y})$, $\delta_r \in \mathbb{R}^s_+$, $\vartheta_r \in \mathbb{R}^p$, $(\zeta, \gamma) \in \mathbb{R}^q \times \mathbb{R}^p$, and $(x^*_F, y^*_F) \in \partial F(\overline{x}, \overline{y})$, $x^*_{G_i} \in \partial G_i(\overline{x})$, with $i \in I$, $r = 1, \cdots, n+m+1$ and $\sum\limits_{r=1}^{n+m+1} \xi_r = 1$ such that:*

$$0 = x^*_F + \sum_{i=1}^{q} \zeta_i x^*_{G_i} + \sum_{j=1}^{p} \gamma_j b_j - \nu \sum_{r=1}^{n+m+1} \xi_r \sum_{j=1}^{p} \vartheta_{r,j} b_j,$$

$$0 = y^*_F + \nu \sum_{k=1}^{s} a_k + \sum_{j=1}^{p} \gamma_j c_j + \sum_{r=1}^{n+m+1} \xi_r \sum_{k=1}^{s} \delta_{r,k} a_k,$$

$$0 = \sum_{k=1}^{s} a_k + \sum_{k=1}^{s} \delta_{r,k} a_k + \sum_{j=1}^{p} \vartheta_{r,j} c_j, \quad \forall r = 1, \cdots, n+m+1,$$

$$\zeta_i \geqslant 0, \ \zeta_i G_i(\overline{x}) = 0, \quad \forall i = 1, \ldots, q,$$

$$\gamma_j \geqslant 0, \ \gamma_j (b_j \overline{x} + c_j \overline{y} - e_j) = 0, \quad \forall j = 1, \ldots, p$$

$$\vartheta_{r,j} \geqslant 0, \ \vartheta_{r,j} (b_j \overline{x} + c_j w_r - e_j) = 0, \quad \forall j = 1, \ldots, p, \quad \forall r = 1, \cdots, n+m+1.$$

(*b*) *Based on the assumptions of Theorem 2.12, there exist* $\nu > 0$, $\xi \in \mathbb{R}^{n+m+1}$, $\delta_r \in \mathbb{R}^s_+$, $\vartheta_r \in \mathbb{R}^p$, $(\zeta, \gamma) \in \mathbb{R}^q \times \mathbb{R}^p$, *and* $(x_F^*, y_F^*) \in \partial F(\overline{x}, \overline{y})$, $x_{G_i}^* \in \partial G_i(\overline{x})$, $i \in I$, $r = 1 \cdots, n+m+1$ *and* $\sum_{r=1}^{n+m+1} \xi_r = 1$ *such that:*

$$
\begin{aligned}
0 &= x_F^* + \sum_{i=1}^{q} \zeta_i x_{G_i}^* + \sum_{j=1}^{p} \gamma_j b_j - \nu \sum_{r=1}^{n+m+1} \xi_r \sum_{j=1}^{p} \vartheta_{r,j} b_j, \\
0 &= y_F^* + \nu \sum_{k=1}^{s} a_k + \sum_{j=1}^{p} \gamma_j c_j + \nu \sum_{r=1}^{n+m+1} \xi_r \sum_{k=1}^{s} \delta_{r,k} a_k, \\
0 &= \sum_{k=1}^{s} a_k + \sum_{k=1}^{s} \delta_{r,k} a_k + \sum_{j=1}^{p} \vartheta_{r,j} c_j, \quad \forall r = 1, \cdots, n+m+1, \\
&\zeta_i \geqslant 0,\ \zeta_i G_i(\overline{x}) = 0, \quad \forall i = 1, \ldots, q, \\
&\gamma_j \geqslant 0,\ \gamma_j (b_j \overline{x} + c_j \overline{y} - e_j) = 0, \quad \forall j = 1, \ldots, p, \\
&\vartheta_{r,j} \geqslant 0,\ \vartheta_{r,j} (b_j \overline{x} + c_j \overline{y} - e_j) = 0, \quad \forall j = 1, \ldots, p, \quad \forall r = 1, \cdots, n+m+1.
\end{aligned}
$$

*Proof.* Let $(\overline{x}, \overline{y})$ be a local optimal solution of the single-level linear program (2.55). According to [72, Proposition 4.1], the parametric linear multiobjective problem (L[x]) satisfies partial calmness at $(\overline{x}, \overline{y})$. By Proposition 2.21, there exists a scalar $\nu > 0$ such that $(\overline{x}, \overline{y})$ is a local solution to the penalized problem (2.29). The remainder of the proof follows analogously to the proof of Theorem 2.26, where necessary optimality conditions were derived under partial calmness. □

# 6 Conclusion

This chapter demonstrated the application of the Charnes-Cooper transformation to derive necessary optimality conditions for semivectorial optimization problems with an efficiently solvable lower-level program (SVBOP). By reformulating the problem as a classical bilevel program, we utilized tools from parametric analysis and sensitivity estimation of the lower-level value function. Specifically, by analyzing Lipschitz continuity and subdifferential properties, we obtained optimality conditions without requiring differentiability assumptions on the value function.

A promising direction for future research is the algorithmic implementation of this value function reformulation approach to numerically solve semivectorial optimization problems. Previous literature

has validated this methodology algorithmically for cases where the lower-level problem involves scalar optimization, as discussed in the survey in [23]. Given the growing importance of multiobjective decision-making, extending these computational techniques to problems with a lower-level vector optimization approach presents an opportunity to develop practical solution methods. Overall, the value function perspective adopted here provides a theoretical foundation for addressing semivectorial programs while preserving their inherent multiobjective structure.

Another direction of the study is the approximate Karush-Kuhn-Tucker optimality conditions and the Karush-Kuhn-Tucker reformulation of the semivectorial bilevel optimization problem.

## Chapter 3

# Optimization problems with nonconvex multiobjective generalized Nash equilibrium problem constraints

## 1 Introduction

In this chapter, we have dealt with a special case of semivectorial bilevel optimization problems, with the lower-level problem being a multiobjective generalized Nash equilibrium problem.

$$\min_{x,y} F(x,y): \ x \in X, \ \ y \in \mathcal{H}_{\text{weff}}(x). \tag{SVBOP}$$

The leader's decision variables are represented by $x \in X = \{x \in \mathbb{R}^n : G_i(x) \leq 0, \quad i \in 1,\ldots,p = I\}$, the follower's decision variables are represented by $y$, and the function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ denotes the leader's objective function, while $\mathcal{H}_{\text{weff}}(x)$ is the set of weakly efficient generalized Nash equilibrium of the multiobjective generalized Nash equilibrium problem parameterized of $x$ composed of $N$ players, where each player $\eta = 1,\ldots,N$ is assumed to minimize

$$\begin{cases} \min\limits_{y^\eta} \ f^\eta\left(x, y^\eta, y^{-\eta}\right) = \left(f_1^\eta\left(x, y^\eta, y^{-\eta}\right), \ldots, f_{s_\eta}^\eta\left(x, y^\eta, y^{-\eta}\right)\right) \\ \text{s.t } y^\eta \in Y^\eta\left(x, y^{-\eta}\right), \end{cases} \tag{$P^\eta[x, y^{-\eta}]$}$$

where $m = m_1 + \ldots + m_N$. We use the notation $(y^\eta, y^{-\eta})$ instead of simply $y$. We assume that each follower player has a vector-valued objective function $f^\eta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{s_\eta}$, and for any $\eta = 1, \ldots, N$ and $y^{-\eta}$ we denote by $\Psi^\eta_{weff}(x, y^{-\eta})$ the weakly efficient solution of problem $(P^\eta[x, y^{-\eta}])$, with

$$Y^\eta\left(x, y^{-\eta}\right) = \left\{y^\eta \in \mathbb{R}^{m_\eta} :\ g^\eta_j\left(x, y^\eta, y^{-\eta}\right) \leq 0,\ j \in J^\eta = \{1, \ldots, q_\eta\}\right\},$$

where, the mapping $g^\eta(x, ., y^{-\eta}) : \mathbb{R}^n \to \mathbb{R}^{q_\eta}$, and $q = q_1 + \ldots + q_N$.

In this research, we refer to problems such as "nonconvex mathematical programs with multiobjective generalized Nash equilibrium problems."

The Nash equilibrium is an important notion in many domains. The importance of it goes beyond its theoretical origins in game theory to practical applications spanning economics, political science, biology, computer science, and many other areas (see [61,64]).

Given that the lower-level decision-maker must solve a multiobjective generalized Nash equilibrium problem for each fixed value of the upper-level variable, the overall weak Pareto front can be computed using the kth-objective weighted-constraint approach. This transforms the original mathematical programming problem into several standard programming problems. This topic was first explored in [48], where the author highlighted the relationship between the surrogate problems and the original mathematical programming problem.

The goal of this chapter is twofold. First, we generalize the existing results from [40] on the generalized Nash equilibrium problem for the nonconvex cases. Second, we introduce a new scalarization technique, the weighted sum method, to extend the current results for bilevel problems to scenarios involving two players, as detailed in [26,34,48]. We develop two approaches. The first approach addresses a mathematical programming problem with a multiobjective generalized Nash equilibrium problem at the lower level. Our method involves several steps:

- We apply the kth-objective weighted constraint approach to transform the scalarization parameters into new upper-level variables.
- We then convert the original mathematical programming form of the generalized Nash equilibrium problem into several standard programming problems, with the Nash equilibrium problem at the lower level.

The primary advantage of this scalarization method is that the scalarization parameters are unique and strictly positive, making it applicable to nonconvex problems. We reformulate the optimization problem into a single-level optimization problem, which is globally (or locally) equivalent to the original problem, using the optimal value function derived from the lower-level generalized Nash equilibrium problems.

The second objective of this chapter is to present optimality conditions using limiting subdifferentials and limiting normal cones based on the weak basic constraint qualification (CQ) condition, which is suitable for mathematical programming with generalized Nash equilibrium optimization problems. We utilize Mordukhovich's generalized differentiation calculus to derive comprehensive first-order necessary optimality conditions in a smooth setting.

## 2 The weighted-constraint scalarization to weakly generalized Nash equilibrium

Consider a multiobjective generalized Nash equilibrium problem ($MGNEP$) composed by N players such that for any $\eta = 1, \ldots, N$, and $y^{-\eta}$, the $\eta$th player chooses $y^{\eta}$ that solves

$$\begin{cases} \min\limits_{y^{\eta}} \; f^{\eta}\left(y^{\eta}, y^{-\eta}\right) = \left(f_1^{\eta}\left(y^{\eta}, y^{-\eta}\right), \ldots, f_{s_{\eta}}^{\eta}\left(y^{\eta}, y^{-\eta}\right)\right) \\ \text{s.t } y^{\eta} \in Y^{\eta}\left(y^{-\eta}\right), \end{cases} \qquad (P^{\eta}[y^{-\eta}])$$

by controling his or her own strategy $y^{\eta} \in \mathbb{R}^{m_{\eta}}$ while taking into account the choices made by opponents, which are represented by the vector $y^{-\eta} \in \mathbb{R}^{m-m_{\eta}}$, where $m = m_1 + \ldots + m_N$ represents the total number of players' decision variables. Sometimes we use the notation $(y^{\eta}, y^{-\eta})$ instead of just $y$ to emphasize the variables of player within the vector $y$. The block components are not implied to be rearranged in any manner by this notation, and $y = (y^{\eta}, y^{-\eta})$ is always the case. Furthermore, each player has an objective $f^{\eta} : \mathbb{R}^{m_{\eta}} \to \mathbb{R}^{s_{\eta}}$, called utility function or payoff function, which depends on his own vaiables $y^{\eta}$ as well as the variables $y^{-\eta}$ of the other players. Moreover, each player's strategies belong to a set $Y^{\eta}\left(y^{-\eta}\right)$ that depends on the rival player's strategie. This continuous function's components are $f_1^{\eta}, \ldots, f_{s_{\eta}}^{\eta} : \mathbb{R}^{m_{\eta}} \to \mathbb{R}$.

Here after, for each $\eta$ we denote by $\Psi_{weff}^{\eta}\left(y^{-\eta}\right)$ the weakly efficient solution set of problem $(P^{\eta}[y^{-\eta}])$.

Assuming that the functions $f_k^{\eta}\left(., y^{-\eta}\right)$ have a lower bound on the constraint set $Y^{\eta}\left(y^{-\eta}\right)$ and that this lower bound is known, it is reasonable to assert that there is no loss of generality in imposing the

condition below.

$$\min_{k=1,\cdots,s_\eta}\left\{\min_{y^\eta\in Y^\eta(y^{-\eta})} f_k^\eta\left(y^\eta,y^{-\eta}\right)\right\}>0,\quad \forall\eta=1,\ldots,N.$$

**Definition 3.1.** [40,65] A point $\bar{y}=\left(\bar{y}^1,\ldots,\bar{y}^N\right)$ is said to be a weakly efficient generalized Nash equilibrium of the multiobjective generalized Nash equilibrium problem $(MGNEP)$ if

$$\bar{y}^\eta\in\Psi^\eta_{\text{weff}}\left(\bar{y}^{-\eta}\right)\quad\forall\ \eta=1,\cdots,N.$$

The set of weakly efficient generalized Nash equilibriums will be denoted by $\mathcal{H}_{\text{weff}}$. That is

$$\mathcal{H}_{\text{weff}}=\left\{\bar{y}=\left(\bar{y}^1,\ldots,\bar{y}^N\right)\ \ \bar{y}^\eta\in\Psi^\eta_{\text{weff}}\left(\bar{y}^{-\eta}\right)\quad\forall\ \eta=1,\cdots,N\right\}.$$

For $\eta=1,\cdots,N$ set $S^\eta=\{1,\ldots,s_\eta\}$, and let

$$\mathbb{W}^\eta=\left\{w\in\mathbb{R}^{s_\eta}:\ w_k\geqslant 0,\ \sum_{k=1}^{s_\eta}w_k=1\right\},\tag{3.1}$$

and

$$\mathbb{W}^\eta_+=\left\{w\in\mathbb{R}^{s_\eta}_+:\ w_k>0,\ \sum_{k=1}^{s_\eta}w_k=1\right\}.\tag{3.2}$$

Our goal now is to transform the multiobjective generalized Nash equilibrium problem $(MGNEP)$ into a single-objective generalized Nash equilibrium problem $(GNEP)$. To do this, for each $\eta\in\{1,\ldots,N\}$, fix $k\in S^\eta$ and $w^\eta\in\mathbb{W}^\eta_+$, and then consider the following scalarized problem:

$$\begin{cases}\min\limits_{y^\eta}\ f_k^\eta\left(y^\eta,y^{-\eta}\right)\\ \text{s.t}\ y^\eta\in Y_k^\eta\left(y^{-\eta},w^\eta\right),\end{cases}\tag{$P_k^\eta[y^{-\eta},w^\eta]$}$$

where for any $y^{-\eta}$ and $w^\eta$ the feasible set and the solution set of the problem $(P_k^\eta[y^{-\eta},w^\eta])$ is given by

$$Y_k^\eta\left(y^{-\eta},w^\eta\right)=\left\{y^\eta\in Y\left(y^{-\eta}\right):w_r^\eta\ f_r^\eta\left(y^\eta,y^{-\eta}\right)\leq w_k^\eta\ f_k^\eta\left(y^\eta,y^{-\eta}\right),\forall\ r\in S^\eta\backslash\{k\}\right\},\tag{3.3}$$

and $\Psi_k^\eta\left(y^{-\eta},w^\eta\right)$.

For each fixed $w^\eta\in\mathbb{W}^\eta_+$, one can easily check that

$$Y^\eta\left(y^{-\eta}\right)=\bigcup_{k=1}^{s_\eta}Y_k^\eta\left(y^{-\eta},w^\eta\right).$$

In the sequel, for $w\in\mathbb{W}_+=\mathbb{W}^1_+\times\ldots\times\mathbb{W}^N_+$ with $w=\left(w^1,\ldots,w^N\right)$, we need the following construction:

$$\mathbb{W}^\eta_+\left(y^{-\eta}\right)=\left\{w^\eta\in\mathbb{W}^\eta_+:\ y^\eta\in\Psi_k^\eta\left(y^{-\eta},w^\eta\right),\ \text{for all}\ k=1,\ldots,s_\eta\right\},\ \ \text{and}\tag{3.4}$$

$$\mathcal{H}(w)=\left\{y\in\mathbb{R}^m\ :\ y^\eta\in\bigcap_{k=1}^{s_\eta}\Psi_k^\eta\left(y^{-\eta},w^\eta\right),\quad \text{holds for each } \eta=1,\cdots,N\right\}. \tag{3.5}$$

where, for each $\eta=1,\ldots,N:\ \bigcap_{k=1}^{s_\eta}\Psi_k^\eta\left(y^{-\eta},w_0^\eta\right)$ denote the commun solution.

The next result gives a relationship between the set of weakly efficient generalized Nash equilibrium $\mathcal{H}_{weff}$ and $\mathcal{H}(w)$. Before that, some comments are in order. For fixed $\eta=1,\cdots,N$, and for all $w_0^\eta\in\mathbb{W}_+^\eta$ we can see from [14] that

$$\bigcap_{k=1}^{s_\eta}\Psi_k^\eta\left(y^{-\eta},w_0^\eta\right)\subseteq\Psi_{\text{weff}}^\eta\ \left(y^{-\eta}\right)\subset\bigcup_{w\in\mathbb{W}_+^\eta}\left[\bigcap_{k=1}^{s_\eta}\Psi_k^\eta\left(y^{-\eta},w^\eta\right)\right]. \tag{3.6}$$

We deduce from the last inclusion that if for some $w_0^\eta\in\mathbb{W}_+^\eta$ we have that $\bigcap_{k=1}^{s_\eta}\Psi_k^\eta\left(y^{-\eta},w_0^\eta\right)\neq\emptyset$, then the left side of the above inclusion furnishes a method for computing weak efficient points. However, if $\bigcap_{k=1}^{s_\eta}\Psi_k^\eta\left(y^{-\eta},w_0^\eta\right)=\emptyset$, we need the following assumptions to ensure the nonemptiness of $\mathbb{W}_+^\eta\left(y^{-\eta}\right)$ and consequently the existence of a common solution of $(P_k^\eta[y^{-\eta},w^\eta])$.

$(\mathcal{A}_1)$**:** There exists an element $w^\eta\in\mathbb{W}_+^\eta$ such that the sets $\Psi_r^\eta\left(y^{-\eta},w^\eta\right)\neq\emptyset$, for all $r\in S^\eta$, and there exists $k\in S^\eta$ and $y_k^\eta\in\Psi_k^\eta\left(y^{-\eta},w^\eta\right)$ such that $\forall r\neq k$, there exists an element $y_r^\eta\in\Psi_r^\eta\left(y^{-\eta},w^\eta\right)$ satisfying the inequality

$$f_r^\eta\left(y_k^\eta,y_k^{-\eta}\right)\leq f_r^\eta\left(y_r^\eta,y_r^{-\eta}\right).$$

$(\mathcal{A}_2)$**:** There exists $w^\eta$ an element of $\mathbb{W}_+^\eta$ such that for any $k\in S^\eta$, we can find $\left(y_1^\eta,\ldots,y_{s_\eta}^\eta\right)$ an element of the cartesian product of $\Psi_1^\eta\left(y^{-\eta},w^\eta\right)\times\ldots\times\Psi_{s_\eta}^\eta\left(y^{-\eta},w^\eta\right)$ satisfying the following inequality

$$\forall r,k\in S^\eta,\quad f_r^\eta\left(\bar{y}_k^\eta,\bar{y}_k^{-\eta}\right)\leq f_r^\eta\left(y_r^\eta,y_r^{-\eta}\right).$$

Consequently, using [14, Proposition 3.3], under the hypothesis $(\mathcal{A}_1)$, we get that $y_k^\eta\in\Psi_{\text{weff}}^\eta\left(y_k^{-\eta}\right)$. Under the hypothesis $(\mathcal{A}_2)$, using [14, Corollary 3.1] instead of [14, Proposition 3.3], we also find that $y_k^\eta\in\Psi_{\text{weff}}^\eta\left(y_k^{-\eta}\right)$. Consequently, by [14, Proposition 3.2],

$$\emptyset\neq\mathbb{W}_+^\eta\left(y_k^\eta\right)=\{w_0^\eta\}\text{ and }y_k^\eta\in\bigcap_{r=1}^{s_\eta}\Psi_r^\eta(\bar{w},y_k^{-\eta}).$$

**Remark 3.2.** As a result, if we suppose that for all $\eta=1,\ldots,N$, $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold true, then the set $\mathcal{H}(w)$ is nonempty.

Now, we are really going to state the link between $\mathcal{H}_{weff}$ and $\mathcal{H}(w)$.

**Theorem 3.3.** *Let $w \in \mathbb{W}_+$. Suppose that $\mathbb{W}_+^\eta(y^{-\eta}) \neq \emptyset$ for any $\eta = 1, \ldots, N$. Then, we have*

$$\mathcal{H}_{weff} = \bigcup_{w \in \mathbb{W}_+} \mathcal{H}(w).$$

*Proof.* Let $\bar{y} \in \mathcal{H}_{weff}$, then $\bar{y}$ is a weakly efficient generalized Nash equilibrium. Then, from the definition 3.1, we have $\bar{y}^\eta \in \Psi_{\text{weff}}^\eta(\bar{y}^{-\eta})$ for all $\eta = 1, \cdots, N$. Moreover, from the right inclusion of (3.6), for each $\eta = 1, \cdots, N$ it follows that there exists $w^\eta \in \mathbb{W}_+^\eta$ such that $\bar{y}^\eta \in \bigcap_{k=1}^{s_\eta} \Psi_k^\eta(\bar{y}^{-\eta}, w^\eta)$, which implies that $\bar{y}^\eta \in \mathcal{H}(w)$.

Conversely, fix an arbitrary $w \in \mathbb{W}_+$ satisfying $w^\eta \in \mathbb{W}_+^\eta$ for all $\eta = 1, \cdots, N$. Let $\bar{y} \in \mathcal{H}(w)$. Then, from the definition, we get $\bar{y}^\eta \in \bigcap_{k=1}^{s_\eta} \Psi_k^\eta(\bar{y}^{-\eta}, w^\eta)$ for all $\eta = 1, \cdots, N$. Using the left inclusion of (3.6), it follows that $\bar{y}^\eta \in \Psi_{\text{weff}}^\eta(\bar{y}^{-\eta})$ for all $\eta = 1, \cdots, N$. This implies that $\bar{y} \in \mathcal{H}_{weff}$. □

# 3 Necessary optimality conditions applying weighted-constraint scalarization

This chapter deals with the mathematical program with a multiobjective generalized Nash equilibrium problem in the constraint $(P^\eta[x, y^{-\eta}])$-(SVBOP).

