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Abstract

The aim of this report is to prove new results of the strong law of large numbers in both Bochner integration and Pettis integration cases, as well as to present some of its applications. First of all, we provide a new result of the strong law of large numbers for adapted sequences. In this context, we establish a relationship between the strong law of large numbers and amart and we prove the strong law of large numbers for mils and for a set valued supermartingale. We also prove some convergence results for the conditional expectation of Pettis integrable random sets. This result allows us to establish a new version of reversed martingale and dominated convergence theorem in Pettis integration. Finally, we provide a new result of strong law of large numbers for random sets Pettis integrables. In this context several auxiliary results are showed.

Keywords: Strong law of large numbers, adapted sequences, mils, set valued supermartingale, reversed martingale, amart, random sets, Pettis conditional expectation, Mosco convergence, dominated convergence theorem, Levy's theorem, Bochner integration, Pettis integration, Monte Carlo method.

Résumé

Le but de ce projet de thèse est, d'une part, de présenter et démontrer de nouveaux résultats de la loi forte des grands nombres que ce soit dans le cas de Bochner intégration ou dans le cas de Pettis intégration. D'autre part, de donner quelques applications concrètes de la loi forte des grands nombres. Dans ce manuscrit, nous avons contribué à la résolution de trois types de problèmes dans la théorie des probabilités et statistique en général et la loi forte des grands nombres en particulier à savoir; la relation entre la loi forte des grands nombres et les suites adaptées, ainsi que la convergence des espérances conditionnelles des ensembles aléatoires Pettis intégrables et finalement la loi forte des grands nombres dans le cadre de Pettis intégration. Dans ce contexte, nous avons démontré la loi forte des grands nombres (LFGN) pour les suites adaptées. Dans ce sens nous avons établi la relation entre la loi forte des grands nombres et amarts. Comme application, nous avons donné la loi forte des grands nombres pour les mils et les surmartingales multivoques. Nous avons également étudié la convergence des espérances conditionnelles des ensembles aléatoires Pettis intégrables. Cette étude a abouti à de nouvelles versions du théorème de type Levy pour des suites de tribus décroissantes et du théorème de la convergence dominée pour les espérances conditionnelles toujours dans le cadre de Pettis intégration. Par la suite nous avons utilisé les résultats de la convergence des espérances conditionnelles Pettis intégrables, ainsi que des techniques des martingales inverses pour démontrer la loi forte des grands nombres pour des ensembles aléatoires Pettis-intégrables.

Dans ce contexte plusieurs résultats auxiliaires sont abordés.

Mots-clés: loi forte des grands nombres, suites adaptées, mils, amarts, martingales, martingale inverse, ensemble aléatoire, espérance conditionnelle, Mosco convergence, théorème de convergence dominée, théorème de Levy, Pettis intégration, Bochner intégration, Méthode de Monte Carlo.

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Acronyms

- $SLLN$: Strong law of large numbers
- $cwk(\mathbb{E})$: the family of all nonempty convex and weakly compact subsets of $\mathbb{E}$.
- $ccb(\mathbb{E})$: the family of all nonempty closed convex and bounded subsets of $\mathbb{E}$.
- $L^1_{\mathbb{E}}(\mathcal{F})$ is the set of all $\mathcal{F}$ -measurable and integrable random variable from Ω to $\mathbb{E}$.
- $L^1_{ccb(\mathbb{E})}(\mathcal{F})$ is the set of all $\mathcal{F}$ -measurable and integrably bounded multifunctions from Ω to $ccb(\mathbb{E})$.
- $L^1_{cwk(\mathbb{E})}(\mathcal{F})$ is the family of all $\mathcal{F}$ -measurable and Bochner-integrable multifunctions from Ω to $cwk(\mathbb{E})$.
- $E^{\mathcal{B}}X$ is the conditional expectation of X relative to $\mathcal{B}$.
- $\overline{B_{\mathbb{E}}}$ is the closed unit ball of $\mathbb{E}$.
- $P^1_{\mathbb{E}}(\mathfrak{F})$ is the space of all $\mathfrak{F}$ -measurable and Pettis-integrable functions defined on $(\Omega, \mathfrak{F}, P)$ to $\mathbb{E}$.
- $P^1_{cwk(\mathbb{E})}(\mathfrak{F})$ is the set of all $cwk(\mathbb{E})$ -valued Pettis-integrable random set.
- $P^1_{ccb(\mathbb{E})}(\mathfrak{F})$ is the set of all $ccb(\mathbb{E})$ -valued Pettis-integrable random set.
- $S^1_X(\mathfrak{F})$ is the set of all $\mathfrak{F}$ -measurable and integrable selections of X .
- $S^{Pe}_X(\mathfrak{F})$ is the set of all $\mathfrak{F}$ -measurable and Pettis-integrable selections of X .

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Thesis summary in french

0.1 Introduction

Dans une expérience aléatoire, nous nous sommes souvent intéressés à la réalisation ou non d'un événement A . Notons p la probabilité de la réalisation de cet événement. Supposons que notre expérience est répétée d'une façon indépendante et une infinité de fois. Notons par 1 la réalisation de A et zéro si A est non réalisé. Ce que nous appelons l'expérience de Bernoulli. Par simulation nous remarquons que plus le nombre de répétitions augmente plus que la moyenne arithmétique des résultats obtenus se stabilise autour de p . Si nous modélisons mathématiquement cette expérience par la réalisation d'une suite de variables aléatoires indépendantes et ayant la même loi, alors $\frac{S_n}{n} = \frac{\sum_{i=1}^n X_i}{n}$ converge vers $EX_1 = p$. C'est ce qu'on appelle la loi forte des grands nombres (LFGN).

La loi forte des gands nombres a attiré l'intention de plusieurs scientifiques, elle a été démontrée en première fois par le mathématicien Suisse Jacobi Bernoulli en 1713. Vue ses applications dans plusieurs domaines; pour estimer par exemple l'intégrale d'une fonction (ce qu'on appelle la méthode de Monte Carlo), l'assurance et la mutualisation du risque, l'évaluation de la fiabilité d'un système par l'estimation des bornes de probabilité de défaillance, en statistique inférentielle, pour déterminer une loi de probabilité à partir d'une série d'expériences et en finance pour évaluer le prix des produits dérivés. Plusieurs mathématiciens se sont joints à Bernoulli pour étudier la loi des grands nombres.

En théorie des probabilités, la loi forte des grands nombres est un théorème fondamental qui assure la convergence presque sûre de la moyenne d'une suite de variables aléatoires sous

certaines hypothèses vers une valeur déterministe. Autrement dit; étant donnée une suite de variables aléatoires $(X_n)_{n \geq 1}$ définie sur un espace de probabilité $(\Omega, \mathcal{F}, P)$ et $S_n = \sum_{i=1}^n X_i$ pour tout $n \geq 1$. On dit que $(X_n)_{n \geq 1}$ satisfait la loi forte des grands nombres s'il existe deux suites réelles $(a_n)_{n \geq 1}$ et $(b_n)_{n \geq 1}$ telles que $(b_n)_{n \geq 1}$ est positive et croissante vers l'infini et

$$\frac{S_n - a_n}{b_n} \longrightarrow 0 \text{ p.s.} \quad (1)$$

Généralement, on distingue entre deux grands types de la loi forte des grands nombres; le premier type concerne les variables aléatoires indépendantes et identiquement distribuées, cependant que le second type traite le cas où les variables aléatoires $(X_n)_{n \geq 1}$ ne sont pas obligatoirement indépendantes et n'ont pas obligatoirement la même moyenne.

Le premier type de la loi forte des grands nombres est généralement valide pour les variables aléatoires à valeurs dans un espace de Banach. Tandis que le second type est difficile à traiter dans un espace de Banach quelconque. Nous allons voir dans ce cas que la LFGN est satisfaite mais sous certaines conditions géométriques sur l'espace de Banach considéré.

Les résultats les plus célèbres dans le cas d'une suite de variables aléatoires indépendantes identiquement distribuées sont la LFGN de Kolmogorov ($a_n = ES_n$ et $b_n = n$) et la LFGN de Marcinkiewicz-Zygmund ($a_n = ES_n$, $b_n = n^{\frac{1}{p}}$ où $1 < p < 2$). Pour une suite de variables aléatoires indépendantes et non-identiquement distribuées. Kolmogorov a établi que la condition $\sum_{n=1}^{\infty} \frac{\text{Var}(X_n)}{b_n^2} < \infty$ implique la relation (1) avec $a_n = ES_n$.

([\[34\]](#), [\[35\]](#) and [\[27\]](#)) ont prouvé que dans la LFGN de Kolmogorov, la condition de l'indépendance des variables aléatoires peut être remplacée par la condition de l'indépendance deux à deux des variables aléatoires. Cependant [\[66\]](#) ont obtenu la LFGN pour une suite de variables aléatoires uniformément intégrables indépendantes deux à deux.

Au cours des dernières années, de nombreux auteurs ont présenté la LFGN en écartant l'hypothèse de l'indépendance des variables aléatoires (voir [\[21\]](#), [\[71\]](#), [\[87\]](#), [\[69\]](#)), [\[28\]](#), [\[65\]](#)).

[\[23\]](#), [\[10\]](#), [\[37\]](#), [\[49\]](#), [\[50\]](#), [\[48\]](#) ont prouvé diverses généralisations de la LFGN, dont $(X_n)_{n \geq 1}$ est une suite adaptée à valeurs dans un espace de Banach au lieu d'une suite de variables aléatoires réelles.

Ces dernières années, la théorie des ensembles aléatoires a été largement étudiée car elle

généralise la notion des variables aléatoires et des vecteurs aléatoires. ([88], [89]) est le premier qui a fourni le concept des ensembles aléatoires, et ses premiers travaux ont traité les relations entre les ensembles aléatoires et les probabilités géométriques. Plus tard, [63] et [73] ont construit une théorie mathématique complète des ensembles aléatoires qui a été grandement influencée par la perspective de probabilité géométrique. Le cadre proposé exerce une forte influence sur les théorèmes limites en probabilités développés au cours des dernières décennies.

Avec le développement de la théorie des ensembles aléatoires et leur applications (voir [9], [62], [68], [73], [18], [52], [53], [38], [55]), plusieurs variantes de la loi forte des grands nombres ont été construites.

La première version de la LFGN multivoque a été donné par [7] pour une suite de variables aléatoires indépendantes et identiquement distribuées (i.i.d) dont les valeurs sont des parties compactes de $\mathbb{R}^n$. Ce type de la LFGN dont la convergence est établie au sens de la distance de Hausdorff a été traité par plusieurs auteurs (voir [26], [51], [6], [60], [40]) et [85].

En d'autre part, une nouvelle notion d'intégrale s'est développée pour des vecteurs aléatoires et ensembles aléatoires non forcément mesurables et intégrables mais uniquement scalairement mesurable et scalairement intégrable, c'est l'intégrale de Pettis. Une notion plus générale que l'intégrale de Bochner pour les variables aléatoires à valeurs dans un espace de Banach de dimension infinie. Si l'espace est de dimension finie les deux intégrables coïncident. Cette notion d'intégrale a été l'origine du développement de la théorie de l'intégration de Pettis qui a trouvé un grand succès ces dernières années. En effet cette théorie s'est développée suivant plusieurs directions; étude des nouvelles propriétés des intégrales de Pettis, convergence des processus Pettis intégrables, existence et convergence des espérances conditionnelles Pettis intégrables et autres. Il est bien connu dans la littérature que l'espérance conditionnelle des variables aléatoires Bochner intégrables existe toujours ([82], [61], [57], [62], [36], [18]). Cependant ce n'est pas le cas pour les variables aléatoires (ou ensembles aléatoires) Pettis intégrables (voir les contre exemples de [90] et [92]). Comme notre intérêt dans ce travail se porte également sur la loi forte des grands nombres des processus aléatoires Pettis intégrable, notre attention est attirée par l'existence de l'espérance conditionnelle des vecteurs aléatoires Pettis intégrables et ensembles aléa-

toires Pettis intégrables comme étant la notion principale dans la définition d'un processus aléatoire. En exploitant les résultats récemment développés dans ce sens ([33], [2], [1], [4], [38], [79] et [55]), nous avons démontré des nouvelles versions de la loi forte des grands nombres aussi bien dans le cas de Bochner intégration et dans le cas de Pettis intégration. En d'autre part nous présentons quelques applications concrètes de la LFGN. Nous contribuons avec quelques résultats dans trois types de problèmes en théorie des probabilités et statistique à savoir; la relation entre la LFGN et les suites adaptées ou plus précisément la relation entre la loi forte des grands nombres et les martingales asymptotiques, la convergence des espérances conditionnelles d'ensembles aléatoires Pettis intégrables et finalement la LFGN dans le cadre de Pettis intégration.

Dans ce travail nous avons présenté et démontré des nouveaux résultats sur la loi forte des grands nombres (LFGN). Notre intérêt est focalisé sur trois types de problèmes. Le premier traite la loi forte des grands nombres pour les suites adaptées en général et la relation entre cette dernière et la nature des suites adaptées considérées; en effet nous démontrons la relation entre LFGN et les amarts. Comme application nous prouvons la loi forte des grands pour les mils et les surmartingales multivoques.

Le deuxième problème traite la convergence des ensembles aléatoires Pettis intégrables, une notion qui sera exploité dans le dernier travail de cette thèse pour établir la loi forte des grands nombres pour les ensembles aléatoires Pettis intégrables. Une notion qui généralise la loi forte des grands nombres pour les ensembles aléatoires intégrablement bornés.

Dans cette partie nous donnons un résumé des principaux résultats de ce projet de thèse comme suit: dans la section 0.2, nous traitons la loi forte des grands nombres pour les suites adaptées et dans la section 0.3 un théorème de la convergence des espérances conditionnelles Pettis intégrables est proposé. L'utilisation des techniques de convergence des martingales inverses pour prouver la LFGN pour les ensembles aléatoires Pettis intégrables est donnée en section 0.4 et dans la section 0.5, une application de la LFGN est discutée.

0.2 Loi forte des grands nombres et suites adaptées

Soit $(\Omega, \mathcal{F}, P)$ un espace de probabilité et $(X_n)_{n \geq 1}$ une suite de variables aléatoires intégrables à valeurs dans $\mathbb{R}$.

Soit $\mathcal{F}_n$ la sous σ -algèbre de $\mathcal{F}$ engendré par $X_1, X_2, X_3, \dots, X_n$. Posons $S_n = \sum_{i=1}^n X_i$, $\forall n \geq 1$. $L_{\mathbb{R}}^1(\mathcal{F})$ (resp. $L_{cwk(\mathbb{E})}^1(\mathcal{F})$, resp. $L_{ccb(\mathbb{E})}^1(\mathcal{F})$) désignent l'espace des fonctions $\mathcal{F}$ -mesurables et intégrables de Ω dans $\mathbb{R}$ (resp. l'espace des multifonctions $\mathcal{F}$ -mesurables et intégrablement bornées définie sur Ω à valeurs dans l'ensemble des parties convexes et faiblement compactes de $\mathbb{E}$, resp. l'espace des multifonctions $\mathcal{F}$ -mesurables et intégrablement bornées définies sur Ω à valeurs dans l'ensemble des parties convexes fermées de $\mathbb{E}$).

On dit que $(X_n, \mathcal{F}_n)_{n \geq 1}$ est une suite adaptée si pour tout $n \geq 1$, $X_n \in L_{\mathbb{R}}^1(\mathcal{F}_n)$.

Définition 0.1 Une suite adaptée $(X_n, \mathcal{F}_n)_{n \geq 1}$ est une martingale (resp. surmartingale, resp. sous martingale) si, pour tout $n \geq 1$, $E^{\mathcal{F}_n} X_{n+1} = X_n$ p.s. (resp. pour tout $n \geq 1$, $E^{\mathcal{F}_n} X_{n+1} \geq X_n$ p.s., resp. pour tout $n \geq 1$, $E^{\mathcal{F}_n} X_{n+1} \leq X_n$ p.s.).

où $E^{\mathcal{F}_n} X$ désigne l'espérance conditionnelle de $X \in L_{\mathbb{R}}^1(\mathcal{F})$ par rapport à $\mathcal{F}_n$

Définition 0.2 Une application $\tau : \Omega \longrightarrow \mathbb{N}$ est un temp d'arrêt borné, si τ prend un nombre fini de valeurs de $\mathbb{N}$ et si, $\forall n \geq 1$, $[\tau = n] \in \mathcal{F}_n$.

Soit Γ l'ensemble des temps d'arrêt bornés.

Définition 0.3 $(X_n, \mathcal{F}_n)_{n \geq 1}$ est un amart, si on a

$$\limsup_{\sigma} \int_{\tau \geq \sigma} |E^{\mathcal{F}_\sigma} X_\tau - X_\sigma| dP = 0, \quad \tau, \sigma \in \Gamma.$$

Où $\mathcal{F}_\sigma = \{A \in \mathcal{F} / A \cap [\sigma = n] \in \mathcal{F}_n, \forall n \geq 1\}$.

Définition 0.4 $(X_n, \mathcal{F}_n)_{n \geq 1}$ est un mil si on a

$$\limsup_m \sup_{n \geq m} |E^{\mathcal{F}_m} X_n - X_m| = 0 \quad p.p.$$

Définition 0.5 • La multifonction $X : \Omega \longrightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ est mesurable si, pour chaque ouvert U de $\mathbb{R}$,

$$X^{-1}(U) = \{\omega \in \Omega / X(\omega) \cap U \neq \emptyset\}$$

est un élément de $\mathcal{F}$.

- la fonction mesurable f de Ω à valeurs dans $\mathbb{R}$ est dite sélection de X , si $f(\omega) \in X(\omega)$ pour chaque $\omega \in \Omega$. on note par $S_X^1(\mathcal{F})$ l'ensemble de toutes les sélections intégrables et $\mathcal{F}$ -mesurables de X . Il est bien connu d'après [57] que toute multifonction X à valeur fermée et convexe est mesurable, s'il existe une famille dénombrable de sélections mesurables $(f_n)_{n \geq 1}$ telle que, pour chaque $\omega \in \Omega$, on a

$$X(\omega) = \overline{\{f_n(\omega), n \in \mathbf{N}\}}.$$

- La multifonction X est dite intégrable au sens d'Aumann si, $S_X^1(\mathcal{F}) \neq \emptyset$. L'intégrale d'Aumann de la multifonction X est définie par:

$$\int_A X dP = \left\{ \int_A f dP, f \in S_X^1(\mathcal{F}) \right\}.$$

Définition 0.6 Une suite $(X_n)_{n \geq 1} \in L_{ccb(\mathbb{R})}^1(\mathcal{F})$ est une surmartingale multivoque si, $X_n \in L_{ccb(\mathbb{R})}^1(\mathcal{F}_n)$ et $E^{\mathcal{F}_n} X_{n+1}(\omega) \subseteq X_n(\omega)$ pour chaque $n \geq 1$ et presque sûrement $\omega \in \Omega$.

Définition 0.7 ([14])

Soit $(A_n)_{n \geq 1}$ un ensemble non vide. La limite inférieure forte et la limite supérieure faible de A_n respectivement sont données par:

$$s - li A_n = \{x \in E, x_n \longrightarrow x, x_n \in A_n, \forall n \geq 1\}$$

$$w - ls A_n = \{y \in E, y_k \xrightarrow{w} y, y_k \in A_{n_k}, \forall k \geq 1\},$$

dont $(A_{n_k})_{k \geq 1}$ est une sous suite de $(A_n)_{n \geq 1}$.

Lemma 0.1 ([57])

Toute surmartingale multivoque $(X_n)_{n \geq 1} \in L_{ccb(\mathbb{R})}^1(\mathcal{F})$ admet au moins une martingale sélectionnée.

tion (c'est-à-dire qu'il existe une fonction f_n telle que $E^{\mathcal{F}_n} f_{n+1} = f_n$ et $f_n(\omega) \in X_n(\omega)$, pour tout $n \geq 1$ et presque sûrement $\omega \in \Omega$).

Lemma 0.2 ([31])

Soit $(X_n, \mathcal{F}_n)_{n \geq 1}$ une suite adaptée. Alors il existe $(Y_n, \mathcal{F}_n)_{n \geq 1}$ une martingale et $(Z_n, \mathcal{F}_n)_{n \geq 1}$ dans $L^1_{\mathbb{R}}(\mathcal{F}_{n-1})$ telle que $X_n = Y_n + Z_n$ où $Y_1 = X_1$, $Z_1 = 0$ et

$$Y_{n+1} - Y_n = S_{n+1} - E^{\mathcal{F}_n}(S_{n+1}),$$

$$Z_{n+1} - Z_n = E^{\mathcal{F}_n}(S_{n+1}) - S_n.$$

L'un des premiers résultats classiques sur la LFGN est le théorème de Kolmogorov suivant:

Théorème 0.1 ([64])

Soit $(X_n)_{n \geq 1}$ une suite de variables aléatoires indépendantes et identiquement distribuées à valeurs dans $\mathbb{R}$, telle que $E(X_1) < \infty$. Alors

$$\lim_{n \rightarrow +\infty} \frac{S_n}{n} = E(X_1) \text{ p.s.}$$

L'extension du théorème 0.1 dans le cas où les variables aléatoires $(X_n)_{n \geq 1}$ sont indépendantes mais non identiquement distribuées est:

Théorème 0.2 ([64]) Soit $(X_n)_{n \geq 1}$ une suite de variables aléatoires indépendantes à valeurs dans $\mathbb{R}$, telle que $E(X_i) = 0$ et $E(X_i^2) \leq M \quad \forall i \geq 1$ et $M \in \mathbb{R}$. Alors

$$\lim_{n \rightarrow +\infty} \frac{S_n}{n} = 0 \text{ p.s.}$$

[23] a montré un résultat plus général, il a pris $S_n = \sum_{k=1}^n X_k$ une martingale au lieu d'une suite de variables aléatoires indépendantes avec $E(X_k) = 0$ et au lieu du coefficient $\frac{1}{n}$, le coefficient C_n où $(C_n)_{n \geq 1}$ est une suite positive décroissante vers zéro. Son résultat est le suivant:

Théorème 0.3 ([23])

Soit $(X_n)_{n \geq 1}$ une suite de variables aléatoires à valeurs dans $\mathbb{R}$ telle que $(S_n = \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$

est une martingale et $(C_n)_{n \geq 1}$ une suite positive et décroissante avec $\lim_{n \rightarrow +\infty} C_n = 0$. Soit A , α et β telle que $\alpha > 1$, $2\alpha \geq \beta > 0$ et $A > 0$. Si pour chaque $i \geq i_0$ on a :

$$(1) \quad E|S_i|^{2\alpha} \leq A \cdot E(\sum_{k=1}^i X_k^2)^\alpha.$$

$$(2) \quad i^{\alpha-1} C_i^{2\alpha-\beta} \leq A, \sum_{k=i}^{\infty} C_k^{2\alpha} k^{\alpha-2} \leq A C_i^\beta.$$

$$(3) \quad \sum_{k=1}^{+\infty} C_k^\beta E|X_k|^{2\alpha} < \infty.$$

Alors

$$\lim_{n \rightarrow +\infty} C_n S_n = 0 \quad p.s.$$

Dans ce contexte, en se basant sur l'inégalité de [15], la décomposition de Doob (Lemme 1.2, p. 33), une version de l'inégalité de Chow (Proposition 1.2, p. 29), l'inégalité de Hölder et le lemme de Kronecker (Lemme 1.1, p. 32), nous proposons un résultat qui généralise le théorème 0.3 (nous considérons une suite adaptée au lieu d'une martingale) avec moins d'hypothèses.

C'est le résultat principal de cette partie.

Théorème 0.4 (Theorem 1.6, p. 33) Soit $(X_n)_{n \geq 1}$ une suite de variables aléatoires à valeurs dans $\mathbb{R}$ telle que $(S_n = \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$ est une suite adaptée et $(C_n)_{n \geq 1}$ une suite positive et décroissante avec $\lim_{n \rightarrow +\infty} C_n = 0$. Soit A , α et β telle que $\alpha > 1$, $2\alpha \geq \beta > 0$ and $A > 0$. Si pour chaque $i \geq i_0$ on a:

$$(i) \quad i^{\alpha-1} C_i^{2\alpha-\beta} \leq A, \sum_{k=i}^{\infty} C_k^{2\alpha} k^{\alpha-2} \leq A C_i^\beta.$$

$$(ii) \quad \sum_{k=1}^{+\infty} C_k^\beta E|X_k|^{2\alpha} < \infty.$$

$$(iii) \quad \sum_{k=1}^{+\infty} C_k (E^{\mathcal{F}_k} S_{k+1} - S_k) < \infty.$$

Alors

$$\lim_{n \rightarrow +\infty} C_n S_n = 0 \quad p.s.$$

En utilisant le résultat de ce dernier théorème, nous établissons trois applications de la LFGN:

- La loi forte des grands nombres appliquée aux mils

Avant de donner ce résultat, présentons d’abord un théorème important due à [12] dans le cas d’une suite de variables aléatoires indépendantes:

Théorème 0.5 ([12]) *Soit $(X_n)_{n \geq 1}$ une suite de variables aléatoires indépendantes à valeurs dans $L^1_{\mathbb{R}}(\mathcal{F})$ et $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$. Alors les assertions suivantes sont équivalentes*

- (i) $(S_n = \sum_{k=1}^n X_k, \mathcal{F}_n)_{n \geq 1}$ est un mil.
- (ii) La suite $(E(S_n))_{n \geq 1}$ converge dans $\mathbb{R}$.
- (iii) $(S_n = \sum_{k=1}^n X_k, \mathcal{F}_n)_{n \geq 1}$ est un amart.

Dans ce contexte, en se basant sur ce résultat et les mêmes techniques de la démonstration du théorème 0.4, nous obtenons le résultat suivant:

Théorème 0.6 (Theorem 1.8, p. 39) *Soit $(X_n)_{n \geq 1}$ une suite de variables aléatoires indépendantes à valeurs dans $L^1_{\mathbb{R}}(\mathcal{F})$. Soit $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ et supposons que $(S_n = \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$ est un mil intégrable. Soit A , α et β telle que $\alpha > 1$, $2\alpha \geq \beta > 0$ et $A > 0$. Si pour chaque $i \geq i_0$ on a:*

- (i) $i^{\alpha-1} C_i^{2\alpha-\beta} \leq A$, $\sum_{k=i}^{\infty} C_k^{2\alpha} k^{\alpha-2} \leq A C_i^{\beta}$.
- (ii) $\sum_{k=1}^{+\infty} C_k^{\beta} E|X_k|^{2\alpha} < \infty$.

Alors

$$\lim_{n \rightarrow +\infty} C_n S_n = 0 \text{ p.s.}$$

- Loi forte des grands nombres et amart:

Dans la deuxième application, nous établissons une relation entre la LFGN et amart. Mais mentionnons d’abord la proposition suivante due à [43], qui donne une condition suffisante pour qu’une suite de variables aléatoires soit un amart.

Proposition 0.1 ([43]) *Soit $(Z_n)_{n \geq 1}$ une suite adaptée dans $L^1_{\mathbb{R}}(\Omega, \mathcal{F}, P)$ telle que,*

$$E \sup_n |Z_n| < +\infty.$$

Alors les deux assertions suivantes sont équivalentes:

1. $(Z_n)_{n \geq 1}$ converge presque sûrement.
2. $(Z_n)_{n \geq 1}$ est un amart.

Maintenant, nous proposons le théorème suivant :

Théorème 0.7 (Theorem 1.9, p. 42) Soit $(X_n)_{n \geq 1}$ une suite de variables aléatoires dans $L^1_{\mathbb{R}}(\mathcal{F})$ telle que $(S_n = \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$ est une suite adaptée. Soit A , α et β telle que $\alpha > 1$, $2\alpha \geq \beta > 0$ et $A > 0$. supposons que les conditions suivantes sont satisfaites.

- (i) $i^{\alpha-1} C_i^{2\alpha-\beta} \leq A$, $\sum_{k=i}^{\infty} C_k^{2\alpha} k^{\alpha-2} \leq A C_i^{\beta}$, $\forall i \geq i_0$.
- (ii) $\sum_{k=1}^{+\infty} C_k^{\beta} E|X_k|^{2\alpha} < \infty$.
- (iii) $\sum_{k=1}^{+\infty} C_k E|X_k| < +\infty$ a.s.

Alors $(C_n \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$ est un amart.

Le dernier résultat de cette partie concerne la LFGN pour les surmartingales multivoque. Dans cette application, nous démontrons l'existence d'une sélection intégrable de $(S_n)_{n \geq 1}$ telle que $C_n g_n$ converge presque sûrement.

- Loi forte des grands nombres pour les surmartingales multivoque.

Avant d'énoncer la loi forte des grands nombres pour les surmartingales multivoque, rappelons la proposition suivante due à [37].

