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RIEMANNIAN GEOMETRY OF VECTOR BUNDLES

Dedication

To the memory of my beloved mother...

To my father, brother and sisters;

To all my friends;

To all my professors and teachers;

To those who pursue the truth at any price.

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Résumé

La géométrie des fibrés vectoriels constitue un champs fertile de recherche en géométrie Riemannienne. Les fibrés vectoriels forment une large classe de variétés qui possèdent des propriétés géométriques intéressantes et un grand nombre d'applications à la fois en géométrie différentielle et en physique mathématique. Les efforts dans cette direction ont commencé dans les années 1950, lorsque S. Sasaki a introduit sa fameuse métrique 'naturelle' sur le fibré tangent d'une variété Riemannienne [68].

Par la suite, la métrique de Sasaki a été intensément étudiée par plusieurs géomètres. Cependant la métrique a montré une 'rigidité extrême' (cf. [44]). Par conséquent, plusieurs géomètres ont introduit d'autres (classes de) métriques, dont la plus large est la classe des métriques g-naturelles sur le fibré tangent, généralisant la métrique de Sasaki (cf. [1–3,45]), et obtenue suite à une classification utilisant les théories des jets et des fibrés vectoriels naturels (cf. [45]).

Le succès universel de la géométrie Riemannienne du fibré tangent a conduit certains géomètres à généraliser des métriques connues dans le contexte du fibré tangent au contexte des fibrés vectoriels. Parmi les métriques qui ont été généralisées on trouve la métrique de Sasaki, la métrique de Cheeger-Gromoll, et les métriques de Cheeger-Gromoll généralisées [43,49,50,83]. Etant donné un fibré vectoriel (E, π, M), au dessus d'une variété Riemannienne (M, g), muni d'une métrique fibré h et d'une connexion compatible D, R. Albuquerque a récemment introduit une classe de métriques, à deux poids φ_1 et φ_2, dépendant du carré de la norme, qui généralise la métrique de Sasaki. Ces métriques ont été appelées les *métriques à symétrie sphérique*. Cette thèse est consacrée à l'étude des métriques à symétrie sphérique. Une étape clé été réalisée lorsque on a montré que les projections des fibrés vectoriels, munis des métriques à symétrie sphérique, sont des *submersions horizontalement conformes* avec une fonction de *dilatation* donnée par $\lambda = e^{-\varphi_1}$ (cf. [11]). Ceci nous a permis d'utiliser la machinerie technique forte, contenue dans le travail de B. O'Neill dans le contexte des submersions Riemanniennes, et sa généralisation, par S. Gudmundsson, dans le cadre des submersions horizontalement conformes (cf. [29,30,61,62]).

On commence par l'étude des géodésiques et de la complétude des métriques à symétrie sphérique. On adopte la même approche utilisée par O'Neill dans son traitement des géodésiques des submersions Riemanniennes (cf. [62]). En particulier, on donne les EDOs qui décrivent les géodésiques et en déduire quelques exemples. Ensuite, on traite la question de complétude. Dans [13], R.Albuquerque a conjecturé que l'espace total d'un fibré vectoriel, muni d'une métrique à symétrie sphérique, est complet si et seulement si la base et les fibres sont complets. On répond affirmativement à cette question sans même supposer la complétude des fibres, mais sous une condition de bornétude sur la fonction de dilatation. En effet, on montre que

> *Si l'espace total E est muni d'une métrique Riemannienne telle que la projection $\pi : (E, G) \longrightarrow (M, g)$ est une submersion horizontalement conforme à dilatation bornée, alors (E, G) est complet si et seulement si la variété de base (M, g) est complète.*(cf. [11])

En particulier, on applique le résultat précédent aux métriques à symétrie sphérique et aux métriques de Cheeger-Gromoll généralisées. Précisément, on montre que

Supposons que l'espace total E est muni d'une métrique G appartenant à l'une des classes suivantes:

(i) métriques à symétrie sphérique avec des poids φ_1 et φ_2, tels que φ_1 est bornée;

(ii) métriques de Cheeger-Gromoll généralisées.

Alors, la variété Riemannienne (E, G) est complète si et seulement la variété de base (M, g) est complète.

De plus, notre résultat principal s'applique aussi à une sous classe de métriques g-naturelles sur le fibré tangent et le fibré tangent unitaire, à savoir *les métriques de type Kaluza-Klein.*

Par la suite, on étudie les symétries infinitésimales relativement aux métriques à symétrie sphériques, notamment les champs de vecteurs fermés, conformes et de Killing. On construit plusieurs exemples de champs de vecteurs conformes. En particulier, on explore les relèvements horizontaux des champs de vecteurs de la base ainsi que les relèvements verticaux des sections du fibré considéré.

Dans le cas des relèvements de horizontaux, on démontre le résultat suivant

Soit X un champ de vecteurs sur la base. Alors les assertions suivantes sont équivalentes:

(i) X^h est conforme sur E;

(i) X^h est un champs de Killing sur E;

(iii) X est un champs de Killing sur M et $R^E(X, .)$,

avec X^h est le relèvement horizontal de X par rapport à D et R^E est la courbure de D.

En effet, on montre une forme plus générale du résultat précédent pour des champs de vecteurs horizontaux, pas nécessairement des relèvements horizontaux.

De l'autre coté, on caractérise les champs de vecteurs verticaux conformes. En particulier, pour les relèvements verticaux des sections, on montre le résultat suivant

Soit σ une section non-nulle de E. Alors le relèvement vertical σ^v est un champ conforme de fonction potentille f si et seulement si les assertions suivantes sont vérifiées

(i) $f(e) = \varphi_1'(r)h((\pi(e)), e)$, pour tout $e \in E$, avec $r = h(e, e)$;

(ii) $\varphi_1' = \varphi_2'$ sur $\mathbb{R}^+$;

(iii) σ est parallèle.

En plus, on donne une classification des champs de vecteurs de Killing horizontaux, et on se focalise sur les champs de vecteurs gradients de certaines classes de fonctions.

Les courbures des métriques à symétrie sphérique sont, ensuite, intensément étudiées. On commence par la question d'hérédité. Précisément, on montre que

Si l'espace total E muni d'une métrique à symétrie sphérique est plat, de courbure sectionnelle constante, d'Einstein ou de courbure scalaire constante, alors la variété de base possède la même propriété, respectivement.

Plus intéressante encore est l'étude de la question inverse, c'est-à-dire voir quand l'espace total possède une propriété géométrique remarquable sachant que la base la possède. Premièrement, on caractérise la platitude des métriques à symétrie sphérique. Précisément, on montre que

> *Une métrique à symétrie sphérique est plate si et seulement si la base est une variété Riemannienne plate, la connexion est plate et les fonctions poids sont constantes.*

De plus, on montre plusieurs résultats de rigidité des métriques à symétrie sphérique concernant la courbure sectionnelle. Par exemple, on montre

> *Soit G une métrique à symétrie sphérique avec fonctions poids φ_1 et φ_2. Supposons que φ_1 atteint son minimum à zero (e.g. φ_1 est croissante). Si l'espace total (E, G) est à une courbure sectionnelle constante, alors la connexion est plate, la base est plate et les fonctions poids sont constantes. Ce qui est équivalent à la platitude*

En conséquence, nous étions forcés à étudiér le signe de la courbure sectionnelle ou à chercher des métriques à symétrie sphérique à courbure sectionnelle bornée. On construit, dans le cas des connexions plates, des exemples de métriques à symétrie sphérique à courbure sectionnelle bornée.

Puis, dans le même esprit, on étudie les propriétés géométriques reliées au tenseur de courbure de Ricci. En particulier, on construit, lorsque la connexion est plate, une famille à deux paramètres de métrique d'Einstein parmi les métriques à symétrie sphériques.

D'autres résultats de rigidité, relativement à la courbure de Ricci, sont démontrés. Par exemple, on a

> *Supposons que E est un fibré en droites muni d'une métrique à symétrie sphérique telle que $\varphi_1 = 0$. Alors (E, G) est une variété d'Einstein si et seulement si elle est Ricci-plate, (M, g) est Ricci-plate et D est plate.*

et

> *Supposons que le rang $k \geq 2$ et (E, G) est Ricci-plate. Si la fonction $r \mapsto r\varphi_2'(r)$ est décroissante, alors D est plate, (M, g) est Ricci-plate et φ_2 est constante. La réciproque est aussi vraie.*

Ensuite, on montre d'autres résultats de rigidité concernant la courbure de Ricci dans le cas d'une connexion arbitraire et dans le cas d'une connexion plate.

Concernant la courbure scalaire, on montre une multitude de résultats de rigidité. Par exemple, dans le cas d'un fibré en droites, on montre le résultat suivant

> *Soit (E, G) un fibré en droites muni d'une métrique à symétrie sphérique G, alors (E, G) est à courbure scalaire constante $\widetilde{Sc}$ si et seulement si (M, g) est à courbure scalaire constante $Sc = \widetilde{Sc}$ et la connexion D est plate.*

On montre encore, pour des fibrés vectoriels de rang 2, que

> *Soit E un fibré vectoriel de rang $k = 2$. Supposons que E est muni d'une métrique à symétrie sphérique G avec fonctions poids φ_1 et φ_2, telles que la fonction $r \mapsto r\varphi_2'(r)$ est croissante et $\varphi_2'(0) = 0$. Si (E, G) est à courbure scalaire constante, alors (M, g) est à courbure scalaire constante, D est plate et φ_2 est constante.*

Des résultats similaires, pour les rangs supérieurs, sont établis sous des conditions sur les signes et la monotonie des fonctions poids. Particulièrement, on généralise plusieurs résultats de rigidité connus dans le cas de la métrique de Sasaki.

Une propriété géométrique qui est étudiée dans le cas connexions plates est le fait d'être localement symétrique. On montre le résultat suivant

Supposons que le rang $k \geq 2$, la connexion D est plate et G une métrique à symétrie sphérique sur E. Alors (E, G) est localement symétrique si et seulement si les conditions suivantes sont vérifiées

(i) φ_1 *est constante;*

(ii) ou bien φ_2 *est constante ou* $\varphi_2(r) = \ln(r + c)$, *pour tout* $r \in \mathbb{R}^+$, *pour une constante* $c \geq 0$;

(iii) (M, g) *est localement symétrique.*

Finalement, on se focalise sur les sections harmoniques. On considère alors les trois problèmes suivants d'harmonicité pour les sections:

- sections harmoniques comme applications (cf. [24,25,77,86]);
- points critiques de la fonctionnelle d'énergie mais avec des variations constituées par des sections.(cf. [55])
- l'harmonicité verticale, c'est-à-dire le fait d'être un point critique de la fonctionnelle d'énergie(cf. [83]).

Pour chaqu'un de ces problèmes, on donne les équations d'Euler-Lagrange. A cause de la complexité de ces équations, on est forcé de donner des conditions nécessaires et suffisantes pour les sections parallèles et celles de longueurs constantes. En particulier, on généralise un résultat de J. Konderak ([43]) comme suit

Supposons que M est compacte et E est muni d'une métrique à symétrie sphérique avec fonctions poids φ_1 et φ_2. Si φ_1' est positive et $(1 + 2r\varphi_2')$ est strictement positive sur $\Sigma_\sigma := \{r = h(\sigma, \sigma)(x) : x \in M\}$ (resp. φ_1' est négative et $(1 + 2r\varphi_2')$ est strictement négative sur Σ_σ). Si une section σ de E est harmonique alors elle est parallèle et sa norme constante est $r = h(\sigma, \sigma)$. De plus, si $\varphi_1' \neq 0$, alors $\sigma = 0$.

Enfin, on donne des exemples de sections harmoniques.

Abstract

The geometry of vector bundles constitutes an interesting research area in Riemannian geometry. Vector bundles form a large class of manifolds with very tempting geometric properties along with a large number of potential applications. They lay at the heart of many current problems of differential geometry and mathematical physics. Hence the importance of investigating the Riemannian geometry of those spaces. The endeavours in this direction could be traced back to the 1950s, when Shigeo Sasaki introduced his famous 'natural' metric in the case of tangent bundles [68].

Subsequently, the Sasaki metric was extensively studied, but it turns out to be very *rigid* (cf. [44]). Next, many geometers introduced generalizations of the *Sasaki metric*. Particularly, a classification of all 'natural' metrics was accomplished based on *jet theory* and *natural vector bundles* (cf. [45]), which leads eventually to the discovery of the large class of g-natural metrics (cf. [1–3, 45]).

The universal success of the Riemannian geometry of tangent bundles lead many geometers to generalize many known metrics to the framework of vector bundles in general. Among those metrics, we find the Sasaki metric, the Cheeger-Gromoll metric and the generalized Cheeger-Gromoll metrics [43, 49, 50, 83]. Recently, on a vector bundle (E, π, M), over a Riemannian base (M, g), endowed with a fiber metric h and a compatible connection D, R. Albuquerque introduced a class of metrics which generalize the Sasaki metric, with two weights φ_1 and φ_2 depending only on the square norm of the vector. They were called *spherically symmetric metrics*. This thesis is devoted to the study of spherically symmetric metrics. A key step is made when we prove that the projections of vector bundles with spherically symmetric metrics are *horizontally conformal submersions* with *dilation function* $\lambda = e^{-\varphi_1}$ (cf. [11]). This provides us with a very rigid technical machinery, namely the results of B. O'Neill on Riemannian submersions and their generalizations to the case of horizontally conformal submersions in the work of S. Gudmundsson (cf. [29, 30, 61, 62]).

We shall begin by geodesics and completeness of spherically symmetric metrics. We adopt the same approach used by O'Neill in his investigations of geodesics of Riemannian submersions (cf. [62]). So, we will give the ODEs describing geodesics and we will derive some examples. Then, we tackle the question of completeness. In [13], R.Albuquerque conjectured that the total space endowed with a spherically symmetric metric is complete if and only if the base and fibers are complete. We will answer the question in affirmative without assuming the completeness of fibers, but under a condition on one of the weight functions. In fact, we prove that

> *If the total space E is endowed with a Riemannian metric for which the projection $\pi : (E, G) \longrightarrow (M, g)$ is a horizontally conformal submersion with bounded dilation, then (E, G) is complete if and only if (M, g) is complete.*(cf. [11])

In particular, this applies to spherically symmetric metrics as well as generalized Cheeger-Gromoll metrics. More precisely, we will prove the following

> *Assume the total space is endowed with a metric G belonging to one of the following classes*

(i) Spherically symmetric metrics with weights φ_1 and φ_2, such that φ_1 is bounded;

(ii) Generalized Cheeger-Gromoll metrics.

Then, the Riemannian manifold (E, G) is complete if and only if the base manifold (M, g) is complete.

The main theorem applies also to a subclass of g-natural metrics on tangent bundles and tangent sphere bundles, namely *Kaluza-Klein type metrics*.

We will investigate the symmetries of spherically symmetric metrics. We will study closed, conformal and Killing vector fields. We also construct many examples of conformal vector fields. In particular, we will study horizontal lifts of vector fields on the base and vertical lifts of sections.

In the case of horizontal lifts, we will prove the following result

Let X be a vector field on the base. Then the following are equivalent

(i) X^h is conformal on E;

(i) X^h is a Killing vector field on E;

(iii) X is a Killing vector field on M and $R^E(X, .)$.

Where X^h denotes horizontal lift of X with respect to D and R^E is the curvature of D.

In fact, we will prove a general form of the previous result for horizontal vector field not necessarily horizontal lifts.

On the other hand, we will characterize conformal vertical vector fields. In particular, for vertical lifts of sections, we will prove the following result

Let σ be a non-zero section. Then the vertical lift σ^v is a conformal vector field with potential function f if and only if the following assertions hold

(i) $f(e) = \varphi_1'(r)h((\pi(e)), e)$, for all $e \in E$, where $r = h(e, e)$;

(ii) $\varphi_1' = \varphi_2'$ on $\mathbb{R}^+$;

(iii) σ is parallel.

Moreover, we will give a classification result of horizontal Killing vector fields. At the end we focus on gradient vector fields of certain classes of functions.

The curvature properties of spherically symmetric metrics will be heavily studied. We begin by the question of heredity. Precisely, we will prove that

If the total space endowed with a spherically symmetric metric is flat, of constant sectional curvature, an Einstein metric or of constant scalar curvature, then the base space possesses the same property, respectively.

More importantly, we will study the inverse problem, that is when the total space possesses one of the previous geometric properties provided the base the same property. First, we will characterize the flatness of spherically symmetric. Precisely, we prove that

A spherically symmetric metric is flat if and only if the base is a flat Riemannian manifold, the connection is flat and the weight functions are constant.

Furthermore, we will prove many rigidity results of spherically symmetric metrics with respect to sectional curvatures. Interestingly, we shall prove the following

Let G be a spherically symmetric metric with weights φ_1 and φ_2. Assume that φ_1 attains it minimum at zero (i.e. φ_1 is increasing). If the total space (E,G) has constant sectional curvature, then the connection is flat, the base is flat and the weight functions are constant. Which is equivalent to flatness

Thus, we have been compelled to study the sign of sectional curvatures and to look for spherically symmetric metrics with bounded sectional curvatures. We have succeeded to construct examples of spherically symmetric metrics in the case of flat connections.

Next, we study in the same spirit the geometric properties related to the Ricci curvature tensor. In particular, we have considered the problem of finding Einstein spherically symmetric metrics. Interestingly, we will construct a two parameter family of Einstein metrics when the connection is flat.

Many rigidity results with respect to the Ricci curvature tensor will be given, for example we will prove

Assume that E is a line bundle endowed with a spherically symmetric metric such that $\varphi_1 = 0$. Then (E,G) is an Einstein manifold if and only it is Ricci flat, (M,g) is Ricci flat and D is flat.

and

Assume that the rank $k \geq 2$ and (E,G) is Ricci flat. If the function $r \mapsto r\varphi_2'(r)$ is decreasing, then D is flat, (M,g) is Ricci flat and φ_2 is constant. The converse also holds

Subsequently, we will give other rigidity results concerning the Ricci curvature tensor both in the case of an arbitrary connection or a flat one.

Concerning the scalar curvature, we will prove many rigidity results. For example, in the case of line bundles, we will prove the following

Let (E,G) be a line bundle endowed with a spherically symmetric metric G, then (E,G) is of constant scalar curvature $\widetilde{Sc}$ if and only if (M,g) is of constant scalar curvature $Sc = \widetilde{Sc}$ and the connection D is flat.

Moreover, we will prove, for vector bundles with rank 2, that

Let E be a vector bundle of rank $k = 2$. Assume E is endowed with a spherically symmetric metric G, with weights φ_1 and φ_2, such that the function $r \mapsto r\varphi_2'(r)$ is decreasing and $\varphi_2'(0) = 0$. If (E,G) is of constant scalar curvature, then (M,g) is of constant scalar curvature, D is flat and φ_2 is constant.

Similar results, for higher ranks, will be proved under sign and monotonicity assumptions on the weight functions. Particularly, we will generalize known rigidity results in the case the Sasaki metric.

Another geometric property that will be studied in the case of flat connections is being locally symmetric. We will prove that

Assume that the rank $k \geq 2$, the connection D is flat and let G be a spherically symmetric metric on E. Then (E,G) is locally symmetric if and only if the following conditions hold

(i) φ_1 *is constant;*

(ii) either φ_2 *is constant or* $\varphi_2(r) = \ln(r+c)$, *for all* $r \in \mathbb{R}^+$, *for a constant* $c \geq 0$;

- (M, g) *is locally symmetric.*

Finally, we draw attention to harmmonic sections. We consider three harmonicity problems for sections. Precisely, the :

- Harmonic sections as maps (cf. [24,25,77,86]);
- Critical points of the energy functional but with variations through sections.(cf. [55])
- Vertical harmonicity, that is being a critical point of the vertical energy functional (cf. [83]).

For each of these problems, we will derive the Euler-Lagrange equations. Facing the complexity of the equations, we shall give necessary and sufficient conditions for constant length, and eventually parallel sections. In particular, we will generalize a result of J. J. Konderak ([43]) as follows

> *Assume that M is compact and E is endowed with a spherically symmetric metric with weights φ_1 and φ_2. If φ_1' is non-negative and $(1+2r\varphi_2')$ is positive on $\Sigma_\sigma := \{r = h(\sigma,\sigma)(x) : x \in M\}$(resp. φ_1' is non-positive and $(1+2r\varphi_2')$ is negative on Σ_σ), then σ is parallel and has constant norm $r = h(\sigma,\sigma)$. Moreover, if $\varphi_1' \neq 0$, then $\sigma = 0$.*

At the end, we will give some examples of harmonic sections.

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Introduction

Vector bundles are present in almost every area of mathematics, ranging from algebraic topology to algebraic geometry, differential geometry and mathematical physics. Part of the reason is the rich structure that they enjoy. In classical differential geometry, the *tangent bundle* to a manifold is an example of vector bundles. It arises naturally as the 'habitat' for velocities. Through this lens, vector bundles emerge in a straightforward way, since one can replace the tangent space by an arbitrary vector space while maintaining the 'clutching property' of local trivializations. Though, it is difficult to trace back the origins of the concept of a vector bundle, and generally bundle theory, we can say without a lot of risk that the physical idea of a *field* was one of the greatest motivations. It was necessary for mathematicians, and eventually physicists, to introduce mathematical objects capable of capturing this idea. Other motivations came from differential geometry itself e.g. Hamiltonian formalisms [58, 66]. In this respect, vector bundles provide a unifying language for differential geometry in which most of the geometric objects and structures can be expressed, and a 'universal' language for physics, especially *fields theories* both *classical* and *quantum* e.g. *Yang-Mills theories* and their quantized versions [27, 33, 51, 74, 80].

The tangent bundle to a manifold arises in a natural way from basic differential geometry. It is perhaps the first example of a non-trivial vector bundle. The manifold structure on tangent bundles were defined in the most straightforward of ways, but their Riemannian geometry was not given birth until 1958, when Shegiu Sasaki introduced his famous metric known as the *Sasaki metric* (cf. [68]). After Sasaki, many geometers attempted to 'lift' a Riemannian metric on the base to a metric on its tangent bundle (cf. [89]), but they didn't make much progress. The Sasaki metric was extensively studied from a multitude of points of view (cf. [44] for the curvature properties, [22] for a complex geometric perspective and [60] for *harmonic maps*). Based on an analysis of the results that relates the geometries of the base and total space, some geometers remarked that the Sasaki metric is extremely *rigid* in the sense that any interesting geometric property requires the flatness of the base manifold. For instance, O. Kowalski showed that the Sasaki metric is never locally symmetric unless the base manifold is flat [44]. Further, E. Musso and F. Tricerri proved that the tangent bundle endowed with the Sasaki metric cannot have constant scalar curvature unless the metric on the base is flat [56]. The flatness of the base is equivalent to the flatness of the total space. So, in some sense, the Sasaki metric was not 'the best' choice of metric one wish to consider.

Subsequently, many attempts have been made to generalize Sasaki's construction while overcoming the rigidity. In 1998, E. Musso and F. Tricceri constructed the metric known as the *Cheeger-Gromoll metric*. In [69], M. Sekizawa proved that the Cheeger-Gromoll metric can never have constant scalar curvature provided that the base manifold has constant sectional curvature. Then, in [5], M.T.K Abbassi and M. Sarih showed that the Cheeger-Gromoll metric cannot have a constant sectional curvature. The Cheeger-Gromoll metric itself was generalized in the 2000s by Benyouness & *al* [50] in the framework of *harmonic sections* [49].

Going back to the 1950s, C. Ehresmann developed his theory of *espaces généralisés* and *prolongations of manifolds* (cf. [35]). This was the fruit of a long lasting work on the theory of connections and 'prolongations of manifolds'. The central objects in his theory were *jets* and

jet manifolds. Those ideas allowed the introduction of many generalizations of the tangent bundle. If we adopt the kenetic point of view of tangent vectors, that is considering them as velocities of curves in the manifold, then one can define 'higher order velocities' using the concept of a jet (cf. [35]). Analogically, *higher order frames* were introduced, and generalizations of the *general linear group* were also defined using jets, namely *the differential groups*.

On the other side, the first half of the twentieth century was the golden age of algebraic topology. The interest in categories and functors as a unifying language was increasing. Tangent bundles give a functor on the category of smooth manifolds with smooth maps as arrows. Generalizing this fact to many other examples, one can define many vector bundles by making suitable choices of categories and functors. This has given rise to the theory that we know today as *natural vector bundle theory* (cf. [35,47,53]). Back at the time, most of the known examples of vector bundles were natural vector bundles, hence a classification problem was on the horizon. Interestingly, in 1976, R.S. Plais and C.L. Terneg showed that all natural bundles which have *finite order*, a condition on jets, are in fact *associated bundles* to the *principal bundles* of higher order frames [63].

By the 1980s, jet and natural vector bundle theories were mature enough to generate a revolution in our understanding of bundle theory. Based on two remarks, firstly most of the lifts of Riemannian metrics on a base manifold to metrics on its tangent bundle can be seen as *natural transformations* of certain functors, and secondly all this lifts depend on the first jet of the metric seen as a section of a certain vector bundle, O. Kowalski and M. Sekizawa have given a classification of those natural transformations, and hence a classification of all the *natural lifts* of metrics to tangent bundles (cf. [45]). Their work revived the interest in the Riemannian geometry of tangent bundles and lead to the discovery of the vast class of g-natural metrics on tangent bundles [2, 3, 7, 45]. g-natural metrics is a class of metrics depending on six smooth functions of the square norm of vectors. The Riemannian geometry of those metrics was the subject of dozens of works by many renewed geometers. For details on their properties, we refer to [2, 3, 7, 44, 45] and the references therein. Furthermore, all the metrics constructed so far on tangent bundles were particular cases of g-natural metrics. Moreover, every g-natural metric on a tangent bundle induces a metric on *tangent sphere bundles*. For details, we refer to [2–4, 7] and the references therein.

As mentioned earlier, vector bundles arise as immediate generalizations of tangent bundles. So, it is totally natural to generalize what has been done to the framework of vector bundles, since the Riemannian geometry of tangent bundles was a great source of examples of interesting geometries as well as a 'lab' for testing many geometric results. On top of this, since Einstein's first attempt to construct a *unified field theory* and the introduction of *Yang-Mills theories*, it became clear that vector bundles are indispensable as a language for physics to write the equations of field theories. One could say that the twentieth century physics is a physics of unification, and unification requires vector bundles. Many current physical problems concerns *gauge field theories* and *Yang-Mills theories*, which are equations written in some specific vector bundles [33,80]. As many physicists suggest, in particular Lee Somlin, that we should deepen our understanding of the structure of space-time and eventually the spaces where we write down our physical equations [71]. Furthermore, line bundles in particular, and vector bundles in general, are increasingly playing a role in algebraic geometry, especially elliptic curves theory, in complex geometry, Calabi-Yau geometry (e.g. the *Picard group* [40]). The Riemannian geometry of vector bundles is a contribution to deepen our understanding of vector bundles, which may lead to a solution to problems related with the areas listed above as well as in Riemannian and differential geometry.

Now, consider a vector bundle (E, π, M) with a Riemannian base. Assume that E is endowed with a *fiber metric* h and a *connection* D *compatible* with h. Using this data, many authors have generalized some of the metrics cited above. For example, the Sasaki metric has been generalized to a Riemnannian metric on vector bundles. It is defined in a straightforward

manner and was called also the *Sasaki metric*. This was considered in the context of *harmonic sections* by J. J. Konderak (cf. [43]). Other examples of metrics that were generalized to the case of vector bundles in the study of harmonicity are the *generalized Cheeger-Gromoll metrics* (cf. [49]). The generalized Cheeger-Gromoll metrics were studied by M. Boucetta and H. Soufi in the context of Euclidean Lie algebroids (cf. [19]).

Though tangent bundles are special cases of vector bundles, they enjoy a very rich structure. This can be explained by three facts. Firstly, the local coordinates of the tangent bundle manifolds, and hence the local trivialisations of the tangent bundles, stem directly from local coordinates of the base manifold. Secondly, an affine connection on the base manifold is, by definition, a connection on the total space. Thirdly, a Riemannian metric on the base gives inner products on fibers. This is a double-egged sword because the geometry of the total space will always be closely related to that of the total space, and this prevent us from constructing interesting geometries without first assuming that the base manifold possesses that property. This is also due to the existence of the zero section. In the case of a vector bundle in general both the inner products on fibers as well as the connection are extra-ingredients which we provide. So, there is no reason *a priori* that the geometries will be closely related as in the case of tangent bundles. As matter of fact, one can hope to overcome many rigidities encountered in the case of tangent bundles. This is another sufficient reason to pursue a research on the Riemannian geometry of vector bundles in full generality.

The construction of all the metrics on tangent bundles, and eventually vector bundles in the large, rely on the decomposition of the tangent spaces to the total space, thanks to the connection, into *horizontal* and *vertical* parts. In the realm of tangent bundles, a *horizontal* (resp. *vertical*) vector is a *horizontal* (resp. *vertical*) *lift* of a vector tangent to the base. It is well known that *horizontal lifts* exist always thanks to the connection. But when it comes to *vertical lifts*, one should be careful. In the case of tangent bundles, the identification of one forms on M with functions on TM allows us to lift tangent vectors to the base into vertical tangent vectors to the total space. In this case, since the inner products on fibers are nothing but the Riemannian metric on the base manifold, one could make easily 'combinations' of horizontal and vertical vectors, which explains the largeness of the class of g-natural metrics. Unlike the previous case, for a general vector bundle, one could only lift vertically elements of the fibers and not tangent vectors to the base. This is a hurdle, since there is no way to make the horizontal and vertical lifts 'interact' between them.

Undoubtedly, tangent bundles are particular cases of vector bundles. But, there is a fundamental difference between the Riemannian geometries of tangent bundles and vector bundles in general. This is due to the arbitrariness in the choice of the necessary geometric data to the construction of metrics on vector bundles. More precisely, while the connection on a tangent bundle comes from the Riemannian metric on the base, it is an extra-data we provide for a vector bundle in general. Indeed, while for a tangent bundle, the inner products on fibers come from the Riemannian metric on the base, it is an extra-structure on a vector bundle in general. All at all, the Riemannian geometry of a vector bundle is based on the following data:

- The connection;
- The Rimannian metric on the base;
- The weight functions of the metrics.

Hence the manipulation of the previous geometric data, both in computations and analysis, is the only way to study the geometry of vector bundle manifolds, and that's what we will be doing in this thesis.

Motivated by the search of manifolds with *special holonomy* and the general form of g-natural metrics, R. Albuquerque introduced a class of Riemannian metrics on vector bundle

manifolds, the so-called *spherically symmetric metrics*(cf. [13]). This is a class of Riemannain metrics with two weight functions depending only on the square norm of vectors. This thesis is devoted to the study of different aspects of the class of spherically symmetric metrics. The author of [13] has accomplished some computations of curvature in the case of flat connections and also computations of the curvature on the zero section in the general case. He derived the ODEs describing geodesics and he tried to find some of them. R. Albuquerque has given illuminations and left many problems for further study. Finally, he computed the holonomy of a special spherically symmetric metric on the spaces of self-dual and anti-self dual 2-forms.

To the best of our knowledge, the Riemannian geometry of vector bundles is still unexplored except some partial results mostly in the context of harmonicity. In this thesis, we will study different aspects of spherically symmetric metrics. Unlike the case of tangent bundles, we will literally develop from scratch the necessary technical machinery for computations and analysis.

We will scrutinize *spherically symmetric metrics*. The study will be divided into three lines. First of all, we will study the completeness and geodesics. The second line deals with 'first order geometry', that is the symmetries of spherically symmetric metrics. Finally, we come to 'second order geometry', that is curvature. This thesis is divided into five chapters. The first chapter contains the necessary notions of vector bundles, connections on them and a short survey of Riemannian metrics on vector bundles. The other four chapters are the synthesis of the results from our papers [8–12].

It is noteworthy that for the general framework of vector bundles, it was very difficult to make the computations using the basic formulas. Hence we had to search for technical alternatives. Fortunately, the projections of vector bundles when endowed with spherically symmetric metrics are *horizontally conformal submersions*, that is a generalization of *Riemannian submersions* in the following sense: instead of assuming that the projection is an isometry, we multiply by a conformal factor called the *dilation function*. This concept comes from the theory of harmonic morphisms. Indeed, Ishihara (cf. [37]) and Fuglede (cf. [26]), have independently characterised harmonic morphisms, under a dimensionality assumption, as harmonic maps which are *horizontally conformal maps*. In [61], B. O'Neill studied Riemannian submersions in the same spirit of sub-Riemannian geometry. He introduced two tensors associated with a submersion, the so-called *fundamental tensors of a submersion*. Further, O'Neill derived the equations that relate the curvatures of the total space and base of a Riemannian submersion. He gave the equations analogous to Gauss and Codazzi equations, such equations were called the *fundamental equations*. Then, in his doctoral thesis, S. Gudmundsson generalized O'Neill's work to the case of horizontally conformal submersions. Our computations relay in a fundamental way on the approach of O'Neill and Gudmundsson (cf. [29,30,61,62]).

As mentioned above, the first chapter is devoted to recall the concepts of vector bundles and connections on them. Other relevant notions are given in the appendices. We have tried to make this thesis self-contained while making it as concise as possible. We avoided to load it with too much notions from the literature. So, any relevant notion which will not be needed in the core is sent to the appendices.

The second chapter deals with the questions of geodesics and completeness. In the light of O'Neill's approach to geodesics [62], we have derived the equations describing geodesics and we have given some of their classes. Concerning the completeness of spherically symmetric metrics, R. Albuquerque conjectured that the total space when endowed with a spherically symmetric metric is complete if and only if the fibers and base are complete (cf. [13]). We have given an affirmative answer without assuming the completeness of fibers but under a boundedness assumption on the dilation function. The proof of our main theorem is based on the Hopf-Rinow theorem and techniques of metric topology (cf. [11]).

The third chapter is devoted to the study of symmetries of spherically symmetric metrics.

It is a summary of our results in [8]. We begin first by studying closed vector fields. Then we give a detailed exposition of different infinitesimal transformations: Killing and conformal vector fields. Finally, we specialize to gradient vector fields. Moreover, we give some classification results of horizontal Killing vector fields and parallel vector fields under some assumptions on the weight functions.

The fourth chapter is a synthesis of our results in [9, 12]. We give the formulas of the curvature tensor, the Ricci tensor, the sectional curvature and the scalar curvature both in the general case and in the case of a flat connection. We will analyse the formulas in general, then we will give the corresponding results in the special case of flat connections. We have adopted an approach from general to particular: first arbitrary connections, then flat connections. It is the natural order of things. We begin by proving that many geometric properties are hereditary, that is once the total space has one of those properties, the base manifold necessarily possesses the same property. Next, we try to answer the converse, that is assuming the base manifold has a remarkable geometric property, one searches the necessary and sufficient conditions for the total space to have the same property. The aim is to find interesting geometries among spherically symmetric metrics. We proceed from stronger to weaker geometric properties. So, we begin by a study of sectional curvature. We characterize flat and spherically symmetric metrics with constant sectional curvatures, then we try to find those with bounded sectional curvatures. Next, we study the Ricci curvature and spaces with special Ricci tensors e.g. Einstein metrics. In the case of flat connections, we will construct a 1-parameter family of Einstein metrics. Finally, we give some families of spherically symmetric metrics with constant scalar curvature.

The last chapter deals with harmonic sections. It is a summary of our results in [10]. At the beginning we will review the concepts of harmonicity and related notions. Then, we proceed directly to a detailed study of harmonic sections. We shall consider three situations:

- Harmonic sections as maps between two Riemannian manifolds;
- Harmonic sections as critical points of the energy functional with variations through sections;
- Vertically harmonic section, that is the critical points of the vertical energy.

In each one of these situations, we give the Euler-Lagrange equations and we give necessary and/or sufficient conditions for some sections to be harmonic. One of the aims is the generalization of Konderak's results as well as some of Benyounes *&al'*. We give also concrete examples of harmonic sections.

It is worth mentioning that we didn't include the actual computations for the sake of being concise. But, most of the computations are either peer reviewed [8, 11, 12], or they are confirmed, since they agree with the computations made in the case of flat connections [12, 13] and on the zero section [13].

Part I

Preliminaries

Chapter 1

Preliminaries

This chapter is a brief exposition of the theory of smooth vector bundles, it contains the necessary ingredients for the subsequent sections. Most of the material can be found in classical textbooks and treatises on the subject for example [35, 42, 53, 59, 65, 72, 78] for vector bundle theory in the large, for natural vector bundles and jet manifolds, much of the material can be found in [35,46,47,53]. This chapter contains also a precise and concise account of connection theory on vector bundles. Moreover, the last section is a short survey of Riemannian metrics on vector bundle manifolds as well as O'Neill's theory of fundamental tensors and fundamental equations in both frameworks: Riemannian submersions and horizontally conformal submersions (cf. [29,30,61,62]).

All objects are assumed to be smooth and smooth means infinitely differentiable. Manifolds are assumed to be connected and without boundary.

1.1 Vector bundle theory

1.1.1 Vector bundles in the large

Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$.

Definition 1. A $\mathbb{K}$-vector bundle (or vector bundle over $\mathbb{K}$) of rank k and standard fiber V is a triple (E, π, M) where E and M are smooth manifolds, V is a k-dimensional $\mathbb{K}$-vector space such that:

1. $\pi : E \longrightarrow M$ is a smooth mapping,
2. for all $x \in M$, $E_x = \pi^{-1}(x)$ is a $\mathbb{K}$-vector space of dimension k,
3. E is locally trivial i.e. for all $m \in M$, there is a neigborhood U of m and a fiber preserving diffeomorphism $\psi : \pi^{-1}(U) \longrightarrow U \times V$, that is a diffeomorphism such that the following diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times V \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & U & \end{array}$$

is commutative with $\psi_{|E_x} : E_x \longrightarrow V$ is a vector space isomorphism, with $pr_1 : U \times V \longrightarrow U$ is the projection on the first factor.

In this case, E is called the *total space*, M is the *base*, V is the *typical fiber* or the *standard fiber* and π is called the *projection*. Finally, the subset $E_x = \pi^{-1}(x)$ is called the fiber of E over $x \in M$.

We shall refer to a vector bundle only by its total space and we will omit the rank as long as we don't need it in the context. When $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{K} = \mathbb{C}$), we speak about *real* (resp. *complex*) vector bundles .

Remarks 1. Let (E, π, M) be a $\mathbb{K}$-vector bundle of rank k, then

1. The pairs (U, ψ) in the previous definition are called *vector bundle charts* or *local trivializations*. We will make use of both of them indifferently. Further, a collection of vector bundle charts covering the base is called a *vector bundle atlas*.

2. Since E is *locally trivial*, the projection is a surjective submersion. Indeed, surjectivity is clear. To prove that π is a submersion, let $e \in E$, $m = \pi(e)$ and (U, ψ) a vector bundle chart around m, then $pr_1 \circ \psi = \pi$, by passing to differentials on both sides we conclude that π is a submersion since pr_1 is a submersion.

3. Since the projection is a surjective submersion, the fibers are closed submanifolds of the total space.

4. When the rank $k = 1$, we speak about a *line bundle*.

5. The dimension of the total space is given by $\dim(E) = n + k$, with $\dim(M) = n$.

Because we aim to study the Riemannian geometry of vector bundles, it is worthy to consider only real vector bundles. Furthermore, we can always pass from a complex vector bundle to a real one. More precisely, let (E, π, M) be a complex vector bundle, since every complex vector space is canonically a real vector space of real dimension twice the complex dimension, and $\mathbb{C}$ is identified with $\mathbb{R}^2$, each fiber of E is a real vector space and each complex vector bundle chart induces a real vector bundle chart of E. This process is called the *realification* of E (cf. [65] for details). Henceforward, we consider only real vector bundles.

It is worth mentioning that vector bundles may be introduced through the language of vector bundle charts and vector bundle atlases with a relation of compatibility. This formalism is developed in [35] and [53].

Definitions 1. Let E and M be two smooth manifolds.

(i) Let $\pi : E \longrightarrow M$ be a smooth mapping between E and M, and let V be a k-dimensional real vector space. A vector bundle chart on (E, π, M) is a pair (U, ψ), where U is an open subset in M and ψ is a fiber preserving diffeomorphism i.e. a diffeomorphism such that the following diagram:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times V \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & U & \end{array}$$

is commutative.

(ii) Two vector bundle charts (U_1, ψ_1) and (U_2, ψ_2) are said to be compatible if $\psi_1 \circ \psi_2^{-1}$ is a fiber linear isomorphism, i.e. $(\psi_1 \circ \psi_2^{-1})(x, v) = (x, \psi_{1,2}(x)v)$ for some smooth mapping $\psi_{1,2} : U_1 \cap U_2 \longrightarrow GL(V)$ and for all $x \in U_1 \cap U_2$ and $v \in V$. The mapping $\psi_{1,2}$ is then unique and smooth and is called the *transition function* between the two vector bundle charts.

(iii) A vector bundle atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ for (E, π, M) is a set of pairwise compatible vector bundle charts (U_α, ψ_α) such that $(U_\alpha)_{\alpha \in I}$ is an open cover of M. Two vector bundle atlases are said to be equivalent if their union is again a vector bundle atlas.

(iv) A vector bundle (E, π, M) consists of two manifolds E (called *the total space*) and M(called *the base*), and a smooth projection map $\pi : E \longrightarrow M$ together with an equivalence class of vector bundle atlases.

It is not hard to see the equivalence between the two languages, but we prefer the first definition since it makes clear the local triviality and allows one to stay in touch with all of its elements, besides the fact that the existence of local trivializations matters more than a particular choice of a vector bundle atlas.

Remark 1. In practice, vector bundles are constructed by exhibiting a set E which will eventually be the total space together with a 'base manifold' M, a surjective map $\pi : E \longrightarrow M$ and pairs (U, ψ) where U is open in M and $\psi : \pi^{-1}(U) \longrightarrow U \times V$ which is fiberwise linear, where V is a fixed finite dimensional vector space, then showing the compatibility of the exhibited pairs will give at the same time a smooth structure on E and a vector bundle structure on (E, π, M).

Examples 1.

1. **The trivial vector bundle:** let M be a smooth manifold and V be a k-dimensional vector space, then $(M \times V, pr_1, M)$ is a vector bundle of rank k and standard fiber V, where pr_1 is the projection on the first factor of $M \times V$.

2. **The universal line bundle over $\mathbb{R}P^n$:** the real projective space $\mathbb{R}P^n$ is the quotient of $\mathbb{R}^{n+1}$ by the equivalence relation $\sim$ defined as: $u \sim v$ if and only if there exists $\lambda \in \mathbb{R}\backslash\{0\}$ such that $u = \lambda v$. Let $p : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}P^n$ be the natural projection. Denote by U_j the subset of $\mathbb{R}P^n$ defined by $U_j = \{p(z_1, ..., z_{n+1}) : z_j \neq 0\}$. Define $\psi_j : p^{-1}(U_j) \longrightarrow \mathbb{R}$ the map given by $\psi_j(z_1, ..., z_{n+1}) = z_j$, hence $\mathbb{R}P^n$ possesses a structure of a line bundle for which the vector bundle charts are $(U_j, (p, \psi_j))$, for $j = 1, ..., n + 1$.

3. **The tangent bundle to a manifold:** Let M be an n-dimensional manifold, $x \in M$ and T_xM the tangent space to M at x. Consider

$$TM = \bigsqcup_{x \in M} T_xM,$$

and the map $\pi : TM \longrightarrow M$ with $\pi(v) = x$, for all $v \in T_xM$. For a manifold chart (U, φ) on M, define $\widetilde{\varphi} : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ with $\widetilde{\varphi}(x, v) = (\varphi(x), d_x\varphi(v))$. The pairs $\{(\pi^{-1}(U), \widetilde{\varphi})\}$ induce a topology and a smooth structure on TM for which (TM, π, M) is a vector bundle with standard fiber $\mathbb{R}^n$.

It is natural to ask how many vector bundles with a fixed standard fiber could be constructed over a fixed base. The attempt to answer such a question has led to the important *reconstruction process* that we aim to recall in what follows. It is worth mentioning that the topological part of the question has led to K-Theory and characteristic classes, but we are interested here in the classification of smooth vector bundles, so the topological study is out of the scope.

First, let (E, π, M) be a vector bundle with standard fiber V and let $\{U_\alpha\}$ be a covering of M such that $\{(U_\alpha, \psi_\alpha)\}$ is a vector bundle atlas for E. Then, for α, β such that $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, we have $\psi_\alpha \circ \psi_\beta^{-1}(x, v) = (x, \psi_{\alpha\beta}(x)(v))$, for all $x \in M$ and $v \in V$, for a smooth map $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(V)$ (cf. [35]). This family of transition functions satisfies the following identities:

$$\begin{cases} \psi_{\alpha\beta}(x)\psi_{\beta\alpha\gamma}(x) = \psi_{\alpha\gamma}(x), \text{ for each } x \in U_\alpha \cap U_\beta \cap U_\gamma, \\ \psi_{\alpha\alpha}(x) = \mathrm{Id}_V, \text{ for each } x \in U_\alpha. \end{cases} \tag{1.1.1}$$

We say that the collection $(\psi_{\alpha\beta})$ satisfies the *cocycle condition*, and we call it the *cocycle of transition functions* for the vector bundle atlas $\{(U_\alpha, \psi_\alpha)\}$.

Suppose that $\{(U_\alpha, \varphi_\alpha)\}$ is another vector bundle atlas over the same cover $\{U_\alpha\}$. The compatibility of the charts $(U_\alpha, \varphi_\alpha)$ and (U_α, ψ_α) implies that there exists a smooth map $\tau_\alpha : U_\alpha \longrightarrow GL(V)$ such that $(\varphi_\alpha \circ \psi_\alpha^{-1})(x, v) = (x, \tau_\alpha(x)(v))$ for all $x \in U_\alpha$ and $v \in V$. A simple computation gives

$$\tau_\alpha(x)\psi_{\alpha\beta}(x) = \varphi_{\alpha\beta}(x)\tau_\beta(x), \text{ for all } x \in U_{\alpha\beta}. \tag{1.1.2}$$

We say that the two cocyles of transition functions $(\varphi_{\alpha\beta})$ and $(\psi_{\alpha\beta})$ are *cohomologuous*. The resulting *cohomology classes* over the open cover with values in the sheaf $\underline{GL}(V) = C^\infty(., GL(V))$ combine to a set $\check{H}^1((U_\alpha), \underline{GL}(V))$ called the first Čech cohomology of the open cover (U_α) with values in the sheaf $\underline{GL}(V)$ (cf. [35, 53]).

For a refinement $(V_i)_{i\in I}$ of $(U_\alpha)_\alpha$, with $V_i \subseteq U_{\varepsilon(i)}$, for some refinement mapping $\varepsilon : I \longrightarrow A$, any cocycle $(\psi_{\alpha\beta})$ over (U_α) induces the cocycle $\varphi_{ij} = \left(\psi_{\varepsilon(i)\varepsilon(j)}\right)_{|W_{ij}}$ denoted by $\varepsilon^*(\psi_{\alpha\beta})$. The mapping ε^* respects the cohomology relations and induces a mapping $\varepsilon^\sharp : \check{H}^1((U_\alpha), \underline{GL}(V)) \longrightarrow \check{H}^1((W_i), \underline{GL}(V))$. This allows one to pass to the inductive limit $\varinjlim \check{H}^1(\mathcal{U}, \underline{GL}(V)) =: \check{H}^1(M, \underline{GL}(V))$ over all open covers of M. This gives the so-called *first Čech cohomology* of M with values in the sheaf $\underline{GL}(V)$, see [35] for more details.

So, two vector bundle atlases of the same vector bundle give cohomologuous cocycles of transition functions. In what follows, we will investigate the inverse process, that is using a finite dimensional vector space, an open cover of a manifold with a collection of functions such as those in (1.1.1), one can construct a vector bundle for which the given collection of functions is its cocycle of transition functions.

Let V be a real finite dimensional vector space. Given an open cover $\mathcal{U} = \{U_\alpha\}$ of the manifold M together with a collection of smooth mappings $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(V)$ defined for α, β with $U_\alpha \cap U_\beta \neq \emptyset$. We assume that the collection $(\psi_{\alpha\beta})$ satisfies the cocycle condition given by equations (1.1.1).

Consider the set $\bigcup_\alpha \left(\{\alpha\} \times U_\alpha \times V\right)$ and define on it the relation

$$(\alpha, x_1, v) \sim (\beta, x_2, w) \text{ if and only if } x_1 = x_2 \text{ and } \psi_{\beta\alpha}(x_1)(v) = w.$$

By virtue of the cocycle condition (1.1.1), it follows that the relation $\sim$ is an equivalence relation. The equivalence class of (α, x, v) is denoted by $[\alpha, x, v]$, and the set of equivalence classes is denoted by $E = VB((\psi_{\alpha\beta}))$ and it is endowed with the quotient topology. Consider the map $\pi : E \longrightarrow M$ defined by $\pi([\alpha, x, v]) = x$. For each α, consider the map $\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times V$ with $\psi_\alpha([\alpha, x, v]) = (x, v)$, for all $[\alpha, x, v] \in E$. Then, for α, β with $U_\alpha \cap U_\beta \neq \emptyset$, we have $\psi_\alpha \circ \psi_\beta^{-1}(x, v) = (x, \psi_{\alpha\beta}(x)(v))$ which is clearly smooth. Hence (E, π, M) is a smooth vector bundle with standard fiber V for which $\{(U_\alpha, \psi_\alpha)\}$ is a vector bundle atlas. We refer to this process as *reconstruction*.

The previous construction shows that the structure of a vector bundle is totally encoded within a vector space (it would be the standard fiber), a cover of a manifold and a cocycle of transition functions satisfying (1.1.1).

Using arguments from topological dimension theory, it has been proved that:

Theorem 1. *(see [35]) Every vector bundle has a finite vector bundle atlas.*

Theorem 2. *(see [35]) For every vector bundle (E_1, π_1, M), there is a second vector bundle (E_2, π_2, M) such that $(E_1 \oplus E_2, \pi, M)$ is the trivial vector bundle.*

1.1.2 Vector bundle morphisms

Definition 2. Let (E_1, π_1, M_1) and (E_2, π_2, M_2) be two vector bundles. A smooth map $\Phi : E_1 \longrightarrow E_2$ is called a *vector bundle morphism* if for all $x \in M_1$, there exists a point $y \in M_2$ such that:

1. $\Phi(E_{1,x}) \subset E_{2,y}$;
2. The induced mapping $\Phi_x : E_{1,x} \longrightarrow E_{2,y}$ is a linear map.

In particular, Φ is a *vector bundle isomorphism* if it is invertible and its inverse is also a vector bundle morphism.

If $\Phi : E_1 \longrightarrow E_2$ is a vector bundle morphism, then Φ induces a map $\varphi : M_1 \longrightarrow M_2$ such that $\varphi \circ \pi_1 = \pi_2 \circ \Phi$, one says that Φ *covers* φ. Further, since Φ is smooth, φ is also smooth.

The composition of vector bundle morphisms is again a vector bundle morphism. Moreover, inverse bundle isomorphisms induce inverse diffeomorphisms between the base manifolds.

For a vector bundle morphism Φ to be an isomorphism it is sufficient for Φ to be a diffeomorphism. Furthermore, vector bundle isomorphisms are characterized as follows:

Proposition 1. *(see [78]) Let (E_i, π_i, M_i), for $i = 1, 2$, be vector bundles and let $\Phi : E_1 \longrightarrow E_2$ be a vector bundle morphism inducing $\varphi : M_1 \longrightarrow M_2$ between the base manifolds. Then Φ is an isomorphism if and only if*

1. *φ is a diffeomorphism,*
2. *$\Phi_x : E_{1,x} \longrightarrow E_{2,\varphi(x)}$ is a linear isomorphism, for all $x \in M$.*

Corollary 1. *Let (E_i, π_i, M), for $i = 1, 2$, be vector bundles over the same base and let $\Phi : E_1 \longrightarrow E_2$ be a vector bundle morphism covering the identity. Then Φ is an isomorphism if and only if each fiber $\pi_1^{-1}(p)$ is mapped isomorphically to $\pi_2^{-1}(p)$.*

Remark 2. Let E_1 and E_2 be two vector bundles over a manifold M. Denote by $\mathrm{Hom}(E_1, E_2)$ the set of vector bundle morphisms covering the identity map of M, and let $\Phi, \Psi \in \mathrm{Hom}(E_1, E_2)$. Then, for all $e \in E_1$ and $\lambda \in \mathbb{R}$, set

$$(\lambda\Phi + \Psi)(e) = \lambda\Phi(e) + \Psi(e).$$

Hence, $\mathrm{Hom}(E_1, E_2)$ is endowed with the structure of a real vector space.

Example 1. An important example of vector bundle morphism is the tangent mapping to a smooth map. Let $\varphi : M_1 \longrightarrow M_2$ be a map, then its tangent map $d\varphi : TM_1 \longrightarrow TM_2$ defined $d\varphi(x, v) = (\varphi(x), d_x\varphi(v))$, is a vector bundle morphism between the tangent bundles TM_1 and TM_2.

The correspondence between vector bundles with a fixed standard fiber over a fixed base and cocycles of transition functions may be formulated in the language of vector bundle morphisms as follows:

Theorem 3. *(see [35, 53]) There exists a bijective correspondence between the first Čech cohomology $\check{H}^1(M, \underline{GL}(V))$ and the set of isomorphism classes of vector bundles over M with standard fiber V*

Theorem 4. *(see [35]) Let (E, π_1, M) and (F, π_2, N) be vector bundles, and let $\Phi : E \longrightarrow F$ be a vector bundle morphism covering φ such that $\mathrm{rank}(\Phi_x : E_x \longrightarrow F_{\varphi(x)})$ is constant in $x \in M$. Then the kernel $\ker(\Phi)$, defined by $\ker(\Phi)_x := \ker(\Phi_x)$, for all $x \in M$, is a vector subbundle of E.*

1.1.3 Sections and frames

For the tangent bundle to a manifold M, vector fields are defined as smooth maps $X : M \longrightarrow TM$ such that $\pi_M \circ X = \mathrm{Id}_M$, where π_M is the canonical projection of TM onto M. Sections of arbitrary vector bundles are just generalizations of vector fields in the case of tangent bundles. Often physicist use the term 'field' to indicate sections of vector bundles regardless of the bundle under consideration.

Given a vector bundle (E, π, M) and an open set $U \subset M$. Set $E_U := \pi^{-1}(U)$ and consider the restriction of the projection $\pi_{|U} : \pi^{-1}(U) \longrightarrow U$, then $(E_U, \pi_{|U}, U)$ is a vector bundle.

Definition 3. Let (E, π, M) be a vector bundle. A (global) *section* of E is a map $\sigma : M \longrightarrow E$ such that $\pi \circ \sigma = \mathrm{Id}_M$. Accordingly, a *local section* is a map $\rho : U \longrightarrow E_U$, for some open subset in M, such that $\pi \circ \rho = \mathrm{Id}_U$.