Assuming that the functions $f_k^\eta$ have a lower bound on the constraint set $Y^\eta(x, y^{-\eta})$ and that those lower bounds are known, it is reasonable to assert that there is no loss of generality in imposing the condition below.

$$\min_{k=1,\cdots,s_\eta} \left\{ \min_{y^\eta \in Y^\eta(x, y^{-\eta})} f_k^\eta\left(x, y^\eta, y^{-\eta}\right) \right\} > 0, \quad \forall \eta = 1, \ldots, N, \ \forall x \in X.$$

The feasible region of problem (SVBOP) is represented by

$$\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : \ x \in X, \ y \in \mathcal{H}_{\text{wef}}(x)\}.$$

## 3.1 Single-level reformulation

One option is to deal with hierarchical problems by transforming the optimization problem (SVBOP) into a single-level optimization problem. To accomplish this, we apply the weighted-sum technique

introduced in section 3 for reformulating each lower-level problem $(P^\eta[x, y^{-\eta}])$ of player $\eta$ as $s_\eta$ standard one-level optimization problem. We begin by applying the following scalarization technique to the problem $(P^\eta[x, y^{-\eta}])$ for each $\eta \in 1, \ldots, N$ and $w^\eta \in \mathbb{W}^\eta$, $\mathbb{W}^\eta$ is the sets stated in (3.1)

$$\begin{cases} \min\limits_{y^\eta} \; f_k^\eta\left(x, y^\eta, y^{-\eta}\right) \\ \text{s.t } y^\eta \in Y_k^\eta\left(x, y^{-\eta}, w^\eta\right), \end{cases} \qquad (P_k^\eta[x, y^{-\eta}, w^\eta])$$

where for any $y^{-\eta}$ and $w^\eta \in \mathbb{W}^\eta$, the feasible set and solution set of problem $(P_k^\eta[x, y^{-\eta}, w^\eta])$ are given by

$$Y_k^\eta\left(x, y^{-\eta}, w^\eta\right) = \left\{y^\eta \in Y\left(y^{-\eta}\right) : w_r^\eta \; f_r^\eta\left(x, y^\eta, y^{-\eta}\right) \leq w_k^\eta \; f_k^\eta\left(x, y^\eta, y^{-\eta}\right), \forall \; r \in S^\eta \backslash \{k\}\right\}, \qquad (3.7)$$

$$\Psi_k^\eta\left(x, y^{-\eta}, w^\eta\right) := \left\{y^\eta \in Y_k^\eta\left(x, y^{-\eta}, w^\eta\right) : \; f_k^\eta\left(x, y^\eta, y^{-\eta}\right) - \varphi_k^\eta\left(x, y^{-\eta}, w^\eta\right) \leq 0, \quad k \in S^\eta\right\},$$

with

$$\varphi_k^\eta\left(x, y^{-\eta}, w^\eta\right) = \min_{y^\eta}\left\{f_k^\eta\left(x, y^\eta, y^{-\eta}\right) : y^\eta \in Y_k^\eta\left(x, y^{-\eta}, w^\eta\right), \quad k \in S^\eta\right\}, \text{ and } S^\eta = \{1, \ldots, N\}.$$

In the sets stated above (3.4) and (3.5), we give the following sets parametrized by $x$. For $\eta = 1, \cdots, N$, let $w = \left(w^1, \ldots, w^N\right) \in \mathbb{W}_+ = \mathbb{W}_+^1 \times \ldots \times \mathbb{W}_+^N$, and set

$$\mathcal{H}(x, w) = \left\{y : \; y^\eta \in \bigcap_{k=1}^{s_\eta} \Psi_k^\eta\left(x, y^{-\eta}, w^\eta\right), \text{ for each } \eta = 1, \ldots, N\right\},$$

$$\mathbb{W}_+^\eta(x, y^\eta) = \left\{w^\eta \in \mathbb{W}_+^\eta : \; y^\eta \in \Psi_k^\eta\left(x, y^{-\eta}, w^\eta\right)\right), \text{ for all } k = 1, \ldots, s_\eta\right\} \text{ and}$$

$$\mathbb{W}_+(x, y) = \prod_{\eta=1}^{N} \mathbb{W}_+^\eta(x, y^\eta).$$

**Remark 3.4.** Let $k \in S^\eta$ and $w \in \mathbb{W}^\eta$. In case $w_k^\eta = 0$, we get $Y_k^\eta(x, y^{-\eta}, w^\eta) = \emptyset$.

Let $x \in X$, $y^\eta \in Y\left(x, y^{-\eta}\right)$ and $w_0^\eta \in \mathbb{W}_+^\eta$. Similarly to section 3, we have

$$\bigcap_{k=1}^{s_\eta} \Psi_k^\eta\left(x, y^{-\eta}, w_0^\eta\right) \subseteq \Psi_{\text{weff}}^\eta \quad \left(x, y^{-\eta}\right) \subset \bigcup_{w \in \mathbb{W}_+^\eta} \left[\bigcap_{k=1}^{s_\eta} \Psi_k^\eta\left(x, y^{-\eta}, w^\eta\right)\right]. \qquad (3.8)$$

Under the assumptions

$(\mathcal{B}_1)$ : There exists $w^\eta \in \mathbb{W}_+^\eta$ such that the sets $\Psi_r^\eta\left(x, y^{-\eta}, w^\eta\right) \neq \emptyset$, for all $r \in S^\eta$ and there exists $k \in S^\eta$ and $y_k^\eta \in \Psi_k^\eta\left(x, y^{-\eta}, w^\eta\right)$ such that $\forall r \neq k$, there exists $y_r^\eta \in \Psi_r^\eta\left(x, y^{-\eta}, w^\eta\right)$ satisfying the inequality

$$f_r^\eta\left(\bar{x}, \bar{y}_k^\eta, \bar{y}_k^{-\eta}\right) \leq f_r^\eta\left(\bar{x}, \bar{y}_r^\eta, \bar{y}_r^{-\eta}\right).$$

$(\mathcal{B}_2)$**:** There exists $w^\eta \in \mathbb{W}_+^\eta$ such that $\left(y_1^\eta, \ldots, y_{s_\eta}^\eta\right) \in \Psi_1^\eta\left(x, y^{-\eta}, w^\eta\right) \times \ldots \times \Psi_{s_\eta}^\eta\left(x, y^{-\eta}, w^\eta\right)$ satisfying the following inequality

$$\forall r, k \in S^\eta, \quad f_r^\eta\left(\bar{x}, \bar{y}_k^\eta, \bar{y}_k^{-\eta}\right) \leq f_r^\eta\left(\bar{x}, \bar{y}_r^\eta, \bar{y}_r^{-\eta}\right),$$

we have $\mathbb{W}_+^\eta(x, y) \neq \emptyset$ and consequently the existence of a common solution for $(P_k^\eta[x, y^{-\eta}, w^\eta])$ is guaranteed.

**Remark 3.5.** Considering that, $(\mathcal{B}_1)$ and $(\mathcal{B}_2)$ are true for all $\eta = 1, \ldots, N$, the set $\mathcal{H}(x, w)$ is nonempty.

**Remark 3.6.** By [14, Proposition 3.3], under the hypothesis $(\mathcal{B}_1)$, we get that $y_k^\eta \in \Psi_{\text{weff}}^\eta(x, y_k^{-\eta})$. Under the hypothesis $(\mathcal{B}_2)$, using [14, Corollary 3.1] instead of [14, Proposition 3.3], we find also that $y_k^\eta \in \Psi_{\text{weff}}^\eta(x, y_k^{-\eta})$. Consequently, by [14, Proposition 3.2],

$$\emptyset \neq \mathbb{W}_+^\eta\left(x, y_k^\eta\right) = \{w_0^\eta\} \text{ and } y_k^\eta \in \bigcap_{r=1}^{s_\eta} \Psi_r(x, y_k^{-\eta}, w_0^\eta).$$

Now, we are really going to state the link between $\mathcal{H}_{weff}(x)$ and $\mathcal{H}(x, w)$.

**Theorem 3.7.** *Let $x \in X$ and $w \in \mathbb{W}_+$. Suppose that $\mathbb{W}_+^\eta(x, y^\eta) \neq \emptyset$ for any $\eta = 1, \ldots, N$. Then, we have*

$$\mathcal{H}_{weff}(x) = \bigcup_{w \in \mathbb{W}_+} \mathcal{H}(x, w).$$

*Proof.* It follows the path of that of Theorem 3.3. □

Hence, the optimization problem (SVBOP) with a multiobjective Nash equilibruim problem at the lower level can be replaced by the following optimization problem where each follower has a single objective

$$\begin{cases} \min\limits_{x,y,w} F(x, y) \\ \text{s.t} \quad x \in X, \ w \in \mathbb{W}, \\ \qquad y \in \mathcal{H}(x, w). \end{cases} \tag{3.9}$$

The following result shows the equivalence between problem (SVBOP) and problem (3.9).

**Proposition 3.8.** *Consider problems* (SVBOP) *and* (3.9).

*(1) Let $(\bar{x},\bar{y})$ be a local (resp. global) optimal solution of problem* (SVBOP). *Then, for any $\bar{w}\in \mathbb{W}_+(\bar{x},\bar{y})$, the point $(\bar{x},\bar{y},\bar{w})$ is a local (resp. global) optimal solution of problem* (3.9).

*(2) Let $\bar{w}\in \mathbb{W}_+(\bar{x},\bar{y})$. Assume that $(\bar{x},\bar{y},\bar{w})$ is a global optimal solution of problem* (3.9). *Then, the point $(\bar{x},\bar{y})$ is a global optimal solution of problem* (SVBOP).

*(3) Assume that $(\bar{x},\bar{y},\bar{w})$ is a local optimal solution of* (3.9) *for all $\bar{w}\in \mathbb{W}_+(\bar{x},\bar{y})$ and suppose that the set-valued mappings $\mathcal{H}$, is closed at all points from $\{\bar{x}\}\times\mathbb{W}_+$. Then, $(\bar{x},\bar{y})$ is a local optimal solution of problem* (SVBOP).

*Proof.* (1) Suppose that there exists $\widehat{w}\in \mathbb{W}_+(\bar{x},\bar{y})$ such that $(\bar{x},\bar{y},\widehat{w})$ is not a local solution to problem (3.9). Then, there exists a sequence $(x_\varepsilon,y_\varepsilon,w_\varepsilon)$ with $x_\varepsilon\longrightarrow\bar{x}$, $y_\varepsilon\longrightarrow\bar{y}$, and $\widehat{w}_\varepsilon\longrightarrow\widehat{w}$ such that

$$F(x_\varepsilon,y_\varepsilon)<F(\bar{x},\bar{y}),\ x_\varepsilon\in X,\ \widehat{w}_\varepsilon\in\mathbb{W}=\mathbb{W}^1\times\ldots\times\mathbb{W}^N,\ y_\varepsilon\in\mathcal{H}(x_\varepsilon,\widehat{w}_\varepsilon),\ \text{for all } \varepsilon\in\mathbb{N}. \quad (3.10)$$

We claim that $\widehat{w}_\varepsilon\in\mathbb{W}_+$. Indeed, by the contrary, suppose that there exists $\eta=1,\ldots,N$ such that $\widehat{w}_\varepsilon^\eta\notin\mathbb{W}_+^\eta$. ie there exists $k_\eta\in S^\eta$ such that $\widehat{w}_\varepsilon^{k_\eta}=0$. Letting $\varepsilon\to+\infty$, we obtain $\widehat{w}^{k_\eta}=0$ a contradiction with $\widehat{w}^\eta\in\mathbb{W}_+^\eta(\bar{x},\bar{y})$.

Now, by (3.10), we get $\widehat{w}_\varepsilon\in\mathbb{W}_+$, $y_\varepsilon\in\mathcal{H}(x_\varepsilon,\widehat{w}_\varepsilon),\forall\varepsilon\in\mathbb{N}$. Then, $\widehat{w}_\varepsilon\in\mathbb{W}_+(x_\varepsilon,y_\varepsilon)$, $\forall\varepsilon\in\mathbb{N}$, which ensure that $\mathbb{W}_+(x_\varepsilon,y_\varepsilon)\neq\emptyset$. From Theorem 3.7 we get $y_\varepsilon\in\mathcal{H}_{\text{weff}}(x_\varepsilon)$ for all $\varepsilon\in\mathbb{N}$ . As a conclusion, there exists a sequence $(x_\varepsilon,y_\varepsilon)$ converging to $(\bar{x},\bar{y})$ with $x_\varepsilon\in X$, $y_\varepsilon\in\mathcal{H}_{\text{weff}}(x_\varepsilon)$ such that

$$F(x_\varepsilon,y_\varepsilon)<F(\bar{x},\bar{y}),\ \text{for all } \varepsilon\in\mathbb{N},$$

which is contrary to the fact that $(\bar{x},\bar{y})$ is a local optimal solution of problem (SVBOP).

(2) Let $\bar{w}\in\mathbb{W}_+$, and let $(\bar{x},\bar{y},\bar{w})$ be a global optimal solution of problem (3.9). Assume that $(\bar{x},\bar{y})$ is not a global optimal solution of problem (SVBOP). Then, we can find $(x,y)$ with $x\in X$, and $y\in\mathcal{H}_{weff}(x)$ such that

$$F(x,y)<F(\bar{x},\bar{y}).$$

From Theorem 3.7, there exists $w\in\mathbb{W}_+$ such that $y\in\mathcal{H}(x,w)$. Consequently $(x,y,w)$ is a feasible point of problem (3.9) such that

$$F(x,y)<F(\bar{x},\bar{y}),$$

which contrary to the fact that $(\bar{x},\bar{y},\bar{w})$ is a global optimal solution of problem (3.9).

(3) Let $\bar{w} \in \mathbb{W}_+(\bar{x}, \bar{y})$ such that $(\bar{x}, \bar{y}, \bar{w})$ is a local optimal solution of (3.9). Suppose that $(\bar{x}, \bar{y})$ is not a local optimal solution of problem (SVBOP). Then, there exists a sequence $(x_\varepsilon, y_\varepsilon)$ with $x_\varepsilon \longrightarrow \bar{x}$, and $y_\varepsilon \longrightarrow \bar{y}$, with $x_\varepsilon \in X$, and $y_\varepsilon \in \mathcal{H}_{\text{weff}}(x_\varepsilon)$ such that

$$F(x_\varepsilon, y_\varepsilon) < F(\bar{x}, \bar{y}), \text{ for all } \varepsilon \in \mathbb{N}.$$

For any $(x, y)$, we have $\mathbb{W}_+(x, y) \subset \mathbb{B}(0, 1)$. Hence, $\mathbb{W}_+$ is uniformly bounded, then $\mathbb{W}_+$ inner semicompact at $(\bar{x}, \bar{y})$ (see Remark 1.15).

Since $y_\varepsilon \in \mathcal{H}_{\text{weff}}(x_\varepsilon)$, using Theorem 3.7, there exists $w_\varepsilon \in \mathbb{W}_+$ such that $y_\varepsilon \in \mathcal{H}(x_\varepsilon, w_\varepsilon)$. Consequently, $w_\varepsilon \in \mathbb{W}_+(x_\varepsilon, y_\varepsilon)$, and $\mathbb{W}_+(x_\varepsilon, y_\varepsilon) \neq \emptyset$. By the inner semicompactness of $\mathbb{W}_+$ at $(\bar{x}, \bar{y})$, we can find a sequence $w_\varepsilon \in \mathbb{W}_+(x_\varepsilon, y_\varepsilon)$ which has an accumulation point $w_0 \in \mathbb{W}$. Then, $w_\varepsilon \in \mathbb{W}_+$, $y_\varepsilon \in \mathcal{H}(x_\varepsilon, w_\varepsilon)$, for all $\varepsilon \in \mathbb{N}$.

Since the mappings $\mathcal{H}$ is closed at $(\bar{x}, w_0)$, we have $\bar{y} \in \mathcal{H}(\bar{x}, w_0)$.

We claim that $w_0 \in \mathbb{W}_+$. Indeed, suppose that there exists $\eta = 1, \ldots, N$ such that $w_0^\eta \notin \mathbb{W}_+^\eta$. By the opposite, if there are $k_\eta \in S^\eta$ such that $w_0^{k_\eta} = 0$, we get $\mathcal{H}(\bar{x}, w_0) = Y_k^\eta(\bar{x}, \bar{y}^{-\eta}, w_0) = \emptyset$. A contradiction to $\bar{y} \in \mathcal{H}(\bar{x}, w_0)$.

Then, $(\bar{x}, \bar{y}, w_0)$ is a feasible point of problem (3.9). Therefore, there exists $(x_\varepsilon, y_\varepsilon, w_\varepsilon)$ converging to $(\bar{x}, \bar{y}, w_0)$ such that

$$F(x_\varepsilon, y_\varepsilon) < F(\bar{x}, \bar{y}),\ x_\varepsilon \in X,\ w_\varepsilon \in \mathbb{W}_+,\ y_\varepsilon \in \mathcal{H}(x_\varepsilon, w_\varepsilon),$$

which contradicts the fact that $(\bar{x}, \bar{y}, \bar{w})$ is a local optimal solution of problem (3.9).

□

For $\eta = 1, \ldots, N$, $k \in S^\eta$, setting

$$\psi_k^\eta(x, y, w^\eta) = f_k^\eta\left(x, y^\eta, y^{-\eta}\right) - \varphi_k^\eta\left(x, y^{-\eta}, w^\eta\right), \text{and}$$

$$h_{k,r}^\eta(x, y, w^\eta) = w_r^\eta f_r^\eta(x, y) - w_k^\eta f_k^\eta(x, y),\ r \in S^\eta \setminus \{k\}.$$

and using the optimal value reformulation, (3.9) is equivalent to the one-level optimization problem.

$$\begin{cases} \min\limits_{x,y,w} F(x,y) \\ x \in X,\ w^{\eta} \in \mathbb{W}^{\eta}, \eta = 1,\ldots,N, \\ g^{\eta}(x,y) \leq 0,\ \forall \eta = 1,\ldots,N, \\ h^{\eta}_{k,r}(x,y,w^{\eta}) \leq 0,\ \forall \eta = 1,\ldots,N,\ \forall r \in S^{\eta} \setminus \{k\},\ \forall k \in S^{\eta}, \\ f^{\eta}_{k}(x,y^{\eta},y^{-\eta}) - \varphi^{\eta}_{k}(x,y^{-\eta},w^{\eta}) \leq 0,\ \forall \eta = 1,\ldots,N,\ \forall k \in S^{\eta}. \end{cases} \tag{3.11}$$

## 3.2 Necessary optimality conditions

In the second part of this section, we employ the results obtained above to derive new forms of necessary optimality conditions for the original mathematical program problem ($SVBOP$). In order to make the following as short as possible, we introduce the following notations: for any $\eta \in \{1,\cdots,N\}$, $k \in \mathbb{T} = \{1,\ldots,t\}$ with $t = \max\limits_{\eta=1}^{N} s_{\eta}$ and $y^{-\eta}$, we define

$$\Theta_k = \left\{(x,y,w):\ x \in X,\ w \in \mathbb{W},\ y = \left(y^1,\ldots,y^N\right) \in Y^1_k\left(x,y^{-1},w^1\right) \times \ldots \times Y^N_k\left(x,y^{-N},w^N\right)\right\}.$$

$$\Omega_k = \left\{(x,y,w) \in \Theta_k:\ \psi_k(x,y,w) = \left(\psi^1_k(x,y,w), \ldots,\psi^N_k(x,y,w)\right) \leq 0\right\}.$$

$$\Xi = \left\{(x,y,w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{s_1+\ldots+s_N}:\ x \in X,\ w \in \mathbb{W},\ y \in Y^1\left(x,y^{-1}\right) \times \ldots \times Y^N\left(x,y^{-N}\right)\right\},$$

$$\Pi_k = \left\{(x,y,w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{s_1+\ldots+s_N} : \Upsilon_k(x,y,w) \in \mathbb{R}^{s_1+\ldots+s_N}_{-}\right\},$$

where

$$\Upsilon^{\eta}_k(x,y,w^{\eta}) = \Big(h^{\eta}_{k,1}(x,y,w^{\eta}),\cdots,h^{\eta}_{k,(k-1)}(x,y,w^{\eta}),\psi^{\eta}_k(x,y,w^{\eta}), h^{\eta}_{k,(k+1)}(x,y,w^{\eta}),\cdots,h^{\eta}_{k,s_{\eta}}(x,y,w^{\eta})\Big),$$

$$\Upsilon_k(x,y,w) = \left(\Upsilon^1_k\left(x,y,w^1\right),\ldots,\Upsilon^N_k\left(x,y,w^N\right)\right).$$

**Remarks 3.9.** *1) Under the above notations, we have*

$$\Omega = \bigcap_{k=1}^{t} \Omega_k \text{ and } \Omega_k = \Xi \cap \Pi_k.$$

*2) Note that for $\eta = 1, \ldots, N$, when $k \notin S^\eta$ we have $Y_k^\eta(x, y^{-\eta}, w^\eta) = \mathbb{R}^{m_\eta}$ and $f^\eta = 0$. In this case, the sets $\Omega_k$, $\Theta_k$ are well defined.*

To provide necessary optimalty condition for (SVBOP), we will make some hypotheses throughout the paper, as follows:

$(\mathcal{R}_1)$**:** The set $\Xi$ is regular and semismooth at $(\bar{x}, \bar{y}, \bar{w})$ such that

$$D^*\Upsilon_k(\bar{x}, \bar{y}, \bar{w})(\sigma) \cap (-\text{bd } N_\Xi(\bar{x}, \bar{y}, \bar{w}) = \emptyset, \text{ for all } \sigma \in N\left(\Upsilon_k(\bar{x}, \bar{y}, \bar{w}), \mathbb{R}^s_-\right) \setminus \{0\}.$$

$(\mathcal{R}_2)$**:** Suppose that

$$[x_1^* + \cdots + x_t^* = 0, x_k^* \in N_{\Omega_k}(\bar{x}, \bar{y}, \bar{w})] \Longrightarrow x_k^* = 0, \ k \in \mathcal{T}.$$

$(\mathcal{R}_3)$**:** The upper-level regularity of (SVBOP) at $\bar{x} \in X$ is given by

$$\left.\begin{array}{c} \sum_{i=1}^{p} \alpha_i \nabla G_i(\bar{x}) = 0 \\ \alpha_i \geq 0, \quad \alpha_i G_i(\bar{x}) = 0, \quad i \in I, \end{array}\right\} \Longrightarrow [\alpha_i = 0, i \in I].$$

$(\mathcal{R}_4)$**:** For $\eta = 1, \ldots, N$, the lower-level regularity of $(P^\eta[x, y^{-\eta}])$ at $(\bar{x}, \bar{y}^\eta) \in X \times Y^\eta(\bar{x}, \bar{y}^{-\eta})$

$$\left.\begin{array}{c} \sum_{j=1}^{q_\eta} \beta_j^\eta \nabla_{y^\eta} g_j^\eta(\bar{x}, \bar{y}) = 0, \\ \beta_j^\eta \geq 0, \quad \beta_j^\eta g_j^\eta(\bar{x}, \bar{y})) = 0, \quad j \in J^\eta, \end{array}\right\} \Longrightarrow \left[\beta_j^\eta = 0, j \in J^\eta.\right]$$

$(\mathcal{R}_5)$**:** For $\eta = 1, \ldots, N$, the lower-level regularity of $(P_k^\eta[x, y^{-\eta}, w^\eta])$ ,$k \in S^\eta$, at $(\bar{x}, \bar{y}^\eta, \bar{w}^\eta) \in X \times Y_k^\eta(\bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta) \times \mathbb{W}^\eta$.