Proposition 0.2 ([37]) Soit $(X_n)_{n \geq 1}$ une suite dans $L^1_{ccb(\mathbb{R})}(\mathcal{F})$ telle que $(S_n = \sum_{k=1}^n X_k, \mathcal{F}_n)$ est une surmartingale. Alors, S_n peut se réécrire sous la forme

$$S_n = X_1 + f_2 + f_3 + \dots + f_n, \quad f_i \in L^1_{\mathbb{R}}(\mathcal{F}), (i \geq 2).$$

En utilisant cette proposition et le théorème 0.4, nous avons démontré le théorème suivant:

Théorème 0.8 (Theorem 1.10, p. 44) Soit $(X_n)_{n \geq 1}$ une suite dans $L^1_{ccb(\mathbb{R})}(\mathcal{F})$ telle que $(S_n = \sum_{k=1}^n X_k, \mathcal{F}_n)$ est une surmartingale et $(C_n)_{n \geq 1}$ une suite positive décroissante avec $\lim_{n \rightarrow +\infty} C_n = 0$. Soit A , α et β telle que $\alpha > 1$, $2\alpha \geq \beta > 0$ et $A > 0$. Si pour chaque $i \geq i_0$ on a :

$$(i) \quad i^{\alpha-1} C_i^{2\alpha-\beta} \leq A, \quad \sum_{k=i}^{\infty} C_k^{2\alpha} k^{\alpha-1} \leq A C_i^{\beta}.$$

$$(ii) \quad \sum_{k=1}^{+\infty} C_k^{\beta} E|X_k|^{2\alpha} < \infty.$$

Alors $0 \in s - li C_n S_n$ p.s.

Remarque 0.1 *Ce dernier résultat peut être vu comme une généralisation du théorème 0.4.*

0.3 Convergence des espérances conditionnelles Pettis intégrables

La convergence des espérances conditionnelles des variables aléatoires (ou des ensembles aléatoires) joue un rôle indispensable dans la théorie des probabilités notamment dans la théorie des martingales, la géométrie stochastique, l'économie mathématique, l'optimisation et le contrôle.

L'objectif de ce chapitre est d'utiliser les outils récents établis autour des espérances conditionnelles (voir [38], [33], [2] et [1]), pour prouver une extension des résultats de [82], [61] et [18] dans le cadre de Pettis intégration. Plus précisément, nous étudions la convergence au sens de Mosco de deux suites d'ensembles aléatoires $E^{\mathfrak{B}_n} X$ et $E^{\mathfrak{B}_n} X_n$, où $(\mathfrak{B}_n)_{n \geq 1}$ est une suite décroissante de sous- σ -algèbres de $\mathcal{F}$ et $(X_n)_{n \geq 1}$ est une suite d'ensembles aléatoires Pettis-intégrables à valeurs convexes et faiblement compactes d'un espace de Banach séparable.

Donnons d'abord quelques notations et définitions nécessaires, par la suite.

On se donne un espace probabilisé $(\Omega, \mathcal{F}, P)$ et un espace de Banach séparable $\mathbb{E}$ dont la norme sera notée $\|\cdot\|$. Notons $\mathbb{E}^*$ le dual de E . Soit $(\mathfrak{B}_n)_{n \geq 1}$ une suite décroissante de sous- σ -algèbres de $\mathcal{F}$.

Etant donné un sous ensemble B dans $cwk(\mathbb{E})$, la fonction distance et la fonction d'appui de B sont respectivement définies par:

$$d(x, B) = \inf\{\|x - y\|, y \in B\}, \quad (x \in \mathbb{E}).$$

$$\delta^*(x^*, B) = \sup\{\langle x^*, y \rangle, y \in B\}, \quad (x^* \in \mathbb{E}^*).$$

Pour chaque B dans $cwk(\mathbb{E})$, on a

$$|B| = \sup\{\|x\|, \quad x \in B\}.$$

Une suite $(A_n)_{n \geq 1}$ à valeurs $cwk(\mathbb{E})$ est dite Mosco convergente vers $A_\infty \in cwk(\mathbb{E})$ (voir. [78]), et on note $M - \lim A_n = A_\infty$ si, $A_\infty = s - li A_n = w - ls A_n$, où

$$s - li A_n = \{y \in \mathbb{E}, \quad y_n \longrightarrow y, \quad y_n \in A_n, \quad \forall n \geq 1\}$$

$$w - ls A_n = \{y \in \mathbb{E}, \quad y_k \xrightarrow{w} y, \quad y_k \in A_{n_k}, \quad \forall k \geq 1\},$$

où $(A_{n_k})_{k \geq 1}$ est une sous suite de $(A_n)_{n \geq 1}$.

Une fonction mesurable $g : \Omega \longrightarrow \mathbb{E}$ est dite Pettis-intégrable, si g est scalairement intégrable (i. e. $\langle x^*, g \rangle$ est intégrable), en plus pour toute $A \in \mathcal{F}$, il existe un x_A dans $\mathbb{E}$, telle que

$$\int_A \langle x^*, g \rangle dP = \langle x^*, x_A \rangle, \quad \forall x^* \in \mathbb{E}^*.$$

$P_{\mathbb{E}}^1(\mathcal{F})$ désigne l'espace des fonctions $\mathcal{F}$ -mesurable et Pettis intégrables définies de $(\Omega, \mathcal{F}, P)$ dans $\mathbb{E}$. L'ensemble aléatoire $X : \Omega \longrightarrow cwk(\mathbb{E})$ est scalairement intégrable si, pour tout $x^* \in \mathbb{E}^*$, la fonction $\delta^*(x^*, X(\cdot))$ est intégrable.

On dit que l'ensemble aléatoire X est Pettis intégrable, si X est scalairement intégrable et pour toute $A \in \mathcal{F}$, il existe $K_A \in cwk(\mathbb{E})$ telle que

$$\int_A \delta^*(x^*, X) dP = \delta^*(x^*, K_A), \quad \forall x^* \in \mathbb{E}^*.$$

Une fonction g de Ω dans $\mathbb{E}$ est appelée sélection de X si pour tout $\omega \in \Omega$, $g(\omega) \in X(\omega)$. Notons $S_X^{Pe}(\mathcal{F})$ l'ensemble des sélections $\mathcal{F}$ -mesurables et Pettis intégrables de X . L'ensemble aléatoire X est dit Aumann Pettis intégrable si $S_X^{Pe}(\mathcal{F}) \neq \emptyset$ et on a

$$\int_{\Omega} X dP = \left\{ \int_{\Omega} f dP, f \in S_X^{Pe}(\mathcal{F}) \right\}.$$

Nous noterons par $P_{cwk(\mathbb{E})}^1(\mathcal{F})$ l'espace des multifonctions $\mathcal{F}$ -mesurables et Pettis intégrables définies de $(\Omega, \mathcal{F}, P)$ à valeurs dans $cwk(\mathbb{E})$.

Soit X une variable aléatoire dans $L^1_{\mathbb{E}}(\mathcal{F})$ et $\mathcal{B}$ une sous σ -algèbre de $\mathcal{F}$. Il est connu dans la littérature (voir [82]) que l'espérance conditionnelle de X relative à $\mathcal{B}$ est l'unique variable aléatoire Bochner intégrable et $\mathcal{B}$ -mesurable vérifiant

$$\int_A E^{\mathcal{B}} X dP = \int_A X dP, \forall A \in \mathcal{B}.$$

L'extension de ce résultat pour les ensembles aléatoires intégrablement borné est donnée par ([62]) comme suit;

Etant donnée une multifonction X dans $L^1_{cwk(\mathbb{E})}(\mathcal{F})$ et une sous- σ -algèbre $\mathcal{B}$ de $\mathcal{F}$, alors, il existe un unique ensemble aléatoire intégrablement borné de Ω à valeurs dans $cwk(\mathbb{E})$ qu'on note $E^{\mathcal{B}} X$ et satisfaisant aux propriétés suivantes:

$$S^1_{E^{\mathcal{B}} X}(\mathcal{B}) = cl\{E^{\mathcal{B}} f, f \in S^1_X(\mathcal{F})\},$$

$$\delta^*(x^*, E^{\mathcal{B}} X(.)) = E^{\mathcal{B}} \delta^*(x^*, X(.)),$$

où la fermeture est prise au sens de la norme dans $L^1_{\mathbb{E}}(\mathcal{F})$. Avant de présenter les résultats principaux de cette partie, nous rappelons les deux corollaires suivants; le premier est un théorème de type Levy et le deuxième est connu sous le nom de théorème de la convergence dominée pour les espérances conditionnelles de variables aléatoires Bochner intégrables relativement à $(\mathcal{B}_n)_{n \geq 1}$ où $(\mathcal{B}_n)_{n \geq 1}$ est une suite décroissante de sous σ -algèbres de $\mathcal{F}$.

Corolaire 0.1 ([99]) *Soit $(\mathcal{B}_n)_{n \geq 1}$ une suite décroissante de sous σ -algèbres de $\mathcal{F}$ et $f \in L^1_{\mathbb{E}}(\mathcal{F})$, posons $\mathcal{B}_{\infty} = \cap_{n \geq 1} \mathcal{B}_n$. Alors*

$$\lim_{n \rightarrow +\infty} E^{\mathcal{B}_n} f(.) = E^{\mathcal{B}_{\infty}} f(.) \text{ p.s.}$$

Corolaire 0.2 ([18]) *Soit $(\mathcal{B}_n)_{n \geq 1}$ une suite décroissante de sous σ -algèbres de $\mathcal{F}$ et posons $\mathcal{B}_{\infty} = \cap_{n \geq 1} \mathcal{B}_n$. Si Y une variable aléatoire positive vérifiant $E^{\mathcal{B}_{\infty}} Y \in [0, +\infty[$ et $(X_n)_{n \geq 1}$ dans $L^1_{\mathbb{E}}(\mathcal{F})$ telle que:*

1. $\forall n \geq 1, |X_n| \leq Y \text{ p.s.}$
2. $\lim_{n \rightarrow +\infty} X_n(.) = X_{\infty}(.) \text{ p.s.}$

Alors

$$\lim_{n \rightarrow +\infty} E^{\mathcal{B}_n} X_n(\cdot) = E^{\mathcal{B}_\infty} X_\infty(\cdot) \text{ p.s.}$$

Après avoir introduit tous les outils nécessaires, nous présentons maintenant les résultats principaux de cette partie. Dans un premier temps nous donnons une extension du Corollaire 0.1 et du Corollaire 0.2 dans le cas de l'intégration de Pettis, puis nous généralisons ces deux résultats aux ensembles aléatoires à valeurs $\text{cwk}(\mathbb{E})$ Pettis intégrables.

Théorème 0.9 (Theorem 2.1, p. 53) Soit $(\mathcal{B}_n)_{n \geq 1}$ une suite décroissante de sous σ -algèbres de $\mathcal{F}$ et posons $\mathcal{B}_\infty = \cap_{n \geq 1} \mathcal{B}_n$. Soit $f \in P_{\mathbb{E}}^1(\mathcal{F})$ telle que $E^{\mathcal{B}_\infty} |f| \in [0, +\infty[$. Alors

$$\lim_{n \rightarrow +\infty} E^{\mathcal{B}_n} f(\cdot) = E^{\mathcal{B}_\infty} f(\cdot) \text{ p.s.}$$

Théorème 0.10 (Theorem 2.2, p. 54) Soit $(\mathcal{B}_n)_{n \geq 1}$ une suite décroissante de sous σ -algèbres de $\mathcal{F}$ et posons $\mathcal{B}_\infty = \cap_{n \geq 1} \mathcal{B}_n$. Si Y est une variable aléatoire positive vérifiant $E^{\mathcal{B}_\infty} Y \in [0, +\infty[$. Soit $(X_n)_{n \geq 1}$ une suite dans $P_{\mathbb{E}}^1(\mathcal{F})$ telle que

1. $\forall n \geq 1, |X_n(\cdot)| \leq Y(\cdot) \text{ p.s.}$
2. $\lim_{n \rightarrow +\infty} X_n(\cdot) = X_\infty(\cdot) \text{ p.s.}$

Alors

$$\lim_{n \rightarrow +\infty} E^{\mathcal{B}_n} X_n(\cdot) = E^{\mathcal{B}_\infty} X_\infty(\cdot) \text{ p.s.}$$

Le théorème suivant présente une généralisation du théorème 0.9 aux ensembles aléatoires Pettis intégrables à valeurs dans $\text{cwk}(\mathbb{E})$.

Théorème 0.11 (Theorem 2.3, p. 55) Soit X un ensemble aléatoire Pettis-intégrable défini de Ω dans $\text{cwk}(\mathbb{E})$ et $(\mathcal{B}_n)_{n \geq 1}$ une suite décroissante de sous σ -algèbres de $\mathcal{F}$. posons $\mathcal{B}_\infty = \cap_{n \geq 1} \mathcal{B}_n$. Si $E^{\mathcal{B}_\infty} |X| \in [0, +\infty[$, on a

$$M - \lim_{n \rightarrow \infty} E^{\mathcal{B}_n} X(\cdot) = E^{\mathcal{B}_\infty} X(\cdot) \text{ p.s.}$$

[1] ont démontré le même résultat que le Théorème 0.11 dans le cas où $(\mathcal{B}_n)_{n \geq 1}$ est une suite croissante de sous σ -algèbres de $\mathcal{F}$.

Maintenant, nous donnons une nouvelle version du lemme de Fatou pour l'espérance conditionnelle d'un ensemble aléatoire Pettis intégrable et à valeur dans $cwk(\mathbb{E})$. Ce théorème peut être vu également comme une généralisation du Théorème 0.10 ou comme une extension du théorème 4.6 dans [36].

Théorème 0.12 (*Theorem 2.4, p. 56*) Soit $(\mathcal{B}_n)_{n \geq 1}$ une suite décroissante de sous σ -algèbres de $\mathcal{F}$. Posons $\mathfrak{B}_\infty = \cap_{n \geq 1} \mathfrak{B}_n$. Si Y est une variable aléatoire positive vérifiant $E^{\mathfrak{B}_\infty} Y \in [0, +\infty[$ et $(X_n)_{\mathbb{N}^* \cup \{+\infty\}}$ est une suite d'ensembles aléatoires Pettis intégrables à valeurs dans $cwk(\mathbb{E})$ telle que

1. $\forall n \in \mathbb{N}^* \cup \{+\infty\}, |X_n(\cdot)| \leq Y(\cdot) \text{ p.s.}$
2. $X_\infty(\cdot) \subset s\text{-}li \ X_n(\cdot) \text{ p.s.}$

Alors

$$E^{\mathfrak{B}_\infty} X_\infty(\cdot) \subset s\text{-}li E^{\mathfrak{B}_n} X_n(\cdot) \text{ p.s.}$$

Le dernier résultat de cette section est la Mosco convergence de l'espérance conditionnelle d'une suite d'ensembles aléatoires Pettis intégrables à valeurs dans $cwk(\mathbb{E})$.

Théorème 0.13 (*Theorem 2.5, p. 58*) Soit $(\mathcal{B}_n)_{n \geq 1}$ une suite décroissante de sous σ -algèbres de $\mathcal{F}$. Soit $\mathfrak{B}_\infty = \cap_{n \geq 1} \mathfrak{B}_n$. Si Y est une variable aléatoire positive vérifiant $E^{\mathfrak{B}_\infty} Y \in [0, +\infty[$ et $(X_n)_{\mathbb{N}^* \cup \{+\infty\}}$ est une suite d'ensembles aléatoires Pettis intégrables à valeurs dans $\mathbb{E}$ telle que

1. $\forall n \in \mathbb{N}^* \cup \{+\infty\}, |X_n(\cdot)| \leq Y(\cdot) \text{ p.s.}$
2. $X_\infty(\cdot) \subset s\text{-}li \ X_n(\cdot) \text{ p.s.}$
3. $\lim_{n \rightarrow +\infty} \delta^*(x^*, X_n(\cdot)) = \delta^*(x^*, X_\infty(\cdot)) \text{ p.s. } \forall x^* \in E^*.$

Alors

$$M\text{-}\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} X_n = E^{\mathfrak{B}_\infty} X_\infty \text{ p.s.}$$

0.4 Loi forte des grands nombres pour les multifonctions Pettis-intégrables

L'objectif de cette partie, est de prouver la LFGN pour les ensembles aléatoires indépendants Pettis intégrables et à valeurs convexes faiblement compactes. Dans ce contexte, nous démontrons que pour une suite d'ensembles aléatoires indépendants Pettis intégrable donnée, il existe une suite décroissante $(\mathfrak{B}_n)_{n \geq 1}$ de sous σ -algèbres de $\mathcal{F}$ telle que

$$\frac{S_n}{n} = E^{\mathfrak{B}_n} X_1,$$

de sorte que la loi forte des grands nombres pour la suite $(X_n)_{n \geq 1}$ soit équivalente à la convergence de la martingale inverse $(E^{\mathfrak{B}_n} X_1)_{n \geq 1}$.

Il est bien connu dans la littérature que le théorème classique de la loi forte des grands nombres est une conséquence d'autres théorèmes ([95]) comme le théorème de convergence des martingales inverses ([29]) et le théorème ergodique de Birkhof, en relation avec ce travail il est important d'indiquer un article de [99].

Avant de présenter les résultats essentiels de cette section, nous donnons quelques notations et rappels qui seront utiles par la suite.

Soit $(\Omega, \mathcal{F}, P)$ un espace probabilisé et $\mathbb{E}$ un espace de Banach muni de sa tribu borélienne $\mathcal{B}(\mathbb{E})$. La norme de $\mathbb{E}$ sera notée par $\|\cdot\|$ et notons $\mathbb{E}^*$ son dual. Soit $(\mathfrak{B}_n)_{n \geq 1}$ une suite décroissante de sous- σ -algèbres de $\mathcal{F}$. Notons que l'espace des fonctions $\mathcal{F}$ -mesurables et Pettis intégrables $P_E^1(\mathcal{F})$ peut être muni de l'une des topologies suivantes:

- La topologie de la norme de Pettis

$$\|g\|_{Pe} = \sup_{x^* \in B_{E^*}} \int_{\Omega} | \langle x^*, g \rangle | dP.$$

- La topologie induite par la dualité $(P_E^1(X), L^\infty \otimes \mathbb{E}^*)$. Rappelons qu'une suite $(g_n)_{n \geq 1}$ dans $P_E^1(X)$ converge vers g au sens de cette topologie, si pour chaque $h \in L^\infty(\mathcal{F})$ et

pour tout $x^* \in \mathbb{E}^*$, on a

$$\lim_{n \rightarrow +\infty} \int_{\Omega} h(\omega) \langle x^*, g_n(\omega) \rangle dP = \int_{\Omega} h(\omega) \langle x^*, g(\omega) \rangle dP.$$

Cette dernière topologie est connue sous le nom de la topologie faible de l'espace $P_E^1(\mathcal{F})$.

Il faut signaler que $\|\cdot\|_{P_E}$ et $\|\cdot\|_1$ sont équivalentes sur $L_{\mathbb{E}}^1$ si et seulement si $\mathbb{E}$ est de dimension finie (p. 14 dans [29]) et sur $P_{\mathbb{E}}^1$ la norme $\|\cdot\|_{P_E}$ est équivalente à la norme $g \mapsto \sup_{A \in \mathcal{F}} \|\int_A g dP\|$ ([75]).

La tribu d'Effros $\mathfrak{B}$ sur l'ensemble des parties convexes et fermées de E est engendrée par les parties de la forme

$$U^- = \{F \in cc(E) / F \cap U \neq \emptyset\},$$

dont U parcourt l'ensemble des ouverts de E . La mesurabilité de X est donc relative à cette tribu. On peut définir une tribu sur l'ensemble des parties convexes et faiblement compactes de $\mathbb{E}$, appelée la tribu trace de $\mathfrak{B}$ sur $cwk(\mathbb{E})$, cette tribu est engendrée par les parties de la forme $\{cwk(E) \cap B / B \in \mathfrak{B}\}$ et sera notée par ξ .

Soit $\Gamma : \Omega \rightarrow cc(\mathbb{E})$ une multifonction mesurable, la distribution de Γ sur l'espace mesurable $(cc(\mathbb{E}), \mathfrak{B})$ est définie par:

$$P_{\Gamma}(B) = P(\Gamma^{-1}(B)), \forall B \in \mathfrak{B}.$$

Deux multifonctions mesurables Γ and Δ sont dits indépendantes si, l'égalité suivante est satisfaite

$$P_{(\Gamma, \Delta)} = P_{\Gamma} \otimes P_{\Delta}.$$

Pour plus des détails sur l'indépendance des multifonctions nous renvoyons le lecteur aux travaux de ([52]).

Avant de présenter, le résultat principal de cette partie. Nous donnons une proposition qui démontre la linéarité de l'espérance conditionnelle d'un ensemble aléatoire Pettis intégrable.

Proposition 0.3 (*Proposition 3.3, p. 72*) Soient X et Y deux multifonctions mesurables et Pettis intégrables à valeurs dans $cwk(\mathbb{E})$. Soit $\mathfrak{B}$ une sous σ -algèbre de $\mathcal{F}$ telle que $E^{\mathfrak{B}}|X| \in$

$[0, +\infty[$ et $E^{\mathfrak{B}}|Y| \in [0, +\infty[$. Alors

$$E^{\mathfrak{B}}(X + Y) = E^{\mathfrak{B}}X + E^{\mathfrak{B}}Y \text{ p. s.}$$

En utilisant le lemme de factorisation de Doob et le théorème classique de transfert, nous prouvons le résultat suivant:

Théorème 0.14 (Theorem 3.3, p. 73) Soient X et Y dans $P_{cwk(\mathbb{E})}^1(\mathfrak{F})$, Z une variable aléatoire à valeurs dans un espace mesurable (Σ, Γ) telle que (Z, X) et (Z, Y) ont la même distribution. Si $E^{\sigma(Z)}|X| \in [0, +\infty[$ et $E^{\sigma(Z)}|Y| \in [0, +\infty[$. Alors

$$E^{\sigma(Z)}X(\omega) = E^{\sigma(Z)}Y(\omega) \text{ p.s.}$$

L'extension de ce dernier résultat est donnée en [17] pour les ensembles aléatoires convexes fermés Bochner intégrables et dans [99] pour les ensembles aléatoires indépendants identiquement distribués toujours Bochner intégrable.

Nous aurions également besoin du théorème classique suivant ([75]).

Théorème 0.15 (Theorem 3.4, p. 75) Soient Φ_1 et Φ_2 deux sous σ -algèbres de $\mathcal{F}$, $\Phi = \sigma(\Phi_1, \Phi_2)$ est la σ -algèbre engendrée par Φ_1 et Φ_2 et soit f_1 une fonction réelle intégrable et mesurable. Si f_1 et Φ_1 sont indépendantes de Φ_2 . Alors

$$E^{\Phi}f_1(\omega) = E^{\Phi_1}f_1(\omega) \text{ p.s.}$$

Maintenant, nous présentons le théorème principal de cette partie:

Théorème 0.16 (Theorem 3.5, p. 75) Soit $(X_n)_{n \geq 1}$ une suite de multifonctions mesurables et indépendantes dans $P_{cwk(\mathbb{E})}^1(\mathfrak{F})$.

Soient $S_n = X_1 + X_2 + X_3 + \dots + X_n$, $\mathfrak{B}_n = \sigma(S_n, X_{n+1}, X_{n+2}, \dots)$, $\mathfrak{B}_\infty = \bigcap_{n \geq 1} \mathfrak{B}_n$ telle que les conditions suivantes soient satisfaites:

(i) $\forall n, \forall j \in \{1, 2, \dots, n\}$, (S_n, X_1) et (S_n, X_j) ont même loi.

(ii) $\forall n, \forall j \in \{1, 2, \dots, n\}$, $E^{\mathfrak{B}_n}|X_j| \in [0, +\infty[$.

Alors, nous avons les affirmations suivantes:

1. $\forall n, \forall j \in \{1, 2, \dots, n\}, E^{\sigma(S_n)} X_j(\cdot) = E^{\sigma(S_n)} X_1(\cdot) \text{ p.s.}$
2. $E^{\mathfrak{B}_n} X_1(\cdot) = E^{\sigma(S_n)} X_1(\cdot) \text{ p.s.}$
3. $M - \lim_{n \rightarrow \infty} \frac{S_n}{n} = M - \lim_{n \rightarrow \infty} E^{\mathfrak{B}_n} X_1(\cdot) = E^{\mathfrak{B}_\infty} X_1(\cdot) = \int X_1(\cdot) dP \text{ p.s.}$

0.5 Applications de loi forte des grands nombres

Les méthodes de Monte-Carlo constituent une application très riche de la loi des grands nombres. Ces méthodes se caractérisent par l'utilisation du hasard ou des procédés aléatoires pour résoudre des problèmes centrés sur le calcul d'une valeur numérique.

L'idée de la méthode Monte-Carlo est de mettre la quantité recherchée sous la forme d'une espérance de variable aléatoire, c'est souvent simple (calcul d'intégrale par exemple) mais peut être plus compliqué (équations différentielles stochastiques). A l'issue de cette étape, il reste à calculer une quantité de la forme $E(X)$, c'est à dire l'espérance d'une variable aléatoire X . Pour calculer $E(X)$, il convient de savoir simuler une variable aléatoire selon la loi de X . On dispose alors d'une suite $(x_i)_{i \geq 1}$ de N réalisations de la variable aléatoire X , puis d'approximer cette quantité en utilisant la loi forte des grands nombres. Nous pouvons résumer cette méthode en trois étapes, premièrement nous écrivons la solution de problème traité sous la forme d'une espérance d'une variable aléatoire simple à simuler. Ensuite, nous devons calculer une quantité de la forme $E(X)$ où X est une variable aléatoire. Nous supposons que l'on sait simuler une suite de réalisations $(x_i)_i$ de X indépendantes suivant la même loi que X . Nous simulons donc cette suite, puis nous approximations $E(X)$ par :

$$\frac{1}{n}(x_1 + x_2 + \dots + x_n).$$

La qualité de cette approximation dépend directement de σ^2 , la variance de la quantité à calculer. Afin d'améliorer cette approximation, plusieurs méthodes dites de réduction de variance se sont développées. Trois d'entre eux sont les plus utilisées; la méthode de l'échantillonnage préférentiel, la méthode de la variable de contrôle et la méthode des variables antithétiques.

Dans le but de comparer ces trois méthodes, nous proposons un exemple d'application dont nous souhaitons estimer la quantité C suivante:

$$C = E[(e^G - 1)_+].$$

Où G est une variable aléatoire de loi normale centrée réduite. La quantité C provient de la formule du prix de l'option à l'équilibre, voir la formule (4.6) la section 4.4. Nous calculons cette quantité 1000 fois avec les trois méthodes de réduction de variance ainsi qu'avec la méthode de Monte-Carlo sans réduction de variance. Avec les résultats obtenus nous traçons le boxplot comparatif des différentes méthodes.

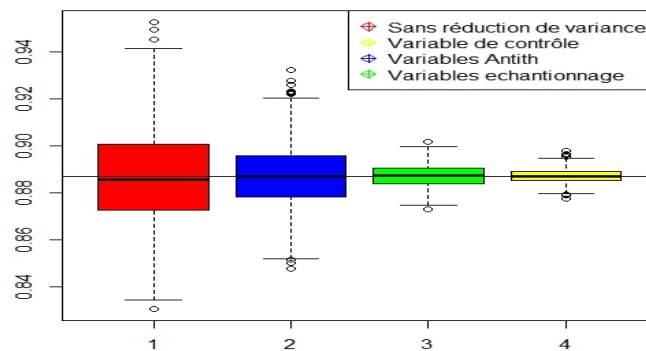


Figure 1: Boxplot comparatif des différentes méthodes

Nous voyons sur ce boxplot l'intérêt d'utiliser une méthode de réduction de variance, en effet, en utilisant l'une de ces méthodes nos calculs de C sont beaucoup plus précis.

General introduction

Let $(X_n)_{n \geq 1}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and put $S_n = \sum_{i=1}^n X_i$ for $n \geq 1$. Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be sequences of real constants such that $(b_n)_{n \geq 1}$ is an increasing and positive sequence that converges to ∞ . We say that the sequence $(X_n)_{n \geq 1}$ satisfies the strong law of large numbers (SLLN) with centering constants $(a_n)_{n \geq 1}$ and norming constants $(b_n)_{n \geq 1}$ if

$$\frac{S_n - a_n}{b_n} \longrightarrow 0 \text{ a.s.} \quad (2)$$

The strong law of large numbers has been extensively studied in the literature, for its effectiveness it continues to play an important role in probability theory, especially in probability limit theorems and in statistics, areas of economics and insurance.

Generally, we may distinguish two types of strong laws of large numbers. The first one is the strong law for independent identically distributed (i.i.d.) random variables, and the second is for independent random variables with the same mean and some L^p -norm conditions. The first type of strong law is generally valid for Banach space valued random variables. On the other hand, the second type of strong law for Banach space valued random variables is not necessarily valid and is closely connected with some geometric conditions of a Banach space.

There are two famous SLLNs for a sequence of independent identically distributed random variables: The Kolmogorov's SLLN and Marcinkiewicz–Zygmund SLLN ($p = 1$ and $1 < p < 2$, respectively) state that $E|X_1|^p < \infty$ if and only if the relation (2) holds with $a_n = ES_n$ and $b_n = n^{\frac{1}{p}}$. For a sequence of independent and not necessarily identically

distributed random variables, Kolmogorov states that $\sum_{n=1}^{\infty} \frac{\text{Var}(X_n)}{b_n^2} < \infty$ implies (2) with $a_n = ES_n$.

([34], [35]) showed in an elementary way that in the classical SLLN of Kolmogorov, the condition "independent and identically distributed (i.i.d) random variables" can be replaced with the condition "pairwise independent and identically distributed random variables". [27] proved, by a novel extension of arguments of [34], an analogue of the Kolmogorov's SLLN for pairwise independent random variables. On the other hand, [66] have obtained the SLLN for a certain class of uniformly integrable random variables which are also pairwise independent.

