UNIVERSITÉ SIDI MOHAMED BEN ABDELLAH FACULTÈ DES SCIENCES ET TECHNIQUES FÈS

THÈSE DE DOCTORAT

Présentée par

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Discipline: Mathématiques

Spécialité : Algèbre

Sujet de la thèse:

Autour des idéaux n-absorbants et des idéaux (m,n)-fermés d'anneaux commutatifs

Date prévue: 16 Novembre 2019, devant le jury composé de:

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ON n-ABSORBING IDEAL AND (m,n)-CLOSED IDEAL IN A COMMUTATIVE RING

Dédicace

A ma très chère mère lalla Khadija Mejdaoui

Affable, honorable, aimable : Tu représentes pour moi le symbole de la bonté par excellence, la source de tendresse et l'exemple du dévouement qui n'a pas cessé de m'encourager et de prier pour moi. Ta prière et ta bénédiction m'ont été d'un grand secours pour mener à bien mes études. Aucune dédicace ne saurait être assez éloquente pour exprimer ce que tu mérites pour tous les sacrifices que tu n'as cessé de me donner depuis ma naissance, durant mon enfance et même à l'âge adulte.

Je te dédie ce travail en témoignage de mon profond amour. Puisse Dieu, le tout puissant, te préserver et t'accorder santé, longue vie et bonheur.

A la mémoire de mon père Issoual Abdessalam

Aucune dédicace ne saurait exprimer l'amour, l'estime, le dévouement et le respect que j'ai toujours eu pour vous. Rien au monde ne vaut les efforts fournis jour et nuit pour mon éducation et mon bien être. Ce travail est le fruit de tes sacrifices que tu as consentis pour mon éducation et ma formation.

A mes chèrs frères Abdellatif, Aziz et Karim

Avec tous mon amour je vous dédie ce modeste travail en vous souhaitant une vie pleine de bonheure et de réussite.

A mes chères soeurs Amina, Bouchra, Saida et Najat

En témoignage de lattachement, de l'amour et de l'affection que je porte pour vous. Je vous dèdie ce travail avec tous mes voeux de bonheur, de santé et de réussite.

A mes adorables nièces Zineb et Ferdaous

Pour toute l'ambiance dont vous m'avez entouré, pour toute la spontanéité et votres élans chaleureux, Je vous dédie ce travail . Puisse Dieu le tout puissant exhausser tous vos voeux.

A tous les membres de l'équipe de recherche Algèbre commutative et Aspect Homologique de la Faculté des Sciences et Techniques de Fès

Najib Mahdou, Aziza Rahmouni Hassani, Mohammed Tamekkante, Chahrazade Bakkari, Mohamed Chhiti.

Remerciements

► Je voudrais tout d'abord remercier **Najib Mahdou**, mon directeur de thèse qui est à l'origine de ce travail. C'est un honneur pour moi de travailler avec lui et je ne peux qu'admirer son talent. Je lui suis infiniment reconnaissant non seulement parce qu'il a accepté de me prendre en thèse, mais aussi parce qu'il a partagé ses idées avec moi. Il a dirigé ma thèse avec beaucoup de patience et m'a énormément encouragé tout en me laissant la liberté dans mes recherches.

► Depuis le début de ma thèse **Ayman Badawi** a été trés présent. Cette présence a été couronnée par une collaboration sur un travail qu'on a fait ensemble lui, mon directeur de thèse et moi même. Je le remercie énormémemt pour son ecout à mes questions au cours des rencontres aux congrés internationales et aussi pour sa gentillesse qui ébranle des montagnes.

► Je remercie aussi vivement le professeur **Abdelmoujib Benkirane**, pour l'honneur qu'il me fait en acceptant de présider ce jury. Votre sourire et votre bonne humeur allégeront le poids de la soutenance et contribueront à ensoleiller des souvenirs que je garderai et enrichiseront les souvenirs que j'ai de vous en tant que mon professeur de Licence (Théorie des distributions et équations différentielles) et de maîtrise(Analyse Fonctionnelle Appliquée).

► C'est également avec plaisir que je remercie chaleureusement les professeurs **Abdellatif Rochdi**, **Lahcen Oukhtite** et **Mohammed Tamekkante**, pour l'attention avec laquelle ils ont lu ce manuscrit et à l'élaboration de leurs rapports ainsi que l'interêt qu'ils ont porté à mon travail. Merci d'avoir accepté cette charge.

► J'adresse aussi mes remerciements les plus sincères aux examinateurs de ce travail, les professeurs **Mohamed Chhiti, Abdellah Mamouni, Aziza Rahmouni Hassani, Mohammed Sobrani** pour l'honneur que vous me faites de participer à mon jury.

► Mes remerciements vont également à **Moutu Abdou Salam Moutui** et **Mohammed**

Tamekkante pour leurs collaborations dans trois de mes articles. Je remercie tous les membres du laboratoire d'algèbre commutative et non commutative, en particulier **Lahcen oukhtite** pour son accueille très chaleureux.

► Presque par un coup du hasard que j'ai fait ma thèse au sein de l'équipe d'algèbre commutative du laboratoire "Algèbre, Analyse Fonctionnelle et Applications" et il n'est pas nécessaire d'en faire partie pour constater qu'on ne peut travailler dans une ambiance plus stimulante surtout au cours des séminaires organisés au sein du laboratoire. Je voudrais remercier **Rachida El Khalfaoui, Abdelhak El Khalfi, Sanae Moussaoui et Youssef Zahir** pour leurs remarques et questions au cours des séminaires, dont beaucoup ont engendré pas mal d'idées et développé d'autres. Ils ont souvent entendu des répétitions de mes exposés et j'ai profité de beaucoup de conversations avec eux.

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Author's papers involved in this thesis/Articles de l'auteur inclus dans cette thèse:

(1) *Trivial Extension defined by 2-absorbing-like conditions* (with N. Mahdou), Journal of Algebra and its Applications,Vol.17, N. 11 (2018), 1850208, 10 pp.

(2) *Amalgamated algebras along an ideal defined by 2-absorbing like conditions* (with N. Mahdou), accepted for publication in Italian Journal of Pure and Applied Mathematics,2020.

(3) *Rings in which every 2-absorbing ideal is a prime*(with N.Mahdou), accepted for publication in Volume of Recent Results in Pure mathematics, Springer.

(4) *On n-absorbing ideals and (m,n)-closed ideals in trivial ring extensions of commutative ring* (with A. Badawi and N. Mahdou), Journal of Algebra and its Applications, Vol.18, N. 7 (2019), 1950123, 19 pp.

(5) *On n-absorbing and strongly n-absorbing ideals in amalgamated algebras* (with N. Mahdou and M.A. Moutui), Journal of Algebra and its Applications 2020.

(6) *On (m,n)-closed ideals in amalgamated algebras* (with N. Mahdou and M.A. Moutui), submitted for publication.

Other papers/Autres articles:

(1) *On weakly (m,n)-submodule*, (with N. Mahdou and M. Tamekkante), submitted for publication.

(2) *On n-AB ring and on $\Omega(R)$ where R is a commutative ring*(with D.F. Anderson and N. Mahdou), submitted for publication.

(3) *Two-absorbing factorisation for trivial extension and amalgamated duplication*, (with Malik Tusif, Azeem Khadam and Najib Mahdou), to be submitted for publication.

(4) *Commutative ring with N-factorisation*, (with Malik Tusif, Azeem Khadam and Najib Mahdou), to be submitted for publication.

(5) *On n-Rings* (with N. Mahdou), in preparation.

(6) *On graded (m,n)-submodule* (with N. Mahdou), to be submitted for publication.

(7) *On graded weakly (m,n)-submodules* (with N. Mahdou and A. El Khalfi), in preparation.

(8) *Ring in which every (m,n)-closed ideal is prime*, in preparation

Résumé

La théorie des idéaux en algèbre commutative a connu plusieurs développements depuis **Noether, Kummer, Dedekind** et d'autres. Dans ce sens **D. F. Anderson et A. Badawi** ont introduit la notion d'idéal n-absorbant comme généralisation d'idéal premier comme suit: Soit I un idéal propre d'un anneau commutatif R et $n \geq 1$ un entier positif. On dit que I est un idéal n-absorbant (resp., fortement n-absorbant) si pour tout $a_1, ..., a_{n+1} \in R$ avec $a_1....a_{n+1} \in I$, (resp., pour tous idéaux $I_1, ..., I_{n+1}$ de R tel que $I_1...I_{n+1} \subseteq I$), alors I contient au moins un produit de n éléments des $a_i, 1 \leq i \leq n+1$(resp., contient au moins un produit de n idéaux parmi les idéaux $I_i, 1 \leq i \leq n+1$).

Ils ont établit plusieurs propriétés d'idéaux n-absorbants et ont formulé trois conjectures:

• La première conjecture: Soit I est un idéal d'un anneau R, alors I est un idéal n-absorbant de R si et seulement si I est un idéal fortement n-absorbant de R.

• La deuxième conjecture: Si I est un idéal n-absorbant, alors $(\mathrm{Rad}(I))^n \subseteq I$.

• La troisième conjecture: Si I est un idéal d'un anneau R, alors I est un idéal n-absorbant si et seulement si $I[X]$ est un idéal n-absorbant de $R[X]$.

Les travaux de cette thèse se situent dans la théorie des idéaux en algèbre commutative suivant deux axes en s'appuyant sur les travaux de D. F. Anderson et A. Badawi notamment les travaux sur les idéaux n-absorbants, les idéaux (m,n)-fermés et les travaux sur les anneaux réguliers au sens de Von Neumann.

Dans un premier axe nous avons étudié la notion d'idéal 2-absorbant dans l'extension triviale et l'amalgamation et nous avons aussi étudié les anneaux commutatifs dans lesquels tout idéal 2-absorbant est un idéal premier. Puis nous avons élargi le domaine de validité des conjectures de D. F. Anderson et A. Badawi qui sont satisfaites dans les domaines de Prüfer, et arithmétiques à des anneaux avec diviseurs de zéro et d'autres extensions d'anneaux et classes d'anneaux.

Dans un deuxième axe on a étudié la notion d'idéal (m,n)-fermé notion variante de la notion n-absorbant. L'objectif est de construire des exemples de tels idéaux en utilisant l'extension triviale et amalgamée dans des cas particuliers. On a profité de cette étude

pour transférer des notions tel que anneau régulier au sens Von Neumann, anneau π-régulier et anneau semi-simple aux extensions amalgamées qui peuvent être caractérisées par les idéaux (m,n)-fermés.

Summary

The theory of ideals in commutative algebra has undergone several developments since **Noether, Kummer, Dedekind** and others. In this sense **D. F. Anderson and A. Badawi** introduced the notion of n -absorbing ideal and establishing several properties and formulating three conjectures:

• The first conjecture: If I is a ideal of a commutative ring R, then I is a n-absorbing ideal of R if and only if I is strongly n-absorbing ideal of R.

• The second conjecture: If I is an n -absorbing ideal, $(\sqrt{I})^n \subseteq I$.

• The third conjecture: If I is a ideal of a commutative ring R, then I is n -absorbing if and only if $I[X]$ is a n-absorbing ideal of $R[X]$.

In a first axis we have widened the domain of validity of Anderson-Badawi conjectures which are satisfied in a Prü fer domain, and arithmetic rings to rings with divisors of zero and other ring extensions and other categories of rings.

In a second axis, we have studied the notion of the (m,n)-closed ideal. The objective is to build examples of such ideals using the trivial extension and amalgamated extension in simple cases to overcome the difficulty of the shape of ideal in such extensions. We take advantage of this study to transfer notions such as Von Neumann ring, π-regular and semi-simple ring to the amalgamated extensions that can be charted by (m,n)-closed ideals.

Introduction

Introduction

Au cours de la conférence qui a eu lieu à Cortona en Italy En 2004, A. Badawi a définit la notion d'idéal 2-absorbant généralisant celle d'idéal premier comme suit: Si I est un idéal propre de R, on dit que I est un idéal 2-absorbant (resp., fortement 2-absorbant) si $abc \in R$ (resp., $I_1I_2I_3 \subseteq I$) pour $a,b,c \in R$ (resp.,pour des idéaux I_1, I_2, I_3 de R), alors $ab \in I$ ou $bc \in I$ ou $ac \in I$ (resp., $I_1I_2 \subseteq I$ ou $I_1I_3 \subseteq I$ ou $I_2I_3 \subseteq I$) puis A. Badawi a montré dans [9] en 2007 que les deux notions sont identiques. Il a donné beaucoup de propriétés d'idéal 2-absorbant ainsi que des exemples d'anneaux qui contiennent des idéaux 2-absorbants non premiers puisque un idéal premier est clairement un idéal 2-absorbant. Notamment dans un anneau principal R les idéaux 2-absorbants non premiers sont de la forme p^2R ou pqR avec p et q des éléments premiers de R. Suite à l'article [9] beaucoup d'autres travaux ont vu le jour traitant des idéaux 2-absorbants voir [16],[17],[11],[14],[26],[28],[40] [41],[66],[72], et d'autres.

En 2011 D. F. Anberson et A. Badawi ont généralisé la notion d'idéal 2-absorbant en définissant la notion d'idéal n-absorbant et d'idéal fortement n-absorbant pour un entier naturel $n \geq 1$ comme suit:

Definition 0.0.1 ***Idéal n-absorbant et idéal fortement n-absorbant:***
Soient I un idéal propre d'un anneau commutatif R, et $n \geq 1$ un entier positif.

1. *I est dit un idéal n-absorbant si pour tout $a_1, ..., a_{n+1} \in R$ avec $a_1....a_{n+1} \in I$, alors I contient au moins un produit de n éléments des $a_i, 1 \leq i \leq n+1$.*

2. *I est dit un idéal fortement n-absorbant si pour tous idéaux $I_1, ..., I_{n+1}$ de R tel que $I_1...I_{n+1} \subseteq I$, alors I contient au moins un produit de n idéaux parmi les idéaux $I_i, 1 \leq i \leq n+1$.*

D. F. Anderson et A. Badawi ont donné des résultats basiques sur les idéaux n-absorbants. De plus ils ont prouvé dans [5] que si I est un idéal 2-absorbant de l'anneau

R, alors $I[X]$ est un idéal 2-absorbant de $R[X]$ et ont formulé trois conjectures comme ci dessous:

Soit $n \geq 1$ un entier positif.

• Première conjecture

Si I est un idéal de R, alors I est un idéal n-absorbant de R si et seulement si I est un idéal fortement n-absorbant de R.

• Deuxième conjecture

Si I est un idéal n-absorbant de R, alors $(\mathrm{Rad}(I))^n \subseteq I$.

• Troisième conjecture

Si I est un idéal n-absorbant de R, alors $I[X]$ est un idéal n-absorbant de $R[X]$.

Plusieures auteurs ont contribué à la validité des trois conjectures, voir par exemple [45], [43],[68],[77],[78].

La deuxième conjecture est démontrée d'une manière complète et élégante dans [78]. On trouve aussi dans [45] et [44] une élégante preuve de la deuxième conjecture en se ramenant à un anneau Noethérian et en utilisant une récurrence.

Le but de mon travail est de valider les conjectures de D. F. Anderson et A. Badawi dans les extensions triviales et les extensions amalgamées et voir même prouver une partie des conjectures dans certains classes d'anneaux. Pour cela j'ai commencé par transférer la notion d'idéal 2-absorbant puis n-absorbants pour n entier positif ≥ 2 dans l'extension triviale et l'extension amalgamée et obtenir ainsi des exemples qui vont servir de bases pour la suite.

La notion d'idéal premier minimal est liée aux idéaux n-absorbants selon le théorème.

Theorem 0.0.2 *[5, Theorem 2.15]*
Si I est un idéal n-absorbant de R. Alors I admet au plus n idéaux premiers minimaux.

Les idéaux premiers minimaux sont caractérisés par le théorème suivant.

Lemma 0.0.3 ***idéal premier minimal****[54, Theorem 2.1]*
Soient $I \subseteq P$ des idéaux de R avec P idéal premier. On a l'équivalence entre:

1. *P est premier minimal sur I.*
2. *pour tout $x \in P$, il existe $y \in R \setminus P$ et un entier n positif tel que $yx^n \in I$.*

Si I est un idéal propre n-absorbant d'un anneau R, et si $\mathrm{Min}_R(I)$ désigne l'ensemble des idéaux premiers minimaux sur I et

$$\omega_R(I) = \min\{k \in \mathbb{N} / \quad I \text{ est un idéal } k\text{-absorbant dans } R\}$$

alors $|\mathrm{Min}_R(I)| \leq \omega_R(I)$.

Definition 0.0.4 ***Anneau divisible et idéal premier divisible***

1. *Soit P un idéal premier d'un anneau commutatif R. On dit que P est un idéal premier divisible, si $P \subseteq (x)$ pour tout $x \in R \setminus P$.*

2. *L'anneau R est dit divisible si tout idéal premier est divisible.*

Definition 0.0.5 ***L'extension triviale***

Soient A un anneau commutatif, M un A-module et $R := A \propto M = \{(a,e)/a \in A, e \in M\}$ muni de l'addition composante par composante et la multiplication : $(a,e).(b,f) = (ab, af+be)$ pour tout $(a,e),(b,f)$ de R. R muni de ces deux lois est un anneau commutatif dit extension triviale de A par M.

On a $\mathrm{Spec}(R) = \{P \propto M / P \in \mathrm{Spec}(A)\}$ et $\mathrm{Max}(R) = \{P \propto M / P \in \mathrm{Max}(A)\}$.

Pour tout idéal I de A, on a $I \propto M$ est un idéal de R. Et si N est un sous-module de M, alors $I \propto N$ est un idéal de R si et seulement si $IM \subseteq N$.

En général on ne connaît pas la forme des idéaux de R à l'exception de quelques cas, par exemple le cas où A est intègre et M un A-module divisible, sous ces deux conditions les idéaux de R sont $I \propto M$ où I un idéal de A et $0 \propto N$ où N est un sous-module de M.

Definition 0.0.6 ***L'amalgamation de A et B le long l'idéal J en respectant f.***

• *Soient $f : A \to B$ un morphisme d'anneaux commutatifs, J un idéal de B. L'amalgamée de A et B le long J en respectant f est le sous-anneau de $A \times B$ donné par $A \bowtie^f J := \{(a, f(a)+j); a \in A, j \in J\}$.*

• *Si I est un idéal de A la duplication de A suivant I est le sous-anneau de $A \times A$ donné par: $A \bowtie I := \{(a, a+i)/a \in A, i \in I\}$.*

On rappelle que si P est un idéal premier de A, alors $P'^f := P \bowtie^f J$ est un idéal premier de $A \bowtie^f J$ et que si Q est un idéal premier de B, alors $\overline{Q}^f := \{(a, f(a)+j) : a \in A, j \in J, f(a)+j \in Q\}$ est un idéal premier de $A \bowtie^f J$ et que les idéaux premiers de $A \bowtie^f J$ sont de la forme $P'^f \cup \overline{Q}^f$ où $Q \in \mathrm{Spec}(B) \backslash V(J)$. De même $\mathrm{Max}(A \bowtie^f J) = \{M \bowtie^f J / M \in \mathrm{Max}(A)\} \cup \{\overline{Q}^f / Q \in \mathrm{Max}(B) \backslash V(J)\}$ où $V(J) := \{Q \in \mathrm{Spec}(B) / J \subseteq Q\}$.

En 2016 D. F. Anderson et A. Badawi ont introduit la notion d'idéal (m,n)-fermé [6] comme suit:

Definition 0.0.7 • *Soit R un anneau commutatif avec* $1 \neq 0$. *On rappelle d'aprés [6] qu'un idéal propre I de R est dit* (m,n)*-fermé si* $x^m \in I$ *pour* $x \in R$ *implique* $x^n \in I$.

• *Soit R un anneau commutatif avec* $1 \neq 0$. *On rappelle d'aprés [6] qu'un idéal propre I de R est dit fortement* (m,n)*-fermé si* $J^m \subseteq I$ *pour J un idéal de R implique* $J^n \subseteq I$.

• *Soit R un anneau commutatif avec* $1 \neq 0$. *On rappelle d'aprés [6] qu'un idéal propre I de R est dit semi n-absorbant si* $x^{n+1} \in I$ *pour* $x \in R$ *implique* $x^n \in I$.

• *Soit R un anneau commutatif avec* $1 \neq 0$. *On rappelle d'aprés [6] qu'un idéal propre I de R est dit fortement semi n-absorbant si* $J^{n+1} \subseteq I$ *pour un idéal J de R implique* $J^n \subseteq I$.

Dans [6] D. F. Anderson et A. Badawi ont fournit des propriétés basiques des idéaux (m,n)-fermés, notamment une caractérisation des anneaux réguliers au sens de Von Neumann et les anneaux π-réguliers à l'aide des idéaux (m,n)-fermés.

Definition 0.0.8 *Soit R un anneau commutatif. R est dit régulier au sens de Von Neumann, si pour tout* $x \in R$ *il existe* $y \in R$ *tel que* $x = x^2y$. *R est dit* π*-régulier si pour tout* $x \in R$ *il existe* $y \in R$ *et un entier positif n tel que* $x^n = x^{2n}y$. *On rappelle que De plus R est* π*-régulier (resp., régulier au sens de Von Neumann) si et seulement si* $dim(R) = 0$*(resp., R est réduit et dim(R)=0.) [54, Theorem 3.1].*

On a aussi R est π*-régulier si et seulement si* $R/Nil(R)$ *est régulier au sens de Von Neumann.*

Notre contribution dans cette partie est de voir comment la notion d'idéal (m,n)-fermés se transfert à l'extension triviale et à l'amalgamation. On donne une condition pour que les idéaux homogénes de la forme $I \propto N$ où I un idéal de A et N un sous-module propre de M soient des idéaux (m,n)-fermés de l'extension triviale $A \propto M$. Ainsi on obtient des exemples d'idéaux (m,n)-fermés dans des anneaux avec diviseurs de zéro.

0n va utiliser les idéaux (m,n)-fermés pour transférer à l'amalgamation la notion d'anneau semi-simple et aussi d'anneau régulier au sens de Von Neumann et d'anneau π-régulier.

Part I

Partie I: Les chapitres en français

CHAPTER 1

CHAPITRE 1
NOTION D'IDÉAL 2-ABSORBANT DANS L'EXTENSION TRIVIALE

Sommaire

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1.1 Introduction

Dans ce chapitre * tous les anneaux considérés sont unitaires et commutatifs. En 2007 A. Badawi a défini dans [9] la notion d'idéal 2-absorbant comme suit: Si I un idéal propre d'un anneau R. On dit que I est un idéal 2-absorbant (resp., fortement 2-absorbant) de R si pour $a,b,c \in R$ (resp., pour tous idéals I_1,I_2,I_3 de R) tels que $abc \in I$ (resp., $I_1I_2I_3 \subseteq I$), alors $ab \in I$ ou $bc \in I$ ou $ac \in I$ (resp., $I_1I_2 \subseteq I$ ou $I_1I_3 \subseteq I$ ou $I_2I_3 \subseteq I$). A. Badawi a prouvé que les deux notions sont identiques. La notion d'idéal 2-absorbant généralise la notion d'idéal premier, les deux notions sont distinctes comme on le voit pour $6\mathbb{Z}$ qui est un idéal 2-absorbant non premier dans $\mathbb{Z}$.
Un idéal I est dit faiblement 2-absorbant (resp., faiblement fortement 2-absorbant) de

*J.Algebra and its Appl.Vol.17, N. 11 (2018). (En collaboration avec N. Mahdou).

R si pour $a,b,c \in R$(resp., pour tous idéaux I_1,I_2,I_3 de R) tels que $0 \neq abc \in I$(resp., $0 \neq I_1I_2I_3 \subseteq I$), alors $ab \in I$ ou $bc \in I$ ou $ac \in I$(resp., $I_1I_2 \subseteq I$ ou $I_1I_3 \subseteq I$ ou $I_2I_3 \subseteq I$). Par définition $\{0\}$ est un idéal faiblement 2-absorbant de R.

Un idéal I est dit idéal 2-absorbant primaire (resp., faiblement 2-absorbant primaire) de R. Si $a,b,c \in R$ tels $abc \in I$(resp.,$0 \neq abc \in I$,) alors $ab \in I$ ou $bc \in \mathrm{Rad}(I)$ ou $ac \in \mathrm{Rad}(I)$. Cette notion généralise celle d'idéal primaire.

On rappelle que si A est un anneau commutatif et E un A-module, l'extension triviale de A par E est le groupe $A \times E$ muni de l'addition composante par composante et de la multiplication $(a,e).(b,f) = (ab, af+be)$ ces deux lois font de

$$R := A \propto E = \{(a,e); a \in A, e \in E\}$$

un anneau commutatif tel que A s'injecte de manière canonique dans R via le morphisme $a \in A \mapsto (a,0) \in R$. On remarque que $A \propto E$ est un anneau avec diviseur de zéro de plus $0 \propto E$ est un idéal qui vérifie $(0 \propto E)^2 = 0$.

On rappelle que $\mathrm{Spec}(A \propto E) = \{P \propto E; P \in \mathrm{Spec}(A)\}$, et que $\mathrm{Max}(A \propto E) = \{M \propto E; M \in \mathrm{Max}(A)\}$. Et que si I est un idéal de A et F un sous-module de E, alors $I \propto F$ est un idéal de $A \propto E$ si et seulement si $IE \subseteq F$.

1.2 Idéal 2-absorbant dans l'extension triviale

On va étudier les idéaux 2-absorbants dans l'extension triviale.

Définition 1.2.1 *Un idéal propre I d'un anneau commutatif R est dit idéal 2-absorbant si pour tout $a,b,c \in R$ tel que $abc \in I$, on a: $ab \in I$ ou $bc \in I$ ou $ac \in I$.*

Théorème 1.2.2 *Soient A un anneau commutatif, E un A- module, $R := A \propto E$ et I un idéal de A. Alors, $I \propto E$ est un idéal 2-absorbant de R si et seulement si I est un idéal 2-absorbant de A.*

Soient A est un anneau intègre de corps de fractions K, E un K-espace vectoriel, on sait que les idéaux de $A \propto E$ sont de la forme $I \propto E$ ou $0 \propto F$ où I est idéal de A et F un A-sous module de E.

Théorème 1.2.3 *Soient A un anneau intègre de corps de fractions K et E un K-espace vectoriel. Alors les idéaux 2-absorbants de $R := A \propto E$ sont de la forme $I \propto E$ ou $0 \propto F$ avec I un idéal de A et F est un K-sous-espace vectoriel de E.*

La preuve de ce théorème nécessite le lemme suivant:

Lemme 1.2.4 *Soient A un anneau intègre de corps de fractions K, E un K-espace vectoriel et F un A-sous-module de E. Alors $0 \propto F$ est un idéal 2-absorbant de $R := A \propto E$ si et seulement si F est un K-sous-espace vectoriel de E.*

Remarque 1.2.5 *Le théorème 1.2.2 montre que si I est un idéal 2-absorbant de A, alors $I \propto E$ est un idéal 2-absorbant de $A \propto E$. Ceci tombe en défaut pour les idéaux homogènes de la forme $I \propto N$ où N est A-sous-module propre de E tel que $IE \subseteq N$ comme le montre le contre-exemple suivant. On prend $A = E = \mathbb{Z}$ et $N = 6\mathbb{Z}$ et $I = \{0\}$. On a $0 \propto 6\mathbb{Z}$ est un idéal de $\mathbb{Z} \propto \mathbb{Z}$ non 2-absorbant, puisque $(0,1)(2,2)(3,3) = (0,6)$, mais $(0,1)(2,2) = (0,2), (2,2)(3,3) = (6,12)$ et $(0,1)(3,3) = (0,3)$ aucun de ces trois produit n'est dans $0 \propto 6\mathbb{Z}$.*

Dans ce sens on a le théorème suivant qui donne une caractérisation pour que l'idéal $I \propto N$ soit 2-absorbant de $R = A \propto E$.

Théorème 1.2.6 *On considère A un anneau commutatif, E un A-module et $R := A \propto E$. Si I un idéal de A, alors on a les assertions suivantes:*

1. *Si $I \propto N$ est un idéal 2-absorbant de R, alors I est un idéal 2-absorbant de A.*
2. *Si I est un idéal 2-absorbant et P-primaire de A et $[N : E] = P$ avec P un idéal premier de R. Alors $I \propto N$ est un idéal 2-absorbant de R.*

Définition 1.2.7 ***Idéal premier divisible***

Un idéal P premier d'un anneau commutatif A est dit idéal premier divisible si $P \subseteq Ax$ pour tout x dans $A \setminus P$.

L'anneau A est dit divisible si tout idéal premier est divisible.

Définition 1.2.8 ***Idéal P-primaire et sous-module P-primaire***

• *Un idéal I d'un anneau commutatif A est dit P-primaire où P un idéal premier de A si pour tout $a, b \in A$ avec $ab \in I$ on a: $a \in I$ ou $b \in Rad(I)$.*

• *Un sous-module N un d'un A-module M est dit P- primaire où P un idéal premier de A anneau commutatif si, $(N :_A M)$ est un idéal P-primaire de A.*

Corollaire 1.2.9 *On considère A un anneau commutatif, E un A-module et $R := A \propto E$. Soit P un idéal premier divisible non nul de A et I un idéal 2-absorbant de A dont le radical est P et N un A-sous-module propre P-primaire de E tel que $IE \subseteq N$. Alors $I \propto N$ est un idéal 2-absorbant de R vérifiant $(P \propto E)^2 \subseteq I \propto N$.*

Corollaire 1.2.10 *Soient A est un anneau commutatif et P un idéal premier propre de A. Alors $P \propto P$ est un idéal 2-absorbant de $A \propto A$ qui n'est pas un idéal premier de A.*

Finalement on a le résultat suivant dans le cas des anneaux locaux

Proposition 1.2.11 *On suppose que A un anneau commutatif local d'idéal maximal M tel que $M^2 = 0$ et E un A-module tel que $ME = 0$. Alors tout idéal de $A \propto E$ est un idéal 2-absorbant de $A \propto E$.*

1.3 Idéal 2-absorbant primaire dans l'extension triviale

Dans cette section nous étudions le transfert de la notion d'idéal primaire 2-absorbant dans des extensions triviales.

Définition 1.3.1 *Un idéal propre I d'un anneau commutatif R est dit un idéal 2-absorbant primaire si pour tout $a,b,c \in R$ tel que $abc \in I$ on a: $ab \in I$ ou $bc \in Rad(I)$ ou $ac \in Rad(I)$.*

Théorème 1.3.2 *Soient A un anneau commutatif, E un A-module et $R := A \propto E$. Considérons I un idéal de A et N un A-sous-module de E tel que $IE \subseteq N$. Alors on a l'équivalence suivante:*

$I \propto N$ est un idéal 2-absorbant primaire de R si et seulement si I est un idéal 2-absorbant primaire de A.

Corollaire 1.3.3 *Soient A un anneau commutatif, E un A-module, $R := A \propto E$ et I un idéal de A. Alors I est un idéal 2-absorbant primaire de A si et seulement si $I \propto E$ est un idéal 2-absorbant primaire de R.*

Proposition 1.3.4 *Si A un anneau commutatif, E un A-module et $R := A \propto E$. Soit I un idéal de A et N un A-sous-module de E tel que $IE \subseteq N$. On suppose que $Rad(I)$ est un idéal premier de A, alors $I \propto N$ est un idéal 2-absorbant primaire de R.*

On sait que si I est idéal d'un anneau commutatif A tel que $\mathrm{Rad}(I) = P$ est un idéal 2-absorbant de A, mais I n'est pas forcément 2-absorbant primaire comme le montre l'exemple suivant:

On prend $A = \mathbb{Z}[Y] + 3X\mathbb{Z}[Y,X]$, soit $P = YA$ et $Q = 3X\mathbb{Z}[Y,X]$ qui sont des idéaux premiers de A. Et soit $I = PQ^2$ on a $\mathrm{Rad}(I) = P \cap Q$ est un idéal 2-absorbant de A et on a de plus $3X^2.Y.3 \in I$, mais $3X^2.Y \notin I, 3X^2.3 = 9X^2 \notin \mathrm{Rad}(I)$ et $Y.3 = 3Y \notin \mathrm{Rad}(I)$. Donc I n'est pas un idéal 2-absorbant primaire de A.

Proposition 1.3.5 *Soient A un anneau commutatif divisible, E un A-module et $R := A \propto E$. On considère I un idéal de A dont le radical est un idéal premier de A et N un A-sous module de E tel que $IE \subseteq N$. Alors $I \propto N$ est un idéal 2-absorbant primaire de R.*

On montre dans [5, Theorem 4.10] que si P est idéal premier d'un anneau intègre A, alors $0 \propto P$ est un idéal 2-absorbant de $A \propto A$. On va donner quelques résultats similaires dans le cas des idéaux 2-absorbants primaires.

Théorème 1.3.6 *Soient P est un idéal premier d'un anneau commutatif A et $R = A \propto A$. Alors on a les assertions suivantes:*

1. *L'idéal $P \propto P$ est 2-absorbant primaire de R.*
2. *L'idéal $0 \propto P$ est 2-absorbant primaire de R.*

1.4 Idéal faiblement 2-absorbant dans l'extension triviale

On va étudier la notion d'idéal faiblement 2-absorbant dans l'extension triviale. On verra que plusieurs résultats obtenus sur les idéaux 2-absorbants ne restent plus vraies pour les idéaux faiblement 2-absorbants. Il faut noter que les deux notions sont distinctes puisque, l'idéal nul est faiblement 2-absorbant dans A par définition et pas forcément 2-absorbant. Par contre elles sont identiques sur les anneaux sans diviseurs de zéro.

Définition 1.4.1
1. *Un idéal I d'un anneau commutatif A est dit idéal faiblement 2-absorbant, si pour tout $,a,b,c \in A$ avec $0 \neq abc \in I$ on a:$ab \in I$ ou $ac \in I$ ou $bc \in I$.*
2. *On dit que $(a,b,c) \in A^3$ est un zéro-triple de I si $abc = 0$ et aucun des produits ab,ac et bc n'est dans I.*

Théorème 1.4.2 *Soient A un anneau commutatif, E un A-module, $R := A \propto E$ et I un idéal propre de A. Alors on a l'équivalence entre les deux assertions suivantes:*

1. *$I \propto E$ est un idéal faiblement 2-absorbant de R,*
2. *I est un idéal faiblement 2-absorbant de A et pour tout zéro-triple (a,b,c) de I, on a: $ab \in ann(E)$, $ac \in ann(E)$ et $bc \in ann(E)$.*

Corollaire 1.4.3 *Soient A un anneau commutatif réduit, E un A-module. Alors $0 \propto E$ est le seul idéal de $A \propto E$ de la forme $I \propto E$ (où I est un idéal de A) qui soit un idéal faiblement 2-absorbant non 2-absorbant dans $A \propto E$.*

Remarque 1.4.4 *Si (A,M) un anneau local d'idéal maximal M vérifiant $M^3 = 0$ et E un A-module tel que $ME = 0$. Alors tout idéal de $A \propto E$ est faiblement 2-absorbant.*

Proposition 1.4.5 *Si (A,M) un anneau local d'idéal maximal M, E un A-module tel que $ME = 0$. Alors tout idéal de $(A \propto E) \times (A \propto E)$ est faiblement 2-absorbant si et seulement si $M^2 = 0$.*

Remarque 1.4.6 *On montre dans [14, Theorem 2.4] que si I est un idéal faiblement 2-absorbant non 2-absorbant, alors $I^3 = 0$. Mais la réciproque est faux c'est à dire: si $I^3 = 0$ et I non 2-absorbant, ceci n'implique pas que I est faiblement 2-absorbant, comme le montre l'exemple suivant: On a $0 \propto 2\mathbb{Z}$ est un idéal non 2-absorbant de $\mathbb{Z} \propto \mathbb{Q}$ et on a $I^3 = 0$, mais I est non faiblement 2-absorbant, puisque $ann(\mathbb{Q}) = \{0\}$.*

1.5 Idéal faiblement 2-absorbant primaire dans l'extension triviale

Définition 1.5.1 *Un idéal I d'un anneau commutatif A est dit I idéal faiblement 2-absorbant primaire, si pour tout $a,b,c \in A$ avec $0 \neq abc \in I$, on a: $ab \in I$ ou $ac \in Rad(I)$ ou $bc \in Rad(I)$.*

Théorème 1.5.2 *On considère A un anneau commutatif, E un A-module et $R := A \propto E$. Soit I un idéal de A, alors les deux assertions suivantes sont équivalentes:*

1. *$I \propto E$ est un idéal faiblement 2-absorbant primaire de R,*

2. *I est un idéal faiblement 2-absorbant primaire de A et pour tout $(a,b,c) \in A^3$ tel que $abc = 0$, et $ab \notin I, bc \notin Rad(I)$ et $ac \notin Rad(I)$, alors $ab \in ann(E), ac \in ann(E)$ et $bc \in ann(E)$.*

On rappelle la définition d'idéal faiblement premier.

Définition 1.5.3 *Un idéal I d'un anneau commutatif R est dit faiblement premier, si $0 \neq ab \in I$ pour $a,b \in R$ alors $a \in I$ ou $b \in I$.*

Théorème 1.5.4 *Soit I un idéal propre d'un anneau commutatif A tel que $Rad(I)$ est un idéal faiblement premier de A. Si N un A-sous-module de E tel que $IE \subseteq N$ et pour tout $(a,b) \in A^2$ vérifiant $ab = 0$ et $a \notin Rad(I)$ et $b \notin Rad(I)$, implique $a \in ann(E)$ et $b \in ann(E)$. Alors $I \propto N$ est un idéal faiblement 2-absorbant primaire de R.*

La preuve de ce thérème nécessite le lemme suivant.

Lemme 1.5.5 *Si I est un idéal d'un anneau commutatif A tel que $Rad(I)$ est un idéal faiblement premier de A. Alors I est un idéal faiblement 2-absorbant primaire de A.*

1.6 Idéal semi-2-absorbant dans l'extension triviale

Définition 1.6.1 *Un idéal I d'un anneau commutatif A est dit semi-2-absorbant si pour tout $a \in A$ tel que $a^3 \in I$, on a: $a^2 \in I$.*

Théorème 1.6.2 *Soit I un idéal d'un anneau commutatif A, E un A-module et $R := A \propto E$. Alors $I \propto E$ est un idéal semi 2-absorbant de R si et seulement si I est un idéal semi 2-absorbant de A.*

Définition 1.6.3 *Un idéal I d'un anneau commutatif A est dit semi-premier si pour tout $a \in A$ tel que $a^2 \in I$, on a: $a \in I$.*

Proposition 1.6.4 *Soit A un anneau commutatif et I un idéal semi-premier de A. Alors on a:*

1. *$I \propto I$ est un idéal semi 2-absorbant de $A \propto A$ lequel est n'est un idéal semi- premier.*
2. *Si A est réduit, alors $0 \propto I$ est un idéal semi 2-absorbant non semi-premier.*

Finalement on caractérise dans un cas particulier quand un idéal homogène de $A \propto E$ est semi 2-absorbant

Proposition 1.6.5 *Si A un anneau commutatif de caractéristique égale à 2, E un A-module et $R := A \propto E$. Soit I un idéal de A, et N un A-sous-module de E tel que $IE \subseteq N$. Alors $I \propto N$ est un idéal semi-2-absorbant de R si et seulement si I est un idéal semi 2-absorbant de A.*

CHAPTER 2

CHAPITRE 2
NOTION D'IDÉAL 2-ABSORBANT DANS L'AMALGAMATION

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2.1 Introduction

L'objectif de ce chapitre * est d'étudier les idéaux 2-absorbants et les idéaux 2-absorbants primaires dans l'amalgamation.

Soit $f : A \to B$ un morphisme d'anneaux commutatifs, J un idéal de B. On définit l'amalgamée de A par B le long l'idéal J en respectant le morphisme f, le sous-anneau de $A \times B$ donné par:

$$A \bowtie^f J := \{(a, f(a) + j); a \in A, j \in J\}.$$

On rappelle que si P est un idéal premier de A, alors $P'^f := P \bowtie^f J$ est un idéal premier de $A \bowtie^f J$ et que si Q est un idéal premier de B, alors $\overline{Q}^f := \{(a, f(a) + j) : a \in A, j \in$

*Italian Journal of Pure and Applied Mathematics, accepté pour publication (En collaboration avec N. Mahdou).

$J, f(a)+j \in Q\}$ est un idéal premier de $A \bowtie^f J$ et que les idéaux premiers de $A \bowtie^f J$ sont de la forme $P'^f \cup \overline{Q}^f$ où $Q \in \mathrm{Spec}(B)\backslash V(J)$. De même $\mathrm{Max}(A \bowtie^f J) = \{M \bowtie^f J; M \in \mathrm{Max}(A)\} \cup \{\overline{Q}^f; Q \in \mathrm{Max}(B)\backslash V(J)\}$.

2.2 Idéal 2-absorbant dans l'amalgamation

Théorème 2.2.1 *On considère $f : A \to B$ un morphisme d'anneaux commutatifs, J un idéal de B et $R := A \bowtie^f J$. Alors on a les assertions suivantes:*

1. *Soit I est un idéal de A, alors I est un idéal 2-absorbant de A si et seulement si $I \bowtie^f J$ est un idéal 2-absorbant de R.*
2. *Soit P un idéal premier de A et Q un idéal premier de B, alors l'ensemble*

$$\{(a, f(a)+j)\ a \in P,\ j \in J,\ f(a)+j \in Q\}$$

   est un idéal 2-absorbant de R.
3. *On suppose que $Nil(A) \neq P$ sont des idéaux premiers divisibles, alors $P^2 \bowtie^f J$ est un idéal 2-absorbant de R.*
4. *On se donne $m \in max(A)$ et $Q \in max(B) \setminus V(J)$. Soit $M = m \bowtie^f J$ où $M := \{(a, f(a)+j); a \in A, j \in J, f(a)+j \in Q\}$. Alors M^2 est un idéal 2-absorbant de R.*

Exemple 2.2.2 *Soit $R := \mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$ qui est un anneau intègre et $f : \mathbb{Z} \hookrightarrow \mathbb{Q}[[X]]$ l'injection canonique. L'idéal $P = p\mathbb{Z}$ est premier divisible de $\mathbb{Z}$, où p est un entier premier. Donc, selon le théorème 2.2.1 l'idéal $p^2\mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$ est 2-absorbant de R.*

Exemple 2.2.3 *Si $A \subset B$ sont des anneaux commutatifs, $X := \{X_1, \cdots, X_n\}$ ensemble fini d'indéterminées sur B. Dans l'anneau des séries entières formelles, on considère le sous-anneau*

$$A + XB[[X]] := \{h \in B[[X]]; h(0) \in A\}$$

si $\sigma : A \hookrightarrow B[[X]]$ est l'injection canonique et $J := XB[[X]]$, il est donc clair que $A \bowtie^\sigma J$ est isomorphe à $A + XB[[X]]$ via l'isomorphisme $u(a, \sigma(a)+j) = \sigma(a)+j = a+j$.

Si A est intègre de corps de fractions $K, f : A \hookrightarrow K[[X]]$ l'injection canonique et $J := XK[[X]]$. Alors les idéaux 2-absorbants de $A \bowtie^f J$ sont de la forme $I \bowtie^f J$ avec I est un idéal 2-absorbant de A ou $0 \bowtie^f J^k$ avec $k \in \{1, 2\}$.

Exemple 2.2.4 *Soit* $R := \mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$, *où* $f : \mathbb{Z} \hookrightarrow \mathbb{Q}[[X]]$ *est l'injection canonique. Il est clair que R est un anneau de valuation. L'idéal* $I := 0 \bowtie^f (X\mathbb{Z} + X^2\mathbb{Q}[[X]])$ *est non 2-absorbant dans R puisque* $(2,2)^2(0, \frac{1}{2^2X} = (0,X) \in I$, *mais ni* $(2,2)^2 \in I$ *ni* $(2,2)(0, \frac{1}{2^2}X) = (0, \frac{1}{2}X) \in I$. *On remarque que les idéaux* $pq\mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$ *et* $p^2\mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$ *sont 2-absorbants dans R avec le Théorème 2.2.1 et puisque les idéaux 2-absorbants de* $\mathbb{Z}$ *sont de la forme* $pq\mathbb{Z}$ *pour des entiers premiers distincts* $p,q \in \mathbb{Z}$ *ou* $p^2\mathbb{Z}$ *pour un entier premier p. Et par l'exemple 2.3, les idéaux* $0 \bowtie^f X^i\mathbb{Q}[[X]]$ *où* $1 \leq i \leq 2$ *sont des idéaux 2-absorbants de R. On conclut que les seuls idéaux 2-absorbants de R sont exactement* $pq\mathbb{Z} \bowtie^f X\mathbb{Q}[[X]], p^2\mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$ *et* $0 \bowtie^f X^i\mathbb{Q}[[X]]$ *où* $1 \leq i \leq 2$.

Exemple 2.2.5 *Soient* $R := \mathbb{Z} \bowtie^f X\mathbb{R}[[X]]$, $f : \mathbb{Z} \hookrightarrow \mathbb{R}[[X]]$ *l'injection canonique et* $I_1 := 0 \bowtie^f (X\mathbb{Z} + X^2\mathbb{R}[[X]])$, $I_2 := 0 \bowtie^f (X\mathbb{Q} + X^2\mathbb{R}[[X]])$. *Alors:*

1. I_1 *est un idéal qui n'est pas 2-absorbant de R.*
2. I_2 *est un idéal 2-absorbant de R.*

2.3 Idéal faiblement 2-absorbant dans l'amalgamation

Théorème 2.3.1 *Soient* $f : A \to B$ *un morphisme d'anneaux commutatifs, J un idéal de B et* $R := A \bowtie^f J$.

