ROYAUME DU MAROC

Université Mohammed V - RABAT - جامعة محمد الخامس - الرباط -

Faculté des sciences كلية العلوم

CENTRE D'ETUDES DOCTORALES - SCIENCES ET TECHNOLOGIES

N° d'ordre : 3431

THESE

En vue de l'obtention du : ***DOCTORAT***

Centre de Recherche : *Centre de Recherche de Mathématiques et Applications de Rabat*
Structure de Recherche : *Laboratoire Analyse Mathématique et Applications*
Discipline : *Mathématiques*
Spécialité : *Analyse mathématique*

Présentée et soutenue le : 20/02/2021 par :

Zakariya CHAOUAI

Qualitative analysis of nonlinear elliptic and parabolic equations with divergence form

JURY

Zine El Abidine GUENNOUN	*P.E.S, Faculté des sciences, Université Mohammed V, Rabat.*	*Président*
Zine El Abidine ABDELALI	*P.E.S, Faculté des sciences, Université Mohammed V, Rabat.*	*Rapporteur / Examinateur*
Mohemed OUANNASSER	*P.E.S, Faculté des sciences, Université Mohammed V, Rabat.*	*Rapporteur / Examinateur*
Jaouad IGBIDA	*P.H, centre régional des métiers de l'éducation et de la formation, El Jadida.*	*Rapporteur / Examinateur*
Abdelilah LAMRANI ALAOUI	*P.H, centre régional des métiers de l'éducation et de la formation, Fès.*	*Examinateur*
Abderrahmane EL HACHIMI	*P.E.S, Faculté des sciences, Université Mohammed V, Rabat.*	*Directeur de thèse*

Année Universitaire : 2020-2021

✉ Faculté des Sciences, avenue Ibn Battouta, BP. 1014 RP, Rabat –Maroc
☎ 00212(0)5 37 77 18 76 🖷 00212(0)5 37 77 42 61 ; http://fsr.um5.ac.ma/

To the memory of my dears whom I've lost during the last couple of years
– My grand-parents Khadija, Driss, Fatima, Abdessalem.
– My aunt Aziza.

ACKNOWLEDGEMENTS

The work of this Ph.D. thesis was performed in the Laboratory of Mathematical Analysis and Applications, Faculty of Sciences, Mohammed V University in Rabat, under the direction of Professor Abderrahmane EL HACHIMI.

Firstly, I would like to express my sincere gratitude to my supervisor **Abderrahmane EL HACHIMI**, Professor at the Faculty of Sciences in Rabat, Mohammed V University, for his unfailing patience, dedication, and enthusiasm in helping me with this project. He offered lots of valuable advice and instructions, and without his guidance and persistent help, this thesis would not have been possible.

My sincere thanks to Professor **Zine El abidine GUENNOUN**, Professor at the Faculty of Sciences in Rabat, Mohammed V University. It is an honor for me that he chairs the examining board.

I want to thank Professor **Mohamed OUANNASSER**, Professor at the Faculty of Sciences in Rabat, Mohammed V University, for agreeing to perform an report on this thesis and participate on my examining board. I thank him very much for his availability, for accepting to give their precious remarks in order to ameliorate my modest work.

I make a point of warmly thanking to Professor **Zine El Abidine ABDELALI**, Professor at the Faculty of Sciences in Rabat, Mohammed V University, for agreeing to perform an report on this thesis and participate on my examining board. I thank him very much for his many remarks and suggestions to improve the quality of this thesis.

I want to express special thanks to Professor **Jaouad IGBIDA**, P.H at Regional Center for Education and Training in El Jadida who agreed without hesitation to perform an report on this thesis and participate on my examining board. I thank him very much for his availability, his patience, and his scientific rigor.

I extend sincere thanks to Professor **Abdelilah LAMRANI ALAOUI**, P.H at Regional Center for Education and Training in Fes, for agreeing to participate on my examining board. I would like to thank him very much for his availability.

I would like to express here my acknowledgments to all the professors of the laboratory. I also want to thank all members of the laboratory for the benefice discussions we have had on our different areas of mathematical researches. Also, I thank every person who contributes to the elaboration of this work.

I want to express my thanks to all my friends. In particular, Omar Dwimi, Hamza El Azhar, Aymane El Fardi, Mohammed El Hadiri, Ali Mahmoud, Soufiane Matook, and Mohamed Ziyat.

Finally, I would like to thank the long-standing strong support team : my family. I especially thank my parents (Zoubida and Said, thank you for EVERYTHING, there will never be enough words to thank her parents!!!), my brothers Badr and Hamza (thank you for the encouragement and the warm and weekly calls) and my aunt Nadia (thank you for your douae and gods heal you).

And praise to God in the beginning and the end for guiding me in achieving this work.

RÉSUMÉ

Cette thèse s'inscrit dans le domaine mathématique de l'analyse des équations aux dérivées partielles non linéaires. Plus précisément, on s'intéresse d'abord, dans ce travail, à l'étude de l'existence et de la multiplicité des solutions pour une classe de problèmes elliptiques de type p-laplacien avec un terme d'ordre p sur le gradient en dimension $N \geq 3$, en utilisant des méthodes variationnelles. Ensuite on se propose d'étudier un problème d'évolution de type parabolique avec un terme diffusion donnée par l'opérateur p-Laplacien, une source non-locale et une absorption de type gradient. On établit pour ce type de problème, l'existence de la solution locale et on montre un principe de comparaison faible qui jouera un rôle crucial pour prouver quelques propriétés qualitatives des solutions. Dans le dernier chapitre de cette thèse, on étudiera les propriétés qualitatives des solutions positives pour un problème doublement non linéaire avec des exposants variables et notamment avec une diffusion de type $p(x,t)$-Laplacien. Les résultats obtenus dans ce chapitre reposent sur la construction de sous et sur-solutions.

Mots Clés : problèmes elliptiques ; problèmes paraboliques ; p-Laplacien ; principe de comparaison ; sur-solutions ; sous-solutions ; exposants variables ; comportements en temps fini.

ABSTRACT

This thesis deals with the mathematical field of nonlinear partial differential equations analysis. More precisely, at first we are interested in this work to study the existence and the multiplicity of solutions for a class of elliptic problems of p-Laplacian type with a p-gradient term in dimension $N \geq 3$, by using variational methods. Then, we will study an evolution problem of parabolic type with a diffusion term given by the p-Laplacian operator , a nonlocal source and an absorption term with gradient type. For this type of problems, we will establish the existence of local solutions and a weak comparison principle which will play a crucial role in studying some qualitative properties of solutions. In the last chapter of this thesis we will study qualitative properties of positive solutions for a doubly nonlinear problem with variable exponents, in particular with a diffusion of $p(x,t)$-Laplacian type. The results obtained in this chapter are based on the construction of sub and super-solutions.

Keywords: elliptic problems; parabolic problems; p-Laplacian; comparison principle; super-solutions; sous-solutions; variable exponents; finite time behaviors.

RÉSUMÉ DÉTAILLÉ DE LA THÈSE

Les équations aux dérivées partielles (en abrégé EDP) constituent la généralisation naturelle des équations différentielles au cas où l'inconnue dépend de plusieurs variables. Elles modélisent de très nombreux phénomènes physiques, chimiques ou biologiques. Une équation aux dérivées partielles peut avoir plusieurs types de solutions. Dans cette thèse, nous sommes intéressés par ce qu'on appelle les solutions faibles. Classiquement, une solution d'une EDP d'ordre p est une fonction p fois continument différentiable. Cependant cette notion est trop restrictive et si l'on veut prendre compte, de manière satisfaisante, de certains phénomènes. Il conviendrait, alors, d'affaiblir la notion de solution afin d'y inclure des concepts plus singuliers. c'est ainsi que dans les années 1930, sous l'impulsion notamment des mathématiciens Jean Leray et Leonid Sobolev, est apparue la notion de solution faible.

Notons que les problèmes qui ont été abordés dans ce travail apparaissent dans de nombreux modèles issus de la physique, la dynamique des populations, ainsi que dans certains modèles d'écoulement de fluides non-newtoniens. On pourra consulter à ce sujet, les articles de Ph. Souplet [86, 87], l'ouvrage de P. Quittner et Ph. Souplet [80], ainsi que l'ouvrage de S. Antontsev et S. Shmarev [11], où une présentation de ces modèles ainsi qu'une bibliographie abondante relative aux problèmes quasi-linéaire y sont données.

L'objectif de cette thèse est l'étude de certains problèmes elliptiques et paraboliques, quasi-linéaires. Dans les modèles considérés au cours de ce travail, nous considérons un ensemble borné de frontière assez régulière de $\mathbb{R}^N$, $(N \geq 3)$, avec des conditions aux limites, homogènes, de type Dirichlet .

Précisément, dans le Chapitre 2, nous abordons la question de la multiplicité de solutions pour une classe de problèmes elliptiques de type p-laplacien avec un terme de p-gradient en dimension $N \geq 3$. Pour cela, nous utilisons le théorème de Passe-Montagne, sur un problème équivalent possédant une structure variationnelle. La difficulté principale est que le terme de non-linéarité considéré ne satisfait pas la condition d'Ambrosetti et de Rabinowitz (AR). L'idée clé est alors de remplacer cette condition par une condition de non-quadraticité à l'infini. Les résultats obtenus dans ce chapitre, généralisent certains résultats précédemment obtenus par L. Jeanjean, H. R. Quoirin [63] et L. Jeanjean B. Sirakov [64].

Dans le chapitre 3, nous commençons par étudier l'existence locale et l'unicité de la solution d'un problème parabolique de type p-Laplacien avec une source non-locale et un terme absorption de type gradient. Pour les problèmes paraboliques de ce type, on sait qu'une solution peut cesser d'exister dans un temps fini du fait que sa norme L^∞ tende vers l'infini, c'est à dire que la solution explose en temps fini. Nous donnerons alors des conditions suffisantes d'existence et de non-existence de la solution globale en norme L^∞. Ainsi, en fonction des choix des données initiales, des valeurs des coefficients et des exposants, et de la mesure du domaine, nous montrons que la solution globale positive, lorsqu'elle existe, s'annulle après un temps fini; Nous utilisons pour cela la méthode dite d'estimation de la norme intégrale L^2 ainsi que la méthode de sous et sur-solutions.

Enfin, dans le Chapitre 4,nous nous intéressons aux propriétés qualitatives des solutions positives pour un problème doublement non linéaire avec des exposants variables. Pour ce faire, nous exploitons les résultats d'existence obtenus par S. Antontsev et S. Shamarev, dans [8]. Nous commençons par monter un principe de comparaison faible qui jouera un rôle crucial pour étudier les propriétés qualitatives de la solution. La première propriété étudiée dans ce chapitre est l'extinction de la solution en temps fini en L^∞ norme,sous l'hypothèse que les conditions initiales sont assez petites, dans un sens à préciser. Ensuite, par l'argument de recouvrement fini et sans aucune hypothèse supplémentaire sur les conditions initiales, nous montrons que la solution unique est strictement positive à partir d'un certain temps t. En outre, nous prouvons une propriété de propagation à partir des données initiales. Pour cela, nous donnons des estimations précises du support de la solution à partir de diverses hypothèses sur les coefficients et les exposants figurant dans l'équation.

CONTENTS

Acknowledgements **ii**

Résumé détaillé de la thèse **vi**

General introduction **1**
Motivation et présentation générale . . . 1
État de l'art et résultats du chapitre 2 . . . 3
État de l'art et résultats du chapitre 3 . . . 5
État de l'art et résultats du chapitre 4 . . . 8

1 Lebesgue-Sobolev spaces with variable exponent **12**
1.1 A brief history of function spaces with variable exponents . . . 12
1.2 Motivation . . . 13
1.3 Lebesgue spaces with variable exponents . . . 14
1.4 Sobolev spaces with variable exponents . . . 16
1.5 Some difficulties related to the Lebesgue-Sobolev spaces with variable exponents . . . 18

2 Multiplicity of solutions for a class of elliptic problem of p-Laplacian type with a p-Gradient term **20**
2.1 Introduction and main result . . . 20
2.2 Preliminaries and geometry of the functional I . . . 23
2.3 Proof of Theorem 2.1.1 . . . 26
2.3.1 First critical point: Palais-Smale condition . . . 26
2.3.2 Second critical point . . . 32
2.3.3 Boundedness of solutions . . . 33

3 Qualitative properties of weak solutions for p-Laplacian equations with nonlocal source and gradient absorption **34**
3.1 Introduction . . . 34
3.2 Local existence of weak solutions . . . 35
3.3 Global and non global existence of solutions . . . 46
3.3.1 Blow-up in finite time . . . 47
3.3.2 Global existence of the solution . . . 49
3.4 Extinction in finite time . . . 52

4 Qualitative properties of nonnegative solutions for a doubly nonlinear problem with variable exponents **57**
4.1 Introduction 57
4.2 Assumptions and results 59
4.3 Finite time extinction and non-extinction 63
4.3.1 Finite time extinction. 63
4.3.2 Non-extinction of solutions 66
4.4 Finite speed of propagation property 70

Bibliography **75**

GENERAL INTRODUCTION

Comme tout être vivant, pour ne pas mourir, la mathématique doit se recréer sans cesse. Nous ne pouvons l'apprendre, nous devons la redécouvrir. Ainsi, la mort de la recherche mathématique serait la mort de la pensée mathématique.

(Jean Leray)

Motivation et présentation générale

Depuis des décennies, les chercheurs s'intéressent à étudier l'existence et les propriétés qualitatives des solutions des équations aux dérivées partielles pour lesquelles la diffusion est combinée avec d'autres effets, notamment la réaction, l'absorption et (ou) la convection. Les équations auxquelles on s'intéresse dans cette thèse, sont des équations d'évolution du type parabolique ou des équations stationnaires du type elliptique. Une forme générale de telles équations est la suivante :

$$\mathcal{P} : \begin{cases} \partial_t b(x,u) - \mathrm{div} A(x,t,\nabla u) = f(x,t,u) - g(x,u,\nabla u) & \text{dans } Q_T = \Omega \times (0,T), \\ u = 0 & \text{sur } \partial\Omega \times (0,T), \\ b(x,u\,(x,0)) = b(x,u_0\,(x)) & \text{dans } \Omega. \end{cases}$$

Les problèmes de type $\mathcal{P}$ modélisent plusieurs phénomènes que ce soit dans la physique, la chimie ou dans la biologie. Pour ces différents phénomènes, les termes b, A, f et g ont diverses formes.

Par exemple le terme $\partial_t b(x,u)$ est un terme d'accumulation qui apparait dans les processus thermiques, lorsque la capacité calorifique du fluide dépend de la température. C'est le cas, par exemple, lorsque l'eau et la glace sont simultanément présentes, alors $b(x,u) = b(u)$ est une fonction strictement croissante ayant une discontinuité en $u = 0$, (voir par exemple le problème de Stefan dans [89]). Lorsque u n'est pas une température mais l'humidité du sol, différents choix sont possibles : dans l'étude des sols non saturés, b est indépendante de x et est strictement croissante par rapport à u, mais dans le cas où les sols sont partiellement saturés, b est supposée être non décroissante mais devient constante dans $(s,+\infty)$ pour certains $s > 0$, (voir par exemple G. Bayada et M. Chambat [20]). D'autre part, dans la théorie de filtration d'un fluide dans un milieu poreux, $b(x,u) = b(u)$ est supposée continue sur $\mathbb{R}$ et croissante; cependant si le fluide est un gaz barotrope idéal, alors $b(x,u) = |u|^{m(x)-1}\,u$, (voir l'ouvrage de S. Antontsev, S. Shmarev [11]).

Le terme div$A(x,t,\nabla u)$, quant à lui, est un terme de diffusion qui peut prend aussi plusieurs formes. Par exemple, l'étude des écoulements unidirectionnels de certains fluides non Newtoniens (comme par exemple, les peintures, le sang, le miel, les shampooings, etc.) conduit à des opérateurs de diffusion non linéaires du type

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\,\nabla u), \quad \text{pour certain } p > 1.$$

Le cas où $1 < p < 2$ correspond aux fluides pseudo-plastiques (comme, l'essence, l'huile lubrifiante, etc.); tandis que $p > 2$ correspond aux fluides dilatants (comme, les glaciers, la lave volcanique, etc.). On remarque que si $p = 2$, alors $\Delta_2 = \Delta$ est l'opérateur Laplacien qui apparait dans l'étude des fluides Newtoniens.

D'autre part, signalons un type important d'opérateur non-linaires qui est le $p(x,t)$-Laplacien

$$\Delta_{p(x,t)} u = \operatorname{div}(|\nabla u|^{p(x,t)-2}\,\nabla u),$$

où $p(x,t) > 1$ est une fonction mesurable sur $\Omega \times (0,+\infty)$. Ce type d'opérateurs modélise plusieurs phénomènes physiques; notamment: les fluides électrorhéologiques ou la restauration d'image, voir [82, 29].

Les terme $f(x,t,u)$ et $g(x,u,\nabla u)$ sont respectivement, un terme source et un terme d'absorption. Ces deux termes apparaissent dans de nombreux problèmes que ce soit en chimie ou en biologie. Rappelons que Ph. Souplet a proposé dans [86] un modèle de dynamique des populations. Le modèle est le suivant : on considère une population d'une espèce biologique vivant sur un territoire Ω. la solution $u(x,t)$ représente la densité d'une espèce en position x et au temps t, l'évolution de cette densité est le résultat de trois types de mécanismes: déplacements, naissances et décès. Les déplacements sont mesurés par div$A(x,t,\nabla)$, la contribution des décès accidentels est représentée par $g(x,t,\nabla)$ où g est croissante par rapport à u et ∇u, (par exemple $g(x,u,\nabla) = C_1 u\,|\nabla u|^q$) et finalement, la contribution des naissances est supposée être proportionnelle au nombre de couples (ou généralement m-tuples) et est représentée par $f(x,t,u) = C_2 u^m$. Cependant, si l'on suppose que l'évolution des espèces dans un domaine de l'espace ne dépend pas seulement de la densité proche mais aussi de la quantité totale d'espèces à cause des effets de l'inhomogénéité spatiale alors $f(x,t,u) = C_2 u^m \int_\Omega u^k$.

Dans cette thèse, nous commençons par donner un chapitre de préliminaires qui contient des rappels sur les espaces de Lebesgue et Sobolev à exposants variables.

Dans le chapitre 2, nous étudieons la multiplicité des solutions pour le problème elliptique quasilinéaire suivant :

$$\begin{cases} -\Delta_p u = c(x)|u|^{q-1}u + \mu|\nabla u|^p + h(x) & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

Dans ce chapitre, nous utilisons une méthode variationnelle. Pour cela, nous construisons une fonctionnelle I dont les points critiques sont les solutions faibles de notre problème. Nous montrons ainsi, l'existence d'un première point critique u au niveau de Palais-Smale, c'est à dire vérifiant $I(u) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}(\gamma(t)) = c > 0$, le deuxième point critique sera un point v dans $B(0,\rho)$, tel que $I(v) < 0$.Nous terminons ce chapitre par montrer la bornitude des solutions fiables.

Le chapitre 3 est consacré à étudier l'existence, l'unicité et les propriétés qualitatives de solutions du problème suivant :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda|u|^{k-1}u \int_\Omega |u|^s dx - \mu|u|^{l-1}u|\nabla u|^q & \text{dans } \Omega \times (0,\infty), \\ u = 0 & \text{sur } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{dans } \Omega. \end{cases}$$

Sous certaines conditions particulières, nous commençons par montrer que notre problème est bien posé. En supposant la condition $u_0 \in W^{1,\infty}(\Omega)$, nous prouvons qu'il existe une unique solution $u \in L^\infty(0, T; W_0^{1,\infty}(\Omega))$. Ainsi, nous montrons l'existence globale ou l'explosion de la solution en norme L^∞, sous différentes conditions. Nous montrons également que si l'on a existence globale de la solution unique, alors il y a extinction de la solution en temps fini. Nous utilisons pour cela deux outils: la méthode dite d'estimation de la norme intégrale L^2 et la méthode de sous et sur-solutions.

Nous terminons cette thèse par étudier les propriétés qualitatives des solutions du problème de Dirichlet à exposants variables suivant :

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t} - \Delta_{p(x,t)}u = a(x,t)|u|^{q(x,t)-1}u & \text{dans } Q_T = \Omega \times (0,T), \\ u = 0 & \text{sur } \partial\Omega \times (0,T), \\ b(x, u(x,0)) = b(x, u_0(x)) & \text{dans } \Omega, \end{cases}$$

En se basant sur les résultats d'existence obtenus par S. Antontsev et S. Shamarev [8], nous montrons, d'abord, un principe de comparaison faible qui implique directement l'unicité de la solution, puis nous étudions les propriétés d'extinction et de non-extinction de la solution. On conclut ce chapitre par donner des conditions suffisantes pour que la solution ait un support compact, localement et globalement par rapport au temps t.

État de l'art et résultats du chapitre 2

Dans ce chapitre, nous étudions la multiplicité des solutions du problème elliptique suivant

$$(P) \begin{cases} -\Delta_p u = c(x)|u|^{q-1}u + \mu|\nabla u|^p + h(x) & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

où

- Ω est un ensemble borné dans $\mathbb{R}^N$, $N \geq 3$, de frontière assez régulière.
- $1 < p < N, q > 0, \mu \in \mathbb{R}^*$.
- c et h appartiennent à $L^k(\Omega)$ avec $k > \frac{N}{p}$.

L'étude du problème de type (P) a été et reste jusqu'à maintenant un champ très actif de recherche. En effet, à notre connaissance, l'étude du problème de type (P) a débuté par Boccardo, Murat et Puel dans les années $'80$ [21, 22], sous l'hypothèse $c(x) \leq -\alpha_0$ p.p. dans Ω, désormais appelé le cas coercif. Ces auteurs ont montré l'existence d'au moins une solution bornée par la méthode des sous et sur-solutions. Ensuite, l'unicité de la solution a été montrée par Barles, Murat [18] et Barles et at. [17]. Feront et Murat [50] ont ensuite montré l'unicité de la solution dans le cas où $c(x) \leq 0$ p.p. dans Ω, au sens où $c(x) \equiv 0$ dans une partie de Ω mais $c(x) < 0$ dans une autre partie. Récemment, dans le même cas mais avec $p = 2$ et $q = 1$, et sous des hypothèses de régularité plus fortes sur les coefficients, Arcoya et al [13] ont obtenu des conditions d'existence et d'unicité de la solution, faisant ansi le lien entre les résultats de Boccardo, Murat et Puel [21] et ceux de Ferone et Murat [51]. Plus précisément, ils ont supposé les conditions suivantes :

$$\begin{cases} \mu \in L^\infty(\Omega) \text{ et } meas(\Omega \backslash Supp\ c) > 0, \\ \inf\limits_{u \in W_c, \|u\|_{H_0^1(\Omega)}} \displaystyle\int_\Omega \left(|\nabla u|^2 - \|\mu^+\|_{L^\infty(\Omega)} h^+(x) u^2 \right) > 0, \\ \inf\limits_{u \in W_c, \|u\|_{H_0^1(\Omega)}} \displaystyle\int_\Omega \left(|\nabla u|^2 - \|\mu^-\|_{L^\infty(\Omega)} h^-(x) u^2 \right) > 0. \end{cases}$$

où $W_c := \{w \in H_0^1(\Omega) : \ c(x)w(x) = 0, \text{ a.e. in } \Omega\}$, a joué un rôle crucial pour avoir la coercivité et implique l'unicité de la solution. Ensuite, dans le cas non coercif; c'est à dire $c(x) \gneqq 0$ p.p dans Ω, Jeanjean and Sirakov dans [64] ont étudié le problème (P) où $p = 2$, $q = 1$ et ont montré l'existence d'au moins deux solutions sous la condition que les fonctions $c(x)$ et $h(x)$ sont assez petites. Ce résultat a été complété dans plusieurs directions. Par exemple, De Coster et Jeanjean [34] ont, d'une part, montré la multiplicité des solutions dans le cas où $0 < \mu_2 \leq \mu(x) \leq \mu_1$, μ_1 et μ_2 sont deux constantes, et avec des hypothèses fortes sur les coefficients et, d'autre part, ils ont dévoilé l'impact du signe de $h(x)$ sur le signe des solutions. Ensuite, Jenajean et Quoirin [63] ont monté l'existence d'au moins deux solutions positives sous la condition où $h(x)$ est positive, μ est une constante mais avec $c(x)$ qui change de signe.

Dans ce chapitre, qui est un travail en collaboration avec S. Maatouk, nous avons réussi a montrer l'existence d'au moins deux solutions du problème (P) dans le cas où $c(x) \gneqq 0$ p.p. dans Ω, $h(x)$ change de singe et avec l'hypothèse que les deux dernières fonctions sont assez petites, bien sûr dans un sens approprié. L'idée de la démonstration consiste premièrement à trouver une formulation variationnelle associée au problème (P). Cela à été fait grâce au changement de variable de Kazdan-Kramer, $v = (e^{\frac{\mu u}{p-1}} - 1)/\mu$, qui transforme notre problème (P) en le problème suivant

$$(P') \begin{cases} -\Delta_p v = c(x)g(v) + h(x)f(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

où

$$g(s) = \frac{(p-1)^{q-p+1}}{\mu^q}(1+\mu s)^{p-1}|\ln(1+\mu s)|^{q-1}\ln(1+\mu s), \ \text{ with } s > \frac{-1}{\mu}, \tag{0.1}$$

et

$$f(s) = \frac{(1+\mu s)^{p-1}}{(p-1)^{p-1}}. \tag{0.2}$$

Le problème (P') admet, maintenant, une formule variationnelle. Alors l'existence d'une solution faible du probléme (P) est équivalente à l'existence d'une point critique da la fonctionnelle $I \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ suivante,

$$I(v) = \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega c(x)G(v) - \int_\Omega h(x)F(v), \tag{0.3}$$

où $G(s) = \int_0^s g(t)dt$ et $F(s) = \int_0^s f(t)dt$.

Par ailleurs, du fait que la fonction g est légèrement super-linéaire à l'infini, la condition d'Ambrosetti-Rabinowitz (AR_c):

$$\text{il existe } \ \theta > p \ \text{ et } \ s_0 > 0 \ \text{ tels que } \ 0 < \theta G(s) \leq sg(s), \ \text{ pour } \ |s| > s_0,$$

n'est pas satisfaite. Il est donc difficile de prouver directement que la fonctionnelle I admet un point critique par la méthode de Palais-Smale qui en même temps consiste à montrer la bornitude des suites de Palais-Smale. Ainsi, pour monter que la suite de Palais-Smale est bornée, nous prouvons que g satisfait la condition de non quadraticité à l'infini suivante,

$$(NQ) \qquad H(s) = g(s)s - pG(s) \to +\infty, \ \text{ where } \ s \to +\infty.$$

Le résultat principal de ce chapitre est le suivant,

Theorem 0.0.1. *Supposons que $\|c\|_k$ et $\|h\|_k$ sont assez petits. Alors, la fonctionnelle I admet au moins deux points critiques. Par conséquent, le problème (P) a au moins deux solutions.*

Signalons que pour montrer que le problème (P) a au moins deux solutions, nous avons, d'abord, prouvé qu'il y a une première solution u au niveau de $\tilde{c} > 0$ c'est à dire $I^{'}(u) = 0$ et $I(u) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) = \tilde{c}$, où $\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0,\ \gamma(1) = v_0\}$. Et, pour prouver l'existence d'une deuxième solution v, nous avons montré que $I(v) = \inf_{w\in B(0,\rho)} I(w) < 0$. Les solutions u et v sont, par conséquent, totalement différentes.

État de l'art et résultats du chapitre 3

Le chapitre 3 est consacré à étudier le problème parabolique suivant,

$$\mathcal{P} : \begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{k-1}u \int_{\Omega} |u|^s dx - \mu |u|^{l-1}u|\nabla u|^q & \text{dans } \Omega \times (0,\infty), \\ u = 0 & \text{sur } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{dans } \Omega, \end{cases}$$

- Ω est un domaine borné dans $\mathbb{R}^N$, $N \geq 1$, de frontière $\partial\Omega$ assez régulière.
- $p > 1$, $q > 0$, les paramètres k, s, l, q, λ, et μ sont positives.
- $u_0 \not\equiv 0$, telle que $u_0 \in W^{1,\infty}(\Omega)$.

L'opérateur p-laplacien; $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ est un opérateur non dégénéré et singulier aux points $u = 0$ et $\nabla u = 0$, pour $p > 2$ et pour $1 < p \leq 2$, respectivement. Alors, le problème $\mathcal{P}$ n'admet pas en général une solution classique. Il est par suite raisonnable de chercher des solutions faibles associées lorsqu'on cherche à résoudre le problème $\mathcal{P}$. Nous définissons donc, en premier lieu, la notion de solution faible pour le problème $\mathcal{P}$.

Definition 0.1. *Soit $\alpha = \max\{p, k+s, q+l\}$. On dit que $u(x,t)$ est une solution faible de $\mathcal{P}$ sur $Q_T = \Omega \times (0,T)$, si*

1. $u \in C(\overline{\Omega} \times [0,T)) \cap L^{\alpha}(0,T; W_0^{1,\alpha}(\Omega))$ *et* $\frac{\partial u}{\partial t} \in L^2(Q_T)$.
2. *Pour toute fonction test* $\varphi \in C(\overline{Q_T}) \cap L^p(0,T; W_0^{1,p}(\Omega))$, *on a*
$$\int_{Q_T} (\varphi \frac{\partial u}{\partial t} + |\nabla u|^{p-2}\, \nabla u \nabla \varphi) dxdt = \int_{Q_T} (\lambda |u|^{k-1}u \int_{\Omega} |u|^s - \mu |u|^{l-1}u|\nabla u|^q)\varphi dxdt.$$
3. $u\,(.,0) = u_0$ *dans* Ω, *et* $u = 0$ *sur* $\partial\Omega \times (0,T)$.

Évidement, il y a d'autres notions que l'on peut utiliser pour résoudre le problème $\mathcal{P}$. Parmi ces notions, signalons les solutions de viscosité utilisées pour résoudre les équations de Hamilton-Jacobi. En effet, la théorie des solutions de viscosité fournit un cadre plus général pour les problèmes d'existence et d'unicité, mais celui-ci est moins adapté pour l'étude des problèmes avec des singularités. La notion de solution faible qui fait intervenir les espaces de Sobolev et qui demande un peu plus de régularité est certainement une meilleure alternative. Notre premier résultat concernant l'existence d'au moins une solution faible locale est le suivant,

Theorem 0.0.2. *On suppose que $k+s \geq 1$, u_0 satisfait* (3.1) *et $\|\nabla u_0\|_\infty \leq M$, où M est une constante positive. Alors, il existe $T_0 = T_0(p,k,s,q,l,\Omega,u_0)$ tel que le problème $\mathcal{P}$ admet au moins une solution faible $u \in L^\infty(0,T_0, W_0^{1,\infty}(\Omega))$.*

La méthode classique pour montrer l'existence locale d'au moins une solution faible consiste à introduire un problème régularisé uniformément parabolique possédant des solutions classiques. La plus grande difficulté est de montrer que ces suites sont uniformément bornées dans l'espace $W_0^{1,\infty}$. L'idée de la démonstration consiste alors à contrôler ∇u_ϵ près du bord pour un temps petit par des fonctions barrières bien choisies. L'invariance par translation en la variable d'espace du problème $\mathcal{P}$, nous permet ensuite d'obtenir un contrôle du gradient sur tout le domaine Ω en utilisant le principe du maximum. Grâce à ces estimations, on peut extraire une sous-suite convergente, notée encore u_ϵ, par le théorème d'Ascoli. Mais, puisque $p > 1$, la méthode utilisée dans [16] n'est pas adaptée à notre cas. Ainsi, une convergence faible-* des ∇u_ϵ seule est insuffisante pour passer à la limite pour les termes non-linéaires. Pour cela, nous montrons d'abord, que les suites ∇u_ϵ sont des suites de Cauchy dans un espace de Banach L^m, puis on en déduit directement la convergence forte des ∇u_ϵ. Ce qui nous facilite de passer du problème régularisé vers notre problème $\mathcal{P}$.

Dans la deuxième partie de ce travail, nous supposons que les données initiales sont positives mais sont non nulles dans une partie mesurable de Ω, c'est à dire $u_0 \gneqq 0$. Nous commençons par montrer un principe de comparaison faible (voir proposition 3.3.1), qui jouera un rôle important pour obtenir quelques propriétés qualitatives des solutions positives du problème $\mathcal{P}$. Ce principe de comparaison implique également l'unicité de la solution positive. Ainsi, dans ce cas, le problème $\mathcal{P}$ est bien posé.

Ensuite, la première propriété qualitative que nous abordons dans ce chapitre est l'explosion de la solution positive en norme L^∞ en temps fini T_{max}. Rappelons que l'explosion de la solution se produit si la solution devient infinie à certains (ou plusieurs) points t proche d'un certain temps fini T. c'est à dire qu'il existe T, appelé temps d'explosion, tel que la solution est bien définie pour tout $0 < t < T$ tandis que $\|u(x,t)\|_{L^\infty(\Omega)} \to \infty$ lorsque $t \to T^-$. Maintenant, nous donnons les conditions suffisantes pour garantir que la solution du problème $\mathcal{P}$ explose en temps fini avec des données initiales de taille assez grande.

Theorem 0.0.3. *Soient $q \geq \frac{p}{2}$ et $q + l > 1$. On suppose que $k, s \geq 1$ sont tels que $k + s > \max\{p-1, q+l\}$. Alors, la solution faible du probléme $\mathcal{P}$ explose en norme L^∞ en temps fini T_{max}, si u_0 est assez grande.*

L'idée de la démonstration consiste à construire une sous-solution à support compact qui explose en temps fini et à utiliser le principe de comparaison, pour atteindre notre objectif. Cette méthode a été utilisée à notre connaissance, pour la première fois par Ph. Souplet et F. B. Weissler dans [88] pour p = 2. Li and Xie ont ensuite développé cette méthode dans [69] pour le cas où $p > 2$.

La deuxième propriété qui nous intéresse dans ce chapitre est l'étude de l'existence globale par rapport au temps, c'est à dire étudier les conditions suffisantes pour que la solution soit bornée en L^∞ pour tout $0 < t < \infty$. Cependant, nous sommes obligés de supposer que $q \leq p - 1$ pour pouvoir exclure la possibilité d'explosion du gradient. Nous signalons que, d'après le théorème d'existence locale ci-dessus, les solutions du problème $\mathcal{P}$ existent globalement par rapport au temps si les solutions sont uniformément bornées en norme $W^{1,\infty}$. Ainsi, comme indiqué dans le chapitre 3, section 3.2, dans notre situation, nous ne sommes pas intéressés par l'étude de la bornitude ou l'explosion de $\|\nabla u\|_{L^\infty}$. Signalons par ailleurs que J. N. Zhao, dans [98], a montré que le gradient ne peut pas exploser si $q \leq p - 1$. Le résultat suivant est notre première contribution d'existence globale des solutions du problème $\mathcal{P}$.

Theorem 0.0.4. *Soient $k, s \geq 1$ et $q \geq \frac{p}{2}$. On suppose que l'une des hypothèses suivantes est satisfaite*

1. $q + l > \max\{p-1, k+s\}$,
2. $q + l = k + s > p - 1$ *et* $|\Omega|$ *(ou* λ *) est assez petit,*
3. $q + l = k + s > p - 1$ *et* μ *est assez grand.*

Alors, la solution faible du $\mathcal{P}$ est uniformément bornée en norme L^∞.

Le théorème ci-dessus nous dévoile que si le terme d'absorption est dominant dans notre équation, alors la solution existe globalement par rapport au temps dans l'espace L^∞. La démonstration se base sur la construction d'une sur-solution uniformément bornée, ensuite avec le principe de comparaison on en déduit notre résultat. Cette méthode a été utilisée par P. Quittner et Ph. Souplet [80] dans le cas où $p = 2$.

Notre deuxième résultat concernant l'existence globale de la solution par rapport au temps du problèmes $\mathcal{P}$ est le suivant,

Theorem 0.0.5. *Soit $k, s \geq 1$. On suppose que l'une des hypothèses suivantes est satisfaite*

1. $k + s < p - 1$.
2. $k + s > p - 1$ *et* $\|u_0\|_\infty$ *est assez petit.*
3. $k + s = p - 1$ *et* $|\Omega|$ *(or* λ *) est assez petit.*

Alors, la solution faible du problème $\mathcal{P}$ est uniformément bornée en norme L^∞.