Over the last years, many authors have presented SLLN discarding the independence assumption of the random variables (see [21], [71], [87], [69]), [28], [65] among others).

[23], [10], [37], [49], [50], [48] showed various generalizations of the SLLN, where the sequence $(X_n)_{n \geq 1}$ is a random adapted sequence with values in Banach space instead of real random variables.

In recent years, the theory of random sets has been extensively studied because is both important theoretical (since they generalize random variables and random vectors) and practical (since they represent geometrical objects in certain models of growth). ([88], [89]) appeared to be the first author who provided the concept of random sets, and his early works investigated the relationships between random sets and geometric probabilities. Later, [63] and [73] provided a comprehensive mathematical theory of random sets which was greatly influenced by the geometric probability prospective. Their proposed framework exerted a strong influence on the limit theorems developed in recent decades.

With the development of the theory and application of random sets (see. [9], [62], [68], [73], [18], [52], [53], [38], [55]), several variants of SLLNs were built. Among them. A multivalued strong law of large numbers was first proved by [7] for independent, identically distributed (i.i.d) random variables whose values are compact subsets of $\mathbb{R}^d$. This strong law with the Hausdorff distance convergence has been extended by several authors (see [26], [51], [60] and [40]) and was recently completed in [5], [85] and [6].

With the development of the Pettis integral theory which is more general than de Bochner integral and the recent proof of the existence of the conditional expectation of random vectors

and random sets provided by several authors (see. [79], [55], [38], [4], [33], [1] and [2]). It is interesting to state SLLN in Pettis integration.

The aim of this work is to propose a new results of the strong law of large numbers whether in the case of Bochner and in the case of Pettis integration, and to present some of its applications. In the first, the relation between the SLLN and the adapted sequences is given, we also, prove some convergence of conditional expectation Pettis integrable. Finally, we propose a new result of strong law of large numbers for random sets Pettis integrable. In this context several auxiliary results are showed.

Our Contribution is organized as follows:

Chapitre 1- SLLN for adapted sequences and application to multivalued supermartingale: In this chapter, we propose a new result of the SLLN for an adapted sequences. And in the last section of this chapter we establish the relation between SLLN and amart. Finally we prove the SLLN for mils and set valued supermartingales.

Chapitre 2- Mosco convergence of conditional expectation for sequences of Pettis integrable random sets: In this chapter, we prove the convergence of the conditional expectation for sequences of Pettis-integrable vector valued random variables. We state a new version of type Levy theorem for a decreasing sequence of sub- σ -algebras and dominated convergence theorem in Pettis integration. At the last section of this chapter, we present the almost sure Mosco convergence of Pettis-integrable random sets $(E^{\mathfrak{B}_n}X)_{n \geq 1}$ and $(E^{\mathfrak{B}_n}X_n)_{n \geq 1}$, where $(\mathfrak{B}_n)_{n \geq 1}$ is a decreasing sequence of sub- σ -algebras of $\mathfrak{F}$ and $(X_n)_{n \geq 1}$ as well as X are the convex and weakly compact Pettis-integrable random sets.

Chapitre 3- SLLN for Pettis integrable random sets: In this chapter, by using the property of Pettis integrable reversed martingale stated in chapter 2, we prove the SLLN for Pettis-integrable multifunctions with convex weakly compact values. Which generalise the concept of Bochner integrable random sets.

Chapitre 4- Simulation and Monte Carlo Methods: In this chapter, we present the Monte Carlo method and its application to evaluate option pricing. At the last section of this chapter, the addresses issues of simulating random sets and application to reliability analysis are presented.

SLLN for adapted sequences and application to multivalued supermartingale

1.1 Introduction

There are two main types of strong law of large numbers (SLLN) for random variables. The first one deals with independent and identically distributed random variables. See for example [34], [67], [44], [83]. And the second is concerned with independent random variables with some additional conditions (see. [23], [37], [77], [22]) and so on. In this chapter, we are interested in second type of strong law of large numbers. This chapter is organized as follows.

In section 2, we recall some definitions and results that will be used after. In section 3, we prove the SLLN for an adapted sequence which extends SLLN for martingale. The last section of this chapter establishes the relation between SLLN and amart and provides the SLLN for mils and set valued supermartingales.

1.2 Notations and preliminaries

Throughout this paper, we assume that $(\Omega, \mathcal{F}, P)$ is a probability space, $(\mathcal{F}_n)_{n \geq 1}$ is an increasing sequence of sub- σ -algebras of $\mathcal{F}$, $(C_n)_{n \geq 1}$ is a decreasing positive real sequence that converges to 0. Let $\mathbb{E}$ be a Banach space, we denote by $ccb(\mathbb{R})$ (resp. $ccb(\mathbb{E})$) the family

of all nonempty closed convex and bounded subsets of $\mathbb{R}$ (resp. closed convex and bounded subsets of $\mathbb{E}$). $L_{\mathbb{R}}^1(\mathcal{F})$ (resp. $L_{ccb(\mathbb{R})}^1(\mathcal{F})$) is the set of all $\mathcal{F}$ -measurable and integrable random variable from Ω to $\mathbb{R}$ (resp. is the set of all $\mathcal{F}$ -measurable and integrably bounded random set from Ω to $ccb(\mathbb{R})$). Let us recall some definitions and results that will be used in the sequel.

Definition 1.1 Let $(X_n)_{n \geq 1}$ be a sequence of random variables defined from Ω to $\mathbb{R}$, let $(\mathcal{F}_n)_{n \geq 1}$ be an increasing sequence of sub- σ -algebras of $\mathcal{F}$. $(X_n, \mathcal{F}_n)_{n \geq 1}$ is an adapted sequence if, for all $n \geq 1$, X_n is $\mathcal{F}_n$ -measurable.

Definition 1.2 An adapted sequence $(X_n, \mathcal{F}_n)_{n \geq 1}$, such that, for all $n \geq 1$, $X_n \in L_{\mathbb{R}}^1(\mathcal{F})$ is a martingale (resp. supermartingale, resp. submartingale) if, for all $n \geq 1$, $E^{\mathcal{F}_n} X_{n+1} = X_n$ a.s. (resp. for all $n \geq 1$, $E^{\mathcal{F}_n} X_{n+1} \leq X_n$ a.s., resp. for all $n \geq 1$, $E^{\mathcal{F}_n} X_{n+1} \geq X_n$ a.s.)

Definition 1.3 An adapted sequence $(X_n, \mathcal{F}_n)_{n \geq 1}$ in $L_{\mathbb{R}}^1(\mathcal{F})$ is an amart, if

$$\lim_{\sigma} \sup_{\tau \geq \sigma} \int_{\Omega} |E^{\mathcal{F}_{\sigma}} X_{\tau} - X_{\sigma}| dP = 0, \quad \tau, \sigma \in \Gamma.$$

Where $\mathcal{F}_{\sigma} = \{A \in \mathcal{F} / A \cap [\sigma = n] \in \mathcal{F}_n, \forall n \geq 1\}$.

Definition 1.4 An adapted sequence $(X_n, \mathcal{F}_n)_{n \geq 1}$ in $L_{\mathbb{R}}^1(\mathcal{F})$ is a mil, if

$$\lim_m \sup_{n \geq m} |E^{\mathcal{F}_m} X_n - X_m| = 0 \text{ a.s.}$$

Definition 1.5 ([19])

- A set valued function $X : \Omega \rightarrow 2^{\mathbb{E}} \setminus \{\emptyset\}$ is measurable, if for every open subset U of $\mathbb{E}$

$$X^{-1}(U) = \{\omega \in \Omega / X(\omega) \cap U \neq \emptyset\}$$

is an element of $\mathcal{F}$. A measurable multifunction is called a random set.

- A measurable function f from Ω to $\mathbb{E}$ is called a selection of X , if $f(\omega) \in X(\omega)$ for any $\omega \in \Omega$. We denote by $S_X(\mathcal{F})$ the set of all $\mathcal{F}$ -measurable selection of X . It is well known that every measurable random set X with values in $cc(\mathbb{E})$ admits at least one

measurable selection. Furthermore, a function $X : \Omega \rightarrow cc(\mathbb{E})$ is measurable, if, there is a countable family of measurable selections $(f_n)_{n \geq 1}$ such that, for each $\omega \in \Omega$

$$X(\omega) = \overline{\{f_n(\omega), n \in \mathbf{N}\}},$$

where the closure is taken with respect to the norm in $\mathbb{E}$.

- Let $C \in 2^{\mathbb{E}} \setminus \{\emptyset\}$, the support function and radius of C are defined as follows:

$$s^*(x^*, C) = \sup_{x \in C} \langle x^*, x \rangle,$$

$$|C| = \sup_{x \in C} \|x\|.$$

- A random set X is integrably bounded, if $|X(\cdot)|$ is integrable.
- A random set with values in $cc(\mathbb{E})$ is integrable if $S_X^1 \neq \emptyset$, the Aumann integral of X is defined by

$$\int_A X dP = \left\{ \int_A f dP, f \in S_X^1(\mathcal{F}) \right\}.$$

Definition 1.6 A sequence $(X_n)_{n \geq 1} \in L_{ccb(\mathbb{R})}^1(\mathcal{F})$ is a multivalued supermartingale (resp. submartingale multivoque) if, $X_n \in L_{ccb(\mathbb{R})}^1(\mathcal{F}_n)$ and $E^{\mathcal{F}_n} X_{n+1}(\omega) \subseteq X_n(\omega)$ (resp. $E^{\mathcal{F}_n} X_{n+1}(\omega) \supseteq X_n(\omega)$) for all $n \geq 1$ and almost surely $\omega \in \Omega$.

Definition 1.7 ([14])

Let $(A_n)_{n \geq 1}$ be nonempty sets in $\mathbb{E}$.

$$s - li A_n = \{x \in E, x_n \longrightarrow x, x_n \in A_n, \forall n \geq 1\}.$$

$$w - ls A_n = \{y \in E, y_k \xrightarrow{w} y, y_k \in A_{n_k}, \forall k \geq 1\},$$

where $(A_{n_k})_{k \geq 1}$ is a subset of $(A_n)_{n \geq 1}$.

One of the first classical results on the SLLN is Kolmogorov's strong law of large numbers

Theorem 1.1 ([64]) *Let $(X_n)_{n \geq 1}$ be a sequence of independent and identically distributed random variables with values in $\mathbb{R}$, such that $E(X_1) < \infty$. Then*

$$\lim_{n \rightarrow +\infty} \frac{S_n}{n} = E(X_1) \text{ a.s.}$$

The extension of the Theorem 1.1 in the case of random variables independent but not identically distributed is given by the following theorem

Theorem 1.2 *Let $(X_n)_{n \geq 1}$ be a sequence of independent random variables with values in $\mathbb{R}$ such that $E(X_i) = 0$ and $E(X_i^2) \leq M \ \forall i \geq 1$ et $M \in \mathbb{R}$. Then*

$$\lim_{n \rightarrow +\infty} \frac{S_n}{n} = 0 \text{ a.s.}$$

[25] proved a SLLN under the condition

$$\sum_{k=1}^{+\infty} \frac{E|X_k|^{2\alpha}}{k^{\alpha+1}} < \infty,$$

Theorem 1.3 ([25]) *Let $(X_n)_{n \geq 1}$ be a sequence of independent random variables with values in $\mathbb{R}$ such that $E(X_i) = 0 \ \forall i \geq 1$ and*

$$\sum_{k=1}^{+\infty} \frac{E|X_k|^{2\alpha}}{k^{\alpha+1}} < \infty,$$

for some $\alpha \geq 1$. Then

$$\lim_{n \rightarrow +\infty} \frac{S_n}{n} = 0 \text{ a.s.}$$

[23] proved the following inequality

Proposition 1.1 ([23]) *Let $(X_k, \mathcal{F}_k)_{k \geq 1}$ be a submartingale in $L^1_{\mathbb{R}}$ and $(C_k)_{k \geq 1}$ be a positive decreasing sequence to 0, then for all $\varepsilon > 0$ and $m \geq 1$, we have*

$$\varepsilon P\left[\max_{1 \leq k \leq m} C_k X_k \geq \varepsilon\right] \leq C_1 E(X_1^+) + \sum_{k=2}^m C_k E(X_k^+ - X_{k-1}^+),$$

where $X_i^+ = \max(X_i, 0)$.

Using the same Chow's techniques, we prove an another version of the previous proposition that will be used in the proof of the main result.

Proposition 1.2 *Let $(X_k, \mathcal{F}_k)_{k \geq 1}$ be a martingale in $L^1_{\mathbb{R}}$ and $(C_k)_{k \geq 1}$ be a positive decreasing sequence to 0, then for all $\varepsilon > 0$, $\alpha > 1$ and $n \geq 1$, we have*

$$\varepsilon P[\max_{k \geq n} C_k^{2\alpha} |X_k|^{2\alpha} \geq \varepsilon] \leq C_n^{2\alpha} E|X_n|^{2\alpha} + \sum_{k=n+1}^{\infty} C_k^{2\alpha} E(|X_k|^{2\alpha} - |X_{k-1}|^{2\alpha}),$$

Proof Let $A = \{\max_{k \geq n} C_k^{2\alpha} |X_k|^{2\alpha} \geq \varepsilon\}$,

and $A_k = \{C_k^{2\alpha} |X_k|^{2\alpha} \geq \varepsilon \text{ and for all } j > k; C_j^{2\alpha} |X_j|^{2\alpha} < \varepsilon\}$, then $A = \bigcup_{k \geq n} A_k$

Therefore

$$\begin{aligned} \varepsilon P(A) &= \varepsilon \sum_{k=n}^{\infty} P(A_k) \leq \sum_{k=n}^{\infty} C_k^{2\alpha} \int_{A_k} |X_k|^{2\alpha} dP \\ &\leq C_n^{2\alpha} \int_{A_n} |X_n|^{2\alpha} dP + \sum_{k=n+1}^{\infty} C_k^{2\alpha} \int_{A_k} |X_k|^{2\alpha} dP \\ &\leq C_n^{2\alpha} E|X_n|^{2\alpha} - C_n^{2\alpha} \int_{\Omega \setminus A_n} |X_n|^{2\alpha} dP + C_{n+1}^{2\alpha} \int_{A_{n+1}} |X_{n+1}|^{2\alpha} dP + \sum_{k=n+2}^{\infty} C_k^{2\alpha} \int_{A_k} |X_k|^{2\alpha} dP. \end{aligned}$$

On the other hand, we have

$$\Omega \setminus A_n = A_{n+1} \cup \Omega \setminus A_n \cup A_{n+1}$$

Then

$$\begin{aligned} \varepsilon P(A) &\leq C_n^{2\alpha} E|X_n|^{2\alpha} - C_n^{2\alpha} \int_{\Omega \setminus A_n} |X_n|^{2\alpha} dP + C_{n+1}^{2\alpha} \int_{\Omega \setminus A_n} |X_{n+1}|^{2\alpha} dP - \\ &\quad - C_{n+1}^{2\alpha} \int_{\Omega \setminus A_n \cup A_{n+1}} |X_{n+1}|^{2\alpha} dP + \sum_{k=n+2}^{\infty} C_k^{2\alpha} \int_{A_k} |X_k|^{2\alpha} dP. \\ &\leq C_n^{2\alpha} E|X_n|^{2\alpha} + C_{n+1}^{2\alpha} \int_{\Omega \setminus A_n} |X_{n+1}|^{2\alpha} - |X_n|^{2\alpha} dP - \\ &\quad - C_{n+1}^{2\alpha} \int_{\Omega \setminus A_n \cup A_{n+1}} |X_{n+1}|^{2\alpha} dP + \sum_{k=n+2}^{\infty} C_k^{2\alpha} \int_{A_k} |X_k|^{2\alpha} dP. \\ &\leq C_n^{2\alpha} E|X_n|^{2\alpha} + C_{n+1}^{2\alpha} E(|X_{n+1}|^{2\alpha} - |X_n|^{2\alpha}) - C_{n+1}^{2\alpha} \int_{A_n} |X_{n+1}|^{2\alpha} - |X_n|^{2\alpha} dP - \\ &\quad - C_{n+1}^{2\alpha} \int_{\Omega \setminus A_n \cup A_{n+1}} |X_{n+1}|^{2\alpha} dP + \sum_{k=n+2}^{\infty} C_k^{2\alpha} \int_{A_k} |X_k|^{2\alpha} dP. \end{aligned}$$

And by the submartingale propriety, one has $\int_{A_n} |X_{n+1}|^{2\alpha} - |X_n|^{2\alpha} dP \geq 0$ then

$$\begin{aligned} \varepsilon P(A) &\leq C_n^{2\alpha} E|X_n|^{2\alpha} + C_{n+1}^{2\alpha} E(|X_{n+1}|^{2\alpha} - |X_n|^{2\alpha}) - C_{n+1}^{2\alpha} \int_{\Omega \setminus A_n \cup A_{n+1}} |X_{n+1}|^{2\alpha} dP \\ &\quad + C_{n+2}^{2\alpha} \int_{A_{n+2}} |X_{n+2}|^{2\alpha} dP + \sum_{k=n+3}^{\infty} C_k^{2\alpha} \int_{A_k} |X_k|^{2\alpha} dP. \end{aligned}$$

And since

$$\Omega \setminus A_n \cup A_{n+1} = A_{n+2} \cup \Omega \setminus A_n \cup A_{n+1} \cup A_{n+2}.$$

Therefore

$$\begin{aligned} \varepsilon P(A) &\leq C_n^{2\alpha} E|X_n|^{2\alpha} + C_{n+1}^{2\alpha} E(|X_{n+1}|^{2\alpha} - |X_n|^{2\alpha}) - C_{n+2}^{2\alpha} \int_{\Omega \setminus A_n \cup A_{n+1}} |X_{n+1}|^{2\alpha} dP + \\ &\quad + C_{n+2}^{2\alpha} \int_{\Omega \setminus A_n \cup A_{n+1}} |X_{n+2}|^{2\alpha} dP - C_{n+2}^{2\alpha} \int_{\Omega \setminus A_n \cup A_{n+1} \cup A_{n+2}} |X_{n+2}|^{2\alpha} dP + \sum_{k=n+3}^{\infty} C_k^{2\alpha} \int_{A_k} |X_k|^{2\alpha} dP. \end{aligned}$$

Then

$$\begin{aligned} \varepsilon P(A) &\leq C_n^{2\alpha} E|X_n|^{2\alpha} + C_{n+1}^{2\alpha} E(|X_{n+1}|^{2\alpha} - |X_n|^{2\alpha}) + C_{n+2}^{2\alpha} E(|X_{n+2}|^{2\alpha} - |X_{n+1}|^{2\alpha}) \\ &\quad - C_{n+2}^{2\alpha} \int_{A_n \cup A_{n+1}} |X_{n+2}|^{2\alpha} - |X_{n+1}|^{2\alpha} dP - C_{n+3}^{2\alpha} \int_{\Omega \setminus A_n \cup A_{n+1} \cup A_{n+2}} |X_{n+2}|^{2\alpha} dP + \\ &\quad + C_{n+3}^{2\alpha} \int_{A_{n+3}} |X_{n+3}|^{2\alpha} dP + \sum_{k=n+4}^{\infty} C_k^{2\alpha} \int_{A_k} |X_k|^{2\alpha} dP. \end{aligned}$$

And by the submartingale propriety, one has $C_{n+2}^{2\alpha} \int_{A_n \cup A_{n+1}} |X_{n+2}|^{2\alpha} - |X_{n+1}|^{2\alpha} dP \geq 0$ and since

$$\Omega \setminus A_n \cup A_{n+1} \cup A_{n+2} = A_{n+3} \cup \Omega \setminus A_n \cup A_{n+1} \cup A_{n+2} \cup A_{n+3}.$$

Then

$$\begin{aligned} \varepsilon P(A) &\leq C_n^{2\alpha} E|X_n|^{2\alpha} + C_{n+1}^{2\alpha} E(|X_{n+1}|^{2\alpha} - |X_n|^{2\alpha}) + C_{n+2}^{2\alpha} E(|X_{n+2}|^{2\alpha} - |X_{n+1}|^{2\alpha}) + \\ &\quad C_{n+3}^{2\alpha} E(|X_{n+3}|^{2\alpha} - |X_{n+2}|^{2\alpha}) - C_{n+3}^{2\alpha} \int_{\Omega \setminus A_n \cup A_{n+1} \cup A_{n+2} \cup A_{n+3}} |X_{n+3}|^{2\alpha} dP + \sum_{k=n+4}^{\infty} C_k^{2\alpha} \int_{A_k} |X_k|^{2\alpha} dP. \end{aligned}$$

Proceeding in the same manner, we obtain the desired result

$$\varepsilon P[\max_{k \geq n} C_k^{2\alpha} |X_k|^{2\alpha} \geq \varepsilon] \leq C_n^{2\alpha} E|X_n|^{2\alpha} + \sum_{k=n+1}^{\infty} C_k^{2\alpha} E(|X_k|^{2\alpha} - |X_{k-1}|^{2\alpha}).$$

[23] extended the SLLN of [25] to martingale and furthermore C_n instead of $\frac{1}{n}$, where $(C_n)_{n \geq 1}$ is a positive decreasing sequence such that $\lim_{n \rightarrow +\infty} C_n = 0$.

Theorem 1.4 ([23]) *Let $(X_n)_{n \geq 1}$ be a sequence of random variables with values in $\mathbb{R}$ such that $(S_n = \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$ is a martingale and let $(C_n)_{n \geq 1}$ be a positive decreasing sequence such that $\lim_{n \rightarrow +\infty} C_n = 0$. Let A, α et β such that $\alpha > 1, 2\alpha \geq \beta > 0$ and $A > 0$. If for every $i \geq i_0$ one has:*

- (1) $E|S_i|^{2\alpha} \leq A.E(\sum_{k=1}^i X_k^2)^\alpha$.
- (2) $i^{\alpha-1} C_i^{2\alpha-\beta} \leq A, \sum_{k=i}^{\infty} C_k^{2\alpha} k^{\alpha-2} \leq A C_i^\beta$.
- (3) $\sum_{k=1}^{+\infty} C_k^\beta E|X_k|^{2\alpha} < \infty$.

Then

$$\lim_{n \rightarrow +\infty} C_n S_n = 0 \text{ a.s.}$$

[10] proved an another version of theorem 1.3. This result can be seen as a special case of the theorem 1.4 ($C_n = \frac{1}{n}$ and $\beta = \alpha + 1$).

Theorem 1.5 *Let $(X_n)_{n \geq 1}$ be a sequence of random variables with values in $\mathbb{R}$ such that $(S_n = \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$ is a martingale. If the following condition is satisfied*

$$\sum_{k=1}^{+\infty} \frac{E|X_k|^{2\alpha}}{k^{\alpha+1}} < \infty.$$

Then

$$\lim_{n \rightarrow +\infty} \frac{S_n}{n} = 0 \text{ a.s.}$$

[15] proved the following proposition

Proposition 1.3 ([15])

Let $1 < \alpha < \infty$, then there exists two positive constants A_α and B_α such that, for all integrable martingale $(X_n, \mathcal{F}_n)_{n \geq 1}$ in $L^1_{\mathbb{R}}(\mathcal{F})$, we have

$$A_\alpha E\left(\sum_{k=1}^n d_k^2\right)^{\frac{\alpha}{2}} \leq E|X_n|^\alpha \leq B_\alpha E\left(\sum_{k=1}^n d_k^2\right)^{\frac{\alpha}{2}},$$

where $d_k = X_k - X_{k-1}$ and $d_1 = X_1$.

Before giving our main results, we first recall the following lemma, which will be used later on.

Lemma 1.1 ([70])

If $(x_n)_{n \geq 1}$ is a real sequence such that the series $\sum x_n$ converge to l finite and $(b_n)_{n \geq 1}$ is a positive increasing sequence to $+\infty$. Then

$$\lim_{n \rightarrow +\infty} \frac{1}{b_n} \sum_{k=1}^n b_k x_k = 0.$$

Proof Let $S_n = \sum_{k=1}^n x_k$ et $S = \sum_{k=1}^\infty x_k$, then there exists an integer N such that, for every $k \geq N$, $|S_n - S| < \varepsilon$.

And by the Abel's transformation, we have

$$\begin{aligned} \frac{1}{b_n} \sum_{k=1}^n b_k x_k &= S_n - \frac{1}{b_n} \sum_{k=1}^{n-1} (b_{k+1} - b_k) S_k \\ &= S_n - \frac{1}{b_n} \sum_{k=1}^{N-1} (b_{k+1} - b_k) S_k - \frac{1}{b_n} \sum_{k=N}^{n-1} (b_{k+1} - b_k) S_k \\ &= S_n - \frac{1}{b_n} \sum_{k=1}^{N-1} (b_{k+1} - b_k) S_k - \frac{1}{b_n} \sum_{k=N}^{n-1} (b_{k+1} - b_k) S - \frac{1}{b_n} \sum_{k=N}^{n-1} (b_{k+1} - b_k) (S_k - S) \\ &= S_n - \frac{1}{b_n} \sum_{k=1}^{N-1} (b_{k+1} - b_k) S_k - \frac{S}{b_n} (b_n - b_N) + \frac{1}{b_n} \sum_{k=N}^{n-1} (b_{k+1} - b_k) (S - S_k) \\ &= (S_n - S) + \frac{1}{b_n} [b_N S - \sum_{k=1}^{N-1} (b_{k+1} - b_k) S_k] + \frac{1}{b_n} \sum_{k=N}^{n-1} (b_{k+1} - b_k) (S - S_k) \end{aligned}$$

And since, when n tends to infinity, the three above terms converge to 0. Therefore

$$\left| \frac{1}{b_n} \sum_{k=1}^n b_k x_k \right| \longrightarrow 0.$$

Lemma 1.2 ([31]) Let $(X_k, \mathcal{F}_k)_{k \geq 1}$ be an adapted sequence in $L^1_{\mathbb{R}}(\mathcal{F})$. There exists two adapted sequences $(Y_k, \mathcal{F}_k)_{k \geq 1}$ and $(Z_k, \mathcal{F}_k)_{k \geq 1}$ in $L^1_{\mathbb{R}}(\mathcal{F})$ such that, for all $k \geq 1$, $X_k = Y_k + Z_k$ where $(Y_k, \mathcal{F}_k)_{k \geq 1}$ is a martingale and Z_k is $\mathcal{F}_{k-1}$ -measurable for all $k \geq 2$ and $Z_1 = 0$. $(Z_k, \mathcal{F}_k)_{k \geq 1}$ is named predictable sequence.

Lemma 1.3 ([57]) Let $\mathbb{E}$ be a Banach space, any multivalued supermartingale $(X_n)_{n \geq 1}$ with values in $cc(\mathbb{E})$ admits at least a martingale selection (i.e. there exists a function f_n such that $E^{\mathcal{F}_n} f_{n+1} = f_n$ and $f_n(\omega) \in X_n(\omega)$, for all $n \geq 1$ and almost surely $\omega \in \Omega$).

1.3 Strong law of large numbers for adapted sequences

The following result proves SLLN for adapted sequence, which extends the Theorem 1.4.

It is the main result in this section.

Theorem 1.6 Let $(X_n)_{n \geq 1}$ be a sequence of random variable with values in $\mathbb{R}$. Assume that $(S_n = \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$ is an adapted sequence and $(C_n)_{n \geq 1}$ is a positive decreasing sequence such that $\lim_{n \rightarrow +\infty} C_n = 0$. Let A , α and β such that $\alpha > 1$, $2\alpha \geq \beta > 0$ and $A > 0$. If the following conditions are satisfied

$$(i) \quad i^{\alpha-1} C_i^{2\alpha-\beta} \leq A, \quad \sum_{k=i}^{\infty} C_k^{2\alpha} k^{\alpha-2} \leq A C_i^{\beta}, \quad \forall i \geq 1.$$

$$(ii) \quad \sum_{k=1}^{+\infty} C_k^{\beta} E|X_k|^{2\alpha} < \infty.$$

$$(iii) \quad \sum_{k=1}^{+\infty} C_k (E^{\mathcal{F}_k} S_{k+1} - S_k) < \infty.$$

Then

$$\lim_{n \rightarrow +\infty} C_n S_n = 0 \quad a.s.$$

Proof Since $(S_n, \mathcal{F}_n)_{n \geq 1}$ is an adapted sequence, then by [31], there exists a martingale $(Y_n, \mathcal{F}_n)_{n \geq 1}$ and a predictable sequence $(Z_n, \mathcal{F}_n)_{n \geq 1}$, such that,

$$\forall n \geq 1, S_n = Y_n + Z_n,$$

where, $S_1 = Y_1$, $Z_1 = 0$.

On the other hand,

$$Y_{n+1} - Y_n = Y_{n+1} - E^{\mathcal{F}_n}(Y_{n+1}) = S_{n+1} - Z_{n+1} - E^{\mathcal{F}_n}(S_{n+1} - Z_{n+1}).$$

Moreover $(Z_n)_{n \geq 1}$ is a predictable sequence, so

$$E^{\mathcal{F}_n}(Z_{n+1}) = Z_{n+1},$$

therefore

$$Y_{n+1} - Y_n = S_{n+1} - E^{\mathcal{F}_n}(S_{n+1}).$$

Proceeding in the same manner, we have

$$Z_{n+1} - Z_n = E^{\mathcal{F}_n}(Z_{n+1}) - Z_n = E^{\mathcal{F}_n}(S_{n+1} - Y_{n+1}) - (S_n - Y_n),$$

by using the fact that $(Y_n)_{n \geq 1}$ is a martingale, we obtain

$$Z_{n+1} - Z_n = E^{\mathcal{F}_n}(S_{n+1}) - S_n.$$

- We prove that $\lim_{n \rightarrow +\infty} C_n Y_n = 0$ a.s.