1. *Si I un idéal de A tel que* $I \bowtie^f J$ *est un idéal faiblement 2-absorbant de R, alors I est faiblement 2-absorbant de A.*
2. *Si I un idéal faiblement 2-absorbant qui n'est pas 2-absorbant de A. Alors les assertions suivantes sont équivalentes:*
   (a) $I \bowtie^f J$ *est un idéal faiblement 2-absorbant de* $A \bowtie^f J$ *qui n'est pas 2-absorbant de* $A \bowtie^f J$.
   (b) *Pour tout zéro-triplet* $(a,b,c) \in A^3$ *de I, on a* $f(a)f(b)J = f(a)f(c)J = f(b)f(c)J = 0$ *et* $J^3 = 0$.

La preuve du théorème nécessite le lemme suivant

Lemme 2.3.2 *On considère* $f : A \to B$ *un morphisme d'anneaux commutatifs, P et Q des idéaux de A et J et K des idéaux de B. Alors on a:*

$$(P \bowtie^f J)(Q \bowtie^f K) = PQ \bowtie^f (f(P)K + f(Q)J + JK).$$

En particulier :

$$(I \bowtie^f J)^2 = I^2 \bowtie^f \left(f(I)J + J^2\right)$$

et

$$(I \bowtie^f J)^3 = I^3 \bowtie^f \left(f(I)^2 J + f(I)J^2 + J^3\right).$$

Corollaire 2.3.3 *On considère $f : A \to B$ un morphisme d'anneaux commutatifs, J un idéal de B et I un idéal faiblement 2-absorbant non 2-absorbant de A. Si pour tout zéro-triplet $(a,b,c) \in A^3$ de I on a: $f(a), f(b)$ et $f(c)$ dans $ann(J)$. Alors $I \bowtie^f J$ est un idéal faiblement 2-absorbant non 2-absorbant de $R = A \bowtie^f J$.*

Corollaire 2.3.4 *Soient (A,m) un anneau local d'idéal maximal m, $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B tel que $f(m)J = 0$. Alors les propriétés suivantes sont satisfaites:*

1. *Soit I est un idéal de A. Alors $I \bowtie^f J$ est un idéal faiblement 2-absorbant non 2-absorbant de R si et seulement si I est un idéal faiblement 2-absorbant non 2-absorbant de A et $J^3 = 0$.*
2. *On suppose que $m^3 = 0$. Alors tout idéal de R est faiblement 2-absorbant.*

2.4 Idéal 2-absorbant primaire dans l'amalgamation

Dans cette section, nous étudions le transfert de la notion d'idéal 2-absorbant primaire aux algèbres amalgamées le long d'un idéal.

Théorème 2.4.1 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs, J un idéal de B. Alors*

1. *Un idéal I de A est un idéal 2-absorbant primaire si et seulement si $I \bowtie^f J$ est un idéal 2-absorbant primaire de $A \bowtie^f J$.*
2. *On suppose que I est un idéal de A tel que $Rad(I)$ est un idéal premier de A. Alors $I \bowtie^f J$ est un idéal 2-absorbant primaire de $A \bowtie^f J$.*
3. *Soit P est un idéal premier de A. Alors $P^n \bowtie^f J$ est un idéal 2-absorbant primaire pour tout entier positif n.*
4. *On suppose que A est un anneau divisible et I un idéal de A. Alors $I \bowtie^f J$ est un idéal 2-absorbant primaire de R.*

5. *On suppose que A est un anneau local d'idéal maximal M avec $f^{-1}(Q) \neq M$ pour tout idéal $Q \in Spec(B) \setminus V(J)$ et I un idéal de A dont le radical est M. Alors $I \bowtie^f J$ et $I^e := I(A \bowtie^f J)$ sont des idéaux 2-absorbants primaires de R.*

Pour la preuve du théorème on aura besoin du lemme suivant:

Lemme 2.4.2 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs, J un idéal de B et I un idéal de A. Alors $Rad(I \bowtie^f J) = Rad(I) \bowtie^f J$.*

Corollaire 2.4.3 *On considère M un idéal maximal d'un anneau intègre T et D un sous-anneau de T tel que $M \cap D = \{0\}$. Si I un idéal de D. Alors I est un idéal faiblement 2-absorbant primaire de D si et seulement si $I + M$ est un idéal faiblement 2-absorbant primaire de $D + M$.*

Le résultat suivant permet de déterminer les idéaux 2-absorbants primaires de $D \bowtie^f XK[[X]]$ où D un anneau intègre de corps de fractions K et $f : D \hookrightarrow K[[X]]$ l'injection canonique.

Théorème 2.4.4 *Soient A un anneau intègre de corps de fractions K et $f : A \hookrightarrow K[[X]]$ l'injection canonique et $J := XK[[X]]$. Les idéaux 2-absorbants primaires de $A \bowtie^f J$ sont de la forme $I \bowtie^f J$ où I est un idéal 2-absorbant primaire de A ou $0 \bowtie^f (WX^m + X^{m+1}K[[X]])$ où W est A-sous module non nul de K et m un entier naturel positive.*

Pour démontrer ce théorème on aura besoin du lemme suivant:

Lemme 2.4.5 *Soit D un anneau intègre de corps de fractions K. Alors les idéaux 2-absorbants primaires de $D + XK[[X]]$ sont de la forme $I + XK[[X]]$ où I est un idéal 2-absorbant primaire de A, ou $WX^m + X^{m+1}K[[X]]$ où m un entier positif et W un A-sous module non nul de K.*

2.5 Idéal faiblement 2-absorbant primaire dans l'amalgamation

Dans cette section on rappelle la définition suivante:

Définition 2.5.1 *1. Un idéal propre I d'un anneau commutatif A est dit idéal faiblement 2-absorbant primaire si pour tout $(a,b,c) \in A^3$ tel que $0 \neq abc \in I$, on a: $ab \in I$ ou $ac \in Rad(I)$ ou $bc \in Rad(I)$.*

2. *Un zéro-triplet de I est tout* $(a,b,c) \in A^3$ *tel que* $abc = 0$ *et* $ab \notin I$ *et* $ac \notin Rad(I)$ *et* $bc \notin Rad(I)$.

Théorème 2.5.2 *Soient* $f : A \to B$ *un morphisme d'anneaux commutatifs,* J *un idéal de* B. *Alors on a:*

1. *Si I est idéal de A tel que* $I \bowtie^f J$ *est un idéal faiblement* 2*-absorbant primaire de R alors, I est un idéal faiblement* 2*-absorbant primaire de* A.
2. *Si I est un idéal faiblement* 2*-absorbant primaire non* 2*-absorbant primaire de* A, *alors les propriétés suivantes sont équivalentes:*
   (a) $I \bowtie^f J$ *est un idéal faiblement* 2*-absorbant primaire de R.*
   (b) *Pour tout* $(a,b,c) \in A^3$ *zéro-triplet de I, on a:* $f(a)f(b)J = f(a)f(c)J = f(b)f(c)J = 0$ *et* $J^3 = 0$.

Corollaire 2.5.3 *Soient* $f : A \to B$ *un morphisme d'anneaux commutatifs et J un idéal de B tel que* $J^3 = 0$. *On considère I un idéal faiblement* 2*-absorbant primaire de A tel que si* $(a,b,c) \in A^3$ *est un zéro-triplet de I, on ait* $f(a), f(b)$ *et* $f(c)$ *dans* $ann(J)$. *Alors* $I \bowtie^f J$ *est un idéal faiblement* 2*-absorbant de R.*

Le théorème suivant donne un lien entre la notion de faiblement premier (resp., faiblement primaire) et faiblement 2-absorbant (resp., faiblement 2-absorbant primaire).

Théorème 2.5.4 *Soient* $f : A \to B$ *un morphisme d'anneaux commutatifs,* J *un idéal de B et I un idéal de A tel que* $Rad(I)$ *est faiblement premier de A et pour tout* $a,b \in A$ *avec* $ab = 0, a \notin I, b \notin I$, *on a* $a,b \in ann(J)$. *Alors* $I \bowtie^f J$ *est un idéal faiblement* 2*-absorbant primaire de* $A \bowtie^f J$.

La preuve du théorème nécessite le lemme suivant:

Lemme 2.5.5 *Soient* $f : A \to B$ *un morphisme d'anneaux commutatifs et J un idéal de B. Si I un idéal de* A, *alors les assertions suivantes sont équivalentes:*

1. *L 'idéal I est faiblement premier de* A, $J^2 = 0$, *et* $a,b \in A$ *avec* $ab = 0, a \notin I, b \notin I$, *on a* $a,b \in ann(J)$.
2. $I \bowtie^f J$ *est un idéal faiblement premier de* $A \bowtie^f J$.

CHAPTER 3

CHAPITRE 3
LES ANNEAUX DANS LESQUELS TOUT IDÉAL 2-ABSORBANT EST UN IDÉAL PREMIER

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3.1 Introduction

L'objectif de chapitre * est d'étudier les anneaux dans lesquels tout idéal 2-absorbant est un idéal premier et d'étudier le transfert de la notion de 2-AB anneaux dans l'extension triviale et l'amalgamée.

Bennis et Fahid ont définit un 2-AB anneau comme un anneau dans lequel tout idéal 2-absorbant est un idéal premier. Puis ils ont conjecturé que dans un 2-AB anneau tout idéal premier est idempotent, et ont prouvé la conjecture dans le cas d'un anneau de valuation.

*Volume of recent results in pure mathematics, Springer, accepté pour publication (En collaboration avec N. Mahdou) .

Dans ce chapitre on étudie le transfert de la notion de 2-AB anneau dans l'extension triviale et l'extension amalgamée. On donne également une réponse positive à la conjecture dans certains classes d'anneaux.

Définition 3.1.1 *Un anneau A est dit 2-AB anneaux si tout idéal 2-absorbant de A est un idéal premier de A.*

Exemple 3.1.2 *Soit R un anneau de valuation de dimension 2 avec les idéaux premiers $0 \subset P \subset M$ et groupe valué $G = \mathbb{Q} \bigoplus \mathbb{Q}$. Alors R est un 2-AB anneau. En effet, montrons d'abord que $M^2 = M$ and $P^2 = P$. En effet, soit $(q,q') \in \mathbb{Q} \bigoplus \mathbb{Q}$ tel que $(q,q') > (0,0)$ si et seulement si $q > 0$ ou $q = 0$ et $q' > 0$. Dans le premier cas $(q,q') = (q/2,q'/2) + (q/2,q'/2)$, dans le second cas $(0,q') = (0,q'/2) + (0,q'/2)$. Donc $M^2 = M$. avec le même argument on a $P^2 = P$. Maintenant soit$I \neq M$ un idéal 2-absorbant de R, donc $Rad(I) = P$, et $P^2 = P \subset I \subset P$ ce qui donne $I = P$ est premier de R. D'ou R est un 2-AB anneau.*

3.2 Les 2-AB anneaux dans l'extension triviale

Théorème 3.2.1 *Soient A un anneau commutatif, E un A-module et $R := A \propto E$. Alors on a les assertions suivantes:*

1. *Si R est un 2-AB anneau, alors A est un 2-AB anneau.*
2. *Si J un idéal 2-absorbant de R et A est un 2-AB anneau, alors J est soit un idéal premier de R soit un idéal semi- primaire de R.(i.e $Rad(I)$ est premier)*
3. *Si A est un anneau intègre de corps de fractions K et E un K-espace vectoriel, alors $A \propto E$ est 2-AB anneau si et seulement si A est 2-AB anneau et $E = \{0\}$.*
4. *Si E est un A-module de type fini, alors $A \propto E$ est un 2-AB anneau si et seulement si A est un 2-AB anneau et $E = \{0\}$.*

Remarque 3.2.2 *D 'aprés le chapitre précédent, si I est un idéal d'un anneau A tel que $Rad(I)$ est premier, alors I est 2-absorbant primaire.*

La preuve du théorème utilise le lemme suivant.

Lemme 3.2.3
1. *Toute image homomorphe d'un 2-AB anneau est un 2-AB anneau.*
2. *Toute localisation d'un 2-AB anneau est un 2-AB anneau.*

Corollaire 3.2.4 *Soient A un anneau Noethérien et E un A-module de type fini, alors R est 2-AB anneau si et seulement si A est 2-AB anneau et* $E = \{0\}$.

En générale la réciproque (1) du théorème 3.2.1 est fausse comme le montrent les exemples suivants.

Exemple 3.2.5 *On considère A un anneau intègre ,*$R := A \propto A$ *et* $I := 0 \propto P$ *où P est un idéal premier de A. Alors I est un idéal 2-absorbant non premier de R [5].*

Exemple 3.2.6 *Si P est un idéal premier d'un anneau A, alors* $P \propto P$ *est un idéal 2-absorbant de* $A \propto A$ *et non premier [55].*

3.3 Les 2-AB anneaux dans l'amalgamation

Théorème 3.3.1 *Soient* $f : A \to B$ *un morphisme d'anneaux commutatifs et J un idéal de B. Alors les assertions suivantes sont satisfaites:*

1. *Si* $A \bowtie^f J$ *est 2-AB anneau, alors A et* $f(A) + J$ *sont 2-AB anneaux.*
2. *Si J est de type fini comme A-module, alors* $A \bowtie^f J$ *est 2-AB anneau si et seulement si A est 2-AB anneau et* $J = 0$.
3. *Si* $f^{-1}(J) = 0$, *alors* $A \bowtie^f J$ *est 2-AB anneau si et seulement si* $f(A) + J$ *est 2-AB anneau.*
4. *On suppose que J un idéal non nul et idempotent et tel que* $J \subset Nil(B)$. *Alors* $A \bowtie^f J$ *est un 2-AB anneau si et seulement si A est 2-AB anneau.*

Pour démontrer le théorème précédent on aura besoin des deux lemmes suivants:

Lemme 3.3.2 *Soient* $f : A \to B$ *un morphisme d'anneaux commutatifs, J un idéal de B. Si* $A \bowtie^f J$ *est un 2-AB anneau, alors A est un anneau local d'idéal maximal M de plus* $J \subset Jac(B)$ *et* $(f(M) + J)J = J$.

Pour le deuxième lemme on rappelle la notion d'idéal 2-absorbant minimal d'un idéal.

Définition 3.3.3 *Soit I un idéal 2-absorbant de R. I est dit idéal 2-absorbant minimal d'un idéal J si* $J \subset I$ *et si il n'existe aucun idéal 2-absorbant* I' *tel que* $J \subseteq I' \subset I$. *Par idéal 2-absorbant minimal on désigne un idéal 2-absorbant minimal de* $\{0\}$. *On note* $2 - Min_R(J)$ *l'ensemble des idéaux 2-absorbants minimaux de J.*

Le point (4) du théorème utilise le le lemme suivant.

Lemme 3.3.4 *Un anneau R est un 2-AB anneau si et seulement si les deux assertions suivantes sont satisfaites:*

1. *Les idéaux premiers de R sont comparables, en particulier R est local,*
2. *Pour tout idéal premier P, on a: $2-Min(P^2)=\{P\}$.*

Corollaire 3.3.5 *Soit $f:A\to B$ un morphisme d'anneaux commutatifs, on suppose que B est local d'idéal maximal M et J un idéal non idempotent de B. Alors $A\bowtie^f J$ est un 2-AB anneau si et seulement si A est un 2-AB anneau.*

Exemple 3.3.6 *Soient A un anneau intègre de corps de fractions K et E un K-espace vectoriel. On considère $f:A\to B:=A\propto B$ le morphisme canonique défini par $f(a)=(a,0)$ pour tout $a\in A$. Soit $J:=0\propto E$. Alors $A\bowtie^f J$ est un 2-AB anneau si et seulement si A est un 2-AB anneau et $E=0$.*

Exemple 3.3.7 *Soient $f:A\to B$ un morphisme d'anneaux commutatifs, J un idéal de B tel que $J\subset Jac(B)$ et A un 2-AB anneau, alors on a pas nécessairement $A\bowtie^f J$ un 2-AB anneau, comme le montre l'exemple suivant:*

Soient A est un anneau local d'idéal maximal M, E un A-module non nul tel que $ME=0$, $f:A\to B:=A\propto E$ le morphisme canonique défini plus haut et $J=0\propto E$. On a $J^2=f(M)J=0$ mais $J\neq 0$. Donc $A\bowtie^f J$ n'est pas 2-AB anneau, voir l'exemple précédent.

Dans le théorème suivant on va déterminer dans un cas particulier les idéaux 2-absorbants de $A\bowtie^f XK[[X]]$ lorsque A est un 2-AB anneau.

Théorème 3.3.8 *Soient A un anneau intègre de corps de fractions K et $f:A\hookrightarrow K[[X]]$ l'injection canonique et $J:=XK[[X]]$. On suppose que A est un 2-AB anneau, alors les idéaux 2-absorbants de $R:=A\bowtie^f J$ sont soit les idéaux premiers de R soit les idéaux de la forme $0\bowtie^f X^2K[[X]]$. Autrement dit le seul idéal 2-absorbant non premier de R est $0\bowtie^f X^2K[[X]]$.*

3.4 Autour de la ($\star$)-conjecture

Bennis et Fahid ont conjecturé que dans un 2-AB anneau tout idéal premier est idempotent. dans ce qui va suivre on donnera une réponse positive à la conjecture dans des cas particuliers.

($\bigstar$) Conjecture: Si R est un 2-AB anneau, alors pour tout idéal premier P de R, on a: $P^2 = P$.

Proposition 3.4.1 *Soit R un anneau commutatif, alors:*

1. *Si R est un anneau intègre divisible, alors R est un 2-AB anneau si et seulement si $P^2 = P$ pour tout idéal premier de R.*
2. *Si R est un 2-AB anneau et si tout idéal de radical premier est primaire, alors tout idéal premier est idempotent.*

Remarque 3.4.2 *Un anneau R est dit vérifie la propriété (*) si tout idéal dont le radical est premier est un idéal primaire voir [?]. On suppose que R vérifie $(*)$, d'aprés [?] si P est un idéal premier non maximal, alors $P^2 = P$. Maintenant si M est un idéal maximal de R, il est connu que M^2 est un idéal 2-absorbant. Si R est un 2-AB anneau, alors R est un anneau local d'idéal maximal M vérifiant $M^2 = M$ On conclut alors pour tout idéal premier I de R on a $I^2 = I$.*

CHAPTER 4

CHAPITRE 4
LES IDÉAUX n-ABSORBANTS ET LES IDÉAUX (m,n)-FERMÉS DANS L'EXTENSION TRIVIALE

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4.1 Introduction

Dans ce chapitre, * on va étudier les conjectures de D. F. Anderson et A. Badawi dans l'extension triviale.

Soient N un A-sous-module d'un A- module M, P un idéal premier de A. On dit que N est un sous-module P-primaire , si $[N :_A M] := \{x \in A; xM \subseteq N\}$ est un idéal P-primaire. On considère $n \geq 1$ un entier positif et I un idéal propre d'un anneau commutatif A. D. F. Anderson et A. Badawi proposent dans [5] (voir aussi [25]) les trois conjectures suivantes:

*J. Algebra and its Appl.Vol.18, N. 7 (2019), 1950123, 19 pp.(En collaboration avec N. Mahdou et A.Badawi).

- **Première Conjecture** : I est un idéal n-absorbant de A si et seulement si I est un idéal fortement n-absorbant de A
- **Deuxième Conjecture** : Si I est un idéal n-absorbant de A, alors $(\mathrm{Rad}(I))^n \subseteq I$. Une réponse positive pour cette conjecture est donnée de manière élégante dans [78].
- **Troisième Conjecture** : Si I est un idéal n-absorbant de A, alors $I[X]$ est un idéal n-absorbant de $A[X]$.

Dans ce chapitre on étudie la validité de ces trois conjectures dans l'extension triviale $A \propto M$. On donne également une réponse positive à la première conjecture dans le cas des u-anneaux.

On rappelle que si A est un anneau intègre, M un A-module divisible (i.e: pour tout $m \in M$ pour tout $\lambda \in A$ non diviseur de zéro, il existe $m' \in M$ tel que $m = \lambda m'$), alors les idéaux de $R := A \propto M$ sont de la forme $0 \propto N$ où N un A-sous-module de M ou $I \propto M$ où I est un idéal de A.

Dans le théorème suivant on regroupe quelques résultats sur les idéaux n-absorbants et (m,n)-fermés d'une extension triviale.

4.2 Idéal n-absorbant dans l'extension triviale

Théorème 4.2.1 *Soient A est un anneau commutatif, I un idéal propre de A et M un A-module. Alors on a:*

1. *I est un idéal n-absorbant de A si et seulement si $I \propto M$ est un idéal n-absorbant de R.*
2. *I est un idéal fortement n-absorbant de A si et seulement si $I \propto M$ est un idéal fortement n-absorbant de R.*
3. *I est un idéal (m,n)-fermé de A si et seulement si $I \propto M$ est un idéal (m,n)-fermé de R.*

Exemple 4.2.2 *On suppose que A un corps et M un K-espace vectoriel. Il est clair que $R := A \propto M$ est local d'idéal maximal $m = 0 \propto M$. Puisque $m^2 = 0$, on conclut que tout idéal de R est 2-absorbant, donc fortement 2-absorbant. Ainsi tout idéal de R est fortement n-absorbant pour tout entier $n \geq 2$ [9, Theorem 2.13].*

On rappelle les résultats suivants [5].

Théorème 4.2.3 *1. Si I est un idéal n-absorbant d'un anneau A pour un entier $n \geq 1$, alors $(Rad(I))^n \subseteq I$ [78].*

2. *Soient P un idéal premier d'un anneau A et I un idéal P-primaire tel que $P^n \subseteq I$ pour un entier $n \geq 1$. Alors I est un idéal n-absorbant de A.*

3. *Soient P un idéal premier de A et I un idéal P-primaire de A. Alors I est un idéal fortement n-absorbant de A si et seulement si I est un idéal n-absorbant de A et $P^n \subseteq I$.*

4. *Soit P un idéal premier divisible d'un anneau A et I un idéal n-absorbant de A avec $Rad(I) = P$. Alors I est un idéal P-primaire de A.*

5. *On suppose $Rad(0) \subseteq P$ sont des idéaux premiers divisibles d'un anneau A et $n \geq 1$ un entier naturel. Donc P^n est P-primaire de A, et par suite un idéal n-absorbant de A.*

En vue du théorème 4.2.3, on a le résultat suivant:

Corollaire 4.2.4 *1. Soient P est un idéal premier d'un anneau A, $n \geq 1$ un entier et I un idéal P-primaire de A. Alors I est un idéal n-absorbant de A si et seulement si $P^n \subseteq I$ si et seulement si I est un idéal fortement n-absorbant de A.*

2. *Si P un idéal premier divisible de A et I un idéal propre tel que $Rad(I) = P$. Alors I est un idéal n-absorbant de A si et seulement si I est P-primaire et $P^n \subseteq I$ si et seulement si I est un idéal fortement n-absorbant de A.*

3. *On suppose que $Rad(0) \subseteq P$ sont des idéaux premiers divisibles d'un anneau A et $n \geq 1$ un entier. Alors P^n est un idéal fortement n-absorbant de A.*

Théorème 4.2.5 *Soient A est un anneau commutatif, M un A-module, $R = A \propto M$, $n \geq 1$ un entier, I un idéal propre de A et N un sous-module de M tel que $IM \subseteq N$. Alors:*

1. *Si $I \propto N$ est un idéal n-absorbant de R, alors I est un idéal n-absorbant de A.*

2. *Si P un idéal premier de A, I un idéal P-primaire de A et N un P-primaire sous-module de M. Alors I est un idéal n-absorbant de A si et seulement si $I \propto N$ est un idéal n-absorbant de R.*

3. *Si P un idéal premier de A, I un idéal P-primaire de A et N un P-primaire sous-module de M. Alors $I \propto N$ est un idéal n-absorbant de R si et seulement si $I \propto N$ est un idéal fortement n-absorbant de R.*

4. *Si P un idéal premier divisible, I un idéal n-absorbant de A tel que $Rad(I) = P$ et N un P-primaire sous-module de M. Alors l 'idéal $I \propto N$ est fortement n-absorbant de R.*

5. *On suppose que $Rad(0) \subseteq P$ sont des idéaux premiers divisibles de A tel que $P^n M \subseteq N$ pour un entier $n \geq 1$. Si N est P-primaire sous-module de M, alors $P^n \propto N$ est fortement n-absorbant de R.*

6. *Si A est un domaine de Prüfer et $J = I \propto M$. Alors J est un idéal n-absorbant de R si et seulement si J est un idéal fortement n-absorbant de R. De plus $\omega_R(J) = \omega_R^*(J)$.*

4.3 La première conjecture dans l'extension triviale

Soit $n \geq 1$ un entier et I un idéal propre d'un anneau A. D. F. Anderson et A. Badawi proposent les conjectures suivantes voir [5] et [25].

Première conjecture : I est un idéal n-absorbant de A si et seulement si I est un idéal fortement n-absorbant de A.

Troisième conjecture : I est un idéal n-absorbant de A si et seulement si $I[X]$ est un idéal n-absorbant de $A[X]$.

A. Laradji a prouvé la première conjecture dans certains classes d'anneaux, en particulier Il a montré que la troisième conjecture entraine la première conjecture voir [68].

On a le lemme suivant:

Lemme 4.3.1 *Soient A est un anneau intègre de corps de fractions K, M un K-espace vectoriel, F un K-sous-espace vectoriel de M et $R := A \propto M$. Alors $0 \propto F$ est un idéal 2-absorbant de R si et seulement si $0 \propto F$ est un idéal fortement n-absorbant pour tout entier $n \geq 2$.*

Théorème 4.3.2 *Soient A est un anneau intègre de corps de fractions K, M un K-espace vectoriel, F un A-sous-module de M et $R := A \propto M$. Alors $0 \propto F$ est un idéal n-absorbant pour un certain entier $n \geq 1$ si et seulement si F est un K-sous-espace vectoriel de M.*

Corollaire 4.3.3 *Soient A est un anneau intègre qui n'est pas un corps et de corps de fractions K et $R := A \propto K$. Alors $0 \propto A$ est un idéal non n-absorbant de R pour tout entier $n \geq 1$.*

Théorème 4.3.4 *Soient A un anneau intègre de corps de fractions K, M un K-espace vectoriel et $R := A \propto M$. Alors La première conjecture est vraie dans R si et seulement si la première conjecture est vraie dans A.*

Corollaire 4.3.5 *Soient A un domaine de Prüfer de corps de fractions K, M un K-espace vectoriel. Alors la première conjecture est vraie dans $R := A \propto M$.*

Théorème 4.3.6 *Si R est un anneau Noethérien. Alors tout idéal propre de R est fortement n-absorbant pour un certain entier $n \geq 1$.*

Théorème 4.3.7 *Soient A un anneau Noethérien, M un A-module, $R := A \propto M$ et I un idéal propre de A. Alors $J := I \propto M$ est un idéal fortement n-absorbant de R pour un certain entier $n \geq 1$.*

Théorème 4.3.8 *Soient A un anneau Noethérien, M un A-module de type fini et $R := A \propto M$. Alors tout idéal de R est fortement n-absorbant pour un certain entier $n \geq 1$.*

Question 1 *En vue du théorème 4.3.6, au cours du congrés internationale d'Algèbre et ses applications (ICAA) à Errachidia au Maroc, El Amin El Kaidi a posé la question suivante:*

Étant donné A un anneau dans lequel tout idéal est n-absorbant pour un certain entier $n \geq 1$ a-t-on l'anneau A Noethérien. L'exemple suivant donne une réponse négative à cette question.

Exemple 4.3.9 *Soit $A \subseteq K$ des corps tels que K n'est pas de type fini comme A-module (par exemple $A = \mathbb{Q}, K = \mathbb{R}$) et $R := A \propto K$. Puisque R est un anneau local d'idéal maximal $M := 0 \propto K$ et $M^2 = 0$, on conclut que tout idéal de R est 2-absorbant et donc fortement n-absorbant pour tout entier $n \geq 2$. L'idéal $M := 0 \propto K$ n'est pas de type fini puisque K n'est pas de type fini comme A-module. Donc R n'est pas Noethérien.*

Remarque 4.3.10 *Si A est un anneau et n un entier ≥ 1 tels que tout idéal de A est n-absorbant. Alors, $dim(R) = 0$ et R admet au plus n idéaux maximaux.*

On a les résultats suivants qui traitent des classes d'anneaux dans lesquels tout idéal est n-absorbant pour un certain entier $n \geq 2$.

Théorème 4.3.11 *Soient A un anneau intègre de corps de fractions K, M un K-espace vectoriel de dimension finie et $R := A \propto M$. Alors tout idéal de R est n-absorbant pour un certain entier $n \geq 1$ si et seulement si $A = K$.*

Corollaire 4.3.12 *Soient A est un anneau intègre de corps de fractions K, M un K-espace vectoriel de dimension finie et $R := A \propto M$. Alors les assertions suivantes sont équivalentes:*

1. *Tout idéal de R est n-absorbant pour un certain entier $n \geq 1$.*
2. *Tout idéal de R est fortement n-absorbant pour un certain entier $n \geq 1$.*
3. *$A = K$.*
4. *R est un anneau Noethérien.*

Théorème 4.3.13 *Soient A un anneau Noethérien intègre de corps de fractions K, M un K-espace vectoriel et $R = A \propto M$. Un idéal propre J de R est n-absorbant pour un certain entier $n \geq 1$ si et seulement si J est fortement m-absorbant pour un certain entier $m \geq 1$.*

4.4 La troisième conjecture dans l'extension triviale

D.F. Anderson et A. Badawi conjecturent que si I est un idéal propre n-absorbant d'un anneau commutaif A, alors $I[X]$ est un idéal n-absorbant de $A[X]$.

Soient A un anneau commutatif, M un A-module et $R = A \propto M$, on sait que $(A \propto M)[X]$ est naturellement isomorphe à $A[X] \propto M[X]$. Si I est un idéal propre de A, alors $(I \propto M)[X]$ est naturellement isomorphe à $I[X] \propto M[X]$. On rappelle le résultat suivant.

Théorème 4.4.1 *[5, Theorem 4.15]*
Si I est un idéal propre d'un anneau commutatif A. Alors I est un idéal 2-absorbant si et seulement si $I[X]$ est un idéal 2-absorbant de $A[X]$.

Théorème 4.4.2 *Soient A un anneau intègre de corps de fractions K, M un K-espace vectoriel et $R = A \propto M$. Alors la troisième conjecture est vraie dans R si et seulement si la troisième conjecture est vraie dans A.*

A. Laradji a montré que la troisième conjecture est vraie dans un anneau arithmétique, puisque un domaine de Prüfer est à la fois un anneau arithmétique et un anneau de Gauss.

Lemme 4.4.3 *[68]*
Soient A est un domaine de Prüfer et I un idéal propre n-absorbant de A où $n \geq 1$ un entier. Alors $I[X]$ est un idéal n-absorbant de $A[X]$.

Au court des résultats suivants on construit des anneaux avec diviseurs avec de zéro non arithmétiques dans lesquels la troisième conjecture est vérifiée.

Théorème 4.4.4 *Si A est un domaine de Prüfer de corps de quotient K, M un K-espace vectoriel, $n \geq 1$ un entier et J un idéal propre de $R = A \propto M$. (Noter que si $M = K[X]$, alors R n'est pas arithmétique). Si J est un idéal n-absorbant de R, alors $J[X]$ est un idéal n-absorbant de $R[X]$ et $\omega_R(J) = \omega_{R[X]}(J[X])$.*

Dans l'exemple suivant on construit un anneau non-arithmétique qui vérifie la troisième conjecture.

Exemple 4.4.5 *Soit A un domaine de Prüfer de corps de fractions K, $M = K[X]$ et $R = A \propto M$. Alors on a:*

1. *La troisième conjecture est vraie dans R d'aprés le théorème 4.4.4.*
2. *R n'est pas un anneau arithmétique.*

Remarque 4.4.6 *Soient I est un idéal propre d'un anneau A et $n \geq 1$ un entier . On montre que si I est fortement n-absorbant de A, alors $(Rad(I))^n \subseteq I$ voir[78] et que si $I[X]$ est un idéal n-absorbant de $A[X]$, alors I est un idéal fortement n-absorbant de A voir [68]. On montre également que si I est un idéal n-absorbant et A arithmétique, alors $I[X]$ est un idéal n-absorbant de $A[X]$. Donc dans un anneau arithmétique les trois conjectures de D. F. Anderson et A. Badawi sont vérifiées voir [68].*

Maintenant, on va construire un anneau avec diviseurs de zéro non-arithmétique dans lequel les trois conjectures sont vérifiées.

Théorème 4.4.7 *Soient A un domaine de Prüfer de corps de fractions K, M un K-espace vectoriel , $n \geq 1$ un entier et $R = A \propto M$ (par exemple si $M = K[X]$ on a R non arithmétique). Soit J un idéal propre n-absorbant de R. Alors les assertions suivantes sont vérifiées:*

1. *J est un idéal fortement n-absorbant de R.*
2. *$J[X]$ est un idéal n-absorbant de $A[X]$.*
3. *$(Rad(J))^n \subseteq J$.*

4.5 La première conjecture dans un *u*-anneau

On rappelle d'aprés [80] qu'un anneau commutatif R est dit un u-anneau si tout idéal I de R contenu dans une union finie d'idéaux de R est contenu dans au moins l'un de ces idéaux. Il est connu que tout anneau de Bezout est un u-anneau voir [80] et tout domaine de Prüfer est u-domaine voir[80]. S. Smach et S.Hizem ont montré que la première conjecture est vérifiée dans un u-anneau voir[77]. Dans cette section on propose de prouver que la première conjecture est vraie dans un u-anneau avec une approche différente que celle proposée par les deux auteurs dans [77]. Pour se faire on a besoin des notations suivantes. Soit R un anneau commutatif. Si $x_1,\cdots,x_n \in R$, alors le produit $x_1...\widehat{x_k}...x_n$ désigne le produit $x_1..x_n$ qui ne contient pas le terme x_k. de manière similaire si $I_1,...,I_n$ des idéaux de R, le produit $I_1..\widehat{I_k}...I_n$ désigne le produit $I_1...I_n$ où on a omit I_k.

On commence par les lemmes suivants. Dans ces lemmes $n \geq 1$ est un entier.

Lemme 4.5.1 *Soit R un anneau commutatif. On suppose qu'il existe des idéaux $I_1,\cdots,I_{n+1}$ de R tels que le produit $I_1....I_{n+1}$ est l'idéal nul et aucun produit des n de I_i' s n'est nul. Alors il existe $J_1,\cdots,J_{n+1}$ des idéaux de type fini tel que $J_1.\cdots.J_{n+1}=0$ et aucun produit des n de J_i' s n'est nul.*

Lemme 4.5.2 *Supposons que dans n'importe quel anneau commutatif R, l'idéal $\{0\}$ est fortement n-absorbant si et seulement si $\{0\}$ est un idéal n-absorbant de cet anneau. Alors tout idéal n-absorbant d'un anneau R est un idéal fortement n-absorbant de R.*

Lemme 4.5.3 *Soit R un u-anneau tel que $\{0\}$ est un idéal n-absorbant de R. Alors $\{0\}$ est un idéal fortement n-absorbant de R*

Théorème 4.5.4 *Soit R un u-anneau. Alors la première conjecture est vérifiée dans R à savoir tout idéal n-absorbant de R est un idéal fortement n-absorbant de R.*

On rappelle qu'un anneau est dit *um*-anneau, si tout A-module qui sécrit comme union finie de A-sous-modules est égale à l'un des ses sous-modules.

Remarque 4.5.5 *Soit R un anneau commutatif contenant une partie S infinie telle que $x-y$ soit inversible pour tout $x \neq y$ de S. Alors R est un um-anneau [80]. Dans [80] on montre qu'un anneau R est un um-anneau si et seulement si R/M est infinie ou R_M est un anneau de Bezout pour tout idéal maximal M de R. On conclut que tout um-anneau est un u-anneau. La réciproque n'est pas vraie, par exemple $R=\mathbb{Z}$ est un u-anneau mais pas un um-anneau puisque pour tout idéal maximal M de $\mathbb{Z}$, on a $\mathbb{Z}/M$ est finie.*

En vue de la remarque 4.5.5, on a les résultats suivants:

Théorème 4.5.6 *Si R est un um-anneau. Alors R est un u-anneau.*

Théorème 4.5.7 *Soient R un anneau commutatif avec $1 \neq 0$, n un entier positif et I un idéal propre de R. On suppose que R contient une partie infinie S telle que $x-y$ est inversible dans R pour tout $x \neq y \in S$. Alors R est un u-anneau, et donc I est un idéal fortement n-absorbant de R si et seulement si I est un idéal n-absorbant de R.*

Remarque 4.5.8 *On peut donner une preuve alternative du théorème 4.5.7. Noter que si R contient une partie infinie S telle que $x-y$ est inversible dans R pour tout $x \neq y$ de S, alors R est un um-anneau d'aprés [80]. Donc R est un u-anneau d'aprés le théorème 4.5.6.*

Théorème 4.5.9 *Soient A un u-anneau intègre de corps de fractions K, M un K-espace vectoriel et $R = A \propto M$. Alors la première conjecture est vraie dans R.*

L'exemple suivant illustre le cas d'un anneau qui vérifie la première conjecture sans qu'il soit un *u*-anneau.

Exemple 4.5.10 *Soit $R = \mathbb{Z}_3 \propto \mathbb{Z}_3[X]$, on a R vérifie la premiŕe conjecture. Il est clair que $M = 0 \propto \mathbb{Z}_3[X]$ est le seul idéal maximal de R. Puisque ni R/M est infinie ($R/M \simeq \mathbb{Z}_3$) ni R_M est un anneau de Bezout ($R_M = R$), alors R n'est pas un u-anneau. Noter que R n'est pas un um-anneau non plus d'apés le théorème 4.5.6.*

Théorème 4.5.11 *Soient A un um-anneau commutatif, M un A-module et $R = A \propto M$. Alors La première conjecture est vraie dans R.*

4.6 Idéal (m,n)-fermé dans l'extension triviale

Soit R un anneau commutatif avec $1 \neq 0$. On rappelle d'aprés [6] qu'un idéal propre I de R est dit (m,n)-fermé si $x^m \in I$, pour $x \in R$ implique $x^n \in I$ où $0 < n \leq m$ des entiers naturels.

Théorème 4.6.1 *Soit A un anneau commutatif, M un A-module et $R = A \propto M$. On considére $J = I \propto N$ un idéal propre de R où I est idéal propre de A et N un sous-module de M tel que $IM \subseteq N$. Si I est un idéal (m,n)-fermé de A où $0 < n < m$ des entiers, alors J est un idéal $(m,n+1)$-fermé de R.*

En vue du théorème 4.6.1 on donne un exemple d'idéal $(3,2)$-fermé dans $\mathbb{Z}$, mais $I \propto I$ est un idéal non $(3,2)$-fermé de $\mathbb{Z} \propto Z$.

Exemple 4.6.2 *Soient $R = \mathbb{Z} \propto \mathbb{Z}$, $p \neq 2$ un entier premier, $I = p^4\mathbb{Z}$ et $J = I \propto I$. J est un idéal propre de R et I est $(3,2)$-fermé . Soit $x = (p^2, p) \in R$, alors $x^3 = (p^6, 3p^5) \in J$. Puisque $p \neq 2$, on a $x^2 = (p^4, 2p^3) \notin J$. Donc J est non $(3,2)$-fermé de R.*

Lemme 4.6.3 *Soient A est un anneau commutatif, M un A-module, $R = A \propto M$, $J = I \propto N$ un idéal propre de R, où I est un idéal (m,n)-fermé de R pour $0 < n < m$ des entiers et N un sous-module de M tel que $IM \subseteq N$. Étant donné $x = (a,c) \in R$ pour $a \in A$ et $c \in M$. Alors $x^m \in J$ si et seulement si $a^m \in I$.*

Théorème 4.6.4 *Soient A un anneau commutatif, M un A-module, $R = A \propto M$. On considère $J = I \propto N$ un idéal propre de R où I est un idéal propre de A et N un sous-module de M tel que $IM \subseteq N$. Pour $0 < n < m$ des entiers. Les assertions suivantes sont équivalentes:*

1. *J un idéal (m,n)-fermé de R.*
2. *I est un idéal (m,n)-fermé de A et si $a^m \in I$ pour un certain $a \in A$, alors $na^{n-1}M \subseteq N$.*

Théorème 4.6.5 *Soient A un anneau commutatif, M un A-module et $0 < n < m$ des entiers. On considère I un idéal propre de A et $R = A \propto M$. Si $Car(A)|n$. Alors les assertions suivantes sont équivalentes:*

1. *$J = I \propto N$ est un idéal (m,n)-fermé de R pour tout sous-module N de M tel que $IM \subseteq N$.*
2. *I est un idéal (m,n)-fermé de A.*

Dans ce qui va suivre on va étudier les idéaux (m,n)-fermés principaux.

Théorème 4.6.6 *Soient D un anneau intègre, $R = D \propto D$, , des entiers $1 \leq n < m$ et $I = p^kD$ où p un élément premier de D et k un entier positif. On suppose que $m > k$ et $Car(R) \neq n$, alors les assertions suivantes sont équivalentes:*

1. *$J = I \propto p^iD$ est un iéal (m,n)-fermé de R pour un certain entier $i \geq 1$.*
2. *L'un des trois cas suivants est satisfait*

(a) $k < n < m$ *et* $i \leq k$.

(b) $n = k$, *et* $1 \leq i < k$.

(c) $n = i = k$, *et* $p \mid k \cdot 1_D$ *(dans D), où* 1_D *est l'unité de D.*

Définition 4.6.7 *Soit p un élément premier d'un anneau intègre D. On suppose que* $p^w \mid d$ *pour un certain* $d \in D$ *et un entier positif w mais* $p^{w+1} \nmid d$. *On écrit alors* $p^w \mid\mid d$

Théorème 4.6.8 *Soient D est un anneau intègre,* $R = D \propto D$, m,n *des entiers tels que* $1 \leq n < m, I = p^k D$, *où p un élément premier de D et k entier positif. Si de plus* $m < k$ *et* $Car(D) \neq n$ *et en notant* $v = \lceil \frac{k}{m} \rceil$, $u = \lceil \frac{k}{v} \rceil$. *Alors Les assertions suivantes sont équivalentes:*

1. $J = I \propto p^i D$ *est un idéal* (m,n)*-fermé de R pour un certain entier* $i \geq 1$.
2. *L'un des trois cas suivants est satisfait:*
   (a) $u < n < m$ *et* $i \leq k$.
   (b) $u = n, p \nmid n \cdot 1_D$ *(dans D), et* $i \leq v(n-1) < k$.
   (c) $u = n$, $p^w \mid\mid n \cdot 1_D$ *(dans D), et* $i \leq min\{v(n-1)+w, k\}$.

Soient D un anneau intègre, m et n des entiers fixes tels que $1 \leq n < m$. D. F. Anderson et A. Badawi ont déterminé l'ensemble $\{k \in \mathbb{N}; p^k D \quad \text{est} \quad (m,n) - \text{fermé}\}$, ce qu'on va faire dans le cas de l'extension triviale $R := D \propto D$, on rappelle le résultat suivant.

Théorème 4.6.9 *Soient D un anneau inègre, m et n des entiers positifs avec* $1 \leq n < m$, $I = p^k D$, *où p un élément premier de D et k un entier positif. Alors les assertions suivantes sont équivalentes:*

1. *I est un idéal* (m,n)*-fermé de D.*
2. *Si* $m = bn + c$ *pour les entiers b et c avec* $b \geq 2$ *et* $0 \leq c \leq n-1$, *alors* $k \in \{1, \ldots, n\}$. *Si* $m = n + c$ *pour les entiers c avec* $1 \leq c \leq n-1$, *alors* $k \in \bigcup_{h=1}^{n} \{mi + h \mid i \in \mathbb{Z}$ *and* $0 \leq ic \leq n-h\}$.

En vue des théorèmes 4.6.6, 4.6.8 and 4.6.9, on a le résultat suivant.

Théorème 4.6.10 *Soient D est un anneau inègre,* $R = D \propto D$, m *et n entiers avec* $1 \leq n < m$, $I = p^k D$, *où p est un élément premier de D et k un entier positif. On suppose que* $Car(D) \neq n$. *Alors les assertions suivantes sont équivalentes:*

1. $J = I \propto p^i D$ *est un idéal* (m,n)*-fermé de R pour un certain entier* $i \geq 1$.

2. *Si $m = bn+c$ pour les entiers b et c avec $b \geq 2$ et $0 \leq c \leq n-1$, alors $k \in \{1,\ldots,n\}$ et un seul cas des cas suivants est satisfait:*

   (a) $k < n < m$ *and* $i \leq k$.

   (b) $n = k$, *and* $1 \leq i < k$.

   (c) $n = i = k$, *and* $p \mid k \cdot 1_D$ *(dans D), où* 1_D *l 'identité de D.*

   Si $m = n+c$ pour un entier c avec $1 \leq c \leq n-1$, alors $k \in \bigcup_{h=1}^{n}\{mi+h \mid i \in \mathbb{Z}$ alors $0 \leq ic \leq n-h\}$ et un seul cas des cas suivants est satisfait:

   Let $v = \lceil \frac{k}{m} \rceil$ et $u = \lceil \frac{k}{v} \rceil$. Alors

   (a) $u < n < m$ *et* $i \leq k$.

   (b) $u = n, p \nmid n \cdot 1_D$ *(dans D), et* $i \leq v(n-1) < k$.

   (c) $u = n$, $p^w \mid\mid n \cdot 1_D$ *(dans D), et* $i \leq min\{v(n-1)+w, k\}$.

