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Dérivations et applications en relation avec les dérivations dans certains types d'anneaux et d'algèbres

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DEDICATION

To my parents and my parents-in-Law

To my wife Fatiha,
whose encouragement and constant support have been invaluable,

and

my lovely children Nourellhouda, Hamza and Asmae

Avant Propos

Les travaux présentés dans cette thèse ont été effectués au laboratoire Mathématiques, Statistiques et Applications de la Faculté des Sciences de Rabat. Cette thèse a été réalisée sous la direction du professeur Driss BENNIS et codirection du professeur Zine El Abidine ABDELALI.

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RESUME

Cette thèse présente notre contribution à l'étude des dérivations et des applications en relation avec les dérivations dans certains types d'anneaux et d'algèbres. Ce domaine regroupe plusieurs axes de recherche très actifs. Dans notre étude doctorale, nous nous sommes intéressés aux axes suivants:

- Commutativité des anneaux vérifiant certaines identités différentielles concernant les dérivations et des applications additives en relation avec les dérivations.
- Etude des propriétés algébriques des anneaux qui établissent des relations entre diverses sortes de dérivations comme la dérivation de Jordan, la dérivation généralisée, la dérivation généralisée de Jordan, la dérivation généralisée de Lie, la (m, n)-dérivation de Jordan généralisée.
- Etude de la forme de diverses sortes de dérivations sur des constructions d'algèbres.

L'étude de quelques problèmes dans ces axes de recherches a donné lieu à quelques articles. Nous présentons cinq d'entre eux dans cette thèse.

Mots Clés.
Dérivation; dérivation intérieure; dérivation de Jordan; dérivation généralisée; dérivation généralisée de Jordan; f-dérivation généralisée; dérivation généralisée de Lie; (m, n)-dérivation de Jordan généralisée; (m, n)-multiplicateur de Jordan généralisé; extension triviale d'une algèbre .

SUMMARY

This thesis presents our contribution on the investigation of derivation and related maps on some rings and algebras. This domain of research brings together several very active axes of research. In our doctoral study, we are interested in the following axes:

- Commutativity of the rings satisfying certain differential identities concerning derivations and related maps.
- Study of the algebraic properties of rings establishing relations between various kinds of derivations such as Jordan derivation, generalized derivation, Jordan generalized derivation, Lie generalized derivation, generalized (m, n)-Jordan derivation.
- Study of the form of various kinds of derivations in algebra constructions.

The study of some problems in these axes of research gives rise to some articles. We present five of them in this thesis.

Key Words.
Derivation; inner derivation; Jordan derivation; generalized derivation; Jordan generalized derivation; f-generalized derivation; Lie generalized derivation; generalized (m, n)-Jordan derivation; generalized (m, n)-Jordan centralizer; trivial extension algebras.

Contents

Author's papers involved in this thesis :

1. D. Bennis and B. Fahid, Derivations and the first cohomology group of trivial extension algebras, Mediterr. J. Math. 14, (2017) doi:10.1007/s00009-017-0949-z
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1. D. Bennis, D. Anderson, B. Fahid, A. Shaiea, On n-trivial extensions of rings, Accepted for publication in Rocky Mountain Journal of Mathematics.
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5. Z. Abdelali, D. Bennis and B. Fahid, 2-trivial extension Banach algebras, In preparation.

Introduction

Cette thèse présente notre contribution à l'étude des dérivations et des applications en relation avec les dérivations sur certains types d'anneaux et d'algèbres. Rappelons qu'une application $d : R \longrightarrow R$ est une dérivation d'un anneau R si d est additive et satisfait la règle de Leibnitz (i.e., $d(ab) = d(a)b + ad(b)$ pour tous $a, b \in R$). La dérivation intérieure $d : R \longrightarrow R$ associée à un élément $a \in R$, définie par $d(x) = [x, a]$ pour tout $x \in R$, est un exemple classique de dérivation (où $[x, a] = xa - ax$ est le commutateur de x et a).

D'après [6] et [55], l'origine des dérivations remonte aux années trente du siècle dernier avec Hasse [56], Jacobson [63] et Teichmüller [97] (voir aussi les références classiques suivantes [68, 69, 95] de Kaplansky, Kolchin et Ritt, respectivement). Cependant, il semble difficile d'affirmer avec certitude lequel d'entre eux était le pionnier de ce domaine. Dans les années cinquante, la notion de dérivation a connu un nouvel essor considérable surtout après le travail de Posner [93] où il a prouvé que, sur un anneau premier[1] sans 2-torsion[2], si la composition de deux dérivations est une dérivation, alors l'une d'entre elles est nulle. Ce résultat, connu par le nom du premier théorème de Posner, a été développé et généralisé à différents contextes (voir, par exemple, Bergen [24], Chebotar [33], Chuang [36], Chuang et Lee [37], Hirano, Tominaga et Trezepizur [59], Hvala [61], Jensen [64], Lanski [74], Martindale and Miers [81] et Ye et Luh [106]). Dans le même article [93], Posner a prouvé que l'existence d'une dérivation centralisante non nulle f sur un anneau premier R (i.e., $[f(x), x] \in Z(R)$ pour tout $x \in R$, où $Z(R)$ désigne le centre de R) force l'anneau R à être commutatif. Ce résultat, connu par le nom du second théorème de Posner, a attiré l'attention de plusieurs chercheurs qui l'ont étendu à divers contextes. Par exemple, dans [10], Awtar a étendu le second théorème de Posner au cas des dérivations centralisantes sur des idéaux de Jordan non nuls ou des idéaux de Lie[3] non nuls stables par le carré. Dans [91], Oukhtite a

[1]Un anneau R est dit premier si pour tous $a, b \in R$, $aRb = \{0\}$ implique que $a = 0$ ou $b = 0$.

[2]Un anneau R est dit sans n-torsion, où $n \in \mathbb{N}^\star$, si pour tout $a \in R$, $na = 0$ implique que $a = 0$.

[3]Un sous-groupe additif U de R est dit un idéal de Lie (resp., de Jordan) de R si $[u, r] \in U$

étendu le second théorème de Posner au contexte des anneaux avec involution. Dans [27, Théorème B], Brešar a initié l'étude d'un contexte plus général en montrant que, pour un idéal à gauche non nul U d'un anneau premier R, s'il existe des dérivations $d \neq 0$ et g de R vérifiant $ud(u) - g(u)u \in Z(R)$ pour tout $u \in U$, alors R est commutatif. Voir aussi les articles de Mayne [82, 83, 84], l'article de Oukhtite et Mamouni [92] et celui de Vukman [99].

A partir des années 1990, la relation entre les identités différentielles et la commutativité des anneaux a connu un nouveau développement en utilisant d'autres extensions et généralisations de la notion de dérivation comme la dérivation généralisée introduite par Brešar dans [26]. Rappelons qu'une application additive $F : R \longrightarrow R$ est appelée une dérivation généralisée s'il existe une dérivation $d : R \longrightarrow R$, appelée dérivation associée à F telle que $F(xy) = F(x)y + xd(y)$ pour tous $x, y \in R$. Comme exemple classique d'une dérivation généralisée, on cite la dérivation intérieure généralisée $f : R \longrightarrow R$ définie par $f(x) = ax + xb$ avec a, b des éléments de R. En 1998, Hvala [61] a étudié des identités liées aux dérivations généralisées (voir aussi [62]). D'autres résultats similaires peuvent être trouvés dans Komatsu et Nakajima [70], Nakajima [89] et Nakajima et Sapanci [90]. Récemment, Ali, De Filippis et Shujat [3] ont étudié d'autres identités avec des dérivations généralisées sur les idéaux unilatéraux d'un anneau semi-premier[4]. Précisément, ils ont montré le résultat suivant: Soit R un anneau semi-premier et U un idéal à gauche non nul de R. Si R admet une dérivation généralisée F, d est la dérivation associée à F tel que $d(U) \neq (0)$ alors R contient un idéal non nul centralisant, si l'une des conditions suivantes est vérifiée :

- R est sans 2-torsion et $F(xy) \in Z(R)$ pour tous $x, y \in U$, sauf si $F(U)U = UF(U) = Ud(U) = (0)$.
- $F(xy) \pm yx \in Z(R)$ pour tous $x, y \in U$.
- $F(xy) \pm [x, y] \in Z(R)$ pour tous $x, y \in U$.
- $F \neq 0$ et $F([x, y]) = 0$ pour tous $x, y \in U$, sauf si $Ud(U) = (0)$.
- $F \neq 0$ et $F([x, y]) \in Z(R)$ pour tous $x, y \in U$, sauf si $d(Z(R))U = (0)$ ou $Ud(U) = (0)$.

Dans notre recherche doctorale, nous avons étudié la commutativité d'un anneau premier satisfaisant certaines identités différentielles sur un idéal à gauche non nul (ceci sera présenté dans le chapitre 5). Nous nous sommes également

(resp., $u \circ r \in U$) pour tout $u \in U$ et $r \in R$ (où $u \circ r = ur + ru$ est l'anti-commutateur de u et r).

[4]Un anneau R est dit semi-premier si pour tout $a \in R$, $aRa = \{0\}$ implique que $a = 0$.

intéressés à l'étude de la relation entre certains types de dérivations (ceci sera présenté dans le chapitre 4). Ce domaine de recherche a été initié par Herstein [57] qui a prouvé que toute dérivation de Jordan sur un anneau premier sans 2-torsion est une dérivation. Rappelons qu'une dérivation de Jordan d'un anneau A dans un A-bimodule M est une application additive $d : A \longrightarrow M$ telle que $d(x \circ y) = d(x) \circ y + x \circ d(y)$ pour tous $x, y \in A$. Dans son article [57] , Herstein a défini une dérivation de Jordan d'un anneau A dans un A-bimodule M comme étant une application additive $d : A \longrightarrow M$ qui vérifie $d(a^2) = d(a)a + ad(a)$ pour tout $a \in A$. Il est facile de vérifier que toute dérivation de Jordan au sens de Herstein est une dérivation de Jordan. Cependant, la réciproque n'est pas vraie en générale. Il le sera dans le cas des modules sans 2-torsion.

Le problème de description des types de dérivations sur des constructions d'algèbres a aussi été l'un des objectifs de notre sujet de doctorat. Il aide principalement à construire de nouveaux exemples intéressants satisfaisant à des conditions préassignées. En particulier, l'extension triviale d'algèbres, qui est une généralisation d'algèbres des matrices triangulaires, a suscité l'intérêt de nombreux auteurs (voir Section 1.1 où nous rappelons les définitions des extensions triviales d'algèbres et les algèbres des matrices triangulaires "généralisées"). Par exemple, en 2002, Zhang [108] a travaillé sur une question ouverte concernant la moyennabilité faible des algèbres, et c'est en utilisant les extensions triviales d'algèbres qu'il a pu aboutir à une réponse "négative" à la question étudiée (voir aussi [9, 48, 85, 108]). Il importe aussi de mentionner que plusieurs auteurs ont étudié les groupes de cohomologie des extensions triviales d'algèbres et ainsi établi des conditions sur lesquelles l'algèbre satisfait la propriété "toute dérivation est intérieure" (voir par exemple [9, 14, 34, 39, 41, 49]). L'un des objectifs de notre sujet de doctorat était l'approfondissement de l'étude des dérivations et le premier groupe de cohomologie des extensions triviales d'algèbres (ceci sera présenté dans le chapitre 1). Dans le même contexte, nous avons aussi étudié la forme des dérivations généralisées de Jordan et des dérivations généralisées de Lie sur des extensions triviales d'algèbres (ceci sera présenté dans les chapitres 2 et 3, respectivement).

La présente thèse regroupe cinq chapitres couvrant cinq articles [19, 20, 21, 22, 23]. Elle incarne notre contribution à la théorie des dérivations et des applications en relation avec les dérivations dans certains types d'anneaux et d'algèbres. La thèse est organisée comme suit:

Dans le chapitre 1, nous étudions en détail la notion de la dérivation et le premier groupe de cohomologie des extensions triviales d'algèbres. Notre étude conduit à des généralisations des résultats connus et des résultats récents sur les dérivations et le premier groupe de cohomologie des extensions triviales d'algèbres et les algèbres des matrices triangulaires.

Ce chapitre est organisé comme suit:

Dans la section 1.1, nous rappelons la définition des extensions triviales d'algèbres et des algèbres des matrices triangulaires (généralisées), puis nous étudions la relation entre les extensions triviales d'algèbres et les algèbres des matrices triangulaires. Plus précisément, nous donnons des conditions pour lesquelles une extension triviale est isomorphe à une algèbre des matrices triangulaires (voir Proposition 1.1.1 et Corollaires 1.1.2 et 1.1.3).

Dans la section 1.2, nous étudions des dérivations sur des extensions triviales d'algèbres (voir Lemme 1.2.1, Définitions 1.2.4 et 1.2.6), ce qui nous permet de décrire la forme des dérivations intérieures sur des extensions triviales d'algèbres (Théorème 1.2.12).

La section 1.3 est consacrée à l'étude du premier groupe de cohomologie des extensions triviales d'algèbres. Dans ce contexte, la notion du premier groupe restreint de cohomologie est introduite (voir Section 1.3). Dans la Proposition 1.1.1, nous rattachons le premier groupe de cohomologie restreint au premier groupe de cohomologie classique, et nous remarquons que l'étude du premier groupe de cohomologie des extensions triviales d'algèbres est basée sur l'étude des premiers groupes de cohomologie restreints (Théorème 1.3.2). Ceci conduit à une généralisation à la fois du résultat classique [38, Théorème 5.5] et du résultat récent [9, Théorème 4.4] qui ont utilisé des arguments purement homologique (voir Corollaire 1.3.3). En conséquence, nous obtenons une caractérisation des extensions triviales d'algèbres sur lesquelles chaque dérivation est intérieure (voir Corollaires 1.3.8 et 1.3.9).

Dans le chapitre 2, nous présentons un travail où on a étudié la forme des dérivations généralisées de Jordan sur des extensions triviales d'algèbres. Ce type d'applications a été introduit par Li et Benkovič, dans [76], comme analogue de la notion de dérivation généralisée pour les dérivations de Jordan comme suit: Une application linéaire $f : A \longrightarrow M$ est appelée une dérivation généralisée de Jordan s'il existe une application linéaire $d : A \longrightarrow M$ telle que $f(x \circ y) = f(x) \circ y + x \circ d(y)$ pour tous $x, y \in A$. Notons que chaque dérivation de Jordan est une dérivation généralisée de Jordan, mais les dérivations généralisées ne sont pas nécessairement des dérivations généralisées de Jordan. Li et Benkovič [76] ont montré que, sous certaines conditions, toute dérivation généralisée de Jordan d'une algèbre des matrices triangulaires est une dérivation généralisée. L'un de nos résultats principaux (Théorème 2.2.19) généralise ce dernier résultat comme suit: Pour une algèbre sans 2-torsion A et un A-bimodule M sans 2-torsion, s'il existe un idempotent non trivial e

dans A tel que $eme' = m$ pour tout $m \in M$, où $e' = 1 - e$, et tel que $eAe'Ae = \{0\} = e'AeAe'$ et $e'r.Ann_A(M)e' = \{0\} = el.Ann_A(M)e$, alors chaque dérivation généralisée de Jordan sur l'extension triviale $A \ltimes M$ d'une algèbre A peut être écrite comme une somme d'une dérivation généralisée et d'une antidérivation[5]. Ce résultat peut être aussi considéré comme une généralisation du résultat principal de Ghahramani donné dans [53].

Chapitre 2 est organisé comme suit:

Dans la section 2, nous caractérisons la forme générale des dérivations généralisées de Jordan, des dérivations généralisées et des antidérivations sur des extensions triviales d'algèbres (voir Lemmes 2.2.1, 2.2.2 et 2.2.3). Ensuite, nous caractérisons en termes de leurs composantes quand chaque dérivation généralisée de Jordan sur une extension triviale d'une algèbre peut être écrite comme une somme d'une dérivation généralisée et d'une antiderivation (voir Théorème 2.2.5). Cette approche nous permet de traiter séparément chaque composante d'une dérivation généralisée de Jordan sur une extension triviale d'une algèbre. La méthode suivie nous a permis de trouver et traiter de nouvelles situations si on compare avec [53], et dans lesquelles des dérivations généralisées de Jordan sont décrites (voir Théorèmes 2.2.7 et 2.2.21).

Dans [12], Benkovič a introduit la notion de f-dérivations qui unifie plusieurs sortes de dérivations (voir Section 2.3 pour plus de détails). Il a prouvé, sous certaines conditions, que chaque f-dérivation est une dérivation de Jordan [12, Théorème 1.3]. Naturellement, on peut se demander s'il existe une version généralisée de la f-dérivation. Dans la section 2.3, nous proposons une définition d'une f-dérivation généralisée et nous étendons le resultat de Benkovič [12, Théorème 1.3] à ce nouveau contexte (voir Théorème 2.3.1).

Dans le chapitre 3, nous étudions le problème de la description de la forme des dérivations généralisées de Lie sur des extensions triviales d'algèbres. Une application linéaire $T : A \longrightarrow A$ est dite une *L-dérivation généralisée de Lie* (ou simplement une dérivation généralisée de Lie), où $L : A \longrightarrow A$ est une application linéaire, si

$$T([a, b]) = [T(a), b] + [a, L(b)] \quad (a, b \in A). \tag{1}$$

Alors, quand $L = T$, T est la dérivation de Lie classique, et quand $L = 0$, T est un multiplicateur de Lie (i.e., $T([a, b]) = [T(a), b]$ pour tous $a, b \in A$). Ainsi, la dérivation généralisée de Lie, introduite comme étant un cas particulier de la f-dérivation généralisée, peut être vue comme une version "généralisée" de la

[5]Rappelons qu'une application additive $d : R \longrightarrow R$ est dite antidérivation de l'anneau R si d vérifie $d(ab) = d(b)a + bd(a)$ pour tous $a, b \in R$.

dérivation de Lie. Dans le résultat principal du chapitre 3, nous montrons, sous certaines conditions, que toute dérivation généralisée de Lie sur une extension triviale d'une algèbre est une somme d'une dérivation généralisée et d'une application centralisante qui s'annule sur tous les commutateurs (voir Théorème 3.2.16). Ce résultat étend l'étude des dérivations de Lie sur des extensions triviales d' algèbres faite dans [87]. En conséquence, nous caractérisons la dérivation généralisée de Lie sur une algèbre des matrices triangulaires (voir Corollaire 3.2.17).

Dans le chapitre 4, nous donnons une réponse affirmative aux deux conjectures sur la (m,n)-dérivation de Jordan généralisée et le (m,n)-multiplicateur de Jordan généralisé. Rappelons que le concept de la (m,n)-dérivation de Jordan généralisée a été introduit par Ali et Fošner dans [4] comme suit: Soit $m,n \geq 0$ deux entiers avec $m+n \neq 0$. Une application additive $F : R \longrightarrow R$ est une (m,n)-dérivation de Jordan généralisée s'il existe une (m,n)-dérivation de Jordan $d : R \longrightarrow R$ telle que $(m+n)F(x^2) = 2mF(x)x + 2nxd(x)$ pour tous $x, y \in R$. En outre, Fošner [50] a introduit le concept du (m,n)-multiplicateur de Jordan généralisé comme suit: Soit $m,n \geq 0$ deux entiers avec $m+n \neq 0$. Une application additive $T : R \longrightarrow R$ est un (m,n)-multiplicateur de Jordan généralisé s'il existe un (m,n)-multiplicateur de Jordan $T_0 : R \longrightarrow R$ tel que $(m+n)T(x^2) = mT(x)x + nxT_0(x)$ pour tout $x \in R$.

Ce chapitre est organisé comme suit:

Dans la section 4.1, nous présentons quelques définitions et quelques résultats qui ont mené aux deux conjectures suivantes:

Conjecture. ([4], Conjecture 1) Soient R un anneau semi-premier avec des restrictions de torsion appropriées et $F : R \longrightarrow R$ une (m,n)-dérivation de Jordan généralisée non nulle (où $m,n \geq 1$ deux entiers). Alors F est une dérivation de R à images dans $Z(R)$.

Conjecture. ([50], Conjecture 1) Soient R un anneau semi-premier avec des restrictions de torsion appropriées et $T : R \longrightarrow R$ un (m,n)-multiplicateur de Jordan généralisé (où $m,n \geq 1$ deux entiers). Alors T est un multiplicateur.

Ensuite, nous présentons nos résultats principaux où nous donnons une réponse affirmative aux deux conjectures ci-dessus (Théorèmes 4.1.3 et 4.1.6):

La section 4.2 est consacrée à la preuve de nos théorèmes principaux.

Dans le dernier chapitre, nous étudions la commutativité d'un anneau premier satisfaisant certaines identités différentielles sur un idéal à gauche non nul. Comme mentionné précédemment, motivés par le succès qu'a connu le second théorème de Posner, plusieurs auteurs ont introduit un nouveau type d'identités différentielles. Dans ce contexte, Ashraf et Rehman [8] prouvent qu'un anneau premier R avec un idéal non nul I doit être commutatif, si R admet une dérivation non nulle d vérifiant $d(xy) - xy \in Z(R)$ pour tous $x, y \in I$ ou $d(xy) + xy \in Z(R)$ pour tous $x, y \in I$. Ashraf, A. Ali et S. Ali [5] ont étudié ces identités dans le cas où d est une dérivation généralisée. Dhara, Rehman et Raza [46] ont utilisé des identités différentielles plus générales. Précisément, ils ont montré que pour un idéal de Lie non nul stable par le carré U d'un anneau premier R, si R admet des dérivations généralisées non nulles F, G et H satisfaisant $F(x)G(y) \pm H(xy) \in Z(R)$ ou $F(x)F(y) \pm H(yx) \in Z(R)$ pour tous $x, y \in U$, alors $U \subseteq Z(R)$. Pour d'autres travaux traitant des types similaires à ces identités, voir aussi [2, 43, 44, 45, 98]. Naturellement, on peut se demander si nous obtenons la même conclusion que celle de Dhara, Rehman et Raza si nous remplaçons l'idéal de Lie non nul stable par le carré de l'anneau premier R par d'autres sous-ensembles particuliers de R à savoir, les idéaux de Jordan, les idéaux et les idéaux unilatéraux. Cependant, l'étude des identités sur les idéaux de Jordan et sur les idéaux peut être considérée comme un cas particulier de l'étude de ces identités sur les idéaux de Lie stables par le carré. En effet, il est clair que chaque idéal est un idéal de Lie stable par le carré et, d'après [58, Théorème1.1], tout idéal de Jordan non nul d'un anneaux semi-premier sans 2-torsion contient un idéal non nul. Dans le chapitre 5, nous montrons que tout idéal de Lie stable par le carré d'un anneau premier sans 2-torsion R contient aussi un idéal non nul de R (Proposition 5.1.5). Cela signifie que l'étude des identités sur les idéaux unilatéraux peut être considérée comme le cas optimal. Dans ce contexte, nous montrons que R sera commutatif si nous considérons, dans les identités de Dhara, Rehman et Raza, seulement des idéaux à gauche au lieu des idéaux de Lie stables par le carré (voir Théorèmes 5.2.1 et 5.2.6).

Introduction

This thesis presents our contribution on the domain of derivations and related mappings on some rings and algebras. Recall that a map $d : R \longrightarrow R$ is a derivation of a ring R if d is additive and satisfies the Leibnitz' rule; that is $d(ab) = d(a)b + ad(b)$ for all $a, b \in R$. The inner derivation $d : R \longrightarrow R$ associated to an element $a \in R$, defined by $d(x) = [x, a]$ for all $x \in R$, is a classical example of a derivation on a ring (where $[x, a] = xa - ax$ is called the commutator of x and a).

Following [6] and [55], the origin of derivations dates back to the thirties of the last century with Hasse [56], Jacobson [63] and Teichmüller [97] (see also the following classical references [68, 69, 95] of Kaplansky, Kolchin and Ritt, respectively). However, it seems difficult to affirm with certainty which of them was the pioneer of this domain. In the fifties, the notion of derivation has got an over tremendous development after Posner's work [93] in which he proved that, in a 2-torsion free[6] prime ring[7], if the iterate of two derivations is a derivation, then one of them must be zero. This result, known by Posner's first theorem, have been generalized to different contexts (see, for example, Bergen [24], Chebotar [33], Chuang [36], Chuang and Lee [37], Hirano, Tominaga and Trezepizur [59], Hvala [61], Jensen [64], Lanski [74], Martindale and Miers [81] and Ye et Luh [106]). In the same paper [93], Posner proved that the existence of a nonzero centralizing derivation f on a prime ring R (i.e.,$[f(x), x] \in Z(R)$ for all $x \in R$, where $Z(R)$ denotes the center of R) forces the ring R to be commutative. This result is now known by Posner's second theorem. Since then, several authors have been interested in extending or generalizing this result to different contexts. For instance, Awtar, in [10], extended Posner's second theorem to the case of derivations centralizing on either nonzero Jordan ideals or nonzero square closed Lie ideals[8]. In [91], Oukhtite extended Posner's

[6] A ring R is said to be n-torsion free, where $n \in \mathbb{N}^\star$, if for all $a \in R$, $na = 0$ implies that $a = 0$.

[7] A ring R is prime if for all $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$.

[8] An additive subgroup U of R is said to be a Lie (resp., Jordan) ideal of R if $[u, r] \in U$ (resp., $u \circ r \in U$) for all $u \in U$ and $r \in R$ (where $u \circ r = ur + ru$ is the anticommutator of u and r).

second theorem to the context of rings with involution. In [27, Theorem B], Brešar initiated the study of a more general context by proving, for a nonzero left ideal U of a prime ring R, if there are derivations $d \neq 0$ and g of R that satisfy $ud(u) - g(u)u \in Z(R)$ for all $u \in U$, then R is commutative. See also Mayne's papers [82, 83, 84], Oukhtite and Mamouni's paper [92] and Vukman's one [99].

Since the nineteen of the last century the relation of identities related to derivation and the commutativity of rings has know new developments, namely using other types of derivations including the generalized derivation introduced by Brešar in [26]. Recall that an additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$, called the associated derivation of F, such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. A generalized inner derivation (i.e., a mapping of the form $f(x) = ax + xb$, with $a, b \in R$ are two fixed elements) is a classical example of a generalized derivation. The notion of generalized derivations covers both the notions of a derivation and of a left multiplier (i.e., an additive mapping $f : R \longrightarrow R$ satisfying $f(xy) = f(x)y$ for all $x, y \in R$). In 1998, Hvala [61] studied identities related to generalized derivations (see also [62]). Other similar results can be found in Komatsu and Nakajima [70], Nakajima [89] and Nakajima and Sapanci [90]. Recently, Ali, De Filippis and Shujat [3] studied other identities with generalized derivations on one sided ideals of a semiprime ring[9]. They proved the following result: Let R be a semiprime ring and U be a nonzero left ideal of R. If R admits a generalized derivation F, d is the derivation associated with F such that $d(U) \neq (0)$ then R contains some nonzero central ideal, if one of the following conditions holds :

- R is 2-torsion free and $F(xy) \in Z(R)$ for all $x, y \in U$, unless $F(U)U = UF(U) = Ud(U) = (0)$.
- $F(xy) \pm yx \in Z(R)$ for all $x, y \in U$.
- $F(xy) \pm [x, y] \in Z(R)$ for all $x, y \in U$.
- $F \neq 0$ and $F([x, y]) = 0$ for all $x, y \in U$, unless $Ud(U) = (0)$.
- $F \neq 0$ and $F([x, y]) \in Z(R)$ for all $x, y \in U$, unless either $d(Z(R))U = (0)$ or $Ud(U) = (0)$.