Ce résultat nous montre que si le terme de diffusion a plus d'effet sur l'équation que le terme source, alors on a existence globale de la solution. Dans le cas contraire, il faut que la donnée initiale u_0, la mesure de Ω ou le coefficient λ soient assez petits. La démonstration de ce résultat se base sur le lemme de Hopf et la construction d'une sur-solution du problème $\frac{\partial v}{\partial t} - \Delta_p v = \lambda v^k \int_\Omega v^s dx$. Encore une fois, par le principe de comparaison, on en déduit notre résultat.

Puisque nous avons trouvé des conditions suffisantes pour que les solutions soient uniformément bornées, nous allons encore plus loin. Nous allons donner des conditions suffisantes pour que les solutions faibles du problème $\mathcal{P}$ s'annulent en temps fini. Bien sûr, la propriété d'extinction de la solution peut être réduite à l'explosion pour un problème parabolique associé par la transformation $u \to 1/u$. Cette transformation n'est généralement pas pratique et il est préférable d'étudier chaque comportement séparément. Notre première contribution concernant l'extinction des solutions est la suivante,

Theorem 0.0.6. *Soient $1 \leq k + s \leq p - 1$, $l + q < 1$ et $q + l(p-1) < 1$. On suppose que $|\Omega|$ (ou λ) est assez petit. Alors, il existe $T_e > 0$ tel que $u = 0$ dans $\Omega \times [T_e, \infty)$.*

La preuve de ce théorème se base sur l'utilisation d'un lemme technique (voir (3.4.3)) et de l'inégalité de Gagliardo-Nirenberg voir (3.4.1). Ensuite, nous montrons l'extinction des solutions en norme L^2. Le théorème précédent est obtenu de façon presque similaire dans [95], mais, dans le cas où le problème $\mathcal{P}$ contient uniquement le terme local (c-a-d $s = 0$ et $\lambda = 1/|\Omega|$) et sans terme de gradient (c-a-d $q = 0$).

Contrairement au théorème ci dessus, le théorème suivant montre que l'extinction des solutions en temps fini peut également se produire pour $1 < p < 2$ mais avec des données initiales u_0 assez petites.

Theorem 0.0.7. *Soient $1 < p < 2$, $q \geq \frac{p}{2}$ et $\|u_0\|_\infty$ assez petit. On suppose que l'une des hypothèses suivantes est satisfaite*

1. $q + l < k + s$,
2. $q + l = k + s$ *et* μ *est assez petit.*
3. $q + l = k + s$ *et* $|\Omega|$ *(ou* λ*) est assez petit.*

Alors, il existe $T_e > 0$, telle que $u = 0$ dans $\Omega \times [T_e, \infty)$.

Nous terminons ce chapitre par montrer le théorème ci dessus à l'aide de la construction d'une sur-solution qui s'annule en un temps fini T_e qui dépend de $\|u_0\|_{L^\infty}$. Nous signalons que le résultat d'extinction de la solution obtenu dans ce théorème concerne la norme L^∞.

État de l'art et résultats du chapitre 4

Dans ce chapitre, nous nous sommes intéressés au problème parabolique quasi-linéaire suivant

$$\mathcal{P}:\begin{cases}\dfrac{\partial b(x,u)}{\partial t}-\Delta_{p(x,t)}u=a(x,t)|u|^{q(x,t)-1}u & \text{dans } Q_T=\Omega\times(0,T),\\ u=0 & \text{sur } \partial\Omega\times(0,T),\\ b(x,u\,(x,0))=b(x,u_0\,(x)) & \text{dans } \Omega,\end{cases}$$

- Ω est un domaine borné dans $\mathbb{R}^N$, $N\geq 1$, de frontière assez régulière.
- $\Delta_{p(x,t)}u=\operatorname{div}\left(|\nabla u|^{p(x,t)-2}\nabla u\right)$, est le $p(x,t)$-Laplacien, où $p(x,t)$ est une fonction continue sur $\overline{Q}_T$.
- $b(x,u)=|u|^{m(x)-1}u$, où $m(x)$ est une fonction mesurable positive.
- $q(x,t)$ et $a(x,t)$ sont deux fonctions mesurables où $q(x,t)$ est une fonction positive.
- $u_0\in L^\infty(\Omega)$ et $u_0\geq 0$ p.p. dans Ω.

Durant tout ce chapitre, nous supposons que les hypothèses suivantes sur les exposants $p(x,t)$, $q(x,t)$, $m(x)$ et le coefficient $a(x,t)$, sont satisfaites,

$$\begin{cases}\text{il existe des constantes strictement positives } p^\pm, q^\pm, m^\pm \text{ et } a_1 \text{telle que, pour tout } (x,t) \text{ dans } Q_T\\ \quad 1<p^-\leq p(x,t)\leq p^+,\ \ 0<q^-\leq q(x,t)\leq q^+,\\ 0<m^-\leq m(x)\leq m^+,\ \ |a(x,t)|\leq a_1,\end{cases}\tag{0.4}$$

Le problème $\mathcal{P}$ a été étudié par Antontsev et Shamarev dans [8]. Ces auteurs ont montré, sous différentes conditions, que le problème $\mathcal{P}$ a au moins une solution bornée au sens de la définition (0.2). Ils ont défini l'espace où les solutions du problème $\mathcal{P}$ existent comme suivant :

$$\mathcal{U}=\left\{u\in L^\infty(Q_T)\cap W(Q_T),\frac{\partial}{\partial t}b(x,u)\in W^{'}(Q_T)\right\},$$

où

$$W(Q_T)=\left\{u\in L^{m(x)+1}(Q_T):\ \ |\nabla u|^{p(x,t)}\in L^1(Q_T),\ \ u(.,t)\in V_t(\Omega)\text{ p.p. } t\in(0,T)\right\},$$

et $W^{'}(Q_T)$ est l'espace dual de $W(Q_T)$ et pour tout $t\in(0,T)$ fixé,

$$V_t(\Omega)=\left\{u\in L^{m(x)+1}(\Omega)\cap W_0^{1,1}(\Omega):|\nabla u|\in L^{p(x,t)}(\Omega)\right\}.$$

Notons que plus d'informations sur les espaces de Lebesgue et de Sobolev à exposent variables sont données au chapitre 1, on peut également voir [2, 11, 49, 76, 81]. Ainsi, la définition de la solution faible du problème $\mathcal{P}$ a été introduite dans [8] comme suivant,

Definition 0.2. *On dit que $u(x,t)$ est une solution faible du problème $\mathcal{P}$ sur Q_T si*

1. $u\in\mathcal{U}$.

2. *Pour toute fonction test $\phi \in \mathcal{U}$, on a*

$$\int_{Q_T} \phi \frac{\partial}{\partial t} b(x,u) + |\nabla u|^{p(x,t)-2} \nabla u \nabla \phi dxdt = \int_{Q_T} a(x,t)|u|^{q(x,t)-1} u\phi dxdt,$$

3. $b(x,u(.,0)) = b(x,u_0)$ *p.p. dans* Ω, *et* $u = 0$ *sur* $\partial\Omega \times (0,T)$.

Definition 0.3. *On dit que $u(x,t)$ est une solution forte de $\mathcal{P}$, si u est une solution faible et satisfait*

$$\frac{\partial}{\partial t} b(x,u) \in L^1(Q_T).$$

Nous signalons que l'existence d'au moins une solution faible ou forte bornée a été démontrée par S. Antontsev et S. Shmarev dans [8] par la méthode de Galerkin. Dans notre travail, nous nous sommes intéressés par l'étude des propriétés qualitatives des solutions positives de $\mathcal{P}$. Comme nous l'avons noté ci-dessus, ce type de problèmes modélise quelques phénomènes physiques. Les solutions positives de ce type de problèmes possèdent en effet, des propriétés intéressantes ayant une signification physique. Un aspect particulier qui mérite d'être étudié est la propriété de la propagation finie; c'est à dire que si les données initiales sont à support compact alors les solutions ont aussi un support compact pour tout $t > 0$. En plus, pour ce type de propriétés, une frontière libre apparaît pour séparer les ensembles $\{(x,t) : u > 0\}$ et $\{(x,t) : u = 0\}$. Dans l'étude de la filtration turbulente d'un gaz (ou d'un liquide) à travers un milieu poreux, une solution est strictement positive signifie que le gaz finit par se propager dans tout le domaine. Ainsi, une solution à support compact signifie que le gaz restera à jamais confiné dans une région limitée de l'espace. D'autre part, l'extinction de la solution à partir d'un temps fini signifie tout simplement que le gaz disparait de tout l'espace à partir d'un certain temps T.

Avant de commencer à traiter les propriétés qualitatives des solutions fortes de $\mathcal{P}$, nous commençons par énoncer un principe de comparaison qui jouera un rôle important dans nos démonstrations.

Proposition 0.0.8. *Soit $()u,v)$ un couple de sous et sur-solution de $\mathcal{P}$. On suppose que $\frac{\partial}{\partial t}b(x,u)$, $\frac{\partial}{\partial t}b(x,v) \in L^1(Q_T)$. Si $a(x,t) \leq 0$ p.p. dans Q_T, ou $m^+ \leq q^-$. Alors, on a $u \leq v$ p.p. dans Q_T.*

Remarquons que les hypothèses de notre principe de comparaison sont bien plus faibles que les hypothèses utilisées par S. Antontsev et S. Shmarev dans [8]. En plus, notre démonstration est encore plus simple. La démonstration de cette proposition s'appuie sur la considération de la fonction

$$\phi_\eta(s) = \begin{cases} 1, & \text{if } s > \eta, \\ \frac{s}{\eta}, & \text{if } |s| \leq \eta, \\ 0, & \text{if } s < -\eta, \end{cases}$$

où η est une constante positive et assez petite, comme une fonction test.

Maintenant, nous donnons les propriétés qualitatives des solutions du problème $\mathcal{P}$ obtenues dans ce chapitre. Les deux théorèmes suivants traitent la propriété d'extinction de la solution en temps fini. Dans le premier théorème, nous n'imposerons aucune condition sur le coefficient $a(x,t)$. Cependant, nous supposons que $p^+ - 1 < m^-$; c'est à dire que dans ce cas nous sommes dans le cas d'une diffusion rapide. Contrairement au premier théorème, nous n'imposerons aucune condition sur les exposants $p(x,t)$ et $m(x)$ dans le deuxième théorème. Mais, nous supposerons qu'il existe une constante $c > 0$ telle que $a(x,t) < -c$, c'est à dire que le terme d'absorption est assez fort. Bien sûr, les temps d'extinction trouvés dans chaque résultat sont différents.

Theorem 0.0.9. *Soit u une solution forte de $\mathcal{P}$. Supposons que $m^+ < q^-$, $p^+ - 1 < m^-$, $\sup_{Q_T} |\nabla p(x,t)| < \infty$ et $\|u_0\|_\infty$ assez petit. Alors, il existe un temps fini T_1 telle que pour tout $t \geq T_1$*

$$u(x,t) = 0, \ \ p.p. \ x \in \Omega.$$

Theorem 0.0.10. *Soit u une solution forte de $\mathcal{P}$. Supposons que $m^- > q^+$, $a(x,t) \leq -c < 0$ p.p. dans Q_T, $\sup_{Q_T} |\nabla p(x,t)| < \infty$ et $\|u_0\|_\infty$ est assez petit. Alors, il existe un temps fini T_2 telle que pour tout $t \geq T_2$*

$$u(x,t) = 0, \ \ p.p. \ x \in \Omega.$$

La démonstration de chacun de ces deux résultats ont basée sur de la construction de sur-solutions et de l'application du proposition (0.0.8).

Notre deuxième contribution dans ce chapitre concerne la non-extinction des solutions du problème $\mathcal{P}$. Nous allons donner des conditions suffisantes pour que les solutions de $\mathcal{P}$ ne s'annulent jamais lorsque t est assez grand. Notons que la non-extinction de la solution n'a jamais été traitée dans le cas où les équations sont à exposants variables. Le résultat suivant va être établi dans le cas où $m^+ < p^- - 1$, c'est à dire que le terme de diffusion est lent. Ainsi, on n'imposera aucune condition sur le coefficient $a(x,t)$.

Theorem 0.0.11. *Soit u une solution forte de $\mathcal{P}$, et u_0 non identiquement nul. Supposons $m^+ < q^-$, $m^+ < p^- - 1$, et $\sup_{Q_T} |\nabla p(x,t)| < \infty$. Alors, il existe un temps fini T_p pour tout $t \geq T_p$*

$$u(x,t) > 0, \ \ p.p. \ x \in \Omega.$$

La démonstration de ce théorème est divisée en deux étapes. La première étape consiste à démontrer la croissance du support de la solution par rapport au temps t, c'est à dire que supp $u(.,s) \subset$ supp $u(.,t)$, pour tout $0 < s < t$. Dans la deuxième étape, nous construisons une sous-solution $\underline{u}(x,t)$ de $\mathcal{P}$ positive dans une boule $B(x_0,R) \subset\subset \Omega$, où $R < 1/2$, et pour tout $t > T_0$. Grâce au principe de comparaison (0.0.8), on en déduit que les solutions de $\mathcal{P}$ sont strictement positives dans $B(x_0,R)$, pour tout $t > T_0$. On conclut la preuve de ce théorème par la technique de recouvrement fini.

Finalement, la dernière propriété traitée dans ce chapitre réside dans le fait que chaque solution de $\mathcal{P}$ possède la propriété d'une vitesse finie de propagation des perturbations à partir des données initiales. Plus précisément, nous montrons que si les conditions initiales u_0 sont à support compact, alors il existe une unique solution également à support compact. Ainsi, nous estimons le support de la solution forte dans chaque résultat.

Dans le théorème ci-dessous, nous n'imposerons aucune condition sur le coefficient $a(x,t)$. Rappelons que S. Antontsev et S. Shamarev, dans [11], on signalé que la solution peu exploser en temps fini T^* dans le cas où $a(x,t)$ est positive. Nous nous proposons alors d'estimer le support de la solution localement par rapport aux temps, c'est dire pour tout $t \in (0,T)$, où $T < T^*$. Notons que dans les deux théorèmes suivants le terme de diffusion est lent.

Theorem 0.0.12. *Soit u une solution faible de $\mathcal{P}$. Soit $0 < R < +\infty$ telle que supp $u_0 \subset \Omega_R$. Supposons que $m^+ < p^- - 1$, $q(x,t) = m(x)$ et $\sup_{Q_T} |\nabla p(x,t)| < \infty$. Alors, pour tout $t \in (0,T)$, il existe une unique solution de $\mathcal{P}$ à support compact telle que*

$$supp \ u(t,.) \subset \{x \in \bar{\Omega} : |x| \leq \rho_1(t)\},$$

où $\rho_1(t)$ est donné explicitement au chapitre 4.

Ensuite, dans le théorème ci dessous, nous supposons $a(x,t) < 0$, puis nous établissons une estimation du support de la solution pour tout $t \in (0,T)$, où $T < \infty$, car d'après [11], la solution ne peut exploser dans ce cas.

Theorem 0.0.13. *Soit u une solution forte de $\mathcal{P}$. Soit $0 < R < +\infty$ telle que supp $u_0 \subset \Omega_R$. On suppose que $m^+ < p^- - 1$, $\sup_{Q_T} |\nabla p(x,t)| < \infty$, et $a(x,t) \leq 0$. Alors, pour tout $t \in (0,T)$, il existe une unique solution de $\mathcal{P}$ à support compact telle que*

$$supp\ u(t,.) \subset \{x \in \bar{\Omega} : |x| \leq \rho_2(t)\},$$

où $\rho_2(t)$ est donné explicitement au chapitre 4.

Dans ce dernier théorème, nous montrons que le phénomène sur la vitesse de propagation finie se produit également lorsque l'équilibre entre le terme la diffusion et le terme de l'absorption est réalisé. Nous donnons, dans ce cas, une estimation explicite du support de la solution. Ensuite on peut déduire aussi que ce support reste borné même si t tend vers $+\infty$.

Theorem 0.0.14. *Soit u une solution forte de $\mathcal{P}$. Soit $0 < R < +\infty$ telle que supp $u_0 \subset \Omega_R$. On suppose que $q^+ < p^- - 1$, $\sup_{Q_T} |\nabla p(x,t)| < \infty$ et $a(x,t) \leq a_2 < 0$. Alors, pour tout $t \in (0,\infty)$,il existe une unique solution de $\mathcal{P}$ à support compact telle que*

$$supp\ u(t,.) \subset \{x \in \bar{\Omega} : |x| \leq R_1\},$$

où R_1 est une constante strictement positive donnée explicitement au chapitre 4.

Les démonstrations de toutes les propriétés étudiées dans le chapitre 4 se basent sur la construction de sous et sur-solutions de $\mathcal{P}$ et l'utilisation du principe de comparaison.

CHAPTER 1

LEBESGUE-SOBOLEV SPACES WITH VARIABLE EXPONENT

In the past few years the subject of variable exponent spaces has undergone a vast development. Nevertheless, the standard reference for basic properties has been the article [67] by Kovacik and Rakosuik from 1991. (The same properties were derived by different methods by Fan and Zhao [49] 10 years later.) In this chapter we shall introduce Lebesgue and Sobolev spaces with variable exponent and state some of their basic properties, such as reflexivity, separability, duality and first results concerning embeddings and density of smooth functions.

1.1 A brief history of function spaces with variable exponents

Lebesgue spaces with variable exponents appeared in the literature in 1931 in the paper by Orlicz [77]. In that paper, the author considered the question of Hölder's inequality in the space $\ell^{q(\cdot)}$ and then he generalized the same question to the Lebesgue space with variable exponent $L^{q(\cdot)}$ on the real line. After this, he was interested in the study of function spaces L^{Φ} that contain all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\rho(\lambda u) = \int_{\Omega} \Phi(\lambda |u(x)|) dx < \infty,$$

for some $\lambda > 0$ and Φ satisfying some natural assumptions, where Ω is an open set in $\mathbb{R}^N$ (see [74]). We point out that in [74] the case $|u(x)|^{q(x)}$ corresponding to variable exponents was not included. In the 1950's these problems were systematically studied by Nakano [76], who developed the theory of "modular function spaces". Nakano, in the appendix [p. 284], explicitly mentioned Lebesgue spaces with variable exponents as an example of more general spaces he considered. Lebesgue spaces with variable exponents on the real line reappeared independently in the Russian literature, where they were studied as spaces of interest in their own right, notably Tsenov in 1961 [90] and Sharapudinov in 1979 [83]. The question raised by Tsenov and solved by Sharapudinov is the minimization of the integral

$$\int_a^b |u(x) - v(x)|^{q(x)} dx,$$

where u is a given function and v varies over a finite dimensional subspace of $L^{q(x)}([a,b])$. In [83] Sharapudinov also introduced the Luxemburg norm for the Lebesgue space and showed some classical results such as the separability and reflexivity. In the mid-1980s Zhikov [99] started a new direction of investigation, which applied the Lebesgue spaces with variable exponents to problems in the calculus variational integrals with non-standard growth conditions. Precisely, he was concerned with minimizing the functionals

$$F(u) = \int_{\Omega} f(x, \nabla u) dx,$$

when f satisfies the non-standard growth condition

$$-c_0 + c_1|t|^p \leq f(x,t) \leq c_0 + c_2|t|^s,$$

where the c_i are positive real constants and $0 < p < s$. A particular example of such function is $f(x,t) = |t|^{q(x)}$, where $p \leq q(x) \leq s$.

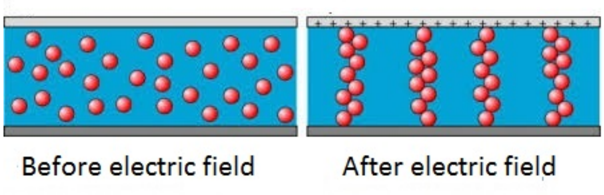
The paper by Kovacik and Rakosuik in the early 1990s [67], was considered as a major step in the investigation of Lebesgue spaces with variable exponents. This paper established several basic properties of spaces $L^{q(x)}(\Omega)$ and $W^{1,q(x)}(\Omega)$ with variable exponents in $\mathbb{R}^N$. After ten years, Fan and Zhao [49] established the same properties in [67] by different methods. At the beginning of millennium, many efforts were made to understand these spaces, for example, how to establish the connection between these spaces, conditions of coercivity and the variational integrals with nonstandard growth. Density of smooth functions in $W^{m,q(x)}(\Omega)$ and related Sobolev embedding properties are due to Edmunds and Rakosnik [45]. Pioneering regularity results for functionals with nonstandard growth are due to Acerbi and Mingione [2]. The abstract theory of Lebesgue and Sobolev spaces with variable exponents was developed in the monographs by Diening, Harjulehto, Hästö, and Ruzicka [43] and by Cruz-Uribe, Fiorenza [35].

These spaces constitute the most appropriate functional framework for solving several nonlinear partial differential equations such as the problems involving the operator $p(x)$-Laplacian. These problems model many physical phenomena such as electrorheological fluids [82] and image restoration [29]. In the forthcoming section, we discuss these phenomena in details.

1.2 Motivation

In this section, we give two relevant examples that justify the mathematical study of models involving variable exponents.

Electrorheological fluids: Electrorheological fluids are colloidal suspensions of a certain type, consisting of dielectric particles dispersed in an insulating oil. The marvelous feature of such fluids is that they can solidify into a jelly-like state almost instantaneously when subjected to an externally applied electric field with moderate strength, with a stiffness varying proportionally to the field strength. The liquid-solid transformation is reversible. Once the applied field is removed, the original flow state is recovered. This phenomenon is known as the Winslow effect, and we can represent it as follows



Electrorheological fluids have the quality and potential for a wide field of applications. These include for example robotics, aircraft and aerospace applications. We refer the reader to [81, 82] for more information.

There exist several possibilities for modeling the physics of electrorheological fluids. In [81], Rajagopal and Růžička have developed a model that takes into account the complex interaction of electromagnetic fields and the moving liquid. The constitutive equation for the motion of an electrorheological fluid is given by

$$\partial_t u + \mathrm{div} S(u) + (u \cdot \nabla)u + \nabla \pi = f, \tag{1.1}$$

where $u : \mathbb{R}^{3+1} \to \mathbb{R}^3$ is the velocity of the fluid at a point, $\nabla = (\partial_1, \partial_2, \partial_3)$ is the gradient operator, π is the pressure, f represents external forces and the stress tensor S is of the form

$$S(u)(x) = \mu(x)(1 + |Du(x)|^2)^{\frac{p-2}{2}} Du(x),$$

where $p = p(x)$ and $Du = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetric part of the gradient of u. Rajagopal and Růžička established an existence result for problem (1.1) in variable exponent spaces.

Image restoration: Image restoration is the operation of taking a corrupt/noisy image and estimating the clean, original image. Corruption may come in many forms such as motion blur, noise and camera mis-focus. Chen, Levine and Rao [29], proposed a new model for image restoration. The diffusion resulting from the model was proposed is a combination of the Gaussian smoothing and regularization based on the total variation. Precisely, given an original image f, it is assumed that it has been corrupted by some additive noise n. Then the problem is to recover the true image u

$$f = u + n.$$

The adaptive model was proposed is:

$$\min_{u \in BV(\Omega) \cup L^2(\Omega)} \int_\Omega \phi(x, \nabla u) + \frac{\lambda}{2} \int_\Omega (u - f)^2, \tag{1.2}$$

where

$$\phi(x,t) = \begin{cases} \frac{1}{q(x)} |t|^{q(x)}, & |t| \le \beta, \\ |t| - \frac{\beta q(x) - \beta^{q(x)}}{q(x)}, & |t| > \beta, \end{cases}$$

$q(x) = 1 + \frac{1}{1 + k|\nabla G_\sigma * f(x)|}$, $G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} exp(\frac{|x|^2}{2\sigma^2})$ is the Gaussian kernel, $k > 0, \sigma > 0$ are fixed parameters and $\beta > 0$ is a user-defined threshold. For problem (1.2), Chen, Levine and Rao established the existence and uniqueness of the solution and the long-time behavior of the associated flow of the proposed model. The effectiveness of the model in image restoration is illustrated by some experimental results included in their paper.

1.3 Lebesgue spaces with variable exponents

In this part, we define Lebesgue spaces with variable exponents, $L^{q(\cdot)}$. The variable Lebesgue spaces, as their name implies, are a generalization of the classical Lebesgue L^q spaces, replacing the constant exponent q with a variable exponent function $q(\cdot)$. The resulting Banach function spaces $L^{q(\cdot)}$ have many properties similar to the L^q spaces, but they also differ in surprising and subtle ways. By virtue of types of problems studied in this thesis, throughout this work, we restrict to define the Lebesgue spaces with variable exponents $L^{q(\cdot)}$ only for a continuous function $q : \overline{\Omega} \to (1, +\infty)$. However, these spaces can be defined for any measurable function $q : \Omega \to [1, +\infty]$. For more details on the basic properties of these spaces, we refer the reader to the papers [35, 43, 49, 67].

Set

$$\mathcal{C}_+(\overline{\Omega}) := \{h \in \mathcal{C}(\overline{\Omega}) : h(x) > 1 \text{ for any } x \in \overline{\Omega}\}.$$

For any $h \in \mathcal{C}_+(\overline{\Omega})$ we define

$$h^- := \min_{x \in \overline{\Omega}} h(x), \quad h^+ := \max_{x \in \overline{\Omega}} h(x).$$

Definition 1.1. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. For a measurable function $u : \Omega \to \mathbb{R}$, the mapping*

$$\rho_{q(\cdot)}(u) = \int_\Omega |u(x)|^{q(x)} dx,$$

is called modular of u with respect to $q(\cdot)$.

Remark 1.3.1. *In view of the definition in [75, p. 1], $\rho_{q(\cdot)}$ is a convex modular, that means, $\rho_{q(\cdot)}$ verifies the following properties: for any measurable functions $u, v : \Omega \to \mathbb{R}$, we have*

- $\rho_{q(\cdot)}(u) = 0 \Leftrightarrow u = 0,$
- $\rho_{q(\cdot)}(u) = \rho_{q(\cdot)}(-u),$
- $\rho_{q(\cdot)}(\alpha u + \beta v) \le \alpha \rho_{q(\cdot)}(u) + \beta \rho_{q(\cdot)}(v), \quad \forall \alpha, \beta \ge 0,\ \alpha + \beta = 1.$

Using the modular $\rho_{q(\cdot)}$, Lebesgue spaces with variable exponent are defined as follows.

Definition 1.2. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. The set*

$$L^{q(x)}(\Omega) := \left\{u : u : \Omega \to \mathbb{R} \text{ is measurable with } \rho_{q(\cdot)}(u) < \infty\right\},$$

is called Lebesgue space with variable exponent $q(\cdot)$ or generalized Lebesgue space.

Now we introduce the so-called Luxemburg norm on $L^{q(x)}(\Omega)$.

Definition 1.3. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. For $u \in L^{q(x)}(\Omega)$ we define its Luxemburg norm with respect to $q(\cdot)$ by*

$$\|u\|_{L^{q(x)}(\Omega)} = \|u\|_{q(x)} := \inf\left\{\lambda > 0 : \ \rho_{q(\cdot)}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

Proposition 1.3.2 ([49, 67])**.** *Let $q \in \mathcal{C}_+(\overline{\Omega})$. Then, the variable exponent Lebesgue spaces $(L^{q(x)}(\Omega), \|\cdot\|_{q(x)})$ are separable Banach spaces.*

The norm $\|u\|_{q(x)}$ is in close relation with the modular $\rho_{q(\cdot)}(u)$. We have

Proposition 1.3.3 ([98])**.** *Let $u, (u_n) : \Omega \to \mathbb{R}$ be measurable functions; then,*

$$\min\{\|u\|_{q(x)}^{q-}, \|u\|_{q(x)}^{q+}\} \le \rho_{q(\cdot)}(u) \le \max\{\|u\|_{q(x)}^{q-}, \|u\|_{q(x)}^{q+}\}, \tag{1.3}$$

$$\|u\|_{q(x)} < 1 (= 1, > 1) \Leftrightarrow \rho_{q(\cdot)}(u) < 1 (= 1, > 1), \tag{1.4}$$

$$\|u_n - u\|_{q(x)} \to 0 \Leftrightarrow \rho_{q(\cdot)}(u_n - u) \to 0, \tag{1.5}$$

$$\|u_n\|_{q(x)} \to +\infty \Leftrightarrow \rho_{q(\cdot)}(u_n) \to +\infty. \tag{1.6}$$

Definition 1.4. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. The variable exponent Lebesgue space $L^{q'(x)}(\Omega)$ is called the conjugate space of $L^{q(x)}(\Omega)$, where $q'(x)$ is the conjugate exponent of $q(x)$, that means, $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$.*

Hölder's inequality is another important tool that can also be retrieved.

Proposition 1.3.4 ([49, 67]). *Let $q \in \mathcal{C}_+(\overline{\Omega})$ and $q'(x)$ its conjugate exponent. Then, for any $u \in L^{q(x)}(\Omega)$ and $v \in L^{q'(x)}(\Omega)$, the Hölder type inequality*

$$\int_\Omega |uv|dx \leq \left(\frac{1}{q^-} + \frac{1}{q'^-}\right) \|u\|_{q(x)} \|v\|_{q'(x)} \leq 2\|u\|_{q(x)} \|v\|_{q'(x)}, \tag{1.7}$$

holds true.

Among the main properties in the study of Banach spaces is the description of their dual space $(L^{q(x)}(\Omega))^*$, and closely related to this, the question of reflexivity. For this purpose we give

Proposition 1.3.5 ([49, 67]). *Let $q \in \mathcal{C}_+(\overline{\Omega})$ and $q'(x)$ its conjugate exponent. Then, the space $(L^{q(x)}(\Omega), \|\cdot\|_{q(x)})$ is reflexive, and that the mapping $I : L^{q'(x)}(\Omega) \to (L^{q(x)}(\Omega))^*$ defined by*

$$\langle I(v), w\rangle = \int_\Omega v(x)w(x)dx, \quad \forall v \in L^{q'(x)}(\Omega), \forall w \in L^{q(x)}(\Omega),$$

is a linear isomorphism and $\|I(v)\|_{(L^{q(x)}(\Omega))^*} \leq 2\|v\|_{L^{q'(x)}(\Omega)}$.

The use of different variable exponents induces different generalized Lebesgue spaces. But, since $|\Omega| < \infty$, then we can recapture classic embedding properties.

Proposition 1.3.6 ([49, 67]). *Let $q_1(x), q_2(x) \in \mathcal{C}_+(\overline{\Omega})$. Then, $q_1(x) \leq q_2(x)$ almost everywhere in Ω, if and only if, $L^{q_2(x)}(\Omega)$ embedded continuously in $L^{q_1(x)}(\Omega)$.*

Remark 1.3.7. *For a single $q \in \mathcal{C}_+(\overline{\Omega})$, obviously, we have the following chain of continuous embeddings:*

$$L^\infty(\Omega) \hookrightarrow L^{q^+(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \hookrightarrow L^{q^-(x)}(\Omega) \hookrightarrow L^1(\Omega).$$

We finish this section by recalling the following proposition.

Proposition 1.3.8 ([45]). *Let p and q be measurable functions such that $p \in L^\infty(\Omega)$ and $1 < p(x)q(x) \leq \infty$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$. Then*

$$\|u\|_{p(x)q(x)} \leq 1 \Rightarrow \|u\|^{p^+}_{p(x)q(x)} \leq \||u|^{p(x)}\|_{q(x)} \leq \|u\|^{p^-}_{p(x)q(x)},$$
$$\|u\|_{p(x)q(x)} \geq 1 \Rightarrow \|u\|^{p^-}_{p(x)q(x)} \leq \||u|^{p(x)}\|_{q(x)} \leq \|u\|^{p^+}_{p(x)q(x)}.$$

In particular when $p(x) = p$ is a constant, then

$$\||u|^p\|_{q(x)} = \|u\|^p_{pq(x)}.$$

1.4 Sobolev spaces with variable exponents

In this part, we give some basic results on the generalized Sobolev spaces $W^{1,q(x)}(\Omega)$. One motivation of studying these spaces is that solutions of partial differential equations belong naturally to Sobolev spaces.

Definition 1.5. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. We define the Sobolev space with variable exponent $q(x)$ by*

$$W^{1,q(x)}(\Omega) = \{u \in L^{q(x)}(\Omega) : \ |\nabla u| \in L^{q(x)}(\Omega)\},$$

Definition 1.6. *Let $q \in \mathcal{C}_+(\overline{\Omega})$ and $u \in W^{1,q(x)}(\Omega)$. We define the Sobolev norm by*

$$\|u\|_{W^{1,q(x)}(\Omega)} = \|u\|_{1,q(x)} := \|u\|_{q(x)} + \|\nabla u\|_{q(x)},$$

where $\|\nabla u\|_{q(x)} = \||\nabla u|\|_{q(x)}$.

Proposition 1.4.1 ([49, 67]). *The space $(W^{1,q(x)}(\Omega), \|\cdot\|_{1,q(x)})$ is separable and reflexive Banach space.*

An immediate consequence of Proposition 1.3.6 is

Proposition 1.4.2 ([49, 67]). *Let $q_1(x), q_2(x) \in \mathcal{C}_+(\overline{\Omega})$. If $q_1(x) \leq q_2(x)$ almost everywhere in Ω, then $W^{1,q_2(x)}(\Omega)$ can be embedded into $W^{1,q_1(x)}(\Omega)$.*

Now let us generalize the well-known Sobolev embedding theorem of $W^{1,q}(\Omega)$ to $W^{1,q(x)}(\Omega)$. Let us start by the following definition

Definition 1.7. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. The variable exponent defined by*

$$\frac{Nq(x)}{N - q(x)} := q^*(x), \quad \forall x \in \overline{\Omega},$$

is called Sobolev conjugate exponent.

Now we can state the Sobolev embedding result

Proposition 1.4.3 ([49]). *Let $q, r \in \mathcal{C}_+(\overline{\Omega})$. Assume that*

$$q(x) < N, \quad r(x) < q^*(x), \quad \forall x \in \overline{\Omega}.$$

Then, there is a continuous and compact embedding $W^{1,q(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$.

A natural question which may be asked is: is there a continuous embedding of $W^{1,q(x)}(\Omega)$ into $L^{q^*(x)}(\Omega)$. The following example shows that, in general, this cannot be expected.

Example 1.4.4. *Let $\Omega = \{x = (x_1, x_2) : 0 < x_1 < 1,\ 0 < x_2 < 1\} \subset \mathbb{R}^2, q(x) = 1 + x_2, u(x) = (2 + x_2)^{\frac{1}{1+x_2}}$. Then, we have $u(x) \in W^{1,q(x)}(\Omega)$ and $q^*(x) = 2(1 + x_2)/(1 - x_2)$. It is easy to check that $u \notin L^{q^*(x)}(\Omega)$.*

By assuming that the exponent $q(\cdot)$ is regular, we can get the embedding $W^{1,q(x)}(\Omega) \hookrightarrow L^{q^*(x)}(\Omega)$. To this end, we give the following definition

Definition 1.8. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. We say that $q(\cdot)$ satisfies the log-Hölder condition if,*

$$|q(x) - q(y)| \leq \frac{c}{-log|x - y|}, \quad \forall\ |x - y| < \frac{1}{2},\ x, y \in \overline{\Omega},$$

for some constant $c := c(q(\cdot)) > 0$.

Now, we can extend the Sobolev embedding theorem to variable exponents Sobolev spaces. We have

Proposition 1.4.5. *[35, 38] Let $q \in \mathcal{C}_+(\overline{\Omega})$. If $q(\cdot)$ satisfies the log-Hölder condition, then $W^{1,q(x)}(\Omega)$ can be embedded into $L^{q^*(x)}(\Omega)$.*

Since $W^{1,q(x)}(\Omega)$ is a separable Banach space, then there exists a countable dense subset. Now, we would like to identify particular families of functions that are dense. Because weak derivatives coincide with classical derivatives for smooth functions, it is natural to consider the question of when such functions are dense. We begin by defining two subspaces of $W^{1,q(x)}(\Omega)$.

Definition 1.9. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. Let $W_0^{1,q(x)}(\Omega)$ be the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,q(x)}(\Omega)$, and let $H_0^{1,q(x)}(\Omega) = W^{1,q(x)}(\Omega) \cap W_0^{1,1}(\Omega)$.*

Remark 1.4.6.