En vue des théorèms 4.6.1 and 4.6.9, on a le résultat suivant.

Théorème 4.6.11 *Soient D est un anneau intègre, $I = p^kD$, où p est un élément premier de D, k est un entier positif, M est un D-module, $R = D \propto M$, $J = I \propto N$ est un idéal propre de R, où N est sous-module de M tel que $IM \subseteq N$ et m et n des entiers avec $1 \leq n < m$. Alors les assertions suivantes sont équivalenes:*

1. *I est un idéal (m,n)- fermé de D et J est un idéal $(m,n+1)$-fermé de R.*

2. *Si $m = bn+c$ pour les entiers b et c avec $b \geq 2$ et $0 \leq c \leq n-1$, alors $k \in \{1,\ldots,n\}$. Si $m = n+c$ pour un entier c avec $1 \leq c \leq n-1$, alors $k \in \bigcup_{h=1}^{n}\{mi+h \mid i \in \mathbb{Z}$ et $0 \leq ic \leq n-h\}$.*

Théorème 4.6.12 *Soient A est un anneau intègre de corps de fractions K, M est un K-espace vectoriel et $R = A \propto M$. Alors les assertions suivantes sont équivalentes:*

1. *Tout idéal propre de A est un idéal (m,n)-fermé de A pour certains entiers positifs $1 \leq n < m$.*

2. *Tout idéal propre de R est un idéal (m,n)-fermé de R pour certains entiers positifs $1 \leq n < m$.*

CHAPTER 5

CHAPITRE 5
LES IDÉAUX n-ABSORBANTS ET FORTEMENT n-ABSORBANTS DANS L'AMALGAMATION

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5.1 Introduction

Dans ce chapitre,* on étudie les conjectures de D. F. Anderson et A. Badawi dans l'amalgamation.

Soient A et B deux anneaux commutatifs, $f : A \to B$ un morphisme d'anneaux et J un idéal de B. On considére le sous anneau de $A \times B$ défini par:

$$A \bowtie^f J := \{(a, f(a) + j); a \in A, j \in J\}$$

dit l'amalgamée de A par B le long l'idéal J en respectant f.

*J. Algebra and Its Appl, On line 2020 (en collaboration avec N. Mahdou et M. A. Moutui)

Le but de ce chapitre est d'étudier la notion d'idéal n-absorbant et d'idéal fortement n-absorbant de l'amalgamée $A \bowtie^f J$ et étudier la validité des conjectures de D. F. Anderson et A. Badawi.

Le long du chapitre si R est un anneau commutatif, on note $\mathrm{Spec}(R)$(resp: $\mathrm{Max}(R)$) l'ensemble des idéaux premiers(resp., idéaux maximaux) de R. Si I est un idéal de R, on note $\overline{\mathrm{Spec}(R,I)}$(resp.,$\overline{\mathrm{Max}(R,I)}$ l'ensemble des idéaux premiers(resp., des idéaux maximaux) ne contenant pas I.

5.2 Idéal n-absorbant et fortement n-absorbant d'une amalgamation

Soient $f : A \to B$ un morphisme d'anneaux commutatifs, J un idéal de B, l'anneau $A \bowtie^f J$ est l'amalgamation de A avec B le long J en respectant f. Si I un idéal de A et K un idéal de $f(A)+J$, on pose:

$$I \bowtie^f J = \{(a, f(a)+j); a \in I, j \in J\},$$

$$\overline{K}^f = \{(a, f(a)+j); a \in A, j \in J, f(a)+j \in K\},$$

$$\overline{I \times K}^f = \{(a, f(a)+j); a \in I, j \in J, f(a)+j \in K\}.$$

Clairement $I \bowtie^f, \overline{I \times K}^f$ et $\overline{I \times K}^f$ sont des idéaux de $A \bowtie^f J$.

Notre premier résultat est de voir quand $I \bowtie^f J$, $\overline{I \times K}^f$ et $\overline{I \times K}^f$ sont des idéaux n-absorbants de $A \bowtie^f J$

Théorème 5.2.1 *Soit n un entier positif. Avec les notations précédentes, les assertions suivantes sont satisfaites:*

1. *$I \bowtie^f J$ est un idéal n-absorbant de $A \bowtie^f J$ si et seulement si I est idéal n-absorbant de A.*
2. *$I \bowtie^f J$ est un idéal fortement n-absorbant de $A \bowtie^f J$ si et seulement si I est idéal fortement n-absorbant de A.*
3. *Si I est un idéal n_1-absorbant A et K est un idéal n_2-absorbant de $f(A)+J$, alors $\overline{I \times K}^f$ est un idéal $n = (n_1+n_2)$-absorbant de $A \bowtie^f J$.*
4. *Si I est un idéal fortement n_1-absorbant A et K est un idéal fortement n_2-absorbant de $f(A)+J$, alors $\overline{I \times K}^f$ est un idéal fortement $n = (n_1+n_2)$-absorbant de $A \bowtie^f J$.*

5. $\overline{K}^f$ *est un idéal n-absorbant de* $A \bowtie^f J$ *si et seulement si K est un idéal n-absorbant* $f(A)+J$.

6. $\overline{K}^f$ *est un idéal fortement n-absorbant de* $A \bowtie^f J$ *si et seulement si K est un idéal fortement n-absorbant* $f(A)+J$.

Lemme 5.2.2 *Soit n un entier naturel, avec les notations précédentes on a:*

1. *Si J est un idéal fortement n-absorbant de B, alors* $f^{-1}(J)$ *est un idéal fortement n-absorbant de A.*

2. *Si f est surjective et I un idéal de A contenant* $\ker(f)$. *Alors* $f(I)$ *est un idéal fortement n-absorbant de B si et seulement si I est un idéal fortement n-absorbant de A.*

Lemme 5.2.3 *1. Si* $A \subseteq B$ *est une extension d'anneaux et J un idéal fortement n-absorbant de B. Alors* $J \cap A$ *est un idéal fortement n-absorbant de A.*

2. *Si* $I \subseteq J$ *sont des idéaux de A. Alors* J/I *est fortement n-absorbant de* A/I *si et seulement J est fortement n-absorbant de A si et seulement si* $\{0\}$ *est fortement n-absorbant de* A/I.

Lemme 5.2.4 *Soient* I_1 *un idéal d'un anneau A et* I_2 *un idéal d'un anneau B où* n, p *deux entiers positifs. Alors* $I_1 \times I_2$ *est un idéal fortement* $(n+p)$*-absorbant de* $A \times B$ *si et seulement si* I_1 *est un idéal fortement n-absorbant de A et* I_2 *est un idéal fortement p-absorbant de B.*

Corollaire 5.2.5 *Soient* $f : A \to B$ *un morphisme d'anneaux commutatifs et J un idéal de B. On considère un entier positif n. Alors les assertions suivantes sont satisfaites:*

1. *Si P est un idéal premier de A tel que* P^n *est un P-primary idéal de A. Alors* $P^n \bowtie^f J$ *est un idéal fortement n-absorbant de* $A \bowtie^f J$. *En particulier, Ceci est vrai si A est un anneau de valuation.*

2. *Si* $Nil(A) \subseteq P$ *sont des idéaux premiers divisibles de A. Alors* $P^n \bowtie^f J$ *est un idéal fortement n-absorbant de* $A \bowtie^f J$.

3. *Si P est un idéal premier divisible de A tel que* $Z(A) \subseteq P$, *alors* $P^n \bowtie^f J$ *est fortement n-absorbant de* $A \bowtie^f J$.

4. *Si A est un anneau intègre de Prüfer et $K = I \bowtie^f J$ où I est un idéal de A. Alors K est un idéal n-absorbant de $A \bowtie^f J$ si et seulement si K est un idéal fortement n-absorbant de $A \bowtie^f J$.*

Corollaire 5.2.6 *Soient M un idéal maximal d'un anneau intègre T et D un sous-anneau de T tel que $M \cap D = \{0\}$. Si I est un idéal de D. Alors $I+M$ est un idéal n-absorbant(resp., fortement n-absorbant) de $D+M$ si et seulement si I est un idéal n-absorbant(resp.,fortement n-absorbant) de D.*

Soient $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B. On considère un idéal I (resp., H) de A (resp., de $f(A)+J$) tels que $f(I)J \subseteq H \subseteq J$. On remarque $I \bowtie^f H := \{(i, f(i)+h); i \in I, h \in H\}$ est un idéal de $A \bowtie^f J$. Noter que si I est un idéal propre de A, alors l'extension I^e de I dans $A \bowtie^f J$ coïncide avec $I \bowtie^f (f(I)B)J$. De plus si $f(I)J \subseteq H \subseteq J$ et si I est un idéal n-absorbant de A, alors on a pas nécessairement $I \bowtie^f H$ un idéal n-absorbant de $A \bowtie^f J$ comme le montre l'exemple suivant.

Exemple 5.2.7 *On pose $A := \mathbb{Z}, B := \mathbb{Q}[[X]]$ l'anneau des séries formelles d'indetérminée X, et $f : \mathbb{Z} \hookrightarrow \mathbb{Q}[[X]]$ l'injection canonique. Soit $J = X\mathbb{Q}[[X]]$ idéal maximal de B et $H := \{XP(X); P \in f(A)+J = \mathbb{Z}+X\mathbb{Q}[[X]]\}$ qui est un idéal de $f(A)+J$. L'idéal $I = \{0\}$ est premier, donc n-absorbant pour tout entier positif n et $f(I)J = 0 \subseteq H \subseteq J$. Donc $0 \bowtie^f H$ est un idéal de $A \bowtie^f J$. Mais $0 \bowtie^f H$ n'est pas n-absorbant pour tout entier naturel n. En effet $(2,2)^n(0, \frac{X}{2^n}) = (0,X) \in 0 \bowtie^f H$, mais $(2,2)^{n-1}(0, \frac{X}{2^n}) = (0, \frac{X}{2}) \notin 0 \bowtie^f H$ et $(2,2)^n = (2^n, 2^n) \notin 0 \bowtie^f H$.*

Maintenant on va voir sous certains conditions que l'idéal $I \bowtie^f K$ est n-absorbant de $A \bowtie^f J$ si I est un idéal n-absorbant de A vérifiant $f(I)J \subseteq H \subseteq J$.

Théorème 5.2.8 *Soient (A,m) un anneau local d'idéal maximal m et $f : A \to B$ un morphisme d'anneau. Étant donné J un idéal de B tel que $f^{-1}(Q) \neq J$ pour tout $Q \in spec(B) \setminus V(J)$ et $f(m)J = 0$. Pour n un entier naturel positif et H un idéal de $f(A)+J$ tels que $J^n \subseteq H \subseteq J$. On a:*

Pour tout idéal I de A dont le radical est m, les assertions suivantes sont satisfaites:

1. *I est un idéal n-absorbant de A si et seulement si $I \bowtie^f H$ est un idéal n-absorbant de $A \bowtie^f J$.*

2. *I est un idéal fortement n-absorbant de A si et seulement si $I \bowtie^f H$ est un idéal fortement n-absorbant de $A \bowtie^f J$.*

La preuve du théoreème 5.2.8 utilise le lemme suivant. On rappelle: $Spec(A \bowtie^f J) = \{P'^f = P \bowtie^f J | P \in Spec(A)\} \cup \{\overline{Q}^f | Q \in \overline{Spec(B,J)}\}$ and $Max(A \bowtie^f J) = \{m'^f = m \bowtie^f J | m \in Max(A)\} \cup \{\overline{Q}^f | Q \in \overline{Max(B,J)}\}$.

Lemme 5.2.9 *Soient (A,m) un anneau local d'idéal maximal m, $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B tel que $f^{-1}(Q) \neq m$ pour tout $Q \in spec(B) \setminus V(J)$. On considère un idéal I(resp:H) de A (resp: de $f(A)+J$)tel que $f(I)J \subseteq H \subseteq J$. Alors les assertions suivantes sont satisfaites:*

1. *If $Rad(I) = m$, alors $Rad(I \bowtie^f H) = m'^f = m \bowtie^f J$.*
2. *I est m-primaire si et seulement si $I \bowtie^f H$ est $m \bowtie^f J$-primaire.*

Corollaire 5.2.10 *Soient (A,m) est un anneau local d'idéal maximal m, $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B tel que $f^{-1}(Q) \neq m$ pour tout $Q \in spec(B) \setminus V(J)$. Si n un entier positif tel que $J^{n-1} \subseteq f(I)$ et $f(m) \subseteq f(I)$. Soit $I^e = I(A \bowtie^f J)$ l'extension de I dans $A \bowtie^f J$. Alors les assertions suivantes sont satisfaites:*

1. *I est un idéal n-absorbant de A si et seulement si I^e est un idéal n-absorbant de $A \bowtie^f J$.*
2. *I^e est un idéal n-absorbant de $A \bowtie^f J$ si et seulement si I^e est un idéal fortement n-absorbant de $A \bowtie^f J$.*

Exemple 5.2.11 *On considère (A,m) un anneau de valuation local qui n'est pas un corps($(A,m) = (\mathbb{Z}_p, p\mathbb{Z}_p)$), K le corps de fractions de A, $B = K[[X]]$ l'anneau des séries entières formelles et X une indetérminée. Soit $f : A \hookrightarrow B$ l'injection canonique et $J := XK[[X]]$ idéal maximal de B. On observe que $spec(B) = \{0,J\}$ et $f^{-1}(0) = \ker(f) = 0 \neq m$. Soit n un entier naturel positif et $H = X^n K[[X]] = J^n$ idéal de $f(A)+J = A+XK[[X]]$. Par le théorème 5.2.8(1), on conclut que $0 \bowtie^f H$ est un idéal n-absorbant de $A \bowtie^f J$.*

Dans le prochain théorème on caractérise les idéaux n-absorbants et fortement n-absorbants de $A \bowtie^f J$ dans des cas particuliers

Théorème 5.2.12 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs, J un idéal de B, K un idéal de $f(A)+J$ et I idéal de A. On se donne n un entier positif.*

1. *Les idéaux n-absorbants de $A \bowtie^f J$ contenant $\{0\} \times J$ sont de la forme $I \bowtie^f J$ où I est n-absorbant de A.*

2. *Les idéaux fortement n-absorbants de $A \bowtie^f J$ contenant $\{0\} \times J$ sont de la forme $I \bowtie^f J$ où I est un idéal fortement n-absorbant de A.*

3. *Les idéaux n-absorbants de $A \bowtie^f J$ contenant $f^{-1}(J) \times \{0\}$ sont de la forme $\overline{K}^f$ où K est un idéal n-absorbant de $f(A)+J$.*

4. *Les idéaux fortement n-absorbants de $A \bowtie^f J$ contenant $f^{-1}(J) \times \{0\}$ sont de la forme $\overline{K}^f$ où K est un idéal fortement n-absorbant de $f(A)+J$.*

Théorème 5.2.13 *Soient (A,m) un anneau local d'idéal maximal m, $f : A \to B$ un morphisme d'anneaux et J un idéal de B. Pour n un entier naturel positif tel que $m^n = 0$. Les assertions suivantes sont satisfaites:*

1. *$I \bowtie^f J$ est un idéal n-absorbant de $A \bowtie^f J$, pour tout idéal I de A.*

2. *Soit $2 \leq q \leq n$ un entier tel que $J^q = 0$, on suppose de plus que $f(m)J = 0$. Alors, tout idéal de $A \bowtie^f J$ est n-absorbant.*

La preuve du théorème utilise le lemme suivant

Lemme 5.2.14 *Si (A,m) un anneau local tel que $m^n = 0$. Alors, tout idéal de A est n-absorbant.*

Exemple 5.2.15 *Soit (A,m) un anneau local avec $m^2 = 0$ et E un A-module tel que $mE = 0$ et $B := A \propto E$ l'extension triviale de A par E et $J = 0 \propto E$ idéal de B. Considérons le morphisme d'anneaux suivant*

$$f : \begin{cases} A \hookrightarrow B \\ a \mapsto (a,0). \end{cases}$$

On remarque que $f(m)J = (m \propto 0)(0 \propto E) = 0$ et $J^2 = 0$. Donc par l'assertion (2) du théorème 5.2.13, tout idéal de $A \bowtie^f J$ est 2-absorbant. Et donc tout idéal de $A \bowtie^f J$ est n-absorbant pour tout entier $n \geq 2$.

L'exemple suivant montre que les conditions $m^n = 0$, $f(m)J = 0$ and $J^q = 0$ ne sont pas nécessaires dans l'assertion (2) du théorème 5.2.13.

Exemple 5.2.16 *Soit $(A,m) := (\mathbb{Z}_p, p\mathbb{Z})$ anneau local, $f : \mathbb{Z}_p \hookrightarrow \mathbb{Q}[[X]]$ l'injection canonique de $\mathbb{Z}_p$ dans $\mathbb{Q}[[X]]$ et $R := A \bowtie^f J$, où $J := X\mathbb{Q}[[X]]$ est l 'idéal maximal de B. Donc:*

1. $J^2 \neq 0$.
2. $m^n \neq 0$ *et* $f(m)J \neq 0$, $\forall n \in \mathbb{N}$.
3. *R est un domaine de Prüfer donc, tout idéal propre de R est k-absorbant pour un certain entier naturel k.*

5.3 La première conjecture dans l'amalgamation

Exemple 5.3.1 *On pose* $A := \mathbb{Z}$, $B := X\mathbb{Q}[X]$, $f : A \hookrightarrow B$ *l'injection canonique. Soit* $J := X\mathbb{Q}[X]$, *on a* $J^2 \neq 0$, $A \bowtie^f J \simeq \mathbb{Z} + X\mathbb{Q}[X]$ *lequel est bien connu qu'il est un domaine de Prüfer, donc la première conjecture y est vérifiée.*

Dans le théorème suivant on va étudier la première conjecture dans certains cas particuliers d'amalgamation.

Théorème 5.3.2 *Soient* $f : A \to B$ *un morphisme d'anneaux commutatifs et J un idéal de B. Alors les assertions suivantes sont satisfaites:*

1. *Si la première conjecture est vraie dans* $A \bowtie^f J$ *elle est aussi vraie dans A.*
2. *Si la première conjecture est vraie dans A. Alors, pour tout idéal I de A, on a:* $I \bowtie^f J$ *est un idéal n-absorbant de* $A \bowtie^f J$ *si et seulement si* $I \bowtie^f J$ *est un idéal fortement n-absorbant de* $A \bowtie^f J$.

Corollaire 5.3.3 *On considère* $f : A \to B$ *un morphisme surjectif d'anneaux commutatifs, J un idéal de B et n un entier positif. On suppose que* $f^{-1}(J) = \{0\}$. *Alors:*
La première conjecture est vraie dans $f(A) + J$ *si et seulement si la première conjecture est vraie dans* $A \bowtie^f J$.

Corollaire 5.3.4 *On considère* $f : A \to B$ *un morphisme surjectif d'anneaux commutatifs, J un idéal de B et n un entier positif. Soit* $f^{-1}(J) \times \{0\} \subseteq K$ *un idéal de* $A \bowtie^f J$. *Si la première conjecture est vraie dans B. Alors, K est un idéal n-absorbant de* $A \bowtie^f J$ *si et seulement si K est un idéal fortement n-absorbant de* $A \bowtie^f J$.

Corollaire 5.3.5 *Soit* $f : A \to B$ *un morphisme surjectif d'anneaux commutatifs, J un idéal de B et n un entier positif. On suppose que* $f^{-1}(J) \subseteq I$ *un idéal de A. Si la premìere conjecture est vraie dans B. Alors, I est un idéal n-absorbant de A si et seulement si I est un idéal fortement n-absorbant de A.*

Soient R un anneau Noethérien et I un idéal de R. On sait que I est fortement n-absorbant pour un certain entier positif n.

Théorème 5.3.6 *Soient $f : A \to, B$ un morphisme d'anneaux commutatifs et J un idéal de B. On suppose que A est $f(A)+J$ sont des anneaux Noethériens. Alors tout idéal de $A \bowtie^f J$ est fortement n-absorbant pour un certain entier positif n.*

5.4 La troisième conjecture dans l'amalgamation

Soit $n \geq 1$ un entier et I un idéal propre de A. D. F. Anderson et A. Badawi proposent la conjecture suivante:

Troisième conjecture: Si I est un idéal propre n-absorbant de A. Alors $I[X]$ est un idéal n-absorbant de $A[X]$.

Il est connu [68] que si A est un anneau arithmétique, alors la troisième conjecture est vraie dans A.
Notre premier résultat est un transfert de la notion de n-absorbant dans $(A \bowtie^f J)[X]$.

Théorème 5.4.1 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs, J un idéal de B et I un idéal de A. Alors $I[X]$ est un idéal n-absorbant de $A[X]$ si et seulement si $(I \bowtie^f J)[X]$ est un idéal n-absorbant de $(A \bowtie^f J)[X]$.*

Exemple 5.4.2 *Soient $A = \mathbb{Z}, B = \mathbb{Z}/9\mathbb{Z}$, $f : A \to B$ la surjection canonique et $J = 3\mathbb{Z}/9\mathbb{Z}$ idéal maximal de B. Soit $I = 5\mathbb{Z}$ lequel est un idéal n-absorbant de A. On a $I[X]$ est un idéal n-absorbant de $A[X]$. Donc selon 5.4.1 on a $(I \bowtie^f J)[X]$ est un idéal n-absorbant de $(A \bowtie^f J)[X]$.*

Proposition 5.4.3 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B. Si la troisième conjecture est vraie dans $A \bowtie^f J$, alors elle est aussi vraie dans A.*

Exemple 5.4.4 *Soient A un anneau intègre de corps de fractions K, E un K-espace vectoriel, $B := A \propto E$, $f : A \hookrightarrow B$ l'injection définie par $f(a) = (a, 0)$ pour tout $a \in A$ et $R = A \bowtie^f J$. Alors La troisième conjecture est vraie dans R si et seulement si elle est vraie dans A.*

Théorème 5.4.5 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B. Alors les assertions suivantes sont satisfaites:*

1. *Si la troisième conjecture est vraie dans A. Alors pour tout idéal n-absorbant K contenant $\{0\} \times J$, on a $K[X]$ est un idéal n-absorbant de $(A \bowtie^f J)[X]$.*

2. *On suppose que la troisième conjecture est vraie dans $f(A)+J$. Soit K un idéal n-absorbant de $f(A)+J$ contenant $f^{-1}(J) \times \{0\}$. Alors $K[X]$ est un idéal n-absorbant de $(A \bowtie^f J)[X]$.*

CHAPTER 6

CHAPITRE 6
LES IDÉAUX (m,n)-FERMÉS DANS L'AMALGAMATION

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6.1 Introduction

Dans ce chapitre, * on compléte l'étude faite dans le chapitre 4 concernant les idéaux (m,n)-fermés dans l'extension triviale en étudiant les idéaux (m,n)-fermés dans l'extension amalgamée.

La notion d'idéal (m,n)-fermé d'un anneau commutatif est introduite par D. F. Anderson et A. Badawi en 2016, voir [6] comme suit: Soit m et n des entiers naturels et I un idéal propre d'un anneau R. On dit que I est un idéal (m,n)-fermé de R si $a^m \in I$ pour un certain a de R implique $a^n \in I$. L'idéal I est dit semi n-absorbant de R si il est $(n+1,n)$-fermé de R. L'idéal I est dit radical si il est $(2,1)$-fermé de R.

Le but de ce chapitre est d'étudier la notion d'idéal (m,n)-fermé dans l'extension amalgamée.

Soient $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B. On pose $R := A \bowtie^f J$ l'extension amalgamée de A par B le long l'idéal J en respectant f.

*Soumis pour publication (en collaboration avec N. Mahdou et M. A. Moutui)

Soit $0 < n < m$ des entiers naturels. Soit I un idéal de A et K un idéal de $f(A)+J$, on a:

$$I \bowtie^f J = \{(a, f(a)+j); a \in I, j \in J\},$$

$$\overline{K}^f = \{(a, f(a)+j); a \in A, j \in J, f(a)+j \in K\},$$

$$\overline{I \times K}^f = \{(a, f(a)+j); a \in I, j \in J, f(a)+j \in K\}.$$

Clairement $I \bowtie^f, \overline{I \times K}^f$ et $\overline{I \times K}^f$ sont des idéaux de $A \bowtie^f J$.

Notre premier résultat est de voir quand $I \bowtie^f$ et $\overline{K}^f$ sont des idéaux (m,n)-fermés de $A \bowtie^f J$

6.2 Sur quelques idéaux (m,n)-fermés de l'amalgamation

Proposition 6.2.1 *Avec les notations précédentes on a les assertions suivantes:*

1. *$I \bowtie^f J$ est un idéal (m,n)-fermé de R si et seulement si I est un idéal (m,n)-fermé de A.*
2. *$\overline{K}^f$ est un idéal (m,n)-fermé de $f(A)+J$ si et seulement si K est un idéal (m,n)-fermé de A.*

Le corollaire suivant est une conséquence simple de la proposition 6.2.1.

Corollaire 6.2.2 *Avec les notations précédentes on a les assertions suivantes:*

1. *$I \bowtie^f J$ est un idéal radical de R si et seulement si I est un idéal radical de A.*
2. *$I \bowtie^f J$ est un idéal semi n-absorbant de R si et seulement si I est un idéal semi n-absorbant de A.*
3. *$\overline{K}^f$ est un idéal radical de $f(A)+J$ si et seulement si K est un idéal radical de A.*
4. *$\overline{K}^f$ est un idéal semi n-absorbant de $f(A)+J$ si et seulement si K est un idéal semi n-absorbant de A.*

Soit I un idéal propre de l'anneau A, la duplication amalgamée de A le long I est:

$$A \bowtie I := \{(a, a+i); a \in A, i \in I\}$$

Le corollaire suivant est une conséquence immédiate de la proposition 6.2.1 et du corollaire 6.2.2.

Corollaire 6.2.3 *Soient A est un anneau, I un idéal propre de A et K un idéal de A. Alors les assertions suivantes sont satisfaites:*

1. *$K \bowtie I$ est un idéal (m,n)-fermé de $A \bowtie I$ si et seulement si K est un idéal (m,n)-fermé de A.*
2. *$K \bowtie I$ est un idéal radical de $A \bowtie I$ si et seulement si K est un idéal radical de A.*
3. *$K \bowtie I$ est semi-n-absorbant de $A \bowtie I$ si et seulement si K est un idéal semi n-absorbant de A.*

Remarque 6.2.4 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B. On considère I (resp.,H) est un idéal de A (resp.,$f(A)+J$) tels que $f(I)J \subseteq H \subseteq J$. Il est claire que si $I \bowtie^f H$ est un idéal (m,n)-fermé de $A \bowtie^f J$, alors I est aussi un idéal (m,n)-fermé de A. Mais la réciproque est en général fausse comme le prouve l'exemple suivant.*

Exemple 6.2.5 *Soit $A := \mathbb{Z}, B = \mathbb{Q}[[X]]$, soit $f : A \hookrightarrow B$ l'injection canonique et $J := X\mathbb{Q}[[X]]$ idéal maximal de B. L 'idéal $I = \{0\}$ est $(2,1)$-fermé de A puisqu'il est premier. D'autre part puisque $f(I)J = 0 \subseteq H \subseteq J$, alors $0 \bowtie^f H$ est un idéal de $f(A)+J$ où H est l'idéal de $f(A)+J = \mathbb{Z}+X\mathbb{Q}[[X]]$ donné par $H := \{XP(X), P \in f(A)+J\}$. On a $0 \bowtie^f H$ non $(2,1)$-fermé de $f(A)+J$. En effet $(0,2X^2) = (0,\sqrt{2}X)^2 \in 0 \bowtie^f H$, mais $(0,\sqrt{2}X) \notin 0 \bowtie^f H$.*

On va étudier maintenant dans quel cas $I \bowtie^f H$ est un idéal (m,n)-fermé de $A \bowtie^f J$ lorsque I est un idéal de A et H un idéal de $f(A)+J$

Proposition 6.2.6 *Soit $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B. Si H un idéal de $f(A)+J$ tel que $f(I)J \subseteq H \subseteq J$ avec I un idéal de A et $1 \leq n \leq m$ des entiers. Alors les assertions suivantes sont satisfaites:*

1. *Si $I \bowtie^f H$ est un idéal (m,n)-fermé de $A \bowtie^f J$, alors I est aussi un idéal (m,n)-fermé de A.*
2. *Si I est un idéal $(m,1)$-fermé de A et pour tout $x \in J$, on a $x^n \in H$. Alors $I \bowtie^f H$ est un idéal (m,n)-fermé de $A \bowtie^f J$.*

Le corollaire suivant est une conséquence simple de la proposition 6.2.6.

Corollaire 6.2.7 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B. On considère I un idéal $(m,1)$-fermé et que $x^n \in (f(I)B)J$ pour tout $x \in J$ où $n \geq 1$ entier. Alors l'extension de l'idéal I dans $A \bowtie^f J$ noté I^e est un idéal (m,n)-fermé de $A \bowtie^f J$ pour tout entier $m \geq n \geq 1$.*

Dans la suite, nous montrons comment en utilisant la proposition 6.2.6 on construit un exemple d'idéal (m,n)-fermé de la forme $I \bowtie^f H$ de l'amalgamation $A \bowtie^f J$.

Exemple 6.2.8 *Soient A un anneau, E un A-module et $B = A \propto E$ l'extension triviale de A par E. On considère I un idéal de A et F un sous-module de E tel que $IE \subseteq F$ et $J = I \propto E$ un idéal de B. Soit $f : A \to B$ le morphisme d'anneaux défini par $f(a) = (a,0)$ pour tout $a \in A$. On remarque que $H := I \propto F$ est un idéal de $f(A) + J = A \propto E$ et $f(I)J = (I \propto 0)(I \propto E) \subseteq I \propto F \subseteq J$, d 'ou $I \bowtie^f H$ est un idéal de $A \bowtie^f J$. Soit $n \in \mathbb{N}^*$ et $(i,e) \in J$, on a $(i,e)^n = (i^n, ni^{n-1}e)$. Par la proposition 6.2.6, on conclut que si I est un idéal $(m,1)$-fermé de A alors $I \bowtie^f H$ est un idéal (m,n)-fermé de $A \bowtie^f J$ pour tous entiers $1 \leq n \leq m$. En particulier si I est un idéal premier, on a $I \bowtie^f H$ est un idéal radical de $A \bowtie^f J$.*

Maintenant on va donner une caractérisation pour que $I \bowtie^f H$ soit (m,n)-fermé de $A \bowtie^f J$ lorsque J idéal de B et A est local d'idéal maximal M vérifiant $f(M)J = 0$.

Théorème 6.2.9 *Soient A un anneau local d'idéal maximal M, $f : A \to B$ un morphisme d'anneaux et J un idéal de B tel que $f(M)J = 0$. On considère I un idéal de A et H un idéal de $f(A)+J$ tel que, pour tout $x \in J$, on ait $x^n \in H \subseteq J$. Alors les assertions suivantes sont équivalentes:*

1. *$I \bowtie^f H$ est un idéal (m,n)-fermé de $A \bowtie^f J$ pour tous entiers $1 \leq n < m$.*
2. *I est un idéal (m,n)-fermé de A pour tous entiers $1 \leq n < m$.*

Avant d'illustrer le théorème 6.2.9 nous prouvons le résultat suivant que sera utile pour la suite

Lemme 6.2.10 *Soit (A,M) un anneau local d'idéal maximal M tel que $M^n = 0$ où n un entier positif. Alors tout idéal de A est n-absorbant, en particulier tout idéal de A est (m,n)-fermé pour tout entier m tel que $n < m$.*

Maintenant, nous montrons comment utiliser le théorème 6.2.9 et le lemme 6.2.10 pour construire des exemples d'idéaux (m,n)- fermés de la forme $I \bowtie^f H$.

Exemple 6.2.11 *On considère (A,M) un anneau local d'idéal maximal M tel que $M^n = 0$ où $2 \leq n$ un entier, E un A/M-espace vectoriel et $f : A \to B := A \propto E$ le morphisme d'anneaux défini par $f(a) = (a,0)$. Soit $J := M \propto E$ et $H := I \propto E$ idéal de $f(A) + J$ où $I \subsetneq M$ un idéal de A. Clairement on a $J^n = 0$ et $f(I)J = (I \propto E)(M \propto E) \subseteq H \subseteq J$ et $J^n \subseteq H$. Par le théorème 6.2.9 on a $I \bowtie^f H$ est un idéal (m,n)-fermé de $A \bowtie^f J$ pour tous entiers $2 \leq n < m$, puisque I est un idéal n-absorbant de A selon le lemme 6.2.10.*

Nous fermons cette section avec la proposition suivante donnant un résultat qui fait appel à la caractéristique de l'anneau R. Pour cela, on note $\mathrm{Car}(R)$ la caractéristique de l'anneau R.

Proposition 6.2.12 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B. On suppose que $Char(f(A)+J) = n$, et H un idéal de $f(A)+J$ tel que $f(I)J \subseteq H \subseteq J$. Alors $I \bowtie^f H$ est un idéal (m,n)-fermé de $A \bowtie^f J$ si et seulement si I est un idéal (m,n)-fermé de A pour tous entiers $0 < n \leq m$.*

6.3 Quand un idéal de l'amalgamation est (m,n)-fermé

Dans le résultat suivant, on donnera une caractérisation pour que tout idéal de l'amalgamation $A \bowtie^f J$ soit un idéal(m,n)-fermé pour tous entiers $0 < n \leq m$.

Théorème 6.3.1 *On suppose que $f^{-1}(J)$ soit un idéal radical de A. Alors tout idéal de $A \bowtie^f J$ est un idéal (m,n)-fermé de $A \bowtie^f J$ si et seulement si les assertions suivantes sont satisfaites:*

1. *Tout idéal de A est un idéal (m,n)-fermé de A.*
2. *Tout idéal de $f(A)+J$ est un idéal (m,n)-fermé de $f(A)+J$.*

La preuve de ce théorème nécessite les lemmes suivants.

Lemme 6.3.2 *Soient R un anneau commutatif et m et n des entiers tels que $1 \leq n < m$. Alors les assertions suivantes sont équivalentes:*

1. *Tout idéal de R est un idéal (m,n)-fermé de R.*
2. *$dim(R) = 0$ et $w^n = 0$ pour tout $w \in Nil(R)$.*

Lemme 6.3.3 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B. Alors:*

$$Nil(A \bowtie^f J) := \{(a, f(a)+j)/a \in Nil(A), j \in Nil(B) \cap J\}$$

Lemme 6.3.4 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B. Alors: $dim(A \bowtie^f J) = Max(dim(A), dim(f(A)+J))$.*

Corollaire 6.3.5 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B. Les assertions suivantes sont satisfaites:*

1. *Tout idéal de $A \bowtie^f J$ est un idéal radical de $A \bowtie^f J$ si et seulement si les assertions suivantes sont vérifiées:*

   (i) *Tout idéal de A est un idéal radical de A,*
   (ii) *Tout idéal de $f(A)+J$ est un idéal radical de $f(A)+J$.*

2. *Si $f^{-1}(J)$ est un idéal radical de A. Alors tout idéal de $A \bowtie^f J$ est un idéal semi n-absorbant de $A \bowtie^f J$ si et seulement si les assertions suivantes sont vérifiées:*

   (i) *Tout idéal de A est un idéal semi n-absorbant de A,*
   (ii) *Tout idéal de $f(A)+J$ est un idéal semi n-absorbant de $f(A)+J$.*

Remarque 6.3.6 *On observe que la condition " $f^{-1}(J)$ est radical de A" est omis dans le corollaire 6.3.5. En effet si tout idéal de $A \bowtie^f J$ est un idéal radical, alors $f^{-1}(J) \times \{0\}$ est un idéal radical de $A \bowtie^f J$ or $f^{-1}(J) \simeq f^{-1}(J) \times \{0\}$, d' où $f^{-1}(J)$ est un idéal radical de A. Réciproquement si les assertions (i) et (ii) du corollaire 6.3.5(1) sont satisfaites, alors $f^{-1}(J)$ est un idéal radical de A.*

Le corollaire suivant étudie le cas de la duplication

Corollaire 6.3.7 *Si A un anneau commutatif et I un idéal propre de A. Alors les assertions suivantes sont satisfaites:*

1. *Si I est un idéal radical de A. Alors tout idéal de $A \bowtie I$ est un idéal (m,n)-fermé de $A \bowtie I$ si et seulement si tout idéal de A est un idéal (m,n)-fermé de A.*
2. *Tout idéal de $A \bowtie I$ est radical si et seulement si tout idéal de A est radical.*
3. *Si I est un idéal radical de A. Alors tout idéal de $A \bowtie I$ est un idéal semi n-absorbant de $A \bowtie I$ si et seulement si tout idéal de A est un idéal semi n-absorbant de A.*

Á l'aide du théorème 6.3.1 on retrouve le théorème [15, Theorem 6.12].

Corollaire 6.3.8 *Soient A un anneau intègre de corps de fractions K, E un K-espace vectoriel et $B := A \propto E$ l'extension triviale de A par E. Alors les assertions suivantes sont équivalentes:*

1. *Tout idéal de A est un idéal (m,n)-fermé de A pour m,n des entiers tes que $1 \leq n \leq m$.*

2. *Tout idéal de B est un idéal (m,n)-fermé de B pour m,n des entiers tes que $1 \leq n \leq m$.*

Comme conséquence du théorème 6.3.1 on donne une caractérisation compléte des idéaux (m,n)-fermés de l'anneau $D+M$.

Corollaire 6.3.9 *Soient M un idéal maximal d'un anneau intègre T et D un sous-anneau de T tel que $D \cap M = \{0\}$. Alors tout idéal de $D+M$ est un idéal (m,n)-fermé de $D+M$ si et seulement si tout idéal de D est un idéal (m,n)-fermé de D pour tous entiers $1 \leq n \leq m$.*

Comme application du théorème 6.3.1, on donnera une nouvelle caractérisation pour que l'amalgamation $A \bowtie^f J$ soit un anneau régulier au sens de Von Neumann. On rappelle qu'un anneau réduit est dit régulier au sens de Von Neumann si tout idéal est radical.

Corollaire 6.3.10 *Étant donné $f : A \to B$ un morphisme d'anneaux commutatifs, J un idéal de B. Alors $A \bowtie^f J$ est un anneau régulier au sens de Von Neumann si et seulement si A et $f(A)+J$ sont des anneaux réguliers au sens de Von Neumann.*

Le résultat suivant est une conséquence du corollaire 6.3.10, il permet le transfert de la notion π-régulier à l'amalgamation. On rappelle qu'un anneau R est π-regulier si $R/\mathrm{Nil}(R)$ est un anneau régulier au sens de Von Neumann.

Corollaire 6.3.11 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B. Alors $A \bowtie^f J$ est un anneau π-régulier si et seulement si A et $f(A)+J$ sont des anneaux π-réguliers.*

Une autre application du théorème 6.3.1 est une condition nécessaire et suffisante pour que l'amalgamation soit semi-simple

Corollaire 6.3.12 *Soient $f : A \to B$ un morphisme d'anneaux commutatifs et J un idéal de B. Alors $A \bowtie^f J$ est un anneau semi-simple si et seulement si A et $f(A)+J$ sont des anneaux semi-simples.*

Part II

Partie II: Les chapitres en anglais

CHAPTER 1
TRIVIAL EXTENSIONS DEFINED BY 2-ABSORBING-LIKE CONDITIONS

Abstract*

Let R be a commutative ring with $1 \neq 0$. The notions of 2-absorbing ideal and 2-absorbing primary ideal are introduced by Ayman Badawi as generalizations of prime ideal and primary ideal respectively. A proper ideal I of R is called a 2-absorbing ideal of R (resp., 2-absorbing primary ideal) if whenever $a,b,c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$ (resp., $ab \in I$ or $ac \in \mathrm{Rad}(I)$ or $bc \in \mathrm{Rad}(I)$).

In this paper, we investigate the transfer of 2-absorbing-like properties to trivial ring extensions.

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*Journal of Algebra and its Applications,Vol.17, N. 11 (2018), 1850208, 10 pp. (in collaboration with N. Mahdou)

1.1 Introduction

Throughout this paper all rings are commutative with a nonzero identity. A. Badawi in [9] generalized the concept of prime ideal. He defined a nonzero proper ideal I of R to be a 2-absorbing ideal of R, if whenever $a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. D. F. Anderson and A. Badawi [5] generalized the concept of 2-absorbing ideals to n-absorbing ideals. According to their definition, a proper ideal I of R is called an n-absorbing ideal if whenever $a_1a_2\cdots a_{n+1} \in I$ for $a_1,a_2,\cdots,a_{n+1} \in R$ then there are n of the $a_i's$ whose product is in I. A 1-absorbing ideal is just a prime ideal. The concept 2-absorbing ideal has another generalization, called weakly 2-absorbing ideals, which has studied in [14]. A proper ideal I of R is called weakly 2-absorbing ideal of R if whenever $a,b,c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. The concept of 2-absorbing primary ideals, a generalization of primary ideals was introduced by A. Badawi in [16]. A proper ideal I of commutative ring R is called a 2-absorbing primary ideal of R if whenever $a,b,c \in R$ and $abc \in I$ then either $ab \in I$ or $ac \in \mathrm{Rad}(I)$ or $bc \in \mathrm{Rad}(I)$. In [14] A. Badawi defined a proper ideal I of a commutative ring R to be a weakly 2-absorbing primary ideal if whenever $a,b,c \in R$ and $0 \neq abc \in I$ then either $ab \in I$ or $ac \in \mathrm{Rad}(I)$ or $bc \in \mathrm{Rad}(I)$. In [9, Theorem 2.3], A. Badawi shows that if I is 2-absorbing, then there are at most two prime ideals of R that are minimal over I. We say that a prime ideal P is a minimal prime (see [54]) ideal of I, if for each $x \in P$, there is $y \in R \setminus P$ and nonnegative integer n such that $yx^n \in I$. It is also proved in [16, theorem 2.8], that if I is an ideal of a commutative ring R such that $\mathrm{Rad}(I) \neq I$ and $\mathrm{Rad}(I)$ is prime, then I is 2-absorbing if and only if the ideal $[I :_R x] := \{y \in R; yx \in I\}$ is prime for each $x \in \mathrm{Rad}(I) \setminus I$.

Let R be a commutative ring with $1 \neq 0$. A prime ideal P of a ring R is said to be *divided prime ideal* if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. An commutative R is said to be a *divided ring* if every prime ideal of R is a divided prime ideal.

Let A be a ring and E an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a,e)(a',e') := (aa', ae' + a'e)$. For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \propto E'$ is an ideal of R. However, prime (resp., maximal) ideals of R have the form $p \propto E$, where p is a prime (resp., maximal) ideal of A.

Trivial ring extensions have been studied extensively. Considerable work, part of is summarized in Glaz's book [50] and Huckaba's book [54], has been concerned with trivial ring extension these extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [4, 19, 50, 54, 64].

In this paper, we investigate the transfer of 2-absorbing-like properties to trivial ring extensions.

1.2 A 2-absorbing ideal in trivial extension

Now, we investigate the transfer of 2-absorbing ideal in trivial ring extensions.

Theorem 1.2.1 *Let A a commutative ring, E an A-module, $R := A \propto E$ be the trivial ring extension of A by E, and let I be a proper ideal of A. Then I is a 2-absorbing ideal of A if and only if $I \propto E$ is a 2-absorbing ideal of $A \propto E$.*

Proof. Assume that $I \propto E$ is a 2-absorbing ideal of $A \propto E$ and let $abc \in I$ for some $a,b,c \in A$. We have $(a,0)(b,0)(c,0) \in I \propto E$ and so $(a,0)(b,0) \in I \propto E$ or $(a,0)(c,0) \in I \propto E$ or $(b,0)(c,0) \in I \propto E$ since $I \propto E$ 2-absorbing ideal. Therefore, $ab \in I$ or $bc \in I$ or $ac \in I$, as desired.

Conversely, let $(a,m)(b,n)(c,p) \in I \propto E$ for some $(a,m),(b,n),(c,p) \in A \propto E$. Then $abc \in I$ and so, for example $ab \in I$ (since I is a 2-absorbing ideal). Therefore, $(a,m)(b,n) = (ab, an + bm) \in I \propto E$ which completes the proof of Theorem 2.1.

Let A be an integral domain with quotient field K, E be a K-vector space, and $R := A \propto E$. In this context, we have:

Theorem 1.2.2 *Let A be an integral domain with quotient field K, E be a K-vector space, and $R := A \propto E$. Then the 2-absorbing ideals of R has the form $I \propto E$ where I is a 2-absorbing ideal of A, or $0 \propto F$ where F is a K-vector subspace of E.*

To prove this Theorem, we need the following Lemma.

Lemma 1.2.3 *Let A be an integral domain with quotient field K, E be a K-vector space, F be a A-submodule of E and $R := A \propto E$. Then $0 \propto F$ is a 2-absorbing ideal of R if and only if F is a K-vector subspace of E.*

Proof.

Suppose that $J := 0 \propto F$ is a 2-absorbing ideal of R. Our aim is to show that F is a K-vector subspace of E. It suffices to show that $\frac{1}{a} f \in F$ for every nonzero element a of A and for every $f \in F$.

Let $a \in A - \{0\}$ and $f \in F$. Then, $(a,0)^2(0, \frac{f}{a^2}) = (0,f) \in 0 \propto F$ and so $(a,0)(0, \frac{f}{a^2}) = (0, \frac{f}{a}) \in 0 \propto F$ since $0 \propto F$ is a 2-absorbing ideal of $A \propto E$ and $(a,0)^2 = (a^2,0) \notin 0 \propto F$. Hence, $\frac{1}{a} f \in F$, as desired.