Every vector bundle has a distinguished global section, namely the zero section $O : M \longrightarrow E$ such that $O(x) = 0_x$, with 0_x is the zero element of the fiber E_x.

Examples 2. Let M be a manifold and p, q be positive integers.

1. If $E = M \times \mathbb{R}$ is the trivial line bundle bundle, then sections of E are identified with real valued functions on M;

2. If $E = T^{p,q}M$ is the vector bundle of (p, q)-tensors on M, then sections are tensor fields of type (p, q);

3. If $E = \bigwedge^p T^*M$ is the vector bundle of skew-symmetric p-linear maps, then sections are differential p-forms.

Definition 4. Let σ be a section of E. The *support* of σ is the closure of the set $\{x \in M : \sigma(x) \neq 0_x\}$ and it is denoted by $\mathrm{supp}(\sigma)$.

The space of smooth sections of a vector bundle (E, π, M) is denoted $\Gamma(E)$. The pointwise addition and multiplication induce naturally on $\Gamma(E)$ the structure of a vector space. Furthermore, if $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$, then the product defined by

$$(f\sigma)(x) = f(x).\sigma(x),$$

for all $x \in M$, gives rise to a structure of $C^\infty(M)$-module on $\Gamma(E)$.

A vector bundle has 'many' sections. Indeed, consider a vector bundle atlas $\{(U_\alpha, \psi_\alpha)\}$ for E and let V be the standard fiber of E. For any smooth function $f_\alpha : U_\alpha \longrightarrow V$ the map $x \longmapsto \psi_\alpha^{-1}(x, f_\alpha(x))$ is a local smooth section of E. If (g_α) is a partition of unity subordinate to the open cover (U_α), then the map $\sigma : M \longrightarrow E$, with $\sigma(x) = \sum_\alpha g_\alpha(x)\psi_\alpha^{-1}(x, f_\alpha(x))$, is a global section of E.

Remark 3. It is important to note that a local section need not to be extendable to a global section. Indeed, consider the simplest case of the trivial line bundle $E = \mathbb{R} \times \mathbb{R}$ and let $\sigma : (0, +\infty) \longrightarrow E$ be the local section defined by $\sigma(x) = (x, \frac{1}{x})$, thus it is clear that σ can not be extended to a global section.

Proposition 2. *(see [66]) Let (E, π, M) be a vector bundle and let $\sigma \in \Gamma(E)$. Then (M, σ) is an embedded submanifold of E.*

In the case of the tangent bundles to manifolds M and N, if $f : M \longrightarrow N$ is a map, then two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be f-related if $df \circ X = Y \circ f$. This relation extends to arbitrary vector bundles as follows:

Definition 5. Let (E_1, π_1, M_1) and (E_1, π_2, M_2) be two vector bundles and let $\Phi : E_1 \longrightarrow E_2$ be a vector bundle morphism covering a map $\varphi : M_1 \longrightarrow M_2$. Two sections $\sigma_1 \in \Gamma(E_1)$ and $\sigma_2 \in \Gamma(E_2)$ are said to be Φ-*related* if $\Phi \circ \sigma_1 = \sigma_2 \circ \varphi$.

Example 2. Let M, N be two manifolds and let $f : M \longrightarrow N$ be a map. Then, two vector fields X and Y are f-related if and only they are df-related in the sense of Definition 5, where df is the differential of f.

In the case of tangent bundles, if $(U, x^1, ..., x^n)$ is a local coordinate system with $\frac{\partial}{\partial x^i}$ are the coordinate vector fields, then by their very definition, the system of vectors $\{\left(\frac{\partial}{\partial x^i}\right)_x\}$ is a basis for T_xM, for all $x \in U$. For arbitrary vector bundles, there exists the analogous notion of *frame*.

Definition 6. Let (E, π, M) be a vector bundle of rank k and let U be an open subset of M and $\mathcal{S} = \{\sigma_i\}_{1 \leq i \leq l}$, with $l \leq k$, a system of local sections of E over U. The system $\mathcal{S}$ is said to be *pointwise linearly independent* if $\{\sigma_i(x)\}_{1 \leq i \leq l}$ is a linearly independent system of vectors in E_x for all $x \in U$, in this case $\mathcal{S}$ is said to be an l-*frame*. If $l = k$, we speak about k-frame or just a local frame.

Thus local frames provide bases for the fibers, and hence they allow one to expand local sections. Local frames are ultimately related to vector bundle charts. Indeed, let E be a vector bundle with standard fiber V, if (U, ψ) is a vector bundle chart of E and $\{e_i\}_{1 \leq i \leq k}$ is a basis for V, then the maps defined by $\sigma_i : x \mapsto \psi_\alpha^{-1}(x, e_i)$ are local sections of E over U_α. Moreover, the system $\{\sigma_i\}$ is a local frame on U_α. Hence, local trivialisations generate local frames.

Conversely, if $\{\sigma_i\}$ is a local frame on U and $\{e_i\}$ is a basis for V, then consider the map $\psi : \pi^{-1}(U) \longrightarrow U \times V$ with $\psi(e) = (\pi(e), \sum_i \lambda_i e_i)$ where $e = \sum_i \lambda_i \sigma_i(\pi(e))$. It is not hard to see that ψ is a local trivialization of E.

Hence the equivalence between giving local frames and local trivializations. This correspondence shows that a vector bundle is trivial if and only if it has a global frame.

Definition 7. A manifold is said to be *parallelizable* if its tangent bundle is trivial.

Example 3. The spheres S^1, S^3 and S^7 are parallelizable. See [66] for a detailed proof.

Proposition 3. *(see [66]) Let (E, π, M) be a real vector bundle of rank k, and let $x \in M$ and $r \leq k$ be an integer.*

(i) Let $\{e_1, ..., e_k\}$ be a basis of E_x, then there exists an open neighbourhood U of x and a local frame $\{\sigma_i\}$ over U such that $\sigma_i(x) = e_i$ for all $i = 1, ..., k$.

(ii) Let $\sigma_1, ..., \sigma_r$ be local sections defined on neighbourhoods $U_1, ..., U_r$ of x, respectively, such that the system $\{\sigma_i(x)\}$ is linearly independent in E_x. Then, there exists an open neighbourhood $U \subset U_1 \cap ... \cap U_r$ such that the system $\{(\sigma_1)_{|U_1}, ..., (\sigma_r)_{|U_r}\}$ is a local frame in E over U.

(iii) Let $\{\sigma_1, ..., \sigma_r\}$ be a local r-frame over a neighbourhood U of x, then there exists local sections $\sigma_{r+1}, ..., \sigma_k$ over $V \subset U$ such that the system $\{(\sigma_1)_{|V}, ..., (\sigma_r)_{|V}, \sigma_{r+1}, ..., \sigma_k\}$ is a local frame over V.

1.1.4 The vertical subbundle

Theorem 2 implies that every vector bundle is 'included' in a trivial bundle. Further, many examples of vector bundles arise as 'included' in a 'bigger' vector bundle. The following definition makes it precise.

Definition 8. Let (E, π, M) be a vector bundle. A *vector subbundle* of E is a vector bundle F with the same base such that each fiber F_x is a linear subspace of E_x and for which the inclusion $i : F \longrightarrow E$ of total spaces is smooth.

In this section we will see that to each vector bundle corresponds a vector subbundle of the tangent bundle to the total space, namely the *vertical subbundle* defined as follows. For each $e \in E$ with $\pi(e) = x$, the fiber E_x is a closed submanifold of E. Denote by $i_x : E_x \longrightarrow E$ the embedding of E_x into E, and consider the subspace $\mathcal{V}_e := \mathrm{Im}(d_e i_x)$ of $T_e E$ called the *vertical subspace*. It is not hard to see that $\mathcal{V}_e = \ker(d_e \pi)$. The vertical subspaces combine into

$$\mathcal{V} = \bigsqcup_{e \in E} \mathcal{V}_e.$$

Consider the restriction of the natural projection $\pi_E : TE \longrightarrow E$ of the tangent bundle TE to the set $\mathcal{V}$ which is also denoted by π_E. Then $(\mathcal{V}, \pi_E, E)$ is a vector subbundle of TE (cf. [65]).

The elements of a vertical subspace are said to be *vertical vectors*. Accordingly, a vector field that lies completely in $\mathcal{V}$ is said to be a *vertical vector field*. Moreover, a curve in E is said to be *vertical* if it lies completely in one of the fibers of E, hence a vector is vertical if and only if it is tangent to a vertical curve. For $x \in M$ and $e, u \in \pi^{-1}(x)$, set $\gamma_{e,u}(t) = e + tu$ with $t \in (-\epsilon, \epsilon)$. Then $\gamma_{e,u}(t) \in \pi^{-1}(x)$, for any $|t| < \epsilon$, that is $\gamma_{u,e}$ is a vertical curve. Hence the *vertical lift of u at e* is the vertical vector given by $u^v_e = \dot{\gamma}_{e,u}(0)$. Using vertical lifts of elements of E to vertical vectors, sections of E can be lifted. Indeed, if σ is a section of E, consider the vertical vector field σ^v defined by

$$\sigma^v(e) = \sigma(\pi(e))^v_e,$$

where $\sigma(\pi(e))^v_e$ is the vertical lift of $\sigma(\pi(e))$ at e. For a section σ of E, the vertical vector field σ^v is called the vertical lift of σ. The *vertical lifts* of sections are vertical vector fields.

1.1.5 Operations on vector bundles

Almost all the operations on vector spaces are carried to vector bundles in a natural way. In this section, we review such generalizations which allows one to construct many new examples of vector bundles from old ones.

A direct approach

The dual vector bundle: let (E, π, M) be a vector bundle with standard fiber V. Consider the set

$$E^* = \bigsqcup_{x \in M} E^*_x,$$

and the map $\pi^* : E^* \longrightarrow M$ such that $\pi^*(\alpha) = x$ for all $\alpha \in E^*_x$, where E^*_x is the dual vector space of E_x. For a vector bundle chart (U, ψ) of E, set

$$\psi^* : (\pi^*)^{-1}(U) \longrightarrow U \times V^* \quad \text{with} \quad \psi^*(x, \alpha) = (x, \alpha \circ \varphi^{-1}_{|E_x}),$$

where V^* stands for the dual of V. Clearly any two vector bundle charts with overlapping domains are compatible. Hence, a vector bundle atlas on E induces a vector bundle atlas on E^* for which (E^*, π^*, M) is a vector bundle with standard fiber V^*, it is called the *dual vector bundle* of E.

Further, let $\{\sigma_i\}$ be a local frame of E over an open subset $U \subseteq M$. For every i, let $\sigma^i : M \longrightarrow E^*$ the map defined as follows: $\{\sigma^i(x)\}$ is the dual basis to $\{\sigma_i(x)\}$. Using the

local trivializations described above, one checks that the maps σ^i are in fact smooth local sections of E^*, which give a local frame of E^* called the *dual frame* of $\{\sigma_i\}$.

The pointwise evaluation mappings on fibers, that is the mappings $E_x \times E_x^* \longrightarrow \mathbb{R}$ such that $\langle v, \alpha \rangle = \alpha(v)$, combine to a natural pairing on local sections

$$\langle ., . \rangle : \Gamma(E_U) \times \Gamma(E_U^*) \longrightarrow C^\infty(U, \mathbb{R}) \quad \text{with} \quad (\sigma, s) \mapsto \langle s, \sigma \rangle = s(\sigma).$$

The direct sum: let (E_1, π_1, M) and (E_2, π_2, M) be two vector bundles with standard fibers V_1 and V_2, respectively. For each point $x \in M$ the vector space $E_{1,x} \oplus E_{2,x}$ denotes the direct sum of the two vector spaces $E_{1,x}$ and $E_{2,x}$, those spaces combine to a set

$$E_1 \oplus E_2 = \bigsqcup_{x \in M} E_{1,x} \oplus E_{2,x},$$

and consider the map $\pi^\oplus : E_1 \oplus E_2 \longrightarrow M$ with $\pi^\oplus(v) = x$, for $v \in E_{1,x} \oplus E_{2,x}$. For local trivializations (U, φ) and (U, ψ) of E_1 and E_2, respectively. Define the map

$$\chi : (\pi^\oplus)^{-1}(U) \longrightarrow U \times (V_1 \oplus V_2),$$

by

$$\chi(x, (v_1, v_2)) = (x, (\varphi_{|E_{1,x}}(v_1), \psi_{|E_{2,x}}(v_2)),$$

for all $(x, (v_1, v_2)) \in E$. It is not hard to see the compatibility between those induced charts. Hence they define a smooth structure on $E_1 \oplus E_2$ for which $(E_1 \oplus E_2, \pi^\oplus, M)$ is a vector bundle with standard fiber $V_1 \oplus V_2$ and rank $k = k_1 + k_2$ where k_i is the rank of E_i, for $i = 1, 2$. The vector bundle $E_1 \oplus E_2$ is called the *direct sum* (or the *Whitney sum*) of E_1 and E_2.

If σ_1 and σ_2 are smooth sections of E_1 and E_2, respectively. Then, consider the map $\sigma_1 \oplus \sigma_2 : M \longrightarrow E$ defined by $\big(\sigma_1 \oplus \sigma_2\big)(x) = \sigma_1(x) \oplus \sigma_2(x)$, for all $x \in M$. The map $\sigma_1 \oplus \sigma_2$ is a smooth section of $E_1 \oplus E_2$. Furthermore, given local frames $\{\sigma_i\}$ and $\{\rho_j\}$ of E_1 and E_2 over the same open subset of M, respectively, then $\{\sigma_i \oplus \rho_j\}_{i,j}$ is a local frame of $E_1 \oplus E_2$.

The tensor product: let (E_1, π_1, M) and (E_2, π_2, M) be two vector bundles with standard fibers V_1 and V_2, respectively. For each point $x \in M$, the vector space $E_{1,x} \otimes E_{2,x}$ denotes the tensor product of the two vector spaces, which yield the set

$$E_1 \otimes E_2 = \bigsqcup_{x \in M} E_{1,x} \otimes E_{2,x},$$

and define the map $\pi^\otimes : E_1 \otimes E_2 \longrightarrow M$ by $\pi^\otimes(v_1 \otimes v_2) = x$, for all $v_1 \otimes v_2 \in E_{1,x} \otimes E_{2,x}$. Now, let (U, φ) and (V, ψ) are local trivializations of E_1 and E_2, respectively, the map

$$\chi : (\pi^\otimes)^{-1}(U) \longrightarrow U \times (V_1 \otimes V_2)$$

defined by

$$\chi(x, (v_1 \otimes v_2)) = (x, (\varphi_{|E_{1,x}}(v_1) \otimes \psi_{|E_{2,x}}(v_2))),$$

for all $x \in M$ and $(v_1 \otimes v_2) \in E_{1,x} \otimes E_{2,x}$. Clearly the compatibility between two charts of the above type follows from the compatibility of the charts of E_1 and E_2. Consequently $E_1 \otimes E_2$ is endowed with a smooth structure for which the triple $(E_1 \otimes E_2, \pi^\otimes, M)$ is a vector bundle with standard fiber $V_1 \otimes V_2$ and rank $k_1 k_2$ where k_i is the rank of E_i, for $i = 1, 2$.

For sections σ_1 and σ_2 of E_1 and E_2, respectively, consider the map $\sigma_1 \otimes \sigma_2 : M \longrightarrow E_1 \otimes E_2$ defined by $\big(\sigma_1 \otimes \sigma_2\big)(x) = \sigma_1(x) \otimes \sigma_2(x)$. The smoothness of $\sigma_1 \otimes \sigma_2$ follows from that of σ_i, $i = 1, 2$, and $\sigma_1 \otimes \sigma_2$ becomes a smooth section of $E_1 \otimes E_2$. Furthermore, if $\{\sigma_i\}$ and $\{\rho_j\}$ are local frames of E_1 and E_2 on the same open set, respectively, then $\{\sigma_i \otimes \rho_j\}_{i,j}$ is a local frame of $E_1 \otimes E_2$.

A functorial approach to vector bundle operations

For two vector spaces V and W, we denote by $Hom(V,W)$ the vector space of linear maps from V to W, and for each vector space V, the map Id_V denotes the identity mapping of V.

Let $\mathcal{C}$ be the category of real finite dimensional vector spaces together with linear maps as morphisms. We shall restrict ourselves to a particular kind of functors, namely smooth functors. For basic definitions of categories and functors see Appendix **A**.

Definition 9. A *covariant smooth functor* $\mathcal{F}$ of one variable is a map $\mathcal{F}:\mathcal{C}\longrightarrow\mathcal{C}$ such that:

(i) for each $V\in Ob(\mathcal{C})$, $\mathcal{F}(V)\in Ob(\mathcal{C})$,

(ii) for all $V,W\in Ob(\mathcal{C})$ and $h\in Hom(V,W)$, $\mathcal{F}(h)\in Hom(\mathcal{F}(V),\mathcal{F}(W))$. Furthermore, this correspondence is smooth and satisfies the following:

(a) $\mathcal{F}(\mathrm{Id}_V)=\mathrm{Id}_{\mathcal{F}(V)}$, for all $V\in Ob(\mathcal{C})$;

(b) $\mathcal{F}(f\circ g)=\mathcal{F}(f)\circ\mathcal{F}(g)$, for all $f\in Hom(V,W)$, $g\in Hom(W,Z)$ and $V,W,Z\in Ob(\mathcal{C})$.

Examples 3. There is a lot of examples of smooth covariant functors:

1. For each $V\in Ob(\mathcal{C})$, assign the direct sum $V\oplus ...\oplus V$ of k copies of V, and for each $f\in Hom(V,W)$ assign the map defined by

$$f\oplus ...\oplus f(v_1\oplus ...\oplus v_k)=f(v_1)\oplus ...\oplus f(v_k).$$

This yields a smooth covariant functor called the *direct sum functor*.

2. For each $V\in Ob(\mathcal{C})$, assign $V\otimes ...\otimes V$ the tensor product of k copies of V, and to each $f\in Hom(V,W)$ assign the map defined as

$$f\otimes ...\otimes f(v_1,...,v_k)=f(v_1)\otimes ...\otimes f(v_k).$$

Hence, we obtain a smooth covariant functor called the *tensor product functor* .

3. For each $V\in Ob(\mathcal{C})$ assign the exterior algebra $\bigwedge(V)$, and for each $f\in Hom(V,W)$ assign the map defined by

$$\bigwedge(f)(v_1\wedge ...\wedge v_k)=f(v_1)\wedge ...\wedge f(v_k).$$

This yields a smooth covariant functor called the *exterior algebra functor* .

Definition 10. A *contravariant smooth functor* $\mathcal{F}$ of one variable is a map $\mathcal{F}:\mathcal{C}\longrightarrow\mathcal{C}$ such that:

(i) for each $V\in Ob(\mathcal{C})$, $\mathcal{F}(V)\in Ob(\mathcal{C})$,

(ii) for all $V,W\in Ob(\mathcal{C})$ and $h\in Hom(V,W)$, $\mathcal{F}(h)\in Hom(\mathcal{F}(W),\mathcal{F}(V))$. Furthermore, this correspondence is smooth and satisfies the following:

(a) $\mathcal{F}(\mathrm{Id}_V)=\mathrm{Id}_{\mathcal{F}(V)}$, for all $V\in Ob(\mathcal{C})$;

(b) $\mathcal{F}(f\circ g)=\mathcal{F}(g)\circ\mathcal{F}(f)$, for all $f\in \mathrm{Hom}(V,W)$, $g\in Hom(W,Z)$ and $V,W,Z\in Ob(\mathcal{C})$.

Examples 4. The basic example of smooth contravariant functor is the *duality functor*. More precisely, let $V\in Ob(\mathcal{C})$ and let V^* be the dual of V. For each $f\in Hom(V,W)$, assign the transpose map $f^*\in \mathrm{Hom}(W^*,V^*)$.

There is a third kind of smooth functors, namely mixed smooth functors of more than one variable, that is smooth functors which are covariant in some arguments and contravariant in others. Their definition is straightforward and comes in a natural way. It can be easily verified that $(V,W) \longrightarrow V^* \otimes W$ and $(f,g) \mapsto f^* \otimes g$, for all $V,W \in Ob(\mathcal{C})$ and $f,g \in \mathrm{Hom}(V,W)$, is a functor of this type. Another example is the functor $\mathrm{Hom}(.,.)$ itself, which is naturally isomorphic to $V^* \otimes W$ by the universal property of the tensor product.

Proposition 4. *(see [65]) Let $\mathcal{F}$ be a smooth functor of k variables on $\mathcal{C}$ and let (E_i, π_i, M), for $i = 1, ..., k$, be k vector bundles with standard fibers V_i, then*

$$\mathcal{F}(E_1, ..., E_k) = \bigsqcup_{x \in M} \mathcal{F}(E_{1,x}, ..., E_{k,x}),$$

is naturally a vector bundle over M with standard fiber $\mathcal{F}(V_1, ..., V_k)$.

Examples 5. Let $\mathcal{F}$ be one of the following functors:

1. The duality functor,
2. The tensor product functor,
3. The direct sum functor,
4. The functor $Hom(.,.)$,
5. The exterior algebra functor;

Given a vector bundle (E, π, M), proposition 4 gives a vector bundle structure on $\mathcal{F}(E)$. This way we recover all the operations on vector bundles described in the previous section.

It is worth mentioning that we may use the reconstruction process combined with this functorial approach in order to achieve the previous constructions. Indeed, as before let (E, π, M) be a vector bundle and $\{(U_\alpha, \psi_\alpha)\}$ is a vector bundle atlas for E with $(\psi_{\alpha\beta})_{\alpha,\beta}$ is its cocycle of transition functions. Let $\mathcal{F}$ be a covariant smooth functor on the category of finite dimensional spaces, then consider $\mathcal{F}(\psi_{\alpha\beta}) : U_\alpha \cap U_\beta \longrightarrow GL(\mathcal{F}(V))$ defined by $\mathcal{F}(\psi_{\alpha\beta})(x) = \mathcal{F}(\psi_{\alpha\beta}(x))$ for all $x \in U_\alpha \cap U_\beta$. It is clear that $\big(\mathcal{F}(\psi_{\alpha\beta})\big)_{\alpha,\beta}$ satisfies the cocycle condition given in (1.1.1) and the functor $\mathcal{F}$ respects the cohomology of cocycles, so there exist a unique vector bundle $(\mathcal{F}(E) = VB(\mathcal{F}(\psi_{\alpha\beta}), p, M)$, where p is the natural projection.

If we consider a contravariant smooth functor, then the construction remains the same provided that we take $\mathcal{F}(\psi_{\alpha\beta}^{-1})$ instead of $\mathcal{F}(\psi_{\alpha\beta})$. In the case of mixed smooth functors we mix those cocycles in a natural manner.

Hence all the operations on vector bundles may be constructed through the use of functors, cocycles of transition functions and the reconstruction process.

The induced vector bundle

Let (E, π, M) be a vector bundle with rank k and standard fiber V and let $\varphi : N \longrightarrow M$ be a map from some manifold N. Set

$$\varphi^* E = \{(x, v) \in N \times E : \ \varphi(x) = \pi(v)\}. \tag{1.1.3}$$

Consider the map $p_1 : \varphi^* E \longrightarrow N$ with $p_1(x, v) = x$, for $(x, v) \in \varphi^* E$, that is p_1 is the projection on the first factor. Thus $p_1^{-1}(x) = \{x\} \times E_{\varphi(x)}$, for all $x \in N$.

Proposition 5. *Let (E, π, M) be a vector bundle with standard fiber V, and let N be a manifold and $\varphi : N \longrightarrow M$ be a map, then*

(i) The triple (φ^*E, p_1, N) is a vector bundle with standard fiber V,

(ii) The vector bundle φ^*E is isomorphic to E via the projection on the second factor of φ^*E,

(iii) Every local section $\sigma : U \longrightarrow E$ induces a local section of φ^*E defined by

$$(\varphi^*\sigma)(x) = (x, \sigma(\varphi(x))),$$

for all $x \in \varphi^{-1}(U)$.

Definition 11. The vector bundle (φ^*E, p_1, M) is called the *induced vector bundle* or the *pullback* of E by φ. Further, for a section $\sigma \in \Gamma(E)$, the section $\varphi^*\sigma$ is called the *pullback section* of σ by φ.

Remark 4. Let (E, π, M) be a vector bundle and let $\varphi : N \longrightarrow M$ be a map, then the tangent bundle to the pullback bundle φ^*E is given by

$$T(\varphi^*E) = \{(u, v) \in TN \times TE : \ \varphi_* u = \pi_* v\}.$$

Examples 6.

(i) Let M be a manifold and TM be its tangent bundle. If $c : I \longrightarrow M$ is a smooth curve in M, then the pullback c^*TM is just the set of tangent vectors to c and its sections are vector fields along the curve c.

(ii) Let $E \longrightarrow S^1$ be the Möbius strip over S^1 given as the quotient of $\mathbb{R}^2 \simeq \mathbb{C}$ by the equivalence relation defined by: $(s_1, t_1) \sim (s_2, t_2)$ if and only if there exists $l \in \mathbb{Z}$ such that $s_2 = s_1 + 2\pi l$ and $t_2 = (-1)^l t_1$. Realize S^1 as a circle in $\mathbb{C}$ and let $\varphi_n : S^1 \longrightarrow S^1$ be the n-fold covering map, that is the map defined by $\varphi_n(z) = z^n$, for all $z \in S^1$. Since φ_n^*E is also a line bundle over S^1, we get

$$\varphi_n^*E = \begin{cases} E & \text{if } n \text{ is odd,} \\ S^1 \times \mathbb{R} & \text{if } n \text{ is even.} \end{cases}$$

For more details, we refer to [65].

(iii) If N is a manifold and (M, φ) is a submanifold, then φ^*TN is isomorphic to the tangent bundle to N.

Proposition 6. *Let (E, π, M) be a vector bundle and let $\varphi : N \longrightarrow M$ be a map. Then the vector bundles $\mathcal{V}\varphi^*E$ and $p_2^*\mathcal{V}E$ are isomorphic, where $\mathcal{V}E$ and $\mathcal{V}\varphi^*E$ are the vertical subbundles of E and φ^*E, respectively.*

Using vertical lifts of elements of E to vertical vectors, the vertical bundle can be seen as a pullback. More precisely, we have:

Proposition 7. *If (E, π, M) is a vector bundle, then the vertical subbundle $\mathcal{V}E$ is isomorphic to the pullback $\pi^\star E$ of E by its projection $\pi : E \longrightarrow M$. Further, $\pi^\star E$ is isomorphic to the direct sum $E \oplus E$.*

Those isomorphisms are essential tools, they will be used throughout this thesis. They allow one to identify vertical vectors with elements of the considered vector bundle which will be a crucial step in the definition of Riemannian metrics on vector bundles using inner products on fibers and Riemannian metrics on the base manifold. The structure that gives to a vector bundle inner products on fibers which vary smoothly with base points is the object of the next subsection.

1.1.6 Riemannian vector bundles

The fibers of a vector bundle possess a linear structure, hence it is possible to have additional structures on fibers e.g. inner products, complex structures or symplectic structures. In what follows we shall be interested in vector bundles with inner products on fibers which vary smoothly with the base points. Such vector bundles are called *Riemannian vector bundles* (or *Euclidean vector bundles*). This subsection is devoted to recall the classical theory of Riemannian vector bundles as well as an explicit construction of a fiber metric on a class of vector bundles with an additional structure.

Definition 12. A *Riemannian fiber metric* (or simply a *fiber metric*) on a vector bundle is a section of the vector bundle $(E \otimes E)^*$ such that for all $x \in M$, h_x is an inner product on E_x. A vector bundle endowed with a Riemannian fiber metric is said to be a *Riemannian vector bundle*.

An equivalent definition is that a fiber metric on a vector bundle E is the assignment to each point $x \in M$ of an inner product h_x in a differentiable manner in the following sense: for local sections $\sigma, \rho \in \Gamma(E_U)$, the function $U \ni x \longmapsto h_x(\sigma(x), \rho(x))$ is a smooth function on U.

Proposition 8. *(see [42, 65]) Every vector bundle E has a fiber metric.*

Given a vector bundle (E, π, M) with standard fiber $\mathbb{R}^n$, the existence of a fiber metric on a vector bundle E is equivalent to the existence of a reduction of the structure group from $GL(n, \mathbb{R})$ to the orthogonal group $O(n)$.

Remark 5.

(i) Proposition 8 works when the base is paracompact (cf. [42]), but since we required manifolds to be Hausdorff and second countable, paracompactness follows.

(ii) The $C^\infty(M)$-module structure on the space of sections of E plays a decisive role in the proof of Proposition 8, see [65] for details.

1.2 Connections on vector bundles

Given a smooth manifold M, a smooth function f on M may be differentiated in a natural way. The differentiation of smooth functions does need no additional structure other than the smooth structure of the manifold itself. When one tries to track 'variations' of other geometric objects, vector fields for example, there is no canonical way to do it and the introduction of an additional structure is in order.

A natural approach consist in generalizing the already known notion of derivative. For example, for a vector field X on M, one would like to define a derivative $\nabla X(x_0)$ by taking the value of X along a curve starting at x_0 at a given time t and then take the 'difference', then divide by t and take the limit at $t = 0$, by analogy with classical derivatives. But the difference does not make sense since the vectors $X_{x(0)}$ and $X_{x(t)}$ belongs to different vector spaces. One way to overcome this difficulty is to add a structure that allow a 'displacement' of vectors in a 'preferred way' so as to bring them to the same vector space in order to take the desired difference. This gives birth to the so-called connections or parallel displacements which are tools serving mainly to 'connect' fibers. What was said for manifolds and tangent vector fields holds for vector bundles and their sections.

Since vector bundles and principal bundles are closely linked to each other, more precisely each vector bundle arise as an associated bundle to its principal bundle of frames, connections arise as a particular manifestation of much deeper and more general structures, namely

Ehersmann connections on principal bundles. For the link between those notions, we refer the reader to the Appendix **D**.

This section serves as a precise and concise account of the theory of connections on vector bundles. Much of the material given here can be found in classical treatises on the subject(e.g. [35,42,65]). Though there are many ways to present connection theory on vector bundles, we choose the direct one. We define connections on vector bundles as covariant derivatives, then we derive parallel displacement and we link this definition to the more general one in the case of fiber bundles in the large. For an exposition of connections as covariant derivatives, we refer the reader to [39].

1.2.1 Generalities

Let (E, π, M) be a vector bundle. Recall that $\Gamma(E)$ denotes the space of smooth sections. It is a real vector space and a $C^\infty(M)$-module.

Definition 13. A *connection* (or a *covariant derivative*) on E is a map

$$D : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E), \quad (X, \sigma) \mapsto D_X\sigma$$

satisfying the following:

(i) $D_{X+Y}\sigma = D_X\sigma + D_Y\sigma$,

(ii) $D_{fX}\sigma = fD_X\sigma$,

(iii) $D_X(\sigma + \delta) = D_X\sigma + D_X\delta$,

(iv) $D_X(f\sigma) = (Xf)\sigma + fD_X\sigma$,

for all $X, Y \in \mathfrak{X}(M)$ and $\sigma, \delta \in \Gamma(E)$.

Examples 7.

1. Let $(E = M \times \mathbb{R}^k, \pi, M)$ be the trivial bundle. Sections of E are identified with $\mathbb{R}^k$-valued functions. For a function $f : M \longrightarrow \mathbb{R}^k$ and $X \in \mathfrak{X}(M)$, the map $D_X f := X(f)$ is a connection on the trivial bundle, it is called the *trivial connection*.

2. Let (M, g) be a Riemannian manifold. Then the Levi-Civita connection is a connection on the tangent bundle TM to M.

3. Let (E, π, M) be a vector bundle and $U \subset M$ be an open set. $E_{|U} = \pi^{-1}(U)$ is the restriction of E to U. If D is a connection on E, then D induces a connection on $E_{|U}$.

Let $\varphi : N \longrightarrow M$ be a map and φ^*E be the pullbcak of the vector bundle E. If σ is a section of E, then $\varphi^*\sigma = \sigma \circ \varphi$ is a section of φ^*E. Further, if D is a connection in E, then D induces the map $\varphi^*D : \mathfrak{X}(N) \times \Gamma(\varphi^*E) \longrightarrow \Gamma(\varphi^*E)$ defined by

$$(\varphi^*D)_X\varphi^*\sigma = \varphi^*(D_{\varphi_*X}\sigma)$$

for all $X \in \mathfrak{X}(N)$ and $\sigma \in \Gamma(E)$.

Proposition 9. *The map φ^*D is a connection on φ^*E called the* pullback connection *on φ^*E.*

Theorem 5. *Every vector bundle admits a connection.*

It is clear from (*iv*) of Definition 13 that connections does not behave like tensors. Nevertheless, for a connection D on E and a vector field $X \in \mathfrak{X}(M)$, the operator D_X is 'local' in the sense of the following:

Proposition 10. *Let X be a vector field on M. Then, for each section $\sigma \in \Gamma(E)$, the value of $D_X\sigma$ at a point $p \in M$ depends only on the values of σ in a open neighbourhood of p.*

A connection D on E satisfies a 'tensoriality' in X, hence the value of $D_X\sigma$ at a point $p \in M$, for a fixed section σ, depends on the value of X at p only. Indeed, this result stems from a more general fact on $C^\infty(M)$-linear maps between spaces of sections of vector bundles.

Proposition 11. *Let E and F be two vector bundles over the same base M and let*

$$\Psi : \Gamma(E) \longrightarrow \Gamma(F)$$

be a $C^\infty(M)$-linear map. Then there exists a unique vector bundle morphism $\Phi : E \longrightarrow F$ such that $\Psi(\sigma) = \Phi \circ \sigma$, for all $\sigma \in \Gamma(E)$.

Remark 6. It is worth mentioning that the space of sections of E is a finitely generated projective $C^\infty(M)$-module (cf. [78]). The last proposition establishes a link between morphisms of $C^\infty(M)$-modules and vector bundle morphisms covering the identity map of the base. This correspondence is one-to-one. In this sense, the structure of a vector bundle is totally encoded in its module of smooth sections.

Corollary 2. *Let D be a connection on E, and let σ be a section of E and $p \in M$. Then, for vector fields $X, Y \in \mathfrak{X}(M)$ such that $X_p = Y_p$, we have $(D_X\sigma)(p) = (D_Y\sigma)(p)$.*

To sum up, given a connection D on E, a section σ of E and a vector field X on M. Then the value of $D_X\sigma$ at a point $p \in M$ depends only on the value X_p and the values of σ in a neighbourhood of p.

Definition 14. Let D be a connection on E. A section of E is said to be *parallel* if $D_X\sigma = 0$, for all $X \in \mathfrak{X}(M)$.

More examples of connections can be constructed using the operations on vector bundles seen in Section 1. Indeed, to each operation, it corresponds canonically a connection constructed from the given connections.

Proposition 12. *Let (E, π_1, M) and (F, π_2, M) be two vector bundles. Assume that E and F are endowed with connections D^E and D^F, respectively. Then the maps defined by*

1. $$\big(D^E\big)^*_X\xi = X.\xi(\sigma) - \xi\big(D^E_X\sigma\big),$$
   for all $\sigma \in \Gamma(E)$, $\xi \in \Gamma(E^*)$ and $X \in \mathfrak{X}(M)$,

2. $$\big(D^E \oplus D^F\big)_X(\sigma \oplus \delta) = (D^E_X\sigma) \oplus (D^F_X\delta),$$
   for all $\sigma \in \Gamma(E)$, $\delta \in \Gamma(F)$ and $X \in \mathfrak{X}(M)$,

3. $$\big(D^E \otimes D^F\big)_X\big(\sigma \otimes \delta\big) = D^E_X\sigma \otimes \delta + \sigma \otimes D^F_X\delta,$$
   for all $\sigma \in \Gamma(E)$, $\delta \in \Gamma(F)$ and $X \in \mathfrak{X}(M)$;

are connections on the vector bundle E^*, $E \oplus F$ and $E \otimes F$, respectively.

Proposition 13. *Given two connections D^1 and D^2 on a vector bundle E, for smooth functions $\rho_1, \rho_2 \in C^\infty(M)$ such that $\rho_1 + \rho_2 = 1$, the map given by*

$$D = \rho_1 D^1 + \rho_2 D^2,$$

is also a connection on E.

Locally, a connection D may be represented by matrices. Indeed, given $\{e_p : p = 1, ..., k\}$ a local frame of E over an open set $U \subset M$, for every $X \in \mathfrak{X}(M)$ and for each $p = 1, ..., k$, one has

$$D_X e_p = \sum_{p=1}^{k} \omega_p^q(X) e_q, \tag{1.2.1}$$

where each $\omega_p^q \in \Omega^1(U)$. Thus a connection may be represented locally by a matrix $\omega = (\omega_p^q)_{p,q}$ with entries in $\Omega^1(U)$. Such a matrix is called a *connection matrix*.

Restricting to a smaller open set U if necessary, let $(U, x^1, ..., x^n)$ be a coordinate neighbourhood. Taking $X = \frac{\partial}{\partial x^i}$, we get $\omega_p^q(\frac{\partial}{\partial x^i}) = \Gamma_{ip}^q$ the Christoffel symbols of the connection D which are functions on U. The Christoffel symbols (Γ_{ip}^q) may be defined directly as the components of $D_{\frac{\partial}{\partial x^i}} e_p$, that is

$$D_{\frac{\partial}{\partial x^i}} e_p = \sum_q \Gamma_{ip}^q e_q,$$

for $i = 1, ..., n$ and $p = 1, ..., k$.

Given a matrix f with entries in $C^\infty(M)$, denote by df the matrix for which every entry is the differential of the corresponding entry of f. The following proposition gives how the connection matrix changes under a change of frames.

Proposition 14. *Let D be a connection on E, and let $\{e_p\}$ and $\{e'_p\}$ be two local frames of E on U and U', respectively, with $U \cap U' \neq \emptyset$. Denote by ω and ω' the connection matrices in those frames, respectively. Take $f = (f_p^q)_{p,q}$ to be the matrix where the entries are defined by*

$$e'_p = \sum_q f_p^q e_q,$$

for all $p, q = 1, ..., k$. Then the connection matrices on $U \cap U'$ are related by

$$\omega' = f.df + f^{-1}.\omega.f.$$

A more geometric vision of connections is given by *parallel displacement* or *parallel transport*. To define parallel transport along curves in M, we need first to deepen our understanding of connections. Though the value of $D_X\sigma$ at p depends only on the value of X at p and the values of σ in a neighbourhood of p, one can prove more. In fact the value of $D_X\sigma$ at p depends on the value of X at p and the values of σ along a curve γ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X_p$. Indeed, let γ be such a curve and σ be a section such that $\sigma(\gamma(t)) = 0$ near $t = 0$. Locally the section σ splits as

$$\sigma = \sum_k f_k e_k,$$

where $\{e_k\}$ is a local frame of E around p and (ω_l^k) is the connection matrix with respect to it. Then

$$D_X\sigma = \sum_k df_k(X) e_k + \sum_{k,l} f_k \omega_k^l(X) e_l,$$

which clearly vanishes at points such that $f_k(\gamma(t)) = 0$, for each t near 0.

Thus the covariant derivative D gives a way to differentiate sections along curves. If $\sigma \in \Gamma(E)$ and $\gamma : I \longrightarrow M$ is a curve in M, set $\sigma(t) = \sigma \circ \gamma(t)$, for all $t \in I$, which gives a section of E along γ. If $\{e_p\}$ is a local frame of E and $\{x^i\}$ is a local coordinate system, then locally we have $\sigma(t) = \sum_p a_p(t) e_p(\gamma(t))$.

Furthermore, let $V(t) = \dot{\gamma}(t) = \frac{d}{dt}\gamma(t) = \sum_i \dot{\gamma}^i(t)\frac{\partial}{\partial x^i}$, then

$$(D_V\sigma)(t) = \sum_p \Big(\dot{a_p}(t) e_p(\gamma(t)) + \sum_{i,q} \big(\dot{\gamma}^i(t) a_p(t) \Gamma^q_{ip}(\gamma(t)) e_q(\gamma(t)) \big) \Big), \tag{1.2.2}$$

where (Γ^q_{ip}) are the Christoffel symbols with respect to the chosen local frame.

The Equation (1.2.2) gives an ODE in the components a_p's. Therefore, for given initial conditions, equation (1.2.2) has local solutions. Such a solution is called a *parallel curve* in E.

Definition 15. Given a curve $\gamma : I \longrightarrow M$ and let $t_1, t_2 \in I$. Then *parallel transport along* γ (with respect to D) from time t_1 to time t_2 is the map

$$\mathbb{P}_\gamma^{t_1,t_2} : E_{\gamma(t_1)} \longrightarrow E_{\gamma(t_2)},$$

which associates to each $u \in E_{\gamma(t_1)}$ the element $\gamma^*(t_2) \in E_{\gamma(t_2)}$, where γ^* is the unique parallel curve above γ with the initial condition $\gamma^*(t_1) = u$.

Proposition 15. *For each curve $\gamma : I \longrightarrow M$ and for all $t_1, t_2, t_3 \in I$, the parallel transport along γ satisfies the following properties:*

1. $\mathbb{P}_\gamma^{t_1,t_2}$ *is a linear isomorphism,*
2. $\mathbb{P}_\gamma^{t_1,t_2} \circ \mathbb{P}_\gamma^{t_2,t_3} = \mathbb{P}_\gamma^{t_1,t_3}$.

Conversely, the connection D may be recovered from the parallel transport. Indeed, for a section $\sigma \in \Gamma(E)$ and a vector $X \in T_pM$, we have

$$D_X\sigma = \lim_{t\to 0} \frac{\mathbb{P}_\gamma^{t,0}(\sigma(\gamma(t))) - \sigma(p)}{t},$$

where $\gamma : I \longrightarrow M$ is a curve such that $0 \in I$, $\gamma(0) = p$ and $\dot{\gamma}(0) = X$. It is important to remark that this does not depend on the choice of the curve as long as $\dot{\gamma}(0) = X$ (cf. [39,42,59] for details).

1.2.2 More on connections

Connections on vector bundles are particular cases of the general notion of Ehersmann connections on fiber bundles. Further, connections on vector bundles may be studied in the framework of induced connections from Ehersmann connections on principal fiber bundles. For details on those topics and relationships with Ehersmann connections, we refer to Appendix **D**. For now we give just an equivalent definition of connections as a horizontal distribution and how it is linked to the covariant derivative and parallel transport.

The terminology 'connection' comes from the fact that this structure connects fibers through parallel transport. Indeed, for each $e \in E$, there is a distinguished subspace of the tangent space T_eE, namely the tangent space at e to the fiber through e, that is the vertical subspace $\mathcal{V}_e$ at e. Nevertheless, there is no privileged complementary subspace to $\mathcal{V}_e$ unless a connection is given. Indeed, given a connection D on E and $e \in E$. Along each curve

$\gamma : I \longrightarrow M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = X$, let σ be a parallel section along γ. The subspace of T_eE spanned by elements of the form

$$\frac{d}{dt}_{|t=0}\sigma(\gamma(t)),$$

is called the *horizontal subspace* and it is complementary to $\mathcal{V}_e$.

This suggest the following equivalent definition of connections which is nothing but the definition of an Ehresmann connection on a fiber bundle in the large with a slight modification to suit the linear structure on fibers, which suggest the terminology: *linear connection.*

Recall that at each $e \in E$, the vertical subspace $\mathcal{V}_e$ is the tangent space $T_e\pi^{-1}(\pi(e))$ which coincides with the subspace $\ker(d_e\pi)$. For each $\lambda \in \mathbb{R}$, there is a natural multiplication

$$m_\lambda : E \longrightarrow E, \quad m_\lambda(e) = \lambda e,$$

which is a vector bundle morphism.

Definition 16. A *(linear) connection* on E is a smooth distribution $\mathcal{H}$ on the total space E such that $TE = \mathcal{H} \oplus \mathcal{V}$ and for each $\lambda \in \mathbb{R}$ and $e \in E$ we have

$$\mathcal{H}_{\lambda e} = (m_\lambda)_*\mathcal{H}_e.$$

Given a connection as a horizontal distribution, one can define a map $K : TE \longrightarrow E$ which induces a linear map on fibers $K_e : T_eE \longrightarrow E_{\pi(e)}$ and satisfying $K_e(u_e^v) = u$ for all $u \in E_{\pi(e)}$. The map K is called the *connection map* (or the *connector*) (cf. [23, 35, 53]). Conversely, a map K satisfying the previous requirements, by setting $\mathcal{H} = \ker(K)$, gives rise to a horizontal distribution. Thus, it is not hard to see the equivalence between giving a connection and a connection map as defined above. In fact it may be taken as an equivalent definition. For details on this approach see [35, 53].

Covariant derivatives (or connections) allow one to define the horizontal distribution through parallel transport.

Proposition 16. *Let D be a covariant derivative on E, then, for each $p \in M$ and $e \in E_p$, there is a unique injection*

$$Hor_e : T_pM \longrightarrow T_eE$$

such that $Hor_e(\dot{\gamma}(0)) = \frac{d}{dt}_{|t=0}\mathbb{P}_\gamma^{0,t}(e)$ for all paths with $\gamma(0) = p$, where $\mathbb{P}_\gamma^{0,t}$ is the parallel transport along γ. Moreover the image of $Hor_e(T_pM)$ is complementary to $\mathcal{V}_e$.

Since the parallel transport is linear, a covariant derivative gives a horizontal distribution that satisfies the requirements of definition 16. For a more general link between covariant derivatives and connections in the large we refer the reader to Appendix **D**.

1.2.3 Curvature

The Levi-Civita connection on a Riemannian manifold has an associated tensor field of type $(1, 3)$ which measures the flatness of the Riemannian manifold, that is the failure of being locally isometric to the Euclidean space. Analogously, a connection on a vector bundle has an associated curvature tensor which measures its 'flatness'.

As exterior differential calculus plays a central role in differential geometry and global analysis, it is important to relate connections, if possible, to those objects. Given a vector bundle, one can generalize the notion of differential forms to vector valued differential forms defined as follows.

Definition 17. A *differential E-valued form* of degree r (or E-valued r-form) on M is a section of the bundle $\bigwedge^r T^*M \otimes E$. The set of those forms is denoted by $\Omega^r(M;E)$

Since $\bigwedge^r T^*M \otimes E$ is a vector bundle, the set $\Omega^r(M;E)$ is a vector space and a $C^\infty(M)$-module. It is clear that $\Omega^r(M, M\times\mathbb{R}) = \Omega^r(M)$. Analogously, an E-valued r-form ω may be seen as an r-linear skew symmetric map

$$\omega : \mathfrak{X}(M) \times ... \times \mathfrak{X}(M) \longrightarrow \Gamma(E).$$

Now, given a connection D on E, we consider the map $R : \mathfrak{X}(M)\times\mathfrak{X}(M)\times\Gamma(E) \longrightarrow \Gamma(E)$ defined by

$$R(X,Y)\sigma = D_X D_Y\sigma - D_Y D_X\sigma - D_{[X,Y]}\sigma, \tag{1.2.3}$$

for all $X, Y \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(E)$.

Proposition 17. *The map R is $C^\infty(M)$-linear in each of its arguments and skew-symmetric in X and Y, hence it defines an element $R \in \Omega^2(M; End(E))$ called the* curvature tensor *of D, where $End(E) = Hom(E,E)$.*

Let (E,π_1,M) and (F,π_2,M) be two vector bundles, and let D^E and D^F be connections on E and F, respectively. Denote by R^E and R^F their respective curvature tensors. In the same fashion, denote by R^{E^*}, $R^{E\oplus F}$ and $R^{E\otimes F}$ the curvature tensors of the connections $(D^E)^*$, $D^E \oplus D^F$ and $D^E \otimes D^F$, respectively. Then:

Proposition 18. *For $X, Y \in \mathfrak{X}(M)$, we have:*

(i) $R^{E^*}(X,Y) = R^E(X,Y)^*$,

(ii) $R^{E\oplus F}(X,Y) = R^E(X,Y) \oplus R^F(X,Y)$

(iii) $R^{E\otimes F}(X,Y) = Id_E \otimes R^F(X,Y) + R^E(X,Y) \otimes Id_F$.

For the pullback of a connection, we have:

Proposition 19. *Given a connection D on E with curvature tensor R^E, then for all $X, Y \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(E)$, we have*

$$R^{\varphi^*E}(X,Y)(\varphi^*\sigma) = \varphi^*\big(R^E(X,Y)\sigma\big),$$

where R^{φ^*E} is the curvature of the pullback connection φ^*D by a map φ.

The curvature of a connection on a vector bundle may be interpreted geometrically in a multitude of ways. For instance, curvature may be interpreted as the failure of a hodge-like operator to be a 'differential' in the sense of homological algebra.

On smooth manifold, the differentiation of exterior differential forms arise naturally from the smooth structure of the manifold. Furthermore, the differential d of forms satisfies the equation $d^2 = 0$, that is d is a 'differential' in the sense of homological algebra, which gives the De Rham cohomology, a necessary tool for geometry and analysis on smooth manifolds.

A connection D on a vector bundle E allows one to define an exterior differentiation of forms with values in the given vector bundle and hence gives rise to a general theory of exterior differential forms in such a way that, tacking the trivial line bundle $M\times\mathbb{R}$, we recover the classical theory of differential forms.

For a form $\omega \in \Omega^r(M;E)$ and vector fields $X_1, ..., X_{r+1} \in \mathfrak{X}(M)$, set

$$d_D(X_1, ..., X_{r+1}) := \sum_{k=1}^{r+1} (-1)^k (D_{X_k}\omega)\big(X_0, ..., \hat{X}_k, ..., X_{r+1}\big),$$

where the 'hat' over a vector means that it is deleted, and $(D_{X_k}\omega)$ is the covariant derivative of ω defined by

$$(D_X\omega)(X_1,...,X_r) := D_X(\omega(X_1,...,X_r)) - \sum_j \omega(X_1,...,D_XX_j,...,X_r).$$

The wedge product is also generalized to this setting. Let $K \in \Omega^r(M;\mathrm{End}(E))$, where $\mathrm{End}(E)$ is bundle of endomorphisms of E, and let $\eta \in \Omega^l(M;E)$, then define

$$(K\wedge\eta)(X_1,...,X_{r+l}) = \sum_\alpha \mathrm{sign}(\alpha)K(X_{\alpha(1)},...,X_{\alpha(r)})\big(\eta(X_{\alpha(r+1)},...,X_{\alpha(r+l)})\big),$$

for all $X_1,...,X_{r+l} \in \mathfrak{X}(M)$. Hence we have:

Proposition 20. *The map $d_D^2 = d_D \circ d_D : \Omega^*(M;E) \longrightarrow \Omega^{*+1}(M;E)$ is given by*

$$d_D^2(\eta) = R\wedge\eta,$$

where R is the curvature of D.

As matter of fact, the curvature R is an obstruction for d_D to be a differential. This shades more light on the classical case, for the equation $d^2 = 0$ is just a manifestation of the flatness of the trivial connection on the trivial line bundle $M\times\mathbb{R} \longrightarrow M$.

Remark 7. As in the case of the curvature of the Levi-Civita connection on a Riemannian manifold, curvature may also be interpreted as the failure of the mixed second covariant derivatives on two parameter families of curves [48].

1.2.4 Metric connections

Now assume that the vector bundle E is endowed with a fiber metric, that is E is a Riemannian vector bundle. It is important to have connections that preserves this additional structure. For the rest of this subsection we assume that E is endowed with a fiber metric h.

Definition 18. A connection D on E is said to be *metric* if it satisfies the following

$$X.h(\sigma,\delta) = h(D_X\sigma,\delta) + h(\sigma,D_X\delta),$$

for all $X \in \mathfrak{X}(M)$ and $\sigma,\delta \in \Gamma(E)$.

A metric connection on (E,h) is also said to be *compatible* with h. Since a fiber metric can be defined as a section of the vector bundle $E^*\otimes E^*$, compatibility is equivalent to h being parallel with respect to the connection $D^{E^*}\otimes D^{E^*}$ induced from D.

Proposition 21. *(see [65]) Every Riemannian vector bundle admits a metric connection.*

Taking $\sigma,\delta : I \longrightarrow E$ two parallel sections along a curve $\gamma : I \longrightarrow M$, compatibility implies that

$$\frac{d}{dt}_{|t=0} h(\sigma(t),\delta(t)) = 0.$$

Compatibility may also be interpreted as follows.

Proposition 22. *The parallel transport induced by a metric connection on a vector bundle preserves inner product on fibers, that is parallel transports are isometries.*

Remark 8. Fiber metrics induce on vector bundles, constructed naturally from other vector bundles, fiber metrics. Furthermore, compatibility with the 'natural' connections given in Proposition 12 is also inherited when the initial vector bundles possess metric connections.

Compatibility between the metric and the connection induces a symmetry of the curvature tensor.

Proposition 23. *Let D be a metric connection on a Riemannian vector bundle (E, h). The curvature tensor R satisfies the identity*

$$h(R(X,Y)\sigma,\delta) + h(R(X,Y)\delta,\sigma) = 0,$$

for all $X, Y \in \mathfrak{X}(M)$ and $\sigma, \delta \in \Gamma(E)$.

1.3 Riemannian metrics on vector bundles

This section is devoted to survey the major classes of Riemannian metrics on vector bundle manifolds existing in literature so far. From now on, (E, π, M) is a vector bundle with standard fiber $\mathbb{R}^k$ over a Riemannian manifold (M, g). Assume that E is endowed with a fiber metric h and a compatible connection D. Denote by K the connection map associated with the connection D and by ∇ the Levi-Civita connection of (M, g).

1.3.1 The building blocks

We shall prepare the ground for the subsequent definitions. We begin by giving an account of some vector bundle isomorphisms necessary for an alternative definition of metrics.

As in Section 1, let $\mathcal{V}$ be the vertical subbundle, which is naturally isomorphic to $\pi^\star E \longrightarrow E$ the pullback of (E, π, M) by $\pi : E \longrightarrow M$.

The connection D gives a horizontal distribution that is complementary to $\mathcal{V}$. Indeed, at each $e \in E$, the horizontal subspace $\mathcal{H}_e = \ker(K_e)$ is complementary to the vertical subspace, that is $T_eE = \mathcal{H}_e \oplus \mathcal{V}_e$. The *horizontal subbundle* is given as $\mathcal{H} = \sqcup_{e\in E}\mathcal{H}_e$. A vector field is said to be a *horizontal vector field* if it lies completely in $\mathcal{H}$. The horizontal subbundle is naturally isomorphic to the vector bundle $\pi^*TM \longrightarrow E$, the pullback of the tangent bundle to M by the map $\pi : E \longrightarrow M$. Thus, we have

$$TE = \mathcal{H} \oplus \mathcal{V} \simeq \pi^*TM \oplus \pi^\star E. \tag{1.3.1}$$

For $x \in M$ and $X \in T_xM$, *the horizontal lift* of X at e is the unique horizontal vector $X^h_e \in \mathcal{H}_e$ satisfying $d_e\pi(X^h_e) = X$. Vector fields on M can also be horizontally lifted. Indeed, if $X \in \mathfrak{X}(M)$, then the horizontal lift of X is the horizontal vector field $X^h \in \mathfrak{X}(E)$ such that X^h_e is the horizontal lift of $X_{\pi(x)}$ at e.