Since $(|Y_k|, \mathcal{F}_k)_{k \geq 1}$ is a submartingale in $L^1_{\mathbb{R}}(\mathcal{F})$, then by [23], we have

$$\varepsilon P\left[\max_{1 \leq k \leq m} C_k |Y_k| \geq \varepsilon\right] \leq C_1 E|Y_1| + \sum_{k=2}^m C_k E(|Y_k| - |Y_{k-1}|).$$

On the other hand $(|Y_k|^{2\alpha}, \mathcal{F}_k)_{k \geq 1}$ is a submartingale in $L^1_{\mathbb{R}}(F)$, then

$$\begin{aligned} \varepsilon^{2\alpha} P\left[\sup_{k \geq n} C_k |Y_k| \geq \varepsilon\right] &= \varepsilon^{2\alpha} P\left[\sup_{k \geq n} C_k^{2\alpha} |Y_k|^{2\alpha} \geq \varepsilon^{2\alpha}\right] \\ &\leq C_n^{2\alpha} E|Y_n|^{2\alpha} + \sum_{k=n+1}^{\infty} C_k^{2\alpha} E(|Y_k|^{2\alpha} - |Y_{k-1}|^{2\alpha}) \end{aligned}$$

So, we have to prove that

$$\lim_{n \rightarrow +\infty} C_n^{2\alpha} E|Y_n|^{2\alpha} + \sum_{k=n+1}^{\infty} C_k^{2\alpha} E(|Y_k|^{2\alpha} - |Y_{k-1}|^{2\alpha}) = 0.$$

Also by [15], there exists a constant $M > 0$ such that

$$E|Y_k|^{2\alpha} \leq ME\left(\sum_{i=1}^k (Y_i - Y_{i-1})^2\right)^\alpha \quad (1.1)$$

On the other hand, by Holder's inequality, we have

$$E\left(\sum_{i=1}^k (Y_i - Y_{i-1})^2\right)^\alpha \leq k^{\alpha-1} E \sum_{i=1}^k (Y_i - Y_{i-1})^{2\alpha}. \quad (1.2)$$

Therefore

$$\begin{aligned} \int_{\Omega} (Y_n - Y_{n-1})^{2\alpha} dP &= \int_{\Omega} |Y_n - Y_{n-1}|^{2\alpha} dP \\ &= \int_{\Omega} |(S_n - S_{n-1}) - (Z_n - Z_{n-1})|^{2\alpha} dP \\ &\leq \int_{\Omega} (|X_n| + |Z_n - Z_{n-1}|)^{2\alpha} dP = \int_{\Omega} (|X_n| + |E^{F_{n-1}} S_n - S_{n-1}|)^{2\alpha} dP \end{aligned}$$

We obtain

$$\begin{aligned} \int_{\Omega} |Y_n - Y_{n-1}|^{2\alpha} dP &\leq 2^{2\alpha-1} \left(\int_{\Omega} (|X_n|^{2\alpha} + |E^{F_{n-1}} S_n - S_{n-1}|^{2\alpha}) dP \right) \\ &= 2^{2\alpha-1} \left(\int_{\Omega} |X_n|^{2\alpha} + \int_{\Omega} |S_n - S_{n-1}|^{2\alpha} dP \right) \\ &= 2^{2\alpha} \left(\int_{\Omega} |X_n|^{2\alpha} dP \right). \end{aligned}$$

So

$$E \sum_{i=1}^k (Y_i - Y_{i-1})^{2\alpha} \leq 2^{2\alpha} E \left(\sum_{i=1}^k |X_i|^{2\alpha} \right) \quad (1.3)$$

then, by combining (1.1), (1.2) and (1.3), we obtain

$$E|Y_k|^{2\alpha} \leq 2^{2\alpha} M k^{\alpha-1} E\left(\sum_{i=1}^k |X_i|^{2\alpha}\right) \quad (1.4)$$

According to the first condition of Theorem 1.6 and (1.4), we obtain

$$C_k^{2\alpha} E|Y_k|^{2\alpha} \leq M 2^{2\alpha} A C_k^\beta E\left(\sum_{i=1}^k |X_i|^{2\alpha}\right) \quad (1.5)$$

So, by using the second condition of Theorem 1.6 and Lemma 1.1, we obtain

$$\lim_{k \rightarrow +\infty} C_k^{2\alpha} E|Y_k|^{2\alpha} = 0 \quad (1.6)$$

On the other hand, we have

$$\begin{aligned} \sum_{k=1}^{\infty} C_k^{2\alpha} E(|Y_k|^{2\alpha} - |Y_{k-1}|^{2\alpha}) &\leq \sum_{k=1}^{\infty} (C_k^{2\alpha} - C_{k+1}^{2\alpha}) E|Y_k|^{2\alpha} \\ &\leq \sum_{k=1}^{\infty} (C_k^{2\alpha} - C_{k+1}^{2\alpha}) 2^{2\alpha} M k^{\alpha-1} E\left(\sum_{i=1}^k |X_i|^{2\alpha}\right) \\ &\leq M 2^{2\alpha} \left(\sum_{i=1}^{\infty} E|X_i|^{2\alpha}\right) \sum_{k=i}^{\infty} (C_k^{2\alpha} - C_{k+1}^{2\alpha}) k^{\alpha-1} \end{aligned} \quad (1.7)$$

and

$$\begin{aligned} \sum_{k=i}^{\infty} (C_k^{2\alpha} - C_{k+1}^{2\alpha}) k^{\alpha-1} &= i^{\alpha-1} C_i^{2\alpha} + \sum_{k=i+1}^{\infty} C_k^{2\alpha} ((k+1)^{\alpha-1} - k^{\alpha-1}) \\ &\leq i^{\alpha-1} C_i^{2\alpha} + \sum_{k=i+1}^{\infty} C_k^{2\alpha} A_\alpha k^{\alpha-2} \end{aligned}$$

So, by the condition (i) and (ii) of Theorem 1.6, we obtain

$$\begin{aligned} \sum_{k=i}^{\infty} (C_k^{2\alpha} - C_{k+1}^{2\alpha}) k^{\alpha-1} &\leq A C_i^\beta + A_\alpha A_\alpha C_i^\beta \\ &\leq (1 + A_\alpha) A C_i^\beta \end{aligned} \quad (1.8)$$

Again by (1.7) and (1.8)

$$\sum_{k=1}^{\infty} C_k^{2\alpha} E(|Y_k|^{2\alpha} - |Y_{k-1}|^{2\alpha}) \leq M 2^{2\alpha} (1 + A_\alpha) A \sum_{i=1}^{\infty} C_i^\beta E|X_i|^{2\alpha} \quad (1.9)$$

So, by the condition (ii) of Theorem 1.6, we obtain

$$\sum_{k=1}^{\infty} C_k^{2\alpha} E(|Y_k|^{2\alpha} - |Y_{k-1}|^{2\alpha}) < \infty. \quad (1.10)$$

And by (1.6) and (1.10), we have

$$\lim_{n \rightarrow +\infty} C_n^{2\alpha} E|Y_n|^{2\alpha} + \sum_{k=n+1}^{\infty} C_k^{2\alpha} E(|Y_k|^{2\alpha} - |Y_{k-1}|^{2\alpha}) = 0$$

Therefore

$$\lim_{n \rightarrow +\infty} C_n Y_n = 0 \quad a.s. \quad (1.11)$$

- Showing that $\lim_{n \rightarrow +\infty} C_n Z_n = 0 \quad a.s.$

Since

$$|C_n Z_n| = C_n |Z_n| = C_n \left| \sum_{k=1}^{n-1} (Z_{k+1} - Z_k) \right| = C_n \left| \sum_{k=1}^{n-1} (E^{F_k} S_{k+1} - S_k) \right| \quad (1.12)$$

Then by using (1.12), condition (iii) of Theorem 1.6 and Lemma 1.1, we obtain

$$\lim_{n \rightarrow +\infty} C_n Z_n = 0 \quad a.s.$$

Therefore

$$\lim_{n \rightarrow +\infty} C_n S_n = \lim_{n \rightarrow +\infty} C_n Y_n + C_n Z_n = 0 \quad a.s.$$

Remark 1.1 In the previous theorem, if $(S_n = \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$ is a martingale, the condition (iii) is omitted.

As immediate result we get the following corollary.

Corollary 1.1 *Let $(X_n)_{n \geq 1}$ be a sequence of random variable with values in $\mathbb{R}$. Assume that $(S_n = \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$ is a martingale and $(C_n)_{n \geq 1}$ is a positive decreasing sequence such that $\lim_{n \rightarrow +\infty} C_n = 0$. Let A , α and β such that $\alpha > 1$, $2\alpha \geq \beta > 0$ and $A > 0$. If the following conditions are satisfied*

- (i) $i^{\alpha-1} C_i^{2\alpha-\beta} \leq A$, $\sum_{k=i}^{\infty} C_k^{2\alpha} k^{\alpha-2} \leq A C_i^{\beta}$, $\forall i \geq 1$.
- (ii) $\sum_{k=1}^{+\infty} C_k^{\beta} E|X_k|^{2\alpha} < \infty$.

Then

$$\lim_{n \rightarrow +\infty} C_n S_n = 0 \text{ a.s.}$$

1.4 Applications

1.4.1 Strong law of large numbers for mils

In the first application, we establish the SLLN for mils. Before that, let us present an useful theorem (see. [12])

Theorem 1.7 *Let $(X_n)_{n \geq 1}$ be a sequence of independent random variable in $L^1_{\mathbb{R}}(\mathcal{F})$. the following statements are equivalent*

- (i) $(S_n = \sum_{k=1}^n X_k, \mathcal{F}_n)_{n \geq 1}$ is a mil.
- (ii) The sequence $(E(S_n))_{n \geq 1}$ converges.
- (iii) $(S_n = \sum_{k=1}^n X_k, \mathcal{F}_n)_{n \geq 1}$ is an amart.

Proof • Showing that (i) $\Leftrightarrow$ (ii)

For every $m \leq n$, we have

$$E^{\mathcal{F}_m} S_n - S_m = E^{\mathcal{F}_m} (S_m + (S_n - S_m)) - S_m = E^{\mathcal{F}_m} (S_n - S_m),$$

Then

$$E^{\mathcal{F}_m} S_n - S_m = E^{\mathcal{F}_m} (X_{m+1} + X_{m+2} + \dots + X_n) =$$

$$= E(X_{m+1} + X_{m+2} + \dots + X_n) = E(S_n) - E(S_m).$$

Therefore

$$E^{\mathcal{F}_m} S_n - S_m = E(S_n) - E(S_m). \quad (1.13)$$

Then by (1.13), it is easy to see that the statements (i) and (ii) are logically equivalent

- Showing that (iii) $\Rightarrow$ (ii)

Since $(S_n = \sum_{k=1}^n X_k, \mathcal{F}_n)_{n \geq 1}$ is a real amart, then

$$\lim_{n \geq m \rightarrow \infty} \int |E^{\mathcal{F}_m} S_n - S_m| = 0 \text{ a.s.}$$

So by (1.13) $\lim_{n \geq m \rightarrow \infty} |E(S_n) - E(S_m)| = 0$ a.s. Then $(E(S_n))_{n \geq 1}$ is a Cauchy sequence in $\mathbb{R}$. Therefore the sequence $(E(S_n))_{n \geq 1}$ converges.

- Showing that (ii) $\Rightarrow$ (iii)

Let $Y_n = S_n + \sum_{k=n+1}^{\infty} E(X_k) = S_n + Z_n$, for every $n \geq 1$. On the other hand, one has

$$\begin{aligned} E^{\mathcal{F}_n} Y_{n+1} &= E^{\mathcal{F}_n} (S_{n+1} + \sum_{k=n+2}^{\infty} E(X_k)) \\ &= S_n + E^{\mathcal{F}_n} X_{n+1} + E^{\mathcal{F}_n} \left(\sum_{k=n+2}^{\infty} E(X_k) \right) = \\ &= S_n + \sum_{k=n+1}^{\infty} E(X_k) = Y_n. \end{aligned}$$

This implies that $(Y_n, \mathcal{F}_n)_{n \geq 1}$ is a martingale and since $Z_n \xrightarrow{n \rightarrow \infty} 0$. Then by [31]

$(S_n = \sum_{k=1}^n X_k, \mathcal{F}_n)_{n \geq 1}$ is an amart.

Now, we have the first application

Theorem 1.8 Let $(X_n)_{n \geq 1}$ be a sequence of independent random variable in $L^1_{\mathbb{R}}(\mathcal{F})$. Let $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ and suppose that $(S_n = \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$ is an integrable mil. Let A, α and β such that $\alpha > 1$, $2\alpha \geq \beta > 0$ and $A > 0$. Assume that the following conditions are satisfied

$$(i) \quad i^{\alpha-1} C_i^{2\alpha-\beta} \leq A, \quad \sum_{k=i}^{\infty} C_k^{2\alpha} k^{\alpha-2} \leq A C_i^{\beta}, \quad \forall i \geq 1.$$

$$(ii) \quad \sum_{k=1}^{+\infty} C_k^{\beta} E|X_k|^{2\alpha} < \infty.$$

Then

$$\lim_{n \rightarrow +\infty} C_n S_n = 0 \quad a.s.$$

Proof Since $(S_n = \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$ is an integrable mil, then by [31], there exists a martingale $(Y_n, \mathcal{F}_n)_{n \geq 1}$ and a predictable sequence $(Z_n, \mathcal{F}_n)_{n \geq 1}$, such that, $\forall n \geq 1, S_n = Y_n + Z_n$ where, $S_1 = Y_1, Z_1 = 0$.

By condition (i), (ii) and Theorem 1.6 we conclude that $\lim_{n \rightarrow +\infty} C_n Y_n = 0$ a.s. It is enough to check that $\lim_{n \rightarrow +\infty} C_n Z_n = 0$ a.s. We have

$$\begin{aligned} C_n Z_n &= C_n \sum_{k=1}^{n-1} (Z_{k+1} - Z_k) = \\ &= C_n \sum_{k=1}^{n-1} (E^{\mathcal{F}_k} S_{k+1} - S_k). \end{aligned}$$

On the other hand, the random variable $(X_n)_{n \geq 1}$ are independent, so

$$E^{\mathcal{F}_k} S_{k+1} - S_k = E^{\mathcal{F}_k} S_k + E^{\mathcal{F}_k} X_{k+1} - S_k.$$

Since S_k is $\mathcal{F}_k$ -measurable, then

$$E^{\mathcal{F}_k} S_{k+1} - S_k = E^{\mathcal{F}_k} X_{k+1} = EX_{k+1}.$$

Then

$$C_n Z_n = C_n (E(S_n) - EX_1).$$

By Theorem 1.7 and Lemma 1.1, we have $\lim_{n \rightarrow +\infty} C_n Z_n = 0$ a.s. Therefore

$$\lim_{n \rightarrow +\infty} C_n S_n = 0 \quad a.s.$$

1.4.2 Strong law of large numbers and amart

In the second application, we establish a relation between the SLLN and amart. Before that, let us mention the following proposition due to [43] which gives a sufficient conditions of adapted sequence of random variables to be an amart.

Proposition 1.4 *Let $(Z_n)_{n \geq 1}$ be an adapted real sequence in $L^1(\Omega, \mathcal{F}, P)$ such that,*

$$E \sup_n |Z_n| < +\infty.$$

The following statements are equivalent:

1. $(Z_n)_{n \geq 1}$ converges almost surely.
2. $(Z_n)_{n \geq 1}$ is an amart.

Proof • *Showing the first implication $1 \Rightarrow 2$.*

Suppose that $Z_n \rightarrow Z$ a.s. Let $(\tau_n)_{n \in \mathbb{N}}$ be a sequence of bounded stopping times increasing to infinity. So $Z_{\tau_n} \rightarrow Z$.

On the other hand, we have $|Z_{\tau_n}| \leq \sup_n |Z_n|$. Then, by using dominated convergence theorem, we obtain

$$EZ_{\tau_n} \rightarrow EZ.$$

Therefore (Z_n, F_n) is an amart.

• *Showing the second implication $2 \Rightarrow 1$.*

Let $Z^ = \limsup_{n \rightarrow +\infty} Z_n$ and $Z^{**} = \liminf_{n \rightarrow +\infty} Z_n$. According to Lemma 3.1 in [52], there exists two sequences of increasing bounded stopping times $(\tau_n)_{n \in \mathbb{N}}$ and $(\sigma_n)_{n \in \mathbb{N}}$ such that $Z_{\tau_n} \rightarrow Z^*$ a.s and $Z_{\sigma_n} \rightarrow Z^{**}$ a.s.*

Since $|Z_{\tau_n} - Z_{\sigma_n}| \leq 2 \sup_n |Z_n|$ which is integrable it follows by dominated convergence theorem and the amart property that

$$0 \leq E(Z^* - Z^{**}) = \lim_{n \rightarrow +\infty} E(Z_{\tau_n} - Z_{\sigma_n}) = 0.$$

And hence that $Z^ = Z^{**}$, therefore $(Z_n)_{n \geq 1}$ converges almost surely.*

Now, we have the second application:

Theorem 1.9 *Let $(X_n)_{n \geq 1}$ be a sequence of random variable in $L^1_{\mathbb{R}}(\mathcal{F})$ and let $(S_n = \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$ be an adapted sequence. Let A , α and β such that $\alpha > 1$, $2\alpha \geq \beta > 0$ and $A > 0$. Assume that the following conditions are satisfied*

$$(i) \quad i^{\alpha-1} C_i^{2\alpha-\beta} \leq A, \quad \sum_{k=i}^{\infty} C_k^{2\alpha} k^{\alpha-2} \leq A C_i^{\beta}, \quad \forall i \geq 1.$$

$$(ii) \quad \sum_{k=1}^{+\infty} C_k^{\beta} E|X_k|^{2\alpha} < \infty.$$

$$(iii) \quad \sum_{k=1}^{+\infty} C_k E|X_k| < +\infty.$$

Then $(C_n \sum_{i=1}^n X_i, F_n)_{n \geq 1}$ is an amart.

Proof • *Step1: we prove that $C_n S_n$ converges almost surely*

Since $(S_n = \sum_{i=1}^n X_i, \mathcal{F}_n)_{n \geq 1}$ be an adapted sequence, so by [31] there exists a martingale $(Y_n, \mathcal{F}_n)_{n \geq 1}$ and a predictable sequence $(Z_n, \mathcal{F}_n)_{n \geq 1}$, such that, $\forall n \geq 1, S_n = Y_n + Z_n$ where $S_1 = Y_1, Z_1 = 0$.

By condition (i), (ii) and Theorem 1.6 we conclude that $\lim_{n \rightarrow +\infty} C_n Y_n = 0$ a.s. So, it is enough to check that $\lim_{n \rightarrow +\infty} C_n Z_n = 0$ a.s.

Since

$$\begin{aligned} |C_n Z_n| &= |C_n \sum_{k=1}^n (Z_k - Z_{k-1})| = \\ &= |C_n \sum_{k=1}^n (E^{\mathcal{F}_{k-1}} S_k - S_{k-1})| \leq \\ &\leq C_n \sum_{k=1}^n |E^{\mathcal{F}_{k-1}} X_k|. \end{aligned}$$

On the other hand, by monotone convergence theorem, we have

$$E\left(\sum_{k=1}^{\infty} C_k E^{\mathcal{F}_{k-1}} |X_k|\right) = \sum_{k=1}^{\infty} C_k E E^{\mathcal{F}_{k-1}} |X_k| = \sum_{k=1}^{\infty} C_k E |X_k|$$

So by the third condition of Theorem 1.9 we have

$$\sum_{k=1}^{\infty} C_k E^{\mathcal{F}_{k-1}} |X_k| < +\infty \text{ a.s.}$$

Using the Lemma 1.1, we obtain

$$C_n Z_n \longrightarrow 0 \text{ a.s.}$$

Then

$$\lim_{n \rightarrow +\infty} C_n S_n = \lim_{n \rightarrow +\infty} (C_n Y_n + C_n Z_n) = 0 \text{ a.s.}$$

we deduce that $C_n S_n$ converges almost surely to 0.

- Step 2: we show that $(C_n \sum_{i=1}^n X_i)_{n \geq 1}$ is an amart.

We have

$$C_n \sum_{k=1}^n |X_k| \leq \sum_{k=1}^n C_k |X_k|, \forall n \geq 1.$$

Then

$$\sup_n C_n \sum_{k=1}^n |X_k| \leq \sum_{k=1}^{\infty} C_k |X_k|.$$

Then

$$E \sup_n C_n \sum_{k=1}^n |X_k| \leq E \left(\sum_{k=1}^{\infty} C_k |X_k| \right).$$

On the other hand, by monotone convergence theorem, we have

$$\sum_{k=1}^{\infty} C_k E |X_k| = E \left(\sum_{k=1}^{\infty} C_k |X_k| \right)$$

so

$$E \sup_n C_n \sum_{k=1}^n |X_k| \leq \sum_{k=1}^{\infty} C_k E |X_k|.$$

Then by the assumption (iii), we have

$$E \sup_n \left| C_n \sum_{i=1}^n X_i \right| < \infty.$$

Then, according to the Proposition 1.4, we have $(C_n \sum_{i=1}^n X_i)_{n \geq 1}$ is an amart.

The third application concerns the strong law of large numbers for multivalued supermartingale in $L^1_{ccb}(\mathbb{R})(\mathcal{F})$. We prove the existence of selection $(g_n)_{n \geq 1}$ of a set valued $(S_n)_{n \geq 1}$ such that $(C_n g_n)_{n \geq 1}$ converges almost surely to 0, where $(C_n)_{n \geq 1}$ is a positive decreasing sequence such that $\lim_{n \rightarrow +\infty} C_n = 0$.

1.4.3 Strong law of large numbers for multivalued supermartingale

Before stating the strong law of large numbers for multivalued supermartingale, let us recall the following proposition due to [37].

Proposition 1.5 *Let $(X_n)_{n \geq 1}$ be a sequence in $L^1_{ccb(\mathbb{R})}(\mathcal{F})$ such that $(S_n = \sum_{k=1}^n X_k, \mathcal{F}_n)$ is a supermartingale. Then, S_n has the form*

$$S_n = X_1 + f_2 + f_3 + \dots + f_n, \quad f_i \in L^1_{\mathbb{R}}(\mathcal{F}), (i \geq 2).$$

Proof Since $(S_n, \mathcal{F}_n)_{n \geq 1}$ is supermartingale, then, $\forall n \geq 1$ $E^{F_n} S_{n+1} \subset S_n$ and $\forall x^* \in E^*$, we have

$$\int_{\Omega} \delta^*(x^*, E^{\mathcal{F}_n} S_{n+1}) dP(\omega) \leq \int_{\Omega} \delta^*(x^*, S_n) dP(\omega),$$

then

$$\delta^*(x^*, \int_{\Omega} E^{\mathcal{F}_n} S_{n+1} dP(\omega)) - \delta^*(x^*, \int_{\Omega} S_n dP(\omega)) \leq 0$$

hence

$$\delta^*(x^*, \int_{\Omega} E^{\mathcal{F}_n} X_{n+1} dP(\omega)) \leq 0.$$

Therefore

$$\int_{\Omega} X_{n+1} dP(\omega) = \{0\}.$$

Then, by [60], for all $n \geq 1$, there exists an integrable random variable f_{n+1} such that $X_{n+1}(\omega) = \{f_{n+1}(\omega)\}$ a.s.

Then

$$S_n = X_1 + f_2 + f_3 + \dots + f_n.$$

The last application is the strong law of large numbers for multivalued supermartingale. It is the multivalued version of Theorem 1.6.

Theorem 1.10 *Let $(X_n)_{n \geq 1}$ be a sequence in $L^1_{ccb(\mathbb{R})}(\mathcal{F})$ such that $(S_n = \sum_{k=1}^n X_k, \mathcal{F}_n)$ is a supermartingale. Assume that $(C_n)_{n \geq 1}$ is a positive decreasing sequence such that $\lim_{n \rightarrow +\infty} C_n = 0$. Let A , α and β such that $\alpha > 1$, $2\alpha \geq \beta > 0$ and $A > 0$. If the two following conditions are satisfied*

$$(i) \quad i^{\alpha-1} C_i^{2\alpha-\beta} \leq A, \quad \sum_{k=i}^{\infty} C_k^{2\alpha} k^{\alpha-1} \leq A C_i^{\beta}, \quad \forall i \geq 1.$$

$$(ii) \quad \sum_{k=1}^{+\infty} C_k^{\beta} E|X_k|^{2\alpha} < \infty.$$

Then $0 \in s - li C_n S_n$ a.s.

Proof Since $(S_n, \mathcal{F}_n)_{n \geq 1}$ is a supermartingale in $L_{ccb(\mathbb{R})}^1(\mathcal{F})$ then by Proposition 1.5 S_n has the form

$$S_n = X_1 + f_2 + f_3 + \dots + f_n, \quad f_i \in L_{\mathbb{R}}^1(\mathcal{F}), (i \geq 2).$$

On the other hand by Lemma 1.3, there exists an integrable martingale selection $(g_n)_{n \geq 1}$ for $(S_n)_{n \geq 1}$. Then this implies the existence of an integrable selection h_1^n of X_1 such that for all $n \geq 1$, we have

$$g_n = h_1^n + f_2 + f_3 + f_4 + \dots + f_n. \quad a.s. \quad (1.14)$$

And since $(g_n, \mathcal{F}_n)_{n \geq 1}$ is a martingale, Then by the relation (1.14), the condition (i) and (ii) of the Theorem 1.10, it is easy to see that the martingale $(g_n)_{n \geq 1}$ satisfied all conditions of Corollary 1.1.

Therefore

$$\lim_{n \rightarrow +\infty} C_n g_n = 0 \quad a.s.$$

This implies that

$$0 \in s - li C_n S_n \quad a.s.$$

Mosco convergence of conditional expectation for sequences of Pettis integrable random sets

2.1 Introduction

The convergence of conditional expectation of random variables (resp. random sets) plays a crucial role in the probability theory. This concept was used in many concrete problems, more precisely in the theory of martingales, optimization and control, stochastic geometry and in the mathematical economics.

The conditional expectation of random variables (resp. random sets) Bochner-integrable always exists and has been treated by several authors (see. for example [82], [61], [56], [62], [36], [97], [18] and others). However, the conditional expectation of Pettis-integrable random variables (resp. random sets) doesn't generally exist (see. [90] and [92]). Recently, several authors have been studied the existence of this operator (see. [33], [2], [1]), [4], [38], [79] and [55].

By using the recent tools established in this theory, we prove an extension of results stated by [61] and by [18] in Pettis integration. More precisely the aim of this chapter is the study of almost sure Mosco convergence for sequences of random sets $(E^{\mathfrak{B}_n} X_n)_{n \geq 1}$, where $(\mathfrak{B}_n)_{n \geq 1}$ is a decreasing sequence of sub- σ -algebra of $\mathfrak{F}$ and $(X_n)_{n \geq 1}$ is a sequence of Pettis-integrable

convex and weakly compact random sets in a separable Banach space. It is interesting to know that the same result for vector valued Pettis-integrable function is not established yet. This chapter is organized as follows. In section 2, we present some definitions, preliminaries and needed results. Section 3 is devoted to the convergence of the conditional expectation for sequences of Pettis-integrable vector valued random variables; we state a new version of Neveu's theorem and dominated convergence theorem. In section 4, we present the almost sure Mosco convergence of Pettis-integrable random sets $(E^{\mathfrak{B}_n}X)_{n \geq 1}$ and $(E^{\mathfrak{B}_n}X_n)_{n \geq 1}$, where $(\mathfrak{B}_n)_{n \geq 1}$ is a decreasing sequence of sub- σ -algebras of $\mathfrak{F}$ and $(X_n)_{n \geq 1}$ as well as X are the convex and weakly compact Pettis-integrable random sets. At the last section of this chapter, we present the convergence of conditional expectation for a $cc(E)$ - valued Pettis-integrable random sets.

2.2 Notations and preliminaries

Throughout this paper $(\Omega, \mathfrak{F}, P)$ is a complete probability space, $(\mathfrak{B}_n)_{n \geq 1}$ is a decreasing sequence of sub- σ -algebras of $\mathfrak{F}$ and $\mathfrak{B}_\infty = \cap_{n \geq 1} \mathfrak{B}_n$. Let E be a separable Banach space with the dual space E^* . Let $cwk(E)$ (resp. $cc(E)$) be the family of all nonempty convex and weakly compact subsets of E (resp. the family of all nonempty closed convex subsets of E). Let $L_E^1(\mathfrak{F})$ (resp. $L_{cwk(E)}^1(\mathfrak{F})$) be the family of all $\mathfrak{F}$ -measurable and Bochner-integrable multifunctions $X : \Omega \longrightarrow E$ (resp. the family of integrably bounded random sets with values in $cwk(E)$). Denoted by $\overline{B_E}$ (resp. $\overline{B_{E^*}}$) the closed unit ball of E (resp. E^*).