Conversely, without loss of generality, we may assume that $F \subsetneq E$ and let $e \in E - F$. It is clear to see that $[0 \propto F :_R (0,e)] = 0 \propto E$ which is a prime ideal of R and $\mathrm{Rad}(0 \propto F) = 0 \propto E \neq 0 \propto F$. Hence, the result follows from [9, Theorem 2.8].

Proof of Theorem 1.2.2.

Since A is an integral domain with quotient field K, and E be a K-vector space, the ideals of $R := A \propto E$ has the form $I \propto E$ where I si an ideal of A or $0 \propto F$ where F is a A-submodule of E from [4, Theorem 3.2]. Now the result follows from Theorem 2.1 and Lemma 2.3.

Remark 1.2.4 *Let I be a nonzero ideal of A. Theorem 1.2.1 shows that I is a 2-absorbing ideal of A if and only if $I \propto E$ is a 2-absorbing ideal of $R := A \propto E$.*

If I is a 2-absorbing proper ideal of A and $N(\neq E)$ is a submodule of E such that $IE \subseteq N$, in general $I \propto N$ is not a 2-absorbing ideal of R. For example in $\mathbb{Z} \propto \mathbb{Z}$, we have $Rad(\{0\} \propto 6\mathbb{Z}) =$

$\{0\} \propto \mathbb{Z}$ which is a prime ideal, and $[\{0\} \propto 6\mathbb{Z} : (0,1)] = 6\mathbb{Z} \propto \mathbb{Z}$ which is not a prime ideal. Hence, $\{0\} \propto 6\mathbb{Z}$ is not a 2-absorbing ideal of $\mathbb{Z} \propto \mathbb{Z}$ by [9, Theorem 2.8].

In this sense we have the following results.

Theorem 1.2.5 *Let A be a commutative ring, E be an A-module, $R := A \propto E$ be the trivial ring extension of A by E, I be a proper ideal of A, and $N \subsetneq E$ be a submodule of E such that $IE \subseteq N$. Then:*

1. *If $I \propto N$ is a 2-absorbing ideal of R, then I is a 2-absorbing ideal of A.*
2. *Assume that I is a 2-absorbing ideal and P-primary ideal of A, where $Rad(I) = P$ is a prime ideal of A and $[N : E] = P$. Then $I \propto N$ is a 2-absorbing ideal of R.*

Proof.

1. The proof is similar to the proof of Theorem 1.2.1.
2. Let $(a,i)(b,j)(c,k) \in I \propto N$, whenever $(a,i),(b,j),(c,k) \in A \propto E$. Hence, $abc \in I$ and we assume that neither $(a,i)(b,j) \in I \propto N$ nor $(a,i)(c,k) \in I \propto N$ nor $(b,j)(c,k) \in I \propto N$. Since $I \propto N$ is a $(P \propto E)$-primary ideal of R by [54, theorem 25.2], we have $a \in P$, $b \in P$ and $c \in P$. Since I is a 2-absorbing ideal of A, if we assume that $ab \in I$, then $(a,i)(b,j) = (ab, aj + bi) \in I \propto N$ since $aj \in N$ and $bi \in N$, a desired contradiction. We conclude that $I \propto N$ is a 2-absorbing ideal of R.

Definition 1.2.6 *A proper prime ideal P of A is called a divided prime ideal if $P \subseteq Ax$, for each $x \in A \setminus P$.*

Corollary 1.2.7 *Let A be a commutative ring, E be an A-module, and let $R := A \propto E$. Let P be a nonzero divided prime ideal of A and I be a 2-absorbing ideal of A such that $Rad(I) = P$. Let $N \subsetneq E$ be a P-primary submodule of E such that $IE \subseteq N$. Then $I \propto N$ is a 2-absorbing ideal of R and $(P \propto E)^2 \subseteq I \propto N$.*

Proof. Since P is a divided prime ideal of A such that $\mathrm{Rad}(I) = P$, and I is a 2-absorbing ideal of A, so I is a P-primary ideal of A and $P^2 \subset I$ from [9, Theorem 3.6]. On the other hand, we have $I \propto N$ is a 2-absorbing ideal of $A \propto E$ by the last Theorem 2.5 since $[N : E] = P$. Therefore, by [9, Theorem 3.6], we conclude that $I \propto E$ is a $(P \propto E)$-primary ideal of E and $(P \propto E)^2 \subset I \propto N$.

Corollary 1.2.8 *Let A be a commutative ring, let P be a (proper) prime ideal of A. Then $P \propto P$ is a 2-absorbing ideal of $A \propto A$ which is not a prime ideal of $A \propto A$.*

Proof. Let $b \in A \setminus P$ and $a \in P$. We have $(a,b)(0,1) = (0,a) \in P \propto P$ but $(a,b) \notin P \propto P$ and $(0,1) \notin P \propto P$. Hence, $P \propto P$ is a non-prime ideal of $A \propto A$. On the other hand, $P \propto P$ is a 2-absorbing ideal of $A \propto A$ by Theorem 2.5 and since $[P : A] = P$ and $\mathrm{Rad}(P) = P$ is a P- primary ideal of A.

Proposition 1.2.9 *Let (A,M) be a local ring and E be an A-module such that $ME = 0$ and $M^2 = 0$. Then every proper ideal of $A \propto E$ is 2-absorbing.*

Proof. The ring $A \propto E$ is local with maximal ideal $M \propto E$ satisfying $(M \propto E)^2 = 0$. Now the result follows from [14, Corollary 3.3].

1.3 A 2-absorbing primary in trivial extension

Now, we investigate the transfer of a 2-absorbing primary ideal in trivial ring extensions.

Theorem 1.3.1 *Let A be a commutative ring, E be an A-module, and $R := A \propto E$ be the trivial ring extension of A by E. Let I be a proper ideal of A and N be a submodule of E such that $IE \subseteq N$.*

Then $I \propto N$ is a 2-absorbing primary ideal of R if and only I is a 2-absorbing primary ideal of A.

Proof. Assume that $I \propto N$ is a 2-absorbing primary ideal of R. Let $a,b,c \in A$ such that $abc \in I$. Hence, $(a,0)(b,0)(c,0) \in I \propto N$ and so $(ab,0) \in I \propto N$ or $(ac,0) \in \mathrm{Rad}(I \propto N)$ or $(bc,0) \in \mathrm{Rad}(I \propto N)$ since $I \propto N$ is a 2-absorbing primary ideal of R. Therefore, $ab \in I$ or $bc \in \mathrm{Rad}(I)$ or $ac \in \mathrm{Rad}(I)$, as desired.

Conversely, assume that I is a 2-absorbing ideal of A, and let $(a,i),(b,j),(c,k) \in A \propto E$ such that $(a,i)(b,j)(c,k) \in I \propto N$. Two cases are then possible:

Case 1: $ab \in I$ and $bc \in I$ and $ac \in I$.

It is clear that $(a,i)(b,j) \in \mathrm{Rad}(I \propto N) = \mathrm{Rad}(I) \propto E$.

Case 2: $ab \notin I$ or $ac \notin I$ or $bc \notin I$.

Without loss of generality, we may assume that $ab \notin I$. Since I is a 2-absorbing primary ideal of A, we have $ac \in \mathrm{Rad}(I)$ or $bc \in \mathrm{Rad}(I)$ and so $(a,i)(c,k) \in \mathrm{Rad}(I \propto N)$ or $(b,j)(c,k) \in \mathrm{Rad}(I \propto N)$. Then $I \propto N$ is a 2-absorbing primary ideal of R.

Corollary 1.3.2 *Let A be a commutative ring, E be an A-module, $R := A \propto E$ be the trivial ring extension of A by E, and let I be a proper ideal of A. Then $I \propto E$ is a 2-absorbing primary ideal of R if and only I is a 2-absorbing primary ideal of A.*

Proposition 1.3.3 *Let A be a commutative ring, E be an A-module, $R := A \propto E$ be the trivial ring extension of A by E, and let I be a proper ideal of A and N be a submodule of E such that $IE \subseteq N$. If $\mathrm{Rad}(I)$ is a prime ideal of A, then $I \propto N$ is a 2-absorbing primary ideal of R.*

Proof.

Clearly $\mathrm{Rad}(I \propto N) = \mathrm{Rad}(I) \propto E$ is a prime ideal of R. Now the result follows from [16, Theorem 2.8].

We know that if J is an ideal of a commutative ring R such that $\mathrm{Rad}(J)$ is a 2-absorbing ideal of R, then J is not necessarily a 2-absorbing primary ideal of R (See [16, Example 2.9]).

Recall that a commutative ring R is said to be a *divided ring* if every prime ideal of R is a divided prime ideal. In the context of trivial ring extension we have:

Proposition 1.3.4 *Let $R := A \propto E$, where A is a commutative divided ring and E is an A-module. Let I be a proper ideal of A and N submodule of E such that $IE \subseteq N$. If I is a 2-absorbing ideal of A, then $I \propto N$ is a 2-absorbing primary ideal of R.*

Proof. We know that $\mathrm{Rad}(I) = P$ is a prime ideal or $\mathrm{Rad}(I) = P_1 \cap P_2$ where P_1 and P_2 are the only distinct prime ideals of A that are minimal from [9, Theorem 2.4]. Now since A is a divided ring, $\mathrm{Rad}(I)$ is prime ideal of a ring A. Using [16, Theorem 2.8], we obtain that $\mathrm{Rad}(I) \propto E$ is a prime ideal of a ring R, or $\mathrm{Rad}(I \propto N) = \mathrm{Rad}(I) \propto E$. Hence, $I \propto N$ is a 2-absorbing primary ideal of R, as desired.

In [5, Theorem 4.10], it's proved that if P is a prime ideal of an integral domain A, then $0 \propto P$ is a 2-absorbing ideal of $A \propto A$. Now, we prove the same assertion with the notion of 2-absorbing primary ideal.

Theorem 1.3.5 *Let A be a commutative ring, $R := A \propto A$, and let P be a proper prime ideal of A. Then:*

1. *$P \propto P$ is a 2-absorbing primary ideal of R.*
2. *Assume that A is an integral domain. Then $0 \propto P$ is a 2-absorbing primary ideal of R.*

Proof.

1. We know that $P \propto P$ is a 2-absorbing ideal of R from the corollary 1.2.8 then it is 2-absorbing primary ideal of R.
2. Since A is an integral domain, the radical of the ideal $0 \propto P$ is $0 \propto A$ which is a prime ideal of R. Now the result follows from [16, Theorem 2.8].

1.4 Weakly 2-absorbing ideal in trivial extension

Definition 1.4.1 *Let A be a commutative ring. A proper ideal I is called a weakly 2-absorbing ideal of A if whenever $a,b,c \in A$ and $0 \neq abc \in I$, then either $ab \in I$ or $bc \in I$ or $ac \in I$.*

Let $a,b,c \in A$. (a,b,c) is called a triple-zero of I if $abc = 0$ and $ab \notin I, ac \notin I, bc \notin I$.

Now, we investigate the transfer of a weakly 2-absorbing ideal in trivial ring extensions.

Theorem 1.4.2 *Let A be a commutative ring, E be an A-module, $R := A \propto E$, and let I be a proper ideal of A. Then $I \propto E$ is a weakly 2-absorbing ideal of R if and only if I is a weakly 2-absorbing ideal of A and for any triple-zero (a,b,c) of I, we have $ab \in ann(E), bc \in ann(E)$ and $ac \in ann(E)$.*

Proof. Assume that $I \propto E$ is a weakly 2-absorbing ideal of R, and let $0 \neq abc \in I$, where $a,b,c \in A$. Then $(0,0) \neq (a,0)(b,0)(c,0) \in I \propto E$ which imply that $(a,0)(b,0) \in I \propto E$ or $(b,0)(c,0) \in I \propto E$ or $(a,0)(c,0) \in I \propto E$ and so $ab \in I$ or $bc \in I$ or $ac \in I$.

On the other hand, let (a,b,c) be a triple-zero of I, that is $abc = 0$, $ab \notin I$, $ac \notin I$, and $bc \notin I$. We claim that $ab \in \mathrm{ann}(E), bc \in \mathrm{ann}(E)$ and $ac \in \mathrm{ann}(E)$. Deny. Without loss of generality,

we may assume that $ab \notin \text{ann}(E)$. So there exist $m \in E$ with $abm \neq 0$. Then $(a,0)(b,0)(c,m) = (abc,abm) = (0,abm) \in I \propto E$, but $(a,0)(b,0) = (ab,0) \notin I \propto E$ and $(a,0)(c,m) = (ac,am) \notin I \propto E$ and $(b,0)(c,m) \notin I \propto E$, a desired contradiction.

Conversely, assume that $(0,0) \neq (a,m)(b,n)(c,p) \in I \propto E$. Hence, $abc \in I$. Two cases are then possible.

Case 1: $abc \neq 0$. Then I is a weakly 2-absorbing ideal of A gives $ab \in I$ or $bc \in I$ or $ac \in I$. Hence $(a,m)(b,n) \in I \propto E$ or $(a,m)(c,p) \in I \propto E$ or $(b,n)(c,p) \in I \propto E$, as desired.

Case 2: $abc = 0$. By Case 1, we may assume that $ab \notin I$, $ac \notin I$ and $bc \notin I$. Then by hypothesis ab, ac and bc are in $\text{ann}(E)$. Then $(a,m)(b,n)(c,p) = (0,0)$, a desired contradiction.

Therefore, $I \propto E$ is a weakly 2-absorbing ideal of R and this completes the proof.

Corollary 1.4.3 *Let A be a reduced commutative ring and E be an A-module. Then the unique a weakly 2-absorbing ideal which is not a 2-absorbing ideal of $R := A \propto E$ which has the form $I \propto E$, where I is an ideal of A, is the ideal $0 \propto E$.*

Proof. Let I be an ideal of A and $J := I \propto E$ such that J is weakly 2-absorbing which is not a 2-absorbing ideal of R. Then I is a weakly 2-absorbing ideal of A which is not a 2-absorbing ideal of A and we conclude that $I \subset \text{Nil}(A) = \{0\}$. This means that $J = 0 \propto E$.

The zero ideal $\{0\}$ is a weakly 2-absorbing ideal of A from Theorem 4.2. Let $(a,b,c) \in A^3$ a triple-zero of $\{0\}$, that is $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$. We claim that $ab \in \text{ann}(E), bc \in \text{ann}(E)$ and $ac \in \text{ann}(E)$. Deny. Without loss of generality, we may assume that $ab \notin ann(E)$. Then, there exists $n \in E$ such that $abn \neq 0$ or $(0,0) \neq (a,0)(b,0)(c,n) = (0,abn) \in J$ but neither $(a,0)(b,0) = (ab,0) \in J$ neither $(a,0)(c,n) = (ac,an) \in J$ neither $(b,0)(c,n) = (bc,bn) \in J$, a desired contradiction since J is a weakly 2-absorbing ideal of R.

On the other hand, by [14, Theoem 2.4] and since $J^3 = 0$, J is not a 2-absorbing ideal of $A \propto E$, which completes the proof of Corollary 1.4.3

Remark 1.4.4 *Let (A,M) be a local ring and E be an A-module such that $ME = 0$. If $M^3 = 0$, then every ideal of $A \propto E$ is a weakly 2-absorbing.*
Indeed $A \propto E$ is a local ring with maximal ideal $M \propto E$, we have $(M \propto E)^3 = 0$. Now the result follows from [14, Theorem 3.2].

Proposition 1.4.5 *Let (A,M) be a local ring and E be an A-module such that $ME = 0$. Then, every ideal of $(A \propto E) \times (A \propto E)$ is a weakly 2-absorbing if and only if $M^2 = 0$.*

Proof. We know that $R := A \propto E$ is a local ring with maximal ideal $M \propto E$. Since R is not a field, and $(M \propto E)^2 = M^2 \propto 0$, using [14, Theorem 3.4], we conclude that an ideal in $R \times R$ is a weakly 2-absorbing if and only if $M^2 = 0$.

Remark 1.4.6 *In [14, Theorem 3.2], it's proved that if I is a weakly 2-absorbing ideal of a commutative ring R which is not 2-absorbing, then $I^3 = 0$. But if $I^3 = 0$ and I is not a 2-absorbing ideal of R, we have not necessarily I is a weakly 2-absorbing ideal of R.*

For example, it's known that $I = 0 \propto 2\mathbb{Z}$ is not a 2-absorbing ideal of $\mathbb{Z} \propto \mathbb{Q}$ by [5, Example 4.12.] and $I^3 = 0$, but I is not a weakly 2-absorbing ideal since $ann(\mathbb{Q}) = 0$.

1.5 Weakly 2-absorbing primary ideal in trivial extension

Definition 1.5.1 *Let A be a commutative ring. A proper ideal I of A is called weakly 2-absorbing primary if whenever $a,b,c \in A$ such that $0 \neq abc \in I$, then $ab \in I$ or $ac \in Rad(I)$, or $bc \in Rad(I)$.*

Now, we investigate the transfer of weakly 2-absorbing primary ideal in trivial ring extensions.

Theorem 1.5.2 *Let A be a commutative ring, E be an A-module, $R := A \propto E$ be the trivial ring extension of A by E, and let I be a proper ideal of A. Then, $I \propto E$ is a weakly 2-absorbing primary ideal of R if and only if I is a weakly 2-absorbing primary ideal of A and whenever $a,b,c \in A, abc = 0, ab \notin I, ac \notin Rad(I), bc \notin Rad(I)$, we have $ab \in ann(E), ac \in ann(E)$ and $bc \in ann(E)$.*

Proof. Assume that $I \propto E$ is a weakly 2-absorbing primary of R and let $0 \neq abc \in I$, where $a,b,c \in A$ and assume that $ab \notin I$. Then $(0,0) \neq (a,0)(b,0)(c,0) \in I \propto N$ and $(ab,0) = (a,0)(b,0) \notin I \propto E$. Hence, $(a,0)(c,0) \in \text{Rad}(I \propto E) = \text{Rad}(I) \propto E$, or $(b,0)(c,0) \in \text{Rad}(I \propto E) = \text{Rad}(I) \propto E$ since $I \propto E$ is weakly 2-absorbing primary. Therefore, $ac \in \text{Rad}(I)$ or $bc \in \text{Rad}(I)$, that is I is a weakly 2-absorbing primary ideal of A.

Now, assume that $abc = 0$ (where $a,b,c \in A$) and $ab \notin I, bc \notin \text{Rad}(I), ac \notin \text{Rad}(I)$. If $ab \notin \text{ann}(E)$, so there exist $n \in E$ with $abn \neq 0$. Then $(0,0) \neq (a,0)(b,0)(c,n) = (abc, abn) = (0, abn) \in I \propto E$, but $(a,0)(b,0) = (ab,0) \notin I \propto E$ and $(a,0)(c,n) = (ac,an) \notin \text{Rad}(I) \propto E = \text{Rad}(I \propto N)$ and $(b,0)(c,n) \notin \text{Rad}(I) \propto E = \text{Rad}(I \propto E)$, a desired contradiction.

Conversely, assume that I is a weakly 2-absorbing primary ideal of A and let $(0,0) \neq (a,m)(b,n)(c,k) \in I \propto E$, whenever $(a,n),(b,m),(c,k) \in R$, so $abc \in I$. Two case are then possible:

Case one: $abc \neq 0$.

Since I is a weakly 2-absorbing primary ideal of A, we have $ab \in I$ or $ac \in \text{Rad}(I)$ or $bc \in \text{Rad}(I)$. Then $(a,m)(b,n) \in I \propto E$ or $(a,m)(c,k) \in \text{Rad}(I) \propto E = \text{Rad}(I \propto E)$ or $(b,n)(c,k) \in \text{Rad}(I) \propto E$.

This means that $I \propto E$ is weakly 2-absorbing primary.

Case two: $abc = 0$.

By hypothesis ab, bc and ac are in $\text{ann}(E)$ and so $(a,m)(b,n)(c,k) = (0,0)$, a desired contradiction.

Theorem 1.5.3 *Let A be a commutative ring, E be an A-module, $R := A \propto E$ be the trivial ring extension of A by E, I be a proper ideal of A such that $Rad(I)$ is weakly prime of A, and let N be a submodule of E such that $IE \subseteq N$ and whenever $a,b \in A, ab = 0, a \notin Rad(I), b \notin Rad(I), a,b \in ann(E)$. Then $I \propto N$ is a weakly 2-absorbing primary ideal of R*

To prove this Theorem, we need the following Lemma.

Lemma 1.5.4 *Let A be a commutative ring and I be a proper ideal of A such that $Rad(I)$ is a weakly prime ideal of A. Then I is a weakly 2-absorbing primary ideal of A.*

Proof of Lemma.

Let $0 \neq abc \in I$, whenever $a,b,c \in A$. Suppose $ab \notin I$. Since $0 \neq abc \in \mathrm{Rad}(I)$ and $\mathrm{Rad}(I)$ is a weakly prime ideal of A, we have $ab \in \mathrm{Rad}(I)$ or $c \in \mathrm{Rad}(I)$, then $ab \in \mathrm{Rad}(I)$ or $ac \in \mathrm{Rad}(I)$.

Proof of Theorem

Since $\mathrm{Rad}(I \propto N) = \mathrm{Rad}(I) \propto E$ and since $\mathrm{Rad}(I) \propto E$ is a weakly prime ideal of $A \propto E$ by [3, Theorem 17], we conclude according the Lemma 1.5.4.

1.6 Semi 2-absorbing ideal in trivial extension

Definition 1.6.1 *Let A be a commutative ring. A proper ideal I of A is called a semi 2-absorbing if whenever $a \in A$, such that $a^3 \in I$ we have $a^2 \in I$.*

If I is a 2-absorbing ideal of A, then I is a semi 2-absorbing ideal of A. The converse is false since for example, $12\mathbb{Z}$ is not a 2-absorbing ideal of $\mathbb{Z}$ but it is semi 2-absorbing ideal of $\mathbb{Z}$.

Theorem 1.6.2 *Let A be a commutative ring, E be an A-module, and let I be a proper ideal of A. Then $I \propto E$ is a semi 2-absorbing ideal of $A \propto E$ if and only if I is a semi 2-absorbing ideal of A.*

Proof. Suppose that $I \propto E$ is a semi 2-absorbing ideal of $A \propto E$ and let $a \in A$ such that $a^3 \in I$. Hence, $(a,0)^3 = (a^3,0) \in I \propto A$ since $I \propto A$ is a semi 2-absorbing ideal of $A \propto E$, we conclude that $(a,0)^2 = (a^2,0) \in I \propto A$ and so $a^2 \in I$.

Conversely, suppose that I is a semi 2-absorbing ideal of A and let $(a,e) \in A \propto E$ such that $(a,e)^3 = (a^3,3a^2e) \in I \propto E$. Then $a^3 \in I$ and so $a^2 \in I$ (since I is a semi 2-absorbing ideal of A). Therefore, $(a,e)^2 = (a^2,2ae) \in I \propto E$, as desired.

Proposition 1.6.3 *Let A be a commutative ring and I be a proper semi-prime ideal of A. Then:*

1. *$I \propto I$ is a semi 2-absorbing ideal of $A \propto A$ which is not a semi-prime.*
2. *If A is reduced, $0 \propto I$ is a semi 2-absorbing which is not a semi-prime.*

Proof.

1. Let $(a,b) \in A \propto A$ such that $(a,b)^3 = (a^3,3a^2b) \in I \propto I$. Hence, $a^3 \in I$ and so $a^2 \in I$ (since I is semi-prime), then $(a,b)^2 = (a^2,2ab) \in I \propto I$ (since $a \in I$). Therefore, $I \propto I$ is a semi 2-absorbing ideal of $A \propto A$. On the other hand, $I \propto I$ is not semi-prime, since $(0,1)^2 \in I \propto I$ but $(0,1) \notin I \propto I$.
2. Straightforward.

.

Proposition 1.6.4 *Let A be a commutative ring such that $ch(A) = 2$ (where $ch(A)$ is the characteristic of A) and let N be a submodule of E such that $IE \subseteq N$. Then, I is a semi 2-absorbing ideal of A if and only if $I \propto N$ is a semi 2-absorbing ideal of $A \propto E$.*

Proof.

Suppose that I is a semi 2-absorbing ideal of A and let $(a,e) \in A \propto E$ such that $(a,e)^3 = (a^3, 3a^2e) \in I \propto N$. Then $a^2 \in I$, since $(a,e)^2 = (a^2,0) \in I \propto N$. We conclude that $I \propto N$ is a semi 2-absorbing ideal.

The converse is clear and this completes the proof.

CHAPTER 2
AMALGAMATED ALGEBRAS ALONG AN IDEAL DEFINED BY 2-ABSORBING LIKE CONDITIONS

Abstract*

Let R be a commutative ring with $1 \neq 0$. The notion of 2-absorbing ideal and 2-absorbing primary ideal are introduced by Ayman Badawi as a generalization of prime ideal and primary ideal respectively. A proper ideal I of R is called a 2-absorbing ideal (resp., 2-absorbing primary ideal) if whenever $a,b,c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$ (resp., $ab \in I$ or $ac \in \mathrm{Rad}(I)$ or $bc \in \mathrm{Rad}(I)$).

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*Italian Journal of Pure and Applied Mathematics, accepted for publication (in collaboration with N. Mahdou

2.1 Introduction

Throughout this paper all rings are commutative with a nonzero identity. In [9], A. Badawi generalized the concept of prime ideals. He defined a nonzero proper ideal I of R to be a 2-absorbing ideal of R, if whenever $a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In [5], D. Anderson and A. Badawi generalized the concept of 2-absorbing ideals to n-absorbing ideals. According to their definition, a proper ideal I of R is called an n-absorbing ideal if whenever $a_1a_2\cdots a_{n+1} \in I$ for $a_1,a_2,\cdots,a_{n+1} \in R$, then there are n of the $a_i's$ whose product is in I. A 1-absorbing ideal is just a prime ideal. The concept of a 2-absorbing ideal has another generalization, called a weakly 2-absorbing ideal, which has studied in [14]. A proper ideal I of R is called a weakly 2-absorbing ideal of R if whenever $a,b,c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. The concept of a 2-absorbing primary ideal, a generalization of primary ideal was introduced by A. Badawi in [16]. A proper ideal I of a ring R is called a 2-absorbing primary ideal of R if whenever $a,b,c \in R$ and $abc \in I$ then either $ab \in I$ or $ac \in \mathrm{Rad}(I)$ or $bc \in \mathrm{Rad}(I)$. In [14], A. Badawi defined a proper ideal I of a ring R to be a weakly 2-absorbing primary ideal if whenever $a,b,c \in R$ and $0 \neq abc \in I$ then either $ab \in I$ or $ac \in \mathrm{Rad}(I)$ or $bc \in \mathrm{Rad}(I)$. Recall from [3] (resp., [47]) that a proper ideal I of R is called a weakly prime ideal (resp., weakly primary ideal) if whenever $a,b \in R$ and $0 \neq ab \in I$, then $a \in I$ or $b \in I$ (resp., $a \in I$ or $b \in \mathrm{Rad}(I)$).

Let A and B be two rings, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a)+j) / a \in A, j \in J\}$$

called the amalgamation of A with B along J with respect to f (introduced and studied by D'Anna, Finocchiaro, and Fontana in [36],[37]). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [45, 33, 34] and denoted by $A \bowtie I$). Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation [36, Examples 2.5 & 2.6] and other classical constructions, such as the Nagata's idealization and the CPI extensions (in the sense of Boisen and Sheldon [24]) are strictly related to it (see [36, Example 2.7 & Remark 2.8]).

One of the key tools for studying $A \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [36, Section 4]. This point of view allows the authors in [36, 37] to provide an ample description of various properties of $A \bowtie^f J$, in connection with the properties of A, J and f. Namely, in [36], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation.

Recall that a prime ideal P of a ring R is said a *divided prime ideal* if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. A ring R is said to be a *divided ring*

if every prime ideal of R is a divided prime ideal.

In this paper we will study the transfer of the 2-absorbing ideal notion to amalgamated algebras along an ideal.

2.2 A 2-absorbing ideal on amalgamated algebras along ideal

Now, to the first main result:

Theorem 2.2.1 *Let A and B be two rings, J be an ideal of B, $f : A \to B$ be a ring homomorphism, and set $R := A \bowtie^f J$. Then:*

1. *Let I be an ideal of A. Then, $I \bowtie^f J$ is a 2-absorbing ideal of $A \bowtie^f J$ if and only if I is a 2-absorbing ideal of A.*
2. *Let P be a prime ideal of A and Q be a prime ideal of B. The set $\{(a, f(a)+j)/a \in P, j \in J, f(a)+j \in Q\}$ is a 2-absorbing ideal of $A \bowtie^f J$.*
3. *Assume that $Nil(A) \neq P$ are a divided prime ideals of A. Then $P^2 \bowtie^f J$ is a 2-absorbing ideal of $A \bowtie^f J$.*
4. *Let $m \in max(A)$ and $Q \in max(B) \setminus V(J)$, and set $M := m \bowtie^f J$, or $M := \{(a, f(a)+j)/a \in A, j \in J, f(a)+j \in Q\}$. Then M^2 is a 2-absorbing ideal of $A \bowtie^f J$.*

Proof. (1) Clear since an ideal I is a 2-absorbing ideal of A if and only if 0 is a 2-absorbing ideal in A/I and since $\dfrac{A \bowtie^f J}{I \bowtie^f J} \cong \dfrac{A}{I}$ (by [35, Proposition 5.1] and [5, Theorem 4.2]).

(2) The ideal $P \times Q$ is a 2-absorbing ideal of $A \times B$ and so $(P \times Q) \cap (A \bowtie^f J)$ is a 2-absorbing ideal of $A \bowtie^f J$ by [5, Theorem 4.7] and [5, Corollary 4.3]. On the other hand, $\{(a, f(a)+j)/a \in P, j \in J, f(a)+j \in Q\} = (P \times Q) \cap (A \bowtie^f J)$. Hence, we can conclude.

(3) Since $\mathrm{Nil}(A) \neq P$ are a divided prime ideals of A, we see by [9, Theorem 3.7] that P^2 is a 2-absorbing ideal of A and so $P^2 \bowtie^f J$ is a 2-absorbing ideal of $A \bowtie^f J$ from Theorem 2.1(1), as desired.

(4) In both cases, M is a maximal ideal of $A \bowtie^f J$ and so M^2 is a 2-absorbing ideal from [5, Lemma 2.8].

Example 2.2.2 *Let $R := \mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$ which is an integral domain, where $f : \mathbb{Z} \hookrightarrow \mathbb{Q}[[X]]$ be the natural embedding. The ideal $P = p\mathbb{Z}$ is a divided prime ideal of $\mathbb{Z}$, where p is a prime integer. Hence, by the Theorem 2.2.1, the ideal $p^2\mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$ is a 2-absorbing ideal of R.*

Example 2.2.3 *Let $A \subset B$ be a rings and $X := \{X_1, \cdots, X_n\}$ be a finite set of indeterminates over B. In the formal power series ring $B[[X]]$, we consider the following subring:*

$$A + XB[[X]] := \{h \in B[[X]] | h(0) \in A\}.$$

If $\sigma : A \hookrightarrow B[X]$ is the natural embedding and $J := XB[X]$, then it is easy to check that $A \bowtie^{\sigma} J$ is isomorphic to $A + XB[[X]]$ via the isomorphism $u(a, \sigma(a) + j) = \sigma(a) + j = a + j$.

Let A be an integral domain with quotient field K, $f : A \hookrightarrow K[[X]]$ the natural embedding, and let $J := XK[[X]]$. The 2-absorbing ideals of $A \bowtie^{f} J$ are of the form $I \bowtie^{f} J$, where I is a 2-absorbing ideal of A, or $0 \bowtie^{f} J$ or $0 \bowtie^{f} J^2$.

Proof.

First, we show that $0 \bowtie^{f} J^2$ is a 2-absorbing ideal of R. Let $x_i = (a_i, a_i + XS_i(X)) \in R$, where $S_i \in K[[X]], 1 \leq i \leq 3$ and assume that $x_1x_2x_3 \in 0 \bowtie^{f} J^2$. Since A is an integral domain, we may assume that $a_3 = 0$. Two cases are then possible:

Case 1: $a_1a_2 = 0$.

Then $x_1x_3 = (0, X^2S_1(X)S_3(X)) \in 0 \bowtie^{f} J^2$ or $x_2x_3 \in 0 \bowtie^{f} J^2$.

Case 2: $a_1a_2 \neq 0$.

Then $x_1x_2x_3 = (0, XS_3(X)(a_1, a_1 + XS_1(X))(a_2, a_2 + XS_2(X)) \in 0 \bowtie^{f} J^2$ since J^2 is an ideal of $K[[X]]$ and $a_1a_2 \neq 0$. Therefore, $XS_3(X) \in J^2$ and so $x_1x_3 = (0, XS_3(X)(a_1, a_1 + XS_1(X)) \in 0 \bowtie^{f} J^2$.

Hence, $0 \bowtie^{f} J^2$ is a 2-absorbing ideal of R. On the other hand by Theorem 2.1, we have $0 \bowtie^{f} J, I \bowtie^{f} J$ are 2-absorbing ideals of $A \bowtie^{f} J$, as desired.

Conversely, let L be a 2-absorbing ideal of $A \bowtie^{f} J$. Then $u(L)$ is a 2-absorbing ideal of $A + XK[[X]]$ and so $u(L) = I + XK[[X]]$ for some 2-absorbing ideal I of A, or $u(L) = XK[[X]]$, or $u(L) = X^2K[[X]]$ by [5, Theorem.4.17]. Therefore, $L = u^{-1}(I + J) = I \bowtie^{f} J$ or $L = u^{-1}(J) = 0 \bowtie^{f} J$ or $L = u^{-1}(J^2) = 0 \bowtie^{f} J^2$, as desired.

Example 2.2.4 *Let $R := \mathbb{Z} \bowtie^{f} X\mathbb{Q}[[X]]$, where $f : \mathbb{Z} \hookrightarrow \mathbb{Q}[[X]]$ is the natural embedding. It's clear that R is not a valuation domain.*

The ideal $I := 0 \bowtie^{f} (X\mathbb{Z} + X^2\mathbb{Q}[[X]])$ is not a 2-absorbing ideal of R since $(2,2)^2(0, \frac{1}{2^2}X) = (0,X) \in I$, but neither $(2,2)^2 \in I$ nor $(2,2)(0, \frac{1}{2^2}X) = (0, \frac{1}{2}X) \in I$. Remark that the ideals $pq\mathbb{Z} \bowtie^{f} X\mathbb{Q}[[X]]$ and $p^2\mathbb{Z} \bowtie^{f} X\mathbb{Q}[[X]]$ are 2-absorbing ideals of R by Theorem 2.2.1 and since the 2-absorbing ideals of $\mathbb{Z}$ are of the form $pq\mathbb{Z}$ for distinct positive primes $p, q \in \mathbb{Z}$ or $p^2\mathbb{Z}$ for a positive prime p. And by example 2.2.3, the ideals $0 \bowtie^{f} X^i\mathbb{Q}[[X]]$ where $1 \leq i \leq 2$ are the 2-absorbing ideals of R. We conclude that the only 2-absorbing ideals of R are $pq\mathbb{Z} \bowtie^{f} X\mathbb{Q}[[X]]$, $p^2\mathbb{Z} \bowtie^{f} X\mathbb{Q}[[X]]$ and $0 \bowtie^{f} X^i\mathbb{Q}[[X]]$ where $1 \leq i \leq 2$.

Example 2.2.5 *Let $R := \mathbb{Z} \bowtie^{f} X\mathbb{R}[[X]]$, where $f : \mathbb{Z} \hookrightarrow \mathbb{R}[[X]]$ the natural embedding, and set $I_1 := 0 \bowtie (X\mathbb{Z} + X^2\mathbb{R}[[X]])$ and $I_2 := 0 \bowtie^{f} (X\mathbb{Q} + X^2\mathbb{R}[[X]])$. Then:*

1. *I_1 is not a 2-absorbing ideal of R.*
2. *I_2 is a 2-absorbing ideal of R.*

Proof. 1) The ideal I_1 is not 2-absorbing since $(2,2)^2(0, \frac{1}{2^2}X) = (0,X) \in I_1$, but neither $(2,2)^2 \in I_1$ nor $(2,2)(0, \frac{1}{2^2}X) = (0, \frac{1}{2}X) \in I_1$.

2) We claim that I_2 is a 2-absorbing ideal of R. Indeed, let $x_i = (a_i, a_i + Xs_i(X))$, where $1 \leq i \leq 3$ and $a_i \in \mathbb{Z}, s_i(X) \in \mathbb{R}[X]$ such that $x_1x_2x_3 \in I_2$. Hence, $a_1a_2a_3 = 0$ and we may assume that $a_1 = 0$. Two cases are then possible:

Case 1: $a_2a_3 = 0$. In this case, x_1x_2 or $x_1x_3 \in I_2$, as desired.

Case 2: $a_2a_3 \neq 0$. In this case, $x_1x_2x_3 = (0, Xs_1(X)(a_2, a_2 + Xs_2(X)(a_3, a_3 + Xs_3(X)) \in I_2$ and so $a_2a_3s_1(X) \in \mathbb{Q} + X\mathbb{R}[X]$. Hence, $s_1(X) \in \mathbb{Q} + X\mathbb{R}[X]$ (since $a_2a_3 \in \mathbb{Z}^*$) and so $x_1x_2 \in I_2$, as desired.

Therefore, I_2 is a 2-absorbing ideal of R.

2.3 A Weakly 2-absorbing on amalgamated algebras along an ideal

We recall the following definition.

Definition 2.3.1 *1) A proper ideal I of a commutative ring R is called weakly 2-absorbing ideal, if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $bc \in I$ or $ac \in I$,*

2) A $(a, b, c) \in R^3$ is called triple zero of I if $abc = 0$, and $ab \notin I, bc \notin I$ and $ac \notin I$.

A 2-absorbing ideal is a weakly 2-absorbing ideal. The following example shows that the converse is false.

Example 2.3.2 *Let $A = B = \mathbb{Z}_8$, $J = 4\mathbb{Z}_8$, $f = Id_A$, $R := \mathbb{Z}_8 \bowtie^f J$, and set $I := 0 \bowtie^f J = \{0,0), (0,4)\}$. Then,*

1) I is a weakly 2-absorbing ideal of R since for every $a, b, c \in R \setminus I$, $abc \in I$ if and only if $abc = 0$.

2) I is not a 2-absorbing ideal of R since $(2,0)(2,0)(2,0) = (0,0) \in I$ and $(2,0)(2,0) = (4,0) \notin I$.

We next show how to construct examples of a weakly 2-absorbing ideals using the method of amalgamated algebras along an ideal.

Theorem 2.3.3 *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B, and let I be an ideal of A. Then:*

1. *If $I \bowtie^f J$ is a weakly 2-absorbing ideal of $A \bowtie^f J$, then I is a weakly 2-absorbing ideal of A.*

2. *Assume that I is a weakly 2-absorbing ideal which is not a 2-absorbing ideal of A. Then the following statements are equivalent:*
   i) $I \bowtie^f J$ is a weakly 2-absorbing ideal of $A \bowtie^f J$.
   ii) For every triple zero $(a, b, c) \in A^3$ of I, we have $f(a)f(b)J = f(a)f(c)J = f(b)f(c)J = 0$ and $J^3 = 0$.

Before proving Theorem 2.3.3, we establish the following lemma.

Lemma 2.3.4 *Let $f : A \to B$ be an homomorphism of rings, P,Q are ideals of A, and J,K are ideals of B. Then:*

$$(P \bowtie^f J)(Q \bowtie^f K) = PQ \bowtie^f (f(P)K + f(Q)J + JK).$$

In particular:

$$(I \bowtie^f J)^2 = I^2 \bowtie^f \left(f(I)J + J^2\right)$$

and

$$(I \bowtie^f J)^3 = I^3 \bowtie^f \left(f(I)^2J + f(I)J^2 + J^3\right).$$

Proof.

To simplify the notation, we will not write the index terms on finite sums in this proof.

Any element of $(P \bowtie^f J)(Q \bowtie^f K)$ can be written as follows: $\sum(p_i, f(p_i)+j_i)(q_i, f(q_i)+k_i) = (\sum(p_iq_i), f(\sum p_iq_i), \sum f(p_i)k_i + f(q_i)j_i + j_ik_i)$
$=(\sum p_iq_i, f(\sum p_iq_i) + \sum f(p_i)k_i + \sum f(q_i)j_i + \sum j_ik_i) \in (PQ \bowtie^f f(P)K + f(Q)J + JK)$. Hence, $(P \bowtie^f J)(Q \bowtie^f K) \subset (PQ \bowtie^f f(P)K + f(Q)J + JK)$.

Conversely, let $(\sum p_iq_i, f(\sum p_iq_i) + \sum f(p_l)k_l + \sum f(q_m)j_m + \sum j_sk_s) \in (PQ \bowtie^f f(P)K + f(Q)J + JK)$. This element can be written as $\sum(p_i, f(p_i)(q_i, f(q_i) + \sum(p_l, f(p_l))(0,k_l) + \sum(q_m, f(q_m))(0,j_m) + \sum(0,j_s)(0,k_s)$ which is in $(P \bowtie^f J)(Q \bowtie^f K)$ since $(p_i, f(p_i))(q_i, f(q_i)) \in (P \bowtie^f J)(Q \bowtie^f K)$, $(p_l, f(p_l))(0,k_l) \in (P \bowtie^f J)(Q \bowtie^f K)$, $(0,j_m)(q_m, f(q_m)) \in (P \bowtie^f J)(Q \bowtie^f K)$ and $(0,j_s)(0,k_s) \in (P \bowtie^f J)(Q \bowtie^f K)$.

In particular, we can see that

$$(I \bowtie^f J)^2 = I^2 \bowtie^f \left(f(I)J + J^2\right)$$

and

$$(I \bowtie^f J)^3 = I^3 \bowtie^f \left(f(I)^2J + f(I)J^2 + J^3\right).$$

Proof of Theorem 2.3.3.

1) Assume that $I \bowtie^f J$ is a weakly 2-absorbing ideal of $A \bowtie^f J$, and let $a,b,c \in A$ such that $0 \neq abc \in I$. Then $0 \neq (a, f(a))(b, f(b))(c, f(c)) \in I \bowtie^f J$. Since $I \bowtie^f J$ is a weakly 2-absorbing ideal of $A \bowtie^f J$, one of the products $(a, f(a))(b, f(b)), (a, f(a))(c, f(c)), (b, f(b))(c, f(c))$ is in $I \bowtie^f J$ and so $ab \in I$ or $ac \in I$ or $bc \in I$. We conclude that I is a weakly 2-absorbing ideal of A.

2) Let I be a weakly 2-absorbing ideal of A which is not 2-absorbing. Assume that $I \bowtie^f J$ is a weakly 2-absorbing ideal of $A \bowtie^f J$ and let (a,b,c) be a triple zero of I. We claim that $f(a)f(b)J = f(a)f(c)J = f(b)f(c)J = 0$. Deny. Then, we may assume that $f(a)f(b) \notin \text{ann}(J)$ and so there exists $j \in J$ such that $f(a)f(b)j \neq 0$. Hence, $(0,0) \neq (a, f(a))(b, f(b)(c, f(c) + j) = (0, f(a)f(b)j) \in I \bowtie^f J$, a desired contradiction since $(a, f(a)(b, f(b)) \notin I \bowtie^f J$, $(a, f(a)(c, f(c) + j) \notin I \bowtie^f J$, and $(b, f(b)(c, f(c) + j) \notin I \bowtie^f J$.

Now, we claim that $J^3 = 0$. By Theorem 2.1, $I \bowtie^f J$ is not a 2-absorbing ideal of $A \bowtie^f J$ and so $(I \bowtie^f J)^3 = I^3 \bowtie^f (f(I)^2J + f(I)J^2 + J^3) = 0$, by lemma 3.4 and [14, Theorem 2.4]. Therefore, $J^3 = 0$.

Conversely, let $(a,f(a)+j),(b,f(b)+k),(c,f(c)+l) \in A \bowtie^f J$ such that $(0,0) \neq (a,f(a)+j)(b,f(b)+k)(c,f(c)+l) = (abc,f(abc)+m) \in I \bowtie^f J$, where $m \in J$. Two cases are then possible:

Case one: $abc \neq 0$.

Since I is a weakly 2-absorbing ideal of A, we may assume for example that $ab \in I$. Hence, $(a,f(a)+j)(b,f(b)+k) \in I \bowtie^f J$.

Case two: $abc = 0$.

If (a,b,c) is a triple zero of I, by the above hypothesis, $f(a)f(b)J = f(a)f(c)J = f(b)f(c)J = 0$. Hence, $(a,f(a)+j)(b,f(b)+k)(c,f(c)+l) = (abc,f(abc)+f(a)f(b)l+f(a)f(c)k+f(b)f(c)j+jkm) = (0,jkm) = (0,0)$ since $J^3 = 0$, a desired contradiction.

Therefore, $I \bowtie^f J$ is a weakly 2-absorbing ideal of $A \bowtie^f J$ and this completes the proof of Theorem 2.3.3.

Corollary 2.3.5 *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B such that $J^3 = 0$, and let I be a weakly 2-absorbing ideal of A which is not 2-absorbing such that for every $(a,b,c) \in A^3$ a triple zero of I, $(f(a),f(b),f(c)) \in ann(J)^3$. Then, $I \bowtie^f J$ is a weakly 2-absorbing ideal of $A \bowtie^f J$ which is not a 2-absorbing ideal.*

Corollary 2.3.6 *Let A be a local ring with a maximal ideal m, $f : A \to B$ be a ring homomorphism, J be an ideal of B such that $f(m)J = 0$. Then, the following statements holds:*

1. *Let I be an ideal of A. Then, $I \bowtie^f J$ is a weakly 2-absorbing ideal of $A \bowtie^f J$ which is not 2-absorbing if and only if I is a weakly 2-absorbing ideal of A which is not 2-absorbing and $J^3 = 0$.*

2. *Assume that $m^3 = 0$. Then every ideal of $A \bowtie^f J$ is a weakly 2-absorbing ideal which is not 2-absorbing.*

Proof.