A curve in E is said to be *horizontal* if its tangent vector is a horizontal vector at each point.

Definition 19. Let γ be a curve in M, a horizontal lift of γ is a horizontal curve γ^* in E such that $\pi(\gamma^*(t)) = \gamma(t)$ for all t.

Horizontal curves are ultimately related to horizontal lifts of vector fields. More precisely, the integral curves of the horizontal lift X^h of a vector field X are the horizontal lifts of integral curves of X. The following result guarantees the existence of horizontal lifts of curves in the base. For a detailed proof in the general context of fiber bundles, we refer the reader to [42].

Proposition 24. *Let $\gamma : [0,1] \longrightarrow M$ be a curve starting at x and let $e \in E$ a point in E such that $\pi(e) = x$, then there exists a unique horizontal lift γ^* of γ starting at e.*

The splitting (1.3.1) induces a decomposition of vectors. In fact, for all $X \in T_eE$, we have $X = X^H + X^V$, where $X^H \in \mathcal{H}_e$ (resp. $X^V \in \mathcal{V}_e$) is the *horizontal* (resp. *vertical*) component of X; further, this gives rise to a decomposition of vector fields. Indeed, if $X \in \mathfrak{X}(E)$, then there exist a horizontal vector field X^H and a vertical vector filed X^V such that $X = X^H + X^V$.

1.3.2 Building on Sasaki: Riemannian metrics on vector bundle manifolds

In this section we review different classes of Riemannian metrics on vector bundle manifolds. Riemannian geometry of vector bundles was brought into existence when S. Sasaki introduced his famous metric on the tangent bundle to a Riemannian manifold (cf. [68]). This metric generalizes to a vector bundle E as follows

$$G^s_e(X,Y) = g_{\pi(e)}(\pi_*X, \pi_*Y) + h_{\pi(e)}(K_eX, K_eY), \tag{1.3.2}$$

for all $X, Y \in T_eE$ and $e \in E$ (cf. [23, 43]).

In the context of tangent bundles to a Riemannian manifold, it has been proved that the Sasaki metric is too *rigid* in the sense that almost every geometric property (e.g. being flat, being of constant sectional curvature, being an Einstein metric...etc) requires the flatness of the base and hence of the connection. For more details, the reader is referred to [2, 3, 44]. In order to overcome such a rigidity, many generalizations of the Sasaki metric were introduced. One of those generalizations is the *Cheeger-Gromoll metric* introduced in the case of tangent bundles (cf. [2, 3, 56]). For vector bundles, this metric takes the form

$$\begin{aligned} G^{cg}_e(X,Y) = g(d\pi(X), d\pi(Y)) + \frac{1}{1+r}\Big(h(K_eX, K_eY) \\ + h(K_eX, e)h(K_eY, e)\Big), \end{aligned} \tag{1.3.3}$$

for $e \in E$ and $X, Y \in T_eE$, with $r = h(e,e)$.

The Cheeger-Gromoll metric has also showed some rigidity. For example, it is never of constant scalar curvature, hence it turns to be unpromising of any interesting geometric property. So, the Cheeger-Gromoll metric itself was generalized to what is known as *generalized Cheeger-Gromoll metrics* and defined as

$$\begin{aligned} G^{p,q}_e(X,Y) = g(d\pi(X), d\pi(Y)) + \big(\frac{1}{1+r}\big)^p\big(h(K_e(X), K_e(Y)) \\ + qh(K_eX, e)h(K_eY, e)\big), \end{aligned} \tag{1.3.4}$$

for $e \in E$ and $X, Y \in T_eE$, with $r = h(e,e)$. Those metrics were called (p,q)-metrics (cf. [49] for more details). For, $p = q = 1$, one recovers the Cheeger-Gromoll metric G^{cg}.

Generalized Cheeger-Gromoll metrics has been studied from many points of view. For example, in [49], the authors studied harmonic sections when the total space is endowed with a (p,q)-metric. In [19], (p,q)-metrics were studied in the context of Euclidean Lie algebroids. Another instance is the *Kaluza-Klein metrics* on principal fiber bundles and their associated vector bundles, studied by C.M. Wood in the context of harmonic sections.

All of those metrics generalize the Sasaki metric, it also stems from their analogous in the case of tangent bundles. In the following section we shall focus on a particular class of Riemannian metrics which also generalizes the Sasaki metric and is related, in a multitude of ways, to the metrics reviewed in this subsection.

1.3.3 Spherically symmetric metrics

R. Albuquerque introduced an important class of Riemannian metrics on vector bundle manifolds (cf. [13]) which is closely related to the previous classes of Riemannian metrics and inspired by the general form of g-natural metrics on tangent bundles (cf. [2, 3, 7]).

Based on the decomposition (1.3.1) and using the connection map K, define the metrics

$$G_e(X,Y) = e^{2\varphi_1(r)}g_{\pi(e)}(\pi_*(X), \pi_*(Y)) + e^{2\varphi_2(r)}h_{\pi(e)}(K_e(X), K_e(Y)), \tag{1.3.5}$$

for all $e \in E$ and $X, Y \in T_eE$, where φ_1, φ_2 are smooth scalar functions on E depending only on the norm $r = h(e,e)$ and smooth at $r = 0$ on the right as well as all their successive derivatives. Those metrics were called *spherically symmetric metrics* (we shall use **SS-metrics** for short) (cf. [13]).

One may use the pullbacks sketched in the previous section in order to give an alternative expression of those metrics, more precisely an SS-metric may be expressed as follows

$$G = e^{2\varphi_1}\pi^* g \oplus e^{2\varphi_2}\pi^\star h, \tag{1.3.6}$$

where φ_1, φ_2 are smooth scalar functions on E depending only on the norm $r = h(e,e)$ and smooth at $r = 0$ on the right as well as all their successive derivatives.

Remark 9. The choice of weight functions is motivated by the long and rich history of g-natural metrics on tangent bundles, for the classification of such metrics yields metrics with weight functions depending only on the square of the norm of tangent vectors. For details on g-natural metrics and their classification, we refer the reader to [2, 3, 7, 45] and for the motivation of such a choice, we refer to [13].

For the rest of this thesis, we shall always endow the total space E with a spherically symmetric metric of the form (1.3.5) (or equivalently (1.3.6)). In the sequel, we will make use of the following notations:

(i) $\pi^* g = \langle ., . \rangle_M$,

(ii) $\pi^\star h = \langle ., . \rangle_E$,

(iii) $\pi^* g \oplus \pi^\star h = \langle ., . \rangle$.

We recall the expression of the Levi-Civita connection of an SS-metric, but first we fix the following notations. The vector bundle π^*TM is endowed with the pullback connection $\pi^*\nabla$ and the vector bundle $\pi^\star E$ is endowed with $\pi^\star D$, this gives rise to the connection $\tilde{D} = \pi^*\nabla \oplus \pi^\star D$ on the vector bundle TE. The connection $\tilde{D}$ is torsion free but it is not a metric connection with respect to SS-metrics. Under this considerations, the Levi-Civita connection was computed in [13] and it is given by:

Proposition 25. *(see [13]) Let X and Y be two vector fields on E, then*

$$\tilde{\nabla}_X Y = \tilde{D}_X Y + C_X Y + S_X Y - \frac{1}{2}\mathcal{R}^\xi(X,Y),$$

where

i) $\tilde{D} = \pi^*\nabla \oplus \pi^\star D$, with ∇ is the Levi-Civita connection of (M, g);

ii) $C(.,.)$ is the vector valued form given by

$$\begin{aligned} C_X Y = a(\xi^\flat(X)Y^H + \xi^\flat(Y)X^H) + c_1\langle X^H, Y^H\rangle\xi + c_2\langle X^V, Y^V\rangle\xi \\ + b(\xi^\flat(X)Y^V + \xi^\flat(Y)X^V), \end{aligned}$$

with

$$\begin{cases} a = 2\varphi_1', & c_1 = -2\varphi_1' e^{2(\varphi_1 - \varphi_2)}, \\ b = 2\varphi_2', & c_2 = -2\varphi_2', \end{cases}$$

for all X, $Y \in \mathfrak{X}(E)$, ξ being the tautological section of the vertical sub-bundle defined by $\xi_e = e \in \pi^\star E$ and the '$\flat$' is taken w.r.t h;

iii) $\mathcal{R}^\xi(X,Y) = \pi^\star R^E(X,Y)\xi$, for all X, $Y \in \mathfrak{X}(E)$;

iv) $S(.,.)$ is the π^*TM-valued form defined

$$e^{2\varphi_1}\langle S(X,Y),Z\rangle_M = \frac{e^{2\varphi_2}}{2}\Big(\langle \mathcal{R}^{\xi}(X,Z),Y\rangle_E + \langle \mathcal{R}^{\xi}(Y,Z),X\rangle_E\Big),$$

for all X, Y and $Z \in \mathfrak{X}(E)$.

1.3.4 Horizontally conformal submersions

Horizontally conformal maps in the large

Harmonic morphisms between Riemannian manifolds are maps which pullback germs of harmonic functions to germs of harmonic functions. It was proved, independently, by T. Ishihara (cf. [37]) and B. Fuglede (cf. [26]) that those maps are harmonic maps which satisfy the additional property of being *horizontally conformal* in a sense that we will make precise later. Recalling some basic facts about horizontally conformal maps is the aim of this subsection. For details on those maps the reader is referred to [30] and the references therein.

Generally speaking, let (N,k) and (M,g) be two Riemannian manifolds and let $\phi : (N,k) \longrightarrow (M,g)$ be a map, then for any point $x \in N$, set $\mathcal{V}_x(\phi) = \ker(d_x\phi)$, which is called the *vertical subspace* at x associated with ϕ, and denote by $\mathcal{H}_x(\phi) = \mathcal{V}_x(\phi)^{\perp}$ its orthogonal complement with respect to k_x, which is called the *horizontal subspace* at x associated with ϕ.

Definition 20. Let $\phi : (N,k) \longrightarrow (M,g)$ be a non-constant smooth map. Consider the sets $C_\phi := \{x \in N : d_x\phi \equiv 0\}$ and $\tilde{N} = N \setminus C_\phi$, then ϕ is said to be *horizontally conformal* if there exists $\lambda : \tilde{N} \longrightarrow \mathbb{R}^+$ such that

$$\lambda(x)^2 k_x(X,Y) = g_{\phi(x)}(d_x\phi(X), d_x\phi(Y)),$$

for all $X, Y \in \mathcal{H}_x(\phi)$ and $x \in \tilde{N}$. The function λ is then extended to the whole of N by putting $\lambda_{|C_\phi} = 0$. The extended function $\lambda : N \longrightarrow \mathbb{R}^+$ is called the *dilation* of ϕ.

It turns out that the dilation function of a horizontally conformal map is smooth (cf. [30]). If $\phi : (N,k) \longrightarrow (M,g)$ is a horizontally conformal submersion (**HCS** for short), then $C_\phi = \emptyset$. The gradient vector field $grad(\lambda^2)$ can be decomposed, with respect to the decompositions $T_xM = \mathcal{H}_x(\phi) \oplus \mathcal{V}_x(\phi)$, as

$$grad(\lambda^2) = grad_{\mathcal{H}}(\lambda^2) + grad_{\mathcal{V}}(\lambda^2).$$

Definition 21. A non-constant map $\phi : (N,k) \longrightarrow (M,g)$ is said to be *horizontally homothetic* if it is horizontally conformal with dilation function λ such that $grad_{\mathcal{H}}(\lambda^2) = 0$ on $\tilde{N}$.

It is true that horizontal conformal maps arise naturally from the theory of harmonic morphisms (cf. [26,37]). Nevertheless, horizontally conformal submersions arise also as a natural generalization of Riemannian submersions. Indeed, a horizontally conformal submersion with dilation function $\lambda \equiv 1$ is nothing but a Riemannian submersion. It will become clear that HCS's may be studied in exactly the same way used to study Riemannian submersions.

The geometry of Riemannain submersions: a survey

It is classically known that the geometry of an isometric immersion is totally determined by a single tensor field, namely the second fundamental form. The relation between the geometries of the manifold and that of the submanifold under consideration is totally encoded in the Gauss and Codazzi equations.

In the same spirit, B. O'Neill studied Riemannian submersions. He introduced two fundamental tensor fields associated with a Riemannian submersion. Furthermore, he derived the equations which relate the geometries of the total space and base manifold. This section is devoted to survey some of the important and basic facts in the theory of Riemannian submersions with a special focus on O'Neill's fundamental tensors.

As before, let (N,k) and (M,g) be two Riemannian manifolds, and let $\pi : (N,k) \longrightarrow (M,g)$ be a Riemannian submersion, that is a map of maximal rank which preserves the length of horizontal vectors (cf. [17,61]). At each point $x \in N$, $H_x(\pi)$ (resp. $V_x(\pi)$) is the horizontal (resp. vertical) subspace at x associated with π. We denote by $\mathcal{H}(.)$ (resp. $\mathcal{V}(.)$) the horizontal (resp. vertical) part of a vector with respect to the orthogonal decomposition.

Let T be the second fundamental form of all fibres of π which is defined, for arbitrary vector fields E and F, by

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F. \tag{1.3.7}$$

By dualizing the definition of T, that is reversing $\mathcal{H}$ and $\mathcal{V}$ in (1.3.7), we have a second tensor A defined by

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F. \tag{1.3.8}$$

The $(1,2)$-tensor fields T and A on N are called the *fundamental tensors* of π.

The fundamental tensors of a submersion are introduced for any submersion for which the total space is endowed with a Riemannian metric, not necessarily for Riemannian submersions.

The fundamental tensor T satisfies the following basic properties which stem from the definition.

Proposition 26. *(see [61]) Let $\pi : (N,k) \longrightarrow M$ be a submersion and let T be its fundamental tensors, then*

1. *For all $x \in N$ and $E \in T_xM$, the linear operator $T_E : T_xM \longrightarrow T_xM$ is skew-symmetric;*
2. *T is vertical, that is $T_{\mathcal{V}E} = T_E$, for all $E \in \mathfrak{X}(M)$;*
3. *For vertical vectors V, W, $T_V W = T_W V$;*
4. *The restriction of T to the vertical distribution $T_{\mathcal{V}\times\mathcal{V}}$ is exactly the second fundamental form of the fibres of π;*
5. *$T = 0$ if and only if π has totally geodesic fibers;*

The fundamental tensor A of a Riemannian submersion satisfies the following properties which also come from its definition.

Proposition 27. *(see [61]) Let $\pi : (N,k) \longrightarrow M$ be a submersion and let A be its fundamental tensor, then*

1. *For all $x \in N$ and $E \in T_xM$, the linear operator $A_E : T_xM \longrightarrow T_xM$ is skew-symmetric;*
2. *A is horizontal, that is $A_{\mathcal{H}E} = A_E$, for all $E \in \mathfrak{X}(M)$;*
3. *For horizontal vectors X, Y, $A_X Y = -A_Y X$;*
4. *If $\pi : (N,k) \longrightarrow (M,g)$ is a Riemannian submersion, then for horizontal vector fields X, Y, we have*
$$A_X Y = \frac{1}{2}\mathcal{V}[X,Y],$$

The equations which relate the curvatures of the total space and base, known as the *fundamental equations*, rely essentially on the fundamental tensors and their covariant derivatives. In what follows, we shall recall those components of the covariant derivatives of the fundamental tensors which appear in the fundamental equations.

Lemma 1. *(see [61]) Let $\pi : (N,k) \longrightarrow M$ be a submersion, T and A be the fundamental tensors of π, then for horizontal vectors X,Y and vertical vectors V,W, we have*

1. $(\nabla_V A)_W = -A_{T_V W}$,
2. $(\nabla_X T)_Y = -T_{A_X Y}$,
3. $(\nabla_X A)_W = -A_{A_X W}$,
4. $(\nabla_V T)_Y = -T_{T_V Y}$.

Denote by $\hat{\nabla}_V W = \mathcal{V}(\nabla_V W)$, for vertical vectors V,W, where ∇ is the Levi-Civita connection of (N,k).

In [61], B. O'Neill gave the fundamental equations of a Riemannian submersion which we will not recall here but we will recall their generalizations to horizontally conformal submersions in the following section. For more details on Riemannian submersions and their fundamental equations, we refer the reader to O'Neill's papers [61,62].

On the geometry of horizontally conformal submersions

The approach reviewed in the previous subsection applies in a natural way to the case of horizontally conformal submersions with a slit modification of the fundamental equations. This generalization to the framework of horizontally conformal submersions is due to S. Gudmundsson in his doctoral thesis (cf. [29,30]).

The definition of fundamental tensors as well as their properties are the same since all what is needed is a Riemannian metric on the total space. The fundamental equations in the case of a HCS are given in the following:

Theorem 6. *(see [30]) Assume $n > m \geq 2$ and let $(N^n,k,\nabla,R),(M^m,g,\nabla^*,R^*)$ be two Riemannian manifolds with their Levi-Civita connections and their corresponding curvature tensors. Let $\pi : (N,k) \longrightarrow (M,g)$ be a horizontally conformal submersion with dilation $\lambda : N \longrightarrow \mathbb{R}^+$. If X,Y,Z,H are horizontal and U,V,W,F are vertical vectors, then*

$$k(R(U,V)W,F) = k(R^{\mathcal{V}}(U,V)W,F) + k(T_U W, T_V F) - k(T_V W, T_U F), \tag{1.3.9}$$

$$k(R(U,V)W,X) = k((\nabla_U T)_V W,X) - k((\nabla_V T)_U W,X), \tag{1.3.10}$$

$$\begin{aligned} k(R(U,X)Y,V) &= k((\nabla_U A)_X Y,V) + k(A_X U, A_Y V) \\ &\quad - k((\nabla_X T)_U Y,V) - k(T_V Y, T_U X) \\ &\quad + \lambda^2 k(A_X Y,U) k(V, grad_{\mathcal{V}}(\frac{1}{\lambda^2})), \end{aligned} \tag{1.3.11}$$

$$\begin{aligned} k(R(X,Y)Z,U) &= k((\nabla_X A)_Y Z,U) - k((\nabla_Y A)_X Z,U) \\ &\quad - k(T_U Z, \mathcal{V}[X,Y]), \end{aligned} \tag{1.3.12}$$

$$
\begin{aligned}
k(R(X,Y)Z,H) &= \frac{1}{\lambda^2}k(\pi^*R^*(X,Y)Z,H) \\
&+ \frac{1}{4}\Big(k(\mathcal{V}[X,Z],\mathcal{V}[Y,H]) - k(\mathcal{V}[Y,Z],\mathcal{V}[X,H]) \\
&+ 2k(\mathcal{V}[X,Y],\mathcal{V}[Z,H])\Big) + \frac{\lambda^2}{2}\Big(k(X,Z)k(\nabla_Y grad(\frac{1}{\lambda^2}),H) \\
&- k(Y,Z)k(\nabla_X grad(\frac{1}{\lambda^2}),H) + k(Y,H)k(\nabla_X grad(\frac{1}{\lambda^2}),Z) \\
&- k(X,H)k(\nabla_Y grad(\frac{1}{\lambda^2}),Z)\Big) + \frac{\lambda^4}{4}\Big(k(X,H)k(Y,Z) \\
&- k(Y,H)k(X,Z))\|grad(\frac{1}{\lambda^2})\|2 \\
&+ k(X(\frac{1}{\lambda^2})Y - Y(\frac{1}{\lambda^2})X, H(\frac{1}{\lambda^2})Z - Z(\frac{1}{\lambda^2})H)\Big).
\end{aligned}
\tag{1.3.13}
$$

Using Theorem 6, one can relate the corresponding sectional curvatures, K_k and K_g, of the Riemannian manifolds (N,k) and (M,g), respectively. For X,Y two linearly independent tangent vectors to N, $K_k(X\wedge Y)$ denotes the sectional curvature of the plane generated by X and Y. The corresponding notation is used also for K_g.

Theorem 7. *(see [30]) Let $n > m \geq 2$ and $\pi : (N^n,k) \longrightarrow (M^m,g)$ be a horizontally homothetic submersion, with dilation $\lambda : M \longrightarrow \mathbb{R}^+$. Then, for orthonormal horizontal vectors X and Y , we have*

$$K_N(X\wedge Y) = \lambda^2 K_M(d\pi(X)\wedge d\pi(Y)) - \frac{3}{4}\|\mathcal{V}[X,Y]\|^2 - \frac{\lambda^4}{4}\|grad_{\mathcal{V}}(\frac{1}{\lambda^2})\|^2.$$

Lemma 2. *(see [61]) If X and Y are horizontal, V is vertical and E is an arbitrary vector, then $g((\nabla_E A)_X Y,V)$ is alternate in X and Y.*

Proposition 28. *Assume that $n > m \geq 2$ and let $\pi : (N^n,k) \longrightarrow (M^m,g)$ be a horizontally homothetic submersion with totally geodesic fibers and dilation $\lambda : N \longrightarrow \mathbb{R}^+$. Then*

1. *For orthonormal vertical vectors U,V, we have*

$$K_k(U\wedge V) = k(R^{\mathcal{V}}(U,V)V,U).$$

2. *If X is a horizontal vector and V is a vertical vector such that $\|X\| = \|V\| = 1$, then*

$$K_k(X\wedge V) = \|A_X V\|^2.$$

Proof: 1. is obvious. For 2., let X be a horizontal vector and V be a vertical vector such that $\|X\| = \|V\| = 1$, then

$$
\begin{aligned}
K_k(X\wedge V) &= k(R(V,X)X,V) \\
&= k((\nabla_V A)_X X,V) + k(A_X V, A_X V) - k((\nabla_X T)_V X,V) \\
&- k(T_V X, T_V X) + \lambda^2 k(A_X X,V)k(V,grad_{\mathcal{V}}(\frac{1}{\lambda^2})).
\end{aligned}
$$

$T = 0$ since π has totally geodesic fibers, and $g((\nabla_V A)_X X,V) = 0$, by virtue of Lemma 2. Further, $A_X X = 0$ since A is skew-symmetric on horizontal vectors. Thus

$$K_k(X\wedge V) = k(A_X V, A_X V) = \|A_X V\|^2.$$

■

1.3.5 Spherically symmetric metrics again

The aim of this subsection is to prove that, when the total space of a vector bundle with a Riemannian base is endowed with an SS-metric, the projection is a horizontally homothetic submersion.

Let G be a spherically symmetric metric on E, with weight functions φ_1 and φ_2. The projection $\pi : (E, G) \longrightarrow (M, g)$ is a submersion.

At each point $e \in E$, the vertical (resp. horizontal) subspace associated to π are exactly the vertical subspace of T_eE (resp. the horizontal subspace with respect to the connection D).

Lemma 3. *(see [13]) Let φ be a function on E depending only on $r = h(e, e)$, and let X be a horizontal vector field and V be a vertical vector field, then*

(i) $X.\varphi = 0$,

(ii) $V.\varphi = 2\varphi'\xi^\flat(V)$.

Proposition 29. *(see [11]) Let G be an SS-metric on E of the form (1.3.5), then the projection $\pi : (E, G) \longrightarrow (M, g)$ is a horizontally homothetic submersion with dilation function $\lambda : E \longrightarrow \mathbb{R}$ given by $\lambda(e) = e^{-\varphi_1(r)}$, where $r = h(e, e)$.*

Proof: Since π is a submersion, $C_\pi = \emptyset$, hence $\tilde{E} = E \backslash C_\pi = E$. Furthermore, $V_e(\pi) = \mathcal{V}_e$ and since the horizontal and vertical distributions are orthogonal with respect to G, we have $H_e(\pi) = \mathcal{H}_e = \ker(K_e)$. As a matter of fact, for all $e \in E$ and $Y_1 = X_1^h, Y_2 = X_2^h \in \mathcal{H}_e$, we have

$$\begin{aligned} g_{\pi(e)}(d\pi(Y_1), d\pi(Y_2)) &= g_{\pi(e)}(X_1, X_2) \\ &= e^{-2\varphi_1(r)} G_e(Y_1, Y_2), \end{aligned}$$

whence π is horizontally conformal with dilation function $\lambda : e \mapsto e^{-\varphi_1(r)}$, with $r = h(e, e)$.

Denote by n the dimension of M end let $\{e_i\}_{1 \leq i \leq n}$ be a local linear orthonormal frame of M and $\{\sigma_p\}$ be a local orthonormal frame of E. Denote by

$$E_i = e^{-\varphi_1} e_i^h \quad \text{and} \quad E_{n+p} = e^{-\varphi_2} \sigma_p^v,$$

for $i = 1, ..., n$ and $p = 1, ..., k$. Hence $\{E_I : I = 1, ..., n + k\}$ is a local linear orthonormal frame of E. Since, by virtue of Lemma 3, we have $E_i.\lambda = e^{-\varphi_1} e_i^h.\lambda = 0$, for all $i = 1, ..., n$, then $grad_{\mathcal{H}}(\lambda) = 0$, thus π is horizontally homothetic. ■

Proposition 29 implies that spherically symmetric metrics could be studied in the realm of horizontally conformal submersions. In fact, we adopt this approach since it allows one to overcome the lengthy computations. This approach also allows one to study more effectively many geometric questions e.g. the completeness problem and geodesics.

Part II

Results

Chapter 2

On the completeness of spherically symmetric metrics

Motivation

The study of geodesics lies at the heart of Riemannian geometry. On the other hand, completeness is an important geometric property to investigate, since complete Riemannian manifolds possess many useful geometric features. Concerning the geodesics of spherically symmetric metrics, the author of [13] has derived the equations describing geodesics of vector bundle manifolds endowed with an SS-metric, but his computational approach leads to a complicated system of ODE's which makes very difficult either an analysis or a construction of examples of geodesics. We shall follow a different approach here, namely that of [62] used to study the geodesics of Riemannian submersions. Regarding the completeness of spherically symmetric metrics, R. Albuquerque conjectured that the total space of a vector bundle, when endowed with an SS-metric, is complete if and only if the base and fibers are complete. In what follows, we shall refer to this problem as the Completeness Problem (**CP**).

It is a classical fact of Riemannian geometry that the metric tensor induces a distance on a Riemannian manifold; further, the Hopf-Rinow theorem establishes the link between the (geodesic) completeness and completeness as a metric space. To the best of our knowledge, the study of the completeness of total spaces of vector bundles, and Riemannian submersions in general, has not been accomplished and only one partial result has been established in the context of Riemannian submersions. Indeed, R. Hermann proved that if the total space of a Riemannian submersion is complete then so is the base (cf. [34]). The aim of this chapter is to present an answer to the CP in some situations, but without assuming the completeness of fibers. Our main results are based on topological techniques and the Hopf-Rinow theorem, they rely on a comparison of the geodesic distances of the base manifold and total space.

The projections of vector bundles endowed with spherically symmetric metrics are horizontally conformal submersions (Proposition 29). Further, the projection maps of vector bundles endowed with the *Cheeger-Gromoll metric* (cf. [56]), or more generally (p,q)*-metrics* (cf. [49]) are Riemannian submersions. Moreover, the projection maps of a subclass of g-natural metrics are also horizontally conformal submersions. For a detailed account of g-natural metrics and their properties, we refer the reader to [2,3,7,45,55]. So, the **CP** may be studied in the realm of this general framework, this approach allows us to apply the results to different classes of metrics on vector bundle manifolds and in different situations. In particular, the completeness of tangent bundles to Riemannian manifolds with some g-natural metrics is studied.

For the rest of this chapter we assume that G is a spherically symmetric metric with weight functions φ_1 and φ_2. In this chapter, we will make use of the Einstein summation convention when we sum over repeated indices.

2.1 Geodesics

In this section, we aim to find geodesics or at least some of them in this context using techniques and results from the theory of geodesics of Riemannian submersions established in [34,57,62]. But first we review Albuquerque's results on geodesics of spherically symmetric metrics (cf. [13]).

Let $U \subseteq M$ be an open set and let $\{x^i\}$ be a local coordinate system on U, and consider a trivialization of E over U. Consider a local frame $\{\sigma_\alpha\}$ of E induced by the chosen trivialization. A curve $\delta : I \longrightarrow E$ $(0 \in I)$ be a curve in E, can be written locally as

$$\delta = (\delta^1, ..., \delta^n, y^1, ..., y^k).$$

The curve δ defines a section $\sigma = y^p\sigma_p$ along the curve $\gamma = \pi \circ \delta$, then along this curve we have

$$D_{\dot\gamma}\sigma = \dot{y}^p\sigma_p + \dot{\delta}^i y^q \Gamma^{E,p}_{iq}\sigma_p = z^p\sigma_p,$$

where $\Gamma^{E,p}_{i,q}$ are the Christoffel symbols of D and

$$z^p = \dot{y}^p + \dot{\delta}^i y^q \Gamma^{E,p}_{i,q}\sigma_p = z^p,$$

Theorem 8. *(see [13]) The curve δ is a geodesic in (E, G) if and only if it satisfies the following system*

$$\begin{cases} \ddot{\delta}^k + \dot{\delta}^i\dot{\delta}^j\Gamma^k_{ij} + 2a\dot{\delta}^k z^p y^p + e^{2\varphi_2 - 2\varphi_1}\dot{\delta}^i z^p y^r R^E_{pril} g^{lk} &= 0 \\ \ddot{\delta}^p + c_1\dot{\delta}^i\dot{\delta}^j g_{ij}\dot{y}^p + \dot{\delta}^i z^q \Gamma^{E,p}_{iq} - bz^q y^q y^p + 2bz^p z^q y^q &= 0. \end{cases}$$

for all $i = 1, ..., n$ and $p = 1, ..., k$, with R^E_{pril} are the components of the curvature of D, Γ^k_{ij} (resp. $\Gamma^{E,q}_{ip}$) are the Christoffel symbols of (M, g) (resp. D), and a, c_1 and b are the coefficients given in Proposition 25.

As the reader may agree, the system of ODE's given in Theorem 8 is not easy to analyse, so an alternative approach is in order. In the context of Riemannian submersions, B. O'Neill found the equations describing geodesics of the total space using only the fundamental tensors discussed earlier. Working with this approach has two advantages: firstly it avoids local expressions and hence analysis becomes feasible, and secondly it relies only on fundamental tensors, so using the splitting of vectors into horizontal and vertical components allows one to exhibit examples of geodesics.

Let (N, k, ∇^k) and (M, g, ∇^g) be two Riemannian manifolds together with their Levi-Civita connections, and let $\pi : (N, k) \longrightarrow (M, g)$ be a Riemannian submersion. For a curve α in N and a vector field X along α, the covariant derivative $\nabla^k_{\dot\alpha}X$ is simply denoted X', where $\dot\alpha$ is the velocity vector field of α. Denote by X_* the projection of X via π, which is a vector field along $\pi \circ \alpha$ and X'_* denotes its covariant derivative along $\pi \circ \alpha$. In [62], O'Neill proved the following:

Theorem 9. *Let $\pi : (N, k) \longrightarrow (M, g)$ be a Riemannian submersion, and let $X = H + V$ be a vector field along a curve α in N, then*

$$\mathcal{H}(X') = X'_* + A_H(\mathcal{V}\dot\alpha) + A_{\mathcal{H}(\dot\alpha)}V + T_{\mathcal{V}\dot\alpha}(V), \tag{2.1.1}$$

$$\mathcal{V}(X') = A_{\mathcal{H}(\dot\alpha)}(H) + T_{\mathcal{V}(\dot\alpha)}(H) + \mathcal{V}(V'); \tag{2.1.2}$$

where A and T are the fundamental tensors of π.

Taking $X = \dot{\alpha}$, the velocity vector field itself, O'Neill gave the equations describing a geodesic of the total space. For a curve α in N, denote by $\ddot{\alpha}$ the acceleration vector field and $\ddot{\alpha}_*$ the horizontal lift of the acceleration of $\pi \circ \alpha$.

Corollary 3. *(see [62]) Let α be a curve in N with $X = \mathcal{H}\dot{\alpha}$ and $U = \mathcal{V}\dot{\alpha}$, then*

$$\mathcal{H}(\ddot{\alpha}) = \ddot{\alpha}_* + 2A_X U + T_U U, \tag{2.1.3}$$

$$\mathcal{V}(\ddot{\alpha}) = T_U X + \mathcal{V}(U'). \tag{2.1.4}$$

Using Theorem 9, O'Neill also gave a proof of the following result which is originally due to R. Hermann (cf. [34]).

Corollary 4. *Let $\pi : (N, k) \longrightarrow (M, g)$ be a Riemannian submersion. If α is a geodesic which is horizontal at some point, then it is a horizontal geodesic, and hence $\pi \circ \alpha$ is a geodesic in M.*

The proof of Theorem 9, and hence the corollaries, does not use the fact that π is a Riemannian submersion, since only the definitions of the fundamental tensors A and T are needed. Hence, the same formulas hold when π is a horizontally conformal submersion. Of course the equations, if expanded, will change since A changes.

Now, let $\pi : (E, G) \longrightarrow (M, g)$ be the projection of the vector bundle E when endowed with G. We have proved, in Proposition 29, that π is a horizontally conformal submersion with dilation function $\lambda = e^{-\varphi_1}$. Furthermore, π has totally geodesic fibers, hence $T \equiv 0$. Thus Corollary 3 yields the following:

Lemma 4. *Let α be a curve in E with $X = \mathcal{H}\dot{\alpha}$ and $U = \mathcal{V}\dot{\alpha}$, then*

$$\mathcal{H}(\ddot{\alpha}) = \ddot{\alpha}_* + 2A_X(U), \tag{2.1.5}$$

$$\mathcal{V}(\ddot{\alpha}) = \mathcal{V}(U'). \tag{2.1.6}$$

Consequently, one easily proves the following:

Proposition 30. *Let α be a horizontal geodesic in E, then $\pi \circ \alpha$ is a geodesic.*

Proposition 31. *Let α be a geodesic in E. If φ_1 is constant and $\mathcal{R}^{\xi}(X, .) = 0$, where $X = \mathcal{H}(\dot{\alpha})$, then $\pi \circ \alpha$ is a geodesic in the base M.*

Though it is hard to give the general form of geodesics, we exhibit the following families: horizontal lifts of geodesics in M and geodesics in fibers.

Proposition 32. *Let γ be a geodesic in M, then the horizontal lift of γ, with respect to D, is a geodesic in E. In particular, if (E, G) is complete, then (M, g) is complete.*

Proposition 33. *A vertical curve is a geodesic in (E, G) if and only if it is a geodesic in the fiber that contains it.*

Proof: If α is a vertical geodesic with $\pi \circ \alpha = x$, then $\mathcal{H}(\dot{\alpha}) = 0$, and hence α is a geodesic if and only if $\mathcal{V}(U') = 0$, which implies that α is a geodesic in $\pi^{-1}(x)$. The converse follows from the fact that π has totally geodesic fibers. ■

2.2 Completeness of SS-metrics

In virtue of Proposition 29, the projection of a vector bundle endowed with a spherically symmetric metric is a horizontally conformal submersion with dilation $\lambda : E \longrightarrow \mathbb{R}$ such that $\lambda(e) = e^{-\varphi_1(r)}$, with $r = h(e,e)$. If $\varphi_1 = 0$, then the projection is a Riemannian submersion.

First, we prove the following comparison result of the geodesic distances of the total space and base of a horizontally conformal submersion with bounded dilation function.

Lemma 5. *([11]) Let $\pi : (N,k) \longrightarrow (M,g)$ be a horizontally conformal submersion with dilation function λ. Assume that M is complete and the square of the dilation function is bounded from above by $b > 0$ i.e. $\lambda^2 \leq b$. Then, for all $e_1, e_2 \in N$, we have*

$$d_g(\pi(e_1), \pi(e_2) \leq \sqrt{b} d_k(e_1, e_2),$$

where d_g and d_k are the (geodesic) distances induced by the metric tensors g and k, respectively.

Proof: Let $e_1, e_2 \in N$, with $x_i = \pi(e_i)$ for $i = 1, 2$. For each curve $\delta : [a,b] \longrightarrow N$ relating e_1 to e_2, the map $\pi \circ \delta : [a,b] \longrightarrow M$ is a curve relating x_1 to x_2. Since (M,g) is complete, there exists a geodesic $c : I \longrightarrow M$ relating x_1 to x_2, such that $l(c) = d_g(x_1, x_2)$, where $l(c)$ denotes the length of c, with respect to g.

For a curve $\delta : I \longrightarrow N$, we have

$$\begin{aligned} k(\dot{\delta}(t), \dot{\delta}(t)) &= k(\mathcal{H}(\dot{\delta}(t)), \mathcal{H}(\dot{\delta}(t))) + k(\mathcal{V}(\dot{\delta}(t)), \mathcal{V}(\dot{\delta}(t))) \\ &= \lambda(\delta(t))^{-2} g\big(\pi_* \dot{\delta}(t), \pi_* \dot{\delta}(t)\big) + k(\mathcal{V}(\dot{\delta}(t)), \mathcal{V}(\dot{\delta}(t))), \end{aligned}$$

where $\mathcal{H}$ (resp. $\mathcal{V}$) denotes the horizontal (resp. vertical) part of a vector. If $L(\delta)$ denotes the length of δ, with respect to k, then

$$\begin{aligned} L(\delta) &= \int_a^b \big(k(\dot{\delta}(t), \dot{\delta}(t)\big)^{\frac{1}{2}} dt \\ &= \int_a^b \Big(\lambda(\delta(t))^{-2} g(\dot{\overbrace{\pi \circ \delta}}(t), \dot{\overbrace{\pi \circ \delta}}(t)) + k(\mathcal{V}(\dot{\delta}(t)), \mathcal{V}(\dot{\delta}(t)))\Big)^{\frac{1}{2}} dt \\ &\geq \frac{1}{\sqrt{b}} \int_a^b \big(g(\dot{\overbrace{\pi \circ \delta}}(t), \dot{\overbrace{\pi \circ \delta}}(t))\big)^{\frac{1}{2}} dt \\ &\geq \frac{l(\pi \circ \delta)}{\sqrt{b}}. \end{aligned}$$

Hence, for any curve (potentially piecewise smooth) $\delta : [a,b] \longrightarrow N$, we have

$$L(\delta) \geq \frac{l(\pi \circ \delta)}{\sqrt{b}} \geq \frac{l(c)}{\sqrt{b}} = \frac{1}{\sqrt{b}} d_g(x_1, x_2),$$

whence the result. ∎

Theorem 10. *([11]) Let G be a Riemannian metric on a vector bundle (E, π, M) for which the projection $\pi : (E,G) \longrightarrow (M,g)$ is a horizontally conformal submersion such that the dilation function λ satisfies $a \leq \lambda^2 \leq b$, for some reals $a, b > 0$. If the base (M,g) is complete, then so is the total space (E,G).*

Proof: By virtue of the Hopf-Rinow theorem, it suffices to show that (E, d_G) is a complete metric space. Let $(e_n)_n$ be a Cauchy sequence of the metric space (E, d_G). By virtue of Lemma 5, we have $d_g(\pi(e_{n+p}), \pi(e_n)) \leq \sqrt{b} d_G(e_{n+p}, e_n)$, for all $n, p \in \mathbb{N}$, we conclude that

the sequence (x_n), with $x_n = \pi(e_n)$, is a Cauchy sequence of the metric space (M, d_g). By virtue of the Hopf-Rinow theorem, the sequence (x_n) converges to a limit $x_0 \in M$.

Let $\epsilon > 0$, since the sequence $(x_n)_n$ converges to x_0, there exists $N_\epsilon \in \mathbb{N}$ such that $d(x_n, x_0) < \epsilon$ for all $n \geq N_\epsilon$. So, for $n \geq N_\epsilon$, let $c : [0,1] \longrightarrow M$ be a geodesic relating x_n to x_0 and let c^* be the horizontal lift of c starting at e_n as given in Proposition 24.

We get

$$\|\dot{\widehat{c^*}}(t)\|_G^2 = G(\dot{\widehat{c^*}}(t), \dot{\widehat{c^*}}(t)) = \lambda(c^*(t))^{-2} g(\dot{c}(t), \dot{c}(t)),$$

thus

$$L(c^*) \leq \frac{l(c)}{\sqrt{a}} = \frac{1}{\sqrt{a}} d_g(x_n, x_0).$$

In particular, we have $d_G(e_n, c^*(1)) \leq \frac{1}{\sqrt{a}} d_g(x_n, x_0) \leq \frac{\epsilon}{\sqrt{a}}$. Thus, the sequence $(e_n)_n$ converges to the limit point $e = c^*(1)$. Once again, the Hopf-Rinow theorem gives the result. ■

Remark 10. In Lemma 5 and the proof of Theorem 10, we have proved that if (M, g) is complete and G is a Riemannian metric on E such that the projection $\pi : (E, G) \longrightarrow (M, g)$ is a horizontally conformal submersion with dilation function λ satisfying $a \leq \lambda^2 \leq b$, for some real numbers $a, b > 0$, then we have

$$\sqrt{a} d_G(e_1, e_2) \leq d_g(\pi(e_1), \pi(e_2)) \leq \sqrt{b} d_G(e_1, e_2),$$

where $e_1, e_2 \in E$ are the ends of the lift of a minimizing geodesic relating x_1 and x_2, with $x_i = \pi(e_i)$ for $i = 1, 2$.

As an application we present an answer to the **CP** for SS-metrics without assuming the completeness of fibers, as well as for the Cheeger-Gromoll and generalized Cheeger-Gromoll metrics:

Corollary 5. *Assume that E is endowed with a metric G belonging to one of the following classes:*

(i) SS-metrics such that φ_1 is bounded,

(ii) (p, q)-metrics $G^{p,q}$.

Then, the Riemannian manifold (E, G) is complete if and only if (M, g) is complete.

Proof: It follows from Proposition 32 and Theorem 10. ■

Example 4. Take $\varphi_1(r) = cos(r)$, and φ_2 an arbitrary weight function as in section 1. Then, (E, G) is complete if and only if (M, g) is complete, where G is the spherically symmetric metric with weights φ_1 and φ_2.

2.3 Application: the tangent bundle to a Riemannian manifold

Though tangent bundles are examples of vector bundles, they constitute a particular exception. Indeed, they can be endowed with a far more large family of naturally constructed metrics for many good reasons, firstly a Riemannian metric on the base gives at the same time a linear connection on the tangent bundle and inner products on fibers, secondly local frames of the tangent bundle are nothing but the coordinate frames, that is local trivializations of the tangent bundle arise naturally from the smooth structure of the base. For all of

these considerations, it becomes quite natural to have a huge class of metrics on the tangent bundle to a Riemannian manifold, namely g-natural metrics which are defined as follows.

Let (M, g) be an n-dimensional Riemannian manifold and ∇ its Levi-Civita connection. At each point (x, u) of its tangent bundle TM, the tangent space to TM splits into the horizontal and vertical subspaces, with respect to ∇, as

$$T_{(x,u)}TM = \mathcal{H}_{(x,u)}TM \oplus \mathcal{V}_{(x,u)}TM,$$

where $\mathcal{H}_{(x,u)}TM$ (resp. $\mathcal{V}_{(x,u)}TM$) is the horizontal (resp. vertical) subspace of $T_{(x,u)}TM$.

For any vector $X \in T_xM$, there exists a unique vector $X^h \in \mathcal{H}_{(x,u)}TM$, called the *horizontal lift* of X to $(x, u) \in TM$, such that $\pi_* X^h = X$, where $\pi : TM \to M$ is the natural projection. Moreover, *vertical lift* of $X \in T_xM$ to $(x, u) \in TM$ is the vertical vector $X^v \in \mathcal{V}_{(x,u)}TM$ such that $X^v(df) = Xf$, for all functions f on M (we consider the 1-forms df on M as functions on TM via the equation $df(x, u) = uf$). The map $X \mapsto X^h$ is an isomorphism between the vector spaces T_xM and $\mathcal{H}_{(x,u)}TM$. Similarly, the map $X \mapsto X^v$ is an isomorphism between T_xM and $\mathcal{V}_{(x,u)}TM$. Each tangent vector $Z \in T_{(x,u)}TM$ can be written in the form $Z = X^h + Y^v$, where $X, Y \in T_xM$ are uniquely determined tangent vectors. Horizontal and vertical lifts of vector fields on M can be defined in an obvious way and are uniquely defined vector fields on TM.

Classically, the total space of the tangent bundle was endowed with the Sasaki metric, but, as mentioned above, this metric presents a strong rigidity in the sense of [44] and [56]. In order to overcome such a rigidity, many generalizations of the Sasaki metric were introduced (e.g. the Cheeger-Gromoll metric, Oproiu metrics...etc). We refer the reader to the paper [2] for a detailed survey of the geometry of the tangent bundle. The introduction of g-natural metrics comes from the description of all second order natural transformations of Riemannian metrics on manifolds into metrics on their tangent bundles [45]. For more details about the concept of naturality and related notions, we refer the reader to [35, 45, 47]. The family of g-natural metrics (possibly pseudo-Riemannian) has been characterized as follows:

Proposition 34. *(see [7]) Let (M, g) be a Riemannian manifold and G be a g-natural metric on TM. Then there are six smooth functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, such that for every u, X, $Y \in T_xM$, we have*

$$\begin{cases} G_{(x,u)}(X^h, Y^h) &= (\alpha_1 + \alpha_3)(r)g_x(X, Y) + (\beta_1 + \beta_3)(r)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) &= \alpha_2(r)g_x(X, Y) + \beta_2(r)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^h) &= \alpha_2(r)g_x(X, Y) + \beta_2(r)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) &= \alpha_1(r)g_x(X, Y) + \beta_1(r)g_x(X, u)g_x(Y, u), \end{cases}$$

where $r = g_x(u, u)$. For $n = 1$, the same holds with $\beta_i = 0$, for $i = 1, 2, 3$.

Notations 1. In the sequel, we shall use the following notations:

(i) $\phi_i = \alpha_i(t) + t\beta_i(t)$,

(ii) $\alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t)$,

(iii) $\phi(t) = \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t)$,

for all $t \in \mathbb{R}^+$.

Some of the metrics described in Proposition 34 could possibly be pssudo-Riemannian in general, hence Riemannian g-natural metrics among them are characterized as follows:

Proposition 35. *(see [7]) The necessary and sufficient conditions for a g-natural metric G on the tangent bundle to a Riemannian manifold (M,g) to be Riemannian are that the functions of Proposition 34, defining G, satisfy the inequalities*

$$\begin{cases} \alpha_1(t) > 0, & \phi_1(t) > 0, \\ \alpha(t) > 0, & \phi(t) > 0, \end{cases} \tag{2.3.1}$$

for all $t \in \mathbb{R}^+$.

For $n = 1$, the system reduces to $\alpha_1(t) > 0$ and $\alpha(t) > 0$, for all $t \in \mathbb{R}^+$.

The *unit tangent sphere bundle* over a Riemannian manifold (M,g) is the hypersurface $T_1M = \{(x,u) \in TM : g_x(u,u) = 1\}$ in TM. The tangent space to T_1M, at a point $(x,u) \in T_1M$, is given by

$$T_{(x,u)}T_1M = \big\{X^h + Y^v : X \in T_xM, Y \in \{u\}^\perp \subset T_xM\big\}.$$

By a (pseudo)-Riemannian g-natural metric $\tilde{G}$ on T_1M we mean a (pseudo)-Riemannian metric induced by a g-natural metric G on TM. It's well known that a g-natural metric on T_1M is completely determined by the values of four real constants, namely:

$$a = \alpha_1(1); \quad b = \alpha_2(1); \quad c = \alpha_3(1); \quad d = (\beta_1 + \beta_3)(1);$$

where α_i and β_i are the weight functions of a g-natural metric as given in Proposition 34. In the same way, a g-natural metric $\tilde{G}$ on T_1M is Riemannian if and only if

$$a > 0; \quad \alpha = a(a+c) - b^2 > 0; \quad \phi = a(a+c+d) - b^2 > 0.$$

If $\tilde{G}$ is a g-natural metric on T_1M, then, at each point $(x,u) \in T_1M$, the metric $\tilde{G}$ is completely determined by

$$\begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) = (a+c)g_x(X,Y) + dg_x(X,u)g_x(Y,u), \\ \tilde{G}_{(x,u)}(X^h, Y^v) = \tilde{G}_{(x,u)}(Y^v, X^h) = bg_x(X,Y), \\ \tilde{G}_{(x,u)}(X^v, Y^v) = cg_x(X,Y). \end{cases}$$

It is worth mentioning that the tangent sphere bundle is not a vector bundle, it is a fiber bundle. So our results does not work directly in this case, but we shall extend them without troubles nor extra assumptions.

Let G be a Riemannian g-natural metric on TM with weight functions α_i and β_i, for $i = 1,2,3$, such that:

$$\alpha_2 = 0, \beta_2 = 0, \beta_1 + \beta_3 = 0.$$

This way, we obtain a class of metrics known as *Kaluza-Klein metrics* (**KK-metrics**). Let G be a KK-metric, then using similar techniques to those used in the proof of Proposition 29, one can prove that the projection $\pi : (TM,G) \longrightarrow (M,g)$ (resp. $\pi' : (T_1M, \tilde{G}) \longrightarrow (M,g)$) is a horizontally homothetic submersion with dilation function $\lambda = \sqrt{(\alpha_1 + \alpha_3)}$ (resp. a horizontally homothetic with constant dilation function $\lambda = \sqrt{(a+c)}$).

Consequently, necessary conditions for the completeness of KK-metrics on tangent bundles and unit tangent bundle can be derived from Theorem 10. Indeed, we have:

Proposition 36. *Let (M,g) be a Riemannian manifold and let TM be its tangent bundle endowed with a KK-metric G with weight functions α_i and β_i, for $i = 1,2,3$. Assume that $\alpha_1 + \alpha_3$ satisfies $a \leq \alpha_1 + \alpha_3 \leq b$, for some real numbers $a, b > 0$. If (M,g) is complete, then so is (TM,G). Furthermore, if $\beta_3 = 0$, then the equivalence holds.*

Proof: It follows from Lemma 5 and Theorem 10 and Proposition 32. ∎

Proposition 37. *Let (M, g) be a Riemannian manifold and let T_1M be its unit tangent sphere bundle endowed with $\tilde{G}$ induced from a KK-metric G with weight functions α_i and β_i, for $i = 1, 2, 3$. Then $(T_1M, \tilde{G})$ is complete if and only if (M, g) is complete.*

Proof: We give only a sketch of the proof since it stems from the proof of Theorem 10. Since the projection $\pi : (T_1M, G) \longrightarrow (M, g)$ is a horizontally conformal submersion with constant dilation function $\lambda = \sqrt{(a + c)}$ with $a = \alpha_1(1)$ and $c = \alpha_3(1)$, Lemma 5 implies that the projection of a Cauchy sequence of the total space is a Cauchy sequence of the base M. Moreover, the horizontal lift of a geodesic, with respect to the Levi-Civita connection of $\tilde{G}$, exists by virtue of a more general version of Proposition 24 in the case of fiber bundles (cf. [42]). Hence, one obtains a limit point for the considered sequence, which implies at once that (E, G) is complete. The converse follows easily since the dilation function is constant. ■

Remark 11. The class of *Kaluza-Klein metrics* on (unit) tangent bundles to Riemannian manifolds includes the Sasaki metric [68], the spherically symmetric metrics, the Cheeger-Gromoll metric and the Cheeger-Gromoll type metrics $G^{p,q}$ defined in [49].

Chapter 3

Symmetries of spherically symmetric metrics

Motivation

In the context of Riemannian geometry, it is well known that many geometric features of a Riemannian manifold are encoded in the so-called symmetries of the metric tensor like isometries, conformal, affine and projective transformations (cf. [32,39,42,59,79]). Symmetries are generally difficult to describe globally, but they can be described locally using 1-parameter groups of transformations of vector fields which give rise to the notions of Killing, conformal, affine and projective vector fields, respectively, which are called *infinitesimal symmetries* and contain many geometric informations about the manifold under consideration. For instance, the set of Killing vector fields forms a Lie subalgebra of the Lie algebra of smooth vector fields on a Riemannian manifold. Furthermore, when the manifold is complete, the Lie subalgebra of killing vector fields is the Lie algebra of the isometry group.

Another instance that makes the investigation of conformal vector fields extremely important is the fact that certain Riemannian manifolds can be 'classified' using conformal gradient vector fields [21].

This chapter is devoted to the study of conformal as well as closed and Killing vector fields on vector bundle manifolds when endowed with a *spherically symmetric metric*.

3.1 Preliminaries

Generally speaking, let (M, g) be an n-dimensional Riemannian manifold, ∇ its Levi-Civita connection and $\mathfrak{X}(M)$ its Lie algebra of smooth vector fields.

Two Riemannian metrics g_1 and g_2 on the manifold M are said to be *conformally equivalent* if there exists a function $\varphi \in C^\infty(M)$ such that $g_1 = e^\varphi g_2$. A *conformal manifold* is a manifold endowed with a conformal equivalence class of Riemannian metrics. Two conformally equivalent Riemannian metrics induce the same angle measure between vectors, so a conformal class on M is equivalent to the specification of an angle measure on each tangent space to M which vary smoothly with $x \in M$.

A map between conformal manifolds is said to be *conformal* if it preserves the angle measures between vectors; in other words, if M_1 and M_2 are conformal manifolds with g_1 and g_2 any representative Riemannian metrics chosen on them respectively, then a mapping $f : M_1 \longrightarrow M_2$ is conformal if $f^* g_2$ is conformally equivalent to g_1. A conformal diffeomorphism of a conformal manifold is called a *conformal transformation*. Finally, a vector field is said to be a *conformal vector field* if its local 1-parameter group consists of local conformal transformations.

If X is a vector field on (M, g), then the Lie derivative of g along X is defined by

$$\mathcal{L}_X g = \frac{d}{dt}\Big|_{t=0} \varphi_t^* g,$$

where $\{\varphi_t\}_t$ is the local 1-parameter group of X.

Conformal vector fields are infinitesimally characterized using Lie derivatives. Indeed, a vector field X on (M, g) is conformal if and only if there exists a smooth function f on M such that

$$\mathcal{L}_X g = 2fg,$$

the function f being called the *potential function* of X. A conformal vector field with potential function f, such that $f \neq 0$, is said to be a *non-trivial conformal vector field*. On the other hand, a conformal vector field on M with vanishing potential function is said to be a *Killing vector field*. So, a vector field on M is a Killing vector field if and only if its local 1-parameter group consists of local isometries. The Lie bracket of two Killing vector fields is a Killing vector field, thus the set of Killing vector fields forms a Lie subalgebra of $\mathfrak{X}(M)$ which we denote by $\hat{I}(M)$.