- In case $s_\eta \geq 2$, $\eta = 1, \ldots, N$ it is given by

$$\left.\begin{array}{c} \sum_{j=1}^{q_\eta} \beta_j^\eta \nabla_{y^\eta} g_j^\eta(\bar{x}, \bar{y}) + \sum_{r \in S \setminus \{k\}} \gamma_r^k \nabla_{y^\eta} h_{k,r}^\eta(\bar{x}, \bar{y}, \bar{w}^\eta) = 0 \\ \beta_j^\eta \geq 0, \quad \beta_j^\eta g_j^\eta(\bar{x}, \bar{y}) = 0, \quad j \in J^\eta, \\ \gamma_r^k \geq 0, \gamma_r^k h_{k,r}^\eta(\bar{x}, \bar{y}, \bar{w}^\eta) = 0, r \in S^\eta \setminus \{k\} \end{array}\right\}$$
$$\Longrightarrow \left[\gamma_r^k = 0, \beta_j^\eta = 0, r \in S^\eta \setminus \{k\}, j \in J^\eta\right]$$

In case $s_\eta = 1$, $\eta = 1, \ldots, N$ it is given by

$$\left.\begin{array}{c} \sum\limits_{j=1}^{q_\eta} \beta_j^\eta \nabla_{y^\eta} g_j^\eta(\bar{x}, \bar{y}) = 0, \\ \beta_j^\eta \geq 0, \quad \beta_j^\eta g_j^\eta(\bar{x}, \bar{y})) = 0, \quad j \in J^\eta, \end{array}\right\} \Longrightarrow \left[\beta_j^\eta = 0, j \in J^\eta.\right]$$

**Theorem 3.10.** *Let $(\bar{x}, \bar{y}, \bar{w})$ be a local optimal solution of problem (3.11). For $\eta = 1, \ldots, N$, assume that*

$(\mathcal{T}_1)$ *The functions $f^\eta$ and $g^\eta$ are strictly differentiable at $(\bar{x}, y^\eta)$ for all $y^\eta \in \bigcap\limits_{k=1}^{s} \Psi_k^\eta(\bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta)$.*

$(\mathcal{T}_2)$ *The functions $F$ and $G$ are strictly differentiable at $(\bar{x}, \bar{y})$ and $\bar{x}$, respectively.*

$(\mathcal{T}_3)$ *The solution set mappings $\Psi_k^\eta, k \in S^\eta$ are inner semicompact at $(\bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta)$ and for all $y^\eta \in \bigcap\limits_{k=1}^{s} \Psi_k^\eta(\bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta)$ the point $(\bar{x}, y^\eta, \bar{w}^\eta)$ is lower level regular.*

$(\mathcal{T}_4)$ *The hypotheses $(\mathcal{R}_1)$, $(\mathcal{R}_2)$, $(\mathcal{R}_3)$ and $(\mathcal{R}_4)$ are satisfied at $(\bar{x}, \bar{y}, \bar{w})$, $(\bar{x}, \bar{y}, \bar{w})$, $\bar{x}$ and $(\bar{x}, \bar{y}^\eta)$ respectively.*

*Then there are scalars $\sigma_k^\eta \in \mathbb{R}_+^{s_\eta}$, $\alpha_i \in \mathbb{R}_+^p$, $\beta_{k,l}^\eta, \beta_j^\eta \in \mathbb{R}_+^{q_\eta}$, $\pi_{k,l}^\eta \geq 0, \gamma_{k,l}^\eta \in \mathbb{R}_+^{s_\eta}$ and $y_{k,l}^\eta \in \Psi_k^\eta\left(\bar{x}, y_{l,k}^{-\eta}, \bar{w}^\eta\right)$, with $k \in S^\eta$ and $l = 1, \ldots, n+1$ such that $\sum\limits_{l=1}^{n+1} \pi_{l,k}^\eta = 1$ is satisfied with*

$$\begin{aligned} 0 \;=\; & \nabla_x F(\bar{x}, \bar{y}) + \sum_{\eta=1}^{N} \left( \sum_{k=1}^{s_\eta} \left( \sum_{r \in S^\eta \setminus \{k\}} \sigma_{k,r}^\eta \left( \bar{w}_r^\eta \nabla_x f_r^\eta(\bar{x}, \bar{y}) - \bar{w}_k^\eta \nabla_x f_k^\eta(\bar{x}, \bar{y}) \right) \right) \right) \\ & + \sum_{\eta=1}^{N} \left( \sum_{k=1}^{s_\eta} \sigma_k^\eta \left[ \nabla_x f_k^\eta(\bar{x}, \bar{y}) - \sum_{l=1}^{n+1} \pi_{k,l}^\eta \left( \nabla_x f_k^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right) + \sum_{j=1}^{q_\eta} \beta_{j,k}^{\eta,l} \nabla_x g_j^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right) \right.\right.\right. \\ & \left.\left.\left. + \sum_{r \in S^\eta \setminus \{k\}} \gamma_{k,r}^{\eta,l} \left( \bar{w}_r^\eta \nabla_x f_r^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right) - \bar{w}_k^\eta \nabla_x f_k^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right) \right) \right) \right] + \sum_{j=1}^{q_\eta} \beta_j^\eta \nabla_x g_j^\eta(\bar{x}, \bar{y}) \right) \\ & + \sum_{i=1}^{p} \alpha_i \nabla G_i(\bar{x}), \end{aligned}$$

$$\begin{aligned} 0 \;=\; & \nabla_y F(\bar{x}, \bar{y}) + \sum_{\eta=1}^{N} \left[ \sum_{k=1}^{s_\eta} \left( \sum_{r \in S^\eta \setminus \{k\}} \sigma_{k,r}^\eta \left( \bar{w}_r^\eta \nabla_{y^\eta} f_r^\eta(\bar{x}, \bar{y}) - \bar{w}_k^\eta \nabla_{y^\eta} f_k^\eta(\bar{x}, \bar{y}) \right) \right) \right. \\ & \left. + \sum_{k=1}^{s_\eta} \sigma_k^\eta \; \nabla_{y^\eta} f_k^\eta(\bar{x}, \bar{y}) + \sum_{j=1}^{q_\eta} \beta_j^\eta \nabla_{y^\eta} g_j^\eta(\bar{x}, \bar{y}) \right], \end{aligned}$$

$$\begin{aligned} 0 &= \nabla_{y^\eta} f_k^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right) + \sum_{j=1}^{q_\eta} \beta_{j,k}^{\eta,l} \nabla_{y^\eta} g_j^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right) + \sum_{r \in S^\eta \backslash \{k\}} \gamma_{k,r}^{\eta,l}\left(\bar{w}_r^\eta \nabla_{y^\eta} f_r^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right)\right. \\ &\quad \left. - \bar{w}_k^\eta \nabla_{y^\eta} f_k^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right)\right), \quad \eta = 1, \ldots, N, \forall k \in S^\eta, \forall l = 1, \ldots, n+s+1. \end{aligned}$$

$$\begin{aligned} 0 &= \gamma_{k,r}^{\eta,l}\left(\bar{w}_r^\eta f_r^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right) - \bar{w}_k^\eta f_k^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right)\right), \\ 0 &= \beta_{j,k}^{\eta,l} g_j^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right), \quad \eta = 1, \ldots, N, \ \forall\ k \in S^\eta, \forall\ j \in J^\eta, \forall\ r \in S^\eta \backslash \{k\}, \forall\ l = 1, \ldots, n+s+1. \end{aligned}$$

$$0 = \beta_j^\eta g_j^\eta\left(\bar{x}, \bar{y}\right), \quad 0 = \alpha_i G_i\left(\bar{x}\right), \quad \eta = 1, \ldots, N, \ \forall\ j \in J^\eta, \forall\ i \in I.$$

$$0 = \gamma_{k,r}^\eta\left(\bar{w}_r^\eta f_r^\eta\left(\bar{x}, y_k^\eta, \bar{y}^{-\eta}\right) - \bar{w}_k^\eta f_k^\eta\left(\bar{x}, y_k^\eta, \bar{y}^{-\eta}\right)\right), \quad \eta = 1, \ldots, N, \ \forall\ r \in S^\eta \backslash \{k\}, \forall\ k \in S^\eta.$$

$$0 = \sigma_{k,r}^\eta\left(\bar{w}_r^\eta f_r^\eta\left(\bar{x}, \bar{y}\right) - \bar{w}_k^\eta f_k^\eta\left(\bar{x}, \bar{y}\right)\right), \quad \eta = 1, \ldots, N, \ \forall\ r \in S^\eta \backslash \{k\}, \forall\ k \in S^\eta.$$

*Proof.* Let $(\bar{x}, \bar{y}, \bar{w})$ be a local optimal solution of problem (3.11). Hence, from [53, Proposition 5.1], and using the strict differentiability assumption of the function F at $(\bar{x}, \bar{y})$, we have

$$0 \in \nabla F(\bar{x}, \bar{y}) + N_\Omega\left(\bar{x}, \bar{y}, \bar{w}\right).$$

According to the regularity hypothesis $(\mathcal{R}_2)$, using [54, Corollary 3.37], we get

$$0 \in \nabla F(\bar{x}, \bar{y}) + \sum_{k=1}^{t} N_{\Omega_k}\left(\bar{x}, \bar{y}, \bar{w}\right).$$

Using the inner semicompactness of $\Psi_k^\eta, \eta = 1, \ldots, N,\ k \in S^\eta$ at $(\bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta)$, and the lower level regularity $(\mathcal{R}_5)$ at $(\bar{x}, y^\eta, \bar{w})$ for all $y^\eta \in \Psi_k^\eta\left(\bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta\right),\ \eta = 1, \ldots, N$, we deduce from [55, Theorem 5.2(ii)], that for $\eta = 1, \ldots, N,\ \varphi_k^\eta$ and $\eta_k$ are Lipschitz continuity around $(\bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta)$ and $(\bar{x}, \bar{y}, \bar{w})$ respectively. Hence, applying [25, Lemma 3.3] there exist $\sigma_k \geqslant 0$ with $\sigma_k = \left(\sigma_k^1, \ldots, \sigma_k^N\right)$, $k \in \mathbb{T}$ where $\left(\sigma_k^1, \ldots, \sigma_k^N\right) = \left(\left(\sigma_{k,1}^1, \ldots, \sigma_{k,s_1}^1\right), \ldots, \left(\sigma_{k,1}^N, \ldots, \sigma_{k,s_N}^N\right)\right)$ such that

$$0 \in \nabla F(\bar{x}, \bar{y}) + \sum_{k=1}^{t} \partial\langle \sigma_k, \Upsilon_k \rangle\left(\bar{x}, \bar{y}, \bar{w}\right) + N_\Xi\left(\bar{x}, \bar{y}, \bar{w}\right),$$

while taking into account the regularity and semi-smoothness property of $\Xi$, the satisfaction of $(\mathcal{R}_1)$ at $(\bar{x}, \bar{y}, \bar{w})$ and the fact that $\Omega_k = \Xi \cap \Pi_k$,(see Remarks 3.9).

Let us first calculate $A = \sum_{k=1}^{s} \partial\langle \sigma_k, \Upsilon_k \rangle (\bar{x}, \bar{y}, \bar{w})$. From [41, Theorem 4.1] one has

$$\partial\langle \sigma_k, \Upsilon_k \rangle (\bar{x}, \bar{y}, \bar{w}) = \sum_{\eta=1}^{N} \sum_{r \in S^\eta \backslash \{k\}} \sigma_{k,r}^{\eta} \nabla_{x,y^\eta,w^\eta} h_{k,r}^{\eta} (\bar{x}, \bar{y}, \bar{w}^\eta) + \sigma_k^\eta \partial_{x,y^\eta,w^\eta} \psi_{k,k}^{\eta} (\bar{x}, \bar{y}, \bar{w}^\eta).$$

By summing over $k$, we obtain

$$A = \sum_{k=1}^{t} \Big[ \sum_{\eta=1}^{N} \sum_{r \in S^\eta \backslash \{k\}} \sigma_{k,r}^{\eta} \nabla_{x,y^\eta,w^\eta} h_{k,r}^{\eta} (\bar{x}, \bar{y}, \bar{w}^\eta) + \sigma_k^\eta \partial_{x,y^\eta,w^\eta} \psi_{k,k}^{\eta} (\bar{x}, \bar{y}, \bar{w}^\eta) \Big].$$

Since $t = \max_{\eta=1}^{N} s_\eta$, reordering the summation, we get

$$A = \sum_{\eta=1}^{N} \left( \sum_{k=1}^{s_\eta} \left( \left( \sum_{r \in S^\eta \backslash \{k\}} \sigma_{k,r}^{\eta} \nabla_{x,y^\eta,w^\eta} h_{k,r}^{\eta} (\bar{x}, \bar{y}, \bar{w}^\eta) \right) + \sigma_k^\eta \partial_{x,y^\eta,w^\eta} \psi_k^\eta (\bar{x}, \bar{y}, \bar{w}^\eta) \right) \right).$$

Consequently

$$\begin{aligned} 0 \in \nabla F(\bar{x}, \bar{y}) \; + \; & \sum_{\eta=1}^{N} \left( \sum_{k=1}^{s_\eta} \left( \left( \sum_{r \in S^\eta \backslash \{k\}} \sigma_{k,r}^{\eta} \nabla_{x,y^\eta,w^\eta} h_{k,r}^{\eta} (\bar{x}, \bar{y}, \bar{w}^\eta) \right) \right. \right. \\ & \left. \left. + \sigma_k^\eta \partial_{x,y^\eta,w^\eta} \psi_k^\eta (\bar{x}, \bar{y}, \bar{w}^\eta) \right) \right) + N_\Xi (\bar{x}, \bar{y}, \bar{w}), \end{aligned} \tag{3.12}$$

and

$$\sigma_k^\eta \geq 0, \; \sigma_{r,k}^\eta \geq 0 \quad h_{k,r}^{\eta} (\bar{x}, \bar{y}, \bar{w}^\eta) = 0, \qquad \eta = 1, \ldots, N, \; \forall \; r \in S^\eta \backslash \{k\}.$$

For any $\eta = 1, \ldots, N$, $k \in S^\eta$ and $r \in S^\eta \backslash \{k\}$ we have

$$\nabla h_{k,r}^{\eta} (\bar{x}, \bar{y}, \bar{w}^\eta) = \left\{ \begin{array}{l} \bar{w}_r^\eta \nabla_x f_r^\eta (\bar{x}, \bar{y}) - \bar{w}_k^\eta \nabla_x f_k^\eta (\bar{x}, \bar{y}) \\ \bar{w}_r^\eta \nabla_{y^\eta} f_r^\eta (\bar{x}, \bar{y}) - \bar{w}_k^\eta \nabla_{y^\eta} f_k^\eta (\bar{x}, \bar{y}) \\ \nabla_{w^\eta} h_{k,r}^{\eta} (\bar{x}, \bar{y}, \bar{w}^\eta) \end{array} \right\}. \tag{3.13}$$

Now, let $\Phi(x, y, w) = [G_i(x), \; i \in I, \; g_j^\eta(x, y), \eta = 1, \ldots, N, j \in J^\eta, \; -w]^\top$, and $\phi(x, y, w) = w^\top e_{\mathbb{R}^s} - 1$. Starting with $N_\Xi(\bar{x}, \bar{y}, \bar{w})$, let us note that $\Xi$ can be rewritten as $\Xi = \{(x, y, w) : \Phi(x, y, w) \leq 0, \phi(x, y, w) = 0\}$. Now, the fulfillment of both the upper-level ($\mathcal{R}_3$) and lower-level ($\mathcal{R}_4$) regularity conditions implies the satisfaction of the following implication:

$$\left. \begin{array}{r} \langle u, \nabla \Phi(\bar{x}, \bar{y}, \bar{w}) \rangle + \langle v, \nabla \phi(\bar{x}, \bar{y}, \bar{w}) \rangle = 0 \\ u \geq 0, \langle u, \Phi(\bar{x}, \bar{y}, \bar{w}) \rangle = 0 \end{array} \right\} \Longrightarrow [u = 0, v = 0].$$

Hence, from [59, Theorem 6.14], and for $\eta = 1, \ldots, N$, we obtain

$$N_{\Xi}(\bar{x}, \bar{y}, \bar{w}) \subseteq \left\{ \begin{array}{c} \left( \begin{array}{c} \sum_{i=1}^{p} \alpha_i \nabla G_i(\bar{x}) + \sum_{\eta=1}^{N} \sum_{j=1}^{q_\eta} \beta_j^\eta \nabla_x g_j^\eta(\bar{x}, \bar{y}) \\ \sum_{\eta=1}^{N} \sum_{j=1}^{q_\eta} \beta_j^\eta \nabla_{y^\eta} g_j^\eta(\bar{x}, \bar{y}) \\ -\nu + \iota e \end{array} \right) : \\ \mu_r \geq 0, \ \mu_r \bar{w}_r = 0, \quad r \in S^\eta \\ \alpha_i \geq 0, \quad \alpha_i G_i(\bar{x}) = 0, \quad i \in I, \\ \eta = 1, \ldots, N, \ \beta_j^\eta \geq 0, \quad \beta_j^\eta g_j^\eta(\bar{x}, \bar{y}) = 0, \quad j \in J^\eta. \end{array} \right\} \tag{3.14}$$

On the other hand, for $\eta = 1, \ldots, N$, we have the following convex hull property:

$$\partial \left( -\varphi_k^\eta \left( \bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta \right) \right) \subseteq co \left( -\partial \varphi_k^\eta \left( \bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta \right) \right) \subseteq \ -co \ \partial \varphi_k^\eta \left( \bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta \right).$$

Then

$$\partial \ \psi_k^\eta \left( \bar{x}, \bar{y}, \bar{w}^\eta \right) \subseteq \left\{ \begin{array}{c} \nabla_x f_k^\eta(\bar{x}, \bar{y}) \\ \nabla_{y^\eta} f_k^\eta(\bar{x}, \bar{y}) \\ 0 \end{array} \right\} - co \ \partial_{x, w^\eta} \varphi_k^\eta \left( \bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta \right) \times \{0_{\mathbb{R}^{m_\eta}}\}.$$

According to [56, Theorem 7], because the solution set-valued mappings $\Psi_k^\eta, k \in S^\eta, \ \eta = 1, \ldots, N$, are inner semicompact at $(\bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta)$ and the point $(\bar{x}, y_k^\eta, \bar{w}^\eta)$ is lower-level regular for all $y_k^\eta \in \Psi_k^\eta \left( \bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta \right), \ k \in S^\eta$, we get

$$\partial \varphi_k^\eta \left( \bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta \right) \subset \bigcup_{y_k^\eta \in \Psi_k^\eta(\bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta)} \ \bigcup_{(\beta_k^\eta, \gamma_k^\eta) \in \Lambda_k^\eta \left( \bar{x}, y_k^\eta, \bar{y}^{-\eta}, \bar{w}^\eta \right)} \tag{3.15}$$

$$\left\{ \begin{array}{c} \left( \begin{array}{c} \nabla_x f_k^\eta \left( \bar{x}, y_k^\eta, \bar{y}^{-\eta} \right) + \sum_{j=1}^{q_\eta} \beta_{j,k}^\eta \nabla_x g_j^\eta(\bar{x}, y_k^\eta, \bar{y}^{-\eta}) \\ \sum_{r \in S^\eta \setminus \{k\}} \gamma_{k,r}^\eta \nabla_{w^\eta} h_{k,r}^\eta \left( \bar{x}, y_k^\eta, \bar{y}^{-\eta}, \bar{w}^\eta \right) \end{array} \right) + \\ \left[ \begin{array}{c} \sum_{r \in S^\eta \setminus \{k\}} \gamma_{k,r}^\eta \left( \bar{w}_r^\eta \nabla_x f_r^\eta \left( \bar{x}, y_k^\eta, \bar{y}^{-\eta} \right) - \bar{w}_k^\eta \nabla_x f_k^\eta \left( \bar{x}, y_k^\eta, \bar{y}^{-\eta} \right) \right) \\ 0 \end{array} \right] \end{array} \right\} \tag{3.16}$$

where $\Lambda_k^\eta\left(\bar{x}, y_k^\eta, y_k^{-\eta}, \bar{w}^\eta\right)$ is the collection of lower-level Lagrange multipliers defined by

$$\Lambda_k^\eta\left(\bar{x}, y_k^\eta, \bar{y}^{-\eta}, \bar{w}^\eta\right) = \left\{ \begin{array}{c} (\beta_k^\eta, \gamma_k^\eta) \in \mathbb{R}^{s_\eta + q_\eta} : \\ \beta_{j,k}^\eta \geq 0,\ \beta_{j,k}^\eta \nabla_{y^\eta} g_j^\eta(\bar{x}, y_k^\eta, \bar{y}^{-\eta}) = 0, \quad \eta = 1, \ldots, N, j \in J^\eta, \\ \gamma_{k,r}^\eta \geq 0,\ \gamma_{k,r}^\eta\ (\bar{w}_r^\eta \nabla_{y^\eta} f_r^\eta\left(\bar{x}, y_k^\eta, \bar{y}^{-\eta}\right) - \bar{w}_k^\eta \nabla_{y^\eta} f_k^\eta\left(\bar{x}, y_k^\eta, \bar{y}^{-\eta}\right)) = 0, \\ \eta = 1, \ldots, N,\ r \in S^\eta \backslash \{k\}, \\ \nabla_{y^\eta} f_k^\eta\left(x, y_k^\eta, \bar{y}^{-\eta}\right) + \sum_{j=1}^{q_\eta} \beta_j^{\eta,k} \nabla_{y^\eta} g_j^\eta(\bar{x}, y_k^\eta, \bar{y}^{-\eta}) \\ + \sum_{r \in S^\eta \backslash \{k\}} \gamma_{k,r}^\eta\left(\bar{w}_r^\eta \nabla_{y^\eta} f_r^\eta\left(\bar{x}, y_k^\eta, \bar{y}^{-\eta}\right) - \bar{w}_k^\eta \nabla_{y^\eta} f_k^\eta\left(\bar{x}, y_k^\eta, \bar{y}^{-\eta}\right)\right) = 0 \end{array} \right\} \tag{3.17}$$

For any pair $\eta = 1, \ldots, N$, $k \in S^\eta$ and $(a_k^\eta, b_k^\eta) \in \operatorname{co} \partial \varphi_k^\eta\left(\bar{x}, \bar{y}^{-\eta}, \bar{w}^\eta\right)$. Now, Carathéodory's theorem gives us from (3.27), and (3.17) $\pi_{k,l}^\eta$, $\gamma_{r,k}^{\eta,l}$, $r \in S^\eta$, $\beta_j^{\eta,l}$, $j \in J^\eta$, with $l = 1, \cdots n+1$, such that

$$\sum_{l=1}^{n+1} \pi_{k,l}^\eta = 1 \tag{3.18}$$

$$b_k^\eta = \sum_{l=1}^{n+1} \pi_{k,l}^\eta \sum_{r \in S^\eta \backslash \{k\}} \gamma_{k,r}^{\eta,l}\ \nabla_w h_{k,r}^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}, \bar{w}^\eta\right). \tag{3.19}$$

$$\begin{aligned} a_k^\eta &= \sum_{l=1}^{n+1} \pi_{k,l}^\eta \left( \nabla_x f_k^\eta(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}) + \sum_{j=1}^{q_\eta} \beta_{j,k}^{\eta,l} \nabla_x g_j^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right) \right. \\ &\quad \left. + \sum_{r \in S^\eta \backslash \{k\}} \gamma_{k,r}^{\eta,l}\left(\bar{w}_r^\eta \nabla_x f_r^\eta\left(\bar{x}, y_{k,l}^\eta, y_{k,l}^{-\eta}\right) - \bar{w}_k^\eta \nabla_x f_k^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right)\right) \right). \end{aligned}$$

$$\begin{aligned} 0 &= \nabla_{y^\eta} f_k^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right) + \sum_{j=1}^{q_\eta} \beta_{j,k}^{\eta,l} \nabla_{y^\eta} g_j^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right) \\ &\quad + \sum_{r \in S^\eta \backslash \{k\}} \gamma_{k,r}^{\eta,l}\left(\bar{w}_r^\eta \nabla_{y^\eta} f_r^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right) - \bar{w}_k^\eta \nabla_{y^\eta} f_k^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right)\right) \end{aligned} \tag{3.20}$$

$$\begin{aligned} 0 &= \beta_{j,k}^{\eta,l} \nabla_{y^\eta} g_j^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right),\ \beta_{j,k}^{\eta,l} \geqslant 0, \\ 0 &= \gamma_{k,r}^{\eta,l}\left(\bar{w}_r^\eta \nabla_{y^\eta} f_r^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right) - \bar{w}_k^\eta \nabla_{y^\eta} f_k^\eta\left(\bar{x}, y_{k,l}^\eta, \bar{y}^{-\eta}\right)\right),\ \gamma_{k,r}^{\eta,l} \geqslant 0. \end{aligned} \tag{3.21}$$

Combining (3.24),(3.13)-(3.26)-(3.27)-(3.29)-(3.21), we obtain the result. □

In this paragraph, we give necessary optimality conditions in the case where the lower-level generalized Nash equilibrium problem has a single cost $s_\eta = 1$ and $\eta = 1, 2$.