(1) Let I be an ideal of A. If $I \bowtie^f J$ is a weakly 2-absorbing ideal of $A \bowtie^f J$ which is not 2-absorbing, then I is a weakly 2-absorbing ideal of A which is not 2-absorbing and $J^3 = 0$ by Theorem 2.3.3.

Conversely, assume that I is a weakly 2-absorbing ideal of A which is not 2-absorbing and let $(a,b,c) \in A^3$ be a triple zero of I. We claim that $a,b,c \in M$. Deny. We may assume that $a \notin M$. Hence, a is invertible and so $bc = 0 \in I$ since $abc = 0$, a desired contradiction.

Hence, $(a,b,c) \in M^3$, and so $f(a)f(b)J = f(a)f(c)J = f(b)f(c)J = 0$ since $f(M)J = 0$. The result follows from Theorem 2.3.3.

(2) Since $J^3 = 0$, then $J \subset \mathrm{Jac}(B)$. Also, $A \bowtie^f J$ is a local ring with a maximal ideal $M = m \bowtie^f J$ (since A is local with maximal ideal m) and $M^3 = m^3 \bowtie^f f(m)^2J + f(m)J^2 + J^3 = 0 \bowtie^f J^3 = 0$. Hence, every ideal of $A \bowtie^f J$ is a weakly 2-absorbing ideal by [14, Theorem.3.2] and since $J^3 = 0$.

2.4 A 2-absorbing primary ideal on amalgamated algebras along ideal

In this section we study the transfer of the 2-absorbing primary ideal notion to amalgamated algebras along an ideal.

Theorem 2.4.1 *Let A and B be rings, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B. Then:*

1. *An ideal I of A is a 2-absorbing primary ideal of A if and only if $I \bowtie^f J$ is a 2-absorbing primary ideal of $A \bowtie^f J$.*
2. *Let I be an ideal of A such that $Rad(I)$ is a prime ideal of A. Then $I \bowtie^f J$ is a 2-absorbing primary ideal of $A \bowtie^f J$.*
3. *Let P be a prime ideal of A. Then $P^n \bowtie^f J$ is a 2-absorbing primary ideal for every positive integer n.*
4. *Assume that A is a divided ring and let I be an ideal of A. Then $I \bowtie^f J$ is a 2-absorbing primary ideal.*
5. *Assume that A is a local ring with maximal ideal M, $f^{-1}(Q) \neq M$ for each ideal $Q \in Spec(B) \setminus V(J)$, and let I be an ideal of A whose radical is M. Then $I \bowtie^f J$ and $I^e := I(A \bowtie^f J)$ are 2-absorbing primary ideals of $A \bowtie^f J$.*

Before proving Theorem 2.4.1, we establish the following lemma.

Lemma 2.4.2 *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B, and let I be an ideal of A. Then $Rad(I \bowtie^f J) = Rad(I) \bowtie^f J$.*

Proof.

Let $(a, f(a)+j) \in \mathrm{Rad}(I \bowtie^f J)$. Then there exists a nonzero integer n such that $(a, f(a)+j)^n = (a^n, f(a^n)+j') \in I \bowtie^f J$, where $j' = \sum_{k=0}^{n-1} \binom{n}{k} f(a^k) j^{n-k} \in J$. Then $a^n \in I$ and so $a \in \mathrm{Rad}(I)$ and $(a, f(a)+j) \in \mathrm{Rad}(I) \bowtie^f J$, as desired.

Conversely, let $(a, f(a)+j) \in \mathrm{Rad}(I) \bowtie^f J$. Then there exists $n \in \mathbb{N}^*$ such that $a^n \in I$ and so $(a, f(a)+j)^n = (a^n, f(a^n)+j') \in I \bowtie^f J$. Hence, $(a, f(a)+j) \in \mathrm{Rad}(I \bowtie^f J)$ which completes the proof of Lemma 2.4.2.

Proof of Theorem 2.4.1.

1) Assume that I is a 2-absorbing ideal of A, and let $(a_i, f(a_i)+k_i) \in A \bowtie^f J$, for $i = 1,2,3$, such that $\prod_{i=1}^{3}(a_i, f(a_i)+k_i) = (a_1a_2a_3, f(a_1a_2a_3)+l) \in I \bowtie^f J$. Then $a_1a_2a_3 \in I$ and so $a_1a_2 \in I$ or $a_1a_3 \in \mathrm{Rad}(I)$ or $a_2a_3 \in \mathrm{Rad}(I)$. Therefore, $(a_1, f(a_1)+k_1)(a_2, f(a_2)+k_2) \in I \bowtie^f J$ or $(a_1, f(a_1)+k_1)(a_3, f(a_3)+k_3) \in \mathrm{Rad}(I \bowtie^f J)$ or $(a_2, f(a_2)+k_2)(a_3, f(a_3)+k_3) \in \mathrm{Rad}(I \bowtie^f J)$ Using the previous lemma.

Conversely, assume that $I \bowtie^f J$ is a 2-absorbing primary ideal of $A \bowtie^f J$, and let $a,b,c \in A$ such that $abc \in I$. Hence, $(a,f(a))(b,f(b)(c,f(c)) \in I \bowtie^f J$ and so $(a,f(a))(b,f(b))(=(ab,f(ab)) \in I \bowtie^f J$ or $(a,f(a))(c,f(c)) \in \text{Rad}(I \bowtie^f J)$ or $(b,f(b))(c,f(c)) \in \text{Rad}(I \bowtie^f J)$. Then $ab \in I$ or $ac \in \text{Rad}(I)$ or $bc \in \text{Rad}(I)$, as desired.

2) Since $\text{Rad}(I \bowtie^f J) = \text{Rad}(I) \bowtie^f J$ and $\text{Rad}(I)$ is a prime ideal of A, then $\text{Rad}(I \bowtie^f J)$ is a prime ideal of $A \bowtie^f J$. Using [16, Theorem 2.8], we conclude that $I \bowtie^f J$ is a 2-absorbing primary ideal of $A \bowtie^f J$.

3) Let P be a prime ideal of A and n be a positive integer. Then P^n is a 2-absorbing primary ideal of A and so $P^n \bowtie^f J$ is a 2-absorbing primary ideal of $A \bowtie^f J$ by 1).

4) Clear since $\text{Rad}(I \bowtie^f J) = \text{Rad}(I) \bowtie^f J$, $\text{Rad}(I)$ is a prime ideal of A and since A is a divided ring.

5) We know from [38, Corollary 3.2] that the radical of $I(A \bowtie^f J)$ is $M \bowtie^f J$ which is a maximal ideal of $A \bowtie^f J$. Then the result follows from [16, Theorem 2.8] and this completes the proof of Theorem 2.4.1.

Corollary 2.4.3 *Let M be a maximal ideal of an integral domain T, D be a subring of T such $D \cap M = \{0\}$, and let I be an ideal of D. Then $I+M$ is a 2-absorbing primary ideal of $D+M$ if and only if I is a 2-absorbing primary ideal of D.*

Proof. It is clear that $D+M$ is naturally isomorphic to $D \bowtie^f M$, where $f : D \hookrightarrow T$ the natural embedding, and let I be a proper ideal of D. Then $I+M$ is a 2-absorbing primary ideal of $D+M$ (if and only if $I \bowtie^f M$ is a 2-absorbing primary ideal of $D \bowtie^f M$) if and only if I is a 2-absorbing ideal of D.

Now we will determine the 2-absorbing primary ideals in a $D \bowtie^f XK[[X]]$, where D is an integral domain with quotient field K and $f : D \hookrightarrow K[[X]]$ is the natural embedding. Recall that $D \bowtie^f XK[[X]]$ is naturally isomorphic to $D + XK[[X]]$.

Theorem 2.4.4 *Let A be an integral domain with quotient field K, $f : A \hookrightarrow K[[X]]$ be the natural embedding, and set $J := XK[[X]]$. The 2-absorbing primary ideals of $A \bowtie^f J$ are of the form $I \bowtie^f J$, where I is a 2-absorbing primary ideal of A, or $0 \bowtie^f (WX^m + X^{m+1}K[[X]])$, where W is a nonzero A-submodule of K and m a positive integer.*

Before proving Theorem 2.4.4, we establish the following lemma.

Lemma 2.4.5 *Let D be an integral domain with quotient field K. Then, the nonzero 2-absorbing primary ideal of $R := D + XK[[X]]$ has the form $I + XK[[X]]$, where I is a 2-absorbing primary ideal of D, or $WX^m + X^{m+1}K[[X]]$ where m is a positive integer and W a nonzero A-submodule of K.*

Proof. Let L be a nonzero 2-absorbing primary ideal of $R := D + XK[[X]]$. Then L is comparable with $J := XK[[X]]$.

If $J \subset L$, then $L = I + J$ where I is a 2-absorbing primary ideal of A.

If $L \subset J$, then $L = WX^m + J^{m+1}$ for some a nonzero A-submodule W of K and positive integer m. Remark that $J^{m+1} \subset L$ and so $J \subset \text{Rad}(L)$ which means that $\text{Rad}(L) = J$ is a prime ideal of R (since $L \subset J$). The proof is now clear.

Proof of Theorem.

Let L be a 2-absorbing primary ideal of $A \bowtie^f J$. Then $u^{-1}(L)$ is a 2-absorbing primary ideal of $A+J := A+XK[[X]]$. By Lemma 4.5, two cases are then possible:

Case 1: $u^{-1}(L) = I+J$.

In this case, I is a 2-absorbing primary ideal of A and $L = I \bowtie^f J$.

Case 2: $u^{-1}(L) = WX^m + J^{m+1}$ for some a nonzero A-submodule W of K and a positive integer m.

In this case, $L = 0 \bowtie^f (WX^m + J^{m+1})$.

The converse is straightforward which completes the proof of Theorem.

2.5 A weakly 2-absorbing primary ideals of amalgamated algebras along ideals

Now, we study the transfer of the 2-absorbing primary notion to the amalgamated algebras along an ideal. First, recall the following definition:

Definition 2.5.1 *1) A proper ideal I of a ring R is called a weakly 2-absorbing primary ideal if whenever $a,b,c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in Rad(I)$ or $bc \in Rad(I)$.*

2) Let I be a weakly 2-absorbing ideal of R. We say that $(a,b,c) \in A^3$ is a triple-zero of I if $abc = 0, ab \notin I, ac \notin Rad(I)$ and $bc \notin Rad(I)$.

Now, we can study the transfer of the 2-absorbing primary notion to the amalgamated algebras along an ideal.

Theorem 2.5.2 *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then:*

1. *Let I be an ideal of A. If $I \bowtie^f J$ is a weakly 2-absorbing primary ideal of $A \bowtie^f J$, then I is a weakly 2-absorbing primary ideal of A.*
2. *Let I be a weakly 2-absorbing primary ideal of A which is not 2-absorbing primary. The following statements are equivalent:*

   a) $I \bowtie^f J$ is a weakly 2-absorbing primary ideal of $A \bowtie^f J$.

   b) For every a triple zero $(a,b,c) \in A^3$ of I, we have $f(a)f(b)J = f(a)f(c)J = f(b)f(c)J = 0$ and $J^3 = 0$.
3. *Assume that $Nil(A)$ is a prime ideal (resp., primary ideal) and $J \subseteq Nil(B)$. An ideal of $A \bowtie^f J$ is weakly 2-absorbing primary if and only if it is a 2-absorbing primary ideal.*

Proof.

(1) Assume that $I \bowtie^f J$ is weakly 2-absorbing primary and let $(a,b,c) \in A^3$ such that $0 \neq abc \in I$. Then $0 \neq (a,f(a))(b,f(b))(c,f(c)) \in I \bowtie^f J$ and so I is a weakly 2-absorbing primary by lemma 4.2 and since $I \bowtie^f J$ is a weakly 2-absorbing primary.

(2) $a) \Rightarrow b)$. Let I be a weakly 2-absorbing primary ideal of A which is not 2-absorbing primary and assume that $I \bowtie^f J$ is a weakly 2-absorbing primary ideal of $A \bowtie^f J$. Let $(a,b,c) \in A^3$ be a triple zero of I. We claim that $f(b)f(c)J \neq 0$. Deny. Then, $f(b)f(c)J \neq 0$ and so there exists $j \in J$ such that $f(b)f(c)j \neq 0$. Hence, $0 \neq (a, f(a)+j)(b, f(b))(c, f(c)) = (0, f(b)f(c)j) \in I \bowtie^f J$ but neither $(a, f(a)+j)(b, f(b)) \in I \bowtie^f J$, nor $(a, f(a)+j)(c, f(c)) \in \text{Rad}(I \bowtie^f J)$, nor $(b, f(b))(c, f(c)) \in \text{Rad}(I \bowtie^f J)$. This is a desired contradiction.

On the other hand, the ideal I is not 2-absorbing primary and so $I^3 = 0$ by [14, Theorem 2.10]. Hence, $I \bowtie^f J$ is not a 2-absorbing primary ideal of $A \bowtie^f J$ so $(I \bowtie^f J)^3 = 0$, which means that $J^3 = 0$ by Lemma 2.3.4.

$b) \Rightarrow a)$. Conversely, let $0 \neq (a, f(a)+i)(b, f(b)+j)(c, f(c)+k) \in I \bowtie^f J$ for $(a, f(a)+i), (b, f(b)+j)$ and $(c, f(c)+k)$ in $A \bowtie^f J$. Two cases are then possible:

Case 1: $abc \neq 0$.

Since I is a weakly 2-absorbing primary ideal of A, then $ab \in I$, or $ac \in \text{Rad}(I)$, or $bc \in \text{Rad}(I)$. Hence, $(a, f(a)+j)(b, f(b)+k) \in I \bowtie^f J$, or $((a, f(a)+i)(c, f(c)+k) \in \text{Rad}(I) \bowtie^f J$ or $(b, f(b)+j)(c, f(c)+k) \in \text{Rad}(I) \bowtie^f J$, as desired.

Case 2: $abc = 0$.

Without loss of generality, we may assume that $ab \notin I$, $ac \notin \text{Rad}(I)$, and $bc \notin \text{Rad}(I)$. Then, (a,b,c) is a triple zero of I and $f(a)f(b)J = f(a)f(c)J = f(b)f(c)J = 0$. The fact that $J^3 = 0$ gives $(a, f(a)+j)(b, f(b)+k)(c, f(c)+l) = (abc, f(abc)+f(a)f(b)l+f(a)f(c)k+f(b)f(c)j+jkm) = (0, jkm) = (0,0)$ for every $i, j, k \in J$.

(3) First, we show that $\text{Nil}(A \bowtie^f J) = \{(a, f(a)+j)/a \in \text{Nil}(A), j \in \text{Nil}(B) \cap J\}$. Indeed, let $(a, f(a)+j) \in \text{Nil}(A \bowtie^f J)$. Then, there exists a positive integer n such that $a^n = 0$ and $(f(a)+j)^n = 0$ and so $a \in \text{Nil}(A)$. Hence, $f(a)$ is a nilpotent element in B as well as $f(a)+j$. Hence, $j \in \text{Nil}(B)$ and thus $\text{Nil}(A \bowtie^f J) \subseteq \{(a, f(a)+j)/a \in \text{Nil}(A), j \in \text{Nil}(B) \cap J\}$. On the other hand, it is not difficult to check that $\{(a, f(a)+j)/a \in \text{Nil}(A), j \in \text{Nil}(B) \cap J\} \subseteq \text{Nil}(A \bowtie^f J)$. Finally $\{(a, f(a)+j)/a \in \text{Nil}(A), j \in \text{Nil}(B) \cap J\} = \text{Nil}(A \bowtie^f J)$.

Hence $J \subseteq \text{Nil}(B)$ gives $\text{Nil}(A \bowtie^f J) = \text{Nil}(A) \bowtie^f J$ is a prime ideal of $A \bowtie^f J$ and so by [14, Theorem 2.14], an ideal of $A \bowtie^f J$ is a weakly 2-absorbing primary ideal if and only if it is a 2-absorbing primary ideal.

Corollary 2.5.3 *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B such that $J^3 = 0$, let I be a weakly 2-absorbing primary ideal of A, and assume that for every triple zero $(a,b,c) \in A^3$ of I, we have $(f(a), f(b), f(c)) \in ann(J)^3$. Then $I \bowtie^f J$ is a weakly 2-absorbing primary ideal of $A \bowtie^f J$.*

Proof.

Let I be a 2-absorbing primary ideal of A. It is clear that $f(a)f(b)J = f(a)f(c)J = f(b)f(c)J = 0$ for any $(a,b,c) \in A^3$ such that $abc = 0$, and $ab \notin I, bc \notin \text{Rad}(I), ac \notin \text{Rad}(I)$. Now the result follows from Theorem 2.5.2.

In the next theorem, we will see the relation between the notion of a weakly prime ideal (resp., weakly primary ideal) and a weakly 2-absorbing primary ideal.

Definition 2.5.4 *An ideal I of a commutative ring A is called a weakly prime ideal of A, if for every $(a,b) \in I^2$ such that $0 \neq ab \in I$, then $a \in I$ or $b \in I$.*

We end this work by the following result.

Theorem 2.5.5 *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. Let I be an ideal of A such that $Rad(I)$ is a weakly prime ideal of A, and whenever $a,b \in A$ such that $ab = 0$ and $a \notin Rad(I), b \notin Rad(I)$ we have $a,b \in ann(J)$. Then, $I \bowtie^f J$ is a weakly 2-absorbing primary ideal of $A \bowtie^f J$.*

Before proving Theorem 2.5.5, we establish the following lemma.

Lemma 2.5.6 *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B, and let I be an ideal of A. The following statements are equivalent:*

1. *I is a weakly prime ideal of A, $J^2 = 0$, and for each $a,b \in A$ such that $ab = 0, a \notin I, b \notin I$, we have $f(a)J = f(b)J = 0$.*
2. *$I \bowtie^f J$ is a weakly prime ideal of $A \bowtie^f J$.*

Proof.

$(2) \Rightarrow (1)$:

Assume that $I \bowtie^f J$ is a weakly prime ideal of $A \bowtie^f J$ and let $a,b \in A$ such that $0 \neq ab \in I$. Hence, $0 \neq (a, f(a))(b, f(b)) \in I \bowtie^f J$ which is a weakly prime ideal, then $(a, f(a)) \in I \bowtie^f J$ or $(b, f(b)) \in I \bowtie^f J$. We conclude that $a \in I$ or $b \in I$. On the other hand, let $a,b \in A$ such that $ab = 0$ and neither $a \in I$ nor $b \in I$. Let $(j,k) \in J^2$. We have $(a, f(a)+j)(b, f(b)+k) = (0, f(a)k + f(b)j + jk) \in I \bowtie^f J$. Since $I \bowtie^f J$ is a weakly prime ideal and neither $(a, f(a)+j)$ nor $(b, f(b)+k)$ are in $I \bowtie^f J$, we conclude that $f(a)k + f(b)j + jk = 0$, for all $(j,k) \in J^2$. Hence, $f(a)J = f(b)J = J^2 = 0$.

$(1) \Rightarrow (2)$:

Let $(a, f(a)+j), (b, f(b)+k) \in A \bowtie^f J$ such that $0 \neq (a, f(a)+j)(b, f(b)+k) = (ab, f(ab) + f(a)k + f(b)j + jk) \in I \bowtie^f J$,. Hence, $ab \in I$. Two cases are then possible:

Case 1: $ab \neq 0$.

Since I is a weakly prime ideal, we may assume that $a \in I$. Hence, $(a, f(a)+j) \in I \bowtie^f J$, as desired.

Case 2: $ab = 0$. By hypothesis, two cases are then possible:

If $a \in I$ or $b \in I$, then $(a, f(a)+j) \in I \bowtie^f J$ or $(b, f(b)+k) \in I \bowtie^f J$, as desired.

If $f(a)J = f(b)J = 0$, then $(a, f(a)+j)(b, f(b)+k) = 0$, as desired.

Hence, $I \bowtie^f J$ is a weakly prime ideal of $A \bowtie^f J$.

Proof of Theorem 2.5.5.

By Lemma 2.5.6, we have $\mathrm{Rad}(I \bowtie^f J) = \mathrm{Rad}(I) \bowtie^f J$. Hence, $\mathrm{Rad}(I \bowtie^f J)$ is a weakly prime ideal of $A \bowtie^f J$ by lemma 2.5.6 and since $\mathrm{Rad}(I)$ is a weakly prime ideal of A. Now the result follows the [14, Theorem 2.7] and this completes the proof of Theorem 2.5.5.

CHAPTER 3
RING IN WHICH EVERY 2-ABSORBING IDEAL IS A PRIME IDEAL

Abstract*

In this article, we study those ring in which every 2-absorbing ideal is prime. We investigate the stability of this property under homorphic image and localization, and its transfer to various contexts of constructions such as direct products, trivial ring extensions and amalgamated algebra along an ideal.

Mathematics Subject Classification (2010): 13Axx, 13Bxx, 13Cxx.
Keywords: 2-absorbing ideal, 2-absorbing-like properties, amalgamated algebras along an ideal.

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3.1 Introduction

Throughout this work, all rings are commutative with identity element, and all modules are unitary.

*Volume of recent results in pure mathematics, Springer, accepted for publication (in collaboration with N. Mahdou)

Over the past several years, there has been considerable attention in the literature to n-absorbing ideals of commutative rings and their generalizations, for example see [5, 9, 16, 17, 18, 15, 27, 26, 40, 41, 22, 55, 63, 66, 67, 68, 72]. We recall from [5] that a proper ideal I of R is called a *2-absorbing ideal* of R if $a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Clearly, every prime ideal is 2-absorbing ideal. That a 2-absorbing ideal need not be a prime ideal as shown by an example in [9].

In [20], Bennis and Fahid introduced and studied a ring R satisfying the following condition: every 2-absorbing ideal of R is prime. We call a such ring a 2-AB ring. They showed that a ring R is a 2-AB ring if and only if (1) every prime ideals of R are comparable; in particular, R is quasi-local with maximal ideal M and (2) if P is minimal prime over a 2-absorbing ideal I, then $IM = P$. They also characterize under valuation domain, that a ring R is a 2-AB ring if and only if $P = P^2$ for every prime ideal P of R.

Recall that a non-zero prime ideal P of a ring R is called *divided prime* if $P \subset (x)$ for every $x \in R \setminus P$. An integral domain R is said to be *divided domain* if every prime ideal of R is a divided prime ideal. An integral domain R is said to be a *valuation domain* if $x \mid y$ (in R) or $y \mid x$ (in R) for every nonzero $x, y \in R$. It is known that a valuation domain is a divided domain. If I is an ideal of R, I is said semi-primary if $\mathrm{Rad}(I)$ is a prime ideal of R. The set of minimal 2-absorbing ideals over an ideal I will be denoted by $2-\mathrm{Min}_R(I)$.

In [20], the author ask the following question [20, Question 2]: Is it true that $P^2 = P$ for any prime ideal P of a $2-AB$ ring. In section three, we study this conjecture which we call a $(\star)$ conjecture.

Let A be a ring and E an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R = A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a,e)(a',e') = (aa', ae' + a'e)$. Recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J = I \propto E'$ is an ideal of R. However, prime (resp., maximal) ideals of R have the form $P \propto E$, where P is a prime (resp., maximal) ideal of A [4, Theorem 3.2]. Suitable background on commutative trivial ring extensions is [4, 15, 19, 54, 55, 64, 50].

Let A and B be two rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$

$$A \bowtie^f B = \{(a, f(a) + j) | a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f.

Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation (see [36, Examples 2.5 and 2.6]) and other classical constructions, such as and The Nagata idealization (see [75, p:2]) and the CPI extensions (in the sense of Boisen and Sheldon [24]) are strictly related to it (see [36, Example 2.7 and Remark 2.8]). A particular case of this construction is the amalgamated duplication of a ring along an ideal I (introduced and studied by D'Anna and Fontana in [33, 34]). Let A be a ring, and let I be an ideal of A. $A \bowtie I := \{(a, a+i) : a \in A, i \in I\}$ is called the amalgamated duplication of A along the ideal I. See for instance [23, 36, 37, 38, 33, 34].

In this article, we study those ring in which every 2-absorbing ideal is prime. We investigate the stability of this property under trivial ring extensions and amalgamated algebra along an ideal in section 2 and section 3 respectively. In section 3, we discuss the validity of the equation $P^2 = P$ for any prime ideal P in a $2-AB$ ring.

3.2 Trivial ring extension defined by 2-AB ring property

Recall that a ring R is said to be a 2-AB ring if every 2-absorbing ideal of R is prime. Our first result studies the possible transfer of the $2AB$-ring property between a ring A and a trivial ring extension $A \propto E$.

Theorem 3.2.1 *Let A be a ring, E be an A-module and $R = A \propto E$ be a trivial ring extension of A by E. Then:*

1. *If R is a 2-AB ring, then so is A.*
2. *Assume that A is a 2-AB ring and let J be a 2-absorbing ideal of R. Then J is either a prime ideal or a semi-primary ideal of R.*
3. *Assume that A is an integral domain with quotient field K and E is a K-vector space. Then $R := A \propto E$ is a 2-AB ring if and only if so is A and $E = 0$.*

The proof of this theorem involves the following lemma with independent interest.

Lemma 3.2.2 1. *Any homomorphic image of a $2-AB$ ring is a $2-AB$ ring.*

2. *Any localization of a $2-AB$ ring is a $2-AB$ ring.*

Proof. Let A be a commutative ring and I be an ideal of A.

1. Let K be a 2-absorbing ideal of A/I, then there exists a 2-absorbing ideal J of R such that $J \supset I$ and $K = J/I$. Hence, J is a prime ideal of R (since R is 2-AB ring) and so K is a prime ideal of A/I.
2. Let S be a multiplicative subset of R and let $S^{-1}I$ be a 2-absorbing ideal of $S^{-1}R$, such that $I \cap S = \emptyset$. Then I is a prime ideal of A and so $S^{-1}I$ is a prime ideal of $S^{-1}A$.

Proof of Theorem 3.2.1.

1. Assume that $A \propto E$ is a 2-AB ring. Hence, A is a 2-AB ring by the lemma 3.2.2(1) (since $A \propto E/0 \propto E \simeq A$).

2. Assume that A is a 2-AB ring and let J be a 2-absorbing ideal of R. Two cases are then possible:

   Case 1: $0 \propto E \subseteq J$.
   In this case, $J = I \propto E$ for some ideal I of A from the [4, Theorem 3.1]. By [55, Theorem 2.1], I is a 2-absorbing ideal of A and hence I is a prime ideal of A and consequentially J is a prime ideal of R.

   Case 2: $0 \propto E \nsubseteq J$.
   By [9, Theorem 2.1], $\mathrm{Rad}(J)$ is a 2-absorbing ideal of R which containing $0 \propto E$, by the [4, Theorem] there exists some ideal K of A such that $\mathrm{Rad}(J) = K \propto E$. The ideal K is a 2-absorbing ideal of A, therefore K is a prime and hence $\mathrm{Rad}(J)$ is prime ideal of R.

3. Recall that if A is an integral domain with quotient field K and E be a divisible A-module, the ideals of $A \propto E$ are of the form $I \propto E$ where I is an ideal of A, or $0 \propto F$ where F is a A-submodule of E.

   If A is a 2-AB ring and $E = O$, then $R \cong A$ is a $2AB$ ring. Conversely, assume that R is a $2-AB$ ring. It is clear that A is a 2-AB ring by 1). We know from [55, Theorem 2.2] that if F is a A-submodule of E, then $0 \propto F$ is 2-absorbing ideal of R if and only if F is a K-vector subspace of E. Thus for $F = 0$, we get $0 \propto 0$ is a 2-absorbing ideal, thus a prime ideal of R. We conclude that R is an integral domain, thus $E = 0$.

4. Assume that E is finitely generated. If A is a $2AB$ ring and $E = O$, then $R \cong A$ is a 2-AB ring. Conversely, assume that R is a $2-AB$ ring. Then A is a $2-AB$ ring by 1) and A is a quasi-local ring with a maximal ideal M such that $M^2 = M$ by [20, Corollary 2.4]. On the other hand R is a quasi-local ring with maximal ideal $M \propto E$ such that $(M \propto E)^2 = M \propto E$, so $ME = E$, and by Nakayama's lemma, we conclude that $E = 0$. This completes the proof of Theorem.

Corollary 3.2.3 *Let A be a Noetherian module, E be a finitely generated A-module and $R = A \propto E$. Then, R is a 2-AB ring if and only if so is A and $E = 0$.*

Proof.
If A is a 2-AB ring and $E = O$, then $R \cong A$ is a 2-AB ring. Conversely, assume that R is a 2-AB ring. Then, A is a field from [20, Corollary 2.5] and so E is a A-vector space. The result follows from Theorem 3.2.1.

In general, the converse of Theorem 3.2.1(1) is false even if $E = A$, as the following examples shows:

Example 3.2.4 *Let A be an integral domain, $R = A \propto A$ be the trivial ring extension of A by A, and set $I = 0 \propto P$, where P is a prime ideal of A. Then, I is a non-prime 2-absorbing ideal of $A \propto A$ by [5, Theorem 4.10].*

Example 3.2.5 *Let P be a prime ideal of a ring A. Then, $P \propto P$ is a non-prime 2-absorbing ideal of $A \propto A$ by [55, corollary 2.8].*

3.3 Amalgamated algebras along an ideal defined by 2-AB ring

In this section we study the transfer of a $2-AB$ ring property in amalgamated algebras along an ideal.

Theorem 3.3.1 *Let $f : A \to B$ be a rings homomorphism, J be an ideal of B, and $A \bowtie^f J$ be the amalgamated algebra. Then the following statements hold:*

1. *If $A \bowtie^f J$ is a $2-AB$ ring, then A and $f(A)+J$ are a $2-AB$ rings.*
2. *Assume that J is a finitely generated A-module. Then $A \bowtie^f J$ is a $2-AB$ ring if and only if so is A and $J = 0$.*
3. *Assume that $f^{-1}(J) = \{0\}$. Then $A \bowtie^f J$ is a $2-AB$ ring if and only if so is $f(A)+J$.*
4. *Assume that J is a nonzero idempotent ideal of B such that $J \subset Nil(B)$. Then $A \bowtie^f J$ is $2-AB$ ring if and only if so is A.*

The proof of this theorem involves the following lemmas.

Lemma 3.3.2 *Let $f : A \to B$ be a rings homomorphism and let J be an ideal of B. If $A \bowtie^f J$ is a $2-AB$ ring, then A is a local ring with maximal ideal M, $J \subset Jac(B)$ and $(f(M)+J)J = 0$.*

Proof.
Since $A \bowtie^f J$ is a $2-AB$ ring, then $A \bowtie^f J$ is a local ring by [20, Theorem 2.3] and so A is a local ring with maximal ideal M and $J \subset \mathrm{Jac}(B)$. On the other hand, the maximal ideal of $A \bowtie^f J$ is $M \bowtie^f J$. Hence, by [20, corollary 2.4], $(M \bowtie^f J)^2 = M^2 \bowtie^f (f(M)J + J^2) = M \bowtie^f J$ and so $f(M)J + J^2 = J$.

Recall that the set of minimal 2-absorbing ideals over an ideal I will be denoted by $2-\mathrm{Min}_R(I)$.

Lemma 3.3.3 *([20, Theorem 2.9])*
A ring R is a 2-AB ring if and only if the two following conditions holds:

1. *A prime ideals of R are comparable. In particular, R is quasi-local.*
2. *For every prime ideal P of $R, 2-Min_R(P^2) = P$.*

Proof of Theorem 3.3.1

1. Clear by Lemma 3.2.2(1) since $\frac{A \bowtie^f J}{\{0\} \times J} \simeq A$ and $A \bowtie^f J (\simeq A)$ is a $2-AB$ ring. Conversely, assume that

2. Assume that J is a finitely generated A-module. If A is a $2-AB$ ring and $J = 0$, then $\frac{A \bowtie^f J}{\{0\} \times J} \simeq A$. Conversely, assume that $A \bowtie^f J$ is a $2-AB$ ring. Hence, A is a $2-AB$ ring by 1). On the other hand, A is a local ring with maximal ideal M such that $f(M)J + J^2 = J$ by Lemma 3.2.3. It is well known that $f(M) + J$ is a maximal ideal of the quasi-local ring $f(A) + J$. On the other hand, since J is a finitely generated A-module, it is a finitely generated $(f(A) + J)$-module. By Nakayam's lemma, we conclude that $J = 0$.

3. Clear since $A \bowtie^f J \simeq f(A) + J$ (since $f^{-1}(J) = 0$).

4. If $A \bowtie^f J$ is a $2-AB$-ring, then so is A by 1). Conversely, assume that A is a $2-AB$ ring and J a nonzero idempotent ideal of B such that $J \subseteq Nil(B)$. Hence, A is a local ring with maximal ideal M and so $A \bowtie^f J$ is a local ring with maximal ideal $M \bowtie^f J$ (since $J \subset \mathrm{Jac}(B)$). On the other hand, since $J \subset \mathrm{Nil}(B)$, the prime ideals of $A \bowtie^f J$ are of the form $P \bowtie^f J$, where P is a prime ideal of A. Since the prime ideals are comparable; it is clear that the prime ideals of $A \bowtie^f J$ are comparable.

   Now we will shows that $2 - \min_{A \bowtie^f J} \left((P \bowtie^f J)^2\right) = P \bowtie^f J$. Indeed let K be a 2-absorbing ideal of $A \bowtie^f J$ and suppose that there exists a 2-absorbing ideal L of $A \bowtie^f J$ such that $(P \bowtie^f J)^2 \subset L \subset K$. But $(P \bowtie^f J)^2 = P^2 \bowtie^f (f(P)J + J^2) = P^2 \bowtie^f J \subset L \subset K$. Hence, $\{0\} \times J \subset K$ and $\{0\} \times J \subset L$, thus there exists a prime ideals K_1 and L_1 of A such that $L = L_1 \bowtie^f J$ and $K = K_1 \bowtie^f J$. Then $P^2 \subset L_1 \subset K_1$ and so $K_1 = L_1 = P$ (since A is a $2-AB$ ring), thus $K = L = P \bowtie^f J$. Then $A \bowtie^f J$ is a $2-AB$ ring by Lemma 3.3 and this completes the proof of Theorem 3.3.1.

Corollary 3.3.4 *Let $f : A \to B$ be a rings homomorphism, B be a local ring with maximal ideal M, and let J be a nonzero idempotent ideal of B. Then $A \bowtie^f J$ is a 2-AB ring if and only if A is a 2-AB ring.*

Proof. Clearly by theorem 3.3.1 since $J \subset M = \mathrm{Jac}(B) = \mathrm{Nil}(B)$.

Example 3.3.5 *Let A be an integral domain with quotient field K, and E be a K-vector space. Set $B := A \propto E$, and $J := 0 \propto E$. We consider $f : A \to B$ the canonical homomorphism defined by $f(a) = (a,0)$ for every $a \in A$.*

Then $A \bowtie^f J$ is a 2-AB ring if and only if A so is and $J = 0$.

Proof. Indeed, assume that $A \bowtie^f J$ be a 2-AB ring, then A so is by the theorem 3.3.1. On the other hand we have $f(M)J = (M \propto 0)(0 \propto E) = 0$ and $J^2 = 0$, thus $J = 0$. The converse is clear.

Let $f : A \to B$ be a rings homomorphism, J be an ideal of B. If A is 2-AB ring and $J \subset \mathrm{Jac}(B)$, the ring $A \bowtie^f J$ need not be a $2-AB$ ring as the following example shows.

Example 3.3.6 *Let A be a local ring with maximal ideal M and E be a nonzero A-module such that $ME = 0$. Set $B := A \propto E$ and $f : A \to B$ be the canonical homomorphism defined by $f(a) = (a, 0)$ for every $a \in A$ and set $J = 0 \propto E$. Then $A \bowtie^f J$ is not a $2-AB$ ring.*

Proof. It is clear that J is an ideal of B and $J \subset \mathrm{Jac}(B)$ since $J^2 = 0$. On the other hand, $f(M)J = 0$ but $J \neq 0$. Therefore, $A \bowtie^f J$ is not a $2-AB$ ring.

Now we will determine the 2-absorbing ideals in a particular case $A \bowtie^f XK[[X]]$, where A is an integral domain with quotient field K.

Theorem 3.3.7 ***(The 2-absorbing ideal in** $A \bowtie^f J$ **in case** $J := XK[[X]]$.)*

Let A be an integral domain with quotient field K, $f : A \hookrightarrow K[[X]]$ be the natural embedding and $J := XK[[X]]$. Assume that A is a 2-AB ring, and let L be a 2-absorbing ideal of $A \bowtie^f J$. Then one of the following statements hold:

1. *L is a prime ideal of $A \bowtie^f J$.*
2. *$L = 0 \bowtie^f X^2K[[X]]$ which is a non-prime 2-absorbing ideal of $A \bowtie^f J$.*

Proof.

First we show that $0 \bowtie^f J^2$ is a 2-absorbing ideal of R. Let $x_i = (a_i, a_i + XS_i(X)) \in R$, where $S_i \in K[[X]], 1 \leq i \leq 3$. Suppose $x_1x_2x_3 \in 0 \bowtie^f J^2$. Since A is an integral domain, we may assume that $a_3 = 0$. Two cases are then possible:

Case 1: Assume that $a_1a_2 = 0$. Then $x_1x_3 = (0, X^2S_1(X)S_3(X)) \in 0 \bowtie^f J^2$ or $x_2x_3 \in 0 \bowtie^f J^2$.

Case 2: $a_1a_2 \neq 0$. Then $x_1x_2x_3 = (0, XS_3(X)(a_1, a_1 + XS_1(X))(a_2, a_2 + XS_2(X)) \in 0 \bowtie^f J^2$, and so $XS_3(X) \in J^2$ (since J^2 is an ideal of $K[[X]]$,and $a_1a_1 \neq 0$). Hence, $x_1x_3 = (0, XS_3(X)(a_1, a_1 + XS_1(X)) \in 0 \bowtie^f J^2$.

Therefore, $0 \bowtie^f J^2$ is a 2-absorbing ideal of R. On the other hand, $0 \bowtie^f J$ and $I \bowtie^f J$ are 2-absorbing ideals of $A \bowtie^f J$ by Theorem 3.2.1.

Conversely, let L be a 2-absorbing ideal of $A \bowtie^f J$. Then $u(L)$ is an 2-absorbing ideal of $A + XK[[X]]$. Using Theorem[5, Theorem.4.17], $u(L) = I + XK[[X]]$ for some 2-absorbing ideal I of A, or $u(L) = XK[[X]]$, or $u(L) = X^2K[[X]]$, where $u : A \bowtie^f J \to A + XK[[X]]$ is a canonical isomorphism. Hence, $L = u^{-1}(I + J) = I \bowtie^f J$ or $L = u^{-1}(J) = 0 \bowtie^f J$ or $L = u^{-1}(J^2) = 0 \bowtie^f J^2$. Assume that $L = I \bowtie^f J$, where I is a 2-absorbing ideal of A. Then I is a prime ideal of A (since A is a $2-AB$ ring) and so L is a prime ideal of $A \bowtie^f J$. Now, if $L = 0 \bowtie^f X^2K[[X]]$, by [5, Theorem 4.17] and the fact that $L \simeq X^2K[[X]]$, we conclude that L is a non prime 2-absorbing ideal of $A \bowtie^f J$.

3.4 About $(\star)$ conjecture

In [20] Bennis and Fahid conjecture that in every 2-AB ring A, we have $P^2 = P$ for every prime ideal P of A which we call a $(\star)$ conjecture. It is proved in [20, Proposition 2.6] that a valuation domain satisfy $(\star)$ conjecture. In the next proposition, we give a positive answer to $(\star)$ conjecture in particular case.

Proposition 3.4.1 *Let R be a ring. Then :*

1. *Assume that R is a divided domain. Then R is a $2-AB$ ring if and only if $P^2 = P$ for every prime ideal of R.*
2. *Assume that every semi-primary ideal of R is a primary ideal of R and R is a 2-AB ring. Then for every prime ideal P of R, we have $P^2 = P$.*

Proof.

1. Assume that R is a 2-AB divided domain and let P be a nonzero prime ideal. Hence, P^2 is a 2-absorbing ideal of R (by [9, Corollary 3.8]) and so it is a prime ideal of R. Then $P^2 = P$.

   Conversely, let I be a nonzero 2-absorbing proper ideal of R. Since R is divided domain, we have $\mathrm{Rad}(I) = P$ is a nonzero divided prime ideal, and by [9, Theorem 3.6], I is P-primary and $P^2 \subset I \subset P$. Thus $I = P$ is a prime ideal of R, as desired.

2. Assume that R is a 2-AB ring and let P be a prime ideal of R. By the assumption P^2 is P-primary ideal of R, P^2 is a 2-absorbing ideal of R by [5, Theorem 3.1]. We conclude that P^2 is a prime ideal of R, and $P^2 = P$, as desired.

In [51], Gilmer studies the ring in which every semi-primary ideal is a primary ideal.

Remark 3.4.2 *Let R be a ring in which every semi-primary is primary. By [51, Corollary 2.2], for every no maximal prime ideal P of R, we have $P^2 = P$. Now Let M be a maximal ideal of R, it is well known that M^2 is a 2-absorbing ideal of R by [9, Theorem 3.1]. If R is a 2-AB ring, then R is quasi-local ring with a maximal ideal M by [20, Theorem 2.9]. We conclude that M^2 is a prime ideal of R, and $M^2 = M$. Thus for every prime ideal I of R we have $I^2 = I$.*

CHAPTER 4
ON n-ABSORBING IDEALS AND (m,n)-CLOSED IDEALS IN TRIVIAL RING EXTENSIONS OF COMMUTATIVE RINGS

Abstract*

Let R be a commutative ring with $1 \neq 0$. Recall that a proper ideal I of R is called a *2-absorbing ideal* of R if $a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. A more general concept than 2-absorbing ideals is the concept of n-absorbing ideals. Let $n \geq 1$ be a positive integer. A proper ideal I of R is called an *n-absorbing ideal* of R if $a_1, a_2, ..., a_{n+1} \in R$ and $a_1, a_2 \cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I. The concept of n-absorbing ideals is a generalization of the concept of prime ideals (note that a prime ideal of R is a 1-absorbing ideal of R). Let m and n be integers with $1 \leq n < m$. A proper ideal I of R is called an (m,n)*-closed ideal* of R if whenever $a^m \in I$ for some $a \in R$ implies $a^n \in I$. Let A be a commutative ring with $1 \neq 0$ and M be an A-module. In this paper, we study n-absorbing ideals and (m,n)-closed ideals in the trivial ring extension of A by M (or idealization of M over A) that is denoted by $A \propto M$.

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*Journal of Algebra and its Applications, Vol.18, N. 7 (2019), 1950123, 19 pp. (in collaboration with A. Badawi and N. Mahdou

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4.1 Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. Over the past several years, there has been considerable attention in the literature to n-absorbing ideals of commutative rings and their generalizations, for example see ([5], [25], [53], [63], [76], and [77]). We recall from [9] that a proper ideal I of R is called a *2-absorbing ideal* of R if $a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. A more general concept than 2-absorbing ideals is the concept of n-absorbing ideals. Let $n \geq 1$ be a positive integer. A proper ideal I of R is called an *n-absorbing ideal* of R as in [5] if $a_1,a_2,...,a_{n+1} \in R$ and $a_1,a_2 \cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I. A proper ideal of R is called a *strongly n-absorbing ideal* of R as in [5] if whenever $I_1...I_{n+1} \subseteq I$ for ideals $I_1,\cdots,I_{n+1}$ of R, then the product of some n of the I_j's is contained in I. The concept of n-absorbing ideals is a generalization of the concept of prime ideals (note that a prime ideal of R is a 1-absorbing ideal of R). Let m and n be positive integers with $1 \leq n < m$. We recall from [6] that a proper ideal I of R is called an (m,n)-*closed ideal* of R if whenever $a^m \in I$ for some $a \in I$ implies $a^n \in I$.

Let A be a commutative ring and M be an A-module. *The trivial ring extension of A by M* (or the idealization of M over A) is the ring $R = A \propto M$ whose underlying group is $A \times M$ with multiplication given by $(a,b)(c,d) = (ac, ad+bc)$ (for example see [54]). In this paper, we study n-absorbing ideals, strongly n-absorbing ideals, and (m,n)-closed ideals in the ring $R = A \propto M$. We start by recalling some background materials. We say A is a *quasilocal* ring if A has exactly one maximal ideal. If I is a primary ideal of a ring A with $\mathrm{Rad}(I) = P$ (a prime ideal of A), then we say that I is a *P-primary* ideal of A. A prime ideal P of a ring A is called *divided* if $P \subset x$ for every $x \in A \setminus P$. Suppose that I is a n-absorbing ideal of a ring A for some integer $n \geq 1$. Then, as in [5], we put $w_A(I) = \min\{n \in \mathbb{N} \mid I \text{ is } n\text{-absorbing ideal of A}\}$, and $w_A^*(I) = \min\{n \in \mathbb{N} \mid I \text{ is a strongly } n\text{-absorbing ideal of A}\}$. Let A be a commutative ring and M be an A-module. Then a submodule N of M is called a *P-primary submodule* of M for some prime ideal P of A if $(N:M) = \{x \in A \mid xM \subseteq N\}$ is a primary ideal of A with $\sqrt{(N:M)} = \{a \in A \mid a^nM \subseteq N \text{ for some integer } n \geq 1\} = P$.