The Lie derivative of the metric tensor is given by

$$\mathcal{L}_X g(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X), \tag{3.1.1}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

In virtue of the Koszul formula, we have

$$2g(\nabla_Y X, Z) = \mathcal{L}_X g(Y, Z) + dX^\flat(Y, Z), \tag{3.1.2}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where $X^\flat$ denotes the dual 1-form to X with respect to g. A vector field $X \in \mathfrak{X}(M)$ is said to be *closed* if the dual 1-form $X^\flat$ is closed.

Given a vector field $X \in \mathfrak{X}(M)$, the exterior differential of $X^\flat$ can be expressed as

$$dX^\flat(Y, Z) = 2g(\varphi_X(Y), Z), \tag{3.1.3}$$

for all $Y, Z \in \mathfrak{X}(M)$, where φ_X is a skew-symmetric $(1,1)$-tensor field on M called the *associated tensor field* to the vector field X. Using equations (3.1.2) and (3.1.3), we conclude that if X is a conformal vector field with potential function f, then

$$\nabla_Y X = fY + \varphi_X(Y), \tag{3.1.4}$$

for all $Y \in \mathfrak{X}(M)$.

3.2 Symmetries of Spherically symmetric metrics

Let G be a spherically symmetric metric with weights φ_1 and φ_2 on the vector bundle (E, π, M). This section is devoted to investigate the different classes of symmetries of SS-metrics. In order to avoid confusion, we will use the following notations: vector fields on (E, G) are denoted by X, Y, Z or X^H, Y^V...etc (when we want to specify whether they are horizontal or vertical). On the other hand, we use $\underline{X}, \underline{Y}, \underline{Z}$...etc for vector fields on the base.

First we recall some local expressions of the covariant derivative of a vector field on E, with respect to an appropriate local frame, as well as the expression of the Lie derivative of G given in [13]. We use the Einstein summation convention.

Let (U, x^i) be a local coordinate system on M, $\{\sigma_p\}$ be a local orthonormal frame of E, and let $\{\partial_i^h, \sigma_p^v\}$ be the local linear frame of E, induced from $\{\partial_i\}$ and $\{\sigma_p\}$, where ∂_i^h is the

horizontal lift of $\dfrac{\partial}{\partial x^i}$ and σ_p^v is the vertical lift of σ_p. Denote by $\{x^i, v^\alpha\}$ the induced local coordinate system on E. If $Y = Y^i\partial_i^h + Y_\alpha\sigma_\alpha^v$ is a vector field, then $\tilde{\nabla}Y$ is totally determined by the following equations (cf. [13]):

$$\big(\tilde{\nabla}_{\partial_i^h}Y\big)^j = \frac{\partial Y^j}{\partial x^i} + Y^l\Gamma^{M,j}_{il} + av^\alpha Y_\alpha\delta^i_j + \frac{e^{2(\varphi_2-\varphi_1)}}{2}v^\alpha Y_\beta R^E_{\beta\alpha im}g^{mj}, \tag{3.2.1}$$

$$\big(\tilde{\nabla}_{\sigma_\alpha^v}Y\big)^j = \frac{\partial Y^j}{\partial v^\alpha} + av^\alpha Y^j + \frac{e^{2(\varphi_2-\varphi_1)}}{2}v^\beta Y^l R^E_{\alpha\beta lm}g^{mj}, \tag{3.2.2}$$

$$\big(\tilde{\nabla}_{\partial_i^h}Y\big)_\alpha = \frac{\partial Y_\alpha}{\partial x^i} + Y_\beta\Gamma^{E,\alpha}_{i\beta} + c_1v^\alpha Y^j g_{ij} - \frac{1}{2}\frac{e^{2(\varphi_2-\varphi_1)}}{2}v^\beta Y^j R^E_{\alpha\beta ij}, \tag{3.2.3}$$

$$\big(\tilde{\nabla}_{\sigma_\alpha^v}Y\big)_\beta = \frac{\partial Y_\beta}{\partial v^\alpha} + c_2v^\beta Y_\alpha + bv^\alpha Y_\alpha + bv^\gamma Y_\gamma\delta^\alpha_\beta, \tag{3.2.4}$$

where $\Gamma^{M,i}_{jk}$ and $\Gamma^{E,\alpha}_{i\beta}$ are respectively the connection components of the connections ∇ and D, and $R^E_{\alpha\beta ij}$ are the components of the curvature of the connection D given by $R^E_{\beta\alpha ij} = h(R^E(\partial_i,\partial_j)\sigma_\alpha,\sigma_\beta)$.

Proposition 38. *Let $X \in \mathfrak{X}(E)$ be a vector field, then the Lie derivative of G along X is given by*

$$\begin{aligned}\mathcal{L}_X G(Y,Z) = {} & e^{2\varphi_1}\mathcal{L}_{X^H}\pi^*g(Y,Z) + e^{2\varphi_2}\mathcal{L}_{X^V}\pi^\star h(Y,Z) + 2ae^{2\varphi_1}\xi^\flat(X)\langle Y,Z\rangle_M \\ & + 2be^{2\varphi_2}\xi^\flat(X)\langle Y,Z\rangle_E + e^{2\varphi_2}\langle\mathcal{R}^\xi(X,Y),Z\rangle_E + e^{2\varphi_2}\langle\mathcal{R}^\xi(X,Z),Y\rangle_E,\end{aligned}$$

where

$$\mathcal{L}_{X^H}\pi^*g(Y,Z) = \pi^*g(\pi^*\nabla_Y X^H, Z) + \pi^*g(\pi^*\nabla_Z X^H, Y), \tag{3.2.5}$$

and

$$\mathcal{L}_{X^V}\pi^\star h(Y,Z) = \pi^\star h(\pi^\star D_Y X^V, Z) + \pi^\star h(\pi^\star D_Z X^V, Y). \tag{3.2.6}$$

3.2.1 Closed vector fields

Our aim in this subsection is to study closed vector fields on (E,G), that is vector fields with closed dual 1-forms. We shall denote the dual 1-form of a vector field X on E (resp. $\underline{X}$ on M), with respect to G (resp. g), by $X^\flat$ (resp. $\underline{X}^\flat$).

Given $X \in \mathfrak{X}(E)$, we define a 1-form $\psi_X \in \Omega^1(E)$ by

$$\psi_X(Y) = 2\big(ae^{2\varphi_1}\langle X,Y\rangle_M + be^{2\varphi_2}\langle X,Y\rangle_E\big),$$

for all $Y \in \mathfrak{X}(E)$, with $a = 2\varphi_1'$ and $b = 2\varphi_2'$.

Long but routine computations give:

Proposition 39. *Let $X \in \mathfrak{X}(E)$, then*

$$\begin{aligned}dX^\flat(Y,Z) = {} & e^{2\varphi_1}(\pi^*g(\pi^*\nabla_Y X^H, Z) - \pi^*g(\pi^*\nabla_Z X^H, Y)) \\ & + e^{2\varphi_2}(\pi^\star h(\pi^\star D_Y X^V, Z) - \pi^\star h(\pi^\star D_Z X^V, Y)) \\ & + \xi^\flat(Y)\psi_X(Z) - \xi^\flat(Z)\psi_X(Y) + e^{2\varphi_2}\langle\mathcal{R}^\xi(Y,Z),X\rangle_E,\end{aligned}$$

where X^H and X^V are, respectively, the horizontal and vertical components of X.

We will determine when horizontal lifts of vector fields on M, vertical lifts of sections of E and the tautological vector field ξ are closed vector fields.

Lemma 6. *Let X^H be a non-zero horizontal vector field on E. If X^H is closed, then φ_1 is constant.*

Proof. If X^H is closed, then we have, by virtue of Proposition 39,

$$0 = d(X^H)^\flat(Y^H, Z^V) = -2ae^{2\varphi_1}\xi^\flat(Z^V)\langle X^H, Y^H\rangle_M,$$

for vertical vector fields Z^V and horizontal vector fields Y^H on E. In particular, for $Y^H = X^H \neq 0$ and $Z^V = \xi$, we get

$$r\varphi_1' = 0,$$

thus $\varphi_1' = 0$ on $\mathbb{R}_*^+$, and by continuity $\varphi_1' = 0$ on $\mathbb{R}^+$. □

Theorem 11. *Let $\underline{X}$ be a vector field on M and $\underline{X}^h$ be its horizontal lift. Then, $\underline{X}^h$ is a closed vector field on E if and only if φ_1 is constant and $\underline{X}$ is a closed vector field on M.*

Proof. For any vector fields $\underline{Y}$ and $\underline{Z}$ on M, we have, by Proposition 39,

$$d(\underline{X}^h)^\flat(\underline{Y}^h, \underline{Z}^h) = e^{2\varphi_1} d\underline{X}^\flat(\underline{Y}, \underline{Z}).$$

If $\underline{X}^h$ is closed, then $\underline{X}$ is closed and φ_1 is constant by virtue of Lemma 6.

Conversely, assume φ_1 is constant, i.e. $a = 0$, and $\underline{X}$ is a closed vector field on M. For $e \in E$, every $Y, Z \in T_eE$ can be expressed as $Y = \underline{Y}^h + Y^V$ and $Z = \underline{Z}^h + Z^V$, where $\underline{Y} = (d_e\pi)Y$, $\underline{Z} = (d_e\pi)Z$ and Y^V and Z^V are the vertical components of Y and Z, respectively. Using Proposition 39, we get

$$\begin{aligned} d(\underline{X}^h)^\flat(Y, Z) &= e^{2\varphi_1}\left[\pi^* g(\pi^*\nabla_{\underline{Y}^h}\underline{X}^h, \underline{Z}^h) - \pi^* g(\pi^*\nabla_{\underline{Z}^h}\underline{X}^h, \underline{Y}^h)\right] \\ &= e^{2\varphi_1}\left[g(\nabla_{\underline{Y}}\underline{X}, \underline{Z}) - g(\nabla_{\underline{Z}}\underline{X}, \underline{Y})\right] \\ &= e^{2\varphi_1} d\underline{X}^\flat(\underline{Y}, \underline{Z}) = 0. \end{aligned}$$

Thus $\underline{X}^h$ is a closed vector field on E. □

Now, we shall study some special cases of vertical vector fields on E. For the tautological vector field ξ, a straightforward computation, using Proposition 39, gives the following

Proposition 40. *ξ is a closed vector field.*

Concerning vertical lifts of smooth sections of E, we have

Theorem 12. *Assume that $k \geq 2$ and let σ be a smooth section of E. The vertical lift σ^v of σ is a closed vector field on E if and only if the following assertions hold:*

1. *σ is a parallel section of E,*
2. *φ_2 is constant.*

Proof. If σ^v is closed, then we have, by Proposition 39,

$$0 = d(\sigma^v)^\flat(Y^V, \underline{Z}^h) = -e^{2\varphi_2}\pi^\star h(\pi^\star D_{\underline{Z}^h}\sigma^v, Y^V) = -e^{2\varphi_2}\pi^\star h((D_{\underline{Z}}\sigma)^v, Y^V),$$

for every vector field $\underline{Z}$ on M and every vertical vector field Y^V on E. Thus $D_{\underline{Z}}\sigma = 0$ for all $\underline{Z} \in \mathfrak{X}(M)$, hence σ is parallel.

On the other hand, by definition of the pullback connection we have $\pi^\star D_{Y^V}\sigma^v = 0$. Then for vertical vector fields Y^V and Z^V on E, we have by Proposition 39

$$\begin{aligned} 0 =& d(\sigma^v)^\flat(Y^V, Z^V) \\ =& 2be^{2\varphi_2}\left(\xi^\flat(Y^V)\langle\sigma^v, Z^V\rangle_E - \xi^\flat(Z^V)\langle\sigma^v, Y^V\rangle_E\right). \end{aligned}$$

Taking $\xi_e = Y_e^V \perp Z_e^V = \sigma_e^v$ in the last identity, for $e \in E$, we get $b = 0$ and hence $\varphi_2' = 0$.

Conversely, if σ is parallel, then we have, by Proposition 39,

$$d(\sigma^v)^\flat(Y^V, Z^H) = d(\sigma^v)^\flat(Y^H, Z^V) = 0,$$

and if φ_2 is constant, then we get

$$d(\sigma^v)^\flat(Y^V, Z^V) = 0,$$

for all vertical vector fields Y^V and Z^V on E.

Finally, to show that $d(\sigma^v)^\flat(Y^H, Z^H) = 0$, for all horizontal vector fields Y^H and Z^H on E, let $e \in E$ with $\pi(e) = x$ and $\underline{Y}, \underline{Z} \in \mathfrak{X}(M)$ such that $\underline{Y}_x = (d_e\pi)Y_e^H$ and $\underline{Z}_x = (d_e\pi)Z_e^H$, so that $\underline{Y}_e^h = Y_e^H$ and $\underline{Z}_e^h = Z_e^H$. It is easy to see that

$$\begin{aligned}\big(d(\sigma^v)^\flat(Y^H, Z^H)\big)_e &= e^{2\varphi_2}\langle \mathcal{R}^\xi(\underline{Y}^h, \underline{Z}^h), \sigma^v\rangle_E(e) \\ &= -e^{2\varphi_2} h((R^E(\underline{Y}, \underline{Z})\sigma)(x), e) = 0.\end{aligned}$$

□

3.2.2 Conformal horizontal vector fields

In this section we will study conformal horizontal vector fields.

Lemma 7. *If X^H is a horizontal vector field on E, then the Lie derivative of G along X^H is given as follows:*

1. $\mathcal{L}_{X^H} G(Y^H, Z^H) = e^{2\varphi_1}\mathcal{L}_{X^H}\pi^* g(Y^H, Z^H)$,
2. $\mathcal{L}_{X^H} G(Y^H, Z^V) = e^{2\varphi_2}\langle \mathcal{R}^\xi(X^H, Y^H), Z^V\rangle_E$,
3. $\mathcal{L}_{X^H} G(Y^V, Z^H) = e^{2\varphi_2}\langle \mathcal{R}^\xi(X^H, Z^H), Y^V\rangle_E$,
4. $\mathcal{L}_{X^H} G(Y^V, Z^V) = 0$;

for all horizontal vector fields Y^H, Z^H and vertical vector fields Y^V, Z^V on E.

Proposition 41. *Let X^H be a horizontal vector field on E. Then the following assertions are equivalent:*

1. *X^H is a conformal vector field,*
2. *X^H is a Killing vector field,*
3. *$\mathcal{R}^\xi(X^H, .) = 0$ and $\mathcal{L}_{X^H}\pi^* g(Y^H, Z^H) = 0$, for all horizontal vector fields Y^H, Z^H on E.*

Proof. Assume that X^H is a horizontal conformal vector field on E, then there exists $f \in C^\infty(E)$ such that

$$\mathcal{L}_{X^H} G(Y, Z) = 2fG(Y, Z),$$

for all $Y, Z \in \mathfrak{X}(E)$. Let $e \in E$, and let Y^V be a vertical vector field which does not vanish in a neighbourhood of e (e.g. $Y^V = \xi$), thus

$$0 = \big(\mathcal{L}_{X^H} G\big)_e(Y^V, Y^V) = 2f(e)G_e(Y_e^V, Y_e^V) = 2f(e)e^{2\varphi_2(r)}\pi^\star h_e(Y_e^V, Y_e^V),$$

which implies that $f(e) = 0$, hence $f \equiv 0$, thus X^H is a Killing vector field. Hence 1. implies 2.. The converse is obvious. The equivalence between (2) and (3) follows directly from Lemma 7. □

So, there is no non-trivial horizontal conformal vector fields, as a matter of fact, the problem of finding horizontal conformal vector fields reduces to the problem of finding horizontal vector fields which are Killing vector fields.

Now, we shall focus on the case of horizontal lifts of vector fields of the base manifold.

Proposition 42. *Let $\underline{X} \in \mathfrak{X}(M)$ and $\underline{X}^h$ its horizontal lift. Then, $\underline{X}^h$ is a Killing vector field on E if and only if the following assertions hold:*

1. *$\underline{X}$ is a Killing vector field on M,*
2. *$R^E(\underline{X}, .) = 0$.*

Proof. If $\underline{X}^h$ is a Killing vector field on (E, G), then by virtue of (3) of Proposition 41, we have

$$\mathcal{L}_{\underline{X}^h}\pi^* g(Y^H, Z^H) = 0,$$

for all horizontal vector fields Y^H, Z^H. In particular, if $Y^H = \underline{Y}^h$ and $Z^H = \underline{Z}^h$, where $\underline{Y}, \underline{Z} \in \mathfrak{X}(M)$, then we obtain

$$\begin{aligned} 0 = \mathcal{L}_{\underline{X}^h}\pi^* g(Y^H, Z^H) &= \mathcal{L}_{\underline{X}^h}\pi^* g(\underline{Y}^h, \underline{Z}^h) \\ &= (\mathcal{L}_{\underline{X}} g)(\underline{Y}, \underline{Z}), \end{aligned}$$

thus $\underline{X}$ is a Killing vector field on (M, g). Further, (2) follows again from (3) of Proposition 41.

Conversely, let $e \in E$, then any $Y, Z \in T_eE$ can be expressed as $Y = \underline{Y}^h + Y^V$ and $Z = \underline{Z}^h + Z^V$, where $\underline{Y} = (d_e\pi)Y$, $\underline{Z} = (d_e\pi)Z$, with Y^V and Z^V are the vertical components of Y and Z, respectively. Using Lemma 7 and the fact that $R^E(\underline{X}, .) = 0$, we get

$$\begin{aligned} \mathcal{L}_{\underline{X}^h}\pi^* g(Y, Z) &= \pi^* g(\pi^* \nabla_{\underline{Y}^h} \underline{X}^h, \underline{Z}^h) + \pi^* g(\pi^* \nabla_{\underline{Z}^h} \underline{X}^h, \underline{Y}^h) \\ &= g(\nabla_{\underline{Y}} \underline{X}, \underline{Z}) + g(\nabla_{\underline{Z}} \underline{X}, \underline{Y}) \\ &= \mathcal{L}_{\underline{X}} g(\underline{Y}, \underline{Z}) = 0, \end{aligned}$$

hence $\underline{X}^h$ is a Killing vector field on E. □

So, it follows that:

Corollary 6. *If the connection D is flat (i.e. $R^E = 0$), then $\underline{X} \in \mathfrak{X}(M)$ is a Killing vector field on (M, g) if and only if $\underline{X}^h$ is a Killing vector field on (E, G).*

The following result generalises a similar one in the case of φ_1 and φ_2 being constant (see [13]).

Corollary 7. *Assume that φ_1 is constant, and let $\underline{X}$ be a vector field on M such that:*

1. *$\underline{X}$ is a parallel vector field,*
2. *$R^E(\underline{X}, .) = 0$,*

then $\underline{X}^h$ is a parallel vector field on (E, G). Furthermore, all parallel horizontal vector field are obtained in this way.

Proof. It follows from Theorem 11 and Proposition 42. Further, by virtue of equation (3.2.2), since φ_1 is constant and hence $a = 0$, a horizontal parallel vector field must be a horizontal lift. □

In what follows, we give a classification of horizontal Killing vector fields if φ_1 is constant.

Theorem 13. *Assume that φ_1 is constant. Then, the only horizontal Killing vector fields on (E, G) are the horizontal lifts of Killing vector fields on (M, g).*

Proof. Let $Y = Y^i\partial_i^h$ be a horizontal Killing vector field on (E, G). Then, by virtue of Proposition 41, we have $\mathcal{R}^\xi(Y, .) = 0$. On the other hand, we have

$$2G(\tilde{\nabla}_{\sigma_\alpha^v} Y, \partial_i^h) = d(Y^\flat)(\sigma_\alpha^v, \partial_i^h),$$

and $d(Y^\flat)(\sigma_\alpha^v, \partial_i^h) = ae^{2\varphi_1}\xi^\flat(\sigma_\alpha^v)\langle Y, \partial_i^h\rangle_M$, thus, by virtue of equation (3.2.2), we have

$$2e^{2\varphi_1}\big(\tilde{\nabla}_{\sigma_\alpha^v} Y\big)^i = 2e^{2\varphi_1}\big(\frac{\partial Y^i}{\partial v^\alpha} + av^\alpha Y^i\big) = ae^{2\varphi_1} v^\alpha Y^i. \tag{3.2.7}$$

We conclude that

$$\frac{\partial Y^i}{\partial v^\alpha} = -av^\alpha Y^i = 0, \tag{3.2.8}$$

for all $i = 1, ..., n$ and $\alpha = 1, ..., k$, since φ_1 is constant and hence $a = 0$, it follows that Y must be a horizontal lift. □

The equation $\dfrac{\partial Y^i}{\partial v^\alpha} = 0$, for all $i = 1, ..., n$ and $\alpha = 1, ..., k$, is the local characterization of the fact that Y preserves the fibers of E. Recall that a vector field $Y \in \mathfrak{X}(E)$ is said to be *fiber-preserving* if the Lie bracket $[Y, Z]$ is a vertical vector field on E for any vertical vector field $Z \in \mathfrak{X}(E)$. It is then easy to see that all vertical vector fields on E are fiber-preserving and a horizontal vector field on E is fiber-preserving if and only if it is the horizontal lift of a vector field on M.

Consequently, Theorem 13 can be stated as follows: if φ_1 is constant, then the only Killing horizontal vector fields on E are those which are fiber-preserving. Actually, the identity (3.2.8) gives the converse:

Proposition 43. *If a non-zero fiber-preserving horizontal vector field on E is Killing, then φ_1 is constant.*

3.2.3 Conformal vertical vector fields

Lemma 8. *Let X^V be a vertical vector field on E, then the Lie derivative of G along X^V is given as follows:*

1. $\mathcal{L}_{X^V} G(Y^H, Z^H) = 2ae^{2\varphi_1}\xi^\flat(X^V)\langle Y^H, Z^H\rangle_M$,
2. $\mathcal{L}_{X^V} G(Y^H, Z^V) = e^{2\varphi_2}\pi^\star h(\pi^\star D_{Y^H} X^V, Z^V)$,
3. $\mathcal{L}_{X^V} G(Y^V, Z^H) = e^{2\varphi_2}\pi^\star h(\pi^\star D_{Z^H} X^V, Y^V)$,
4. $\mathcal{L}_{X^V} G(Y^V, Z^V) = e^{2\varphi_2}\mathcal{L}_{X^V}\pi^\star h(Y^V, Z^V) + 2be^{2\varphi_2}\xi^\flat(X^V)\langle Y^V, Z^V\rangle_E$;

where Y^H, Z^H are horizontal vector fields and Y^V, Z^V are vertical vector fields.

Lemma 9. *Let X^V be a vertical vector field. Then X^V is a conformal vector field on E with potential function f if and only if the following conditions hold:*

1. $f = a\xi^\flat(X^V)$,
2. $\mathcal{L}_{X^V}\pi^\star h(Y^V, Z^V) = 2(a - b)\xi^\flat(X^V)\pi^\star h(Y^V, Z^V)$, *where Y^V, Z^V are vertical vector fields.*

3. $\pi^\star D_{Y^H} X^V = 0$, *for any horizontal vector field* Y^H.

Proof. It follows directly from the expression of the Lie derivative. □

Proposition 44. *If φ_1 is constant, then every vertical conformal vector field on E is a Killing vector field.*

Proof. If X^V is a conformal vector field with potential function f, then $f = a\xi^\flat(X^V)$. The function φ_1 being constant yields $a = 2\varphi_1' = 0$, whence $f = 0$, thus X^V is a Killing vector field. □

The first example of vertical vector fields to examine is the tautological vector field ξ.

Theorem 14. *The vector field ξ can not be a conformal vector field on (E, G).*

Proof. If ξ is a conformal vector field, then by virtue of Lemma 9, a necessary condition for ξ to be a conformal vector field with potential f is the equation $f = a\xi^\flat(\xi) = ra$ with $a = 2\varphi_1'$. On the other hand, we have

$$\begin{aligned}\mathcal{L}_\xi \pi^\star h(Y^V, Z^V) &= \pi^\star h(\pi^\star D_{Y^V}\xi, Z^V) + \pi^\star h(Y^V, \pi^\star D_{Z^V}\xi)\\ &= 2\pi^\star h(Y^V, Z^V),\end{aligned}$$

for all vertical vector fields Y^V, Z^V on E. Hence

$$\mathcal{L}_\xi \pi^\star h(Y^V, Z^V) = 2\langle Y^V, Z^V\rangle_E. \tag{3.2.9}$$

Lemma 9 again implies that we must have

$$2(a-b)\xi^\flat(\xi) = 2, \tag{3.2.10}$$

which is equivalent to

$$\varphi_1' - \varphi_2' = \frac{1}{2r}. \tag{3.2.11}$$

The regularity, of the weight functions, required in the definition of spherically symmetric metrics implies that the LHS is differentiable at 0 on the right, but clearly the RHS does not fulfil this requirement; as a matter of fact, ξ can not be a conformal vector field. □

Remarks 2.

1. Let O be the zero section of E. If we restrict ourselves to the 'slit' vector bundle defined as $E^* = E \setminus \{o\}$ endowed with the induced spherically symmetric metric G^*. Then $\xi_{|E^*}$ is a conformal vector field with potential function $f = 2r\varphi_1'$ if and only if φ_1 and φ_2 satisfy the equation $\varphi_1 - \varphi_2 = \frac{1}{2}\ln(r) + C$, with C is a constant.

2. In an attempt to find geometrically interesting examples of vertical vector fields, a large class of vertical vector fields which generalizes the vector field ξ can be constructed using vector bundle morphisms. Indeed, let $\Phi : E \longrightarrow E$ be a vector bundle morphism covering the identity of M i.e. $\pi \circ \Phi = \pi$. The morphism Φ gives rise to the vector field defined by
$$l\Phi(e) = \big(\Phi(e)\big)^v_e,$$
for all $e \in E$. Clearly $l\Phi$ is a smooth vertical vector field on E. This construction is a generalization of a construction in [73] which was used to classify Killing vector fields in the case of Cheeger-Gromoll metric (cf. [3]). By virtue of Lemma 9, $l\Phi$ is a conformal vector field with potential function f if and only if:

(i) $f(e) = 2\varphi_1'(r)h(\Phi(e), e)$ with $r = \|e\|^2$,

(ii) $\mathcal{L}_{l\Phi}\pi^\star h(Y^V, Z^V) = 2(a-b)\xi^\flat(l\Phi)\pi^\star h(Y^V, Z^V)$, where Y^V, Z^V are vertical vector fields,

(iii) $\pi^\star D_{X^H}(l\Phi)$ for all horizontal vector field X^H.

For $\Phi = \mathrm{Id}_E$, we recover the tautological vector field ξ. Since, it does not work for the case of Id_E, as the reader may agree, it is non trivial to find a vector bundle morphism Φ such that $l\Phi$ is conformal in the full generality.

Proposition 45. *Let σ be a smooth section of E. If the vector field σ^v is conformal, then σ is a parallel section.*

Proof. If σ^v is a conformal vector field, then by virtue of Lemma 9 we have $\pi^\star D_{X^H}\sigma^v = 0$ for any horizontal vector field X^H. In particular for all $X \in \mathfrak{X}(M)$, we have

$$0 = \pi^\star D_{X^h}\sigma^v = (D_X\sigma)^v,$$

which implies that σ is parallel. □

Theorem 15. *Let σ be a non-zero smooth section of E. Then σ^v is a conformal vector field with potential function f if and only if the following assertions hold:*

1. *$f(e) = 2\varphi_1'(r)h(\sigma(\pi(e)), e)$, for all $e \in E$, where $r = h(e, e)$;*
2. *$\varphi_1' = \varphi_2'$ on $\mathbb{R}_+$;*
3. *σ is parallel.*

Proof. It suffices to prove that the three assertions of the theorem are equivalent to the corresponding ones in Lemma 9, when X^V is replaced by σ^v. It is clear that the first assertion of the theorem is equivalent to the first one in Lemma 9, since $\xi^\flat_e(\sigma^v) = h(\sigma(\pi(e)), e)$, for all $e \in E$.

On the other hand, since $(\pi^* D_{Y^H}\sigma^v)_e$ depends only on Y^H_e, then we can suppose that $Y^H = \underline{Y}^h$, where $\underline{Y}$ is a vector field on M such that $\underline{Y}^h_e = Y^H_e$. Then $(\pi^* D_{Y^H}\sigma^v)_e = \left(\pi^* D_{\underline{Y}^h}\sigma^v\right)_e = \left(D_{\underline{Y}}\sigma\right)^v_e$. Thus $\pi^* D_{Y^H}\sigma^v = 0$, for every horizontal vector field Y^H on E, which is equivalent to $D_{\underline{Y}}\sigma = 0$, for every vector field $\underline{Y}$ on M. We deduce that the third assertion of the theorem is equivalent to the third one in Lemma 9.

Finally, we know that for any arbitrary vertical vector fields Y^V, Z^V on E, we have

$$\mathcal{L}_{\sigma^v}\pi^\star h(Y^V, Z^V) = \pi^\star h(\pi^\star D_{Y^V}\sigma^v, Z^V) + \pi^\star h(Y^V, \pi^\star D_{Z^V}\sigma^v) = 0,$$

since $\pi^\star D_{Y^V}\sigma^v = 0$ by definition of the pullback connection $\pi^\star D$. Hence the second assertion of Lemma 9 is equivalent to

$$(a-b)\xi^\flat(\sigma^v) = 0. \tag{3.2.12}$$

Fix $r \in \mathbb{R}_+^*$, and let $x \in M$ such that $\sigma(x) \neq 0$. Putting $e := \frac{\sqrt{r}}{\|\sigma(x)\|}\sigma(x)$, we have $h(e, e) = r$, then $\xi^\flat_e(\sigma^v) = h(\sigma(x), e) = \sqrt{r}\|\sigma(x)\| \neq 0$. Thus (3.2.12) implies that $a(r) = b(r)$, i.e. $\varphi_1'(r) = \varphi_1'(r)$, for all $r \in \mathbb{R}_+^*$, and by continuity for all $r \in \mathbb{R}_+$. Conversely, if $\varphi_1' = \varphi_2'$ on $\mathbb{R}_+$, then $a = b$, and consequently (3.2.12) is satisfied. We deduce that the second assertion of the theorem is equivalent to the second one in Lemma 9. □

Corollary 8. *Let σ be a non-zero smooth section of E. Then σ^v is a Killing vector field on E if and only if the following assertions hold:*

1. *φ_1 and φ_2 are constant;*

2. *σ is parallel.*

Proof. Since Killing vector fields correspond to conformal vector fields with potential functions $f = 0$, then the first assertion of Theorem 15 is equivalent, in this case, to $\varphi_1' = 0$, i.e. φ_1 is a constant. We conclude that the second assertion of Theorem 15 is equivalent, in this situation, to φ_2 is constant. □

As a corollary of Theorems 12 and 15 and equation (3.2.4), we recover the following result due to Albuquerque (see [13]):

Corollary 9. *Let G be the Sasaki metric, and let σ be a parallel section with respect to D, then σ^v is a parallel vector field on (E, G). Furthermore, all parallel vertical vector fields are obtained in this way.*

Remark 12. A necessary condition for the existence of parallel sections is that $\chi(E) = 0$, so the non-vanishing of the Euler-Poincaré characteristic of E is a topological obstruction to the existence of parallel sections and hence to the existence of conformal vector fields of the form σ^v, where $\sigma \in \Gamma(E)$.

Now, combining Proposition 42 and Theorem 15, we obtain the following class of conformal vector fields:

Theorem 16. *Let $\sigma \in \Gamma(E)$ be a parallel section, assume that $\varphi_1' = \varphi_2'$ on $\mathbb{R}_+$, and let $X \in \mathfrak{X}(M)$ be a Killing vector field on (M, g) such that $R^E(X, .) = 0$, then the vector field $Z = X^h + \sigma^v$ is a conformal vector field on (E, G) with potential function $f(e) = 2\varphi_2'(r)h(\sigma(\pi(e)), e)$. In particular, if D is flat, then, for a Killing vector field X on M, the vector field $Z = X^h + \sigma^v$ is a conformal vector field with potential function $f(e) = 2\varphi_2'(r)h(\sigma(\pi(e)), e)$.*

Remark 13. If $\varphi_1' = \varphi_2' \neq 0$ on $\mathbb{R}^+$, then every vector field of the form $Z = X^h + \sigma^v$, where $\sigma \in \Gamma(E)$ is a parallel section and X a Killing vector field on (M, g), such that $R^E(X, .) = 0$, is a non-trivial conformal vector field on (E, G). A sufficient condition to have $\varphi_1' = \varphi_2'$ is to take $\varphi_1 = \varphi_2$, which gives a metric conformally equivalent to the Sasaki metric $\pi^* g \oplus \pi^\star h$ (cf. [23, 43]).

Example 5. Take $E = TM$, $D = \nabla$ the Levi-Civita connection of (M, g), $h = g$ and R the curvature tensor of g. Assume $\varphi_1(r) = \arctan(r)$ and $\varphi_2(r) = 1 + \arctan(r)$, and let $X \in \mathfrak{X}(M)$ be a Killing vector field on M such that $R(X, .) = 0$, and Y be a parallel vector field on M, then $Z = X^h + Y^v$ is a non-trivial conformal vector field with potential function $f(x, u) = \dfrac{2}{1 + \|u\|^2} g(Y_x, u)$, where $\|u\|^2 = g(u, u)$ and X^h and Y^v are the horizontal and vertical lifts of X and Y, respectively.

Example 6. Take $E = T^*M$, D the natural connection induced from the Levi-Civita connection ∇ of (M, g), and h the fiber metric induced on T^*M from g by musical isomorphisms. Set $\varphi_1(r) = \varphi_2(r) = P(r)$ where P is a non-constant polynomial with real coefficients. Assume that (M, g) is flat and let $X \in \mathfrak{X}(M)$ be a Killing vector field on M and $\omega \in \Omega^1(M)$ with $D\omega = 0$, then the vector field given by $Z = X^h + \omega^v$, is a non-trivial conformal vector field on T^*M with potential function $f(x, \alpha) = 2P'(r)g(\omega_x^\sharp, \alpha^\sharp)$ with $r = h(\alpha, \alpha)$, where X^h is the horizontal lift of X with respect to D and ω^v is the vertical lift of ω.

3.2.4 Conformal gradient vector fields

In this section we will focus on the case of gradient vector fields. First, recall that given $U \subset M$ an open set, and let $\{e_i : \ i = 1, ..., n\}$ be a linear local orthonormal frame on U. Restricting to a smaller U if necessary, we assume that $\{\sigma_p : \ p = 1, ..., k\}$ is a local orthonormal frame of E on U. Then $\{e^{-\varphi_1} e_i^h, e^{-\varphi_2}\sigma_p^v\}$ is a linear orthonormal frame on (E, G).

We will begin by functions which depend only on r, for a real valued function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$, f is said to be smooth if it is smooth on $\mathbb{R}_+^*$ and smooth at zero on the right as well as all its successive derivatives; so there is the associated function $F : E \longrightarrow \mathbb{R}$ given by $F(e) = f(r)$, for $e \in E$ and $r = h(e, e)$.

Proposition 46. *Assume that $k \geq 2$, let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a non-constant smooth real valued function, and let $F : E \longrightarrow \mathbb{R}$ be the function induced from f, that is $F(e) = f(r)$ with $r = \|e\|^2$. Then $\tilde{\nabla} F$ can not be a conformal vector field on E.*

Proof. We have $\tilde{\nabla} F = \lambda\xi$ with $\lambda = 2e^{-2\varphi_2} f'$. Indeed, by virtue of Lemma 3, we have

$$\begin{aligned}
\big(\tilde{\nabla} F\big)_e &= \sum_{i=1}^{n} \Big\{ \big(e^{-\varphi_1(r)} e_i^h . f(r)\big)\big(e^{-\varphi_1(r)} e_i^h\big) \Big\}_e \\
&\quad + \sum_{p=1}^{k} \Big\{ \big(e^{-\varphi_2(r)} \sigma_p^v . f(r)\big)\big(e^{-\varphi_2(r)} \sigma_p^v\big) \Big\}_e \\
&= 2e^{-2\varphi_2(r)} f'(r) \sum_{p=1}^{k} \Big\{ \xi^\flat(\sigma_p^v)\sigma_p^v \Big\}_e \\
&= 2e^{-2\varphi_2(r)} f'(r) \sum_{p=1}^{k} \Big\{ h(\sigma_p, e)\sigma_p^v \Big\}_e \\
&= 2e^{-2\varphi_2(r)} f'(r) e_e^v.
\end{aligned}$$

On the other hand, a simple computation gives

$$\mathcal{L}_{\tilde{\nabla} F} G(Y, Z) = (Y.\lambda) e^{2\varphi_2} \xi^\flat(Z) + (Z.\lambda) e^{2\varphi_2} \xi^\flat(Y) + \lambda \mathcal{L}_\xi G(Y, Z). \tag{3.2.13}$$

Since $Y.\lambda = 2\lambda'(r)\xi^\flat(Y)$, equation (3.2.13) is equivalent to

$$\mathcal{L}_{\tilde{\nabla} F} G(Y, Z) = 4\lambda' e^{2\varphi_2} \xi^\flat(Y)\xi^\flat(Z) + \lambda \mathcal{L}_\xi G(Y, Z). \tag{3.2.14}$$

So, for Y^H and Z^H horizontal vector fields, equation (3.2.14) implies that

$$\begin{aligned}
\mathcal{L}_{\tilde{\nabla} F} G(Y^H, Z^H) &= \lambda \mathcal{L}_\xi G(Y^H, Z^H) \\
&= 2a\lambda e^{2\varphi_1} \xi^\flat(\xi) \langle Y^H, Z^H \rangle_M \\
&= 2ar\lambda e^{2\varphi_1} \langle Y^H, Z^H \rangle_M \\
&= 2ar\lambda G(Y^H, Z^H).
\end{aligned}$$

Thus, if $\tilde{\nabla} F$ is a conformal vector field with potential function β, then $\beta = ar\lambda$.

Now, for vertical vector fields Y^V and Z^V, we have

$$\begin{aligned}
\mathcal{L}_\xi G(Y^V, Z^V) &= 2e^{2\varphi_2} \langle Y^V, Z^V \rangle_E + 2rbe^{2\varphi_2} \langle Y^V, Z^V \rangle_E \\
&= 2(1 + br) e^{2\varphi_2} \langle Y^V, Z^V \rangle_E,
\end{aligned}$$

thus

$$\mathcal{L}_{\tilde{\nabla} F} G(Y^V, Z^V) = 2(1 + br)\lambda e^{2\varphi_2} \langle Y^V, Z^V \rangle_E + 4\lambda' e^{2\varphi_2} \xi^\flat(Y^V)\xi^\flat(Z^V). \tag{3.2.15}$$

Taking $Y^V = Z^V = \xi$ into equation (3.2.15) and using the conformality of $\tilde{\nabla}F$, we get

$$2(1+br)\lambda e^{2\varphi_2}r + 4\lambda' e^{2\varphi_2}r^2 = 2\beta e^{2\varphi_2}r,$$

which gives

$$(1+br)\lambda + 2\lambda' r = a\lambda r.$$

Thus λ satisfies the following first order ODE:

$$\lambda' = \frac{1}{2}(a - b - \frac{1}{r})\lambda. \tag{3.2.16}$$

The solutions of the ODE (4.3.2) are of the form $\lambda(r) = \frac{A}{\sqrt{r}}e^{(\varphi_1-\varphi_2)(r)}$ which, if non-trivial, does not meet the regularity requirement on f and the weight functions. Hence $\lambda = 0$, that is f is constant. Which implies that $\tilde{\nabla}F$ is not a gradient conformal vector field. □

Proposition 47. *Assume that $\varphi_1 = 0$ and let $f \in C^\infty(M)$. If $X = \nabla f$ is a gradient Killing vector field such that $R^E(\nabla f, .) = 0$, then $(\nabla f)^h$ is a gradient Killing vector field on E. In particular, if D is flat, then for every gradient Killing vector field X on M, X^h is a gradient Killing vector field on E.*

For a smooth function $f : M \longrightarrow \mathbb{R}$, the *vertical lift* of f is the smooth function $f^v : E \longrightarrow \mathbb{R}$ with $f^v = f \circ \pi$.

Proposition 48. *Let f be a non-constant smooth function on M such that ∇f is a gradient Killing vector field satisfying $R^E(\nabla f, .) = 0$, then $\tilde{\nabla} f^v$ is a gradient Killing vector field on E if and only if φ_1 is constant.*

Proof. We have

$$\begin{aligned}
\big(\tilde{\nabla} f^v\big)_e &= \sum_{i=1}^{n}\Big\{\big(e^{-\varphi_1(r)}e_i^h.f^v\big)\big(e^{-\varphi_1(r)}e_i^h\big)\Big\}_e + \sum_{p=1}^{k}\Big\{\big(e^{-\varphi_2(r)}\sigma_p^v.f^v\big)\big(e^{-\varphi_2(r)}\sigma_p^v\big)\Big\}_e \\
&= e^{-2\varphi_1(r)}\sum_{i=1}^{n}\Big\{(e_i.f)^v e_i^h\Big\} \\
&= e^{-2\varphi_1(r)}(\nabla f)_e^h,
\end{aligned}$$

and since ∇f is a gradient Killing vector field satisfying $R^E(\nabla f, .) = 0$, then $(\nabla f)^h$ is a Killing vector field on E. On the other hand, we have

$$\begin{aligned}
\mathcal{L}_{e^{-2\varphi_1}(\nabla f)^h}G(Y,Z) = \big(Y.e^{-2\varphi_1}\big)G((\nabla f)^h, Z) + \big(Z.e^{-2\varphi_1}\big)G((\nabla f)^h, Y) \\
+ e^{-2\varphi_1}\mathcal{L}_{(\nabla f)^h}G(Y,Z),
\end{aligned}$$

where Y and Z are arbitrary vector fields. Thus

$$\mathcal{L}_{e^{-2\varphi_1}(\nabla f)^h}G(Y,Z) = \big(Y.e^{-2\varphi_1}\big)G((\nabla f)^h, Z) + \big(Z.e^{-2\varphi_1}\big)G((\nabla f)^h, Y). \tag{3.2.17}$$

If $\tilde{\nabla} f^v$ is a Killing vector field, then taking equation (3.2.17) for Y^H a horizontal vector field and Z^V a vertical vector field, we obtain

$$(Z^V.e^{-2\varphi_1})G(Y^H, (\nabla f)^h) = -4\varphi_1'(r)e^{-2\varphi_1}\xi^\flat(Z^V)G(Y^H, (\nabla f)^h) = 0.$$

In particular, for $Y^H = (\nabla f)^h$ and $Z^V = \xi$, we get $r\varphi_1' = 0$, thus $\varphi_1' = 0$ on $(0, +\infty)$, and by continuity on $\mathbb{R}^+$. Conversely, if φ_1 is constant, then obviously $\tilde{\nabla} f^v$ is a horizontal Killing vector field. □

Notation 1. For a section $\sigma \in \Gamma(E)$, define a smooth function $F_\sigma : E \longrightarrow \mathbb{R}$ by $F_\sigma(e) = h(\sigma(\pi(e)), e)$.

Lemma 10. *Let σ be a smooth section of E, then*

$$\tilde{\nabla} F_\sigma = e^{-2\varphi_1} \sum_{i=1}^{n} F_{(D_{e_i}\sigma)} e_i^h + e^{-2\varphi_2} \sigma^v,$$

where $\{e_i\}$ is any orthonormal frame of M.

Proof.

$$\begin{aligned}
(\tilde{\nabla} F_\sigma)_e &= \sum_{i=1}^{n} \left(e^{-\varphi_1} e_i^h . F_\sigma(e^{-\varphi_1} e_i^h)\right)_e + \sum_{p=1}^{k} \left(e^{-\varphi_2} \sigma_p^v . F_\sigma(e^{-\varphi_2} \sigma_p^v)\right)_e \\
&= e^{-2\varphi_1} \sum_{i=1}^{n} \Big(e_i^h . (\pi^\star h(\sigma^v, \xi))\Big) e_i^h + e^{-2\varphi_2} \sum_{p=1}^{k} \Big(\sigma_p^v . (\pi^\star h(\sigma^v, \xi))\Big) \sigma_p^v \\
&= e^{-2\varphi_1} \sum_{i=1}^{n} \Big(\pi^\star h(\pi^\star D_{e_i^h} \sigma^v, \xi) + \pi^\star h(\sigma^v, \pi^\star D_{e_i^h} \xi)\Big) e_i^h \\
&+ e^{-2\varphi_2} \sum_{p=1}^{k} \Big(\pi^\star h(\pi^\star D_{\sigma_p^v} \sigma^v, \xi) + \pi^\star h(\sigma^v, \pi^\star D_{\sigma_p^v} \xi)\Big) \sigma_p^v \\
&= e^{-2\varphi_1} \sum_{i=1}^{n} \pi^\star h(\pi^\star D_{e_i^h} \sigma^v, \xi) e_i^h + e^{-2\varphi_2} \sum_{p=1}^{k} \pi^\star h(\sigma^v, \sigma_p^v) \sigma_p^v \\
&= e^{-2\varphi_1} \sum_{i=1}^{n} \pi^\star h((D_{e_i}\sigma)^v, \xi) e_i^h + e^{-2\varphi_2} \sigma^v \\
&= e^{-2\varphi_1} \sum_{i=1}^{n} h(D_{e_i}\sigma, e) e_i^h + e^{-2\varphi_2} \sigma^v \\
&= e^{-2\varphi_1} \sum_{i=1}^{n} F_{(D_{e_i}\sigma)}(e) e_i^h + e^{-2\varphi_2} \sigma^v.
\end{aligned}$$

□

Since the vector field σ^v arises in the expression of the gradient vector field of functions of the form F_σ, and since the parallelism of a section σ is a necessary condition for its vertical lift σ^v to be conformal, we assume that the section σ is parallel, thus by virtue of Lemma 10, we get

$$\tilde{\nabla} F_\sigma = e^{-2\varphi_2} \sigma^v.$$

Lemma 11. *Assume $\varphi_1' = \varphi_2'$ and let σ be a non-zero parallel section. If the vector field $X^V = \lambda(r)\sigma^v$ is a conformal vector field, then λ is constant.*

Proof. Let $Y, Z \in \mathfrak{X}(E)$, then

$$\mathcal{L}_{X^V} G(Y, Z) = \lambda \mathcal{L}_{\sigma^v} G(Y, Z) + (Y.\lambda) G(\sigma^v, Z) + (Z.\lambda) G(\sigma^v, Y).$$

But σ^v is conformal with potential function $f = a\xi^\flat(\sigma^v)$. Then, for horizontal vector fields Y^H and Z^H, we have

$$\mathcal{L}_{X^V} G(Y^H, Z^H) = \lambda \mathcal{L}_{\sigma^v} G(Y^H, Z^H) = 2a\lambda \xi^\flat(\sigma^v) G(Y^H, Z^H),$$

thus, if X^V is conformal, then its potential function must be $\beta = a\lambda\xi^\flat(\sigma^v)$. Furthermore, we have

$$\mathcal{L}_{X^V}G(Y^V,Z^V) = \lambda\mathcal{L}_{\sigma^v}G(Y^V,Z^V) + (Y^V.\lambda)G(\sigma^v,Z^V) + (Z^V.\lambda)G(\sigma^v,Y^V),$$

with

$$\mathcal{L}_{\sigma^v}G(Y^V,Z^V) = 2be^{2\varphi_2}\xi^\flat(X^V)\langle Y^V,Z^V\rangle_E.$$

On the other hand, since $\varphi_1' = \varphi_2'$, we have $a = b$, then

$$\mathcal{L}_{X^V}G(Y^V,Z^V) = \beta e^{2\varphi_2}\langle Y^V,Z^V\rangle_E + (Y^V.\lambda)G(\sigma^v,Z^V) + (Z^V.\lambda)G(\sigma^v,Y^V),$$

but $Y^V.\lambda(r) = 2\lambda'(r)\xi^\flat(Y^v)$, whence

$$\begin{aligned}\mathcal{L}_{X^V}G(Y^V,Z^V) &= 2\beta G(Y^V,Z^V)\\ &= 2\beta e^{2\varphi_2}\langle Y^V,Z^V\rangle_E + (Y^V.\lambda)G(\sigma^v,Z^V) + (Z^V.\lambda)G(\sigma^v,Y^V)\\ &= 2\beta e^{2\varphi_2}\langle Y^V,Z^V\rangle_E + 4\lambda'(r)e^{2\varphi_2(r)}\langle\sigma^v,Y^V\rangle_E\langle\sigma^v,Z^V\rangle_E.\end{aligned}$$

Thus

$$4\lambda'(r)e^{2\varphi_2(r)}\langle\sigma^v,Y^V\rangle_E\langle\sigma^v,Z^V\rangle_E = 0.$$

Taking the last equation for $Y^V = Z^V = \sigma^v$, we get $4\lambda'(r)\|\sigma\|^2 = 0$, whence $\lambda' = 0$. □

To sum up we have:

Theorem 17. *Assume that the functions φ_1 and φ_2 are constant, and let σ be a parallel section of E, then $\tilde{\nabla}F_\sigma$ is a gradient Killing vector field on (E,G).*

Proof. It follows directly from Lemmas 10 and 11 and Theorem 45. □

Example 7. Take for instance $\varphi_1 = \varphi_2 = 0$, i.e. G is the Sasaki metric on E, and let σ be a parallel section of (E,G), then $\tilde{\nabla}F_\sigma$ is a gradient Killing vector field.

Chapter 4

Curvatures of spherically symmetric metrics

Motivation

The curvature tensor, as well as its traces, constitute the most important geometric invariants of a Riemannian manifold for they encode the essential geometric information that measures how the given manifold 'differ' from a flat space; in other words, curvature is an obstruction for a Riemannian manifold to be locally isometric to the Euclidean space.

Concerning the curvature of spherically symmetric metrics, the author of [13] computed the curvature of SS-metrics in the case of flat connections, as well as the curvature on the zero section. This allows him to establish many geometric results of spherically symmetric metrics. Our aim is to accomplish the computations of curvature in the general setting and to exploit the formulas to establish several geometric properties of SS-metrics. In this chapter we present our results in [12] and [9].

4.1 On hereditary properties of SS-metrics

For the rest of this chapter (E, π, M) is a vector bundle for which the total space is endowed with a spherically symmetric metric G with weights φ_1 and φ_2. We mean by a hereditary property any geometric property inherited by the base manifold (M, g) once the total space (E, G) possesses the same property. This section is devoted to study such properties.

We will make use of the following notations:

(i) R and $\widetilde{R}$ are the Riemann curvature tensors of (M, g) and (E, G) (we use the same notation for the corresponding curvature $(0, 4)$-tensor fields), respectively;

(ii) Ric and $\widetilde{\text{Ric}}$ are the Ricci tensors of (M, g) and (E, G), respectively;

(iii) K_g and K_G are the sectional curvatures of of (M, g) and (E, G), respectively;

(iv) Sc and $\widetilde{Sc}$ are the scalar curvatures of (M, g) and (E, G), respectively.

(v) The norms $\|.\|_g^2 = g(.,.)$, $\|.\|_G^2 = G(.,.)$ and $\|.\|^2 = h(.,.)$.

In [13], the author computed the curvature tensor, the Ricci tensor and the scalar curvature on the zero section of E, we summarize the results in the following:

Proposition 49. *Let us denote O_M the zero section of (E, π, M), and let $x \in M$ and $o \in O_M$ such that $\pi(o) = x$. Then, for all $X^H, Y^H, Z^H, W^H \in \mathcal{H}_o$, we have*

1. $\widetilde{R}_o(X^H,Y^H,Z^H,W^H) = e^{2\varphi_1(0)}\langle \pi^* R_x(X^H,Y^H)Z^H, W^H\rangle_M$,
2. $\widetilde{\mathrm{Ric}}_o(X^H,Y^H) = \mathrm{Ric}_x(d\pi(X^H),d\pi(Y^H)) - 2k\varphi_1'(0)e^{2(\varphi_1-\varphi_2)(0)}\langle X^H,Y^H\rangle_M$,
3. $\widetilde{Sc}_o = e^{-2\varphi_1(0)}Sc_x + 4ke^{-2\varphi_2(0)}((1-k)\varphi_2'(0) - n\varphi_1'(0))$.

Furthermore, we prove the following result concerning the sectional curvature:

Proposition 50. *([12]) For all $\{X^H,Y^H\} \subseteq \mathcal{H}_o$ an orthonormal system, we have*

$$\big(K_G\big)_o(X^H,Y^H) = e^{-2\varphi_1(0)}(K_g)_x(d\pi(X^H),d\pi(Y^H)).$$

Proof: By the first equation of Proposition 49, we have

$$\begin{aligned}(K_G)_o(X^H,Y^H) &= \tilde{R}_o(X^H,Y^H,Z^H,W^H)\\ &= e^{2\varphi_1(0)}\langle \pi^* R_x(X^H,Y^H)Z^H,W^H\rangle_M\\ &= e^{-2\varphi_1(0)}(K_g)_x(d\pi(X^H),d\pi(Y^H))\end{aligned}$$

because

$$\|X^H\wedge Y^H\|_G^2 = e^{4\varphi_1}\|X^H\wedge Y^H\|^2.$$

■

Using the computations of Propositions 49 and 50, we obtain:

Theorem 18. *If the manifold (E,G) is flat, of constant sectional curvature, of constant scalar curvature, or an Einstein manifold, respectively, then the manifold (M,g) possesses the same property.*

Theorem 18 summarizes the hereditary properties of spherically symmetric metrics. In the forthcoming sections we shall be interested in the inverse problem, that is the necessary and sufficient conditions under which (E,G) possesses one of the previous geometric properties provided the base has the same property.

4.2 Curvatures of SS-metrics

We proved in Proposition 29 that the projection $\pi : (E,G) \longrightarrow (M,g)$ is a horizontally homothetic submersion with dilation function $\lambda : E \longrightarrow \mathbb{R}$ such that $\lambda(e) = e^{-\varphi_1(r)}$ and $r = h(e,e)$, for all $e \in E$. Moreover, we reviewed in the first chapter O'Neill's theory of fundamental tensors and fundamental equations of a submersion along with their generalizations to the framework of horizontally conformal submersions. Consequently, the curvatures of SS-metrics may be computed using the fundamental equations of horizontally conformal submersion. Indeed, we will adopt this approach to compute the different curvatures of (E,G).

4.2.1 The Riemann curvature tensor

Recall that we denote by A and T the fundamental tensors of π. The tensor $T \equiv 0$ because π has totally geodesic fibers (cf. [13]). So, using the formulas of Theorem 6, we compute the different components of $\widetilde{R}$. Since the vector field $\mathrm{grad}(\frac{1}{\lambda^2})$, as well as some components of the covariant derivatives of A, appear in the fundamental equations, we begin by the following preparatory lemmas.