Given B in $cwk(E)$, the distance function and the support function of B are defined by

$$d(x, B) = \inf\{\|x - y\|, y \in B\}, \quad (x \in E).$$

$$\delta^*(x^*, B) = \sup\{\langle x^*, y \rangle, y \in B\}, \quad (x^* \in E^*).$$

For any B in $cwk(E)$, we get

$$|B| = \sup\{\|x\|, x \in B\}.$$

A $ckw(E)$ -valued sequence $(A_n)_{n \geq 1}$ is called Mosco convergent to a closed convex and weakly compact set A_∞ (see. [78]), if $A_\infty = s - li A_n = w - ls A_n$, where

$$s - li A_n = \{y \in E, y_n \longrightarrow y, y_n \in A_n, \forall n \geq 1\}$$

$$w - ls A_n = \{y \in E, y_k \xrightarrow{w} y, y_k \in A_{n_k}, \forall k \geq 1\},$$

where $(A_{n_k})_{k \geq 1}$ is a subsequence of $(A_n)_{n \geq 1}$. If $(A_n)_{n \geq 1}$ Mosco converges to A_∞ in $ckw(E)$, we write $M - \lim A_n = A_\infty$.

The topology determined by convergence of support functional is denoted by τ_s . A sequence A_n is τ_s -convergent to subset A_∞ if $\lim_{n \rightarrow +\infty} \delta^*(x^*, A_n) = \delta^*(x^*, A_\infty)$ for all $x^* \in E^*$.

The topology determined by the convergence of distance functionals is called the Wijsman topology and is denoted by τ_w . A sequence Y_n is τ_w convergent to subset Y if $\lim_{n \rightarrow +\infty} d(x, Y_n) = d(x, Y)$ for all $x \in E$.

The linear topology τ_L (see. [11]) on $cc(E)$ is the upper bounded of the two previous topologies τ_s and τ_w .

Proposition 2.1 ([11]) *Let $(A_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ be a sequence in $cc(E)$, then $A_\infty = \tau_L - \lim_{n \rightarrow +\infty} A_n$, if and only if*

- $\lim_{n \rightarrow +\infty} \delta^*(x^*, A_n) = \delta^*(x^*, A_\infty)$, for all $x^* \in E^*$.
- $\lim_{n \rightarrow +\infty} d(x, A_n) = d(x, A_\infty)$, for all $x \in E$.

A measurable function $g : \Omega \longrightarrow E$ is Pettis-integrable, if g is scalarly integrable (i.e. $\langle x^*, g \rangle$ is integrable), and for each $A \in \mathfrak{F}$, there exists x_A in E , such that

$$\int_A \langle x^*, g \rangle dP = \langle x^*, x_A \rangle, \forall x^* \in E^*.$$

x_A is called the Pettis-integral of g over A . We will denote by $P_E^1(\mathfrak{F})$ the space of all $\mathfrak{F}$ -measurable and Pettis-integrable E -valued function defined on $(\Omega, \mathfrak{F}, P)$. We consider the space $P_E^1(\mathfrak{F})$ provided with the following topologies:

- The topology of the usual Pettis norm

$$\|g\|_{Pe} = \sup_{x^* \in B_{E^*}} \int_{\Omega} |\langle x^*, g \rangle| dP.$$

- The topology induced by the duality $(P_E^1(\mathfrak{F}), L^\infty \otimes E^*)$. Recall that a sequence $(g_n)_{n \geq 1}$ in $P_E^1(\mathfrak{F})$ converges to g in this topology, if for each $h \in L^\infty(\mathfrak{F})$ and for all $x^* \in E^*$, one has

$$\lim_{n \rightarrow +\infty} \int_{\Omega} h(\omega) \langle x^*, g_n(\omega) \rangle dP(\omega) = \int_{\Omega} h(\omega) \langle x^*, g(\omega) \rangle dP(\omega).$$

This topology is known as the weak topology and is denoted by $w-Pe$.

A multifunction $X : \Omega \longrightarrow cc(E)$ is said to be measurable, if for every open set U of E the subset

$$X^-(U) = \{\omega \in \Omega / X(\omega) \cap U \neq \emptyset\}.$$

is an element of $\mathfrak{F}$. The Effros σ -field ξ on $cc(E)$ is generated by the subsets $U^- = \{F \in cc(E), / F \cap U \neq \emptyset\}$, so the multifunction $X : \Omega \longrightarrow cc(E)$ is measurable if, for any $B \in \xi$, one has $X^-(B) \in \mathfrak{F}$. A measurable multifunction is called a random set.

A measurable function $f : \Omega \longrightarrow E$ is said to be a selection of X , if, for any $\omega \in \Omega$, $f(\omega) \in X(\omega)$. We denote by $S_X^1(\mathfrak{F})$ the set of all $\mathfrak{F}$ -measurable and integrable selection of X . It is known that if E is a complete space (see. theorem 1.0 in [62]) a closed and convex valued multifunction X is measurable if and only if $dom(X) \in \mathfrak{F}$ and X has a Castaing representation (i. e. there exists a sequence $(f_n)_{n \geq 1}$ of measurable selection of X such that for all $\omega \in \Omega$, $X(\omega) = cl\{f_n(\omega), n \geq 1\}$), or if and only if the real function $d(x, X(\cdot))$ is measurable for any x in E .

The random set $X : \Omega \longrightarrow cwk(E)$ is scalarly integrable, if for any $x^* \in E^*$, the real function $\delta^*(x^*, X(\cdot))$ is integrable. We say that the random set X is Pettis-integrable, if X is scalarly integrable and for each $A \in \mathfrak{F}$, there exists $K_A \in cwk(E)$ such that

$$\int_A \delta^*(x^*, X) dP = \delta^*(x^*, K_A), \forall x^* \in E^*.$$

$K_A := \int_A X dP$ is called the Pettis-integral of X over A .

A random set X is said to be Aumann-Pettis integrable if $S_X^{Pe} \neq \emptyset$ and the multivalued Aumann Pettis-integral of a Pettis-integrable random set X is defined by

$$\int_{\Omega} X dP = \left\{ \int_{\Omega} f dP, f \in S_X^{Pe}(\mathfrak{F}) \right\}.$$

We will denote by $P_{cwk(E)}^1(\mathfrak{F})$ the set of all $cwk(E)$ -valued Pettis-integrable random set.

Let X be in $L_E^1(\mathfrak{F})$ and $\mathcal{B}$ a sub- σ -algebra of $\mathfrak{F}$. It is known in the literature (see. [82]) that the conditional expectation of X relative to $\mathcal{B}$ exists and is the unique almost surely $\mathcal{B}$ -measurable and Bochner integrable random variable such that

$$\int_A E^{\mathcal{B}} X dP = \int_A X dP, \forall A \in \mathcal{B}.$$

The extension of this result is stated by [62] of integrable random sets;

If X is an integrable random set and $\mathcal{B}$ is a sub- σ -algebra of $\mathfrak{F}$, then there exists a unique almost surely $\mathcal{B}$ -measurable and integrable random set denoted $E^{\mathcal{B}} X$ such that

$$S_{E^{\mathcal{B}} X}^1(\mathcal{B}) = cl\{E^{\mathcal{B}} f, f \in S_X^1(\mathfrak{F})\},$$

where the closure is taken with respect to the norm in $L_E^1(\mathfrak{F})$.

The conditional expectation for Pettis-integrable random variable not always exists (see. [90] and [92]). Recently ([1], [2]), [33] and [93] gave a sufficient conditions of the existence of this operator for Pettis-integrable vector random variables and Pettis-integrable random sets. For the convenience of the reader, we recall the following propositions which will be used after.

Proposition 2.1 ([1]) *Let $\mathcal{B}$ be a sub- σ -algebra of $\mathfrak{F}$ and let X be a Pettis-integrable E -valued function such that $E^{\mathcal{B}}|X| \in [0, +\infty[$. Then there exists a unique $\mathcal{B}$ -measurable Pettis-integrable E -valued function denote by $E^{\mathcal{B}} X$ which enjoys the following property, for every $h \in L^{\infty}(\mathcal{B})$, one has*

$$\int_{\Omega} h E^{\mathcal{B}} X dP = \int_{\Omega} h X dP.$$

$E^{\mathcal{B}} X$ is called the Pettis conditional expectation of X relative to $\mathcal{B}$.

Proposition 2.2 ([1], [33]) *Let $\mathcal{B}$ be a sub- σ -algebra of $\mathfrak{F}$ and let X be a $\text{cw}k(E)$ -valued Pettis-integrable random set such that $E^{\mathcal{B}}|X| \in [0, +\infty[$. Then there exists a unique $\mathcal{B}$ -measurable $\text{cw}k(E)$ -valued Pettis-integrable random set which enjoys the following property, for every $h \in L^\infty(\mathcal{B})$, one has*

$$\int_{\Omega} h E^{\mathcal{B}} X dP = \int_{\Omega} h X dP.$$

$E^{\mathcal{B}}X$ is called the Pettis conditional expectation of X relative to $\mathcal{B}$ such that

$$S_{E^{\mathcal{B}}X}^{\text{Pe}}(\mathcal{B}) = \{E^{\mathcal{B}}f, f \in S_X^{\text{Pe}}(\mathfrak{F})\}.$$

[2] extended the previous proposition to $cc(E)$ -valued Pettis-integrable multifunction.

Proposition 2.3 *Assume that E^* is separable. Let $\mathcal{B}$ be a sub- σ -algebra of $\mathfrak{F}$ and let X be a $cc(E)$ -valued Pettis-integrable random set satisfying the following conditions*

- $E^{\mathcal{B}}|X(\cdot)| \in [0, +\infty[$ a.s.
- The set $\{\delta^*(x^*, X), x^* \in B_{E^*}\}$ is uniformly integrable.

Then there exists a unique $\mathcal{B}$ -measurable $cc(E)$ -valued Pettis-integrable random set denote by $E^{\mathcal{B}}X$ such that

$$S_{E^{\mathcal{B}}X}^{\text{Pe}}(\mathcal{B}) = cl\{E^{\mathcal{B}}f, f \in S_X^{\text{Pe}}(\mathfrak{F})\}.$$

It is interesting to note that, if X is a $cc(E)$ -valued Pettis-integrable random set and $\mathcal{B}$ is a sub- σ -algebra of $\mathfrak{F}$. Then the two following statements are equivalents

- $E^{\mathcal{B}}|X(\cdot)| \in [0, +\infty[$ a.s.
- There exists a partition $(B_m)_{m \geq 1}$ of Ω $\mathfrak{F}$ -measurable, such that for each $m \geq 1$, $\int_{B_m} |X| dP < +\infty$.

We close this section with the following lemmas. The first one is known as Levy theorem for decreasing sequence of sub- σ -algebras $(\mathfrak{B}_n)_{n \geq 1}$ and the second is the dominated convergence theorem for the conditional expectation of Bochner integrable random variables.

Lemma 2.1 ([99]) *Let $(\mathfrak{B}_n)_{n \geq 1}$ be a decreasing sequence of a sub- σ -algebra of $\mathfrak{F}$ and let $f \in L_E^1(\mathfrak{F})$, set $\mathfrak{B}_\infty = \cap_{n \geq 1} \mathfrak{B}_n$. Then*

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f(\cdot) = E^{\mathfrak{B}_\infty} f(\cdot) \text{ a.s.}$$

Proof Since $f \in L_E^1(\mathfrak{F})$ so f is bounded and $\mathfrak{F}$ -measurable then, there exists a $\mathfrak{F}$ -measurable simple functions $(f_m)_{m \geq 1}$ such that $\lim_{m \rightarrow +\infty} f_m = f$ a.s.

On the other hand

$$\|E^{\mathfrak{B}_n} f - E^{\mathfrak{B}_\infty} f\| = \|E^{\mathfrak{B}_n}(f - f_m) + E^{\mathfrak{B}_n} f_m - E^{\mathfrak{B}_\infty}(f - f_m) - E^{\mathfrak{B}_\infty} f_m\|$$

And by the triangular inequality, we obtain

$$\|E^{\mathfrak{B}_n} f - E^{\mathfrak{B}_\infty} f\| \leq \|E^{\mathfrak{B}_n}(f - f_m)\| + \|E^{\mathfrak{B}_\infty}(f - f_m)\| + \|E^{\mathfrak{B}_n} f_m - E^{\mathfrak{B}_\infty} f_m\|$$

By the Jensen's inequality $\|E^{\mathfrak{B}_n}(f - f_m)\| \leq E^{\mathfrak{B}_n}\|f - f_m\|$ and $\|E^{\mathfrak{B}_\infty}(f - f_m)\| \leq E^{\mathfrak{B}_\infty}\|f - f_m\|$. Therefore

$$\|E^{\mathfrak{B}_n} f - E^{\mathfrak{B}_\infty} f\| \leq E^{\mathfrak{B}_n}\|f - f_m\| + E^{\mathfrak{B}_\infty}\|f - f_m\| + \|E^{\mathfrak{B}_n} f_m - E^{\mathfrak{B}_\infty} f_m\|.$$

For a fixed m , applying the Neveu's Corollary (corollary II-2-12 in [82]), we obtain

$$\limsup_{n \rightarrow \infty} \|E^{\mathfrak{B}_n} f - E^{\mathfrak{B}_\infty} f\| \leq 2E^{\mathfrak{B}_\infty}\|f - f_m\|$$

And by the dominated convergence theorem, the second term of inequality converges to zero as $m \rightarrow +\infty$. Which implies the desired result

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f(\cdot) = E^{\mathfrak{B}_\infty} f(\cdot) \text{ a.s.}$$

Lemma 2.2 ([18]) *Let $(\mathfrak{B}_n)_{n \geq 1}$ be a decreasing sequence of a sub- σ -algebras of $\mathfrak{F}$ and set $\mathfrak{B}_\infty = \cap_{n \geq 1} \mathfrak{B}_n$. Let Y be a positive random variable satisfying $E^{\mathfrak{B}_\infty} Y \in [0, +\infty[$ and $(X_n)_{n \geq 1}$ in $L_E^1(\mathfrak{F})$ such that:*

1. $\forall n \geq 1, |X_n| \leq Y$.
2. $\lim_{n \rightarrow +\infty} X_n(\cdot) = X_\infty(\cdot) \text{ a.s.}$

Then the following equality holds

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} X_n(\cdot) = E^{\mathfrak{B}_\infty} X_\infty(\cdot) \text{ a.s.}$$

2.3 Convergence of conditional expectation for a sequences of E -valued Pettis-integrable random variables

In this section, we give an extension of Lemma 2.1 and Lemma 2.2 in Pettis integration case.

Theorem 2.1 *Let $(\mathfrak{B}_n)_{n \geq 1}$ be a decreasing sequence of a sub- σ -algebras of $\mathfrak{F}$ and set $\mathfrak{B}_\infty = \bigcap_{n \geq 1} \mathfrak{B}_n$. Let $f \in P_E^1(\mathfrak{F})$ such that $E^{\mathfrak{B}_\infty} |f| \in [0, +\infty[$. Then*

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f(\cdot) = E^{\mathfrak{B}_\infty} f(\cdot) \text{ a.s.}$$

Proof Since $E^{\mathfrak{B}_\infty} |f| \in [0, +\infty[$, so is $E^{\mathfrak{B}_n} |f|$ for each $n \geq 1$.

Then by the Proposition 2.1, $E^{\mathfrak{B}_n} f$ exists and is in $P_E^1(\mathfrak{B}_n)$. The condition $E^{\mathfrak{B}_\infty} |f| < +\infty$ provides a $\mathfrak{B}_\infty$ -measurable partition $(A_k)_{k \geq 1}$ of Ω such that $f_k = f 1_{A_k} \in L_E^1(\mathfrak{F})$. Using the Lemma 2.1, we obtain

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f_k = E^{\mathfrak{B}_\infty} f_k \text{ a.s.}$$

As $A_k \in \mathfrak{B}_\infty \subset \mathfrak{B}_n$, for every $n \in \mathbf{N}^*$ and for every $k \in \mathbf{N}^*$, then by using the Proposition 2.1, it is easy to see that

$$E^{\mathfrak{B}_n} f_k = E^{\mathfrak{B}_n} 1_{A_k} f = 1_{A_k} E^{\mathfrak{B}_n} f \text{ a.s.}$$

On the other hand

$$\begin{aligned} E^{\mathfrak{B}_\infty} f &= \sum_{k=1}^{+\infty} E^{\mathfrak{B}_\infty} f_k = \sum_{k=1}^{+\infty} \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f_k \text{ a.s.} \\ &= \sum_{k=1}^{+\infty} 1_{A_k} \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f = \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f \text{ a.s.} \end{aligned}$$

We conclude that

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f(\cdot) = E^{\mathfrak{B}_\infty} f(\cdot) \text{ a.s.}$$

Now, we give a new version of dominated convergence theorem for a sequences of E -valued Pettis-integrable random variables.

Theorem 2.2 *Let $(\mathfrak{B}_n)_{n \geq 1}$ be a decreasing sequence of a sub- σ -algebras of $\mathfrak{F}$ and set $\mathfrak{B}_\infty = \bigcap_{n \geq 1} \mathfrak{B}_n$. Let Y be a positive random variable satisfying $E^{\mathfrak{B}_\infty} Y \in [0, +\infty[$. Let $(X_n)_{n \geq 1}$ be a sequence in $P_E^1(\mathfrak{F})$ such that*

1. $\forall n \geq 1, |X_n(\cdot)| \leq Y(\cdot)$
2. $\lim_{n \rightarrow +\infty} X_n(\cdot) = X_\infty(\cdot)$ a.s.

Then

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} X_n(\cdot) = E^{\mathfrak{B}_\infty} X_\infty(\cdot) \text{ a.s.}$$

Proof By assumption (i), the condition $E^{\mathfrak{B}_\infty} Y < +\infty$ and Proposition 2.1, we have $E^{\mathfrak{B}_n} X_n$ exists and it is in $P_E^1(\mathfrak{B}_n)$. Also, there exists a $\mathfrak{B}_\infty$ -measurable partition $(A_m)_{m \geq 1}$ of Ω such that $\int_{A_m} Y dP < +\infty$ for all $m \geq 1$.

On the other hand

- $|X_n| 1_{A_m} \leq Y 1_{A_m}$.
- $\lim_{n \rightarrow +\infty} X_n 1_{A_m} = X_\infty 1_{A_m}$ a.s.

Then $X_n 1_{A_m}$ and $Y 1_{A_m}$ satisfied all conditions of the Lemma 2.2, consequently

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} (X_n 1_{A_m}) = E^{\mathfrak{B}_\infty} (X_\infty 1_{A_m}) \text{ a.s.} \quad (2.1)$$

Since $A_m \in \mathfrak{B}_\infty \subset \mathfrak{B}_n$, for every $n \in \mathbf{N}^*$ and for every $m \in \mathbf{N}^*$, then

$$E^{\mathfrak{B}_n} (1_{A_m} X_n) = 1_{A_m} E^{\mathfrak{B}_n} X_n \text{ a.s.} \quad (2.2)$$

$$\begin{aligned} \text{By (2.1) and (2.2), we have } E^{\mathfrak{B}_\infty} X_\infty &= \sum_{m \geq 1} E^{\mathfrak{B}_\infty} (X_\infty) 1_{A_m} = \sum_{m \geq 1} E^{\mathfrak{B}_\infty} (X_\infty 1_{A_m}) \text{ a.s.} \\ &= \sum_{m \geq 1} \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} (X_n 1_{A_m}) = \sum_{m \geq 1} \lim_{n \rightarrow +\infty} 1_{A_m} E^{\mathfrak{B}_n} (X_n) = \\ &= \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} X_n \text{ a.s.} \end{aligned}$$

We conclude that

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} X_n(\cdot) = E^{\mathfrak{B}_\infty} X_\infty(\cdot) \text{ a.s.}$$

2.4 Convergence of conditional expectation for a $ckw(E)$ -valued Pettis-integrable random sets

Now, we provide an extension of the Neveu's theorem for a $ckw(E)$ -valued Pettis-integrable random set which extends Theorem 2.1.

Theorem 2.3 *Assume that E is separable. Let X be a $ckw(E)$ -valued Pettis-integrable random set and let $(\mathfrak{B}_n)_{n \geq 1}$ be a decreasing sequence of a sub- σ -algebra of $\mathfrak{F}$. Set $\mathfrak{B}_\infty = \bigcap_{n \geq 1} \mathfrak{B}_n$. If $E^{\mathfrak{B}_\infty}|X| \in [0, +\infty]$, we have*

$$M - \lim_{n \rightarrow \infty} E^{\mathfrak{B}_n} X(\cdot) = E^{\mathfrak{B}_\infty} X(\cdot) \text{ a.s.}$$

Proof • *Step1: we prove that $E^{\mathfrak{B}_\infty} X(\cdot) \subset s - li E^{\mathfrak{B}_n} X(\cdot)$ a.s.*

Since X is $ckw(E)$ -valued Pettis-integrable random set, then by [32], $S_X^{Pe}(\mathfrak{F}) \neq \emptyset$. On the other hand $E^{\mathfrak{B}_\infty}|X| \in [0, +\infty]$, then by Proposition 2.2, $E^{\mathfrak{B}_n} X$ exists and it is in $P_{ckw(E)}^1(\mathfrak{B}_n)$. Let $h \in S_X^{Pe}(\mathfrak{F})$, then $E^{\mathfrak{B}_n} h$ exists and by [33], $E^{\mathfrak{B}_n} h(\cdot) \in E^{\mathfrak{B}_n} X(\cdot)$ a.s. Using Theorem 2.1, we have

$$\lim_{n \rightarrow \infty} E^{\mathfrak{B}_n} h(\cdot) = E^{\mathfrak{B}_\infty} h(\cdot) \text{ a.s.}$$

Since $E^{\mathfrak{B}_\infty} h$ is Pettis-integrable, then

$$E^{\mathfrak{B}_\infty} h \in S_{s-li E^{\mathfrak{B}_n} X}^{Pe}(\mathfrak{B}_\infty), \forall h \in S_X^{Pe}(\mathfrak{F}).$$

By Proposition 2.2, we have $S_{E^{\mathfrak{B}_\infty} X}^{Pe}(\mathfrak{B}_\infty) = \{E^{\mathfrak{B}_\infty} f, f \in S_X^{Pe}(\mathfrak{F})\}$.

Therefore $S_{E^{\mathfrak{B}_\infty} X}^{Pe}(\mathfrak{B}_\infty) \subset S_{s-li E^{\mathfrak{B}_n} X}^{Pe}(\mathfrak{B}_\infty)$.

Consequently

$$E^{\mathfrak{B}_\infty} X(\omega) \subset s - li E^{\mathfrak{B}_n} X(\omega) \text{ a.s.}$$

• *Step2: we show that $w - ls E^{\mathfrak{B}_n} X(\cdot) \subset E^{\mathfrak{B}_\infty} X(\cdot)$ a.s.*

Let $(e_k^*)_{k \geq 1}$ be a dense sequence in E^* for the Mackey topology, so by [2] for all $k \geq 1$ and for all $n \geq 1$, we have

$$\delta^*(e_k^*, E^{\mathfrak{B}_n} X(\cdot)) = E^{\mathfrak{B}_n} \delta^*(e_k^*, X(\cdot)) \text{ a.s. and } \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X(\cdot)) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X(\cdot)) \text{ a.s.}$$

This yields the existence of a negligible set N'_1 such that, for every $\omega \in \Omega \setminus N'_1$, for every $n \geq 1$

and for every $k \geq 1$ one has

$$\delta^*(e_k^*, E^{\mathfrak{B}_n} X(\omega)) = E^{\mathfrak{B}_n} \delta^*(e_k^*, X(\omega)) \text{ and } \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X(\omega)) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X(\omega)).$$

Using, the Lemma 2.1 applied to integrable real function $\delta^*(e_k^*, X)$, so there exists a negligible set N_2' such that for every $k \geq 1$ and $\omega \in \Omega \setminus N_2'$

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} \delta^*(e_k^*, X(\omega)) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X(\omega)). \quad (2.3)$$

So, let $\omega \in \Omega \setminus N_1' \cup N_2'$ and $x \in w - ls E^{\mathfrak{B}_n} X(\omega)$, then there exists $(x_m)_{m \geq 1}$ in $(E^{\mathfrak{B}_{n_m}} X)_{m \geq 1}$ such that $x_m \xrightarrow{w} x$, where $(E^{\mathfrak{B}_{n_m}} X)_{m \geq 1}$ is a subsequence of $(E^{\mathfrak{B}_n} X)_{n \geq 1}$. Then by (2.3)

$$\begin{aligned} \langle e_k^*, x \rangle &= \lim_{m \rightarrow +\infty} \langle e_k^*, x_m \rangle \leq \limsup_{m \rightarrow \infty} \delta^*(e_k^*, E^{\mathfrak{B}_{n_m}} X(\omega)) = \limsup_{m \rightarrow \infty} E^{\mathfrak{B}_{n_m}} \delta^*(e_k^*, X(\omega)) = \\ &= E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X(\omega)) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X(\omega)), \quad \forall k \geq 1. \end{aligned}$$

Therefore, for all $k \geq 1$ and for almost surely $\omega \in \Omega$, $\langle e_k^*, x \rangle \leq \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X(\omega))$.

Then $x \in E^{\mathfrak{B}_\infty} X(\omega)$, which implies that

$$w - ls E^{\mathfrak{B}_n} X(\cdot) \subset E^{\mathfrak{B}_\infty} X(\cdot) \text{ a.s.}$$

This yields

$$M - \lim_{n \rightarrow \infty} E^{\mathfrak{B}_n} X(\cdot) = E^{\mathfrak{B}_\infty} X(\cdot) \text{ a.s.}$$

Now, we give a new version of Fatou's lemma for the conditional expectation of the strong lower limit of a $ckw(E)$ -valued Pettis-integrable random sets. The following theorem is an extension of Theorem 2.2 to multivalued case.

Theorem 2.4 Let $(\mathfrak{B}_n)_{n \geq 1}$ be a decreasing sequence of a sub- σ -algebras of $\mathfrak{F}$ and set $\mathfrak{B}_\infty = \bigcap_{n \geq 1} \mathfrak{B}_n$. Let Y be a positive random variable satisfying $E^{\mathfrak{B}_\infty} Y \in [0, +\infty[$. Let $(X_n)_{n \in \mathbb{N}^* \cup \{+\infty\}}$ be a sequence of a $ckw(E)$ -valued Pettis-integrable random sets such that,

1. $\forall n \in \mathbb{N}^* \cup \{+\infty\}, |X_n(\cdot)| \leq Y(\cdot)$.
2. $X_\infty(\cdot) \subset s - li X_n(\cdot) \text{ a.s.}$

Then

$$E^{\mathfrak{B}_\infty} X_\infty(\cdot) \subset s - li E^{\mathfrak{B}_n} X_n(\cdot) \text{ a.s.}$$

Proof By (1) and the condition $E^{\mathfrak{B}_\infty}Y \in [0, +\infty]$, $E^{\mathfrak{B}_n}X_n$ exists and it is in $P_{ckw(E)}^1(\mathfrak{F})$ and by [33], we have

$$S_{E^{\mathfrak{B}_n}X_n}^{Pe}(\mathfrak{B}_n) = \{E^{\mathfrak{B}_n}g, \quad g \in S_{X_n}^{Pe}(\mathfrak{F})\} \quad (2.4)$$

Since X_n is $ckw(E)$ -valued Pettis-integrable random sets, then by [42] there exists a sequence $(f_k)_{k \geq 1}$ of Pettis-integrable selection of X_n such that, $\forall \omega \in \Omega$, $X_n(\omega) = cl\{f_k(\omega), k \geq 1\}$.

Let $g \in S_{X_\infty}^{Pe}(\mathfrak{F})$ and $\omega \in \Omega$, set

$$H_n(\omega) = \{y \in X_n(\omega), \quad \|g(\omega) - y\| \leq d(g(\omega), X_n(\omega)) + \frac{1}{n}\}.$$

On the other hand, we have

$$d(g(\omega), X_n(\omega)) = \inf\{\|g(\omega) - z\|, \quad z \in X_n(\omega)\} = \inf_{k \geq 1} d(g(\omega), f_k(\omega)).$$

Then, according to the lower bound property, there exists $p \in \mathbb{N}^*$ such that

$$\|g(\omega) - f_p(\omega)\| \leq d(g(\omega), X_n(\omega)) + \frac{1}{n}.$$

Which implies that $H_n(\omega) \neq \emptyset$. Moreover X_n is closed and convex, so is H_n and since H_n is weakly compact and $H_n(\omega) \subset X_n(\omega)$, then $H_n(\omega) \in ckw(E)$. Consequently, it follows from theorem III.41 in [20] that H_n is measurable and by theorem III.6 in [20], there is a measurable selection g_n of H_n . So g_n is also a measurable selection of X_n , then by theorem 5.4 in [32], g_n is Pettis-integrable and for all $\omega \in \Omega$, we have

$$\|g(\omega) - g_n(\omega)\| \leq d(g(\omega), X_n(\omega)) + \frac{1}{n}.$$

By (2), we have $X_\infty(\cdot) \subset s - li \quad X_n(\cdot)$ a.s. and since $g(\cdot) \in X_\infty(\cdot)$ a.s.

then $\lim_{n \rightarrow +\infty} d(g(\cdot), X_n(\cdot)) = 0$ a.s. Therefore

$$\lim_{n \rightarrow +\infty} \|g(\cdot) - g_n(\cdot)\| = 0 \text{ a.s.}$$

From the conditions (1), we deduce that the sequence $(g_n)_{n \geq 1}$ satisfied all conditions of the Theorem 2.2, which implies that

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} g_n(\cdot) = E^{\mathfrak{B}_\infty} g(\cdot) \text{ a.s.}$$

By the definition of the strong upper limit (s -li) and (2.4), we conclude that

$$E^{\mathfrak{B}_\infty} g(\cdot) \in s\text{-li } E^{\mathfrak{B}_n} X_n(\cdot) \text{ a.s.}$$

Then

$$E^{\mathfrak{B}_\infty} g \in S_{s\text{-li } E^{\mathfrak{B}_n} X_n}^{Pe}(\mathfrak{B}_\infty), \forall g \in S_{X_\infty}^{Pe}(\mathfrak{F}).$$

On the other hand $S_{E^{\mathfrak{B}_\infty} X_\infty}^{Pe}(\mathfrak{B}_\infty) = \{E^{\mathfrak{B}_\infty} f, f \in S_{X_\infty}^{Pe}(\mathfrak{F})\}$.