Let $n \geq 1$ be an integer and I be a proper ideal of A. Anderson and Badawi in [5] (also see [25]) proposed the following three conjectures.

1. Conjecture one: I is an n-absorbing ideal of A if and only if I is a strongly n-absorbing ideal of A.

2. Conjecture two: If I is an n-absorbing ideal of A, then $(\mathrm{Rad}(I))^n \subseteq I$. An affirmative answer to this conjecture is given in [78].

3. Conjecture three: If I is an n-absorbing ideal of A, then $I[X]$ is an n-absorbing ideal of $A[X]$.

In this paper, we study the validity of the above three conjectures in the ring $R = A \propto M$.

4.2 n-absorbing ideals in trivial ring extensions

We recall ([4, Corollary 3.4]) that if A is an integral domain and M is a divisible A-module, then every ideal of $A \propto M$ has the form $I \propto M$ for some proper ideal I of A or $0 \propto N$ for some submodule N of M.

In the following result, we collect some trivial facts about n-absorbing ideals and (m,n)-closed ideals in $R = A \propto M$ and hence we omit the proof.

Theorem 4.2.1 *Let A be a commutative ring, I be a proper ideal of A, M be an A-module, and $R = A \propto M$. Then*

1. *I is an n-absorbing ideal of A if and only if $I \propto M$ is an n-absorbing ideal of R.*
2. *I is a strongly n-absorbing ideal of A if and only if If $I \propto M$ is a strongly n-absorbing of R.*
3. *I is an (m,n)-closed ideal of A if and only if $I \propto M$ is an (m,n)-closed ideal of R.*

Example 4.2.2 *Let A be a field and M be an A-vector space. It is clear that $R = A \propto M$ is a quasilocal ring with the maximal is $m = \{0\} \propto M$. Since $m^2 = \{0\}$, we conclude that every ideal of R is a 2-absorbing ideal of R and hence a strongly 2-absorbing ideal of R by [9, Theorem 2.13]. Thus every ideal of R is a strongly n-absorbing ideal of R for every $n \geq 2$.*

We recall the following results.

Theorem 4.2.3
1. *(**[78]**) If I is an n-absorbing ideal of a ring A for some integer $n \geq 1$, then $(Rad(I))^n \subseteq I$.*
2. *(**[5, Theorem 3.1]**) Let P be a prime ideal of a ring A, and let I be a P-primary ideal of A such that $P^n \subseteq I$ for some positive integer n (for example, if A is a Noetherian ring). Then I is an n-absorbing ideal of A.*
3. *(**[5, Theorem 6.6]**) Let P be a prime ideal of a ring A, I be a P-primary ideal of A, and $n \geq 1$ be an integer. Then I is a strongly n-absorbing ideal of A if and only if $P^n \subseteq I$ and I is an n-absorbing ideal of R.*
4. *(**[5, Theorem 3.2]**) Let P be a divided prime ideal of A, and let I be an n-absorbing ideal of A with $Rad(I) = P$. Then I is a P-primary ideal of A.*
5. *(**[5, Theorem 3.3]**) Assume that $Rad(0) \subset P$ are divided prime ideals of A and $n \geq 1$ be an integer. Then P^n is a P-primary ideal of A, and thus P^n is an n-absorbing ideal of A.*

In view of Theorem 4.2.3, we have the following result.

Corollary 4.2.4 *1. Let P be a prime ideal of a ring A, $n \geq 1$ be an integer, and let I be a P-primary ideal of A. Then I is an n-absorbing ideal of A if and only if $P^n \subseteq I$ if and only if I is a strongly n-absorbing ideal of A.*

2. *Let P be a divided prime ideal of A, and let I be a proper ideal of A with $Rad(I) = P$. Then I is an n-absorbing ideal of A if and only if I is a P-primary ideal of A and $P^n \subseteq I$ if and only if I is a strongly n-absorbing ideal of A.*

3. *Assume that $Rad(0) \subset P$ are divided prime ideals of A and $n \geq 1$ be an integer. Then P^n is a strongly n-absorbing ideal of A.*

Proof.

1. By Theorem 4.2.3[(1), (2), (3)], the claim follows.
2. By Theorem 4.2.3[(4), (1), (2), (3)], the claim follows.
3. By Theorem 4.2.3[(5), (2), (3)], the claim follows.

Theorem 4.2.5 *Let A be a commutative ring, M be an A-module, $R = A \propto M$, $n \geq 1$ be an integer, I be a proper ideal of A, and N be a submodule of M such that $IM \subseteq N$. Then:*

1. *If $I \propto$ is an n-absorbing ideal of R, then I is an n-absorbing ideal of A.*
2. *Let P be a prime ideal of A, I be a P-primary ideal of A, and N be a P-primary submodule of M. Then I is an n-absorbing ideal of A if and only if $I \propto N$ is an n-absorbing ideal of R.*
3. *Let P be a prime ideal of A, I be a P-primary ideal of A, and N be a P-primary submodule of M. Then $I \propto N$ is an n-absorbing ideal of R if and only if $I \propto N$ is a strongly n-absorbing ideal of R.*
4. *Let P be a divided prime ideal of A, I be an n-absorbing ideal of A with $Rad(I) = P$, and N be a P-primary submodule of M. Then $I \propto N$ is a strongly n-absorbing ideal of R.*
5. *Assume that $Rad(0) \subset P$ are divided prime ideals of A such that $P^n M \subseteq N$. If N is a P-primary submodule of M, then $P^n \propto N$ is a strongly n-absorbing ideal of R.*
6. *Assume that A is a Prüfer domain and let $J = I \propto M$. Then $J = I \propto M$ is an n-absorbing ideal of R if and only if J is a strongly n-absorbing ideal of R. Moreover $w(J) = w^*(J)$.*

Proof.

1. No comments.
2. Since I is a P-primary ideal of A and N is a P-primary submodule of M, we conclude that $I \propto N$ is a $P \propto M$-primary ideal of R by [4, Theorem 3.6]. Suppose that I is an n-absorbing ideal of A. Then $(\mathrm{Rad}(I))^n = P^n \subseteq I$ by Theorem 4.2.3(1). Hence $(\mathrm{Rad}(I \propto N))^n = (P \propto M)^n \subseteq P^n \propto N \subseteq I \propto N$. Thus $I \propto N$ is an n-absorbing ideal of R by Corollary 4.2.4(1). Conversely, suppose that $I \propto N$ is an n-absorbing ideal of R. Then $(\mathrm{Rad}(I \propto N))^n = (P \propto M)^n \subseteq I \propto N$ by Theorem 4.2.3(1). In particular, $P^n \subseteq I$. Since I is a P-primary ideal of A and $P^n \subseteq P$, we conclude that I is an n-absorbing ideal of A by Corollary 4.2.4(1).
3. Since $I \propto N$ is a $P \propto M$-primary ideal of R by [4, Theorem 3.6] and $(\mathrm{Rad}(I \propto N))^n = (P \propto M)^n \subseteq I \propto N$ by Theorem 4.2.3(1), the claim follows by Theorem 4.2.3(3).

4. By Corollary 4.2.4(2), we conclude that I is a P-primary ideal of A. Hence we are done by (2) and (3).
5. By Theorem 4.2.3, we conclude that P^n is a P-primary ideal of A and hence an n-absorbing ideal of A. Thus we are done by (2) and (3).
6. Suppose that $J = I \propto M$ is an n-absorbing ideal of R. Then I is an $n-$ absorbing ideal of A. Since A is a Prüfer domain, we conclude that I is a strongly n-absorbing ideal of A by [5, Corollary 6.9]. Hence $J = I \propto M$ is s strongly n-absorbing ideal of R. The converse is clear. It is clear that $w(J) = w^*(J)$.

4.3 Conjecture one in trivial ring extension

Let $n \geq 1$ be an integer and I be a proper ideal of a ring A. Anderson and Badawi in [5] (also see [25]) proposed the following conjecture:

Conjecture one: I is an n-absorbing ideal of A if and only if I is a strongly n-absorbing ideal of A.

Laradji in [68] proved that conjecture one holds in some rings that satisfy certain conditions. In particular, he proved that Conjecture three implies Conjecture one. We have the following lemma.

Lemma 4.3.1 *Let A be an integral domain with quotient field K, M be a K-vector space, F be a K-subspace of M, and $R = A \propto M$. Then $J = \{0\} \propto F$ is a strongly 2-absorbing ideal of R, and thus J is a strongly n-absorbing ideal of R for every $n \geq 2$.*

Proof. First we show that J is a 2-absorbing ideal of R. Let $x_i = (a_i, e_i) \in R$, where $1 \leq i \leq 3$. Suppose that $x_1x_2x_3 \in \{0\} \propto F$. Since A is an integral domain, we may assume that $a_3 = 0$. Suppose that $a_1a_2 = 0$. Then $x_1x_3 \in J$ or $x_2x_3 \in J$. Suppose that $a_1a_2 \neq 0$. Then $x_1x_2x_3 = (0, a_1a_2e_3)$. Since F is a K-subspace of M, we conclude that $a_2^{-1}a_1^{-1}(a_1a_2e_3) = e_3 \in F$. Hence $x_3 = (0, e_3) \in J$, and thus $x_1x_3 \in J$. Hence J is a 2-absorbing ideal of R. Thus J is a strongly 2-absorbing ideal of R by [9, Theorem 2.13], and hence J is a strongly n-absorbing ideal of R for every $n \geq 2$.

Theorem 4.3.2 *Let A be an integral domain with quotient field K, M be a K-vector space, F be an A-submodule of M, and $R = A \propto M$. Then $\{0\} \propto F$ is an n-absorbing ideal of R for some $n \geq 2$ if and only if F is a K-subspace of M.*

Proof. Suppose that $J = \{0\} \propto F$ is an n-absorbing ideal of R for some $n \geq 2$. Let a be a nonzero element of A and $f \in F$. We show $\frac{1}{a}f \in F$. Let $x = (a, 0), y = (0, \frac{f}{a^n}) \in R$. Then $x^n y = (0, f) \in J$. Since $a \neq 0$ and J is an n-absorbing ideal of R, we conclude that $x^{n-1}y = (0, \frac{f}{a}) \in J$. Thus $\frac{1}{a}f \in F$. Now let $h \in K$ and $v \in F$. Then $h = \frac{b}{c} \in K$ for some $b, c \in A$ with $c \neq 0$. Since $\frac{1}{c}v \in F$ and F is an A-submodule of M, we conclude that $hv = \frac{b}{c}v \in F$. Thus F is a K-subspace of M. The converse is clear by Lemma 4.3.1.

Corollary 4.3.3 *Let A be an integral domain that is not a field with quotient field K, and $R = A \propto K$. Then $J = \{0\} \propto A$ is not an n-absorbing ideal of R for every $n \geq 1$.*

Proof. Since A is not a field, we conclude that A is not a K-subspace of K. Hence we are done by Theorem 4.3.2.

Theorem 4.3.4 *Let A be an integral domain with quotient field K, M be a K-vector space, and $R = A \propto M$. Then Conjecture one holds in R if and only if Conjecture one holds in A.*

Proof. First, observe that M is a divisible A-module. Hence every ideal of $R = A \propto M$ has the form $I \propto M$ for some proper ideal I of A or $0 \propto N$ for some submodule N of M by [4, Corollary 3.4].

Suppose that Conjecture one holds in R. Let I be a proper n-absorbing ideal of A for some integer $n \geq 1$. Then $J = I \propto M$ is a n-absorbing ideal of $R = A \propto M$, and hence a strongly n-absorbing ideal R by hypothesis. Thus I is a strongly n-absorbing ideal of A by Theorem 4.2.1(2).

Conversely, suppose that Conjecture one holds in A. Let J be a proper n-absorbing ideal of $R = A \propto M$ for some $n \geq 1$. Hence J is the form $I \propto M$ where I is a proper ideal of A or $0 \propto F$ where F is a K-subspace of M.

Case 1: $J = I \propto M$, where I is a proper ideal of A. Since J is an n-absorbing ideal of R, we conclude that I is an n-absorbing ideal of A by Theorem 4.2.1(1), and hence I is a strongly n-absorbing ideal of A by hypothesis. Thus $J = I \propto M$ is a strongly n-absorbing ideal of $R = A \propto M$ by Theorem 4.2.1(2).

Case 2: $J = \{0\} \propto F$, where F is an A-submodule of M. If $n = 1$, then $F = M$ and we are done. Hence assume that $n \geq 2$. Since J is an n-absorbing ideal of R, we conclude that F is a K-subspace of M by Theorem 4.3.2. Hence J is a strongly n-absorbing ideal of R for every $n \geq 2$ by Lemma 4.3.1. Thus Conjecture one holds in $R = A \propto M$.

Corollary 4.3.5 *Let D be a Prüfer domain with quotient field K, M be K-vector space, and $R = D \propto M$. Then Conjecture one holds in R.*

Proof. Since A is a Prüfer domain, Conjecture one holds in A by [5, Corollary 6.9]. Thus Conjecture one holds in R by Theorem 4.3.4.

We recall the following result.

Theorem 4.3.6 *(***[5, Corollary 6.8]***) Let R be a Noetherian ring. Then every proper ideal of R is a strongly n-absorbing ideal of R for some positive integer n.*

Theorem 4.3.7 *Let A be a Noetherian ring, M be an A-module, $R = A \propto M$, and I be a proper ideal of A. Then $J = I \propto M$ is a strongly n-absorbing ideal of R for some positive integer n.*

Proof. Since I is a strongly n-absorbing ideal of A for some positive integer n by Theorem 4.3.6, we conclude that $J = I \propto M$ is a strongly n-absorbing ideal of R.

Theorem 4.3.8 *Let A be a Noetherian ring, M be a finitely generated A-module, and $R = A \propto M$. Then every ideal of R is a strongly n-absorbing ideal of R for some positive integer n.*

Proof. Since A be a Noetherian ring and M is a finitely generated A-module, we conclude that R is a Noetherian ring by [4, Theorem 4.8]. Hence the claim follows from Theorem 4.3.6.

Question 2 *In view of Theorem 4.3.6, El Amin El Kaidi asked the following question: Let A be a ring and assume that every ideal of A is an n-absorbing ideal of R for some integer $n \geq 1$. Does it imply that A is a Noetherian ring.*

The following example gives a negative answer to the above question.

Example 4.3.9 *Let $A \subset K$ be fields such that K is not a finitely generated A-module (for example, let $A = \mathbb{Q}$ and $K = \mathbb{R}$) and $R = A \propto K$. Since R is a quasilocal ring with maximal ideal $M = \{0\} \propto K$ and $M^2 = \{(0,0)\}$, we conclude that every ideal of R a 2-absorbing ideal of R (and hence every ideal of R is a strongly n-absorbing ideal of R for every $n \geq 2$ by [5, Theorem 2.13]). Since K is not a finitely generated A-module, we conclude that $\{0\} \propto K$ is not a finitely generated of R. Thus R is not a Noetherian ring.*

Remark 4.3.10 *Let R be a ring and n a positive integer such that every proper ideal of R is an n-absorbing ideal of R. Then by [5, Theorem 5.9], we have $dim(R) = 0$ and R has at most n maximal ideals.*

We have the following result

Theorem 4.3.11 *Let A be an integral domain with quotient field K, M be a finite dimensional vector space over K, and $R = A \propto M$. Then every proper ideal of R is an n-absorbing ideal of R for some $n \geq 1$ if and only if $A = K$.*

Proof. Suppose that $A = K$. Since M is a finite dimensional vector space over K, we conclude that R a Noetherian ring by [4, Theorem 4.8]. Hence every proper ideal of R is an n-absorbing ideal of R for some $n \geq 1$ by Theorem 4.3.6. Conversely, suppose that every proper ideal of R is an n-absorbing ideal of R for some $n \geq 1$. Since M is a finite dimensional vector space over K, we may assume that that $M = K \times \cdots \times K$ (m times, where $m = dim_K(M) < \infty$). Hence $N = A \times \cdots \times A$ is a an A-submodule of M and $J = \{0\} \times N$ is a 2-absorbing ideal of R. Since $J = \{0\} \times N$ is a 2-absorbing ideal of R, we conclude that N is a K-subspace of M by Theorem 4.3.2. Thus $A = K$.

In light of Theorem 4.3.6 and Theorem 4.3.11, we have the following result.

Corollary 4.3.12 *Let A be an integral domain with quotient field K, M be a finite dimensional vector space over K, and $R = A \propto M$. Then the following statements are equivalent.*

1. *Every proper ideal of R is a strongly n-absorbing ideal of R for some $n \geq 1$.*
2. *Every proper ideal of R is an n-absorbing ideal of R for some $n \geq 1$.*
3. *$A = K$.*
4. *A is a Noetherian ring.*
5. *R is a Noetherian ring.*

Theorem 4.3.13 *Let A be a Noetherian domain with quotient field K, M be a K-vector space, and $R = A \propto M$. Then a proper ideal J of R is an n-absorbing ideal of R for some $n \geq 1$ if and only if J is a strongly m-absorbing ideal of R for some $m \geq 1$.*

Proof. If $n = 1$ or $m = 1$. Then J is a prime ideal of R, and hence the claim is clear. Let J be a proper ideal of R. Since M is a divisible A-module, we conclude that $J = I \propto M$ for some proper ideal I of A or $J = \{0\} \propto F$ for some A-submodule F of M by [4, Corollary 3.4]. Suppose that J is n-absorbing ideal of R for some $n \geq 2$. Assume that $J = I \propto M$ for some proper ideal I of A. Since I is a strongly m-absorbing ideal of A for some positive integer m by Theorem 4.3.6, we conclude that $J = I \propto M$ is a strongly m-absorbing ideal of R. Suppose that $J = \{0\} \propto F$ for some A-submodule F of M. Then F is a K-subspace of M by Theorem 4.3.2. Thus J is a strongly k-absorbing ideal of R for every integer $k \geq 2$ by Lemma 4.3.1. The converse is clear.

4.4 Conjecture three in trivial ring extension

Let A be a commutative ring, and M an A-module, let $R = A \propto M$, we know $(A \propto M)[X]$ is naturally isomorphic to $A[X] \propto M[X]$. If I is a ideal of A, then $(I \propto M)[X]$ is naturally isomorphic to $I[X] \propto M[X]$.

We recall the following result.

Theorem 4.4.1 *(***[5, Theorem 4.15]***) Let I be a proper ideal of a ring A. Then $I[X]$ is a 2-absorbing ideal of $R[X]$ if and only if I is a 2-absorbing ideal of R.*

Theorem 4.4.2 *Let A be an integral domain with quotient field K, M be a K-vector space, and $R = A \propto M$. Then Conjecture three holds in R if and only if Conjecture three holds in A.*

Proof. Suppose the Conjecture three holds in A. Let J be a proper n-absorbing ideal of R for some $n \geq 1$. Hence $J = I \propto M$ for some proper ideal I of A or $J = \{0\} \propto F$ for some K-subspace F of M by [4, Corollary 3.4] and Theorem 4.3.2.

Case one. Suppose that $J = I \propto M$ for some proper ideal I of A. Then I is an n-absorbing ideal of A. Thus $I[X]$ is an n-absorbing ideal of $A[X]$ by hypothesis. Hence $w_A(I) = w_{A[X]}(I[X])$. Since $J[X]$ is isomorphic to $I[X] \propto M[X]$, we conclude that $J[X]$ is an n-absorbing ideal of $R[X]$. Since $w_{R[X]}(J[X]) = w_{A[X] \propto M[X]}(I[X] \propto M[X]) = w_{A[X]}(I[X]) = w_A(I)$. Hence $w_{R[X]}(J[X]) = w_R(J)$.

Case two. Suppose that $J = 0 \propto F$ for some K-subspace F of M.

Since J is a 2-absorbing ideal of R, we conclude that $J[X]$ is a 2-absorbing absorbing ideal of $R[X]$ by Theorem 4.4.1. Hence Conjecture three holds in R

Conversely, suppose that Conjecture three holds in R. Let I be an n-absorbing ideal of A. Then $I \propto M$ is n-absorbing ideal of R. Hence $(I \propto M)[X]$ is n-absorbing ideal of $R[X]$ by hypothesis. Since $(I \propto M)[X] \cong I[X] \propto M[X]$, we conclude that $I[X]$ is an n-absorbing ideal of $A[X]$

Laradji [68, Corollary 2.11] showed that Conjecture three holds in arithmetical rings. Since a Prüfer domain is both arithmetical and Gaussian ring, the following result is an immediate consequence of [68, Corollary 2.11] and [77, Theorem 2.6].

Lemma 4.4.3 *([68, Corollary 2.11] and [77, Theorem 2.6]) Let A be a Prüfer domain and I be a proper n-absorbing ideal of A for some integer $n \geq 1$. Then $I[X]$ is an n-absorbing ideal of $A[X]$.*

In the following result, we construct rings with zero-divisors that satisfy Conjecture three but they do not need be arithmetical rings.

Theorem 4.4.4 *Let A be a Prüfer domain with quotient field K, M be K-vector space, n be a positive integer, and J be a proper ideal of $R = A \propto M$ (note that if $M = K[X]$, then R is not an arithmetical ring by [19, Theorem 2.1(2)]). If J is an n-absorbing ideal of R, then $J[X]$ is an n-absorbing ideal of $R[X]$ and $w_R(J) = w_{R[X]}(J[X])$.*

Proof. Since A is a Prüfer domain, Conjecture three holds in A by Lemma 4.4.3. Thus Conjecture three holds in R by Theorem 4.4.2. Thus If J is an n-absorbing ideal of R, then $J[X]$ is an n-absorbing ideal of $R[X]$ and $w_R(J) = w_{R[X]}(J[X])$.

In the following example, we construct a non-arithmetical ring that satisfies Conjecture three.

Example 4.4.5 *Let A be a Prüfer domain with quotient field K, $M = K[X]$, and $R = A \propto M$. Then:*

1. *R satisfies Conjecture three by Theorem 4.4.4.*
2. *R is a non-arithmetical ring by [19, Theorem 2.1(2)].*

Remark 4.4.6 *Let I be a proper ideal of a ring A and $n \geq 1$. It is shown [5, Theorem 6.1] that if I is a strongly n-absorbing ideal of A, then $(Rad(I))^n \subseteq I$. It is shown [68, Proposition 2.9(1)] that if $I[X]$ is an n-absorbing ideal of $A[X]$, then I is a strongly n-absorbing ideal of A. It is shown [68, Corollary 2.11] that if I is an n-absorbing ideal of an arithmetical ring A, then $I[X]$ is an n-absorbing ideal of $A[X]$. Hence if A is an arithmetical ring, then all three Conjectures holds in A.*

In the following result, we construct rings with zero-divisors that satisfy all three conjectures but they do not need be arithmetical rings.

Theorem 4.4.7 *Let A be a Prüfer domain with quotient field K, M be K-vector space, n be a positive integer, and $R = A \propto M$ (note that if $M = K[X]$, then R is not an arithmetical ring by [19, Theorem 2.1(2)]). Suppose that J is an n-absorbing ideal of R. Then the following statements holds:*

1. *J is a strongly n-absorbing ideal of R.*
2. *$J[X]$ is an n-absorbing ideal of R.*
3. *$(Rad(J))^n \subseteq J$.*

Proof. (**1**). It is clear by Corollary 4.3.5.
(**2**). It is clear by Theorem 4.4.4.
(**3**). It is clear by [78].

4.5 Conjecture one in u-rings

We recall from [80] that commutative ring R is called a u-ring if whenever an ideal I of R is contained in a finite union of ideals of R, then I is contained in at least one of those ideals. It is known that every Bezout ring is a u-ring and every Prüfer domain is a u-domain. In [77, Theorem 2.4], S. Smach and S. Hizem showed that Conjecture one holds in u-rings. In this section, we

propose a proof of this result that is different from that in [77, Theorem 2.4]. We need the following notation. Let R be a commutative ring. If $x_1,...,x_n \in R$, then $x_1...,\widehat{x_k}...x_n$ denotes the product $x_1...x_n$ that omits x_k. Similarly, if $I_1,...,I_{n+1}$ are ideals of R, then $I_1...\widehat{I_k}...I_{n+1}$ denotes the product $I_1....I_{n+1}$ that omits I_k. We start with the following lemmas.

Lemma 4.5.1 *Let R be a commutative ring. Suppose there are ideals $I_1,\cdots,I_{n+1}$ of R such that $I_1....I_{n+1}=\{0\}$ and no product of n of the I_j's is equal to $\{0\}$. Then there are finitely generated ideals $J_1,\cdots,J_{n+1}$ of R such that $J_1...J_{n+1}=\{0\}$ and no product of n of the J_i's is equal to $\{0\}$.*

Proof.

Suppose there are ideals $I_1,\cdots,I_{n+1}$ of R such that $I_1....I_{n+1}=\{0\}$ and no product of n of the I_j's is equal to $\{0\}$.

Let $j\in\{1,....,n+1\}$. Since $\prod_{i=1,i\neq j}^{n+1} I_i \neq \{0\}$ for all $i\neq j$, there exist $a_{i,j}\in I_i$ such that $\prod_{i=1,i\neq j}^{n+1} a_{i,j}\neq \{0\}$. Let $J_j=(a_{1,j},...,\widehat{a_{j,j}},...,a_{n+1,j})$ the ideal generated by $\{a_{i,j}, i\neq j, i=1,...,n+1\}$. Since $J_j\subseteq I_j$, we have $J_1...J_{n+1}=\{0\}$. Thus $\prod_{i=1,i\neq j}^{n+1} J_i\neq\{0\}$, for every $j\in\{1,...,n+1\}$, as desired.

Lemma 4.5.2 *Suppose that in any ring $\{0\}$ is a strongly n-absorbing ideal if and only if $\{0\}$ is an n-absorbing ideal. Then every n-absorbing ideal in an arbitrary ring R is a strongly n-absorbing ideal of R.*

Proof. Suppose I is n-absorbing ideal in a ring R and let the canonical homomorphism $f:R\to R/I$. Then $\{0\}$ is an n-absorbing ideal of $A'=A/I$ by [5, Theorem 4.2] and thus $\{0\}$ is a strongly n-absorbing ideal of A'. Let $I_1,\cdots,I_{n+1}$ are ideals of A such that $\prod_{i=1}^{n+1} I_i\subset I$, then $\prod_{i=1}^{n+1} f(I_i)=\{0\}$. Since $\{0\}$ is a strongly n-absorbing ideal of A', there exist $j\in\{1,...,n+1\}$ such that $\prod_{i=1,i\neq j}^{n+1} f(I_i)=\{0\}$ and so $\prod_{i=1,i\neq j}^{n+1} I_i\subset I$. Therefore, I is a strongly n-absorbing ideal of A.

Lemma 4.5.3 *Let R be a commutative u-ring such that $\{0\}$ is an n-absorbing ideal. Then $\{0\}$ is a strongly n-absorbing of R.*

Proof. Let $I_1,...,I_{n+1}$ be ideals of R such that $I_1\cdots I_{n+1}=\{0\}$. Assume that there is no product of n ideals of the I_j's equals to zero. By Lemma 4.5.2, there are finitely generated ideals $J_1,\cdots,J_{n+1}$ of R such that $J_1...J_{n+1}=\{0\}$ and no product of n of the J_i's equals to $\{0\}$. Let n_j be the minimal number of generators for J_j, and $\varphi(J_1,\cdots,J_{n+1})=\sum_{i=1}^{n+1} n_j$. It is clear that $\varphi(J_1,\cdots,J_{n+1})\in\{n+1,\cdots,n(n+1)\}$.

We will show by induction that there exists a product of n ideals of the J_i's equals to zero, which is the desired contradiction.

Suppose that $\varphi(J_1,\cdots,J_{n+1}) = \sum_{i=1}^{n+1} n_j = n+1$. Then for every $j = 1,...,n+1$, there exists an element $a_j \in R$ such that $J_j = Ra_j$. Hence, $J_1....J_{n+1} = \{0\}$. Since $\{0\}$ is an n-absorbing ideal of R, there exists one product $a_1 \cdots \widehat{a_k} \cdots a_{n+1} = \{0\}$ and hence $J_1...\widehat{J_k} \cdots J_{n+1} = \{0\}$.
Now assume that whenever $L_1L_2 \cdots L_{n+1} = \{0\}$ for some ideals $L_1,...,L_{n+1}$ of R and $\varphi(L_1,\cdots,L_{n+1}) < \varphi(J_1,...,J_{n+1})$, there exists a $k \in \{1,....,n+1\}$ such that $L_1 \cdots \widehat{L_k} \cdots L_{n+1} = \{0\}$. Since $\sum_{j=1}^{n+1} n_j > n+1$, without loss of generality, suppose $n_1 > 1$, and let $a_1 \in J_1$. Then $a_1J_2...J_{n+1} = \{0\}$. Let $L_1 = Ra_1$, and for $j \geq 2$, let $L_j = J_j$. Hence $L_1...L_{n+1} = \{0\}$ and $\varphi(L_1,\cdots,L_{n+1}) = 1 + \sum_{k=2}^{n+1} n_k < \varphi(J_1,...,J_{n+1})$. By induction there exists some $j \in \{2,...,n+1\}$ such that $L_1J_2...\widehat{J_j}...J_{n+1} = \{0\}$. Since $J_2....J_{n+1} \neq \{0\}$ by hypothesis, we have $a_1 \in \text{ann}(Q_j)$, where $Q_j = J_2...\widehat{J_j}...J_{n+1}$. Thus $J_1 \subset \bigcup_{i=1}^{n+1} \text{ann}(Q_j)$. Since R is a u-ring, there exists $j \in \{1,...,n+1\}$ such that $J_1 \subset \text{ann}(Q_j)$. Thus $J_1...\widehat{J_j}...J_{n+1} = \{0\}$, a contradiction. Therefore, there exists $j \in \{1,....,n+1\}$ such that $I_1...\widehat{I_j}...I_{n+1}$ equals to zero. Hence $\{0\}$ is a strongly n-absorbing of R.

Theorem 4.5.4 *Let R be a commutative u-ring. Then R satisfies Conjecture one, that is every n-absorbing ideal of R is a strongly n-absorbing ideal of R.*

Proof. Let R be a commutative u-ring. Suppose that I is a proper n-absorbing ideal of R. Then the quotient ring R/I is a u-ring by [80, Proposition 1.3] and $\{0\}$ is an n-absorbing ideal of R/I. Therefore, $\{0\}$ is a strongly n-absorbing of R/I by Lemma 4.5.3. Hence I is a strongly n-absorbing ideal of R. We recall from [80] that a ring A is called a *um-ring* if whenever an R-module equal to a finite union of submodules must be equal to one of them.

Remark 4.5.5 *Let R be a commutative ring and assume that R contains an infinite set S such that $x-y$ is a unit for all $x \neq y$ in S. Then R is a um-ring by [80, Proposition 1.7]. It is shown [80, Theorem 2.3] that a ring R is a um-ring if and only if R/M is infinite for every maximal ideal M of R. It is shown [80, Theorem 2.6] that a ring R is an u-ring if and only if R/M is infinite or R_M is a Bezout ring for every maximal ideal M of R. Hence in view of [80, Theorem 2.3] and [80, Theorem 2.6], we conclude that every um-ring is a u-ring. The converse is not true, for let $R = \mathbb{Z}$. Then R is a u-ring. Since R/M is finite for every maximal ideal M of R, we conclude that R is not a um-ring.*

In view of Remark 4.5.5, we have the following result.

Theorem 4.5.6 *Let R be a um-ring. Then R is a u-ring.*

The proof of the following result is similar to the proof of [80, Proposition 1.7].

Theorem 4.5.7 *Let R be a commutative ring with $1 \neq 0$, n be a positive integer, and I be a proper ideal of R. Suppose that R contains an infinite set S such that $x-y$ is a unit for all $x \neq y$ in S. Then R is a u-ring, and hence I is a strongly n-absorbing of R if and only if I is an n-absorbing ideal of R.*

Proof. Suppose that R contains an infinite set S such that $x-y$ is a unit for all $x \neq y$ in S. We show that R is a u-ring. Deny. Let I be an ideal of R and $p \geq 1$ be an integer such that $I \subset \bigcup_{i=1}^{p} I_i$, and

suppose that for every $i \in \{1,...,p\}$, we have $I \nsubseteq I_i$. We may assume that for each $i \in \{1,...,p\}$, we have $I \nsubseteq \bigcup_{j\neq i} I_j$. Hence for each $1 \leq i \leq 2$, there exists $a_i \in I$ such that $a_i \notin \bigcup_{j\neq i} I_j$. Consider the set $H = \{a_1 + xa_2 \mid x \in S\}$. Then for every $x \in S$, we have $a_1 + xa_2 \in I$ and $a_1 + xa_2 \notin I_2$. Since $H \subseteq I$ and $H \cap I_2 = \emptyset$, we have $H \subset \bigcup_{j\neq 2} I_j$. Since H is infinite, there exist $x_1 \neq x_2$ in S such that $a_1 + x_1a_2$ and $a_1 + x_2a_2 \in I_i$ for some $i \neq 2$. Hence $(x_1 - x_2)a_2 \in I_i$, and thus $a_2 \in I_i$, which is a contradiction. Thus R is a u-ring.

Remark 4.5.8 *One can give an alternative proof of Theorem 4.5.7. Note that since R contains an infinite set S such that $x - y$ is a unit for all $x \neq y$ in S, we conclude that R is a um-ring by [80, Proposition 1.7]. Hence R is a u-ring by Theorem 4.5.6.*

Theorem 4.5.9 *Let A be a u-domain with quotient field K, M be a K-vector space, and $R = A \propto M$. Then Conjecture one holds in R.*

Proof. Since A satisfies Conjecture one by Theorem 4.5.4, we conclude that R satisfies Conjecture one by Theorem 4.3.4.

The following is an example of a ring that is not a u-ring but it satisfies Conjecture one.

Example 4.5.10 *Let $R = \mathbb{Z}_3 \propto \mathbb{Z}_3[X]$. Then R satisfies Conjecture one by Theorem 4.5.9. It is clear that $M = \{0\} \propto \mathbb{Z}_3[X]$ is the only maximum ideal of R. Since neither R/M is infinite (note that $R/M \cong \mathbb{Z}_3$) nor R_M (note that $R_M = R$) is a Bezout ring, we conclude that R is not a u-ring by [80, Theorem 2.6]. Note that R is not a um-ring by Theorem 4.5.6.*

Theorem 4.5.11 *Let A be a commutative um-ring, M be an A-module, and $R = A \propto M$. Then Conjecture one holds in R.*

Proof. Let H be a maximal ideal of R. Then $H = L \propto M$ for some maximal ideal L of A. Since $R/H \cong A/L$ and A is a *um*-ring, we conclude that A/L is infinite, and thus R/H is infinite. Hence R is a *um*-ring by [80, Theorem 2.3]. Thus R is a u-ring by Theorem 4.5.6. Hence R satisfies Conjecture one by Theorem 4.5.4.

4.6 (m,n)-closed ideals in trivial ring extension

Let R be a commutative ring with $1 \neq 0$. We recall from [6] that a proper ideal I of R is called an (m,n)-*closed* ideal if $x^m \in I$ for $x \in R$ implies $x^n \in I$.

Theorem 4.6.1 *Let A be a ring, M be an A-module, and $R = A \propto M$. Suppose that $J = I \propto N$ is a proper ideal of R, where I is a proper ideal of A and N is a submodule of M such that $IM \subseteq N$. If I is an (m,n)-closed ideal of A for some integers $0 < n < m$, then J is an $(m,n+1)$-closed ideal of R.*

Proof. Suppose that I is an (m,n)-closed ideal of A for some integers $0 < n < m$. Let $x = (a,c) \in R$ and suppose that $x^m = (a^m, ma^{m-1}c) \in J$. Since I is an (m,n)-closed ideal of A, we conclude that $(a^{n+1}, (n+1)a^n c) = x^{n+1} \in J$. Thus J is an $(m,n+1)$-closed ideal of R.

In view of Theorem 4.6.1, the following is an example of an $(3,2)$-closed ideal I of Z but the proper ideal $J = I \propto I$ of $R = Z \propto Z$ is not an $(3,2)$-closed ideal of R.

Example 4.6.2 *Let $R = Z \propto Z$, $p \neq 2$ be a positive prime number of Z, $I = p^4Z$ a proper ideal of Z, and $J = I \propto I$. Then J is a proper ideal of R and I is an $(3,2)$-closed ideal of Z by [6, Corollary 3.3]. Let $x = (p^2, p) \in R$. Then $x^3 = (p^6, 3p^5) \in J$. Since $p \neq 2$, we have $x^2 = (p^4, 2p^3) \notin J$.*

Lemma 4.6.3 *Let A be a ring, M be an A-module, and $R = A(+)M$. Suppose that $J = I \propto N$ is a proper ideal of R, where I is an (m,n)-closed ideal of A for some integers $0 < n < m$, and N is a submodule of M such that $IM \subseteq N$. Let $x = (a,c) \in R$ for some $a \in A$ and $c \in M$. Then $x^m \in J$ if and only if $a^m \in I$.*

Proof. Suppose that $x^m = (a^m, ma^{m-1}c) \in J$. Then it is clear that $a^m \in I$.

Conversely, suppose that $a^m \in I$. Since I is an (m,n)-closed ideal of R, $a^n \in I$. Since $n \leq m-1$, we conclude that $a^{m-1} \in I$. Since $IM \subseteq N$ and $a^{m-1} \in I$, we conclude that $x^m = (a^m, ma^{m-1}c) \in J$.

Theorem 4.6.4 *Let A be a ring, M be an A-module, and $R = A \propto M$. Suppose that $J = I \propto N$ is a proper ideal of R, where I is a proper ideal of A and N is a submodule of M such that $IM \subseteq N$. Let $0 < n < m$ be integers. The following statements are equivalent.*

1. *J is an (m,n)-closed ideal of R.*
2. *I is an (m,n)-closed ideal of A and whenever $a^m \in I$ for some $a \in A$ implies $na^{n-1}M \subseteq N$.*

Proof. $(1) \Rightarrow (2)$. Suppose that J is an (m,n)-closed ideal of R. Then it is clear that I is an (m,n)-closed ideal of A. Assume that $a^m \in I$ for some $a \in A$. Let $c \in M$ and $x = (a,c)$. Since $a^m \in I$, we have $x^m \in R$ by Lemma 4.6.3. Since J is an (m,n)-closed ideal of R, we conclude that $x^n = (a^n, na^{n-1}c) \in R$. Thus $na^{n-1}M \subseteq N$.

$(2) \Rightarrow (1)$. Suppose that I is an (m,n)-closed ideal of A and whenever $a^m \in I$ for some $a \in A$ implies $na^{n-1}M \subseteq N$. Let $x = (a,c) \in R$ for some $a \in A$ and $c \in M$ and suppose that $x^m = (a^m, ma^{m-1}c) \in J$. Since $a^m \in I$ and I is an (m,n)-closed ideal of A, we conclude that $a^n \in A$ and $na^{n-1}c \in N$. Thus $x^n = (a^n, na^{n-1}c) \in J$. Hence J is an (m,n)-closed ideal of R.

Theorem 4.6.5 *Let A be a ring, M be an A-module, m and n integers with $1 \leq n < m$, I be a proper ideal of A, and $R = A \propto M$. Suppose that $char(A) \mid n$. Then the following statements are equivalent.*

1. *$J = I \propto N$ is an (m,n)-closed ideal of R for every submodule N of M where $IM \subseteq N$.*
2. *I is an (m,n)-closed ideal of A.*

Proof. $(1) \Rightarrow (2)$. It is clear by Theorem 4.6.4.

$(2) \Rightarrow (1)$. Let N be a submodule of M such that $IM \subseteq N$. Since $char(A) \mid n$, we conclude that whenever $a^m \in I$ for some $a \in A$ implies $na^{n-1}M = 0_M \subseteq N$, where 0_m is the additive identity of M. Thus $J = I \propto N$ is an (m,n)-closed ideal of R by Theorem 4.6.4.

Theorem 4.6.6 *Let D be an integral domain, $R = D \propto D$, m and n integers with $1 \leq n < m$, and $I = p^kD$, where p is a prime element of D and k is a positive integer. Suppose that $m > k$ and $char(D) \neq n$. Then the following statements are equivalent.*

1. *$J = I \propto p^iD$ is an (m,n)-closed ideal of R for some integer $i \geq 1$.*
2. *One of the following three cases must hold:*
   (a) *$k < n < m$ and $i \leq k$.*
   (b) *$n = k$, and $1 \leq i < k$.*
   (c) *$n = i = k$, and $p \mid k \cdot 1_D$ (in D), where 1_D is the identity of D.*

Proof. $(1) \Rightarrow (2)$. Suppose that $J = I \propto p^iD$ is an (m,n)-closed ideal of R for some integer $i \geq 1$. Since J is an ideal of R, we conclude that $I \subseteq p^iD$. Hence $i \leq k$. Since $J = I \propto p^iD$ is an (m,n)-closed ideal of R, we conclude that I is an (m,n)-closed ideal of D and whenever $a^m \in I$ for some $a \in D$ implies $na^{n-1}D \subseteq p^iD$ by Theorem 4.6.4. Since $m > k$, $p^m \in I$ and hence $p^n \in I$ and $np^{n-1}D \subseteq p^iD$. In particular, $np^{n-1} \in p^iD$. Since $p^n \in I$, we conclude that $n \geq k$. Suppose that $n = k$. Then $np^{n-1} = kp^{k-1} \in p^iD$ if and only if either $1 \leq i < k$ or $i = k$ and $p \mid k \cdot 1_D$.

$(2) \Rightarrow (1)$. In view of proof $(1) \Rightarrow (2)$ above, one can easily verify that if (a) or (b) or (c) holds, then I is an (m,n)-closed ideal of D and whenever $a^m \in I$ for some $a \in D$ implies $na^{n-1}D \subseteq p^iD$. Hence J is an (m,n)-closed ideal of R by Theorem 4.6.4.

Definition 4.6.7 *Let p be a prime element of an integral domain D. Suppose that $p^w \mid d$ for some $d \in D$ and a positive integer w but $p^{w+1} \nmid d$. Then we write $p^w \mid\mid d$*

Theorem 4.6.8 *Let D be an integral domain, $R = D \propto D$, m and n integers with $1 \leq n < m$, and $I = p^kD$, where p is a prime element of D and k is a positive integer. Suppose that $m < k$ and $char(D) \neq n$. Let $v = \lceil \frac{k}{m} \rceil$ and $u = \lceil \frac{k}{v} \rceil$. Then the following statements are equivalent.*

1. *$J = I \propto p^iD$ is an (m,n)-closed ideal of R for some integer $i \geq 1$.*
2. *One of the following three cases must hold:*
   (a) *$u < n < m$ and $i \leq k$.*
   (b) *$u = n, p \nmid n \cdot 1_D$ (in D), and $i \leq v(n-1) < k$.*
   (c) *$u = n$, $p^w \mid\mid n \cdot 1_D$ (in D), and $i \leq min\{v(n-1)+w, k\}$.*

Proof. $(1) \Rightarrow (2)$. Suppose that $J = I \propto p^iD$ is an (m,n)-closed ideal of R for some integer $i \geq 1$. Since J is an ideal of R, we conclude that $I \subseteq p^iD$. Hence $i \leq k$. It is clear that $v = \lceil \frac{k}{m} \rceil$ is the smallest positive integer where $(p^v)^m \in I$. Also, it is clear that u is the smallest positive integer where $(p^v)^u \in I$. Since $J = I \propto p^iD$ is an (m,n)-closed ideal of R and $1 \leq n < m$, we conclude that $u \leq n < m$. Since $J = I(+)p^iD$ is an (m,n)-closed ideal of R, we conclude that I is an (m,n)-closed ideal of D and whenever $a^m \in I$ for some $a \in D$ implies $na^{n-1}D \subseteq p^iD$ by Theorem 4.6.4. Hence since $(p^v)^m \in I$, we conclude that $n(p^v)^{n-1} \in p^iD$ by Theorem 4.6.4. If $u < n < m$, then $u \leq n-1$ and thus $n(p^v)^{n-1} \in p^kD = I$ (note that $(p^v)^u \in I$) and $i \leq k$. Suppose that $n = u$ and $p \nmid n \cdot 1_D$ (in D). Since u is the smallest positive integer where $(p^v)^u \in I$ and $p \nmid n \cdot 1_D$, we conclude that $v(n-1) < k$ and $n(p^v)^{n-1} \in p^iD$ if and only if $i \leq v(n-1) < k$. Suppose that $u = n$ and $p^w \mid\mid n \cdot 1_D$ (in D). Since $i \leq q$, we conclude that $n(p^v)^{n-1} \in p^iD$ if and only if $i \leq min\{v(n-1)+w, k\}$.

$(2) \Rightarrow (1)$. In view of proof $(1) \Rightarrow (2)$ above, one can easily verify that if (a) or (b) or (c) holds, then I is an (m,n)-closed ideal of D and whenever $a^m \in I$ for some $a \in D$ implies $na^{n-1}D \subseteq p^iD$. Hence J is an (m,n)-closed ideal of R by Theorem 4.6.4.

Let R be an integral domain, $I = p^kR$, where p is a prime element of R and k is a positive integer, and m and n be fixed positive integers with $1 \leq n < m$. The authors in [6, Theorem 3.1] determined the set $\{k \in \mathbb{N} \mid p^kR \text{ is } (m,n)\text{-closed}\}$. We recall the following result.