First of all, recall that given $\{e_i\}_{1\leq i\leq n}$ a linear orthonormal frame on $U \subset M$, shrinking U if necessary, and $\{\sigma_p\}_{1\leq p\leq k}$ a local orthonormal frame of E on U, then set

$$E_i = e^{-\varphi_1}e_i^h \quad , \quad E_{n+p} = e^{-\varphi_2}\sigma_p^v.$$

Hence $\{E_I\}$, with $I = 1,...,n,...,n+k$, is a linear orthonormal frame of E. We refer to such a frame as an *induced linear frame.*

Lemma 12. *([9])* $\operatorname{grad}(e^{2\varphi_1}) = \operatorname{grad}(\frac{1}{\lambda^2}) = 4\varphi_1' e^{2\varphi_1 - 2\varphi_2}\xi$.

Proof: Let $\{E_I\}$ be an induced local orthonormal frame, then by virtue of Lemma 3, we have

$$\begin{aligned}
\operatorname{grad}(e^{2\varphi_1}) &= \sum_{I=1}^{n+k} E_I.(e^{2\varphi_1})E_I \\
&= \sum_{i=1}^{n} \big(e^{-\varphi_1}e_i^h.e^{2\varphi_1}\big)(e^{-\varphi_1}e_i^h) + \sum_{p=1}^{k} \big(e^{-\varphi_2}\sigma_p^v.e^{2\varphi_1}\big)(e^{-\varphi_2}\sigma_p^v) \\
&= \sum_{p=1}^{k} e^{-2\varphi_2}\big(\sigma_p^v.e^{2\varphi_1}\sigma_p^v\big) \\
&= 4\varphi_1' e^{2\varphi_1-2\varphi_2}\sum_{p=1}^{k}\xi^\flat(\sigma_p^v)\sigma_p^v \\
&= 4\varphi_1' e^{2\varphi_1-2\varphi_2}\xi.
\end{aligned}$$

■

Lemma 13. *([9]) Let A be the fundamental tensor of the projection $\pi : (E, G) \longrightarrow (M, g)$. If X, Y, Z are horizontal vectors and V, W are vertical vectors, then*

(i) $(\tilde{\nabla}_V A)_W = 0$,

(ii) $(\tilde{\nabla}_X A)_W = -\Big(a\xi^\flat(W)A_X + A_{S(X,W))}\Big)$,

(iii) $\mathcal{V}\Big((\tilde{\nabla}_X A)_Y Z\Big) = \frac{1}{2}\Big(\mathcal{R}^\xi(\pi^*\nabla_X Y, Z) + \mathcal{R}^\xi(Y, \pi^*\nabla_X Z) - \pi^\star D_X\mathcal{R}^\xi(Y, Z)\Big)$,

(iv)

$$\begin{aligned}
\mathcal{V}\Big((\tilde{\nabla}_U A)_X Y\Big) &= \pi^\star D_U A_X Y + c_2\langle U, A_X Y\rangle_E\xi + b\big(\xi^\flat(U)A_X Y + \xi^\flat(A_X Y)U\big) \\
&\quad - 2a\xi^\flat(U)A_X Y - A_{S(U,X)}Y - A_X S(U, Y),
\end{aligned}$$

where S is the $\mathcal{H}$-valued $(1,2)$-tensor field defined in Proposition 25.

Proof: (i) follows using Lemma 1 and the fact that $T = 0$. To prove (ii), we must compute explicitly the expression of $A_X W$. By virtue of Proposition 25, we have

$$\begin{aligned}
A_X W &= \mathcal{V}(\tilde{\nabla}_{\mathcal{H}X}\mathcal{H}W) + \mathcal{H}(\tilde{\nabla}_{\mathcal{H}X}\mathcal{V}W) \\
&= \mathcal{H}(\tilde{\nabla}_X W) \\
&= \mathcal{H}\Big(\pi^\star D_X W + a\xi^\flat(W)X + S(X, W)\Big) \\
&= a\xi^\flat(W)X + S(X, W),
\end{aligned}$$

where $a = 2\varphi_1'$ and S is the $\mathcal{H}$-valued $(0,2)$-tensor given by

$$e^{2\varphi_1}\langle S(X,Y), Z\rangle_M = \frac{e^{2\varphi_2}}{2}\Big(\langle\mathcal{R}^\xi(X,Z), Y\rangle_E + \langle\mathcal{R}^\xi(Y,Z), X\rangle_E\Big). \tag{4.2.1}$$

Then, since A is horizontal, we have

$$\begin{aligned}(\tilde{\nabla}_X A)_W E &= \tilde{\nabla}_X(A_W E) - A_{\tilde{\nabla}_X W}E - A_W(\tilde{\nabla}_X E)\\ &= -A_{\tilde{\nabla}_X W}E.\end{aligned}$$

Now, we prove (iii) using just the expression of the Levi-Civita connection $\tilde{\nabla}$, we have

$$\begin{aligned}(\tilde{\nabla}_X A)_Y Z &= \tilde{\nabla}_X(A_Y Z) - A_{\tilde{\nabla}_X Y}Z - A_Y(\tilde{\nabla}_X Z)\\ &= \tilde{\nabla}_X\big(c_1\langle Y,Z\rangle_M\xi - \frac{1}{2}\mathcal{R}^\xi(Y,Z)\big)\\ &\quad - A_{\mathcal{H}\tilde{\nabla}_X Y}Z - A_Y(\tilde{\nabla}_X Z).\end{aligned}$$

By virtue of Propositions 25 and 27, we have:

- $\tilde{\nabla}_X Y = \pi^*\nabla_X Y + c_1\langle X,Y\rangle_M\xi - \frac{1}{2}\mathcal{R}^\xi(X,Y)$, thus $\mathcal{H}\tilde{\nabla}_X Y = \pi^*\nabla_X Y$,
- $\tilde{\nabla}_X\xi$ is horizontal,
- $A_Y\mathcal{R}^\xi(X,Z)$ is horizontal;

hence

$$\begin{aligned}\mathcal{V}\Big(\tilde{\nabla}_X\big(c_1\langle Y,Z\rangle_M\xi - \frac{1}{2}\mathcal{R}^\xi(Y,Z)\big)\Big) &= \mathcal{V}\Big(c_1X.\langle Y,Z\rangle_M\xi + c_1\langle Y,Z\rangle_M\tilde{\nabla}_X\xi\\ &\quad - \frac{1}{2}\big[\pi^\star D_X\mathcal{R}^\xi(Y,Z) + a\xi^\flat(\mathcal{R}^\xi(Y,Z))\\ &\quad + S(X,\mathcal{R}^\xi(Y,Z))\big]\Big)\\ &= c_1X.\langle Y,Z\rangle_M\xi - \frac{1}{2}\pi^\star D_X\mathcal{R}^\xi(Y,Z),\end{aligned}\tag{4.2.2}$$

$$\begin{aligned}\mathcal{V}\big(A_{\mathcal{H}\tilde{\nabla}_X Y}Z\big) &= \mathcal{V}\big(A_{\pi^*\nabla_X Y}Z\big) = \mathcal{V}\big(\tilde{\nabla}_{\pi^*\nabla_X Y}Z\big)\\ &= c_1\langle\pi^*\nabla_X Y,Z\rangle_M\xi - \frac{1}{2}\mathcal{R}^\xi(\pi^*\nabla_X Y,Z),\end{aligned}\tag{4.2.3}$$

and

$$\begin{aligned}\mathcal{V}\big(A_Y(\tilde{\nabla}_X Z)\big) &= \mathcal{V}\big(\tilde{\nabla}_Y(\mathcal{H}\tilde{\nabla}_X Z)\big) = \mathcal{V}\big(\tilde{\nabla}_Y(\pi^*\nabla_X Z)\big)\\ &= c_1\langle Y,\pi^*\nabla_X Z\rangle_M\xi - \frac{1}{2}\mathcal{R}^\xi(Y,\pi^*\nabla_X Z).\end{aligned}\tag{4.2.4}$$

Then

$$\begin{aligned}\mathcal{V}\Big((\tilde{\nabla}_X A)_Y Z\Big) &= c_1X.\langle Y,Z\rangle_M\xi - \frac{1}{2}\pi^\star D_X\mathcal{R}^\xi(Y,Z)\\ &\quad - c_1\langle\pi^*\nabla_X Y,Z\rangle_M\xi + \frac{1}{2}\mathcal{R}^\xi(\pi^*\nabla_X Y,Z)\\ &\quad - c_1\langle Y,\pi^*\nabla_X Z\rangle_M\xi + \frac{1}{2}\mathcal{R}^\xi(Y,\pi^*\nabla_X Z)\\ &= \frac{1}{2}\Big(\mathcal{R}^\xi(\pi^*\nabla_X Y,Z) + \mathcal{R}^\xi(Y,\pi^*\nabla_X Z) - \pi^\star D_X\mathcal{R}^\xi(Y,Z)\Big),\end{aligned}\tag{4.2.5}$$

and

$$\begin{aligned}\mathcal{V}\Big((\tilde{\nabla}_U A)_X Y\Big) &= \mathcal{V}\Big[\tilde{\nabla}_U(A_X Y) - A_{\tilde{\nabla}_U X}Y - A_X(\tilde{\nabla}_U Y)\Big]\\ &= \pi^\star D_U A_X Y + c_2\langle U, A_X Y\rangle_E \xi + b\big(\xi^\flat(U)A_X Y + \xi^\flat(A_X Y)U\big)\\ &- A_{a\xi^\flat(U)X + S(U,X)}Y - A_X(a\xi^\flat(U)Y + S(U,Y))\\ &= \pi^\star D_U A_X Y + c_2\langle U, A_X Y\rangle_E \xi + b\big(\xi^\flat(U)A_X Y + \xi^\flat(A_X Y)U\big)\\ &- 2a\xi^\flat(U)A_X Y - A_{S(U,X)}Y - A_X S(U,Y),\end{aligned}$$

where $a = 2\varphi_1'$, $b = 2\varphi_2'$, $c_2 = -2\varphi_2$. ∎

Theorem 19. *([9])Let $\pi : (E, G) \longrightarrow (M, g)$ be a vector bundle endowed with a spherically symmetric metric G. If X, Y, Z, H are horizontal and U, V, W, F are vertical vectors, then*

$$\begin{aligned}G(\tilde{R}(X,Y)Z,H) &= e^{2\varphi_1} g(\pi^* R(X,Y)Z, d\pi(H)) + \frac{e^{2\varphi_2}}{4}\Big(\langle\mathcal{R}^\xi(X,Z),\mathcal{R}^\xi(Y,H)\rangle_E\\ &- \langle\mathcal{R}^\xi(Y,Z),\mathcal{R}^\xi(X,H)\rangle_E + 2\langle\mathcal{R}^\xi(X,Y),\mathcal{R}^\xi(Z,H)\rangle_E\Big)\\ &+ 2\varphi_1' e^{4\varphi_1 - 2\varphi_2}\Big(\langle X\wedge S(Y,\xi), Z\wedge H\rangle_M + \langle S(X,\xi)\wedge Y, Z\wedge H\rangle_M\\ &+ 2ar\langle X\wedge Y, Z\wedge H\rangle_M\Big)\\ &- 4r(\varphi_1')^2 e^{4\varphi_1 - 2\varphi_2}\langle X\wedge Y, Z\wedge H\rangle_M,\end{aligned} \tag{4.2.6}$$

$$\begin{aligned}G(\tilde{R}(X,Y)Z,U) &= \frac{e^{2\varphi_2}}{2}\Big(\langle\mathcal{R}^\xi([X,Y],Z) + \mathcal{R}^\xi(Y,\pi^*\nabla_X Z) - \mathcal{R}^\xi(X,\pi^*\nabla_Y Z), U\rangle_E\\ &+ \langle\pi^\star D_Y\mathcal{R}^\xi(X,Z) - \pi^\star D_X\mathcal{R}^\xi(Y,Z), U\rangle_E\Big),\end{aligned} \tag{4.2.7}$$

$$\begin{aligned}G(\tilde{R}(U,X)Y,V) &= 4e^{2\varphi_1}\big(2\varphi_1'\varphi_2' - (\varphi_1')^2 - \varphi_1''\big)\langle X,Y\rangle_M\xi^\flat(U)\xi^\flat(V)\\ &- 2\varphi_1'\big(2r\varphi_2' + 1\big)e^{2\varphi_1}\langle X,Y\rangle_M\langle U,V\rangle_E\\ &+ e^{2\varphi_2}\Big(-2\varphi_1'\langle\mathcal{R}^\xi(X,Y),U\rangle_E\xi^\flat(V) - \frac{1}{2}\langle\pi^\star D_U\mathcal{R}^\xi(X,Y),V\rangle_E\\ &- \frac{b}{2}\xi^\flat(\mathcal{R}^\xi(X,Y))\langle U,V\rangle_E + a\xi^\flat(U)\langle\mathcal{R}^\xi(X,Y),V\rangle_E\\ &- \langle A_{S(U,X)}Y + A_X S(U,Y), V\rangle_E\Big)\\ &+ e^{2\varphi_1}\Big(2\varphi_1'\big(\xi^\flat(U)\langle X, S(Y,V)\rangle_M\\ &+ \xi^\flat(V)\langle Y, S(X,U)\rangle_M\big) + \langle S(U,X), S(V,Y)\rangle_M\Big),\end{aligned} \tag{4.2.8}$$

$$G(\tilde{R}(U,V)W,X) = 0, \tag{4.2.9}$$

$$G(\tilde{R}(U,V)W,F) = G(R^{\mathcal{V}}(U,V)W,F), \tag{4.2.10}$$

where

$$A_X Y = c_1\langle X,Y\rangle_M\xi - \frac{1}{2}\mathcal{R}^\xi(X,Y)$$

and

$$\begin{aligned}R^{\mathcal{V}}(U,V)W &= R^{\pi^\star D}(U,V)W + 4(\varphi_2'' - (\varphi_2')^2)\big(\xi^\flat(W)(U\wedge V)\xi - \langle U\wedge V, \xi\wedge W\rangle\xi\big)\\ &+ 4(\varphi_2' + r(\varphi_2')^2)(U\wedge V)W.\end{aligned}$$

Proof: $T = 0$ because π has totally geodesic fibers. The projection π is a horizontally homothetic submersion with dilation function $\lambda(e) = e^{-\varphi_1(r)}$ with $r = \|e\|_h^2 = h(e,e)$. Since λ depends only on r, then by virtue of Lemma 12, we have

$$\operatorname{grad}(\frac{1}{\lambda^2}) = 4\varphi_1' e^{2\varphi_1 - 2\varphi_2}\xi, \tag{4.2.11}$$

hence

$$\begin{aligned} \|grad(\frac{1}{\lambda^2})\|^2 &= G(grad(\frac{1}{\lambda^2}), grad(\frac{1}{\lambda^2})) \\ &= G\big(4\varphi_1' e^{2\varphi_1-2\varphi_2}\xi, 4\varphi_1' e^{2\varphi_1-2\varphi_2}\xi\big) \\ &= 16 r(\varphi_1')^2 e^{4\varphi_1 - 2\varphi_2}. \end{aligned} \tag{4.2.12}$$

Moreover, using Proposition 25 and the expression of $grad(\frac{1}{\lambda^2})$, we have

$$\begin{aligned} \tilde\nabla_X grad(\frac{1}{\lambda^2}) &= \tilde\nabla_X\big(4\varphi_1' e^{2\varphi_1-2\varphi_2}\xi\big) \\ &= \big(4\varphi_1' e^{2\varphi_1-2\varphi_2}\big)\tilde\nabla_X(\xi) \\ &= \big(4\varphi_1' e^{2\varphi_1-2\varphi_2}\big)(S(X,\xi) + aX\xi^\flat(\xi)), \end{aligned} \tag{4.2.13}$$

where S is the $\mathcal{H}$-valued $(1,2)$-tensor defined in Proposition 25 and $a = 2\varphi_1'$. Combining equations (4.2.12),(4.2.13) and (1.3.13), equation (4.2.6) follows.

By virtue of Lemma 13, we have

$$\begin{aligned} G(\tilde R(X,Y)Z,U) &= G((\tilde\nabla_X A)_Y Z, U) - G((\tilde\nabla_Y A)_X Z, U) \\ &= \frac{1}{2}G\Big(\mathcal{R}^\xi(\pi^*\nabla_X Y, Z) + \mathcal{R}^\xi(Y, \pi^*\nabla_X Z) - \pi^\star D_X\mathcal{R}^\xi(Y,Z), U\Big) \\ &- \frac{1}{2}G\Big(\mathcal{R}^\xi(\pi^*\nabla_Y X, Z) + \mathcal{R}^\xi(X, \pi^*\nabla_Y Z) - \pi^\star D_Y\mathcal{R}^\xi(X,Z), U\Big) \\ &= \frac{e^{2\varphi_2}}{2}\Big(\langle\mathcal{R}^\xi([X,Y],Z) + \mathcal{R}^\xi(Y,\pi^*\nabla_X Z) - \mathcal{R}^\xi(X,\pi^*\nabla_Y Z), U\rangle_E \\ &+ \langle\pi^\star D_Y\mathcal{R}^\xi(X,Z) - \pi^\star D_X\mathcal{R}^\xi(Y,Z)\rangle_E\Big), \end{aligned}$$

which is nothing but equation (4.2.7).

Again Lemma 13 implies

$$G(\tilde R(U,X)Y,U) = G((\tilde\nabla_X A)_Y Z, U) - G((\tilde\nabla_Y A)_X Z, U) + \lambda^2 g(A_X Y, U) g(V, grad_{\mathcal{V}}(\frac{1}{\lambda^2})). \tag{4.2.14}$$

We have

$$grad(\frac{1}{\lambda^2}) = grad_{\mathcal{V}}(\frac{1}{\lambda^2}) = 4\varphi_1' e^{2\varphi_1-2\varphi_2}\xi, \tag{4.2.15}$$

on the other hand, we have

$$A_X U = \mathcal{H}(\tilde\nabla_X U) = a\xi^\flat(U)X + S(X,U), \tag{4.2.16}$$

and by virtue of Lemma 13, we have

$$\begin{aligned} \mathcal{V}(\tilde\nabla_U A)_X Y &= \pi^\star D_U A_X Y + c_2\langle U, A_X Y\rangle_E\xi + b\big(\xi^\flat(U)A_X Y + \xi^\flat(A_X Y)U\big) \\ &- 2a\xi^\flat(U)A_X Y - A_{S(U,X)}Y - A_X S(U,Y). \end{aligned} \tag{4.2.17}$$

Substituting from equations (4.2.15), (5.2.3), and (5.2.4) into (4.2.14), we get

$$\begin{aligned}G(\tilde{R}(U,X)Y,V) &= e^{2\varphi_2}\langle \pi^\star D_U A_X Y + c_2\langle U, A_X Y\rangle_E \xi + b\big(\xi^\flat(U)A_X Y + \xi^\flat(A_X Y)U\big)\\ &\quad - 2a\xi^\flat(U)A_X Y - A_{S(U,X)}Y - A_X S(U,Y), V\rangle_E\\ &\quad + e^{2\varphi_1}\Big(4(\varphi_1')^2\xi^\flat(U)\xi^\flat(V)\langle X,Y\rangle_M + 2\varphi_1'\big(\xi^\flat(U)\langle X, S(Y,V)\rangle_M\\ &\quad + \xi^\flat(V)\langle Y, S(X,U)\rangle_M\big) + \langle S(U,X), S(V,Y)\rangle_M\Big)\\ &\quad + 4\varphi_1' e^{2\varphi_2}\xi^\flat(V)\langle A_X Y, U\rangle_E,\end{aligned}$$

but

$$\begin{aligned}\pi^\star D_U A_X Y &= \pi^\star D_U(c_1\langle X,Y\rangle_M\xi - \frac{1}{2}\mathcal{R}^\xi(X,Y))\\ &= -4\xi^\flat(U)\big(\varphi_1'' + 2\varphi_1'(\varphi_1' - \varphi_2')\big)e^{2\varphi_1-2\varphi_2}\langle X,Y\rangle_M\xi + c_1\langle X,Y\rangle_M U\\ &\quad - \frac{1}{2}\pi^\star D_U\mathcal{R}^\xi(X,Y)).\end{aligned}$$

Thus

$$\begin{aligned}G(\tilde{R}(U,X)Y,V) &= e^{2\varphi_2}\Big(-4\big(\varphi_1'' + 2\varphi_1'(\varphi_1' - \varphi_2')\big)e^{2\varphi_1-2\varphi_2}\langle X,Y\rangle_M\xi^\flat(U)\xi^\flat(V)\\ &\quad + c_1\langle X,Y\rangle_M\langle U,V\rangle_E - \frac{1}{2}\langle \pi^\star D_U\mathcal{R}^\xi(X,Y), V\rangle_E\\ &\quad + c_1c_2\langle X,Y\rangle_M\xi^\flat(U)\xi^\flat(V) - \frac{c_2}{2}\langle\mathcal{R}^\xi(X,Y),U\rangle_E\xi^\flat(V)\\ &\quad + bc_1\langle X,Y\rangle_M\xi^\flat(V)\xi^\flat(U) - \frac{b}{2}\langle\mathcal{R}^\xi(X,Y),V\rangle_E\xi^\flat(U)\\ &\quad + b\big(rc_1\langle X,Y\rangle_M - \frac{1}{2}\xi^\flat(\mathcal{R}^\xi(X,Y))\big)\langle U,V\rangle_E\\ &\quad - 2ac_1\langle X,Y\rangle_M\xi^\flat(U)\xi^\flat(V) + a\xi^\flat(U)\langle\mathcal{R}^\xi(X,Y),V\rangle_E\\ &\quad - \langle A_{S(U,X)}Y + A_X S(U,Y), V\rangle_E\Big)\\ &\quad + e^{2\varphi_1}\Big(4(\varphi_1')^2\xi^\flat(U)\xi^\flat(V)\langle X,Y\rangle_M + 2\varphi_1'\big(\xi^\flat(U)\langle X,S(Y,V)\rangle_M\\ &\quad + \xi^\flat(V)\langle Y,S(X,U)\rangle_M\big) + \langle S(U,X),S(V,Y)\rangle_M\Big)\\ &\quad + 4\varphi_1' e^{2\varphi_2}\xi^\flat(V)\Big(c_1\langle X,Y\rangle_M\xi^\flat(U) - \frac{1}{2}\langle\mathcal{R}^\xi(X,Y),U\rangle_E\Big),\end{aligned}$$

hence

$$\begin{aligned}G(\tilde{R}(U,X)Y,V) &= 4e^{2\varphi_1}\big(2\varphi_1'\varphi_2' - (\varphi_1')^2 - \varphi_1''\big)\langle X,Y\rangle_M\xi^\flat(U)\xi^\flat(V)\\ &\quad - 2\varphi_1'\big(2r\varphi_2' + 1\big)e^{2\varphi_1}\langle X,Y\rangle_M\langle U,V\rangle_E\\ &\quad + e^{2\varphi_2}\Big(-2\varphi_1'\langle\mathcal{R}^\xi(X,Y),U\rangle_E\xi^\flat(V) - \frac{1}{2}\langle\pi^\star D_U\mathcal{R}^\xi(X,Y),V\rangle_E\\ &\quad - \frac{b}{2}\xi^\flat(\mathcal{R}^\xi(X,Y))\langle U,V\rangle_E + a\xi^\flat(U)\langle\mathcal{R}^\xi(X,Y),V\rangle_E\\ &\quad - \langle A_{S(U,X)}Y + A_X S(U,Y), V\rangle_E\Big)\\ &\quad + e^{2\varphi_1}\Big(2\varphi_1'\big(\xi^\flat(U)\langle X,S(Y,V)\rangle_M\\ &\quad + \xi^\flat(V)\langle Y,S(X,U)\rangle_M\big) + \langle S(U,X),S(V,Y)\rangle_M\Big).\end{aligned}$$

Finally, a long but routine computation, using Proposition 25, gives equation (4.2.10). ∎

When the connection D is flat, the expressions of the curvature tensor become substantially simpler. In this case, the author of [13] computed the curvature $(1,3)$-tensor field in a direct manner. Before we state Albuquerque's results, let's fix the following notations:

1. $\langle .,. \rangle_M = \pi^* g$, $\langle .,. \rangle_E = \pi^\star h$ and $\langle .,. \rangle = \pi^* g \oplus \pi^\star h$;

2. For vectors $X, Y, Z \in TE$, set

$$(X \wedge Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y;$$

3. For vectors $X, Y, Z, W \in TE$, set

$$\langle X \wedge Y, Z \wedge W \rangle = \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle.$$

Proposition 51. *Assume that D is flat, and let $X, Y, Z \in \mathfrak{X}(E)$. Then the curvature tensor of (E, G) is given by:*

$$\begin{aligned}
1.\ \widetilde{R}(X^H, Y^H)Z^H =& \pi^* R(X^H, Y^H)Z^H + 4r(\varphi_1')^2 e^{2(\varphi_1 - \varphi_2)}(X^H \wedge Y^H)Z^H;\\
2.\ \widetilde{R}(X^H, Y^H)Z^V =& 0;\\
3.\ \tilde{R}(X^H, Y^V)Z^H =& e^{2(\varphi_1 - \varphi_2)}\langle X^H, Z^H \rangle \big[4(\varphi_1'' + (\varphi_1')^2 - 2\varphi_1'\varphi_2')\xi^\flat(Y^V)\xi\\
&+ 2(2r\varphi_1'\varphi_2' + \varphi_1')Y^V\big];\\
4.\ \widetilde{R}(X^H, Y^V)Z^V =& \big[4(2\varphi_1'\varphi_2' - (\varphi_1')^2 - \varphi_1'')\xi^\flat(Y^V)\xi^\flat(Z^V)\\
&- 2(2r\varphi_1'\varphi_2' + \varphi_1')\langle Y^V, Z^V \rangle\big]X^H;\\
5.\ \widetilde{R}(X^V, Y^V)Z^H =& 0;\\
6.\ \widetilde{R}(X^V, Y^V)Z^V =& 4(\varphi_2'' - (\varphi_2')^2)\big[\xi^\flat(Z^V)(X^V \wedge Y^V)\xi\\
&- \langle X^V \wedge Y^V, \xi \wedge Z^V \rangle \xi\big] + 4(\varphi_2' + r(\varphi_2')^2)(X^V \wedge Y^V)Z^V.
\end{aligned}$$

The latter formulas will be of great use in the study of spherically symmetric metrics on vector bundles endowed with flat connections. Fortunately, the computations accomplished in Theorem 19 agree with the formulas of curvature on the zero section found in Proposition 49, as well as the expressions of the curvature tensor in the case of flat connections (cf. [13]).

4.2.2 The Ricci curvature tensor

This section is devoted to the computation of Ricci Curvature which will be analysed in the subsequent sections in order to find, if possible, Einstein spherically symmetric metrics.

let $\{E_I\}$, for $I = 1, ..., n, n+1, ..., n+k$, be the linear orthonormal frame on E induced from a linear orthonormal frame $\{e_i\}$ of M and an orthonormal frame $\{\sigma_p\}$ of E, given by

$$E_i = e^{-\varphi_1} e_i^h \quad , \quad E_{n+p} = e^{-\varphi_2}\sigma_p^v.$$

The Ricci curvature tensor $\widetilde{\mathrm{Ric}}$ is expressed in this local frame as

$$\widetilde{\mathrm{Ric}}(X, Y) = \sum_{I=1}^{n+k} G(\tilde{R}(X, E_I)E_I, Y), \tag{4.2.18}$$

then

Proposition 52. *([12])The Ricci curvature of (E,G) at any point $e\in E$ is totally determined by*

$$
\begin{aligned}
&\widetilde{\mathrm{Ric}}(X^H,Y^H)\\
&=Ric(X,Y)-4\Big(r\big\{(k-2)\varphi_1'\varphi_2'+\varphi_1''+n(\varphi_1')^2\big\}+\frac{k}{2}\varphi_1'\Big)e^{2\varphi_1-2\varphi_2}\langle X^H,Y^H\rangle_M\\
&+\big((1-n)\varphi_1'-k\varphi_2'e^{2\varphi_2}+\frac{1}{2}\big)\xi^\flat(\mathcal{R}^\xi(X^H,Y^H))\\
&+\frac{3e^{2\varphi_2-2\varphi_1}}{4}\sum_{i=1}^n\langle\mathcal{R}^\xi(X^H,e_i^h),\mathcal{R}^\xi(e_i^h,Y^H)\rangle_E\\
&-\frac{1}{2}\sum_{p=1}^k\Big(\langle\pi^\star D_{\sigma_p^v}\big(\mathcal{R}^\xi(X^H,Y^H)\big),\sigma_p^v\rangle_E+2\langle A_{S(\sigma_p^v,X^H)}Y^H+A_{X^H}\big(S(\sigma_p^v,Y^H)\big),\sigma_p^v\rangle_E\\
&-2e^{2\varphi_1-2\varphi_2}\langle S(\sigma_p^v,X^H),S(\sigma_p^v,Y^H)\rangle_M\Big)
\end{aligned}
\tag{4.2.19}
$$

$$
\begin{aligned}
\widetilde{\mathrm{Ric}}(X^H,Y^V)=\frac{e^{2\varphi_2-2\varphi_1}}{2}\sum_i\Big\{&\langle\mathcal{R}^\xi([X^H,e_i^h],e_i^h)+\mathcal{R}^\xi(e_i^h,\pi^*\nabla_{X^H}e_i^h)\\
&-\mathcal{R}^\xi(X^H,(\nabla_{e_i}e_i)^h),Y^V\rangle_E+\langle\pi^\star D_{e_i^h}\big(\mathcal{R}^\xi(X^H,e_i^h)\big),Y^V\rangle_E\Big\}
\end{aligned}
\tag{4.2.20}
$$

$$
\begin{aligned}
&\widetilde{\mathrm{Ric}}(X^V,Y^V)\\
&=4\Big\{(1-k)(\varphi_2'+r(\varphi_2')^2)-r(\varphi_2''-(\varphi_2')^2)-\frac{n}{2}\varphi_1'\big(2r\varphi_2'+1\big)\Big\}\langle X^V,Y^V\rangle_E\\
&+4\Big\{n\big(2\varphi_1'\varphi_2'-(\varphi_1')^2-\varphi_1''\big)+(2-k)\big(\varphi_2''-(\varphi_2')^2\big)\Big\}\xi^\flat(X^V)\xi^\flat(Y^V)\\
&+\sum_{i=1}^n\langle S(X^V,e_i^h),S(Y^V,e_i^h)\rangle_M
\end{aligned}
\tag{4.2.21}
$$

for all $X^H,Y^H\in\mathcal{H}_e$ and $Z^V,T^V\in\mathcal{V}_e$, where Ric *is the Ricci tensor of (M,g), $X=(d\pi)_eX^H$ and $Y=(d\pi)_eY^H$.*

In the case of flat connections, we recover the same formulas found in [12]. More precisely, we have:

Proposition 53. *([12]) Assume that D is flat, then the Ricci curvature of (E,G) at an arbitrary point $e\in E$ is totally characterized by*

$$
\begin{aligned}
&1.\ \widetilde{\mathrm{Ric}}(X^H,Y^H)=\mathrm{Ric}(X,Y)-4e^{2(\varphi_1-\varphi_2)}\Big[\frac{k}{2}\varphi_1'\\
&\qquad\qquad+r\big[\varphi_1''+n(\varphi_1')^2+(k-2)\varphi_1'\varphi_2'\big]\Big]g(X,Y),\\
&2.\ \widetilde{\mathrm{Ric}}(X^H,Z^V)=\widetilde{\mathrm{Ric}}(Z^V,X^H)=0,\\
&3.\ \widetilde{\mathrm{Ric}}(Z^V,T^V)=-\Big[2n(2r\varphi_1'\varphi_2'+\varphi_1')+4(k-1)(\varphi_2'+r(\varphi_2')^2)\\
&\qquad\qquad+4r(\varphi_2''-(\varphi_2')^2)\Big]\langle Z^V,T^V\rangle-\Big[4n(\varphi_1''+(\varphi_1')^2-2\varphi_1'\varphi_2')\\
&\qquad\qquad+4(k-2)(\varphi_2''-(\varphi_2')^2)\Big]\xi^\flat(Z^V)\xi^\flat(T^V),
\end{aligned}
$$

for all $X^H,Y^H\in\mathcal{H}_e$ and $Z^V,T^V\in\mathcal{V}_e$, $X=(d\pi)_eX^H$ and $Y=(d\pi)_eY^H$.

4.2.3 The sectional curvatures

This subsection concerns the computation of the sectional curvature of (E, G). Using Theorem 7 and Proposition 28, we obtain:

Proposition 54. *Let $\pi : (E, G) \longrightarrow (M, g)$ be a vector bundle endowed with a spherically symmetric metric, then:*

i) If X and Y are orthonormal horizontal vectors, then:

$$K_G(X \wedge Y) = e^{-2\varphi_1} K_g(d\pi(X) \wedge d\pi(Y)) - 4r(\varphi_1')^2 e^{-2\varphi_2} - \frac{3}{4}\|\mathcal{R}^\xi(X,Y)\|^2.$$

ii) For X a horizontal vector and V a vertical vector such that $\|X\| = \|V\| = 1$, then

$$K_G(X \wedge V) = 4(\varphi_1')^2(\xi^\flat(V))^2 + \|S(X,V)\|^2.$$

iii) For orthonormal vertical vectors U, V, we have

$$\begin{aligned} K_G(U \wedge V) = K_0(U \wedge V) &+ 4(\varphi_2'' - (\varphi_2')^2)\big(\big(\xi^\flat(V)\big)^2 - \big(\xi^\flat(U)\big)^2\big) \\ &- 4(\varphi_2' + r(\varphi_2')^2)e^{-2\varphi_2}, \end{aligned}$$

where

$$K_0(U \wedge V) = G(R^{\pi^\star D}(U,V)V, U),$$

and K_g is the sectional curvature of (M, g).

Proof:

1. Let X and Y be orthonormal horizontal vectors, then by virtue of Theorem 7, we have

$$\begin{aligned} K_G(X \wedge Y) &= \lambda^2 K_g(d\pi(X) \wedge d\pi(Y)) - \frac{3}{4}\|\mathcal{V}[X,Y]\|^2 - \frac{\lambda^4}{4}\|grad_{\mathcal{V}}(\frac{1}{\lambda^2})\|^2 \\ &= e^{-2\varphi_1} K_g(d\pi(X) \wedge d\pi(Y)) - \frac{3}{4}\|\mathcal{R}^\xi(X,Y)\|^2 \\ &- \frac{\lambda^4}{4}\big(G(4\varphi_1' e^{2\varphi_1 - 2\varphi_2}\xi, 4\varphi_1' e^{2\varphi_1 - 2\varphi_2}\xi)\big) \\ &= e^{-2\varphi_1} K_g(d\pi(X) \wedge d\pi(Y)) - \frac{3}{4}\|\mathcal{R}^\xi(X,Y)\|^2 - 4r(\varphi_1')^2 e^{-2\varphi_2}. \end{aligned}$$

2. For X a horizontal vector and V a vertical vector such that $\|X\| = \|V\| = 1$, by virtue of Theorem 19, we have

$$K_G(X \wedge V) = \|A_X V\|^2,$$

where

$$A_X V = \mathcal{H}(\tilde{\nabla}_X V) = a\xi^\flat(V)X + S(X,V),$$

hence

$$\begin{aligned} \|A_X V\|^2 &= G(A_X V, A_X V) \\ &= G\big(a\xi^\flat(V)X + S(X,V), a\xi^\flat(V)X + S(X,V)\big) \\ &= 4(\varphi_1')^2(\xi^\flat(V))^2 + 2a\xi^\flat(V)G(S(X,V),X) + \|S(X,V)\|^2. \end{aligned}$$

By virtue of Proposition 25, we have

$$G(S(X,V),X) = e^{2\varphi_1}\langle S(X,V), X\rangle_M = \frac{e^{2\varphi_2}}{2}\big(\langle \mathcal{R}^\xi(X,X), V\rangle_E\big) = 0,$$

hence the formula follows.

3. Equation (iii) is just Proposition 28.

■

In the case of flat connections, one can make a direct computation using Proposition 19 without any assumptions on vectors. Indeed, we have:

Proposition 55. *Let $X, Y \in T_eE$ be such that $\{X, Y\}$ is an orthonormal system of vectors, and set $X = X^H + X^V$, $Y = Y^H + Y^V$. Then*

$$\begin{aligned}
K_G(X,Y) =& e^{2\varphi_1}\left[K_g((d\pi)_e X,(d\pi)_e Y) - 4r(\varphi_1')^2 e^{2\varphi_1-2\varphi_2}\right]\|X^H \wedge Y^H\|^2 \\
&+ 4(\varphi_1'' + (\varphi_1')^2 - 2\varphi_1'\varphi_2')e^{2\varphi_1}\Big[2\xi^\flat(X^V)\xi^\flat(Y^V)\langle X^H, Y^H\rangle \\
&- (\xi^\flat(Y^V))^2\|X^H\|^2 - (\xi^\flat(X^V))^2\|Y^H\|^2\Big] \\
&+ 2\varphi_1'(1 + r\varphi_2')e^{2\varphi_1}\Big[2\langle X^V, Y^V\rangle\langle X^H, Y^H\rangle \\
&- \|Y^V\|^2\|X^H\|^2 - \|X^V\|^2\|Y^H\|^2\Big] \\
&+ 4(\varphi_2'' - (\varphi_2')^2)e^{2\varphi_2}\left[2\xi^\flat(X^V)\xi^\flat(Y^V)\langle X^V, Y^V)\xi, X^V)\rangle\right. \\
&\left.- \xi^\flat(X^V)^2\|Y^V\|^2 - \xi^\flat(Y^V)^2\|X^V\|^2\right] \\
&- 4\varphi_2'(1 + r\varphi_2')e^{2\varphi_2}\|X^V \wedge Y^V\|^2,
\end{aligned}$$

where K_g is the sectional curvature of (M, g).

4.2.4 The Scalar curvature

We compute the scalar curvature of the manifold (E, G). For the sake of precision, we will use the following notation: $\|.\|_M^2 = \langle ., .\rangle_M$ and $\|.\|_E^2 = \pi^\star h(., .)$.

Proposition 56. *The scalar curvature of the Riemannian manifold (E, G) is given by*

$$\begin{aligned}
\widetilde{Sc} =& e^{-2\varphi_1}Sc + 4e^{-2\varphi_2}\Big[-n\Big(r\big\{(k-2)\varphi_1'\varphi_2' + \varphi_1'' + n(\varphi_1')^2\big\} + \frac{k}{2}\varphi_1'\Big) \\
&+ r\Big\{n\big(2\varphi_1'\varphi_2' - (\varphi_1')^2 - \varphi_1''\big) + (2-k)\big(\varphi_2'' - (\varphi_2')^2\big)\Big\} \\
&+ k\Big\{(1-k)(\varphi_2' + r(\varphi_2')^2) - r(\varphi_2'' - (\varphi_2')^2) - \frac{n}{2}\varphi_1'\big(2r\varphi_2' + 1\big)\Big\} \\
&+ \frac{3}{16}\|\mathcal{R}^\xi\|_E^2 + \frac{1}{4}\|S\|_M^2,\Big],
\end{aligned} \tag{4.2.22}$$

where Sc is the scalar curvature of (M, g), $\|S\|_M^2 = 2\sum_{i=1}^n\sum_{p=1}^k \|S(\sigma_p^v, e_i^h)\|_M^2$ and $\|\mathcal{R}^\xi\|_E^2 = \sum_{i,j=1}^n \|\mathcal{R}^\xi(e_i^h, e_j^h)\|^2$.

Proof: Let $\{E_I\}_{I=1,\dots,n+k}$ be an induced local orthonormal frame, then

$$\begin{aligned}
\widetilde{Sc} &= \sum_{I=1}^{n+k}\widetilde{Ric}(E_I, E_I) \\
&= \sum_{i=1}^{n} e^{-2\varphi_1}\widetilde{Ric}(e_i^h, e_i^h) + \sum_{p=1}^{k} e^{-2\varphi_2}\widetilde{Ric}(\sigma_p^v, \sigma_p^v).
\end{aligned}$$

By virtue of Proposition 52, we obtain

$$\begin{aligned}
\widetilde{Ric}(e_i^h, e_i^h) =& Ric(e_i, e_i) - 4e^{2\varphi_1-2\varphi_2}\Big(r\big\{(k-2)\varphi_1'\varphi_2' + \varphi_1'' + n(\varphi_1')^2\big\} + \frac{k}{2}\varphi_1'\Big) \\
&+ \frac{3e^{2\varphi_2-2\varphi_1}}{4}\sum_{j=1}^{n}\|\mathcal{R}^\xi(e_i^h, e_j^h)\|_E^2 + e^{2\varphi_1-2\varphi_2}\sum_{p=1}^{k}\|S(\sigma_p^v, e_i^h)\|_M^2.
\end{aligned}$$

We have also

$$\begin{aligned}
&\widetilde{\mathrm{Ric}}(\sigma_p^v, \sigma_p^v) \\
&= 4\Big\{ n\big(2\varphi_1'\varphi_2' - (\varphi_1')^2 - \varphi_1''\big) + (2-k)\big(\varphi_2'' - (\varphi_2')^2\big)\Big\}\xi^\flat(\sigma_p^v)^2 \\
&+ 4\Big\{ (1-k)(\varphi_2' + r(\varphi_2')^2) - r(\varphi_2'' - (\varphi_2')^2) - \frac{n}{2}\varphi_1'\big(2r\varphi_2' + 1\big)\Big\} \\
&+ \sum_{i=1}^{n} \|S(\sigma_p^v, e_i^h)\|_M^2 + \sum_q \langle R^{\pi^\star D}(\sigma_p^v, \sigma_q^v)\sigma_q^v, \sigma_p^v\rangle_E.
\end{aligned}$$

But $R^{\pi^\star D}(\sigma_p^v, \sigma_q^v)\sigma_q^v = 0$ by definition of the pullback connection.

Hence, the scalar curvature is given by

$$\begin{aligned}
\widetilde{Sc} = e^{-2\varphi_1} Sc &+ e^{-2\varphi_2}\Big[-4n\Big(r\big\{(k-2)\varphi_1'\varphi_2' + \varphi_1'' + n(\varphi_1')^2\big\} + \frac{k}{2}\varphi_1'\Big) \\
&+ 4r\Big\{ n\big(2\varphi_1'\varphi_2' - (\varphi_1')^2 - \varphi_1''\big) + (2-k)\big(\varphi_2'' - (\varphi_2')^2\big)\Big\} \\
&+ 4k\Big\{ (1-k)(\varphi_2' + r(\varphi_2')^2) - r(\varphi_2'' - (\varphi_2')^2) - \frac{n}{2}\varphi_1'\big(2r\varphi_2' + 1\big)\Big\} \\
&+ \frac{3}{4}\sum_{i,j=1}^{n} \|\mathcal{R}^\xi(e_i^h, e_j^h)\|_E^2 + 2\sum_{i=1}^{n}\sum_{p=1}^{k} \|S(\sigma_p^v, e_i^h)\|_M^2\Big].
\end{aligned}$$

■

Again we recover our previous computations in the case of flat connections (cf. [12]). In fact we found the following:

Proposition 57. *The scalar curvature of (E, G) is given by*

$$\begin{aligned}
\widetilde{\mathrm{Sc}} = e^{-2\varphi_1}\mathrm{Sc} + e^{-2\varphi_2}\Big\{ &-4n\Big[r\big(2\varphi_1'' + (n+1)(\varphi_1')^2 + 2(k-2)\varphi_1'\varphi_2'\big) + k\varphi_1'\Big] \\
&+ 4(1-k)\Big[r\big(2\varphi_2'' + (k-2)(\varphi_2')^2\big) + k\varphi_2'\Big]\Big\}.
\end{aligned}$$

4.3 Curvature properties of SS-metrics

This section is devoted to exploit the computations accomplished so far in order to study curvature properties of (E, G), that is to determine under which conditions the manifold (E, G) is flat, of constant sectional curvature, an Einstein manifold or of constant scalar curvature, provided that the base has the respective property. We distinguish two cases: the general case when the connection is arbitrary and the case of a flat connection. We shall begin by the general case then we will specialize to the flat case.

4.3.1 The general case

We begin by characterizing flatness of SS-metrics. By virtue of Proposition 49, the flatness of the total space implies that of the base. The following theorem gives a definitive characterization of flatness of the total space.

Theorem 20. *Let (E, G) be a vector bundle endowed with an SS-metric. If (E, G) is of constant sectional curvature κ_G, then κ_G is non-negative. Furthermore, (E, G) is flat if and only if the following hold:*

1. *the base (M, g) is flat,*

2. *the connection D is flat,*

3. *the weight functions φ_1 and φ_2 are constant.*

Proof: If (E, G) is of constant sectional curvature κ_G, then (ii) of Proposition 54 implies that $\kappa_G \geq 0$.

Now, if (E, G) is flat, then (M, g) is also flat by virtue of Proposition 49. Moreover, equation (ii) of Proposition 54 implies that for a horizontal vector X and a vertical vector V with $\|X\| = \|V\| = 1$, we have

$$4(\varphi_1')^2(\xi^\flat(V))^2 + \|S(X,V)\|^2 = 0,$$

which yields $(\varphi_1')^2(\xi^\flat(V))^2 = 0$, for any unit vertical vector field V and $S = 0$. Hence $\varphi_1' = 0$ and $\mathcal{R}^\xi = 0$, which implies at once that φ_1 is constant and D is flat.

By virtue of equation (4.2.10), we have

$$4(\varphi_2'' - (\varphi_2')^2)\big(\xi^\flat(W)(U \wedge V)\xi - \langle U \wedge V, \xi \wedge W)\xi\big) + 4(\varphi_2' + r(\varphi_2')^2)(U \wedge V)W = 0,$$

for vertical vector fields U, V, W. Taking $W = V = \xi$ and $U \perp \xi$, we get

$$4\big((\varphi_2'' - (\varphi_2')^2)r + (\varphi_2' + r(\varphi_2')^2)\big)(U \wedge V)\xi = 0, \tag{4.3.1}$$

which yields the ODE

$$(\varphi_2'' - (\varphi_2')^2)r + (\varphi_2' + r(\varphi_2')^2) = r\varphi_2'' + \varphi_2' = 0. \tag{4.3.2}$$

Set $\psi = \varphi_2'$, then we get the equation $r\psi' + \psi = 0$. The general from of the non-trivial solutions of the later ODE is $\psi = ln(\frac{1}{r}) + c$ for real constants c. Those solutions are not smooth at zero on the right, and hence the only solution to this ODE which meets the requirements of a weight function for an SS-metric is the trivial solution. As matter of fact $\psi = 0$, which implies that φ_2 is constant. The converse is obvious. ∎

Remark 14. In [13], R. Albuquerque studied the holonomy of the spaces of self-dual and anti-self dual 2-forms when endowed with a class of SS-metrics. Let M be a self-dual Einstein (resp. anti-self dual Enistein) manifold, and let $E = \wedge^2_- T^*M$ (resp. $E = \wedge^2_+ T^*M$) endowed with the connection D induced from the Levi-Civita connection of (M, g) and let h be the metric fiber endowed from g. The author of [13] endowed E with the spherically symmetric metric given by

$$G_{c_0,d_0} = \sqrt{2c_0^2 sr + d_0}\,\pi^* g \oplus \frac{c_0^2}{\sqrt{2c_0^2 sr + d_0}}\pi^\star h, \tag{4.3.3}$$

where $c_0, d_0 > 0$ are constants and $s = \dfrac{1}{12}Sc$, with Sc is the scalar curvature of M. The spherically symmetric metric weights are

$$\varphi_1(r) = \frac{1}{4}\ln\big(2c_0^2 sr + d_0\big), \quad \varphi_2(r) = \ln(c_0) - \frac{1}{4}\ln\big(2c_0^2 sr + d_0\big).$$

It has been proved that when $s \neq 0$, the holonomy of (E, G_{c_0,d_0}) is G_2, the author of [13] also proved that when the base is a K_3 surface, the holonomy is $SU(2)$.

We give here another example in the case $s = 0$, for which the holonomy is not G_2. Indeed, let (M, g) be any flat Riemannian manifold, hence $s = 0$ and the two functions φ_1 and φ_2 are constant. Furthermore, the connection D is also flat since it stems from the Levi-Civita connection of (M, g). Therefore, (E, G_{c_0,d_0}) is flat by virtue of Theorem 20.

A much stronger rigidity result concerning sectional curvature may be derived. More precisely, we prove that, under some monotony conditions on φ_1, an SS-metric could not be of constant sectional curvature unless it is flat.

Proposition 58. *Assume that φ_1 attains its minimum at $r = 0$ (in particular if φ_1 is increasing). If (E, G) has constant sectional curvature, then it is flat.*

Proof: Assume that (E, G) has constant sectional curvature κ_G, then by virtue of Proposition 49, (M, g) has also constant sectional curvature given by $\kappa_g = e^{2\varphi_1(0)}\kappa_G$.

The first equation of Proposition 54, implies that, for arbitrary orthonormal horizontal vectors X, Y, we have

$$\kappa_G = e^{-2\varphi_1(r)}\big(e^{2\varphi_1(0)}\kappa_G\big) - 4r(\varphi_1')^2 e^{-2\varphi_2} - \frac{3}{4}\|\mathcal{R}^\xi(X,Y)\|^2, \tag{4.3.4}$$

which yields

$$\kappa_G\big(1 - e^{2(\varphi_1(0)-\varphi_1(r))}\big) + 4r(\varphi_1')^2 e^{-2\varphi_2} = -\frac{3}{4}\|\mathcal{R}^\xi(X,Y)\|^2. \tag{4.3.5}$$

Since $\varphi_1(r) \geq \varphi_1(0)$, for all $r \in [0, +\infty)$, and $\kappa_G \geq 0$ by virtue of Theorem 20, then equation (4.3.18) yields $\|\mathcal{R}^\xi(X,Y)\|^2 = 0$ for horizontal vector fields X and Y, thus D is flat. Again (4.3.5) implies that $\varphi_1' = 0$.

Thus, equation (ii) of Proposition 54 implies that $\kappa_G = 0$, which implies at once that (E, G) is flat. ∎

Since SS-metrics show a strong rigidity regarding constancy of the sectional curvature, we are compelled to look at the sign of the sectional curvature. We have the following:

Corollary 10. *If (M, g) has non-positive sectional curvature, then the sectional curvature of (E, G) changes its sign.*

Proof: The first equation of Proposition 54 implies that $K_G(X \wedge Y) \leq 0$ for orthonormal horizontal vectors X, Y. On the other hand, equation (ii) of Proposition 54 implies that $K(X \wedge V) \geq 0$ for any vertical vector V and any horizontal vector X such that $\{X, V\}$ is an orthonormal system, hence the result. ∎

We fix the following terminology: a 2-plane is said to be *horizontal* (resp. *vertical*) if it lies completely in the horizontal distribution (resp. vertical distribution).

Corollary 11. *The sectional curvature K_g has the same sign of K_G on horizontal 2-planes. In particular, if K_G is non-negative then K_g is non-negative.*

Proof: It follows from Proposition 49. ∎

Proposition 59. *If the sectional curvature K_G is non-positive, then the following hold:*

(i) D is flat,

(ii) K_g is non-positive,

(iv) φ_1 is constant.

Proof: Equation (ii) of Proposition 54 implies that $S = 0$ and $\varphi_1' = 0$. Furthermore, Proposition 49 implies that K_g has the same sign as K_G on horizontal 2-planes. ∎

Summing up, flatness of spherically symmetric metrics is characterized. Further, if φ_1 is increasing, then spherically symmetric metrics of constant sectional curvature are necessarily flat. Moreover, the sectional curvature of the base manifold has the same sign as the sectional curvature of the total space on horizontal 2-planes. On top of this, the non-positivity of K_G implies that of K_g, the flatness of the connection and φ_1 is constant. Under all of this considerations, one would like to investigate SS-metrics G with bounded sectional curvatures. There are strong reasons to believe that the same rigidity regarding sectional curvature arises in this context. To illustrate more, we consider the situations described below.

Proposition 60. *Assume the base (M, g) is flat and the sectional curvature of (E, G) satisfies $|K_G| \leq C_G$, and the weight functions φ_1 and φ_2 are given as follows*

(i) $\varphi_1(r) = \sqrt{C_G}r + 1$,

(ii) $\varphi_2(r) \leq \dfrac{1}{2}\ln(2r)$,

then D is flat.

Proof: From equation (i) of Proposition 54, we get

$$\frac{3}{4}\|\mathcal{R}^{\xi}(X,Y)\|^2 = -\big(K_G(X \wedge Y) + 4r(\varphi_1')^2 e^{-2\varphi_2}\big),$$

but

$$K_G(X \wedge Y) + 4r(\varphi_1')^2 e^{-2\varphi_2} \geq -C_G + 4rC_G e^{-2\varphi_2(r)} = C_G\big(4re^{-2\varphi_2} - 1\big).$$

Under the hypotheses on the weight functions, we have $\big(4re^{-2\varphi_2} - 1\big) \geq 0$. Thus $\big(K_G(X \wedge Y) + 4r(\varphi_1')^2 e^{-2\varphi_2}\big) \geq 0$, hence $\|\mathcal{R}^{\xi}(X,Y)\|^2 = 0$, thus D is flat. ∎

Remark 15. The previous example indicates that spherically symmetric metrics with bounded sectional curvature are unlikely to exist. In the case of flat connections, the present authors constructed examples of bounded geometries (cf. [12]).

As the reader may agree, the expressions of the curvature tensor, Ricci tensor, sectional curvatures and scalar curvature are so complicated that their analysis seems to be impossible in full generality, hence the restriction to a subclass of SS-metrics is in order. A subclass of particular interest is given by spherically symmetric metrics with weights $\varphi_1 = 0$ and φ_2 is arbitrary, that is those SS-metrics for which the projection maps are Riemannian submersions.

Henceforward, assume that G is an SS-metric with $\varphi_1 = 0$. Since having constant scalar curvature is a weak geometric property, we begin by analysing scalar curvature in this case.

It is clear from the expression of scalar curvature that if the connection D is flat, φ_2 is constant and (M, g) is of constant scalar curvature, then (E, G) has constant scalar curvature coinciding with the scalar curvature of the base. Investigating the converse is the aim of the following result.

We begin by an interesting result in the case of line bundles, that is a vector bundle with rank $k = 1$.