1. *It is clear that if $q(x) \equiv q$ is a constant, then $H_0^{1,q}(\Omega) = W_0^{1,q}(\Omega)$. In this case, the space $\mathcal{C}^\infty(\Omega)$ is dense in $W^{1,q}(\Omega)$.*
2. *For a general function $q \in \mathcal{C}_+(\overline{\Omega})$, from the definition, we have $W_0^{1,q(x)}(\Omega) \subset H_0^{1,q(x)}(\Omega)$, and $H_0^{1,q(x)}(\Omega)$ is a closed linear subspace of $W^{1,q(x)}(\Omega)$.*
3. *In general, $H_0^{1,q(x)}(\Omega) \neq W_0^{1,q(x)}(\Omega)$. Indeed, let $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\}, 1 < \alpha_1 < 2 < \alpha_2$. Define the variable exponent q by*
$$q(x) = \begin{cases} \alpha_1, & \text{if } \ x_1 x_2 > 0, \\ \alpha_2, & \text{if } \ x_1 x_2 < 0, \end{cases}$$
then $H_0^{1,q(x)}(\Omega) \neq W_0^{1,q(x)}(\Omega)$. This example was given by Zhikov in [99].
4. *The identity $H_0^{1,q(x)}(\Omega) = W_0^{1,q(x)}(\Omega)$, means that, the space $\mathcal{C}_0^\infty(\Omega)$ is dense in $(H_0^{1,q(x)}(\Omega), \|\cdot\|_{W^{1,q(x)}(\Omega)})$.*

Using the log-Hölder condition on the variable exponent q, we get the following density result.

Proposition 1.4.7 ([35, 38, 49])**.** *Let $q \in \mathcal{C}_+(\overline{\Omega})$ and q satisfies the log-Hölder condition. Then, we have*

1. *$\mathcal{C}^\infty(\Omega)$ is dense in $W^{1,q(x)}(\Omega)$.*
2. *$H_0^{1,q(x)}(\Omega) = W_0^{1,q(x)}(\Omega)$.*

Now, we give the Poincaré inequality in generalized Sobolev space.

Proposition 1.4.8. *[35, 38] Let $q \in \mathcal{C}_+(\overline{\Omega})$ and q satisfies the log-Hölder condition. Then, we have*

$$\|u\|_{q(x)} \leq C\|\nabla u\|_{q(x)}, \quad \forall u \in W_0^{1,q(x)}(\Omega),$$

for some constant $C > 0$ depending only on the dimension N, $diam(\Omega)$, and $c(q(\cdot))$ defined in Definition 1.8. Moreover, $\|u\|_{W_0^{1,q(x)}(\Omega)} = \|\nabla u\|_{q(x)}$ is a norm in $W_0^{1,q(x)}(\Omega)$.

1.5 Some difficulties related to the Lebesgue-Sobolev spaces with variable exponents

As each functional space has their difficulties, in this part, we give some important results and techniques from classical Lebesgue-Sobolev spaces which do not hold in the Lebesgue-Sobolev spaces with variable exponents even when the exponent is very regular, i.e., q satisfies the log-Hölder condition, or $q \in \mathcal{C}^\infty(\overline{\Omega})$.

1. The space $L^{q(x)}(\Omega)$ is not rearrangement invariant; the translation operator $T_h : L^{q(x)}(\Omega) \to L^{q(x)}(\Omega)$, $T_h u(x) := u(x+h)$ is not bounded; Young's convolution inequality
$$\|u * v\|_{q(x)} \leq c\|u\|_1 \|v\|_{q(x)},$$
does not hold. Moreover, the map T_h is bounded if and only if q is a constant (see Proposition 3.6. 1 in [38]).

2. The formula
$$\int_\Omega |u(x)|^q dx = q\int_0^\infty t^{q-1}|\{x\in\Omega : |u(x)|>t\}|dt$$
has no variable exponent analogue.

3. In the constant exponent case there is an obvious connection between modular and norm versions of the inequality, which does not hold in the variable exponent context. In other words, the modular Poincaré inequality
$$\rho_q(u)\le c\rho_q(\nabla u)$$
can not, in general, hold in a modular form $\rho_{q(\cdot)}(u)$. In fact, taking the following example: let $q:(-2,2)\to[2,3]$ be a Lipschitz continuous exponent that equals 3 in $(-2,-1)\cup(1,2),$ 2 in $(-\frac{1}{2},\frac{1}{2})$ and is linear elsewhere. Let u_λ be a Lipschitz function such that $u_\lambda(\pm 2)=0; u_\lambda=\lambda$ in $(-1,1)$ and $|u'_\lambda|=\lambda$ in $(-2,-1)\cup(1,2)$. Then,
$$\frac{\rho_{q(x)}(u_\lambda)}{\rho_{q(x)}(u'_\lambda)}=\frac{\int_{-2}^{2}|u_\lambda|^{q(x)}dx}{\int_{-2}^{2}|u'_\lambda|^{q(x)}dx}\ge\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}}\lambda^2dx}{2\int_{-2}^{-1}\lambda^3dx}=\frac{1}{2\lambda}\to\infty,$$
as $\lambda\to 0^+$.

4. Solutions of the $p(x)$-Laplacian equation are not scalable, i.e. λu need not be a solution even if u is.

CHAPTER 2

MULTIPLICITY OF SOLUTIONS FOR A CLASS OF ELLIPTIC PROBLEM OF P-LAPLACIAN TYPE WITH A P-GRADIENT TERM

We consider the following problem

$$(P)\begin{cases}-\Delta_p u = c(x)|u|^{q-1}u + \mu|\nabla u|^p + h(x) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where Ω is a bounded set in $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary, $1 < p < N$, $q > 0$, $\mu \in \mathbb{R}^*$, and c and h belong to $L^k(\Omega)$ for some $k > \frac{N}{p}$. In this paper, we assume that $c \gneqq 0$ a.e. in Ω and h without sign condition, then we prove the existence of at least two bounded solutions under the condition that $\|c\|_k$ and $\|h\|_k$ are suitably small. For this purpose, we use the Mountain Pass theorem, on an equivalent problem to (P) with variational structure. Here, the main difficulty is that the nonlinearity term considered does not satisfy Ambrosetti and Rabinowitz condition. The key idea is to replace the former condition by the **nonquadraticity condition at infinity**.

2.1 Introduction and main result

Let Ω be a bounded set in $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary $\partial\Omega$. In this chapter, we are concerned with the following elliptic problem

$$(P)\begin{cases}-\Delta_p u = c(x)|u|^{q-1}u + \mu|\nabla u|^p + h(x) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $1 < p < N$, $q > 0$, $\mu \in \mathbb{R}^*$, and c and h belong to $L^k(\Omega)$ for some $k > \frac{N}{p}$.

In the literature, there are many results concerning the existence, the uniqueness, and the multiplicity of solutions for models like (P) under various assumptions on c and h. At first, it is important to mention that the sign of c plays a crucial role in the problem (P) regarding uniqueness, as well as existence, of bounded solutions. In this setting, we refer to ([64]) for more details.

In the coercive case, that is $c(x) \leq -\alpha_0$ a.e. in Ω for some $\alpha_0 > 0$, Boccardo, Murat and Puel ([21, 23, 22]), proved the existence of bounded solutions for more general divergence form problems with quadratic growth in the gradient by using the sub and supersolution method. Moreover, Barles and Murat ([6]) and Barles et at. ([5]) have treated the uniqueness question for similar problems. Notice that, if we allow $c(x) \leq 0$ a.e. in Ω, then Ferone and Murat ([50],[51]) observed that finding solutions to (P) becomes rather complex without imposing some strong regularity conditions on the data. For the particular case $c \equiv 0$, there had been many contributions ([1, 72, 78]). However, for $c \leq 0$ that may vanish only on some parts of Ω, the uniqueness of solutions was left open until the recent paper authored by Arcoya et at. ([14]). This last result was proved for $p = 2$, $q = 1$, and under the following condition

$$\begin{cases} c, h \text{ belong to } L^k(\Omega) \text{ for some } k > \frac{N}{2}, \mu \in L^\infty(\Omega) \text{ and } meas(\Omega \backslash Supp\ c) > 0, \\ \inf\limits_{u \in W_c, \|u\|_{H_0^1(\Omega)}} \int_\Omega \left(|\nabla u|^2 - \|\mu^+\|_{L^\infty(\Omega)} h^+(x) u^2 \right) > 0, \\ \inf\limits_{u \in W_c, \|u\|_{H_0^1(\Omega)}} \int_\Omega \left(|\nabla u|^2 - \|\mu^-\|_{L^\infty(\Omega)} h^-(x) u^2 \right) > 0. \end{cases}$$

where $W_c := \{w \in H_0^1(\Omega) : \ c(x)w(x) = 0, \text{ a.e. in } \Omega\}$. For a related uniqueness result see also Arcoya et at. ([13]).

The case where $c(x) \gneqq 0$ a.e. in Ω, the question of non-uniqueness has been being an open problem given by Sirakov ([85]) and it has received considerable attention by many authors. Moreover, it should be pointed out that the sign of h and whether μ is a function or a constant, generate additional difficulties for solving (P). In this setting, Jeanjean and Sirakov ([64]) showed the existence of two bounded solutions assuming that $\mu \in \mathbb{R}^*$, c and h are in $L^k(\Omega)$ for some $k > \frac{N}{2}$ and satisfying

$$\|[\mu h]^+\|_{L^{\frac{N}{2}}(\Omega)} < C_N,$$

$$\max\{\|c\|_{L^k(\Omega)}, \|[\mu h]^-\|_{L^k(\Omega)}\} < \bar{c},$$

where $\bar{c} > 0$ depends only on $N, k, meas(\Omega), |\mu|, \|[\mu h]^+\|_{L^k(\Omega)}$, and C_N is the optimal constant in Sobolev's inequality. Here, h is allowed to change sign. Shortly after, this result was extended by Coster and Jeanjean ([34]) for μ is a bounded function such that $\mu(x) \geq \mu_1 > 0$ by using the degree topological method.

Finally, in the case where c is allowed to change sign and with $c(x) \gneqq 0$ a.e. in Ω, Jenajean and Quoirin ([63, Theorem 1.1]) showed the existence of two bounded positive solutions when $h \gneqq 0$, μ is a positive constant, and c^+ and μh are suitably small.

We would also like to mention that all the above quoted multiplicity results were restricted to the Laplacian operator with quadratic growth in the gradient, i.e. $p = 2$, and for $q = 1$. Moreover, it is interesting to mention that when c is allowed to change sign the solutions are positive.

In this chapter, we prove the multiplicity of bounded solutions for the problem (P) by assuming the following assumption

$$(H) \begin{cases} c, h \text{ belongs to } L^k(\Omega) \text{ for some } k > \frac{N}{p}, h \text{ is allowed to change sign}, \\ c \gneqq 0 \text{ a.e. in } \Omega, q > 0, \text{ and } \mu \in \mathbb{R}^*. \end{cases}$$

Now, we give a brief exposition of the proof of our multiplicity result. At first, without loss of generality, we solve the problem (P) by restricting it to the case μ is a positive constant. For μ is a negative constant, we replace u by $-u$ in (P), then we conclude. Next, we observe that the problems of type (P) do not have a variational formulation due to the presence of the p-gradient term. To overcome this difficulty, we perform the Kazdan-Kramer change of variable, that is, $v = (e^{\frac{\mu u}{p-1}} - 1)/\mu$. Thus, we obtain the following equivalent problem (P')

$$(P')\begin{cases} -\Delta_p v = c(x)g(v) + h(x)f(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$g(s) = \frac{(p-1)^{q-p+1}}{\mu^q}(1+\mu s)^{p-1}|\ln(1+\mu s)|^{q-1}\ln(1+\mu s), \text{ with } s > \frac{-1}{\mu}, \tag{2.1}$$

and

$$f(s) = \frac{(1+\mu s)^{p-1}}{(p-1)^{p-1}}. \tag{2.2}$$

We mean by bounded weak solutions of (P'), the functions $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla u = \int_\Omega c(x)g(v)u + \int_\Omega h(x)f(v)u,$$

for any $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Obviously, if $v > \frac{-1}{\mu}$ is a solution of (P'), then $u = \frac{p-1}{\mu}\ln(1+\mu v)$ is a solution of (P). Hence, the solutions obtained here are not necessarily positive (compare with ([63])).

One of the most fruitful ways to deal with (P') is the variational method, which takes into account that the weak solutions of (P') are critical points in $W_0^{1,p}(\Omega)$ of the C^1-functional

$$I(v) = \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega c(x)G(v) - \int_\Omega h(x)F(v), \tag{2.3}$$

with $G(s) = \int_0^s g(t)dt$ and $F(s) = \int_0^s f(t)dt$.

In this chapter, to obtain the two critical points for I, we use the Mountain Pass Theorem to show one critical point and the standard lower semicontinuity argument to show the other. For the first one, according to the famous paper by Ambrosetti and Rabinowitz ([5]), the most important step is to show that I satisfies the Palais-Smale condition at the level $\tilde{c}$ (see Definition 2.2). The fulfillment of this condition relies on the well-known Ambrosetti-Rabinowitz condition ((AR_c) for short), namely

$$\text{there exist } \theta > p \text{ and } s_0 > 0 \text{ such that } 0 < \theta G(s) \leq sg(s), \text{ as } |s| > s_0.$$

Unfortunately, this condition is somewhat restrictive and not being satisfied by many nonlinearities g. However, many researches have been made to drop the (AR_c). We refer, for instance, to [33, 92, 73, 52, 62]. Notice that, the nonlinearity g considered here does not satisfy (AR_c). Moreover, since we do not assume any sign condition on h, the fulfillment of the Palais-Smale condition turns out more delicate (see eg. [53, 63]). To the best of our knowledge, only Jenajean and Quoirin ([63]), recently, proved the Palais-Smale condition under the assumptions c changes sign, h is positive, and without assuming (AR_c). In their proof, for $p = 2$ and $q = 1$, the authors based one of the arguments on the positivity of h and the explicit determination of a function H;

$$H(s) = g(s)s - 2G(s).$$

In our situation, as h is allowed to change sign and the analog of their function H can not be computed explicitly, due to our general consideration of p and q ($1 < p < N$ and $q > 0$), hence, their arguments can not be adapted.

The key point to show the Palais-Smale condition in this chapter is to prove that g, among other conditions, satisfies the following (see Lemma 2.3.1),

$$(NQ) \qquad H(s) = g(s)s - pG(s) \to +\infty, \ \text{ where } \ s \to +\infty.$$

The condition (NQ) is a variant of the well known **nonquadraticity condition at infinity**, which was introduced by Costa and Malgalhães ([33]), and is given as follows

$$(CM) \quad \text{there exist } a > 0 \text{ and } \nu \geq \nu_0 > 0 \text{ such that } \liminf_{|s|\to\infty} \frac{H(s)}{|s|^{\nu}} \geq a.$$

Observe that, since $\nu > 0$, then (NQ) is weaker than (CM). Moreover, it should be noted that (NQ) was considered by Furtado and Silva in their recent paper ([52]). Our result follows by using similar arguments.

Concerning the existence of the second critical point handled by the standard lower semicontinuity argument, we look for a local minimum in $W_0^{1,p}(\Omega)$ for the functional I. Indeed, we observe that I takes positive values in a large sphere, due to its geometrical structure (see Proposition 2.2.3), and $I(0) = 0$.

Now we state the main result of this chapter

Theorem 2.1.1. *Assume that (H) is satisfied. If $\|c\|_k$ and $\|h\|_k$ are suitably small, then the functional I has at least two critical points. Hence, the problem (P) has at least two bounded weak solutions.*

The chapter is organized as follows. In Section 2.2 we recall some preliminary results and show that the functional I has a geometrical structure. In Section 2.3 we prove our main result, Theorem 2.1.1.

Notation

Through this chapter, we use the following notations.

1) The Lebesgue norm $(\int_\Omega |u|^p)^{\frac{1}{p}}$ in $L^p(\Omega)$ is denoted by $\|.\|_p$ for $p \in [1, +\infty[$. The norm in $L^\infty(\Omega)$ is denoted by $\|u\|_{L^\infty(\Omega)} := ess\sup_{x\in\Omega} |u(x)|$. The Hölder conjugate of p is denoted by p'.

2) The spaces $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ are equipped with Poincaré norm $\|u\| := (\int_\Omega |\nabla u|^p)^{\frac{1}{p}}$ and the dual norm $\|\cdot\|_* := \|\cdot\|_{W^{-1,p'}(\Omega)}$ respectively.

3) We denote by $B(0, R)$ the ball of radius R centered at 0 in $W_0^{1,p}(\Omega)$ and $\partial B(0, R)$ its boundary.

4) We denote by $C_i, c_i > 0$ any positive constants that are not essential in the arguments and that may vary from one line to another.

2.2 Preliminaries and geometry of the functional I

In this section, we recall the standard definitions of Palais-Smale sequence at the level $\tilde{c}$ and Palais-Smale condition at the level $\tilde{c}$ for I, and we prove that the functional I defined in (2.3) has a geometrical structure.

Let us define the level at $\tilde{c}$ as follows

$$\tilde{c} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0,\ \gamma(1) = v_0\}$ is the set of continuous paths joining 0 and v_0, where $v_0 \in W_0^{1,p}(\Omega)$ is defined in Proposition 2.2.3 below.

Definition 2.1. *Let E be a Banach space with dual space E^* and (u_n) is a sequence in E. We say that (u_n) is a Palais-Smale sequence at the level $\tilde{c}$ for I if*

$$I(u_n) \to \tilde{c}, \quad \text{and} \quad \|I'(u_n)\|_{E^*} \to 0.$$

Definition 2.2. *We say that I satisfies the Palais-Smale condition at the level $\tilde{c}$ if any Palais-Smale sequence at the level $\tilde{c}$ for I possesses a convergent subsequence.*

In order to prove that I has a geometrical structure, we need some properties of g, which we gather in the following lemma without proof

Lemma 2.2.1.

1. $\frac{g(s)}{|s|^{p-2}s} \to c$ *as* $s \to 0$, *where* $c = 0$ *if* $q > p-1$ *and* $c = 1$ *if* $q = p-1$.
2. $\frac{g(s)}{|s|^{q-1}s} \to (p-1)^{q-p+1}$ *as* $s \to 0$, *for all* $q > 0$.
3. $\frac{g(s)}{s^{p-1}} \to +\infty$ *and* $\frac{G(s)}{s^p} \to +\infty$ *as* $s \to +\infty$, *for all* $q > 0$.

Lemma 2.2.2.

1. *If $q \geq p-1$, then we have*
$$|g(s)| \leq c_0|s|^r + c_1|s|^{p-1},$$
for all $s > -\frac{1}{\mu}$, and for all $r \in (p-1, p)$.

2. *If $0 < q < p-1$, then we have*
$$|g(s)| \leq c_1|s|^r + c_2|s|^q,$$
for all $s > -\frac{1}{\mu}$, and for all $r \in (p-1, p)$.

Proof. By using Lemma 2.2.1 (1), there exists $\eta > 0$ such that for all $|s| < \eta$ we have

$$|g(s)| \leq c_1|s|^{p-1}.$$

Let $\delta \in (0,1)$. If $s \geq \eta$, then we have

$$g(s) \leq c_2(\eta, \mu, \delta)s^{p-1+\delta}. \tag{2.4}$$

Moreover, simple calculation yield

$$g'(s) = \frac{(p-1)^{q-p+1}}{\mu^{q-1}}(1+\mu s)^{p-2}|\ln(1+\mu s)|^{q-1}\left[(p-1)\ln(1+\mu s) + q\right].$$

Now, if $-\frac{1}{\mu} < s \leq -\eta$, then we have $|g(s)| \leq |g(T)|$, where $T = (e^{\frac{-q}{p-1}} - 1)/\mu$. Hence,

$$|g(s)| \leq c_3(\eta, \mu, \delta)|s|^{p-1+\delta}. \tag{2.5}$$

By combining (2.4) and (2.5), (1) holds. To prove the property (2), we use Lemma 2.2.1 (2) and the same previous argument. $\square$

Proposition 2.2.3. *Assume that (H) holds. If $\|c\|_k$ and $\|h\|_k$ are suitably small, then the functional I has a geometrical structure, that is, I satisfies the following properties*

i) there exists $\rho > 0$ such that for all v in $\partial B(0,\rho)$, $I(v) \geq \beta$, where $\beta > 0$.

ii) there exists $v_0 \in W_0^{1,p}(\Omega)$ such that $\|v_0\| > \rho$ and $I(v_0) \leq 0$.

Proof. i) To prove this lemma we distinguish two cases on q . Firstly, if $0 < q < p-1$, then by using Lemma 2.2.2 (2) and Hölder's inequality, we get

$$\int_\Omega c(x)G(v) \leq c_1\|c\|_k\|v^{r+1}\|_{k'} + c_2\|c\|_k\|v^{q+1}\|_{k'}.$$

We choose $r > p-1$ with r close to $p-1$ such that $(r+1)k' < \frac{pN}{N-p}$, which exists due to the assumption $k > \frac{N}{p}$. Obviously, $(q+1)k' < \frac{pN}{N-p}$. Thus, by using Sobolev's embedding we get

$$\int_\Omega c(x)G(v) \leq C_1\|c\|_k\|v\|^{r+1} + C_2\|c\|_k\|v\|^{q+1}.$$

Moreover, from the definition of the function f in (2.2), we have

$$|f(v)| \leq c(1+|v|^{p-1}), \text{ for some } c > 0. \tag{2.6}$$

Using Sobolev's embedding, we get

$$\int_\Omega h(x)F(v) \leq C_3\|h\|_k + C_4\|h\|_k\|v\|^p.$$

By the definition of I in (2.3), we deduce that

$$I(v) \geq \frac{1}{p}\|v\|^p - C_1\|c\|_k\|v\|^{r+1} - C_2\|c\|_k\|v\|^{q+1} - C_3\|h\|_k - C_4\|h\|_k\|v\|^p.$$

Now, let v in $\partial B(0,\rho)$. Then, we have

$$I(v) \geq \frac{1}{p}\rho^p - \|c\|_k(C_1\rho^{r+1} + C_2\rho^{q+1}) - \|h\|_k(C_3 + C_4\rho^p).$$

We take ρ sufficiently large, and such that $\|c\|_k \leq \rho^{-r-2+p}$ and $\|h\|_k \leq \rho^{-1}$
(which are sufficiently small by hypothesis), then

$$I(v) \geq \frac{1}{p}\rho^p - C\rho^{p-1} \geq \rho^{p-1}\left(\frac{1}{p}\rho - C\right) = \beta_1.$$

Secondly, that is $q \geq p-1$, we choose again r as above such that $pk' < (r+1)k' < \frac{pN}{N-p}$. Then, by using Lemma 2.2.2 (1) and Sobolev's embedding, we get

$$\int_\Omega c(x)G(v) \leq c_1\|c\|_k\|v\|^{r+1} + c_2\|c\|_k\|v\|^p.$$

Now, as the first case, we get

$$I(v) \geq \frac{1}{p}\rho^p - C'\rho^{p-1} \geq \rho^{p-1}\left(\frac{1}{p}\rho - C'\right) = \beta_2.$$

Finally, we summarize the two cases and get

$$I(v) \geq \beta, \quad \text{where} \quad \beta = \min(\beta_1, \beta_2).$$

ii) To prove the second property, we show that $I(tv) \to -\infty$ as $t \to +\infty$. For this, let $v \in C_0^\infty(\Omega)$ be a positive function such that $cv \gneqq 0$. By the definition of I in (2.3), we have

$$\begin{aligned} I(tv) &= \frac{t^p}{p}\int_\Omega |\nabla v|^p - \int_\Omega c(x)G(tv) - \int_\Omega h(x)F(tv) \\ &= t^p\left(\frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega c(x)\frac{G(tv)}{t^p v^p}v^p - \int_\Omega h(x)\frac{F(tv)}{t^p v^p}v^p\right). \end{aligned}$$

From inequality (2.6), we get

$$\int_\Omega |h(x)\frac{F(tv)}{t^p v^p}v^p| \leq c \quad \text{as} \quad t \to +\infty.$$

Moerever, by Lemma 2.2.1 (3), we get

$$\int_\Omega c(x)\frac{G(tv)}{t^p v^p}v^p \to +\infty \quad \text{as} \quad t \to +\infty.$$

Thus, we deduce the desired result. □

Finally, we stress that since I has a geometrical structure, then the existence of a Palais-Smale sequence at the level $\tilde{c}$ for I is ensured. This can be observed directly from the proof given in ([5]), or alternatively using Ekeland's variational principle ([46]).

2.3 Proof of Theorem 2.1.1

Recall from introduction that the proof of our main result is divided into two steps as follows. In the first step, we show the existence of the first critical point for the C^1-functional I by using the Mountain Pass Theorem due to Ambrosetti-Rabinowitz ([5]). Precisely, we show that the functional I satisfies the Palais-Smale condition at the level $\tilde{c}$. In the second step, we show the existence of the second critical point of I on $B(0,\rho)$ (which is a local minimum) by using the lower semicontinuity argument. Moreover, we are going to see that these critical points are not the same. Finally, we show that any solution of problem (P) is bounded.

2.3.1 First critical point: Palais-Smale condition

In this subsection, we prove that I satisfies the Palais-Smale condition at the level $\tilde{c}$. Precisely, we show that any Palais-Smale sequence at the level $\tilde{c}$ for I is bounded in $W_0^{1,p}(\Omega)$, and then, it has a strongly convergent subsequence.

They key point to prove the boundedness of the Palais-Smale sequence at the level $\tilde{c}$ in $W_0^{1,p}(\Omega)$, is to show that g verifies the nonquadraticity condition at infinity (NQ). Indeed, we have the following lemma

Lemma 2.3.1. *The function g defined in* (2.1) *verifies the nonquadraticity condition at infinity (NQ);*

$$(NQ) \qquad H(s) = g(s)s - pG(s) \to +\infty, \quad \text{where} \quad s \to +\infty.$$

Proof. To prove (NQ), we show that H is increasing and unbounded for s sufficiently large. We recall that $H(s) = g(s)s - pG(s)$. Then, by simple calculations, we get

$$H'(s) = C\mu s(1+\mu s)^{p-2}(\ln(1+\mu s))^{q-1}[(1-p)\frac{\ln(1+\mu s)}{\mu s} + q],$$

where $C = (p-1)^{q-p+1}/\mu^q$. Thus, H is increasing for s large enough. Moreover, H is unbounded. Indeed, by contradiction, if H is bounded, then there exists a positive constant M such that

$$H(s) \leq M, \quad \text{for s large enough.}$$

In addition, from the definition of H and using integration by parts on G, we get

$$H(s) = -\frac{C}{\mu}(\ln(1+\mu s))^q(1+\mu s)^{p-1} + qC\int_0^s (1+\mu t)^{p-1}(\ln(1+\mu t))^{q-1}dt.$$

By choosing $\delta \in (p-1, p)$, we obtain

$$\frac{H(s)}{s^\delta} = -\frac{C}{\mu}\frac{(\ln(1+\mu s))^q(1+\mu s)^{p-1}}{s^\delta} + qC\frac{\int_0^s(1+\mu t)^{p-1}(\ln(1+\mu t))^{q-1}}{s^\delta} \leq \frac{M}{s^\delta}.$$

When $s \to +\infty$, we obtain $\frac{H(s)}{s^\delta} \to +\infty$ and $\frac{M}{s^\delta} \to 0$. Hence, we have a contradiction. As a conclusion, the function g verifies (NQ). □

Lemma 2.3.2. *Let (u_n) be a Palais-Smale sequence at the level $\tilde{c}$ for I in $W_0^{1,p}(\Omega)$. Then, (u_n) is bounded in $W_0^{1,p}(\Omega)$.*

Proof. Let (u_n) be a Palais-Smale sequence at the level $\tilde{c}$ for I in $W_0^{1,p}(\Omega)$. We prove by contradiction that (u_n) is bounded in $W_0^{1,p}(\Omega)$. We assume that (u_n) is unbounded in $W_0^{1,p}(\Omega)$, that is, $\|u_n\| \to +\infty$.
For all integer $n \geq 0$, we define

$$I(z_n) := \max_{0\leq t\leq 1} I(tu_n), \quad \text{where} \quad z_n = t_n u_n \text{ and } t_n \in [0,1].$$

We are going to prove that $I(z_n) \to +\infty$ and also $(I(z_n))$ is bounded, which is the desired contradiction.

a) Showing that $I(z_n) \to +\infty$: We set $v_n := \frac{u_n}{\|u_n\|}$, then (v_n) is bounded in $W_0^{1,p}(\Omega)$. Hence, there exists a subsequence denoted again (v_n) such that v_n converges weakly and strongly to v in $W_0^{1,p}(\Omega)$ and in $L^s(\Omega)$ for some $1 \leq s < p^*$ respectively. Moreover, v_n also converges to v almost everywhere in Ω. Recall that $p^* := \frac{Np}{N-p}$, is Sobolev conjugate.

Now, we claim by contradiction that $v \equiv 0$ a.e. in Ω.

Since (u_n) is Palais-Smale type sequence, then we have

$$I(u_n) \to \tilde{c} \quad \text{and} \quad \|I'(u_n)\|_* \to 0. \tag{2.7}$$

Hence,

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi - \int_\Omega c(x)g(u_n)\varphi - \int_\Omega h(x)f(u_n)\varphi = \epsilon_n, \tag{2.8}$$

for all $\varphi \in W_0^{1,p}(\Omega)$ and for some $\epsilon_n \to 0$ as $n \to +\infty$. We divide both sides of (2.8) by $\|u_n\|^{p-1}$, to obtain

$$\int_\Omega c(x)\frac{g(u_n)}{\|u_n\|^{p-1}}\varphi = \frac{\epsilon_n}{\|u_n\|^{p-1}} + \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla\varphi + \int_\Omega h(x)\frac{f(u_n)}{\|u_n\|^{p-1}}\varphi. \tag{2.9}$$

On the one hand, since v_n converges weakly to v in $W_0^{1,p}(\Omega)$ and by the inequality (2.6), then for n large enough the second and the third terms of the right-hand side of (2.9) are bounded.
On the other hand, if $v \not\equiv 0$ in Ω, then $cv \not\equiv 0$ in Ω. Now, we choose $\varphi \in W_0^{1,p}(\Omega)$ such that $cv\varphi > 0$ in Ω_φ and $cv\varphi \equiv 0$ in $\Omega\backslash\Omega_\varphi$, with $|\Omega_\varphi| > 0$. Since $v_n\|u_n\| = u_n$ in Ω, then by using Lemma 2.2.1 (3), we obtain

$$\liminf c(x)\frac{g(u_n)}{\|u_n\|^{p-1}}\varphi = \liminf c(x)(v_n)^{p-1}\frac{g(v_n\|u_n\|)}{(v_n\|u_n\|)^{p-1}}\varphi = +\infty \ \text{ in } \ \Omega_\varphi.$$

Hence, by using the Fatou's lemma in (2.9) we obtain the unbounded term in the left-hand side of (2.9). Hence, the claim (*i.e* $v \equiv 0$ a.e. in Ω.)

Since $\|u_n\| \to +\infty$, then there exists $M > 0$ such that $\|u_n\| > M$, for n large enough. Moreover, we have

$$I(z_n) \geq I\left(M\frac{u_n}{\|u_n\|}\right) = I(Mv_n) = \frac{M^p}{p} - \int_\Omega c(x)G(Mv_n) - \int_\Omega h(x)F(Mv_n).$$

In what follows, we treat only the case $0 < q < p-1$. The other case follows with similar arguments. From Lemma 2.2.2 (2), we have $|G(s)| \leq c_1|s|^{r+1} + c_2|s|^{q+1}$, where $p-1 < r < p$. Since $c \in L^k(\Omega)$, for some $k > \frac{N}{p}$ and v_n converges strongly to v in $L^s(\Omega)$ with $1 \leq s < p^*$, then, we obtain

$$\int_\Omega c(x)G(Mv_n) \to 0 \ \text{ as } n \to +\infty,$$

due to $v \equiv 0$ a.e. in Ω. By Hölder's inequality, we get

$$\int_\Omega h(x)F(Mv_n) \leq C \ \text{ as } \ n \to +\infty.$$

Hence, by choosing $M > 0$ large enough, we deduce that $I(z_n) \to +\infty$, as $n \to +\infty$.

b) Showing that $I(z_n)$ is bounded : To prove that $(I(z_n))$ is bounded, we distinguish two cases: $t_n \leq \frac{2}{\|u_n\|}$ and $t_n > \frac{2}{\|u_n\|}$.

The case $t_n \leq \frac{2}{\|u_n\|}$:

Here, we only handle the proof for $q \in (0, p-1)$. The other case follows as in the proof of Proposition 2.2.3 i). By the definition of (z_n) and $I \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$, we have $\langle I'(t_nu_n), t_nu_n\rangle = 0$, which means that

$$t_n^p\|u_n\|^p = \int_\Omega c(x)g(t_nu_n)t_nu_n + \int_\Omega h(x)f(t_nu_n)t_nu_n.$$

By the definition of I in (2.3), we have

$$\begin{aligned} pI(t_nu_n) &= t_n^p\|u_n\|^p - p\int_\Omega c(x)G(t_nu_n) - p\int_\Omega h(x)F(t_nu_n) \\ &= \int_\Omega c(x)H(t_nu_n) + \int_\Omega h(x)K(t_nu_n), \end{aligned} \tag{2.10}$$

where the function H is defined in (NQ) and $K(s) := f(s)s - pF(s)$. Moreover, from Lemma 2.2.2 (2), we have

$$\begin{aligned}\int_\Omega c(x)H(t_n u_n) &\leq \int_\Omega |c(x)||g(t_n u_n)t_n u_n| + p\int_\Omega |c(x)||G(t_n u_n)| \\ &\leq c_1 \int_\Omega |c(x)||t_n u_n|^{r+1} + c_2 \int_\Omega |c(x)||t_n u_n|^{q+1}.\end{aligned}$$

By choosing r and q as in the proof of Proposition 2.2.3 i), we get

$$\int_\Omega c(x)H(t_n u_n) \leq C_1\|c\|_k\|t_n u_n\|^{r+1} + C_2\|c\|_k\|t_n u_n\|^{q+1}. \tag{2.11}$$

By inequality (2.6) and Sobolev's embedding, we get

$$\begin{aligned}\int_\Omega h(x)K(t_n u_n) &\leq \int_\Omega |h(x)||f(t_n u_n)t_n u_n| + p\int_\Omega |h(x)||(F(t_n u_n)t_n u_n| \\ &\leq c_1\|h\|_k + c_2\|h\|_k\|t_n u_n\| + c_3\|h\|_k\|t_n u_n\|^p.\end{aligned} \tag{2.12}$$

Then, by (2.10), (2.11), and (2.12), we obtain

$$I(t_n u_n) \leq C,$$

for all $n \geq 0$, where C is independent of n. Thus, $(I(z_n))$ is bounded, which contradicts the fact that $(I(z_n))$ is unbounded (see **a)**).