In this direction we have contributed by a work that investigates commutativity of a prime ring satisfying certain differential identities on a nonzero left ideal (this will be presented in Chapter 5). We have also interested in studying

[9] A ring R is semiprime if for all $a \in R$, $aRa = \{0\}$ implies that $a = 0$.

relations between some sorts of derivations (this will be presented in chapter 4). This area of research was initiated by Herstein who proved in [57] that any Jordan derivation on a 2-torsion free prime ring is a derivation. Recall that a Jordan derivation d of a ring R into an R-bimodule M is an additive map $d : R \longrightarrow M$ such that $d(x \circ y) = d(x) \circ y + x \circ d(y)$ for all $x, y \in R$. In his paper [57], Herstein defined Jordan derivation as an additive map $d : R \longrightarrow M$ which satisfies $d(a^2) = d(a)a + ad(a)$ for all $a \in R$. It is easy to show that every Jordan derivation in the sense of Herstein is a Jordan derivation. However, the converse is not true in general. It holds true in the case of 2-torsion free modules.

The problem of describing various kind of derivation on some algebra constructions has also been one of our aims in this research doctoral. It mainly helps to construct new interesting examples of algebras satisfying preassigned conditions. In particular, the trivial extension algebras, which can be seen as a generalization of triangular algebras, has been of interest by many authors (see Section 1.1 in which we recall the definitions of the trivial extension algebra and the triangular matrix algebra "generalized"). For instance, in 2002, Zhang [108] worked on an open question about the weak amenability of algebras, and using trivial extension algebras, he was able to come up with a "negative" response to the open question (see also [9, 48, 85, 108]). It is also important to mention that several authors have studied the cohomology groups of trivial extension algebras and so established conditions on which algebra satisfies the property "every derivation is inner" (see for example [9, 14, 34, 39, 41, 49]). One of the objectives of our doctoral subject was to deepen the study of derivations and the first group of cohomology of trivial extension algebras (see Chapter 1). In the same context, we have also studied the form of Jordan generalized derivations and Lie generalized derivations on some trivial extension algebras (this will be presented in chapters 2 and 3, respectively).

The present thesis falls into five chapters covering five papers [19, 20, 21, 22, 23]. It embodies our contribution of derivation and related maps on some rings and algebras. This thesis is organized as follows:

In chapter 1, we investigate in details derivation on trivial extension algebras. Our study leads to generalizations of both known and recent results on the first cohomology group of both trivial extension algebras and triangular matrix algebras.

This chapter is organized in the following way:

In Section 1.1, we recall the definitions of the trivial extension algebras and the (generalized) triangular matrix algebras, then we study the relation between trivial extension algebras and triangular matrix algebras. More precisely, we give conditions under which a trivial extension algebra is isomorph to a triangular matrix algebra (see Proposition 1.1.1 and Corollaries 1.1.2 and 1.1.3).

In Section 1.2, we investigate derivations on trivial extension algebras (see Lemma 1.2.1 and Definitions 1.2.4 and 1.2.6). As a consequence, we describe the form of inner derivations on trivial extension algebras (Theorem 1.2.12).

Section 1.3 is devoted to the study of the first cohomology group of trivial extension algebras. In this context, the notion of restricted first cohomology groups is introduced (see Section 1.3). In Proposition 1.1.1, we relate the restricted first cohomology group with the classical first cohomology group, and as a consequence we notice that the study of the first cohomology group of trivial extension algebras is based on the investigation of the restricted first cohomology groups. This leads to the following generalization of both the classical result [38, Theorem 5.5] and the recent result [9, Theorem 4.4] which used a purely homological argument (see Corollary 1.3.3). As a consequence, we get a characterization of trivial extension algebras on which every derivation is inner (see Corollaries 1.3.8 and 1.3.9).

In chapter 2, we mainly deal with the problem of describing the form of Jordan generalized derivations on trivial extension of algebras. This kind of mapping was introduced by Li and Benkovič, in [76], as a "Jordan mapping[10]" counterpart of the generalized derivations as follows: A linear map $f : A \longrightarrow M$ is called a Jordan generalized derivation if there exists a linear map $d : A \longrightarrow M$ such that $f(x \circ y) = f(x) \circ y + x \circ d(y)$ for all $x, y \in A$, where d is called an associated linear map of f. Notice that each Jordan derivation is a Jordan generalized derivation, but generalized derivations are not necessarily Jordan generalized derivations. Li and Benkovič showed, under some conditions, that every Jordan generalized derivation of a triangular algebra is a generalized derivation. One of our main results (Theorem 2.2.19) generalizes Li and Benkovič's result as follows: For a 2-torsion free algebra A and a 2-torsion free A-bimodule M, if there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, where $e' = 1 - e$, and such that $eAe'Ae = \{0\} = e'AeAe'$ and $e'r.Ann_A(M)e' = \{0\} = el.Ann_A(M)e$, then every Jordan generalized derivation on $A \ltimes M$ can be written as the sum of

[10]Recall that a Jordan derivation d of a ring A is a linear map $d : A \longrightarrow M$ such that $d(x \circ y) = d(x) \circ y + x \circ d(y)$ for all $x, y \in A$. Notice that every Jordan derivation in the sense of Herstien is a Jordan derivation. The converse is true in the case of 2-torsion free rings.

a generalized derivation and an antiderivation[11]. This result can be also considered as a "generalized" counterpart of Ghahramani's main result given in [53].

This chapter is organized in the following way:

In Section 2, we characterize the general form of Jordan generalized derivations, generalized derivations and antiderivations on trivial extension algebras (see Lemmas 2.2.1, 2.2.2 and 2.2.3). Then, we characterize in terms of the form of their components when every Jordan generalized derivation on a trivial extension algebra can be written as a sum of a generalized derivation and an antiderivation (see Theorem 2.2.5). This approach allows us to treat each component of a Jordan generalized derivation on a trivial extension algebra separately. The method followed led us to establish other new situations than those presented in [53] and in which Jordan generalized derivations are described (see Theorems 2.2.7 and 2.2.21).

In [12], Benkovič introduced the notion of f-derivations which unifies several kind of derivations including the classical derivations (see Section 2.3 for more details). Also Benkovič in [12, Theorem 1.3], proved (under some conditions) that every f-derivation is a Jordan derivation. Naturally one can ask whether there is a "generalized" counterpart of the f-derivations. In Section 2.3, we introduced the notion of a f-generalized derivation and generalized the result of Benkovič in [12, Theorem 1.3] (see Theorem 2.3.1).

In chapter 3, we investigate the problem of describing the form of Lie generalized derivations on trivial extension algebras. A linear map $T : A \longrightarrow A$ is said to be a *Lie generalized L-derivation* (or simply a Lie generalized derivation), where $L : A \longrightarrow A$ is a linear map, if

$$T([a, b]) = [T(a), b] + [a, L(b)] \quad (a, b \in A). \tag{2}$$

Then, when $L = T$, T is just the classical Lie derivation, and when $L = 0$, T is just a Lie centralizer; that is $T([a, b]) = [T(a), b]$ (or equivalently, $T([a, b]) = [a, T(b)]$) for all $a, b \in A$ (see [66]). Thus Lie generalized derivation, introduced as a particular case of the f-generalized derivation, can be seen as a "generalized" counterpart of Lie derivations. As a main result of chapter 3, we show, under some conditions, that every Lie generalized derivation on a trivial extension algebra is an sum of a generalized derivation and a central map which vanishes on all commutators (see Theorem 3.2.16). This result extends the study of Lie derivations on trivial extension algebras done in [87].

[11]Recall that a map $d : R \longrightarrow R$ is an antiderivation of a ring R if d is additive and satisfies $d(ab) = d(b)a + bd(a)$ for all $a, b \in R$.

As an application we characterize Lie generalized derivation on a triangular algebra (see Corollary 3.2.17).

In chapter 4, we give an affirmative answer to two conjectures of generalized (m,n)-Jordan derivations and generalized (m,n)-Jordan centralizers. Recall that the concept of generalized (m,n)-Jordan derivations was introduced by Ali and Fošner in [4] as follows: Let $m,n \geq 0$ be two fixed integers with $m+n \neq 0$. An additive mapping $F : R \longrightarrow R$ is called a generalized (m,n)-Jordan derivation if there exists an (m,n)-Jordan derivation $d : R \longrightarrow R$ such that $(m+n)F(x^2) = 2mF(x)x + 2nxd(x)$ holds for all $x, y \in R$. Also, Fošner [50] introduced the concept of generalized (m,n)-Jordan centralizers as follows: Let $m,n \geq 0$ be two fixed integers with $m+n \neq 0$. An additive mapping $T : R \longrightarrow R$ is called a generalized (m,n)-Jordan centralizer if there exists an (m,n)-Jordan centralizer $T_0 : R \longrightarrow R$ such that $(m+n)T(x^2) = mT(x)x + nxT_0(x)$ holds for all $x \in R$.

This chapter is organized in the following way:

In Section 4.1, we present some definitions and some results that led to the following two conjectures:

Conjecture. ([4], Conjecture 1) Let $m,n \geq 1$ be two fixed integers, let R be a semiprime ring with suitable torsion restrictions, and let $F : R \longrightarrow R$ be a nonzero generalized (m,n)-Jordan derivation. Then F is a derivation which maps R into $Z(R)$.
Conjecture. ([50], Conjecture 1) Let $m,n \geq 1$ be two fixed integers, let R be a semiprime ring with suitable torsion restrictions, and let $T : R \longrightarrow R$ be a generalized (m,n)-Jordan centralizer. Then T is a two-sided centralizer.

As mains results, we give an affirmative answer to the two conjectures mentioned above (Theorems 4.1.3 and 4.1.6).

Section 4.2 is devoted to the proof of our main theorems.

In the last chapter, we investigate commutativity of a prime ring satisfying certain differential identities on a nonzero left ideal. As mentioned before, motivated by the success that known Posner's second theorem, several authors have introduced new kind of differential identities. In this context, Ashraf and Rehman proved in [8] that a prime ring R with a nonzero ideal I must be commutative, if R admits a nonzero derivation d satisfying $d(xy) - xy \in Z(R)$ for all $x, y \in I$ or $d(xy) + xy \in Z(R)$ for all $x, y \in I$. In [5], Ashraf, A. Ali and S. Ali studied these identities in the case where d is a generalized derivation. In [46], Dhara, Rehman and Raza used more general differential identities. Precisely, they showed that for a nonzero square closed Lie ideal U of a prime ring R, if R admits nonzero generalized derivations F, G and

H satisfying $F(x)G(y) \pm H(xy) \in Z(R)$ or $F(x)F(y) \pm H(yx) \in Z(R)$ for all $x, y \in U$, then $U \subseteq Z(R)$. For other works dealing with similar identities, see [2, 43, 44, 45, 98]. Naturally, one can ask whether we get the same conclusion as the one of Dhara, Rehman and Raza's results if we replace the nonzero square closed Lie ideal of the prime ring R by other particular subsets of R such as Jordan ideals, (both sided) ideals and one sided ideals. But the study of the identities on Jordan ideals and (both sided) ideals can be considered as particular case of the study of these identities on square closed Lie ideals. Indeed, it is clear that every ideal is a square closed Lie ideal and, by [58, Theorem 1.1], every nonzero Jordan ideal of 2-torsion free semiprime rings contains a nonzero ideal. Then, only the "one sided ideals" case could be of interest. Our aim in this chapter is to show that R will be commutative if we consider, in Dhara, Rehman and Raza's identities, only nonzero left ideals instead of square closed Lie ideals (see Theorems 5.2.1 and 5.2.6). This is indeed can be seen as a generalization of Dhara, Rehman and Raza's results since we prove that any square closed Lie ideal L of a 2-torsion free prime ring R contains a nonzero ideal of R (see Proposition 5.1.5).

Chapter 1

Derivations and the first cohomology group of trivial extension algebras

Abstract. In this chapter, we investigate, in detail, derivations on trivial extension algebras. We obtain generalizations of both known results on derivations on triangular matrix algebras and a known result on first cohomology group of trivial extension algebras. As a consequence, we get the characterization of trivial extension algebras on which every derivation is inner. We show that, under some conditions, a trivial extension algebra on which every derivation is inner has necessarily a triangular matrix representation. This chapter starts with detailed study (with examples) of the relation between the trivial extension algebras and the triangular matrix algebras.

Throughout this chapter, $\mathcal{R}$ will denote a commutative ring with unity, $\mathcal{A}$ will be a unital $\mathcal{R}$-algebra with center $Z(\mathcal{A})$ and $\mathcal{M}$ will be a unital $\mathcal{A}$-bimodule.

1.1 Trivial extension algebras and triangular matrix algebras

This section is devoted to a discussion on the relation between trivial extension algebras and triangular matrix algebras. First, we recall the definition of these classical constructions and we give some basic known results.

Let A and B be two $\mathcal{R}$-algebras and let M be an (A, B)-bimodule. The set

$$\mathrm{Tri}(A; M; B) = \{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mid a \in A, m \in M, b \in B\}$$

endowed with the usual matrix operations is an $\mathcal{R}$-algebra called a triangular matrix $\mathcal{R}$-algebra (see [25]and [34] for more details about this construction). Following [25, Chapter 5], an algebra S is said to have a triangular matrix representation if S is isomorphic to a triangular matrix algebra. By [25, Theorem 5.1.4], a unital algebra $\mathcal{A}$ has a triangular matrix representation if there exists a non-trivial idempotent $e \in \mathcal{A}$ such that $(1-e)\mathcal{A}e = 0$. Namely, in this case, $\mathcal{A}$ is isomorphic to $\mathrm{Tri}(e\mathcal{A}e; e\mathcal{A}(1-e); (1-e)\mathcal{A}(1-e))$. Clearly, a triangular matrix algebra is an example of a noncommutative algebra; its center $Z(\mathrm{Tri}(A; M; B))$ is the set of elements $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a \in Z(A)$ and $b \in Z(B)$, such that $am = mb$ for all $m \in M$. As interesting examples of triangular matrix algebras one can cite the (classical) upper triangular matrix algebras, the block upper triangular matrix algebras, one-point extension algebras and the nest algebras (see, for instance, [14]).

As shown below, a triangular algebra can be obtained as a special case of trivial extension algebra. Recall that the direct product $\mathcal{A} \times \mathcal{M}$ together with the addition pairwise scalar product and the algebra multiplication defined by $(a, m)(b, n) = (ab, an + mb)$ for all $a, b \in \mathcal{A}$ and $m, n \in \mathcal{M}$, is a unital algebra which is called a trivial extension of $\mathcal{A}$ by $\mathcal{M}$ and will be denoted by $\mathcal{A} \ltimes \mathcal{M}$. The center of a trivial extension algebra $\mathcal{A} \ltimes \mathcal{M}$ is determined as follows:

$$Z(\mathcal{A}\ltimes\mathcal{M}) = \{(a, m) \mid a \in Z(\mathcal{A}) \text{ and } [b, m] = 0 = [a, y] \text{ for all } b \in \mathcal{A} \text{ and } y \in \mathcal{M}\}.$$

Note also that $Z(\mathcal{A} \ltimes \mathcal{M}) = \pi_{\mathcal{A}}(Z(\mathcal{A} \ltimes \mathcal{M})) \times \pi_{\mathcal{M}}(Z(\mathcal{A} \ltimes \mathcal{M}))$, where $\pi_{\mathcal{A}} : \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A}$ and $\pi_{\mathcal{M}} : \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{M}$ are the natural projections given by $\pi_{\mathcal{A}}(a, m) = a$ and $\pi_{\mathcal{A}}(a, m) = m$ for all $(a, m) \in \mathcal{A} \ltimes \mathcal{M}$ (see [87]).

It is well known that every triangular matrix algebra can be viewed as a trivial extension algebra. Indeed, $\mathrm{Tri}(A; M; B)$ is isomorphic to $(A \times B) \ltimes M$ as an $\mathcal{R}$-algebra where M is viewed as an $A \times B$-bimodule via the module actions given by $(a, b)m = am$ and $m(a, b) = mb$ for all $(a, b) \in A \times B$ and $m \in M$. However, a trivial extension algebra has not necessarily a triangular matrix representation. To give an appropriate example, we set the following obvious but important result.

Proposition 1.1.1 *A trivial extension algebra $\mathcal{A} \ltimes \mathcal{M}$ has a triangular matrix representation if and only if there exists a non-trivial idempotent e of $\mathcal{A}$ such that $(1-e)\mathcal{A}e = 0$ and $(1-e)\mathcal{M}e = 0$. In particular, if $\mathcal{A} \ltimes \mathcal{M}$ has a triangular matrix representation, then so has $\mathcal{A}$.*

Proof. If $\mathcal{A} \ltimes \mathcal{M}$ has a triangular matrix representation, then there exists a non-trivial idempotent $E = (e, m) \in \mathcal{A} \ltimes \mathcal{M}$ such that $(1 - E)\mathcal{A} \ltimes \mathcal{M}E = 0$. Clearly, e is a non-trivial idempotent of $\mathcal{A}$ which satisfies $(1-e)\mathcal{A}e = 0$ and $(1-e)\mathcal{M}e = 0$.
The converse holds by considering the non-trivial idempotent $(e, 0)$ of $\mathcal{A} \ltimes \mathcal{M}$.
■

As noted above the decision of whether an algebra S has a triangular matrix representation depends on the existence of an appropriate idempotent. Moreover, if such an idempotent exists, say e ($fSe = 0$, where $f = 1 - e$), then, for $M := eSf$, $emf = m$ for every $m \in M$. This condition on $\mathcal{M}$ plays a crucial role in proving some interesting results (see, for instance, [14, 53, 87]). In [53, Example 3.13] and [87, Example 2.6], examples of trivial extension algebras $\mathcal{A} \ltimes \mathcal{M}$ with the particular kind of idempotent of $\mathcal{A}$ without having a triangular matrix representation are given. A deep observation of these examples leads to consider the following particular case of Proposition 1.1.1.
In the following result, we use the following well-known fact: If $\mathcal{N}$ is an $\mathcal{A}$-bimodule, then it can be used to define an $\mathcal{A} \ltimes \mathcal{M}$-bimodule via the module actions given by $(a, m)n = an$ and $n(a, m) = na$ for all $(a, m) \in \mathcal{A} \ltimes \mathcal{M}$ and $n \in \mathcal{N}$.

Corollary 1.1.2 *For an $\mathcal{A}$-bimodule $\mathcal{N}$, the following assertions are equivalent.*

1. *The trivial extension algebra $(\mathcal{A} \ltimes \mathcal{M}) \ltimes \mathcal{N}$ has a triangular matrix representation.*
2. *There exists a non-trivial idempotent e of $\mathcal{A}$ such that $(1 - e)\mathcal{A}e = 0$, $(1 - e)\mathcal{M}e = 0$ and $(1 - e)\mathcal{N}e = 0$.*

In particular, when $\mathcal{A} \ltimes \mathcal{M}$ is a triangular matrix algebra $\text{Tri}(A; M; B)$, we get the following result.

Corollary 1.1.3 *Let A and B be two $\mathcal{R}$-algebras and M be an (A, B)-bimodule. Let $\mathcal{S} = \text{Tri}(A; M; B)$ and $\mathcal{N}$ be an $\mathcal{S}$-bimodule. The following assertions are equivalent.*

1. *The trivial extension algebra $\mathcal{S} \ltimes \mathcal{N}$ has a triangular matrix representation.*
2. *There exists a non-trivial idempotent $e = (e_a, e_b) \in A \times B$ such that $(1 - e_a)Ae_a = 0$, $(1 - e_b)Be_b = 0$, $(1 - e_a)Me_b = 0$ and $(1 - e)\mathcal{N}e = 0$.*

In particular, if in addition, A and B have only trivial idempotents, then the idempotent e (in the second assertion) is necessarily $(1, 0)$.

Now, we can easily construct the desired example (compare with [53, Example 3.13] and [87, Example 2.6]).

Example 1.1.4 *Let A and B be two $\mathcal{R}$-algebras which have only trivial idempotents. Let M be both an (A, B)-bimodule and a (B, A)-bimodule. Let $\mathcal{S} :=$*

$\mathrm{Tri}(A; M; B)$. *We make M into an $\mathcal{S}$-bimodule by defining $((a,b),m)m' := bm'$ and $m'((a,b),m) := m'a$ for all $(a,b) \in A \times B$ and $m, m' \in M$. As an $\mathcal{S}$-bimodule, M will be denoted as $\mathcal{N}$.*
Then, using Corollary 1.1.3, one can show that the trivial extension of $\mathcal{S}$ by $\mathcal{N}$ does not have a triangular matrix representation, because the idempotent $F = ((0,1),0)$ of $\mathcal{S}$ satisfies $Fn(1-F) = n$ for every $n \in \mathcal{N}$.

The last condition is equivalent to some other interesting conditions as shown in the following result which can be proven easily (compare with the remark given before [87, Theorem 2.2]).

Proposition 1.1.5 *Consider a non-trivial idempotent e of an algebra S and set $f = 1 - e$. For an S-bimodule N, the following assertions are equivalent:*

1. *For every $m \in N$, $emf = m$.*
2. *For every $m \in N$, $fm = 0 = me$.*
3. *For every $m \in N$, $em = m = mf$.*
4. *For every $m \in N$ and $a \in S$, $am = eaem$ and $ma = mfaf$.*

In this chapter, we sometimes consider a trivial extension $\mathcal{A} \ltimes \mathcal{M}$ which satisfies the following property (satisfied by the ring of Example 1.1.4): There is a non-trivial idempotent e of $\mathcal{A}$ such that $e\mathcal{A}f = 0$ (where $f = 1-e$) and $emf = m$ for every $m \in \mathcal{M}$ (thus, $\mathcal{A} \ltimes \mathcal{M}$ satisfies all the equivalent properties of Proposition 1.1.5). We refer to such an algebra by a trivial extension of type $(\star)$ (this trivial extension algebras were first considered in [87]). Note that in this case, we have $(e,0)(a,m)(e,0) = (eae,0)$ and $(f,0)(a,m)(f,0) = (faf,0)$. Then, $(e,0)(\mathcal{A} \ltimes \mathcal{M})(e,0) \cong e\mathcal{A}e$ and $(f,0)(\mathcal{A} \ltimes \mathcal{M})(f,0) \cong f\mathcal{A}f$. In addition, using assertion (4) of Proposition 1.1.5, one can see that $\mathcal{M}$ is a left $e\mathcal{A}e$-module and a right $f\mathcal{A}f$-module via the module actions $am = eaem$ and $ma = mfaf$ for all $m \in \mathcal{M}$ and $a \in \mathcal{A}$.

1.2 Derivations on trivial extension algebras

In this section, we investigate derivations on trivial extension algebras.

We begin with [87, Lemma 2.1] which describes derivations on trivial extensions (see also [85, Proposition 2.2]). We state [87, Lemma 2.1] using the following terminology:

Recall that an $\mathcal{R}$-linear map $\mathcal{D}$ from $\mathcal{A}$ into $\mathcal{M}$ is said to be a derivation if $\mathcal{D}(ab) = \mathcal{D}(a)b + a\mathcal{D}(b)$ for all $a, b \in \mathcal{A}$. It is known that the sum of two derivations in $\mathcal{A}$ with values in $\mathcal{M}$ is also a derivation. This defines the structure of a group on the set of all derivations in $\mathcal{A}$ with values in $\mathcal{M}$ denoted by $\mathrm{Der}(\mathcal{A},\mathcal{M})$. In particular, when $\mathcal{M}=\mathcal{A}$, we simply set $\mathrm{Der}(\mathcal{A}) := \mathrm{Der}(\mathcal{A},\mathcal{A})$. A derivation $\mathcal{D}\in\mathrm{Der}(\mathcal{A},\mathcal{M})$ is said to be inner if it is of the form $\mathcal{D}(a)=[m_0,a]$ for some $m_0\in\mathcal{M}$. In addition, it is known that the set of all inner derivations in $\mathcal{A}$ with values in $\mathcal{M}$ is a subgroup of $\mathrm{Der}(\mathcal{A},\mathcal{M})$. It will be denoted by $\mathrm{Innder}(\mathcal{A},\mathcal{M})$, and when $\mathcal{M}=\mathcal{A}$, we simply set $\mathrm{Innder}(\mathcal{A}) := \mathrm{Innder}(\mathcal{A},\mathcal{A})$. It is a well-known fact that a derivation need not be inner. Namely, the well-known first cohomology group $H^1(\mathcal{A}) := \mathrm{Der}(\mathcal{A})/\mathrm{Innder}(\mathcal{A})$ measures how much the group of all derivations on $\mathcal{A}$ differs from the group of inner derivations.

The set of all $\mathcal{A}$-bimodule homomorphisms $f:\mathcal{M}\longrightarrow\mathcal{N}$ is an $\mathcal{R}$-module denoted by $\mathrm{Hom}_{\mathcal{A}-\mathcal{A}}(\mathcal{M},\mathcal{N})$ and, when $\mathcal{M}=\mathcal{N}$, it is denoted by $\mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})$. Following the notation in [96], let $\mathcal{E}(\mathcal{M})$ denote the $\mathcal{R}$-submodule of $\mathrm{Hom}_{\mathcal{A}-\mathcal{A}}(\mathcal{M},\mathcal{A})$ consisting of all $\mathcal{A}$-bimodule homomorphisms f such that $f(m)n+mf(n)=0$ for all $m,\ n\in\mathcal{M}$. A linear map $S:\mathcal{M}\longrightarrow\mathcal{M}$ is called a *module generalized d-derivation* for some $d\in\mathrm{Der}(\mathcal{A})$ if $S(am)=aS(m)+d(a)m$ and $S(ma)=S(m)a+md(a)$ for all $m\in\mathcal{M}$ and $a\in\mathcal{A}$. If there is no ambiguity about the associated derivation d, S will be simply called a module generalized derivation. Clearly, the set of all module generalized derivations $S:\mathcal{M}\longrightarrow\mathcal{M}$ is a group denoted by $\mathrm{GDer}(\mathcal{M})$. Compare this notion with the notion of the generalized derivation on modules introduced in [1] and the classical generalized derivation on rings [26]. In addition, it is clear that $\mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})$ is a subgroup of $\mathrm{GDer}(\mathcal{M})$. Namely, every $\mathcal{A}$-bimodule homomorphism can be considered as a module generalized derivation associated with the zero derivation.