Proposition 61. *Let (E, G) be a line bundle endowed with an SS-metric, then (E, G) is of constant scalar curvature $\widetilde{Sc}$ if and only if (M, g) is of constant scalar curvature $Sc = \widetilde{Sc}$ and D is flat.*

Proof: If (E, G) is of constant scalar curvature $\tilde{Sc}$, then by virtue of Proposition 49, the Riemannian manifold (M, g) has also constant scalar curvature $Sc_0 = \widetilde{Sc}$. Now, taking

$\varphi_1 = 0$ in equation (4.2.22) yields the following

$$\begin{aligned}\widetilde{Sc} &= Sc + 4e^{-2\varphi_2}\Big[r\Big\{(2-k)\big(\varphi_2'' - (\varphi_2')^2\big)\Big\} \\ &\quad + k\Big\{(1-k)(\varphi_2' + r(\varphi_2')^2) - r(\varphi_2'' - (\varphi_2')^2)\Big\} \\ &\quad + \frac{3}{16}\|\mathcal{R}^\xi\|_E^2 + \frac{1}{4}\|S\|_M^2\Big] \\ &= Sc_0 + 4e^{-2\varphi_2}\Big[2r(1-k)(\varphi_2'' - (\varphi_2')^2) + k(1-k)(\varphi_2' + r(\varphi_2')^2) \\ &\quad + \frac{3}{16}\|\mathcal{R}^\xi\|_E^2 + \frac{1}{4}\|S\|_M^2\Big],\end{aligned}$$

where

$$\|S\|_M^2 = 2\sum_{i=1}^{n}\sum_{p=1}^{k}\|S(\sigma_p^v, e_i^h)\|_M^2, \tag{4.3.6}$$

$$\|\mathcal{R}^\xi\|_E^2 = \sum_{i,j=1}^{n}\|\mathcal{R}^\xi(e_i^h, e_j^h)\|^2. \tag{4.3.7}$$

Since $k = 1$ and $Sc = \widetilde{Sc}$, we obtain

$$\frac{3}{16}\|\mathcal{R}^\xi\|_E^2 + \frac{1}{4}\|S\|_M^2 = 0,$$

which implies that $\mathcal{R}^\xi = 0$, and hence D is flat. The converse is straightforward. ∎

Proposition 62. *Let (E, G) be a vector bundle of rank $k \geq 2$. Assume that the weight function φ_2 satisfies the following:*

1. *φ_2 is an increasing function,*
2. *φ_2' is a decreasing function.*

Then, (E, G) is of constant scalar curvature if and only if (M, g) is of constant scalar curvature, D is flat and φ_2 is constant.

Proof: Assume that (E, G) is of constant scalar curvature $\widetilde{S}$, then so is (M, g) with scalar curvature $Sc_0 = \widetilde{Sc} - 4k(1-k)e^{-2\varphi_2(0)}\varphi_2'(0)$.

On the other hand, from the proof of Proposition 61, we have

$$\begin{aligned}\widetilde{Sc} &= Sc_0 + 4e^{-2\varphi_2}\Big[2r(1-k)(\varphi_2'' - (\varphi_2')^2) + k(1-k)(\varphi_2' + r(\varphi_2')^2) \\ &\quad + \frac{3}{16}\|\mathcal{R}^\xi\|_E^2 + \frac{1}{4}\|S\|_M^2\Big].\end{aligned}$$

Thus

$$\begin{aligned}&- 4k(1-k)e^{-2\varphi_2(0)}\varphi_2'(0) + 4e^{-2\varphi_2}\Big[2r(1-k)(\varphi_2'' - (\varphi_2')^2) + k(1-k)(\varphi_2' + r(\varphi_2')^2) \\ &+ \frac{3}{16}\|\mathcal{R}^\xi\|_E^2 + \frac{1}{4}\|S\|_M^2\Big] = 0,\end{aligned}$$

hence

$$\begin{aligned}&8r(1-k)e^{-2\varphi_2(r)}(\varphi_2''(r) - (\varphi_2'(r))^2) + 4k(1-k)e^{-2\varphi_2(r)}\big(\varphi_2'(r) - e^{2(\varphi_2(r)-\varphi_2(0))}\varphi_2'(0) \\ &\quad + r(\varphi_2'(r))^2\big) + 4e^{-2\varphi_2(r)}\Big[\frac{3}{16}\|\mathcal{R}^\xi\|_E^2 + \frac{1}{4}\|S\|_M^2\Big] = 0,\end{aligned} \tag{4.3.8}$$

Since φ_2 is increasing, we have $\varphi_2'(r) \geq 0$ and $e^{2(\varphi_2(r)-\varphi_2(0))} \geq 1$, thus $\varphi_2'(r) - e^{2(\varphi_2(r)-\varphi_2(0))}\varphi_2'(0) \leq \varphi_2'(r) - \varphi_2'(0)$ which is negative since φ_2' is decreasing. Further, $(\varphi_2'' - (\varphi_2')^2) \leq 0$. Thus all terms of equation (4.3.8) are positive, hence they vanish.

Thus $\|\mathcal{R}^\xi\|_E^2 = 0$, whence $\mathcal{R}^\xi = 0$ which yields D is flat. Again equation (4.3.8) becomes

$$8r(1-k)e^{-2\varphi_2(r)}(\varphi_2''(r) - (\varphi_2'(r))^2) + 4k(1-k)\big(e^{2\varphi_2(r)}\varphi_2'(r) - e^{-2\varphi_2(0)}\varphi_2'(0) + re^{-2\varphi_2(r)}(\varphi_2'(r))^2\big) = 0.$$

In particular, we obtain $\varphi_2' = 0$. ∎

Example 8. An example of metrics that satisfies the two requirement of Proposition 62 can be obtained by taking the weights $\varphi_1 = 0$ and $\varphi_2(r) = \alpha e^{\beta r}$ for $r \in [0, +\infty)$, where α and β are negative real constants.

In particular, we get a generalization of a well known rigidity result of the Sasaki metric on tangent bundles to Riemannian manifolds (c.f [2,3,44,68]).

Proposition 63. *Let (E, G^s) be a vector bundle of rank $k \geq 1$ endowed with the Sasaki metric G^s. Then (E, G^s) is of constant scalar curvature $\widetilde{Sc}$ if and only if (M, g) is of constant scalar curvature $Sc = \widetilde{Sc}$ and D is flat.*

Another instance which goes in the same direction as before is the case of vector bundles of rank $k = 2$.

Proposition 64. *Let (E, G) be a vector bundle of rank $k = 2$. Assume E is endowed with a spherically symmetric metric such that the function $r \mapsto r\varphi_2'(r)$ is decreasing and $\varphi_2'(0) = 0$. If (E, G) is of constant scalar curvature, then (M, g) is of constant scalar curvature, D is flat and φ_2 is constant.*

Proof: If (E, G) is of constant scalar curvature, then by virtue virtue of Proposition 49, the base is also of constant scalar curvature Sc with $Sc = \widetilde{Sc}$, since $\varphi_2'(0) = 0$. On the other hand, taking equation (4.2.22) for $k = 2$ yields

$$2(r\varphi_2'' + \varphi_2') + \frac{3}{16}\|\mathcal{R}^\xi\|_E^2 + \frac{1}{4}\|S\|_M^2 = 0. \tag{4.3.9}$$

Since the function $r \mapsto r\varphi_2'(r)$ is increasing, equation (4.3.9) implies that $\mathcal{R}^\xi = 0$, and hence D is flat and $(r\varphi_2'' + \varphi_2') = 0$, thus φ_2 is constant for the same reasons as those in the proof of Theorem 20. ∎

Those results indicate to which extent SS-metrics are rigid regarding the connection D. Now, we shall investigate the existence of Einstein spherically symmetric metrics. The expressions of the Ricci tensor suggest that Einstein spherically symmetric metrics are unlikely to exist if the connection D is not flat. On the other hand, the present authors proved in [12] that if the total space is an Einstein manifold, then necessarily the base is also an Einstein manifold. Moreover, we proved, in the case of flat connections, that φ_1 being constant is a necessary condition for a spherically symmetric metric with weights φ_1 and φ_2 to be an Einstein metric. The present authors also constructed a two-parameters family of spherically symmetric Einstein metrics when $k \geq 2$, the connection is flat and (M, g) is an Einstein manifold, where the weights φ_1 and φ_2 are given as one of the following pairs:

(i) φ_1 and φ_2 are constant,

(ii) φ_1 is constant and $\varphi_2(r) = -ln(r + c)$ with $c = \dfrac{\lambda}{4(k-1)}e^{-2\varphi_1}$ where λ is the Einstein constant of the Einstein manifold (M, g).

Motivated by the previous considerations, we assume that $\varphi_1 = 0$ and we search for Einstein spherically symmetric metrics. We prove that under a monotonicity condition on φ_2, Einstein spherically symmetric metrics with negative Einstein constant can not exist unless the connection D is flat.

Proposition 65. *Assume that (E, G) is an Einstein manifold with constant $\mu \leq 0$. If the function $r \mapsto r\varphi_2'(r)$ is decreasing, then D is flat, and consequently (M, g) and (E, G) are Ricci flat.*

Proof: Taking equation (4.2.21) for $X^V = Y^V = \xi$ gives

$$\mu r e^{2\varphi_2} = \widetilde{\mathrm{Ric}}(\xi, \xi) = 4r(1-k)(r\varphi_2'' + \varphi_2') + \sum_{i=1}^{n} \langle S(\xi, e_i^h), S(\xi, e_i^h) \rangle_M.$$

We conclude that $S(\xi, e_i^h) = 0$ for all i. Hence $S(\sigma_p^v, e_i^h) = 0$, thus $\mathcal{R}^\xi = 0$. As matter of fact D is flat. Again equation (4.2.21) implies that $\mu = 0$, and by virtue of Proposition 49, (M, g) is also an Einstein manifold with constant $\lambda = \mu e^{2\varphi_1}$, thus (M, g) is also Ricci flat. ∎

Example 9. The functions $\varphi_2(r) = -\ln(r + c)$, with $c > 0$, belong to the class of functions such that the function $r \mapsto r\varphi_2(r)$ is decreasing.

Corollary 12. *Assume that E is a line bundle. Then (E, G) is an Einstein manifold if and only it is Ricci flat, (M, g) is Ricci flat and D is flat.*

Proof: Assume that (E, G) is an Einstein manifold, then by virtue of Proposition 61 the connection D is flat. Hence from the proof of Proposition 65, it follows that (E, G) is Ricci flat, and hence by virtue of Proposition 49, the Riemannian manifold (M, g) is also Ricci flat. The converse is obvious. ∎

Corollary 13. *Let G^s be the Sasaki metric. The manifold (E, G^s) is Ricci flat if and only if D is flat and (M, g) is Ricci flat.*

The latter result generalizes a well known rigidity result on the Sasaki metric in the case of tangent bundles to Riemannian manifolds (cf. [44]).

Corollary 14. *Assume that $k \geq 2$ and (E, G) is Ricci flat. If the function $r \mapsto r\varphi_2'(r)$ is decreasing, then D is flat, (M, g) is Ricci flat and φ_2 is constant. The converse also holds.*

Proof: By virtue of the proof of Proposition 65, one gets D flat and consequently $r\varphi_2'' + \varphi_2' = 0$. Setting $\psi = \varphi_2'$, which gives $r\psi' + \psi = 0$. The latter ODE has solutions of the form $\psi = \dfrac{A}{r}$ which is not smooth at $r = 0$ on the right, hence the only admissible solution is $\psi = 0$. The conclusions (M, g) being Ricci flat come from Proposition 49. The converse follows at once from the expression of the Ricci tensor. ∎

4.3.2 The case of flat connections

Throughout this subsection, we assume that the connection D is flat and we derive geometric properties of the manifold (E, G).

In this context we begin by investigating locally symmetric SS-metrics, and we will give a full characterization of them. Our purpose is to prove Theorem 21. For this, we need some technical formulas and some intermediate results. Long but routine computations, using Lemma 3 and Propositions 25 and 51, give the following:

Proposition 66. *Let $e \in E$, $X^H, Y^H, Z^H, T^H \in \mathcal{H}_e$ and $X^V, Y^V, Z^V, T^V \in \mathcal{V}_e$. Then*

$$\begin{aligned}
&i)\ \left(\tilde{\nabla}_{T^H}\tilde{R}\right)(X^H,Y^H)Z^H \\
&\quad = (\pi^*\nabla_{T^H}\pi^*R)(X^H,Y^H)Z^H) - 2\varphi_1' e^{2(\varphi_1-\varphi_2)}\Big[\pi^*R(X^H,Y^H,Z^H,T^H) \\
&\qquad - 2e^{2(\varphi_1-\varphi_2)}\Big(2r(\varphi_1''-\varphi_1'\varphi_2')+\varphi_1'\Big)\langle X^H\wedge Y^H, Z^H\wedge T^H\rangle\Big]\xi; \\
&ii)\ \left(\tilde{\nabla}_{T^H}\tilde{R}\right)(X^H,Y^V)Z^V \\
&\quad = 4\varphi_1' e^{2(\varphi_1-\varphi_2)}\left[2(r(\varphi_2''-\varphi_1'\varphi_2')+\varphi_2')-\varphi_1')\right]\langle X^H,T^H\rangle(\xi\wedge Y^V)Z^V; \\
&iii)\ \left(\tilde{\nabla}_{T^H}\tilde{R}\right)(X^V,Y^V)Z^V \\
&\quad = 4\varphi_1'\left[-2r(\varphi_2''-(\varphi_2')^2)+\varphi_1'(1+2r\varphi_2')\right]\langle X^V\wedge Y^V,\xi\wedge Z^V\rangle T^H; \\
&iv)\ \left(\tilde{\nabla}_{T^V}\tilde{R}\right)(X^V,Y^V)Z^V = \\
&\quad = -4(\varphi_2''-(\varphi_2')^2)(1+2r\varphi_2')\Big[\langle X^V\wedge Y^V,T^V\wedge Z^V\rangle\xi \\
&\qquad + \langle X^V\wedge Y^V,\xi\wedge Z^V\rangle T^V - 2\xi^\flat(T^V)(X^V\wedge Y^V)Z^V \\
&\qquad - \xi^\flat(Z^V)(X^V\wedge Y^V)T^V - \langle T^V,Z^V\rangle(X^V\wedge Y^V)\xi\Big] \\
&\qquad + 8\Big[\varphi_2^{(3)} - 2\varphi_2'\varphi_2'' - 4(\varphi_2''-(\varphi_2')^2)\varphi_2'\Big]\xi^\flat(T^V)\Big[\xi^\flat(Z^V)(X^V\wedge Y^V)\xi \\
&\qquad - \langle X^V\wedge Y^V,\xi\wedge Z^V\rangle\xi\Big].
\end{aligned}$$

Theorem 21. *([12]) Assume that $k \geq 2$ and let G be a spherically symmetric metric E. Then (E,G) is locally symmetric if and only if the following conditions hold*

i) φ_1 is constant;

ii) either φ_2 is constant or $\varphi_2(r) = -\ln(r+c)$, for all $r \in \mathbb{R}_+$, for a constant $c > 0$;

iii) (M,g) is locally symmetric.

In particular, (E,G) is of constant sectional curvature if and only if φ_1 and φ_2 are constant and (M,g) is of constant sectional curvature.

Proof. Assume that (E,G) is locally symmetric. Let $e \in E$, such that $r = \|e\|^2 \neq 0$. Taking at first $X^V = Z^V = T^V = \xi_e$ and $0 \neq Y^V \perp \xi_e$, in the fourth formula of Proposition 66, we obtain

$$0 = r^2\left\{2(\varphi_2''-(\varphi_2')^2)(1+2r\varphi_2') + r\Big[\varphi_2^{(3)} - 2\varphi_2'\varphi_2'' - 4(\varphi_2''-(\varphi_2')^2)\varphi_2'\Big]\right\}Y^V,$$

which gives, by virtue of $Y^V \neq 0$,

$$r(\varphi_2^{(3)} - 2\varphi_2'\varphi_2'') + 2(\varphi_2''-(\varphi_2')^2) = 0, \tag{4.3.10}$$

on $\mathbb{R}_+^*$ and by continuity on $\mathbb{R}_+$. Putting $\psi = \varphi_2''-(\varphi_2')^2$, equation (4.3.10) can be written in the form $r\psi' + 2\psi = 0$, whose general solution is $\psi = \dfrac{A}{r^2}$, where $A \in \mathbb{R}$. Since φ_2 is C^∞ on $\mathbb{R}_+$, so is ψ, and consequently $A = 0$, i.e. $\psi = 0$. It follows that $\varphi_2''-(\varphi_2')^2 = 0$. Then, by regularity considerations, it is easy to see that either φ_2' vanishes identically on $\mathbb{R}_+$ or $\varphi_2'(r) = -\frac{1}{c+r}$, for all $r \in \mathbb{R}^+$, where $c \in \mathbb{R}_+^*$ is a constant.

Case 1: $\varphi_2' = 0$ identically. In this case, the second equation of Proposition 66 becomes

$$\left(\tilde{\nabla}_{T^H}\tilde{R}\right)(X^H, Y^V)Z^V = -4(\varphi_1')^2 e^{2(\varphi_1-\varphi_2)}\langle X^H, T^H\rangle(\xi \wedge Y^V)Z^V. \tag{4.3.11}$$

Putting in (4.3.11) $0 \neq T^H = X^H$ and $0 \neq Y^V = Z^V \perp \xi_e$, we have

$$4(\varphi_1')^2 e^{2(\varphi_1-\varphi_2)}\|X^H\|^2\|Y^V\|^2\xi_2 = 0,$$

and consequently $\varphi_1' = 0$.

Case 2: $\varphi_2'(r) = -\frac{1}{c+r}$, for all $r \in \mathbb{R}_+$, where $c \in \mathbb{R}_+$ is a constant. In this case, the third equation of Proposition 66 becomes

$$\left(\tilde{\nabla}_{T^H}\tilde{R}\right)(X^V, Y^V)Z^V = 4(\varphi_1')^2\frac{c-r}{c+r}\langle X^V \wedge Y^V, \xi \wedge Z^V\rangle T^H. \tag{4.3.12}$$

Putting in (4.3.12) $0 \neq X^V = Z^V \perp Y^V = \xi_e$, and supposing that $T^H \neq 0$, we have by local symmetry

$$0 = r(\varphi_1')^2\frac{c-r}{c+r}.$$

We deduce that φ_1' vanishes identically on $\mathbb{R}_+ \setminus \{0, c\}$, and by continuity on $\mathbb{R}_+$.

In both cases, using the first equation of Proposition 66 and the fact that $\varphi_1' = 0$, we deduce from $\tilde{\nabla}_{T^H}\tilde{R}(X^H, Y^H)Z^H = 0$ that (M, g) is also locally symmetric.

Conversely suppose we have *i)*, *ii)* and *iii)* of Theorem 21. Then $\varphi_1' = 0$ and $\varphi_2'' - (\varphi_2')^2 = 0$. Substituting into *i)*, *iii)* and *iv)* of Proposition 66, we find respectively $(\tilde{\nabla}_{T^H}\tilde{R})(X^H, Y^H)Z^H = (\pi^*\nabla_{T^H}\pi^*R)(X^H, Y^H)Z^H) = 0$, by local symmetry of (M, g), and $(\tilde{\nabla}_{T^H}\tilde{R})(X^V, Y^V)Z^V = (\tilde{\nabla}_{T^V}\tilde{R})(X^V, Y^V)Z^V = 0$. On the other hand, substituting from $\varphi_1' = 0$ into Proposition 51, we find

$$\tilde{R}(X^H, Y^H)Z^V = \tilde{R}(X^H, Y^V)Z^H = \tilde{R}(X^H, Y^V)Z^V = \tilde{R}(X^V, Y^V)Z^H = 0.$$

We deduce that $\tilde{\nabla}\tilde{R} = 0$, which completes our proof. □

Now, we shall be interested in sectional curvature. We begin by answering when spherically symmetric metrics have constant sectional curvature. Using Proposition 55, we prove the following rigidity result, of sectional curvature, similar to that of Theorem 58 but without monotonicity assumptions on φ_1.

Theorem 22. *([12]) The manifold (E, G) is of constant sectional curvature if and only if φ_1 and φ_2 are constant and (M, g) is flat. In this case, (E, G) is also flat.*

Proof. Suppose that (E, G) is of constant sectional curvature K_G. Then it is locally symmetric. We claim that φ_2 is constant. If not, by Theorem 21, φ_2 is constant or $\varphi_2(r) = -\ln(r+c)$, for all $r \in \mathbb{R}_+$, for a constant $c > 0$. We have also φ_1 is constant. Using Proposition 58, we have

$$K_G(X, Y) = e^{2\varphi_1}K_g((d\pi)_e X, (d\pi)_e Y)\|X^H \wedge Y^H\|^2 + \frac{4c}{(r+c)^4}\|X^V \wedge Y^V\|^2, \tag{4.3.13}$$

for any orthonormal system of vectors $\{X, Y\}$ in T_eE. If we suppose X horizontal and Y vertical in (4.3.17), in such a way that $X^V = Y^H = 0$, then we have $K_G = 0$. On the other hand, taking both X and Y vertical in (4.3.17), in such a way that (by orthonormality) $\|X^V \wedge Y^V\|^2 = e^{-4\varphi_2} = (r+c)^4$, then we have $c = 0$, which is a contradiction. We conclude that φ_2 is constant, so by (4.3.17), we have

$$K_G(X, Y) = e^{2\varphi_1}K_g((d\pi)_e X, (d\pi)_e Y)\|X^H \wedge Y^H\|^2. \tag{4.3.14}$$

Using the same arguments as before, we obtain $K_G = 0$ and then the sectional curvature of (M, g) is also constant equal to 0.

The converse in the Theorem is trivial. □

Though in the general case there are strong indications that bounded geometries are not likely to exist when the connection is non-flat, there exist examples of bounded geometries in the case of flat connections. Indeed, we have:

Theorem 23. *If the sectional curvature of (E, G) is bounded, then so is the sectional curvature of (M, g).*

Proof: From Proposition 55, for any orthonormal system $\{X^H, Y^H\}$ in $\mathcal{H}_e$ (i.e a horizontal 2-plane), we have

$$K_G(X^H, Y^H) = e^{-2\varphi_1} K_g((d\pi)_e(X^H), (d\pi)_e(Y^H)) - 4r(\varphi_1')^2 e^{-2\varphi_2}, \tag{4.3.15}$$

which, restricted to the zero section, yields

$$K_G(X^H, Y^H) = e^{-2\varphi_1(0)} K_g((d\pi)_{0_x}(X^H), (d\pi)_{0_x}(Y^H)),$$

where o_x is the zero vector of E_x, $x \in M$. Using the isomorphism $(d\pi)_{0_x} : \mathcal{H}_{0_x} \to M_x$ and the fact that K_G is bounded, we deduce that K_g is also bounded. ∎

The converse of Theorem 23 is not true in the general setting, but we shall prove that it remains true in some situations.

Proposition 67. *([12]) Assume $\varphi_2 \geq 0$ is constant and take $\varphi_1(r) = \arctan(r)$. If the sectional curvature of (M, g) is bounded, then the sectional curvature of (E, G) is also bounded.*

Proof: We assume that K_g is bounded and $|K_g| \leq K_0$. Let $X, Y \in T_eE$ be such that $\{X, Y\}$ is an orthonormal system of vectors, and set $X = X^H + X^V$, $Y = Y^H + Y^V$. Under the condition $\varphi_2' = 0$, we find that

$$\begin{aligned}
\tilde{K}(X, Y) =& e^{2\varphi_1} \left[K((d\pi)_e X, (d\pi)_e Y) - 4r(\varphi_1')^2 e^{2\varphi_1 - 2\varphi_2}\right] \|X^H \wedge Y^H\|^2 \\
&+ 4(\varphi_1'' + (\varphi_1')^2) e^{2\varphi_1} \Big[2\xi^\flat(X^V)\xi^\flat(Y^V)\langle X^H, Y^H\rangle \\
&- (\xi^\flat(Y^V))^2 \|X^H\|^2 - (\xi^\flat(X^V))^2 \|Y^H\|^2\Big] \\
&+ 2\varphi_1' e^{2\varphi_1} \Big[2\langle X^V, Y^V\rangle \langle X^H, Y^H\rangle \\
&- \|Y^V\|^2 \|X^H\|^2 - \|X^V\|^2 \|Y^H\|^2\Big].
\end{aligned}$$

Using the orthonormality conditions, we have

$$\|X^H \wedge Y^H\|^2 = e^{-4\varphi_1}(\tilde{g}(X^H, X^H)\tilde{g}(Y^H, Y^H) - \tilde{g}(X^H, Y^H)^2). \tag{4.3.16}$$

By definition of the metric, the Cauchy-Schwartz inequality with respect to G and the fact that we have $0 \leq \varphi_1 \leq \dfrac{\pi}{2}$, since $\varphi_1(r) = \arctan(r)$, we conclude that

$$e^{2\varphi_1} \|X^H \wedge Y^H\|^2 \leq 2e^{-2\varphi_1} \leq 2. \tag{4.3.17}$$

Again using the Cauchy-Schwartz inequality with respect to h and G and the fact that $\varphi_1, \varphi_2 \geq 0$, we prove easily that

$$|2\xi^\flat(X^V)\xi^\flat(Y^V)\langle X^H, Y^H\rangle - (\xi^\flat(Y^V))^2 \|X^H\|^2 - (\xi^\flat(X^V))^2 \|Y^H\|^2| \leq 4r^2, \tag{4.3.18}$$

$$|2\langle X^V, Y^V\rangle \langle X^H, Y^H\rangle - \|Y^V\|^2 \|X^H\|^2 - \|X^V\|^2 \|Y^H\|^2| \leq 4, \tag{4.3.19}$$

and

$$e^{2\varphi_1-2\varphi_2} \leq e^{\pi}. \tag{4.3.20}$$

So, by virtue of (4.3.17)–(4.3.20), we have:

$$|K_G(X,Y)| \leq 2\big(K_0 + 4r|\varphi_1'|^2 e^{\pi}\big) + 16(|\varphi_1'' + (\varphi_1')^2|)e^{\pi}r^2 + 8|\varphi_1'|e^{\pi}. \tag{4.3.21}$$

Since $\varphi_1(r) = \arctan(r)$, we have $0 \leq \varphi_1 \leq \frac{\pi}{2}$, $\varphi_1'(r) = \frac{1}{1+r^2}$ and $\varphi_1''(r) = \frac{-2r}{(1+r^2)^2}$. Thus, $\varphi_1' \leq 1$, $r\varphi_1' \leq \frac{1}{2}$ and $|(\varphi_1')^2 + \varphi_1''| \leq 2$. Furthermore, $\varphi_1'' + (\varphi_1')^2 = \frac{-2r}{(1+r^2)^2} + \frac{1}{(1+r^2)^2} = \frac{1-2r}{(1+r^2)^2}$, thus $r^2(|\varphi_1'' + (\varphi_1')^2|) \leq \frac{r^2(1+2r)}{(1+r^2)^2}$ which is a bounded function since $\frac{r^2(1+2r)}{(1+r^2)^2} \longrightarrow 0$ when $r \longrightarrow +\infty$. Denoting by K_1 the bound of the function $r \mapsto r^2(|\varphi_1'' + (\varphi_1')^2|)$, we conclude that

$$|K_G| \leq 2K_0 + 4e^{\pi}(3 + 4K_1),$$

which means that K_G is bounded. ∎

We consider the problem of finding Einstein spherically symmetric metrics. We begin by giving a characterization and then we give necessary and sufficient conditions for their existence.

Theorem 24. *([12]) (E,G) is an Einstein manifold with constant μ if and only if*

a- (M,g) *is also an Einstein manifold with constant* λ,

b- $2e^{-2\varphi_2}\Big[n(2r\varphi_1'\varphi_2' + \varphi_1') + 2(k-1)(\varphi_2' + r(\varphi_2')^2) + 2r(\varphi_2'' - (\varphi_2')^2)\Big] = -\mu$,

c- $n(\varphi_1'' + (\varphi_1')^2 - 2\varphi_1'\varphi_2') + (k-2)(\varphi_2'' - (\varphi_2')^2) = 0$,

d- $e^{2(\varphi_1-\varphi_2)}\Big[4r\big[\varphi_1'' + n(\varphi_1')^2\big] + \varphi_1'[2k + (k-2)r\varphi_2']\Big] + \mu e^{2\varphi_2}\Big] = \lambda$.

Proof: Suppose that (E,G) is an Einstein manifold with constant μ, i.e. $\widetilde{\mathrm{Ric}} = \mu G$. Then by *1.* of Proposition 53, we have $\mathrm{Ric}(X,Y) = \lambda g(X,Y)$, for all $x \in M$ and $X, Y \in T_xM$, where λ is the function defined, on E, by

$$\lambda = e^{2(\varphi_1-\varphi_2)}\Big[4r\big[\varphi_1'' + n(\varphi_1')^2\big] + \varphi_1'[2k + (k-2)r\varphi_2']\Big] + \mu e^{2\varphi_2}\Big]. \tag{4.3.22}$$

Since Ric and g depend only on x, λ will depend only on x, i.e. λ is constant on each fiber of E. But it is obvious, from (4.3.22), that λ depends only on r. Therefore λ is constant, so (M,g) is an Einstein manifold, and (4.3.22) is exactly condition *d-* of the theorem. On the other hand, taking $0 \neq Z^V = T^V \perp \xi_e$ in *3.* of Proposition 53, we obtain by virtue of $\widetilde{\mathrm{Ric}}(Z^V,T^V) = \mu\tilde{g}(Z^V,T^V) = \mu e^{2\varphi_2}\langle Z^V,T^V\rangle$, the following equation

$$-2\Big[n(2r\varphi_1'\varphi_2' + \varphi_1') + 2(k-1)(\varphi_2' + r(\varphi_2')^2) + 2r(\varphi_2'' - (\varphi_2')^2)\Big] = \mu e^{2\varphi_2},$$

which gives condition *b-* of the theorem. Now, *3.* of Proposition 53 becomes, by virtue of *b-* of the theorem

$$\Big[4n(\varphi_1'' + (\varphi_1')^2 - 2\varphi_1'\varphi_2') + 4(k-2)(\varphi_2'' - (\varphi_2')^2)\Big]\xi^{\flat}(Z^V)\xi^{\flat}(T^V) = 0, \tag{4.3.23}$$

for all $Z^V, T^V \in \mathcal{V}_e$. Taking, in (4.3.23), $Z^V = T^V = \xi_e$, we obtain the equality *c-* of the theorem on $\mathbb{R}_+^*$, and by continuity on $\mathbb{R}_+$.

The converse follows immediately from Proposition 53. ∎

Generally speaking, it is not easy to solve the system of differential equations under conditions *b*-, *c*- and *d*- of Theorem 24. We shall then investigate the special situations when φ_1 or φ_2 are constant.

Case 1: φ_2 is constant. In this case we have

Proposition 68. *([12]) Assume that φ_2 is constant, then (E, G) is Einstein if and only if (E, G) and (M, g) are Ricci-flat, and φ_1 is constant.*

Proof: Since $\varphi_2' = 0$, then we get, from *b*- of Theorem 24, $\varphi_1' = \frac{-\mu e^{2\varphi_2}}{2n}$. Substituting into *c*- of Theorem 24, we obtain $\mu = 0$, and consequently $\varphi_1' = 0$. Hence *d*- of Theorem 24 becomes $\lambda = \mu e^{2\varphi_1} = 0$. The converse follows immediately from Theorem 24. ∎

Case 2: φ_1 is constant.
Subcase 2.1: for $k \neq 2$, *c*- of Theorem 24 is equivalent to $\varphi_2'' = (\varphi_2')^2$, whose solutions are given by φ_2 is constant and

$$\varphi_2(r) = -\ln(r + c), \text{ for all } r \in \mathbb{R}_+, \text{ where } c > 0 \text{ is a constant.} \tag{4.3.24}$$

On the other hand, *b*- of Theorem 24 is equivalent to

$$4(1-k)\varphi_2'(1 + r\varphi_2') = \mu e^{2\varphi_2}. \tag{4.3.25}$$

If $k = 1$ then by the preceding equation we have $\mu = 0$, i.e. $(E, \tilde{g})$ is Ricci-flat, and we deduce from *d*- of Theorem 24 that $\lambda = 0$.

If $k \neq 1$, then we have two eventualities

- For φ_2 constant, we obtain $\mu = \lambda = 0$.
- For φ_2 given by (4.3.24), equation (4.3.25) gives $c = \frac{\mu}{4(k-1)}$. In particular $\mu > 0$.

Subcase 2.1: for $k = 2$, it is easy to check that the function φ_2 given by (4.3.24) satisfy conditions *b*-, *c*- and *d*- of Theorem 24.

Summarizing, we have just proved the following:

Proposition 69. *([12]) Assume that (M, g) be an Einstein manifold with constant $\lambda \geq 0$ and consider a spherically symmetric metric G on E, such that φ_1 is constant.*

a- If $k = 1$, then (E, G) is Einstein if and only if it is Ricci-flat if and only if (M, g) is Ricci-flat and

either φ_2 is constant,

or $\varphi_2(r) = -\ln(r + c)$, for all $r \in \mathbb{R}_+$, where $c > 0$ is a constant.

b- If $k \geq 3$, then (E, G) is Einstein with constant μ if and only if

either φ_2 is constant and (M, g) and $(E, \tilde{g})$ are Ricci-flat.

or $\mu = \lambda e^{2\varphi_1} > 0$ and $\varphi_2(r) = -\ln(r + \frac{\mu}{4(k-1)})$, for all $r \in \mathbb{R}_+$.

c- If $k \neq 1$ and $\varphi_2(r) = -\ln(r + c)$, for all $r \in \mathbb{R}_+$, with $c = \frac{\lambda}{4(k-1)} e^{-2\varphi_1}$, then (E, G) is an Einstein manifold, which is not Ricci flat unless (M, g) is Ricci flat.

Theorem 25. *([12]) Let (M, g) be a locally symmetric Einstein manifold of positive constant scalar curvature, then there is a 1-parameter family of locally symmetric Einstein metrics on E, which are not Ricci-flat.*

Proof. The result follows from Proposition 69 and Theorem 21 by taking the family of metrics $G_a := a\pi^*g + \frac{16(k-1)^2a^2}{(4(k-1)ar+\lambda)^2}\pi^*h$, parameterized by $a > 0$. The Ricci tensor of G_a is given by $\widetilde{\mathrm{Ric}} = \frac{\lambda}{a}G_a$, and its scalar curvature is constant equal to $\frac{(n+k)\lambda}{a} > 0$. □

Notice that the metric G_a depends on the rank k of E. But if we require the solutions of the system in Theorem 24 to be independent of the dimension n of M and the rank k of E, then the only solutions are φ_1 and φ_2 are constant.

Finally we draw attention to the scalar curvature of (E, G) in the context of flat connections. Theorem 18 implies that if (E, G) has constant scalar curvature, then so is (M, g). In some special cases, we have the converse.

Proposition 70. *Assume that φ_1 and φ_2 are constant. Then $\widetilde{Sc} = e^{-2\varphi_1}Sc$, thus $\widetilde{Sc}$ is constant on each fiber of E.*

Proof: Proposition 57, for $\varphi_1' = \varphi_2' = 0$, gives $\tilde{S}c = e^{-2\varphi_1}Sc$. ■

Concerning the sign of the scalar curvature of $(E, \tilde{g})$, we have:

Proposition 71. *Assume that φ_1 (resp. φ_2) is constant and φ_2 and φ_2' (resp. φ_1 and φ_1') are increasing functions. If S is negative, then $\tilde{S}$ is negative.*

Proof: If φ_1 (resp. φ_2) is constant, then Proposition 57 gives the following equation

$$\widetilde{Sc} = e^{-2\varphi_1}Sc - 4(k-1)\Big[r\big(2\varphi_2'' + (k-2)(\varphi_2')^2\big) + k\varphi_2'\Big]$$
$$(\text{resp. } \widetilde{Sc} = e^{-2\varphi_1}Sc - 4n\Big[r\big(2\varphi_1'' + (n+1)(\varphi_1')^2\big) + k\varphi_1'\Big])$$

which implies that, if $\varphi_2', \varphi_2'' \geq 0$ (resp. $\varphi_1', \varphi_1'' \geq 0$) and Sc is negative, then $\widetilde{Sc}$ is negative. ■

Chapter 5

On harmonic sections

Motivation

In the realm of Riemannian/differential geometry, many important geometric objects arise as solutions to variational problems (e.g. Einstein Metrics, Yang-Mills connections,...etc) [77]. Harmonic maps also arise from a natural variational problem on Riemannian manifolds associated with the energy functional (cf. [24,25]). In the context of general vector bundles, a class of mappings is of special interest, namely sections. The harmonicity of sections, as smooth maps, has been studied by many authors e.g. [36,55,60] in the context of the tangent bundle, and [43] in the context of vector bundles.

In his theory of harmonic sections, C.M. Wood introduced another concept of harmonicty in the framework of sections of fibre bundles which generalizes, in some sense, the notion of harmonic maps [82,83,85], that he called *vertical harmonicity*. Wood defined the so-called vertical energy and vertical tension field from which we recover the classical energy functional and tension field when we restrict ourselves to the trivial bundle.

Harmonicity of sections was, in almost all works, studied when the total space of the considered vector bundle is endowed with the Sasaki metric [36, 43, 60]. Nevertheless, some works were performed with more general classes of metrics e.g. g-natural metrics on tangent bundles [55] or the Cheeger-Gromoll type metrics on vector bundles in the large [49]. To the best of our knowledge, for general vector bundles, the study of harmonicity of sections, as smooth maps, is done when the total space is endowed with the generalized Cheeger-Gromoll metric. We investigate the harmonicity of sections of vector bundles when the total spaces are endowed with spherically symmetric metrics. More precisely, we shall be interested in the following three variational problems:

Problem 1: finding harmonic sections as smooth maps, that is finding critical points of the energy functional with variations through smooth maps.

Problem 2: finding vertically harmonic sections, that is finding critical points of the vertical energy functional.

Problem 3: finding critical points of the energy functional with variations through smooth sections.

In this chapter, we give the necessary and sufficient conditions under which the previous problems admit solutions, and we will also present the solutions when they exist. Moreover, we consider the stability problem of solutions. Finally, in order to illustrate the results found, we give a multitude of examples.

5.1 Preliminaries

Let (M, g) and (N, k) be two Riemannian manifolds, and let $\phi : M \longrightarrow N$ be a smooth map. The pullback of k by ϕ is the $(0,2)$-tensor field on M given by

$$(\phi^* k)_x(X, Y) = k_{\phi(x)}((d\phi)_x X, (d\phi)_x Y),$$

for all $x \in M$, $X, Y \in T_x M$. The differential of ϕ is the smooth map $d\phi : TM \longrightarrow TN$ given by $d\phi(X) = (d_{\pi(X)}\phi)(X)$, for all $X \in TM$. Thus $d\phi$ is a section of the bundle $T^*M \otimes \phi^*TN$ i.e. $d\phi \in \Gamma(T^*M \otimes \phi^*TN)$, where ϕ^*TN is the pullback of the vector bundle TN by ϕ.

Set $e(\phi) = \frac{1}{2}\|d\phi\|^2$, where $\|.\|$ is the *Hilbert-Schmidt norm* of ϕ, then $e(\phi)$ is a smooth function on M called the *energy density* of ϕ. It is locally expressed as follows

$$e(\phi) = \frac{1}{2} tr_g(\phi^* k) = \frac{1}{2} \sum_{i=1}^{n} \phi^* k(E_i, E_i),$$

where $\{E_i\}$ is any linear orthonormal frame of M and $n = dim(M)$.

If the manifold M is compact, then the *energy functional* of ϕ is defined to be

$$\mathcal{E}(\phi) = \frac{1}{2} \int_M \|d\phi\|^2 \upsilon_g, \tag{5.1.1}$$

where υ_g is the Riemannian volume element of (M, g). The energy functional can be written as $\mathcal{E}(\phi) = \int_M e(\phi)\upsilon_g$. The energy functional is generally denoted by $E(.)$, but to avoid confusion with vector bundles E, we have chosen to use the notation $\mathcal{E}(.)$ for the energy functional and to maintain the use of E for vector bundles.

Let us denote by ∇^g and ∇^k the Levi-Civita connections of (M, g) and (N, k), respectively. The Pullback vector bundle ϕ^*TN inherits naturally the pullback connection, from the Levi-Civita connection of N, which we denote by $\phi^*\nabla^k$, and T^*M is naturally endowed with the connection $(\nabla^M)^*$, whence the vector bundle $T^*M \otimes \phi^*TN$ is naturally endowed with the connection given by

$$\overline{\nabla} = (\nabla^g)^* \otimes \phi^*\nabla^k. \tag{5.1.2}$$

We denote by $C^\infty(M, N)$ the set of all smooth mappings from M to N. A *smooth variation* of ϕ is a collection of maps $\{\phi_t\}_{-\epsilon<t<\epsilon} \subset C^\infty(M, N)$ which

i) depends smoothly on t, i.e. the mapping $F : (-\epsilon, \epsilon) \times M \longrightarrow N$ defined by $F(t, x) = \phi_t(x)$, $-\epsilon < t < \epsilon$ and $x \in M$, is smooth,

ii) satisfies $\phi_0(x) = F(0, x) = \phi(x)$, for all $x \in M$.

Any smooth variation $\{\phi_t\}_{-\epsilon<t<\epsilon}$ of ϕ determines a smooth vector field along ϕ, that is a section of the vector bundle ϕ^*TN, defined by

$$V(x) = \frac{d}{dt}\Big|_{t=0} \phi_t(x), \tag{5.1.3}$$

for all $x \in M$.

Definition 22. A mapping $\phi \in C^\infty(M, N)$ is said to be *harmonic* if it is a critical point of the energy functional on $C^\infty(M, N)$, i.e. for any smooth variation $\{\phi_t\} \subset C^\infty(M, N)$ of ϕ with $-\epsilon < t < \epsilon$, we have

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}(\phi_t) = 0.$$

Definition 23. The *tension field* of a smooth mapping ϕ is the vector field along ϕ given by

$$\tau(\phi) = tr_g(\nabla d\phi),$$

where tr_g is the trace with respect to the metric g.

The first variation formula of the energy functional is given by:

Theorem 26. *(see [24, 25, 77, 87]) Let $\phi \in C^\infty(M,N)$, then for any smooth variation $\{\phi_t\}_{-\epsilon<t<\epsilon}$ of ϕ, we have*

$$\frac{d}{dt}\Big|_{t=0}\mathcal{E}(\phi_t) = -\int_M h(V,\tau(\phi))v_g,$$

where V is the variation vector field associated with the variation $\{\phi_t\}$ and $\tau(\phi)$ is the tension field of ϕ.

It follows from Theorem 26 that harmonic maps are exactly the mappings with identically vanishing tension fields, and since the condition $\tau(\phi) = 0$ is a tensorial one, we can overcome the assumption of compactness of M.

Definition 24. A mapping $\phi : (M,g) \longrightarrow (N,k)$, with M not necessarily compact, is said to be a *harmonic map* if $\tau(\phi) = 0$.

The equation $\tau(\phi) = 0$ is called the *Euler-Lagrange equation* of the variational problem defined by the energy functional (5.1.1).

Now, let $\phi : (M,g) \longrightarrow (N,k)$ be a harmonic map and let $\{\phi_t\}_t$ be a variation of ϕ through smooth maps from M to N, the second variation formula is given by

$$\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{E}(\phi_t) = \int_M k(\Delta_\phi V - \mathcal{R}_\phi V, V)v_g, \tag{5.1.4}$$

where V is the variational vector field associated with the variation $\{\phi_t\}_t$, Δ_ϕ is the rough Laplacian of V with respect to the connection $\phi^*\nabla^N$, locally expressed as

$$\Delta_\phi V = -\sum_i \phi^*\nabla^N_{E_i}\phi^*\nabla^N_{E_i}V - \phi^*\nabla^N_{\nabla_{E_i}E_i}V,$$

and $\mathcal{R}_\phi$ is the tensor defined locally by

$$\mathcal{R}_\phi(V) = \sum_i R^N(\phi_* E_i, V)\phi_* E_i, \tag{5.1.5}$$

$\{E_i\}$ being a local orthonormal frame on M.

The operator $\mathcal{J}_\phi$ defined by

$$\mathcal{J}_\phi(V) = \Delta_\phi V - \mathcal{R}_\phi V \tag{5.1.6}$$

is an operator on vector fields along ϕ called *the Jacobi operator of the harmonic map ϕ*. The *index form of ϕ* is defined to be the bilinear form given by:

$$I(V,W) = \int_M k(\mathcal{J}_\phi V, W)v_g. \tag{5.1.7}$$

Definition 25. A harmonic map $\phi : (M,g) \longrightarrow (N,k)$ is said to be *stable* if $I(V,V) \geq 0$, for all $V \in \Gamma(\phi^*TN)$, otherwise we say that ϕ is *unstable*.

It follows that, if N has non-positive sectional curvatures, then every harmonic map is stable. Indeed,

$$\int_M k(\Delta_\phi V, V)\upsilon_g = \int_M k(\phi^*\nabla^N V, \phi^*\nabla^N V)\upsilon_g, \tag{5.1.8}$$

where $k(\phi^*\nabla^N V, \phi^*\nabla^N V) = \sum_{i=1}^n k((\phi^*\nabla^N)_{E_i}V, (\phi^*\nabla^N)_{E_i}V)$, for any local orthonormal frame $\{E_i\}$ of M. Furthermore, $k(R^N(u,v)v,u) \leq 0$ for any pair $\{u,v\}$ of linearly independent vectors, together with equation (5.1.8), imply that ϕ is stable.

The stability of a harmonic map ϕ can be described using eigenvalues of the Jacobi operator $\mathcal{J}_\phi$. A real number λ is said to be an *eigenvalue* of the Jacobi operator $\mathcal{J}_\phi$ if there exists a section $V \in \Gamma(\phi^*TN)$, with $V \neq 0$, such that

$$\mathcal{J}_\phi V = \lambda V.$$

If λ is an eigenvalue of $\mathcal{J}_\phi$, then the vector space

$$V_\lambda(\phi) = \{V \in \Gamma(\phi^*TN)/\mathcal{J}_\phi V = \lambda V\}$$

is called the *eigenspace associated with* λ, and its dimension is called the *multiplicity of* λ.

The operator $\mathcal{J}_\phi$ is a second order elliptic self-adjoint differential operator, whose spectrum $Spec(\mathcal{J}_\phi)$ consists of an infinite discrete set of eigenvalues with finite multiplicities and without accumulation points (see [76, 77]). Thus, we define, respectively, the *index* and *nullity* of the harmonic map ϕ as:

$$Index(\phi) = \sum_{\lambda<0} \dim(V_\lambda(\phi)),$$

and

$$Nullity(\phi) = \dim(V_0(\phi)).$$

In this terminology, we have

$$\phi \text{ is stable } \iff Spec(\mathcal{J}_\phi) \subset \mathbb{R}_+ \iff Index(\phi) = 0.$$

It's worth mentioning that $Index(\phi)$ and $Nullity(\phi)$ are exactly the index and nullity of the index form I, defined in (5.1.7), as a bilinear map.

5.2 Sections which are harmonic maps

Henceforward, we suppose that (E, π, M) is a vector bundle with an oriented Riemannian base (M, g), endowed with a fiber metric h and a compatible connection D. The total space E will be implicitly endowed with a spherically symmetric metric G of the form (1.3.5) (or equivalently (1.3.6)) with weights φ_1 and φ_2.

In what follows, we shall use the following notations:

1. $\{E_i\}$ is an orthonormal linear frame of M. Sometimes, we will use the same notation for an orthonormal basis on some tangent space of M.

2. For a section σ of E, we set

   i) $r = h(\sigma, \sigma)$,

   ii) $\Sigma_\sigma = \{r(x) = h(\sigma(x), \sigma(x))/x \in M\}$,

   iii) $\|D\sigma\|^2 = \sum_i h(D_{E_i}\sigma, D_{E_i}\sigma)$,

3. υ_g the Riemannian volume element of (M, g).

The differential of a smooth section is given as follows:

Proposition 72. *For all $x \in M$, $X \in T_xM$ and $\sigma \in \Gamma(E)$, we have*

$$(d\sigma)_x(X) = X^h_{\sigma(x)} + (D_X\sigma)^v_{\sigma(x)}.$$

In this section, we shall be interested in solving **Problem 1** which consists in determining sections which are harmonic maps $(M, g) \longrightarrow (E, G)$. Sections of the vector bundle E are immersions of M into E. Furthermore, a parallel section $\sigma : (M, g) \longrightarrow (E, G)$ is an isometric immersion if and only if $\varphi_1 = 0$. Indeed, if σ is a parallel section which is isometric, i.e. $G((d\sigma)_x(X), (d\sigma(Y))_x) = g(X, Y)$, for all $x \in M$ and $X, Y \in T_xM$, then for $X = Y$ a unit vector, we obtain $e^{2\varphi_1} = 1$, and hence $\varphi_1 = 0$.

Proposition 73. *Assume that M is compact and let $\sigma \in \Gamma(E)$, then*

$$\mathcal{E}(\sigma) = \frac{1}{2}\int_M \Big(ne^{2\varphi_1(r)} + e^{2\varphi_2(r)}\|D\sigma\|^2\Big)v_g.$$

Proof: By virtue of Proposition 72, we have

$$\begin{aligned}
\|(d\sigma)_x\|^2 &= \sum_i^n G((d\sigma)_x(E_i), (d\sigma)_x(E_i)) \\
&= \sum_i^n G(E_i^h + (D_{E_i}\sigma)^v_{\sigma(x)}, E_i^h + (D_{E_i}\sigma)^v_{\sigma(x)}) \\
&= \sum_i^n \Big(e^{2\varphi_1} g(E_i, E_i) + e^{2\varphi_2} h_{\sigma(x)}((D_{E_i}\sigma)_{\sigma(x)}, (D_{E_i}\sigma)_{\sigma(x)})\Big) \\
&= ne^{2\varphi_1(r)} + e^{2\varphi_2(r)}\|D\sigma\|^2_x.
\end{aligned}$$

Thus the energy functional is given by

$$\mathcal{E}(\sigma) = \frac{1}{2}\int_M \big(ne^{2\varphi_1} + e^{2\varphi_2}\|D\sigma\|^2\big)v_g. \tag{5.2.1}$$

■

It is worth mentioning that, for each section $\sigma \in \Gamma(E)$, we have

$$\mathcal{E}(\sigma) \geq \frac{n}{2}\int_M e^{2\varphi_1(r)} v_g,$$

and equality holds, that is σ is an absolute minimum of the energy integral $\mathcal{E}$, if and only if σ is parallel.

Theorem 27. *Assume that M is compact and σ is a harmonic section of E. If φ_1' is non-negative and $(1 + 2r\varphi_2')$ is positive on Σ_σ (resp. φ_1' is non-positive and $(1 + 2r\varphi_2')$ is negative on Σ_σ), then σ is parallel and has constant norm $r = \|\sigma\|^2$. Moreover, if $\varphi_1'(r) \neq 0$, then $\sigma \equiv 0$.*

Proof: Let $\sigma \in \Gamma(E)$ be a harmonic section. Consider the variation given by $\sigma_t = (1+t)\sigma$. By virtue of Propositions 72 and 73, we have

$$\mathcal{E}(\sigma_t) = \frac{1}{2}\int_M \|d\sigma_t\|^2 v_g = \frac{1}{2}\int_M \Big(ne^{2\varphi_1(r_t)} + e^{2\varphi_2(r_t)}\|D\sigma_t\|^2\Big)v_g, \tag{5.2.2}$$

where

$$r_t = \|\sigma_t\|^2 = (1+t)^2 r \quad \text{and} \quad r = \|\sigma\|^2.$$

Straightforward computations give:

1. $\|D\sigma_t\|^2 = (1+t)^2\|D\sigma\|^2$,
2. $\dfrac{d}{dt}(e^{2\varphi_i(r_t)}) = 4\varphi_i'(r_t)r(1+t)e^{2\varphi_i(r_t)}$, for $i = 1, 2$.

Substituting from the above equations into (5.2.2) and using the fact that σ is harmonic, that is σ is a critical point of the energy functional, we must have

$$\begin{aligned} 0 = \frac{d}{dt}_{|t=0}\mathcal{E}(\sigma_t) &= \frac{1}{2}\frac{d}{dt}_{|t=0}\int_M \Big(ne^{2\varphi_1(r_t)} + (1+t)^2e^{2\varphi_2(r_t)}\|D\sigma\|^2\Big)\upsilon_g \\ &= \frac{1}{2}\int_M \Big(4nr\varphi_1'(r)e^{2\varphi_1(r)} + (2e^{2\varphi_2(r)} + 4r\varphi_2'(r)e^{2\varphi_2(r)})\|D\sigma\|^2\Big)\upsilon_g \\ &= \int_M \Big(2nr\varphi_1'(r)e^{2\varphi_1(r)} + e^{2\varphi_2(r)}(1+2r\varphi_2'(r))\|D\sigma\|^2\Big)\upsilon_g, \end{aligned}$$

which implies, by hypotheses, that

$$2nr\varphi_1'(r)e^{2\varphi_1(r)} = 0, \tag{5.2.3}$$

and

$$e^{2\varphi_2(r)}(1+2r\varphi_2'(r))\|D\sigma\|^2 = 0. \tag{5.2.4}$$

Equation (5.2.4) implies, under our assumptions, that $\|D\sigma\| = 0$, hence σ is parallel, thus $r = \|\sigma\|^2$ is constant since M is connected. By virtue of (5.2.3), we have $r\varphi_1'(r) = 0$. So, if $\varphi_1'(r) \neq 0$, we get $r = 0$, and hence $\sigma \equiv 0$. ∎

Remark 16. Theorem 27 does not work for the non compact case. The following counter-example was given in [43]: for $M = \mathbb{R}^n$ and $E = M \times \mathbb{R}$ with the natural projection p and assume that $\varphi_1 = \varphi_2 = 0$ on $\mathbb{R}^+$. It's easy to see that $\Gamma(E) = C^\infty(M)$. We suppose that (E, p, M) is endowed with the canonical connection, that is the connection defined by $\nabla_X f = Xf$ for $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, thus harmonic sections are harmonic functions on $\mathbb{R}^n$, but parallel sections are just constant functions.

Lemma 14. *(see [43], [23]) Let $X, Y \in \mathfrak{X}(M)$, and $\sigma_1, \sigma_2 \in \Gamma(\mathcal{E})$. Then*

1. $[\sigma_1^v, \sigma_2^v] = 0$,
2. $[X^h, \sigma^v] = (D_X\sigma)^v$,
3. $[X^h, Y^h] = [X, Y]^h - (R^E(X, Y)\xi)^v$.