The case $t_n > \frac{2}{\|u_n\|}$:

Here, we are proceeding the technique inspired by [52]. To this end, we need the following technical lemma

Lemma 2.3.3. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ the nonnegative function defined as*

$$\Phi(s) = \begin{cases} e^{-\epsilon/s^2}, & \text{if } s \neq 0, \\ 0, & \text{if } s = 0, \end{cases}$$

with $\epsilon > 0$. Then, we have

i) $\lim_{s\to 0} \Phi(s) = \lim_{s\to 0} \Phi'(s) = 0.$

ii) for any positive function $z \in L^k$ for some $k > \frac{N}{p}$ we have

$$\lim_{\epsilon\to 0} \int_\Omega \int_s^t \frac{z(x)}{\tau^{p+1}} \left(\frac{1 - \Phi_\epsilon(|\tau u_n|)}{\|u_n\|^p}\right) d\tau dx = 0, \quad \textit{uniformly in } n \in \mathbb{N}.$$

Proof. Obviously we have i). To prove ii), we follow the same approach given in [52] for the case $p = 2$ and $z(x) = 1$, which can be immediately generalized for any positive function $z \in L^k$ for some $k > \frac{N}{p}$ and $p > 1$. □

Now, we resume the proof of Lemma 2.3.2. From Lemma 2.3.1, we have $H(s) \geq \sigma$, for s large enough and some $\sigma > 0$ (which will be chosen later). Moreover, if $0 < q < p-1$, then from Lemma 2.2.1 (2), we have for s sufficiently small,

$$H(s) \geq -C_1|s|^{q+1}.$$

Then, by the continuity of H, we have for all $s > -\frac{1}{\mu}$,

$$H(s) \geq \sigma \Phi_\epsilon(s) - C_2|s|^{q+1}. \tag{2.13}$$

Let $0 < s < t$, then we have

$$\begin{aligned} \frac{I(tu_n)}{t^p\|u_n\|^p} - \frac{I(su_n)}{s^p\|u_n\|^p} &= -\int_\Omega c(x)\left[\frac{G(tu_n)}{t^p\|u_n\|^p} - \frac{G(su_n)}{s^p\|u_n\|^p}\right] \\ &\quad - \int_\Omega h(x)\left[\frac{F(tu_n)}{t^p\|u_n\|^p} - \frac{F(su_n)}{s^p\|u_n\|^p}\right] \\ &= \underbrace{-\int_\Omega c(x)\int_s^t \frac{d}{d\tau}\left(\frac{G(\tau u_n)}{\tau^p\|u_n\|^p}\right)d\tau dx}_{A} \\ &\quad + \underbrace{\int_\Omega -h(x)\left[\frac{F(tu_n)}{t^p\|u_n\|^p} - \frac{F(su_n)}{s^p\|u_n\|^p}\right]}_{B}. \end{aligned} \tag{2.14}$$

Let us handle the two terms A and B respectively.

$$\begin{aligned} A &= -\int_\Omega\int_s^t c(x)\frac{\tau^p u_n g(\tau u_n) - p\tau^{p-1}G(\tau u_n)}{\tau^{2p}\|u_n\|^p}d\tau dx \\ &= -\int_\Omega\int_s^t \frac{c(x)}{\|u_n\|^p}\frac{H(\tau u_n)}{\tau^{p+1}}d\tau dx. \end{aligned}$$

By using (2.13), we get

$$\begin{aligned} A &\leq \int_\Omega\int_s^t \frac{c(x)}{\|u_n\|^p}\left(C_2\frac{|u_n|^{q+1}}{\tau^{p-q}} - \sigma\frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}}\right)d\tau dx \\ &\leq \int_\Omega \frac{c(x)}{\|u_n\|^p}\left(\frac{C_2}{p-q-1}\frac{|u_n|^{q+1}}{s^{p-q-1}} - \sigma\int_s^t \frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}}d\tau\right)dx. \end{aligned} \tag{2.15}$$

For the term B, we have

$$\begin{aligned} B &\leq C\left(\int_\Omega |h(x)|\frac{(1+|tu_n|)^p}{t^p\|u_n\|^p} + \int_\Omega |h(x)|\frac{(1+|su_n|)^p}{s^p\|u_n\|^p}\right) \\ &\leq C\left(\int_\Omega |h(x)|\left(\frac{1}{t_n\|u_n\|} + \frac{|u_n|}{\|u_n\|}\right)^p + \int_\Omega |h(x)|\left(\frac{1}{s_n\|u_n\|} + \frac{|u_n|}{\|u_n\|}\right)^p\right). \end{aligned} \tag{2.16}$$

By setting $s := \frac{1}{\|u_n\|}$, we obtain

$$\begin{aligned}
\frac{I(tu_n)}{t^p\|u_n\|} &\leqslant I(v_n) + \int_\Omega c(x)\left(\frac{C_2}{p-q-1}|v_n|^{q+1} - \sigma\int_s^t \frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}\|u_n\|^p}\right)d\tau dx \\
&\quad + C\left(\int_\Omega |h(x)|(\frac{1}{2}+|v_n|)^p + \int_{\Omega^+}|h(x)|(1+|v_n|)^p\right) \\
&\leqslant I(v_n) + C\left[\int_\Omega c(x)|v_n|^{q+1} + \int_\Omega |h(x)| + 2\int_\Omega |h(x)||v_n|^p\right] \\
&\quad - \sigma\int_\Omega \frac{c(x)}{p}\left(1-\frac{1}{t_n^p\|u_n\|^p}\right) + \sigma\int_\Omega \frac{c(x)}{p}\left(1-\frac{1}{t_n^p\|u_n\|^p}\right) \\
&\quad - \sigma\int_\Omega\int_s^t c(x)\frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}\|u_n\|^p}d\tau dx \\
&\leqslant I(v_n) + C\left[\int_\Omega c(x)|v_n|^{q+1} + \int_\Omega |h(x)| + 2\int_\Omega |h(x)||v_n|^p\right] \\
&\quad - \sigma\int_\Omega \frac{c(x)}{p}\left(1-\frac{1}{t_n^p\|u_n\|^p}\right) - \sigma\int_\Omega\int_s^t \frac{c(x)}{\tau^{p+1}}\left(\frac{1-\Phi_\epsilon(|\tau u_n|)}{\|u_n\|^p}\right)d\tau dx.
\end{aligned}$$

By the technical Lemma 2.3.3, we have

$$\lim_{\epsilon\to 0}\int_\Omega\int_s^t \frac{c(x)}{\tau^{p+1}}\left(\frac{1-\Phi_\epsilon(|\tau u_n|)}{\|u_n\|^p}\right)d\tau dx = 0, \quad \text{uniformly in } n \in \mathbb{N}.$$

Then,

$$\begin{aligned}
\frac{I(tu_n)}{t^p\|u_n\|^p} &\leqslant \frac{1}{p} - \int_\Omega c(x)G(v_n) - \int_\Omega h(x)F(v_n) + C\left[\int_\Omega c(x)|v_n|^{q+1}\right. \\
&\qquad \left. + \int_\Omega |h(x)| + 2\int_\Omega |h(x)||v_n|^p\right] - \sigma\int_\Omega \frac{c(x)}{p}\left(1-\frac{1}{2^p}\right).
\end{aligned}$$

We choose σ such that

$$\sigma > \frac{2^p(1+pC\|h\|_k)}{(2^p-1)\int_\Omega c(x)}.$$

which gives,

$$\frac{1}{p} + C\|h\|_k - \sigma\int_\Omega \frac{c(x)}{p}\left(1-\frac{1}{2^p}\right)dx < 0.$$

Since v_n converges to 0 almost everywhere in Ω, weakly in $W_0^{1,p}(\Omega)$, and strongly in $L^s(\Omega)$ for some $1 \leq s < p^*$, then, we have

$$I(t_n u_n) < 0, \text{ in } \Omega \text{ for } n \text{ large enough.}$$

Hence, $(I(z_n))$ is bounded. Therefore, this contradicts the fact that $(I(z_n))$ is unbounded (see **a)**).

Now, If $q \geq p-1$, then from Lemma 2.2.1 (1) and the continuity of $H(s)$, we have for all $s > -\frac{1}{\mu}$,

$$H(s) \geq \sigma\Phi_\epsilon(s) - C_1|s|^{p-1}. \tag{2.17}$$

Following the computations as in (4.34), we find exactly the same terms A and B. The term B is handled as in (3.35), whereas A is handled as follows

$$\begin{aligned} A &\leq \int_\Omega \int_s^t \frac{c(x)}{\|u_n\|^p} \left(C_1 \frac{|u_n|^{p-1}}{\tau^2} - \sigma \frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}} \right), \\ &\leq \int_\Omega \int_s^t \frac{c(x)}{\|u_n\|^p} \left(C_1 \frac{|u_n|^{p-1}}{s} - \sigma \frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}} \right). \end{aligned}$$

Moreover, since $(p-1)k' < pk' < \frac{Np}{N-p}$, then by using Sobolev embedding, the rest of the proof is similar to the case $q \in (0, p-1)$. Hence, we have also the contradiction with the fact that I is unbounded (see **a)**). □

To finish the proof of the Palais-Smale condition for I, we only need to show the following lemma

Lemma 2.3.4. *Any Palais-Smale sequence at the level $\tilde{c}$ of $W_0^{1,p}(\Omega)$ has a strongly convergent subsequence.*

Proof. Let (u_n) be a Palais-Smale sequence at the level $\tilde{c}$, then $I'(u_n) \to 0$ in $W^{-1,p'}(\Omega)$, which means that

$$-\Delta_p u_n - c(x)g(u_n) - h(x)f(u_n) \to 0 \;\text{ in }\; W^{-1,p'}(\Omega).$$

By Lemma 2.3.2, (u_n) is bounded in $W_0^{1,p}(\Omega)$. Hence, u_n converges weakly to u in $W_0^{1,p}(\Omega)$ and strongly in $L^s(\Omega)$ for some $1 \leq s < p^*$. Therefore,

$$-\Delta_p u_n \to c(x)g(u) + h(x)f(u) \;\text{ in }\; W^{-1,p'}(\Omega). \tag{2.18}$$

We know that the operator $-\Delta_p : W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)$ is a homeomorphism ([44]). Hence, from (2.18) we get

$$u_n \to (-\Delta_p)^{-1}(c(x)g(u) + h(x)f(u)) \;\text{ in }\; W_0^{1,p}(\Omega).$$

Therefore, by the uniqueness of the limit we have

$$u_n \to u, \text{ in } W_0^{1,p}(\Omega).$$

□

2.3.2 Second critical point

In this subsection, we use the geometrical structure of I (see Proposition 2.2.3) and the standard lower semicontinuity argument, we show the existence of the second critical point. We state the result as follows

Theorem 2.3.5. *Assume that $\|c\|_k$ and $\|h\|_k$ are suitably small to ensure Proposition 2.2.3. Then, the functional I possesses a critical point $v \in B(0,\rho)$ with $I(v) \leq 0$.*

Proof. Since $I(0) = 0$, then $\inf_{v \in B(0,\rho)} I(v) \leq 0$. Moreover, if $h \not\equiv 0$, then we obtain that $\inf_{v \in B(0,\rho)} I(v) < 0$. Indeed, we choose $v \in C_0^\infty(\Omega)$ a positive function that satisfies $cv > 0$ and $hv > 0$. From the definition of I in (2.3), we have for $t > 0$

$$I(tv) = t^p \left(\frac{1}{p} \int_\Omega |\nabla v|^p - \int_\Omega c(x) \frac{G(tv)}{t^p v^p} v^p - \int_\Omega h(x) \frac{F(tv)}{t^p v^p} v^p \right). \tag{2.19}$$

If $q \geq p-1$, then from Lemma 2.2.1 (2), we have $G(s)/s^p \to c < +\infty$ as $s \to 0^+$. If $0 < q < p-1$, obviously, we have $G(s)/s^p \to +\infty$ as $s \to 0^+$. In addition, in both cases, we have $\frac{F(s)}{s^p} \to +\infty$ as

$s \to 0^+$. Hence, by using these limits, we get from (2.19) that $I(tv) < 0$ for $t > 0$ small enough.

Now, we set $m := \inf_{v \in B(0,\rho)} I(v)$. Then, by Proposition 2.2.3 i), we have $I(v) \geq \beta > 0$ for $\|v\| = \rho$. Moreover, there exists a sequence $(v_n) \subset B(0,\rho)$ such that $I(v_n)$ converges to m. Since (v_n) is bounded in $W_0^{1,p}(\Omega)$, then there exists a subsequence denoted again (v_n) such that v_n converges to v weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^s(\Omega)$ for some $1 \leq s < p^*$ respectively. Hence, we get

$$\int_\Omega h(x)F(v_n) \to \int_\Omega h(x)F(v) \quad \text{and} \quad \int_\Omega c(x)G(v_n) \to \int_\Omega c(x)G(v) \quad \text{as} \quad n \to +\infty.$$

In addition, since $\|v\|^p \leq \liminf_{n\to\infty} \|v_n\|^p$, then $I(v) \leq m = \inf_{v \in B(0,\rho)} I(v)$. Hence, we conclude that v is a local minimum of I in $B(0,\rho)$.

□

Remark 2.3.6. *By the subsection 2.3.1, I has a critical point at the level $\tilde{c}$, that is, there exists w in $W_0^{1,p}(\Omega)$ such that $I(w) = \tilde{c}$ and $I'(w) = 0$. Since $I(w) = \tilde{c} > 0 \geq I(v)$, where $v \in B(0,\rho)$ is the second critical point given in the previous theorem, then w is different from v. Hence, we have two distinct solutions for the problem (P).*

2.3.3 Boundedness of solutions

Now, to finish the proof of our main result, it remains to show the boundedness of the solutions. Therefore, we show the following result

Proposition 2.3.7. *Any solution u of the problem (P') belongs to $L^\infty(\Omega)$.*

Proof. If $|u| \leq 1$, it is over. Otherwise, we begin by writing the problem (P') as follows

$$-\Delta_p u = a(x)(1 + |u|^{p-1}),$$

where

$$a(x) = \frac{c(x)g(u) + h(x)f(u)}{1 + |u|^{p-1}}.$$

Then, by Theorem 2.4 in [79], we can deduce the boundedness of u if we show that a belongs to $L^{\frac{p}{N(1-\epsilon)}}(\Omega)$, for some $\epsilon \in]0,1[$. Indeed, from (2.6) and Lemma 2.2.2, we obtain

$$|a(x)| \leq C\left[|c(x)|(|u|^{r-p+1} + 1) + |h(x)|\right]. \tag{2.20}$$

Let $m > 1$ and m' it's conjugate. By using Hölder's inequality in (2.20), we obtain

$$\int_\Omega |a(x)|^{\frac{p}{N(1-\epsilon)}} \leq C\left[\|c(x)^{\frac{p}{N(1-\epsilon)}}\|_m \|u^{(r-p+1)\frac{p}{N(1-\epsilon)}}\|_{m'} + \|h^{\frac{p}{N(1-\epsilon)}}\|_m + 1\right].$$

By choosing $0 < \epsilon < 1 - \frac{(N-p)(r-p+1)}{N^2} - \frac{p}{kN}$, we have

$$\frac{p}{N(1-\epsilon)} m \leq k \quad \text{and} \quad (r-p+1)\frac{p}{N(1-\epsilon)} m' < \frac{Np}{N-p}.$$

Hence, the terms $\|c(x)^{\frac{p}{N(1-\epsilon)}}\|_m$, $\|h(x)^{\frac{p}{N(1-\epsilon)}}\|_m$, and $\|u^{(r-p+1)\frac{p}{N(1-\epsilon)}}\|_{m'}$ are finite (recall that $c, h \in L^k(\Omega)$ for some $k > \frac{N}{p}$). □

CHAPTER 3

QUALITATIVE PROPERTIES OF WEAK SOLUTIONS FOR *P*-LAPLACIAN EQUATIONS WITH NONLOCAL SOURCE AND GRADIENT ABSORPTION

We consider the following Dirichlet initial boundary value problem with a gradient absorption and a nonlocal source

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda u^k \int_\Omega u^s dx - \mu u^l |\nabla u|^q$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, where $p > 1$, the parameters $k, s, l, q, \lambda > 0$ and $\mu \geq 0$. Firstly, we establish local existence for weak solutions; the aim of this part is to prove a crucial a priori estimate on $|\nabla u|$. Then, we give appropriate conditions in order to have existence and uniqueness or nonexistence of a global solution in time. Finally, depending on the choices of the initial data, ranges of the coefficients and exponents and measure of the domain, we show that the non-negative global weak solution, when it exists, must extinct after a finite time.

3.1 Introduction

This chapter is devoted to studying existence and qualitative properties of weak solutions for the following problem

$$\mathcal{P}: \begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{k-1}u \int_\Omega |u|^s dx - \mu |u|^{l-1}u|\nabla u|^q & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, $p > 1$, the parameters $k, s, l, q, \lambda > 0$, and $\mu \geq 0$. Throughout this chapter we assume that the initial data u_0 satisfy

$$u_0 \not\equiv 0, \quad u_0 \in W^{1,\infty}(\Omega). \tag{3.1}$$

This equation was first introduced in [31] for the case $p = 2$, $l = 0$, when the source term is local, i.e $s = 0$ and $\lambda = 1/meas(\Omega)$, in order to study the effect of a damping gradient term on global existence in time or nonexistence of solutions. In fact, the dynamics of this equation reflect the competition between the gradient term $|u|^{l-1}u|\nabla u|^q$ which has a dissipative effect and may prevent blow-up and the reaction term $|u|^{k-1}u \int_\Omega |u|^s dx$, which may produce blow-up. Problems of this form appear in various models of application. For instance, in models describing the evolution of population density of biological species, under the effect of certain natural mechanisms [86].

It is well known that solutions of $\mathcal{P}$ exhibit various qualitative properties, see [25, 70, 94, 95] , which reflect natural phenomena, according to various conditions on the exponents p, q, l, k and s, the coefficients λ and μ, the initial data u_0 and the domain Ω. For instance, when $p = 2$, Souplet obtained in [87] interesting results concerning qualitative properties of the solutions of problem $\mathcal{P}$. In the last decade however, many authors (see [19, 58, 59, 60]) studied problem $\mathcal{P}$, with $p > 1$ in the case where $\lambda = l = 0$, in the framework of viscosity solutions. Most of them investigated large-time behavior, finite speed of propagation and proved decay estimates. Otherwise, for the case where $\mu = 0$, we highlight for example the works in [69] and [47], where the authors dealt with existence and nonexistence of global weak solutions. However, the nonlocal models are more close to the practical problems than the local models, and now many local theories are no longer holding. Therefore, they are more difficult and challenging. To the best of our knowledge, problem $\mathcal{P}$ has not been treated in the case $p \neq 2$. In the present chapter, we shall first show the existence of a local weak solution (in time) and, under additional assumptions, we discuss the global existence and blow-up in finite time of the weak solution. Moreover, the critical blow-up exponent is specified. On the other hand, in the end of this chapter, we exploit our global existence result to deduce the extinction of the non-negative solution after a finite time by using different methods depending on the range of p. Indeed, for $p \geq 2$ we use the so-called L^2-integral norm estimate method which was also used in [11, 38, 70] and the references therein; while in case where $1 < p < 2$, the proof is based on the construction of suitable super-solutions and the use of a comparison principle.

We point out that the operator Δ_p is degenerate and singular at the points $u = 0$ or $\nabla u = 0$ for $p > 2$ and $1 < p < 2$, respectively. Hence, generally there is no classical solutions of $\mathcal{P}$, that is why it is reasonable to work on weak solutions which can be considered as limits of the solutions for some appropriate regularized non-degenerate and non-singular problems associated with $\mathcal{P}$. The main difficulty is to prove a priori estimates in $W_0^{1,\infty}$ of solutions of the approximated problems without restriction on the exponents p, q, l, k, s and the boundary of Ω ([30]). For this purpose, by using similar arguments from [16] based on the construction of suitable barrier super-solution and sub-solution, we get uniform estimates on the gradients near the boundary for small time. However, as $p > 1$ leads obviously to a considerable amount of additional technical difficulties, the argument to pass to the limit used in [16] does not work in our situation. For that we exploit the method used by [28, 93] based on the construction of a sequence of solutions for the corresponding approximation problems and prove the strong convergence of a subsequence of the weak solutions and we show that the limit function of the subsequence is the desired weak solution of problem $\mathcal{P}$.

The chapter is organized as follows, in the next section we establish existence and uniqueness of weak solutions, while Section 3.3 is devoted to proving the existence and nonexistence of a unique global solution. Finally, the extinction properties of the solution are considered in section 3.4.

3.2 Local existence of weak solutions

Now, we state the definition of weak solutions of problem $\mathcal{P}$. We denote $Q_T = \Omega \times (0, T)$, $\partial Q_T = \{\partial\Omega \times [0, T]\} \cup \{\bar{\Omega} \times \{0\}\}$ and $|\Omega| = meas(\Omega)$.

Definition 3.1. *Let $\alpha = \max\{p, k + s, q + l\}$. We say that $u(x, t)$ is a super-(sub-) solution of $\mathcal{P}$ on Q_T if*

1. *$u \in C(\overline{\Omega} \times [0,T)) \cap L^{\alpha}(0,T; W_0^{1,\alpha}(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2(Q_T)$.*

2. *for every non-negative test-function $\varphi \in C(\overline{Q_T}) \cap L^p(0,T; W_0^{1,p}(\Omega))$, we have*

$$\int_{Q_T} (\varphi \frac{\partial u}{\partial t} + |\nabla u|^{p-2} \nabla u \nabla \varphi) dxdt \geq (\leq) \int_{Q_T} (\lambda |u|^{k-1} u \int_{\Omega} |u|^s - \mu |u|^{l-1} u |\nabla u|^q) \varphi dxdt.$$

3. *$u(.,0) \geq (\leq) u_0$ in Ω, and $u \geq (\leq) 0$ on $\partial\Omega \times (0,T)$.*

A function u is a weak solution of $\mathcal{P}$ if it is simultaneously a super-solution and a sub-solution.

The following result deals with the local existence of weak solutions.

Theorem 3.2.1. *We assume $k + s \geq 1$. Let u_0 satisfy* (3.1) *and $\|\nabla u_0\|_\infty \leq M$, where M is a positive constant. Then there exists $T_0 = T_0(p,k,s,q,l,\Omega,u_0)$ such that the problem $\mathcal{P}$ has at least a weak solution $u \in L^\infty(0,T_0,W_0^{1,\infty}(\Omega))$.*

Proof. We consider the following approximate problems associated with $\mathcal{P}$

$$\mathcal{P}_\epsilon : \begin{cases} \dfrac{\partial u_\epsilon}{\partial t} - \Delta_p^\epsilon u_\epsilon = \lambda f_\epsilon(u_\epsilon) \int_\Omega g_\epsilon(u_\epsilon) dx - \mu H_\epsilon(u_\epsilon, |\nabla u_\epsilon|) & \text{in } \Omega \times (0,\infty), \\ u_\epsilon = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u_\epsilon(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

for $\epsilon \in (0,1/2)$, where $\Delta_p^\epsilon u_\epsilon = \operatorname{div}((|\nabla u_\epsilon|^2 + \epsilon)^{(p-2)/2} \nabla u_\epsilon)$, and $H_\epsilon(s,\zeta) = F_\epsilon(s) G_\epsilon(\zeta)$ with $G_\epsilon(\zeta) = (\zeta^2 + \epsilon)^{q/2} - \epsilon^{q/2}$ for some $q > 0$, and $F_\epsilon(s) = (|s| + \epsilon)^{l-1} s$ if $l < 1$, whereas $F_\epsilon(s) = |s|^{l-1} s$ if $l \geq 1$. By the same way, we introduce similar approximations $f_\epsilon(s)$ and $g_\epsilon(s)$ for $|u|^{k-1} u$ and $|u|^s$, respectively.
By applying well known results of [68], problem $\mathcal{P}_\epsilon$ has a unique classical solution, for any fixed ϵ . Furthermore, a weak solution of $\mathcal{P}$ can be obtained as a limit of a sequence of solutions of $\mathcal{P}_\epsilon$, as ϵ tends to 0. To prove this last fact, we shall show first some uniform a priori estimates.

Lemma 3.2.2. *Let $Q_T = \Omega \times (0,T)$. There exists $T_0 > 0$ such that*

$$\|u_\epsilon\|_{L^\infty(Q_{T_0})} \leq C_1, \tag{3.2}$$

$$\|\nabla u_\epsilon\|_{L^\infty(Q_{T_0})} \leq C_2, \tag{3.3}$$

and

$$\|\frac{\partial u_\epsilon}{\partial t}\|_{L^2(Q_{T_0})} \leq C_3, \tag{3.4}$$

where C_i, $i = 1, ..., 3$ are positive constants independent of ϵ.

Proof. Firstly, let T^* be the maximal existence time for solutions of $\mathcal{P}_\epsilon$. Let us prove the uniform boundedness of u_ϵ. We divide the proof into two cases, $k + s > 1$ and $k + s = 1$. For the first case, we consider the following function

$$\hat{u}(t) = (k+s-1)^{1/(1-k-s)} (T_1 - \lambda|\Omega| t)^{1/(1-k-s)},$$

where

$$T_1 = \frac{1}{k+s-1} \|u_0\|_{L^\infty(\Omega)}^{1-k-s}.$$

We prove that $\hat{u}$ is a super-solution on Q_{T_0}, where $T_0 \le \min(T^*, \frac{T_1}{2\lambda|\Omega|})$. We have by simple calculations

$$\begin{aligned}\frac{\partial \hat{u}}{\partial t} - \Delta_p^{\epsilon}\hat{u} &= \lambda \hat{u}^k \int_{\Omega} \hat{u}^s dx - \mu H_{\epsilon}(\hat{u}, |\nabla \hat{u}|)\\ &\ge \lambda f_{\epsilon}(\hat{u}) \int_{\Omega} g_{\epsilon}(\hat{u}) dx - \mu H_{\epsilon}(\hat{u}, |\nabla \hat{u}|).\end{aligned}$$

It is easy to see that $\hat{u}(0) \ge u_0$ a.e in Ω and $\hat{u} \ge 0$ on $\partial\Omega \times (0, T_0)$. Hence, by the the maximum principle [68] we get

$$\|u_{\epsilon}\|_{L^{\infty}(Q_{T_0})} \le C(T_0).$$

Whereas, if $k + s = 1$, we take

$$\hat{u}(t) = M e^{\lambda|\Omega|t},$$

where $M = \|u_0\|_{L^{\infty}(\Omega)}$. Again, by simple calculations $\hat{u}$ is a super-solution of $\mathcal{P}_{\epsilon}$ on Q_{T_0}, for any $T_0 > 0$. Hence

$$\|u_{\epsilon}\|_{L^{\infty}(Q_{T_0})} \le C(T_0).$$

Next, we shall show the uniform boundedness of $|\nabla u_{\epsilon}|$. For that, we follow the same argument used in [16] and [95]. We need first to show that

$$\sup_{x\in\Omega, \delta(x)\le\eta} \frac{|u_{\epsilon}(x,t) - u_{\epsilon}(\tilde{x},t)|}{\delta(x)} \le C_1, \quad 0 < t \le T_0, \tag{3.5}$$

where $\delta(x)$ denotes the distance from x to $\partial\Omega$ and $\tilde{x}$ is the projection of x into $\partial\Omega$, which is well defined and unique. Recall that there exists $\eta_0 > 0$ small enough so that, for any $x \in \overline{\Omega}$ with $\delta(x) \le \eta_0$. In order to show (3.5) we construct, as in [16, 95], a local barrier function under the exterior sphere condition, for any x near $\partial\Omega$. Let $\rho > 0$ be such that for any $x \in \partial\Omega$, $\overline{B_{\rho}(x + \rho\nu_x)} \cap \overline{\Omega} = \{x\}$, where ν_x is the unit outward normal vector on $\partial\Omega$. For fixed arbitrary $x_0 \in \Omega$ such that $\delta(x_0) \le \eta$ where $\eta \in (0, \eta_0)$ will be specified later, define $x_1 = \tilde{x}_0 + \rho\nu_{\tilde{x}_0}$. Without loss of generality, we may assume that $x_1 = 0$. Moreover, we set $\Gamma := B_{\rho}(x + \rho\nu_x) \cap \Omega$ and $r = |x|$. For fixed $\tau > 0$, we define for any $s > 0$, $\phi(s) = s(s+\tau)^{-\beta}$, where $\beta \in (0,1)$ will be chosen later. We can easily see that ϕ is an increasing and concave function. Denote $\overline{u}(x) = \phi(r - \rho)$, for any $x \in \Omega$. Our goal here is to prove that $\overline{u}$ is a super-solution of $\mathcal{P}_{\epsilon}$ in $\Gamma \times (0, T_0)$, for some T_0 small enough that will be determined later. Firstly let us show that

$$\frac{\partial \overline{u}}{\partial t} - \Delta_p^{\epsilon}\overline{u} \ge \lambda f_{\epsilon}(\overline{u}) \int_{\Omega} g_{\epsilon}(\overline{u}) dx - \mu H_{\epsilon}(\overline{u}, |\nabla \overline{u}|).$$

Since

$$-\mu H_{\epsilon}(\overline{u}, \nabla \overline{u}) \le 0,$$

it is sufficient to show that

$$\frac{\partial \overline{u}}{\partial t} - \Delta_p^{\epsilon}\overline{u} \ge \lambda f_{\epsilon}(\overline{u}) \int_{\Omega} g_{\epsilon}(\overline{u}) dx. \tag{3.6}$$

On the other hand, we have

$$\frac{\partial \overline{u}}{\partial x_i}(x) = \phi'(r - \rho)\frac{x_i}{|x|},$$

and

$$\Delta\overline{u}(x) = \phi''(r - \rho) + \frac{N-1}{r}\phi'(r - \rho).$$

Then, by simple calculations we obtain

$$-\Delta_p^{\epsilon}\overline{u} = [(\phi'(r-\rho))^2+\epsilon]^{\frac{p-2}{2}}[-\Delta\overline{u}-(p-2)\frac{(\phi'(r-\rho))^2}{(\phi'(r-\rho))^2+\epsilon}\phi''(r-\rho)].$$

If $p \geq 2$, we have

$$-\Delta_p^{\epsilon}\overline{u} \geq [(\phi'(r-\rho))^2+\epsilon]^{\frac{p-2}{2}}[-\phi''(r-\rho)-\frac{N-1}{r}\phi'(r-\rho)];$$

while if $1 < p < 2$, we get

$$-\Delta_p^{\epsilon}\overline{u} \geq [(\phi'(r-\rho))^2+\epsilon]^{\frac{p-2}{2}}[-(p-1)\phi''(r-\rho)-\frac{N-1}{r}\phi'(r-\rho)].$$

Hence

$$-\Delta_p^{\epsilon}\overline{u} \geq [(\phi'(r-\rho))^2+\epsilon]^{\frac{p-2}{2}}[-\kappa\phi''(r-\rho)-\frac{N-1}{r}\phi'(r-\rho)], \tag{3.7}$$

where $\kappa = 1$, if $p \geq 2$ and $\kappa = p-1$, if $1 < p < 2$. Thanks to (3.6) and (3.7) it suffices to show that

$$\begin{aligned} &-\kappa\phi''(r-\rho)-\frac{N-1}{r}\phi'(r-\rho) \\ &\geq \lambda[(\phi'(r-\rho))^2+\epsilon]^{\frac{2-p}{2}}(\phi(r-\rho))^k\int_\Omega(\phi(r-\rho))^s dx \end{aligned} \tag{3.8}$$

Moreover, we have

$$[(\phi'(r-\rho))^2+\epsilon]^{\frac{2-p}{2}} \leq \begin{cases} [(\phi'(r-\rho))^2+1]^{\frac{2-p}{2}}, & \text{if } 1 < p < 2, \\ (\phi'(r-\rho))^{2-p}, & \text{if } p \geq 2, \end{cases}$$

from which (3.8) holds if

$$\begin{aligned} &-\kappa\phi''(r-\rho)-\frac{N-1}{r}\phi'(r-\rho) \\ &\geq \lambda[(\phi'(r-\rho))^2+\sigma]^{\frac{2-p}{2}}(\phi(r-\rho))^k\int_\Omega(\phi(r-\rho))^s dx \end{aligned} \tag{3.9}$$

where,

$$\sigma = \begin{cases} 0, & \text{if } p \geq 2, \\ 1, & \text{if } 1 < p < 2. \end{cases}$$

Furthermore, using the fact that Ω is bounded there exists $R > 0$ such that $\Omega \subset B(x_1, R)$. Then, from the definition of ϕ it follows that

$$\int_\Omega(\phi(r-\rho))^s dx \leq |\Omega|R^{(1-\beta)s} = C. \tag{3.10}$$

From now on, we take $\tau = \eta$. Thus, for all $x \in \Gamma$ we have

$$\phi(r-\rho) \leq \eta(r-\rho+\eta)^{-\beta}, \tag{3.11}$$

and

$$\begin{aligned} &(r-\rho+\eta)^{-\beta-2}[2\kappa\beta\eta-\frac{N-1}{\rho}(2\eta)^2] \\ &\qquad \leq -\kappa\phi''(r-\rho)-\frac{N-1}{r}\phi'(r-\rho). \end{aligned} \tag{3.12}$$

Moreover, since $\beta \in (0,1)$ we get

$$(1-\beta)(r-\rho+\eta)^{-\beta} \leq \phi^{'}(r-\rho) \leq (r-\rho+\eta)^{-\beta}. \tag{3.13}$$

Thanks to (3.10), (3.11), (3.12) and (3.13), to get (3.9) it suffices to have

$$\begin{aligned}(r-\rho+\eta)^{-\beta-2}&[2\kappa\beta\eta - \frac{N-1}{\rho}(2\eta)^2] \\ &\geq C\lambda[\theta(r-\rho+\eta)^{-2\beta}+\sigma]^{\frac{2-p}{2}}(\eta(r-\rho+\eta)^{-\beta})^k,\end{aligned} \tag{3.14}$$

where

$$\theta = \begin{cases} (1-\beta)^2, & \text{if } p \geq 2, \\ 1, & \text{if } p < 2. \end{cases}$$

We suppose that η is small enough so that

$$\begin{cases} (r-\rho+\eta)^{-2\beta} \geq (2\eta)^{-2\beta} \geq 1, \\ 2\kappa\beta\eta - \frac{N-1}{\rho}(2\eta)^2 \geq \kappa\beta\eta. \end{cases}$$

Then, the inequality (3.14) holds if

$$(r-\rho+\eta)^{-\beta-2}\kappa\beta\eta \geq C\lambda[\theta(r-\rho+\eta)^{-2\beta}+\sigma]^{\frac{2-p}{2}}(\eta(r-\rho+\eta)^{-\beta})^k,$$

Consequently, we just need to get

$$(r-\rho+\eta)^{-\beta-2}\kappa\beta\eta \geq 2^{\sigma^{\frac{2-p}{2}}}C\lambda\theta^{\frac{2-p}{2}}(r-\rho+\eta)^{-\beta(2-p)}(\eta(r-\rho+\eta)^{-\beta})^k. \tag{3.15}$$

Now, we are looking for conditions on β and η to obtain the last inequality. To this end, using the fact that $\eta \leq r-\rho+\eta \leq 2\eta$, the inequality (3.15) is satisfied if we choose η small enough such that

$$\beta \geq \frac{1}{\kappa}\max\{1, 2^{\beta(p-2)}\}2^{\beta+2+\sigma^{\frac{2-p}{2}}}C\lambda\theta^{\frac{2-p}{2}}\eta^{\beta(p-1)+1+k(1-\beta)}.$$

Hence, we deduce that $\overline{u}$ is a super-solution on $\Gamma \times (0,T)$, for any $T > 0$. Furthermore, by the same argument we can deduce also that $-\overline{u}$ is a sub-solution of $\mathcal{P}_\epsilon$ on $\Gamma \times (0,T)$, for any $T > 0$. Now, we need to have a control on the parabolic boundary of $\Gamma \times (0,T_0)$, where T_0 will be determined below. To this end, we consider the following function

$$v(r,t) = (4A^2K^2+1)^\alpha t + A(1-e^{-K(r-\rho)}),$$

where A, K and α are positive constants which will be specified later. We claim that

$$u_\epsilon \leq v \quad \text{on } Q_{T_0}.$$

For this purpose, since

$$\lambda v^k \int_\Omega v^s dx \geq \lambda f_\epsilon(v)\int_\Omega g_\epsilon(v)dx,$$

we shall show that

$$\frac{\partial v}{\partial t} - \Delta_p^\epsilon v \geq \lambda v^k \int_\Omega v^s dx - \mu H_\epsilon(v,|\nabla v|). \tag{3.16}$$

Since $-\mu H_\epsilon(v,|\nabla v|) \leq 0$, it thus suffices to prove that

$$\frac{\partial v}{\partial t} - \Delta_p^\epsilon v \geq \lambda v^k \int_\Omega v^s dx. \tag{3.17}$$

It is clear that for all $t \in (0, T_0)$, $r \to v(r,t)$ is an increasing and concave function. Then, by the same calculations as for (3.7) we obtain

$$-\Delta_p^\epsilon v \geq [|\nabla v|^2 + \epsilon]^{p-2/2}[-\kappa \frac{\partial^2 v}{\partial r^2} - \frac{N-1}{r}\frac{\partial v}{\partial r}].$$

Now, since

$$-\kappa \frac{\partial^2 v}{\partial r^2} - \frac{N-1}{r}\frac{\partial v}{\partial r} = AKe^{-K(r-\rho)}(\kappa K - \frac{N-1}{r}).$$