**Remark 3.11.**

$$\begin{cases} \min\limits_{y^\eta} \ f^\eta\left(x, y^\eta, y^{-\eta}\right) \\ \text{s.t } y^\eta \in \Lambda^\eta\left(x, y^{-\eta}\right). \end{cases} \qquad (P_1^\eta[x, y^{-\eta}])$$

where the solution set of problem $(P_1^\eta[x, y^{-\eta}])$ for any $y^{-\eta}$ is denoted by

$$\Psi^\eta\left(x, y^{-\eta}\right) := \left\{y^\eta \in \Lambda^\eta\left(x, y^{-\eta}\right) : \ f^\eta\left(x, y^\eta, y^{-\eta}\right) - \varphi^\eta\left(x, y^{-\eta}\right) \leq 0, \right\}, \qquad (3.22)$$

with

$$V\left(x, y\right) = f^\eta\left(x, y^\eta, y^{-\eta}\right) - \varphi^\eta\left(x, y^{-\eta}\right), \quad \varphi^\eta\left(x, y^{-\eta}\right) = \min_{y^\eta}\left\{f^\eta\left(x, y^\eta, y^{-\eta}\right) : y^\eta \in \Lambda^\eta\left(x, y^{-\eta}\right)\right\}.$$

**Theorem 3.12.** *For $\eta = 1, 2$, assume that $(\bar{x}, \bar{y})$ is a local optimal solution of problem (3.11).*

**(1)** *The functions $f^\eta$ and $g^\eta$ be strictly differentiable at $(\bar{x}, y^\eta)$ for all $y^\eta \in \Psi^\eta\left(\bar{x}, y^{-\eta}\right)$, whereas $F$ and $G$ are strictly differentiable at $(\bar{x}, \bar{y})$ and $\bar{x}$, respectively.*

**(2)** *The solution set mappings $\Psi^\eta$, are inner semicompact at $\bar{x}$ while for all $y^\eta \in \Psi^\eta\left(\bar{x}, y^{-\eta}\right)$ the point $(\bar{x}, y^\eta)$ is lower level regular $\mathcal{H}_6$.*

**(3)** *The hypotheses $(\mathcal{H}_3)$, $(\mathcal{H}_4)$ and $(\mathcal{H}_5)$ are satisfied at $(\bar{x}, \bar{y}, \bar{z})$ and $\bar{x}$, respectively.*

*Then, there exist scalars $\sigma^\eta \in \mathbb{R}_+$, $\alpha_i \in \mathbb{R}_+^p$, $\beta_l^\eta, \beta_j^\eta \in \mathbb{R}_+^{q_\eta}$, $\pi_l^\eta \geq 0, \gamma_l^\eta \in \mathbb{R}_+$ and $y_{k,l}^\eta \in \Psi^\eta\left(\bar{x}, y^{-\eta}\right)$, with $l = 1, \ldots, n+1$ such that $\sum\limits_{l=1}^{n+1} \pi_l^\eta = 1$ is satisfied with*

$$\begin{aligned} 0 \ = \ & \nabla_x F(\bar{x}, \bar{y}) + \sigma^\eta \left[\nabla_x f^\eta(\bar{x}, \bar{y}) - \sum_{l=1}^{n+1} \pi_l^\eta \left(\nabla_x f^\eta\left(\bar{x}, y_l\right) + \sum_{j=1}^{q_\eta} \beta_j^{\eta,l} \nabla_x g_j^\eta\left(\bar{x}, y_l\right)\right)\right] \\ & + \sum_{i=1}^{p} \alpha_i \nabla G_i(\bar{x}) + \sum_{j=1}^{q_\eta} \beta_j^\eta \nabla_x g_j^\eta(\bar{x}, \bar{y}), \end{aligned}$$

$$
\begin{aligned}
0 &= \nabla_y F(\bar{x},\bar{y}) + \sigma^{\eta}\ \nabla_{y^{\eta}} f^{\eta}(\bar{x},\bar{y}) + \sum_{j=1}^{q_{\eta}} \beta_j^{\eta} \nabla_{y^{\eta}} g_j^{\eta}(\bar{x},\bar{y}),\\
0 &= \nabla_{y^{\eta}} f^{\eta}\left(\bar{x}, y_l\right) + \sum_{j=1}^{q_{\eta}} \beta_j^{\eta,l} \nabla_{y^{\eta}} g_j^{\eta}\left(\bar{x}, y_l\right),\ \eta = 1,\ldots,N\\
0 &= \beta_j^{l} g_j^{\eta}\left(\bar{x}, y_l\right), \qquad \forall\ j \in J^{\eta}, \forall\ l = 1,\ldots,n+s_{\eta}+1.\\
0 &= \beta_j^{\eta} g_j^{\eta}\left(\bar{x},\bar{y}\right),\ \ 0 = \alpha_i G_i\left(\bar{x}\right), \qquad \forall\ j \in J^{\eta}, \forall\ i \in I,\ \eta = 1,\ldots,N.
\end{aligned}
$$

*Proof.* Let $(\bar{x},\bar{y})$ be a local optimal solution of problem (3.11). Hence, from the strict differentiability assumption of the function F at $(\bar{x},\bar{y})$. By [53, Proposition 5.1], we have

$$0 \in \nabla F(\bar{x},\bar{z}) + N_{\Omega^{\eta}}\left(\bar{x},\bar{y}\right). \tag{3.23}$$

The Local Lipschitz continuity of all functions and inner semicompactness of $\Psi^{\eta}$ at $\bar{x}$, and the lower level regularity $(\mathcal{H}_3)$ at $(\bar{x},\bar{y}^{\eta})$ for all $y^{\eta} \in \Psi^{\eta}\left(x, y^{-\eta}\right)$, it follows from [55, Theorem 5.2(ii)], for $\eta = 1,\ldots,N$ we obtain the Lipschitz continuity of $\varphi^{\eta}$ and $\Psi^{\eta}$, around $(\bar{x},\bar{z})$ and $(\bar{x},\bar{y})$. Hence, applying [25, Lemma 3.3] there exist scalars $\sigma^{\eta} \geqslant 0$ such that

$$0 \in \nabla F(\bar{x},\bar{z}) + \sigma^{\eta}\ \partial V^{\eta}\left(\bar{x},\bar{y}\right) + N_{\Xi}\left(\bar{x},\bar{y}\right). \tag{3.24}$$

Let $\Upsilon(x,y,z) = [G_i(x),\ i \in I,\ g_j^{\eta}(x,y), j \in J^{\eta}$. Starting with $N_{\Xi}(\bar{x},\bar{y})$, let us note that $\Xi$ can be rewritten as $\Xi = \{(x,y) : \Upsilon(x,y,z) \leq 0\}$. Now, the fulfillment of both the lower-level $(\mathcal{H}_5)$ and upper-level $(\mathcal{H}_6)$ regularity conditions implies the satisfaction of the following implication.

$$\left.\begin{array}{r} \langle u, \nabla\Upsilon(\bar{x},\bar{y})\rangle = 0 \\ u \geq 0, \langle u, \nabla\Upsilon(\bar{x},\bar{y})\rangle = 0 \end{array}\right\} \Longrightarrow [u = 0].$$

Hence, from [59, Theorem 6.14],and for $\eta = 1,\ldots,N$ we obtain

$$N_{\Xi}(\bar{x},\bar{y},\bar{z}) \subseteq \left\{ \begin{array}{c} \left( \begin{array}{c} \sum_{i=1}^{p} \alpha_i \nabla G_i(\bar{x}) + \sum_{j=1}^{q_{\eta}} \beta_j^{\eta} \nabla_x g_j^{\eta}(\bar{x},\bar{y}) \\ \sum_{j=1}^{q_{\eta}} \beta_j^{\eta} \nabla_{y^{\eta}} g_j^{\eta}(\bar{x},\bar{y}) \end{array} \right) : \\ \alpha_i \geq 0, \quad \alpha_i G_i(\bar{x}) = 0, \quad i \in I, \\ \beta_j^{\eta} \geq 0, \quad \beta_j^{\eta} g_j^{\eta}(\bar{x},\bar{y}) = 0, \quad j \in J^{\eta}. \end{array} \right\} \tag{3.25}$$

On the other hand, for $\eta = 1, \ldots, N$ we have the following convex hull property

$$\partial\left(-\varphi^{\eta}\left(\bar{x}, \bar{y}^{-\eta}\right)\right) \subseteq co\left(-\partial\varphi^{\eta}\left(\bar{x}, \bar{y}^{-\eta}\right)\right) \subseteq \ -co\ \partial\varphi^{\eta}\left(\bar{x}, \bar{y}^{-\eta}\right).$$

Then

$$\partial\ V^{\eta}\left(\bar{x}, \bar{y}\right) \subseteq \left\{ \begin{array}{l} \nabla_x f^{\eta}(\bar{x}, \bar{y}) \\ \nabla_{y^{\eta}} f^{\eta}(\bar{x}, \bar{y}) \\ 0 \end{array} \right\} - co\ \partial\varphi^{\eta}\left(\bar{x}, \bar{y}^{-\eta}\right) \times \{0\}.$$

According to [56, Theorem 7], because the solution set-valued mappings $\Psi^{\eta},\ \eta = 1, \ldots, N$, are inner semicompact at $\bar{x}$ and the point $(\bar{x}, y)$ is lower-level regular for all $y^{\eta} \in \Psi^{\eta}\left(\bar{x}, y^{-\eta}\right)$, we get

$$\partial\varphi^{\eta}\left(\bar{x}, \bar{y}^{-\eta}\right) \subset \bigcup_{y_k^{\eta} \in \Psi^{\eta}(\bar{x}, \bar{y}^{-\eta})} \bigcup_{\beta^{\eta} \in \Lambda(\bar{x}, y)} \left\{ \nabla_x f^{\eta}\left(x, y\right) + \sum_{j=1}^{q_{\eta}} \beta_j^{\eta} \nabla_x g_j^{\eta}(\bar{x}, y) \right\} \tag{3.26}$$

where $\Lambda\left(\bar{x}, y\right)$ is the collection of lower-level Lagrange multipliers defined by

$$\Lambda\left(\bar{x}, y\right) = \left\{ \begin{array}{c} \beta^{\eta} \in \mathbb{R}^{q_{\eta}}: \\ \beta_j^{\eta, k} \geq 0,\ \beta_j^{\eta} \nabla_{y^{\eta}} g_j^{\eta}(\bar{x}, y) = 0, \quad j \in J^{\eta}, \\ \nabla_{y^{\eta}} f^{\eta}\left(x, y\right) + \sum_{j=1}^{q_{\eta}} \beta_j^{\eta} \nabla_{y^{\eta}} g_j^{\eta}(\bar{x}, y) = 0 \end{array} \right\} \tag{3.27}$$

For any $\eta$ and $a^{\eta} \in \operatorname{co} \partial\varphi^{\eta}\left(\bar{x}, \bar{y}^{-\eta}\right)$. Carathéodory's theorem give us from (3.26), and (3.27) we obtain $\pi_l^{\eta},\ \beta_j^{\eta, l},\ j \in J^{\eta}$, with $l = 1, \cdots n+1$, such that

$$\sum_{l=1}^{n+1} \pi_l^{\eta} = 1 \tag{3.28}$$

$$\begin{aligned} a^{\eta} &= \sum_{l=1}^{n+1} \pi_l^{\eta} \left( \nabla_x f^{\eta}(\bar{x}, y_l) + \sum_{j=1}^{q_{\eta}} \beta_{j,l}^{\eta} \nabla_x g_j^{\eta}\left(\bar{x}, y_l\right) \right) \\ 0 &= \nabla_{y^{\eta}} f^{\eta}\left(\bar{x}, y_l\right) + \sum_{j=1}^{q_{\eta}} \beta_{j,l}^{\eta} \nabla_{y^{\eta}} g_j^{\eta}\left(\bar{x}, y_l\right), \\ 0 &= \beta_{j,l}^{\eta} \nabla_{y^{\eta}} g_j^{\eta}\left(\bar{x}, y_l\right),\ \beta_{j,l}^{\eta} \geqslant 0. \end{aligned} \tag{3.29}$$

Combining (3.24),(3.13)-(3.26)-(3.27)-(3.29), we obtain the result. □

The upcoming example applies the preceding results to mathematical programs with multiobjective Nash equilibrium lower-level problems.

**Example 3.13.** Consider the mathematical program with multiobjective Nash equilibrium problem

$$\min_{x,y} F(x,y) = x^2 + \left(y^1\right)^2 : \ x \in X, \ \ y \in \mathcal{H}_{\text{weff}}(x), \tag{SVBOP}$$

with $y = (y^1, y^2)$.

The leader's decision variables are represented by $x \in X = \{x \in \mathbb{R} : G_1(x) = -x \leq 0\} = \mathbb{R}_+$, and $\mathcal{H}_{\text{weff}}(x)$ is the set of weakly efficient generalized Nash equilibrium of the multiobjective generalized Nash equilibrium problem parameterized by $x$ composed by two players represented below:

$$\begin{cases} \min\limits_{y^\eta} \ f^\eta\left(x, y^\eta, y^{-\eta}\right) \\ \text{s.t } y^\eta \in Y^\eta\left(x, y^{-\eta}\right). \end{cases} \tag{$P^\eta[x, y^{-\eta}]$}$$

For $\eta = 1$, we have

$$\begin{cases} \min\limits_{y^1} f^1(x, y^1; y^2) = (-y^1 - y^2, y^1 - y^2) \\ g^1(x, y^1, y^2) = (y^1 - 2; -y^1; -y^2 + 3) \leq 0, \end{cases} \tag{$P^1[x, y^2, w^1]$}$$

and for $\eta = 2$ we have

$$\begin{cases} \min\limits_{y^2} f^2(x, y^2, y^1) = (-2y^1 - y^2; -2y^1 + y^2), \\ g^2(x, y^1, y^2) = (y^2 - 3; -y^2; -y^1 + 2) \leq 0. \end{cases} \tag{$P^2[x, y^1, w^2]$}$$

Applying the weighted-sum technique, we get

$$\begin{cases} \min\limits_{y^1} f_1^1(x, y^1, y^2) = -y^1 - y^2, \\ y^1 \in Y_1^1(x, y^2, w_1^1, 1 - w_1^1), \\ y^2 \in [3, +\infty[. \end{cases} \quad \text{and} \quad \begin{cases} \min\limits_{y^1} f_2^1(x, y^1; y^2) = y^1 - y^2, \\ y^1 \in Y_2^1(x, y^2, w_1^1, 1 - w_1^1), \\ y^2 \in [3, +\infty[. \end{cases}$$

$$\begin{cases} \min\limits_{y^2} f_1^2(x, y^2, y^1) = -2y^1 - y^2, \\ y^2 \in Y_1^2(x, y^1, w_1^2, 1 - w_1^2), \\ y^1 \in [2, +\infty[. \end{cases} \quad \text{and} \quad \begin{cases} \min\limits_{y^2} f_2^2(x, y^2, y^1) = -2y^1 + y^2, \\ y^2 \in Y_1^2(x, y^1, w_1^2, 1 - w_1^2), \\ y^1 \in [2, +\infty[. \end{cases}$$

where

$$Y_1^1\left(x, y^2, w_1^1, 1-w_1^1\right) = \begin{cases} [0,2] & \text{if } 0 \leq w_1^1 < \dfrac{y^2-2}{2y^2} \\ [0, y^2(1-2w_1^1)] & \text{if } \dfrac{y^2-2}{2y^2} \leq w_1^1 \leq \dfrac{1}{2} \\ \emptyset & \text{if } \dfrac{1}{2} < w_1^1 \leq 1. \end{cases}$$

$$Y_2^1\left(x, y^2, w_1^1, 1-w_1^1\right) = \begin{cases} \emptyset & \text{if } 0 \leq w_2^1 < \dfrac{y^2-2}{2y^2}, \\ [y^2(1-2w_1^1), 2] & \text{if } \dfrac{y^2-2}{2y^2} \leq w_1^1 \leq \dfrac{1}{2}, \\ [0,1] & \text{if } \dfrac{1}{2} < w_1^1 \leq 1. \end{cases}$$

$$Y_1^2\left(x, y^1, w_1^2, 1-w_1^2\right) = \begin{cases} [0,3] & \text{if } 0 \leq w_1^2 < \dfrac{2y^1-3}{4y^1}, \\ [0, y^1(2-4w_1^2)] & \text{if } \dfrac{2y^1-3}{4y^1} \leq w_1^2 \leq \dfrac{1}{2}, \\ \emptyset & \text{if } \dfrac{1}{2} < w_1^2 \leq 1. \end{cases}$$

$$Y_2^2\left(x, y^1, w_1^2, 1-w_1^2\right) = \begin{cases} \emptyset & \text{if } 0 \leq w_2^2 < \dfrac{2y^1-3}{4y^1}, \\ [y^1(2-4w_1^2), 3] & \text{if } \dfrac{2y^1-3}{4y^1} \leq w_2^2 \leq \dfrac{1}{2}, \\ [0,3] & \text{if } \dfrac{1}{2} < w_2^2 \leq 1. \end{cases}$$

It is easy to check that the optimal values are, respectively:

$$\varphi_1^1\left(x, y^2, w_1^1, 1-w_1^1\right) = \begin{cases} -2-y^2 & \text{if } 0 \leq w_1^1 < \dfrac{y^2-2}{2y^2} \\ -y^2(1-2w_1^1)-y^2 & \text{if } \dfrac{y^2-2}{2y^2} \leq w_1^1 \leq \dfrac{1}{2} \\ +\infty & \text{if } \dfrac{1}{2} < w_1^1 \leq 1. \end{cases}$$

$$\varphi_2^1\left(x, y^2, w_1^1, 1-w_1^1\right) = \begin{cases} +\infty & \text{if } 0 \leq w_1^1 < \dfrac{y^2-2}{2y^2}, \\ y^2(1-2w_1^1)-y^2 & \text{if } \dfrac{y^2-2}{2y^2} \leq w_1^1 \leq \dfrac{1}{2}, \\ -y^2 & \text{if } \dfrac{1}{2} < w_1^1 \leq 1. \end{cases}$$

$$\varphi_1^2\left(x, y^1, w_1^2, 1-w_1^2\right) = \begin{cases} -3-2y^1 & \text{if } 0 \leq w_1^2 < \dfrac{2y^1-3}{4y^1}, \\ -y^1(2-4z_1^2)-2y^1 & \text{if } \dfrac{2y^1-3}{4y^1} \leq w_1^2 \leq \dfrac{1}{2}, \\ +\infty & \text{if } \dfrac{1}{2} < w_1^2 \leq 1. \end{cases}$$

$$\varphi_2^2\left(x, y^1, w_1^2, 1-w_1^2\right) = \begin{cases} +\infty & \text{if } 0 \leq w_1^2 < \dfrac{2y^1-3}{4y^1}, \\ y^1(2-4w_1^2)-2y^1 & \text{if } \dfrac{2y^1-3}{4y^1} \leq w_1^2 \leq \dfrac{1}{2}, \\ -2y^1 & \text{if } \dfrac{1}{2} < w_1^2 \leq 1, \end{cases}$$

supported by the solution sets provided by:

$$\Psi_1^1\left(x, y^2, w_1^1, 1-w_1^1\right) = \begin{cases} 2 & \text{if } 0 \leq w_1^1 < \dfrac{y^2-2}{2y^2} \\ y^2(1-2w_1^1) & \text{if } \dfrac{y^2-2}{2y^2} \leq w_1^1 \leq \dfrac{1}{2} \\ \emptyset & \text{if } \dfrac{1}{2} < w_1^1 \leq 1. \end{cases}$$

$$\Psi_2^1\left(x, y^2, w_1^1, 1-w_1^1\right) = \begin{cases} \emptyset & \text{if } 0 \leq w_1^1 < \dfrac{y^2-2}{2y^2}, \\ y^2(1-2w_1^1) & \text{if } \dfrac{y^2-2}{2y^2} \leq w_1^1 \leq \dfrac{1}{2}, \\ 0 & \text{if } \dfrac{1}{2} < w_1^1 \leq 1. \end{cases}$$

$$\Psi_1^2\left(x, y^1, w_1^2, 1-w_1^2\right) = \begin{cases} 3 & \text{if } 0 \leq w_1^2 < \dfrac{2y^1-3}{4y^1}, \\ y^1(2-4w_1^2) & \text{if } \dfrac{2y^1-3}{4y^1} \leq w_1^2 \leq \dfrac{1}{2}, \\ \emptyset & \text{if } \dfrac{1}{2} < w_1^2 \leq 1. \end{cases}$$

$$\Psi_2^2\left(x, y^2, w_1^2, 1-w_1^2\right) = \begin{cases} \emptyset & \text{if } 0 \leq w_1^2 < \dfrac{2y^1-3}{4y^1}, \\ y^1(2-4w_1^2) & \text{if } \dfrac{2y^1-3}{4y^1} \leq w_1^2 \leq \dfrac{1}{2}, \\ 0 & \text{if } \dfrac{1}{2} < w_1^2 \leq 1. \end{cases}$$

Hence, the optimization problem (SVBOP) with the multiobjective Nash equilibruim problem can be

replaced by the following optimization problem of the form:

$$\begin{cases} \min\limits_{x,y,w} F(x,y) = x^2 + \left(y^1\right)^2 \\ \text{s.t} \;\; x \in X, \; y \in \mathcal{H}(x,w). \end{cases} \tag{3.30}$$

with

$$\mathcal{H}(x,w) = \left\{ y = (y^1, y^2) : \; y^1 \in \Psi_1^1\left(x, y^2, w_1^1\right) \cap \Psi_2^1\left(x, y^2, w_2^1\right) \;\; , y^2 \in \Psi_1^2\left(x, y^1, w_1^2\right) \cap \Psi_2^2\left(x, y^2, w_2^2\right) \right\}$$

Using the optimal value reformulation, (3.30) is equivalent to the one-level optimization problem.

$$\begin{cases} \min\limits_{x,y,w} F(x,y) = x^2 + (y^1)^2 \\ (x, y, w_1^1, w_2^1, w_1^2, w_2^2) \in \Omega. \end{cases} \tag{3.31}$$