Therefore

$$S_{E^{\mathfrak{B}_\infty} X_\infty}^{Pe}(\mathfrak{B}_\infty) \subset S_{s\text{-li } E^{\mathfrak{B}_n} X_n}^{Pe}(\mathfrak{B}_\infty).$$

Consequently

$$E^{\mathfrak{B}_\infty} X_\infty(\cdot) \subset s\text{-li } E^{\mathfrak{B}_n} X_n(\cdot) \text{ a.s.}$$

Now, let us state the Mosco convergence of conditional expectation for a $ckw(E)$ -valued Pettis-integrable random sets.

Theorem 2.5 Assume that E is separable. Let $(\mathcal{B}_n)_{n \geq 1}$ be a decreasing sequence of a sub- σ -algebra of $\mathfrak{F}$ and set $\mathfrak{B}_\infty = \cap_{n \geq 1} \mathfrak{B}_n$. If Y is a positive random variable satisfying $E^{\mathfrak{B}_\infty} Y \in [0, +\infty[$. Let $(X_n)_{\mathbb{N}^* \cup \{+\infty\}}$ be a sequence of a $ckw(E)$ -valued Pettis-integrable random sets such that

1. $\forall n \in \mathbb{N}^* \cup \{+\infty\}, |X_n(\cdot)| \leq Y(\cdot)$.
2. $X_\infty(\cdot) \subset s\text{-li } X_n(\cdot) \text{ a.s.}$
3. $\lim_{n \rightarrow +\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \text{ a.s. } \forall x^* \in E^*$.

Then

$$M\text{-}\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} X_n = E^{\mathfrak{B}_\infty} X_\infty \text{ a.s.}$$

Proof By condition (1), (2) and Theorem 2.4, we conclude that

$$E^{\mathfrak{B}_\infty} X_\infty(.) \subset s-ls E^{\mathfrak{B}_n} X_n(.) \quad a.s.$$

So, it is enough to check that $w-ls E^{\mathfrak{B}_n} X_n(.) \subset E^{\mathfrak{B}_\infty} X_\infty(.)$ a.s.

By the condition (1) and (3), it is easy to check that, the sequence $(\delta^*(x^*, X_n))_{n \geq 1}$ satisfied all conditions of Lemma 2.2, then

$$\forall x^* \in E^*, \quad \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} \delta^*(x^*, X_n(.)) = E^{\mathfrak{B}_\infty} \delta^*(x^*, X_\infty(.)) \quad a.s.$$

Let $(e_k^*)_{k \geq 1}$ be a dense sequence in E^* for the Mackey topology, so by [2] for all $k \geq 1$ and for every $n \geq 1$, we have

$$E^{\mathfrak{B}_n} \delta^*(e_k^*, X_n(.)) = \delta^*(e_k^*, E^{\mathfrak{B}_n} X_n(.)) \quad a.s. \text{ and } E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X_\infty(.)) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X_\infty(.)) \quad a.s.$$

This yields the existence of a negligible set N'_3 such that, for every $\omega \in \Omega \setminus N'_3$ and every $k \geq 1$, one has

$$E^{\mathfrak{B}_n} \delta^*(e_k^*, X_n(\omega)) = \delta^*(e_k^*, E^{\mathfrak{B}_n} X_n(\omega)), \quad E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X_\infty(\omega)) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X_\infty(\omega)) \text{ and}$$

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} \delta^*(x^*, X_n(\omega)) = E^{\mathfrak{B}_\infty} \delta^*(x^*, X_\infty(\omega)).$$

Let $\omega \in \Omega \setminus N'_3$ and $x \in w-ls E^{\mathfrak{B}_n} X_n(\omega)$, then there exists $(x_m)_{m \geq 1}$ in $(E^{\mathfrak{B}_{n_m}} X_{n_m})_{m \geq 1}$ such that $x_m \xrightarrow{w} x$, where $(E^{\mathfrak{B}_{n_m}} X_{n_m})_{m \geq 1}$ is a subsequence of $(E^{\mathfrak{B}_n} X_n)_{n \geq 1}$. Then

$$\begin{aligned} \langle e_k^*, x \rangle &= \lim_{m \rightarrow +\infty} \langle e_k^*, x_m \rangle \leq \limsup_{m \rightarrow \infty} \delta^*(e_k^*, E^{\mathfrak{B}_{n_m}} X_{n_m}(\omega)) = \\ &= \limsup_{m \rightarrow \infty} E^{\mathfrak{B}_{n_m}} \delta^*(e_k^*, X_{n_m}(\omega)) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X_\infty(\omega)) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X_\infty(\omega)), \forall k \geq 1. \end{aligned}$$

Therefore $x \in E^{\mathfrak{B}_\infty} X_\infty(\omega)$. This yields $w-ls E^{\mathfrak{B}_n} X_n(.) \subset E^{\mathfrak{B}_\infty} X_\infty(.)$ a.s. Hence

$$M-\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} X_n(.) = E^{\mathfrak{B}_\infty} X_\infty(.) \quad a.s.$$

2.5 Convergence of conditional expectation for a $cc(E)$ -valued Pettis integrable random sets

In this section, we give a new version of the dominated convergence theorem for the conditional expectation of a closed convex and integrably bounded random sets and we extend this theorems of a $cc(\mathbb{E})$ -valued Pettis integrable random sets. The notion of convergence taken in this section is the linear topology.

Theorem 2.6 *Assume that $\mathbb{E}^*$ is strongly separable. Let $(\mathfrak{B}_n)_{n \geq 1}$ be a decreasing sequence of a sub- σ -algebras of $\mathfrak{F}$ and set $\mathfrak{B}_\infty$ such that $\mathfrak{B}_\infty = \cap_{n \geq 1} \mathfrak{B}_n$. If Y is a positive random variable satisfying $\sup_n E^{\mathfrak{B}_n} Y < \infty$. Let $(X_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ be a sequence of a closed convex and integrably bounded random sets. Assume that*

1. $\forall n \in \mathbb{N}^*, \|X_n(\cdot)\| < Y(\cdot) \text{ a.s.}$
2. $\lim_{n \rightarrow +\infty} \delta^*(x^*, X_n(\cdot)) = \delta^*(x^*, X_\infty(\cdot)) \text{ a.s.}$

Then

$$\tau_L - \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} X_n(\cdot) = E^{\mathfrak{B}_\infty} X_\infty(\cdot) \text{ a.s.}$$

Proof • *Claim 1: chow that $\forall x^* \in E^*, \lim_{n \rightarrow +\infty} \delta^*(x^*, E^{\mathfrak{B}_n} X_n(\omega)) = \delta^*(x^*, E^{\mathfrak{B}_\infty} X_\infty(\omega)) \text{ a.s.}$*

Since $X_n \in L^1_{cc(E)}(\mathcal{F})$ then $\forall x^* \in E^*, \delta^*(x^*, X_n) \in L^1_{\mathbb{R}}(\mathcal{F})$. On the other hand by the condition 1, we have $\forall x^* \in E^*, \forall n \in \mathbb{N}^*,$

$$|\delta^*(x^*, X_n)| \leq \|x^*\| \cdot \|X_n\| \leq \|x^*\| \cdot Y \text{ a.s.}$$

Furthermore, by the condition 2, we have

$$\lim_{n \rightarrow +\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \text{ a.s.}$$

Then the sequence $(\delta^*(x^*, X_n))_n$ satisfied all conditions of Lemma 2.2, which implies

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} \delta^*(x^*, X_n) = E^{\mathfrak{B}_\infty} \delta^*(x^*, X_\infty) \text{ a.s.}$$

Let $(e_k^*)_{k \geq 1}$ be a dense sequence in E^* with respect to the norm topology in E^* , so by Proposition 3.1 in [18], for all $k \geq 1$ and for every $n \geq 1$, we have

$E^{\mathfrak{B}_n} \delta^*(e_k^*, X_n) = \delta^*(e_k^*, E^{\mathfrak{B}_n} X_n)$ a.s. and $E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X_\infty) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X_\infty)$ a.s. Then

$$\lim_{n \rightarrow +\infty} \delta^*(e_k^*, E^{\mathfrak{B}_n} X_n(\cdot)) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X_\infty(\cdot)) \text{ a.s.}$$

And since $\sup_n |E^{\mathfrak{B}_n} X_n| \leq \sup_n E^{\mathfrak{B}_n} |X_n| \leq \sup_n E^{\mathfrak{B}_n} Y < +\infty$ a.s. Then for almost surely $\omega \in \Omega$, the sequence $(\delta^*(\cdot, E^{\mathfrak{B}_n} X_n))_{n \geq 1}$ is uniformly continuous with respect to the norm topology in E^* . Therefore for almost surely $\omega \in \Omega$ and for all $x^* \in \mathbb{E}^*$, we have

$$\lim_{n \rightarrow +\infty} \delta^*(x^*, E^{\mathfrak{B}_n} X_n(\omega)) = \delta^*(x^*, E^{\mathfrak{B}_\infty} X_\infty(\omega)) \text{ a.s.}$$

- Claim 2: show that $\forall x \in E, \lim_{n \rightarrow +\infty} d(x, E^{\mathfrak{B}_n} X_n(\omega)) = d(x, E^{\mathfrak{B}_\infty} X_\infty(\omega))$ a.s.

By the Lemma 2.5 in [98], there exists an integrable sequence $(f_m)_{m \geq 1}$ in $S_{X_n}^1$ such that

$$S_{E^{\mathfrak{B}_\infty} X_\infty}^1(\mathfrak{B}_\infty) = cl\{E^{\mathfrak{B}_\infty} f_m, m \geq 1\}.$$

And by the Lemma 2.1, we have

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f_m(\cdot) = E^{\mathfrak{B}_\infty} f_m(\cdot) \text{ a.s.}$$

Let $x \in \mathbb{E}$, then for every $\varepsilon > 0$ there exists an integer $m_\varepsilon \geq 1$ such that

$$\|x - E^{\mathfrak{B}_\infty} f_{m_\varepsilon}\| \leq d(x, E^{\mathfrak{B}_\infty} X_\infty) + \varepsilon.$$

And since

$$\limsup_{n \rightarrow +\infty} d(x, E^{\mathfrak{B}_n} X_n) \leq \lim_{n \rightarrow +\infty} \|x - E^{\mathfrak{B}_n} f_{m_\varepsilon}\| = \|x - E^{\mathfrak{B}_\infty} f_{m_\varepsilon}\| \text{ a.s.}$$

Therefore

$$\limsup_{n \rightarrow +\infty} d(x, E^{\mathfrak{B}_n} X_n) \leq d(x, E^{\mathfrak{B}_\infty} X_\infty) \text{ a.s.} \quad (2.5)$$

On the other hand $\forall n \in \mathbb{N}^*$,

$$d(x, E^{\mathfrak{B}_n} X_n) = \sup_{x^* \in B_{E^*}} (\prec x^*, x \succ -\delta^*(x^*, E^{\mathfrak{B}_n} X_n))$$

Then, $\forall x^* \in B_{E^*}$, $\prec x^*, x \succ -\delta^*(x^*, E^{\mathfrak{B}_n} X_n) \leq d(x, E^{\mathfrak{B}_n} X_n)$. By passing to the lower limit, we obtain

$$\prec x^*, x \succ -\delta^*(x^*, E^{\mathfrak{B}_\infty} X_\infty) \leq \liminf_{n \rightarrow +\infty} d(x, E^{\mathfrak{B}_n} X_n) \text{ a.s.}$$

Which implies

$$\sup_{x^* \in B_{E^*}} (\prec x^*, x \succ -\delta^*(x^*, E^{\mathfrak{B}_\infty} X_\infty)) \leq \liminf_{n \rightarrow +\infty} d(x, E^{\mathfrak{B}_n} X_n) \text{ a.s.}$$

Therefore

$$d(x, E^{\mathfrak{B}_\infty} X_\infty(.)) \leq \liminf_{n \rightarrow +\infty} d(x, E^{\mathfrak{B}_n} X_n(.)) \text{ a.s.} \quad (2.6)$$

By (2.5), (2.6) and using the fact that the family of distance functionals $(d(., E^{\mathfrak{B}_n} X_n))_{n \geq 1}$ is equicontinuous and the separability of $\mathbb{E}$, the claim 2 holds, which proves the theorem.

Theorem 2.7 Assume that $\mathbb{E}^*$ is strongly separable. Let $(\mathfrak{B}_n)_{n \geq 1}$ be a decreasing sequence of a sub- σ -algebra of $\mathfrak{F}$ and set $\mathfrak{B}_\infty$ such that $\mathfrak{B}_\infty = \cap_{n \geq 1} \mathfrak{B}_n$. If Y is a positive random variable and let $(X_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ be a sequence of $cc(E)$ -valued Pettis-integrable random sets such that

1. $\sup_n E^{\mathfrak{B}_n} Y(.) < \infty$ a.s.
2. $\forall n \geq 1$, the set $\{\delta^*(x^*, X_n), x^* \in B_{\mathbb{E}^*}\}$ is uniformly integrable.
3. $\forall n \in \mathbb{N}^*, \|X_n(.)\| < Y(.)$ a.s.
4. $\lim_{n \rightarrow +\infty} \delta^*(x^*, X_n(.)) = \delta^*(x^*, X_\infty(.))$ a.s.

Then

$$\tau_L - \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} X_n(.) = E^{\mathfrak{B}_\infty} X_\infty(.) \text{ a.s.}$$

Proof By the condition 1, 2, 3 and Proposition 2.3, we have $E^{\mathfrak{B}_n}X_n$ exists and is in $P_{cc(E)}^1(\mathfrak{B}_n)$. So for all $m \geq 1$ set $B_m = \{\omega, E^{\mathfrak{B}_\infty}Y \leq m\}$. It is clair that $(B_m)_m$ is a partition of Ω , $\mathfrak{B}_\infty$ measurable and $\forall m \geq 1$ we have

$$\int_{B_m} \|X_n\| dP \leq \int_{B_m} Y dP \leq \int_{B_m} E^{\mathfrak{B}_\infty}Y dP \leq m.P(B_m) < +\infty.$$

Which implies that the random set $X_n 1_{B_m}$ is in $L_{cc(\mathbb{E})}^1(\mathfrak{F})$, Furthermore, by the condition 4 we have

$$\lim_{n \rightarrow +\infty} \delta^*(x^*, X_n(\cdot)) = \delta^*(x^*, X_\infty(\cdot)) \text{ a.s.}$$

Then $X_n 1_{B_m}$ satisfied all conditions of the Theorem 2.6. Therefore

$$\tau_L - \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n}(X_n 1_{B_m}) = E^{\mathfrak{B}_\infty}(X_\infty 1_{B_m}) \text{ a.s.} \quad (2.7)$$

On the other hand, $B_m \subset \mathfrak{B}_\infty \subset \mathfrak{B}_n$, for every $n \in \mathbb{N}^*$ and for every $m \geq 1$ then

$$E^{\mathfrak{B}_n}(1_{B_m} X_n) = 1_{B_m} E^{\mathfrak{B}_n} X_n \quad (2.8)$$

By (2.7) and (2.8)

$$\begin{aligned} E^{\mathfrak{B}_\infty} X_\infty &= \sum_{m \geq 1} E^{\mathfrak{B}_\infty}(X_\infty) 1_{B_m} = \sum_{m \geq 1} E^{\mathfrak{B}_\infty}(X_\infty 1_{B_m}) = \sum_{m \geq 1} \tau_L - \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n}(X_n 1_{B_m}) \\ &= \sum_{m \geq 1} 1_{B_m} \cdot \tau_L - \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n}(X_n) = \tau_L - \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n}(X_n) \end{aligned}$$

Consequently,

$$\tau_L - \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} X_n = E^{\mathfrak{B}_\infty} X_\infty \text{ a.s.}$$

Remark 2.1 If the sequence of random sets $(X_n)_n$ does not depend on n , we obtain the following corollary. It is the Neveu's theorem of the conditional expectation for a $cc(\mathbb{E})$ -valued Pettis-integrable random sets.

Corollary 2.1 Assume that $\mathbb{E}^*$ is strongly separable. Let $(\mathfrak{B}_n)_{n \geq 1}$ be a decreasing sequence of a sub- σ -algebra of $\mathfrak{F}$ and set $\mathfrak{B}_\infty$ such that $\mathfrak{B}_\infty = \bigcap_{n \geq 1} \mathfrak{B}_n$ and let X be a sequence of $cc(\mathbb{E})$ -valued Pettis integrable random sets such that

1. $\sup_n E^{\mathfrak{B}_n} |X| < \infty$ a.s.
2. The set $\{\delta^*(x^*, X), x^* \in \mathbb{E}^*\}$ is uniformly integrable

Then

$$\tau_L - \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} X = E^{\mathfrak{B}_\infty} X \text{ a.s.}$$

Proof *it is sufficient to take $Y = |X|$ and applying the previous theorem to $X_n = X = X_\infty$.*

SLLN for Pettis integrable random sets

3.1 Introduction

The strong law of large numbers (SLLN) is used in a variety of fields including statistics, probability theory, and areas of economics and insurance. In recent years, SLLN has been extensively studied by several researchers. Let's mention [6], [34], [60], [58], [17] and [37]. In the theory of integration in infinite dimensional spaces, Pettis-integrability is a more general concept than that of Bochner-integrability. The purpose of this chapter is to prove the SLLN for measurable and Pettis-integrable multifunctions by using the techniques of reversed martingale. The proof is based on the recent properties of Pettis-integrable multifunctions. See for example [1], [24], [32], [39], [42] and [80].

In this context, we show that, if $(X_n)_{n \geq 1}$ is a sequence of independent random sets Pettis-integrable, then there exists a decreasing sequence of sub- σ -algebra of $\mathcal{F}$ such that

$$\frac{S_n}{n} = E^{\mathfrak{B}_n} X.$$

Which implies that the strong law of large numbers for $(X_n)_{n \geq 1}$ is equivalent to the convergence of the reversed martingale $(E^{\mathfrak{B}_n} X)_{n \geq 1}$.

It is well known in the literature that the classical theorem of the SLLN is a consequence of other theorems ([95]), it can be obtained by reversed martingale convergence theorem ([29]) or also by the ergodic theorem of Birkhof. In relation to this work, it is important to state an article of [99]. More recently, many of work has been obtained in finite dimension ([7]) for

bounded random sets and ([6]) in the unbounded case. We also mention the work of [54], [60] and [17] for unbounded random sets with values in infinite-dimensional space. The idea of reducing SLLN for independent closed convex random sets and independent random lower semicontinuous integrands to Mosco convergence of integrand reversed martingales was announced in [16] and developed in [98] for convex and weakly compact random sets. This chapter is organized as follows.

In section 2, we recall some definitions and results that will be used after. In section 3, we prove the SLLN for Pettis-integrable multifunctions with convex weakly compact values.

3.2 Notations and definitions

Throughout this chapter, we assume that $(\Omega, \mathfrak{F}, P)$ is a probability space, $(\mathfrak{B}_n)_{n \geq 1}$ is a decreasing sequence of sub- σ -algebra of $\mathfrak{F}$, such that $\mathfrak{B}_\infty = \cap_{n \geq 1} \mathfrak{B}_n$. E is a separable Banach space with the dual space E^* .

Let $(e_k^*)_{k \in \mathbb{N}}$ be a dense sequence in E^* with respect to the Mackey topology $\tau(E^*, E)$, $\overline{B_E}$ (resp. $\overline{B_{E^*}}$) the closed unit ball of E (resp. E^*). Let $cc(E)$ (resp. $cwk(E)$) be the family of all nonempty convex and closed (resp. convex weakly compact) subsets of E .

Given B in $cc(E)$, the distance function and the support function associated with B are defined by

$$d(x, B) = \inf\{\|x - y\|, y \in B\}, \quad (x \in E),$$

$$\delta^*(x^*, B) = \sup\{\langle x^*, y \rangle, y \in B\}, \quad (x^* \in E^*).$$

For any C in $cwk(E)$, we get

$$|C| = \sup\{\|x\|, x \in C\}.$$

A $cc(E)$ -valued sequence $(Y_n)_{n \geq 1}$ is called Mosco convergent to a closed convex set Y_∞ if $Y_\infty = s - li Y_n = w - ls Y_n$, where

$$s - li Y_n = \{y \in E, \exists y_n \longrightarrow y, y_n \in Y_n, \forall n \geq 1\},$$

$$w - ls Y_n = \{y \in E, \exists y_{n_k} \longrightarrow y \text{ weakly}, y_{n_k} \in Y_{n_k}, \forall k \geq 1\}.$$

If $(Y_n)_{n \geq 1}$ Mosco converges to Y_∞ in $cc(E)$, we write $M - \lim Y_n = Y_\infty$.

A measurable function $g : \Omega \longrightarrow E$ is Pettis-integrable, if g is scalarly integrable

(i. e. $\langle x^*, g \rangle$ is integrable), and for each $A \in \mathfrak{F}$, there exists x_A in E , such that

$$\int_A \langle x^*, g \rangle dP = \langle x^*, x_A \rangle, \quad \forall x^* \in E^*.$$

x_A is denoted by $Pe\text{-}\int_A g dP$ and is called the Pettis-integral of g over A . We will denote by $P_E^1(\mathfrak{F})$ the space of all measurable and Pettis-integrable E -valued function defined on $(\Omega, \mathfrak{F}, P)$. We consider the space $P_E^1(\mathfrak{F})$ provided with the following topologies:

- The topology of the usual Pettis norm

$$\|g\|_{Pe} = \sup_{x^* \in B_{E^*}} \int_\Omega |\langle x^*, g \rangle| dP.$$

- The topology induced by the duality $(P_E^1(X), L^\infty \otimes E^*)$. Recall that a sequence $(g_n)_{n \geq 1}$ in $P_E^1(X)$ converges to g in this topology, if for each $h \in L^\infty(\mathfrak{F})$ and for all $x^* \in E^*$, one has

$$\lim_{n \rightarrow +\infty} \int_\Omega h(\omega) \langle x^*, g_n(\omega) \rangle dP = \int_\Omega h(\omega) \langle x^*, g(\omega) \rangle dP.$$

This topology is known as the weak topology of the space $P_E^1(\mathfrak{F})$ and is denoted by $w\text{-}Pe$.

A multifunction $X : \Omega \longrightarrow cc(E)$ is said to be measurable, if for every open set U of E the subset $X^-(U) = \{\omega \in \Omega / X(\omega) \cap U \neq \emptyset\}$ is an element of $\mathfrak{F}$. The Effros σ -field $\mathfrak{B}$ on $cc(E)$ is generated by the subsets $U^- = \{F \in cc(E) / F \cap U \neq \emptyset\}$, so X is measurable if, for any $B \in \mathfrak{B}$, one has $X^{-1}(B) \in \mathfrak{F}$.

Two measurable multifunctions X and Y are said to be equal scalarly almost surely, if the following equality holds:

$$\delta^*(x^*, X(\cdot)) = \delta^*(x^*, Y(\cdot)) \text{ a. s. } \forall x^* \in E^*.$$

For any convex and weakly compact measurable multifunction X , the measurability of X is equivalent to that of its support functional $\delta^*(x^*, X)$. (See. [19]).

The tribe trace of $\mathfrak{B}$ on $cwk(E)$ is defined by the set $\{cwk(E) \cap B / B \in \mathfrak{B}\}$ and is denoted by ξ .

A measurable function $f : \Omega \longrightarrow E$ is said to be a selection of X , if, for any $\omega \in \Omega$, $f(\omega) \in X(\omega)$. We denote by $S_X^1(\mathfrak{F})$ the set of all $\mathfrak{F}$ -measurable and integrable selection of X . It is known that a convex and closed valued multifunction X in separable Banach space is measurable if $dom(X) \in \mathfrak{F}$ and X has a Castaing representation (i.e. there exists a sequence $(f_n)_{n \geq 1}$ of measurable selections of X such that for all $\omega \in \Omega$, $X(\omega) = cl\{f_n(\omega), n \geq 1\}$). The distribution P_Γ of the measurable multifunction $\Gamma : \Omega \longrightarrow cc(E)$ on the measurable space $(cc(E), \mathfrak{B})$ is defined by

$$P_\Gamma(B) = P(\Gamma^{-1}(B)), \forall B \in \mathfrak{B}.$$

Two measurable multifunctions Γ and Δ are said to be independent, if the following equality holds

$$P_{(\Gamma, \Delta)} = P_\Gamma \otimes P_\Delta.$$

The measurable multifunction $X : \Omega \longrightarrow cwk(E)$ is scalarly integrable, if for any $x^* \in E^*$, the real function $\delta^*(x^*, X(\cdot))$ is integrable.

We say that the measurable multifunction X is Pettis-integrable, if it is scalarly integrable and for each $A \in \mathfrak{F}$, there exists $K_A \in cwk(E)$ such that

$$\int_A \delta^*(x^*, X) dP = \delta^*(x^*, K_A), \quad \forall x^* \in E^*.$$

K_A is called the Pettis-integral of X over A . Let $S_X^{Pe}(\mathfrak{F})$ the set of all $\mathfrak{F}$ -measurable and Pettis-integrable selectors of X .

The multivalued Pettis-integral of a $cwk(E)$ -valued multifunction X is defined by

$$\int_\Omega X dP = \left\{ \int_\Omega f dP, f \in S_X^{Pe}(\mathfrak{F}) \right\},$$

and $\int_A X dP$ is convex and $\sigma(E, E^*)$ compact and for each $A \in \mathfrak{F}$, we have

$$\int_A \delta^*(x^*, X) dP = \delta^*(x^*, \int_A X dP), \quad \forall x^* \in E^*.$$

We will denote by $P_{cw(E)}^1(\mathfrak{F})$ the set of all Pettis-integrable $cw(E)$ -valued multifunctions. Given a sub- σ -algebra $\mathcal{B}$ and a $cw(E)$ -valued Pettis-integrable multifunction, the Pettis conditional expectation of X is a $\mathcal{B}$ -measurable $cw(E)$ -valued Pettis-integrable multifunction denoted by $E^{\mathcal{B}}X$ which satisfies

$$\int_A E^{\mathcal{B}}X dP = \int_A X dP, \quad \forall A \in \mathcal{B}.$$

The two following propositions (see. [1]), give a sufficient condition of the existence of the conditional expectation for a Pettis integrable E -valued function and for a $cw(E)$ -valued Pettis-integrable multifunction.

Proposition 3.1 *Let $\mathcal{B}$ be a sub σ -algebra of $\mathfrak{F}$ and let X be a Pettis-integrable E -valued function such that $E^{\mathcal{B}}|X| \in [0, +\infty[$. Then, there exists a unique $\mathcal{B}$ -measurable Pettis-integrable E -valued function denote by $E^{\mathcal{B}}X$, which enjoys the following property, for every $h \in L^\infty(\mathcal{B})$, one has*

$$\int_{\Omega} h E^{\mathcal{B}}X dP = \int_{\Omega} h X dP.$$

Proposition 3.2 *Let $\mathcal{B}$ be a sub- σ -algebra of $\mathfrak{F}$ and let X be a $cw(E)$ -valued Pettis-integrable multifunction such that $E^{\mathcal{B}}|X| \in [0, +\infty[$. Then, there exists a unique $\mathcal{B}$ -measurable $cw(E)$ -valued Pettis-integrable multifunction, which enjoys the following property, for every $h \in L^\infty(\mathcal{B})$, one has*

$$\int_{\Omega} h E^{\mathcal{B}}X dP = \int_{\Omega} h X dP.$$

[2] extended the previous theorem in a $cc(E)$ -valued Pettis-integrable multifunction.

For the convenience of the reader, we also recall the following results, which will be used after.