Theorem 4.6.9 *(**[6, Theorem 3,1]**) Let D be an integral domain, m and n integers with $1 \leq n < m$, and $I = p^kD$, where p is a prime element of D and k is a positive integer. Then the following statements are equivalent.*

1. *I is an (m,n)-closed ideal of D.*
2. *If $m = bn + c$ for integers b and c with $b \geq 2$ and $0 \leq c \leq n-1$, then $k \in \{1,\ldots,n\}$. If $m = n + c$ for an integer c with $1 \leq c \leq n-1$, then $k \in \bigcup_{h=1}^{n}\{mi + h \mid i \in \mathbb{Z} \text{ and } 0 \leq ic \leq n-h\}$.*

In view of Theorems 4.6.6, 4.6.8 and 4.6.9, we have the following result.

Theorem 4.6.10 *Let D be an integral domain, $R = D \propto D$, m and n integers with $1 \leq n < m$, and $I = p^kD$, where p is a prime element of D and k is a positive integer. Suppose that $char(D) \neq n$. Then the following statements are equivalent.*

1. *$J = I \propto p^iD$ is an (m,n)-closed ideal of R for some integer $i \geq 1$.*
2. *If $m = bn + c$ for integers b and c with $b \geq 2$ and $0 \leq c \leq n-1$, then $k \in \{1,\ldots,n\}$ and one of the following three cases must hold:*
   - *(a) $k < n < m$ and $i \leq k$.*
   - *(b) $n = k$, and $1 \leq i < k$.*
   - *(c) $n = i = k$, and $p \mid k \cdot 1_D$ (in D), where 1_D is the identity of D.*

   If $m = n + c$ for an integer c with $1 \leq c \leq n-1$, then $k \in \bigcup_{h=1}^{n}\{mi + h \mid i \in \mathbb{Z} \text{ and } 0 \leq ic \leq n-h\}$ and one of the following three cases must hold:

   Let $v = \lceil \frac{k}{m} \rceil$ and $u = \lceil \frac{k}{v} \rceil$. Then
   - *(a) $u < n < m$ and $i \leq k$.*
   - *(b) $u = n, p \nmid n \cdot 1_D$ (in D), and $i \leq v(n-1) < k$.*
   - *(c) $u = n$, $p^w \mid\mid n \cdot 1_D$ (in D), and $i \leq min\{v(n-1) + w, k\}$.*

Proof. $(1) \Rightarrow (2)$. Suppose that $J = I \propto p^iD$ is an (m,n)-closed ideal of R for some integer $i \geq 1$. Then I is an (m,n)-closed ideal of D by Theorem 4.6.4. Suppose that $m = bn + c$ for integers b and c with $b \geq 2$ and $0 \leq c \leq n-1$. Then $k \in \{1,\ldots,n\}$ by Theorem 4.6.9. Hence $m > k$. Thus we are done by Theorem 4.6.6. Suppose that $m = n + c$ for an integer c with $1 \leq c \leq n-1$. Then $k \in \bigcup_{h=1}^{n}\{mi + h \mid i \in \mathbb{Z} \text{ and } 0 \leq ic \leq n-h\}$ by Theorem 4.6.9. Thus $m < k$. Hence we are done by Theorem 4.6.8.

$(2) \Rightarrow (1)$. Suppose that $k \in \{1,\ldots,n\}$ and (a) or (b) or (c) holds. Since $m > k$, we are done by Theorem 4.6.6. Suppose that $m = n + c$ for an integer c with $1 \leq c \leq n-1$ and $k \in \bigcup_{h=1}^{n}\{mi + h \mid i \in \mathbb{Z} \text{ and } 0 \leq ic \leq n-h\}$ and (a) or (b) or (c) holds. Since $m < k$, we are done by Theorem 4.6.8.

In view of Theorems 4.6.1 and 4.6.9, we have the following result.

Theorem 4.6.11 *Let D be an integral domain, $I = p^k D$, where p is a prime element of D and k is a positive integer, M be a $D-$module, $R = D \propto M$, $J = I \propto N$ is a proper ideal of R, where N is a submodule of M such that $IM \subseteq N$, and m and n integers with $1 \leq n < m$. Then the following statements are equivalent.*

1. *I is an (m,n)-closed ideal of D and J is an $(m,n+1)$-closed ideal of R.*
2. *If $m = bn + c$ for integers b and c with $b \geq 2$ and $0 \leq c \leq n-1$, then $k \in \{1,\ldots,n\}$. If $m = n + c$ for an integer c with $1 \leq c \leq n-1$, then $k \in \bigcup_{h=1}^{n} \{mi + h \mid i \in \mathbb{Z} \text{ and } 0 \leq ic \leq n-h\}$.*

Proof. $(1) \Rightarrow (2)$. Suppose that I is an (m,n)-closed ideal of D and J is an $(m,n+1)$-closed ideal of R. Since I is an (m,n)-closed ideal of D, we are done by Theorem 4.6.9.

$(2) \Rightarrow (1)$. By Theorem 4.6.9, I is an (m,n)-closed ideal of D. Hence J is an $(m,n+1)$-closed ideal of R by Theorem 4.6.1.

Theorem 4.6.12 *Let A be an integral domain with quotient field K, M be a K-vector space, and $R = A \propto M$. Then the following statements are equivalent.*

1. *Every proper ideal of A is an (m,n)-closed ideal of A for some integers $1 \leq n < m$.*
2. *Every proper ideal of R is an (m,n)-closed ideal of R for some integers $1 \leq n < m$.*

Proof. $(1) \Rightarrow (2)$. Suppose that every proper ideal of A is an (m,n)-closed ideal of A for some integers $1 \leq n < m$. Let J be an ideal of R. Since M is a divisible A-module, we have $J = I \propto M$ for some proper ideal I of A or $J = \{0\} \propto N$ for some A-submodule N of M by ([4, Corollary 3.4]). Suppose that $J = I \propto M$ for some proper ideal I. Since I is an (m,n)-closed ideal of A for some integers $1 \leq n < m$, it is clear that $J = I \propto M$ is an (m,n)-closed ideal of R. Suppose that $J = \{0\} \propto N$ for some A-submodule N of M. Since A is an integral domain, we have $J = \{0\} \propto N$ is an $(m,2)$-closed ideal of R for every integer $m \geq 3$. Hence every proper ideal of R is an (m,n)-closed ideal of R for some integers $1 \leq n < m$.

$(2) \Rightarrow (1)$. It is clear.

CHAPTER 5
ON n-ABSORBING AND STRONGLY n-ABSORBING IDEALS IN AMALGAMATED ALGEBRAS

Abstract*

Let R be a commutative ring with $1 \neq 0$. Let $n \geq 1$ be a positive integer. A proper ideal I of R is called *an n-absorbing ideal* (resp., *a strongly n-absorbing ideal*) of R as in [5] if $a_1, a_2, ..., a_{n+1} \in R$ and $a_1.a_2 \cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I (resp., if whenever $I_1 ... I_{n+1} \subseteq I$ for ideals $I_1, \cdots, I_{n+1}$ of R, then the product of some n of the I'_js is contained in I). The concept of n-absorbing ideals is a generalization of the concept of prime ideals (note that a prime ideal of R is a 1-absorbing ideal of R). Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. This paper investigates the n-absorbing and strongly n-absorbing ideals in the amalgamation of A with B along J with respect f denoted by $A \bowtie^f J$.

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*Journal of Algebra and Its Applications, on line 2020), (in collaboration with N. Mahdou and M.A. Moutui)

5.1 Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. Over the past several years, there has been considerable attention in the literature to n-absorbing ideals of commutative rings and their generalizations, (for instance, see [5, 22, 25, 63, 77, 76]). We recall from [9] that a proper ideal I of R is called a *2-absorbing ideal* of R if $a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. A more general concept than 2-absorbing ideals is the concept of n-absorbing ideals. Let $n \geq 1$ be a positive integer. A proper ideal I of R is called an *n-absorbing ideal* of R as in [5] if $a_1, a_2, ..., a_{n+1} \in R$ and $a_1.a_2 \cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I. A proper ideal of R is called a *strongly n-absorbing ideal* of R as in [5] if whenever $I_1 ... I_{n+1} \subseteq I$ for ideals $I_1, \cdots, I_{n+1}$ of R, then the product of some n of the I_j's is contained in I. The concept of n-absorbing ideals is a generalization of the concept of prime ideals (note that a prime ideal of R is a 1-absorbing ideal of R). We start by recalling some background materials. We say that a ring R is a *quasilocal* ring if R has exactly one maximal ideal. If I is a primary ideal of R with $\mathrm{Rad}(I) = P$ (a prime ideal of R), then we say that I is a *P-primary* ideal of R. A prime ideal P of a ring R is called *divided* if $P \subset x$ for every $x \in R \setminus P$. Let $n \geq 1$ be an integer and I be a proper ideal of a ring A. Anderson and Badawi in [5] (also see [25]) proposed the following three conjectures.

1. Conjecture one: I is an n-absorbing ideal of A if and only if I is a strongly n-absorbing ideal of A.
2. Conjecture two: If I is an n-absorbing ideal of A, then $(\mathrm{Rad}(I))^n \subseteq I$. Notice that an affirmative answer to this conjecture is given in [78].
3. Conjecture three: If I is an n-absorbing ideal of A, then $I[X]$ is an n-absorbing ideal of $A[X]$.

Conjecture 1: Let n be a positive integer. Then a proper ideal I of a ring is strongly n-absorbing ideal of R if and only if I is n-absorbing ideal of R.

The conjecture was proved in [5] for $n = 2$. It was also verified for arbitrary n when R is a Prüfer domain. In [15], the authors studied the notions of n-absorbing ideals, strongly n-absorbing ideals and (m,n)-closed ideals in the trivial ring extension.

Let (A,B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be an ideal of B. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f, introduced and studied by D'Anna, Finocchiaro and Fontana in [35, 37]. In particular, they have studied amalgamations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [33]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealizations) (cf. [75, page 2]). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$ and the

$D+M$ constructions) can be studied as particular cases of the amalgamation ([35, Examples 2.5 and 2.6]) and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [24]) are strictly related to it ([35, Example 2.7 and Remark 2.8]). In [35], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [38], they pursued the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.

In this paper, we study n-absorbing ideals and strongly n-absorbing ideals in the amalgamation of A with B along J with respect f denoted by $A \bowtie^f J$. Throughout, for a ring R, $\mathrm{Spec}(R)$ (resp., $\mathrm{Max}(R)$) will denote the set of all prime (resp., maximal) ideals of R, and for any ideal I of R, we denote by $\overline{\mathrm{Spec}(R,I)}$ (resp., $\overline{\mathrm{Max}(R,I)}$ the set of prime (resp., all maximal) ideals of R do not containing I. For any ring R, we denote by $Nil(R)$ (resp., $Jac(R)$), the set of nilpotent elements of R (resp., the Jacobson radical of R).

5.2 On n-absorbing and strongly n-absorbing ideals issued from amalgamation

To avoid unnecessary repetition, let us fix notation for the rest of the paper. Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. All along the paper, $A \bowtie^f J$ denotes the amalgamation of A with B along J with respect to f. With the notation of [38, Corollary 2.5], for an ideal I of A and for an ideal K of $f(A)+J$, set:

$$I \bowtie^f J := \{(i, f(i)+j); i \in I, j \in J\}.$$

$$\overline{K}^f := \{(a, f(a)+j); a \in A, j \in J, f(a)+j \in K\}.$$

and

$$\overline{I \times K}^f := \{(a, f(a)+j); j \in J, a \in I, f(a)+j \in K\}.$$

Clearly, $I \bowtie^f J$, $\overline{K}^f$ and $\overline{I \times K}^f$ are ideals of $A \bowtie^f J$. Our first result investigates about when the ideals $I \bowtie^f J$, $\overline{K}^f$ and $\overline{I \times K}^f$ are $n-$absorbing (resp., strongly n-absorbing) ideals of $A \bowtie^f J$.

Theorem 5.2.1 *Consider a positive integer n. Under the above notations, the following statements hold:*

1. *$I \bowtie^f J$ is an n-absorbing ideal of $A \bowtie^f J$ if and only if I is an n-absorbing ideal of A.*
2. *$I \bowtie^f J$ is a strongly n-absorbing ideal of $A \bowtie^f J$ if and only if I is a strongly n-absorbing ideal of A.*
3. *If I is an n_1-absorbing ideal of A and K is an n_2-absorbing ideal of $f(A)+J$, then $\overline{I \times K}^f$ is an $n=(n_1+n_2)$-absorbing ideal of $A \bowtie^f J$.*

4. *If I is a strongly n_1-absorbing ideal of A and K is a strongly n_2-absorbing ideal of $f(A)+J$, then $\overline{I\times K}^f$ is a strongly $n=(n_1+n_2)$-absorbing ideal of $A\bowtie^f J$.*
5. *$\overline{K}^f$ is an n-absorbing ideal of $A\bowtie^f J$ if and only if K is an n-absorbing ideal of $f(A)+J$.*
6. *$\overline{K}^f$ is a strongly n-absorbing ideal of $A\bowtie^f J$ if and only if K is a strongly n-absorbing ideal of $f(A)+J$.*

The proof of Theorem 5.2.1 requires the following lemmas.

Lemma 5.2.2 *Let n be a positive integer. Under the above notations, the following assertions hold:*

1. *If J is a strongly n-absorbing ideal of B, then $f^{-1}(J)$ is a strongly n-absorbing ideal of A.*
2. *Assume that f is surjective and I is an idéal of A containing $\ker(f)$. Then $f(I)$ is a strongly n-absorbing ideal of B if and only if I is a strongly n-absorbing ideal of A.*

Proof.

1. Let $I_1,\cdots,I_{n+1}$ be ideals of A such that $I_1...I_{n+1}\subseteq f^{-1}(J)$. Then $I_1^e....I_{n+1}^e\subseteq J$ where I_j^e is an extension ideal of I_j in B, for every positive integer j. Since J is a strongly n-absorbing ideal of B, then there exists a product $I_{i_1}^e...I_{i_n}^e$ of some n of the I_j^e 's $\subseteq J$, and so $(I_{i_1}...I_{i_n})^e\subseteq J$ or $f(I_j)$ is contained in I_j^e. Hence, it follows that the product $I_{i_1}...I_{i_n}$ is contained in $f^{-1}(J)$, making $f^{-1}(J)$, a strongly n-absorbing ideal of A, as desired.
2. Let I be a strongly n-absorbing ideal of A. Since f is surjective, then $f(I)$ is an ideal of B. Let $J_1,\cdots,J_{n+1}$ be the ideals of B such that $\prod_{j=1}^{n+1}J_j\subseteq f(I)$, and set $I_j=f^{-1}(J_j)$, for $j=1,...,n+1$. Clearly, $f(I_j)=J_j$ since f is surjective and so $f\left(\prod_{j=1}^{n+1}I_j\right)\subseteq f(I)$. Using the fact that $\ker(f)\subseteq I$, then we claim that $\prod_{j=1}^{n+1}I_j\subseteq I$. Indeed, let $b=\prod_{j=1}^{n+1}a_j\in\prod_{j=1}^{n+1}I_j$. Then $f(b)=\prod_{j=1}^{n+1}f(a_j)\in f(I)$, and so there exists $c\in I$ such that $f(b)=f(c)$. Therefore, $b-c\in\ker(f)\subseteq I$, making $b\in I$. Consequently, $\prod_{j=1}^{n+1}I_j\subseteq I$. Since I is a strongly n-absorbing ideal, then without lost the generality, we may assume that $\prod_{j=1}^{n}I_j\subseteq I$. Hence, it follows that $\prod_{j=1}^{n}J_j\subseteq f(I)$, making, $f(I)$ a strongly n-absorbing ideal of B. Conversely, assume that $f(I)$ is a strongly n-absorbing ideal of B. Let $I_1,\cdots,I_{n+1}$ be ideals of A such that $\prod_{j=1}^{n+1}I_j\subseteq I$. Then $\prod_{j=1}^{n+1}f(I_j)\subseteq f(I)$. Since $f(I)$ is a strongly n-absorbing of B, then without loss the generality, we may assume that $f\left(\prod_{j=1}^{n}I_j\right)=\prod_{j=1}^{n}f(I_j)\subseteq f(I)$. The fact that $\ker(f)\subseteq I$, it follows that $\prod_{j=1}^{n}I_j\subseteq I$. Hence, I is a strongly $n-$absorbing ideal of A, as desired.

Lemma 5.2.3
1. *Let $A\subseteq B$ be an extension of rings and J be a strongly n-absorbing ideal of B. Then $A\cap J$ is a strongly n-absorbing ideal of A.*
2. *Let $I\subseteq J$ be ideals of a ring A. Then J is a strongly n-absorbing ideal of A if and only if J/I is a strongly n-absorbing ideal of A/I. In particular, I is a strongly n-absorbing of A if and only if the zero ideal $\{0\}$ is a strongly n-absorbing ideal of A/I.*

Proof.

1. Let $I_1,...,I_{n+1}$ be $n+1$ ideals of A such that $\prod_{i=1}^{n+1} I_i \subseteq A \cap J$. Since J is an ideal of B, then consider $I_i^e := BI_i$ the extension ideal of I_i in B. Then $\left(\prod_{i=1}^{n+1} I_i\right)^e = \prod_{i=1}^{n+1} I_i^e \subseteq J$. Since J is a strongly n-absorbing ideal of B, then there exist n ideals $I_{i_1}^e,...,I_{i_n}^e$ such that $\prod_{j=1}^{n} I_{i_j}^e \subseteq J$ or $I_{i_j} \subseteq I_{i_j}^e$. Hence, $\prod_{j=1}^{n} I_{i_j} \subseteq J$ and we conclude that $\prod_{j=1}^{n} I_{i_j} \subseteq A \cap J$, as desired.
2. Consider the canonical surjection $\pi : A \to A/I$. Notice that for every ideal J of A such that $I \subseteq J$, we have $\pi(J) = J/I$ and $\ker(\pi) = I$. Hence, by application to the assertion (2) of Lemma 5.2.2, the conclusion is straightforward.

Lemma 5.2.4 *Let I_1 be an ideal of a ring A and I_2 be an ideal of a ring B and let n, p be two positive integers. Then $I_1 \times I_2$ is a strongly $(n+p)$-absorbing ideal of $A \times B$ if and only if I_1 is a strongly n-absorbing ideal of A and I_2 is a strongly p-absorbing ideal of B.*

Proof. Assume that $I_1 \times I_2$ is a strongly $(n+p)$-absorbing ideal of $A \times B$. Then by assertion (2) of Lemma 5.2.3, $I_1 \simeq \frac{I_1 \times I_2}{0 \times I_2}$ (resp., $I_2 \simeq \frac{I_1 \times I_2}{I_1 \times 0}$) is a strongly n-absorbing (resp., a strongly p-absorbing) ideal of A (resp., B). Conversely, assume that I_1 is a strongly n-absorbing ideal of A and I_2 is a strongly p-absorbing ideal of B. We need to show that $I_1 \times I_2$ is a strongly $(n+p)$-absorbing ideal of $A \times B$. Let $K_1, \cdots, K_{n+p+1}$ be ideals of $A \times B$ such that $\prod_{i=1}^{n+p+1} K_i \subseteq I_1 \times I_2$. For every positive integer i, we have $K_i = R_i \times Q_i$, where R_i is an ideal of A and Q_i is an ideal of B. So, $\prod_{i=1}^{n+p+1} R_i \subseteq I_1$ and $\prod_{i=1}^{n+p+1} Q_i \subseteq I_2$. Since I_1 is a strongly n-absorbing of A and I_2 is a strongly p-absorbing ideal of B, then there are $\{i_1, \cdots, i_p\}, \{j_1, \cdots, j_n\} \subseteq \{1,...,n+p+1\}$ such that $\prod_{j=1}^{n} R_{i_j} \subseteq I_1$ and $\prod_{k=1}^{n} Q_{j_k} \subseteq I_2$. Let $H := \{i_1, \cdots, i_p\} \cup \{j_1, \cdots, j_n\}$; so $\mid K \mid \leq n+p$. Thus $\prod_{i \in H} K_i \subseteq I_1 \times I_2$. Therefore, $I_1 \times I_2$ is a strongly $(n+p)$-absorbing ideal of $A \times B$, as desired.

Proof of Theorem 5.2.1.

Recall that $\frac{A \bowtie^f J}{I \bowtie^f J} \cong \frac{A}{I}$ [35, Proposition 5.1(2)]. Consider $u : \frac{A \bowtie^f J}{I \bowtie^f J} \to \frac{A}{I}$ the given ring isomorphism.

1. From [5, Corollary 4.3(b)], I is an n-absorbing ideal of A if and only if $\{0\}$ is an n-absorbing ideal of $\frac{A}{I}$ if and only if $u^{-1}(\{0\}) = \{0\}$ is an n-absorbing ideal of $\frac{A \bowtie^f J}{I \bowtie^f J}$. Since $I \cong \frac{I \bowtie^f J}{0 \bowtie^f J}$, then by [5, Corollary 4.3(b)], it follows that I is an n-absorbing ideal of A if and only if $I \bowtie^f J$ is an n-absorbing ideal of $A \bowtie^f J$.
2. By assertion (2) of Lemma 5.2.3, I is a strongly n-absorbing ideal of A if and only if $\{0\}$ is a strongly n-absorbing ideal of $\frac{A}{I}$ if and only if $u^{-1}(\{0\}) = \{0\}$ is a strongly n-absorbing ideal of $\frac{A \bowtie^f J}{I \bowtie^f J}$. Since $I \cong \frac{I \bowtie^f J}{0 \bowtie^f J}$, then by application to the assertion (2) of Lemma 5.2.3, the conclusion is straightforward.
3. From [5, Theorem 4.7], the ideal $I \times K$ is an (n_1+n_2)-absorbing ideal of $A \times (f(A)+J)$, and so by [5, Corollary 4.3(a)], it follows that $(I \times K) \cap (A \bowtie^f J)$ is an (n_1+n_2)-absorbing ideal of $A \bowtie^f J$. On the other hand it is worthwhile to notice that $\overline{I \times K}^f = (I \times K) \cap (A \bowtie^f J)$. Hence, the conclusion is straightforward.
4. The proof is similar to the proof assertion (3) above, by replacing "[5, Theorem 4.7]" with "Lemma 5.2.4" and "[5, Corollary 4.3(a)]" with "assertion (1) of Lemma 5.2.3".

5. Let K be an ideal of $f(A)+J$. Consider the natural projection $p_B : \overline{K}^f \to K$. Clearly, p_B is surjective and $ker(p_B) = f^{-1}(J) \times \{0\} \subseteq \overline{K}^f$. Hence, by [5, Theorem 4.2], we have the desired result.
6. Let K be an ideal of $f(A)+J$. The proof is similar to the proof of assertion (5) above. Obviously, $ker(p_B) = f^{-1}(J) \times \{0\} \subseteq \overline{K}^f$ and $p_B(\overline{K}^f) = K$. Thus, by assertion (2) of Lemma 5.2.2, the conclusion is trivial.

Let P be a prime ideal of a ring R, I be a P-primary ideal of R and n be a positive integer such that $P^n \subseteq I$. Then I is an n-absorbing ideal of R [5, Theorem 3.3]. In particular, if P^n is a P-primary ideal, then P^n is an n-absorbing ideal of R. Also, recall that if I is a P- primary ideal and n a positive integer, then I is a strongly n-absorbing ideal of R if and only if I is an n-absorbing ideal of R and $P^n \subseteq I$ [5, Theorem 6.6]. For a ring R, we denote by $Z(R)$, the set of zero divisors of R. The next corollary is a consequence of Theorem 5.2.1.

Corollary 5.2.5 *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B and P be an ideal of A. Consider a positive integer n. Then the following assertions hold:*

1. *Assume that P is a prime ideal such that P^n is P-primary for the same positive integer n. Then $P^n \bowtie^f J$ is a strongly n-absorbing ideal of $A \bowtie^f J$. In particular, if A is a valuation domain, then for every prime ideal P of A, $P^n \bowtie^f J$ is a strongly n-absorbing ideal of $A \bowtie^f J$.*
2. *Assume that $Nil(A)$ and P are divided prime ideals of A such that $Nil(A) \subset P$. Then $P^n \bowtie^f J$ is a strongly n-absorbing ideal of $A \bowtie^f J$.*
3. *If P is a divided prime such that $Z(A) \subset P$, then $P^n \bowtie^f J$ is a strongly n-absorbing ideal of $A \bowtie^f J$, for every positive integer $n \geq 1$.*
4. *Assume that A is a Prüfer domain and let $K := I \bowtie^f J$ be an ideal of $A \bowtie^f J$, where I is an ideal of A. Then K is an n-absorbing ideal of $A \bowtie^f J$ if and only if K is a strongly n-absorbing ideal of $A \bowtie^f J$.*

Proof.

1. Since P^n is P -primary, then by [5, Theorem 6.6], P^n is a strongly n-absorbing. Therefore, by assertion (2) of Theorem 5.2.1, it follows that $P^n \bowtie^f J$ is a strongly n-absorbing ideal of $A \bowtie^f J$.
2. Since $\mathrm{Nil}(A)$ and P are divided prime ideals of A such that $\mathrm{Nil}(A) \subset P$, then by [5, Theorem 3.3], P^n is a P-primary ideal of A. Hence, by assertion (1) above, we have the desired result.
3. From [12, Proposition 17], for every positive integer n, P^n is a P-primary ideal, and so by [5, Theorem 6.6], P^n is a strongly n-absorbing ideal of A. Therefore, by application to the assertion (2) of Theorem 5.2.1, it follows that $P^n \bowtie^f J$ is a strongly n-absorbing ideal of $A \bowtie^f J$.
4. Assume that K is an absorbing ideal of $A \bowtie^f J$. Then by assertion (1) of Theorem 5.2.1, I is an $n-$absorbing ideal of A. Since A is a Prüfer domain, then by [5, Corollary 6.9], I is a strongly n-absorbing ideal of A. Hence, by using assertion (2) of Theorem 5.2.1, we deduce that K is a strongly n-absorbing ideal of $A \bowtie^f J$. The converse is trivial.

The next corollary gives a characterization about when the ideal $I+M$ of $D+M$ is an n-absorbing (resp., a strongly n-absorbing) ideal on the $D+M$ ring.

Corollary 5.2.6 *Let M be a maximal ideal of an integral domain T and D a subring of T such that $D\cap M=\{0\}$. Let I be an ideal of D. Then $I+M$ is an n-absorbing (resp., a strongly n-absorbing) ideal of $D+M$ if and only if I is an n-absorbing (resp., a strongly n-absorbing) ideal of D.*

Proof. Let $\sigma : D \hookrightarrow T$ be the natural embedding. Since $D\bowtie^{\sigma} M \cong D+M$ (as $D\cap M=\{0\}$) and $I\bowtie^{\sigma} M \cong I+M$, then $I+M$ is an n-absorbing ideal of $D+M$ if and only if $I\bowtie^{\sigma} M$ is an n-absorbing ideal of $D\bowtie^{\sigma} M$. By assertion (1) of Theorem 5.2.1, it follows that $I+M$ is an n-absorbing ideal of $D+M$ if and only if I is an n-absorbing ideal of D.
Similar proof for strongly n-absorbing by changing "assertion (1)" with "assertion (2)" of Theorem 5.2.1.

Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Consider I (resp., H) be an ideal of A (resp., $f(A)+J$) such that $f(I)J \subseteq H \subseteq J$. Observe that $I\bowtie^{f} H := \{(i, f(i)+h)/i \in I, h \in H\}$ is an ideal of $A\bowtie^{f} J$. Notice that if I is a proper ideal of A, then the extension I^e of I to $A\bowtie^{f} J$ coincides with $I\bowtie^{f} (f(I)B)J$. However, if $f(I)J \subset H \subset J$ and I is an n-absorbing ideal of A, $I\bowtie^{f} H$ is not an n-absorbing ideal of $A\bowtie^{f} J$ in general as shown by the next example which exhibits an ideal I that is an n-absorbing ideal ideal of A such that $f(I)J \subseteq H \subset J$ but $I\bowtie^{f} H$ is not an n-absorbing ideal of $A\bowtie^{f} J$.

Example 5.2.7 *Let $A := \mathbb{Z}$ be the ring of integers, $B := \mathbb{Q}[[X]]$ be the ring of formal power series over $\mathbb{Q}$ in an indeterminate X, $f : \mathbb{Z} \hookrightarrow \mathbb{Q}[[X]]$ be the natural embedding and $J := X\mathbb{Q}[[X]]$ is the maximal ideal of B. Let $H = \{XP(X); P \in f(A)+J = \mathbb{Z}+X\mathbb{Q}[[X]]\}$. Obviously, H is an ideal of $f(A)+J = \mathbb{Z}+X\mathbb{Q}[[X]]$. Observe that $I := 0$ is a prime ideal of A and so is an n-absorbing ideal of A for all positive integer n. Since $f(I)J = 0 \subset H \subset J$, then $0\bowtie^{f} H$ is an ideal of $\mathbb{Z}\bowtie^{f} X\mathbb{Q}[[X]]$. We claim that $0\bowtie^{f} H$ is not an n-absorbing ideal of $\mathbb{Z}\bowtie^{f} X\mathbb{Q}[[X]]$ for all positive integer n. Indeed, $0\bowtie^{f} H = \{(0, XP(X)), P(X) \in \mathbb{Z}+X\mathbb{Q}[[X]]\}$ and we have $(2,2)^n(0,\frac{X}{2^n}) = (0,X) \in 0\bowtie^{f} H$. However, $(2,2)^{n-1}(0,\frac{X}{2^n}) = (0,\frac{X}{2}) \notin 0\bowtie^{f} H$ and $(2,2)^n \notin 0\bowtie^{f} H$.*

Now, we investigate about when the ideal $I\bowtie^{f} H$ is an n-absorbing ideal of $A\bowtie^{f} J$.

Theorem 5.2.8 *Let (A,m) be a local ring, $f : A \to B$ be a ring homomorphism, J be an ideal of B such that $f^{-1}(Q) \neq m$ for every $Q \in Spec(B)\setminus V(J)$. Let n be a positive integer and H be an ideal of $f(A)+J$ such that $0 = f(m)J \subseteq H \subseteq J$ and $J^n \subseteq H$. Consider I an ideal of A which radical is the maximal ideal m. Then the following statements holds:*

1. *I is an n-absorbing ideal of A if and only if $I\bowtie^{f} H$ is an n-absorbing ideal of $A\bowtie^{f} J$.*
2. *$I\bowtie^{f} H$ is an n-absorbing ideal of $A\bowtie^{f} J$ if and only if $I\bowtie^{f} H$ is a strongly n-absorbing ideal of $A\bowtie^{f} J$.*

The proof of Theorem 5.2.8 requires the following lemma. Recall that from [35, Proposition 2.6], $Spec(A\bowtie^{f} J) = \{P'^{f} = P\bowtie^{f} J | P \in Spec(A)\} \cup \{\overline{Q}^{f} | Q \in \overline{Spec(B,J)}\}$ and $Max(A\bowtie^{f} J) = \{m'^{f} = m\bowtie^{f} J | m \in Max(A)\} \cup \{\overline{Q}^{f} | Q \in \overline{Max(B,J)}\}$.

Lemma 5.2.9 *Let (A,m) be a local ring, $f : A \to B$ be a ring homomorphism and J be an ideal of B such that $f^{-1}(Q) \neq m$ for every $Q \in Spec(B) \setminus V(J)$. Consider I (resp., H) an ideal of A (resp., $f(A)+J$) such that $f(I)J \subseteq H \subseteq J$. Then the following statements hold:*

1. *If $Rad(I) = m$, then $Rad(I \bowtie^f H) = m'^f = m \bowtie^f J$.*
2. *I is m-primary ideal of A if and only if $I \bowtie^f H$ is a $m \bowtie^f J$-primary ideal of $A \bowtie^f J$.*

Proof.

1. Consider the ideal $I \bowtie^f H$ of $A \bowtie^f J$. Then two cases are then possible:

   Case 1: $I \bowtie^f H \subseteq P \bowtie^f J$ for some prime ideal P of A.
   Then $I \subseteq P$. Using the fact that $\mathrm{Rad}(I)$ is the intersection of all prime ideals containing I and is also the maximal ideal of A, then it follows that there exists exactly one prime ideal $P = \mathrm{Rad}(I)$. Therefore $P = m$. Thus, $\mathrm{Rad}(I \bowtie^f H) = m'^f = m \bowtie^f J$.
   Case 2: $I \bowtie^f H \subseteq \overline{Q}^f$ where $Q \in \mathrm{Spec}(B) \setminus V(J)$. We deduce that $H \subseteq Q$. Since $f(I)J \subseteq H$, then $f(I) \subseteq Q$ and so $I \subseteq f^{-1}(Q)$. Therefore, by assumption $f^{-1}(Q) = m$ which is a contradiction.

   Hence, it follows that the unique prime ideal of $A \bowtie^f J$ containing $I \bowtie^f H$ is $m'^f = m \bowtie^f J$. Finally, $\mathrm{Rad}(I \bowtie^f H) = m \bowtie^f J$, as desired.

2. Assume that I is m-primary ideal of A. Then $\mathrm{Rad}(I) = m$. And so by assertion (1) above, $\mathrm{Rad}(I \bowtie^f H) = m'^f = m \bowtie^f J$. Therefore, $I \bowtie^f H$ is m'^f-primary ideal of $A \bowtie^f J$. Conversely, suppose that $I \bowtie^f H$ is $m \bowtie^f J$-primary ideal of $A \bowtie^f J$. Since $\mathrm{Rad}(I) = \bigcap_{P \in Spec(A)} P$ and $\mathrm{Rad}(I \bowtie^f H) = \bigcap_{P \in Spec(A)} P \bowtie^f J = m \bowtie^f J$, then it follows that $\mathrm{Rad}(I) = m$, making I, an m-primary ideal of A, as desired.

Proof of Theorem 5.2.8.

1. Since I is m-primary ideal of A which is n-absorbing, we deduce that $(\mathrm{Rad}(I))^n = m^n \subseteq I$ [5, Theorem 3.1]. Using the fact that $f(m)J = 0$, it follows that $(\mathrm{Rad}(I \bowtie^f H))^n = (m \bowtie^f J)^n = m^n \bowtie^f J^n \subseteq I \bowtie^f H$. Thus, by [5, Theorem3.1], $I \bowtie^f H$ is an n-absorbing ideal of $A \bowtie^f J$. Conversely, suppose that $I \bowtie^f H$ is an n-absorbing ideal of $A \bowtie^f J$. Since $\mathrm{Rad}(I) = m$, then by assertion (1) of Lemma 5.2.9, it follows that $(\mathrm{Rad}(I \bowtie^f H))^n = (m \bowtie^f J)^n \subseteq I \bowtie^f H$ [78]. In particular, $m^n \subseteq I$. Using the fact that I is m-primary ideal of A and $m^n \subseteq I$, then by [5, Theorem 3.1], it follows that I is an n-absorbing ideal of A.
2. Assume that $I \bowtie^f H$ is an n-absorbing ideal of $A \bowtie^f J$. Since I is $m-$primary(as $\mathrm{Rad}(I) = m$), then by assertion (2) of Lemma 5.2.9, $I \bowtie^f H$ is $m \bowtie^f J$-primary. From [78], we deduce that $(\mathrm{Rad}(I \bowtie^f H))^n = (m \bowtie^f J)^n \subseteq I \bowtie^f H$. Now, the claim follows from [5, Theorem 6.6]. The converse is trivial.

The next corollary is a consequence of Theorem 5.2.8.

Corollary 5.2.10 *Let (A,m) be a local ring, $f : A \to B$ be a ring homomorphism and J be an ideal of B such that $f^{-1}(Q) \neq m$ for every $Q \in Spec(B) \setminus V(J)$. Let n be a positive integer and I be an*

ideal of A such that $Rad(I) = m$, $J^{n-1} \subseteq f(I)$ and $f(m) \subseteq f(I)B$. Consider the extension ideal $I^e := I(A \bowtie^f J)$ of I to $A \bowtie^f J$. Then the following assertions hold:

1. *I is an n-absorbing ideal of A if and only if the extension I^e is an n-absorbing ideal of $A \bowtie^f J$.*
2. *I^e is an n-absorbing ideal of $A \bowtie^f J$ if and only if I^e is a strongly n-absorbing ideal of $A \bowtie^f J$.*

Proof. Observe that from [38, Proposition 3.1], we have $I^e = I \bowtie^f (f(I)B)J$. Set $H := (f(I)B)J$, which is clearly an ideal of $f(A)+J$. So, $I^e = I \bowtie^f H$.

1. Notice that $J^n \subseteq H$ since $J^{n-1} \subseteq f(I)B$. Hence, by application to the assertion (1) of Theorem 5.2.8, the conclusion is straightforward.
2. By Lemma 5.2.9, $I \bowtie^f H$ is $m \bowtie^f J$-primary and $(\mathrm{Rad}(I \bowtie^f H))^n = (m^n \bowtie^f f(I^{n-1})J + f(I^{n-2})J^2 + \cdots + J^n)$. Since $m^n \subseteq I$ and $f(I^k)J^{n-k} \subseteq f(m^k)J^{n-k} \subseteq H$, then $(\mathrm{Rad}(I \bowtie^f H))^n \subseteq I^e$. Now the claim follows from [5, Theorem6.6]. The converse is straightforward.

Now, we show how one may use Theorem 5.2.9 to construct original examples of n-absorbing ideals of the form $I \bowtie^f H$.

Example 5.2.11 *Let (A,m) be any valuation domain which is not a field (for instance take $(A,m) := (\mathbb{Z}_p, p\mathbb{Z}_p)$), K be the quotient field of A, $B := K[[X]]$ be the formal power series where X is an indeterminate, $f : A \hookrightarrow B$ is the injective ring homomorphism and $J := XK[[X]]$ is the maximal ideal of B. Observe that the spectrum of B is $Spec(B) = \{0, XK[[X]]\}$ and we have $f^{-1}(\{0\}) = ker(f) = \{0\} \neq m$. Consider a positive integer n and let $H := X^nK[[X]]$ be an ideal of $f(A)+J = A + XK[[X]]$ with $J^n \subseteq H$. By assertion (1) of Theorem 5.2.8, it follows that $0 \bowtie^f H$ is an n-absorbing ideal of $A \bowtie^f J$.*

The next theorem characterizes the form of n-absorbing and strongly n-absorbing ideals of $A \bowtie^f J$ in some particular cases.

Theorem 5.2.12 *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Let n be a positive integer, I be an ideal of A and consider an ideal K of $f(A)+J$. Then the following assertions hold:*

1. *The n-absorbing ideals of $A \bowtie^f J$ containing $\{0\} \times J$ are of the form $I \bowtie^f J$ where I is an n-absorbing ideal of A.*
2. *The strongly n-absorbing ideals of $A \bowtie^f J$ containing $\{0\} \times J$ are of the form $I \bowtie^f J$ where I is a strongly n-absorbing ideal of A.*
3. *The n-absorbing ideals of $A \bowtie^f J$ containing $f^{-1}(J) \times \{0\}$ are of the form $\overline{K}^f$ where K is an n-absorbing ideal of $f(A)+J$.*
4. *The strongly n-absorbing ideals of $A \bowtie^f J$ containing $f^{-1}(J) \times \{0\}$ are of the form $\overline{K}^f$ where K is a strongly n-absorbing ideal of $f(A)+J$.*

Proof. Let L be an ideal of $A \bowtie^f J$. Consider the natural projections $p_A : A \bowtie^f J \to A$ defined by $p_A((a, f(a)+j)) = a$ for every $(a, f(a)+j) \in A \bowtie^f J$ and $p_B : A \bowtie^f J \to B$ defined by $p_B((a, f(a)+j)) = f(a)+j$ for every $(a, f(a)+j) \in A \bowtie^f J$.

1. If $L = I \bowtie^f J$ with I an n-absorbing ideal of A, then by assertion (1) of Theorem 5.2.1, it follows that L is an n-absorbing ideal of $A \bowtie^f J$. Obviously, L contains $\{0\} \times J$. Conversely, assume that L is an n-absorbing ideal of $A \bowtie^f J$ containing $\{0\} \times J$. Clearly $ker(p_A) = \{0\} \times J \subseteq L$. Set $I := p_A(L)$. From [5, Theorem 4.2], I is an n-absorbing ideal of A. The fact that L containing $\{0\} \times J$, it follows that L has the form $L = I \bowtie^f J$.
2. Same proof as above, by replacing "assertion (1)" with "assertion (2) of Theorem 5.2.1" and "[5, Theorem 4.2]" with "assertion (2) of Lemma 5.2.2".
3. Notice first that if $L \supseteq \{0\} \times J$, then by assertion (1) above $L = I \bowtie^f J$ where I is an n-absorbing ideal of A. Therefore, we may assume that $L \not\supseteq \{0\} \times J$ and $f^{-1}(J) \times \{0\} \subseteq L$. Clearly $ker(p_B) = f^{-1}(J) \times \{0\} \subseteq L$. Set $K := p_B(L)$. From [5, Theorem 4.2], K is an n-absorbing ideal of $f(A) + J$. One can easily check that $L = \overline{K}^f$. Conversely, let K be an n-absorbing ideal of $f(A) + J$. We claim that $L = \overline{K}^f$ is an n-absorbing ideal of $A \bowtie^f J$ containing $f^{-1}(J) \times \{0\}$. Indeed, let $(a_1, f(a_1) + b_1) \cdots (a_{n+1}, f(a_{n+1}) + b_{n+1}) \in \overline{K}^f$ for every $(a_1, f(a_1) + b_1), \cdots, (a_{n+1}, f(a_{n+1}) + b_{n+1}) \in A \bowtie^f J$. Then $(f(a_1) + b_1) \cdots (f(a_{n+1}) + b_{n+1}) \in K$. Since K is an n-absorbing ideal of $f(A) + J$, then without loss of generality, we may assume that $(f(a_1) + b_1) \cdots (f(a_n) + b_n) \in K$. Therefore, $(a_1, f(a_1) + b_1) \cdots (a_n, f(a_n) + b_n) \in \overline{K}^f$. Hence, $\overline{K}^f$ is an n-absorbing ideal of $A \bowtie^f J$. On the other hand, it is easy to see that $\overline{K}^f \supseteq f^{-1}(J) \times \{0\}$.
4. Assume that $L \supseteq f^{-1}(J) \times \{0\}$ is a strongly n-absorbing ideal of $A \bowtie^f J$. Then by assertion (2) of Lemma 5.2.2, $K := p_B(L)$ is a strongly n-absorbing ideal of $f(A) + J$. One can easily check that $L = \overline{K}^f$. Conversely, assume that K is a strongly n-absorbing ideal of $f(A) + J$. Then by assertion (1) of Lemma 5.2.2, it follows that $p_B^{-1}(K) = \overline{K}^f$ is a strongly n-absorbing ideal of $A \bowtie^f J$, as desired.

In the next theorem, we give a characterization about when every proper ideal of amalgamation is an n-absorbing ideal in the particular case the ring A is local.

Theorem 5.2.13 *Let (A, m) be a local ring, $f : A \to B$ be a ring homomorphism and J be an ideal of B. Let $n \in \mathbb{N}^*$ such that $m^n = 0$. Then the following assertions hold:*

1. *$I \bowtie^f J$ is an n-absorbing ideal of $A \bowtie^f J$ for every proper ideal I of A.*
2. *Let $2 \leq q \leq n$ be a positive integer such that $J^q = 0$ and suppose that $f(m)J = 0$. Then every proper ideal is an n-absorbing ideal of $A \bowtie^f J$.*

The proof of the previous theorem requires the following lemma.

Lemma 5.2.14 *Let (A, m) be local ring such that $m^n = 0$, where n is a positive integer. Then every proper ideal of A is an n-absorbing ideal of A.*

Proof. Let I be a proper ideal of A. Let $a_1, \cdots, a_{n+1} \in A$ such that $\prod_{i=1}^{n+1} a_i \in I$. Two cases are then possible:
Case 1 : There exists $j \in \{1, ..., n+1\}$ such that $a_j \notin m$. Then a_j is invertible and so it follows that $\prod_{i=1, i \neq j}^{n+1} a_i \in I$, as desired.

Case 2 : For every $j \in \{1,...,n+1\}$, $a_j \in m$. So, for every $j \in \{1,...,n+1\}$, a_j is not invertible in A. Therefore, $\prod_{i=1}^{n} a_i = 0 \in I$. Thus in all cases, I is an n-absorbing ideal of A.

Proof of Theorem 5.2.13.

1. Let I be a proper ideal of A. By Lemma 5.2.14, I is an n-absorbing ideal of A. Therefore, by application to the assertion (1) of Theorem 5.2.1, it follows that $I \bowtie^f J$ is an n-absorbing ideal of $A \bowtie^f J$.
2. Since $J^q = 0$ for every positive integer $2 \leq q \leq n$, then $J \subseteq \mathrm{Jac}(B)$. On the other hand, recall that $\mathrm{Max}(A \bowtie^f J) = \{m \bowtie^f J, m \in \mathrm{Max}(A)\} \cup \{\overline{Q}^f\}$, where $Q \in \mathrm{Max}(B)$ not containing $V(J)$ and $\overline{Q}^f := \{(a, f(a)+j)/a \in A, j \in J, f(a)+j \in Q\}$. Since $J \subseteq Q, \forall Q \in \mathrm{Max}(B)$, then $\{\overline{Q}^f\}$ is empty set. Hence $\mathrm{Max}(A \bowtie^f J) = \{m \bowtie^f J, m \in \mathrm{Max}(A)\} = m \bowtie^f J$ since A is local with maximal ideal m. Consequently, $A \bowtie^f J$ is local with maximal ideal $m \bowtie^f J$. One can easily check that $(m \bowtie^f J)^n = m^n \bowtie^f J^n = 0$ as $m^n = 0$ and $f(m)J = 0 = J^q$. Hence, by Lemma 5.2.14, it follows that every proper ideal of $A \bowtie^f J$ is an n-absorbing ideal, as desired.

The next example illustrates Theorem 5.2.13.