We remark that $\bar{u} = (\bar{x}, \bar{y}, \bar{w}_1^1, \bar{w}_2^1, \bar{w}_1^2, \bar{w}_2^2) = (0, 2, 3, \frac{1}{6}, \frac{5}{6}, \frac{1}{8}, \frac{7}{8})$ is a local optimal solution to problem (3.31). Moreover, the set $\Xi$ and the normal cone to $\Xi$ at $\bar{u} \in \Xi$ are given by:

$$\Omega = \Xi = [0, +\infty[ \times \{2\} \times \{3\} \times \left\{ \frac{1}{6}, \frac{5}{6}, \frac{1}{8}, \frac{7}{8} \right\}, \qquad N_\Xi(\bar{u}) = ]-\infty, 0] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

Let

$$\psi_1^1(x, y^1, y^2, w_1^1, 1 - w_1^1) = \begin{cases} -y^1 + 2 & \text{if } 0 \le w_1^1 < \dfrac{y^2 - 2}{2y^2} \\ -y^1 + y^2(1 - 2w_1^1) & \text{if } \dfrac{y^2 - 2}{2y^2} \le w_1^1 \le \dfrac{1}{2} \\ -\infty & \text{if } \dfrac{1}{2} < w_1^1 \le 1. \end{cases}$$

$$\psi_2^1(x, y^1, y^2, w_1^1, 1 - w_1^1) = \begin{cases} -\infty & \text{if } 0 \le w_1^1 < \dfrac{y^2 - 2}{2y^2}, \\ y^1 - y^2(1 - 2w_1^1) & \text{if } \dfrac{y^2 - 2}{2y^2} \le w_1^1 \le \dfrac{1}{2}, \\ y^1 & \text{if } \dfrac{1}{2} < w_1^1 \le 1. \end{cases}$$

$$\psi_1^2(x, y^1, y^2, w_1^2, 1 - w_1^2) = \begin{cases} -y^2 + 3 & \text{if } 0 \le w_1^2 < \dfrac{2y^1 - 3}{4y^1}, \\ -y^2 + y^1(2 - 4w_1^2) & \text{if } \dfrac{2y^1 - 3}{4y^1} \le w_1^2 \le \dfrac{1}{2}, \\ -\infty & \text{if } \dfrac{1}{2} < w_1^2 \le 1. \end{cases}$$

$$\psi_2^2\left(x,y^1,y^2,w_1^2,1-w_1^2\right) = \begin{cases} -\infty & \text{if } 0\leq w_1^2<\dfrac{2y^1-3}{4y^1},\\ y^2-y^1(2-4w_1^2) & \text{if } \dfrac{2y^1-3}{4y^1}\leq w_1^2\leq\dfrac{1}{2},\\ y^2 & \text{if } \dfrac{1}{2}<w_1^2\leq 1. \end{cases}$$

we have

$$D^*\psi\left(\bar{u}\right)(\sigma)=\left\{(0,-\sigma^1+\sigma^2+\frac{3}{2}\sigma^3-\frac{3}{2}\sigma^4,\frac{1}{2}\sigma^1-\frac{1}{2}\sigma^2-\sigma^3+\sigma^4,-6\sigma^1+6\sigma^2,0,-8\sigma^3+8\sigma^4,0)\right\}.$$

Consequently

$$D^*\psi\left(\bar{u}\right)(\sigma)\cap(-bd\ N_\Xi\left(\bar{x},\bar{y}\right))=\emptyset,$$

where $\sigma=(\sigma^1,\sigma^2,\sigma^3,\sigma^4)$ and

$$\begin{aligned}\psi(x,y^1,y^2,w_1^1,1-w_1^1,w_1^2,1-w_1^2)=&\Big(\psi_1^1\left(x,y^2,w_1^1,1-w_1^1\right),\psi_2^1\left(x,y^2,w_1^1,1-w_1^1\right),\\ &\psi_1^2\left(x,y^1,y^2,w_1^2,1-w_1^2\right),\psi_2^2\left(x,y^1,y^2,w_1^2,1-w_1^2\right)\Big).\end{aligned}$$

The set valued mappings $\Psi_1^1$,$\Psi_2^1$, $\Psi_1^2$,$\Psi_2^2$ are inner semicompact at $\left(0,2,3,\frac{1}{6},\frac{5}{6}\right)$, $\left(0,2,3,\frac{1}{6},\frac{5}{6}\right)$, $\left(0,2,3,\frac{1}{8},\frac{7}{8}\right)$ and $\left(0,2,3,\frac{1}{8},\frac{7}{8}\right)$ respectively, and the hypotheses $(\mathcal{R}_3)$, $(\mathcal{R}_4)$ and $(\mathcal{R}_5)$ hold true. Then, there exist $(\sigma_1^1,\sigma_2^1)=(\frac{1}{2},\frac{1}{2})$, $(\sigma_1^2,\sigma_2^2)=(\frac{1}{4},\frac{1}{4})$, $(\sigma_{1,2}^1,\sigma_{2,1}^1)=(\frac{1}{5},\frac{1}{5})$, $(\sigma_{1,2}^2,\sigma_{2,1}^2)=(1,1)$, $\alpha_1=0$, $(\beta_1^1,\beta_2^l)=(\frac{4}{3},\frac{1}{3})$, $(\beta_1^2,\beta_2^2)=(\frac{5}{3},\frac{1}{3})$, $(\gamma_{1,2}^{1,l},\gamma_{2,1}^{1,l})=(\frac{1}{3},\frac{1}{3})$, $(\gamma_{1,2}^{2,l},\gamma_{2,1}^{2,l})=(1,1)$, $(\beta_1^{1,l},\beta_2^{1,l})=(3,3)$, $(\beta_1^{2,l},\beta_2^{2,l})=(2,2)$ and $y_l^{1,l}=y_2^{1,l}=y_l^{2,l}=y_2^{2,l}=\{0\}$, with $l=1\ldots,4$, $\sum_{l=1}^{4}\pi_l^1=\sum_{l=1}^{4}\pi_l^2=1$ such that

$$\begin{aligned}0 &= \nabla_x F(0,0)+\sum_{\eta=1}^{2}\left(\sum_{k=1}^{2}\left(\sum_{r\in\{1,2\}\setminus\{k\}}\sigma_{k,r}^{\eta}\left(\bar{w}_r^{\eta}\nabla_x f_r^{\eta}(0,0)-\bar{w}_k^{\eta}\nabla_x f_k^{\eta}(0,0)\right)\right)\right)\\ &+\sum_{\eta=1}^{2}\left(\sum_{k=1}^{2}\sigma_k^{\eta}\left[\nabla_x f_k^{\eta}(0,0)-\sum_{l=1}^{2}\pi_{k,l}^{\eta}\left(\nabla_x f_k^{\eta}\left(0,0,0\right)+\sum_{j=1}^{2}\beta_{j,k}^{\eta,l}\nabla_x g_j^{\eta}\left(0,0,0\right)\right.\right.\right.\\ &\left.\left.\left.+\sum_{r\in\{1,2\}\setminus\{k\}}\gamma_{k,r}^{\eta,l}\ \left(\bar{w}_r^{\eta}\nabla_x f_r^{\eta}\left(0,0,0\right)-\ \bar{w}_k^{\eta}\nabla_x f_k^{\eta}\left(0,0,0\right)\right)\right)\right]+\sum_{j=1}^{2}\beta_j^{\eta}\nabla_x g_j^{\eta}(0,0)\right)+\alpha_1\nabla G_1(0),\end{aligned}$$

$$\begin{aligned} 0 &= \nabla_y F(0,0) + \sum_{\eta=1}^{2}\left[\sum_{k=1}^{2}\left(\sum_{r\in\{1,2\}\setminus\{k\}} \sigma_{k,r}^{\eta}\left(\bar{w}_r^{\eta}\nabla_{y^\eta} f_r^{\eta}(0,0) - \bar{w}_k^{\eta}\nabla_{y^\eta} f_k^{\eta}(0,0)\right)\right)\right. \\ &\quad \left. + \sum_{k=1}^{2}\sigma_k^{\eta}\,\nabla_{y^\eta} f_k^{\eta}(0,0) + \sum_{j=1}^{2}\beta_j^{\eta}\nabla_{y^\eta} g_j^{\eta}(0,0)\right], \end{aligned}$$

$$\begin{aligned} 0 &= \nabla_{y^\eta} f_k^{\eta}(0,0,0) + \sum_{j=1}^{2}\beta_{j,k}^{\eta,l}\nabla_{y^\eta} g_j^{\eta}(0,0,0) + \sum_{r\in\{1,2\}\setminus\{k\}} \gamma_{k,r}^{\eta,l}\left(\bar{w}_r^{\eta}\nabla_{y^\eta} f_r^{\eta}(0,0,0)\right. \\ &\quad \left. - \bar{w}_k^{\eta}\nabla_{y^\eta} f_k^{\eta}(0,0,0)\right), \quad \eta = 1,2, \forall k \in \{1,2\}, \forall l = 1,\ldots,4. \end{aligned}$$

$$\begin{aligned} 0 &= \gamma_{k,r}^{\eta,l}\left(\bar{w}_r^{\eta} f_r^{\eta}(0,0,0) - \bar{w}_k^{\eta} f_k^{\eta}(0,0,0)\right), \\ 0 &= \beta_{j,k}^{\eta,l} g_j^{\eta}(0,0,0), \quad \eta = 1,2,\ \forall\, k \in \{1,2\}, \forall\, j = 1,2, \forall\, r \in \{1,2\}\setminus\{k\}, \forall\, l = 1,\ldots,4. \end{aligned}$$

$$0 = \beta_j^{\eta} g_j^{\eta}(0,0), \quad 0 = \alpha_1 G_1(0), \quad \eta = 1,2,\ \forall\, j = 1,2.$$

$$0 = \gamma_{k,r}^{\eta}\left(\bar{w}_r^{\eta} f_r^{\eta}(0,0,0) - \bar{w}_k^{\eta} f_k^{\eta}(0,0,0)\right), \quad \eta = 1,2,\ \forall\, r \in \{1,2\}\setminus\{k\}, \forall\, k \in \{1,2\}.$$

$$0 = \sigma_{k,r}^{\eta}\left(\bar{w}_r^{\eta} f_r^{\eta}(0,0) - \bar{w}_k^{\eta} f_k^{\eta}(0,0)\right), \quad \eta = 1,2,\ \forall\, r \in \{1,2\}\setminus\{k\}, \forall\, k \in \{1,2\}.$$

# 4 Conclusion

In this chapter, we address a mathematical programming problem ($SVBOP$). We introduce a concept of equilibrium for generalized Nash equilibrium problems with multiple objectives and provide its scalarization formulation. Using this formulation, we examine the relationship between problem ($SVBOP$) and a related scalar programming problem (3.9), established by applying the kth-objective weighted-constraint technique to the multiobjective lower-level problem of ($SVBOP$). It has been proven that this relationship is stable when considering globally optimal solutions; however, the analysis of locally optimal solutions is more sensitive. We exploit the equivalence of ($SVBOP$) and (3.9) to derive new necessary optimality conditions. Finally, we illustrate our findings with an example.

Other approaches are the study of optimality conditions of the Karush-Kuhn-Tucker approximation and the Karush-Kuhn-Tucker and Charnes-Cooper reformulations of the semivectorial bilevel optimization problem.

## Chapter 4

# Approximate Weak Optimality Conditions in Multiobjective Generalized Nash Equilibrium Problems

## 1 Introduction

We are primarily interested in a multiobjective generalized Nash equilibrium problem (MGNEP). It is a problem in which, given $N$ players, each player $\eta = 1, \cdots, N$ is assumed to minimize

$$\begin{cases} \min\limits_{x^\eta} \ f^\eta\left(x^\eta, x^{-\eta}\right) = \left(f_1^\eta\left(x^\eta, x^{-\eta}\right), \cdots, f_{p_\eta}^\eta\left(x^\eta, x^{-\eta}\right)\right) \\ \text{s.t. } g_j^\eta\left(x^\eta, x^{-\eta}\right) \leq 0, \quad j = 1, \cdots, q_\eta \end{cases} \left(\mathrm{P}^\eta[x^{-\eta}]\right)$$

by controlling his/her own strategy $x^\eta \in \mathbb{R}^{n_\eta}$, while considering the decisions made by the rivals, as indicated by the vector $x^{-\eta} \in \mathbb{R}^{n-n_\eta}$ where $n = n_1 + \cdots + n_N$ stands for the overall number of players' decision variables. In order to highlight the variables of player $\eta$ within the vector $x$, we sometimes use the notation $(x^\eta, x^{-\eta})$ instead of simply $x$. It is important to note that this notation does not imply any reordering of the block components and that we always have $x = (x^\eta, x^{-\eta})$. Moreover, the overall number of constraints is determined by the sum $q = q_1 + \cdots + q_N$. The objective function and constraints of the player $\eta$ are denoted, respectively, by $f^\eta : \mathbb{R}^n \to \mathbb{R}^{p_\eta}$ and $g^\eta : \mathbb{R}^n \to \mathbb{R}^{q_\eta}$. The components of

these functions are $f_i$ with $i \in I^\eta = \{1, \cdots, p_\eta\}$ and $g_j^\eta$ with $j \in J^\eta = \{1, \cdots, q_\eta\}$.

Note that when $p_\eta = 1$ for each player $\eta$, MGNEP reduces to a scalar generalized Nash equilibrium problem (GNEP). In another special case, when $N = 1$, it takes the form of a multiobjective program with inequality constraints.

A point $\overline{x}$ is feasible for MGNEP if $g_j^\eta(\overline{x}) \leq 0$ for any $j \in J^\eta$ and each player $\eta = 1, \cdots, N$. In addition, we will say that the feasible point $\overline{x} = (\overline{x}^\eta, \overline{x}^{-\eta})$ is a weak efficient Nash equilibrium if for each player $\eta = 1, \cdots, N$, $\overline{x}^\eta$ is a weak efficient solution of the constrained multiobjective problem $(P^\eta[x^{-\eta}])$ w.r.t. $\overline{x}^{-\eta}$.

In both scalar and vector/multiobjective problems, there exist approximate optimality conditions that serve as necessary optimality conditions, without requiring any constraint qualification to hold at an optimal point. These conditions, which consider appropriate sequences of points and multipliers, have been extensively studied in various works because of their importance in developing implementable algorithms to detect these types of approximate optimality conditions; see, for instance, [2–4,36,39] for the most recent surveys on the subject.

In particular, we propose investigating a smooth MGNEP (multiobjective generalized Nash equilibrium problem) with inequality constraints and introduce the approximate Karush-Kuhn-Tucker condition for weak efficient solutions of MGNEP. Notably, our findings reveal that a weak efficient Nash equilibrium can potentially satisfy this condition. However, we discovered that these conditions impose excessive constraints for an algorithm to generate correspondence sequences. To address this, we employ a slight modification called the MGNEP-approximate weak Karush-Kuhn-Tucker (MGNEP-AWKKT) condition. Our analysis indicates that the MGNEP-AWKKT condition does not serve as an optimality condition for a general MGNEP. Nonetheless, we demonstrate that this condition holds in the special case of weak variational equilibrium points in a jointly convex MGNEP. Additionally, we establish that the limit point of an MGNEP-AWKKT sequence is a weak Karush-Kuhn-Tucker (WKKT) point for MGNEP, provided the cone continuity property (MGNEP-CCP) condition is met. Finally, to analyze the convergence of this optimality condition, we propose an augmented Lagrangian method for MGNEPs.

The remainder of this chapter is organized as follows. Section 2 summarizes the AKKT concept for standard nonlinear multiobjective optimization. Section 3 defines the notion of standard-AWKKT for MGNEPs and explores its properties. Additionally, we introduce our modified AWKKT condition

tailored for MGNEPs, demonstrating that this MGNEP-AWKKT qualifies as a necessary optimality condition for significant classes of MGNEPs. In Section 4, we describe our augmented Lagrangian method for MGNEPs and present a convergence analysis under the MGNEP-CCP condition, based on the results of Section 3. Finally, in the concluding section, we provide some important remarks and draw our conclusions.

## 2 Preliminaries

In this section, we will review some fundamental nonsmooth analysis constructions and results. For $\overline{x} \in \mathbb{R}^n$ and $\gamma > 0$, we denote by $\mathbb{U}_\gamma(\overline{x})$ (respectively $\mathbb{B}_\gamma(\overline{x})$) the $\gamma$-open (respectively $\gamma$-closed) ball centered at $\overline{x}$. We define $a_+ := \max(0, a)$ and $a_+^2 := (a_+)^2$ for $a \in \mathbb{R}$. We also denote the Euclidean norm by $\|\cdot\|$, and the nonnegative orthant in $\mathbb{R}^n$ by $\mathbb{R}^n_+$. For $y$, $z \in \mathbb{R}^n$, we write as usual $y \leq z$ if $y_i \leq z_i$ for $i = 1, \cdots, n$; $y < z$ if $y_i < z_i$ for $i = 1, \cdots, n$.

Now, we recall the definition related to Clarke's subdifferential [18]. Consider $\phi : \mathbb{R}^n \to \mathbb{R}$ to be a Lipschitz function at $x \in \mathbb{R}^n$. The Clarke subdifferential of $\phi$ at $x$ is denoted by

$$\partial_c \phi(x) := \{\alpha \in \mathbb{R}^n : \ \phi^\circ(x, v) \geq \langle \alpha, v \rangle, \ \ \forall v \in \mathbb{R}^n\},$$

where the term $\phi^\circ(x, v)$ stands for the Clarke directional derivative of $\phi$ at $x$ in the direction $v \in \mathbb{R}^n$ given by

$$\phi^\circ(x, v) = \limsup_{y \to x, t \downarrow 0} \frac{\phi(y + tv) - \phi(y)}{t}.$$

Note that $\partial_c \phi(x) = \{\nabla \phi(x)\}$ is the case when $\phi$ is continuously differentiable. More information about this subdifferential can be found here [18].

Given that $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^q$ are continuously differentiable functions, the following multiobjective optimization problem is taken into consideration.

$$\min_x f(x) = (f_1(x), \cdots, f_p(x)) \quad \text{s.t. } g_j(x) \leq 0, \ j = 1, \cdots, q. \tag{4.1}$$

At a feasible point $\overline{x}$, the set of active indexes is given by $J(\overline{x}) = \{j \in \{1, \cdots, q\} : g_j(\overline{x}) = 0\}$.

Next, we recall the solution concept of multiobjective problems. To proceed, $C \subset \mathbb{R}^n$ should stand for the feasible set of (4.1). A point $\overline{x}$ is an efficient (respectively weak efficient) solution for (4.1) if there is no $x \in C$ such that $f(x) \leq f(\overline{x})$ and $f(x) \neq f(\overline{x})$ (respectively, $f(x) < f(\overline{x})$). Correspondingly, a

point $\overline{x}$ is said to be a local efficient (resp. local weakly efficient) solution of (4.1) if it has a neighborhood $U$ such that $\overline{x}$ is an efficient (respectively, weak efficient) solution on $C \cap U$. In our examination of MGNEP, we will apply the concept of a weakly efficient solution for each problem ($\mathrm{P}^{\eta}[x^{-\eta}]$).

**Definition 4.1.** We say that $\overline{x} \in C$ is a weak Karush-Kuhn-Tucker-point (or satisfies the WKKT condition) of (4.1) if there are vectors $\lambda \in \mathbb{R}^p_+$ and $\mu \in \mathbb{R}^q_+$ such that

$(K_1)$ $\displaystyle\sum_{i=1}^{p} \lambda_i \nabla f_i(\overline{x}) + \sum_{j \in J(\overline{x})} \mu_j \nabla g_j(\overline{x}) = 0,$

$(K_2)$ $\lambda \neq 0,$

$(K_3)$ $\forall j \notin J(\overline{x}),\ \mu_j = 0\ .$

Based on the above stationarity condition, Giorgi et al. [39] introduced the following approximate Karush-Kuhn-Tucker condition.

**Definition 4.2.** We say that $\overline{x} \in C$ satisfies the approximate weak Karush-Kuhn-Tucker (AWKKT) condition of (4.1) if there are sequences $\left(x^k\right) \subset \mathbb{R}^n$ and $\left(\lambda^k, \mu^k\right) \subset \mathbb{R}^p_+ \times \mathbb{R}^q_+$ such that

$(W_1)$ $x^k \to \overline{x},$

$(W_2)$ $\displaystyle\sum_{i=1}^{p} \lambda_i^k \nabla f_i\left(x^k\right) + \sum_{j=1}^{q} \mu_j^k \nabla g_j\left(x^k\right) \to 0,$

$(W_3)$ $\displaystyle\sum_{i=1}^{p} \lambda_i^k = 1,$

$(W_4)$ $\forall j \notin J(\overline{x}),\ \mu_j^k = 0$ (for $k$ large enough).

**Remark 4.3.** The multiplier sequence $\lambda^k$ is in any case non-zero and bounded, while the point sequence $x^k$ need not to be feasible. Moreover, condition $(W_4)$ is clearly equivalent to

$$\forall j \notin J(\overline{x}),\ \mu_j^k g_j\left(x^k\right) \geq 0 \ \text{ (for } k \text{ large enough).}$$

Without any constraint qualification, Georgi et al. [39] proved that the approximate weak Karush-Kuhn-Tucker condition is a necessary optimality condition.

**Proposition 4.4.** *[39, Theorem 3.1] If $\overline{x} \in C$ is a local weak efficient solution of* (4.1)*, then $\overline{x}$ is an approximate weak Karush-Kuhn-Tucker point.*

# 3 Approximate weak Karush-Kuhn-Tucker Condition for Multiobjective generalized Nash equilibruim problem

This section focuses on the investigation of weak sequential optimality conditions for MGNEPs. We will introduce two concepts of approximation weak Karush-Kuhn-Tucker condition called standard-AWKKT and MGNEP-AWKKT, which are inspired by the work of Buenos, Haeser, and Rojas [13]. In order to do so, we will need the following structures. Let $x^{-\eta} \in \mathbb{R}^{n-n_\eta}$, the set of feasible points for player $\eta$ will be provided by

$$\Omega^\eta\left(x^{-\eta}\right) = \{x^\eta \in \mathbb{R}^{n_\eta} \,|\, g_j^\eta\left(x^\eta, x^{-\eta}\right) \leq 0,\ j = 1, \cdots, q_\eta\},$$

and the MGNEP feasible set will be given by

$$\Omega = \{x \in \mathbb{R}^n \,|\, g_j^\eta\left(x^\eta, x^{-\eta}\right) \leq 0,\ j = 1, \cdots, q_\eta,\ \eta = 1 \cdots, N\}.$$

Note that $x$ is an MGNEP feasible point if and only if it is a fixed point of the set-valued map formed by taking the Cartesian product of the feasible sets of all players [66]; that is

$$x \in \Omega \text{ iff } x \in \prod_{\eta=1}^{N} \Omega^\eta\left(x^{-\eta}\right).$$

The set of active constraint indices for player $\eta$ at a feasible point $x \in \Omega$ is determined by

$$J^\eta(x) = \{i \in J^\eta \,|\, g_j^\eta(x) = 0\}.$$

## 3.1 Standard approximate weak Karush-Kuhn-Tucker conditions

**Definition 4.5.** We say that $\overline{x} \in \Omega$ satisfies the standard-AWKKT condition if for each $\eta = 1, \cdots, N$, $\overline{x}^\eta$ satisfies the approximation weak Karush-Kuhn-Tucker condition for the multiobjective optimization problem $(P^\eta[\overline{x}^{-\eta}])$.

Next, we demonstrate that the standard-AWKKT condition is a necessary optimality condition for the weak efficient Nash equilibrium of MGNEP in the absence of additional constraint qualification. The proof technique essentially follows similar argument as those employed in [4,39].