Lemma 1.2.1 ([87], Lemma 2.1) *Every linear map $\mathcal{D}:\mathcal{A}\ltimes\mathcal{M}\longrightarrow\mathcal{A}\ltimes\mathcal{M}$ has the form*

$$\mathcal{D}(a,m)=(\mathcal{D}_{\mathcal{A}}(a)+T(m),\mathcal{D}_{\mathcal{M}}(a)+S(m))\quad(a\in\mathcal{A},\ m\in\mathcal{M})$$

for some linear maps $\mathcal{D}_{\mathcal{A}}:\mathcal{A}\longrightarrow\mathcal{A},\mathcal{D}_{\mathcal{M}}:\mathcal{A}\longrightarrow\mathcal{M},T:\mathcal{M}\longrightarrow\mathcal{A}$ and $S:\mathcal{M}\longrightarrow\mathcal{M}$. We will simply write $\mathcal{D}=(\mathcal{D}_{\mathcal{A}}+T,\mathcal{D}_{\mathcal{M}}+S)$.
Moreover, $\mathcal{D}$ is a derivation if and only if $\mathcal{D}_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{M}}$ are derivations, $T\in\mathcal{E}(\mathcal{M})$ and S is a module generalized $\mathcal{D}_{\mathcal{A}}$-derivation.

Our aim in this section is two fold, the study of the relation between module generalized derivations and their associated derivations, and the innerness of derivations on trivial extension algebras.

Let us start with the following remark which sheds light on some significant differences between the forms of derivations on trivial extension algebras and of those on triangular matrix algebras.

Remark 1.2.2

(i) Note that when $\mathcal{A} \ltimes \mathcal{M} = \mathrm{Tri}(A, \mathcal{M}, B)$ (where $\mathcal{A} = A \times B$), the derivation $\mathcal{D}_{\mathcal{M}}$ is necessarily inner (see [34, Theorem 2.2.1] and [49]). However, for a trivial extension algebra which has not a triangular matrix representation, this does not hold true, in general. To see this, it is sufficient to consider a derivation $d : \mathcal{A} \longrightarrow \mathcal{M}$ which is not inner and consider $\mathcal{D} = (0, d) : \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A} \ltimes \mathcal{M}$ which is a derivation on $\mathcal{A} \ltimes \mathcal{M}$.
Nevertheless, for a trivial extension of type ($\star$), the derivation $\mathcal{D}_{\mathcal{M}}$ is necessarily inner. Indeed, for $a \in \mathcal{A}$, we have

$$\begin{aligned}
\mathcal{D}_{\mathcal{M}}(a) &= \mathcal{D}_{\mathcal{M}}(eae + fae + faf) \\
&= \mathcal{D}_{\mathcal{M}}(eae) + \mathcal{D}_{\mathcal{M}}(fae) + \mathcal{D}_{\mathcal{M}}(faf) \\
&= ea\mathcal{D}_{\mathcal{M}}(e) + \mathcal{D}_{\mathcal{M}}(f)af \\
&= a\mathcal{D}_{\mathcal{M}}(e) + \mathcal{D}_{\mathcal{M}}(f)a \\
&= am - ma,
\end{aligned}$$

where $m = \mathcal{D}_{\mathcal{M}}(e) = -\mathcal{D}_{\mathcal{M}}(f)$.

(ii) Also, when $\mathcal{A} \ltimes \mathcal{M} = \mathrm{Tri}(A, \mathcal{M}, B)$ (where $\mathcal{A} = A \times B$), the map T is zero (see [34, Theorem 2.2.1] and [49]). In fact, this holds for every trivial extension of type ($\star$). Indeed, for $m \in \mathcal{M}$, we have

$$T(m) = T(em) = e\,T(m) \quad \text{and} \quad T(m) = T(mf) = T(m)\,f.$$

Therefore,

$$T(m) = e\,T(m) = e\,T(m)\,f = 0.$$

However, the following example shows that the map T needs not to be zero, in general.
Consider the trivial extension $M_2(\mathbb{Z}/2\mathbb{Z}) \ltimes M_2(\mathbb{Z}/2\mathbb{Z})$ where $M_2(\mathbb{Z}/2\mathbb{Z})$ is the algebra of 2×2 matrices with entries from $\mathbb{Z}/2\mathbb{Z}$. Now, consider the identity map $T : M_2(\mathbb{Z}/2\mathbb{Z}) \to M_2(\mathbb{Z}/2\mathbb{Z})$. Since $M_2(\mathbb{Z}/2\mathbb{Z})$ has characteristic 2, $T \in \mathcal{E}(M_2(\mathbb{Z}/2\mathbb{Z}))$. Then, the linear map $\mathcal{D} : M_2(\mathbb{Z}/2\mathbb{Z}) \ltimes M_2(\mathbb{Z}/2\mathbb{Z}) \to M_2(\mathbb{Z}/2\mathbb{Z}) \ltimes M_2(\mathbb{Z}/2\mathbb{Z})$ defined by $\mathcal{D}((a, b)) = (T(b), 0)$ for all $a, b \in M_2(\mathbb{Z}/2\mathbb{Z})$ is a derivation with $T \neq 0$.

In [34, Corollary 2.2.2], faithfulness of $\mathcal{M}$ was used to show that a derivation $\mathcal{D} = (\mathcal{D}_{\mathcal{A}}, \mathcal{D}_{\mathcal{M}} + S)$ on a triangular matrix algebra $\mathcal{A} \ltimes \mathcal{M} = \mathrm{Tri}(A, \mathcal{M}, B)$ (where $\mathcal{A} = A \times B$) is uniquely determined by S and $\mathcal{D}_{\mathcal{M}}$. Here, we use a condition in terms of annihilators. Recall that the left annihilator, denoted $l.Ann_{\mathcal{A}}(\mathcal{M})$, of $\mathcal{M}$ is the set of all elements r in $\mathcal{A}$ such that $r\mathcal{M} = 0$. Similarly, the right annihilator, $r.Ann_{\mathcal{A}}(\mathcal{M})$, of $\mathcal{M}$ is defined. When $\mathcal{A} \ltimes \mathcal{M} =$

$\mathrm{Tri}(A, \mathcal{M}, B)$ (where $\mathcal{A} = A \times B$), one can show easily that the condition $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$ is equivalent to the fact that $\mathcal{M}$ is faithful as both a left A-module and a right B-module. However, it is worth noting that for trivial extension algebras $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$ does not imply that $\mathcal{M}$ is a faithful $\mathcal{A}$-bimodule.

From the proof of [34, Corollary 2.2.2], we can deduce the following result:

Let A and B be unital algebras over a commutative ring $\mathcal{R}$, and let $\mathcal{M}$ be a unital (A, B)-bimodule, which is faithful as both a left A-module and a right B-module. Consider a linear map $S : \mathcal{M} \longrightarrow \mathcal{M}$. If there are linear maps $d_A : A \longrightarrow A$ and $d_B : B \longrightarrow B$ such that $S(am) = aS(m) + d_A(a)m$ and $S(ma) = S(m)a + md_B(a)$ for all $m \in \mathcal{M}$, $a \in A$ and $b \in B$, then, d_A and d_B are derivations.
Using similar arguments as used in the proof of [34, Corollary 2.2.2], we get the following extension to the context of trivial extension algebras.

Proposition 1.2.3 *Assume that $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$. Consider a linear map $S : \mathcal{M} \longrightarrow \mathcal{M}$. If there is a linear map $d : \mathcal{A} \longrightarrow \mathcal{A}$ such that $S(am) = aS(m) + d(a)m$ and $S(ma) = S(m)a + md(a)$ for all $m \in \mathcal{M}$ and $a \in A$, then d is a derivation. Consequently, S is a module generalized d-derivation.*

Now, we investigate a particular case of module generalized derivations. Let us first give the following observation.

Recall that when $\mathcal{A} \ltimes \mathcal{M} = \mathrm{Tri}(A, \mathcal{M}, B)$ (where $\mathcal{A} = A \times B$), the homomorphism $S : \mathcal{M} \longrightarrow \mathcal{M}$ associated with an inner derivation is of the form $S(m) = a_0m - mb_0$ for fixed $(a_0, b_0) \in A \times B$ and for all $m \in \mathcal{M}$ (see [34, Proposition 2.2.3], [14, Proposition 3.3] and [1]). Note that such a homomorphism can be written in the form similar to that of the classical inner derivation. Indeed, for every $m \in \mathcal{M}$,

$$S(m) = a_0m - mb_0 = (a_0, b_0)m - m(a_0, b_0) = c_0m - mc_0,$$

where $c_0 = (a_0, b_0)$.
In general, for every $c \in \mathcal{A}$, the linear map $S : \mathcal{M} \longrightarrow \mathcal{M}$, defined by $S(m) = cm - mc$ for all $m \in \mathcal{M}$, is a module generalized derivation associated with the inner derivation $d : \mathcal{A} \longrightarrow \mathcal{A}$ defined by $d(a) = ca - ac$ for all $a \in \mathcal{A}$. This observation leads us to introduce the following notion which plays an important role in the sequel.

Definition 1.2.4 A module generalized derivation $S : \mathcal{M} \longrightarrow \mathcal{M}$ is said to be inner if there exists $a_0 \in \mathcal{A}$ such that $S(m) = a_0m - ma_0$ for all $m \in \mathcal{M}$.

Notation. The expression $a_0m - ma_0$ in Definition 1.2.4 above will also be noted by $[a_0, m]$. Namely, we will use the Lie bracket in the following three contexts:

- For an inner derivation on $\mathcal{A}$ associated with an element $a_0 \in \mathcal{A}$:

$$\begin{array}{rccl} [a_0, -]: & \mathcal{A} & \longrightarrow & \mathcal{A} \\ & a & \longmapsto & [a_0, a] = a_0a - aa_0 \end{array}$$

- For an inner derivation from $\mathcal{A}$ to $\mathcal{M}$ associated with an element $m_0 \in \mathcal{M}$:

$$\begin{array}{rccl} [m_0, -]: & \mathcal{A} & \longrightarrow & \mathcal{M} \\ & a & \longmapsto & [m_0, a] = m_0a - am_0 \end{array}$$

- For an inner (module generalized) derivation from $\mathcal{M}$ to $\mathcal{M}$ associated with an element $a_0 \in \mathcal{A}$:

$$\begin{array}{rccl} [a_0, -]: & \mathcal{M} & \longrightarrow & \mathcal{M} \\ & m & \longmapsto & [a_0, m] = a_0m - ma_0 \end{array}$$

Remark 1.2.5 *It is evident that the set of all inner module generalized derivations $\mathcal{M} \longrightarrow \mathcal{M}$ is a group. It will be denoted by* $\mathrm{InnGd}_{\mathcal{A}}(\mathcal{M})$*.*
Let us also denote $\mathrm{InnBi}_{\mathcal{A}}(\mathcal{M}) := \mathrm{InnGd}_{\mathcal{A}}(\mathcal{M}) \cap \mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})$*. The elements of* $\mathrm{InnBi}_{\mathcal{A}}(\mathcal{M})$ *will be called inner $\mathcal{A}$-bimodule homomorphisms. One can show that the group* $\mathrm{InnBi}_{\mathcal{A}}(\mathcal{M})$ *coincide with the set of all inner module generalized derivations $[a_0, -]$, where $a_0 \in \mathcal{A}$ satisfies $[a_0, a] \in l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M})$ for every $a \in \mathcal{A}$.*

It is also clear that, if $c \in Z(\mathcal{A})$, the linear map $S : \mathcal{M} \longrightarrow \mathcal{M}$, defined by $S(m) = cm - mc$ for all $m \in \mathcal{M}$, is an inner bimodule homomorphism. This kind of inner bimodule homomorphisms will be of particular interest, Namely, in the study of the first cohomology group of trivial extension algebras (see Sect.1.3). For this reason, we introduce the following notion. We use the terminology of [49] (see [49, Remark 2.3(ii) and Definition 2.5]).

Definition 1.2.6 A bimodule homomorphism $S : \mathcal{M} \longrightarrow \mathcal{M}$ is said to be a central inner bimodule homomorphism if there exists $a_0 \in Z(\mathcal{A})$ such that $S(m) = a_0m - ma_0$ for all $m \in \mathcal{M}$.

When $\mathcal{A} \ltimes \mathcal{M} = \mathrm{Tri}(A, \mathcal{M}, B)$ (where $\mathcal{A} = A \times B$), the central inner bimodule homomorphisms are called bimodule homomorphisms of the standard form (see [14]). These are exactly the homomorphisms $S : \mathcal{M} \longrightarrow \mathcal{M}$ defined by $S(m) = a_0m - mb_0$ for fixed $(a_0, b_0) \in Z(A) \times Z(B)$ and for all $m \in \mathcal{M}$.

Remark 1.2.7 *1. It is evident that the set of all central inner bimodule homomorphisms* $\mathcal{M} \longrightarrow \mathcal{M}$ *is a subgroup of* $\mathrm{InnBi}_{\mathcal{A}}(\mathcal{M})$. *It will be denoted by* $\mathrm{Innbi}_{\mathcal{A}}(\mathcal{M})$.

2. *It is worth noting that the inclusion* $\mathrm{Innbi}_{\mathcal{A}}(\mathcal{M}) \subseteq \mathrm{InnBi}_{\mathcal{A}}(\mathcal{M})$ *can be strict (see assertion (2) in Example 1.2.10). However, in the next result (Theorem 1.2.8), we show that, when* $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$, *the two groups* $\mathrm{Innbi}_{\mathcal{A}}(\mathcal{M})$ *and* $\mathrm{InnBi}_{\mathcal{A}}(\mathcal{M})$ *coincide.*

3. *Note also that* $\mathrm{Innbi}_{\mathcal{A}}(\mathcal{M}) = 0$ *if and only if the center* $Z(\mathcal{A})$ *of* $\mathcal{A}$ *has a symmetric action on* $\mathcal{M}$ *(that is,* $am = ma$ *for every* $a \in Z(\mathcal{A})$ *and* $m \in \mathcal{M}$*).*

The next result gathers relations that exist between inner module generalized derivations and the associated (ring) derivations under the condition "$l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$". More relations will be given in Proposition 1.2.11 without this condition.

Theorem 1.2.8 *Assume that* $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$. *Consider a module generalized d-derivation* $S : \mathcal{M} \longrightarrow \mathcal{M}$, *where* $d : \mathcal{A} \longrightarrow \mathcal{A}$ *is a derivation.*

1. *If* S *is inner, then* d *is inner. Namely, if for some* $a_0 \in \mathcal{A}$, $S(m) = [a_0, m]$ *for all* $m \in \mathcal{M}$, *then* $d(a) = [a_0, a]$ *for all* $a \in \mathcal{A}$.

2. *If* S *is a bimodule homomorphism, then* $d = 0$. *If in addition,* $S = [a_0, -]$ *is inner, for some* $a_0 \in \mathcal{A}$, *then* a_0 *is in the center* $Z(\mathcal{A})$ *of* $\mathcal{A}$ *(i.e., S is a central inner bimodule homomorphism).*

Proof. 1. Let $a_0 \in \mathcal{A}$ be such that $S(m) = [a_0, m]$ for all $m \in \mathcal{M}$. Using the fact that $S(am) = aS(m) + d(a)m$ and $S(ma) = S(m)a + md(a)$ for all $m \in \mathcal{M}$ and $a \in \mathcal{A}$, we get, for all $m \in \mathcal{M}$ and $a \in \mathcal{A}$,

$$([a_0, a] - d(a))m = 0 \quad \text{and} \quad m([a_0, a] - d(a)) = 0.$$

Then, by hypothesis, $d(a) = [a_0, a]$, as desired.

2. If S is a bimodule homomorphism, then $d(a)m = 0 = md(a)$ for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$. This shows that $d = 0$.
Now, let $a_0 \in \mathcal{A}$ be such that $S(m) = [a_0, m]$ for all $m \in \mathcal{M}$. Using the fact that $S(am) = aS(m)$ and $S(ma) = S(m)a$ for all $m \in \mathcal{M}$ and $a \in \mathcal{A}$, we get, for all $m \in \mathcal{M}$ and $a \in \mathcal{A}$

$$(a_0a - aa_0)m = 0 \quad \text{and} \quad m(a_0a - aa_0) = 0.$$

Then, by hypothesis, $a_0a = aa_0$, as desired. ∎

Next, we give an example of a non-trivial inner bimodule homomorphism and another example showing that, to get assertion (2) of Theorem 1.2.8 above, the condition $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$ cannot be dropped. It is based on the following observation.

Lemma 1.2.9 *Every module generalized derivation $S : \mathcal{M} \longrightarrow \mathcal{M}$ is a bimodule homomorphism if the image of every derivation on $\mathcal{A}$ is in $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M})$.*
In other words, $\mathrm{GDer}(\mathcal{M}) = \mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})$ *if, for every $d \in \mathrm{Der}(\mathcal{A})$, $\mathrm{Im}(d) \subseteq l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M})$, where $\mathrm{Im}(d)$ denotes the image of d.*

Proof. Straightforward. ■

Example 1.2.10 *Consider $\mathcal{R}$, A, B, M, $\mathcal{S}$ and $\mathcal{N}$ as in example 1.1.4. Assume that, as sets, $\mathcal{R} = A = B = M$. Then, the following assertions hold.*

1. *Consider M as an (A, B)-bimodule. Then, every bimodule homomorphism $S : M \longrightarrow M$ is inner. Note that $l.Ann_{A\times B}(M) \cap r.Ann_{A\times B}(M) = 0$.*
2. *Every generalized derivation $S : \mathcal{N} \longrightarrow \mathcal{N}$ is an inner generalized d-derivation for every derivation d on $\mathcal{S}$. Moreover, it is also a bimodule homomorphism.*
   Note that $l.Ann_{\mathcal{S}}(\mathcal{N}) \cap r.Ann_{\mathcal{S}}(\mathcal{N}) = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$.

Proof. 1. Consider a bimodule homomorphism $S : M \longrightarrow M$ and let $x \in M$. Then,

$S(x) = S((x, 0)1) = (x, 0)S(1) = xS(1) = (S(1), 0)x = (S(1), 0)x - x(S(1), 0)$.

This shows that S is inner.
2. First it is clear that every derivation on $\mathcal{S}$ is inner. Thus, consider a derivation $d = [a, -]$ on $\mathcal{S}$ for some $a = \begin{pmatrix} x_a & m_a \\ 0 & y_a \end{pmatrix}$ in $\mathcal{S}$. Then, for every element $b = \begin{pmatrix} x_b & m_b \\ 0 & y_b \end{pmatrix}$ in $\mathcal{S}$,

$$d(b) = [a, b] = \begin{pmatrix} 0 & x_a m_b + m_a y_b - x_b m_a - m_b y_a \\ 0 & 0. \end{pmatrix}$$

This shows that $d(b)n = 0 = nd(b)$ for all $n \in \mathcal{N}$. Then, every generalized derivation $S : \mathcal{N} \longrightarrow \mathcal{N}$ is a bimodule homomorphism. It remains to prove that every bimodule homomorphism is inner. Then, consider a bimodule homomorphism $S : \mathcal{N} \longrightarrow \mathcal{N}$ and let $x \in \mathcal{N}$. Then, $S(x) = S(((0, x), 0)1) = ((0, x), 0)S(1) = xS(1) = ((0, S(1)), 0)x = ((0, S(1)), 0)x - x((0, S(1)), 0)$. Therefore, S is inner. ■

The following result, which is a generalization of [34, Lemma 2.2.5], relates the notion of bimodule homomorphism and the one of a module generalized d-derivation when d is inner. It answers the question concerning the converse implication of the one given in assertion (1) of Theorem 1.2.8.

Proposition 1.2.11 *Consider a module generalized d-derivation $S : \mathcal{M} \longrightarrow \mathcal{M}$, where $d : \mathcal{A} \longrightarrow \mathcal{A}$ is a derivation. If $d = [a_0, -]$ is inner, for some $a_0 \in \mathcal{A}$, then there is a bimodule homomorphism $\Phi : \mathcal{M} \longrightarrow \mathcal{M}$ such that $S = \Phi + [a_0, -]$.*
Moreover, S is inner if and only if Φ is inner.

Proof. As done in [34, Lemma 2.2.5], we simply need to prove that the map $S - [a_0, -]$ is a bimodule homomorphism. ■

Note that for every $a_0 \in Z(\mathcal{A})$ and every $b_0 \in \mathcal{A}$, the derivations $[a_0, -]$ and $[a_0 + b_0, -]$ on $\mathcal{A}$ are the same. Thus, the fact that Φ is a central inner bimodule homomorphism will not affect the corresponding element of the inner derivation d associated with the module generalized derivation S.

We end this section with a characterization of inner derivations on trivial extension algebras. It is a generalization of [34, Proposition 2.2.3] which characterizes inner derivations on triangular matrix algebras (compare it also with [85, Proposition 2.2]).

Theorem 1.2.12 *Consider a derivation $\mathcal{D} : \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A} \ltimes \mathcal{M}$ of the form*

$$\mathcal{D} = (\mathcal{D}_{\mathcal{A}} + T, \mathcal{D}_{\mathcal{M}} + S),$$

where $\mathcal{D}_{\mathcal{A}}$, T, $\mathcal{D}_{\mathcal{M}}$ and S as indicated in Lemma 1.2.1. Then, the following assertions hold.

1. *If $\mathcal{D}$ is inner, then $T = 0$ and both S and $\mathcal{D}_{\mathcal{M}}$ are inner.*

2. *The converse holds if we assume that $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$.*

Proof. $\Rightarrow$) Suppose that $\mathcal{D}$ is inner. Then, there exists $(a_0, m_0) \in \mathcal{A} \ltimes \mathcal{M}$ such that, for every $a \in \mathcal{A}$ and $m \in \mathcal{M}$,

$$\begin{aligned} \mathcal{D}(a, m) &= [(a_0, m_0), (a, m)] \\ &= ([a_0, a], [m_0, a] + [a_0, m]). \end{aligned}$$

Then, for $a = 0$, $D(0, m) = (T(m), S(m)) = (0, [a_0, m])$. This shows that $T(m) = 0$ and $S(m) = [a_0, m]$. Now, for $m = 0$, we get $\mathcal{D}_{\mathcal{M}}(a) = [m_0, a]$, as desired.

$\Leftarrow$) Now, we assume that $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$. Since $\mathcal{D}_{\mathcal{M}}$ and S are inner, there are $a_0 \in \mathcal{A}$ and $m_0 \in \mathcal{M}$ such that, for every $a \in \mathcal{A}$ and $m \in \mathcal{M}$, $\mathcal{D}_{\mathcal{M}}(a) = [m_0, a]$ and $S(m) = [a_0, m]$. By assertion (1) of Theorem 1.2.8, $\mathcal{D}_{\mathcal{A}}(a) = [a_0, a]$. Therefore,

$$\begin{aligned} \mathcal{D}(a, m) &= ([a_0, a], [m_0, a] + [a_0, m]) \\ &= [(a_0, m_0), (a, m)], \end{aligned}$$

as desired. ■

It is worth noting that there are examples of trivial extension algebras showing that the converse implication in Theorem 1.2.12 above does not hold true if we drop the condition $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$.
When $\mathcal{M}$ is assumed to be $\mathcal{A}$, the condition of Theorem 1.2.12 is fulfilled. Thus, we get the following corollary (compare it with [85, Proposition 2.4]).

Corollary 1.2.13 *Assume $\mathcal{M} = \mathcal{A}$ and consider a derivation $\mathcal{D} : \mathcal{A} \ltimes \mathcal{M} \longrightarrow \mathcal{A} \ltimes \mathcal{M}$ of the form*

$$\mathcal{D} = (\mathcal{D}_{\mathcal{A}} + T, \mathcal{D}_{\mathcal{M}} + S)$$

where $\mathcal{D}_{\mathcal{A}}$, T, $\mathcal{D}_{\mathcal{M}}$ and S as indicated in Lemma 1.2.1. Then, $\mathcal{D}$ is inner if and only if $T = 0$, $\mathcal{D}_{\mathcal{M}}$ is inner, $S = \mathcal{D}_{\mathcal{A}}$ and S is inner.
Precisely, the inner derivations on $\mathcal{A} \ltimes \mathcal{A}$ are only of the form $(d_1, d_0 + d_1)$, where d_0 and d_1 are inner derivations on $\mathcal{A}$.

Proof. Only the assertion $S = \mathcal{D}_{\mathcal{A}}$, when $\mathcal{D}$ is inner, merits a proof. Thus, assume that $\mathcal{D}$ is inner. Then, by Theorem 1.2.12, $S = [a_0, -]$ is inner for some $a_0 \in \mathcal{A}$. Then, the associated derivation $\mathcal{D}_{\mathcal{A}} = [a_0, -]$ is also inner (by Theorem 1.2.8). Now, by Proposition 1.2.11, there is a bimodule homomorphism $\Phi : \mathcal{M} \longrightarrow \mathcal{M}$ such that $S = \Phi + [a_0, -]$. Since S is inner, Φ is also inner. Then, using assertion (2) of Theorem 1.2.8, a_0 is in the center $Z(\mathcal{A})$. This implies that $\Phi = 0$, as desired. ■

1.3 First cohomology group of trivial extension algebras

In this section, we study the first cohomology group of trivial extension algebras. For this, we start with the following observation. Let us note by $\mathrm{der}(\mathcal{A} \ltimes \mathcal{M})$ the subgroup of $\mathrm{Der}(\mathcal{A} \ltimes \mathcal{M})$ of derivations on $\mathcal{A} \ltimes \mathcal{M}$ of the form (d, S), where d is a derivation on $\mathcal{A}$ and $S : \mathcal{M} \to \mathcal{M}$ is a module generalized d-derivation. In addition, we use $\mathrm{innder}(\mathcal{A} \ltimes \mathcal{M})$ to denote the inner derivations on $\mathcal{A} \ltimes \mathcal{M}$ of the form $[(a_0, 0), -] = ([a_0, -], [a_0, -])$. Note that

innder($\mathcal{A} \ltimes \mathcal{M}$) is a subgroup of der($\mathcal{A} \ltimes \mathcal{M}$). Thus, we may consider the quotient group der($\mathcal{A} \ltimes \mathcal{M}$)/innder($\mathcal{A} \ltimes \mathcal{M}$) which will be called, by analogy with the classical case, the *restricted first cohomology group* of $\mathcal{A} \ltimes \mathcal{M}$, and denoted by $h^1(\mathcal{A} \ltimes \mathcal{M})$.
Recall the first cohomology group $H^1(\mathcal{A}, \mathcal{M}) = \mathrm{Der}(\mathcal{A}, \mathcal{M})/\mathrm{Innder}(\mathcal{A}, \mathcal{M})$. When $\mathcal{A} = \mathcal{M}$, $H^1(\mathcal{A}, \mathcal{M})$ is simply denoted by $H^1(\mathcal{A})$.

Proposition 1.3.1 *The following assertions hold.*

1. *There is a natural group homomorphism:*

$$\mathrm{Der}(\mathcal{A} \ltimes \mathcal{M}) \cong \mathrm{Der}(\mathcal{A}, \mathcal{M}) \oplus \mathrm{der}(\mathcal{A} \ltimes \mathcal{M}) \oplus \mathcal{E}(\mathcal{M}).$$

2. *There is a natural group homomorphism:*

$$\mathrm{Innder}(\mathcal{A} \ltimes \mathcal{M}) \cong \mathrm{Innder}(\mathcal{A}, \mathcal{M}) \oplus \mathrm{innder}(\mathcal{A} \ltimes \mathcal{M}).$$

Consequently, there is a natural group homomorphism:

$$H^1(\mathcal{A} \ltimes \mathcal{M}) \cong H^1(\mathcal{A}, \mathcal{M}) \oplus h^1(\mathcal{A} \ltimes \mathcal{M}) \oplus \mathcal{E}(\mathcal{M}).$$

Proof. The assertions are simple consequences of Lemma 1.2.1 and Theorem 1.2.12. ■

Now, we give our main result which relates $h^1(\mathcal{A} \ltimes \mathcal{M})$, $H^1(\mathcal{A})$ and the quotient group $\mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})/\mathrm{Innbi}_{\mathcal{A}}(\mathcal{M})$. By analogy with the classical notation, let us denote by $H^1_{\mathcal{A}}(\mathcal{M})$ the quotient group $\mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})/\mathrm{Innbi}_{\mathcal{A}}(\mathcal{M})$. Using Proposition 1.3.1, the main result provides a generalization of the classical result [38, Theorem 5.5] and the recent result [9, Theorem 4.4] (see Corollary 1.3.3). To see this fact, we give a brief discussion of the scope.