Choose $K > \frac{N-1}{\rho\kappa}$. Then, we have $-\Delta_p^\epsilon v \geq 0$. Thus, to get (3.17) it suffices to obtain that

$$(4A^2K^2+1)^\alpha \geq \lambda|\Omega|((4A^2K^2+1)^\alpha t + A)^{k+s},$$

By choosing $\alpha > \frac{k+s}{2}$ and A large enough such that

$$A \geq [\frac{\lambda|\Omega|}{(2K)^{2\alpha}}]^{\frac{1}{2\alpha-k-s}},$$

we can choose T_0 small enough such that

$$0 < t \leq T_0 \leq \frac{1}{J^\alpha}[(\frac{1}{\lambda|\Omega|}J^\alpha)^{\frac{1}{k+s}} - A],$$

where $J = 4A^2K^2+1$. Hence, we deduce the inequality (3.17). On the other hand, by choosing A large enough so that $A(1-e^{-K(r-\rho)}) \geq u_0(x)$, one can prove that $v(x,t) \geq 0$ on $\partial\Omega \times (0,T_0)$. Therefore, by the maximum principle [68] we obtain the claim. Hence, if T_0 and η are small enough we get

$$u_\epsilon(x,t) \leq (4A^2K^2+1)^\alpha t + A(1-e^{-K\eta}) \leq 2^{-\beta}\eta^{1-\beta} = \overline{u}(x,t),$$

on $\{x \in \Omega, |x| = \rho+\eta\} \times [0,T_0]$. Moreover, by choosing also η small enough, we easily have

$$u_0(x) \leq M|x-\tilde{x}| \leq M(r-\rho) \leq (2\eta)^{-\beta}(r-\rho) \leq \overline{u}(x,0).$$

Finally, it is easy to see that $u_\epsilon \leq \overline{u}$ on $\partial\Omega \times [0,T_0]$. Then, by the maximum principle [68] we get $u_\epsilon \leq \overline{u}$ on $\Gamma \times [0,T_0]$. We conclude that $\overline{u}$ is a super-solution on $\Gamma \times [0,T_0]$. By a similar argument, we can show that $\underline{u} = -\phi(r-\rho)$ is a sub-solution on $\Gamma \times [0,T_0]$, we get $\underline{u} \leq u_\epsilon \leq \overline{u}$. Therefore, for all $t \in (0,T_0]$, we have

$$\frac{|u_\epsilon(x_0,t) - u_\epsilon(\tilde{x}_0,t)|}{|x_0-\tilde{x}_0|} \leq \sup_{0\leq s\leq\eta} |\phi'(s)| \leq \eta^{-\beta} = C_1,$$

which allows us to deduce (3.5). Now, let us show that

$$\|\nabla u_\epsilon\|_{L^\infty(Q_{T_0})} \leq C_2. \tag{3.18}$$

We use a similar argument as in ([16, 95]). We denote $\Omega_h = \{x \in \mathbb{R}^N | x-h \in \Omega\}$, where $h \in \mathbb{R}^N$ is such that $|h| \leq \eta$. We observe that, if u_ϵ is a classical solution of $\mathcal{P}_\epsilon$ in $\Omega \times (0,T_0)$, then by the translation invariance of $\mathcal{P}_\epsilon$, $u_\epsilon^h(x,t) = u_\epsilon(x-h,t)$ is a classical solution of $\mathcal{P}_\epsilon$ in $\Omega_h \times (0,T_0)$. Our aim is to prove that

$$|u_\epsilon(x,t) - u_\epsilon^h(x,t)| \leq C_3|h| \quad \text{in } (\Omega\cap\Omega_h) \times [0,T_0]. \tag{3.19}$$

Since $u_0 \in W_0^{1,\infty}(\Omega)$, we have $|u_\epsilon(x,0) - u_\epsilon^h(x,0)| \leq M|h|$ on $\Omega\cap\Omega_h$. Let $t \in [0,T_0]$ and $x \in \partial(\Omega\cap\Omega_h)$. We can suppose $x \in \partial\Omega$, the case where $x+h \in \partial\Omega$ is similar. Hence

$$\begin{aligned}|u_\epsilon(x,t) - u_\epsilon(x+h,t)| &= |u_\epsilon(\tilde{x},t) - u_\epsilon(\widetilde{x+h},t) + u_\epsilon(\widetilde{x+h},t) - u_\epsilon(x+h,t)| \\ &\leq M_2\delta(x+h) \leq C_2|h|.\end{aligned}$$

Then, we have

$$u_\epsilon^h(x,t) - C_2|h| \leq u_\epsilon(x,t) \leq u_\epsilon^h(x,t) + C_2|h| \text{ on } \partial(\Omega \cap \Omega_h) \times [0, T_0].$$

On the other hand, by translation invariance we have $h_\epsilon(t) := \int_\Omega g_\epsilon(u_\epsilon)dx = \int_{\Omega_h} g_\epsilon(u_\epsilon^h)dx$. Moreover, u_ϵ and u_ϵ^h are two solutions of $\mathcal{P}_\epsilon$ in $(\Omega \cap \Omega_h) \times [0, T_0]$. Therefore, it follows that

$$\frac{\partial}{\partial t}(u_\epsilon - u_\epsilon^h) - \Delta_p^\epsilon u_\epsilon + \Delta_p^\epsilon u_\epsilon^h = \lambda h_\epsilon(t)(f_\epsilon(u_\epsilon) - f_\epsilon(u_\epsilon^h)) - \mu(H_\epsilon(u_\epsilon, |\nabla u_\epsilon|) - H_\epsilon(u_\epsilon^h, |\nabla u_\epsilon^h|)).$$

Furthermore, we have

$$H_\epsilon(u_\epsilon, |\nabla u_\epsilon|) - H_\epsilon(u_\epsilon^h, |\nabla u_\epsilon^h|) = G_\epsilon(|\nabla u_\epsilon|)(F_\epsilon(u_\epsilon) - F_\epsilon(u_\epsilon^h)) + F_\epsilon(u_\epsilon^h)(G_\epsilon(|\nabla u_\epsilon|) - G_\epsilon(|\nabla u_\epsilon^h|)).$$

By setting $z = u_\epsilon - u_\epsilon^h$, we obtain that

$$\frac{\partial z}{\partial t} - (a_{i,j}(x,t)z_{x_j})_{x_i} + b(x,t).\nabla z + c(x,t)z = 0, \quad \text{in } (\Omega \cap \Omega_h) \times (0, T_0),$$

where

$$\begin{aligned} a_{i,j}(x,t) = &\int_0^1 (|\nabla(su_\epsilon + (1-s)u_\epsilon^h)|^2 + \epsilon)^{\frac{p-2}{2}} ds\delta_{i,j} + (p-2)\int_0^1 (|\nabla(su_\epsilon + (1-s)u_\epsilon^h)|^2 + \epsilon)^{\frac{p-4}{2}} \\ &\times (su_\epsilon + (1-s)u_\epsilon^h)_{x_i}(su_\epsilon + (1-s)u_\epsilon^h)_{x_j} ds, \end{aligned}$$

$$b(x,t) = 2\mu F_\epsilon(u_\epsilon^h)\int_0^1 G_\epsilon^{'}(|\nabla(su_\epsilon + (1-s)u_\epsilon^h)|)\nabla(su_\epsilon + (1-s)u_\epsilon^h)ds,$$

and

$$c(x,t) = -\lambda h_\epsilon(t)\int_0^1 f_\epsilon^{'}(su_\epsilon + (1-s)u_\epsilon^h)ds + \mu G_\epsilon(\nabla u_\epsilon)\int_0^1 F_\epsilon^{'}(su_\epsilon + (1-s)u_\epsilon^h)ds.$$

Thanks to the smoothness of u_ϵ and u_ϵ^h, the matrices $(a)_{i,j}$ and b and the function c are also smooth. Moreover, it is easy to see that $(a)_{i,j}$ is positive and semidefinite. However, since $\lambda > 0$ we can not use the maximum principle to get the lower and upper bounds of z. For this aim, we define $\bar{z}(x,t) = e^{-\nu t}z(x,t)$, such that $\nu = \sup_{(\Omega\cap\Omega_h)\times[0,T_0]} |c_1(x,t)|$, where

$$c_1(x,t) = \lambda g_\epsilon(t)\int_0^1 f_\epsilon^{'}(su_\epsilon + (1-s)u_\epsilon^h)ds.$$

Then, on the one hand we have

$$\frac{\partial \bar{z}}{\partial t} - (a_{i,j}(x,t)\bar{z}_{x_j})_{x_i} + b.\nabla\bar{z} + (c+\nu)\bar{z} = 0, \quad \text{in } (\Omega \cap \Omega_h) \times (0, T_0),$$

and on the other hand we have $\bar{z}(x,0) \leq M|h|$ in $\Omega \cap \Omega_h$ and $\bar{z}(x,t) \leq C_2|h|$ on $\partial(\Omega \cap \Omega_h) \times [0, T_0]$. Therefore, by the maximum principle $z \leq C_3|h|$, where $C_3 = \max\{M, C_2 e^{\nu T_0}\}$, which means that $u_\epsilon \leq u_\epsilon^h + C_3|h|$ in $(\Omega \cap \Omega_h) \times [0, T_0]$. By the same argument, we can show that $u_\epsilon \geq u_\epsilon^h - C_3|h|$ in $(\Omega \cap \Omega_h) \times [0, T_0]$, which allow us to conclude the proof of (3.18).
Finally, let us prove the boundedness of $\|\frac{\partial u_\epsilon}{\partial t}\|_{L^2(Q_{T_0})}$. To this end, multiplying $\mathcal{P}_\epsilon$ by $\frac{\partial u_\epsilon}{\partial t}$ and integrating over Q_{T_0}, yields

$$\begin{aligned} \int_{Q_{T_0}} (\frac{\partial u_\epsilon}{\partial t})^2 dxdt = &\frac{1}{p}\int_\Omega [(|\nabla u_\epsilon(x,0)|^2 + \epsilon)^{p/2} - (|\nabla u_\epsilon(x,T_0)|^2 + \epsilon)^{p/2}]dx \\ &+ \lambda\int_{Q_{T_0}} h_\epsilon(t)f_\epsilon(u_\epsilon)\frac{\partial u_\epsilon}{\partial t}dxdt - \mu\int_{Q_{T_0}} H_\epsilon(u_\epsilon, |\nabla u_\epsilon|)\frac{\partial u_\epsilon}{\partial t}dxdt. \end{aligned}$$

By the uniform boundedness of u_ϵ and $|\nabla u_\epsilon|$ for all $t \in [0, T_0]$ and by using Young's inequality, we get easily that

$$\int_{Q_{T_0}} (\frac{\partial u_\epsilon}{\partial t})^2 dxdt \leq C_4.$$

□

Now, to end the proof of our theorem we shall use the above lemma in order to pass to the limit. Indeed, since we have the following embedding

$$W^{1,\infty}(\Omega) \hookrightarrow^c C(\overline{\Omega}) \hookrightarrow L^2(\Omega),$$

then from lemma 3.2.2 and the compactness theorem [84, Corollary 4], we deduce that (u_ϵ) is relatively compact in $C([0, T_0]; C(\overline{\Omega})) = C(\overline{\Omega} \times [0, T_0])$. Also, from lemma 3.2.2, the relative compactness of (u_ϵ) in $C(\overline{\Omega} \times [0, T_0])$ and the Ascoli-Arzelà theorem, there exists a subsequence of (u_ϵ), still denoted by (u_ϵ), such that

$$\begin{cases} u_\epsilon \to u, & \text{in } C(\overline{\Omega} \times [0, T_0]), \\ \nabla u_\epsilon \rightharpoonup \nabla u, & \text{weakly star in } L^\infty(Q_{T_0}), \\ \frac{\partial u_\epsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}, & \text{weakly in } L^2(Q_{T_0}). \end{cases} \tag{3.20}$$

Hence, by passing to the limit for any $\varphi \in C_0^1(Q_{T_0})$, we obtain easily that

$$\int_{Q_{T_0}} \varphi \frac{\partial u}{\partial t} dxdt + \int_{Q_{T_0}} \chi \nabla \varphi dxdt = \lambda \int_{Q_{T_0}} h(t) |u|^{k-1} u \varphi dxdt - \mu \int_{Q_{T_0}} \xi \varphi dxdt, \tag{3.21}$$

where, $h(t) = \int_\Omega |u|^s dx$, $\chi = \{\chi_i\ ; 1 \leq i \leq N\}$, with $\chi_i \in L^\infty(Q_{T_0})$ for any $1 \leq i \leq N$, and $\xi \in L^\infty(Q_{T_0})$. In order to prove that u is a weak solution of $\mathcal{P}$, we need to prove that for any function $\varphi \in C_0^1(Q_{T_0})$

$$\int_{Q_{T_0}} |\nabla u|^{p-2} \nabla u \nabla \varphi dxdt = \int_{Q_{T_0}} \chi \nabla \varphi dxdt, \tag{3.22}$$

and

$$\int_{Q_{T_0}} |u|^{l-1} u \, |\nabla u|^q \, \varphi dxdt = \int_{Q_{T_0}} \xi \varphi dxdt. \tag{3.23}$$

Let us show firstly (3.22) by using the well known Minty argument [71]. Let $\psi \in C_0^1(Q_{T_0})$ such that $0 \leq \psi \leq 1$. Multiplying $\mathcal{P}_\epsilon$ by $u_\epsilon \psi$, we get

$$\begin{aligned} \int_{Q_{T_0}} \psi (|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} |\nabla u_\epsilon|^2 dxdt = & - \int_{Q_{T_0}} \psi u_\epsilon \frac{\partial u_\epsilon}{\partial t} + \lambda \int_{Q_{T_0}} u_\epsilon h_\epsilon(t) f_\epsilon(u_\epsilon) \psi dxdt \\ & - \mu \int_{Q_{T_0}} u_\epsilon H_\epsilon(u_\epsilon, |\nabla u_\epsilon|) \psi dxdt \\ & - \int_{Q_{T_0}} u_\epsilon (|\nabla u_\epsilon|^2 + \epsilon)^{p-2/2} \nabla u_\epsilon . \nabla \psi dxdt, \end{aligned}$$

where $h_\epsilon(t) = \int_\Omega g_\epsilon(u_\epsilon) dx$. For any $v \in L^\infty(0, T; W_0^{1,\infty}(\Omega))$, we have

$$\int_{Q_{T_0}} \psi (|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon - |\nabla v|^{p-2} \nabla v) \nabla (u_\epsilon - v) dxdt \geq 0. \tag{3.24}$$

Then, for $p > 1$ we obtain

$$\begin{aligned}
&-\int_{Q_{T_0}} \psi u_\epsilon \frac{\partial u_\epsilon}{\partial t} dxdt + \lambda \int_{Q_{T_0}} u_\epsilon h_\epsilon(t) f_\epsilon(u_\epsilon)\psi dxdt - \mu \int_{Q_{T_0}} u_\epsilon H_\epsilon(u_\epsilon, |\nabla u_\epsilon|)\psi dxdt \\
&-\int_{Q_{T_0}} u_\epsilon(|\nabla u_\epsilon|^2+\epsilon)^{p-2/2}\nabla u_\epsilon.\nabla\psi dxdt + \epsilon^{p/2}|Q_{T_0}| \\
&-\int_{Q_{T_0}} \psi|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \nabla v dxdt - \int_{Q_{T_0}} \psi|\nabla v|^{p-2}\nabla v\nabla(u_\epsilon - v)dxdt \geq 0.
\end{aligned} \tag{3.25}$$

On the other hand, we have

$$(|\nabla u_\epsilon|^2+\epsilon)^{\frac{p-2}{2}}\nabla u_\epsilon = |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon + \frac{\epsilon(p-2)}{2}\int_0^1 (|\nabla u_\epsilon|^2 + s\epsilon)^{\frac{p-4}{2}} ds \nabla u_\epsilon. \tag{3.26}$$

From lemma 3.2.2, it is easy to see that

$$\frac{\epsilon(p-2)}{2}\int_{Q_{T_0}}\int_0^1 u_\epsilon(|\nabla u_\epsilon|^2+s\epsilon)^{\frac{p-4}{2}}\nabla u_\epsilon \nabla\psi ds dxdt \to 0, \quad \text{as} \quad \epsilon \to 0,$$

and by letting $\epsilon \to 0$, in (4.13), it follows that

$$\begin{aligned}
&-\int_{Q_{T_0}} \psi u \frac{\partial u}{\partial t} dxdt + \lambda\int_{Q_{T_0}} h(t)|u|^{k+1}\psi dxdt - \mu\int_{Q_{T_0}} u\xi\psi dxdt \\
&-\int_{Q_{T_0}} u\chi.\nabla\psi dxdt - \int_{Q_{T_0}} \psi\chi\nabla v dxdt - \int_{Q_{T_0}} \psi|\nabla v|^{p-2}\nabla v\nabla(u-v)dxdt \geq 0.
\end{aligned}$$

By taking $\varphi = u\psi$ in (3.21), we deduce from the last inequality that

$$\int_{Q_{T_0}} \psi(\chi - |\nabla v|^{p-2}\nabla v).(\nabla u - \nabla v) \geq 0. \tag{3.27}$$

Moreover, for every $\varepsilon \geq 0$, in (3.27) the choice $v = u - \varepsilon\varphi$, implies that

$$\int_{Q_{T_0}} \psi(\chi - |\nabla(u-\varepsilon\varphi)|^{p-2}\nabla(u-\varepsilon\varphi)).\nabla\varphi dxdt \geq 0,$$

Letting $\varepsilon \to 0$ gives for all $\varphi \in C_0^1(Q_{T_0})$ that

$$\int_{Q_{T_0}} \psi(\chi - |\nabla u|^{p-2}\nabla u)\nabla\varphi dxdt \geq 0.$$

Furthermore, if we choose $\varepsilon \leq 0$ then, for all $\varphi \in C_0^1(Q_{T_0})$, we have

$$\int_{Q_{T_0}} \psi(\chi - |\nabla u|^{p-2}\nabla u)\nabla\varphi dxdt \leq 0.$$

Hence, by choosing ψ such that supp $\varphi \subset$ supp ψ and $\psi = 1$ in supp φ, (3.22) holds. Now, we only have to show (3.23). For this it suffices to show that

$$H_\epsilon(u_\epsilon, |\nabla u_\epsilon|) \to |u|^{l-1}u|\nabla u|^q \quad \text{a.e in } Q_{T_0}. \tag{3.28}$$

Subtracting $\mathcal{P}_\epsilon$ and $\mathcal{P}_\varsigma$, where ϵ and ς are different parameters, yields

$$\begin{aligned}
\frac{\partial}{\partial t}(u_\epsilon - u_\varsigma) - \Delta_p^\epsilon u_\epsilon + \Delta_p^\varsigma u_\varsigma =& \lambda(h_\epsilon(t)f_\epsilon(u_\epsilon) - h_\varsigma(t)f_\varsigma(u_\varsigma)) \\
&- \mu(H_\epsilon(u_\epsilon, |\nabla u_\epsilon|) - H_\varsigma(u_\varsigma, |\nabla u_\varsigma|)) \quad \text{in } Q_{T_0},
\end{aligned} \tag{3.29}$$

$$u_\epsilon - u_\varsigma = 0 \text{ on } \partial\Omega \times (0, T_0),$$
$$u_\epsilon(x,0) - u_\varsigma(x,0) = 0 \text{ in } \Omega.$$

Then, multiplying the equation (3.29) by the test function $u_\epsilon - u_\varsigma$ and integrating over Q_{T_0}, gives

$$\int_{Q_{T_0}} ((|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} \nabla u_\epsilon - (|\nabla u_\varsigma|^2 + \varsigma)^{\frac{p-2}{2}} \nabla u_\varsigma) \nabla (u_\epsilon - u_\varsigma) dxdt =$$
$$- \int_{Q_{T_0}} \frac{\partial (u_\epsilon - u_\varsigma)}{\partial t} (u_\epsilon - u_\varsigma) dxdt + \lambda \int_{Q_{T_0}} (h_\epsilon(t) f_\epsilon(u_\epsilon) - h_\varsigma(t) f_\varsigma(u_\varsigma))(u_\epsilon - u_\varsigma) dxdt$$
$$- \mu \int_{Q_{T_0}} (H_\epsilon(u_\epsilon, |\nabla u_\epsilon|) - H_\varsigma(u_\varsigma, |\nabla u_\varsigma|))(u_\epsilon - u_\varsigma) dxdt.$$

Hence, by lemma 3.2.2 and (3.20) we deduce that

$$\int_{Q_{T_0}} ((|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} \nabla u_\epsilon - (|\nabla u_\varsigma|^2 + \varsigma)^{\frac{p-2}{2}} \nabla u_\varsigma) \nabla (u_\epsilon - u_\varsigma) dxdt \to 0, \tag{3.30}$$

as $\epsilon, \varsigma \to 0$. Moreover from (3.26) it follows that

$$\int_{Q_{T_0}} (|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon - |\nabla u_\varsigma|^{p-2} \nabla u_\varsigma)(\nabla u_\epsilon - \nabla u_\varsigma) dxdt$$
$$= \int_{Q_{T_0}} ((|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} \nabla u_\epsilon - (|\nabla u_\varsigma|^2 + \varsigma)^{\frac{p-2}{2}} \nabla u_\varsigma) \nabla (u_\epsilon - u_\varsigma) dxdt$$
$$- \frac{\epsilon(p-2)}{2} \int_{Q_{T_0}} \int_0^1 (|\nabla u_\epsilon|^2 + s\epsilon)^{\frac{p-4}{2}} ds \nabla u_\epsilon (\nabla u_\epsilon - \nabla u_\varsigma) dxdt$$
$$+ \frac{\varsigma(p-2)}{2} \int_{Q_{T_0}} \int_0^1 (|\nabla u_\varsigma|^2 + s\varsigma)^{\frac{p-4}{2}} ds \nabla u_\varsigma (\nabla u_\epsilon - \nabla u_\varsigma) dxdt.$$

Therefore, from lemma 3.2.2 and (3.30) we obtain

$$\int_{Q_{T_0}} (|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon - |\nabla u_\varsigma|^{p-2} \nabla u_\varsigma)(\nabla u_\epsilon - \nabla u_\varsigma) dxdt \to 0, \tag{3.31}$$

as $\epsilon, \varsigma \to 0$. On the other hand, we note that if $p \geq 2$ then we have

$$\int_{Q_{T_0}} |\nabla u_\epsilon - \nabla u_\varsigma|^p \, dxdt \leq C_5 \int_{Q_{T_0}} (|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon - |\nabla u_\varsigma|^{p-2} \nabla u_\varsigma)(\nabla u_\epsilon - \nabla u_\varsigma) dxdt,$$

where $C_5 > 0$ is a positive constant; while if $1 < p < 2$, thus by Hölder's inequality and lemma 3.2.2, we get

$$\int_{Q_{T_0}} |\nabla u_\epsilon - \nabla u_\varsigma|^p \, dxdt \leq C_6 \left(\int_{Q_{T_0}} \frac{|\nabla u_\epsilon - \nabla u_\varsigma|^2}{|\nabla u_\epsilon|^{2-p} + |\nabla u_\varsigma|^{2-p} + 1} dxdt \right)^{p/2}$$
$$\times \left(\int_{Q_{T_0}} (|\nabla u_\epsilon|^{2-p} + |\nabla u_\varsigma|^{2-p} + 1)^{\frac{p}{2-p}} dxdt \right)^{1-p/2}$$
$$\leq C_7 \int_{Q_{T_0}} (|\nabla u_\epsilon|^{p-2} \nabla u_\epsilon - |\nabla u_\varsigma|^{p-2} \nabla u_\varsigma)(\nabla u_\epsilon - \nabla u_\varsigma) dxdt,$$

where C_6 and C_7 are positive constants. Then (3.31) implies that

$$\int_{Q_{T_0}} |\nabla u_\epsilon - \nabla u_\varsigma|^p \, dxdt \to 0, \tag{3.32}$$

as $\epsilon, \varsigma \to 0$. Therefore, (3.32) and (3.20) imply that

$$|\nabla u_\epsilon| \to |\nabla u| \quad \text{a.e in } Q_{T_0},$$

Moreover, since

$$(|\nabla u_\epsilon|^2+\epsilon)^{\frac{q}{2}} - \epsilon^{q/2} = |\nabla u_\epsilon|^q + \frac{\epsilon(q)}{2}\int_0^1 (|\nabla u_\epsilon|^2 + s\epsilon)^{\frac{q-2}{2}} ds - \epsilon^{q/2},$$

then

$$(|\nabla u_\epsilon|^2+\epsilon)^{\frac{q}{2}} - \epsilon^{q/2} \to |\nabla u|^q \text{ a.e in } Q_{T_0} \quad \text{ as } \epsilon \to 0.$$

Finally, from lemma 3.2.2 and (3.20), we deduce easily (3.28). Hence, by the same argument used in [28, 93] and letting $\epsilon \to 0$ in $\mathcal{P}_\epsilon$ we conclude that u is a weak solution of $\mathcal{P}$. □

Before ending this section, let us point out that the weak solution does not exist globally when the maximal existence time T_{max} of the problem $\mathcal{P}$ is finite. First, let us state the following definition.

Definition 3.2. *We say that u is a maximal weak solution of $\mathcal{P}$ if u is a weak solution on Q_τ, for any $\tau \in (0, T_{max})$ such that u cannot be extended on Q_T for any $T > T_{max}$.*

The following proposition deals with the blow-up alternative. It has been proved in [16], in the case where $\lambda = 0$ and $l = 0$. We will assume that the problem $\mathcal{P}$ has a unique weak solution. We shall give the proof of the uniqueness result under some supplementary conditions, in the next section.

Proposition 3.2.3. *We assume that $\mathcal{P}$ has unique weak solution. Let T_{max} the maximal existence time. If*

$$T_{max} < \infty, \;\; \textit{then} \quad \lim_{t \to T_{max}^-} \|u(.,t)\|_{W^{1,\infty}(\Omega)} = \infty.$$

Proof. Following the proof of the local existence result, suppose that there exist $C > 0$ and $t_k \to T_{max}$ such that for all k

$$\|u(.,t_k)\|_{W^{1,\infty}(\Omega)} \leq C.$$

Then, there exists $\tau = \tau(C)$ independent of k such that the problem :

$$\mathcal{P}_k : \begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{k-1} u \int_\Omega |u|^s dx - \mu |u|^{l-1} u |\nabla u|^q & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u(x,t_k) & \text{in } \Omega, \end{cases}$$

has a unique solution v_k on $[0,\tau)$. For all $x \in \Omega$, we set

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{for } t \in [0,t_k), \\ v_k(x, t-t_k) & \text{for } t \in [t_k, t_k+\tau), \end{cases}$$

which is a weak solution of $\mathcal{P}$ on $[0, t_k + \tau)$. Moreover, we have for k large enough $t_k + \tau > T_{max}$. Then, we have a contradiction with the definition of T_{max}. Thus, the result holds. □

3.3 Global and non global existence of solutions

In this section we restrict ourselves to studying qualitative properties for non-negative solutions of problem $\mathcal{P}$, which have more physical meaning. Such properties are exhibited according to the ranges of the exponents p, k, s, l and q, and to the size of the initial data u_0, the Lebesgue measure of Ω and of the constants λ and μ. In addition to the hypothesis (3.1), we shall assume through the rest of this chapter that

$$u_0 \gneqq 0. \tag{3.33}$$

First, we give the following weak comparison principle for problem $\mathcal{P}$ which plays a major role in obtaining useful properties as uniqueness, for non-negative solutions.

Proposition 3.3.1. *Let $u, v \in L^\infty(0,T;W_0^{1,\infty}(\Omega))$ be non-negative sub and super-solutions of $\mathcal{P}$, respectively. Assume that $k, s \geq 1$ and $q \geq \frac{p}{2}$. Then, we have $u \leq v$ on Q_T.*

The proof of the weak comparison principle is based on the following algebraic lemma, (see, [16]).

Lemma 3.3.2. *Let $\sigma > 1$. For all $a, b \in \mathbb{R}^N$, we have*

$$\langle |a|^{\sigma-2}a - |b|^{\sigma-2}b, a-b\rangle \geq \frac{4}{\sigma^2}||a|^{\frac{\sigma-2}{2}}a - |b|^{\frac{\sigma-2}{2}}b|^2.$$

Proof of Proposition 3.3.1. We choose $\varphi = (u-v)^+ = \max\{u-v, 0\}$ as a test function. Clearly $\varphi = 0$ on $\partial\Omega$ and $\varphi(x,0) = 0$, for all $x \in \Omega$. Then, for any $\tau \in (0,T)$, we have

$$\begin{aligned}
&\int_0^\tau \int_\Omega \varphi \frac{\partial \varphi}{\partial t} dxdt + \underbrace{\int_0^\tau \int_{\{\varphi(.,t)>0\}} [|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v].\nabla \varphi dxdt}_{\mathcal{A}} \\
&\leq \underbrace{\lambda \int_0^\tau \int_{\{\varphi(.,t)>0\}} [u^k \int_\Omega u^s dx - v^k \int_\Omega v^s dx]\varphi dxdt}_{\mathcal{B}} \\
&\quad \underbrace{-\mu \int_0^\tau \int_{\{\varphi(.,t)>0\}} [u^l|\nabla u|^q - v^l|\nabla v|^q]\varphi dxdt}_{\mathcal{C}}.
\end{aligned}$$

From the lemma (3.3.2), we get

$$\begin{aligned}
\mathcal{A} &\geq C\int_0^\tau \int_{\{\varphi(.,t)>0\}} ||\nabla u|^{\frac{p-2}{2}}\nabla u - |\nabla v|^{\frac{p-2}{2}}\nabla v|^2 dxdt \\
&\geq C\int_0^\tau \int_{\{\varphi(.,t)>0\}} ||\nabla u|^{\frac{p}{2}} - |\nabla v|^{\frac{p}{2}}|^2 dxdt.
\end{aligned} \tag{3.34}$$

Concerning the term $\mathcal{B}$, we have

$$\mathcal{B} = \lambda \underbrace{\int_0^\tau \int_{\{\varphi(.,t)>0\}} [u^k\varphi(\int_\Omega u^s dx - \int_\Omega v^s dx)]dxdt}_{\mathcal{D}} + \lambda \int_0^\tau \int_{\{\varphi(.,t)>0\}} [\varphi \int_\Omega v^s dx(u^k - v^k)]dxdt. \tag{3.35}$$

Nevertheless, we have

$$D = \int_0^\tau \int_{\{\varphi(.,t)>0\}} u^k\varphi(\int_\Omega (u^s - v^s)dx)dxdt.$$

By using the mean value theorem and holder's inequality, we obtain

$$\begin{aligned} D &= \int_0^\tau \int_{\{\varphi(.,t)>0\}} u^k \varphi\left(\int_{\{u\geq v\}} (u^s - v^s)dx + \int_{\{u<v\}} (u^s - v^s)dx\right)dxdt \\ &\leq \int_0^\tau \int_{\{\varphi(.,t)>0\}} u^k \varphi\left(\int_{\{u\geq v\}} (u^s - v^s)dx\right)dxdt \\ &= \int_0^\tau \int_{\{\varphi(.,t)>0\}} u^k \varphi\left(\int_{u\geq v} s w^{s-1}(u-v)dx\right)dxdt \qquad \text{where } w \in (v,u) \\ &\leq C_1 \int_0^\tau \int_{\{\varphi(.,t)>0\}} \varphi^2 dxdt. \end{aligned}$$

where the constant C_1 depends on the sup-norms of u and v. Then, for the second term of (3.35) we use the mean value theorem and we deduce that

$$B \leq C_2 \int_0^\tau \int_{\{\varphi(.,t)>0\}} \varphi^2 dxdt. \tag{3.36}$$

For the term $\mathcal{C}$, using the fact that $q \geq \frac{p}{2}$ we get from the mean value theorem and by Young's inequality that

$$\begin{aligned} \mathcal{C} &\leq -\mu \int_0^\tau \int_{\{\varphi(.,t)>0\}} [u^l(|\nabla u|^q - |\nabla v|^q)]\varphi dxdt \\ &\leq C_3\epsilon \int_0^\tau \int_{\{\varphi(.,t)>0\}} ||\nabla u|^{\frac{p}{2}} - |\nabla v|^{\frac{p}{2}}|^2 dxdt + C_4(\epsilon) \int_0^\tau \int_{\{\varphi(.,t)>0\}} \varphi^2 dxdt. \end{aligned} \tag{3.37}$$

Hence, combining inequalities (3.34), (3.36) and (3.37) and choosing ϵ small enough we deduce that

$$\int_\Omega \varphi^2(x,\tau)dx \leq \int_\Omega \varphi^2(x,0)dx + C_5 \int_0^\tau \int_\Omega \varphi^2 dxdt.$$

Then, by Gronwall's lemma, for any $t \in (0,T)$ we have

$$\int_\Omega \varphi^2(x,t)dx \leq e^{C_6 t} \int_\Omega \varphi^2(x,0)dx,$$

which yields the desired result. □

3.3.1 Blow-up in finite time

Now, we give sufficient conditions on the ranges of the exponents p, k, s, l, q and u_0, to get the following blow-up result. The proof of this result is based on the construction of a suitable self-similar sub-solution and on the preceding comparison principle given in proposition (3.3.1).