**Theorem 4.6.** *If $\overline{x} \in \Omega$ is a weak efficient Nash equilibrium of MGNEP, then $\overline{x}$ satisfies the standard-AWKKT condition.*

*Proof.* Let $\overline{x} \in \Omega$ be a weak efficient Nash equilibrium of MGNEP. By definition, for each $\eta = 1, \cdots, N$, there exists $\gamma^\eta > 0$ such that $\overline{x}^\eta$ is a local weak efficient solution of $(P^\eta[\overline{x}^{-\eta}])$ w.r.t. $\gamma^\eta$. Considering the nonsmooth scalarizing functions $\varphi^\eta : \mathbb{R}^{p_\eta} \to \mathbb{R}$ given by $\varphi^\eta(u) = \max_{r=1}^{p_\eta} u_r$, one can deduce that $\overline{x}^\eta$ is a solution to the following scalar problem

$$\min_{x^\eta} \varphi^\eta\left(f^\eta\left(x^\eta, \overline{x}^{-\eta}\right) - f^\eta\left(\overline{x}^\eta, \overline{x}^{-\eta}\right)\right) \quad \text{s.t. } x^\eta \in \Omega^\eta\left(\overline{x}^{-\eta}\right) \cap \mathbb{B}_{\gamma^\eta}\left(\overline{x}^\eta\right).$$

Hence, by setting $\gamma = \min_{\eta=1}^{N} \gamma^\eta$ and adjusting it to a smaller value if needed, we can conclude that $\overline{x}^\eta$ is the unique global minimum point of the problem

$$\begin{array}{ll} \min\limits_{x^\eta} & \varphi^\eta\left(f^\eta\left(x^\eta, \overline{x}^{-\eta}\right) - f^\eta\left(\overline{x}^\eta, \overline{x}^{-\eta}\right)\right) + \dfrac{1}{2} \parallel x^\eta - \overline{x}^\eta \parallel^2 \\ \text{s.t.} & x^\eta \in \Omega^\eta\left(\overline{x}^{-\eta}\right), \parallel x^\eta - \overline{x}^\eta \parallel \leq \gamma. \end{array} \tag{4.2}$$

Let us fix $\eta = 1, \cdots, N$ and let $x^{k,\eta}$ be a solution of

$$\min_{x^\eta} \Psi_k^\eta(x^\eta) \text{ s.t. } \parallel x^\eta - \overline{x}^\eta \parallel \leq \gamma \tag{4.3}$$

where $\Psi_k^\eta(x^\eta) = \varphi^\eta\left(f^\eta\left(x^\eta, \overline{x}^{-\eta}\right) - f^\eta\left(\overline{x}^\eta, \overline{x}^{-\eta}\right)\right) + \dfrac{1}{2} \parallel x^\eta - \overline{x}^\eta \parallel^2 + \sum_{j=1}^{q_\eta} k g_j^\eta\left(x^\eta, \overline{x}^{-\eta}\right)_+^2$.

Since for each $k$ and $\eta$, the function $\Psi_k^\eta$ is continuous and the constraint set $\mathbb{B}_\gamma\left(\overline{x}^\eta\right)$ is nonempty and compact, the existence of $\left(x^{k,\eta}\right)$ is ensured. The proof steps to follow are identical to those in the proof of [39, Theorem 3.1]. We include it here for the reader's convenience.

- We have, from the definition of $\Psi_k^\eta$ that for all $k \in \mathbb{N}$

$$\varphi^\eta\left(f^\eta\left(x^{k,\eta}, \overline{x}^{-\eta}\right) - f^\eta\left(\overline{x}^\eta, \overline{x}^{-\eta}\right)\right) \leq \Psi_k^\eta\left(x^{k,\eta}\right) \leq \Psi_k^\eta\left(\overline{x}^\eta\right) = 0. \tag{4.4}$$

- Let us show that every accumulation point $y^\eta$ of $\left(x^{k,\eta}\right)$ is a feasible point of problem (4.2). To the contrary, suppose that $\sum_{j=1}^{q_\eta} k g_j^\eta\left(y^\eta\right)_+^2 > 0$. By continuity and assuming $x^{k,\eta} \to y^\eta$, there exists a constant $A^\eta > 0$ such that for k large enough, we get $\sum_{j=1}^{q_\eta} k g_j^\eta\left(x^{k,\eta}\right)_+^2 > A^\eta$. Consequently, we obtain:

$$\Psi_k^\eta\left(x^{k,\eta}\right) > \varphi^\eta\left(f^\eta\left(x^{k,\eta}, \overline{x}^{-\eta}\right) - f^\eta\left(\overline{x}^\eta, \overline{x}^{-\eta}\right)\right) + kA^\eta.$$

Therefore, it follows that $\lim_{k\to\infty} \Psi_k^\eta\left(x^{k,\eta}\right) = +\infty$, which is a contradiction to the right inequality in (4.4). Hence, we can conclude that $\sum_{j=1}^{q_\eta} k g_j^\eta\left(y^\eta\right)_+^2 = 0$, implying that $y^\eta \in \Omega^\eta\left(\overline{x}^{-\eta}\right)$. However,

because $x^{k,\eta}$ is feasible to problem (4.2), we get $\| y^\eta - \overline{x}^\eta \| \leq \gamma$. In conclusion, $y^\eta$ is a feasible point of problem (4.2).

- Let us prove that $y^\eta = \overline{x}^\eta$. Indeed, since $\sum_{j=1}^{q_\eta} kg_j^\eta \left(x^{k,\eta}\right)_+^2 \geq 0$, we derive from (4.4) that

$$\varphi^\eta \left(f^\eta \left(x^{k,\eta}, \overline{x}^{-\eta}\right) - f^\eta \left(\overline{x}^\eta, \overline{x}^{-\eta}\right)\right) + \frac{1}{2} \| x^{k,\eta} - \overline{x}^\eta \|^2 \leq 0.$$

Taking limits when $k$ tends to infinity, we get

$$\varphi^\eta \left(f^\eta \left(y^\eta, \overline{x}^{-\eta}\right) - f^\eta \left(\overline{x}^\eta, \overline{x}^{-\eta}\right)\right) + \frac{1}{2} \| y^\eta - \overline{x}^\eta \|^2 \leq 0.$$

Now, since $\overline{x}^\eta$ is the unique vanishing-valued solution to problem (4.2), we obtain $y^\eta = \overline{x}^\eta$. In consequence, $x^{k,\eta} \to \overline{x}^\eta$. Thus, for $(P^\eta[\overline{x}^{-\eta}])$, $(W_1)$ is confirmed.

- We already know that $x^{k,\eta}$ is a solution of (4.3). Furthermore, for sufficiently large $k$, $x^{k,\eta}$ is an interior point of the feasible set of (4.3). Hence, from [18, Proposition 2.3.2] we get $0 \in \partial_c \Psi_k^\eta \left(x^{k,\eta}\right)$. Applying simultanously [18, Proposition 2.3.3, Proposition 2.3.12 and Propodition 2.4.4], we obtain

$$0 \in co\left(\bigcup_{i=1}^{p_\eta} \left\{\nabla_{x^\eta} f_i \left(x^{k,\eta}\right)\right\}\right) + x^{k,\eta} - \overline{x}^\eta + \sum_{j=1}^{q_\eta} kg_j^\eta \left(x^{k,\eta}\right)_+ \nabla_{x^\eta} g_j^\eta \left(x^{k,\eta}\right)$$

where *co* stands for the convex hull. Consequently, there exist scalars $\lambda_i^{k,\eta}$, $i = 1, \cdots, p_\eta$ with $\sum_{i=1}^{p_\eta} \lambda_i^{k,\eta} = 1$ in such a way that

$$\sum_{i=1}^{p_\eta} \lambda_i^{k,\eta} \nabla_{x^\eta} f_i \left(x^{k,\eta}\right) + \sum_{j=1}^{q_\eta} kg_j^\eta \left(x^{k,\eta}\right)_+ \nabla_{x^\eta} g_j^\eta \left(x^{k,\eta}\right) = x^{k,\eta} - \overline{x}^\eta \to 0.$$

Setting for each $k$, $\mu_j^{k,\eta} = kg_j^\eta \left(x^{k,\eta}\right)_+$, with $j = 1, \cdots, q_\eta$, $(W_2)$ and $(W_3)$ hold for $(P^\eta[\overline{x}^{-\eta}])$. In addition, from the continuity of $g_j^\eta$, $(W_4)$ is also satisfied for $(P^\eta[\overline{x}^{-\eta}])$.

Finally, for each player $\eta$ the conditions $(W_1)$-$(W_4)$ from Definition 4.2 hold true. Hence, for each $\eta = 1, \cdots, N$, $\overline{x}^\eta$ satisfies the approximation weak Karush-Kuhn-Tucker condition for $(P^\eta[\overline{x}^{-\eta}])$. Thus $\overline{x}$ satisfies the Standard-AWKKT condition. □

After that, we provide an illustration of Theorem 4.6.

**Example 4.7.** Consider a MGNEP composed of two players with real variables $x^1$ and $x^2$, respectively. The loss functions are defined by

$$f^1\left(x^1,x^2\right)=\left(-\left(x^1\right)^2+3x^2,-3\left(x^1\right)^2+x^2\right)\quad\text{and}\quad f^2\left(x^1,x^2\right)=\left(\left(x^1\right)^2-\left(x^2\right)^2,-x^2\right)$$

and the constraint functions by

$$g^1\left(x^1,x^2\right)=\left(x^1\right)^2-x^2\quad\text{and}\quad g^2\left(x^1,x^2\right)=x^2-1.$$

Observe that MGNEP is a nonconvex problem. For each $\eta\in\{1,2\}$, $\overline{x}^\eta=1$ is a weak efficient solution of

$$\begin{array}{ll}\min\limits_{x^\eta} & f^\eta\left(x^\eta,\overline{x}^{-\eta}\right)=\left(f_1^\eta\left(x^\eta,\overline{x}^{-\eta}\right),f_2^\eta\left(x^\eta,\overline{x}^{-\eta}\right)\right)\\ \text{s.t.} & g^\eta\left(x^\eta,x^{-\eta}\right)\leq 0.\end{array}\tag{4.5}$$

Hence, the point $\overline{x}=(1,1)$ is a weakly efficient Nash equilibrium of the MGNEP. Thus, Theorem 4.6 can be applied.

**For Player** 1**.** We have $\nabla_{x^1}f_1^1\left(x^1,x^2\right)=-2x^1$, $\nabla_{x^1}f_2^1\left(x^1,x^2\right)=-6x^1$, and $\nabla_{x^1}g^1\left(x^1,x^2\right)=2x^1$. Setting, $\left(x^{k,1}\right)=\left(\dfrac{k+1}{k}\right)$, $\left(\lambda_1^{k,1}\right)=\left(\dfrac{k}{4k+1}\right)$, $\left(\lambda_2^{k,1}\right)=\left(\dfrac{3k+1}{4k+1}\right)$ and $\left(\mu^{k,1}\right)=\left(\dfrac{5k+1}{2k+1}\right)$ one has $\left(x^{k,1}\right)\to\overline{x}^1$, $\lambda_1^{k,1}+\lambda_2^{k,1}=1$, and

$$\lambda_1^{k,1}\nabla_{x^1}f_1^1\left(x^{k,1},\overline{x}^2\right)+\lambda_2^{k,1}\nabla_{x^1}f_2^1\left(x^{k,1},\overline{x}^2\right)+\mu^{k,1}\nabla_{x^1}g^1\left(x^{k,1},\overline{x}^2\right)\to 0.$$

**For Player** 2**.** We have $\nabla_{x^2}f_1^2\left(x^1,x^2\right)=-2x^2$, $\nabla_{x^2}f_2^2\left(x^1,x^2\right)=-1$, and $\nabla_{x^2}g^2\left(x^1,x^2\right)=1$. Setting, $\left(x^{k,2}\right)=\left(\dfrac{k}{k-1}\right)$, $\left(\lambda_1^{k,2}\right)=\left(\dfrac{k+1}{3k+1}\right)$, $\left(\lambda_2^{k,2}\right)=\left(\dfrac{2k}{3k+1}\right)$ and $\left(\mu^{k,2}\right)=\left(\dfrac{4k}{3k+1}\right)$ one has $\left(x^{k,2}\right)\to\overline{x}^2$, $\lambda_1^{k,2}+\lambda_2^{k,2}=1$,

$$\lambda_1^{k,2}\nabla_{x^2}f_1^2\left(\overline{x}^1,x^{k,2}\right)+\lambda_2^{k,2}\nabla_{x^2}f_2^2\left(\overline{x}^1,x^{k,2}\right)+\mu^{k,2}\nabla_{x^2}g^2\left(\overline{x}^1,x^{k,2}\right)\to 0.$$

In conclusion, the point $\overline{x}=(1,1)$ satisfies the standard-AWKKT condition.

To conclude this subsection, we extend the definition of the cone continuity property constraint qualification from optimization to GNEPs, taking into account each player's feasible set separately, and provide that it is the constraint qualification to ensure that the standard-AWKKT optimality condition is strong, in the sense that it implies the standard-WKKT point or Not-standard-CCP optimality condition.

**Definition 4.8.** A point $\overline{x} \in \Omega$ is said to satisfy the standard cone continuity property (standard-CCP ) if for each $\eta = 1, \cdots, N$

$$\limsup_{x^\eta \to \overline{x}^\eta} \mathcal{T}_{\Omega^\eta\left(\overline{x}^{-\eta}\right)}\left(x^\eta\right) \subset \mathcal{T}_{\Omega^\eta\left(\overline{x}^{-\eta}\right)}\left(\overline{x}^\eta\right),$$

where $\mathcal{T}_{\Omega^\eta\left(\overline{x}^{-\eta}\right)}\left(x^\eta\right) := \left\{ v^\eta \in \mathbb{R}^{n_\eta} \mid v^\eta = \sum_{j \in J^\eta(\overline{x})} \mu_j^\eta \nabla_{x^\eta} g_j^\eta\left(x^\eta, \overline{x}^{-\eta}\right),\ \mu_j^\eta \in \mathbb{R}_+ \right\}$.

**Theorem 4.9.** *Let $\overline{x} \in \Omega$. Assume that $\overline{x}$ satisfies the standard-AWKKT condition and that the standard-CCP holds at $\overline{x}$. Then, $\overline{x}$ is a standard-WKKT point.*

*Proof.* By the definition of standard-AWKKT and by Theorem 4.1 stated in [36], it is easy to get that $\overline{x}$ is a standard-WKKT point while considering the standard-CCP constraint qualification. □

## 3.2 MGNEP approximate weak KKT conditions

It is important to note that Definition 4.5 serves as a natural extension of the weak approximation Karush-Kuhn-Tucker condition that has been studied in multiobjective optimization problems [39]. Additionally, the standard-AWKKT conditions hold true for any weak efficient Nash equilibrium without any additional requirements. However, the standard-AWKKT condition is too strict to expect an algorithm to produce a sequence of the desired type, namely $\left(x^{k,\eta}\right)$, which can be considered an AWKKT-type sequence for each $\eta = 1, \cdots, N$, only when paired with the exact limit, that is of the form $\left(x^{k,\eta}, \overline{x}^{-\eta}\right)$. To address this limitation, we introduce the MGNEP-AWKKT condition, which is a slight modification of Definition 4.5.

**Definition 4.10.** We say that $\overline{x} \in \Omega$ satisfies the MGNEP-AWKKT if there are sequences $\left(x^k\right) \subset \mathbb{R}^n$ and $\left(\lambda^{k,\eta}, \mu^{k,\eta}\right) \subset \mathbb{R}_+^{p_\eta} \times \mathbb{R}_+^{q_\eta}$ for each $\eta = 1, \cdots, N$ such that

$(W_1)'$ $x^k \to \overline{x}$,

$(W_2)'$ $\sum_{i=1}^{p_\eta} \lambda_i^{k,\eta} \nabla_{x^\eta} f_i^\eta\left(x^k\right) + \sum_{j=1}^{q_\eta} \mu_j^{k,\eta} \nabla_{x^\eta} g_j^\eta\left(x^k\right) \to 0$,

$(W_3)'$ $\sum_{i=1}^{p} \lambda_i^{k,\eta} = 1$,

$(W_4)'$ $\forall j \notin J^\eta\left(\overline{x}\right),\ \mu_j^{k,\eta} = 0$ (for $k$ large enough).

Unlike classical nonlinear problems [39], the MGNEP-AWKKT condition is not a necessary optimality condition for general MGNEPs. As shown in [13, Example 5.3], the GNEP-AKKT condition (which equivalent to MGNEP-AWKKT in the scalar case) may not be an optimality condition, even for a jointly convex MGNEP. Furthermore, the two AWKKT concepts are completely two independent conditions, as neither standard-AWKKT implies necessarily MGNEP-AWKKT nor conversely. An illustration when MGNEP-AWKKT holds but standard-AWKKT does not is presented in the following example.

**Example 4.11.** Consider an MGNEP with $n_1 = n_2 = 1$, $q_1 = 1$, $q_2 = 0$, $p_1 = p_2 = 2$, and two players defined by the following problems

$$\begin{array}{ll} \min\limits_{x^1} & \left(\dfrac{(x^1)^2}{2} + x^2, \dfrac{(x^1)^2}{2} + x^2\right) \\ \text{s.t.} & x^1 x^2 \leq 0. \end{array} \quad \text{and} \quad \begin{array}{ll} \min\limits_{x^2} & \left(\dfrac{(x^2)^2}{2} + x^1, \dfrac{(x^2)^2}{2} + x^1\right) \end{array} \tag{4.6}$$

The unique solution is the point $\overline{x} = (0, 0)$. Let us investigate the properties of the feasible point $x = (-1, 0)$. Consider the sequence $x^k = \left(-1, \dfrac{1}{\sqrt{k}}\right)$, $\mu^{k,1} = \sqrt{k}$, $\lambda_1^{k,1}$ , $\lambda_2^{k,1}$ with $\lambda_1^{k,1} + \lambda_2^{k,1} = 1$ and $\lambda_1^{k,2}$, $\lambda_2^{k,2}$ with $\lambda_1^{k,2} + \lambda_2^{k,2} = 1$. We have

$$\lambda_1^{k,1} x^{k,1} + \lambda_2^{k,1} x^{k,1} + \mu^{k,1} x^{k,2} = 0 \quad \text{and} \quad \lambda_1^{k,2} x^{k,2} + \lambda_2^{k,2} x^{k,2} = \frac{1}{\sqrt{k}} \to 0.$$

As a result, the MGNEP-AWKKT holds. To demonstrate that standard-AWKKT is not satisfied at $x$, consider any sequence $\left(x^k\right)$ in $\mathbb{R}^2$ such that $x^k \to (-1, 0)$, $\lambda_1^{k,1}$ , $\lambda_2^{k,1}$ with $\lambda_1^{k,1} + \lambda_2^{k,1} = 1$ and $\mu^{k,1} \in \mathbb{R}_+$. Then, the first player, we have for $k$ sufficiently large

$$\lambda_1^{k,1} x^{k,1} + \lambda_2^{k,1} x^{k,1} + \mu^{k,1} x^{k,2} = x^{k,1} + \mu^{k,1} x^{k,2} \to -1.$$

Next, using the constraint qualification known as the cone continuity property, we show that a point satisfying the MGNEP-AWKKT condition is also an MGNEP-WKKT point. To proceed, given $\overline{x} \in \Omega$ we define

$$\mathcal{T}^\eta(\overline{x}) := \left\{ v^\eta \in \mathbb{R}^{n_\eta} \mid v^\eta = \sum_{j \in J^\eta(\overline{x})} \mu_j^\eta \nabla_{x^\eta} g_j^\eta(\overline{x}),\ \mu_j^\eta \in \mathbb{R}_+ \right\}$$

and set $\mathcal{T}(\overline{x}) = \prod_{\eta=1}^{N} \mathcal{T}^\eta(\overline{x})$.

**Definition 4.12.** A point $\overline{x} \in \Omega$ is said to satisfy the cone continuity property (MGNEP-CCP) if

$$\left[\forall \eta = 1, \cdots, N, \quad \limsup_{x \to \overline{x}} \mathcal{T}^{\eta}(x) \ \subset \ \mathcal{T}^{\eta}(\overline{x})\right] \quad \text{or} \quad \left[\limsup_{x \to \overline{x}} \mathcal{T}(x) \ \subset \ \mathcal{T}(\overline{x})\right].$$

**Remark 4.13.** In [13], Bueno *et al.* show that the cone continuity property GNEP-CCP is the weakest qualification for a constraint in smooth scalar generalized Nash equilibrium problems. More specifically, achieving GNEP-CCP ensures that regardless of the objective functions of each player, the limits of GNEP-AKKT sequences converge to Karush-Kuhn-Tucker points. Seeking the extension of this to MGNEPs for weak efficient points, we can consider the following scalarized problem of $(\mathrm{P}^{\eta}[x^{-\eta}])$

$$\begin{aligned} \min_{x^{\eta}} \quad & \sum_{i=1}^{p_{\eta}} \lambda_i^{k,\eta} \ f_i^{\eta}\left(x^{\eta}, x^{-\eta}\right) \\ \text{s.t.} \quad & g_j^{\eta}\left(x^{\eta}, x^{-\eta}\right) \leq 0, \quad j = 1, \cdots, q_{\eta} \end{aligned}$$

According to [13, Theorem 5.10], a limit point of an GNEP-AKKT sequence, where the CCP holds, is a GNEP-KKT point for the above mentioned scalar problem. Based on this arguments and the fact that each sequence $\left(\lambda^{k,\eta}\right) \subset \mathbb{R}^{p_{\eta}}$ of Definition 4.10 has at least one non-zero accumulation point one can show that MGNEP-CCP is sufficient for MGNEP to ensure that limits of MGNEP-AWKKT sequences are MGNEP-WKKT points (that is weak Karush-Kuhn-Tucker points for MGNEP). With this, we can conclude that the MGNEP-AWKKT condition is a sufficient optimality condition.

**Theorem 4.14.** *If $\overline{x} \in \Omega$ satisfies the MGNEP-AWKKT condition and MGNEP-CCP holds at $\overline{x}$, then $\overline{x}$ is a MGNEP-WKKT point.*

*Proof.* Let $\overline{x} \in \Omega$ satisfy the MGNEP-CCP and the MGNEP-AWKKT condition. By definition, there are sequences $\left(x^k\right) \subset \mathbb{R}^n$, $\left(\lambda^{k,\eta}\right) \in \mathbb{R}^{p_{\eta}}$ and $\left(\mu^{k,\eta}\right) \subset \mathbb{R}_+^{q_{\eta}}$ with $x^k \to \overline{x}$, $\sum_{i=1}^{p_{\eta}} \lambda_i^{k,\eta} = 1$ and $\mu_j^{k,\eta} = 0$ for $j \notin J^{\eta}(\overline{x})$ such that for each $\eta = 1, \cdots, N$ one has

$$z^{k,\eta} = \sum_{i=1}^{p_{\eta}} \lambda_i^{k,\eta} \nabla_{x^{\eta}} f_i^{\eta}\left(x^k\right) + \sum_{j \in J^{\eta}(\overline{x})} \mu_j^{k,\eta} \nabla_{x^{\eta}} g_j^{\eta}\left(x^k\right) \to 0. \tag{4.7}$$

We define for each $k \in \mathbb{N}$,

$$v^k = \left(v^{k,\eta}\right)_{\eta=1}^{N} \quad \text{where} \quad v^{k,\eta} = \sum_{j \in J^{\eta}(\overline{x})} \mu_j^{k,\eta} \nabla_{x^{\eta}} g_j^{\eta}\left(x^k\right).$$

Now fix $\eta \in \{1, \cdots, N\}$. Since the positive sequence $\left(\lambda^{k,\eta}\right)$ is bounded with $\sum_{i=1}^{p_\eta} \lambda_i^{k,\eta} = 1$, there exists a subsequence $\left(\lambda^{k_s,\eta}\right)$ such that $\lambda^{k_s,\eta} \to \lambda^\eta$ satisfying $\sum_{i=1}^{p_\eta} \lambda_i^\eta = 1$.