In [38] the first cohomology group of a particular case of trivial extensions is studied. Namely, by [38, Theorem 5.5], we have, if $\mathcal{R}$ is a field, that $\mathcal{A}$ is assumed to be a finite dimensional algebra and $\mathcal{M} = D\mathcal{A}$ is the dual $\mathcal{A}$-bimodule of $\mathcal{A}$, then

$$H^1(\mathcal{A} \ltimes \mathcal{M}) \cong H^1(\mathcal{A}, \mathcal{M}) \oplus H^1(\mathcal{A}) \oplus Z(\mathcal{A}) \oplus Alt_{\mathcal{A}}(D\mathcal{A}),$$

where $Alt_{\mathcal{A}}(D\mathcal{A})$ is the set of skew-symmetric bilinear forms β over $D\mathcal{A}$ such that $\beta(fa, g) = \beta(f, ag)$ for all $f, g \in D\mathcal{A}$ and $a \in \mathcal{A}$. As noted above [38, Theorem 5.5], this vector space coincides with $\mathcal{E}(D\mathcal{A})$. In addition, note that, from [38, Proposition 3.3 and Example 3.5], the center $Z(\mathcal{A})$ of $\mathcal{A}$ has a symmetric action on $D\mathcal{A}$ (that is, $af = fa$ for every $a \in Z(\mathcal{A})$ and $f \in D\mathcal{A}$).

This shows that $\mathrm{Innbi}_{\mathcal{A}}(\mathcal{M}) = 0$ and so $H^1_{\mathcal{A}}(\mathcal{M}) = \mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})$. On the other hand,

$$\mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M}) = \mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{A}) = Z(\mathcal{A}).$$

Then, by [38, Theorem 5.5], we deduce that

$$h^1(\mathcal{A} \ltimes \mathcal{M}) = H^1_{\mathcal{A}}(\mathcal{M}) \oplus H^1(\mathcal{A}).$$

Thus, using [9, Corollary 3.7], one can show that the main result provides a generalization of [38, Theorem 5.5]. In fact, it is a generalization of the more general recent result [9, Theorem 4.4] (see Corollary 1.3.3).

Let us denote by $[a]$ the equivalence class of an element a in a given quotient group. We will use the canonical projection $\pi_{\mathcal{A}}$ of $\mathrm{der}(\mathcal{A} \ltimes \mathcal{M})$ onto $\mathrm{Der}(\mathcal{A})$ (i.e., $\pi_{\mathcal{A}}((d, S)) = d$, where d is a derivation on $\mathcal{A}$ and $S : \mathcal{M} \to \mathcal{M}$ is a module generalized d-derivation). In addition, we will use the linear application $\Phi : \mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M}) \longrightarrow \mathrm{der}(\mathcal{A} \ltimes \mathcal{M})$ defined by $\Phi(S) = (0, S)$ for all $S \in \mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})$.

Theorem 1.3.2 *Consider the two maps* $\widehat{\pi}_{\mathcal{A}} : h^1(\mathcal{A} \ltimes \mathcal{M}) \longrightarrow H^1(\mathcal{A})$ *given by* $\widehat{\pi}_{\mathcal{A}}([\mathcal{D}]) = [\pi_{\mathcal{A}}(\mathcal{D})]$ *and* $\widehat{\Phi} : H^1_{\mathcal{A}}(\mathcal{M}) \longrightarrow h^1(\mathcal{A} \ltimes \mathcal{M})$ *given by* $\widehat{\Phi}([S]) := [\Phi(S)]$. *The maps* $\widehat{\pi}_{\mathcal{A}}$ *and* $\widehat{\Phi}$ *are well-defined group homomorphisms.*
Moreover, we have the following exact sequence of group homomorphisms

$$0 \longrightarrow H^1_{\mathcal{A}}(\mathcal{M}) \xrightarrow{\widehat{\Phi}} h^1(\mathcal{A} \ltimes \mathcal{M}) \xrightarrow{\widehat{\pi}_{\mathcal{A}}} H^1(\mathcal{A}).$$

Proof. We prove that $\widehat{\pi}_{\mathcal{A}}$ is well defined. First note that $\pi_{\mathcal{A}}$ sends any inner derivation of $\mathcal{A} \ltimes \mathcal{M}$ to an inner derivation of $\mathcal{A}$. Now, consider $\mathcal{D}_1 = (d_1, S_1), \mathcal{D}_2 = (d_2, S_2) \in \mathrm{der}(\mathcal{A} \ltimes \mathcal{M})$ such that $[\mathcal{D}_1] = [\mathcal{D}_2]$ in $h^1(\mathcal{A} \ltimes \mathcal{M})$. Then, there exists $(a_0, 0) \in \mathcal{A} \ltimes \mathcal{M}$ such that $\mathcal{D}_1 = \mathcal{D}_2 + [(a_0, 0), -]$. Then, $d_1 = d_2 + [a_0, -]$. Therefore, $[\pi_{\mathcal{A}}(\mathcal{D}_1)] = [\pi_{\mathcal{A}}(\mathcal{D}_2)]$.
Now, we show that $\widehat{\Phi}$ is well defined. Also note that Φ sends a central inner bimodule homomorphism to an inner derivation of $\mathcal{A} \ltimes \mathcal{M}$. Consider $f, g \in \mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})$ such that $[f] = [g]$ in $H^1_{\mathcal{A}}(\mathcal{M})$. Then, there exists $a_0 \in Z(\mathcal{A})$ such that $f = g + [a_0, -]$. Then, $(0, f) = (0, g) + [(a_0, 0), -]$ in $\mathrm{der}(\mathcal{A} \ltimes \mathcal{M})$. Therefore, $[(0, f)] = [(0, g)]$ in $h^1(\mathcal{A} \ltimes \mathcal{M})$.
We prove that $\widehat{\Phi}$ is injective. Consider $f \in \mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})$ such that $\widehat{\Phi}([f]) = 0$. Then, $[(0, f)] = 0$ in $h^1(\mathcal{A} \ltimes \mathcal{M})$. This shows that there is $a_0 \in \mathcal{A}$ such that $(0, f) = [(a_0, 0), -]$. Thus, as a derivation on $\mathcal{A}$, $[a_0, -] = 0$. Therefore, $a_0 \in Z(\mathcal{A})$, and so f is a central inner bimodule homomorphism, as desired.
It remains to prove that $\mathrm{Ker}(\widehat{\pi}_A) = \mathrm{Im}(\widehat{\Phi})$. First note that $\widehat{\pi}_A \widehat{\Phi} = 0$ and hence $\mathrm{Im}(\widehat{\Phi}) \subseteq \mathrm{Ker}(\widehat{\pi}_A)$. For the converse, consider $D = (d, S) \in \mathrm{der}(\mathcal{A} \ltimes \mathcal{M})$ such that $\widehat{\pi}_A([D]) = 0$. This means that d is inner, and hence, $d = [a_0, -]$ for some

$a_0 \in \mathcal{A}$. Then, by Proposition 1.2.11, there is $f \in \mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})$ such that $S = f + [a_0, -]$. Then, in $h^1(\mathcal{A} \ltimes \mathcal{M})$, we get

$$[D] = [(d, f + [a_0, -])] = [(0, f) + [(a_0, 0), -]] = [(0, f)] = \widehat{\Phi}([f]),$$

as desired. ■

From [9, Proposition 4.8][1], we have $H^1(\mathcal{A}\ltimes\mathcal{M}, \mathcal{M}) = H^1(\mathcal{A}, \mathcal{M}) \oplus \mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})$. Then, there is a natural surjective group homomorphism

$$H^1(\mathcal{A} \ltimes \mathcal{M}, \mathcal{M}) \longrightarrow H^1(\mathcal{A}, \mathcal{M}) \oplus H^1_{\mathcal{A}}(\mathcal{M}).$$

Obviously, this homomorphism is, in fact, the identity when the center $Z(\mathcal{A})$ of $\mathcal{A}$ has a symmetric action on $\mathcal{M}$. Thus, the following result is a generalization of [9, Theorem 4.4] (compare it also with [85, Theorem 2.5]).

Corollary 1.3.3 *There is an exact sequence of group homomorphisms*

$$0 \longrightarrow H^1_{\mathcal{A}}(\mathcal{M}) \oplus H^1(\mathcal{A}, \mathcal{M}) \oplus \mathcal{E}(\mathcal{M}) \longrightarrow H^1(\mathcal{A} \ltimes \mathcal{M}) \xrightarrow{\widehat{\pi}_{\mathcal{A}}} H^1(\mathcal{A}),$$

where the first (injective) homomorphism is induced from $\widehat{\Phi}$ and the natural isomorphisms of Proposition 1.3.1.

As mentioned in Remark 1.2.2, both $H^1(\mathcal{A}, \mathcal{M}) = 0$ and $\mathcal{E}(\mathcal{M}) = 0$ in the case of triangular matrix algebras $\mathcal{A} \ltimes \mathcal{M} = \mathrm{Tri}(A, \mathcal{M}, B)$ (where $\mathcal{A} = A \times B$). Thus, the following result is a generalization of [34, Theorem 2.3.6].

Corollary 1.3.4 *If $\mathcal{A} \ltimes \mathcal{M} = \mathrm{Tri}(A, \mathcal{M}, B)$ (where $\mathcal{A} = A \times B$), then there is an exact sequence of group homomorphisms*

$$0 \longrightarrow H^1_{\mathcal{A}}(\mathcal{M}) \xrightarrow{\widehat{\Phi}} H^1(\mathrm{Tri}(A, \mathcal{M}, B)) \xrightarrow{\widehat{\pi}_{\mathcal{A}}} H^1(\mathcal{A}).$$

Also as a simple consequence of Theorem 1.3.2, we set the following generalization of [34, Corollary 2.3.8].

Corollary 1.3.5 *If $H^1(\mathcal{A}) = 0$, then we get the following group isomorphisms:*

$$H^1_{\mathcal{A}}(\mathcal{M}) \cong h^1(\mathcal{A} \ltimes \mathcal{M}) \quad \text{and} \quad H^1_{\mathcal{A}}(\mathcal{M}) \oplus H^1(\mathcal{A}, \mathcal{M}) \oplus \mathcal{E}(\mathcal{M}) \cong H^1(\mathcal{A} \ltimes \mathcal{M}).$$

Naturally, one can ask of a relation between $\mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})/\mathrm{InnBi}_{\mathcal{A}}(\mathcal{M})$, $h^1(\mathcal{A} \ltimes \mathcal{M})$ and $H^1(\mathcal{A})$ as done in Theorem 1.3.2. Using, Theorem 1.2.8, we deduce that $\mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})/\mathrm{InnBi}_{\mathcal{A}}(\mathcal{M}) = H^1_{\mathcal{A}}(\mathcal{M})$ when $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$. Then, we get the following result.

[1] The authors would like to thank Professor Rachel Taillefer (Université Blaise Pascal, France) for discussing with us [9, Proposition 4.8].

Corollary 1.3.6 *If* $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$, *then we have the following exact sequence of group homomorphisms*

$$0 \longrightarrow h^1_{\mathcal{A}}(\mathcal{M}) \xrightarrow{\widehat{\Phi}} h^1(\mathcal{A} \ltimes \mathcal{M}) \xrightarrow{\widehat{\pi}_{\mathcal{A}}} H^1(\mathcal{A}),$$

where $h^1_{\mathcal{A}}(\mathcal{M})$ *denotes the quotient group* $\mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})/\mathrm{InnBi}_{\mathcal{A}}(\mathcal{M})$.

The following example shows that the result of Corollary 1.3.6 does not hold true without the condition $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$.

Example 1.3.7 *Consider* $\mathcal{R}$, A, B, M, $\mathcal{S}$ *and* $\mathcal{N}$ *as in Example 1.2.10. Then, using Example 1.2.10 and Remark 1.2.2, we have* $H^1(\mathcal{S}) = 0$, $h^1_{\mathcal{S}}(\mathcal{N}) = 0$, $\mathcal{E}(\mathcal{N}) = 0$ *and* $H^1(\mathcal{S}, \mathcal{N}) = 0$. *However,* $h^1(\mathcal{S} \ltimes \mathcal{N}) \neq 0$ *(so does* $H^1(\mathcal{S} \ltimes \mathcal{N})$*). Indeed, consider the two elements* $a_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $b_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *of* $\mathcal{S}$. *Consider the map* $D : \mathcal{S} \ltimes \mathcal{N} \to \mathcal{S} \ltimes \mathcal{N}$ *defined as follows: For every* $x = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$ *in* $\mathcal{S}$ *and* $n \in \mathcal{N}$,

$$D(x, n) = ([a_0, x], [b_0, n]) = (\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, n).$$

From Example 1.2.10, $[b_0, -]$ *is an* $[a_0, -]$*-generalized derivation. Then,* D *is a derivation on* $\mathcal{S} \ltimes \mathcal{N}$. *However, one can show that* D *is not inner. Indeed, suppose that* D *is inner. Then, there are* $c = \begin{pmatrix} \alpha & e \\ 0 & \beta \end{pmatrix}$ *in* $\mathcal{S}$ *and* $n_0 \in \mathcal{N}$ *such that* $D = [(c, n_0), -]$. *Then, after a simple calculation, using the module actions defined in Example 1.1.4, we deduce that, for every* $m, n \in M = \mathcal{N} = \mathcal{R}$, $\alpha m - m\beta = m$ *and* $\beta n - n\alpha = n$. *This is impossible, as desired.*

As an important consequence of Corollary 1.3.3, we get a result that studies the innerness of derivations on $\mathcal{A} \ltimes \mathcal{M}$.
We use $\overline{\Gamma}$ to denote the subgroup $\pi_{\mathcal{A}}(\mathrm{der}(\mathcal{A} \ltimes \mathcal{M}))$ of $\mathrm{Der}(\mathcal{A})$.

Corollary 1.3.8 *All derivations on* $\mathcal{A} \ltimes \mathcal{M}$ *are inner if and only if the following assertions hold.*

1. *Every derivation in* $\overline{\Gamma}$ *is inner.*
2. *Every derivation in* $\mathrm{Der}(\mathcal{A}, \mathcal{M})$ *is inner.*
3. *Every* $\mathcal{A}$*-bimodule homomorphism in* $\mathrm{End}_{\mathcal{A}-\mathcal{A}}(\mathcal{M})$ *is central inner.*
4. $\mathcal{E}(\mathcal{M}) = 0$.

Note that when $\mathcal{A} \ltimes \mathcal{M} = \mathrm{Tri}(A, \mathcal{M}, B)$ (where $\mathcal{A} = A \times B$) a derivation d of $A \times B$ is of the form $d = (d_1, d_2)$ where d_1 and d_2 are derivations of A and B respectively. Then, a module generalized (d_1, d_2)-derivation $S : \mathcal{M} \longrightarrow \mathcal{M}$ is of the form $S(am) = aS(m) + d(a)m = aS(m) + d_1(a)m$ and $S(ma) = S(m)a + md(a) = S(m)a + md_2(a)$ for all $m \in \mathcal{M}$ and $a \in \mathcal{A}$. Thus, $\overline{\Gamma}$ used in Corollary 1.3.8 is naturally a group product $\overline{\Gamma} = \overline{\Gamma}_1 \times \overline{\Gamma}_2$ where $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ are constituted with the associated derivations on A and B respectively. With these notations and using Corollary 1.3.8 and Remark 1.2.2, we recover Benkovič's result [14, Theorem 3.5].

Corollary 1.3.9 *Let A and B be unital algebras over a commutative ring C and let $\mathcal{M}$ be a unital (A, B)-bimodule. Then, all the derivations on* $\mathrm{Tri}(A, \mathcal{M}, B)$ *are inner if and only if the following assertions hold.*

1. *Every derivation in $\overline{\Gamma}_1$ is inner.*
2. *Every derivation in $\overline{\Gamma}_2$ is inner.*
3. *Every bimodule homomorphism in* $\mathrm{End}_{A\times B}(\mathcal{M})$ *is central inner.*

Triangular matrix algebras can be used as examples of trivial extension algebras satisfying the conditions of Corollary 1.3.8. Take, for instance, the algebra $\mathrm{Tri}(\mathbb{R}, \mathbb{R}, \mathbb{R})$. However, naturally, one may ask for an example of a trivial extension algebra which has not a triangular matrix representation and satisfies the conditions of Corollary 1.3.8. The authors have not been able to give such an example. In fact, every studied example has a triangular matrix representation. This leads to the following natural question.

Question. Does the condition $H^1(\mathcal{A} \ltimes \mathcal{M}) = 0$ imply that $\mathcal{A} \ltimes \mathcal{M}$ has a triangular matrix representation?

Observations of some studied examples show that the key for reaching this target could be the study of the property (3) in Corollary 1.3.8. We end this chapter by showing that this property with some other mild conditions assures that the trivial extension has a triangular matrix representation. This gives a partial affirmative answer to the question above.

The following lemma gives an evident, but important, consequence of the property (3) in Corollary 1.3.8.

Lemma 1.3.10 *If every bimodule homomorphism on $\mathcal{M}$ is central inner, then for every $a \in Z(\mathcal{A})$, there exists $l_a \in Z(\mathcal{A})$ (resp., $r_a \in Z(\mathcal{A})$) such that, for all $m \in \mathcal{M}$, $am = l_a m - m l_a$ (resp., $ma = r_a m - m r_a$).*
In particular, there exists $c \in Z(\mathcal{A})$ such that $m = cm - mc$ for all $m \in \mathcal{M}$.

Proof. The result follows, since $\varphi_r : \mathcal{M} \longrightarrow \mathcal{M}$ defined by $\varphi_r(m) = ma$ and $\varphi_l : \mathcal{M} \longrightarrow \mathcal{M}$ defined by $\varphi_l(m) = am$ are bimodule homomorphisms. ■

Note that, if the element c in Lemma 1.3.10 is a zero-divisor, then $x\mathcal{M}x = 0$ for every $x \in \mathcal{A}$ that annihilates c (either on the left or on the right).

Theorem 1.3.11 *Assume that* $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$.
The trivial extension algebra $\mathcal{A} \ltimes \mathcal{M}$ *has a triangular matrix representation if and only if there exists* $c \in r.Ann(\mathcal{M})$ *(or* $c \in l.Ann(\mathcal{M})$*) such that* $m = cm - mc$ *for all* $m \in \mathcal{M}$.

Proof. One implication is a consequence of the discussion in Sect. 1.1.
We prove the "if" part. We may suppose that $c \in r.Ann(\mathcal{M})$. Similarly, we prove the result if $c \in l.Ann(\mathcal{M})$.
Consider $m \in \mathcal{M}$. Since $c \in r.Ann(\mathcal{M})$, $m(c - c^2) = 0$, and by Lemma 1.3.10, $(c - c^2)m = c(1 - c)m = -cmc = 0$. Then, the fact that $l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$ implies that $c = c^2$. Using Proposition 1.1.1, it remains to show that $(1 - c)\mathcal{A}c = 0$. Let $m \in \mathcal{M}$ and let $a \in \mathcal{A}$. Since $acm \in \mathcal{M}$, $(1 - c)acm = -acmc = 0$. In addition, since $m(1 - c)a \in \mathcal{M}$, $m(1 - c)ac = 0$. Therefore, $(1 - c)ac \in l.Ann_{\mathcal{A}}(\mathcal{M}) \cap r.Ann_{\mathcal{A}}(\mathcal{M}) = 0$. ■

Chapter 2

Jordan generalized derivations on trivial extension algebras

Abstract. In this chapter, we investigate the problem of describing the form of Jordan generalized derivations on trivial extension algebras. One of the main results shows, under some conditions, that every Jordan generalized derivation on a trivial extension algebra is the sum of a generalized derivation and an antiderivation. This result extends the study of Jordan generalized derivations on triangular algebras (see [76]), and also it can be considered as a "generalized" counterpart of the results given on Jordan derivations of a trivial extension algebra (see [53]).

Throughout this chapter $\mathcal{R}$ will denote a commutative ring with identity, A will be a unital $\mathcal{R}$-algebra with center $Z(A)$ and M will be a unital A-bimodule.

2.1 Introduction

Inspired by the concept of generalized derivations given in [26], Jing and Lu [67] introduced the concept of generalized Jordan derivations as follows: An additive map $d : R \longrightarrow R$ of a ring R is called a generalized Jordan derivation, if there is a Jordan derivation $\tau : R \longrightarrow R$ such that $d(a^2) = d(a)a + a\tau(a)$ for each $a \in R$. Jing and Lu, in [67], considered generalized Jordan derivations of prime rings and standard operator algebras. They proved that every generalized Jordan derivation on 2-torsion free prime ring is a generalized derivation. Their results were extended to semiprime rings by Vukman [100]. Hou and Qi [60, 94] described generalized Jordan derivations on nest algebras. Further, Ma and Ji [79] described generalized Jordan derivations of an upper triangular matrix algebra. Moreover, there is another way to extend generalized derivations to the Jordan case given by Li and Benkovič in [76] as follows: A linear map $f : A \longrightarrow M$ is called a Jordan generalized derivation if there exists a linear map $d : A \longrightarrow M$ such that $f(x \circ y) = f(x) \circ y + x \circ d(y)$ for all $x, y \in A$. Obviously, the definition of a generalized Jordan derivation is generally not equivalent to that of Jordan generalized derivation. Notice also that each Jordan derivation is a Jordan generalized derivation, but generalized derivations are not necessarily Jordan generalized derivations. Li and Benkovič showed, under some conditions, that every Jordan generalized derivation of a triangular algebra is a generalized derivation. The aim of this chapter is to study the form of Jordan generalized derivations on trivial extension algebras.

2.2 Jordan generalized derivations on $A \ltimes M$

Our aim is to study a Jordan generalized derivation on a trivial extension algebra. We give conditions under which it is a sum of a generalized derivation and an antiderivation. Let us start with a general description of these kind of mappings on a trivial extension algebra. Clearly, every linear mapping $f : A \ltimes M \longrightarrow A \ltimes M$ can be presented in the form

$$f(a,m) = (f_A(a) + h_1(m), f_M(a) + h_2(m)) \qquad ((a,m) \in A \ltimes M), \tag{2.1}$$

where the linear mappings $f_A : A \longrightarrow A$, $f_M : A \longrightarrow M$, $h_1 : M \longrightarrow A$ and $h_2 : M \longrightarrow M$ are given by $f_A(a) = (\pi_A \circ f)(a,0)$, $f_M(a) = (\pi_M \circ f)(a,0)$, $h_1(m) = (\pi_A \circ f)(0,m)$ and $h_2(m) = (\pi_M \circ f)(0,m)$, respectively. Here $\pi_A : A \ltimes M \longrightarrow A$ and $\pi_M : A \ltimes M \longrightarrow M$ are the natural projections given by $\pi_A(a,m) = a$ and $\pi_M(a,m) = m$, respectively.

In the sequel, we will simply write $f = (f_A + h_1, f_M + h_2)$ for a linear map $f : A \ltimes M \longrightarrow A \ltimes M$ and $d = (d_A + T, d_M + S)$ for a linear map d.

The following three lemmas are obtained using standard arguments.

Lemma 2.2.1 *A linear map f is a Jordan generalized d-derivation if and only if the following conditions hold:*

1. *f_A is a Jordan generalized d_A-derivation.*
2. *f_M is a Jordan generalized d_M-derivation.*
3. *$h_1(a \circ m) = a \circ h_1(m) = a \circ T(m)$ for all $a \in A$ and $m \in M$.*
4. *$h_2(a \circ m) = f_A(a) \circ m + a \circ S(m) = a \circ h_2(m) + d_A(a) \circ m$ for all $a \in A$ and $m \in M$.*
5. *$m \circ h_1(n) + h_1(m) \circ n = 0$ for all $m, n \in M$.*

Lemma 2.2.2 *A linear map f is a generalized d-derivation if and only if the following conditions hold:*

1. *f_A is a generalized d_A-derivation.*

2. *f_M is a generalized d_M-derivation.*

3. *$h_1(am) = ah_1(m)$ and $h_1(ma) = h_1(m)a$ for all $a \in A$ and $m \in M$.*

4. *$h_2(am) = f_A(a)m + aS(m)$ and $h_2(ma) = h_2(m)a + md_A(a)$ for all $a \in A$ and $m \in M$.*

5. *$mh_1(n) + h_1(m)n = 0$ for all $m, n \in M$.*

Lemma 2.2.3 *A linear map f is an antiderivation if and only if the following conditions hold:*

1. *f_A and f_M are antiderivations.*

2. *$h_1(am) = h_1(m)a$ and $h_1(ma) = ah_1(m)$ for all $a \in A$ and $m \in M$.*

3. *$h_2(am) = h_2(m)a + mf_A(a)$ and $h_2(ma) = ah_2(m) + f_A(a)m$ for all $a \in A$ and $m \in M$.*

4. *$mh_1(n) + h_1(m)n = 0$ for all $m, n \in M$.*

Remark 2.2.4

1. *Notice that when $A \ltimes M$ has a triangular matrix representation, $h_1 = 0$ for a Jordan generalized derivation f on $A \ltimes M$. However, in general h_1 is not zero. For this we use the example given in Remark 1.2.2 (ii): Consider the trivial extension $M_2(\mathbb{Z}/2\mathbb{Z}) \ltimes M_2(\mathbb{Z}/2\mathbb{Z})$ where $M_2(\mathbb{Z}/2\mathbb{Z})$ is the algebra of 2×2 matrices with entries from $\mathbb{Z}/2\mathbb{Z}$. Consider the identity map $h_1 : M_2(\mathbb{Z}/2\mathbb{Z}) \to M_2(\mathbb{Z}/2\mathbb{Z})$. Since the map h_1 verified (3) and (5) in Lemma 2.2.1, the linear map $f : M_2(\mathbb{Z}/2\mathbb{Z}) \ltimes M_2(\mathbb{Z}/2\mathbb{Z}) \to M_2(\mathbb{Z}/2\mathbb{Z}) \ltimes M_2(\mathbb{Z}/2\mathbb{Z})$ defined by $f((a, b)) = (h_1(b), 0)$ for all $a, b \in M_2(\mathbb{Z}/2\mathbb{Z})$ is a Jordan generalized derivation with $h_1 \neq 0$. However, using the equation* (5) *in Lemma 2.2.1, we can give a situation where $h_1 = 0$ (see Lemma 2.2.6).*

2. *Note also that if $g : A \to M$ is a Jordan generalized derivation with an associated linear map d_g, then*

$$2g(a) = g(1) \circ a + 2d_g(a) \qquad (a \in A). \tag{2.2}$$

Thus, if $g(1) = 0$ and M is a 2-torsion free A-module, $g = d_g$ is a Jordan derivation. However, as shown in the following example, $g(1)$ is not zero in general: Let A_2 be the algebra of 2×2 upper triangular matrix on $\mathbb{R}$. Consider $\mathbb{R}$ as an A_2-module under the module operations $am = a_{22}m$ and $ma = ma_{11}$ $(a \in A_2, m \in \mathbb{R})$. Fix $0 \neq \alpha \in \mathbb{R}$ and define

$g : A_2 \ltimes \mathbb{R} \longrightarrow A_2 \ltimes \mathbb{R}$ with $g(a, m) = (\alpha a, a_{12} + \alpha m)$. Then g is a Jordan generalized derivation with an associated linear map $d_g(a, m) = (0, a_{12})$ such that $g(I_{22}, 0) \neq 0$.
In Lemma 2.2.6 we give a situation where $g(1) = 0$ for a Jordan generalized derivation $g : A \to M$.