Lemma 3.1 ([84]) *Let $\mathbb{E}$ and $\mathbb{G}$ be two sets. Given a mapping f defined in $\mathbb{E}$ with values in a measurable space $(\mathbb{F}, \mathcal{F})$ and g a function of a set $\mathbb{E}$ in $\mathbb{G}$. Then g is measurable with respect to the σ -algebra $\sigma(f) = f^{-1}(A)$, $A \in \mathcal{F}$ if and only if there exists a measurable function $h : (\mathbb{F}, \mathcal{F}) \longrightarrow \mathbb{G}$ such that $g = h \circ f$*

Theorem 3.1 ([98]) *Let $(\Omega, \mathfrak{F}, P)$ be a probability space and let $X : (\Omega, \mathfrak{F}, P) \longrightarrow \mathbb{E}$ be a random variable, then for every measurable function $\varphi : \mathbb{E} \longrightarrow \mathbb{R}$ one has $\varphi \circ X$ is P -integrable if and only if φ is P_X -integrable. Also we have*

$$\int_{\Omega} \varphi \circ X(\omega) dP(\omega) = \int_{\mathbb{E}} \varphi(x) dP_X(x).$$

We close this section by the following useful corollaries. We begin with the famous Neuveu's corollary ([82])

Corollary 3.1 *Let $(\mathfrak{B}_n)_{n \geq 1}$ be a decreasing sequence of a sub σ -algebra of $\mathfrak{F}$ and let $f \in L^1_{\mathbb{R}}(\mathfrak{F})$, set $\mathfrak{B}_{\infty} = \cap_{m \geq 1} \mathfrak{B}_m$. Then*

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f(\cdot) = E^{\mathfrak{B}_{\infty}} f(\cdot) \text{ a.s.}$$

The following Corollary ([99]) gives an extension of the Corollary 3.1 in the vector case.

Corollary 3.2 ([99]) *Let $(\mathfrak{B}_n)_{n \geq 1}$ be a decreasing sequence of a sub- σ -algebras of $\mathfrak{F}$ and let $f \in L^1_E(\mathfrak{F})$, set $\mathfrak{B}_{\infty} = \cap_{n \geq 1} \mathfrak{B}_n$. Then*

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f = E^{\mathfrak{B}_{\infty}} f \text{ a. s.}$$

We also need the following

Corollary 3.3 *Let $\mathcal{B}$ be a sub- σ -algebra of $\mathfrak{F}$ and let X be a $cwk(E)$ -valued Pettis-integrable multifunction such that $E^{\mathcal{B}}|X| \in [0, +\infty[$. Then for all $f \in S_X^{Pe}(\mathfrak{F})$, the following properties hold*

1. $E^{\mathcal{B}} f(\cdot) \in E^{\mathcal{B}} X(\cdot) \text{ a.s.}$
2. $\delta^*(x^*, E^{\mathcal{B}} X(\cdot)) = E^{\mathcal{B}} \delta^*(x^*, X(\cdot)) \text{ a.s.}$

Proof 1. *Let $f \in S_X^{Pe}(\mathfrak{F})$ and $(e_k^*)_{k \geq 1}$ be a dense sequence in E^* with respect to the Mackey topology $\tau(E^*, E)$, then*

$$\forall A \in \mathcal{B}, \quad \int_A \langle e_k^*, f \rangle dP \leq \int_A \delta^*(e_k^*, X) dP,$$

$$\langle e_k^*, \int_A f dP \rangle \leq \delta^*(e_k^*, \int_A X dP),$$

$$\langle e_k^*, \int_A E^{\mathcal{B}} f dP \rangle \leq \delta^*(e_k^*, \int_A E^{\mathcal{B}} X dP),$$

hence

$$\langle e_k^*, E^{\mathcal{B}} f(\cdot) \rangle \leq \delta^*(e_k^*, E^{\mathcal{B}} X(\cdot)) \text{ a. s.}$$

We conclude that

$$E^{\mathcal{B}} f(\cdot) \in E^{\mathcal{B}} X(\cdot) \text{ a. s.}$$

2. For any $A \in \mathcal{B}$, we have $\delta^*(x^*, \int_A X dP) = \int_A \delta^*(x^*, X) dP$,
 then by Proposition 3.2, we have $\delta^*(x^*, \int_A E^{\mathcal{B}} X dP) = \int_A \delta^*(x^*, X) dP$.
 Since $E^{\mathcal{B}} X$ is a Pettis-integrable multifunction, then

$$\int_A \delta^*(x^*, E^{\mathcal{B}} X) dP = \int_A \delta^*(x^*, X) dP = \int_A E^{\mathcal{B}} \delta^*(x^*, X) dP.$$

And hence by uniqueness of the conditional expectation of $\delta^*(x^*, X)$, we obtain

$$\delta^*(x^*, E^{\mathcal{B}} X(\cdot)) = E^{\mathcal{B}} \delta^*(x^*, X(\cdot)) \text{ a.s.}$$

3.3 Strong Law of large numbers for Pettis-integrable multifunctions

Before stating the main result of this section, we present the following important theorem:

Theorem 3.2 Let $(\mathfrak{B}_n)_{n \geq 1}$ be a decreasing sequence of sub- σ -algebras of $\mathfrak{F}$ and set $\mathfrak{B}_\infty = \bigcap_{n \geq 1} \mathfrak{B}_n$. Let $f \in P_E^1(\mathfrak{F})$ such that $E^{\mathfrak{B}_\infty} |f| \in [0, +\infty[$. Then

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f(\cdot) = E^{\mathfrak{B}_\infty} f(\cdot) \text{ a.s.}$$

Proof Since $E^{\mathfrak{B}_\infty} |f| \in [0, +\infty[$, so by the Proposition 3.1, we have $E^{\mathfrak{B}_n} f$ exists and is in $P_E^1(\mathfrak{B}_n)$, provides a $\mathfrak{B}_\infty$ -measurable partition $(B_k)_{k \geq 1}$ of Ω such that $f_k = f 1_{B_k} \in L_E^1(\mathfrak{F})$.

Using the Corollary 3.2, we obtain

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f_k = E^{\mathfrak{B}_\infty} f_k \text{ a.s.}$$

As $B_k \subset \mathfrak{B}_\infty \subset \mathfrak{B}_n$, for every $n \geq 1$ and for every $k \geq 1$, then

$$E^{\mathfrak{B}_n} f_k = E^{\mathfrak{B}_n} 1_{B_k} f = 1_{B_k} E^{\mathfrak{B}_n} f.$$

On the other hand

$$\begin{aligned} E^{\mathfrak{B}_\infty} f &= \sum_{k=1}^{+\infty} E^{\mathfrak{B}_\infty} f_k = \sum_{k=1}^{+\infty} \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f_k = \\ &= \sum_{k=1}^{+\infty} 1_{B_k} \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f = \lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f. \text{ Therefore} \end{aligned}$$

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} f(\cdot) = E^{\mathfrak{B}_\infty} f(\cdot) \text{ a. s.}$$

Let us prove the following results which will be used after.

Proposition 3.3 *Let X and Y be two $\text{cw}k(E)$ -valued Pettis-integrable multifunctions and $\mathfrak{B}$ be a sub- σ -algebra of $\mathfrak{F}$ such that $E^{\mathfrak{B}}|X| \in [0, +\infty[$ and $E^{\mathfrak{B}}|Y| \in [0, +\infty[$. Then*

$$E^{\mathfrak{B}}(X + Y) = E^{\mathfrak{B}}X + E^{\mathfrak{B}}Y \text{ a.s.}$$

Proof Let $(e_k^*)_{k \geq 1}$ be a dense sequence in E^* with respect to the Mackey topology $\tau(E^*, E)$. Since X and Y are $\text{cw}k(E)$ -valued Pettis-integrable multifunctions, then, for any A in $\mathfrak{B}$, there exist two sets $K_A(X)$ and $K_A(Y)$ in $\text{cw}k(E)$, such that for all $k \geq 1$ $\delta^*(e_k^*, K_A(X)) = \int_A \delta^*(e_k^*, X) dP$ and $\delta^*(e_k^*, K_A(Y)) = \int_A \delta^*(e_k^*, Y) dP$.

By the Theorem 5.1.6 in [24], $X + Y$ is $\text{cw}k(E)$ -valued Pettis-integrable multifunction, then for any A in $\mathfrak{B}$, there exists a set $K_A(X + Y)$ in $\text{cw}k(E)$ such that

$$\delta^*(e_k^*, K_A(X + Y)) = \int_A \delta^*(e_k^*, X + Y) dP.$$

On the other hand

$$\delta^*(e_k^*, K_A(X) + K_A(Y)) = \delta^*(e_k^*, K_A(X)) + \delta^*(e_k^*, K_A(Y)) =$$

$$\begin{aligned}
&= \int_A \delta^*(e_k^*, X) dP + \int_A \delta^*(e_k^*, Y) dP = \int_A \delta^*(e_k^*, X) + \delta^*(e_k^*, Y) dP = \\
&= \int_A \delta^*(e_k^*, X + Y) dP = \delta^*(e_k^*, K_A(X + Y)).
\end{aligned}$$

Then $K_A(X + Y) = K_A(X) + K_A(Y)$ a. s. Therefore

$$\int_A (X + Y) dP = \int_A X dP + \int_A Y dP \quad a. s. \quad (3.1)$$

And since

$$\int_A \delta^*(e_k^*, E^{\mathfrak{B}}(X + Y)) dP = \delta^*(e_k^*, \int_A E^{\mathfrak{B}}(X + Y) dP) = \delta^*(e_k^*, \int_A X + Y dP)$$

By (3.1)

$$\begin{aligned}
\delta^*(e_k^*, \int_A X + Y dP) &= \delta^*(e_k^*, \int_A X dP + \int_A Y dP) = \delta^*(e_k^*, \int_A E^{\mathfrak{B}} X dP + \int_A E^{\mathfrak{B}} Y dP) = \\
&= \int_A \delta^*(e_k^*, E^{\mathfrak{B}} X + E^{\mathfrak{B}} Y) dP,
\end{aligned}$$

then

$$\int_A \delta^*(e_k^*, E^{\mathfrak{B}}(X + Y)) dP = \int_A \delta^*(e_k^*, E^{\mathfrak{B}} X + E^{\mathfrak{B}} Y) dP.$$

Therefore

$$E^{\mathfrak{B}}(X + Y) = E^{\mathfrak{B}} X + E^{\mathfrak{B}} Y \quad a. s.$$

Theorem 3.3 Let X and Y in $P_{cwk(E)}^1(\mathfrak{F})$, let Z be a random variable with values in a measurable space (Σ, Γ) such that (Z, X) and (Z, Y) have the same distribution. If $E^{\sigma(Z)}|X| \in [0, +\infty[$ and $E^{\sigma(Z)}|Y| \in [0, +\infty[$. Then

$$E^{\sigma(Z)}X(.) = E^{\sigma(Z)}Y(.) \quad a. s.$$

Proof Since X is Pettis-integrable with values in $cwk(E)$, then $S_X^{Pe}(\sigma(X)) \neq \emptyset$. So by [42], there exists $(f_n)_{n \geq 1}$ selection Pettis-integrable of X such that

$$X(w) = cl\{f_n, n \geq 1\}.$$

Let $f : \Omega \longrightarrow E$ be a fixed element in $S_X^{Pe}(\sigma(X))$, $h \in L^\infty(\sigma(Z))$ and $x^* \in E^*$. Then, by Doob's factorisation lemma (Lemma 3.1), we can find a ξ -measurable mapping u from $cwk(E)$ into E (i.e. $(cwk(E), \xi) \xrightarrow{u} (E, \mathfrak{B}(E))$) and a Γ -measurable function v from Σ into $\mathbb{R}$ (i.e. $(\Sigma, \Gamma) \xrightarrow{v} (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$), which satisfy

$$\begin{aligned} f : (\Omega, \mathfrak{F}, P) &\xrightarrow{X} (cwk(E), \xi) \xrightarrow{u} (E, \mathfrak{B}(E)), & f(\omega) &= (u \circ X)(\omega). \\ h : (\Omega, \mathfrak{F}, P) &\xrightarrow{Z} (\Sigma, \Gamma) \xrightarrow{v} (\mathbb{R}, \mathfrak{B}(\mathbb{R})), & h(\omega) &= (v \circ Z)(\omega). \end{aligned}$$

On the other hand

$$\int_{\Omega} \delta^*(h \otimes x^*, X) dP = \sup \left\{ \int_{\Omega} h(\omega) \langle x^*, f \rangle dP, f \in S_X^{Pe}(\sigma(X)) \right\} \quad (3.2)$$

And since

$$\int_{\Omega} h(\omega) \langle x^*, f \rangle dP = \int_{\Omega} v \circ Z(\omega) \langle x^*, u \circ X(\omega) \rangle dP.$$

By using the application

$$\begin{aligned} (\Omega, \mathfrak{F}, P) &\xrightarrow{g} (\Sigma \times cwk(E), \Gamma \otimes \xi) \xrightarrow{\varphi} (\mathbb{R}, \mathfrak{B}(\mathbb{R})) \\ \omega &\longrightarrow (Z, X) \longrightarrow v \circ Z(\omega) \langle x^*, u \circ X(\omega) \rangle = h(\omega) \langle x^*, f(\omega) \rangle. \end{aligned}$$

And by the classical transfer theorem, we have

$$\begin{aligned} \int_{\Omega} \varphi \circ g(\omega) dP &= \int_{\Omega} h(\omega) \langle x^*, f(\omega) \rangle dP = \int_{\Omega} v \circ Z(\omega) \langle x^*, u \circ X(\omega) \rangle dP = \\ &= \int_{\Sigma \otimes cwk(E)} \varphi(z, B) dP_{(Z, X)}(z, B) = \int_{\Sigma \otimes cwk(E)} v(z) \langle x^*, u(B) \rangle dP_{(Z, X)}(z, B) = \\ &= \int_{\Sigma \otimes cwk(E)} v(z) \langle x^*, u(B) \rangle dP_{(Z, Y)}(z, B) = \int_{\Sigma \otimes cwk(E)} \varphi(z, B) dP_{(Z, Y)}(z, B) = \\ &= \int_{\Omega} v \circ Z(\omega) \langle x^*, u \circ Y(\omega) \rangle dP = \int_{\Omega} h(\omega) \langle x^*, u \circ Y(\omega) \rangle dP. \end{aligned}$$

Then

$$\int_{\Omega} h(\omega) \langle x^*, f(\omega) \rangle dP = \int_{\Omega} h(\omega) \langle x^*, u \circ Y(\omega) \rangle dP. \quad (3.3)$$

By combining relation (3.2), (3.3) and the fact that $\forall B \in cwk(E)$, $u(B) \in B$, P_Y a. s.

we obtain

$$\int_{\Omega} \delta^*(h \otimes x^*, X) dP = \sup \left\{ \int_{\Omega} h(\omega) \langle x^*, f(\omega) \rangle dP, f \in S_Y^{Pe}(\sigma(Y)) \right\} = \int_{\Omega} \delta^*(h \otimes x^*, Y) dP.$$

Then

$$\int_{\Omega} \delta^*(h \otimes x^*, X) dP = \int_{\Omega} \delta^*(h \otimes x^*, Y) dP.$$

In particular, if $h = 1_A$, for any $A \in \sigma(Z)$, we have

$$\int_{\Omega} 1_A \delta^*(x^*, X) dP = \int_{\Omega} 1_A \delta^*(x^*, Y) dP,$$

then

$$\delta^*(x^*, \int_A X dP) = \delta^*(x^*, \int_A Y dP),$$

therefore

$$\delta^*(x^*, \int_A E^{\sigma(Z)} X dP) = \delta^*(x^*, \int_A E^{\sigma(Z)} Y dP),$$

And hence by uniqueness of the conditional expectation of X relative to $\sigma(Z)$, then

$$E^{\sigma(Z)} X(\cdot) = E^{\sigma(Z)} Y(\cdot) \text{ a. s.}$$

Before giving the principal result, we also need the following classical theorem (see p. 52 in [75]).

Theorem 3.4 *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two sub- σ -algebras of $\mathfrak{F}$, $\Phi = \sigma(\mathfrak{B}_1, \mathfrak{B}_2)$ is a σ -algebra generated by $\mathfrak{B}_1$ and $\mathfrak{B}_2$ and let f_1 an integrable real measurable function. If f_1 and $\mathfrak{B}_1$ are independent of $\mathfrak{B}_2$. Then*

$$E^{\Phi} f_1(\cdot) = E^{\mathfrak{B}_1} f_1(\cdot) \text{ a. s.}$$

Now, we give the main result of this section:

Theorem 3.5 *Let $(X_n)_{n \geq 1}$ be a sequence of independent measurable multifunctions in $P_{cw(E)}^1(\mathfrak{F})$. Let $S_n = X_1 + X_2 + X_3 + \dots + X_n$, $\mathfrak{B}_n = \sigma(S_n, X_{n+1}, X_{n+2}, \dots)$, $\mathfrak{B}_{\infty} = \bigcap_{n \geq 1} \mathfrak{B}_n$ and assume that*

(i) $\forall n, \forall j \in \{1, 2, \dots, n\}$, (S_n, X_1) and (S_n, X_j) have the same distribution.

(ii) $\forall n, \forall j \in \{1, 2, \dots, n\}, E^{\mathfrak{B}_n} |X_j| \in [0, +\infty[.$

Then, we have the following assertions

1. $\forall n, \forall j \in \{1, 2, \dots, n\}, E^{\sigma(S_n)} X_j(.) = E^{\sigma(S_n)} X_1(.)$ a. s.
2. $E^{\mathfrak{B}_n} X_1(.) = E^{\sigma(S_n)} X_1(.)$ a. s.
3. $M - \lim_{n \rightarrow \infty} \frac{S_n}{n} = M - \lim_{n \rightarrow \infty} E^{\mathfrak{B}_n} X_1(.) = E^{\mathfrak{B}_\infty} X_1(.)$ a. s.

Proof • The first equality follows from the Theorem 3.3.

• Now let us prove the second equality.

Let $(e_k^*)_{k \geq 1}$ be a dense sequence in E^* for the Mackey topology. Set $\mathfrak{B}_1 = \sigma(S_n)$, $\mathfrak{B}_2 = \sigma(X_{n+1}, X_{n+2}, \dots)$ and $f_1 = \delta^*(e_k^*, X_1)$.

Since X_1 and S_n are independent of $\mathfrak{B}_2$, so f_1 and S_n are independent of $\mathfrak{B}_2$, then by applying the Theorem 3.4, we have

$$E^{\mathfrak{B}_n} f_1(.) = E^{\mathfrak{B}_1} f_1(.) \text{ a. s.}$$

So, $E^{\mathfrak{B}_n} \delta^*(e_k^*, X_1) = E^{\mathfrak{B}_1} \delta^*(e_k^*, X_1)$ then by using the Corollary 3.3, we obtain

$$\delta^*(e_k^*, E^{\mathfrak{B}_n} X_1) = \delta^*(e_k^*, E^{\mathfrak{B}_1} X_1),$$

and since $E^{\mathfrak{B}_n} X_1(.)$ and $E^{\sigma(S_n)} X_1(.)$ are convex and weakly compact, then

$$E^{\mathfrak{B}_n} X_1(.) = E^{\sigma(S_n)} X_1(.) \text{ a. s.}$$

• Now, we show the last assertion

Step 1: we claim that $\frac{S_n}{n} = E^{\mathfrak{B}_n} X_1(.)$ a. s.

We have $S_n = \sum_{i=1}^n X_i = E^{\sigma(S_n)} (\sum_{i=1}^n X_i)$, so by the Proposition 3.3, the first and the second assertions of the theorem, we obtain

$$E^{\sigma(S_n)} (\sum_{i=1}^n X_i) = \sum_{i=1}^n E^{\sigma(S_n)} X_i = \sum_{i=1}^n E^{\sigma(S_n)} X_1 = n E^{\sigma(S_n)} X_1 = n E^{\mathfrak{B}_n} X_1.$$

Then

$$\frac{S_n}{n} = E^{\mathfrak{B}_n} X_1(.) \text{ a. s.}$$

Step 2: we show that $M - \lim_{n \rightarrow \infty} E^{\mathfrak{B}_n} X_1 = E^{\mathfrak{B}_\infty} X_1(.)$ a. s.

- We begin by proving that $E^{\mathfrak{B}_\infty} X_1(.) \subset s - li E^{\mathfrak{B}_n} X_1(.)$ a. s.

Since X_1 is Pettis-integrable, then there exists $(g_k)_{k \geq 1}$ Pettis-integrable selection of X_1 such that

$$X_1 = cl\{g_k, k \geq 1\}. \quad (3.4)$$

On the other hand $E^{\mathfrak{B}_n} |X_1| \in [0, +\infty]$, then $E^{\mathfrak{B}_n} X_1$ exists and is in $P_{cwk(E)}^1(\mathfrak{B}_n)$. Let $g \in S_{X_1}^{Pe}(\mathfrak{B}_n)$, by Corollary 3.3, $E^{\mathfrak{B}_n} g$ exists and $E^{\mathfrak{B}_n} g(.) \in E^{\mathfrak{B}_n} X_1(.)$ a. s.

Using the theorem 3.2, we have $\lim_{n \rightarrow \infty} E^{\mathfrak{B}_n} g = E^{\mathfrak{B}_\infty} g$ a. s.

Since by [33]

$$S_{E^{\mathfrak{B}_\infty} X_1}^{Pe} = \{E^{\mathfrak{B}_\infty} f, f \in S_{X_1}^{Pe}(\mathfrak{F})\}$$

Then $E^{\mathfrak{B}_\infty} g \in S_{s-li E^{\mathfrak{B}_n} X_1}^{Pe}$ and by (3.4), we obtain $S_{E^{\mathfrak{B}_\infty} X_1(.)}^{Pe} \subset S_{s-li E^{\mathfrak{B}_n} X_1(.)}^{Pe}$ a. s.

Then

$$E^{\mathfrak{B}_\infty} X_1(.) \subset s - li E^{\mathfrak{B}_n} X_1(.) \text{ a. s.}$$

- Now, we show that $w - ls E^{\mathfrak{B}_n} X_1(.) \subset E^{\mathfrak{B}_\infty} X_1(.)$ a. s.

Let $(e_k^*)_{k \geq 1}$ be a dense sequence in E^* for the Mackey topology, then by the Corollary 3.3 we have

$$\delta^*(e_k^*, E^{\mathfrak{B}_n} X_1(.)) = E^{\mathfrak{B}_n} \delta^*(e_k^*, X_1(.)) \text{ a.s.}$$

Then, there exists a negligible set N^* , for all $k \geq 1$ and for every $\omega \in \Omega \setminus N^*$, we have

$$\delta^*(e_k^*, E^{\mathfrak{B}_n} X_1(\omega)) = E^{\mathfrak{B}_n} \delta^*(e_k^*, X_1(\omega)).$$

And by the Corollary 3.1

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} \delta^*(e_k^*, X_1(.)) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X_1(.)) \text{ a.s.}$$

Hence, there exists a negligible set $N = \cup_{k \geq 1} N_k$, for all $k \geq 1$ and for every $\omega \in \Omega \setminus N$, we have

$$\lim_{n \rightarrow +\infty} E^{\mathfrak{B}_n} \delta^*(e_k^*, X_1(\omega)) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X_1(\omega)) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X_1(\omega)).$$

So, for all $k \geq 1$ and for all $\omega \in \Omega \setminus N \cup N^*$

$$\lim_{n \rightarrow +\infty} \delta^*(e_k^*, E^{\mathfrak{B}_n} X_1(\omega)) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X_1(\omega)).$$

Let $\omega \in \Omega \setminus N \cup N^*$ and $x \in w-ls E^{\mathfrak{B}_n} X_1(\omega)$, then there exists $(x_m)_{m \geq 1}$ in $E^{\mathfrak{B}_{n_m}} X_1(\omega)$ such that $x_m \xrightarrow{w} x$, which implies $\lim_{m \rightarrow +\infty} \langle e_k^*, x_m \rangle = \langle x^*, x \rangle$ for every $k \geq 1$. Then

$$\begin{aligned} \langle e_k^*, x \rangle &= \lim_{m \rightarrow +\infty} \langle e_k^*, x_m \rangle \leq \limsup_{m \rightarrow \infty} \delta^*(x^*, E^{\mathfrak{B}_{n_m}} X_1(\omega)) = \\ &= \limsup_{m \rightarrow \infty} E^{\mathfrak{B}_{n_m}} \delta^*(e_k^*, X_1(\omega)) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X_1(\omega)) \end{aligned}$$

Consequently, $x \in E^{\mathfrak{B}_\infty} X_1(\omega)$, then

$$w-ls E^{\mathfrak{B}_n} X_1(\omega) \subset E^{\mathfrak{B}_\infty} X_1(\omega) \text{ a.s.}$$

This yields

$$M-\lim_{n \rightarrow \infty} \frac{S_n}{n} = M-\lim_{n \rightarrow \infty} E^{\mathfrak{B}_n} X_1(\cdot) = E^{\mathfrak{B}_\infty} X_1(\cdot) \text{ a.s.}$$

Corollary 3.4 Under the same hypothesis of the theorem 3.5, we have

$$M-\lim_{n \rightarrow \infty} \frac{S_n}{n} = E^{\mathfrak{B}_\infty} X_1 = \int X_1 dP \text{ a. s.}$$

Proof By the previous theorem, we need only to check that

$$E^{\mathfrak{B}_\infty} X_1(\cdot) = \int_{\Omega} X_1(\cdot) dP \text{ a.s.}$$

Since $\int_{\Omega} X_1 dP$ is convex and weakly compact. Now let $(e_k^*)_{k \geq 1}$ be a dense sequence in E^* for the Mackey topology, we have

$$\lim_{n \rightarrow \infty} \delta^*(e_k^*, E^{\mathfrak{B}_n} X_1) = \lim_{n \rightarrow \infty} E^{\mathfrak{B}_n} \delta^*(e_k^*, X_1) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X_1) = \delta^*(e_k^*, E^{\mathfrak{B}_\infty} X_1) \text{ a.s.} \quad (3.5)$$

On the other hand, $(E^{\mathfrak{B}_n} \delta^*(e_k^*, X_1))_n$ is a Pettis reversed martingale, so for each positive integer m , set, $\mathfrak{F}_m = \sigma(X_m, X_{m+1}, X_{m+2}, \dots)$ and $\mathfrak{F}_\infty = \cap_{m \geq 1} \mathfrak{F}_m$. Hence, for all any $n \geq m$, the multifunction $\sum_{j=m}^n X_j$ is $\mathfrak{F}_m$ -measurable, Moreover by the previous theorem, we have

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \delta^*(e_k^*, X_j) &= \frac{1}{n} \sum_{j=1}^{m-1} \delta^*(e_k^*, X_j) + \frac{1}{n} \sum_{j=m}^n \delta^*(e_k^*, X_j) = \\ &= \frac{1}{n} \delta^*(e_k^*, \sum_{j=1}^{m-1} X_j) + \frac{1}{n} \sum_{j=m}^n \delta^*(e_k^*, X_j) = \frac{m-1}{n} E^{\mathfrak{B}_{m-1}} \delta^*(e_k^*, X_1) + \frac{1}{n} \sum_{j=m}^n \delta^*(e_k^*, X_j). \end{aligned}$$

Then

$$\frac{1}{n} \sum_{j=1}^n \delta^*(e_k^*, X_j) = \frac{m-1}{n} E^{\mathfrak{B}_{m-1}} \delta^*(e_k^*, X_1) + \frac{1}{n} \sum_{j=m}^n \delta^*(e_k^*, X_j) \text{ a.s.} \quad (3.6)$$

Since, for every fixed positive integer m , we have $\lim_{n \rightarrow \infty} \frac{m-1}{n} E^{\mathfrak{B}_{m-1}} \delta^*(e_k^*, X_1) = 0 \text{ a.s.}$

Then by (3.6)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \delta^*(e_k^*, X_j) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \delta^*(e_k^*, X_j) = \lim_{n \rightarrow \infty} \delta^*(e_k^*, E^{\mathfrak{B}_n} X_1) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X_1) \text{ a.s.} \quad (3.7)$$

Since for all fixed integer m , the multifunction $\frac{1}{n} \sum_{j=m}^n X_j(\cdot)$ is $\mathfrak{F}_m$ -measurable and by (3.7), $E^{\mathfrak{B}_\infty} X_1$ is $\mathfrak{F}_m$ -measurable so is $\mathfrak{F}_\infty$ -measurable. Then by the independence of $(\sigma(X_n))_{n \in \mathbb{N}^*}$ and the Kolmogorov's Zero-One law (see. [81]), we conclude that for all $k \in \mathbb{N}^*$

$$\delta^*(e_k^*, E^{\mathfrak{B}_\infty} X_1(\cdot)) = E^{\mathfrak{B}_\infty} \delta^*(e_k^*, X_1(\cdot)) = \int_{\Omega} \delta^*(e_k^*, X_1(\cdot)) dP = \delta^*(e_k^*, \int_{\Omega} X_1(\cdot) dP) \text{ a.s.} \quad (3.8)$$

Since $E^{\mathfrak{B}_\infty} X_1$ and $\int_{\Omega} X_1(\cdot) dP$ are $\text{cwk}(E)$ -valued multifunctions and (3.8) is true, so for all $k \geq 1$, $E^{\mathfrak{B}_\infty} X_1(\cdot) = \int_{\Omega} X_1(\cdot) dP \text{ a.s.}$ Therefore

$$M - \lim_{n \rightarrow \infty} \frac{S_n}{n} = E^{\mathfrak{B}_\infty} X_1(\cdot) = \int_{\Omega} X_1(\cdot) dP \text{ a.s.}$$

Simulation and Monte Carlo Methods

4.1 Introduction

The main idea of the Monte Carlo method is to approximate an expected value $E(X)$ by an arithmetic average of the results of a big number of independent experiments which all have the same distribution as X . The strong law of large numbers ensures that this estimate converges to the correct value as the number of draws increases and the central limit theorem provides information about the error of this estimate after a finite number of draws. The Monte Carlo method is used in several areas including the finance and insurance industry, the reliability analysis of technical systems, the design of telecommunication networks and the estimation of risks.