Theorem 3.3.3. *Let $q \geq \frac{p}{2}$ and $q + l > 1$. Moreover, suppose that $k, s \geq 1$ are such that $k + s > \max\{p-1, q+l\}$. Then, the weak solution of problem $\mathcal{P}$ blows-up in the L^∞ norm in a finite time T_{max}, for u_0 large enough.*

Proof. By translation we can assume without loss of generality that $0 \in \Omega$. We consider the following unbounded function on $[t_0, \frac{1}{\delta}) \times \mathbb{R}^N$

$$v(x,t) = \frac{1}{(1-\delta t)^\alpha} V\left(\frac{|x|}{(1-\delta t)^\beta}\right)$$

where

$$V(y) = \left(1 + \frac{A}{\sigma} - \frac{y^\sigma}{\sigma A^{\sigma-1}}\right)_+, \quad y \geq 0,\ \sigma = \frac{p}{p-1}. \tag{3.38}$$

The positive parameters A, α, β and δ will be specified later. We assume for all $t \in [t_0, \frac{1}{\delta})$ that

$$supp(v(.,t)) = \overline{B(0, R(1-\delta t)^{\beta})} \subset \overline{B(0, R(1-\delta t_0)^{\beta})} \subset \Omega,$$

where $R = (A^{\sigma-1}(A+\sigma))^{1/\sigma}$ is the unique zero of $V(y)$.
Clearly, $v(x,t)$ blows-up in a finite time. By simple calculations, the function $V(y)$ satisfies the following properties

$$\begin{cases} 1 \leq V(y) \leq 1 + \frac{A}{\sigma}, \ \ -1 \leq V'(y) \leq 0 & \text{for } 0 \leq y \leq A, \\ 0 \leq V(y) \leq 1, \ \ -(\frac{R}{A})^{\sigma-1} \leq V'(y) \leq -1 & \text{for } A \leq y \leq R, \\ (p-1)|V'(y)|^{p-2}V''(y) + \frac{N-1}{y}|V'(y)|^{p-2}V'(y) = -\frac{N}{A} & \text{for } 0 < y < R. \end{cases} \tag{3.39}$$

By setting $y = \frac{|x|}{(1-\delta t)^{\beta}}$, let us show that $v(x,t)$ is a sub-solution of problem $\mathcal{P}$. We consider

$$P(v(x,t)) = \frac{\partial v}{\partial t} - \Delta_p v - \lambda v^k \int_{\Omega} v^s dx + \mu v^l |\nabla v|^q.$$

Hence

$$\begin{aligned} P(v(x,t)) = & \frac{\delta(\alpha V(y) + \beta y V'(y))}{(1-\delta t)^{\alpha+1}} - \frac{(p-1)|V'(y)|^{p-2}V''(y) + \frac{N-1}{y}|V'(y)|^{p-2}V'(y)}{(1-\delta t)^{(p-1)(\alpha+\beta)+\beta}} \\ & - \frac{\lambda V^k(y)}{(1-\delta t)^{\alpha(k+s)}} \int_{B(0,R(1-\delta t)^{\beta})} V^s(\frac{|x|}{(1-\delta t)^{\beta}})dx + \mu \frac{V^l(y)}{(1-\delta t)^{l\alpha}} \\ & \times \frac{|V'(y)|^q}{(1-\delta t)^{q(\beta+\alpha)}}. \end{aligned}$$

By setting $K = \int_{B(0,R)} V^s(|\zeta|)d\zeta > 0$, we obtain

$$\int_{B(0,R(1-\delta t)^{\beta})} V^s(\frac{|x|}{(1-\delta t)^{\beta}})dx = \frac{K}{(1-\delta t)^{-N\beta}}.$$

Let us choose

$$\alpha = \frac{N\beta+1}{k+s-1}, \qquad A > \frac{\alpha}{\beta} \qquad \text{and} \qquad 0 < \delta < \frac{\lambda K}{\alpha(1+\frac{A}{\sigma})}.$$

If $p(k+s-1) + N(p-2) > 0$, then we take

$$\beta < \min\{\frac{k+s-l-q}{q(k+s-1)+N(l+q-1)}; \frac{k+s+1-p}{p(k+s-1)+N(p-2)}\};$$

otherwise, if not we take

$$\beta < \frac{k+s-l-q}{q(k+s-1)+N(l+q-1)}.$$

For $t_0 < t < \frac{1}{\delta}$ such that t_0 is sufficiently close to $\frac{1}{\delta}$, we have in the case $0 \leq y \leq A$

$$\begin{aligned} P(v(x,t)) \leq & \frac{1}{(1-\delta t)^{\alpha+1}}[\delta\alpha(1+\frac{A}{\sigma}) + \frac{N}{A}(1-\delta t)^{\alpha+1-(p-1)(\alpha+\beta)-\beta} \\ & - \lambda K + \mu(1+\frac{A}{\sigma})^l(1-\delta t)^{-l\alpha-q(\alpha+\beta)+\alpha+1}] \leq 0. \end{aligned} \tag{3.40}$$

While if $A \leq y < R$, we get

$$P(v(x,t)) \leq \frac{1}{(1-\delta t)^{\alpha+1}}[\delta(\alpha-\beta A) + \frac{N}{A}(1-\delta t)^{\alpha+1-(p-1)(\alpha+\beta)-\beta} + (\frac{R}{A})^{q(\sigma-1)}\mu(1-\delta t)^{-l\alpha-q(\alpha+\beta)+\alpha+1}] \leq 0. \tag{3.41}$$

Besides, thanks to the embedding $W_0^{1,\infty}(\Omega) \hookrightarrow C(\overline{\Omega})$ it follows that $u_0(x) \geq C > 0$ in $B(0,\rho) \subset\subset \Omega$, for some $\rho > 0$. As t_0 is sufficiently close to $\frac{1}{\delta}$, we can assume that $\overline{B(0, R(1-\delta t_0)^{\beta})} \subset B(0,\rho)$ and choose $C > 0$ such that

$$u_0(x) \geq C \geq \frac{V(0)}{(1-\delta t_0)^{\alpha}} \geq v(x,t_0).$$

Furthermore, it is clear that $v = 0$ in $\partial\Omega \times (t_0, \frac{1}{\delta})$. Therefore by the comparison principle, we deduce that

$$u(x,t) \geq v(x,t+t_0), \quad x \in \Omega, \quad 0 < t < \frac{1}{\delta} - t_0.$$

We conclude that $T_{max} \leq \frac{1}{\delta} - t_0$. □

3.3.2 Global existence of the solution

Here, we establish some results concerning the global existence of the non-negative weak solution for problem $\mathcal{P}$. We will show that the uniform boundedness of the non-negative weak solution in L^{∞} norm is sufficient to obtain the global existence. We shall follow the same method used in [94]. By applying [98, theorem 3.1], we deduce that the blow-up of the gradient can not occur, in the case where $q \leq p-1$.

Theorem 3.3.4. *Let $k, s \geq 1$ and $q \geq \frac{p}{2}$. We assume that one of the following conditions hold*

1. *$q + l > \max\{p-1, k+s\}$,*
2. *$q + l = k + s > p - 1$ and $|\Omega|$ (or λ) is small enough,*
3. *$q + l = k + s > p - 1$ and μ is large enough.*

Then, the weak solution of the problem $\mathcal{P}$ is uniformly bounded in L^{∞} norm.

Proof. Let $\rho(\Omega)$ be the diameter of Ω. Since Ω is bounded we have $\rho(\Omega) < \infty$. Let $\epsilon \in (0,1)$ such that there exists a ball with radius ϵ which is included in $B(., \rho(\Omega)+1) \cap \Omega^c$. For any $a \in \Omega$, we choose x_a such that

$$B(x_a, \epsilon) \subseteq B(x_a, \rho(\Omega)+1) \cap \Omega^c, \quad |x_a - a| < \rho(\Omega) + 1. \tag{3.42}$$

We consider the following function

$$v(x,t) = Ke^r \quad \text{with } r = |x - x_a|, \ x \in \Omega, \tag{3.43}$$

where $K \geq 1$ is a constant that will be specified later. It is clear that $\epsilon \leq r \leq 2\rho(\Omega) + 1$. Now, let us show that $v(x,t)$ is a super-solution of problem $\mathcal{P}$. By straightforward calculations, we have

$$\begin{aligned} P(v(x,t)) &= \frac{\partial v}{\partial t} - \Delta_p v - \lambda v^k \int_{\Omega} v^s dx + \mu v^l |\nabla v|^q \\ &= -K^{p-1}e^{(p-1)r}(\frac{N-1}{r} + (p-1)) - \lambda K^k e^{kr} \int_{\Omega} K^s e^{sr} dx \\ &\quad + \mu K^{l+q} e^{r(l+q)}. \end{aligned} \tag{3.44}$$

Therefore, to get $P(v(x,t)) \geq 0$ it suffices to have

$$\mu K^{l+q} e^{r(l+q)} \geq K^{p-1} e^{(p-1)r} \left(\frac{N-1}{\epsilon} + (p-1)\right) + \lambda C |\Omega| K^{k+s} e^{kr}, \tag{3.45}$$

where $C = e^{s(2\rho(\Omega)+1)}$. Thus, (3.45) holds if

$$\begin{cases} \mu K^{l+q-p+1} e^{r(l+q)} \geq 2e^{(p-1)r}\left(\frac{N-1}{\epsilon} + (p-1)\right), \\ \mu K^{l+q-k-s} e^{r(l+q)} \geq 2e^{kr} \lambda C |\Omega| \end{cases} . \tag{3.46}$$

1. If $q + l > \max\{p-1, k+s\}$, then the constant K must verify

$$K \geq \max \left\{ \left[\frac{2}{\mu}\left(\frac{N-1}{\epsilon} + (p-1)\right)\right]^{\frac{1}{l+q-p+1}}, \left[\frac{2\lambda C|\Omega|}{\mu}\right]^{\frac{1}{l+q-k-s}} \right\}.$$

   To ensure that $v(x,0) \geq u_0$, we also need that $K \geq \|u_0\|_\infty$. Then, we choose K such that

$$K = \max \left\{ \left[\frac{2}{\mu}\left(\frac{N-1}{\epsilon} + (p-1)\right)\right]^{\frac{1}{l+q-p+1}}, \left[\frac{2\lambda C|\Omega|}{\mu}\right]^{\frac{1}{l+q-k-s}}, 1, \|u_0\|_\infty \right\}.$$

   It is clear that $v(x,t) \geq u(x,t) = 0$ on $\partial\Omega$. Therefore, by the comparison principle we conclude that $v(x,t)$ is a super-solution of problem $\mathcal{P}$, which means that

$$0 \leq u(x,t) \leq v(x,t) \leq K e^{2\rho(\Omega)+1}.$$

2. If $q + l = k + s > p - 1$, we just need to take

$$K = \max \left\{ \left[\frac{2}{\mu}\left(\frac{N-1}{\epsilon} + (p-1)\right)\right]^{\frac{1}{l+q-p+1}}, 1, \|u_0\|_\infty \right\}, \tag{3.47}$$

   and we choose the Lebesgue measure of Ω (or λ) such that

$$|\Omega| \leq \frac{\mu}{2\lambda C} \quad (\text{ or } \lambda \leq \frac{\mu}{|\Omega| 2C}).$$

   By the same argument as for (1) we deduce the result.

3. Here, we choose K as in (3.47) and $\mu \geq 2\lambda C|\Omega|$. We deduce the result also by the same argument in (1).

□

We end this section by giving a result on global existence of the solution according to the range of the exponents p, k, s and the parameter λ, and to the size of u_0 and the Lebesgue measure of Ω.

Theorem 3.3.5. *Let $k, s \geq 1$. We assume that one of the following conditions hold*

1. *$k + s < p - 1$.*
2. *$k + s > p - 1$ and $\|u_0\|_\infty$ is small enough.*
3. *$k + s = p - 1$ and $|\Omega|$ (or λ) is small enough.*

Then, the weak solution of the problem $\mathcal{P}$ is uniformly bounded in L^∞ norm.

Proof. We consider the following problem

$$\frac{\partial v}{\partial t} - \Delta_p v = \lambda v^k \int_\Omega v^s dx, \tag{3.48}$$

with $v = 0$ on $\partial\Omega \times (0, T_{max})$ and $v_0 = u_0$ in Ω. Then, by the comparison principle we obtain that $u \le v$ in $\Omega \times (0, T_{max})$. On the other hand, let Ω' be a bounded smooth domain such that $\Omega \subset\subset \Omega'$ and ψ be the unique solution of the following elliptic problem

$$\begin{cases} -\Delta_p \psi = 1, & \text{in } \Omega', \\ \psi = 0, & \text{on } \partial\Omega'. \end{cases} . \tag{3.49}$$

It has been proved in [41, 95] that $\psi > 0$ in Ω' and $\frac{\partial \psi}{\partial n} < 0$ where n is the outward unit normal vector. We set $C = \sup_{x \in \overline{\Omega'}} \psi(x)$ and $\delta = \inf_{x \in \overline{\Omega}} \psi(x)$. From the strong maximum principle, it is easy to see that we have $\delta > 0$. Now, we define $w(x,t) = K\frac{\psi(x)}{C}$, where K is a positive constant. By simple calculations, we have

$$\begin{aligned} P(w(x,t)) &= \frac{\partial w}{\partial t} - \Delta_p w - \lambda w^k \int_\Omega w^s dx \\ &= \left(\frac{K}{C}\right)^{p-1} - \lambda (\frac{K}{C})^{k+s} \psi^k \int_\Omega \psi^s dx \\ &\ge \frac{K^{p-1}}{C^{p-1}} - \lambda |\Omega| K^{k+s}, \end{aligned} \tag{3.50}$$

1. For $k + s < p - 1$, we can choose K such that

   $$K = \max \left\{ [\lambda |\Omega| C^{p-1}]^{\frac{1}{p-(1+k+s)}}, \frac{C \|u_0\|_\infty}{\delta} \right\}.$$

   Directly, we deduce that $P(w(x,t)) \ge 0$ and $w(x,0) \ge u_0$ in Ω. Moreover, it is clear that $w(x,t) \ge 0$ on $\partial\Omega \times (0,\infty)$. Hence, by the comparison principle we deduce that $0 \le u(x,t) \le v(x,t) \le w(x,t) \le K$, which yields the result.

2. For $k + s > p - 1$, in order to have $P(w(x,t)) \ge 0$, we can choose K such that

   $$K \le \left[\frac{1}{\lambda |\Omega| C^{p-1}}\right]^{\frac{1}{k+s-p+1}}.$$

   Furthermore, using the fact that u_0 is small enough we can have that

   $$K \ge C\frac{\|u_0\|_\infty}{\delta}.$$

   It is clear that $w(x,t) \ge 0$ on $\partial\Omega \times (0,\infty)$. Therefore, by comparison principle we deduce the result.

3. For $k + s = p - 1$ to have $P(w(x,t)) \ge 0$, we use the fact that Ω is small enough. Then, from (3.50) we can assume that

   $$|\Omega| \le \frac{1}{\lambda C^{p-1}} \quad (\text{ or } \lambda \le \frac{1}{|\Omega| C^{p-1}}).$$

   By choosing $K = C\frac{\|u_0\|_\infty}{\delta}$, we obtain $w(x,0) \ge u_0$. Hence, as for (2) we deduce the result.

□

Remark 3.3.6. *We refer to [47, 69] and the references therein for the existence and comparison principle for problem* (3.48).

3.4 Extinction in finite time

In this section, we establish some results concerning the extinction of the solution in finite time. In this setting, we use two different methods. Firstly, we use the L^2-integral norm estimate method which has been used in [6, 11, 38] and the references therein. The second method is based on the construction of suitable super-solutions and the comparison principle given in proposition (3.3.1).

Before stating the first result, we recall the following Gagliardo-Nirenberg type inequality which can be found in [11, 70], and the following technical lemma (lemma 3.4.3). They are the keys of our proof.

Lemma 3.4.1. *Let $1 < p < \infty$ and $\alpha \in [\beta+1, \infty)$ if $p \geq N$, and $\alpha \in [\beta+1, \frac{Np}{N-p}]$ if $p < N$. Then, there exists a constant $C > 0$, depending only on p, α, β, N and $|\Omega|$, such that for every $u \in W_0^{1,p}(\Omega)$*

$$\|u\|_{L^\alpha(\Omega)} \leq C \|\nabla u\|^\theta_{L^p(\Omega)} \|u\|^{1-\theta}_{L^{\beta+1}(\Omega)},$$

where $\theta = \frac{\frac{1}{\beta+1} - \frac{1}{\alpha}}{\frac{1}{N} - \frac{1}{p} + \frac{1}{\beta+1}} \in [0, 1]$.

Remark 3.4.2. *It is clear that from the value of θ with $\alpha > \beta + 1$ the following inequality*

$$\alpha\left(\frac{\theta}{p} + \frac{1-\theta}{\beta+1}\right) > 1$$

holds. It will play a crucial role for establishing a desired ordinary differential inequality, hereafter.

Lemma 3.4.3. *Let $p \geq 2$ and $1 > q + l$. We assume that $p - 1 > l + q$ and $1 > q + l(p-1)$. Then, there exist $a \geq 1$ and $b > 0$ with $q + l + 1 < a + b < 2$ and $\nu > 1$ such that for all $x, y \geq 0$, $\epsilon > 0$, we have*

$$x^a y^b \leq \frac{\epsilon^\nu}{\nu} x^q y^{l+1} + \frac{\epsilon^{-\nu'}}{\nu'} x^p, \tag{3.51}$$

where $\frac{1}{\nu} + \frac{1}{\nu'} = 1$.

Proof. To prove the inequality (3.51), it is sufficient to prove that

$$y^b \leq \frac{\epsilon^\nu}{\nu} x^{q-a} y^{l+1} + \frac{\epsilon^{-\nu'}}{\nu'} x^{p-a}, \quad \forall x, y > 0. \tag{3.52}$$

Let $\gamma \in \mathbb{R}$ and $\nu > 1$. By Young's inequality, we have

$$\begin{aligned} y^b &= \left(\frac{\epsilon y^b}{x^\gamma}\right)\left(\frac{x^\gamma}{\epsilon}\right) \\ &\leq \frac{1}{\nu}\left(\frac{\epsilon y^b}{x^\gamma}\right)^\nu + \frac{1}{\nu'}\left(\frac{x^\gamma}{\epsilon}\right)^{\nu'} \end{aligned}$$

where, $\frac{1}{\nu} + \frac{1}{\nu'} = 1$. Therefore, to get the inequality (3.52), it suffices to have $b\nu = l + 1$, $\gamma\nu = a - q$ and $\gamma\nu' = p - a$, which imply that $a = \frac{p(\nu-1)+q}{\nu}$ and $b = \frac{l+1}{\nu}$.
If $p = 2$, we choose $\frac{p-q}{p-1} < \nu$ and if $2 < p$ we choose $\frac{p-q}{p-1} < \nu < \frac{p-1-q-l}{p-2}$. Then, using the fact that $1 > q + l(p-1)$ and $1 > q + l$ we can easily see in all cases that $q + l + 1 < a + b < 2$. □

The following theorem is one of the main results of this section.

Theorem 3.4.4. *Assume that $1 \leq k+s \leq p-1$, $l+q<1$ and $1>q+l(p-1)$. Then, there exists $T_e>0$ such that $u=0$ in $\Omega \times [T_e, \infty)$ provided that λ or $|\Omega|$ is small enough.*

Proof. Multiplying the first equation in problem $\mathcal{P}$ by the test function u, integrating over Ω and using Holder's inequality then yields

$$\begin{aligned} \frac{1}{2}\int_\Omega \frac{\partial u^2}{\partial t}dx + \int_\Omega |\nabla u|^p dx &= \lambda \int_\Omega u^{k+1}dx \int_\Omega u^s dx - \mu \int_\Omega u^{l+1}|\nabla u|^q dx \\ &\leq \lambda|\Omega| \int_\Omega u^{k+s+1}dx - \mu \int_\Omega u^{l+1}|\nabla u|^q dx. \end{aligned} \tag{3.53}$$

From lemma (3.4.3), there exist $a \geq 1$ and $b>0$ such that $q+l+1 \leq a+b<2$. Then, by using Poincaré's inequality we get

$$\begin{aligned} -\mu \int_\Omega u^{l+1}|\nabla u|^q dx &\leq -\frac{\mu\nu}{\epsilon^\nu}\int_\Omega u^b|\nabla u|^a dx + \frac{\mu\nu}{\nu'\epsilon^{\nu+\nu'}}\int_\Omega |\nabla u|^p dx \\ &= -\frac{\mu\nu}{\epsilon^\nu}\int_\Omega |u^{\frac{b}{a}}\nabla u|^a dx + \frac{\mu\nu}{\nu'\epsilon^{\nu+\nu'}}\int_\Omega |\nabla u|^p dx \\ &\leq -\frac{C_1\mu\nu}{\epsilon^\nu}\int_\Omega u^{a+b} dx + \frac{\mu\nu}{\nu'\epsilon^{\nu+\nu'}}\int_\Omega |\nabla u|^p dx. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \int_\Omega u^{k+s+1}dx &= \int_{\{u\geq 1\}} u^{k+s+1}dx + \int_{\{u\leq 1\}} u^{k+s+1}dx \\ &\leq \int_{\{u\geq 1\}} u^p dx + \int_{\{u\leq 1\}} u^{a+b}dx \\ &\leq C_2\int_\Omega |\nabla u|^p dx + \int_\Omega u^{a+b}dx. \end{aligned}$$

Hence, form (3.53) it follows that

$$\frac{1}{2}\int_\Omega \frac{\partial u^2}{\partial t}dx + \left(1 - C_2\lambda|\Omega| - \frac{\mu\nu}{\nu'\epsilon^{\nu+\nu'}}\right)\int_\Omega |\nabla u|^p dx \leq \left(\lambda|\Omega| - \frac{C_1\mu\nu}{\epsilon^\nu}\right)\int_\Omega u^{a+b}dx. \tag{3.54}$$

By choosing ϵ such that $\epsilon = \left(\frac{\mu}{\lambda|\Omega|}\right)^{\frac{1}{\nu+\nu'}}$, we deduce from (3.54) that

$$\begin{aligned} &\frac{1}{2}\int_\Omega \frac{\partial u^2}{\partial t}dx + \left(1 - C_2\lambda|\Omega| - \frac{\lambda|\Omega|\nu}{\nu'}\right)\int_\Omega |\nabla u|^p dx \\ &\quad + (\lambda|\Omega|)^{\frac{\nu}{\nu+\nu'}}\left(C_1\mu^{\frac{\nu'}{\nu+\nu'}}\nu - (\lambda|\Omega|)^{\frac{\nu'}{\nu+\nu'}}\right)\int_\Omega u^{a+b}dx \leq 0. \end{aligned} \tag{3.55}$$

On the other hand, if we choose λ such that

$$\lambda < \min\left\{\frac{\nu'}{|\Omega|(C_2\nu'+\nu)}, \frac{(\mu^{\frac{\nu'}{\nu+\nu'}}C_1\nu)^{1+\nu/\nu'}}{|\Omega|}\right\},$$

or $|\Omega|$ small enough such that

$$|\Omega| < \min\left\{\frac{\nu'}{\lambda(C_2\nu'+\nu)}, \frac{(\mu^{\frac{\nu'}{\nu+\nu'}}C_1\nu)^{1+\nu/\nu'}}{\lambda}\right\},$$

then, under the consideration of one of the previous cases and by setting

$$0 < K = \min\left\{1 - C_2\lambda|\Omega| - \frac{\lambda|\Omega|\nu}{\nu'}, (\lambda|\Omega|)^{\frac{\nu}{\nu+\nu'}}(C_1\mu^{\frac{\nu'}{\nu+\nu'}}\nu - (\lambda|\Omega|)^{\frac{\nu'}{\nu+\nu'}})\right\},$$

it follows from (3.55) that

$$\frac{1}{2}\int_\Omega \frac{\partial u^2}{\partial t}dx + K\left[\int_\Omega |\nabla u|^p dx + \int_\Omega u^{a+b}dx\right] \leq 0. \tag{3.56}$$

Now, our goal is to get the extinction result in $L^2(\Omega)$ norm.
Since $p \geq 2$, then we have $p \geq \frac{2N}{N+2}$ and $\frac{Np}{N-p} \geq 2$, when $p < N$. Let us choose $\alpha = 2 \in [a+b, \frac{Np}{N-p}]$ when $p < N$ and $\alpha = 2 \in [a+b, \infty)$ if $p > N$. Then, from Gagliardo-Nirenberg's inequality we obtain

$$\begin{aligned} y(t) = \|u\|^2_{L^2(\Omega)} &\leq \left(C\|\nabla u\|^\theta_{L^p(\Omega)}\|u\|^{1-\theta}_{L^{a+b}(\Omega)}\right)^2 \\ &\leq C^2\left(\int_\Omega(|\nabla u|^p + u^{a+b})dx\right)^{2(\frac{\theta}{p}+\frac{1-\theta}{a+b})}. \end{aligned}$$

We set $\gamma = 2(\frac{\theta}{p} + \frac{1-\theta}{a+b}) > 1$. From (3.56), and we deduce that

$$y'(t) + \kappa y^{1/\gamma}(t) \leq 0,$$

where $\kappa = \frac{2K}{C^{2/\gamma}}$. Integrating the last inequality on $[0,t]$, we get

$$y(t) \leq \left(y^{1-1/\gamma}(0) - \kappa(1-\frac{1}{\gamma})t\right)^{\frac{\gamma}{\gamma-1}},$$

which yields that

$$y(t) \to 0 \quad t \to T_e = \frac{y^{1-1/\gamma}(0)}{\kappa(1-\frac{1}{\gamma})}.$$

Thus, we conclude the desired result. □

Remark 3.4.5. *The previous theorem almost similarly is obtained in [70] in the case where the problem $\mathcal{P}$ contains only the local term (i.e $s = 0$ and $\lambda = \frac{1}{|\Omega|}$) and without gradient term (i.e $q = 0$).*

Different from the last theorem, the following result shows that finite time extinction can also occur if we suppose, among other conditions, that the initial data is small enough .

Theorem 3.4.6. *Let $1 < p < 2$, $q \geq \frac{p}{2}$ and $\|u_0\|_\infty$ small enough. Assume that one of the following conditions hold*

1. *if $q + l < k + s$,*
2. *if $q + l = k + s$ and μ is large enough.*
3. *if $q + l = k + s$ and λ or $|\Omega|$ is small enough.*

Then, there exists $T_e > 0$, such that $u = 0$ in $\Omega \times [T_e, \infty)$.

Proof. We consider the following smooth function

$$v(x,t) = K(T_e - t)_+^{\alpha} \ln(l + x_1 + ... + x_n)$$

where

$$l = \sup_{(x_1,...,x_n)\in\Omega} \{|x_1| + ... + |x_n|\} + 2, \tag{3.57}$$

$$T_e = \left(\frac{\|u_0\|_\infty}{K\ln(2)}\right)^{1/\alpha}, \tag{3.58}$$

and

$$\alpha = \frac{1}{2-p}, \tag{3.59}$$

where $K > 0$ will be specified later. Our goal is to prove that v is a super-solution of $\mathcal{P}$ and by comparison principle (3.3.1), we can deduce the result.
Firstly, it is clear that $v(x,0) \geq \|u_0\|_\infty \geq u_0(x)$, for a.e $x \in \Omega$, and $v(x,t) \geq 0$, for all $x \in \partial\Omega$, $t \geq 0$.
Next, we prove that

$$\frac{\partial v}{\partial t} - \Delta_p v \geq \lambda v^k \int_\Omega v^s dx - \mu v^l |\nabla v|^q, \quad \text{in } Q_T. \tag{3.60}$$

By simple calculations, we have

$$\frac{\partial v}{\partial t} = -K\alpha(T_e - t)^{\alpha-1} ln(l + x_1 + ... + x_n),$$

and since

$$|\nabla v| = K(T_e - t)_+^{\alpha} \frac{N^{1/2}}{l + x_1 + ... + x_n},$$

then, we get

$$-\Delta_p v = \frac{(K(T_e - t)_+^{\alpha})^{p-1} N^{p/2}}{(l + x_1 + ... + x_n)^p}(p-1). \tag{3.61}$$

Hence, to get (3.60) it suffices to have that

$$\begin{aligned} &\frac{(K(T_e - t)_+^{\alpha})^{p-1} N^{p/2}}{(l + x_1 + ... + x_n)^p}(p-1) + \mu(K(T_e - t)_+^{\alpha})^{l+q} \frac{N^{q/2}(\ln(l + x_1 + ... + x_n))^l}{(l + x_1 + ... + x_n)^q} \\ &\geq \lambda(K(T_e - t)_+^{\alpha})^{k+s}(\ln(l + x_1 + ... + x_n))^k \int_\Omega (\ln(l + x_1 + ... + x_n))^s dx \\ &\quad + K\alpha(T_e - t)^{\alpha-1} ln(l + x_1 + ... + x_n), \end{aligned} \tag{3.62}$$

which hold if

$$\frac{(K(T_e - t)_+^{\alpha})^{p-1} N^{p/2}}{(2l)^p}(p-1) \geq K\alpha(T_e - t)^{\alpha-1} ln(2l) \tag{3.63}$$

and

$$\mu(K(T_e - t)_+^{\alpha})^{l+q} \frac{N^{q/2}(\ln(2))^l}{(2l)^q} \geq \lambda(K(T_e - t)_+^{\alpha})^{k+s}(\ln(2l))^{k+s}|\Omega|. \tag{3.64}$$

Otherwise, since $\|u_0\|_\infty$ is small enough, then we can assume that $\|u_0\|_\infty \leq K\ln(2)$. Then $T_e \leq 1$, which implies that $(T_e - t)_+ \leq 1$. Hence

1. If $q + l < k + s$, therefore to obtain (3.63) and (3.64) we need

$$\frac{(p-1)N^{p/2}}{\alpha \ln(2l)(2l)^p} \geq K^{2-p} \tag{3.65}$$

and

$$\frac{\mu N^{q/2}(\ln(2))^l}{\lambda|\Omega|(\ln(2l))^{k+s}(2l)^q} \geq K^{k+s-l-q}. \tag{3.66}$$

Thus, by setting that $E = \frac{(p-1)N^{p/2}}{\alpha \ln(2l)(2l)^p}$ and $F = \frac{\mu N^{q/2}(\ln(2))^l}{\lambda|\Omega|(\ln(2l))^{k+s}(2l)^q}$, we can choose K satisfying that

$$K = \min\{E^{\frac{1}{2-p}}, F^{\frac{1}{k+s-l-q}}\}.$$

2. If $q + l = k + s$, by the same manner used above we get (3.63) simply by choosing K such that

$$K = E^{\frac{1}{2-p}}.$$

While we can see easily that (3.64) holds if

$$\mu \frac{N^{q/2}(\ln(2))^l}{(2l)^q} \geq \lambda(\ln(2l))^{k+s}|\Omega|, \tag{3.67}$$

which is satisfied if we choose μ such that

$$\mu \geq \frac{\lambda(\ln(2l))^{k+s}|\Omega|(2l)^q}{N^{q/2}(\ln(2))^l}.$$

Then, (3.64) follows.

3. If we choose λ or $|\Omega|$ satisfying respectively

$$\mu \frac{N^{q/2}(\ln(2))^l}{(\ln(2l))^{k+s}|\Omega|(2l)^q} \geq \lambda,$$

or

$$\mu \frac{N^{q/2}(\ln(2))^l}{\lambda(\ln(2l))^{k+s}(2l)^q} \geq |\Omega|,$$

then, each of the last inequalities imply directly (3.64), which allow us to complete the proof. □

CHAPTER 4

QUALITATIVE PROPERTIES OF NONNEGATIVE SOLUTIONS FOR A DOUBLY NONLINEAR PROBLEM WITH VARIABLE EXPONENTS

We consider the Dirichlet initial boundary value problem

$$\partial_t u^{m(x)} - div(|\nabla u|^{p(x,t)-2}\nabla u) = a(x,t)u^{q(x,t)}$$

where the exponents $p(x,t) > 1$, $q(x,t) > 0$ and $m(x) > 0$ are given functions. We assume that $a(x,t)$ is a bounded function. The aim of this paper is to deal with some qualitative properties of the solutions. Firstly, we prove that if $ess\sup p(x,t) - 1 < ess\inf m(x)$ then any weak solution will be extinct in finite time when the initial data is small enough. Otherwise, when $ess\sup m(x) < ess\inf p(x,t) - 1$, we get the positivity of solutions for large t. In the second part, we investigate the property of propagation from the initial data. For this purpose, we give a precise estimation of the support of the solution under the conditions that $ess\sup m(x) < ess\inf p(x,t) - 1$ and, either $q(x,t) = m(x)$ or $a(x,t) \leq 0$ .a.e. Finally we give a uniform localization of the support of solutions for all $t > 0$, in the case where $a(x,t) < a_1 < 0$ .a.e and $ess\sup q(x,t) < ess\inf p(x,t) - 1$.

4.1 Introduction

This chapter is devoted to studying qualitative properties of nonnegative weak solutions for the following doubly nonlinear parabolic problem with variable exponents

$$\mathcal{P} : \begin{cases} \dfrac{\partial b(x,u)}{\partial t} - \Delta_{p(x,t)} u = a(x,t)|u|^{q(x,t)-1}u & \text{in } Q_T = \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ b(x,u\,(x,0)) = b(x,u_0\,(x)) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, $b(x,u) = |u|^{m(x)-1}u$ and $\Delta_{p(x,t)}u$ is defined as

$$\Delta_{p(x,t)}u = \text{div}\left(|\nabla u|^{p(x,t)-2}\,\nabla u\right).$$

The exponents p, q, m and the coefficient a are given measurable functions. It will be assumed throughout the chapter that these functions satisfy some specific conditions.

Problems of this form appear in various applications; for instance, in models for gas or fluid flow in porous media [7, 15] and for the spread of certain biological populations [55]. Our motivation to study the problem $\mathcal{P}$ with variable exponents is the fact that it is considered as a model of an important class of non-Newtonian fluids which are well-known as electrorheological fluids, see ([82]). It appears also as a model in image restoration ([29]) and in elasticity ([97]).

It is well known that solutions of problems such as $\mathcal{P}$ exhibit various qualitative properties, which reflect natural phenomena, according to certain conditions on $p(x,t)$, $q(x,t)$, $m(x)$, $a(x,t)$ and u_0, (see for example [3, 4, 10, 12, 11, 56, 57] and the references therein). Among the phenomena that interest us in this work is the finite speed of propagation, which means that if $\rho_0 > 0$ is such that supp $(u_0) \subset B(x_0, \rho_0)$ then supp $(u(x,t)) \subset B(x_0, \rho(t))$, for any $t \in (0,T)$, where $\rho(t)$ is a positive function which depends on ρ_0, (i.e. solutions with compact support). This property has various physical meanings; for instance in the study of turbulent filtration of gas through porous media, a solution with compact support means that gas will remain confined to a bounded region of space, see [24].

The phenomenon of finite speed of propagation was investigated by Kalashnikov in [66]. He considered, for $N = 1$ the equation $\frac{\partial b(u)}{dt} - \Delta u = 0$ in $\mathbb{R} \times (0, \infty)$ and,under specific conditions, proved that if the initial condition u_0 has a compact support, then the condition $\int_{0^+} \frac{1}{b(s)} ds < +\infty$ is necessary and sufficient for solutions to have compact support. This result was extended by Dìaz for $N \geq 1$, in [36]. Later, in [39] Dìaz and Hernández considered the doubly nonlinear problem with absorption term $\frac{\partial b(u)}{\partial t} - \Delta_p u + |u|^{q-1}u = 0$, in $\mathbb{R}^N \times (0, \infty)$ where $b(u) = |u|^{m-1}u$. Under the assumption that u_0 has a compact support and $0 < q < p - 1$, they proved that any solution has a compact support for all $t > 0$. This result was obtained by the construction of a local uniform super-solution. Let us recall that the finite speed of propagation phenomenon has been studied by many authors in the last decades, (see [6, 37, 40, 41]).

Besides, extinction and non-extinction are also important properties for solutions of evolution equations that have attracted many authors in the last few decades. Most of them focused on equations with constant exponents of nonlinearity, (see [32, 38, 48, 65, 54]). For example, Hong et al, dealt in [91] with the homogeneous equation $u_t - \Delta_p u^m = 0$, in $\Omega \times (0, \infty)$, where $p > 1$ and $m > 0$. They proved that the condition $1 < p < 1 + \frac{1}{m}$ is necessary and sufficient for extinction to occur. Moreover, Zhou and Mu [100] studied the extinction behavior of weak solutions for the equation with source term $u_t - \Delta_p u^m = \lambda u^q$, in $\Omega \times (0, \infty)$, where $p > 1$, $m, q, \lambda > 0$ and $m(p-1) < 1$. They proved that $q = m(p-1)$, is a critical extinction exponent.

Otherwise, it is worth noting that problem $\mathcal{P}$ has been treated by Antontsev and Shamarev in several papers. In ([8, 9]), they proved the existence of weak and strong solutions. Moreover, under certain regularity hypotheses on $m(x)$, $p(x,t)$, and under the sign condition $a(x,t) \leq 0$ a.e, they studied properties of finite speed of propagation and extinction in finite time in ([10, 12]). Their results were established by using the local energy method. Here, we shall use the so-called method of sub and super-solutions to extend some of the results in ([10, 12]). To the best of our knowledge, there is few results concerning the study of qualitative properties for parabolic equations with variable exponents by using this method. Furthermore, we shall also extend to the parabolic case, some of the results by Zhang et al. in [96], where radial sub and super-solutions for some elliptic problems with variable exponents are constructed; and some of the results by Chung and Park in [32] and by Yuan et al. in [91], to variable exponents case . In fact, we shall exploit their arguments in our parabolic problem setting with less conditions on the exponents $p(x,t)$, $q(x,t)$, $m(x)$ and the coefficient $a(x,t)$.

The present chapter is organized as follows. In section 4.2, we give assumptions and general definitions; then, we establish a comparison principle which ensures the uniqueness of solutions. In section 4.3, we investigate the extinction and non-extinction properties for the solution of $\mathcal{P}$. Finally in section 4.4, we study the property of finite speed of propagation.

4.2 Assumptions and results

Firstly we refer the readers to Chapter 1 for some elementary results for the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and Sobolev spaces $W^{1,p(x)}(\Omega)$, where is Ω a bounded set of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary.

Let $m(x) > 0$ and $p(x,t) > 1$ be given functions. For $T > 0$ fixed, we denote $Q_T = \Omega \times (0,T)$. Let $m \in C^0(\overline{\Omega})$, we assume that $p(x,t) \in C_+(Q_T)$ satisfies the following log-Hölder condition in Q_T,

$$\forall (x,t),(y,\tau) \in Q_T, \quad \text{such that } |(x,t)-(y,\tau)| = \sqrt{|x-y|^2 + |t-\tau|^2} < 1,$$

we have

$$|p(x,t) - p(y,\tau)| \leq \omega(|(x,t)-(y,\tau)|, \tag{4.1}$$

where ω satisfies

$$\limsup_{\tau \to 0^+} \omega(\tau) ln(\frac{1}{\tau}) < \infty.$$

For every fixed $t \in [0,T]$, we introduce the following Banach space,

$$V_t(\Omega) = \left\{ u \in L^{m(x)+1}(\Omega) \cap W_0^{1,1}(\Omega) : |\nabla u| \in L^{p(x,t)}(\Omega) \right\},$$

endowed with the norm

$$\|u\|_{V_t(\Omega)} = \|u\|_{m(x)+1,\Omega} + \|\nabla u\|_{p(x,t),\Omega}.$$

We denote by $W(Q_T)$ the following Banach space,

$$W(Q_T) = \left\{ u \in L^{m(x)+1}(Q_T) : \ |\nabla u|^{p(x,t)} \in L^1(Q_T), \ u(.,t) \in V_t(\Omega) \text{ a.e. } t \in (0,T) \right\},$$

endowed with the norm

$$\|u\|_{W(Q_T)} = \|u\|_{m(x)+1,Q_T} + \|\nabla u\|_{p(x,t),Q_T}.$$

We denote by $W'(Q_T)$ the dual of $W(Q_T)$.