3. *From the proof of [76, Theorem 2.5], $f(1) \in Z(A \ltimes M)$ when $A \ltimes M$ has a triangular matrix representation and f is a Jordan generalized d-derivation. This was the key of the proof. Indeed, using [76, Theorem 2.3], this implies that the mapping d is a Jordan derivation and $f(x) = f(1)x + d(x)$ for all $x \in A \ltimes M$. However, this does not hold for any trivial extension algebra as shown by [76, Example 2].*

Now we give the first fundamental result.

Theorem 2.2.5 *Every Jordan generalized derivation on $A \ltimes M$ can be written as the sum of a generalized derivation and an antiderivation if and only if the following conditions hold:*

1. *Every Jordan generalized derivation $g : A \longrightarrow M$ is a sum of a generalized derivation and an antiderivation.*

2. *Every linear map $h : M \to A$ such that, for all $a \in A, m, n \in M$, $h(a \circ m) = a \circ h(m)$ and $m \circ h(n) + h(m) \circ n = 0$, is a sum of an A-antihomomorphism δ and an A-homomorphism β which satisfy $m\delta(n) + \delta(m)n = 0 = m\beta(n) + \beta(m)n$ for all $m, n \in M$.*

3. *Every Jordan generalized derivation f on $A \ltimes M$ of the form $f = (f_A, h_2)$ (i.e., $h_1 = 0$ and $f_M = 0$ in (2.1)) can be written as the sum of a generalized derivation and an antiderivation.*

Proof. $\Rightarrow$. We only need to prove (1) and (2).
(1) Let g be a Jordan generalized derivation from A into M. Clearly $(0, g)$ is a Jordan generalized derivation on $A \ltimes M$. Then, by hypothesis, there exists a generalized derivation $(\delta_A + \mathcal{K}', \delta_M + \mathcal{L}')$ and an antiderivation $(D_A + \mathcal{K}, D_M + \mathcal{L})$ such that, for all $a \in A, m \in M$,

$$(0, g(a)) = (D_A(a) + \mathcal{K}(m) + \delta_A(a) + \mathcal{K}'(m), D_M(a) + \mathcal{L}(m) + \delta_M(a) + \mathcal{L}'(m))$$

Take $a = 0$, we get $\mathcal{L}(m) + \mathcal{L}'(m) = 0$. Hence $g = D_M + \delta_M$, we are done.
(2) By hypotheses $(h, 0)$ is a Jordan generalized derivation on $A \ltimes M$. Then, by hypothesis, there exists a generalized derivation $(\delta_A + \mathcal{K}', \delta_M + \mathcal{L}')$ and an antiderivation $(D_A + \mathcal{K}, D_M + \mathcal{L})$ such that, for all $a \in A, m \in M$,

$$(h(m), 0) = (D_A(a) + \mathcal{K}(m) + \delta_A(a) + \mathcal{K}'(m), D_M(a) + \mathcal{L}(m) + \delta_M(a) + \mathcal{L}'(m))$$

Take $m = 0$, we get $D_A + \delta_A = 0$ and $D_M + \delta_M = 0$. Therefore, $h = \mathcal{K} + \mathcal{K}'$, as desired.
$\Leftarrow$. Let $f : A \ltimes M \longrightarrow A \ltimes M$ be a Jordan generalized d-derivation. By hypothesis, h_1 is a sum of an A-antihomomorphism δ and an A-homomorphism β. Also, f_M is a sum of a generalized derivation f_1 and an antiderivation f_2. On the other hand, Lemma 2.2.1 shows that the linear map $(a, m) \longmapsto (f_A(a), h_2(m))$ is a Jordan generalized derivation on $A \ltimes M$. Then, by (3), it can be written as the sum of a generalized derivation Θ and an antiderivation Δ. Then, $f(a, m) = ((\delta(a), f_2(a)) + \Delta(a, m)) + ((\beta(a), f_1(a)) + \Theta(a, m))$, where, using Lemmas 2.2.1, 2.2.2 and 2.2.3, $(a, m) \longmapsto (\delta(a), f_2(a)) + \Delta(a, m)$ is an antiderivation and $(a, m) \longmapsto (\beta(a), f_1(a)) + \Theta(a, m)$ is a generalized derivation. ∎

From [76, Theorem 2.5], triangular algebras are examples of algebras that satisfy the conditions of Theorem 2.2.5. Our second main result (Theorem 2.2.19) generalizes both [76, Theorem 2.5] and [53, Theorem 3.1]. Before giving this result, we treat another situation which is of independent interest. It gives new other examples of algebras outside of the prime ones on which every Jordan generalized derivation is a generalized derivation (see [76, Lemma 2.6] in which it is shown that on prime algebras every Jordan generalized derivation is a generalized derivation).
First we give the following lemma.

Lemma 2.2.6 *Assume that A is 2-torsion free. If a linear map $h : A \to A$ satisfies $h(a \circ b) = a \circ h(b)$ and $a \circ h(b) + h(a) \circ b = 0$ for all $a, b \in A$, then $h = 0$.*

Proof. We have, for every two elements $a, b \in A$,

$$0 = a \circ h(b) + h(a) \circ b = 2h(a \circ b).$$

Then, since A is 2-torsion free, $h(a \circ b) = 0$. This implies that $h = 0$. ∎

Theorem 2.2.7 *Assume that A is a 2-torsion free prime algebra. Then every Jordan generalized d-derivation f on $A \ltimes A$ is a generalized d-derivation of the form $f(x) = f(1)x + d(x)$ for all $x \in A \ltimes A$.*

Proof. Let f be a Jordan generalized d-derivation. Using [76, Lemma 2.6], the Jordan generalized derivations f_A and f_M are generalized derivations (here $M = A$). And, by Lemma 2.2.6, $h_1 = 0$. Now, the relation $h_2(a \circ b) = h_2(a) \circ b + a \circ d_A(b)$ shows that h_2 is a Jordan generalized derivation, and so it is a generalized derivation. Then, $h_2(ab) = h_2(a)b + ad_A(b) = ah_2(b) + d_A(a)b$. Then, $h_2(1) \in Z(A)$. Indeed,

$$h_2(1)a + d_A(a) = h_2(1.a) = h_2(a.1) = ah_2(1) + d_A(a).$$

It remains to prove that $h_2(ab) = f_A(a)b + aS(b)$. We first show that $S(b) = f_A(b) - bh_2(1) + b \circ S(1)$. We have $f_A(a) \circ b + a \circ S(b) = h_2(a \circ b) = a \circ f_A(b) + b \circ S(a)$. Then, $h_2(1) = f_A(1) + S(1)$ and $f_A(1) \circ b + 2S(b) = 2f_A(b) + b \circ S(1)$. Hence, $2(bh_2(1) - b \circ S(1)) = b \circ h_2(1) - 2b \circ S(1) = b \circ (f_A(1) - S(1)) = 2f_A(b) - 2S(b)$. So

$$S(b) = f_A(b) - bh_2(1) + b \circ S(1).$$

Now,

$$\begin{aligned}
h_2(ab) - f_A(a)b - aS(b) &= bf_A(1a) + S(b)a - h_2(ba)\\
&= bf_A(1)a + bd_A(a) + f_A(b)a - bh_2(1)a\\
&+ bS(1)a + S(1)ba - h_2(b)a - bd_A(a)\\
&= f_A(b)a + S(1)ba - h_2(b)a\\
&= f_A(b)a + S(1)ba - (h_2(1)b + d_A(b))a\\
&= f_A(b)a - f_A(1)ba - d_A(b)a\\
&= f_A(b)a - f_A(b)a = 0.
\end{aligned}$$

Finally, using [76, Proposition 2.1], f is of the form $f(x) = f(1)x + d(x)$ for all $x \in A \ltimes A$. This completes the proof. ■

Now we turn to our second aim. We study Jordan generalized derivations on $A \ltimes M$ when there exists a nontrivial idempotent e in A that satisfies $eme' = m$ for all $m \in M$ (where $e' = 1 - e$). To get the second main result we need some lemmas. First recall that the existence of the above idempotent implies the following nice properties which will be used without explicit mention (see also the remark given before [87, Theorem 2.2]).

Lemma 2.2.8 (Proposition 1.1.5) *Consider a non-trivial idempotent e of an algebra A and set $e' = 1 - e$. For an A-bimodule M, the following assertions are equivalent:*

1. *For every $m \in M$, $eme' = m$.*

2. *For every $m \in M$, $e'm = 0 = me$.*

3. *For every $m \in M$, $em = m = me'$.*

4. *For every $m \in M$ and $a \in A$, $am = eaem$ and $ma = me'ae'$.*

We start with the following lemma which shows that the first condition of Theorem 2.2.5 holds when M is a 2-torsion free A-bimodule.

Lemma 2.2.9 *Assume that the A-bimodule M is 2-torsion free. Let $g : A \to M$ be a Jordan generalized derivation with an associated linear map d_g. If there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$ (where $e' = 1 - e$), then $g(1) = 0$, $g = d_g$ is a Jordan derivation and g is a sum of a derivation and an antiderivation.*

Proof. Since g a Jordan generalized derivation, we get

$$2g(e') = g(e' \circ e') = g(e') \circ e' + e' \circ d_g(e') = g(e') + d_g(e').$$

Then $g(e') = d_g(e')$. Now, replacing a by e' in equation (2.2), we get $g(1) = 0$ and so 2-torsion freeness of M implies that $g = d_g$.
Now let $g = f_1 + f_2$, where f_1 and f_2 are defined by $f_1(a) = g(e'ae)$ and $f_2(a) = g(eae+eae'+e'ae')$ for all $a \in A$. We prove that f_1 is an antiderivation. Let $a, b \in A$. We have

$$\begin{aligned} f_1(ab) &= g(e'abe) \\ &= g(e'aebe) + g(e'ae'be) \\ &= g((e'ae) \circ (ebe)) + g((e'ae') \circ (e'be)) \\ &= bg(e'ae) + g(e'be)a \\ &= bf_1(a) + f_1(b)a. \end{aligned}$$

It remains to prove that f_2 is a derivation. To this end, we show that $\Gamma : a \longmapsto g(eae) + g(e'ae')$ is an inner derivation (that is a derivation of the form $\Gamma(a) = ax - xa$ for a fixed $x \in A$) and $d' : a \longmapsto g(eae')$ is a derivation. Note that, for all $a \in A$,

$$\begin{aligned} 0 &= g((eae) \circ (e'ae')) \\ &= ag(e'ae') + g(eae)a. \end{aligned}$$

Hence, replacing a by $e'ae' + e$ in the previous equation, we get

$$g(e'ae') + g(e)a = 0.$$

And replacing a by $eae + e'$ in the same equation, we get

$$ag(e') + g(eae) = 0.$$

Using these relations with the fact that $g(e) = -g(e')$, we get, for every $a \in A$,

$$\begin{aligned} \Gamma(a) &= g(eae) + g(e'ae') \\ &= -ag(e') - g(e)a \\ &= ag(e) - g(e)a. \end{aligned}$$

Then, Γ is an inner derivation.
Now, for every $a, b \in A$,

$$\begin{aligned} d'(ab) &= g(eabe') \\ &= g(eae' \circ e'be') + g(eae \circ ebe') \\ &= g(eae')b + ag(ebe') \\ &= d'(a)b + ad'(b). \end{aligned}$$

This completes the proof. ■

The following lemma shows that also the second condition of Theorem 2.2.5 holds when M is a 2-torsion free A-bimodule.

Lemma 2.2.10 *Let $h : M \to A$ be a linear map such that $h(a \circ m) = a \circ h(m)$ for all $a \in A, m \in M$. If there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$ and $eAe'Ae = \{0\} = e'AeAe'$, where $e' = 1 - e$, then h is a sum of an A-antihomomorphism and an A-homomorphism.*

Proof. First note that $h(m) = h(em) = h(e \circ m) = e \circ h(m) = eh(m) + h(m)e$. Then, $eh(m)e = 0$. Similarly we get $e'h(m)e' = 0$.
This shows that $h = \delta + \beta$ where δ and β are defined by $\delta(m) = e'h(m)e$ and $\beta(m) = eh(m)e'$ (for $m \in M$). We claim that δ is an A-antihomomorphism. Let $a \in A, m \in M$. We have $\delta(am) = e'h(ea \circ m)e = e'h(m)eae + e'h(m)ea = e'h(m)ea = \delta(m)a$. Similarly we prove that $\delta(ma) = a\delta(m)$.
It remains to prove that β is an A-homomorphism. We have

$$\beta(am) = eh(am)e' = eh(ae \circ m)e' = eaeh(m)e' = ea\beta(m).$$

Since $e'a\beta(m) = e'aeh(m)e' = e'(e'ae \circ h(m))e' = e'h(e'ae \circ m)e' = 0$, we get

$$\beta(am) = ea\beta(m) + e'a\beta(m) = a\beta(m).$$

Similarly, we can show that $\beta(ma) = \beta(m)a$. ■

Lemmas 2.2.9 and 2.2.10 show that to get the desired result one should focus on the Jordan generalized d-derivation on $A \ltimes M$ of the form $f = (f_A, h_2)$ (i.e., $h_1 = 0$ and $f_M = 0$ in (2.1)). In the sequel, we will refer to such a particular kind of Jordan generalized d-derivations as a Jordan generalized d-derivation of type Δ. Recall that in this case, f_A is a Jordan generalized d_A-derivation and h_2 satisfies $h_2(a \circ m) = f_A(a) \circ m + a \circ S(m) = a \circ h_2(m) + d_A(a) \circ m$ for all $a \in A$ and $m \in M$ (Lemma 2.2.1).

We use the idea of [53] for the decomposition of a Jordan generalized derivation. First, we give the following observation (compare it with [53, Lemma 3.7]).

Lemma 2.2.11 *Assume that the algebra A and the A-bimodule M are 2-torsion free. Suppose there is a nontrivial idempotent e such that $eAe'Ae = \{0\} = e'AeAe'$, where $e' = 1-e$. Then, for a Jordan generalized d_A-derivation f_A on A, the following assertions hold for all $a \in A$:*

1. $ef_A(e'ae')e = 0$.
2. $e'f_A(eae)e' = 0$.
3. $ef_A(eae')e = 0$.
4. $e'f_A(eae')e' = 0$.
5. $ef_A(e'ae)e = 0$.
6. $e'f_A(e'ae)e' = 0$.

Proof. We prove only (1) and (3). The other assertions are proved similarly.
(1) We have $0 = f_A(e\circ(e'ae')) = e\circ f_A(e'ae')+e'ae'\circ d_A(e)$. Then, $ef_A(e'ae')e = 0$.
(3) We have $f_A(eae') = f_A(e' \circ (eae')) = f_A(e') \circ (eae') + e' \circ d_A(eae')$. This implies that $ef_A(eae')e = 0$. ∎

Thus, using Lemma 2.2.11, a Jordan generalized d-derivation f of type Δ can be decomposed as follows

$$f = J + I + D, \tag{2.3}$$

where, for all $(a, m) \in A \ltimes M$,

$$J(a, m) = (ef_A(e'ae)e' + e'f_A(eae')e, 0), \tag{2.4}$$

$$I(a, m) = (ef_A(eae + e'ae')e' + e'f_A(eae + e'ae')e, 0) \quad \text{and} \tag{2.5}$$

$$D(a, m) = (ef_A(eae)e+ef_A(eae')e'+e'f_A(e'ae)e+e'f_A(e'ae')e', h_2(m)). \tag{2.6}$$

We treat each map separately. For this we use the following lemma.

Lemma 2.2.12 *Suppose there is a nontrivial idempotent e such that $eAe'Ae = \{0\} = e'AeAe'$, where $e' = 1-e$. Then, for a Jordan generalized d-derivation f, the following assertions hold for all $a, b \in A$:*

1. $ef(e'aebe)e' = ebef(e'ae)e'$.
2. $ef(e'ae'be)e' = ef(e'be)e'ae'$.
3. $e'f(eaebe')e = e'f(ebe')eae$.

4. $e'f(eae'be')e = e'be'f(eae')e.$

Proof. We prove only the first assertion. The other ones are proved similarly. Since f is a Jordan generalized d-derivation, we have

$$f(e'aebe) = f(e'ae)ebe + ebef(e'ae) + d(ebe)e'ae + e'aed(ebe).$$

Then, $ef(e'aebe)e' = ebef(e'ae)e'$. ■

Now we prove that J (defined by (2.4)) is an antiderivation. In fact this follows from Lemma 2.2.3 and the following lemma which is a generalization of [53, Lemma 3.3].

Lemma 2.2.13 *Suppose there is a nontrivial idempotent* e *such that* $eAe'Ae = \{0\} = e'AeAe'$*, where* $e' = 1 - e$*. Then, the mapping* $f : A \longrightarrow A$ *defined by* $f(a) = ef(e'ae)e' + e'f(eae')e$ *is an antiderivation.*

Proof. Using Lemma 2.2.12 and the assumption $eAe'Ae = \{0\} = e'AeAe'$, we get

$$\begin{aligned} f(ab) &= ef(e'abe)e' + e'f(eabe')e \\ &= ef(e'aebe)e' + ef(e'ae'be)e' + e'f(eaebe')e + e'f(eae'be')e \\ &= ebef(e'ae)e' + ef(e'be)e'ae' + e'f(ebe')eae + e'be'f(eae')e \\ &= bef(e'ae)e' + ef(e'be)e'a \\ &+ e'f(ebe')ea + be'f(eae')e \\ &= bf(a) + f(b)a. \end{aligned}$$

As desired. ■

Recall that a map $\mathcal{H} : A \to A$ is said to be an inner derivation (resp., a generalized inner derivation) if $\mathcal{H}(x) = ax - xa$ for a fixed $a \in A$ (resp., $\mathcal{H}(x) = ax + xb$ for fixed $a, b \in A$). In [53, Lemma 3.5], it is proved that the first component of our I is an inner derivation when f_A is a Jordan derivation. Here we prove it is a generalized inner derivation when f_A is a Jordan generalized derivation. This helps to show that I is a generalized inner derivation on $A \ltimes M$ (see Lemma 2.2.15).

Lemma 2.2.14 *Suppose there is a nontrivial idempotent* e *such that* $eAe'Ae = \{0\} = e'AeAe'$*, where* $e' = 1 - e$*. Then, the mapping* $I_A : A \longrightarrow A$ *defined by* $I_A(a) = ef(eae + e'ae')e' + e'f(eae + e'ae')e$ *is a generalized inner derivation. Namely,* $I_A(a) = aT - T'a$ *for every* $a \in A$*, where* $T = ef(e)e' - e'f(e)e$ *and* $T' = ed(e)e' - e'f(e)e$*.*

Proof. For all $a \in A$, we have

$$\begin{aligned} 0 &= f((eae)(e'ae') + (e'ae')(eae)) \\ &= eaed(e'ae') + f(eae)e'ae' + e'ae'f(eae) + d(e'ae')eae. \end{aligned}$$

Then, for every $a \in A$,

$$eaed(e'ae')e' + ef(eae)e'ae' = 0 \tag{2.7}$$

and

$$e'ae'f(eae)e + e'd(e'ae')eae = 0. \tag{2.8}$$

For any $a \in A$ replace a by $a + e$ in (2.7). This gives

$$eaed(e'ae')e' + ed(e'ae')e' + ef(eae)e'ae' + ef(e)e'ae' = 0.$$

Hence, replacing a by $e'ae'$ in the previous equation, we get

$$ed(e'ae')e' + ef(e)e'ae' = 0.$$

And also we obtain

$$ef(e'ae')e' + ed(e)e'ae' = 0.$$

Taking $a = e'$, we get

$$ed(e')e' + ef(e)e' = 0.$$

Now, for any $a \in A$, replacing a by $eae + e'$ in (2.7), we obtain

$$eaed(e')e' + ef(eae)e' = 0.$$

Using these relations we obtain

$$-eaef(e)e' + ef(eae)e' = 0.$$

Similarly, we can obtain from relation (2.8) that

$$e'ae'f(e)e + e'f(e'ae')e = 0 \quad \text{and} \quad -e'f(e)eae + e'f(eae)e = 0.$$

These relations and the assumption $eAe'Ae = \{0\} = e'AeAe'$ imply that

$$\begin{aligned} I_A(a) &= ef(eae)e' + ef(e'ae')e' + e'f(eae)e + e'f(e'ae')e \\ &= eaef(e)e' - ed(e)e'ae' + e'f(e)eae - e'ae'f(e)e \\ &= aef(e)e' - ed(e)e'a + e'f(e)ea - ae'f(e)e \\ &= a(ef(e)e' - e'f(e)e) - (ed(e)e' - e'f(e)e)a \\ &= aT - T'a. \end{aligned}$$

As desired. ■

As a consequence of the lemma above, I is a generalized inner derivation on $A \ltimes M$. Namely we have the following lemma.

Lemma 2.2.15 *Suppose there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, and $eAe'Ae = \{0\} = e'AeAe'$, where $e' = 1 - e$. Then, the mapping I (defined by (2.5)) is a generalized inner derivation. Namely, $I(a, m) = (a, m)(T, 0) - (T', 0)(a, m)$ for every $(a, m) \in A \ltimes M$, where $T = ef(e)e' - e'f(e)e$ and $T' = ed(e)e' - e'f(e)e$.*

Now we prove that D is a generalized derivation. For this, we use the following two lemmas. The first one can be of independent interest. It presents some properties of a Jordan generalized d-derivation of type Δ.
Recall that the left annihilator, $l.Ann_A(M)$, of M is the set of all elements r in A such that $rM = 0$. Similarly the right annihilator, $r.Ann_A(M)$, of M is defined.

Lemma 2.2.16 *Assume that the algebra A and the A-bimodule M are 2-torsion free and there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$ (where $e' = 1 - e$). Let f be a Jordan generalized d-derivation of type Δ. Then the following assertions hold:*

1. *$h_2(am) = f_A(a)m + aS(m)$ and $h_2(ma) = mf_A(a) + S(m)a$ for all $a \in A$ and $m \in M$.*
2. *$mf_A(e') = f_A(e)m$ for all $m \in M$.*
3. *$f_A(ab) - f_A(a)b - ad_A(b) \in r.Ann_A(M) \cap l.Ann_A(M)$.*

Proof. (1). We need only to prove the first equality. The second one follows immediately. First we prove that $d_A(e)m = 0 = md_A(e)$.
We have $f_A(2e) = f_A(e \circ e) = f_A(e)e + ef_A(e) + ed_A(e) + d_A(e)e$. Then, $ed_A(e)e = 0$ (since A is 2-torsion free). Then, $d_A(e)m = ed_A(e)em = 0$. On the other hand, since $h_2(m) = h_2(e \circ m) = h_2(m) + d_A(e)m + md_A(e)$, we get

$$md_A(e) = -d_A(e)m = 0. \tag{2.9}$$

Now,

$$\begin{aligned} 2h_2(am) &= h_2((ae + ea)m) \\ &= h_2((ae + ea) \circ m) \\ &= f_A(e \circ a) \circ m + (ae + ea) \circ S(m) \\ &= (f_A(a)e + ef_A(a) + ad_A(e) + d_A(e)a) \circ m + 2aS(m) \\ &= 2f_A(a)m + 2aS(m). \end{aligned}$$

Therefore, the 2-torsion freeness of M implies that $h_2(am) = f_A(a)m + aS(m)$, as desired.
(2). For every $m \in M$, $mf_A(e') + S(m)e' = h_2(me') = h_2(em) = f_A(e)m + eS(m)$. Then, $mf_A(e') = f_A(e)m$.
(3). We have $h_2(am) = f_A(a)m + aS(m)$. Then, $h_2(m) = f_A(e)m + S(m)$ which means that $S(m) = h_2(m) - f_A(e)m$, and so

$$\begin{aligned} S(am) &= h_2(am) - f_A(e)am \\ &= f_A(a)m + aS(m) - f_A(e)am \\ &= (f_A(a) - f_A(e)a)m + aS(m). \end{aligned}$$

Then,

$$\begin{aligned} h_2(abm) &= f_A(a)bm + aS(bm) \\ &= f_A(a)bm + a(f_A(b) - f_A(e)b)m + abS(m). \end{aligned}$$

On the other hand, $h_2(abm) = f_A(ab)m + abS(m)$. Then,

$$f_A(ab)m = (f_A(a)b + a(f_A(b) - f_A(e)b))m. \tag{2.10}$$

We prove that $(f_A(b) - bf_A(e))m = d_A(b)m$.
In (2.10), for $a = e$ we get

$$f_A(eb)m = f_A(b)m. \tag{2.11}$$

And, for $b = e$, we get

$$f_A(ae)m = f_A(a)m. \tag{2.12}$$

Since f_A is a Jordan generalized d_A-derivation,

$$f_A(eb + be) = f_A(e)b + bf_A(e) + ed_A(e) + d_A(e)e.$$

Then, using (2) and equalities (2.11) and (2.12), we get

$$2f_A(b)m = bf_A(e)m + bf_A(e)m + 2d_A(b)m.$$

Then, $(f_A(b) - bf_A(e))m = d_A(b)m$. Hence, (2.10) becomes

$$(f_A(ab) - f_A(a)b - ad_A(b))m = 0. \tag{2.13}$$

The same argument as above, using $h_2(mb) = mf_A(b) + S(m)b$ and taking e' instead of e, shows that

$$m(f_A(ba) - bf_A(a) - d_A(b)a) = 0. \tag{2.14}$$

Finally, since f_A is a Jordan generalized d_A-derivation,

$$f_A(ab) - f_A(a)b - ad_A(b) = -f_A(ba) + bf_A(a) + d_A(b)a.$$

This ends the proof. ■

Lemma 2.2.17 *Assume that the algebra A and the A-bimodule M are 2-torsion free and there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$ (where $e' = 1-e$). Let f be a Jordan generalized d-derivation of type Δ. If f_A is a generalized d_A-derivation, then $h_2(ma) = h_2(m)a + md_A(a)$ for all $a \in A$, $m \in M$.*
Consequently, the Jordan generalized d-derivation f is a generalized d-derivation.