The first publication about Monte Carlo methods was made by [45] while organizing the stochastic process of experimental determination of π number by means of throwing of needle on the sheet of lined paper.

[94] and [74] applied this method to a solution of linear integral equations, in that publication was implicitly revealed the problem of the passing of neutrons through the substance. In this context it is could mention the work of [96]. And side by side this Method obtained their proper place in computational mathematics ([72], [76], [91], ...).

This chapter is organized as follows.

In section 2, we introduce the principle of Monte Carlo method and we concentrate on various methods for obtaining variance reductions. Section 3 is devoted to the simulation of

random variable and application to evaluate option pricing. At the last section of this chapter, the addresses issues of simulating random sets and application to reliability analysis are presented.

4.2 Monte Carlo method

The generic problem for Monte Carlo method is about evaluating the integral

$$E_f[h(X)] = \int_{\mathcal{X}} h(x)f(x)dx \quad (4.1)$$

where $\mathcal{X}$ denotes the set where the random variable X takes its values, which is usually equal to the support of the density f . The principle of the Monte Carlo method for approximating (4.1) is to generate a sample $(X_1, \dots, X_n)$ of random variable X with density f and calculate the empirical average

$$\bar{h}_n = \frac{1}{n} \sum_{i=1}^n h(X_i),$$

which will be the approximation of $E_f[h(X)]$. The empirical average is computed by *mean*($h(x)$) in logiciel R.

And since $E(\bar{h}_n) = E_f[h(X)]$, then the Monte Carlo estimator $\bar{h}_n$ is an unbiased estimator for $E_f[h(X)]$. Although this already ensures that a Monte Carlo estimator is correct in the mean, it does not help us to get a feeling for the absolute value of the error. Therefore, we look at the standard deviation of the difference between $\bar{h}_n$ and $E_f[h(X)]$. As we have

$$Var(\bar{h}_n - E_f[h(X)]) = Var(\bar{h}_n) = \frac{1}{n^2} \sum_{i=1}^n Var(h(X_i)) = \frac{\sigma^2}{n},$$

where $\sigma^2 = Var(h(X))$. So the standard deviation of the error is of order $O(\frac{1}{\sqrt{n}})$ and the asymptotic variance of the approximation is

$$Var(\bar{h}_n) = \frac{1}{n} \int_{\mathcal{X}} (h(x) - E_f[h(x)])^2 f(x)dx,$$

which can also be estimated from the sample $(X_1, \dots, X_n)$ through

$$v_n = \frac{1}{n^2} \sum_{i=1}^n (h(x_i) - \bar{h}_n)^2.$$

More specifically, by the Central Limit Theorem, for large n

$$\sqrt{n} \frac{\bar{h}_n - E_f[h(X)]}{\sqrt{v_n}}$$

is approximately distributed as a $\mathcal{N}(0, 1)$, and this leads to the construction of a confidence interval of the approximation of $E_f[h(X)]$.

4.3 Variance reduction techniques

The main disadvantage of the Monte Carlo method is its slow convergence, in this section, we present the methods of increasing the efficiency of Monte Carlo simulation by reducing the variance of estimator. We discuss control variables method, antithetic variates and importance sampling (for more details see. [8], [41], [46]).

Antithetic variables

The antithetic variables method is the easiest variance reduction method. It is based on the idea to combine a random choice of points with a systematic one. Its main principle is variance reduction by introducing symmetry. Assume that, we want to compute $E(h(X))$ with X a random variable uniformly distributed on $[0, 1]$. While the Monte Carlo estimate would be

$$\bar{h}(X) = \frac{1}{n} \sum_{i=1}^n h(X_i),$$

with X_i being independent copies of X , in the method of antithetic variates we would also use the numbers $1 - X_1, \dots, 1 - X_n$ and introduce the antithetic Monte Carlo estimator

$$\bar{h}_{anti}(X) = \frac{1}{2} \left(\frac{1}{n} \sum_{i=1}^n h(X_i) + \frac{1}{n} \sum_{i=1}^n h(1 - X_i) \right). \quad (4.2)$$

Note that as X and $1 - X$ have the same distribution, both sums on the righthand side of Equation (4.2) are unbiased estimators for $E(h(X))$. Therefore, the antithetic estimator is also unbiased. Let $\sigma^2 = \text{Var}(h(X))$. Then the variance of the antithetic estimator is given by

$$\text{Var}(\bar{h}_{anti}(X)) = \frac{1}{2n}\sigma^2 + \frac{1}{2n}\text{Cov}(h(X), h(1-X)),$$

i.e. we have a reduction of the variance compared to the Monte Carlo estimator based on $2n$ random numbers if $h(X)$ and $h(1-X)$ are negatively correlated.

Control variables

The principle of control variables is based on the idea that if we want to compute $E(X)$, we should try to compute as much as possible exactly and should only compute that part by Monte Carlo simulation that we cannot avoid. More precisely, if we know a random variable Y which is (in some sense) close to X and for which we can compute $E(Y)$ exactly, then this random variable can be chosen as a control variate, i.e. we use the relation

$$E(X) = E(X - Y) + E(Y),$$

which motivates the following control variable Monte Carlo estimator

$$\bar{X}_Y = \frac{1}{n} \sum_{i=1}^n (X_i - Y_i) + E(Y)$$

for $E(X)$ with X_i, Y_i being independent copies of X and Y . By the relation

$$\text{Var}(\bar{X}_Y) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^n (X_i - Y_i)\right) = \frac{1}{n}\text{Var}(X) + \frac{1}{n}\text{Var}(Y) - \frac{2}{n}\text{Cov}(X, Y),$$

we obtain a reduction of the variance for the control variable estimator compared to the crude one if we have

$$\text{Var}(X - Y) \leq \text{Var}(X).$$

So, if we have

$$\frac{1}{n}\text{Var}(Y) - \frac{2}{n}\text{Cov}(X, Y) \leq 0.$$

Importance sampling

In importance sampling method, we change measures to try to give more weight to "important" outcomes thereby increasing sampling efficiency. To make this idea concrete, consider the problem of estimating

$$\alpha = E[h(X)] = \int_{\mathcal{A}} h(x)f(x)dx$$

where X is a random variable with probability density f , the ordinary Monte Carlo estimator is

$$\bar{\alpha} = \frac{1}{n} \sum_{i=1}^n (h(X_i))$$

with $X_1, \dots, X_n$ independent draws from f . Let g be any other probability density satisfying $f(x) > 0 \Rightarrow g(x) > 0$. Then we can alternatively represent α as

$$\alpha = E[h(X)] = \int_{\mathcal{A}} h(x) \frac{f(x)}{g(x)} g(x) dx.$$

This integral can be interpreted as an expectation with respect to the density g , we may therefore write

$$\bar{\alpha} = \bar{E}\left[h(X) \frac{f(X)}{g(X)}\right],$$

where $\bar{E}$ indicating that the expectation is taken with X distributed according to g . If $X_1, \dots, X_n$ are now independent draws from g , the importance sampling estimator associated with g is

$$\bar{\alpha}_g = \frac{1}{n} \sum_{i=1}^n \left[h(X_i) \frac{f(X_i)}{g(X_i)} \right].$$

4.4 Simulation of random variable and option pricing

A formal definition of an option states that it is a type of contract between two parties that provides one party the right but not the obligation to buy or sell the underlying asset at a predetermined price before or at expiration day. There are two major types of options: calls and puts.

- Call is an option contract that gives the right but not the obligation to buy the underly-

ing asset at a predetermined price before or at expiration day.

- Put is an option contract that gives the right but not the obligation to sell the underlying asset at a predetermined price before or at expiration day.

Options may also be classified according to their exercise time:

- European style options may be exercised only at the expiration date.
- American style options can be exercised anytime between purchase and expiration date.

In this section, we focus on evaluating the price of European option.

Let $S(t)$ denote the price of the stock at time t and $C(T, K)$ denote the option price. Consider a call option granting the holder the right to buy the stock at a fixed price K at a fixed time T in the future. In the current time $t = 0$, if at time T the stock price $S(T)$ exceeds the strike price K , the holder exercises the option for a profit of $S(T) - K$, if, on the other hand, $S(T) \leq K$, the option expires worthless. This is a European option, meaning that it can be exercised only at the fixed date T . Then, the payoff to the option holder at time T is thus

$$(S(T) - K)^+ = \max\{S(T) - K, 0\}.$$

To get the present value of this payoff, we multiply by a discount factor e^{-rT} , with r a continuously compounded interest rate. We denote the expected present value by $E[e^{-rT}(S(T) - K)^+]$.

For this expectation to be meaningful, we need to specify the distribution of the random variable $S(T)$, the terminal stock price. In fact, rather than simply specifying the distribution at a fixed time, we introduce a model for the dynamics of the stock price.

The Black-Scholes model describes the evolution of the stock price through the stochastic differential equation (*SDE*).

$$dS(t) = rS(t)dt + \sigma dW(t), \tag{4.3}$$

with W a standard Brownian motion and $dW(t) = W(t + dt) - W(t)$. The parameter σ is the volatility of the stock price.

The solution of the stochastic differential equation (4.3) is

$$S(T) = S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma W(T)}. \quad (4.4)$$

As $S(0)$ is the current price of the stock, we may assume it is known. The random variable $W(T)$ is normally distributed with mean 0 and variance T , this is also the distribution of $\sqrt{T}Z$ if Z is a standard normal random variable (mean 0, variance 1). We may therefore represent the terminal stock price as

$$S(T) = S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}. \quad (4.5)$$

It follows that, the average gain for the holder of this option at time T taking into account the discount factor of the value of the stock is:

$$E[e^{-rT}(S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z} - K)^+] - C(T, K).$$

Taking into account the market equilibrium, the price of European option $C(T, K)$ is thus

$$C(T, K) = E[e^{-rT}(S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z} - K)^+]. \quad (4.6)$$

And using Monte Carlo simulation, we can approach $C(T, K)$ by

$$C(T, K) \approx \frac{1}{N_{smp}} \sum_{i=1}^{N_{smp}} [e^{-rT}(S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x_i} - K)^+],$$

where x_i : samples of a standard normal variable.

With $S_0 = 100$, $K = 120$, $r = 0.02$, $\sigma = 0.2$ et $T = 10$. We have evaluated the quantity $C(T, K)$ and in order to improve the convergence, we have used the importance sampling method seen in the previous section.

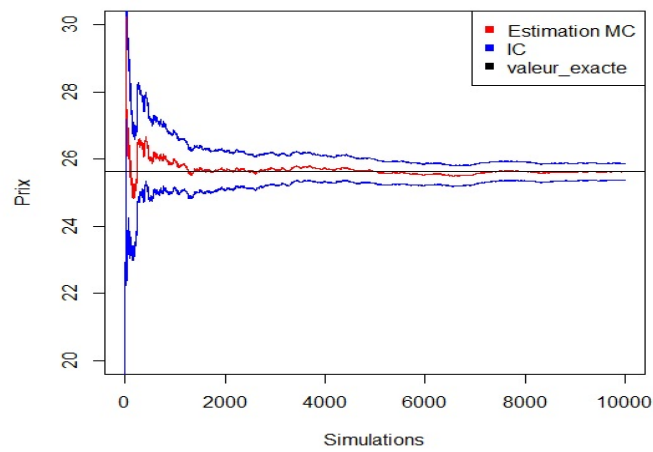


Figure 4.1: Price of Call with $S_0 = 100$, $K = 120$, $r = 0.02$, $T = 10$

- Basket option

A basket option is an option on the weighted average of several underlyings. The price of the underlying asset is written in this case as

$$S_T = \sum_{i=1}^n S_{0i} e^{(r - \frac{1}{2}\sigma_i^2)T + \sigma_i \sqrt{T} G_i}.$$

and the price of the basket option is written:

$$C = E\left(\sum_{i=1}^n e^{-rT} (S_{0i} e^{(r - \frac{1}{2}\sigma_i^2)T + \sigma_i \sqrt{T} G_i} - K)_+\right).$$

Assume that, we have a basket of 5 assets such that $K = 1500$, $T = 10$ and

- Asset 1 : $S_{01} = 100$, $\sigma_1 = 0.3$.
- Asset 2 : $S_{02} = 200$, $\sigma_2 = 0.2$.
- Asset 3 : $S_{03} = 100$, $\sigma_3 = 0.1$.
- Asset 4 : $S_{04} = 200$, $\sigma_4 = 0.4$.
- Asset 5 : $S_{05} = 500$, $\sigma_5 = 0.1$.

we have evaluated the basket option and we obtain the following graph

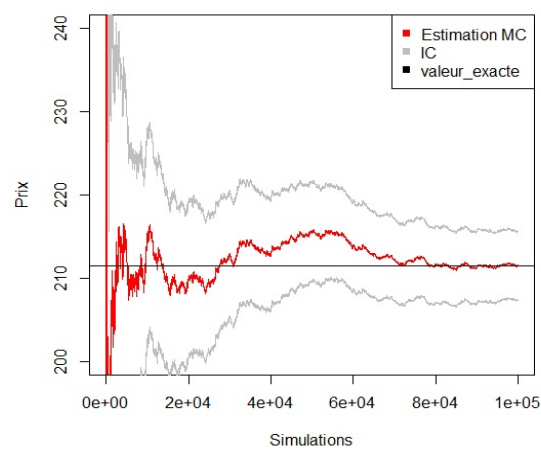


Figure 4.2: Price of call in basket option

In the case of a basket option, the Monte Carlo method is power perfect, because there is no numerical formula for the exact calculation of the option pricing.

4.5 Simulation of random sets and reliability analysis of Structures

The analysis of the reliability of a structure in an input-output model is generally performed by evaluating the probability of failure. This probability can be expressed as

$$P_X(\mathbf{B}) = \int_{\mathbf{B}} dP_X = \int_{\mathbf{B}} f_X(x) dx. \quad (4.7)$$

Such that:

- f_X : is the probability density function (PDF) for the basic random variables X with values in $\mathbb{R}^n$.
- x : vector of basic variables represents the material, loads, shape and size of structural elements.
- $\mathbf{B}$: is the failure region defined by the limit state function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ as follows
 $\mathbf{B} = \{x \in X; g(x) \leq 0\}$.

It is important to mention that the problem of the structural reliability described by (4.7) is just a simple example of calculating the probability of an event when all the basic variables are imprecise. In this section, we suppose the imprecision of the input is described in terms of a family of random variables $(X_\alpha)_{\alpha \in V}$ where V is an index set.

The main purpose of this section is to invest in the theory of random sets: [59], [7], [60], [30], [73], [47] and Monte Carlo simulation [3], [86] and [74] to evaluate the probability of system failure. Precisely;

Given a function $g : \mathbb{R}^n \longrightarrow \mathbb{R} : x \longrightarrow g(x)$ and a family $(X_\alpha)_{\alpha \in V}$ of random variables modelling the uncertainty of variable x . We focus on the evaluation of the upper and lower probability of the event $\{X_\alpha \in \mathbf{B}\}$ i.e. the upper and lower probability of the event $\{g(X_\alpha) \leq 0\}$ (upper/lower probability of failure).

We address this issue by following two different approaches:

- Evaluating the lower and upper probability of failure by Monte-Carlo simulation of $(g(X_\alpha))_{\alpha \in V}$.

- First, forming random sets $\mathcal{X}$ based on the family $(g(X_\alpha))_{\alpha \in V}$ and then evaluating the lower and upper probability as belief and plausibility ([13]) of an event.

Let's start with the first approach: Monte-Carlo simulation of $(g(X_\alpha))_{\alpha \in V}$.

- First approach: approximation of $\bar{P}(g(X_\alpha) \leq 0)$ and $\underline{P}(g(X_\alpha) \leq 0)$.

Let (Ω, Σ, P) be a complete probability space, V be an index set and $(X_\alpha)_{\alpha \in V}$ be a family of random variables. Then the probability of $\{g(X_\alpha) \leq 0\}$ for every fixed α is:

$$P(g(X_\alpha) \leq 0) = \int_{\{g(X_\alpha) \leq 0\}} dP = \int 1_{\{g(X_\alpha) \leq 0\}} dP,$$

And in the similar manner, we define the lower and upper probability of $\{g(X_\alpha) \leq 0\}$ respectively by

$$\bar{P}(g(X_\alpha) \leq 0) = \sup_{\alpha \in V} P(g(X_\alpha) \leq 0) = \sup_{\alpha \in V} \int 1_{\{g(X_\alpha) \leq 0\}} dP(\omega)$$

$$\underline{P}(g(X_\alpha) \leq 0) = \inf_{\alpha \in V} P(g(X_\alpha) \leq 0) = \inf_{\alpha \in V} \int 1_{\{g(X_\alpha) \leq 0\}} dP(\omega).$$

And by the Monte Carlo method, we approach the upper and lower probability of failure respectively by:

$$\bar{P}(g(X_\alpha) \leq 0) = \sup_{\alpha \in V} P(g(X_\alpha) \leq 0) \approx \max_{i=1,2,3,\dots,N_{grid}} P(g(X_{\alpha_i}) \leq 0) \approx$$

$$\approx \max_{i=1,2,3,\dots,N_{grid}} \frac{1}{N_{samp}} \sum_{k=1}^{N_{samp}} 1_{\{g(x_{i_k}) \leq 0\}}$$

And

$$\underline{P}(g(X_\alpha) \leq 0) = \inf_{\alpha \in V} P(g(X_\alpha) \leq 0) \approx \min_{i=1,2,3,\dots,N_{grid}} P(g(X_{\alpha_i}) \leq 0) \approx$$

$$\approx \min_{i=1,2,3,\dots,N_{grid}} \frac{1}{N_{samp}} \sum_{k=1}^{N_{samp}} 1_{\{g(x_{i_k}) \leq 0\}},$$

where x_{i_k} follows the same law as X_{α_i} , $i = 1, 2, 3, \dots, N_{grid}$.

- Second approach

We consider the multifonction $\mathcal{X} : \Omega \longrightarrow cl(\mathbb{R})$ based on the family $(g(X_\alpha))_{\alpha \in V}$ as

$$\mathcal{X}(\omega) = \{g(X_\alpha)(\omega), \alpha \in V\}.$$

Then $\mathcal{X}$ is a random set, if for every Borel set $\mathbf{B}$, the upper and lower inverses

- $\mathcal{X}^-(\mathbf{B}) = \{\omega \in \Omega : \mathcal{X}(\omega) \cap \mathbf{B} \neq \emptyset\}$
- $\mathcal{X}_-(\mathbf{B}) = \{\omega \in \Omega : \mathcal{X}(\omega) \subset \mathbf{B}\},$

are Σ -mesurables.

These upper and lower inverses lead to upper and lower probabilities based on the random set $\mathcal{X}$:

$$\begin{aligned}\tilde{P}(\mathbf{B}) &= P(\mathcal{X}^-(\mathbf{B})) = \int_{\Omega} 1_{\{\mathcal{X}(\omega) \cap \mathbf{B} \neq \emptyset\}} dP(\omega) \\ \underline{P}(\mathbf{B}) &= P(\mathcal{X}_-(\mathbf{B})) = \int_{\Omega} 1_{\{\mathcal{X}(\omega) \subset \mathbf{B}\}} dP(\omega).\end{aligned}$$

The following theorem, gives the sufficient conditions for the set $\mathcal{X}$ generated by the family $(g(X_\alpha))_{\alpha \in V}$ to be a random set.

Theorem 4.1 *Let $(X_\alpha)_{\alpha \in V}$ be a family of random variables. Suppose that V is a compact of a metric space and the applications $\alpha \longrightarrow g(X_\alpha(\omega))$ are continuous for every fixed ω in Ω . Then*

1. *The multifunction $\mathcal{X}$ defined by:*

$$\mathcal{X}(\omega) = \{g(X_\alpha)(\omega); \alpha \in V\}$$

is a compact random set.

2. *The application $\overline{1}_{\mathbf{B}} : \omega \longrightarrow \sup_{\alpha \in V} 1_{\{g(X_\alpha(\omega)) \in \mathbf{B}\}}$ is Σ -measurable.*
3. *The application $\underline{1}_{\mathbf{B}} : \omega \longrightarrow \inf_{\alpha \in V} 1_{\{g(X_\alpha(\omega)) \in \mathbf{B}\}}$ is Σ -measurable.*

where $\mathbf{B}$ is a Borel set.

Proof 1. *It is easy to see that V is separable (as a compact subset of a metric space). Then, there exist a countable dense subset of element $(\alpha_k)_{k \geq 1}$ in V , and since the*

application $\alpha \longrightarrow g(X_\alpha)$ is continuous.

Therefore $\{g(X_{\alpha_k})(\omega), k = 1, 2, 3, \dots\}$ is dense in $\mathcal{X}$ for every fixed ω .

Let $\mathbf{B}$ be an open set in $\mathbb{R}$, Then

$$\mathcal{X}^-(\mathbf{B}) = \{\omega \in \Omega : \mathcal{X}(\omega) \cap \mathbf{B} \neq \emptyset\}$$

$$= \{\omega \in \Omega : \exists k, g(X_{\alpha_k})(\omega) \in \mathbf{B}\}$$

$$= \cup_{k \geq 1} \{\omega \in \Omega : g(X_{\alpha_k})(\omega) \in \mathbf{B}\}$$

is Σ -measurable as a countable union of measurable sets. Moreover $\mathcal{X}$ is the continuous image of a compact set. Therefore $\mathcal{X}$ is a compact random set.

2. Using the following equivalences

$$\overline{1}_{\mathbf{B}}(\omega) = \sup_{\alpha \in V} 1_{\{g(X_\alpha) \in \mathbf{B}\}} = 1 \iff \exists \alpha \in V, g(X_\alpha)(\omega) \in \mathbf{B} \iff \mathcal{X}(\omega) \cap \mathbf{B} \neq \emptyset.$$

We obtain $\mathcal{X}^-(\mathbf{B}) = \overline{1}_{\mathbf{B}}^{-1}(\{1\})$. Then $\overline{1}_{\mathbf{B}}^{-1}(\{1\}) \in \Sigma$. (because $\mathcal{X}$ is Σ -measurable).

And since $\overline{1}_{\mathbf{B}}$ takes only the two values 0 and 1, this implies that $\overline{1}_{\mathbf{B}}$ is measurable.

3. It enough to see

$$\underline{1}_{\mathbf{B}}(\omega) = \inf_{\alpha \in V} 1_{\{g(X_\alpha) \in \mathbf{B}\}} = 1 \iff \forall \alpha \in V, g(X_\alpha)(\omega) \in \mathbf{B} \iff \mathcal{X}(\omega) \subseteq \mathbf{B}.$$

Then

$$\mathcal{X}_-(\mathbf{B}) = \underline{1}_{\mathbf{B}}^{-1}(\{1\}) \in \Sigma.$$

(because $\mathcal{X}$ is Σ -measurable). And since $\underline{1}_{\mathbf{B}}$ takes just two possible values 0 and 1, this implies that $\underline{1}_{\mathbf{B}}$ is measurable.

Under the same assumptions of Theorem 4.1, we have $\mathcal{X}$ is a compact random set with values in $\mathbb{R}$, then $\mathcal{X}(\omega)$ is contained in a random interval of the form $[\underline{\mathcal{X}}(\omega), \overline{\mathcal{X}}(\omega)]$, where $\underline{\mathcal{X}}(\omega) = \min \mathcal{X}(\omega)$ and $\overline{\mathcal{X}}(\omega) = \max \mathcal{X}(\omega)$.

Let $\mathbf{B} = (-\infty, 0]$, we have

$$\tilde{P}(\mathbf{B}) = P((-\infty, 0] \cap [\underline{\mathcal{X}}, \overline{\mathcal{X}}] \neq \emptyset) = P(\underline{\mathcal{X}} \leq 0) = F_{\underline{\mathcal{X}}}(0)$$

And

$$\underline{P}(\mathbf{B}) = P([\underline{\mathcal{X}}, \overline{\mathcal{X}}] \subset (-\infty, 0]) = P(\overline{\mathcal{X}} \leq 0) = F_{\overline{\mathcal{X}}}(0)$$

And by the Monte Carlo method, we approach the upper and lower probability of failure respectively by:

$$\tilde{P}(\mathbf{B}) = F_{\underline{\mathcal{X}}}(0) \approx \frac{1}{N_{samp}} \sum_1^{N_{samp}} 1_{\{\underline{\mathcal{X}}(\omega_k) \leq 0\}}.$$

And

$$\underline{P}(\mathbf{B}) = F_{\overline{\mathcal{X}}}(0) \approx \frac{1}{N_{samp}} \sum_1^{N_{samp}} 1_{\{\overline{\mathcal{X}}(\omega_k) \leq 0\}}.$$

The purpose of the following theorem is to compare the probability bounds obtained by the two approaches

Theorem 4.2 *Let $(X_\alpha)_{\alpha \in V}$ be a family of random variables satisfies the same conditions of the Theorem 4.1 and let $\mathcal{X}$ be a random set such that*

$$\mathcal{X}(\omega) = \{g(X_\alpha)(\omega); \alpha \in V\}.$$

Then

$$\underline{P} \leq \underline{P} \leq \overline{P} \leq \tilde{P}$$

Proof Let $\mathbf{B} \in \mathcal{B}(\mathbb{R})$

- *Let's show that $\overline{P}(\mathbf{B}) \leq \tilde{P}(\mathbf{B})$. By definition, we have*

$$\begin{aligned} \overline{P}(\mathbf{B}) &= \overline{P}(g(X_\alpha) \in \mathbf{B}) = \sup_{\alpha \in V} P(g(X_\alpha) \in \mathbf{B}) = \sup_{\alpha \in V} \int 1_{\{g(X_\alpha) \in \mathbf{B}\}} dP(\omega) \\ &\leq \int \sup_{\alpha \in V} 1_{\{g(X_\alpha) \in \mathbf{B}\}} dP(\omega) = \int \overline{1_{\mathbf{B}}}(\omega) dP(\omega) = \int 1_{\{\mathcal{X}(\omega) \cap \mathbf{B} \neq \emptyset\}} dP(\omega) = \tilde{P}(\mathbf{B}). \end{aligned}$$

- *Proceeding in the same manner, we obtain that $\underline{P}(\mathbf{B}) \geq \underline{P}(\mathbf{B})$.*

The Theorem 4.2, allowed us to deduce that the information given by the upper and lower probabilities induced by the simulation of random set is not equivalent to this given by the class of the probabilities determined by its measurable selections.

Conclusion

La loi forte des grands nombres n'est autre que la convergence presque sûre de la moyenne empirique d'une suite de variables aléatoires (resp. ensembles aléatoires), sous diverses conditions de dépendance, d'intégrabilité portant sur les variables aléatoires et sur les propriétés géométriques de l'espace considéré, vers une valeur déterministe. Dans ce projet de thèse, nous avons démontré des nouveaux résultats de la loi forte des grands nombres. Notre étude a été focalisée sur les variables aléatoires et les ensembles aléatoires dans le cas de Bochner intégration et Pettis intégration.

Dans notre première contribution de ce travail, nous avons étendu le théorème de Chow pour les martingales (Theorem 0.3, p. 7) aux suites de variables aléatoires quelconques adaptées à une filtration (Theorem 1.6, p. 33). Comme conséquence de ce résultat, nous avons établi la loi forte des grands nombres pour les amarts (Theorem 1.9, p. 42), les mils (Theorem 1.8, p. 39) et les surmartingales multivoques (Theorem 1.10, p. 44).

Comme notre intérêt se porte également sur la théorie de l'intégration de Pettis, le deuxième résultat essentiel de ce projet a été dédié à la convergence des martingales inverses multivoques Pettis intégrables. Nous avons établi la convergence au sens de Mosco de ces processus. Ce résultat généralise la convergence des martingales inverses Bochner intégrables. Ce travail nous a également amené à établir des nouvelles versions de quelques théorèmes classiques en probabilités comme le théorème de type Levy (Theorem 2.1, p. 53 et Theorem 2.3, p. 55) le théorème de convergence dominée (Theorem 2.2, p. 54 et Theorem 2.5, p. 58) et le lemme de Fatou (Theorem 2.4, p. 56).

Le champs nous a été ouvert pour établir des nouveaux résultats de la loi forte des grands

nombres pour des ensembles aléatoires Pettis intégrables. Dans ce sens nous avons prouvé la convergence au sens de Mosco de la loi forte des grands nombres pour des ensembles aléatoires Pettis intégrables à valeurs dans l'ensembles des parties convexes et faiblement compactes d'un espace de Banach séparable. Ces résultats ont généralisé la loi forte des grands nombres établie pour les ensembles aléatoires Bochner intégrables.

A la fin de ce projet de thèse nous avons présenté quelques applications de la loi forte des grands nombres.

Perspectives

Ce projet de thèse a donné lieu aux plusieurs problèmes ouverts comme suit:

- L'étude de l'inégalité de Chow pour les sous martingales et l'inégalité de Burkholder pour les martingales dans le cas multivoque.
- La loi forte des grands nombres pour des suites adaptées à valeurs vectorielles et multivoques.
- La loi forte des grands nombres pour une suite de variables aléatoires (ensemble aléatoire) à double indice.
- La convergence des martingales inverses Pettis intégrables à valeurs non bornées.
- La loi forte des grands nombres pour des ensembles aléatoires Pettis intégrables non forcément iid.
- Etendre les méthodes classiques de simulation des variables aléatoires aux ensembles aléatoires.

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