On the other hand, we have from (4.7) and the continuity of the gradients that $v^{k_s,\eta} \to -\sum_{i=1}^{p_\eta} \lambda_i^\eta \nabla_{x^\eta} f_i^\eta(\overline{x})$.

Hence, $v^{k_s} \to -\left(\sum_{i=1}^{p_\eta} \lambda_i^\eta \nabla_{x^\eta} f_i^\eta(\overline{x})\right)_{\eta=1}^{\eta=N}$. Remembring that $v^{k_s} \in \mathcal{T}\left(x^{k_s}\right)$ one deduces that

$$-\left(\sum_{i=1}^{p_\eta} \lambda_i^\eta \nabla_{x^\eta} f_i^\eta(\overline{x})\right)_{\eta=1}^{\eta=N} \in \limsup_{s\to\infty} \mathcal{T}\left(x^{k_s}\right) \subset \limsup_{x\to\overline{x}} \mathcal{T}(x).$$

Now, by using the definition of MGNEP-CCP condition, we have that

$$-\left(\sum_{i=1}^{p_\eta} \lambda_i^\eta \nabla_{x^\eta} f_i^\eta(\overline{x})\right)_{\eta=1}^{\eta=N} \in \mathcal{T}(\overline{x}).$$

Thus, for each $\eta = 1, \cdots N$, we get

$$-\sum_{i=1}^{p_\eta} \lambda_i^\eta \nabla_{x^\eta} f_i^\eta(\overline{x}) = \sum_{j\in J^\eta(\overline{x})} \mu_j^\eta \nabla_{x^\eta} g_j^\eta(\overline{x}), \quad \mu_j^\eta \in \mathbb{R}_+.$$

Consequently, $\overline{x}$ is a MGNEP-WKKT point. □

## 3.3 Case of weak variational equilibrium

Regardless of its bad properties, MGNEP-AWKKT is an optimality condition for specific classes of MGNEPs such as variational equilibrium of a jointly convex MGNEP. Such problems are a significant class of MGNEPs that appears in numerous interesting applications and for which a far more thorough theory exists than for the arbitrary MGNEP [38].

Let us take MGNEP to be jointly convex; that is, we assume the convexity of the objective function $f^\eta(\cdot, x^{-\eta})$ for any player $\eta$ and each $x^{-\eta}$ as well as the constraint $g := g^1 = \cdots = g^N : \mathbb{R}^n \to \mathbb{R}^q$ as a function of the entire vector $x$. Note that in this case the feasible set $\Omega = \{x \in \mathbb{R}^n \mid g(x) \leq 0\}$ is convex. We also set $W^\eta = \{w^\eta \in \mathbb{R}^{p_\eta} | w^\eta \geq 0, w_1^\eta + \cdots, w_{p_\eta}^\eta = 1\}$ and let $w^\eta \in W^\eta$. Consider the variational inequality problem $VI(\Omega, H)$:

$$\text{find } \overline{x} \in \Omega \quad \text{such that} \quad \langle H(\overline{x}), x - \overline{x}\rangle \geq 0 \ \text{ for all } x \in \Omega, \tag{4.8}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$ and the components of $H(x)$ are given by $H_\eta(x) = \sum_{i=1}^{p_\eta} w_i^\eta \nabla_{x^\eta} f_i^\eta(x)$ for $\eta = 1, \cdots, N$.

We now show a relationship between the solution of $VI(\Omega, H)$ and that of the jointly convex MGNEP.

**Proposition 4.15.** *Let a jointly convex MGNEP be given. For each $\eta = 1, \cdots, N$ and $i = 1, \cdots, p_\eta$, assume $f_i^\eta$ is $\mathcal{C}^1$. Then, every solution of the $VI(\Omega, H)$ is also a weak efficient Nash equilibruim of MGNEP.*

*Proof.* Let $\overline{x}$ be a solution of $VI(\Omega, H)$. Since, $f_i^\eta$ are $\mathcal{C}^1$ functions, from [35, Theorem 2.5], we have $\overline{x}$ is a solution of the GNEP with the following scalarized problem:

$$\begin{aligned} &\min_{x^\eta} \langle w^k, f^\eta(x^\eta, \overline{x}^{-\eta}) \rangle \\ &\text{s.t. } (x^\eta, x^{-\eta}) \in \Omega. \end{aligned}$$

Taking into account the definition of $W^\eta$, we deduce from [52, Theorem 3.11] that $\overline{x}^\eta$ is a solution of $(P[\overline{x}^{-\eta}])$ for each player $\eta$. Hence, $\overline{x}$ is also a weak efficient Nash equilibruim of the MGNEP. □

**Remark 4.16.** Take $n_1 = n_2 = 1$, $q_1 = q_2 = 1$, $p_1 = p_2 = 2$, and consider the following MGNEP with two players

$$\begin{aligned} \min_{x^\eta} \quad & (x^1 + x^2, x^1 + x^2) \\ \text{s.t.} \quad & \frac{(x^1)^2}{2} + \frac{(x^2)^2}{2} - x^1 x^2 \leq 0. \end{aligned} \qquad \text{(Player } \eta\text{)}$$

Let us note that the point $\overline{x} = (0, 0)$ is a weak efficient Nash equilibrium of this jointly convex MGNEP. This example illustrates that that the converse in Proposition 4.15 is false. Indeed, we have

$$\nabla_{x^1} f_1^1(\overline{x}) = \nabla_{x^1} f_2^1(\overline{x}) = 1, \quad \nabla_{x^2} f_1^2(\overline{x}) = \nabla_{x^2} f_2^2(\overline{x}) = 1$$

and

$$\langle H(\overline{x}), x - \overline{x} \rangle = \langle \begin{pmatrix} w_1^1 + w_2^1 \\ w_1^1 + w_2^1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rangle = x^1 + x^2.$$

The point $(x^1, x^2) = (-1, -1)$ is a feasible point of MGNEP but $x^1 + x^2 < 0$. Hence, $\overline{x} = (0, 0)$ is not a solution of $VI(\Omega, H)$.

The points for which this converse is true are called weak variational equilibria.

**Definition 4.17.** Let us consider a jointly convex MGNEP with $\mathcal{C}^1$ functions $f_i^\eta$. A weak efficient Nash equilibrium of MGNEP is called a weak variational equilibrium if it is also solution of $VI(\Omega, H)$.

We can show that at each weak variational equilibrium, an MGNEP-AWKKT necessary optimality condition is fulfilled.

**Theorem 4.18.** *If $\overline{x}$ is a weak variational equilibrium point of MGNEP, then $\overline{x}$ satisfies the MGNEP-AWKKT condition.*

*Proof.* Let $\overline{x}$ represent a weak variational equilibrium point of MGNEP. As a result, $\overline{x}$ is a solution of $VI(\Omega, H)$. Thus, $\overline{x}$ is an optimal solution of the following weighted optimization problem:

$$\min_{x\in\Omega} \langle H(\overline{x}), x\rangle.$$

Now, from the scalar case [4], there are sequences $\left(x^k\right) \in \mathbb{R}^n$ and $\left(\mu^k\right) \in \mathbb{R}^{|J(\overline{x})|}$ in such a way that $x^k \to \overline{x}$ and

$$H(\overline{x}) + \sum_{j\in J(\overline{x})} \mu_j^k \nabla g_j\left(x^k\right) \;\to\; 0.$$

Replacing $H(\overline{x})$ by its expression, we get for each $\eta = 1, \cdots, N$

$$\sum_{i=1}^{p_\eta} w_i^\eta \nabla_{x^\eta} f_i^\eta(\overline{x}) + \sum_{j\in J(\overline{x})} \mu_j^k \nabla_{x^\eta} g_j\left(x^k\right) \;\to\; 0.$$

Observing that $w_i^\eta \nabla_{x^\eta} f_i^\eta\left(x^k\right)$ converges to $w_i^\eta \nabla_{x^\eta} f_i^\eta(\overline{x})$ one deduces that

$$\sum_{i=1}^{p_\eta} w_i^\eta \nabla_{x^\eta} f_i^\eta\left(x^k\right) + \sum_{j\in J(\overline{x})} \mu_j^k \nabla_{x^\eta} g_j\left(x^k\right) \;\to\; 0.$$

Setting $\lambda^{k,\eta} = w^\eta$, we obtain the result. □

A summary regarding the satisfaction of standard-AWKKT and MGNEP-AWKKT conditions, as well as their relationship to yielding a WKKT point, is presented in Figure 4.1.

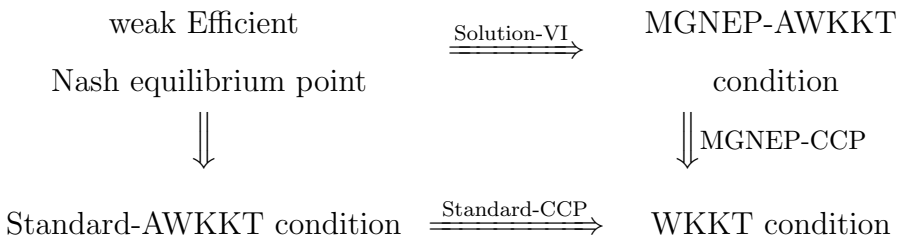


Figure 4.1: *Solution-VI: refers to Definition 4.17 and Theorem 4.18.*
*MGNEP-CCP: refers to the Definition 4.12 and Theorem 4.14.*
*Standard-CCP: refers to the Definition 4.8 and Theorem 4.9.*

# 4 Algorithm and convergence

In this part, we present and explore an enhanced Lagrangian algorithm for MGNEPs. We analyze the optimality finding under MGNEP-CCP setting after presenting its convergence analysis. It is crucial to note that we do not employ any scalarization technique and instead deal directly with MGNEP-AWKKT points during each iteration.

Similarly to the penalty strategies for multiobjective optimization problems, we use a component-wise augmented Lagrangian by adding the penalty term to all objectives and dealing with unconstrained subproblems [19]. For a player $\eta = 1, \ldots, N$, given a vector of Lagrange multipliers $\sigma^\eta \in \mathbb{R}_+^{n_\eta}$ and an external penalty parameter $\eta^\eta > 0$, the augmented Lagrangian function for this player is given by

$$\mathcal{L}_{\eta^\eta}^\eta \left(x, \sigma^\eta\right) = f^\eta \left(x^\eta, x^{-\eta}\right) + \frac{\eta^\eta}{2} \left( \sum_{j=1}^{q_\eta} \left( g_j^\eta \left(x^\eta, x^{-\eta}\right) + \frac{\sigma_j^\eta}{\eta^\eta} \right)_+^2 \right) \mathbb{1}_{p_\eta}.$$

where $\mathbb{1}_{p_\eta} = (1, \cdots, 1) \in \mathbb{R}^{p_n}$. Using the augmented Lagrangian functions, we establish a new unconstrained MGNEP, denoted MGNEP$(\sigma, \eta)$, in which the problem for player $\eta$ is provided by

$$\min_{x^\eta} \mathcal{L}_{\eta^\eta}^\eta \left(x, \sigma^\eta\right).$$

To construct a valid stopping criterion for an algorithm that produces MGNEP-AWKKT sequences, we must first define the $\epsilon$-inexact MGNEP-WKKT concept for a given $\epsilon > 0$.

**Definition 4.19.** For $\epsilon \geq 0$, we say that $(x, \lambda, \mu) \in \mathbb{R}^{n+p+q}$ is an $\epsilon$-inexact MGNEP-WKKT point if for each $\eta = 1, \cdots, N$, we have

$$\left\| \sum_{i=1}^{p_\eta} \lambda_i^\eta \nabla_{x^\eta} f_i^\eta \left(x^\eta, x^{-\eta}\right) + \sum_{j=1}^{q_\eta} \mu_j^\eta \nabla_{x^\eta} g_j^\eta \left(x^\eta, x^{-\eta}\right) \right\| \leq \epsilon, \tag{4.9}$$

$$\sum_{j=1}^{q_\eta} \mid \mu_j^\eta g_j^\eta \left(x^\eta, x^{-\eta}\right) \mid \leq \epsilon, \quad \mu_j^\eta \geq -\epsilon, \ j = 1, \cdots q_\eta, \tag{4.10}$$

$$\sum_{i=1}^{p_\eta} \lambda_i^\eta = 1. \tag{4.11}$$

Note that, when $\epsilon = 0$, the above definition reduces to that of an MGNEP-WKKT point. It is also worth noting that a point $\overline{x}$ satisfies the MGNEP-AWKKT condition if and only if there are sequences $\epsilon_k \geq 0$, $x^k \in \mathbb{R}^n$, $\epsilon_k \to 0$, $x^k \to \overline{x}$ such that $x^k$ is an $\epsilon_k$-inexact MGNEP-KKT point.

The algorithm described below try to find an $\epsilon$-inexact MGNEP-WKKT point in the sense of Definition 4.19 by calculating a sequence of $\epsilon_k$-inexact GNEP-WKKT points corresponding to MGNEP$(\sigma, \eta)$.

**Algorithm 1** AL-MGNEP: Augmented Lagrangian for MGNEPs.

**Step 0**: Let $\overline{\sigma} \in \mathbb{R}_+^q$, $0 < a < 1$, $b > 1$ and $\epsilon \geq 0$. Take $(x^0, \lambda^0, \mu^0) \in \mathbb{R}^n \times \mathbb{R}_+^p \times \mathbb{R}_+^q$ such that $\|\lambda^0\|_1 = 1$, $\sigma^0 \in [0, \overline{\sigma}]$, $\eta^0 \in \mathbb{R}_+^N$, $\epsilon_k \searrow 0$ and set $k := 0$.

**Step 1**: If $\left(x^k, \lambda^k, \mu^k\right)$ is an $\epsilon$-inexact MGNEP-WKKT point of the original MGNEP: STOP.

**Step 2**: Find an $\epsilon_{k+1}$-inexact MGNEP-WKKT point $\left(x^{k+1}, \lambda^{k+1}\right)$ of MGNEP$\left(\sigma^k, \eta^k\right)$

**Step 3**: Update for each player $\eta = 1, \cdots, N$:

$$\begin{aligned} \mu^{k+1,\eta} &= \max\{0, \sigma^{k,\eta} + \eta^{k,\eta} g^\eta \left(x^{k+1}\right)\}, \\ \sigma^{k+1,\eta} &= \text{Proj}_{[0,\overline{\sigma}]}\left(\sigma^{k,\eta} + \eta^{k,\eta} g^\eta \left(x^{k+1}\right)\right). \end{aligned}$$

**Step 4**: For each $\eta = 1, \cdots, N$ define:

$$Q^{k+1,\eta} = \min\{-g^\eta \left(x^{k+1}\right), \frac{\sigma^{k,\eta}}{\eta^{k,\eta}}\}.$$

If $\| Q^{k+1,\eta} \| \leq a \| Q^{k,\eta} \|$ then set $\eta^{k+1,\eta} = \eta^{k,\eta}$, else set $\eta^{k+1,\eta} = b\eta^{k,\eta}$.

**Step 5**: Choose $\sigma^{k+1} \in [0, \overline{\sigma}]$, Set $k := k + 1$ and go to Step 1.

For the sake of simplicity, we will assume that for each $\eta = 1, \cdots, N$, $p_\eta = p$. Also, we will take the same $a$, $b$ for each player.

The following result provides a relationship between Algorithm 1 and the feasibility set of MGNEP. More specifically, we demonstrate that, under specific assumptions, every limit point of our algorithm is a feasible point of MGNEP.

**Theorem 4.20.** *Let $\left(x^k\right)$ be generated by Algorithm 1, $K$ be a subset of $\mathbb{N}$ with $\lim\limits_{k\in K} x^k = \overline{x}$, and suppose that $\Omega \neq \emptyset$. Assume that one of the following statements holds*

1. *For all $\eta = 1, \cdots, N$, the constraints $g^\eta$ are convex.*
2. *$x^{k+1}$ is weak efficient point of MGNEP$\left(\sigma^k, \eta^k\right)$.*

*Then, $\overline{x} \in \Omega$.*

*Proof.* Let a subset $K \subseteq \mathbb{N}$ such that $\lim\limits_{k\in K} x^k = \overline{x}$, and fix $\eta = 1, \cdots, N$. Two cases have to be considered.

Assume the sequence $\left(\eta^{k,\eta}\right)_k$ is bounded, there exist $k_1$, such that for all $k > k_1$, condition $\| Q^{k+1,\eta} \| \leq a \| Q^{k,\eta} \|$ is satisfied. That is to say, $\lim\limits_{k\in K} \| Q^{k,\eta} \| = 0$. Thus,

$$\lim_{k\in K} Q_j^{k+1,\eta} = \lim_{k\in K} \min\{-g_j^\eta\left(x^{k+1}\right), \frac{\sigma_j^{k,\eta}}{\eta^{k,\eta}}\} = 0.$$

Since, $\left(\eta^{k,\eta}\right)_k$ is bounded and $\sigma_j^{k,\eta} \geq 0$, for all $j$ and $k$ we obtain $g_j^\eta\left(\overline{x}\right) = \lim\limits_k g_j^\eta\left(x^{k+1}\right) \leq 0$. Hence, $\overline{x} \in \Omega$.

Suppose now that $\left(\eta^{k,\eta}\right)_k$ is unbounded.

1. Assume that for all $\eta = 1, \cdots, N$, the constraints $g^\eta$ are convex. For $k \in K$, consider the sequence $\epsilon_{k+1}$-inexact MGNEP-WKKT point $\left(x^{k+1}, \lambda^{k+1}\right)$ of MGNEP$\left(\sigma^k, \eta^k\right)$. Using [45, Lemma 13], we have $\overline{x}$ is a GNEP-AKKT point for

$$\min_{x^\eta} \|g_+^\eta\left(x^\eta, x^{-\eta}\right)\|, \quad \eta = 1, \cdots, N.$$

From [4], $\overline{x}$ is a GNEP-KKT point for this scalar convex problem. Consequently, the GNEP-KKT point $\overline{x}$ represents the global minimum of the above minimization problems. Since $\Omega \neq \emptyset$, we obtain that for all $\eta = 1, \cdots, N$, $g_+^\eta\left(\overline{x}^\eta, \overline{x}^{-\eta}\right)$ must equal zero, which means $\overline{x} \in \Omega$.

2. Now assume instead that $x^{k+1}$ is weak efficient solution of MGNEP$\left(\sigma^k, \eta^k\right)$, and take $y \in \Omega$. From the definition of weak efficient solutions, we have

$$\min_{i=1}^{p}\Bigg\{ \quad f_i^{\eta}\left(x^{k+1}\right) + \frac{\eta^{k,\eta}}{2}\Bigg(\sum_{j=1}^{q_\eta}\left(g_j^{\eta}\left(x^{k+1}\right) + \frac{\sigma_i^{\eta}}{\eta^{k,\eta}}\right)_+\Bigg) \\ -f_i^{\eta}(y) - \frac{\eta^{k,\eta}}{2}\Bigg(\sum_{j=1}^{q_\eta}\left(g_j^{\eta}(y) + \frac{\sigma_i^{\eta}}{\eta^{k,\eta}}\right)_+\Bigg)\Bigg\} \quad \leq 0.$$

By multiplying both sides of the inequality by $\left(\eta^{k,\eta}\right)^{-1}$, we obtain

$$\min_{i=1}^{p}\Bigg\{ \quad \frac{f_i^{\eta}\left(x^{k+1}\right) - f_i^{\eta}(y)}{\eta^{k,\eta}} + \frac{1}{2}\Bigg(\sum_{j=1}^{q_\eta}\left(g_j^{\eta}\left(x^{k+1}\right) + \frac{\sigma_i^{\eta}}{\eta^{k,\eta}}\right)_+\Bigg) \\ -\frac{1}{2}\Bigg(\sum_{j=1}^{q_\eta}\left(g_j^{\eta}(y) + \frac{\sigma_i^{\eta}}{\eta^{k,\eta}}\right)_+\Bigg)\Bigg\} \quad \leq 0.$$

Remembering the boundedness of $\sigma_i^{\eta}$, using the continuity of $f^{\eta}$ and the fact that $\lim_k \eta^{k,\eta} = 0$, we get at the limit $\frac{1}{2}\left(\|g_+^{\eta}(\overline{x})\| - \|g_+^{\eta}(y)\|\right) \leq 0$. However, as $y \in \Omega$, $g_+^{\eta}(y) = 0$. Thus $g^{\eta}(\overline{x}) = 0$.

□

The primary global convergence conclusion is shown in the following theorem.

**Theorem 4.21.** *Let $\left(x^k\right)$ be generated by Algorithm 1 and let $K$ be a subset of $\mathbb{N}$ with $\lim_{k\in K} x^k = \overline{x} \in \Omega$. Then, $\overline{x}$ satisfies the MGNEP-AWKKT conditions. If in addition the MGNEP-CCP constraint qualification holds, then $\overline{x}$ is a MGNEP-WKKT point.*

*Proof.* Consider the sequence $\left(x^k\right)$ generated by Algorithm 1 and $K$ a subset of $\mathbb{N}$ such that $\lim_{k\in K} x^k = \overline{x}$, where $\overline{x} \in \Omega$. By Step 2, for each $\eta = 1, \cdots, N$ there exists $\lambda^{k,\eta} \in \mathbb{R}_+^p$ with $\sum_{i=1}^{p} \lambda_i^{k,\eta} = 1$ such that

$$\left\| \sum_{i=1}^{p} \lambda_i^{k,\eta}\left[\nabla_{x^\eta} f_i^{\eta}\left(x^k\right) + \sum_{j=1}^{q_\eta} \mu_j^{k,\eta} \nabla_{x^\eta} g_j^{\eta}\left(x^k\right)\right] \right\| \leq \epsilon_k \tag{4.12}$$

where $\mu_j^{k,\eta} = \max\{0, \sigma_j^{k-1,\eta} + \eta^{k-1,\eta} g_j^{\eta}\left(x^k\right)\}$ with $\mu_j^{k,\eta} \in [0, \overline{\sigma}_j]$. Since, $\sum_{i=1}^{p} \lambda_i^{k,\eta} = 1$ and $\sum_{j=1}^{q_\eta} \mu_j^{k,\eta} \nabla_{x^\eta} g_j^{\eta}\left(x^k\right)$ is independent of indices $i$ we obtain

$$\left\| \sum_{i=1}^{p} \lambda_i^{k,\eta} \nabla_{x^\eta} f_i^{\eta}\left(x^k\right) + \sum_{j=1}^{q_\eta} \mu_j^{k,\eta} \nabla_{x^\eta} g_j^{\eta}\left(x^k\right) \right\| \leq \epsilon_k.$$

On the other hand, by the MGENP feasibility of $\overline{x}$, we obtain from [9, Theorem 4.1] that for each $\eta = 1, \cdots, N$

$$g_j^{\eta}(\overline{x}) < 0 \Rightarrow \mu^{k+1,\eta} = 0, \ j = 1, \cdots, q,$$

for sufficiently large $k$. Consequently, $\overline{x}$ is an MGNEP-AWKKT point. Now, under the MGNEP-CCP constraint qualification, and from Theorem 4.14 one has that $\overline{x}$ is a MGNEP-WKKT point.