Proof. For all $a \in A$ and $m \in M$, we have

$$h_2(ma) - h_2(m)a - md_A(a) = ah_2(m) + d_A(a)m - h_2(am).$$

Then, using the hypothesis and (1) of Lemma 2.2.16, we get

$$\begin{aligned} h_2(ma) - h_2(m)a - md_A(a) &= a(f_A(e)m + eS(m)) + d_A(a)m - f_A(a)m - aS(m) \\ &= af_A(e)m + d_A(a)m - (f_A(e'a) + f_A(ea))m \\ &= af_A(e)m + d_A(a)m - (f_A(e')e'a + e'd_A(e'a) \\ &\quad +f_A(e)a + ed_A(a))m. \end{aligned}$$

Then, using the second assertion of Lemma 2.2.16, we get

$$\begin{aligned} h_2(ma) - h_2(m)a - md_A(a) &= af_A(e)m - amf_A(e') \\ &= af_A(e)m - af_A(e)m = 0. \end{aligned}$$

As desired. ■

Lemma 2.2.18 *Assume that the algebra A and the A-bimodule M are 2-torsion free. Suppose there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, , where $e' = 1 - e$. If $eAe'Ae = \{0\} = e'AeAe'$ and $e'r.Ann_A(M)e' = \{0\} = el.Ann_A(M)e$, then D (defined by (2.6)) is a generalized derivation.*

Proof. One can show easily that D is a Jordan generalized derivation of type Δ. Then, by Lemmas 2.2.16 and 2.2.17, it suffices to prove that the map $f' : A \to A$, defined by $f'(a) = ef_A(eae)e + ef_A(eae')e' + e'f_A(e'ae)e + e'f_A(e'ae')e'$ for all $a \in A$, is a generalized derivation.
By the hypothesis and the assertion (3) of Lemma 2.2.16, we have

$$\begin{aligned} ef_A(eaebe)e &= ef_A(eae)ebe + eaed_A(ebe)e, \\ ef_A(eae'be)e &= ef_A(eae')e'be + eae'd_A(e'be)e = 0 \text{ (since } eAe'Ae = \{0\}), \\ e'f_A(e'ae'be')e' &= e'f_A(e'ae')e'be' + e'ae'd_A(e'be')e', \quad \text{and} \\ e'f_A(e'aebe')e' &= e'f_A(e'ae)ebe' + e'aed_A(ebe')e' = 0 \text{ (since } e'AeAe' = \{0\}). \end{aligned}$$

And, since f_A is a Jordan generalized d-derivation, we get, as done in Lemma 2.2.12, the following equalities:

$$\begin{aligned} ef_A(eaebe')e' &= ef_A(eae)ebe' + eaed_A(ebe')e', \\ ef_A(eae'be')e' &= ef_A(eae')e'be' + eae'd_A(e'be')e', \\ e'f_A(e'aebe)e &= e'f_A(e'ae)ebe + e'aed_A(ebe)e, \quad \text{and} \\ e'f_A(e'ae'be)e &= e'f_A(e'ae')e'be + e'ae'd_A(e'be)e. \end{aligned}$$

These relations with the assumption $eAe'Ae = \{0\} = e'AeAe'$ give us that $f'(ab) = f'(a)b + ad'(b)$ for all $a, b \in A$, where $d'(b) = ed_A(ebe)e + e'd_A(e'be')e' + ed_A(ebe')e' + e'd_A(e'be)e'$. That is, f' is a generalized derivation. ■

Finally, combining the above results we get our second main result which generalizes both [76, Theorem 2.5] and [53, Theorem 3.1]. Notice that, if $A \ltimes M$ has a triangular matrix representation, then using Proposition 1.1.1, the antiderivation f_1 in Lemma 2.2.9, the antihomomorphism δ in Lemma 2.2.10 and the antiderivation J (defined in (2.4)) are zero.

Theorem 2.2.19 *Assume that the algebra A and the A-bimodule M are 2-torsion free. Suppose there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, where $e' = 1 - e$. If $eAe'Ae = \{0\} = e'AeAe'$ and $e'r.Ann_A(M)e' = \{0\} = el.Ann_A(M)e$, then every Jordan generalized derivation on $A \ltimes M$ can be written as the sum of a generalized derivation and an antiderivation.*

Remark 2.2.20 *Assume that the algebra A and the A-bimodule M are 2-torsion free and there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, where $e' = 1 - e$. The following two situations present two particular cases of the trivial extension algebras which satisfy conditions of Theorem 2.2.19.*

1. *When $r.Ann_A(M) \cap l.Ann_A(M) = \{0\}$. In fact, all of the sets $eAe'Ae$, $e'AeAe'$, $e'r.Ann_A(M)e'$ and $el.Ann_A(M)e$ are in $r.Ann_A(M) \cap l.Ann_A(M)$.*
2. *When M is a loyal $(eAe, e'Ae')$-bimodule. Recall that an (A, B)-bimodule M, where A and B are algebras, is said to be loyal if, for every $(a, b) \in A \times B$, $aMb = \{0\}$ implies $a = 0$ or $b = 0$ (see for instance [18, Definition 2.1]).*

Lemmas 2.2.16 and 2.2.17 show that if we assume that every Jordan generalized derivation on A is a generalized derivation (as for the case of prime algebras), we get a new other context where Jordan generalized derivations on $A \ltimes M$ can be written as the sum of a generalized derivation and an antiderivation.

In fact, we show that under this condition we do not need the condition that A is 2-torsion free.

Theorem 2.2.21 *Assume that the A-bimodule M is 2-torsion free and that every Jordan generalized derivation on A is a generalized derivation. Suppose there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, then every Jordan generalized derivation on $A \ltimes M$ can be written as the sum of a generalized derivation and an antiderivation.*

Proof. Lemmas 2.2.9 and 2.2.10 show that we only need to prove the result for the mapping $(a, m) \longmapsto (f_A(a), h_2(m))$. From the prove of Lemmas 2.2.16 and 2.2.17, we can deduce that we need only to prove that $h_2(am) = f_A(a)m + aS(m)$ for all $a \in A$ and $m \in M$.
Note that $d_A(e)m = md_A(e') = 0$. Indeed, by the hypothesis, f_A is a generalized derivation and, by [76, Proposition 2.1], d_A is a derivation. Then, for all $m \in M$, $d_A(e)m = d_A(e)em + ed_A(e)m = d_A(e)m + d_A(e)m$. Then, $d_A(e)m = 0$. Similarly, we prove that $md_A(e') = 0$. Now,

$$\begin{aligned} h_2(am) &= h_2(ae \circ m) \\ &= f_A(ae) \circ m + ae \circ S(m) \\ &= f_A(a)m + ad_A(e) \circ m + aS(m) \\ &= f_A(a)m + aS(m). \end{aligned}$$

As desired. ■

Let $M_n(\mathbb{R})$ (resp., $T_n(\mathbb{R})$) denotes the algebra of all matrices (resp., of all upper triangular matrices) on $\mathbb{R}$. As a consequence of Theorem 2.2.21 and [76, Theorem 2.5] we get the following result.

Corollary 2.2.22 *Let M be a 2-torsion free $M_n(\mathbb{R})$-bimodule (resp., $T_n(\mathbb{R})$ -bimodule). Suppose there exists a nontrivial idempotent e in $M_n(\mathbb{R})$ (resp., $e \in T_n(\mathbb{R})$) such that $eme' = m$ for all $m \in M$. Then every Jordan generalized derivation on $M_n(\mathbb{R}) \ltimes M$ (resp., $T_n(\mathbb{R}) \ltimes M$) can be written as the sum of a generalized derivation and an antiderivation.*

2.3 f-Generalized derivations

In a recent paper [12], Benkovič introduced the notion of f-derivations which unifies several kind of derivations including the classical derivations as follows: Consider a fixed nonzero multilinear polynomial f in noncommuting indeterminates x_i over $\mathcal{R}$:

$$f(x_1, \ldots, x_n) = \sum_{\pi \in S_n} \alpha_\pi x_{\pi(1)} x_{\pi(2)} \ldots x_{\pi(n)} \quad (\alpha_\pi \in \mathcal{R}), \tag{2.15}$$

where S_n denotes the symmetric group of order an integer $n \geq 2$. An $\mathcal{R}$-linear map $\mathcal{D} : A \longrightarrow M$ is called an f-derivation if it satisfies

$$\mathcal{D}(f(x_1, \ldots, x_n)) = \sum_{i=1}^{n} f(x_1, \ldots, x_{i-1}, \mathcal{D}(x_i), x_{i+1}, \ldots, x_n) \tag{2.16}$$

for all $x_1, \ldots, x_n \in A$. Thus, a derivation is an f-derivation for the polynomial $f(x_1, x_2) = x_1x_2$, a Jordan derivation is an f-derivation for the polynomial $f(x_1, x_2) = x_1 \circ x_2 = x_1x_2 + x_2x_1$, a Jordan triple derivation (see for example [51]) is an f-derivation for the polynomial $f(x_1, x_2, x_3) = x_1x_2x_3 + x_3x_2x_1$, a Lie derivation (see [35]) is an f-derivation for the polynomial $f(x_1, x_2) = [x_1, x_2] = x_1x_2 - x_1x_2$, and a Lie triple derivation (see for example [13] and [77]) is an f-derivation for the polynomial $f(x_1, x_2, x_3) = [[x_1, x_2], x_3]$.
In [12, Theorem 1.3], Benkovič proved (under some conditions) that every f-derivation is a Jordan derivation. Then, he used this result to show that (under some conditions) that every f-derivation on a triangular algebra is a derivation [12, Theorem 1.1]. Then, naturally one can ask whether there exists a "generalized" counterpart of Benkovič's results. In this section we answer this natural question positively.

In what follows, we consider a fixed nonzero multilinear polynomial f as defined in 2.15. An $\mathcal{R}$-linear map $F : A \longrightarrow M$ is called an *f-generalized d-derivation* (or simply, an f-generalized derivation), where $d : A \longrightarrow M$ is an $\mathcal{R}$-linear map, if

$$F(f(x_1, \ldots, x_n)) = f(F(x_1), x_2, \ldots, x_n) + \sum_{i=2}^{n} f(x_1, \ldots, x_{i-1}, d(x_i), x_{i+1}, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in A$. Then, obviously every f-derivation F is an f-generalized F-derivation. Also, note that the f-generalized F-derivation unifies various kind of generalized derivations including the generalized derivations and the Jordan generalized derivations (see, for instance, [89] for the notion of d-Lie derivations (a generalized counterpart of Lie derivations), [77] for the notion of generalized Lie triple derivations, and [67] for the notion of generalized Jordan triple derivations).
We say that an element $r \in \mathcal{R}$ is M-regular if, for every $m \in M$, $rm = 0$ implies that $m = 0$. Let

$$\alpha = \sum_{\pi \in S_n} \alpha_\pi \in \mathcal{R}$$

be the sum of coefficients of the polynomial f from (2.15). We start with the generalized counterpart of [12, Theorem 1.3] which needs a similar argument with some suitable modifications.

Theorem 2.3.1 *Let $F : A \longrightarrow M$ be an f-generalized derivation, with $\alpha \neq 0$. If M is $(n-1)$-torsion free and α is M-regular, then F is a Jordan generalized derivation.*

Consequently, as done in [12], Theorem 2.3.1 together with Theorems 2.2.7 and 2.2.19 lead to a characterization of a particular case of f-generalized derivation on some trivial extension algebras.

Corollary 2.3.2 *Assume that A is a 2-torsion free prime algebra. Let $F : A \ltimes A \longrightarrow A \ltimes A$ be an f-generalized derivation, with $\alpha \neq 0$. If A is $2(n-1)$-torsion free and α is A-regular, then F is a generalized derivation.*

Note that the generalized derivation F has the form $F(x) = F(1)x + d(x)$ for all $x \in A \ltimes A$ (by [76, Proposition 2.1]).

Corollary 2.3.3 *Let $F : A \ltimes M \longrightarrow A \ltimes M$ be an f-generalized derivation, where $f \in \mathcal{R}\langle x_1, x_2, ...\rangle$ is a multilinear polynomial of degree $n \geq 2$ with $\alpha \neq 0$. Consider the following conditions:*

(i) A and M are both $2(n-1)$-torsion free.

(ii) α is A-regular and M-regular.

(iii) There exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, where $e' = 1 - e$, and $eAe'Ae = \{0\} = e'AeAe'$ and $e'r.Ann_A(M)e' = \{0\} = el.Ann_A(M)e$.

If (i), (ii) and (iii) hold, then F can be written as the sum of a generalized derivation and an antiderivation.

Also as a generalization of [12, Theorem 1.1], we obtain the following result which characterizes a particular case of f-generalized derivation on triangular algebras.

Corollary 2.3.4 *Let A and B be unital algebras over a 2-torsion free commutative ring R, and M be a unital (A, B)-bimodule that is faithful as both a left A-module and a right B-module. Let $\mathcal{A} = Tri(A, M, B)$ be the triangular algebra. Let $F : \mathcal{A} \longrightarrow \mathcal{A}$ be an f-generalized derivation, where with $\alpha \neq 0$. If $\mathcal{A}$ is $2(n-1)$-torsion free and α is $\mathcal{A}$-regular, then F is a generalized derivation of the form $F(x) = F(1)x + d(x)$ for all $x \in \mathcal{A}$.*

It is worth noting that there are interesting f-generalized derivations with $\alpha = 0$ which deserve investigating. However, even in the case of f-derivations the situation is much more unpredictable as mentioned in [12, Problem 1.2]. Thus the question for this case remains an open interesting question.

CHAPTER 3

Lie generalized derivations on trivial extension algebras

Abstract. In this paper, we investigate the problem of describing the form of Lie generalized derivations on trivial extension algebras. We show, under some conditions, that every Lie generalized derivation on a trivial extension algebra is a sum of a generalized derivation and a center valued map which vanishes on all commutators. As an application we characterize Lie generalized derivation on a triangular algebra.

Throughout this chapter $\mathcal{R}$ will denote a commutative ring with unity, A will be a unital $\mathcal{R}$-algebra with center $Z(A)$ and M will be a unital A-bimodule.

3.1 Introduction

Let $d : A \longrightarrow M$ and $f : A \longrightarrow M$ be linear maps. Recall that f is said to be a *generalized d-derivation* (or simply a generalized derivation) if

$$f(ab) = f(a)b + ad(b) \quad (a, b \in A). \tag{3.1}$$

For $d = f$, a generalized d-derivation f is just the classical derivation (see Brešar's paper [26]). From Gölbaşy and Kaya [54], an additive mapping $F : A \longrightarrow M$ is said to be a right (resp., a left) generalized derivation associated to d if $F(xy) = F(x)y + xd(y)$ (resp., $F(xy) = d(x)y + xF(y)$) for all $x, y \in A$. Thus, in the sense of Gölbaşy and Kaya, F is said to be a generalized derivation associated to d if F is both a left and a right generalized derivation associated to d. In this paper we adopt the definition of generalized derivation in the sense of Gölbaşy and Kaya.

Inspired by Brešar's idea, several authors have introduced "generalized" counterparts of various kind of derivations like for Jordan derivations (see [76]).

A linear map $T : A \longrightarrow A$ is said to be a *Lie generalized L-derivation* (or simply a Lie generalized derivation), where $L : A \longrightarrow A$ is a linear map, if

$$T([a, b]) = [T(a), b] + [a, L(b)] \quad (a, b \in A). \tag{3.2}$$

Then, when $L = T$, T is just the classical Lie derivation and when $L = 0$, T is just a Lie centralizer; that is $T([a, b]) = [T(a), b]$ (equivalently, $T([a, b]) = [a, T(b)]$) for all $a, b \in A$ (see [66]). Also, it is an evident fact that every generalized derivation is a Lie generalized derivation. Moreover, for a generalized derivation $D : A \longrightarrow A$ associated to a map d and a linear map $l : A \longrightarrow Z(A)$, the sum $D+l$ is a Lie generalized derivation (associated to $d+l$) if and only if $l([a, b]) = 0$ for all $a, b \in A$. Following Cheung's terminology (see [35]), a Lie generalized derivation T associated to a map L of this form (i.e. there exist a generalized derivation $D : A \longrightarrow A$ associated to a map d and a linear map $l : A \longrightarrow Z(A)$ such that $T = D + l$ and $L = d + l$) will be called *proper Lie generalized derivation.*

A problem that we are dealing with is studying those conditions on an algebra such that every Lie derivation on it is proper. We say that an algebra A has Lie derivation property if every Lie derivation on A is proper.

It is worth recalling that, Martindale [80] was the first one who showed that every Lie derivation on certain primitive ring is proper. In [34], Cheung initiated the study of the properness of Lie derivations on triangular algebras (see also [15, 34, 47, 65, 86, 105] for a related investigation). Recently, in [87], Mokhtari, Moafian and Ebrahimi Vishki studied Lie derivations on trivial extension algebras. In this chapter, we are interested in describing the form of Lie generalized derivations on trivial extension algebras. As a main result, we give conditions under which every Lie generalized derivation on a trivial extension algebra is a proper Lie generalized derivation (see Theorem 3.2.16). This result extends the study of Lie derivations on trivial extension algebras done in [87]. Also, it allows to show when Lie generalized derivations on triangular algebras is a proper Lie generalized derivation (see Corollary 3.2.17).

3.2 Lie generalized derivations on trivial extension algebras

Our aim is to study a Lie generalized derivation on a trivial extension algebra. We give conditions under which it is proper.

Let us start with a general description of these kind of mapping on a trivial extension algebra.

Clearly, every linear mapping $f : A \ltimes M \longrightarrow A \ltimes M$ has the form

$$f(a, m) = (f_A(a) + h_1(m), f_M(a) + h_2(m)) \qquad ((a, m) \in A \ltimes M), \qquad (3.3)$$

where the linear mappings $f_A : A \longrightarrow A$, $f_M : A \longrightarrow M$, $h_1 : M \longrightarrow A$ and $h_2 : M \longrightarrow M$ are given by $f_A(a) = (\pi_A \circ f)(a, 0)$, $f_M(a) = (\pi_M \circ f)(a, 0)$, $h_1(m) = (\pi_A \circ f)(0, m)$ and $h_2(m) = (\pi_M \circ f)(0, m)$, respectively. Here $\pi_A : A \ltimes M \longrightarrow A$ and $\pi_M : A \ltimes M \longrightarrow M$ are the natural projections given by $\pi_A(a, m) = a$ and $\pi_M(a, m) = m$, respectively.

In the sequel, we will simply write $f = (f_A + h_1, f_M + h_2)$ for a linear map $f : A \ltimes M \longrightarrow A \ltimes M$ and $d = (d_A + T, d_M + S)$ for a linear map d.

The following three Lemmas are obtained using standard arguments.

Lemma 3.2.1 *A linear map f is a Lie generalized d-derivation if and only if, for all $a, b \in A, m, n \in M$, the following conditions hold:*

1. *f_A is a Lie generalized d_A-derivation.*
2. *f_M is a Lie generalized d_M-derivation.*
3. *$h_1([a, m]) = [a, T(m)] = [a, h_1(m)]$ for all $a \in A$ and $m \in M$.*
4. *$h_2([a, m]) = [f_A(a), m] + [a, S(m)]$ and $h_2([m, a]) = [h_2(m), a] + [m, d_A(a)]$ for all $a \in A$ and $m \in M$.*
5. *$[h_1(m), n] + [m, T(n)] = 0$ for all $m, n \in M$.*

Lemma 3.2.2 *A linear map f is a generalized d-derivation if and only if the following conditions hold:*

1. *f_A is a generalized d_A-derivation.*
2. *f_M is a generalized d_M-derivation.*
3. *$h_1(am) = ah_1(m)$ and $h_1(ma) = h_1(m)a$ for all $a \in A$ and $m \in M$.*
4. *$h_2(am) = f_A(a)m + aS(m)$ and $h_2(ma) = h_2(m)a + md_A(a)$ for all $a \in A$ and $m \in M$.*
5. *$mh_1(n) + h_1(m)n = 0$ for all $m \in M$.*

Lemma 3.2.3 *A linear map f is a center valued map vanishing on commutators if and only if the following conditions hold:*

1. *f_A and f_M are center valued map vanishing on commutators.*

2. *$h_1([a,m]) = [h_1(m), a] = 0$ and $[f_A(a), m] = 0$ for all $a \in A$ and $m \in M$.*

3. *$h_2([a,m]) = 0$ and $[h_2(m), a] = 0$ for all $a \in A$ and $m \in M$.*

4. *$[h_1(m), n] = 0$ for all $m, n \in M$.*

Now we give the first fundamental result.

Theorem 3.2.4 *Every Lie generalized derivation on $A \ltimes M$ can be written as a sum of a generalized derivation and a center valued map vanishing on commutators if and only if the following conditions hold:*

1. *Every Lie generalized derivation $g : A \longrightarrow M$ is a sum of a generalized derivation and a center valued map vanishing on commutators.*

2. *Every linear map $h : M \rightarrow A$ such that, for all $a \in A$ and $m, n \in M$, $h([a,m]) = [a, h(m)]$ and $[n, h(m)] = 0$, is a sum of an A-homomorphism δ and a commuting map β which satisfy $m\delta(n) + \delta(m)n = 0 = \beta(m)n - n\beta(m)$ for all $m, n \in M$.*

3. *Every Lie generalized derivation f on $A \ltimes M$ of the form $f = (f_A, h_2)$ (i.e., $h_1 = 0$ and $f_M = 0$ in (3.3)) can be written as the sum of a generalized derivation and a center valued map vanishing on commutators.*

Proof. $\Rightarrow$. We only need to prove (1) and (2).
(1) Let g be a Lie generalized derivation from A into M. Clearly $(0, g)$ is a Lie generalized derivation on $A \ltimes M$. Then, by hypothesis, there exists a generalized derivation $(\delta_A + \mathcal{K}', \delta_M + \mathcal{L}')$ and a center valued map vanishing on commutators $(D_A + \mathcal{K}, D_M + \mathcal{L})$ such that, for all $a \in A, m \in M$,

$$(0, g(a)) = (D_A(a) + \mathcal{K}(m) + \delta_A(a) + \mathcal{K}'(m), D_M(a) + \mathcal{L}(m) + \delta_M(a) + \mathcal{L}'(m))$$

Take $a = 0$, we get $\mathcal{L}(m) + \mathcal{L}'(m) = 0$. Hence $g = D_M + \delta_M$, we are done.
(2) By hypotheses $(h, 0)$ is a Lie generalized derivation on $A \ltimes M$. Then, by hypothesis, there exists a generalized derivation $(\delta_A + \mathcal{K}', \delta_M + \mathcal{L}')$ and a center valued map vanishing on commutators $(D_A + \mathcal{K}, D_M + \mathcal{L})$ such that, for all $a \in A, m \in M$,

$$(h(m), 0) = (D_A(a) + \mathcal{K}(m) + \delta_A(a) + \mathcal{K}'(m), D_M(a) + \mathcal{L}(m) + \delta_M(a) + \mathcal{L}'(m))$$

Take $m = 0$, we get $D_A + \delta_A = 0$ and $D_M + \delta_M = 0$. Therefore, $h = \mathcal{K} + \mathcal{K}'$. We need to show $[\mathcal{K}(m), a] = 0$. Indeed, $[a, \mathcal{K}(m)] = [a, h(m)] - [a, \mathcal{K}'(m)] = h([a,m]) - \mathcal{K}'([a,m]) = \mathcal{K}([a,m]) = 0$.

$\Leftarrow$. Let $f : A \ltimes M \longrightarrow A \ltimes M$ be a Lie generalized d-derivation. By hypothesis, h_1 is a sum of an A-homomorphism δ and a commuting map β. Also, f_M is a sum of a generalized derivation f_1 and a center valued map vanishing on commutators f_2. We show that (β,0) is a center valued map vanishing on commutators. Indeed

$$\begin{aligned}\beta([a,m]) &= (h_1 - \delta)([a,m]) \\ &= h_1([a,m]) - \delta([a,m]) \\ &= [a, h_1(m)] - [a, \delta(m)] \\ &= [a, \beta(m)] \\ &= 0.\end{aligned}$$

On the other hand, it is clear that the linear map $(a, m) \longmapsto (f_A(a), h_2(m))$ is a Lie generalized derivation on $A \ltimes M$. Then, by (3), it can be written as the sum of a center valued map vanishing on commutators Θ and a generalized derivation Δ. Then, $f(a, m) = ((\delta(a), f_2(a)) + \Delta(a, m)) + ((\beta(a), f_1(a)) + \Theta(a, m))$, where, using Lemmas 3.2.1, 3.2.2 and 3.2.3, $(a, m) \longmapsto (\delta(a), f_1(a))+ \Delta(a, m)$ is a generalized derivation and $(a, m) \longmapsto (\beta(a), f_2(a)) + \Theta(a, m)$ is a center valued map vanishing on commutators. ■

Now we turn to our second aim. We study Lie generalized derivations on $A \ltimes M$ when there exists a nontrivial idempotent e in A that satisfies $eme' = m$ for all $m \in M$ (where $e' = 1 - e$). Triangular algebras are examples of this kind of algebras.

To get the second main result we need some lemmas. First recall that the existence of the above idempotent implies the following nice properties which will be used without explicit mention (see also remark given before [87, Theorem 2.2]).

Lemma 3.2.5 (Proposition 1.1.5) *Consider a non-trivial idempotent e of an algebra S and set $e' = 1 - e$. For an S-bimodule N, the following assertions are equivalent:*

1. *For every $m \in N$, $eme' = m$.*
2. *For every $m \in N$, $e'm = 0 = me$.*
3. *For every $m \in N$, $em = m = me'$.*
4. *For every $m \in N$ and $a \in S$, $am = eaem$ and $ma = me'ae'$.*

Lemma 3.2.6 *Suppose there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, where $e' = 1 - e$. Then, for every Lie generalized derivation f_M the following conditions holds:*

1. $2f_M(e'ae) = 0$.
2. $f_M(e') = d_M(e')$.
3. $d_M(1) = 0$.

Proof. (1) Since f_M is a Lie generalized derivations, we have

$$\begin{aligned} f_M(e'ae) &= f_M([e'ae, e]) \\ &= [f_M(e'ae), e] + [e'ae, d_M(e)] \\ &= -ef_M(e'ae) \\ &= -f_M(e'ae). \end{aligned}$$

(2) We have $f_M([a,b]) = [f_M(a), b] + [a, d_M(b)]$. Then, for $a = b = e'$, we get $0 = [f_M(e'), e'] + [e', d_M(e')]$. Hence $f_M(e') = d_M(e')$.
(3) Taking $a = e$ and $b = 1$, we get $0 = [f_M(e), 1] + [e, d_M(1)]$. Hence $d_M(1) = 0$.
■

Lemma 3.2.7 *Assume that the A-bimodule M is 2-torsion free. Let $f_M : A \to M$ be a Lie generalized derivation with an associated linear map d_M. If there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$ (where $e' = 1 - e$), then f_M is a generalized derivation.*

Proof. Let us define a linear map I_M as follows $I_M(a) = f_M(eae) + f_M(e'ae')$ for all $a \in A$. We show that I_M is an inner generalized derivation. We have $0 = f_M(eae)a + ad_M(e'ae') = -af_M(e'ae') - d_M(eae)a$. Then, taking $a = e'ae' + e$ in the above equation, we get $f_M(e'ae') = d_M(e')a$. Similarly if we take $a = eae + e'$, we get $f_M(eae) = -ad_M(e')$. Hence $I_M(a) = -ad_M(e') + d_M(e')a$. It remains to show that g_M defined by $a \longmapsto f_M(eae')$ is a generalized derivation. We have

$$\begin{aligned} g_M(ab) &= f_M(eabe') \\ &= f_M(eaebe') + f_M(eae'be') \\ &= f_M([ea, ebe']) + f_M([ea, e'be']) \\ &= g_M(a)b + ag_M(b). \end{aligned}$$

Hence, by Lemma 3.2.6, this completes the proof. ■

Lemma 3.2.8 *Let $h : M \to A$ be a linear map such that $h([a,m]) = [a, h(m)]$ for all $a \in A, m \in M$. If there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$ (where $e' = 1-e$), then for every map β defined by $m \longmapsto e'h(m)e$, the following assertions hold:*

1. $2\beta(am) = 0$ *and* $2\beta(ma) = 0$ *for all* $a \in A$ *and* $m \in M$.