Throughout this chapter we assume that the coefficients and the exponents of nonlinearity satisfy the following conditions,

$$\begin{cases} \text{there exist positive constants } p^{\pm}, q^{\pm}, m^{\pm} \text{ and } a_1 \text{ such that, for any } (x,t) \text{ in } Q_T \\ 1 < p^- \leq p(x,t) \leq p^+, \ \ 0 < q^- \leq q(x,t) \leq q^+, \\ 0 < m^- \leq m(x) \leq m^+, \ \ |a(x,t)| \leq a_1, \end{cases} \tag{4.2}$$

and the initial data u_0 satisfies

$$u_0 \in L^\infty(\Omega), \quad u_0 \geq 0 \ \text{ a.e. in } \Omega. \tag{4.3}$$

Now, let us state the definition of weak solutions for the problem $\mathcal{P}$.

Definition 4.1. *We say that $u(x,t)$ is a super-(sub)solution of $\mathcal{P}$ on Q_T if*

1. *$u \in L^{\infty}(Q_T) \cap W(Q_T)$ and $\frac{\partial}{\partial t} b(x,u) \in W'(Q_T)$.*

2. *for every nonnegative test-function $\phi \in W(Q_T)$ and $\frac{\partial}{\partial t}\phi \in W'(Q_T)$, we have*

$$\int_{Q_T} \phi \frac{\partial}{\partial t} b(x,u) + |\nabla u|^{p(x,t)-2} \nabla u \nabla \phi dxdt \geq (\leq) \int_{Q_T} a(x,t)|u|^{q(x,t)-1} u\phi dxdt.$$

3. *$b(x, u(.,0)) \geq (\leq) b(x,u_0)$ a.e. in Ω, and $u \geq (\leq) 0$ on $\partial\Omega \times (0,T)$.*

A function u is a weak solution of $\mathcal{P}$ if it is simultaneously a super-solution and a sub-solution.

The following result concerning the local existence of weak solutions of problem $\mathcal{P}$ is established in [8].

Theorem 4.2.1. *Let $m \in C^0(\Omega)$, $p(x,t)$ satisfies the log-Hölder condition in Q_T* (4.1), *and let conditions* (4.2) *and* (4.3) *be fulfilled. Moreover, we assume that*

$$|\nabla(\frac{1}{m(x)})| \in L^{\beta}(\Omega), \text{ with some } \beta > 1,$$

and the exponents m, p satisfy one of the following conditions

1. *p is independent of t, and $m(x) > 0$ in Ω.*

2. *$p(x,t) > 1$, $m(x) > 1$, and $|\nabla(\frac{1}{m(x)})| \in L^{p(x,t)}(Q_T)$.*

3. *$p(x,t) > 1$, $m(x) > 0$, $|\nabla(\frac{1}{m(x)})| \in L^{p(x,t)}(Q_T)$, and*

$$1 > \frac{1}{p(x,t)} + m(x) \text{ in } Q_T.$$

Then, the problem $\mathcal{P}$ has at least one nonnegative weak solution in Q_{T^*}, with

$$T^* = \sup\{\theta : \|u(t)\|_{\infty,\Omega} < \infty, \ \ \forall t \in (0,\theta)\}.$$

Moreover, for small τ the solution satisfies the estimate

$$\|u\|_{\infty,\Omega} \leq \|u_0\|_{\infty,\Omega} e^{At}, \ \ t \in [0,\tau].$$

with a constant A depending only on the data.

The following comparison principle is essential to prove uniqueness and qualitative properties of nonnegative solutions.

Proposition 4.2.2. *Let u (respectively v) be a sub-solution (respectively super-solution) of $\mathcal{P}$, with the initial datum u_0 (respectively v_0), satisfying* (4.3). *We assume that $\frac{\partial}{\partial t} b(x,u)$, $\frac{\partial}{\partial t} b(x,v) \in L^1(Q_T)$, and that conditions* (4.2) *are fulfilled. If either $a(x,t) \leq 0$ a.e. in Q_T,or $m^+ \leq q^-$, then we have $u \leq v$ a.e. in Q_T.*

Remark 4.2.3. *Note that the comparison principle is true for weak solutions u with $\frac{\partial}{\partial t} b(x,u) \in L^1(Q_T) \cap W'(Q_T)$ and recall that in the papers ([8, 9]), the authors gave some conditions on the data of problem $\mathcal{P}$ in order to ensure that this class of solutions is nonempty.*

Proof. We consider the test function $\phi_\eta = sign_\eta(u-v)$, where

$$sign_\eta(s) = \begin{cases} 1, & \text{if } s > \eta, \\ \frac{s}{\eta}, & \text{if } |s| \leq \eta, \\ 0, & \text{if } s < -\eta, \end{cases}$$

and $\eta > 0$ is small. It is easy to see that

$$\phi_\eta(s) \to sign^+(s) \quad \text{as } \eta \to 0,$$

where, $sign_+(s) = 1$, if $s > 0$, and $sign_+(s) = 0$, if $s \leq 0$. Moreover, we claim that for all $u, v \in W(Q_t)$ the function $\phi_\eta(u-v) \in W(Q_T)$. Indeed, it is clear for all $s \in \mathbb{R}$, $|\phi_\eta(s)| \leq 1$. Then, by proposition (1.3.3) of chapter 1 we get

$$\|\phi_\eta(u-v)\|_{m(x)+1,Q_T} < C(T). \tag{4.4}$$

On the other hand, we have

$$\begin{aligned}\int_{Q_T} |\nabla\phi_\eta(u-v)|^{p(x,t)} dxdt &= \int_0^T \int_{\{|u-v|\leq\eta\}} |\nabla\phi_\eta(u-v)|^{p(x,t)} dxdt \\ &\quad + \int_0^T \int_{\{u-v>\eta\}} |\nabla\phi_\eta(u-v)|^{p(x,t)} dxdt \\ &\quad + \int_0^T \int_{\{u-v<-\eta\}} |\nabla\phi_\eta(u-v)|^{p(x,t)} dxdt \\ &= \int_0^T \int_{\{|u-v|\leq\eta\}} |\frac{\nabla(u-v)}{\eta}|^{p(x,t)} dxdt \\ &\leq C(\eta)(\int_{Q_T} |\nabla u|^{p(x,t)} + |\nabla u|^{p(x,t)})dx < \infty,\end{aligned}$$

Since $u, v \in W(Q_T)$, then from proposition (1.3.3) we deduce that

$$\|\nabla\phi_\eta(u-v)\|_{p(x,t),Q_T} < \infty. \tag{4.5}$$

Thus combining (4.4) and (4.5) we conclude the claim. On the other hand, from the Definition 4.1, we obtain

$$\begin{aligned}&\int_{Q_T} \frac{\partial}{\partial t}(b(x,u) - b(x,v))\phi_\eta dxdt + \int_{Q_T} (|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v)\nabla\phi_\eta dxdt \\ &\leq \int_{Q_T} a(x,t)(|u|^{q(x,t)-1}u - |v|^{q(x,t)-1}v)\phi_\eta dxdt.\end{aligned}$$

Due to the monotonicity, we have

$$\int_{Q_T} (|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v)\nabla\phi_\eta dxdt \geq 0,$$

then

$$\int_{Q_T} \frac{\partial}{\partial t}(b(x,u) - b(x,v))\phi_\eta dxdt \leq \int_{Q_T} a(x,t)(|u|^{q(x,t)-1}u - |v|^{q(x,t)-1}v)\phi_\eta dxdt. \tag{4.6}$$

By Lebesgue's dominated convergence theorem we have

$$\lim_{\eta\to 0} \int_{Q_T} \frac{\partial}{\partial t}(b(x,u) - b(x,v))\phi_\eta dxdt = \int_{Q_T} \frac{\partial}{\partial t}(b(x,u) - b(x,v))_+ dxdt. \tag{4.7}$$

Now, we have

$$\begin{aligned}\int_{Q_T} a(x,t)(|u|^{q(x,t)-1}u - |v|^{q(x,t)-1}v)\phi_\eta dxdt &= \int_{Q_T} a(x,t)\int_0^1 \frac{\partial}{\partial s}(|su+(1-s)v|^{q(x,t)-1}\\ &\quad \times (su+(1-s)v))\phi_\eta dsdxdt\\ &= \int_{Q_T} a(x,t)\int_0^1 q(x,t)(|su+(1-s)v|^{q(x,t)-1}\\ &\quad \times (u-v))\phi_\eta dsdxdt.\end{aligned}$$

Then from (4.6) and (4.7), letting $\eta \to 0$, we obtain

$$\int_{Q_T} \frac{\partial}{\partial t}(b(x,u)-b(x,v))_+ dxdt \le \int_{Q_T} a(x,t)\int_0^1 q(x,t)|su+(1-s)v|^{q(x,t)-1}(u-v)_+ dsdxdt. \quad (4.8)$$

Hence, if $a(x,t) \le 0$ a.e. in Q_T, it follows that

$$\int_{Q_T} \frac{\partial}{\partial t}(b(x,u)-b(x,v))_+ dxdt \le 0.$$

Then, by Gronwall's lemma we deduce the desired result. Now, we continue the prove without any sign condition on $a(x,t)$. From (4.8), by using $m^+ \le q^-$ and the Lebesgue's dominated convergence theorem it follows that

$$\begin{aligned}\int_{Q_T} \frac{\partial}{\partial t}(b(x,u)-b(x,v))_+ dxdt \le& a_1 q^+ \int_{Q_T}\int_0^1 |su+(1-s)v|^{q(x,t)-m(x)}\\ &\times |su+(1-s)v|^{m(x)-1}(u-v)_+ dsdxdt\\ \le& C\int_{Q_T} (\|u\|_{\infty,Q_T} + \|v\|_{\infty,Q_T})^{q(x,t)-m(x)}\\ &\times [\int_0^1 |su+(1-s)v|^{m(x)-1}(u-v)_+ ds]dxdt\\ \le& C\lim_{\eta\to 0}\int_{Q_T}\int_0^1 |su+(1-s)v|^{m(x)-1}(u-v)\phi_\eta dsdxdt\\ \le& C\lim_{\eta\to 0}\int_{Q_T}\frac{1}{m(x)}\int_0^1 [\frac{\partial}{\partial s}|su+(1-s)v|^{m(x)-1}\\ &\times (su+(1-s)v)ds]\phi_\eta dxdt\\ \le& \frac{C}{m^-}\int_{Q_T}(b(x,u)-b(x,v))_+ dxdt,\end{aligned}$$

where C is depending on the sup-norms of u and v. Hence we deduce from Gronwall's lemma that,

$$\int_\Omega (b(x,u)-b(x,v))_+ dx \le e^{CT}\int_\Omega (b(x,u(x,0)) - b(x,v(x,0)))_+ dx,$$

which allows us to conclude the result. □

Definition 4.2. *We call $u(x,t)$ a strong solution of $\mathcal{P}$, if u is a weak solution and satisfies*

$$\frac{\partial}{\partial t} b(x,u) \in L^1(Q_T).$$

4.3 Finite time extinction and non-extinction

This section is devoted to studying the extinction and positivity properties for the nonnegative solutions of problem $\mathcal{P}$, without any sign condition on the coefficient $a(x,t)$, and according to the ranges of $p(x,t)$, $q(x,t)$ and $m(x)$. The proof of the results are based on the construction of suitable super and sub-solutions and on the use of the preceding comparison principle given in proposition 4.2.2 .

4.3.1 Finite time extinction.

We state and prove our main extinction result.

Theorem 4.3.1. *Let u be a strong solution of $\mathcal{P}$. Assume that $m^+ < q^-$, $p^+ - 1 < m^-$, $\sup_{Q_T} |\nabla p(x,t)| < \infty$ and $\|u_0\|_\infty$ is small enough. Then, there exists a finite time T_1 such that for all $t \geq T_1$*

$$u(x,t) = 0, \;\; a.e. \; x \in \Omega.$$

Proof. We consider the following function

$$v(x,t) = k(T_1 - t)_+^\alpha \ln(l + x_1 + ... + x_n)$$

where

$$l = \sup_{(x_1,...,x_n)\in\Omega} \{|x_1| + ... + |x_n|\} + 2, \tag{4.9}$$

$$T_1 = (\frac{\|u_0\|_\infty}{k\ln(2)})^{1/\alpha}, \tag{4.10}$$

and

$$\alpha = \frac{1}{m^- - p^+ + 1}, \tag{4.11}$$

where, $k > 0$ will be specified later. Our goal is to prove that v is a super-solution of $\mathcal{P}$ and by comparison principle, we can deduce the result. Firstly, we shall get that

$$v \in L^\infty(Q_T) \cap W(Q_T) \text{ and } \frac{\partial}{\partial t} v^{m(x)} \in W'(Q_T), \text{ for all } T > 0.$$

For all $x \in \Omega$ and $t > 0$, we have

$$k(T_1 - t)^\alpha ln(l + x_1 + ... + x_n) \leq kT_1^\alpha ln(2l) < \infty,$$

and

$$|\nabla u| = \frac{k(T_1 - t)^\alpha N^{1/2}}{l + x_1 + ... + x_n} \leq \frac{KT_1^\alpha N^{1/2}}{2} < \infty,$$

which implies that $v \in L^\infty(Q_T) \cap W(Q_T)$. Moreover, we have

$$\frac{\partial}{\partial t} v^{m(x)} = -\alpha m(x)(kln(l + x_1 + ... + x_n))^{m(x)}(T_1 - t)_+^{\alpha m(x)-1}, \tag{4.12}$$

then

$$|\frac{\partial}{\partial t} v^{m(x)}| \leq \alpha m^+ k^{m^-} \max\{T_1^{\alpha m^+ - 1}, T_1^{\alpha m^- - 1}\}(ln(2l))^{m^+},$$

and hence $\frac{\partial}{\partial t} v^{m(x)} \in L^\infty(Q_T)$. Due to the embedding

$$L^\infty(Q_T) = (L^1(Q_T))' \hookrightarrow W'(Q_T),$$

we get that $\frac{\partial}{\partial t} v^{m(x)} \in W'(Q_T)$.
On the other hand, it is clear that $v(x,0) \geq \|u\|_\infty \geq u_0(x)$, for a.e $x \in \Omega$, and $v(x,t) \geq 0$, for all $x \in \partial\Omega$, $t \geq 0$. Next, we prove that

$$\frac{\partial}{\partial t} v^{m(x)} - \Delta_{p(x,t)} v \geq a\,(x,t)\, v^{q(x,t)}, \quad \text{in } Q_T.$$

Since $|a(x,t)| \leq a_1$, it suffices to prove that

$$\frac{\partial}{\partial t} v^{m(x)} - \Delta_{p(x,t)} v \geq a_1 v^{q(x,t)}, \quad \text{in } Q_T. \tag{4.13}$$

By simple calculations we obtain

$$\begin{aligned} -\Delta_{p(x,t)} v =& \frac{1}{N^{1/2}} \Big(\frac{k(T_1-t)_+^\alpha N^{1/2}}{l+x_1+...+x_n}\Big)^{p(x,t)-1} \Big[\frac{(p(x,t)-1)N}{l+x_1+...+x_n} - \sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \ln\Big(\frac{k(T_1-t)_+^\alpha N^{1/2}}{l+x_1+...+x_n}\Big)\Big] \\ \geq& \frac{1}{N^{1/2}} \Big(\frac{k(T_1-t)_+^\alpha N^{1/2}}{l+x_1+...+x_n}\Big)^{p(x,t)-1} \Big[\frac{(p^- -1)N}{l+x_1+...+x_n} - \sup_{Q_T} |\nabla p| \frac{k(T_1-t)_+^\alpha N}{l+x_1+...+x_n}\Big] \\ \geq& \frac{(k(T_1-t)_+^\alpha)^{p(x,t)-1} N^{p(x,t)/2}}{(l+x_1+...+x_n)^{p(x,t)}} \Big[(p^- -1) - \sup_{Q_T} |\nabla p| k(T_1)^\alpha\Big] \\ =& \frac{(k(T_1-t)_+^\alpha)^{p(x,t)-1} N^{p(x,t)/2}}{(l+x_1+...+x_n)^{p(x,t)}} \Big[(p^- -1) - \sup_{Q_T} |\nabla p| \frac{\|u_0\|_\infty}{\ln(2)}\Big]. \end{aligned} \tag{4.14}$$

We set

$$S = (p^- - 1) - \sup_{Q_T} |\nabla p| \frac{\|u_0\|_\infty}{\ln(2)}.$$

If $\sup_{Q_T} |\nabla p| = 0$, then $S = p^- - 1 > 0$. Otherwise, since $\|u_0\|_\infty$ is small enough, then we can assume that $\|u_0\|_\infty < \frac{(p^- -1)ln(2)}{\sup_{Q_T} |\nabla p|}$, to deduce that $S > 0$.
Now, we are looking for conditions on k to get (4.13). Thanks to (4.12) and (4.14), it is sufficient to have

$$\alpha m(x)(k\ln(l+x_1+...+x_n))^{m(x)}(T_1-t)_+^{\alpha m(x)-1} \geq a_1(k(T_1-t)_+^\alpha \ln(l+x_1+...+x_n))^{q(x,t)}, \tag{4.15}$$

and

$$\frac{(k(T_1-t)_+^\alpha)^{p(x,t)-1} N^{p(x,t)/2}}{(l+x_1+...+x_n)^{p(x,t)}} S \geq 2\alpha m(x)(k\ln(l+x_1+...+x_n))^{m(x)}(T_1-t)_+^{\alpha m(x)-1}. \tag{4.16}$$

As $\|u_0\|_\infty$ is small enough, we can assume also that $\|u_0\|_\infty \leq k\ln(2)$. Then $T_1 \leq 1$, which implies that $(T_1-t)_+ \leq 1$. Since $\alpha = \frac{1}{m^- - p^+ + 1}$ and $q^- > m^+$. Thus (4.15) and (4.16) reduce to

$$(T_1-t)_+^{\alpha m(x)-1} \Big(\alpha m(x)(k\ln(l+x_1+...+x_n))^{m(x)} - a_1(k\ln(l+x_1+...+x_n))^{q(x,t)}\Big) \geq 0,$$

and

$$(T_1-t)_+^{\alpha(p(x,t)-1)} \left(\frac{k^{p(x,t)-1} N^{p(x,t)/2}}{(l+x_1+...+x_n)^{p(x,t)}} S - 2\alpha m(x)(k\ln(l+x_1+...+x_n))^{m(x)}\right) \geq 0,$$

which are satisfied if

$$k^{q(x,t)-m(x)} \leq \frac{\alpha m^-(ln(2))^{m^+}}{a_1(\ln(2l))^{q^+}},$$

and

$$k^{m(x)-p(x,t)+1} \leq \frac{SN^{p^-/2}}{2\alpha m^+(2l)^{p^+}(\ln(2l))^{m^+}}.$$

By setting $E = \frac{SN^{p^-/2}}{2\alpha m^+(2l)^{p^+}(ln(2l))^{m^+}}$ and $F = \frac{\alpha m^-(ln(2))^{m^+}}{a_1(\ln(2l))^{q^+}}$, we can choose k such that

$$k = \min\{(E)^{\frac{1}{m^- - p^+ + 1}}, (E)^{\frac{1}{m^+ - p^- + 1}}, (F)^{\frac{1}{q^- - m^+}}, (F)^{\frac{1}{q^+ - m^-}}\}.$$

Therefore, we get the desired result. □

Next, we will mention an extinction result where there no condition between the ranges of $p(x,t)$ and $m(x)$.

Theorem 4.3.2. *Let u be a strong solution of $\mathcal{P}$. Assume that $m^- > q^+$ and $a(x,t) \leq -c < 0$, $\sup_{Q_T} |\nabla p(x,t)| < \infty$ and $\|u_0\|_\infty$ is small enough. Then, there exists a finite time T_1 such that for all $t \geq T_1$*

$$u(x,t) = 0, \quad a.e.\ x \in \Omega.$$

Proof. To prove this proposition , we have only to consider the same super-solution $v(x,t)$ as in the proof of Theorem (4.3.1) but we choose here the constant $k = 1$, which means

$$v(x,t) = (T_1 - t)_+^\alpha \ln(l + x_1 + ... + x_n)$$

where

$$l = \sup_{(x_1,...,x_n)\in\Omega} \{|x_1| + ... + |x_n|\} + 2, \tag{4.17}$$

$$T_1 = (\frac{\|u_0\|_\infty}{\ln(2)})^{1/\alpha}, \tag{4.18}$$

and

$$\alpha = \frac{1}{m^- - q^+}. \tag{4.19}$$

We have already shown in Theorem (4.3.1) that $-\Delta_{p(x,t)} v \geq 0$. We claim that

$$\frac{\partial}{\partial t} v^{m(x)} \geq a(x,t) u^{q(x,t)},$$

by the same lines as in theorem (4.3.1). Since $q^+ < m^-$, it is therefore sufficient to have

$$(T_1 - t)_+^{\alpha q(x,t)} \left(c(\ln(l + x_1 + ... + x_n))^{q(x,t)} - \alpha m(x)(\ln(l + x_1 + ... + x_n))^{m(x)} \right) \geq 0,$$

which is satisfied if we choose

$$c = \frac{\alpha m^+(\ln(2l))^{m^+}}{(\ln(2))^{q^-}}.$$

Consequently, by the comparison principle we deduce the extinction of the solution in finite time. □

4.3.2 Non-extinction of solutions

The following theorem deals with the positivity of weak solutions.

Theorem 4.3.3. *Let u be a strong solution of $\mathcal{P}$, and u_0 not identically zero. Assume that $m^+ < q^-$, $m^+ < p^- - 1$, and $\sup_{Q_T} |\nabla p(x,t)| < \infty$. Then, there exists a finite time T_p such that for all $t \geq T_p$*

$$u(x,t) > 0, \quad a.e. \ x \in \Omega.$$

The method of proof is inspired from [91] where, the constant exponents case is treated. However, some difficulties arise in the construction of sub-solutions due to the fact that the exponents are variable. The proof of this theorem is divided in two lemmas. In the first lemma, we show by using a comparison function. that the support of weak solution is nondecreasing with respect to time. In the second lemma, we show that the solution is positive locally in Ω; then, by a finite covering argument, we deduce the result.

Lemma 4.3.4. *Let u be a strong solution of $\mathcal{P}$. Assume that $m^+ < q^-$ and the initial condition u_0 is nontrivial. Then*

$$supp\, u(.,s) \subset \ supp\, u(.,t),$$

for all $0 < s < t$.

The proof of lemma 4.3.4 follows the same lines as that of lemma 4.2 in [32] where, the constant exponents case is studied. For completeness, we shall give it here.

Proof. The argument used here is based on a comparison function which with we show that the support of solution is increasing. For that we consider ω an arbitrary set which is a nonzero measure subset of Ω such that $\inf_{x\in\omega} u_0 \neq 0$. We divide the proof in two cases, firstly we treat the case where $m^- \geq 1$, and then the case where $m^+ < 1$. If $m^- \geq 1$, we consider the following function

$$v_1(x,t) = \begin{cases} (\delta^{m^+-q^-} + (q^- - m^+)a_1 t)^{\frac{1}{m^+-q^-}}, & \text{in } \omega \times (0,\infty), \\ 0 & \text{on } \partial\omega \times (0;\infty), \end{cases}$$

where $\delta = \min\{\inf_{x\in\omega} u_0, 1\}$. It is clear that $v_1 \in L^\infty(Q_T) \cap W(Q_T)$, and it is easy to verify that

$$\frac{\partial}{\partial t} v_1^{m(x)} \in L^\infty(Q_T) \hookrightarrow W'(Q_T).$$

On the other hand, by direct calculations we get

$$\frac{\partial}{\partial t} v_1^{m^+} = -a_1 m^+ v_1^{q^-},$$

hence

$$v_1^{m^+-1} \frac{\partial v_1}{\partial t} = -a_1 v_1^{q^-}.$$

Since $m(x) \geq 1$ and $v_1(x,t) \leq 1$ for all $x \in \omega$, $t > 0$, it is follows that

$$\frac{\partial}{\partial t} v_1^{m(x)} - \Delta_{p(x,t)} v_1 - a(x,t) v_1^{q(x,t)} \leq 0, \quad x \in \omega, \quad t > 0.$$

Moreover, from the definition of v_1, we have $v_1(x,0) \leq u_0$, almost everywhere in ω, and $v_1(x,t) = 0$ for all, $x \in \partial\omega$, $t > 0$. Thus by comparison principle we conclude that for any arbitrary ω where $\inf_{x\in\omega} u_0 > 0$, the weak solution of $\mathcal{P}$ satisfies

$$u(x,t) > 0 \quad \text{a.e.} x \in \omega, \ \text{ and all } t > 0.$$

and the result follows in this case.

If $m^+ < 1$, we consider the following function

$$v_2(x,t) = \begin{cases} (\delta^{m^+-q^-} + \frac{(q^- - m^+)}{m^-} a_1 t)^{\frac{1}{m^+-q^-}}, & , \text{ in } \omega \times (0,\infty) \\ 0 & , \text{ on } \partial\omega \times (0;\infty) \end{cases}$$

where $\delta = \min\{\inf_{x\in\omega} u_0, 1\}$. By the same argument used previously we obtain that $v_2 \in L^\infty(Q_T) \cap W(Q_T)$, and $\frac{\partial}{\partial t} v_2^{m(x)} \in W'(Q_T)$. Moreover, by direct calculations we get

$$\frac{\partial}{\partial t} v_2^{m^+} = -\frac{a_1 m^+}{m^-} v_2^{q^-},$$

since $m^+ < 1$ and $v_2(x,t) \leq 1$ for all $x \in \omega$, $t > 0$, hence

$$\frac{\partial}{\partial t} v_2^{m(x)} + a_1 v_2^{q(x,t)} \leq 0,$$

Therefore, we have

$$\frac{\partial}{\partial t} v_2^{m(x)} - \Delta_{p(x,t)} v_2 - a(x,t) v_2^{q(x,t)} \leq 0, \quad x \in \omega, \quad t > 0.$$

Thus, by the same argument used previously we deduce our result. □

Lemma 4.3.5. *Under the same assumptions of theorem* (4.3.3). *Let the initial condition satisfies* $\inf_{x\in B_r(x_0)} u_0 \neq 0$, *for some* $0 < r < R \leq \frac{1}{2}$. *Then, there exists* $T_p > 0$, *such that for any* $t \geq T_p$,

$$u(x,t) > 0, \quad a.e. x \in B_R(x_0),$$

where $B_R(x_0) = \{x \in \Omega : |x - x_0| < R\}$.

Proof. We consider the following function

$$\psi(x,t) = krS^{\frac{-\alpha}{\lambda}}(t) H^{\frac{p^- -1}{\gamma}}(x,t),$$

where

$$H(x,t) = \left[1 - \left(\frac{|x-x_0|}{S^{\frac{1}{\lambda}}(t)}\right)^{\frac{p^-}{p^- -1}}\right]_+,$$

$$S(t) = r^{\frac{\lambda}{\alpha}} + \lambda\sigma k^\beta r^\gamma t, \quad 0 \leq t \leq T_p,$$

and

$$T_p = \frac{R^\lambda - r^{\frac{\lambda}{\alpha}}}{\lambda\sigma k^\beta r^\gamma},$$

where k and α are positive constants small and large enough respectively, γ is a positive constant such that

$$\gamma \leq \frac{p^- - 1}{p^+ - 1}(p^- - 1 - m^+), \tag{4.20}$$

and λ, γ, β are positive constants and will be determined later. By direct calculations, we get

$$|\nabla\psi| = \frac{krp^-}{\gamma} S^{-\frac{\alpha+1}{\lambda}}(t) H^{\frac{p^- -1}{\gamma}-1}(x,t) \left(\frac{|x-x_0|}{S^{\frac{1}{\lambda}}(t)}\right)^{\frac{1}{p^- -1}}, \tag{4.21}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\psi^{m(x)} =& -m(x)\sigma\alpha k^{\beta+m(x)}r^{\gamma+m(x)}S^{\frac{-\alpha m(x)}{\lambda}-1}(t)H^{m(x)\frac{p^--1}{\gamma}}(x,t)\\ &+m(x)\frac{\sigma p^-}{\gamma}k^{\beta+m(x)}r^{\gamma+m(x)}S^{\frac{-\alpha m(x)}{\lambda}-1}(t)H^{m(x)\frac{p^--1}{\gamma}-1}(x,t)\left(\frac{|x-x_0|}{S^{\frac{1}{\lambda}}(t)}\right)^{\frac{p^-}{p^--1}}.\end{aligned} \tag{4.22}$$

We can see easily that $H(x,t)\leq 1$ a.e in $B_R(x_0)\times(0,T_p)$, then by using (4.20), we obtain that

$$\psi\in L^\infty(B_R(x_0)\times(0,T_p))\cap W(B_R(x_0)\times(0,T_p)),$$

and

$$\frac{\partial}{\partial t}\psi^{m(x)}\in L^\infty(B_R(x_0)\times(0,T_p))\hookrightarrow W^{'}(B_R(x_0)\times(0,T_p)).$$

Moreover, from the definition of ψ we have

$$\psi(x,0)=kH^{\frac{p^--1}{\gamma}}(x,0)\leq k\leq u_0(x)\quad \text{a.e. } x\in B_R(x_0).$$

Since for all $t\in[0,T_p]$,

$$S^{\frac{1}{\lambda}}(t)\leq S^{\frac{1}{\lambda}}(T_p)=R,$$

then $H(x,t)=0$ on $\partial B_R(x_0)\times[0,T_p]$, which implies that $\psi(x,t)=0$ on $\partial B_R(x_0)\times[0,T_p]$. Now let us show that

$$\frac{\partial}{\partial t}\psi^{m(x)}-\Delta_{p(x,t)}\psi-a(x,t)\psi^{q(x,t)}\leq 0,\quad \text{in } B_R(x_0)\times(0,T_p).$$

To do so, it suffices to show that

$$\frac{\partial}{\partial t}\psi^{m(x)}-\Delta_{p(x,t)}\psi+a_1\psi^{q(x,t)}\leq 0,\quad \text{in } B_R(x_0)\times(0,T_p). \tag{4.23}$$

Using that (4.21), we obtain by simple calculations

$$\begin{aligned}-\Delta_{p(x,t)}\psi =& \frac{|\nabla\psi|^{p(x,t)-1}}{|x-x_0|}(N-1+\frac{p(x,t)-1}{p^--1})+|\nabla\psi|^{p(x,t)-1}ln(|\nabla\psi|)\sum_{i=1}^{N}\frac{\partial p(x,t)}{\partial x_i}\frac{x_i-x_{0,i}}{|x-x_0|}\\ &-|\nabla\psi|^{p(x,t)-1}(\frac{p^--1}{\gamma}-1)(\frac{p^-}{p^--1})H^{-1}(x,t)\left(\frac{|x-x_0|}{S^{\frac{1}{\lambda}}(t)}\right)^{\frac{1}{p^--1}}(\frac{p(x,t)-1}{S^{\frac{1}{\lambda}}(t)}).\end{aligned} \tag{4.24}$$

We set $C_1=(N-1+\frac{p^+-1}{p^--1})$, $\mu=\sup_{Q_{T_p}}|\nabla p(x,t)|$, and

$$G(x,t)=m(x)\sigma\alpha k^{\beta+m(x)}r^{\gamma+m(x)}S^{\frac{-\alpha m(x)}{\lambda}-1}(t)H^{m(x)\frac{p^--1}{\gamma}}(x,t).$$

To get (4.23), by (4.22) and (4.24) it suffices to have

$$G(x,t)(\alpha-3\frac{p^-}{\gamma}\left(\frac{|x-x_0|}{S^{\frac{1}{\lambda}}(t)}\right)^{\frac{p^-}{p^--1}}H^{-1}(x,t))\geq 0, \tag{4.25}$$

$$G(x,t)\geq 3\frac{C_1}{|x-x_0|}\left[\frac{krp^-}{\gamma}S^{-\frac{\alpha+1}{\lambda}}(t)H^{\frac{p^--1}{\gamma}-1}(x,t)\left(\frac{|x-x_0|}{S^{\frac{1}{\lambda}}(t)}\right)^{\frac{1}{p^--1}}\right]^{p(x,t)-1}, \tag{4.26}$$

$$G(x,t) \geq 3a_1 \left[krS^{\frac{-\alpha}{\lambda}}(t) H^{\frac{p^- -1}{\gamma}}(x,t) \right]^{q(x,t)} \tag{4.27}$$

and

$$\begin{aligned}(\frac{p^- - 1}{\gamma} - 1)(\frac{p^-}{S^{\frac{1}{\lambda}}(t)}) H^{-1}(x,t) \left(\frac{|x - x_0|}{S^{\frac{1}{\lambda}}(t)} \right)^{\frac{1}{p^- -1}} \geq & \mu \frac{krp^-}{\gamma} S^{-\frac{\alpha+1}{\lambda}}(t) \\ & \times H^{\frac{p^- -1}{\gamma} - 1}(x,t) \left(\frac{|x - x_0|}{S^{\frac{1}{\lambda}}(t)} \right)^{\frac{1}{p^- -1}}\end{aligned} \tag{4.28}$$

Now, our goal is to choose the constants γ, λ and β in order to verify each of the inequalities above. Firstly, let us show the inequality (4.25). Thanks to α which is large enough and $R < \frac{1}{2}$, we have

$$\alpha > \frac{\ln(r)}{\ln(2^{\frac{p^- -1}{p^-}} R)},$$

from which we get

$$(\frac{1}{2})^{\frac{p^- -1}{p^-}} > \frac{R}{r^{1/\alpha}} > \frac{|x - x_0|}{S^{\frac{1}{\lambda}}(0)},$$

then

$$H(x,t) \geq \left(\frac{|x - x_0|}{S^{\frac{1}{\lambda}}(t)} \right)^{\frac{p^-}{p^- -1}}, \quad \text{in } B_R(x_0) \times (0, T_p),$$

therefore

$$3\alpha - \frac{p^-}{\gamma} \left(\frac{|x - x_0|}{S^{\frac{1}{\lambda}}(t)} \right)^{\frac{p^-}{p^- -1}} H^{-1}(x,t) \geq 3\alpha - \frac{p^-}{\gamma} \geq 0,$$

hence, (4.25) is satisfied. Next, to get (4.26) and (4.27). We use the fact that $s(t) \leq 1$ and $H(x,t) \leq 1$ a.e in $B_R(x_0) \times (0, T_p)$. By setting

$$\gamma = \min\{ \frac{p^- - 1}{p^+ - 1}(p^- - 1 - m^+), q^- - m^+ \},$$

$$\beta = \min\{ p^- - 1 - m^+, q^- - m^+ \}$$

$$\lambda = \max\{ ((\alpha + \frac{p^-}{p^- - 1})(p^+ - 1) - \alpha m^-), \alpha(q^+ - m^-) \},$$

and using α large enough, so that

$$\alpha > \max\{ Z\frac{3C_1}{\sigma m^-}, \frac{3a_1}{m^- \sigma} \},$$

where

$$Z = \max\{ (\frac{p^-}{\gamma})^{p^+ -1}, (\frac{p^-}{\gamma})^{p^- -1} \},$$

we obtain, the inequalities (4.26), (4.27). Finally, to get the inequality (4.28), it suffices to have that

$$p^-(\frac{p^- - 1}{\gamma} - 1) S^{\frac{\alpha}{\lambda}}(0) \geq \mu \frac{krp^-}{\gamma},$$

which reduces to

$$(\frac{p^- - 1}{\gamma} - 1) \geq \mu \frac{k}{\gamma}.$$

Since k is small enough the last inequality holds, which means that the inequality (4.28) is satisfied and allows us to deduce the inequality (4.23). Then, by comparison principle, for each $t \in [0, T_p]$

$$u(x,t) \geq \psi(x,t) \quad \text{a.e.} x \in B_R(x_0).$$

Therefore the result follows from lemma (4.3.4). □

Proof of Theorem 4.3.3. The proof is similar to that of theorem 1.2, in [91], and we omit the details here. □

4.4 Finite speed of propagation property

In this section we shall give precise estimates for the of support of solution of $\mathcal{P}$, depending to the size of the support of u_0. Let us emphasize that each estimation is obtained under a sign condition on $a(x,t)$ and range of the exponents $p(x,t)$, $q(x,t)$ and $m(x)$. As in [41], the proof is based on the construction of local super-solutions and on the use of comparison principle.
Concerning the construction of super-solutions, we shall proceed as in [96].