□

In the end, we show that the sequence of multipliers generated by the proposed Algorithm 1 is bounded. To proceed, we provide a basic extension of the Mangasarian-Fromovitz constraint qualification (MFCQ) by examining the problem of each player with the constraint $\Omega^{\eta}(x^{-\eta})$ separately.

**Definition 4.22.** We say that $\overline{x} \in \Omega$ satisfies the standard-MFCQ if for each $\eta = 1, \cdots, N$

$$\left\{\nabla_{x^{\eta}} g_j^{\eta}(\overline{x})\right\}, \quad \forall j \in J^{\eta}(\overline{x})$$

is positively linearly independent in $\mathbb{R}^{q_{\eta}}$.

Note that MGNEP-MFCQ and the standard-MFCQ are the same because the MFCQ condition makes no specific use of neighborhoods.

**Theorem 4.23.** *Let $\left(x^k\right)$ be a sequence generated by Algorithm 1 and let $K$ be a subset of $\mathbb{N}$ with $\lim\limits_{k\in K} x^k = \overline{x} \in \Omega$. Assume that the MGNEP-MFCQ holds at $\overline{x}$. Then, the dual sequence $\left(\lambda^k, \mu^k\right) \subset \mathbb{R}_+^p \times \mathbb{R}_+^q$ is bounded on $K$.*

*Proof.* Let $\left(x^k\right)$ be generated by Algorithm 1 and $K$ be a subset of $\mathbb{N}$ such that $\lim\limits_{k\in K} x^k = \overline{x}$, where $\overline{x} \in \Omega$. From Step 2, for each $\eta = 1, \cdots, N$ there exists $\lambda^{k,\eta} \in \mathbb{R}_+^p$ with $\sum\limits_{i=1}^{p} \lambda_i^{k,\eta} = 1$ such that

$$\left\| \sum_{i=1}^{p} \lambda_i^{k,\eta} \left[ \nabla_{x^{\eta}} f_i^{\eta}\left(x^k\right) + \sum_{j=1}^{q_{\eta}} \mu_j^{k,\eta} \nabla_{x^{\eta}} g_j^{\eta}\left(x^k\right) \right] \right\| \leq \epsilon_k \tag{4.13}$$

where $\mu_j^{k,\eta} = \max 0, \sigma^{k-1,\eta} j + \eta^{k-1,\eta} g^{\eta} j\left(x^k\right)$ with $\mu_j^{k,\eta} \in [0, \overline{\sigma} j]$. Since $\sum\limits_{i=1}^{p} \lambda^{k,\eta} i = 1$ and $\sum\limits_{j=1}^{q_{\eta}} \mu_j^{k,\eta} \nabla_{x^{\eta}} g_j^{\eta}\left(x^k\right)$ is independent of indices $i$, we obtain:

$$\left\| \sum_{i=1}^{p} \lambda i^{k,\eta} \nabla_{x^{\eta}} f^{\eta} i\left(x^k\right) + \sum_{j=1}^{q_{\eta}} \mu j^{k,\eta} \nabla_{x^{\eta}} g^{\eta} j\left(x^k\right) \right\| \leq \epsilon k.$$

Now, fix $\eta \in 1, \cdots, N$ and assume that the sequence $\left(\mu^{k,\eta}\right)$ is not bounded. Then, without losing generality, we can assume that $\left|\mu^{k,\eta}\right| \to +\infty$. Hence, $\frac{\mu^{k,\eta}}{\left|\mu^{k,\eta}\right|} \to \mu^{\eta}$ with $|\mu^{\eta}| = 1$. By dividing (4.13) by $\left|\mu^{k,\eta}\right|$, we get

$$\left| \sum_{i=1}^{p} \frac{\lambda i^{k,\eta}}{|\mu^{k,\eta}|} \nabla_{x^{\eta}} f^{\eta} i \left(x^{k}\right) + \sum_{j=1}^{q_{\eta}} \frac{\mu j^{k,\eta}}{|\mu^{k,\eta}|} \nabla_{x^{\eta}} g^{\eta} j \left(x^{k}\right) \right| \leq \frac{\varepsilon k}{|\mu^{k,\eta}|}.$$

Remembering that the sequence $\left(\lambda^{k,\eta}\right)$ is bounded and taking the limit on $K$, we deduce from the above inequality that

$$\sum_{j=1}^{q_{\eta}} \mu_{j}^{\eta} \nabla_{x^{\eta}} g_{j}^{\eta} \left(\overline{x}\right) = 0. \tag{4.14}$$

From the standard-MFCQ condition at $\overline{x}$, we get that $\mu_{j}^{\eta} = 0$ for all $j \in J^{\eta}$, which contradicts the fact that $|\mu^{\eta}| = 1$. In conclusion, for each player $\eta$, the dual sequence $\left(\lambda^{k,\eta}, \mu^{k,\eta}\right)$ is bounded. Hence, the dual sequence $\left(\lambda^{k}, \mu^{k}\right)$ is bounded. □

# 5 Implementation and Experiments

In this section, we provide implementation details of the AL-MGNEP algorithm and present the obtained results.

## 5.1 Implementation

Algorithm 1 was implemented in Python using the NumPy and SciPy libraries. The algorithm parameters are configured with $\overline{\sigma} = 10^{3}$, $a = 0.9$, $b = 4$, and $\epsilon = 10^{-5}$.

The initial guess for the decision variable $x$ is randomly set from the problem feasible set, and the scalarization multiplier vector $\lambda$ has initial entries set to $1/p$ where we have fixed $p = p_1 = \cdots = p_N$ for simplicity. For the initialization of the Lagrange multipliers, $\mu_{j}^{0,\eta}$ is set to 0 if $g_{j}^{\eta}(x) < 0$ for each player $\eta$; otherwise, they are computed through a nonnegative least-squares solution of the KKT condition $(K_1)$ for each player $\eta$. The Python `nnls` function is employed for this computation.

The vector of Lagrange multipliers $\sigma$ and the external penalty parameter $\eta$, used in the augmented Lagrangian function, are initialized as the zero vector and 1, respectively.

Clearly, Step 2 is the most critical stage of the AL-MGNEP algorithm. It involves finding an $\epsilon_{k+1}$-inexact MGNEP-WKKT point $\left(x^{k+1}, \lambda^{k+1}\right)$ corresponding to the unconstrained MGNEP having

$\mathcal{L}^{\eta}_{\eta^{\eta}}(\cdot, \sigma^{\eta})$ as objectives. In this step, the algorithm utilizes an optimization approach to minimize the squared norm of the function $F(x, \lambda)$, which characterizes the gradient of the augmented Lagrangian function for each player $\eta$:

$$
\begin{aligned}
F(x,\lambda) &= \begin{bmatrix} \nabla_{x^1}\mathcal{L}^1_{\eta^1}\left(x,\sigma^1\right) \\ \vdots \\ \nabla_{x^N}\mathcal{L}^N_{\eta^N}\left(x,\sigma^N\right) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^{p_1} \lambda_i^1 \nabla_{x^1} f_i^1\left(x^1, x^{-1}\right) \\ \quad + \sum_{j=1}^{q_1} \left(\eta^1 g_j^1\left(x^1, x^{-1}\right) + \sigma_j^1\right)_+ \nabla_{x^1} g_j^1\left(x^1, x^{-1}\right) \\ \vdots \\ \sum_{i=1}^{p_N} \lambda_i^N \nabla_{x^N} f_i^N\left(x^N, x^{-N}\right) \\ \quad + \sum_{j=1}^{q_N} \left(\eta^N g_j^N\left(x^N, x^{-N}\right) + \sigma_j^N\right)_+ \nabla_{x^N} g_j^N\left(x^N, x^{-N}\right) \end{bmatrix}
\end{aligned}
$$

To obtain an $\epsilon_{k+1}$-inexact MGNEP-WKKT point $\left(x^{k+1}, \lambda^{k+1}\right)$, we aim to approximately minimize $\|F(x,\lambda)\|^2$ subject to the constraints $\sum_{i=1}^{p_\eta} \lambda_i^\eta = 1$ and $\lambda_i^\eta \geq 0$ for all $\eta = 1, \cdots, N$ and $i = 1, \cdots, p_\eta$. The minimization process starts from the initial guess $\left(x^k, \lambda^k\right)$, and halts if the norm of $F$ falls below the threshold $\epsilon_{k+1}$.

The optimization is performed through the SciPy `minimize` function, where the default method is the Sequential Least Squares Programming (SLSQP) algorithm. The method can handle both bounds and equality constraints and wraps the SLSQP optimization subroutine originally implemented by Kraft [46].

## 5.2 Experiments

We define the following measures, denoted as $R_f$, $R_0$, and $R_c$, to assess the feasibility, optimality, and complementarity of the obtained estimation.

$$
\begin{aligned}
R_f &= \max_{\eta=1}^{N} \| g_+^{\eta}(x) \|_{\infty} \\
R_0 &= \max_{\eta=1}^{N} \left\| \sum_{i=1}^{p_\eta} \lambda_i^{\eta} \nabla_{x^\eta} f_i^{\eta}\left(x^{\eta}, x^{-\eta}\right) + \sum_{j=1}^{q_\eta} \mu_j^{\eta} \nabla_{x^\eta} g_j^{\eta}\left(x^{\eta}, x^{-\eta}\right) \right\|_{\infty} \\
R_c &= \max_{\eta=1}^{N} \sum_{j=1}^{q_\eta} \mid \mu_j^{\eta} g_j^{\eta}\left(x^{\eta}, x^{-\eta}\right) \mid .
\end{aligned}
$$

We also compute the norm value of $F$, the gradient of the augmented Lagrangian, at the obtained $\epsilon$-inexact WKKT point. For the computational work needed for convergence, we present the number of outer iterations as well the total number of inner iterations used in Step 2 of Algorithm 1 to determine an $\epsilon_{k+1}$-inexact MGNEP-WKKT point $\left(x^{k+1}, \lambda^{k+1}\right)$. For each experiment, we run Algorithm 1 starting with five different random feasible initial guesses.

We begin with the simple MGNEP from Example 4.11, which has strictly convex objectives and a simple nonconvex constraint for only one player. This problem has the point $\overline{x} = (0, 0)$ as the unique efficient solution.

The MGNEP-AWKKT is also satisfied at this point $(0, 0)$. Indeed, consider the sequence $x^k = \left(\dfrac{-k}{k+1}, \dfrac{1}{\sqrt{k+1}}\right)$, $\mu^{k,1} = \sqrt{k+1}$, $\lambda_1^{k,1}$ , $\lambda_2^{k,1}$ with $\lambda_1^{k,1} + \lambda_2^{k,1} = 1$ and $\lambda_1^{k,2}$, $\lambda_2^{k,2}$ with $\lambda_1^{k,2} + \lambda_2^{k,2} = 1$. We have

$$
\lambda_1^{k,1} x^{k,1} + \lambda_2^{k,1} x^{k,1} + \mu^{k,1} x^{k,2} = 0 \quad \text{and} \quad \lambda_1^{k,2} x^{k,2} + \lambda_2^{k,2} x^{k,2} = \frac{1}{\sqrt{k+1}} \to 0.
$$

The quality of the obtained inexact WKKT point as well the computational work needed to reach it are displayed in Table 4.1.

Algorithm 1 was able to consistently find $\epsilon$-inexact WKKT points for this example. The norm of $F(x, \lambda)$, representing the gradient of the augmented Lagrangian function, decreases throughout the iterations. The optimality metric $R_o$ converges to values close to zero. This convergence pattern verifies the algorithm's effectiveness in satisfying the KKT conditions in only few iterations.

We consider then the nonconvex MGNEP in Example 4.7, which have also $(0, 0)$ as a weak efficient Nash equilibrium. The MGNEP-AWKKT is also satisfied at this point $(0, 0)$ because

| **Test** | $x^0$ | **Inexact WKKT** | $\|F(x,\lambda)\|$ | $R_f$ | $R_o$ | $R_c$ | **Outer** | **Inner** |
|---|---|---|---|---|---|---|---|---|
| 1 | $(0.780, -0.780)$ | (3.24e-07, -4.81e-07) | 7.62e-04 | 0.0 | 4.81e-07 | 3.52e-18 | 4 | 48 |
| 2 | $(0.537, -0.537)$ | (6.40e-07, -4.29e-07) | 8.78e-04 | 0.0 | 6.40e-07 | 0.0 | 1 | 8 |
| 3 | $(0.156, -0.156)$ | (4.37e-07, -4.90e-07) | 8.10e-04 | 0.0 | 4.90e-07 | 0.0 | 1 | 18 |
| 4 | $(0.973, -0.973)$ | (4.80e-07, 1.81e-06) | 1.37e-03 | 8.68e-13 | 1.81e-06 | 8.92e-17 | 3 | 30 |
| 5 | $(0.795, -0.795)$ | (-7.09e-08, 4.26e-07) | 6.57e-04 | 0.0 | 4.26e-07 | 0.0 | 1 | 15 |

Table 4.1: Results of Algorithm 1 applied to Example 4.11 with different initial guesses for $x$.

**For Player** 1. We have $\nabla_{x^1} f_1^1(x^1, x^2) = -2x^1$, $\nabla_{x^1} f_2^1(x^1, x^2) = -6x^1$, and $\nabla_{x^1} g^1(x^1, x^2) = 2x^1$. Setting, $\left(x^{k,1}\right) = \left(\frac{1}{k}\right)$, $\left(\lambda_1^{k,1}\right) = \left(\frac{k}{k+1}\right)$, $\left(\lambda_2^{k,1}\right) = \left(\frac{1}{k+1}\right)$ and $\left(\mu^{k,1}\right) = \left(\frac{k+3}{k+1}\right)$ one has $\left(x^{k,1}\right) \to \overline{x}^1$, $\lambda_1^{k,1} + \lambda_2^{k,1} = 1$, and

$$\lambda_1^{k,1}\nabla_{x^1} f_1^1\left(x^{k,1}, \overline{x}^2\right) + \lambda_2^{k,1}\nabla_{x^1} f_2^1\left(x^{k,1}, \overline{x}^2\right) + \mu^{k,1}\nabla_{x^1} g^1\left(x^{k,1}, \overline{x}^2\right) \to 0.$$

**For Player** 2. We have $\nabla_{x^2} f_1^2(x^1, x^2) = -2x^2$, $\nabla_{x^2} f_2^2(x^1, x^2) = -1$, and $\nabla_{x^2} g^2(x^1, x^2) = 1$. Setting, $\left(x^{k,2}\right) = \left(\frac{1}{k-1}\right)$, $\left(\lambda_1^{k,2}\right) = \left(\frac{k-1}{2k+1}\right)$, $\left(\lambda_2^{k,2}\right) = \left(\frac{k+2}{2k+1}\right)$ and $\left(\mu^{k,2}\right) = \left(\frac{k+4}{2k+1}\right)$ one has $\left(x^{k,2}\right) \to \overline{x}^2$, $\lambda_1^{k,2} + \lambda_2^{k,2} = 1$,

$$\lambda_1^{k,2}\nabla_{x^2} f_1^2\left(\overline{x}^1, x^{k,2}\right) + \lambda_2^{k,2}\nabla_{x^2} f_2^2\left(\overline{x}^1, x^{k,2}\right) + \mu^{k,2}\nabla_{x^2} g^2\left(\overline{x}^1, x^{k,2}\right) \to 0.$$

We display in Table 4.2 the results when Algorithm 1 was able to find a point at which all KKT conditions were approximately satisfied for both players. For this nonconvex problem, the number of failures when the algorithm was not able to reach an $\epsilon$-inexact WKKT point depends on the algorithm parameters subject to the stopping criterion $\epsilon_k < 10^{-5}$. Examples of parameters that have a significant impact on the convergence of the algorithm are the values of $a$ and $b$. For instance, taking $a = 0.9$ and setting values of $b$ equal to 10, 4, and 2 will reduce the number of failures from 50% to 40% and 20%, respectively. Looking at the results shown in Table 4.2, we can see that the computational work required for convergence for this nonconvex problem is obviously more demanding than the convex case with an average of 36 outer iterations and 356 outer iterations.

| **Test** | $x^0$ | **Inexact WKKT** | $\|F(x,\lambda)\|$ | $R_f$ | $R_o$ | $R_c$ | **Outer** | **Inner** |
|---|---|---|---|---|---|---|---|---|
| 1 | $(0.706, 0.776)$ | (-2.61e-06, 4.97e-06) | 3.35e-03 | 0.0 | 9.93e-06 | 0.0 | 35 | 310 |
| 2 | $(0.325, 0.385)$ | (-4.16e-07, 4.83e-06) | 3.11e-03 | 0.0 | 9.66e-06 | 0.0 | 34 | 291 |
| 3 | $(0.646, 0.651)$ | (-2.57e-07, 6.67e-07) | 2.81e-03 | 0.0 | 7.74e-06 | 0.0 | 35 | 408 |
| 4 | $(0.075, 0.089)$ | (2.02e-06, 1.21e-06) | 3.14e-03 | 0.0 | 9.56e-06 | 0.0 | 41 | 381 |
| 5 | $(0.832, 0.718)$ | (-1.45e-06 2.33e-06) | 2.80e-03 | 0.0 | 7.30e-06 | 0.0 | 35 | 389 |

Table 4.2: Results of Algorithm 1 applied to Example 4.7 with different initial guess for $x$.

# 6 Conclusion

This study presented two types of solutions for the differentiable multiobjective generalized Nash equilibrium problem with inequality constraints: standard approximate weak Karush-Kuhn-Tucker (standard-AWKKT) points and MGNEP approximate weak Karush-Kuhn-Tucker (MGENP-AWKKT) points. Our findings revealed that the standard-AWKKT condition, although satisfied by every weak efficient point, proved to be too restrictive for generating the desired sequence required for an algorithm.

On the other hand, we illustrated that while the MGNEP-AWKKT condition does not serve as an optimality condition for general MGNEPs, it does hold true for some important classes such as jointly convex MGNEPs. Additionally, we highlighted the distinction between the MGNEP-AWKKT concept and the optimality-based standard-AWKKT concept. Consequently, algorithms generating MGNEP-AWKKT sequences may identify points that do not meet the true optimality criterion, thereby failing to provide a solution. To ensure that a limit point of an MGNEP-AWKKT sequence qualifies as an MGNEP-WKKT point, we recommended employing the MGNEP-CCP condition (Theorem 4.21). Owing to the generalized aspect of MGNEPs, our results naturally extend the corresponding ones in the literature involving approximate optimality conditions in generalized Nash equilibrium problems and multiobjective optimization.

Moreover, we proposed and investigated an augmented Lagrangian-type algorithm that solves MGNEPs without necessitating scalarization procedures. The convergence properties of this algorithm were demonstrated using the MGNEP-CCP condition. For convex objectives, the AL-MGNEP Algorithm was able to consistently find an $\epsilon$-inexact weak Karush-Kuhn-Tucker point, as exemplified by the ex-

periments on the first example in the experimental section. The algorithm successfully converges across different initial guesses for $x^0$, demonstrating robustness and reliability. For the nonconvex case, a parameters' tuning is required to reduce the number of failures.

Another perspective is the study of the approximate strong optimality conditions in multiobjective generalized Nash equilibrium problems.

# Conclusion and perspectives

In this thesis, we have concentrated on semivectorial bilevel optimization problems, which are a general type of bilevel optimization problem.

We demonstrated how the Charnes-Cooper transformation could be applied to derive necessary optimality conditions for semivectorial optimization problems with an efficiently solvable lower-level program (SVBOP). By reformulating the problem as a classical bilevel program, we were able to utilize tools from parametric analysis and sensitivity estimation of the lower-level value function. Specifically, analyzing Lipschitz continuity and subdifferential properties allowed obtaining optimality conditions without differentiability assumptions on the value function.

A promising avenue for future work is the algorithmic implementation of this value function reformulation approach to solve semivectorial optimization problems numerically. Prior literature had validated this methodology algorithmically when the lower-level involved scalar optimization; see the survey in [23]. With the increasing relevance of multiobjective decision-making, extending these computational techniques to problems where the lower-level involves vector optimization presents an opportunity for developing practical solution methods. Overall, the value function perspective adopted here laid the theoretical groundwork for addressing semivectorial programs while retaining their intrinsic multiobjective structure.

Also, we are concerned with a mathematical programming problem ($SVBOP$). We presented a concept of equilibrium for generalized Nash equilibrium problems with many goals and provided its scalarization formulation. Based on this, we looked at the relationship between problem ($SVBOP$)

and a related scalar programming problem (3.9), which is established by applying the kth-objective weighted-constraint technique to problem ($SVBOP$)'s multiobjective lower level problem. It has been proven that this relationship is non-hazardous when globally optimal solutions are considered; however, the analysis of locally optimal solutions is more sensitive. We exploited the equivalence of ($SVBOP$) and (3.9) in order to derive new necessary optimality conditions. Finally, we showed our discovery using an example.

We introduce two types of solutions to the differentiable multiobjective generalized Nash equilibrium problem with inequality constraints: standard approximate weak Karush-Kuhn-Tucker (standard-AWKKT) points and MGNEP approximate weak Karush-Kuhn-Tucker (MGNEP-AWKKT) points. We established that, while every weakly efficient point satisfies the standard-AWKKT condition, it is too restrictive to generate the sequence necessary by an algorithm.

In contrast, we demonstrated that although the MGNEP-AWKKT condition does not serve as a universal optimality condition for all MGNEPs, it holds true for significant classes such as jointly convex MGNEPs. We also emphasized the distinction between the MGNEP-AWKKT concept and the optimality-based standard-AWKKT concept. Consequently, algorithms generating MGNEP-AWKKT sequences may identify points that do not meet the true optimality criterion, potentially failing to provide a solution. To ensure that a limit point of an MGNEP-AWKKT sequence qualifies as an MGNEP-WKKT point, we recommended using the MGNEP-CCP condition (Theorem 4.21). Our results extend the existing literature on approximate optimality conditions in generalized Nash equilibrium problems and multiobjective optimization due to the generalized nature of MGNEPs.

Furthermore, we proposed and analyzed an augmented Lagrangian-type algorithm designed to solve MGNEPs without requiring scalarization procedures. The convergence properties of this algorithm were established using the MGNEP-CCP condition. For convex objectives, the AL-MGNEP algorithm consistently identified an $\epsilon$-inexact weak Karush-Kuhn-Tucker point, as demonstrated in experiments on the initial example in the experimental section. The algorithm exhibited robustness and reliability by converging across various initial guesses for $x^0$. However, for nonconvex cases, careful parameter tuning is necessary to minimize failures.

The work of this thesis opens the way to many questions, including the following open problems:

(1) Approximate Karush-Kuhn-Tucker Optimality Conditions of the semivectorial bilevel optimization

problem (SVBOP).

(2) The Karush-Kuhn-Tucker reformulation and Optimality Conditions of the Semivectorial Bilevel Optimization Problem (SVBOP).

(3) Charnes-Cooper scalarization to semivectorial bilevel optimization problems ($SVBOP$) with multiobjective generalized Nash equilibrium problem constraints.

(4) Approximate Optimality Conditions of the Strong Type in Generalized Multiobjective Nash Equilibrium Problems ($\mathrm{P}^{\eta}[x^{-\eta}]$) .

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