2. β *is an A-anti-homomorphism.*

Proof. (1) We have, for $a \in A$ and $m \in M$

$$\begin{aligned} \beta(ma) &= e'h(ma)e \\ &= e'h([e, ma])e \\ &= e'[e, h(ma)]e \\ &= -e'h(ma)e \\ &= -\beta(ma). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \beta(am) &= e'h(am)e \\ &= -e'h([e', am])e \\ &= -e'[e', h(am)]e \\ &= -e'h(am)e \\ &= -\beta(am). \end{aligned}$$

(2) We claim that β is an A-antihomomorphism. Let $a \in A$ and $m \in M$. We have $\beta(am) = e'h(am)e = e'h([ea, m])e = -e'h(m)eae = -\beta(m)ae$. By (1), $\beta(am) = \beta(m)ae$. On the other hand we have $\beta(m)ae' = e'h(m)eae' = e'[h(m), eae']e' = e'h([m, eae'])e' = 0$, as desired. Similarly we prove that $\beta(ma) = a\beta(m)$. ■

The following lemma shows that also the second condition of Theorem 3.2.4 holds.

Lemma 3.2.9 *Assume that A is 2-torsion free. Let $h : M \to A$ be a linear map such that $h([a,m]) = [a, h(m)]$ for all $a \in A$ and $m \in M$. If there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$ (where $e' = 1-e$), then h is a sum of an A-homomorphism and a commuting map.*

Proof. First note that $h(m) = h(em) = h([e,m]) = [e, h(m)] = eh(m) - h(m)e$. Then, $eh(m)e = 0$. Similarly we get $e'h(m)e' = 0$.
This shows that $h = \delta + \beta$, where δ and β are respectively defined by $\beta(m) =$

$e'h(m)e$ and $\delta(m) = eh(m)e'$. We claim that δ is an A-homomorphism. We have

$$\delta(am) = eh(am)e' = eh([ae, m])e' = eaeh(m)e' = ea\delta(m).$$

But $e'a\delta(m) = e'aeh(m)e' = e'[e'ae, h(m)]e' = e'h([e'ae, m])e' = 0$, as desired. Similarly, we can show that $\delta(ma) = \delta(m)a$.

It remains to prove that β is a commuting map. By Lemma 3.2.8, we have $[a, \beta(m)] = a\beta(m) - \beta(m)a = \beta(ma) - \beta(am)$. Since A is 2-torsion free and Lemma 3.2.8, we get $[a, \beta(m)] = 0$, as desired. ∎

Lemmas 3.2.7 and 3.2.9 show that, to get the desired result, one should focus on the Lie generalized d-derivation on $A \ltimes M$ of the form $f = (f_A, h_2)$ (i.e., $h_1 = 0$ and $f_M = 0$ in (3.3)). In the sequel, we will refer to such a particular kind of Lie generalized d-derivations as a Lie generalized d-derivation of type Δ. Recall that in this case, f_A is a Lie generalized d_A-derivation and h_2 satisfies $h_2([a, m]) = [f_A(a), m] + [a, S(m)]$ and $h_2([m, a]) = [h_2(m), a] + [m, d_A(a)]$ for all $a \in A$ and $m \in M$ (Lemma 3.2.1).

Lemma 3.2.10 *Assume that there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$ (where $e' = 1 - e$). Let f be a Lie generalized d-derivation of type Δ. If f_A is a generalized d_A-derivation, then the following assertions hold:*

1. *$h_2(am) = f_A(a)m + aS(m)$ and $h_2(ma) = mf_A(a) + S(m)a$ for all $a \in A$ and $m \in M$.*
2. *$mf_A(e') = f_A(e)m$ for all $m \in M$.*
3. *$h_2(ma) = h_2(m)a + md_A(a)$ for all $a \in A$.*

Consequently, the Lie generalized d-derivation f is a generalized d-derivation.

Proof. (1) First we prove that $d_A(e)m = 0 = md_A(e)$. We have $f_A(e) = f_A(e)e + ed_A(e)$. Then, $ed_A(e)e = 0$, hence $d_A(e)m = ed_A(e)em = 0$. On the other hand, $md_A(e) = m(d_A(e)e + ed_A(e)) = 0$ Now, since f is a Lie generalized d-derivation, we have

$$\begin{aligned}
h_2(am) &= h_2([ae, m]) \\
&= [f_A(ae), m] + [ae, S(m)] \\
&= [f_A(a)e + ad_A(e), m] + aS(m) \\
&= f_A(a)m + ad_A(e)m - mad_A(e) + aS(m) \\
&= f_A(a)m + ad_A(e)m - d_A(e)ma + aS(m) \\
&= f_A(a)m + aS(m).
\end{aligned}$$

Similarly we get $h_2(ma) = mf_A(a) + S(m)a$ for all $a \in A$ and $m \in M$.

(2) For every $m \in M$, $mf_A(e') + S(m)e' = h_2(me') = h_2(em) = f_A(e)m + eS(m)$. Then, $mf_A(e') = f_A(e)m$.

(3) Since f is a Lie generalized d-derivation, we have for all $a \in A$ and $m \in M$,

$$h_2(ma) - h_2(m)a - md_A(a) = -ah_2(m) - d_A(a)m + h_2(am).$$

Then, using the hypothesis and (1), we get

$$\begin{aligned} h_2(ma) - h_2(m)a - md_A(a) &= -a(f_A(e)m + eS(m)) - d_A(a)m + f_A(a)m + aS(m) \\ &= -af_A(e)m - d_A(a)m + (f_A(e'a) + f_A(ea))m \\ &= -af_A(e)m - d_A(a)m + (f_A(e')e'a + e'd_A(e'a) \\ &\quad +f_A(e)a + ed_A(a))m. \end{aligned}$$

Then, using the second assertion, we get

$$\begin{aligned} h_2(ma) - h_2(m)a - md_A(a) &= -af_A(e)m + amf_A(e') \\ &= -af_A(e)m + af_A(e)m = 0. \end{aligned}$$

As desired. ■

Lemma 3.2.11 *Suppose there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, and let $g : A \longrightarrow A$ be a generalized d_g-derivation. Then, $[g(eae') + g(e'ae), m] = 0$ for all $a \in A$ and $m \in M$.*

Proof. Since g is a generalized derivation we have

$$\begin{aligned} [g(eae') + g(e'ae), m] &= [g(e)eae' + ed_g(eae') + g(e')e'ae + e'd_g(e'ae), m] \\ &= ad_g(e')m - mad_g(e) \\ &= 0. \end{aligned}$$

We are done. ■

Lemma 3.2.12 *Suppose there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, and let $f_A : A \longrightarrow A$ be a Lie generalized d_A-derivation. Then, $[f_A(eae'), m] = 0 = [f_A(e'ae), m] = 0$ for all $a \in A$ and $m \in M$.*

Proof. Since f_A is a Lie generalized derivation, we have

$$f_A(eae') = f_A([e, eae']) = f_A(e)eae' - eae'f_A(e) + ed_A(eae') - d_A(eae')e.$$

So $[f_A(eae'), m] = 0$ for all $a \in A, m \in M$. Similarly we prove $[f_A(e'ae), m] = 0$ for all $a \in A$ and $m \in M$. ■

Lemma 3.2.13 *Assume there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, then the following assertions hold:*

1. *The center $Z(A \ltimes M)$ is described as follows:*

$$Z(A \ltimes M) = \{(a,0), a \in A, eae \in Z(eAe), e'ae' \in Z(e'Ae'), eaem = me'ae',$$
$$neae = e'ae'n, [a,x] = 0 \text{ for all } m \in eAe', n \in e'Ae, x \in M\}.$$

2. *$[Z(A), m] = 0$, for all $m \in M$, if one of the following holds:*
   - *(i) $Z(eAe) = \pi_{eAe}(Z(A \ltimes M))$ and eAe' is faithful as a right $e'Ae'$-module.*
   - *(ii) $Z(e'Ae') = \pi_{e'Ae'}(Z(A \ltimes M))$ and eAe' is faithful as a right eAe-module.*

Proof. (1) The center $Z(A \ltimes M)$ of $A \ltimes M$ is

$$\begin{aligned} Z(A \ltimes M) &= \{(a,m); a \in Z(A), [b,m] = 0 = [a,y] \text{ for all } b \in A, y \in M\} \\ &= \pi_A(Z(A \ltimes M)) \times \pi_M(Z(A \ltimes M)), \end{aligned}$$

where $\pi_A : A \ltimes M \longrightarrow A$ and $\pi_M : A \ltimes M \longrightarrow M$ are the natural projections given by $\pi_A(a,m) = a$ and $\pi_M(a,m) = m$, respectively. On the other hand, by assumption, $eme' = m$ for all $m \in M$. Then for any $m \in M$, $[e,m] = 0$ implies $m = 0$. This leads to $\pi_M(a,m) = \{0\}$, and so

$$Z(A \ltimes M) = \{(a,0); a \in Z(A), [a,m] = 0 \text{ for all } a \in A, m \in M\} \quad (3.4)$$
$$= \pi_A(Z(A \ltimes M)) \times \{0\}. \quad (3.5)$$

As the algebra A enjoys the pierce decomposition $A = eAe + eAe' + e'Ae + e'Ae'$, we get

$$Z(A) = \{a \in A, eae \in Z(eAe), e'ae' \in Z(e'Ae'), eaem = me'ae', neae = e'ae'n$$
$$\text{for all } m \in eAe', n \in e'Ae\}.$$

Using the last equality we arrive at

$$Z(A \ltimes M) = \{(a,0), a \in A, eae \in Z(eAe), e'ae' \in Z(e'Ae'), eaem = me'ae',$$
$$neae = e'ae'n, [a,x] = 0 \text{ for all } m \in eAe', n \in e'Ae, x \in M\}.$$

(2) We only prove (i). The assertion (ii) is proved similarly. Let $a \in Z(A)$. Since $eae \in Z(eAe)$ and $Z(eAe) = \pi_{eAe}(Z(A \ltimes M))$, there exists an element $(a',0) \in Z(A \ltimes M)$ such that $eae = \pi_A(a',0) = ea'e$. It follows that $me'ae' = eaem = ea'em = me'a'e'$ for each $m \in eAe'$. Since eAe' is a faithful right $e'Ae'$-module, we get $e'ae' = e'a'e'$, and so $a = eae + e'ae' = ea'e + e'a'e' = a'$. In particular, $(a,0) \in Z(A \ltimes M)$ and so $[a,x] = 0$ for all $x \in M$, as claimed. ∎

Theorem 3.2.14 *Assume that A and the A-bimodule M are 2-torsion free and there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, then a Lie generalized derivation f on $A \ltimes M$ of the form*

$$f(a,m) = (f_A(a) + h_1(m), f_M(a) + h_2(m)) \qquad ((a,m) \in A \ltimes M)$$

is proper if and only if there exists a linear map $l_A : A \longrightarrow Z(A)$ satisfying the following conditions:

1. *$f_A - l_A$ is a generalized derivation on A.*
2. *$[l_A(eae), m] = 0 = [l_A(e'ae'), m] = 0$ for all $a \in A$ and $m \in M$.*

Proof. $\Rightarrow$. Suppose that f is proper; that is, $f = D + l$, where D is a generalized derivation and l is a center valued linear map on $A \ltimes M$. Then, from Lemma 3.2.13, we get $l(A \ltimes M) \subseteq \pi_A(Z(A \ltimes M)) \times \{0\}$ and this implies that l has the form $l(a,m) = (l_A(a), 0)$ $((a,m) \in A \ltimes M)$ for some linear map $l_A : A \longrightarrow Z(A)$ with $[l_A(a), m] = 0$ for all $a \in A, m \in M$. On the other hand, $f - l = D$ is a generalized derivation on $A \ltimes M$ and so, by Lemma 3.2.2, $f_A - l_A$ is a generalized derivation.
$\Leftarrow$. Suppose there exists a linear map $l_A : A \longrightarrow Z(A)$ such that $f_A - l_A$ is a generalized derivation. First, we show that $[l_A(a), m] = 0$ for all $a \in A$ and $m \in M$. For this, we have by the condition (2), $[l_A(a), m] = [l_A(eae'), m] + [l_A(e'ae), m]$. Since $f_A - l_A$ is a generalized derivation and by Lemma 3.2.11, $[l_A(a), m] = [f_A(eae'), m] + [f_A(e'ae), m]$. So, by Lemma 3.2.12, $[l_A(a), m] = 0$ for all $a \in A$ and $m \in M$. On the other hand, using Lemma 3.2.10, $\Delta : (a,m) \longmapsto (f_A - l_A(a), h_2(m))$ is a generalized derivation. By Lemma 3.2.9, $(h_1(m), 0) = (\delta(m), 0) + (\beta(m), 0)$ and by Lemma 3.2.7, $(0, f_M)$ is a generalized derivation. By Lemma 3.2.3, the map $(a,m) \longmapsto (l_A(a), 0)$ is a center valued map vanishing on commutators. Therefore, $f(a,m) = \Delta(a,m) + (\delta(m), f_M(m)) + (\beta(m), 0) + (l_A(a), 0)$. ■

It is worth noting that a triangular algebra $Tri(A, M, B)$, seen as a trivial extension $(A \times B) \ltimes M$, satisfies the conditions of Lemma 1.1.5; that is, there exists a nontrivial idempotent e (take $e = (1_A, 0)$) in $A \times B$ such that $eme' = m$ for all $m \in M$.
Furthermore, every Lie generalized derivation f on $Tri(A, M, B)$, identified with $(A \times B) \ltimes M$, is of the form

$$f((a,b), m) = (f_{A\times B}((a,b)), f_M((a,b)) + h_2(m)) \qquad (((a,b), m) \in (A \times B) \ltimes M).$$

That is, $h_1 = 0$. Indeed, by Lemma 3.2.1, $h_1([(a,b), m]) = [(a,b), h_1(m)]$ for all $(a,b) \in A \times B$ and $m \in M$. So, for $a = 1$ and $b = 0$, we get $h_1(m) = 0$ for all $m \in M$.

Also, an observation on the proof of Lemmas 3.2.7 and 3.2.9 shows that, in the case of triangular algebra (where $h_1 = 0$ and $e'(A \times B)e = 0$), we do not need the 2-torsion freeness of $A \times B$ and M as supposed in Lemmas 3.2.7, 3.2.9 and Theorem 3.2.14.

Therefore, as a consequence of Theorem 3.2.14, we obtain the following result.

Corollary 3.2.15 *Let A and B be unital algebras and let M be an (A, B)-bimodule. Then a Lie generalized derivation f on the triangular algebra $Tri(A, M, B)$ of the form*

$$f((a, b), m) = (f_{A\times B}((a, b)), f_M((a, b))+h_2(m)) \qquad (((a, b), m) \in (A\times B)\ltimes M)$$

is proper if and only if there exists a linear map $l_{A\times B} : A \times B \longrightarrow Z(A \times B)$ satisfying the following conditions:

1. *$f_{A\times B} - l_{A\times B}$ is a generalized derivation on $A \times B$.*
2. *$[l_{A\times B}((a, b)), m] = 0$ for all $a \in A$, $b \in B$ and $m \in M$.*

Applying Lemma 3.2.13and Theorem 3.2.14, we come to the second main result providing some sufficient conditions ensuring the Lie generalized derivation property for $A\ltimes M$. Notice that we use the projection maps $\pi_{eAe} : A\ltimes M \longrightarrow A$ and $\pi_{e'Ae'} : A \ltimes M \longrightarrow A$ defined by $\pi_{eAe}(a, m) = eae$ and $\pi_{e'Ae'}(a, m) = e'ae'$, respectively.

In the sequel of this paper, we use W_A, for an algebra A, to denote the smallest subalgebra of A contains all commutators and idempotents.

Theorem 3.2.16 *Assume that A and the A-bimodule M are 2-torsion free and there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, then $A \ltimes M$ has Lie generalized derivation property if the following two conditions are satisfied:*

1. *A has Lie generalized derivation property.*
2. *One of the following three conditions holds:*
   - *(i) $W_{eAe} = eAe$ and $W_{e'Ae'} = e'Ae'$.*
   - *(ii) $Z(eAe) = \pi_{eAe}(Z(A \ltimes M))$ and eAe' is faithful as a right $e'Ae'$-module.*
   - *(iii) $Z(e'Ae') = \pi_{e'Ae'}(Z(A \ltimes M))$ and eAe' is faithful as a right eAe-module.*

Proof. Let f be a Lie generalized derivation on $A \ltimes M$ with a form as given in Theorem 3.2.14. Since f_A is a Lie generalized derivation and A has the Lie derivation property, there exists a linear map $l_A : A \longrightarrow Z(A)$ such that $f_A - l_A$ is a generalized derivation on A (and so l_A vanishes on commutators of A). It is enough to show that, under one of conditions of (2), l_A satisfies the condition (2) of Theorem 3.2.14; that is, $[l_A(eae), m] = 0 = [l_A(e'ae'), m]$ for all $a \in A$ and $m \in M$. For this we consider the subset $A' = \{eae : [l_A(eae), m] = 0, \text{ for all } m \in M\}$ of eAe. First, we prove that A' is a subalgebra of eAe. That A' is an R-submodule of A follows from the linearity of l_A. The following identity confirms that A' is closed under multiplication.

$$[l_A(eaebe), m] = [l_A(eae), bm] + [l_A(ebe), am] \quad (a, b \in A, m \in M). \tag{3.6}$$

To prove (3.6), note that we have

$$\begin{aligned} h_2(am) &= h_2([eae, m]) \\ &= [f_A(eae), m] + [eae, S(m)] \\ &= [(f_A - l_A)(eae), m] + [l_A(eae), m] + [eae, S(m)] \\ &= (f_A - l_A)(a)m + [l_A(eae), m] + aS(m). \end{aligned}$$

So, we get

$$h_2(am) = (f_A - l_A)(a)m + [l_A(eae), m] + aS(m). \tag{3.7}$$

Applying (3.7) for ab we get,

$$h_2(abm) = (f_A - l_A)(ab)m + [l_A(eabe), m] + abS(m). \tag{3.8}$$

On the other hand, since $a[l_A(ebe), m] = [l_A(ebe), am]$,

$$\begin{aligned} h_2(abm) &= ah_2(bm) + (d_A - l_A)(a)bm + [l_A(eae), bm] \\ &= a(f_A - l_A)(b)m + [l_A(ebe), am] + abS(m) + \\ &\quad\ [l_A(eae), bm] + (d_A - l_A)(a)bm. \end{aligned}$$

Using the fact that $f_A - l_A$ is a generalized derivation, a comparison of the last equation and (3.8) leads to

$$[l_A(eabe), m] = [l_A(eae), bm] + [l_A(ebe), am] \quad (a, b \in A, m \in M).$$

And this implies the desired identity (3.6).
Now, we claim that A' contains all idempotents of eAe. First note that, if one puts $a = b$ in (3.6), then

$$[l_A((eae)^2), m] = [l_A(eae), 2am] \quad (a \in A, m \in M). \tag{3.9}$$

This follows that

$$[l_A((eae)^3), m] = [l_A((eae)^2(eae)), am] = [l_A(eae), 3a^2m] \quad (a \in A, m \in M). \tag{3.10}$$

Suppose that $eae \in eAe$ is an idempotent; that is, $(eae)^2 = eae$. By (3.9) and (3.10), we arrive at

$$[l_A(eae), m] = [l_A(3(eae)^2 - 2(eae)^3), m] = 0 \quad (a \in A, m \in M).$$

and this says that the idempotent eae lies in A'.
Furthermore, that A' contains all commutators follows trivially from the fact that l_A vanishes on commutators. Now the assumption $W_{eAe} = eAe$ in (i) gives $A' = eAe$; that is, $[l_A(eae), m] = 0$ for every $a \in A, m \in M$. A similar argument shows that, if $W_{e'Ae'} = e'Ae'$, then $[l_A(e'ae'), m] = 0$ for every $a \in A, m \in M$.
By (2) and Lemma 3.2.13, $[Z(A), m] = 0$ for all $m \in M$. This completes the proof. ∎

If one would ask for examples of algebras A satisfying $W_A = A$, we can cite the full matrix algebra $A = M_n(A')$ $n \geq 2$, where A' is a unital algebra, and also a simple unital algebra A with a nontrivial idempotent (see [15, Page 152] and [35] in which treatment of W_A was first initiated and see also the discussion before [86, Corollary 2.4]).

As an immediate consequence of Theorem 3.2.16, we give the following result. It is a generalization of Cheung's result (see [35, Theorem 11]).

Corollary 3.2.17 *Let A and B be unital algebras and let M be an (A, B)-bimodule. Then the triangular algebra $Tri(A, M, B)$ has the Lie generalized derivation property if the following two conditions are satisfied:*

1. *The algebras A and B have Lie generalized property.*
2. *$W_A = A$ and $W_B = B$.*

Chapter 4

On generalized (m,n)-Jordan derivations and centralizers of semiprime rings

Abstract. In this chapter we give an affirmative answer to two conjectures raised in [4] and [50] on generalized (m, n)-Jordan derivations and generalized (m, n)-Jordan centralizers. Precisely, when R is a semiprime ring, we prove, under some suitable torsion restrictions, that every nonzero generalized (m, n)-Jordan derivation (resp., generalized (m, n)-Jordan centralizer) is a derivation (resp., a two-sided centralizer).

Throughout this chapter, R will represent an associative ring with center $Z(R)$.

4.1 Introduction and main theorems

The study of relations between various sorts of derivations goes back to Herstein's classical result [57] which shows that any Jordan derivation on a 2-torsion free prime ring is a derivation (see also [31] for a brief proof of Herstein's result). In [40], Cusack generalized Herstein's result to 2-torsion free semiprime rings (see also [29] for an alternative proof). Further, Awtar [11] extended the Herstein's theorem to Lie ideals as follows: if U is a Lie ideal of a prime ring R of characteristic different of 2 such that $u^2 \in U$, for every $u \in U$, and $d : R \longrightarrow R$ is an additive mapping such that $d|_U$ is a Jordan derivation of U into R, then $d|_U$ is a derivation of U into R (see also Ashraf and Rehman [7], Brešar and Vukman [32] and Deng [42]).

In the last few years several authors have introduced and studied various sorts of parameterized derivations and gave generalizations of some classical results like Herstein's result. In [4], Ali and Fošner defined the notion of (m, n)-derivations as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $d : R \longrightarrow R$ is called an (m, n)-derivation if

$$(m+n)d(xy) = 2md(x)y + 2nxd(y)$$

holds for all $x, y \in R$.

Obviously, a $(1, 1)$-derivation on a 2-torsion free ring is a derivation.

In the same paper [4], a generalized (m, n)-derivation was defined as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $D : R \longrightarrow R$ is called a generalized (m, n)-derivation if there exists an (m, n)-derivation $d : R \longrightarrow R$ such that

$$(m+n)D(xy) = 2mD(x)y + 2nxd(y)$$

holds for all $x, y \in R$.

Obviously, every generalized $(1, 1)$-derivation on a 2-torsion free ring is a generalized derivation.

In [103], Vukman defined an (m, n)-Jordan derivation as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $d : R \longrightarrow R$ is called an (m, n)-Jordan derivation if

$$(m+n)d(x^2) = 2md(x)x + 2nxd(x)$$

holds for all $x, y \in R$.

Clearly, every $(1, 1)$-Jordan derivation on a 2-torsion free ring is a Jordan derivation.

Recently, in [71], Kosi-Ulbl and Vukman proved the following result.

Theorem 4.1.1 ([71], Theorem 1.5) *Let $m, n \geq 1$ be distinct integers, R a $mn(m+n)|m-n|$-torsion free semiprime ring an $d : R \longrightarrow R$ an (m, n)-Jordan derivation. Then d is a derivation which maps R into $Z(R)$.*

The (m, n)-generalized counterpart of the notion of an (m, n)-Jordan derivation is introduced by Ali and Fošner in [4] as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $F : R \longrightarrow R$ is called a generalized (m, n)-Jordan derivation if there exists an (m, n)-Jordan derivation $d : R \longrightarrow R$ such that

$$(m+n)F(x^2) = 2mF(x)x + 2nxd(x)$$

holds for all $x, y \in R$.

Based on some observations and inspired by the classical results, Ali and Fošner in [4] made the following conjecture.

Conjecture 4.1.2 ([4], Conjecture 1) *Let $m, n \geq 1$ be two fixed integers, let R be a semiprime ring with suitable torsion restrictions, and let $F : R \longrightarrow R$ be a nonzero generalized (m, n)-Jordan derivation. Then F is a derivation which maps R into $Z(R)$.*

The first aim of this chapter is to give an affirmative answer to this conjecture. Namely, our first main result is the following theorem.

Theorem 4.1.3 *Let $m, n \geq 1$ be distinct integers, let R be a k-torsion free semiprime ring, where $k = 6mn(m + n)|m - n|$, and let $F : R \longrightarrow R$ be a nonzero generalized (m, n)-Jordan derivation. Then F is a derivation which maps R into $Z(R)$.*

On the other hand and in parallel, there are similar works which study relations between various sorts of Jordan centralizers and centralizers. Recall that an additive mapping $T : R \longrightarrow R$ is called a left (resp., right) centralizer if $T(xy) = T(x)y$ (resp., $T(xy) = xT(y)$) is fulfilled for all $x, y \in R$, and it is called a left (resp., right) Jordan centralizer if $T(x^2) = T(x)x$ (resp., $T(x^2) = xT(x)$) is fulfilled for all $x \in R$. We call an additive mapping $T : R \longrightarrow R$ a two-sided centralizer (resp., a two-sided Jordan centralizer) if T is both a left as well as a right centralizer (resp., a left and a right Jordan centralizer). Namely, in [107], Zalar proved that any left (resp., right) Jordan centralizer on a 2-torsion free semiprime ring is a left (resp., right) centralizer. In [101], Vukman proved that, for a 2-torsion free semiprime ring R, every additive mapping $T : R \longrightarrow R$ satisfying the relation "$2T(x^2) = T(x)x + xT(x)$ for all $x \in R$" is a two-sided centralizer. Motivated by these results and inspired by his work [101], Vukman in [104] introduced the notion of an (m, n)-Jordan centralizer as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $T : R \longrightarrow R$ is called an (m, n)-Jordan centralizer if

$$(m + n)T(x^2) = mT(x)x + nxT(x)$$

holds for all $x, y \in R$.

Obviously, a $(1, 0)$-Jordan centralizer (resp., $(0, 1)$-Jordan centralizer) is a left (resp., a right) Jordan centralizer. When $n = m = 1$, we recover the maps studied in [101].