Proposition 74. *For any section $\sigma \in \Gamma(E)$, the tension field of σ is given by*

$$\tau(\sigma) = (\tau(\sigma))^V + (\tau(\sigma))^H,$$

where

$$\tau(\sigma)^V = -(\Delta\sigma)^v + (nc_1 + c_2\|D\sigma\|^2)\sigma^v + 2b\sum_{i=1}^n h(D_{E_i}\sigma, \sigma)(D_{E_i}\sigma)^v,$$

and

$$\tau(\sigma)^H = 2\sum_{i=1}^n \Big(ah(D_{E_i}\sigma, \sigma)E_i^h + S((D_{E_i}\sigma)^v, E_i^h)\Big),$$

where $\Delta\sigma$ is the rough Laplacian *of σ given locally by*

$$\Delta\sigma = -\sum_{i=1}^n \big(D_{E_i}D_{E_i}\sigma - D_{\nabla_{E_i}E_i}\sigma\big).$$

Proof: Consider $\sigma \in \Gamma(E)$, and let $\overline{\nabla}$ be the connection induced on the vector bundle $T^*M \otimes \sigma^*TE$. Using the expression of the Levi-Civita connection $\tilde{\nabla}$ of the Riemannian manifold (E, G) given in Proposition 25, and Proposition 72, we have

$$\begin{aligned}
\tau(\sigma) &= tr_g(\overline{\nabla} d\sigma) \\
&= \sum_{i=1}^{n} \left(\tilde{\nabla}_{E_i^h + (D_{E_i}\sigma)^v} (E_i^h + (D_{E_i}\sigma)^v) - ((\nabla_{E_i} E_i)^h + (D_{\nabla_{E_i} E_i}\sigma)^v \right) \\
&= \sum_{i=1}^{n} \Big(\tilde{\nabla}_{E_i^h} E_i^h + \tilde{\nabla}_{E_i^h} (D_{E_i}\sigma)^v + \tilde{\nabla}_{(D_{E_i}\sigma)^v} (E_i^h) + \tilde{\nabla}_{(D_{E_i}\sigma)^v} (D_{E_i}\sigma)^v \\
&- (\nabla_{E_i} E_i)^h - (D_{\nabla_{E_i} E_i}\sigma)^v \Big) \\
&= \sum_{i=1}^{n} \Big(\big((\nabla_{E_i} E_i)^h + c_1 \langle E_i^h, E_i^h \rangle_M \xi + S(E_i^h, E_i^h) - \frac{1}{2} \mathcal{R}^{\xi}(E_i^h, E_i^h) \big) \\
&+ \big(2\tilde{\nabla}_{(D_{E_i}\sigma)^v} (E_i^h) + [E_i^h, (D_{E_i}\sigma)^v] \big) + \big(\pi^\star D_{(D_{E_i}\sigma)^v} (D_{E_i}\sigma)^v \\
&+ 2b(\xi^\flat((D_{E_i}\sigma)^v))(D_{E_i}\sigma)^v + c_2 h(D_{E_i}\sigma, D_{E_i}\sigma)\xi + S((D_{E_i}\sigma)^v, (D_{E_i}\sigma)^v) \\
&- \frac{1}{2} \mathcal{R}^{\xi}((D_{E_i}\sigma)^v, (D_{E_i}\sigma)^v) \big) - (\nabla_{E_i} E_i)^h - (D_{\nabla_{E_i} E_i}\sigma)^v \Big).
\end{aligned}$$

Using the definitions of the tensors S and $\mathcal{R}^{\xi}$ contained in Proposition 25, and the expressions of the Lie bracket given in Lemma 14, we have

$$\begin{aligned}
\tau(\sigma) =& \sum_{i=1}^{n} \Big(c_1 \xi + 2\big(a\xi^\flat((D_{E_i}\sigma)^v) E_i^h + S((D_{E_i}\sigma)^v, E_i^h) + (D_{E_i} D_{E_i}\sigma)^v \big) \\
&+ \pi^\star D_{(D_{E_i}\sigma)^v} (D_{E_i}\sigma)^v + 2b\xi^\flat((D_{E_i}\sigma)^v)(D_{E_i}\sigma)^v \\
&+ c_2 h(D_{E_i}\sigma, D_{E_i}\sigma)\xi - (D_{\nabla_{E_i} E_i}\sigma)^v \Big) \\
=& -(\Delta\sigma)^v + (nc_1 + c_2 \|D\sigma\|^2)\xi \\
&+ \sum_{i=1}^{n} \big(\pi^\star D_{(D_{E_i}\sigma)^v} (D_{E_i}\sigma)^v + 2b\xi^\flat((D_{E_i}\sigma)^v)(D_{E_i}\sigma)^v \big) \\
&+ 2\sum_{i=1}^{n} \Big(a\xi^\flat((D_{E_i}\sigma)^v) E_i^h + S((D_{E_i}\sigma)^v, E_i^h) \Big) \\
=& \tau(\sigma)^V + \tau(\sigma)^H,
\end{aligned}$$

where $\Delta\sigma$ is the section given by

$$\Delta\sigma = -\sum_{i=1}^{n} \big(D_{E_i} D_{E_i}\sigma - D_{\nabla_{E_i} E_i}\sigma \big). \tag{5.2.5}$$

Moreover, by definition of the pullback connection $\pi^\star D$, we have

$$\pi^\star D_{(D_{E_i}\sigma)^v} (D_{E_i}\sigma)^v = 0.$$

Bearing in mind that $\tau(\sigma)$ is a vector field along σ and using the definition of the tautological section ξ, the formulas follow. ∎

Remark 17. The expression of the tension field found in the previous proposition generalizes the formula found in [43] for $\varphi_1 = \varphi_2 = 0$.

As a direct consequence, we have:

Corollary 15. *Let $\sigma \in \Gamma(E)$, and assume that φ_1 is constant on Σ_σ. If σ is parallel, then σ is harmonic.*

Corollary 16. *If σ is a harmonic section of (E, G), then*

$$tr_g\big(R^E(.,X)\sigma, D_.\sigma\big) = 2c_1 h(D_X\sigma, \sigma),$$

for all $X \in \mathfrak{X}(M)$.

Proof: If σ is harmonic, then $\tau(\sigma) = 0$, and in particular $\tau(\sigma)^H = 0$. Thus, by virtue of proposition 25, for each j, we have

$$\begin{aligned}
0 = \langle \tau(\sigma)^H, E_j^h\rangle_M &= 2\langle \sum_{i=1}^n \Big(ah((D_{E_i}\sigma),\sigma)E_i^h + S((D_{E_i}\sigma)^v, E_i^h)\Big), E_j^h\rangle_M \\
&= 2\sum_{i=1}^n \langle S((D_{E_i}\sigma)^v, E_i^h), E_j\rangle_M + 2a\sum_{i=1}^n h((D_{E_i}\sigma),\sigma)\delta_j^i \\
&= 2e^{-2\varphi_1}\sum_{i=1}^n e^{2\varphi_1}\langle S((D_{E_i}\sigma)^v, E_i^h), E_j^h\rangle_M + 2ah(D_{E_j}\sigma,\sigma) \\
&= e^{2\varphi_2-2\varphi_1}\sum_{i=1}^n h(R^E(E_i,E_j)\sigma, D_{E_i}\sigma) + 2ah(D_{E_j}\sigma,\sigma),
\end{aligned}$$

and setting

$$tr_g\big(R^E(.,X)\sigma, D_.\sigma\big) = \sum_{i=1}^n h(R^E(E_i,X)\sigma, D_{E_i}\sigma),$$

we have, for any vector field $X \in \mathfrak{X}(M)$,

$$e^{2\varphi_2-2\varphi_1}\big(tr_g\big(R^E(.,X)\sigma, D_.\sigma\big)\big) = -2ah(D_X\sigma,\sigma),$$

whence

$$\big(tr_g\big(R^E(.,X)\sigma, D_.\sigma\big)\big) = -2ae^{2\varphi_1-2\varphi_2}h(D_X\sigma,\sigma) = 2c_1h(D_X\sigma,\sigma),$$

with $c_1 = -2\varphi_1' e^{2\varphi_1-2\varphi_2} = -ae^{2\varphi_1-2\varphi_2}$. ∎

The rough Laplacian satisfies the following property.

Lemma 15. *Let σ be a section of E and let $F_\sigma = \dfrac{1}{2}\|\sigma\|^2$, then*

$$h(\Delta\sigma,\sigma) = \Delta_g F_\sigma + \|D\sigma\|^2,$$

where Δ_g is the Laplace-Beltrami operator of (M, g).

Proposition 75. *Assume that $R^E = 0$ (i.e. D is flat) and let σ be a harmonic section such that φ_1' does not vanish on Σ_σ. Then σ has constant length k; furthermore, the norm k satisfies*

$$(nc_1 + c_2\|D\sigma\|^2)k - \|D\sigma\|^2 = 0.$$

In particular, if σ is parallel then it's the zero section.

Proof: Let σ be a harmonic section, then $\tau(\sigma)^H = \tau(\sigma)^V = 0$. Since $R^E = 0$, then by virtue of Proposition 25, we have $S = 0$. It follows that

$$0 = \tau(\sigma)^H = 2a \sum_{i=1}^{n} h(D_{E_i}\sigma, \sigma). \tag{5.2.6}$$

Since $a = 2\varphi_1'$, then a does not vanish on Σ_σ, and (5.2.6) yields

$$h(D_{E_i}\sigma, \sigma) = E_i.F_\sigma = 0,$$

for all $i = 1, ..., n$. It follows that σ has constant norm $\|\sigma\|^2 = k$. In particular $\Delta_g F_\sigma = 0$. Hence, by virtue of Lemma 15, we have

$$\begin{aligned}
\langle \tau(\sigma)^V, \sigma^v \rangle_E &= -h(\Delta\sigma, \sigma) + (nc_1 + c_2\|D\sigma\|^2)\|\sigma\|^2 + 2b\sum_{i=1}^{n} h(D_{E_i}\sigma, \sigma)^2 \\
&= -h(\Delta\sigma, \sigma) + (nc_1 + c_2\|D\sigma\|^2)k \\
&= -\Delta_g F_\sigma - \|D\sigma\|^2 + (nc_1 + c_2\|D\sigma\|^2)k, \\
&= (nc_1 + c_2\|D\sigma\|^2)k - \|D\sigma\|^2.
\end{aligned}$$

Furthermore, if σ is parallel, then we must have $nc_1 k = 0$, which implies that $k = 0$, whence $\sigma \equiv 0$. ∎

In what follows, we shall see that the condition of parallelism of sections when φ_1 is constant implies that sections are totally geodesic which is much stronger than Corollary 15.

Theorem 28. *Suppose that $\sigma \in \Gamma(E)$ ($\sigma \neq 0$) is a parallel section, then σ is totally geodesic if and only if φ_1 is constant.*

Proof: Let γ be a geodesic in (M, g). Since σ is a parallel section, by virtue of Proposition 72, we have

$$d\sigma(\dot{\gamma}(t)) = (\dot{\gamma})^h_{\sigma(\gamma(t))}. \tag{5.2.7}$$

Thus $\dot{\widehat{\sigma \circ \gamma}}$ is a horizontal vector field. Hence by Proposition 25, we have

$$\begin{aligned}
\tilde{\nabla}_{\dot{\widehat{\sigma\circ\gamma}}} \dot{\widehat{\sigma \circ \gamma}} &= (\nabla_{\dot{\gamma}}\dot{\gamma})^h + c_1 g(\dot{\gamma}, \dot{\gamma})\xi + S(\dot{\widehat{\sigma \circ \gamma}}, \dot{\widehat{\sigma \circ \gamma}}) - \frac{1}{2}\pi^\star R^D(\dot{\widehat{\sigma \circ \gamma}}, \dot{\widehat{\sigma \circ \gamma}})\xi \\
&= c_1 g(\dot{\gamma}, \dot{\gamma})\xi,
\end{aligned}$$

which implies that σ is totally geodesic if and only if φ_1 is constant. ∎

Remark 18. The latter result generalizes a similar result of J.J. Konderak about totally geodesic sections (see [43]).

5.3 Vertically Harmonic Sections

In this section we shall deal with **Problem 2**, that is the problem of determining vertically harmonic sections considered as critical points of the vertical energy functional with variations through sections. In [82], C.M. Wood introduced vertically harmonic sections as a generalization of harmonic maps, and used them in the study of certain structures on Riemannian manifolds. He proved in [83] an existence theorem of the so-called vertically harmonic sections similar to existence theorem for harmonic maps classically studied in [24, 25].

For $\sigma \in \Gamma(E)$, we denote by $d^v\sigma$ the vertical component of $d\sigma$ with respect to the connection D, and we shall denote by ∇^v the connection in the vertical sub-bundle $\mathcal{V} \longrightarrow E$ obtained by projecting vertically the Levi-Civita connection $\tilde{\nabla}$ of (E, G). The projection π has totally geodesic fibers (see [13]). Set

$$\tau^v(\sigma) = tr_g(\nabla^v d^v\sigma). \tag{5.3.1}$$

The vector field $\tau^v(\sigma)$ is called the *vertical tension field of* σ. When M is compact, the *vertical energy functional* ([49,82,85]) is defined by

$$\mathcal{E}^v(\sigma) = \frac{1}{2}\int_M \|d^v\sigma\|^2 v_g, \tag{5.3.2}$$

where $\|d^v\sigma\| = \sum_{i=1}^n G(d^v\sigma(E_i), d^v\sigma(E_i))$. *Vertically harmonic sections* are defined to be the critical points of the vertical energy functional with variations through smooth sections. In the same manner, it turns out that vertically harmonic sections are those sections with identically vanishing vertical tension field, which allows to give a general definition of vertically harmonic sections even when M is non compact.

Definition 26. A section $\sigma \in \Gamma(E)$ is said to be *vertically harmonic* if

$$\tau^v(\sigma) = 0.$$

By virtue of Proposition 72, we have:

$$d^v\sigma(X) = (D_X\sigma)^v. \tag{5.3.3}$$

Now we shall find the first variation formula for the vertical energy functional. Let $\{\sigma_t\}$ be a variation of $\sigma = \sigma_0$ through sections, that is a map

$$F : M \times \mathbb{R} \longrightarrow E \quad \text{such that} \quad F(x,t) = \sigma_t(x), \tag{5.3.4}$$

and let $pr_1 : M \times \mathbb{R} \longrightarrow M$ be the projection on the first factor, then F is viewed as a section of the pullback vector bundle $pr_1^*E \longrightarrow M \times \mathbb{R}$. Furthermore, the variational vector field $\dfrac{d\sigma_t}{dt}$, which is a vertical vector field, gives rise to the family of sections $\{\rho_t\} \subset \Gamma(E)$ (see [49]) given by

$$\rho_t(x) = K\left(\frac{d\sigma_t(x)}{dt}\right), \quad \text{with} \quad \rho_0 = \rho. \tag{5.3.5}$$

The following lemmas will play an important role in the forthcoming computations.

Lemma 16. *(see [49]) Let ∂_t be the unit vector filed on $M \times \mathbb{R}$ in the positive $\mathbb{R}$-direction, then*

$$pr_1^* D_{\partial_t} F = pr_1^* \rho_t.$$

Lemma 17. *(see [49]) The curvature of the pullback connection pr_1^*D satisfies $R^{pr_1^*D}(TM, T\mathbb{R}) = 0$.*

Lemma 18. *(see [87]) Let σ, ρ be two sections of E, then*

$$h(\Delta\sigma, \rho) = h(D\sigma, D\rho) + div(V),$$

where $div(V)$ is the divergence of the vector field V given locally by

$$V = \sum_{i=1}^n h(D_{E_i}\sigma, \rho)E_i.$$

Note that we use the following sign convention for the divergence $div(V) = tr_g(\nabla V)$ for a vector field V on (M, g).

Set

$$\psi(\sigma_t) = e^{2\varphi_2(r_t)}, \quad \text{where} \quad r_t = \|\sigma_t\|^2.$$

By virtue of (5.3.3), we have

$$\mathcal{E}^v(\sigma_t) = \frac{1}{2}\int_M \psi(\sigma_t)\|D\sigma_t\|^2 \upsilon_g.$$

Thus

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\mathcal{E}^v(\sigma_t) &= \frac{1}{2}\int_M \Big(\frac{d}{dt}\Big|_{t=0}\psi(\sigma_t)\Big)\|D\sigma\|^2\upsilon_g \\ &\quad + \frac{1}{2}\int_M \psi(\sigma)\Big(\frac{d}{dt}\Big|_{t=0}\|D\sigma_t\|^2\Big)\upsilon_g.\end{aligned}$$

Forward computations, based on Lemmas 16, 17 and 18, give the following:

Lemma 19. *For any variation $\{\sigma_t\}$ of σ through sections, we have*

1. $\dfrac{d}{dt}\Big|_{t=0}\psi(\sigma_t) = 4\varphi_2'(r)e^{2\varphi_2(r)}h(\sigma,\rho) = 4\varphi_2'(r)\psi(\sigma)h(\sigma,\rho)$ *with* $r = \|\sigma\|^2$,
2. $\dfrac{d}{dt}\Big|_{t=0}\|D\sigma_t\|^2 = 2h(D\sigma, D\rho)$.

Thus, we obtain the first variation formula as follows:

Proposition 76. *Assume that M is compact and let $\{\sigma_t\}$ be a variation of the section σ through smooth sections, then we have*

$$\frac{d}{dt}\Big|_{t=0}\mathcal{E}^v(\sigma_t) = \int_M \psi(\sigma)\Big(h\big(\Delta\sigma + b\|D\sigma\|^2\sigma - 2bD_{\nabla F_\sigma}\sigma, \rho\big)\Big)\upsilon_g,$$

where $b = 2\varphi_2'$ and $F_\sigma = \dfrac{1}{2}\|\sigma\|^2$.

Proof: Lemma 19 implies that

$$\frac{d}{dt}\Big|_{t=0}\mathcal{E}^v(\sigma_t) = \int_M \Big(2\varphi_2'(r)\psi(\sigma)\|D\sigma\|^2 h(\sigma,\rho) + \psi(\sigma)h(D\sigma, D\rho)\big)\Big)\upsilon_g.$$

By virtue of Lemma 18, we have

$$\frac{d}{dt}\Big|_{t=0}\mathcal{E}^v(\sigma_t) = \int_M \Big(2\varphi_2'(r)\psi(\sigma)\|D\sigma\|^2 h(\sigma,\rho) + \psi(\sigma)\big(h(\Delta\sigma,\rho) + div(V)\big)\Big)\upsilon_g,$$

where $V = \sum_{i=1}^n h(D_{E_i}\sigma,\rho)E_i$. We have:

$$div\big(\psi(\sigma)V\big) = \psi(\sigma)div(V) + V.\psi(\sigma), \tag{5.3.6}$$

where

$$\begin{aligned}V.\psi(\sigma) &= \sum_{i=1}^n h(D_{E_i}\sigma,\rho)E_i.e^{2\varphi_2(\|\sigma\|^2)} \\ &= \sum_{i=1}^n 2h(D_{E_i}\sigma,\rho)E_i.\|\sigma\|^2\psi(\sigma)\varphi_2'(\|\sigma\|^2) \\ &= 4h(D_{\nabla F_\sigma}\sigma,\rho)\psi(\sigma)\varphi_2'(\|\sigma\|^2) \\ &= 2bh(D_{\nabla F_\sigma}\sigma,\rho)\psi(\sigma),\end{aligned}$$

with $F_\sigma = \frac{1}{2}\|\sigma\|^2$, and $b = 2\varphi_2'$ as defined in Proposition 25. Thus

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\mathcal{E}^v(\sigma_t) &= \int_M \Big(2\varphi_2'(r)\psi(\sigma)\|D\sigma\|^2 h(\sigma,\rho) + \psi(\sigma)h(\Delta\sigma,\rho)\\ &\quad + div\big(\psi(\sigma)V\big) - V.\psi(\sigma)\Big)\upsilon_g\\ &= \int_M \Big(2\varphi_2'(r)\psi(\sigma)\|D\sigma\|^2 h(\sigma,\rho) + \psi(\sigma)h(\Delta\sigma,\rho)\\ &\quad - 2bh(D_{\nabla F_\sigma}\sigma,\rho)\psi(\sigma)\Big)\upsilon_g\\ &= \int_M \psi(\sigma)\Big(h\big(\Delta\sigma + b\|D\sigma\|^2\sigma - 2bD_{\nabla F_\sigma}\sigma,\rho\big)\Big)\upsilon_g.\end{aligned}$$

■

Remark 19. In the case of the Sasaki metric ($\varphi_1 = \varphi_2 = 0$), we recover the classical Euler-Lagrange equation $\Delta\sigma = 0$ from the variational **problem 2**.

Therefore, when M is compact, we obtain the following characterization of vertically harmonic sections: a section is vertically harmonic if and only if it satisfies the following equation

$$\Delta\sigma + b\|D\sigma\|^2\sigma - 2bD_{\nabla F_\sigma}\sigma = 0. \tag{5.3.7}$$

Proposition 77. *Assume that M is compact. Let σ be a constant length section with $\|\sigma\|^2 = k$, and assume that $(1 + 2k\varphi_2'(k)) \neq 0$. Then, σ is a vertically harmonic section if and only it is parallel.*

Proof: If σ is vertically harmonic, then, by virtue of (5.3.7), we have

$$\Delta\sigma + 2\varphi_2'(k)\|D\sigma\|^2\sigma = 0,$$

since $\nabla F_\sigma = 0$. Hence

$$\begin{aligned}\int_M h(\Delta\sigma + 2\varphi_2'(r)\|D\sigma\|^2\sigma,\sigma)\upsilon_g &= \int_M \big(\|D\sigma\|^2 + \Delta_g F_\sigma + 2k\varphi_2'(k)\|D\sigma\|^2\big)\upsilon_g\\ &= \int_M \|D\sigma\|^2(1 + 2k\varphi_2'(k))\upsilon_g = 0,\end{aligned}$$

thus $\|D\sigma\|^2 = 0$, whence σ is parallel. ■

It follows from Corollary 15 that:

Proposition 78. *Assume that φ_1 is not constant, then every parallel section is vertically harmonic but not harmonic.*

5.4 G-harmonic sections

In this section we shall be interested in solving **Problem 3**. *G-harmonic sections* are defined to be critical points of the energy functional defined by (5.1.1) restricted to the space of smooth sections, i.e. critical points of $\mathcal{E}|_{\Gamma(E)}$. It is clear that every harmonic section is a G-harmonic section.

Let σ be a smooth section of E. By virtue of the first variation formula, we have

$$\frac{d}{dt}\Big|_{t=0}\mathcal{E}(\phi_t) = -\int_M G(\tau(\sigma),V)\upsilon_g, \tag{5.4.1}$$

where $\{\phi_t\}$ is a variation of the section σ through smooth maps, that is a variation of σ in $C^\infty(M,E)$, V is the variation vector field associated with this variation, and $\tau(\sigma)$ is the tension field of σ which was computed in Theorem 74. In particular, if we choose a variation of σ through $\Gamma(E)$, formula (5.4.1) is still working.

Since the expression of the energy functional is simple, we will find the first variation formula corresponding to **Problem 3** and determine the critical points by a direct method.

Let σ be a section of E. By virtue of Proposition 73, we have

$$\mathcal{E}(\sigma) = \frac{1}{2}\int_M \Big(ne^{2\varphi_1} + e^{2\varphi_2}\|D\sigma\|^2\Big)v_g.$$

For each variation $\{\sigma_t\}$ of σ through smooth sections, we have a family of sections $\{\rho_t\}$, given by (5.3.5), for which we denote $\rho_0 = \rho$. Using Lemma 19, we compute the first variation formula as follows:

Theorem 29. *Assume that M is compact and let $\sigma \in \Gamma(E)$ be a smooth section of E, we have*

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\mathcal{E}(\sigma_t) = \int_M h\Big(e^{2\varphi_2(r)}\big(\Delta\sigma - 2bD_{\nabla F_\sigma}\sigma\big) + 2\big(n\varphi_1'(r)e^{2\varphi_1(r)}\\ + \varphi_2'(r)e^{2\varphi_2(r)}\|D\sigma\|^2\big)\sigma, \rho\Big)v_g,\end{aligned}$$

where $\{\sigma_t\}$ is a variation of σ through smooth sections.

Proof: Let $\sigma \in \Gamma(E)$ and $\{\sigma_t\}$ be a smooth variation of σ through smooth sections. Let us denote by $\rho = K\left(\dfrac{d\sigma_t}{dt}\Big|_{t=0}\right)$ which is a section of E, and $r_t = \|\sigma_t\|^2$ with $r = r_0$, then by virtue of the proof of Proposition 76, we have

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=0}\mathcal{E}(\sigma_t)\\
&= \frac{1}{2}\frac{d}{dt}\Big|_{t=0}\Big(\int_M \Big(ne^{2\varphi_1(r_t)} + e^{2\varphi_2(r_t)}\|D\sigma_t\|^2\Big)v_g\Big)\\
&= \frac{1}{2}\int_M \Big(n\frac{d}{dt}\Big|_{t=0}e^{2\varphi_1(r_t)} + \big(\frac{d}{dt}\Big|_{t=0}e^{2\varphi_2(r_t)}\big)\|D\sigma\|^2 + e^{2\varphi_2(r)}\big(\frac{d}{dt}\Big|_{t=0}\|D\sigma_t\|^2\big)\Big)v_g\\
&= \int_M \big(2n\varphi_1'(r)e^{2\varphi_1(r)}h(\sigma,\rho) + 2\varphi_2'(r)e^{2\varphi_2(r)}h(\sigma,\rho)\|D\sigma\|^2\\
&+ e^{2\varphi_2(r)}h(D\sigma, D\rho)\big)v_g\\
&= \int_M \big(2n\varphi_1'(r)e^{2\varphi_1(r)}h(\sigma,\rho) + 2\varphi_2'(r)e^{2\varphi_2(r)}h(\sigma,\rho)\|D\sigma\|^2\\
&+ e^{2\varphi_2(r)}\big(h(\Delta\sigma - 2bD_{\nabla F_\sigma}\sigma,\rho)\big)\big)v_g\\
&= \int_M h(2\big(n\varphi_1'(r)e^{2\varphi_1(r)} + \varphi_2'(r)e^{2\varphi_2(r)}\|D\sigma\|^2\big)\sigma\\
&+ e^{2\varphi_2(r)}\big(\Delta\sigma - 2bD_{\nabla F_\sigma}\sigma\big), \rho)v_g.\end{aligned}$$

■

By virtue of Theorem 29, a section σ is G-harmonic, i.e. a critical point of $\mathcal{E}|_{\Gamma(E)}$ with variations through $\Gamma(E)$ if and only if σ satisfies the equation

$$e^{2\varphi_2(r)}\big(\Delta\sigma - 2bD_{\nabla F_\sigma}\sigma\big) + 2\big(n\varphi_1'(r)e^{2\varphi_1(r)} + \varphi_2'(r)e^{2\varphi_2(r)}\|D\sigma\|^2\big)\sigma = 0, \tag{5.4.2}$$

which is equivalent to the equation

$$\Delta\sigma + (b\|D\sigma\|^2 - nc_1)\sigma - 2bD_{\nabla F_\sigma}\sigma = 0, \tag{5.4.3}$$

where $b = 2\varphi_2'$ and $c_1 = -2\varphi_1' e^{2\varphi_1 - 2\varphi_2}$ are the functions given in Proposition 25, and $\Delta\sigma$ is the rough laplacian of σ with respect to D.

Remark 20. Using the first variation formula (5.4.1), we find the same characterization of G-harmonic sections. Indeed, for each vertical lift ρ^v of a section ρ, there is a variation of σ through smooth sections such that ρ^v is the variational vector field, thus σ is G-harmonic if and only if $(\tau(\sigma))^V = 0$, that is

$$-\Delta\sigma + (nc_1 + c_2\|D\sigma\|^2)\sigma + 2b\sum_{i=1}^{n} h(D_{E_i}\sigma, \sigma)D_{E_i}\sigma = 0. \tag{5.4.4}$$

Bearing in mind that $D_{\nabla F_\sigma}\sigma = \sum_{i=1}^{n} h(D_{E_i}\sigma, \sigma)D_{E_i}\sigma$, we find the same equation as (5.4.3). As matter of fact, equation (5.4.3) works also for the non-compact case.

Since the variation chosen in Theorem 27 is a variation through smooth sections, we still have the same result for G-harmonic sections. Furthermore, we have:

Proposition 79. *Let σ be a constant length section with $\|\sigma\|^2 = k$, and assume that $\varphi_1'(k) \geq 0$ and $(1 + 2k\varphi_2'(k)) > 0$ (resp. $\varphi_1'(k) \leq 0$ and $(1 + 2k\varphi_2'(k)) < 0$). If σ is G-harmonic then it is parallel. Furthermore, if $\varphi_1'(k) \neq 0$, then $\sigma \equiv 0$.*

Proof: Assume that φ_1 is constant, if σ is a constant length G-harmonic section, then by virtue of equation (5.4.3), we have

$$\Delta\sigma + (b\|D\sigma\|^2 - nc_1)\sigma = 0, \tag{5.4.5}$$

hence

$$\begin{aligned}\int_M h(\Delta\sigma + b\|D\sigma\|^2\sigma - nc_1\sigma, \sigma)\upsilon_g &= \\ &= \int_M \Big(\|D\sigma\|^2 + 2\varphi_2'(k)k\|D\sigma\|^2 + 2n\varphi_1'(k)e^{2(\varphi_1-\varphi_2)(k)}\Big)\upsilon_g \\ &= \int_M \Big(\big(1 + 2k\varphi_2'(k)\big)\|D\sigma\|^2 + 2n\varphi_1'(k)ke^{2(\varphi_1-\varphi_2)(k)}\Big)\upsilon_g,\end{aligned}$$

thus

$$2nk\varphi_1'(k) = 0, \tag{5.4.6}$$

and

$$\big(1 + 2k\varphi_2'(k)\big)\|D\sigma\|^2 = 0, \tag{5.4.7}$$

Under the assumptions on φ_1 and φ_2, equation (5.4.7) gives $\|D\sigma\| = 0$, and hence σ is parallel, then it has constant length $k = \|\sigma\|^2$. If $\varphi_1'(k) \neq 0$, then equation (5.4.6) implies that $k = 0$, and hence $\sigma \equiv 0$. ∎

Corollary 17. *Let σ be a parallel section with length $\|\sigma\|^2 = k(\neq 0)$ and assume that $(1 + 2k\varphi_2'(k)) \neq 0$, then σ is G-harmonic if and only if $\varphi_1'(k) = 0$.*

Proof: If σ is a parallel G-harmonic, then it must satisfy the equation

$$\Delta\sigma + (b\|D\sigma\|^2 - nc_1)k = 0, \tag{5.4.8}$$

thus $nc_1k = 0$, which implies that $\varphi_1'(k) = 0$. The converse follows also from Proposition 79. ∎

In what follows we shall explore the relationships between the solutions to the variational problems 1, 2 and 3.

Proposition 80. *Let σ be a smooth section of E, then σ is harmonic if and only it satisfies the following:*

1. *σ is G-harmonic,*
2. *$\tau(\sigma)^H = 0$.*

In particular, if D is flat, then a constant length section is harmonic if and only it is G-harmonic.

Proof: A section σ is harmonic if and only if $\tau(\sigma)^V = \tau(\sigma)^H = 0$. If D is flat, then the tensor S defined in Proposition 25 is identically vanishing, thus the expression of $\tau(\sigma)^H$, given in proposition 74, is reduced to

$$\tau(\sigma)^H = 2\sum_{i=1}^{n} ah(D_{E_i}\sigma, \sigma)E_i^h.$$

But for constant length sections, $h(D_{E_i}\sigma, \sigma) = 0$ for each i, hence $\tau(\sigma)^H = 0$, thus the equivalence between G-harmonicity and harmonicity. ∎

Proposition 81. *Assume that φ_1 is constant, then a section σ is G-harmonic if and only if it is vertically harmonic.*

Proof: Let σ be a constant length section, if σ is G-harmonic, then by virtue of equation (5.4.3) we have

$$\Delta\sigma + (b\|D\sigma\|^2 - nc_1)\sigma - 2bD_{\nabla F_\sigma}\sigma = 0. \tag{5.4.9}$$

So, since φ_1 is constant, which implies $c_1 = 0$, thus equation (5.4.9) is reduced to the equation

$$\Delta\sigma + b\|D\sigma\|^2\sigma - 2bD_{\nabla F_\sigma}\sigma = 0. \tag{5.4.10}$$

Hence, by virtue of equation (5.3.7), σ is vertically harmonic. The converse also follows from the same equation. ∎

5.5 On the stability of harmonic sections

In what follows, we assume that φ_1 is constant and we will compute the Jacobi operator of parallel sections. Long but routine computations give:

Proposition 82. *Assume that φ_1 is constant and let σ be a parallel section of E and $V \in \Gamma(\sigma^*TE)$, then*

$$\begin{aligned}
\mathcal{J}_\sigma V = \tilde{\Delta}_\sigma V + \frac{1}{2}\sum_{i=1}^{n} \Big(& \mathcal{R}^\xi(E_i^h, \pi^*\nabla_{E_i^h}V^H) + \pi^\star D_{E_i^h}\mathcal{R}^\xi(E_i^h, V^H) \\
& - \mathcal{R}^\xi((\nabla_{E_i}E_i)^h, V^H) + \mathcal{R}^\xi(E_i^h, S(E_i^h, V^V))\Big) \\
& - \sum_{i=1}^{n} \Big(S(E_i^h, \pi^\star D_{E_i^h}V^V) + \pi^*\nabla_{E_i^h}S(E_i^h, V^V) \\
& - S((\nabla_{E_i}E_i)^h, V^V) + S(E_i^h, \mathcal{R}^\xi(E_i^h, V^H))\Big) - \mathcal{R}_\sigma V,
\end{aligned}$$

where $V = V^H + V^V$ is the decomposition of V into its horizontal and vertical parts, and

$$\tilde{\Delta}_\sigma V = -\sum_{i=1}^{n} \big(\tilde{D}_{E_i^h}\tilde{D}_{E_i^h}V - \tilde{D}_{(\nabla_{E_i}E_i)^h}V\big),$$

with $\tilde{D} = \pi^*\nabla \oplus \pi^\star D$.

Corollary 18. *Assume that the connection D is flat, and let σ be a parallel section of E, then, for all $V \in \Gamma(\sigma^*TE)$, we have*

$$\mathcal{J}_\sigma V = \tilde{\Delta}_\sigma V - \sum_{i=1}^{n} \pi^* R(E_i^h, V^H)E_i^h,$$

where R is the curvature tensor of (M, g), and V^H is the horizontal part of V. In particular, if (M, g) is flat, then $\mathcal{J}_\sigma V = \tilde{\Delta}_\sigma V$ for all $V \in \Gamma(\sigma^*TE)$.

Proof: If D is flat, then by virtue of Proposition 25, we have $\mathcal{R}^\xi = 0$ and $S = 0$, which implies that

$$\mathcal{J}_\sigma V = \tilde{\Delta}_\sigma V - \mathcal{R}_\sigma(V).$$

On the other hand,

$$\begin{aligned}\mathcal{R}_\sigma V &= \sum_{i=1}^{n} \tilde{R}(E_i^h, V)E_i^h \\ &= \sum_{i=1}^{n} \big(\tilde{R}(E_i^h, V^H)E_i^h + \tilde{R}(E_i^h, V^V)E_i^h\big).\end{aligned}$$

Using curvature formulas given in section 2.2 of [13], and under the assumption $\varphi_1' = 0$, we have

i) $\tilde{R}(E_i^h, V^V)E_i^h\big) = 0$,

ii) $\tilde{R}(E_i^h, V^H)E_i^h = \pi^* R(E_i^h, V^H)E_i^h$,

from which the result follows. ∎

Theorem 30. *Assume that D is flat, and let σ be a parallel section. If (M, g) has non-positive sectional curvature, then σ is stable.*

Proof: Let σ be a parallel section. Since D is flat, we have $\tilde{\nabla} = \tilde{D} + C$, and since φ_1 is constant and σ is parallel, we have

$$\Delta_\sigma V = \tilde{\Delta}_\sigma V \tag{5.5.1}$$

and

$$\mathcal{R}_\sigma V = \sum_{i=1}^{n} \pi^* R(E_i^h, V^H)E_i^h. \tag{5.5.2}$$

If (M, g) has non positive curvature, then $-G(\mathcal{R}_\sigma V, V) \geq 0$, for all V, thus $I(V, V) \geq 0$, whence σ is stable. ∎

5.6 Some examples

5.6.1 Vector fields

Take $E = TM$, $D = \nabla$ the Levi-Civita connection of (M, g) and $h = g$, we have $\Gamma(E) = \mathfrak{X}(M)$. Let us consider φ_1 and φ_2 two smooth functions as before, and G the spherically symmetric metric with weight functions φ_1 and φ_2. If X is a constant length vector field on M with $\|X\|^2 = k$, then X defines a harmonic map from (M, g) to (TM, G) if and only if the following equations hold:

$$\Delta X + 2\big(n\varphi_1'(k)e^{2(\varphi_1-\varphi_2)(k)} + \varphi_2'(k)\|\nabla X\|^2\big)X = 0, \tag{5.6.1}$$

$$\sum_{i=1}^{n}\Big(S((\nabla_{E_i}X)^v, E_i^h)\Big) = 0; \tag{5.6.2}$$

where S is the tensor field given in Proposition 25, ΔX is the rough Laplacian of X with respect to ∇ and $\{E_i\}$ is a local orthonormal frame of M. Furthermore, X is G-harmonic if and only if it satisfies the following equation

$$\Delta X + 2\big(n\varphi_1'(k)e^{2(\varphi_1-\varphi_2)(k)} + \varphi_2'(k)\|\nabla X\|^2\big)X = 0. \tag{5.6.3}$$

On the other hand, X is vertically harmonic if and only if

$$\Delta X + 2\varphi_2'(k)\|\nabla X\|^2 X = 0, \tag{5.6.4}$$

Now, it is easy to see that any spherically symmetric metric on TM of the form (1.3.5) is a Riemannian g-natural metric, with defining functions (see [2,3])

$$\alpha_1 = e^{2\varphi_2}, \quad \alpha_3 = e^{2\varphi_1} - e^{2\varphi_2}, \quad \alpha_2 = \beta_2 = \beta_1 = \beta_3 = 0. \tag{5.6.5}$$

Substituting from (5.6.5) into (5.6.1)–(5.6.4), we recover the same equations found in [55].

5.6.2 Self-dual and anti-self-dual 2-forms on M

In this subsection, we assume that M is a compact Riemannian 4-manifold, and let $E = \wedge^2_{\pm}T^*M$, D be the connection induced from the Levi-Civita connection of (M, g), and let h be the metric fiber induced from g. Sections of E are self-dual and anti-self-dual 2-forms on M. When $\wedge_-$ is considered, M is assumed to be *a self-dual Einstein manifold*, and when $\wedge_+$ is considered, M is assumed to be an *anti-self-dual Einstein manifold*(see [13]). In [13], the author studied the spherically symmetric metric given by

$$G = \sqrt{2c_0^2 sr + d_0}\,\pi^* g \oplus \frac{c_0^2}{\sqrt{2c_0^2 sr + d_0}}\pi^\star h, \tag{5.6.6}$$

where $c_0, d_0 > 0$ are constants and $s = \dfrac{1}{12}Sc$, with Sc is the scalar curvature of M. More importantly, the author proved that (E, G) has G_2 holonomy. The spherically symmetric metric weights are

$$\varphi_1(r) = \frac{1}{4}\ln\big(2c_0^2 sr + d_0\big),\ \varphi_2(r) = \ln(c_0) - \frac{1}{4}\ln\big(2c_0^2 sr + d_0\big),$$

thus

$$\varphi_1'(r) = \frac{c_0^2 s}{2(2c_0^2 sr + d_0)},\ \varphi_2'(r) = -\frac{c_0^2 s}{2(2c_0^2 sr + d_0)}.$$

Thus, a self-dual (resp. anti-self-dual) 2-form ω is harmonic (as a section) if and only if it satisfies the following equations:

$$\Delta\omega + (ns - \frac{c_0^2 s}{(2c_0^2 sr + d_0)}\|D\omega\|^2)\omega + \frac{c_0^2 s}{(2c_0^2 sr + d_0)} D_{\nabla F_\omega}\omega = 0,$$
$$\sum_{i=1}^{n}\Big(\frac{c_0^2 s}{(2c_0^2 sr + d_0)} h(D_{E_i}\omega, \omega)E_i^h + S((D_{E_i}X)^v, E_i^h)\Big) = 0;$$

where S is the tensor field given in Proposition 25, $F_\omega = \frac{1}{2}\|\omega\|^2$, $\Delta\omega$ is the rough Laplacian of ω with respect to D and $\{E_i\}$ is a local orthonormal frame of M. Moreover, ω is G-harmonic if and only if it satisfies the equation

$$\Delta\omega + \Big(ns - \frac{c_0^2 s}{(2c_0^2 sr + d_0)}\|D\omega\|^2\Big)\omega + \frac{c_0^2 s}{(2c_0^2 sr + d_0)} D_{\nabla F_\omega}\omega = 0, \tag{5.6.7}$$

and ω is vertically harmonic if and only if it satisfies

$$\Delta\omega + \frac{c_0^2 s}{(2c_0^2 sr + d_0)}\Big(D_{\nabla F_\omega}\omega - \|D\omega\|^2\omega\Big) = 0. \tag{5.6.8}$$

In particular, if (M, g) is Ricci-flat, then ω is G-harmonic if and only if it is vertically harmonic if and only if it is harmonic in the sense of Hodge theory.

5.6.3 $(1,1)$-tensors

Take $E = TM \otimes T^*M$, $D = \nabla \otimes \nabla^*$ where ∇ is the Levi-Civita connection of M and ∇^* is the connection naturally induced from ∇ on T^*M, and take h the metric fiber defined as follows

$$h(T, S) = \sum_{i=1}^{n} g(T(E_i), S(E_i)),$$

for all $T, S \in \mathcal{T}_1^1(M)$, with $\{E_i\}$ is a local orthonormal frame of M. Clearly we have $Dh = 0$ (see [87]).

For instance, take φ_1 an arbitrary function that meats the regularity conditions of the definition of spherically symmetric metrics and $\varphi_2 = 0$, then, by virtue of equation (5.3.7), T is a vertically harmonic $(1,1)$-tensor if and only if

$$\Delta T = 0. \tag{5.6.9}$$

In particular, if M is an almost complex manifold and J is an almost complex structure on M, then J is a vertically harmonic almost complex structure if and only if it satisfies (5.6.9). Note here that "harmonicity" of J differs from that of almost complex structures in the sense of [84]. Furthermore, T defines a harmonic map from (M, g) to (E, G) if and only if it satisfies the following equations:

$$\Delta T + 2n\varphi_1'(r)e^{2\varphi_1}T = 0, \tag{5.6.10}$$

$$\sum_{i=1}^{n}\Big(S((D_{E_i}\sigma)^v, E_i^h)\Big) = 0. \tag{5.6.11}$$

Finally, T is G-harmonic if and only if it satisfies the equation

$$\Delta T + 2n\varphi_1'(k)e^{2(\varphi_1 - \varphi_2)(k)}T = 0. \tag{5.6.12}$$

Remark 21. In [49], the authors studied vertically harmonic sections of vector bundles when endowed with Cheeger-Gromoll type metrics. They proved that a section $\sigma : (M, g) \longrightarrow (E, G^{p,q})$ is vertically harmonic if and only if

$$T_p(\sigma) = \phi_{p,q}(\sigma)\sigma, \tag{5.6.13}$$

where

$$T_p(\sigma) = (1 + 2F_\sigma)\Delta\sigma + 2pD_{\nabla F_\sigma}\sigma, \tag{5.6.14}$$

$$\phi_{p,q}(\sigma) = p\|D\sigma\|^2 - pq\|\nabla F_\sigma\|^2 - q(1 + 2F_\sigma)\Delta_g F_\sigma, \tag{5.6.15}$$

with $F_\sigma = \frac{1}{2}\|\sigma\|^2$. On the other hand, by taking $\varphi_1 = 0$ and $\varphi_2(r) = -\frac{p}{2}\ln(1+r)$, we recover $(p, 0)$-metrics as spherically symmetric metrics. By virtue of (5.3.7), a section σ is vertically harmonic if and only if

$$\Delta\sigma + b\|D\sigma\|^2\sigma - 2bD_{\nabla F_\sigma}\sigma = 0, \tag{5.6.16}$$

where $b = 2\varphi_2'$, and since $\varphi_2(r) = -\frac{p}{2}\ln(1 + \|\sigma\|^2)$, we have $b = -\frac{p}{2(1+r)}$, thus equation (5.6.16) becomes

$$(1 + \|\sigma\|^2)\Delta\sigma - p\|D\sigma\|^2\sigma + pD_{\nabla F_\sigma}\sigma = 0. \tag{5.6.17}$$

which is exactly (5.6.13) for $q = 0$.

Part III

Appendices

Appendix A

Categories and functors

Definition 27. A *category* $\mathcal{C}$ consists of the following data

(i) A class of **object**s denoted by $Ob(\mathcal{C})$;

(ii) For every $A, B \in Ob(\mathcal{C})$, there is a collection $\mathrm{Hom}(A, B)$ whose elements are called **arrows** or **morphisms** from A to B.

(iii) For every $A, B, C \in Ob(\mathcal{C})$, there exists a map

$$\begin{aligned} \circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) &\longrightarrow \mathrm{Hom}(A, C) \\ (f, g) &\longmapsto g \circ f, \end{aligned}$$

called the **composition** map which satisfies:

1. **Associativity**: for every $A, B, C, D \in Ob(\mathcal{C})$, and for all $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$ and $h \in \mathrm{Hom}(C, D)$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

2. **Unit law**: for every $A \in Ob(\mathcal{C})$, there exists an element Id_A, such that $f \circ \mathrm{I}_A = f$ and $\mathrm{Id}_A \circ g = g$, for all $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, A)$.

Remarks 3. Given a category $\mathcal{C}$.

1. Arrows between $A, B \in Ob(\mathcal{C})$ are often denoted by $f : A \longrightarrow B$ or $A \xrightarrow{f} B$;

2. We adopt the convention that $\mathrm{Hom}(A, B) \cap \mathrm{Hom}(A', B') = \emptyset$ unless $A = A'$ and $B = B'$.

Most of the 'mathematical structures' can be written in the language of categories. Here are some classical examples.

Examples 8.

1. **Set** in which the objects are sets and morphisms are mappings.

2. **Gr** in which the objects are groups and the morphisms are group homomorphisms.

3. **Top** in which the objects are topological spaces and the morphisms are continuous maps.

4. **Vect** in which the objects are finite dimensional vector spaces and morphisms are linear mappings between them.

Definition 28. Let $\mathcal{C}$ be a category and let $A, B \in Ob(\mathcal{C})$. A morphism $f : A \longrightarrow B$ is said to be *invertible* if there exists a morphism $g : B \longrightarrow A$ such that $f \circ g = \mathrm{Id}_B$ and $g \circ f = \mathrm{Id}_A$. In this case g is called the *inverse*.

An invertible morphism in a category is called an *isomorphism*. Moreover, two objects of a category are said to be isomorphic if there exists an isomorphism between them.

Definition 29. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories. A *covariant functor* $\mathcal{F} : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ consists of

(i) A map $\mathcal{F} : Ob(\mathcal{C}_1) \longrightarrow Ob(\mathcal{C}_2)$;

(ii) A map $\mathcal{F} : \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(\mathcal{F}A, \mathcal{F}B)$, for every $A, B \in Ob(\mathcal{C}_1)$, which preserves composition and the identity element.

Example 10. The functor $\mathcal{F} : Top \longrightarrow Gr$ that assigns to each topological space its fundamental group $\pi_1(X)$ and to each continuous mapping its induced mapping between fundamental groups, this is a covariant functor.

There exist other functors which are not covariant.

Definition 30. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories, a contracovariant functor $\mathcal{F} : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ consists of

(i) A map $\mathcal{F} : Ob(\mathcal{C}_1) \longrightarrow Ob(\mathcal{C}_2)$;

(ii) A map $\mathcal{F} : \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(\mathcal{F}B, \mathcal{F}A)$, for every $A, B \in Ob(\mathcal{C}_1)$, which preserves composition and the identity element.

Example 11. The duality functor is the functor **Vect** $\longrightarrow$ **Vect** that assigns to each vector space its dual, and to each linear mapping between two vector spaces the dual mapping, this is a contravariant functor.

Accordingly, one can define bifunctors which are 'functor' with two arguments, or more generally functors with more than one argument. Their definition extend naturally that of functors.

Appendix B

Principl bundles and associted vector bundles

Though vector bundles are general objects which provide a unified language for coordinate-dependent formalisms both in differential geometry and physics, they form a subclass of a still more general class of objects, namely *fiber bundles*. Fiber bundles arise as a direct generalization of the concept of vector bundle. The idea is quite simple, instead of fibers endowed with a linear structure, one requires only a smooth manifold structure on them. Particularly, when fibers are diffeomorphic to Lie groups, another type of fiber bundles emerges, namely principal fiber bundles.

Interestingly, it turns out that there exists a deep link between the concept of a vector bundle and that of a principal fiber bundle. More precisely, we will see that to each principal bundle with structure group G and a G-module, there is an *associated vector bundle*. Conversely, to each vector bundle with standard fiber V, there is an associated principal bundle called the *bundle of frames*.

B.1 Fiber bundles and principal bundles

Definition 31. A *fiber bundle with standard fiber* S is a triple (E, π, M) where E, M and S are smooth manifolds and $\pi : E \longrightarrow M$ is a surjective map which is locally trivial; that is for every point $p \in M$, there exists a neighbourhood $U \subseteq M$ and a diffeomorphism $\psi : \pi^{-1}(U) \longrightarrow U \times S$ such that the following diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times S \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & U & \end{array}$$

is commutative, i.e. $pr_1 \circ \psi = \pi$, where pr_1 is the projection on the first factor.

Analogously, if (E, π, M) is a fiber bundle with standard fiber S, then E is called the *total space*, M is called the *base* and π is called the *projection* of the fiber bundle. We often refer to a fiber bundle by the total space E only. The pairs (U, ψ) are called *bundle charts*, and a set of bundle charts which cover M is called a *bundle atlas*. The local triviality implies that the projection $\pi : E \longrightarrow M$ is a surjective submersion. Hence, for all $x \in M$, the set $\pi^{-1}(x)$ is a submanifold of E called the *fiber* over x.

By analogy with the case of vector bundles, if (U, φ) and (V, ψ) are two bundle charts with $U \cap V \neq \emptyset$, then for all $p \in U \cap V$, the map $\widetilde{\varphi} \circ \left(\widetilde{\psi}_{|E_p}\right)^{-1}$ is a diffeomorphism of F, where

$\widetilde{\varphi} = pr_2 \circ \varphi$, $\widetilde{\psi} = pr_2 \circ \psi$ and pr_2 is the projection on the second factor. This gives rise to a map

$$f_{\varphi,\psi} : U \cap V \longrightarrow \mathrm{Diff}(S), \quad p \longmapsto \widetilde{\varphi} \circ \left(\widetilde{\psi}_{|E_p}\right)^{-1},$$

with

$$(\varphi \circ \psi^{-1})(x, e) = (x, f_{\varphi\psi}(x)(e))$$

Further, the maps $f_{\varphi,\psi}$ are smooth (c.f [35]) and they satisfy the cocycle condition. The reconstruction process is easily generalized to the framework of fiber bundles. For the details, we refer to [35,53,65].

Examples 9. Fiber bundles are abundant. Here are some examples

1. **The trivial fiber bundle**: Let M and S be two smooth manifolds and $\pi : M \times S \longrightarrow M$ be the projection on the first factor. Hence (E, π, M) is a fiber bundle with standard fiber S.

2. **All vector bundles**.

3. **The Möbius strip:** The circle S^1 is the quotient $\mathbb{R}/\sim$, where for $x \in \mathbb{R}$, $x \sim x + 2n\pi$, for all $n \in \mathbb{Z}$. Define the equivalence relation $\simeq$ on $\mathbb{R} \times (-1, 1)$ by $(x, t) \simeq (x + 2n\pi, (-1)^n t)$, for all $n \in \mathbb{Z}$, and set $E = \mathbb{R} \times (-1, 1)/\simeq$. The result is a fiber bundle called the *Möbius strip* with the projection $\pi : [x, t]_{\simeq} \mapsto [x]_{\sim}$ where the brackets denote the respective equivalence classes.

Sections of fiber bundles are defined analogously to the case of vector bundles. Moreover, sections of a fiber bundle are closely related to its local trivializations.

Definition 32. Let (E, π, M) be a fiber bundle. A *section* of E is a map $\sigma : M \longrightarrow E$ such that $\pi \circ \sigma = \mathrm{Id}_M$.

Local sections are defined in an obvious way. If E is a fiber bundle, then the space of sections of E is denoted $\Gamma(E)$.

The Lie group $\mathrm{Diff}(S)$ is very large, so there is no *a priori* preferable choice of a bundle atlas and hence no preferable cocycle of transition functions. Besides, for a fiber bundle to be of use in geometric problems, one should restrict to special transformations rather than diffeomorphisms of the standard fiber, that is to a subgroup of the group $\mathrm{Diff}(S)$, this is generally accomplished through Lie group of transformations of the standard fiber. The following definitions and constructions deal with that kind of issues.

Let G be a Lie group and S be a smooth manifold, we say that G *acts on the left on* S if there exists a smooth map $\mu : G \times S \longrightarrow S$ such that

(i) $\mu(e, s) = s$, for all $s \in S$, where e is the identity element of G,

(ii) $\mu(gh, s) = \mu(g, \mu(h, s))$, for all $g, h \in G$ and $s \in S$.

In this case, we say that G is a *Lie transformation group* of S and S is said to be a *G-manifold*. An action of G on S may be defined as a group monomorphism $\mu : G \longrightarrow \mathrm{Diff}(S)$.

Actions on the right of S are defined analogously up to a reversing of the elements' order.

Definition 33. Let G be a Lie group acting on the standard fiber S on the left with an action μ. Two bundle charts (U, φ) and (V, ψ) are said to be *G-compatible* if $U \cap V = \emptyset$ or $U \cap V \neq \emptyset$ and there exists a smooth $g : U \cap V \longrightarrow G$ such that $f_{\varphi,\psi}(x) = \mu_{g(x)}$.

The transition functions of two G-compatible bundle charts may be seen as taking values in G which yields the identification $f_{\varphi,\psi}(x)(s) = \mu(f_{\varphi,\psi}(x), s)$, for all $x \in U \cap V$ and $s \in S$.

Definition 34. Let G be a Lie group. A fiber bundle (E, π, M) with standard fiber S is said to be a *fiber bundle with structure group* G or simply a *G-bundle* if G is a Lie group of transformations of S and there is given a bundle atlas formed by pairwise G-compatible bundle charts. Such an atlas is called a *G-bundle atlas*. A bundle chart is said to be *admissible* as a bundle chart for a bundle with structure group G if it is G-compatible with every chart in the given G-bundle atlas.

Remark 22. There may be more than one possible choice of a structure group, but we still refer to a choice among them as *the* structure group. To overcome such a possible ambiguity, one may use the terminology G-bundle to mean a fiber bundle with structure group G. When a structure group is specified, the choice of admissible charts is reduced. Such a process is called the *reduction of the structure group*.