Example 5.2.15 *Let (A,m) be a local ring with $m^2 = 0$ and let E be an A-module such that $mE = 0$ and $B := A \propto E$ be the trivial ring extension of A by E and $J = 0 \propto E$ be an ideal of B. Consider the injective ring homomorphism*

$$f : \begin{cases} A \hookrightarrow B \\ a \mapsto (a,0) \end{cases}$$

Observe that $f(m)J = (m \propto 0)(0 \propto E) = 0$ and $J^2 = 0$. Then by assertion (2) of Theorem 5.2.13, every proper ideal of $A \bowtie^f J$ is a 2-absorbing ideal. And so every proper ideal of $A \bowtie^f J$ is an n-absorbing ideal for every positive integer $n \geq 2$.

The following example shows that the assumptions $m^n = 0$, $f(m)J = 0$ and $J^q = 0$ are not necessary conditions in the assertion (2) of Theorem 5.2.13.

Example 5.2.16 *$(A,m) := (\mathbb{Z}_p, p\mathbb{Z})$ be a local ring, $f : \mathbb{Z}_p \hookrightarrow \mathbb{Q}[[X]]$ be the naturel embedding of $\mathbb{Z}_p$ into $\mathbb{Q}[[X]]$ and $R := A \bowtie^f J$, where $J := X\mathbb{Q}[[X]]$ is the maximal ideal of B. Then:*

1. *$J^2 \neq 0$.*
2. *$m^n \neq 0$ and $f(m)J \neq 0$, $\forall n \in \mathbb{N}$.*
3. *R is a Prüfer domain (see [8, Example 4.3]) and so every proper ideal of R is a k-absorbing for some positive integer k.*

5.3 Conjecture one in amalgamated algebra

Let $n \geq 1$ be an integer and I be a proper ideal of a ring A. Anderson and Badawi in [5] (also see [25]) proposed the following conjecture:

Conjecture one: I is an n-absorbing ideal of A if and only if I is a strongly n-absorbing ideal of A.

This previous Conjecture is true if A is a Prüfer domain [5, Corollary 6.9] and the following example illustrates this fact by exhibiting an amalgamation which is a Prüfer domain.

Example 5.3.1 *Let $A := \mathbb{Z}$, $B := \mathbb{Q}[X]$, $f : A \to B$ the natural embedding of A into B and $J := X\mathbb{Q}[X]$. Observe that $J^2 \neq 0$ and $A \bowtie^f J \simeq \mathbb{Z} + X\mathbb{Q}[X]$ which is well known that it is a Prüfer domain. So, Conjecture one holds in $A \bowtie^f J$.*

The next theorem studies Conjecture one in some particular ideals of amalgamation $A \bowtie^f J$.

Theorem 5.3.2 *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then the following assertions hold:*

1. *If Conjecture one holds in $A \bowtie^f J$, then Conjecture one holds in A.*
2. *Consider an ideal I of A. Set $L := I \bowtie^f J$. If Conjecture one holds in A, then L is an n-absorbing ideal of $A \bowtie^f J$ if and only if L is a strongly n-absorbing of $A \bowtie^f J$.*
3. *If Conjecture one holds in $f(A) + J$. Then for every ideal K of $A \bowtie^f J$ containing $f^{-1}(J) \times \{0\}$, we have : K is an n-absorbing ideal of $A \bowtie^f J$ if and only if K is a strongly n-absorbing ideal of $A \bowtie^f J$.*

Proof.

The proof of assertions (1) and (2) follows readily from (1) and (2) of Theorem 5.2.1.

(3) Assume that K is an n-absorbing ideal of $A \bowtie^f J$ which contains $f^{-1}(J) \times \{0\}$. Since $\frac{A \bowtie^f J}{f^{-1}(J) \times \{0\}} \cong f(A) + J$, then $\frac{K}{f^{-1}(J) \times \{0\}}$ is an n-absorbing ideal of $\frac{A \bowtie^f J}{f^{-1}(J) \times \{0\}}$. Using the fact that Conjecture one holds in $f(A) + J$, we conclude that $\frac{K}{f^{-1}(J) \times \{0\}}$ is a strongly n-absorbing of $\frac{A \bowtie^f J}{f^{-1}(J) \times \{0\}}$. Hence, by assertion (2) of Lemma 5.2.3, K is a strongly n-absorbing ideal of $A \bowtie^f J$. The converse is straightforward.

The following corollary is an immediate consequence of Theorem 5.3.2.

Corollary 5.3.3 *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Let n be a positive integer. Then the following assertions hold:*

1. *Assume that the conjecture one holds in A. Let L be an ideal of $A \bowtie^f J$ containing $\{0\} \times J$. Then L is an n-absorbing ideal of $A \bowtie^f J$ if and only if L is a strongly n-absorbing ideal of $A \bowtie^f J$.*

2. *Assume that the conjecture one holds in $f(A)+J$. If K is an ideal of $A \bowtie^f J$ containing $f^{-1}(J) \times \{0\}$, then K is an n-absorbing ideal of $A \bowtie^f J$ if and only if K is a strongly n-absorbing ideal of $A \bowtie^f J$.*

Recall that if a ring R is Noetherian and I an ideal of R, then there exists some positive integer n such that I is a strongly n-absorbing ideal of R [5, Theorem 6.8]. The next result gives a characterization about when every ideal of amalgamation $A \bowtie^f J$ is a strongly n-absorbing ideal in the special case of Noetherian setting.

Theorem 5.3.4 *Let $f : A \to, B$ be a ring homomorphism and J be an ideal of B. Assume that A and $f(A)+J$ are Noetherian. Then every ideal of $A \bowtie^f J$ is a strongly n-absorbing ideal of $A \bowtie^f J$ for some positive integer n.*

Proof. A combination of [5, Theorem 6.8] with [35, Proposition 5.6] (on the transfer of the Noetherian property) leads to the conclusion.

5.4 Conjecture three in amalgamated algebra

Let $n \geq 1$ be an integer and I be a proper ideal of a ring A. D.F. Anderson and A. Badawi in [5] (also see [25]) proposed the following conjecture:

Conjecture three: If I is an n-absorbing ideal of A, then $I[X]$ is an n-absorbing ideal of $A[X]$.

For instance, if A is an arithmetical ring and I is an n-absorbing ideal of A, then $I[X]$ is an n-absorbing ideal of $A[X]$ [68, Corollary 2.11]. Our first result of this section develops the transfer of n-absorbing ideal property to $(A \bowtie^f J)[X]$.

Theorem 5.4.1 *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Consider I an ideal of A. Then $I[X]$ is an n-absorbing ideal of $A[X]$ if and only if $(I \bowtie^f J)[X]$ is an n-absorbing ideal of $(A \bowtie^f J)[X]$.*

Proof. Notice that $\left(\frac{I \bowtie^f J}{0 \times J}\right)[X] \simeq \frac{(I \bowtie^f J)[X]}{(0 \bowtie^f J)[X]}$. On the other hand $\frac{I \bowtie^f J}{0 \times J} \simeq I$ from [35, Proposition 5.1(2)]. So, we conclude that $\left(\frac{I \bowtie^f J}{0 \times J}\right)[X] \simeq \frac{(I \bowtie^f J)[X]}{(0 \times J)[X]} \simeq I[X]$. Hence, by application to [5, Corollary 4.3 (b)], it follows that $I[X]$ is an n-absorbing ideal of $A[X]$ if and only if $\frac{(I \bowtie^f J)[X]}{(0 \bowtie^f J)[X]}$ is an n-absorbing ideal of $\frac{A \bowtie^f J[X]}{(0 \bowtie^f J)[X]}$ if and only if $(I \bowtie^f J)[X]$ is an n-absorbing ideal of $(A \bowtie^f J)[X]$.

The next example illustrates Theorem 5.4.1 by showing how to construct an n-absorbing ideal of the form $(I \bowtie^f J)[X]$.

Example 5.4.2 *Let A be an arithmetical ring (for instance take $A := \mathbb{Z}$), $B := \mathbb{Z}/9\mathbb{Z}$, $f : A \to B$ is the canonical surjection and $J := 3\mathbb{Z}/9/\mathbb{Z}$ is the maximal ideal of B. Consider the ideal $I := 5\mathbb{Z}$ which is an n-absorbing ideal of A. From [68, Corollary 2.11], it follows that $I[X]$ is an n-absorbing ideal of $A[X]$. By Theorem 5.4.1, we conclude that $(I \bowtie^f J)[X]$ is an n-absorbing ideal of $(A \bowtie^f J)[X]$.*

Proposition 5.4.3 *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. If Conjecture three holds in $A \bowtie^f J$, then Conjecture three holds in A.*

Proof. Let I be an n-absorbing ideal of A, where n is a positive integer. Then by assertion (1) of Theorem 5.2.1, $I \bowtie^f J$ is an n-absorbing ideal of $A \bowtie^f J$. Since Conjecture three holds in $A \bowtie^f J$, then $(I \bowtie^f J)[X]$ is an n-absorbing ideal of $(A \bowtie^f J)[X]$. Hence, by Theorem 5.4.1, $I[X]$ is an n-absorbing ideal of $A[X]$.

The next example illustrates that Conjecture three holds in the case the amalgamation $A \bowtie^f J$ collapses with the special case of trivial extensions of integral domains by vector spaces over their quotient fields.

Example 5.4.4 *Let A be an integral domain with quotient field K and E be a K-vector space. Let $B := A \propto E$ be the trivial ring extension of A by E and $R := A \bowtie^f J$, where $f : A \hookrightarrow B$ is the injective ring homomorphism defined by $f(a) = (a,0)$ for every $a \in A$ and $J = 0 \propto E$ is an ideal of B. Then Conjecture three holds in R if and only if Conjecture three holds in A.*

Proof. First notice that from [35, Proposition 5.1(3)], $R \simeq f(A) + J = A \propto 0 + 0 \propto E = A \propto E = B$ since $f^{-1}(J) = 0$. Hence, by [15, Theorem 4.2], the conclusion is straightforward.

Theorem 5.4.5 *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Set $R := A \bowtie^f J$. Then the following statements hold:*

1. *Assume that Conjecture three holds in A. Then for every n-absorbing ideal K of R containing $\{0\} \times J$, $K[X]$ is an n-absorbing ideal of $R[X]$.*
2. *Assume that Conjecture three holds in $f(A) + J$. Let K be an n-absorbing ideal of $A \bowtie^f J$ containing $f^{-1}\{J\} \times \{0\}$. Then $K[X]$ is an n-absorbing ideal of $R[X]$.*

Proof.

1. Let K be an n-absorbing ideal of R containing $\{0\} \times J$, then there exists an n-absorbing ideal I of A such that $K = I \bowtie^f J$. Since Conjecture three holds in A, then $I[X]$ is an n-absorbing ideal of $A[X]$. Now the result follows from Theorem 5.4.1.
2. Let K be an n-absorbing ideal of R which contains $f^{-1}(J) \times \{0\}$. Then from [5, Corollary 4.3], $\frac{K}{f^{-1}(J)\times\{0\}}$ is an n-absorbing ideal of $\frac{R}{f^{-1}(J)\times\{0\}}$. We conclude that $\frac{K[X]}{(f^{-1}(J)\times\{0\})[X]}$ is an n-absorbing ideal of $\left(\frac{A\bowtie^f J}{f^{-1}(J)\times\{0\}}\right)[X]$ since $\left(\frac{A\bowtie^f J}{f^{-1}(J)\times\{0\}}\right)[X] \simeq (f(A)+J)[X]$ and Conjecture three holds in $f(A)+J$. Now the result follows from [5, Corollary 4.3].

CHAPTER 6
ON (m,n)-CLOSED IDEALS IN AMALGAMATED ALGEBRAS

Abstract*

Let R be a commutative ring with $1 \neq 0$ and let m and n be integers with $1 \leq n < m$. A proper ideal I of R is called an (m,n)-closed ideal of R if whenever $a^m \in I$ for some $a \in R$ implies $a^n \in I$. Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. This paper investigates the concept of (m,n)-closed ideals in the amalgamation of A with B along J with respect f denoted by $A \bowtie^f J$. Namely, section 2 investigates this notion to some extensions of ideals of A to $A \bowtie^f J$. Section 3 features the main result, which examines when each proper ideal of $A \bowtie^f J$ is an (m,n)-closed ideal. This allows us to give necessary and sufficient conditions for the amalgamation to inherit the radical ideal property with applications on the transfer of Von Neumann regular, π-regular and semisimple properties.

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6.1 Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. The notion of (m,n)-closed ideal was introduced and defined by Anderson and Badawi in [6], as follow: Let m and n be positive

*Submitted for publication, (in collaboration with N. Mahdou and M.A. Moutui)

integers with $1 \leq n < m$. a proper ideal I of R is called an (m,n)-*closed ideal* of R if whenever $a^m \in I$ for some $a \in R$ implies $a^n \in I$. They defined I to be a semi-n-absorbing ideal of R if $x^{n+1} \in I$ for $x \in R$ implies $x^n \in I$. Thus, I is a semi-n-absorbing if and only if is a $(n+1,n)$-closed ideal, and I is a radical ideal if and only if I is a $(2,1)$-closed ideal. They gave the basic properties of semi-n-absorbing ideals and (m,n)- closed ideals. They also determined when every proper ideal of R is (m,n)-closed for integers $1 \leq n < m$. Further more they gave several examples to illustrate their results. Over the past several years, there has been considerable attention in the literature to n-absorbing ideals of commutative rings and their generalizations, for example see ([5], [6], [53]) and [68]). We recall from [9] that a proper ideal I of R is called a *2-absorbing ideal* of R if $a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. A more general concept than 2-absorbing ideals is the concept of n-absorbing ideals. Let $n \geq 1$ be a positive integer. A proper ideal I of R is called an *n-absorbing ideal* of R as in [5] if $a_1, a_2, ..., a_{n+1} \in R$ and $a_1, a_2 \cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I. The concept of n-absorbing ideals is a generalization of the concept of prime ideals (note that a prime ideal of R is a 1-absorbing ideal of R). Notice that an n-absorbing ideal is an (m,n)-closed for every positive integer m. In [15], the authors studied the notions of n-absorbing ideals, strongly n-absorbing ideals and (m,n)-closed ideals in the trivial ring extension. Let A be a commutative ring and E be an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a,e)(a',e') = (aa', ae' + a'e)$. Trivial ring extensions have been studied extensively. Considerable work, part of is summarized in Glaz's book [50] and Huckaba's book [54], has been concerned with trivial ring extension these extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [4, 50, 54, 37]. We recall that if I is a proper ideal of A, then $I \propto E$ is an ideal of $A \propto E$. And if F is a submodule of E such that $IE \subset F$, then $I \propto F$ is an ideal of $A \propto E$. The ideals of $A \propto E$ are not all of the form $I \propto E$ or $I \propto F$, but if A is an integral domain, and E an A-module divisible, the ideals of $A \propto E$ are of the form $I \propto E$ or $0 \propto F$, where I is an ideal of A and F a submodule of E. Let (A,B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be an ideal of B. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f, introduced and studied by D'Anna, Finocchiaro and Fontana in [35, 37]. In particular, they have studied amalgamations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [33, 34]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealizations) (cf. [75, page 2]). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$ and the $D + M$ constructions) can be studied as particular cases of the amalgamation ([35, Examples 2.5 and 2.6]) and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [24]) are strictly related to it ([35, Example 2.7 and Remark 2.8]). In [35], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks

that can be expressed as an amalgamation. Moreover, in [37], they pursued the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.

In this paper, we study the concept (m,n)-closed ideals in the amalgamation of A with B along J with respect f denoted by $A \bowtie^f J$. For any ring R, we denote by $Nil(R)$ (resp., $dim(R)$), the set of nilpotent elements of R (resp., the Krull dimension of R).

6.2 On some (m,n)-closed ideals of amalgamation $A \bowtie^f J$

To avoid unnecessary repetition, let us fix notation for the rest of the paper. Let (A,B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be an ideal of B. All along this paper, $A \bowtie^f J$ will denote the amalgamation of A and B along J with respect to f. Let I be an ideal of A and K be an ideal of $f(A)+J$. Notice that $I \bowtie^f J := \{(i, f(i)+j)/i \in I, j \in J\}$ and $\overline{K}^f := \{(a, f(a)+j)/a \in A, j \in J, f(a)+j \in K\}$ are ideals of $A \bowtie^f J$. Our first result gives a characterization when the ideals $I \bowtie^f J$ and $\overline{K}^f$ are (m,n)-closed ideals of $A \bowtie^f J$, for all positive integers m and n, with $1 \leq n < m$.

Proposition 6.2.1 *Under the above notations, the following statements hold:*

1. *$I \bowtie^f J$ is an (m,n)-closed ideal of $A \bowtie^f J$ if and only if I is an (m,n)-closed ideal of A.*
2. *$\overline{K}^f$ is an (m,n)-closed ideal of $A \bowtie^f J$ if and only if K is an (m,n)-closed ideal of $f(A)+J$.*

Proof.

1. Assume that $I \bowtie^f J$ is an (m,n)-closed ideal of $A \bowtie^f J$ for m and n two positive integers with $1 \leq n < m$. Let $a^m \in I$, with $a \in A$. Then $(a, f(a))^m \in I \bowtie^f J$. Since $I \bowtie^f J$ is (m,n)-closed ideal of $A \bowtie^f J$, it follows that $(a, f(a))^n \in I \bowtie^f J$, making $a^n \in I$. Hence, I is an (m,n)-closed ideal of A. Conversely, assume that I is an (m,n)-closed ideal of A. Let $x^m = (a, f(a)+j)^m \in I \bowtie^f J$ with $x = (a, f(a)+j) \in A \bowtie^f J$. Clearly, $a^m \in I$. Since I is an (m,n)-closed ideal of A, then $a^n \in I$. One can easily check that $x^n = (a^n, (f(a)+j)^n) \in I \bowtie^f J$, as desired.
2. Suppose that $\overline{K}^f$ is an (m,n)-closed ideal of $A \bowtie^f J$. We claim that K is an (m,n)-closed ideal of $f(A)+J$. Indeed, let $(f(a)+j)^m \in K$ with $(f(a)+j) \in f(A)+J$. Then $(a^m, (f(a)+j)^m) \in \overline{K}^f$. Since $\overline{K}^f$ is an (m,n)-closed ideal, then $(a, f(a)+j)^n \in \overline{K}^f$. Therefore, $(f(a)+j)^n \in K$. Hence, K is an (m,n)-closed ideal of $f(A)+J$. Conversely, assume that K is an (m,n)-closed ideal of $f(A)+J$. Let $(a, f(a)+j)^m \in \overline{K}^f$ with $(a, f(a)+j) \in A \bowtie^f J$. Obviously, $f(a)+j \in f(A)+J$ and $(f(a)+j)^m \in K$ which is an (m,n)-closed ideal. So, $(a^n, (f(a)+j)^n) \in \overline{K}^f$. Hence, $\overline{K}^f$ is an (m,n)-closed ideal of $A \bowtie^f J$, as desired.

The following corollary is an immediate consequence of Proposition 6.2.1.

Corollary 6.2.2 *Under the above notations, the following statements holds:*

1. *$I \bowtie^f J$ is a radical ideal of $A \bowtie^f J$ if and only if I is a radical ideal of A.*

2. *$I \bowtie^f J$ is a semi-n-absorbing ideal of $A \bowtie^f J$ if and only if I is a semi-n-absorbing ideal of A.*
3. *$\overline{K}^f$ is a radical ideal of $A \bowtie^f J$ if and only if K is a radical ideal of $f(A)+J$.*
4. *$\overline{K}^f$ is a semi-n-absorbing ideal of $A \bowtie^f J$ if and only if K is a semi-n-absorbing ideal of $f(A)+J$.*

Let I be a proper ideal of A. The (amalgamated) duplication of A along I is a special amalgamation given by

$$A \bowtie I := A \bowtie^{id_A} I = \{(a, a+i) \mid a \in A, i \in I\}.$$

The next corollary is an immediate consequence of Proposition 6.2.1 and Corollary 6.3.5 on the transfer of (m,n)-closed ideal to duplications.

Corollary 6.2.3 *Let A be a ring and I be an ideal of A. Consider K an ideal of A. Then the following statements holds:*

1. *$K \bowtie I$ is an (m,n)-closed ideal of $A \bowtie I$ if and only if K is an (m,n)-closed ideal of A.*
2. *$K \bowtie I$ is a radical ideal of $A \bowtie I$ if and only if K is a radical ideal of A.*
3. *$K \bowtie I$ is a semi-n-absorbing ideal of $A \bowtie I$ if and only if I is a semi-n-absorbing ideal of A.*

Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Consider I (resp., H) be an ideal of A (resp., $f(A)+J$) such that $f(I)J \subseteq H \subseteq J$. Observe that $I \bowtie^f H := \{(i, f(i)+h)/i \in I, h \in H\}$ is an ideal of $A \bowtie^f J$.

Remark 6.2.4 *Under the above notations. Let m and n be two integers with $1 \leq n < m$. If $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$, then by using similar argument as statement (1) of Proposition 6.2.1, it follows that I is an (m,n)-closed ideal of A. The converse is not true, in general, as shown by the next example which exhibits an ideal I that is a $(2,1)$-closed ideal of A such that $f(I)J \subseteq H \subset J$ but $I \bowtie^f H$ is not a $(2,1)$-closed ideal of $A \bowtie^f J$.*

Example 6.2.5 *Let $A := \mathbb{Z}$ be the ring of integers, $B := \mathbb{Q}[[X]]$ be the ring of formal power series over $\mathbb{Q}$ in an indeterminate X, $f : \mathbb{Z} \hookrightarrow \mathbb{Q}[[X]]$ be the natural embedding and $J := X\mathbb{Q}[[X]]$. Let $H = \{XP(X); P \in f(A)+J = \mathbb{Z}+X\mathbb{Q}[[X]]\}$. Obviously, H is an ideal of $f(A)+J = \mathbb{Z}+X\mathbb{Q}[[X]]$. Observe that $I := 0$ is a prime ideal of A and so is an $(m,1)$-closed ideal of A for all positive integer m. Since $f(I)J = 0 \subset H \subset J$, then $0 \bowtie^f H$ is an ideal of $\mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$. We claim that $0 \bowtie^f H$ is not a $(2,1)$-closed ideal of $\mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$. Indeed, $0 \bowtie^f H = \{(0, XP(X)), P(X) \in \mathbb{Z}+X\mathbb{Q}[[X]]\}$, we have $(0, \sqrt{2}X)^2 = (0, 2X^2) \in 0 \bowtie^f H$ but $(0, \sqrt{2}X) \notin 0 \bowtie^f H$. Hence, $0 \bowtie^f H$ is not a $(2,1)$-closed ideal of $\mathbb{Z} \bowtie^f X\mathbb{Q}[[X]]$.*

Now, we investigate about when the ideal $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$.

Proposition 6.2.6 *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Let H be an ideal of $f(A)+J$ such that $f(I)J \subseteq H \subseteq J$. Then the following statements hold:*

1. *If $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$, then I is an (m,n)-closed ideal of A, for all positive integers m and n, with $1 \leq n < m$.*

2. *Assume that I is an $(m,1)$-closed ideal of A and $x^n \in H$ for every $x \in J$ with $n \geq 1$ a positive integer. Then $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$.*

Proof.

1. From Remark 6.2.4.
2. Assume that I is an $(m,1)$-closed ideal of A. We claim that $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$. Let $x^m = (a, f(a)+j)^m \in I \bowtie^f H$ for some $x = (a, f(a)+j) \in A \bowtie^f J$. Then $a^m \in I$. Since I is an $(m,1)$-closed ideal of A, then $a \in I$ and so $a^i \in I$ for every positive integer $1 \leq i \leq m$. Let $h = \sum_{i=1}^{n-1} \begin{pmatrix} n \\ i \end{pmatrix} f(a^i) j^{n-i} + j^n$. Observe that for every positive integer $i \leq n-1$, we have $f(a^i) j^{n-i} \in H$ (as $(a, f(a)+j)^m \in I \bowtie^f H$). Using the fact that $j^n \in H$ for every $j \in J$, then by using Binomial Theorem, it follows that $x^n = (a^n, f(a^n)+h) \in I \bowtie^f H$. Therefore, $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$.

The following corollary is a consequence of Proposition 6.2.6.

Corollary 6.2.7 *Let $f : A \to B$ be a homomorphism of rings and J be an ideal of B. Assume that I is a $(m,1)$-closed ideal of A and $x^n \in (f(I)B)J$ for every $x \in J$, with $n \geq 1$ a positive integer. Then the extension ideal of I to $A \bowtie^f J$, denoted by $I^e := I \bowtie^f (f(I)B)J$ is an (m,n)-closed ideal of $A \bowtie^f J$, for all positive integers m and n, with $1 \leq n < m$.*

Proof. Applying Proposition 6.2.6 with $H := (f(I)B)J$, it follows that I^e is an (m,n)-closed ideal of $A \bowtie^f J$.

Next, we show how one may use Proposition 6.2.6 to construct original example of (m,n)-closed ideals of the form $I \bowtie^f H$ of amalgamation $A \bowtie^f J$.

Example 6.2.8 *Let A be a ring, E be an A-module and $B := A \propto E$ be the trivial ring extension of A by E. Let I be an ideal of A and F be a submodule of E such that $IE \subset F$, and $J := I \propto E$ be an ideal of B. Consider the ring homomorphism $f : A \hookrightarrow B$ defined by $f(a) = (a,0)$. Notice that $H := I \propto F$ is an ideal of $f(A)+J = A \propto 0 + I \propto E = (A+I) \propto E = A \propto E$ and $f(I)J = (I \propto 0)(I \propto E) \subset I \propto F \subset J$. So, $I \bowtie^f H$ is an ideal of $A \bowtie^f J$. Let n be a positive integer and let $(i,e) \in J$. Clearly, $(i,e)^n = (i^n, ni^{n-1}e) \in H$. By Proposition 6.2.6, we conclude that I is an $(m,1)$-closed ideal of A if and only if $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$ for every positive integer $1 \leq n < m$. (For instance, if I is a prime ideal of A, then $I \bowtie^f H$ is an $(m,1)$-closed ideal of $A \bowtie^f J$ for every positive integer $m \geq 1$. Therefore, $I \bowtie^f H$ is a radical ideal of $A \bowtie^f J$).*

Now, we give a characterization of (m,n)-closed ideals of the form $I \bowtie^f H$ in the case (A,M) is local and J an ideal of B such that $f(M)J = 0$

Theorem 6.2.9 *Let A be a local ring with maximal ideal M, $f : A \to B$ be a ring homomorphism and J be an ideal of B such that $f(M)J = 0$. Let I be a proper ideal of A. Consider H an ideal of $f(A)+J$ such that $H \subset J$ and $x^n \in H$ for every $x \in J$. Then the following statements are equivalent:*

1. *$I \bowtie^f H$ is an (m,n)- closed ideal of $A \bowtie^f J$ for every positive integer $1 \leq n < m$.*
2. *I is an (m,n)-closed ideal of A for every positive integer $1 \leq n < m$.*

Proof. Notice that $I \bowtie^f H$ is an ideal of $A \bowtie^f J$ since $f(I)J = 0 \subset H$.
$(1) \Rightarrow (2)$ Assume that $I \bowtie^f H$ is an (m,n)- closed ideal of $A \bowtie^f J$ for every positive integer $1 \leq n < m$. Then by Remark 6.2.4, it follows that I is an (m,n)-closed ideal of A for every positive integer $1 \leq n < m$.
$(2 \Rightarrow (1)$ Let $x = (a, f(a) + j) \in A \bowtie^f J$ such that $x^m \in I \bowtie^f H$. Then a^m is an element of I which is an (m,n)-closed ideal of A. Therefore, $a^n \in I$. let $1 \leq l \leq n-1$. Two cases are then possible:
Case 1: $a^l \in M$. Then $f(a^l)j^{n-l} = 0$ and so by using Binomial Theorem, it follows that $(a, f(a) + j)^n = (a^n, f(a^n) + j^n) \in I \bowtie^f H$.
Case 2: $a^l \notin M$, we conclude that a^l is invertible in A which implies that $a^n \in I$ is invertible, a contradiction since I is a proper ideal of A. Hence, $x^n = (a^n, (f(a) + j)^n) \in I \bowtie^f H$. Thus $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$.

Before illustrating Theorem 6.2.9, we prove the next result which will be useful.

Lemma 6.2.10 *Let (A,M) be local ring such that $M^n = 0$, where n is a positive integer. Then every ideal of A is an n-absorbing ideal of A. In particular, every proper ideal of A is an (m,n)-ideal of A, for every positive integer $1 \leq n < m$.*

Proof.

Let I be a proper ideal of A. Let $a_1, \cdots, a_{n+1} \in A$ such that $\prod_{i=1}^{n+1} a_i \in I$. Two cases are then possible:
Case 1 : There exists $j \in \{1, ..., n+1\}$ such that $a_j \notin M$. Then a_j is invertible and so it follows that $\prod_{i=1, i \neq j}^{n+1} a_i \in I$, as desired.
Case 2 : For every $j \in \{1, ..., n+1\}$, $a_j \in M$. So, for every $j \in \{1, ..., n+1\}$, a_j is not invertible in A. Therefore, $\prod_{i=1}^{n} a_i = 0 \in I$. Thus in all cases, I is a n-absorbing ideal of A. In particular, I is an (m,n)-closed ideal of A.

Now, we show how one may use Theorem 6.2.9 and Lemma 6.2.10 to construct original examples of (m,n)-closed ideals of the form $I \bowtie^f H$.

Example 6.2.11 *Let (A,M) be local ring with a maximal ideal M such that $M^n = 0$, and E be an $\frac{A}{M}$-vector space. Consider the ring homomorphism $f : A \hookrightarrow B := A \propto E$ defined by $f(a) = (a, 0)$, for every $a \in A$. Let $J := M \propto E$ be an ideal of B and $H = I \propto E$ be an ideal of $f(A) + J$, where $I \subsetneq M$ is an ideal A with m and n two positive integers such that $2 \leq n \leq m$. One can easily check that $J^n = 0$ and for every ideal I of A, we get $f(I)J = (I \propto E)(M \propto E) \subseteq H$, and $J^n \subseteq H$. Hence, by Theorem 6.2.9, the ideal $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$ since I is a n-absorbing ideal of A (by Lemma 6.2.10).*

We close this section with the following proposition. For this purpose, let $Char(R)$ denote the characteristic of a ring R.

Proposition 6.2.12 *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Assume that $char(f(A) + J) = n$. Let H be an ideal of $f(A) + J$ such that $f(I)J \subset H \subset J$ and for every $j \in J$, $j^n \in H$. Then $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$ if and only if I is an (m,n)-closed ideal of A, for all positive integer $m \geq n$.*

Proof. Assume that I is an (m,n)-closed ideal of A, for all positive integer $m \geq n$. Let $(a, f(a)+j)^m \in I \bowtie^f H$ for every $(a, f(a)+j) \in A \bowtie^f J$. From assumption, it follows that $a^n \in I$. Using the fact that $\text{char}(f(A)+J) = n$ and $j^n \in H$, then $(a, f(a)+j)^n = (a^n, f(a^n)+j^n) \in I \bowtie^f H$. Hence, $I \bowtie^f H$ is an (m,n)-closed ideal of $A \bowtie^f J$. The converse is trivial via Remark 6.2.4.

6.3 When every proper ideal of amalgamation $A \bowtie^f J$ is an (m,n)-closed ideal

The following result gives a characterization about when every proper ideal of the amalgamation $A \bowtie^f J$ is an (m,n)-closed ideal, for some integers $1 \leq n < m$.

Theorem 6.3.1 *Assume that $f^{-1}(J)$ is radical ideal of A. Then every proper ideal of $A \bowtie^f J$ is an (m,n)-closed ideal of $A \bowtie^f J$ if and only if the following statements hold:*
(i) Every proper ideal of A is an (m,n)-closed ideal of A.
(ii) Every proper ideal of $f(A)+J$ is an (m,n)-closed ideal of $f(A)+J$.

The proof of this theorem requires the following lemmas.

Lemma 6.3.2 *[6, Theorem 2.14] Let R be a commutative ring and m and n integers with $1 \leq n < m$. Then the following statements are equivalent.*

1. *Every proper ideal of R is an (m,n)-closed ideal of R.*
2. *$dim(R) = 0$ and $w^n = 0$ for every $w \in Nil(R)$.*

Lemma 6.3.3 *[30, Lemma 2.10] Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then:*

$$Nil(A \bowtie^f J) := \{(a, f(a)+j) / a \in Nil(A), j \in Nil(B) \cap J\}$$

Lemma 6.3.4 *[37, Proposition 4.1] Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then: $dim(A \bowtie^f J) = Max(dim(A), dim(f(A)+J))$.*

Proof. of Theorem 6.3.1:
Assume that every proper ideal of $A \bowtie^f J$ is (m,n)-closed ideal of $A \bowtie^f J$.
(i) By Lemmas 6.3.2 and 6.3.4, it follows that $dim(A \bowtie^f J) = Max(dim(A), dim(f(A)+J)) = 0$. So, $dim(A) = 0$. Next, let $a \in Nil(A)$. Then $(a, f(a)) \in Nil(A \bowtie^f J)$. Using the fact that every proper ideal of $A \bowtie^f J$ is (m,n)-closed ideal of $A \bowtie^f J$, then by Lemma 6.3.2, it follows that $(a, f(a))^n = (0,0)$. Therefore, $a^n = 0$. Hence, every proper ideal of A is an (m,n)-closed ideal of A.
(ii) With similar argument as (i) above, it follows that $dim(f(A)+J) = 0$. Let $f(a)+j \in Nil(f(A)+J)$. Clearly, $f(a)^k \in J$ for some integer $k \geq 1$. So, $a^k \in f^{-1}(J)$ which is radical. Therefore, $a \in f^{-1}(J)$. Consequently, $f(a)+j \in J$. By Lemma 6.3.3, $(0, f(a)+j) \in Nil(A \bowtie^f J)$. Hence, $(0, f(a)+j)^n = (0,0)$ and so $(f(a)+j)^n = 0$. From Lemma 6.3.2, it follows that every

proper ideal of $f(A)+J$ is an (m,n)-closed ideal of $f(A)+J$. Conversely, assume that every proper ideal of A (resp., $f(A)+J$) is an (m,n)-closed ideal of A (resp., $f(A)+J$). We claim that every proper ideal of $A\bowtie^f J$ is (m,n)-closed ideal of $A\bowtie^f J$. Indeed, by Lemma 6.3.4, $dim(A\bowtie^f J) = Max(dim(A), dim(f(A)+J)) = 0$ since $dim(A) = dim(f(A)+J) = 0$. It remains to show that for all $(a, f(a)+j) \in Nil(A\bowtie^f J)$, $(a, f(a)+j)^n = 0$. Consider $(a, f(a)+j) \in Nil(A\bowtie^f J)$. Then $a \in Nil(A)$ and $f(a)+j \in Nil(f(A)+J)$. From assumption, by using Lemma 6.3.2, it follows that $a^n = 0$ and $(f(a)+j)^n$. Therefore, $(a, f(a)+j)^n = 0$. Hence, every proper ideal of $A\bowtie^f J$ is an (m,n)-closed ideal of $A\bowtie^f J$.

The following corollary is an immediate consequence of Theorem 6.3.1.

Corollary 6.3.5 *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then the following statements hold:*

1. *Every proper ideal of $A\bowtie^f J$ is a radical ideal of $A\bowtie^f J$ if and only if the following statements hold:*
   (i) Every proper ideal of A is a radical ideal of A.
   (ii) Every proper ideal of $f(A)+J$ is a radical ideal of $f(A)+J$.
2. *Assume that $f^{-1}(J)$ is a radical ideal of A. Then every proper ideal of $A\bowtie^f J$ is a semi-n-absorbing ideal of $A\bowtie^f J$ if and only if the following statements hold:*
   (i) Every proper ideal of A is a semi-n-absorbing ideal of A.
   (ii) Every proper ideal of $f(A)+J$ is a semi-n-absorbing of $f(A)+J$.

Remark 6.3.6 *Observe that the assumption "$f^{-1}(J)$ is a radical ideal of A" is omitted in Corollary 6.3.5(1). Indeed, if every proper ideal of $A\bowtie^f J$ is a radical ideal of $A\bowtie^f J$, then $f^{-1}(J)\times\{0\}$ is a radical ideal of $A\bowtie^f J$ and so $f^{-1}(J) \simeq f^{-1}(J)\times\{0\}$ is also a radical ideal of A. Conversely if the statements (i) and (ii) hold, then $f^{-1}(J)$ is radical ideal of A.*

Theorem 6.3.1 and Corollary 6.3.5 cover the special case of duplications, as recorded below.

Corollary 6.3.7 *Let A be a ring and I be an ideal of A. Then the following statements hold:*

1. *Assume that I is a radical ideal of A. Then every proper ideal of $A\bowtie I$ is an (m,n)-closed ideal of $A\bowtie I$ if and only every proper ideal of A is an (m,n)-closed ideal of A.*
2. *Every proper ideal of $A\bowtie I$ is a radical ideal of $A\bowtie I$ if and only every proper ideal of A is a radical ideal of A.*
3. *Assume that I is a radical ideal of A. Then every proper ideal of $A\bowtie I$ is a semi-n-absorbing ideal of $A\bowtie I$ if and only every proper a semi-n-absorbing ideal of A.*

Theorem 6.3.1 recovers a known result for trivial ring extensions which is [15, Theorem 6.12].

Corollary 6.3.8 *Let A be an integral domain with quotient field K, E be a K-vector space, and $B := A\propto E$ be the trivial ring extension of A by E. Then the following statements are equivalent:*

1. *Every proper ideal of A is an (m,n)-closed ideal of A for some integers $1 \le n < m$.*
2. *Every proper ideal of B is an (m,n)-closed ideal of B for some integers $1 \le n < m$.*

Proof. Consider $f : A \hookrightarrow B$ the injective ring homomorphism defined by $f(a) = (a,0)$, for every $a \in A$, $J := 0 \propto E$. Clearly, $f^{-1}(J) = 0$. Therefore, from [35, Proposition 5.1 (3)], $f(A)+J = A \propto 0+0 \propto E = A \propto E = B \simeq A \bowtie^f J$. Since A is an integral domain, then $f^{-1}(J) = 0$ is a radical ideal of A. Hence, by Theorem 6.3.1, we have the desired result.

As a consequence of Theorem 6.3.1, we give a complete characterization about when every proper ideal of $D+M$ is an (m,n)-closed ideal on the $D+M$ rings.

Corollary 6.3.9 *Let M be a maximal ideal of an integral domain T, and D be a subring of T such that $D \cap M = \{0\}$. Then every proper ideal of $D+M$ is an (m,n)-closed ideal of $D+M$ if and only if every proper ideal of D is an (m,n)-closed ideal of D for some integers $1 \leq n < m$.*

Proof.

Let $f : D \hookrightarrow T$ be the natural embedding and let $J := M$. Since $f^{-1}(J) = D \cap M = \{0\}$ (which is radical ideal of D, as 0 is prime ideal of D), it is easy to see that $D+M \simeq D \bowtie^f M$. Therefore, by Theorem 6.3.1, it follows that every proper ideal of $D+M$ is an (m,n)-closed ideal of $D+M$ if and only if every proper ideal of $D \bowtie^f M$ is an (m,n)-closed ideal if and only if every proper ideal of D is an (m,n)-closed ideal of D, for some integers $1 \leq n < m$.

As an application Theorem 6.3.1, we give a new characterization for the amalgamation $A \bowtie^f J$ to be Von Neumann regular. Recall that a ring R is Von Neumann regular if and only if every proper ideal of R is a radical ideal [6, Theorem 2.13 (2)]. A combination of this fact and Corollary 6.3.5(1) establishes the transfer of Von Neumann regular property to the amalgamation $A \bowtie^f J$.

Corollary 6.3.10 *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then $A \bowtie^f J$ is Von Neumann regular if and only if A and $f(A)+J$ are Von Neumann regular.*

The next result is an application of Corollary 6.3.10 on the transfer of π-regular property to the amalgamation. Recall that a ring R is π-regular if and only if $R/Nil(R)$ is Von Neumann regular.

Corollary 6.3.11 *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then $A \bowtie^f J$ is π-regular if and only if A and $f(A)+J$ are π-regular.*

Proof.

Set $\overline{A} = A/Nil(A)$, $\overline{B} = B/Nil(B)$, $p : B \to \overline{B}$ the canonical projection and $\overline{J} = p(J)$ and let $\overline{f} : \overline{A} \to \overline{B}$, defined by $\overline{f}(\overline{x}) = \overline{f(x)}$. Observe that $\overline{f}$ is well defined. From [30, Proof of Theorem 2.9], we get $A \bowtie^f J/Nil(A \bowtie^f J) \simeq \overline{A} \bowtie^{\overline{f}} \overline{J}$. Consequently, $A \bowtie^f J$ is a π-regular ring if and only if $\overline{A} \bowtie^{\overline{f}} \overline{J}$ is Von Neumann regular if and only if $\overline{A}$ and $\overline{f(A)+J}$ are Von Neumman regular (by Corollary 6.3.10). Hence, the conclusion is straightforward.

As another application of Theorem 6.3.1, we get necessary and sufficient conditions for an amalgamation to be semisimple.

Corollary 6.3.12 *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then $A \bowtie^f J$ is semisimple if and only if A and $f(A)+J$ are semisimple.*

Proof. It is well known that semisimple rings collapse with Von Neumann regular rings that are Noetherian. A combination of this fact with Corollary 6.3.10 and [35, Proposition 5.6] (on the transfer of the Noetherian property) leads to the conclusion.

CHAPTER 7

PERSPECTIVES

To end this thesis, we introduce the following open questions :

1. About the ring in which every 2-absorbing idéa which is no prime ideal is finitely generated
2. Characterization of rings in which if I is a n-absorbing idéal, then $|\mathrm{Min}(I)| = \omega_R(I)$.
3. Study the n-Ring,i.e the ring in which every n-absorbing idéal is $n-1$-absorbing idéal where $n > 1$ is a positive integer.
4. Give a answer to the N.Mahdou's question what about the rings in which every (m,n)-closed idéal is prime, where $1 \leq n < m$ are the positive integers.

CHAPTER 8

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Université Sidi Mohamed Ben Abdellah

Centre d'Etudes Doctorales : Sciences et Techniques de l'Ingénieur

N° d'ordre: 69/2019 *Fès, le 16/11/2019*

THESE DE DOCTORAT

Présentée par

MOHAMMED ISSOUAL

Discipline : Mathématiques

Spécialité : Algèbre

Sujet de la thèse :

Autour des idéaux n-absorbants et des idéaux (m,n)-fermés d'anneaux commutatifs

Thèse présentée et soutenue le samedi 16 Novembre 2019 devant le jury composé de :

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Titre : Autour des idéaux n-absorbants et des idéaux (m, n)-fermés d'anneaux commutatifs

Résumé

La théorie des idéaux en algèbre commutative a connu plusieurs développements depuis, **Noether, Kummer, Dedekind** et d'autres. Dans ce sens **D. F. Anderson et A. Badawi** ont introduit la notion d'idéal n-absorbant comme généralisation de celle d'idéal premier comme suit : Soit I un idéal d'un anneau commutatif R et $n > 0$ un entier. On dit que l'idéal I est un idéal n-absorbant (resp., fortement n-absorbant) si pour tous $a_1, \dots, a_{n+1} \in R$ avec $a_1 \dots a_{n+1} \in I$ (resp., pour tous idéaux $I_1, \dots I_{n+1}$ de R, avec $I_1, \dots I_{n+1} \subset I$), alors I contient au moins un produit de n éléments des $a_i, 1 \leq i \leq n+1$ (resp., un produit de n idéaux parmi les idéaux $I_i, 1 \leq i \leq n+1$).

D. F. Anderson et A. Badawi ont établi plusieurs propriétés sur les idéaux n-absorbants et ont formulé trois conjectures :

- La première conjecture : Si I est un idéal n-absorbant d'un anneau commutatif R, alors I est un idéal fortement n-absorbant de R.
- La deuxième conjecture : Si I est un idéal n-absorbant d'un anneau commutatif$R$, alors $(Rad(I))^n \subseteq I$.
- La troisième conjecture : Si I est un idéal n-absorbant d'un anneau commutatif R, alors $I[X]$ est un idéal n-absorbant de $R[X]$.

Les travaux de cette thèse se situent dans la théorie des idéaux en algèbre commutative suivant deux axes en s'appuyant sur les travaux de **D. F. Anderson et A. Badawi,** surtout la théorie des idéaux n-absorbants et des idéaux (m, n)-fermés et les travaux sur les anneaux réguliers au sens de Von Neumann.

Dans un premier axe nous avons étudié la notion d'idéal 2-absorbant dans l'extension triviale et l'amalgamation et nous avons aussi étudié les anneaux dans lesquels tout idéal 2-absorbant est premier. Puis nous avons validé les conjectures de **D. F. Anderson et A. Badawi** dans l'extension triviale et l'amalgamation. On a aussi démontré la première conjecture dans certaines classes d'anneaux.

Dans un deuxième axe on a étudié la notion d'idéal (m, n)-fermé notion variante de celle d'idéal n-absorbant. Cette notion est étudiée dans l'extension triviale et l'amalgamation. Puis à l'aide de la notion d'idéal (m, n)-fermé on a transféré des notions comme celle d'anneau régulier au sens de Von Neumann, d'anneau π-régulier et d'anneau semi simple à l'amalgamation.

Mathematics Subject Classification (2010) : 13Axx, 13Bxx, 13Cxx, 13Gxx, 13Hxx.

Mots clés : Extension triviale, amalgamation, idéal 2-absorbant, idéal n-absorbant, idéal (m, n)-fermé.