Based on some observations and results, Vukman conjectured that, on semiprime rings with suitable torsion restrictions, every (m, n)-Jordan centralizer is a two-sided centralizer (see [104]). Recently, this conjecture was solved affirmatively by Kosi-Ulbl and Vukman in [72]. Namely, they proved the following result.

Theorem 4.1.4 ([72], Theorem 1.5) *Let $m, n \geq 1$ be distinct integers, let R be an $mn(m+n)$-torsion free semiprime ring, and let $T : R \longrightarrow R$ be an (m, n)-Jordan centralizer. Then T is a two-sided centralizer.*

Inspired by the work of Vukman [101, 104], Fošner [50] introduced more generalized version of (m, n)-Jordan centralizers as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $T : R \longrightarrow R$ is called a generalized (m, n)-Jordan centralizer if there exists an (m, n)-Jordan centralizer $T_0 : R \longrightarrow R$ such that

$$(m+n)T(x^2) = mT(x)x + nxT_0(x)$$

holds for all $x \in R$.

Thus, a generalized $(1, 0)$-Jordan centralizer is a left Jordan centralizer.

In [50], Fošner showed that, on a prime ring with a specific torsion condition, every generalized (m, n)-Jordan centralizer is a two-sided centralizer. This led Fošner to make the following conjecture.

Conjecture 4.1.5 ([50], Conjecture 1) *Let $m, n \geq 1$ be two fixed integers, let R be a semiprime ring with suitable torsion restrictions, and let $T : R \longrightarrow R$ be a generalized (m, n)-Jordan centralizer. Then T is a two-sided centralizer.*

The second aim of this chapter is to give an affirmative answer to Fošner's conjecture. Namely, our second main result is the following theorem.

Theorem 4.1.6 *Let $m, n \geq 1$ be two fixed integers, let R be an $6mn(m+n)(2n+m)$-torsion free semiprime ring, and let $T : R \longrightarrow R$ be a nonzero generalized (m, n)-Jordan centralizer. Then T is a two-sided centralizer.*

4.2 Proof of the main theorems

In the proof of our main results, Theorems 4.1.3 and 4.1.6, we shall use the following results.

Lemma 4.2.1 ([4], Lemma 1) *Let $m, n \geq 0$ be distinct integers with $m+n \neq 0$, let R be a 2-torsion free ring, and let $F : R \longrightarrow R$ be a nonzero generalized (m, n)-Jordan derivation. Then, $(m+n)^2F(xyx) = m(n-m)F(x)xy + m(3m+n)F(x)yx + m(m-n)F(y)x^2 + 4mnxd(y)x + n(n-m)x^2d(y) + n(m+3n)xyd(x) + n(m-n)yxd(x)$ for all $x, y \in R$.*

Lemma 4.2.2 ([17], Theorem 3.3) *Let $n \geq 2$ be a fixed integer and let R be a prime ring with $char(R) = 0$ or $char(R) \geq n$. If $T : R \longrightarrow R$ is an* *additive mapping satisfying the relation* $T(x^n) = T(x)x^{n-1}$ *for all* $x \in R$, *then* $T(xy) = T(x)y$ *for all* $x, y \in R$.

Lemma 4.2.3 ([50], Lemma 1) *Let* $m, n \geq 0$ *be distinct integers with* $m + n \neq 0$, *let* R *be a ring, and let* $T : R \longrightarrow R$ *be a nonzero generalized* (m, n)*-Jordan centralizer. Then,* $2(m+n)^2T(xyx) = mnT(x)xy + m(2m+n)T(x)yx - mnT(y)x^2 + 2mnxT_0(y)x - mnx^2T_0(y) + n(m+2n)xyT_0(x) + mnyxT_0(x)$ *for all* $x, y \in R$.

Lemma 4.2.4 ([102], Lemma 3) *Let* R *be a semiprime ring and let* $T : R \longrightarrow R$ *be an additive mapping. If either* $T(x)x = 0$ *or* $xT(x) = 0$ *holds for all* $x \in R$, *then* $T = 0$.

We shall use the relation between semiprime rings and prime ideals. Namely, it is well-known that a ring R is semiprime if and only if the intersection of all prime ideals of R is zero if and only if R has no nonzero nilpotent (left, right) ideals (see for instance Lam's book [73] or the recent book of Brešar [30]). Due to the classical Levitzki's paper [75], several authors prefer to refer to this result by Levitzki's lemma.
Let I be an ideal of R. For an element $x \in R$, we use $\overline{x}$ to denote the equivalence class of x modulo I.

Lemma 4.2.5 *Let* R *be both a* 2*-torsion free and a* 3*-torsion free semiprime ring and let* $T : R \longrightarrow R$ *be an additive map such that* $T(x)x^3 = 0$ *and* $T(x^4) = 0$ *for all* $x \in R$. *Then* $T(xy) = T(x)y$ *for all* $x, y \in R$.

Proof. Let $x, y \in R$. We prove that $T(xy) = T(x)y$. We may assume that x and y are not 0. Let P be a prime ideal of R and set $\overline{R} = R/P$. Consider an element $p \in P$. By hypothesis, $0 = T(x+p)(x+p)^3 = (T(x)+T(p))(x^3 + xpx + px^2 + p^2x + x^2p + xp^2 + pxp + p^3)$. Thus, $0 = T(x)(xpx + px^2 + p^2x + x^2p + xp^2 + pxp + p^3) + T(p)(x^3 + xpx + px^2 + p^2x + x^2p + xp^2 + pxp)$. Hence, $T(p)x^3 \in P$, equivalently $\overline{T(p)}\overline{x}^3 = 0$. By Levitzki's lemma, $\overline{T(p)}\overline{x} = 0$, and then $\overline{T(p)} = 0$ (since $\overline{R}$ is a prime ring). Thus, $T(P) \subseteq P$, which implies that $T(x+P) = T(x) + P$. Now, since $T(\overline{x})\overline{x}^3 = 0$ and $T(\overline{x}^4) = 0$, $T(\overline{x}^4) = T(\overline{x})\overline{x}^3$. This shows, using Lemma 4.2.2, that $T(\overline{xy}) = T(\overline{x})\overline{y}$. Therefore, $T(xy) - T(x)y \in P$. Finally, by the semiprimeness of R, we get the desired result. ■

Now we are ready to prove the first main result.

Proof of Theorem 4.1.3. Let d be the associated (m, n)-Jordan derivation of F. Since R is a semiprime ring, d is a derivation which maps R into $Z(R)$ (by Theorem 4.1.1). Let us denote $F - d$ by D. Then, we have $(m+n)D(x^2) =$

$(m+n)F(x^2)-(m+n)d(x^2) = 2mF(x)x+2nxd(x)-2md(x)x-2nxd(x) = 2mD(x)x$ for all $x \in R$. Thus

$$(m+n)D(x^2) = 2mD(x)x, \ x \in R. \tag{4.1}$$

Replacing x with x^2 in (4.1), we get

$$(m+n)D(x^4) = 2mD(x^2)x^2, \ x \in R. \tag{4.2}$$

Multiplying by $m+n$ and then using (4.1), we get

$$(m+n)^2D(x^4) = 4m^2D(x)x^3, \ x \in R. \tag{4.3}$$

On the other hand, putting x^2 for y in the relation of Lemma 4.2.1 and using the fact that D is a generalized (m,n)-Jordan derivation associated with the zero map as a (m,n)-Jordan derivation, we get

$$(m+n)^2D(x^4) = m(n-m)D(x)x^3+m(3m+n)D(x)x^3+m(m-n)D(x^2)x^2, \ x \in R. \tag{4.4}$$

Multiplying both sides in (4.4) by 2 we get

$$2(m+n)^2D(x^4) = 2m(n-m)D(x)x^3+2m(3m+n)D(x)x^3+2m(m-n)D(x^2)x^2, \ x \in R. \tag{4.5}$$

Combining (4.2) and (4.5), we get

$$2(m+n)^2D(x^4) = 2m(n-m)D(x)x^3+2m(3m+n)D(x)x^3+(m+n)(m-n)D(x^4), \ x \in R, \tag{4.6}$$

which gives

$$(m+n)(m+3n)D(x^4) = 4m(m+n)D(x)x^3, \ x \in R. \tag{4.7}$$

Multiplying both sides in (4.7) by $m+n$, we get

$$(m+n)^2(m+3n)D(x^4) = 4m(m+n)^2D(x)x^3, \ x \in R. \tag{4.8}$$

Multiplying by $m+3n$ in (4.3), we get

$$(m+n)^2(m+3n)D(x^4) = 4m^2(m+3n)D(x)x^3, \ x \in R. \tag{4.9}$$

By comparing (4.8) and (4.9), we get

$$4mn(m-n)D(x)x^3 = 0, \ x \in R. \tag{4.10}$$

Since R is a $2mn|n-m|$-torsion free ring, $D(x)x^3 = 0$ for all $x \in R$. Applying $D(x)x^3 = 0$ in equation (4.3), we get $(m+n)^2D(x^4) = 0$ for all $x \in R$. By using the torsion free restriction, we have $D(x^4) = 0$ for all $x \in R$. Hence, $D(xy) = D(x)y$ for all $x, y \in R$ (by Lemma 4.2.5). Applying this in (4.1), yields $(m+n)D(x)x = 2mD(x)x$ for all $x \in R$, equivalently $(m-n)D(x)x = 0$. Since R is an $|m-n|$-torsion free ring, $D(x)x = 0$ for all $x \in R$. Therefore, by Lemma 4.2.4, $D = 0$. This completes the proof. ■

The second main result is proved similarly. Nevertheless, we include a proof for completeness.

Proof of Theorem 4.1.6. Let T_0 be the associated (m,n)-Jordan centralizer of T. Since R is a semiprime ring, T_0 is a two-sided centralizer (by Theorem 4.1.4). Let us denote $T - T_0$ by D. Then, we have $(m+n)D(x^2) = (m+n)T(x^2) - (m+n)T_0(x^2) = mT(x)x + nxT_0(x) - mT_0(x)x - nxT_0(x) = mD(x)x$ for all $x \in R$. Thus

$$(m+n)D(x^2) = mD(x)x, \ x \in R. \tag{4.11}$$

Replacing x with x^2 in (4.11), we get

$$(m+n)D(x^4) = mD(x^2)x^2, \ x \in R. \tag{4.12}$$

Multiplying by $m+n$ and then using (4.11), we get

$$(m+n)^2D(x^4) = m^2D(x)x^3, \ x \in R. \tag{4.13}$$

On the other hand, if we put $y = x^2$ in the relation of Lemma 4.2.3, we get

$$2(m+n)^2D(x^4) = mnD(x)x^3 + m(2m+n)D(x)x^3 - mnD(x^2)x^2, \ x \in R. \tag{4.14}$$

Multiplying both sides in (4.13) by 2 we get

$$2(m+n)^2D(x^4) = 2m^2D(x)x^3, \ x \in R. \tag{4.15}$$

Combining (4.12) and (4.14), we get

$$2(m+n)^2D(x^4) = mnD(x)x^3 + m(2m+n)D(x)x^3 - n(m+n)D(x^4), \ x \in R, \tag{4.16}$$

which implies

$$(m+n)(2m+3n)D(x^4) = 2m(m+n)D(x)x^3, \ x \in R. \tag{4.17}$$

Multiplying both sides of above relation by $m+n$, we have

$$(m+n)^2(2m+3n)D(x^4) = 2m(m+n)^2D(x)x^3, \ x \in R. \tag{4.18}$$

Multiplying by $(2m+3n)$ in (4.13), we get

$$(m+n)^2(2m+3n)D(x^4) = m^2(2m+3n)D(x)x^3, \ x \in R. \tag{4.19}$$

By comparing (4.18) and (4.19), we get

$$mn(2n+m)D(x)x^3 = 0, \ x \in R. \tag{4.20}$$

Since R is a $mn(2n+m)$-torsion free ring, $D(x)x^3 = 0$ for all $x \in R$. Applying $D(x)x^3 = 0$ in equation (4.13) and then using $(m+n)$-torsion freeness of R, we get $D(x^4) = 0$ for all $x \in R$. Moreover, since R is a 2 and a 3-torsion free ring, by Lemma 4.2.5, we get $D(xy) = D(x)y$ for all $x, y \in R$. Applying this in (4.11), yields $(m+n)D(x)x = mD(x)x$ for all $x \in R$. So $nD(x)x = 0$, which implies that $D(x)x = 0$ for all $x \in R$. Therefore, by Lemma 4.2.4, $D = 0$. This completes the proof. ■

Chapter 5

Dhara-Rehman-Raza's identities on left ideals of prime rings

Abstract. It is known that every nonzero Jordan ideal of a 2-torsion free semiprime ring contains a nonzero ideal. In this chapter we show that also any square closed Lie ideal of a 2-torsion free prime ring contains a nonzero ideal. This can be interpreted by saying that studying identities over one sided ideals is the "optimal" case to study identities. With this fact in mind, we generalize some results of Dhara, Rehman and Raza in [Lie ideals and action of generalized derivations in rings, Miskolc Mathematical, **16** (2015), 769 – 779] to the context of nonzero left ideals.

Throughout this chapter, R will represent an associative ring with center $Z(R)$.

5.1 Preliminaries

In this section we recall and present some properties of left ideals which will be used to prove our main results.

Lemma 5.1.1 *Let R be a prime ring and let I be a nonzero left ideal of R. Then for every $a, b \in R$, $aIb = (0)$, implies that $a = 0$ or $Ib = (0)$.*

Proof. We have $aIb = (0)$ implies $aRIb = (0)$, so by primeness of R we get $a = 0$ or $Ib = (0)$. ∎

Lemma 5.1.2 *Let R be a prime ring and I a nonzero left ideal of R. If F is a generalized derivation of R with associated derivation d such that $F(I) = 0$, then $Id(I) = 0$.*

Proof. We have, for every $u, v \in I$, $F(uv) = F(u)v + ud(v) = ud(v) = 0$. Hence $Id(I) = 0$. ∎

Lemma 5.1.3 *Let R be a prime ring and I be a nonzero left ideal of R. Then, the following assertions are equivalents:*

1. $[I, I] = 0$.
2. $I \subseteq Z(R)$.
3. *R is commutative.*

Proof. (1) $\Rightarrow$ (2) let I be a left ideal such that $[I, I] = 0$. Then, $0 = [I, RI] = [I, R]I$. Which implies that $[I, R] = 0$.
(2) $\Rightarrow$ (3) We have $[I, R] = 0$. This gives $0 = [RI, I] = [R, R]I$. Since a left annihilator of a left ideal is zero, $[R, R] = 0$, that is, R is commutative.
(3) $\Rightarrow$ (1) Obvious. ■

Lemma 5.1.4 *Let R be a noncommutative prime ring and I be a nonzero left ideal of R. Let d be a derivation of R and $z \in Z(R)$, such that $[xy, r]d(z) = 0$ for all $x, y \in I$ and $r \in R$. Then $d(z) = 0$.*

Proof. First we show that $d(z) \in Z(R)$, for this we have $d(zr) = d(z)r + zd(r) = d(r)z + rd(z)$, so $d(z) \in Z(R)$. On other hand we have

$$[xy, r]d(z) = 0 \quad \text{for all } \ x, y \in I; \ \ r \in R. \tag{5.1}$$

Replacing x by $sxd(z)$ in the above equation, where $s \in R$, we get

$$[s, r]xd(z)yd(z) = 0 \quad \text{for all } \ x, y \in I; \ \ r \in R. \tag{5.2}$$

So $Id(z) = 0$ and $d(z) = 0$. ■

It is well-known that every nonzero Jordan ideal of a 2-torsion free semiprime ring contains a nonzero ideal (see, for instance, [58, Theorem 1.1]). Here we give a similar result for square closed Lie ideals.

Proposition 5.1.5 *Let L be a nonzero square closed Lie ideal of a 2-torsion free prime ring R. Then, L contains a non-zero ideal of R.*

Proof. Let L be a Lie ideal of R such that $u^2 \in L$ for all $u \in L$. Therefore, for any $u, v \in L$, we get $uv + vu = (u + v)^2 - u^2 - v^2 \in L$. On the other hand we have $uv - vu \in L$. Combining these two equalities we get $2uv \in L$ for all $u, v \in L$. Then, $2L$ is both a Lie ideal and a subring of R. Also $2L \neq 0$ since R is a 2-torsion free ring. Then, by [58, Lemma 1.3] and [46, Lemma 5], $2L$ contains a non-zero ideal of R and so does L since $2L \subseteq L$. ■

5.2 Main results

We start with the first main result. Using Proposition 5.1.5, it can be seen as a generalization of [46, Theorems 1 and 2] (for more details see Corollary 5.2.3).

Theorem 5.2.1 *Let R be a prime ring and I be a nonzero left ideal of R. Let F, G and H be generalized derivations associated to derivations f, g and h of R, respectively, such that $F(x)G(y) - H(xy) \in Z(R)$ for all $x, y \in I$. Then R is commutative or $If(I) = 0$ or $Ig(I) = 0$.*

Proof. We are given that

$$F(x)G(y) - H(xy) \in Z(R) \quad \text{for all} \ \ x, y \in I. \tag{5.3}$$

Replacing y by yz in (5.3) we get

$$F(x)G(y)z - H(xy)z + F(x)yg(z) - xyh(z) \in Z(R) \quad \text{for all} \ \ x, y, z \in I. \tag{5.4}$$

That is

$$[F(x)yg(z) - xyh(z), z] = 0 \quad \text{for all} \ \ x, y, z \in I. \tag{5.5}$$

Replacing x by xu, where $u \in I$, we get

$$[F(x)(uy)g(z) - x(uy)h(z), z] + [xf(u)yg(z), z] = 0 \quad \text{for all} \ \ u, x, y, z \in I. \tag{5.6}$$

Since $uy \in I$, then (5.5) together with (5.6) force that

$$[xf(u)yg(z), z] = 0 \quad \text{for all} \ \ u, x, y, z \in I. \tag{5.7}$$

Writing rx instead of x in (5.7), where $r \in R$, we obtain

$$[r, z]xf(u)yg(z) = 0 \quad \text{for all} \ \ u, x, y, z \in I; \ \ r, s \in R. \tag{5.8}$$

That is

$$[r, z]If(u)Ig(z) = 0 \quad \text{for all} \ \ u, z \in I; \ \ r, s \in R. \tag{5.9}$$

The primeness of R together with (5.9) imply that $[r, z] = 0$ or $If(u) = (0)$ or $Ig(z) = (0)$. We conclude that R is commutative or $If(I) = 0$ or $Ig(I) = 0$. ■

As a consequence we get the following result.

Corollary 5.2.2 *Let R be a prime ring and I be a nonzero left ideal of R. Let F, G and H be generalized derivations associated to derivations f, g and h of R respectively such that $F(x)G(y) + H(xy) \in Z(R)$ for all $x, y \in I$. Then R is commutative or $If(I) = 0$ or $Ig(I) = 0$.*

Proof. We notice that $-H$ is a generalized derivation of R associated to the derivation $-h$. Hence replacing H by $-H$ in Theorem 5.2.1, we get $F(x)G(y)-(-H)(xy) \in Z(R)$ for all $x, y \in I$, that is $F(x)G(y) + H(xy) \in Z(R)$ all $x, y \in I$. Therefore R is commutative or $If(I) = 0$ or $Ig(I) = 0$. ■

Now we can see, as mentioned before Theorem 5.2.1, that [46, Theorems 1 and 2] can be seen as a consequence of Theorem 5.2.1 as follows.

Corollary 5.2.3 *Let L be a nonzero square closed Lie ideal of 2-torsion free prime ring R. Let F, G, H be generalized derivations with associated derivations f, g, h of R, respectively, such that $f \neq 0$ and $g \neq 0$. If $F(x)G(y) \pm H(xy) \in Z(R)$ for all $x, y \in L$, then R commutative.*

Proof. Using Proposition 5.1.5, L contains a non-zero ideal I of R. So we get $F(x)G(y) \pm H(xy) \in Z(R)$ for all $x, y \in I$. Using Theorem 5.2.1 and Corollary 5.2.2 we get that R is commutative. ■

Also, if we suppose $F = f$ and $G = g$ in Theorem 5.2.1, then we get the following corollary.

Corollary 5.2.4 *Let R be a prime ring and I be a nonzero left ideal of R. Let F, G and H be generalized derivations associated to derivations f, g and h of R respectively such that $f(x)g(y) \pm H(xy) \in Z(R)$ for all $x, y \in I$. Then R is commutative or $If(I) = 0$ or $Ig(I) = 0$.*

When H in Theorem 5.2.1 is the identity map, we get the following result.

Corollary 5.2.5 *Let R be a prime ring and I be a nonzero left ideal of R. Let F, G and H be generalized derivations associated to derivations f, g and h of R respectively such that $F(x)G(y) \pm xy \in Z(R)$ for all $x, y \in I$. Then R is commutative or $If(I) = 0$ or $Ig(I) = 0$.*

Now, we give our second main result. Compare it with [46, Theorem 3].

Theorem 5.2.6 *Let R be a prime ring and I be a nonzero left ideal of R. Let F and H be generalized derivations associated to derivations f and h of R respectively such that $F(x)F(y) - H(yx) \in Z(R)$ for all $x, y \in I$. If $0 \neq If(I) \subseteq I$ and $0 \neq Ih(I) \subseteq I$, then R is commutative.*

Proof. Assume that $I \cap Z(R) = \{0\}$
By the hypothesis, we have

$$F(x)F(y) - H(yx) \in Z(R) \quad \text{for all } x, y \in I. \tag{5.10}$$

Replacing in (5.10) y by yz, where $y, z \in I$, we get, for all $x, y, z \in I$:

$$F(x)F(y)z + F(x)yf(z) - H(yz)x - yzh(x) \in Z(R) \tag{5.11}$$

That is

$$(F(x)F(y) - H(yx))z + H(yx)z + \tag{5.12}$$
$$F(x)yf(z) - H(yz)x - yzh(x) \in Z(R) \quad \text{for all } x, y, z \in I.$$

By the hypothesis we have

$$(F(x)F(y) - H(yx))z + H(yx)z + \tag{5.13}$$
$$F(x)yf(z) - H(yz)x - yzh(x) \in I \quad \text{for all } x, y, z \in I.$$

Since $I \cap Z(R) = 0$, we get

$$(F(x)F(y) - H(yx))z + H(yx)z + \tag{5.14}$$
$$F(x)yf(z) - H(yz)x - yzh(x) = 0 \quad \text{for all } x, y \in I.$$

Replacing z by zx, we obtain, for all $x, y, z \in I$:

$$F(x)yzf(x) - yzh(x)x - yz[x, h(x)] = 0. \tag{5.15}$$

Replacing y by py, we get

$$[F(x), p]yzf(x) = 0 \quad \text{for all } x, y, p, z \in I. \tag{5.16}$$

So, using primeness of R, we get, for all $x, y \in I$: $[F(x), p]y = 0$ or $If(x) = 0$. This leads to

$$F(x) \in Z(R) \quad \text{or} \quad If(x) = 0 \quad \text{for all } x \in I.$$

The fact that a group cannot be a union of its proper subgroups with the condition $If(I) \neq 0$ imply that

$$F(x) \in Z(R) \quad \text{for all } x \in I.$$

Then, F is centralizing on I. Therefore, R is commutative. Then $I = 0$, which contradicts our hypothesis. Therefore, $I \cap Z(R) \neq \{0\}$.
Let $z \in I \cap Z(R) \backslash \{0\}$ and replacing y by yz in our identities hypothesis, we get

$$F(x)yf(z) - yxh(z) \in Z(R) \quad \text{for all } x, y \in I. \tag{5.17}$$

So

$$[F(x)y, r]f(z) - [yx, r]h(z) = 0 \quad \text{for all } x, y \in I; \quad r \in R. \tag{5.18}$$

Replacing x by xz in (5.18), we get, for all $x, y \in I$; $r \in R$:

$$[F(x)(zy), r]f(z) + [xf(z)y, r]f(z) - [y(zx), r]h(z) = 0. \tag{5.19}$$

Using (5.18) we arrive at

$$[xf(z)y, r]f(z) = 0 \quad \text{for all } \ x, y \in I; \ \ r \in R. \tag{5.20}$$

Replacing x by sx in (5.20), where $s \in R$, we get

$$[s, r]xf(z)yf(z) = 0 \quad \text{for all } \ x, y \in I; \ \ s \in R. \tag{5.21}$$

The primeness of R together with equation (5.21) show that R is commutative or $If(z) = 0$. If $f(z) = 0$, (5.18) becomes

$$[yx, r]h(z) = 0 \quad \text{for all } \ x, y \in I; \ \ r \in R. \tag{5.22}$$

So by Lemma 5.1.4 we get $h(z) = 0$.

Replacing now y by tz in (5.10), where $t \in R$, we get

$$F(x)F(t) - H(tx) \in Z(R) \quad \text{for all } \ x, \in I; \ \ t \in R. \tag{5.23}$$

Replacing x by sz in (5.23), where $s \in R$, we get

$$F(s)F(t) - H(ts) \in Z(R) \quad \text{for all } \ s, t \in R. \tag{5.24}$$

Therefore, since R is a square closed Lie ideal of R itself and by [46, Theorem 3] and Lemma 5.1.3, we conclude that R is commutative. ∎

Remark 5.2.7 *It should be pointed out that here we have used [46, Theorem 3] at the end of the proof above. An alternative way is to use the same arguments in [46, Theorem 3]. Thus, using [58, Theorem 1.1] and Proposition 5.1.5, we conclude that studying identities over one sided ideals can be seen as the most "optimal" case to study identities.*

We end this chapter, with some consequences of Theorem 5.2.6 as done for the first main result.

Corollary 5.2.8 *Let R be a prime ring and I be a nonzero left ideal of R. Let F and H be generalized derivations associated to derivations f and h of R respectively such that $F(x)F(y) + H(yx) \in Z(R)$ for all $x, y \in I$. If $0 \neq If(I) \subseteq I$ and $0 \neq Ih(I) \subseteq I$, then R is commutative .*

Proof. We notice that $-H$ is a generalized derivation of R associated to the derivation $-h$. Hence replacing H by $-H$ in Theorem 5.2.6, we get $F(x)F(y) - (-H)(yx) \in Z(R)$ for all $x, y \in I$. That is $F(x)F(y) + H(yx) \in Z(R)$ for all $x, y \in I$. This implies that R is commutative. ∎

When we consider $F = f$ in Theorem 5.2.6, we get the following result.

Corollary 5.2.9 *Let R be a prime ring and I be a nonzero left ideal of R. Let F and H be generalized derivations associated to derivations f and h of R respectively such that $f(x)f(y) \pm H(yx) \in Z(R)$ for all $x, y \in I$. If $0 \neq If(I) \subseteq I$ and $0 \neq Ih(I) \subseteq I$, then R is commutative.*

Also, if we suppose H to be the identity map in Theorem 5.2.6, we get the following result.

Corollary 5.2.10 *Let R be a prime ring and I be a nonzero left ideal of R. Let F be a generalized derivation associated to derivations f of R such that $F(x)F(y) \pm yx \in Z(R)$ for all $x, y \in I$. If $0 \neq If(I) \subseteq I$ and $0 \neq Ih(I) \subseteq I$, then R is commutative.*

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