Note that under some conditions on the data, if $a(x,t)$ is positive, then the solutions will blow up in finite time, (see [12]). For that it requires to construct a super-solution defined locally in time, which means in $(0,T)$ for any $T \leq T^*$ where T^* is the maximal existence time. We denote by $\Omega_R = \overline{\Omega} \cap B(0,R)$.

Theorem 4.4.1. *Let u be a strong solution of $\mathcal{P}$. Let $0 < R < +\infty$ be such that supp $u_0 \subset \Omega_R$. We assume $m^+ < p^- - 1$, $q(x,t) = m(x)$ and $\sup_{Q_T} |\nabla p(x,t)| < \infty$. Then, for any $t \in (0,T)$, there exists a unique compactly supported solution of $\mathcal{P}$ such that*

$$supp\ u(t,.) \subset \{x \in \overline{\Omega} : |x| \leq \rho(t)\},$$

where $\rho(t)$ will be specified in the proof below.

Proof. The idea of the proof is to construct a suitable super-solution $\overline{u}$ with compact support which is not necessarily defined in the whole Ω. Then by comparison principle, we deduce directly that $\{(x,t) \in Q_T : \overline{u} = 0\} \subset \{(x,t) \in Q_T : u = 0\}$, and the result follows. For all $t \in (0,T)$, we define $\overline{u}\,(x,t)$ as follows

$$\overline{u}(r,t) = \begin{cases} K\,(T)\, e^{(m^- +1)\lambda t}\, (c\,(t) + k\,(R-r))^{\alpha} & , R \leq r < R + \frac{c(t)}{k} = \rho(t) \\ 0 & , r \geq R + \frac{c(t)}{k} \end{cases}$$

where $r = |x|$,

$$\alpha = \frac{p^-}{p^- - 1 - m^+},$$

$$M = \max\{1, \|u_0\|_\infty\},$$

$$\lambda = \frac{a_1}{(m^-)^2}, \tag{4.29}$$

$$c\,(t) = \left(\frac{M}{K\,(T)}\right)^{\frac{1}{\alpha}} e^{\lambda t \frac{(p^+ - 1 - m^-)}{\alpha}}, \tag{4.30}$$

and

$$K(T) = \left[\frac{m^- \lambda e^{-(m^-+1)\lambda T(p^+-1-m^-)}}{(k(\alpha-1)(p^+-1)+\epsilon^{n-1})\,k^{p^--1}\alpha^{p^+-1}} \right]^{\frac{1}{p^+-1-m^-}}, \tag{4.31}$$

with sufficiently small $\epsilon > 0$.

Firstly, we denote by $Q_T^R = (\Omega \backslash \Omega_R) \times (0, T)$. It is clear that $\overline{u} \in C^1(Q_T^R)$ and, by direct calculations, we have

$$\frac{\partial}{\partial t}\overline{u}^{m(x)} = (m^- + 1)m(x)\lambda\overline{u}^{m(x)} + \alpha m(x)(K(T)\,e^{(m^-+1)\lambda t})^{m(x)}\,(c(t) + k(R-r))^{\alpha m(x)-1}\,c'(t)\,. \tag{4.32}$$

Thus, it follows that

$$\frac{\partial}{\partial t}\overline{u}^{m(x)} \in L^\infty(Q_T^R) \subset W'(Q_T^R).$$

Moreover, from the definition of $\overline{u}$ we have

$$\overline{u}(R, 0) = M \geq \|u_0\|_\infty \geq u_0.$$

Next, we will show that

$$\frac{\partial}{\partial t}\overline{u}^{m(x)} - \Delta_{p(x,t)}\overline{u} \geq a(x,t)\,\overline{u}^{m(x)} \quad \text{on} \quad Q_T^R.$$

Since $|a(x,t)| \leq a_1$, it suffices to show that

$$\frac{\partial \overline{u}^{m(x)}}{\partial t} - \Delta_{p(x,t)}\overline{u} \geq a_1\overline{u}^{m(x)} \quad \text{on} \quad Q_T^R.$$

Using that $p^+ > m^- + 1$, we have $c'(t) > 0$ for all $t \in (0, T)$. Then, from (4.32) and (4.29) we obtain

$$\frac{\partial}{\partial t}\overline{u}^{m(x)} \geq (m^- + 1)m(x)\lambda\overline{u}^{m(x)} \geq a_1\overline{u}^{m(x)} + m(x)\lambda\overline{u}^{m(x)}. \tag{4.33}$$

Due to the last inequality, our goal now is to prove that

$$-\Delta_{p(x,t)}\overline{u} + m(x)\lambda\overline{u}^{m(x)} \geq 0. \quad \text{on} \quad Q_T^R. \tag{4.34}$$

We have

$$\frac{\partial \overline{u}}{\partial x_i} = -\alpha k K(T)\,e^{(m^-+1)\lambda t}\,(c(t) + k(R-r))^{\alpha-1}\,\frac{x_i}{|x|},$$

and

$$|\nabla \overline{u}| = \alpha k K(T)\,e^{(m^-+1)\lambda t}\,(c(t) + k(R-r))^{\alpha-1}\,.$$

We set

$$\nu_T(t) = \alpha k K(T)\,e^{(m^-+1)\lambda t}.$$

Then

$$\Delta_{p(x,t)}\overline{u} = (c(t) + k(R-r))^{(\alpha-1)(p(x,t)-1)-1}(\nu_T(t))^{p(x,t)-1}[k(\alpha-1)(p(x,t)-1) - h(x,t)],$$

where

$$\begin{aligned}h(x,t) &= \ln(c(t)+k(R-r))(c(t)+k(R-r))(\alpha-1)\sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|}\\&+[\frac{(N-1)}{|x|}+\sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|}ln(\nu_T(t))](c(t)+k(R-r))\\&= n\sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|}\ln((c(t)+k(R-r))^{\frac{1}{n}})(c(t)+k(R-r))^{\frac{1}{n}}(c(t)+k(R-r))^{1-\frac{1}{n}}(\alpha-1)\\&+(\frac{(N-1)}{|x|}+\sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|}ln(\nu_T))(c(t)+k(R-r))^{\frac{1}{n}}(c(t)+k(R-r))^{1-\frac{1}{n}},\end{aligned}$$

for all $n \geq 3$. By straightforward considerations, (boundedness of different functions), there exist positive constants $A, B \geq 1$ such that, for all $t \in (0,T)$ and $R \leq r < \rho(t)$,

$$\left|n\sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|}\ln((c(t)+k(R-r))^{\frac{1}{n}})(c(t)+k(R-r))^{\frac{1}{n}}(\alpha-1)\right| \leq A$$

$$\left|(\frac{(N-1)}{|x|}+\sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|}ln(\nu_T(t)))(c(t)+k(R-r))^{\frac{1}{n}}\right| \leq B.$$

Thus

$$\begin{aligned}|h(x,t)| &\leq (A+B)(c(t)+k(R-r))^{1-\frac{1}{n}}\\&\leq ((A+B)(c(t)+k(R-r))^{\frac{1}{n}})^{n-1}.\end{aligned}$$

In this case we have $k\,(r-R) \leq c\,(t)$, and from (4.31), we can choose k small enough, so that $c\,(t)$ is also small for all $t \in (0,T)$. Hence, we have

$$(A+B)\,(c\,(t)+k\,(R-r))^{\frac{1}{n}} \leq \epsilon,$$

for sufficiently small $\epsilon > 0$; which implies that

$$\Delta_{p(x,t)}\overline{u} \leq (c\,(t)+k\,(R-r))^{(\alpha-1)(p(x,t)-1)-1}\,(\nu_T(t))^{p(x,t)-1}(k(\alpha-1)(p(x,t)-1)+\epsilon^{n-1}). \quad (4.35)$$

Set $W(x,t) = k(\alpha-1)(p(x,t)-1)+\epsilon^{n-1}$. Now, to get (4.34) it suffices to have

$$(c(t)+k(R-r))^{(\alpha-1)(p(x,t)-1)-1}(\nu_T(t))^{p(x,t)-1}W(r,t) \leq (c(t)+k(R-r))^{\alpha m(x)}m(x)\lambda K^{m(x)}e^{(m^-+1)\lambda m(x)t}.$$

Since $c\,(t)+k\,(R-r)$ is small enough, hence from the value of α and $K(T)$, we obtain that

$$(c(t)+k(R-r))^{(\alpha-1)(p(x,t)-1)-1} \leq (c(t)+k(R-r))^{\alpha m(x)},$$

and

$$(\alpha k K(T)e^{(m^-+1)\lambda t})^{p(x,t)-1}W(x,t) \leq m(x)\lambda K(T)^{m(x)}e^{(m^-+1)\lambda m(x)t};$$

which implies (4.34). Therefore we deduce the desired result. □

Theorem 4.4.2. *Let u be a strong solution of $\mathcal{P}$. Let $0 < R < +\infty$ be such that supp $u_0 \subset \Omega_R$. We assume $m^+ < p^- - 1$, $\sup_{Q_T}|\nabla p(x,t)| < \infty$, and $a(x,t) \leq 0$. Then, for any $t \in (0,T)$, there exists a unique compactly supported solution of $\mathcal{P}$ such that*

$$supp\ u(t,.) \subset \{x \in \overline{\Omega} : |x| \leq \rho(t)\},$$

where $\rho(t)$ is specified in the proof above.

Proof. We consider the same super-solution $\overline{u}\,(r,t)$ as in the proof of theorem 4.4.1 and we just need to prove that

$$\frac{\partial \overline{u}^{m(x)}}{\partial t} - \Delta_{p(x,t)}\overline{u} \geq a\,(x,t)\,\overline{u}^{q(x,t)} \quad \text{on} \quad Q_T^R,$$

where, $Q_T^R = (\Omega \backslash \Omega_R) \times (0,T)$. Since $a(x,t) \leq 0$, it is sufficient to prove that

$$\frac{\partial \overline{u}^{m(x)}}{\partial t} - \Delta_{p(x,t)}\overline{u} \geq 0 \quad \text{on} \quad Q_T^R.$$

Combining the same lines as in the of previous theorem and the comparison principle in proposition 4.2.2, we conclude to the result. $\square$

Finally, we state a uniform localization of the support of solution.

Theorem 4.4.3. *Let u be a strong solution of $\mathcal{P}$. Let $0 < R < +\infty$ be such that supp $u_0 \subset \Omega_R$. We assume $q^+ < p^- - 1$, $\sup_{Q_T} |\nabla p(x,t)| < \infty$ and $a(x,t) \leq a_2 < 0$. Then for any $t \in (0,\infty)$, there exists a unique compactly supported solution of $\mathcal{P}$ such that*

$$supp\ u(t,.) \subset \{x \in \overline{\Omega} : |x| \leq R_1\}$$

where R_1 is positive constants determined in the proof below.

Proof. In order to get the desired estimation of the support of the solution, we define a suitable local super-solution associated with the stationary problem related to $\mathcal{P}$. Let $M = \max\left\{\|u_0\|_{L^\infty(\Omega)}, 1\right\}$. We define $\hat{u}$ as follows

$$\hat{u}(r) = \begin{cases} K\,(R_1 - r)^\alpha & R \leq r < R_1 \\ 0 & r \geq R_1. \end{cases}$$

where $r = |x|$,

$$K = \frac{M}{(R_1 - R)^\alpha},$$

where, $R_1 - R \leq 1$ and $\alpha > 1$ is a constant that will be determined here after. It is easy to verify that $\hat{u} \in C^1(Q_T^R)$ for any $T > 0$, where $Q_T^R = (\Omega \backslash \Omega_R) \times (0,T)$. Concerning initial condition, we have $\hat{u}(R,0) = \hat{u}(R) \geq M \geq u_0$. Now, let us show that

$$\frac{\partial b(x,\hat{u})}{\partial t} - \Delta_{p(x,t)}\hat{u} \geq a(x,t)\hat{u}^{q(x,t)}, \ \ (x,t) \in Q_T^R. \tag{4.36}$$

We have

$$\frac{\partial \hat{u}}{\partial x_i} = -\alpha K\,(R_1 - r)^{\alpha - 1}\,\frac{x_i}{r},$$

and

$$|\nabla \hat{u}| = \alpha K\,(R_1 - r)^{\alpha - 1}.$$

Then

$$\Delta_{p(x,t)}\hat{u} = (\alpha K)^{p(x,t)-1}(R_1 - r)^{(\alpha-1)(p(x,t)-1)-1}[(\alpha - 1)(p(x,t) - 1) - H(x,t)],$$

with

$$H(x,t) = (\alpha - 1)(R_1 - r)\ln(R_1 - r)\sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|} - (\frac{N-1}{r} + \sum_{i=1}^{N}\frac{\partial p}{\partial x_i}\frac{x_i}{|x|}\ln(K\alpha))(R_1 - r).$$

By straightforward considerations, there exists a constant $A \geq 1$ such that, for every r with $R \leq r < R_1$, we have

$$|(\alpha-1)(p(x,t)-1)-H(x,t)| \leq A.$$

Hence, we get

$$\Delta_{p(x,t)}\hat{u} \leq A(\alpha K)^{p(x,t)-1}(R_1-r)^{(\alpha-1)(p(x,t)-1)-1}.$$

Now, in order to obtain the inequality (4.36) , we need to show

$$\Delta_{p(x,t)}\hat{u} \leq -a(x,t)(\hat{u}(r))^{q(x,t)}.$$

To do so, we just need to get

$$A(\alpha K)^{p(x,t)-1}(R_1-r)^{(\alpha-1)(p(x,t)-1)-1} \leq -a_2(K)^{q(x,t)}\left(R_1-r\right)^{\alpha q(x,t)}, \tag{4.37}$$

Due to the fact that $R_1 - r \leq R_1 - R \leq 1$, (4.37) holds if

$$(\alpha-1)\left(p(x,t)-1\right)-1-\alpha q(x,t) \geq 0, \quad A\alpha^{p(x,t)-1}K^{p(x,t)-1-q(x,t)} \leq -a_2.$$

Then, it is suffices to take

$$\alpha = \frac{p^-}{p^- - 1 - q^+}, \quad a_2 = -A\alpha^{p^+-1}K^{p^+-1-q^-},$$

to complete the proof. □

Remark 4.4.4. *As in [40, 41], we can assume in this section that Ω is just a set of $\mathbb{R}^N$, not necessarily bounded.*

BIBLIOGRAPHY

[1] B. Abdellaoui, A. Dall'Aglio, I. Peral, *Some remarks on elliptic problems with critical growth in the gradient,* J. Differential Equations, 222, no. 1, 21-62, 2006.

[2] E. Acerbi, G. Mingione, *Regularity results for a class of functionals with nonstandard growth*, Arch. Ration. Mech. Anal., 156, 121–140, 2001.

[3] G. Akagi, *Doubly nonlinear parabolic equations involving variable exponents*, J. Discrete Contin. Dyn. Syst. Ser. 7, 1-16, 2014.

[4] G. Akagi, K. Matsuura, *Nonlinear diffusion equations driven by the $p(.)-$Laplacian*, Nonlinear Differ. Equ. Appl. 20, 37-64, 2013.

[5] A. Ambrosetti, P.H. Rabinowitz, *Dual Variational Methods In Critical Point Theory And Applications,* J. Funct. Anal, 14, no. 4, 349-381, 1973.

[6] S. Antontsev, J. I. Díaz, and S. Shmarev, *Energy Methods for Free Boundary Problems: Applications to Non-linear PDEs and Fluid Mechanics*, Bikhäuser, Boston, Progress in Nonlinear Differential Equations and Their Applications, Vol. 48, 2002.

[7] S. Antontsev, S. Shmarev, *A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solution*, J. Nonlinear Anal. 60 (3), 515-545, 2005.

[8] S. Antontsev, S. Shmarev, *Existence and uniqueness for double nonliner parabolic equations with non standard growth conditions*. J. Differ. Equ. Appl. 4, 67-94, 2012.

[9] S. Antontsev, S. Shmarev, *Doubly degenerate parabolic equations with variable nonlinearity I: existence of bounded strong solutions*. J. Adv. Differ. Equ. 17, 1181-1212, 2012.

[10] S. Antontsev, S. Shmarev, *Doubly degenerate parabolic equations with variable nonlinearity II: Blow-up and extinction in a finite time*, J. Nonlinear Anal. 95, 483-498, 2014.

[11] S. Antontsev, S. Shmarev, *Evolution PDEs with nonstandard growth conditions: existence, uniqueness, localization, blow-up, Atlantis Studies in Differential Equations*, Atlantis Press, 2015.

[12] S. Antontsev, S. Shmarev, *On the localization of solutions of doubly nonlinear parabolic equations with nonstandard growth in filtration theory*, J. Applicable Anal. 95, 2162-2180, 2016.

[13] D. Arcoya, C. De Coster, L. Jeanjean, K. Tanaka, *Remarks on the uniqueness for quasilinear elliptic equations with quadratic growth conditions,* J. Math. Anal. Appl, 420, no. 1, 772-780, 2014.

[14] D. Arcoya, C. De Coster, L. Jeanjean, K. Tanaka, *Continuum of solutions for an elliptic problem with critical growth in the gradient,* J. Funct. Anal, 268, no. 8, 2298-2335, 2015.

[15] D.G. Aronson, *The porous medium equation*, Fasano, Primicerio (Eds.), Nonlinear Diffusion Problems, Springer, Berlin, L.N.M., Vol. 1224, 1986.

[16] A. Attouchi, *Well-posedness and gradient blow-up estimate near the boundary for a Hamilton-Jacobi equation with degenerate diffusion*, J. Differ. Equ., 253, 2474-2492, 2012.

[17] G. Barles, A.P. Blanc, C. Georgelin, M. Kobylanski, *Remarks on the maximum principle for nonlinear elliptic PDE with quadratic growth conditions,* Ann. Sc. Norm. Super. Pisa Cl. Sci, Série 4, Tome 28, no. 3, 381-404, 1999.

[18] G. Barles, F. Murat, *Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions,* Arch. Ration. Mech. Anal, 133, no. 1, 77-101, 1995.

[19] J.Ph. Bartier, Ph. Laurençot, *Gradient estimates for a degenerate parabolic equation with gradient absorption and applications*, J. Funct. Anal., 254, 851-878, 2008.

[20] G. Bayada, M. Chambat, On interface conditions for a thin film, SIAM, J.Math.Analysis, 26, 1113-1129, 1995.

[21] L. Boccardo, F. Murat, J.P. Puel, *Existence de solutions faibles pour des équations elliptiques quasilinaires à croissance quadratique,* Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. IV (Paris, 1981/1982), Res. Notes in Math., vol. 84, Pitman, Boston, Mass.-London, pp. 19-73. MR716511 (84k:35064), 1983.

[22] L. Boccardo, F. Murat, J.P. Puel, *L^∞ estimate for some nonlinear elliptic partial differential equations and application to an existence result,* SIAM J. Math. Anal, 23, no. 2, 326-333, 1992.

[23] L. Boccardo, F. Murat, J. P. Puel, *Résultats d'existence pour certains problèmes elliptiques quasilinéaires,* Ann. Sc. Norm. Super. Pisa Cl. Sci, Série 4, Tome 11, no. 2, 213-235, 1984.

[24] M. Bertsch, R. Kersner , L.A. Peletier, *Positivity versus localization in degenerate diffusion equations*, J. Nonlinear Anal. 9, 987-1008, 1985.

[25] Z. Chaouai, A. El Hachimi, *Qualitative properties of nonnegative solutions for a doubly nonlinear problem with variable exponents*, Abstr. Appl. Anal., 1-14, 2018.

[26] Z. Chaouai, S. Maatouk, *Multiplicity of Solutions for a Class of Elliptic Problem of p-Laplacian Type with a p-Gradient Term*, Int. J. Math. Math. Sci, 1-11, 2019.

[27] Z. Chaouai, A. El Hachimi, *Qualitative properties of weak solutions for p-Lapalcian equations with nonlocal source gradient absorption*, Bull. Korean Math. Soc., V. 57, Issue 4, 1003-1031, 2020.

[28] M. Chen, J. Zhao, *On the cauchy problem of evolution p-Laplacian equation with nonlinear gradient term*, Chin. Ann. Math. Ser. B, 30, 1-16, 2009.

[29] Y. Chen, S. Levine, M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. 66, 1383-1406, 2006.

[30] C. Chen, M. Nakao, Y. Ohara, *Global existence and gradient estimates for quasilinear parabolic equations of the m-Laplacian type with a strong perturbation*, Adv. Math. Sci. Appl., 10, 225-237, 2000.

[31] M. Chipot, F.B. Weissler, *Some blow up results for a nonlinear parabolic problem with a gradient term*, SIAM J. Math. Anal., 20, 886-907, 1989.

[32] Y.S. Chung, J.H. Park, *A complete characterization of nonlinear absorption for the evolution p-Laplacian equations to have positive or extinctive solutions*, J. Comput. Math. Appl. 71, 1624-1635, 2016.

[33] D.G. Costa, C.A. Magalhães, *Variational elliptic problems which are nonquadratic at infinity,* Nonlinear Anal, 23, no. 11, 1401-1412, 1994.

[34] C. De Coster, L. Jeanjean, *Multiplicity results in the non-coercive case for an elliptic problem with critical growth in the gradient,* J. Differential Equations, 262, no. 10, 5231-5270, 2017.

[35] D. V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*. Birkhäuser.

[36] J.I. Dìaz, *Solutions with compact support for some degenerate parabolic problems*, Nonlinear. Anal. 3, 831-847, 1979.

[37] J.I. Dìaz, *Nonlinear Partial Differential Equations and Free Boundaries*, vol. I, Elliptic equations. Research Notes in Mathematics n^o 106, Pitman, Londres, 1985.

[38] J.I. Dìaz, *Qualitative study of nonlinear parabolic equations: An introduction*, Extracta Mathematicae, 16, 303-341, 2001.

[39] J.I. Dìaz, J. Hernandez, *Qualitative properties of free boundaries for some nonlinear degenerate parabolic equations, in Nonlinear Parabolic Equations: Qualitative Properties of Solutions*, L. Boccardo and A. Teseieds, Pitman Research Notes, Longman,85-93, 1987.

[40] J.I. Dìaz, M.A. Herrero, *Proprietés de support compact pour certaines équations elliptiques et paraboliques non linéaires*, C. R. Acad. Sci. Paris Ser. I 286, 815-817, 1978.

[41] J.I. Dìaz, M.A. Herrero, *Estimates on the support of the solutions of some nonlinear elliptic and parabolic problems*, Proc. Roy. Soc. Edinburgh Sect. A 98, 249-258, 1981.

[42] E. DiBenedetto, A. Friedman, *Hölder estimates for nonlinear degenerate parabolic systems*, J. Reine Angew. Math., (CrellesJ.) 357, 1-22, 1985.

[43] L. Diening, P. Harjulehto, P. Hästo, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents, Springer*, Berlin, 2011.

[44] G. Dinca, P. Jebelean, J. Mawhin, *Variational and topological methods for Dirichlet problems with p-Laplacian,* Port. Math, 58, no. 3, 339-378, 2001.

[45] D. Edmunds, J. Rákosník, *Sobolev embeddings with variable exponent, Studia Mathematica*, Vol. 143, No. 3, 267-293, 2000.

[46] I. Ekeland, *On the variational principle,* J. Math. Anal. Appl, 47, no. 2, 324-353, 1974.

[47] Z.B. Fang, J. Zhang, *Global and blow-up solutions for the nonlocal p-Laplacian evolution equation with weighted nonlinear nonlocal boundary condition*, J. Integral Equat. Appl., 26, 171-196, 2014.

[48] Z.B. Fang, G. Li, *Extinction and decay estimates of solutions for a class of doubly degenerate equations*, Appl. Math. Lett., 25, 1795-1802, 2012.

[49] X.L. Fan and Q.H. Zhang, *On the spaces $L^{p(x)}$ and $W^{m,p(x)}$* J. Math. Anal. Appl., vol. 263, no. 2, pp. 424-446, 2001.

[50] V. Ferone, F. Murat, *Quasilinear problems having quadratic growth in the gradient: an existence result when the source term is small,* Equations aux dérivées partielles et applications, Gauthier-Villars, Ed. Sci. Med. Elsevier, Paris, 497-515, 1998.

[51] V. Ferone, F. Murat, *Nonlinear problems having quadratic growth in the gradient: an existence result when the source term is small,* Nonl. Anal. : Theory, Methods and Appl., vol. 42, no. 7, pp. 1309-1326, 2000.

[52] M. Furtado, E.D. Silva, *Superlinear elliptic problems under the nonquadraticity condition at infinity,* J. Proc. Roy. Soc. Edinburgh Sect. A, 145, no. 4, 779-790, 2015.

[53] J.J. García-Melián, L. Iturriaga, H. Quoirin, *A priori bounds and existence of solutions for slightly superlinear elliptic problems,* Adv. Nonlinear Stud, 15, no. 4, 923-938, 2015.

[54] Y.G. Gu, *Necessary and sufficient conditions of extinction of solution on parabolic equations*, Acta. Math. Sin. 37, 73-79, 1994.

[55] M.E. Gurtin, R.C. Mac Cammy, *On the diffusion of biological population*, Math. Biosci. 33, 35-49, 1977.

[56] B. Guo, W.J. Gao, *Study of weak solutions for parabolic equations with nonstandard growth conditions*, J. Math. Anal. Appl. 374(2), 374-384, 2011.

[57] B. Guo, W.J. Gao, *Existence and asymptotic behavior of solutions for nonlinear parabolic equations with variable exponent of nonlinearity*, Acta Math. Sci. 32(3), 1053-1062, 2012.

[58] R.G. Iagar, P. Laurençot, *Positivity, decay, and extinction for a singular diffusion equation with gradient absorption*, J. Funct. Anal., 262, 3186-3239, 2012.

[59] R.G. Iagar, Ph. Laurençot, *Asymptotic behavior for a singular diffusion equation with gradient absorption*, J. Differ. Equ., 256, 2739-2777, 2014.

[60] R.G. Iagar, Ph. Laurençot, C. Stinner, *Instantaneous shrinking and single point extinction for viscous Hamilton-Jacobi equations with fast diffusion*, Math. Ann., 368, 65-109, 2016.

[61] L. Iturriaga, S. Lorca, J. Sánchez, *Existence and Multiplicity Results for the p-Laplacian with a p-Gradient Term,* NoD. Nonl. Diff. Equat. Appl. 15, no. 6, 729-743, 2008.

[62] L. Iturriaga, S. Lorca, P. Ubilla, *A quasilinear problem without the Ambrosetti-Rabinowitz-type condition,* Proc. Roy. Soc. Edinburgh Sect. A, 140, no. 2, 391-398, 2010.

[63] L. Jeanjean, H. Ramos Quoirin, *Multiple solutions for an indefinite elliptic problem with critical growth in the gradient,* Proc. Amer. Math. Soc, 144, no. 2, 575-586, 2016.

[64] L. Jeanjean, B. Sirakov, *Existence and multiplicity for elliptic problems with quadratic growth in the gradient,* Comm. Part. Diff. Eq., 38, no. 2, 244-264, 2013.

[65] C.H. Jin, J.X. Yin, Y.Y. Ke, *Critical extinction and blow-up exponents for fast diffusive polytropic filtration equation with sources*. Proc. Edinb. Math. Soc. 52, 419-444, 2009.

[66] A.S. Kalashnikov, *Some problems of qualitative theory of nonlinear degenerate second-order parabolic equations*, Uspekhi Mat. Nauk., 42, 135-176, 1987.

[67] O. Kovacik, J. Rakosuik, *On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$*, Czechoslovak Math. J. Vol. 41, No. 4, 592-618, 1991.

[68] O. Ladyzhenskaya, V. Solonnikow, N. Ural'ceva, *Linear and Quasi-Linear Equations of Parabolic Type*, American Mathematical Society, 1968.

[69] F.C. Li, C.H. Xie, *Global and blow-up solutions to a p-Laplacian equation with nonlocal source*, Comput. Math. Appl., 46, 1525-1533, 2003.

[70] Y. Li, Z. Zhang, L. Zhu, *Classification of certain qualitative properties of solutions for the quasilinear parabolic equations*, Sci. China Math., 61, 855-868, 2018.

[71] J. L. Lions, *Quelques Méthodes de Résolution des Problémes aux Limites Nonlinéaires*, Dunod, Paris, 1969.

[72] C. Maderna, C. Pagani, S. Salsa, *Quasilinear elliptic equations with quadratic growth in the gradient,* J. Differential Equations, 97, no. 1, 54-70, 1992.

[73] O.H. Miyagaki and M.A.S. Souto, *Supelinear problems without Ambrosetti-Rabinowitz growth condition,* J. Differential Equations, 245, no. 12, 3628-3638, 2008.

[74] J. Musielak,W. Orlicz, *On modular spaces, Studia Math.*, 18:49–65, 1959.

[75] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. Vol. 1034, Spring-Verlag, Berlin, 1983.

[76] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Maruzen Co. Ltd., Tokyo, 1950.

[77] W. Orlicz, *Über konjugierte Exponentenfolgen*, Studia Math., 3:200–211, 1931.

[78] A. Porretta, *The ergodic limit for a viscous Hamilton Jacobi equation with Dirichlet conditions,* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl, 21, no. 1, 59-78, 2010.

[79] P. Pucci, R. Servadei, *Regularity of weak solutions of homogeneous or inhomogeneous quasilinear elliptic equations*, Indiana Univ. Math. J, 57, no. 7, 3329-3363, 2008.

[80] P. Quittner and Ph. Souplet, *Superlinear parabolic problems : blow-up, global existence and steady states*, Birkhauser, 2007.

[81] K.R. Rajagopal, M. Růžička, *Mathematical modeling of electrorheological fluids*, Contin. Mech. Thermodyn. 13, 59–78, 2001.

[82] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Springer-Verlag, Berlin, 2000.

[83] I. Sharapudinov, *On the topology of the space Lp(t)([0; 1])*, Math. Notes, 26:796–806, 1979.

[84] J. Simon, *Compact sets in the space $L^p(0, T; B)$*, Ann. Mat. Pura Appl., 146, 65-96, 1987.

[85] B. Sirakov, *Solvability of uniformly elliptic fully nonlinear PDE,* Arch. Ration. Mech. Anal, 195, no. 2, 579-607, 2010.

[86] Ph. Souplet, *Finite time blow-up for a nonlinear parabolic equation with a gradient term and applications*, Math. Methods Appl. Sci., 19, 1317-1333, 1996.

[87] Ph. Souplet, *Blow up in nonlocal reaction-diffusion equations*, SIAM J. Math. Anal., 29, 1301-1334, 1998.

[88] Ph. Souplet and F.B. Weissler, *Self-similar sub-solutions and blow-up for nonlinear parabolic equations*, J. Math. Anal. Appl., 212, 60–74, 1997.

[89] J. Stefan, *Uber die Theorie der Eisbildung, insbesondere uber die Eisbildung im Polarmeere*, Annalen der Physik und Chemie, 1891.

[90] I. Tsenov, *Generalization of the problem of best approximation of a function in the space* Ls, Uch. Zap. Dagestan Gos. Univ., 7:25–37, 1961.

[91] H.J. Yuan, S.Z. Lian, C.L. Cao, W.J. Gao, X.J. Xu, *Extinction and positivity for a doubly nonlinear degenerate parabolic equation*, Acta Math. Sin. 23, 1751-1756, 2007.

[92] M. Willem, W. Zou, *On a Schrdinger equation with periodic potential and spectrum point zero,* Indiana Univ. Math. J, 52 (2003), no. 1, 109-132. vol. 150, 1996.

[93] H. Zhan, *The global solution of a diffusion equation with nonlinear gradient term*, J. Inequal. Appl. 2013, 125, 1-20, 2013.

[94] Z.C. Zhang, Y. Li, *Blow-up and existence of global solutions to nonlinear parabolic equations with degenerate diffusion*, Electron. J. Diff. Equ., 264, 1-17, 2013.

[95] Z. C. Zhang, Y. Li, *Classification of blowup solutions for a parabolic p-Laplacian equation with nonlinear gradient terms*, J. Math. Anal. Appl., 436, 1266-1283, 2016.

[96] Q.H. Zhang, X.P. Liu, Z.M. Qiu, *On the boundary blow-up solutions of p(x)-Laplacian equations with singular coefficient*, J. Nonlinear. Anal. 70, 4053-4070, 2009.

[97] V.V. Zhikov, *On the density of smooth functions in Sobolev-Orlicz spaces*, J. Math. Sci. (N.Y.) 132, 285-294, 2006.

[98] J.N. Zhao, *Existence and nonexistence of solutions for* $u_t = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(\nabla u, u, x, t)$, J. Math. Anal. Appl., 172, 130-146, 1993.

[99] V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR. Izv. Vol. 29 (1987), No.1, 33-36.

[100] J. Zhou, C.L. Mu, *Critical blow-up and extinction exponents for non-Newton polytropic filtration equation with source*. Bull. Korean Math. Soc. 46, 1159-1173, 2009.

ROYAUME DU MAROC
Université Mohammed V - RABAT - جامعة محمد الخامس – الرباط –
Faculté des sciences
كلية العلوم
CENTRE D'ETUDES DOCTORALES - SCIENCES ET TECHNOLOGIES

Abstract

This thesis deals with the mathematical field of nonlinear partial differential equations analysis. More precisely, at first we are interested in this work to study the existence and the multiplicity of solutions for a class of elliptic problems of p-Laplacian type with a p-gradient term in dimension N≥3, by using variational methods. Then, we will study an evolution problem of parabolic type with a diffusion term given by the p-Laplacian operator, a nonlocal source and an absorption term with gradient type. For this type of problems, we will establish the existence of local solutions and a weak comparison principle which will play a crucial role in studying some qualitative properties of solutions. In the last chapter of this thesis we will study qualitative properties of positive solutions for a doubly nonlinear problem with variable exponents, in particular with a diffusion of p(x,t)-Laplacian type. The results obtained in this chapter are based on the construction of sub and super-solutions.

Keywords: elliptic problems; parabolic problems; p-Laplacian; comparison principle; super-solutions; sous-solutions; variable exponents; finite time behaviors.

Résumé

Cette thèse s'inscrit dans le domaine mathématique de l'analyse des équations aux dérivées partielles non linéaires. Plus précisément, on s'intéresse d'abord, dans ce travail, à l'étudie de l'existence et de la multiplicité des solutions pour une classe de problèmes elliptiques de type p-Laplacien avec un terme d'ordre p sur le gradien en dimension N≥3, en utilisant des méthodes variationnelles. Ensuite on se propose d'étudier un problème d'évolution de type parabolique avec un terme diffusion donnée par l'opérateur p-Laplacien, une source non-locale et une absorption de type gradient. On établit pour ce type de problème, l'existence de la solution locale et on montre un principe de comparaison faible qui jouera un rôle crucial pour prouver quelques propriétés qualitatives des solutions. Dans le dernier chapitre de cette thèse, on étudiera les propriétés qualitatives des solutions positives pour un problème doublement non linéaire avec des exposants variables et notamment avec une diffusion de type p(x,t)-Laplacien. Les résultats obtenus dans ce chapitre reposent sur la construction de sous et sur-solutions.

Mots-clés: problèmes elliptiques; problèmes paraboliques; p-Laplacien; principe de comparaison; sur-solutions; sous-solutions; exposants variables; comportements en temps fini.

Année Universitaire : 2020 - 2021

✉ Faculté des Sciences, avenue Ibn Battouta, BP. 1014 RP, Rabat –Maroc
☎ 00212(0)5 37 77 18 76 📠 00212(0)5 37 77 42 61 ; http://fsr.um5.ac.ma/