Vector bundles are particular cases of fiber bundles. Now, given a fiber bundle E which has a vector space as a standard fiber, since $\mathrm{GL}(V)$ acts canonically on V, this leads to the following link between fiber bundles and vector bundles.

Proposition 83. *Let V be a vector space. A fiber bundle (E, π, M) with standard fiber V is a vector bundle with standard fiber V if and only if E admits $\mathrm{GL}(V)$ as a structure group, that is E has a reduction of structure group to $\mathrm{GL}(V)$.*

Theorem 31. *([39]) The tangent bundle to a Riemannian manifold (M, g) of dimension n has the structure of an $\mathrm{O}(n)$-bundle. Furthermore, if M is oriented, then TM is an $\mathrm{SO}(n)$-bundle.*

A special class of fiber bundles arises when the standard fiber is a Lie group. Since a Lie group G acts canonically on itself by left translations, one requires such a fiber bundle to have G itself as the structure group. This leads to the notion of a principal fiber bundle or principal G-bundle.

A right action μ on a manifold P is said to be *free* if the identity element is the only element of G which fixes a point of P, that is if $\mu(p, g) = p$ for some $p \in P$, then $g = e$, where e is the identity element of G. Moreover, given two right (resp. left) G-manifolds X and Y, a map $\phi : X \longrightarrow Y$ is said to be *G-equivariant* if $\phi(x.g) = \phi(x).g$ (resp. $\phi(g.x) = g.\phi(x)$), for all $x \in X$ and $y \in Y$.

Definition 35. Let (P, π, M) be a fiber bundle for which the standard fiber is a Lie group G. The bundle P is said to be a *principal G-bundle* if G acts freely on the right of P and there is a bundle atlas $\mathcal{A}$ on P such that for each bundle chart $(U, \psi) \in \mathcal{A}$, we have

$$\widetilde{\psi}(p.g) = \widetilde{\psi}(p).g, \quad \text{for all } p \in P,\ g \in G,$$

where $\widetilde{\psi} = pr_2 \circ \psi$. The bundle atlas $\mathcal{A}$ is called a *principal bundle atlas* or simply a *principal G-bundle atlas*.

Additionally, the charts belonging to a principal G-bundle atlas are referred to as *principal G-bundle charts*.

If (P, π, M) is a principal G-bundle, then there exists a natural action of the structure group G on P defined as follows: for all $p \in P$ such that $x = \pi(p)$, and let (U, ψ) be a principal G-bundle chart around x, then for all $g \in G$, set $p.g = \psi^{-1}(x, h).g = \psi^{-1}(x, h.g)$. This right action is well defined by virtue of the compatibility of charts, and it is called the *principal action* of G on P.

Remarks 4. Let (P, π, M) be a principal G-bundle.

(i) It follows from the definition of principal bundles that the fibers are exactly the orbits of the principal action.

(ii) If (U, φ) and (V, ψ) are two bundle charts belonging to the principal bundle atlas such that $U \cap V \neq \emptyset$, then the transition function satisfies $f_{\varphi,\psi}(p) = \widetilde{\varphi}(b)\widetilde{\psi}(b)^{-1} \in G$ and it depends only on the base point. Again the collection of transition functions satisfies the cocycle condition which takes values in the structure group G.

(iii) The reconstruction process holds for principal bundles. More precisely, given a Lie group G and a manifold M together with a open cover $\{U_\alpha\}$ and a cocycle of transition functions $(\psi_{\alpha\beta})$ with $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$, then consider the space $X = \bigcup_\alpha \{\alpha\} \times U_\alpha \times G$ endowed with the equivalence relation defined by: $(\alpha, x, g) \sim (\beta, y, h)$ if and only if $x = y$ and $h = \psi_{\alpha\beta}(x)(g)$. One checks that (P, π, M) is a principal G-bundle, where $P = X/\sim$ and $\pi : [\alpha, x, p] \mapsto x$. For a detailed proof, we refer the reader to [35, 42, 59].

Principal bundles may be defined directly without relying on a reduction of structural group. The following definition is of great technical use, and it is more suitable for practical purposes.

Definition 36. Let M be a smooth manifold and G be a Lie group. A smooth manifold P is said to be a *principal bundle* if the following hold:

(i) G acts freely on P on the right;

(ii) M is the quotient P/G and the canonical projection $\pi : P \longrightarrow M$ is smooth;

(iii) P (or equivalently π) is locally trivial: that is every point $x \in M$ has a neighbourhood U and there is a diffeomorphism $\psi : \pi^{-1}(U) \longrightarrow U \times G$ such that $\psi = (\pi, \phi)$ with $\phi : \pi^{-1}(U) \longrightarrow G$ is a smooth map satisfying $\phi(p.a) = \phi(p).a$, for every $a \in G$ and $p \in \pi^{-1}(U)$.

In this case, M is the *base*, P the *total space*, G the *structure group* and π the *projection*.

Examples 10.

1. **The trivial principal G-bundle:** let M be a manifold and G be a Lie group. Consider $P = M \times G$ and let $\pi : (x, g) \longmapsto x$ be the projection on the first factor of P, then (P, π, M) is a principal G-bundle.

2. **Homogeneous spaces:** let G be a Lie group and K be a closed subgroup of G. The Lie subgroup K acts on G by right translation and the set G/K has the structure of smooth manifold for which $\pi : G \longrightarrow G/K$. Then $(G, \pi, G/K)$ is a principal K-bundle.

3. **The bundle of linear frames:** let M be an n-dimensional manifold and $x \in M$. Consider
$$L_x M = \{\gamma_x : T_x M \longrightarrow \mathbb{R} : \ \gamma_x \text{ is an isomorphism}\}.$$
Those spaces combine into $LM = \sqcup_{x \in M} L_x M$. Let $\pi : LM \longrightarrow M$ be the map which sends all of $L_x M$ to x. The general linear group $GL_n(\mathbb{R})$ acts on LM by compositions. Thus (LM, π, M) is a principal $GL_n(\mathbb{R})$-bundle.

Motivated by the search of smaller structure groups, reduction of structure groups of principal bundles is extremely important, for the simple reason that a lot of geometric structures arise as reductions of structure groups for particular principal fiber bundles. The following proposition characterizes such operations. For a detailed proof, we refer to [35, 42].

Proposition 84. *Let (P, π, M) be a principal G-bundle and let G' be a Lie subgroup of G, then P admits a reduction of structure group of G to G' if and only if M has an open covering $\{U_\alpha\}$ together with a cocycle of transition functions which take their values in G'.*

Principal bundles are distinguished by the fact that fibers are exactly the orbits of the principal right action, hence it is natural to require that morphisms of principal bundles must preserve the action, that is an equivariance with respect to the respective principal actions.

Definition 37. Let (P_i, π_i, M_i) be principal G_i-bundles, for $i = 1, 2$. A *principal bundle homomorphism* is a map $\Phi : P_1 \longrightarrow P_2$ such that there exists a Lie group homomorphism $\rho : G_1 \longrightarrow G_2$ for which $\Phi(u.g) = \Phi(u).\rho(g)$, for all $u \in P_1$ and $g \in G_1$.

Sections of principal bundles are very important. If σ is a section of P, then the map $\varphi : M \times G \longrightarrow P$ defined by $\varphi(x, g) = \sigma(x).g$ is a fiber preserving diffeomorphism. Furthermore, φ is a principal bundle isomorphism. Hence the existence of a global section for a principal bundle is equivalent to being trivial. Moreover, giving a local section or a local trivialization are equivalent.

B.2 Associated vector bundles

As mentioned above, there exists an important link between principal bundles and vector bundles. Indeed, to each principal bundle, there are associated bundles and associated vector bundles.

Let (P, π, M) be a principal G-bundle and S be a smooth manifold. Assume that G acts on the left on S via an action l, then G acts on the product $P \times F$ on the right as follows

$$g.(p, s) = (p.g, g^{-1}.s),$$

for all $(p, s) \in P \times S$. Denote by $\sim$ the resulting equivalence relation and denote by $[p, s]$ the equivalence class of $(p, s) \in P \times S$ with respect to this relation, and denote the quotient space by $E := P \times_G M$. The natural projection $q : P \times S \longrightarrow E$ together with the projection π induce the map $\pi_E : P \times_G M \longrightarrow M$ defined by $\pi_E([p, s]) = \pi(p)$, for all $(p, s) \in P \times S$.

The local triviality of E follows from that of P. Indeed, each point $x \in M$ has a neighbourhood U such that $\pi^{-1}(U)$ is diffeomorphic to U via ψ, where $\psi = (\pi, \phi)$ such that $\phi : \pi^{-1}(U) \longrightarrow G$ is G-equivariant. Identifying $\pi^{-1}(U)$ with $U \times G$, the action of G on the right on $\pi^{-1}(U) \times S$ takes the form

$$(x, a, s).g = (x, a.g, s.g),$$

for all $(x, a, s) \in U \times G \times S$ and $g \in G$. This gives a bijection between $\pi_E^{-1}(U)$ and $U \times S$. Define a smooth structure on E by requiring $\pi_E^{-1}(U)$ to be open. Further, we have

Proposition 85. *(see [35, 42, 53]) The triple (E, π_E, M) is a fiber bundle with standard fiber S and structure group G. It is denoted by $P[S, l]$.*

A fiber bundle constructed as above is said to be an *associated fiber bundle*. An application of particular interest arise when the standard fiber is taken to be a vector space.

Precisely, let (P, π, M) be a principal bundle and let V be a G-module, that is G admits a linear representation $\varrho : G \longrightarrow GL(V)$, then the associated bundle with standard fiber V and structure group G has a G-bundle structure, hence a structure of $GL(V)$-bundle. Thus, by virtue of Proposition 83, the resulting bundle is in fact a vector bundle with standard fiber V and structure group G that we denote by $P[V, \varrho]$, and we call the *associated vector bundle* to P with standard fiber V.

Remark 23. All vector bundles seen so far may be obtained as associated vector bundles to special principal fiber bundles. Indeed, let $\mathcal{F}$ be a smooth functor from the category of finite dimensional vector spaces to itself. If G is a Lie group, V is a G-module and $\varrho : G \longrightarrow GL(V)$ is a linear representation of G on V, then $\mathcal{F} \circ \varrho : G \longrightarrow GL(\mathcal{F}V)$ is a linear representation of

G on $GL(V)$. The corresponding technique applies to contravariant and bifunctor after the necessary reordering and modifications. Hence we obtain the operations on vector bundles described earlier.

Conversely, to each vector bundle (E, π_E, M), a principal bundle could be associated. Indeed, let V be the standard fiber of the vector bundle E and let $x \in M$, then for any isomorphism $\phi : V \longrightarrow E_x$, the map $g \longmapsto g \circ \phi$ is a bijective map from $GL(V)$ to the $P(E_x)$ of all isomorphisms from V to E_x. Define the set

$$P(E) = \bigsqcup_{x \in M} P(E_x),$$

and let $\pi : P(E) \longrightarrow M$ the map that satisfies $\pi(P(E_x)) = x$, for all $x \in M$.

Proposition 86. *([65])Let (E, π_E, M) be a vector bundle with standard fiber V, then the triple $(P(E), \pi, M)$ is a principal $GL(V)$-bundle.*

Let (E, π, M) be a vector bundle with standard fiber V and let $P(E)$ be the principal $GL(V)$-bundle of frames of E. The natural left action of $GL(V)$ on V gives rise to the associated vector bundle $P(E)[V]$ which is easily seen to be isomorphic to E. It turns out that every vector bundle arises as an associated bundle with its bundle of frames. For details see [53, 65].

We conclude that vector bundle theory may be entirely presented in the language of associated vector bundles to principal fiber bundles. This approach has the advantages of shading light on the link between the different kinds of fiber bundles and allowing a description of connection theory on vector bundles using the framework of Ehresmann connections.

Appendix C

Natural vector bundles

APPENDIX C. NATURAL VECTOR BUNDLESAPPENDIX C. NATURAL VECTOR BUNDLES

Many known examples of vector bundles belong to the class of natural vector bundles which we aim to review here. Natural vector bundles arise as functors on the category of smooth manifolds. Moreover, finiteness theorems allows a characterization of natural vector bundles as associated vector bundles to the bundles of higher order frames.

C.1 Jet manifolds and differential groups

We will present here a brief account of the theory of jets and jet manifolds, and we will also recall some facts about the differential groups and their basic properties.

Let M and N be two manifolds, $(x, y) \in M \times N$ and r be a positive integer. Consider the set

$$C^\infty(x, y) = \{f : M \longrightarrow N : \ f \, is \, \text{smooth and} \ \ f(x) = y\}.$$

Definition 38. Let $x \in M$, $y \in N$ and $f_1, f_2 \in C^\infty(x, y)$. We say that f_1 and f_2 are *r-equivalent* if there is a chart (U, φ) at x and a chart (V, ψ) at y such that

$$D^p(\psi f_1 \varphi^{-1})(\varphi(x)) = D^p(\psi f_2 \varphi^{-1})(\varphi(x)),$$

for each $0 \leq p \leq r$.

It is obvious that the relation of r-equivalence is an equivalence relation on $C^\infty(x, y)$.

Definition 39. An equivalence class for this relation is called an *r-jet with source x and target y*, and it is denoted by $j^r_x f$ for some $f \in C^\infty(x, y)$. The space of r-jets with source x and target y defined to be the quotient space of $C^\infty(x, y)$ by the relation of r-equivalence and denoted $J^r_{(x,y)}(M, N)$.

The r-jets with all possible sources and all possible targets combine into the set

$$J^r(M, N) = \bigcup_{x \in M, y \in N} J^r_{(x,y)}(M, N).$$

For $0 \leq s \leq r$ and $p = J^r_x f \in J^r(M, N)$, set $\pi^{r,s}(p) = j^s_x f$. It is clear that $\pi^{r,s}$ is a well defined surjective map from $J^r(M, N)$ onto $J^s(M, N)$. For $s = 0$, the set $J^0(M, N)$ is canonically identified with $M \times N$. This gives the maps $\pi^r = \pi^{r,0}$, $\pi^r_M = pr_1 \circ \pi^r$ and $\pi^r_N = pr_2 \circ \pi^r$, where pr_1 and pr_2 are the projections of $M \times N$ onto M and N respectively. Those mappings are called *canonical jet projections* [35, 47, 53].

Definition 40. Let M, N and Q are three smooth manifolds, then two r-jets $t \in J^r_x(M,N)$ and $s \in J^r_y(N,Q)$ are *composable* if the target of t is the source of s i.e. given $t = j^r_x f$ and $s = j^r_y g$, then t and s are composable if $f(x) = y$. In this case we set $s \circ t = j^r_x(g \circ f)$, which is an r-jet called the *composite of t and s*.

The compositions of jets combine to give the mapping

$$J^r_{(x,y)}(M,N) \times J^r_{(y,z)}(N,Q) \longrightarrow J^r_{(x,z)}(M,Q) \quad \text{with} \quad (s,t) \mapsto t \circ s,$$

which is called the *composition mapping*.

Definition 41. If $\dim(M) = \dim(N)$. An r-jet $s \in J^r_{(x,y)}(M,N)$ is said to be *invertible* if there exists an r-jet $t \in J^r_{(y,x)}(N,M)$ such that $t \circ s = j^r_x \mathrm{Id}_M$, where Id_M is the identity mapping of M.

By virtue of the *inverse function theorem*, an r-jet with source x and target y is invertible if and only it has a representative which is a diffeomorphism on a neighbourhood of the point x onto a neighbourhood of the point y. If an r-jet $s \in J^r_{(x,y)}(M,N)$ is invertible, then there is a unique element satisfying the condition of the previous definition. This element will be called the *inverse* of s and will be denoted by s^{-1}.

Now, we shall restrict ourselves to the Euclidean space $\mathbb{R}^n$. Using Taylor's expansion, the space $J^r_{(0,0)}(\mathbb{R}^n,\mathbb{R}^n)$ is canonically identified with the product

$$L^r_{n,n} = L(\mathbb{R}^n,\mathbb{R}^n) \times L^2_s(\mathbb{R}^n,\mathbb{R}^n) \times ... \times L^r_s(\mathbb{R}^n,\mathbb{R}^n),$$

where $L^m_s(\mathbb{R}^n,\mathbb{R}^n)$ denotes the vector space of m-linear, symmetric mappings of $\mathbb{R}^n \times ... \times \mathbb{R}^n$ into $\mathbb{R}^n$.

The space $L^r_{n,n}$ can be endowed with a smooth structure using the coordinate functions defined as follows. For $A \in L^r_{n,n}$, $A = j^r_0\alpha$ with $\alpha = (\alpha^i) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, set

$$a^i_{j_1 j_2 ... j_k}(A) = D_{j_1} D_{j_2} ... D_{j_k} \alpha^i(0),$$

for $1 \leq i \leq n$, $1 \leq j_1 \leq j_2 \leq ... \leq j_k \leq n$ and $1 \leq k \leq r$, where D_j denotes the partial derivative with respect to x_j. It is classically known that those functions define a global coordinate chart on $L^r_{n,n}$ (see [35,47]).

Consider the subset $L^r_n \subseteq L^r_{n,n}$ which consists of all invertible r-jets in $L^r_{n,n}$. It is straightforward to check that

$$L^r_n = L^r_{n,n} \setminus \{A \in L^r_{n,n} : \det(a^i_j(A)) = 0\}.$$

Thus L^r_n is an open set in $L^r_{n,n}$ and a group for the composition of jets. As a matter of fact L^r_n is endowed with the structure of an open submanifold of $L^r_{n,n}$, hence it's a Lie group.

Definition 42. The Lie group L^r_n is called the *r-th differential group*.

The differential groups appear in different contexts. For example L^1_n is nothing but the general linear group $GL_n(\mathbb{R})$, which happens to be the structure group of the principal bundle of linear frames. In order to uncover the deepness differential groups, let's look at those of higher order. To do so, we shall define bundles of higher order frames which are generalizations of the bundle of linear frames. Those bundles are principal L^r_n-bundles.

Definition 43. Let M be an n-dimensional manifold and let $x \in M$. An *r-frame* at the point x is an invertible r-jet $\zeta \in J^r_{(0,x)}(\mathbb{R}^n, X)$. The set of r-frames at x is denoted by F^r_x.

Consider the set

$$F^rM = \bigsqcup_x F^r_x M,$$

which is the set of all r-frames of M. Denote π^r_M the natural projection which sends $F^r_x M$ to x.

A coordinate system (U, φ) of the manifold M induces a coordinate system of F^rM as follows. Take $W^r = (\pi^r_M)^{-1}(U)$, let $\zeta = J^r_0\mu$ with $\mu(0) = x$, and set

$$\varphi^r(\zeta) = \big(\varphi(\mu(0)), J^r_0(t_{\varphi(x)} \circ \varphi \circ \mu)\big),$$

where $t_{\varphi(x)} : z \longmapsto z - \varphi(x)$ is a translation in $\mathbb{R}^n$. It is not hard to see that $\varphi^r : W^r \longrightarrow \varphi(U) \times L^r_n$ is a bijection. Hence, F^rM has a unique structure of a smooth manifold for which the pairs (W^r, φ^r) are charts on F^rM.

Moreover, the differential group L^r_n acts on the right on F^rM via the composition of jets. More precisely set

$$\zeta.A = \zeta \circ A,$$

for all $\zeta \in F^rM$ and $A \in L^r_n$. It is obvious that this action is free. Moreover

Proposition 87. *([47]) The manifold F^rM is a principal bundle over M with structure group L^r_n and projection π^r_M.*

The principal L^r_n-bundle F^rM is called the *bundle of r-frames* on M. The bundles F^rM, for $r \geq 2$, are called *bundles of higher order frames.*

C.2 Bundle functors

Let $\mathcal{M}f_n$ denote the category of n-dimensional manifolds and local diffeomorphisms between them and $\mathcal{FM}$ the category of *fibred manifolds* and their morphisms. See [35] for the definition and properties of fibred manifolds and their morphisms.

Definition 44. A natural vector bundle or a vector bundle functor F is a covariant functor which associates a vector bundle $(F(M), \pi_M, M)$ to each n-dimensional manifold M and a vector bundle morphism

$$\begin{array}{ccc} F(M) & \xrightarrow{F(f)} & F(N) \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f} & N \end{array}$$

to each morphism $f : M \longrightarrow N$ in $\mathcal{M}f_n$, which covers f and which is a fiberwise linear isomorphism, such that the following *continuity condition* holds: for every smooth mapping $f : \mathbb{R} \times M \to M$, the mapping $\mathbb{R} \times F(M) \ni (t, x) \mapsto F(f_t)(x)$ is also smooth.

Remarks 5.

(i) The continuity condition means that F maps smoothly parametrized families in the category $\mathcal{M}f_n$ to parametrized families of vector bundles with their morphisms.

(ii) Contravariant vector bundle functors are defined in the same manner with a slight modification of the arrows.

Examples 11. We have the following well known examples of natural vector bundles:

1. The tangent bundle functor T, which may be defined as a functor on the whole category of smooth manifolds with smooth maps as arrows.

2. The cotangent bundle Functor T^*.
3. The vector bundle of skew symmetric k-tensors $\Lambda^k T^*$.
4. The tensor bundles functor $\bigotimes^k T \otimes \bigotimes^l T^*$.
5. The vector bundles $\mathcal{F}(TM)$ where $\mathcal{F}$ is any smooth functor defined on the category of finite dimensional real spaces as given in Proposition (4).

Natural vector bundles are particular cases of a more general class of fiber bundle functors [35].

Definition 45. A *natural bundle* or *bundle functor* is a covariant functor $F : \mathcal{M}f_n \longrightarrow \mathcal{FM}$ that associates to each manifold M a fiber bundle $(F(M), p_M, M)$, satisfying the following conditions:

1. (Prolongation) $B \circ F = Id_{\mathcal{M}f_n}$, where $B : \mathcal{FM} \longrightarrow \mathcal{M}f_n$ is the base functor.
2. (Locality) If $i : U \longrightarrow M$ is the inclusion of an open submanifold, then $F(U) = \pi_M^{-1}(U)$ and $F(i)$ is just the inclusion of $\pi_M^{-1}(U)$ into FM.
3. (Regularity) If $f : P \times M \longrightarrow N$ is a smooth mapping such that for all $p \in P$ the maps $f_p = f(p,) : M \longrightarrow M$ are local diffeomorphisms, then $\tilde{F}f : P \times F(M) \longrightarrow F(N)$ defined by $\tilde{F}f(p,.) = Ff_p, p \in P$ are smooth.

Definition 46. A natural bundle $F : \mathcal{M}f_n \longrightarrow \mathcal{FM}$ is said to be *of finite order* r, with $0 \leq r < \infty$, if for all local diffeomorphisms $f, g : M \longrightarrow N$ and every point $x \in M$, the equality $j_x^r f = j_x^r g$ implies $F(f)_{|F_x M} = F(g)_{|F_x M}$.

Remark 24. The order of a vector bundle functor is defined exactly in the same way, that is its order as a particular case of fiber bundle functors.

The classification problem of natural bundles leads to the striking fact that all finite order bundle functors arise as associated bundles to the principal bundle of high order frames. Among the most important *finiteness theorems* and hence classification results, we find the spectacular theorem due to R. Palais and Terng [63] (see also [35]).

If F is a natural vector bundle, let $F_0\mathbb{R}^n$ denote the value of $\mathbb{R}^n$ under F over 0. $F_0\mathbb{R}^n$ is called the *standard fiber of* F.

Theorem 32. *([35]) Let $F : \mathcal{M}f_n \longrightarrow \mathcal{FM}$ be a natural bundle of order r, then for every m-dimensional manifold M, there exists a canonical structure of associated bundle $F_S^r M$ to the r-frame bundle, where $S = F_0\mathbb{R}^m$ is the standard fiber of F.*

Since every natural vector bundle on $\mathcal{M}f_n$ is just a natural bundle F with a canonical structure of vector bundle on each value $\mathcal{FM}$ and the values $F(f)$ of morphisms of $\mathcal{M}f_n$ are morphisms of vector bundles, the study of finite order natural vector bundles may be considerably reduced to that state of affairs. In fact the study of vector bundles may be envisaged exclusively through the formalism of associated bundles.

Appendix D

Principal connections and induced connections

The theory of connections on vector bundles emerged as the fruit of the endeavours of many renewed great geometers. In reality, it is hard to track back the origin of the notion, but we owe Riemann, Christoffel, Ricci, Levi-Civita, Heisenberg, Weyl and Schouten the foundations of this central geometric concept and related concepts of parallel transport. The idea of a connection was so charming that it attracts the attention of geometers. Consequently, many attempts to generalize this notions have been made, and many of those attempts have achieved a universal success.

Among those who succeeded were two masters of the French geometric school: Elie Cartan and Charles Ehresmann. The notion of a connection on a principal fiber bundle and generally on a fiber bundle was coined by Charles Ehresmann in his theory of *espaces généralisés*. But it was E. Cartan who introduced the concept of connections on principal bundles through his method of *moving frames*. Nevertheless, we owe Ehresmann the modern formulation of the concept of a connection in a general fiber bundle.

How the theories of connections are related to each other? To give some partial answers to this question is the aim of this short appendix.

D.1 Connections on principal bundles

Let (P, π, M) be a principal G-bundle. We denote by $\mu_g = \mu(., g)$, for $g \in G$, the principal right action of G on P, and by $\mathfrak{g}$ the Lie algebra of G. At each point $p \in P$, the vertical subspace $\ker(d_p\pi)$ of T_pP is denoted by $V_p(P)$. Denote by Ad the adjoint representation of G on $\mathfrak{g}$.

Definition 47. A *connection* Γ in P is an assignment of a subspace $H_p(P)$ of T_pP to each $p \in P$, such that:

1. $T_pP = V_pP \oplus H_pP$,
2. $(\mu_g)_*(H_pP) = H_{p.g}P$,
3. H_pP depends smoothly on p.

Smoothness in condition 3. is smoothness in the sense of distributions (c.f [79]). The resulting distribution is generally called the *horizontal distribution*. Condition 2. means that the horizontal distribution is invariant under the action of G. A connection on a principal fiber bundle is referred to as a *principal connection* (c.f [35, 53]), but we shall refer to those objects as connections when no confusion may arise.

At each $p \in P$, a vector $X \in T_pP$ is said to be *vertical* (resp. *horizontal*) if it lies in V_pP (resp. H_pP). Vertical (resp. horizontal) vector fields are defined in an obvious way. Every vector $X \in T_pP$ can be uniquely decomposed as

$$X = Y + Z,$$

where $Y \in V_pP$ and $Z \in H_pP$. In this case Y and Z are called the *vertical* (resp. *horizontal*) *component*. Vector fields decompose accordingly.

Each $A \in \mathfrak{g}$ generate a vector field on P in the following way. For every $p \in P$, $\zeta_A(p) = d_e\mu(p,)\big(A\big)$, it is a vector field on P called the *fundamental vector fields* on P generated by A with respect to the principal action of G on P. Furthermore, the map $A \mapsto \zeta_A(p)$ defines a linear isomorphism between $\mathfrak{g}$ and V_pP.

We shall show the equivalence between connections on P and a class of $\mathfrak{g}$-valued 1-forms. Given V a G-module, that is a linear representation $\rho : G \longrightarrow GL(V)$ of G on V. A *pseudotensorial form* φ of degree r on P of type (ρ, V) is a V-valued r-form φ on P such that

$$(\mu_g)_*\varphi = \rho(g^{-1}).\varphi, \quad \text{for all } g \in G.$$

When $V = \mathfrak{g}$ and $\rho = Ad$, we speak of pseudotensorial forms of type $Ad(G)$. A pseudotensorial form φ is said to be a *tensorial form* if it is horizontal in the sense that $\varphi(X_1, ..., X_r) = 0$ whenever one of the X_i's is vertical.

Given a connection Γ on P, we define a $\mathfrak{g}$-valued 1-form on P as follows: for each $X \in T_pP$, $\omega(X)$ is the unique $A \in \mathfrak{g}$ such that $\zeta_A(p)$ is the vertical component of X. Clearly, $\omega(X) = 0$ if and only if X is horizontal. The 1-form ω is called the *connection form* of the connection Γ.

Proposition 88. *(see [42]) The connection form ω of a connection satisfies the following conditions:*

(a) $\omega(\zeta_A) = A$, for all $A \in \mathfrak{g}$;

(b) $(\mu_g)_*\omega = Ad(g^{-1})\omega$, that is $\omega((R_g)_*X) = Ad(g^{-1})\omega(X)$ for every $g \in G$ and every vector field X on P.

Conversely, given a $\mathfrak{g}$-valued 1-form ω on P satisfying conditions (a) and (b), there exists a unique connection Γ in P whose connection form is ω.

Hence giving a connection on P is equivalent to the specification of a $\mathfrak{g}$-valued 1-form satisfying conditions (a) and (b) of Proposition 88. Condition (a) means that ω is a pseudotensorial of type $(Ad, \mathfrak{g})$.

Given a connection Γ on P, the projection $\pi : P \longrightarrow M$ induces linear isomorphisms $\pi_* : H_pP \longrightarrow T_{\pi(p)}M$. Those isomorphisms allows one to 'lift' tangent vectors and vector fields on the base to horizontal vectors and vector fields on P. The *horizontal lift* of a vector field X on M is the unique vector field X^h on P which is horizontal and which is π-related to X, i.e. $\pi_*X^h = X \circ \pi$.

Proposition 89. *Given a connection in P and let X be a vector field on M, there is a unique horizontal lift X^h of X. The horizontal lift X^h is invariant by μ_g for all $g \in G$. Conversely, every horizontal vector field on P invariant by G is the horizontal lift of a vector field X on M.*

Horizontal lifts enjoy the following properties.

Proposition 90. *Let X^h and Y^h be the horizontal lifts of X and Y, respectively. Then*

1. *$X^h + Y^h$ is the horizontal lift of $X + Y$,*

2. *For every function $f \in C^\infty(M)$, $f^v X^h$ is the horizontal lift of fX, with $f^v = f \circ \pi$ is the vertical lift of f,*

3. *The horizontal component of $[X^h, Y^h]$ is the horizontal lift of $[X, Y]$.*

The expression of connections as pseudotensorial 1-forms of type $Ad(G)$ gives a global vision. Nevertheless, it is possible to express connections locally. For more details on the local theory of principal connections, we refer the reader to [42].

Concerning the existence of connections, we have the following result:

Theorem 33. *(see [42]) Every principal fiber bundle admits a connection.*

The latter theorem works even when the base is only a paracompact manifold. But paracompactness follows from our definition of a manifold, namely the assumptions that manifolds must be Hausdorff and second countable.

By analogy with the case of connections on vector bundles, one of the ways to 'connect' fibers is through the introduction of a 'displacement' of the elements of a fiber to another fiber. A connection on a principal bundle gives rise to the notion of parallel displacement in the given bundle.

A curve in P is said to be *horizontal* if each of its tangent vectors is horizontal. If $\gamma(t)$, $a \leq t \leq b$, is a piecewise differentiable curve of class C^1 in the base M, a *horizontal lift* of γ is a piecewise differentiable horizontal curve γ^* of class C^1 in P which projects onto γ.

Proposition 91. *([42]) Let $\gamma(t)$, $0 \leq t \leq 1$, be a curve of class C^1 in M. For every $p_0 \in P$, with $\pi(p_0) = \gamma(0)$, there exists a unique lift γ^* of γ which starts at p_0.*

Horizontal lifts of curves and horizontal lift of vector fields are related in the following way: the horizontal lift of an integral curve of $X \in \mathfrak{X}(M)$ is an integral curve of the horizontal lift X^h.

Now, parallel displacement stems from the existence of the horizontal lift of curves in the base. Indeed, if $\gamma(t)$, $0 \leq t \leq 1$, is a differentiable curve of class C^1 in M, and u_0 is a point of P such that $\pi(p_0) = x_0$, then the unique lift γ^* of γ through p_0 has the end point of p_1 with $\pi(p_1) = \gamma(1)$. By varying p_0 in the fiber $\pi^{-1}(\gamma(0))$, we obtain a mapping of the fiber $\pi^{-1}(\gamma(0))$ onto the fiber $\pi^{-1}(\gamma(1))$. Such a mapping is denoted by $\mathbb{P}_\gamma : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(1)$ and called *parallel displacement along the curve γ* (or *parallel transport along the curve γ*). Furthermore, we have:

Proposition 92. *([42]) The parallel displacement along any curve γ commutes with the action of G, i.e. $\mathbb{P}_\gamma \circ \mu_g = \mu_g \circ \mathbb{P}_\gamma$.*

Parallel displacement can be defined on a piecewise smooth curve in an obvious way. Furthermore, it is independent of its parametrization. Moreover, parallel displacement satisfies the following properties:

Proposition 93. *Let $\gamma(t)$, $a \leq t \leq b$, and $\delta(s)$, $c \leq s \leq d$, be two piecewise differentiable curves of class C^1 in M, then*

1. *Parallel displacement along γ^{-1} is the inverse of parallel displacement along γ, where γ^{-1} is the inverse curve of γ,*

2. *If $\gamma(b) = \delta(c)$, then the parallel displacement along the composite curve $\gamma * \delta$ is the composite of parallel displacement along δ followed by parallel displacement along γ.*

Consequently, parallel displacement along a curve is a diffeomorphism between the corresponding fibers. Parallel displacements along loops combine to subgroups of diffeomorphisms and give rise to holonomy groups. For a detailed exposition see [42].

Generally speaking, given a connection Γ on P. The connection 1-form of Γ is a pseudotensorial 1-form of type $Ad(G)$. Denote by $h : T_pP \longrightarrow H_pP$ the projection.

Lemma 20. *Let V be a G-module. If φ is a pseudotensorial r-form on P of type (ρ, V), then*

1. *The form defined by $(\varphi h)(X_1, ..., X_r) = \varphi(hX_1, ..., hX_r)$, for $X_i \in T_pP$, is a pseudotensorial r-form of type (ρ, V),*
2. *$d\varphi$ is a pseudotensorial form $(r+1)$-form of type (ρ, V),*
3. *The $(r+1)$-form $D\varphi$ defined by $D\varphi = (d\varphi)h$ is a tensorial form of type (ρ, V).*

If φ is a pseudotensorial form, then the form $D\varphi$ is called the *covariant exterior derivative* of φ.

Definition 48. Let ω be the connection 1-form of Γ. The tensorial 2-form

$$\Omega = D\omega,$$

is called the *curvature form* of Γ.

The curvature form Ω and the connection form ω are related through the following:

Theorem 34. *(see [42])(Structure equation) Let ω be the connection form and Ω the curvature form. Then*

$$d\omega(X, Y) = -\frac{1}{2}\big[\omega(X), \omega(Y)\big] + \Omega(X, Y),$$

for all $X, Y \in T_pP$ and $p \in P$, where $\big[.,.\big]$ is the Lie bracket of $\mathfrak{g}$.

Furthermore, we have:

Theorem 35. *([42])(Bianchy's identity) If Ω is the curvature form of Γ, then $D\Omega = 0$.*

D.2 Induced connections

Let F be a left G-manifold, then one has the associated fiber bundle to P with standard fiber F and structure group G, namely the fiber bundle (E, π_E, M) with $E = P \times_G F$. The key ingredient to define parallel displacement is the existence of horizontal lifts of curves in the base. We shall see that a connection on principal bundles allows one to define a horizontal distribution on an associated bundle in such a way the horizontal lift of curves to the associated bundles 'comes' from the horizontal lifts of curves in the base to horizontal curves in P. As matter of fact, a connection on a principal bundle induces connections on associated fiber bundles.

Indeed, let Γ be a connection in P with connection form $\omega \in \Omega^1(P, \mathfrak{g})$ is its connection form. At each point $e = [p, f] \in E$ with $p \in P$ and $f \in F$, the *vertical subspace* V_eP is defined to be the tangent space to the fiber $\pi_E^{-1}(\pi(p))$ at e, which happens to be the subspace $\ker(d_e\pi_E)$. To define the *horizontal subspace* of T_eE, consider $(p, f) \in P \times F$ such that $[p, f] = e$ and consider the map $\varphi_f : P \longrightarrow E$ defined by $\varphi_f(p) = [p, f]$, then define $H_eE := d_p\varphi_f(H_pP)$. Clearly H_eE does not depend on the choice of the representative of e and H_eE is complementary to V_eE. The resulting horizontal distribution is a connection in E, called the *induced connection*. For more details on connections on fiber bundles in full generality, we refer the reader to [35, 42, 53].

In an analogous fashion, a curve is said to be *horizontal* if its velocity vector field is horizontal. A horizontal lift of a curve is a horizontal curve that projects to it. Let γ be a curve in M and let $e = [p, f]$ be a point of E. Denote by γ^* the horizontal lift of γ to P at p, then it is easy to see that the curve $\gamma^{**}(t) = [\gamma^*(t), f]$ is the horizontal lift of γ to E at e. The uniqueness of γ^{**} stems from that of γ^* ([42]). Furthermore, a section of E is said to be *parallel* if the image of T_xM by σ is horizontal for each $x \in M$. If γ is a curve relating x_0 to x_1, then *parallel displacement* (or *parallel transport*) of $\sigma(x_0)$ along γ is defined to be $\gamma(x_1)$.

D.3 Connections on vector bundles as induced connections

Since every vector bundle arise as an associated vector bundle to some principal fiber bundle, connection theory on vector bundles may be derived from the theory of connections on principal fiber bundles.

Let E be an associated vector, with standard fiber V, to the principal bundle P. Assume that P is endowed with a connection Γ with a connection form ω. We shall show that induced connections on associated vector bundles give covariant derivatives directly.

First the induced connection allows one to define parallel displacement (or parallel transport) on E. Indeed, let $\gamma : [0, 1] \longrightarrow M$ be a smooth curve in M and let $\gamma^* : [0, 1] \longrightarrow P$ be the horizontal lift of γ to $p \in P$ with respect to Γ. Then, for every $v \in V$, the curve $\gamma^{**} : [0, 1] \longrightarrow E$ with $\gamma^{**}(t) = [\gamma^*(t), v]$ is the horizontal lift of $e = [p, v] \in E$. This gives a map

$$\mathbb{P}^t_\gamma : \pi^{-1}(\gamma(t)) \longrightarrow \pi^{-1}(\gamma(t + h)), \quad \mathbb{P}^t_\gamma(e) = \gamma^{**}(t + h).$$

Thus the parallel displacement is used to define a differential operator on sections along the curve γ as follows

$$\nabla_{\dot\gamma(t)}\sigma = \lim_{h\to 0} \frac{1}{h}\big(\mathbb{P}^{-1}_\gamma(\sigma(\gamma(t + h))) - \sigma(\gamma(t))\big).$$

The map ∇ satisfies the axioms of a covariant derivative (see [42] for details). This way the induced connection gives rise to a covariant derivative which allows one to differentiate sections and hence to do differential geometry on associated vector bundles.

Consequently, connections on vector bundles constitute just a manifestation of the much deeper concept of Ehresmann connections on principal bundles. For more on this topic and the relevance of this formalism the reader may consult [70].

Appendix E

Conclusion and perspectives

The Riemannian geometry of vector bundles can be seen as a natural extension of the geometry of tangent bundles . For more than seventy years, dozens of papers and notes have been devoted to different aspects of the geometry of tangent bundles. But vector bundles in the large still constitute a fertile field for research. We don't claim that tangent bundles are exhausted, but we certainly claim that vector bundles still need exploration. There is a plenty of research directions to carry.

Spherically symmetric metrics were one of the most serious attempts to explore in depth the geometry of the vast class of vector bundle manifolds. It is a general principle in mathematics, that by passing to more general objects, one can understand better the particular cases. That's exactly what happened with vector bundles and tangent bundles as particular cases. We understand now why the geometry of tangent bundles is very rich. The study of vector bundles in the general has shed light on the difficulties encountered when one tries to generalize the results and constructions accomplished with tangent bundles. So, we understand the relevance of the different ingredients as well as the peculiarity of tangent bundles.

Though the class of spherically symmetric metrics showed a rigidity with respect to sectional curvature, the geometric properties related to Ricci curvature still deserve a deep exploration. It is true that we have indications on rigidity with respect to Ricci and scalar curvatures, but we believe that there is a lot more to do with this class of metrics. The rigidities encountered can be explained by the following facts:

- The projection of the vector bundle, when endowed with a spherically symmetric metric, has totally geodesic fibers;

- The dependence of the weight functions on the square norm only and not on the base point.

Thus, we suggest that the future endeavours should be focused on metrics for which the projections do not have totally geodesic fibers and perhaps with more general weight functions.

Concerning the harmonicity of sections, a similar problem occurs. The Euler-Lagrange equations are non-linear PDEs, hence the difficulty to solve them in the full generality. Because of the complexity of the Euler-Lagrange equations, we have been compelled to consider sections of constant length, and parallel sections in particular. This imposes a topological obstruction on the total spaces, since it requires the total space to have a vanishing Euler-Poincaré characteristic. Once again, it seems that the choice of weight functions plays a decisive role.

The geometry of vector bundles and its generalizations to fiber bundles is very promising. It lies at the heart of current research in mathematical physics. So, if we can establish a bridge with the physical community's efforts, we believe that there is a fair chance to contribute to

the contemporary mathematical and physical endeavours to harness a better understanding of our universe.

We don't claim in this thesis neither a full understanding of the Riemannian geometry of vector bundles nor an exhaustion of the subject. There are countless questions to ask about those spaces. To explore them in more details, we suggest a sample of the questions we wish to carry on in our future research.

E.0.1 In differential and Riemannian geometry

1. **Complex and symplectic geometry:** in a forthcoming paper in collaboration with Prof. M.T.K Abbassi and the PhD student Reda El Masdouri, we will consider complex geometry of vector bundle manifolds. We have constructed an almost complex structure as well as an almost symplectic structure compatible with spherically symmetric metrics provided that the base and fibers possess the respective structures. The aim is to construct new Kähler manifolds from old ones.

2. **Geometric analysis:** it has been clear that the expressions of the Ricci curvature are complicated, which make analysis almost impossible. It is worthwhile to investigate its sign. One of the approaches is geometric analysis. More precisely, harmonic function theory provide an interesting tool to do it. Indeed, a theorem of Yau asserts that in a complete manifold with non-negative Ricci curvature, every positive harmonic function must be constant. By studying some specific classes of functions, one could decide when the total space has positive Ricci curvature. This is the object of a forthcoming paper with Prof. M.T.K Abbassi.

3. **Grassmannians:** the Grassmannians constitute very interesting examples of manifolds and fiber bundles. The study of the Riemannian geometry of those spaces is the object of a forthcoming paper in collaboration with Prof. M.T.K Abbassi and the PhD student Houda KHARCHAL. Moreover, it seems that Finsler geometry is suitable for Grassmannians, since their elements can be identified with projectors and hence fibers could be easily endowed with a norm. So, Finsler geometry of Grassmannians is interesting to be investigated.

4. **Ricci solitons:** we have studied conformal vector fields of spherically symmetric metrics. We have computed the Ricci curvature tensor. Since the search for Einstein spherically symmetric metrics, at least in the case of a non-flat connection, did not reveal interesting results, it would be interesting to look for Ricci solitons instead.

5. **Invariants:** though characteristic classes arise from connections on vector bundles, they are cohomological invariants, thus they have a topological flavour. It is true that, in many situations, they contain geometric informations. But we still luck sensitive 'smooth invariants' that capture the 'smoothness' in the structure of vector bundles. So, it is an interesting problem to look at, if any such invariant exists.

E.0.2 In mathematical physics

1. **A search for unification:** string theories are currently among the major endeavours of the mathematical and physical communities. It is a potential GTU (Grand unified theory). String theorists postulate the existence of hidden dimensions and one of the core conjectures is that at each point the hidden dimensions are wrapped up in a six dimensional Calabi-Yau manifold (seven dimensional in some versions) [71]. Since Calabi-Yau manifolds are compact Kähler manifolds which are Ricci-flat (or equivalently with trivial canonical line bundle). We think that fiber bundles with compact fibers may serve

as model spaces for those theories. Hence a generalization of the studies accomplished so far to this framework is in order. We believe that we must construct metrics on fiber bundles with compact fibers and for which the projection does not have totally geodesic fibers. It is noteworthy that some attempts have been done, for a different purpose from ours, in the case of principal fiber bundles [28,88].

String theories, as approaches to quantum gravity, are based on the quantization of Yang-Mills theories. On the other hand, general relativity, that is the field theory of gravitation, is a gauge theory but physicists still can not make it into a Yang-Mills theory. The problem is that we don't know how to quantize a general gauge field theory. May be we should write down the equations in a more general structure, so we can provide a general process for quantization. Perhaps this can be done by considering a fiber bundle instead of a vector bundle. Since we need some Riemannain structure on fibers, we have to generalize the construction of some of metrics we studied, or create new ones if necessary, in this general framework.

2. **Induced connections:** we know for sure that every vector bundle is associated to some principal fiber bundle. Since the fundamental field of every field theory is the curvature of a connection, e.g. the Faraday tensor for electromagnetism or the curvature of a Yang-Mills connection in general, how can we interpret this in the language of principal fiber bundles and what's the physical interpretation of this link.

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Glossary of notations

1- Let m, n, r, s et i be integers, $m \neq 0$, $n \neq 0$.

$\mathbb{N}$**:** the set of natural numbers,

$\mathbb{R}$**:** the set of real numbers,

$\mathbb{R}^+$**:** the set of non-negative real numbers,

$\mathbb{R}_*^+$**:** the set of positive real numbers,

$\mathbb{R}^*$**:** the set of non vanishing real numbers,

$\mathbb{R}^m$**:** the vector space of m-tuples of real numbers,

$(\mathbb{R}^m)^*$**:** the dual space $\mathbb{R}^m$,

t_x**:** the translation by $x \in \mathbb{R}^m$ in $\mathbb{R}^m$,

$GL(m) = GL(m, \mathbb{R})$**:** the general linear group,

$O(m)$**:** the orthogonal group,

$SO(m)$**:** special orthogonal group,

2- Denote by M et N two smooth manifold, i.e. differentiable of class C^∞, with respective dimensions m et n, and let $\pi : E \to M$ be a vector bundle over M

$C^\infty(M)$**:** the algebra of smooth functions C^∞ on M,

$C^\infty(M, N)$**:** the space of smooth maps from M to N,

$\Gamma(E)$**:** the vector space of smooth sections of $E \to M$,

T_xM**:** tangent space to M at $x \in M$,

TM**:** the total space of the tangent bundle to M,

π_M**:** the natural projection of TM,

$\pi_M : TM \to M$**:** the tangent bundle to M,

$\mathfrak{X}(M)$**:** the Lie algebra of smooth vector fields M,

$\mathbf{T}_s^r(x)(M)$**:** the space of tensor fields of type (r, s) on T_xM,

$\mathbf{T}(x)$**:** the space of all tensors on T_xM,

$\mathfrak{T}_s^r(M)$**:** the vector bundle of tensors of type (r, s) on M,

$\mathfrak{T}_s^r(M)$**:** the space of tensor fields of types (r, s) on M,

$\mathfrak{T}(M)$**:** the algebra of all tensors on M,

$\mathcal{V}_u(TM)$**:** the vertical sub-space of $T_u(TM)$,

$\mathcal{V}(TM)$**:** the vertical subspace of TTM,

3-Let $f : M \to N$ be a smooth map on M:

df**:** the differential f

$d_x f$**:** the differential of f at $x \in M$,

4- Let M and N be a smooth manifolds, and let E and F be two vector bundles with the same base M :

$\Gamma(E)$**:** the set of section of E,

supp(σ)**:** the support of a section σ,

$\otimes$**:** tensor product,

$\oplus$**:** direct sum,

E^***:** dual vector bundle,

$\varphi^* E$**:** the pullback of E by a map $\varphi : N \longrightarrow M$,

$E \otimes F$**:** the tensor product E and F,

$E \oplus F$**:** the direct sum (or the Whitney sum) of E and F,

Hom(E, F)**:** the vector bundle of vector bundle homomorphisms between E and F,

$P(E)$**:** the principal bundle of frames of E,

5- Let (E, π, M) be a vector bundle:

D**:** denote a connection on E,

K**:** the connection map of D,

Δ **or** Δ_D**:** the rough Laplacian associated to D,

R **or** R^D**:** the curvature of D,

Γ_{iq}^p**:** the Christoffel symbols of D,

d_D**:** exterior covariant differentiation of D,

$\mathcal{V}$**:** the vertical subbundle of TE,

$\mathcal{H}$**:** the horizontal distribution with respect to D,

$\mathbb{P}$**:** parallel transport with respect to D,

ξ**:** the tautological vector field of E,

X^h**:** the horizontal lift of X,

σ^v**:** the vertical lift of σ,

6- Let (M, g) be a Riemannian manifold

∇ : Levi-Civita connection of (M, g),

d_g**:** (geodesic) distance associated to g,

υ_g**:** the Riemannian volume element of (M, g),

$L(.)$**:** the length functional,

Γ_{ij}^k**:** Christoffel symbols of ∇,

∇_X**:** covariant differentiation along X,

∇_g**:** the Laplace-Beltrami of (M, g),

T**:** torsion of ∇,

R**:** curvature of ∇,

Ric: Ricci curvature tensor of ∇,

K**:** the sectional curvatures of (M, g),

Sc**:** the scalar curvature of (M, g),

H_uTM**:** horizontal subspace $T_u(TM)$,

HTM**:** horizontal subbundle TTM with respect to ∇ (or horizontal distribution),

VTM**:** vertical subbundle TTM,

7-Let (M, g) be a Riemannian manifold with metric connection ∇:

$\mathcal{L}_X$**:** the Lie derivative with respect to a vector field X.

$\nabla(.)$**:** gradient vector field,

div: divergence operator,

8- Known metrics on vector bundles

G^s**:** the Sasaki metric,

G^{cg}**:** the Cheeger-Gromoll metric,

$G^{(p,q)}$**:** generalized Cheeger-Gromoll metrics.

9- For a spherically symmetric metric G on a vector bundle E

$\widetilde{\nabla}$**:** Levi-Civita connection of G,

$\widetilde{R}$**:** curvature tensor of G,

$\widetilde{\mathbf{Ricc}}$**:** Ricci curvature tensor G,

$\widetilde{K}$**:** sectional curvatures of G,

$\widetilde{S}$**:** scalar curvature of G.

10- Let $\phi : (M, g) \longrightarrow (N, h)$**, and let** (E, π, M) **as before and endowed with a spherically symmetric metric** G

$E(\phi)$**:** the energy functional of ϕ,

$\tau(\phi)$**:** tension field of ϕ,

$E^v(\sigma)$**:** vertical energy functional of σ,

$\tau^v(\sigma)$**:** vertical tension field of σ,

For a multi-index $\alpha = (\alpha_1, ..., \alpha_m)$ and $x = (x^1, ..., x^m) \in \mathbb{R}^m$, set
$|\alpha| = \alpha_1 + ... + \alpha_m$,
$\alpha! = \alpha_1!...\alpha_m!$,
$x^\alpha = (x^1)^{\alpha_1}...(x^m)^{\alpha_m}$,
$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1}...\partial x^m)^{\alpha_m}}$,

11- Jets and jet manifolds

$j^r_x f$ **:** r-jet of a map $f : M \to N$ with source x,

$J^r(M, N)$**:** the set of r-jets from M to N,

π^r_s**:** jet projection of r-jets to s-jets with $(0 \leq s \leq r)$,

$J^r_(x, y)(M, N)$**:** the set of r-jets form M to N with source x and target y,

L^r_m**:** the r-th differential group of $\mathbb{R}^m$,

$L^r_{m,n}$**:** the set of jets of mappings $R^m \to R^m$ with source 0 and target 0,

$\pi^r_s(r > s)$**:** jet projection of $L^r_{m,n}$ on $L^s_{m,n}$.

$L(M)$**:** the bundle of linear frames,

$F^r(M)$**:** the bundle of higher order frames,

12- Let G **be a Lie group and let** $a \in G$**:**

R_a**:** right translation by a,

$\mathfrak{g}$**:** the Lie algebra of G,

$\exp_G$**:** the exponential map $\mathfrak{g} \to G$,

$P(M, G)$ **ou** (P, π, M, G)**:** a principal fiber bundle with structure group G, basis M and projection π,

13- Categories and related notions:

$\mathcal{C}$**:** a category,

$Ob\mathcal{C}$**:** the class of objects of $\mathcal{C}$,

gf **ou** $g \circ f$**:** composition of morphisms f and g in $\mathcal{C}$,

Ens: category of sets and maps,

Top: category of topological spaces and continuous maps,

Gr: category of groups and their homomorphisms,

$T : \mathcal{C} \to \mathcal{D}$ **ou** $\mathcal{C} \overset{T}{\to} \mathcal{D}$**:** functors between two categories $\mathcal{C}$ et $\mathcal{D}$,

TA**:** the image of an object A in $\mathcal{C}$ by a functor T,

Tf**:** the image of a morphism f de $\mathcal{C}$ by a functor T,

14- Useful categories:

$\mathcal{VF}$**:** the category of vector bundles together with vector bundle morphisms,

$\mathcal{VF}_m$**:** the category of vector bundles over m-dimensional manifolds and their morphisms,

$\mathcal{M}f$**:** the category of smooth maps C^∞,

$\mathcal{M}f_m$**:** the category m-dimensional manifolds with local diffeomorphisms,

$\mathcal{PB}$**:** the category of principal fiber bundles,

$\mathcal{PB}_m$**:** the category of principal fiber bundles over m-dimensional manifolds with principal fiber morphisms covering local diffeomorphisms,

$\mathcal{PB}(G)$**:** the category of principal fiber bundles with structure group G,

15- Some natural vector bundles

$F : \mathcal{M}f_m \to \mathcal{FM}$**:** a natural fiber bundle on $\mathcal{M}f_m$,

$F : \mathcal{M}f \to \mathcal{FM}$**:** a natural fiber bundle $\mathcal{M}f$,

$S = F_0\mathbb{R}^m$**:** standard fiber of F,

F_xM**:** the fiber de FM over $x \in M$,

$\mathbf{Id}_{\mathcal{M}f_m} : \mathcal{M}f_m \to \mathcal{M}f_m$**:** the identity functor $\mathcal{M}f_m$,

$B : \mathcal{FM} \to \mathcal{M}f$ **:** the base functor,

$T : \mathcal{M}f_m \to \mathcal{FM}_m$ **or** $T : \mathcal{M}f \to \mathcal{FM}$**:** the tangent bundle functor,

$T^* : \mathcal{M}f_m \to \mathcal{VF}_m$ **or** $T^* : \mathcal{M}f \to \mathcal{VF}$**:** cotangent bundle functor ,

$\bigwedge^r : \mathcal{M}f_m \to \mathcal{VF}_m$**:** the bundle of r-linear alternate mappings.