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Étude de quelques EDP fractionnaires via la théorie de Morse, le degré topologique et les méthodes variationnelles

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Dedication

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Contents

Contents **v**

Abstract **1**

Résumé **3**

Notations **5**

Introduction **6**

1 Preliminaries and auxilaries results **13**
1.1 Riemannian Manifold 13
1.2 Some fundamental results on integration theory 14
1.3 Fractional Sobolev spaces on Riemannian manifolds 15
1.4 Fractional Sobolev spaces with variable exponents 16
1.5 Variational methods 18
1.6 Topological Degree Theory 19
1.7 Morse's theory and local linking 22

2 On a new fractional Sobolev space with variable exponent on complete manifolds **24**
2.1 Fractional Sobolev space with variable exponent on complete manifolds . . 24
2.1.1 Fundamental proprieties of a new space 25
2.1.2 Embedding compact and density 28
2.2 Application: Existence results via variational method 30

3 Fractional Sobolev Spaces with Kernel Function on Compact Riemannian Manifolds **37**
3.1 Fractional Sobolev Spaces with Kernel Function 38
3.1.1 Basic proprieties of the new space 39
3.1.2 Embedding Compact 41

3.2 Application: Existence result via topological degree method 44

4 Existence and infinitely many solutions to double phase problem involving a singular term 51

4.1 Analysis of double phase fractional problems with singularity 51
4.2 The Approximated Problem 52
4.2.1 Computation of group criticals 53
4.2.2 Existence and multiplicity of solutions via Morse's theory 62
4.2.3 Remarks and comments 64

5 Infinitely many Solutions to a Kirchhoff type equation involving logarithmic nonlinearity via Morse's Theory 65

5.1 Auxiliary Inequalities 65
5.2 Analysis of Kirchhoff fractional problems 66
5.2.1 Computation of group critical 68
5.2.2 Fundamental Theorem 76

References 79

ABSTRACT

In this thesis, our main purpose is to introduce some fractional Sobolev space with variable exponent and kernel function on compact Riemannian manifolds $(\mathcal{M}, g)$. In addition, we shall prove the fundamental properties of these spaces, such as completeness, reflexivity, separability, compact embedding, and density. The second goal of this thesis is to study some non-local elliptic problems in the setting of Sobolev spaces generalized on compact Riemannian manifolds $(\mathcal{M}, g)$.

We divided our thesis into five chapters:

- In the **first chapter**, we present a comprehensive state of the art and mathematical tools, then included the main results with a glimpse of the proof.

- In the **second chapter**, we extend Sobolev spaces with variables exponents to cover the fractional case with compact manifolds. By using the variational methods, we study the existence solution to elliptic problem:

$$\begin{cases} (-\Delta_g)^s_{p(z)}\mathrm{u}(z) + \mathcal{V}(z)\,|\mathrm{u}(z)|^{q(z)-2} & = f(z,\mathrm{u}(z)) \quad \text{in} \quad \mathcal{Q}, \\ \mathrm{u} & = 0 \qquad\qquad\quad \text{in} \quad \partial\mathcal{Q}. \end{cases}$$

- The main goal of the **third chapter** is to prove the fundamental proprieties of a new fractional Sobolev space with a kernel function. As an application, using the topological degree methods for solving the following problem:

$$\begin{cases} (\mathcal{L}^K_g)_{p(y,\cdot)}w(y) & = \lambda\beta(y)|w(y)|^{r(y)-2}w(y) + f(y,w(y)) \quad \text{in} \quad \mathcal{U}, \\ w & = 0 \qquad\qquad\qquad\qquad\qquad\quad \text{in} \quad \mathcal{M}\backslash\mathcal{U}. \end{cases}$$

- In the **fourth chapter**, we discuss the existence of infinitely many solutions for fractional double phase problems involving a singular term. Our treatments in this chapter are based on Morse's theory and computation critical group.

- In the **last chapter**, we study the multiplicity of the solution for $p(\mathrm{x},\cdot)-$ fractional -Kirchhoff the following problem involving logarithmic non-linearity:

$$\begin{cases} M\left(J_{s,p(\mathrm{x},\cdot)}\mathrm{u}\right)\Delta^s_{p(\mathrm{x},\cdot)}\mathrm{u} & = \lambda|\mathrm{u}|^{r(\mathrm{x})-2}\mathrm{u}\log|\mathrm{u}| + \lambda f(\mathrm{x},\mathrm{u}) \quad \text{in} \quad \mathcal{U}, \\ \mathrm{u} & = 0 \qquad\qquad\qquad\qquad\qquad\quad \text{in} \quad \mathbb{R}^N\backslash\mathcal{U}, \end{cases}$$

  The approach is based on the computation the critical group and Morse's theory.

Keywords: Fractional elliptic problems, kernel function, fractional Sobolev space, Morse's theory, Kirchhoff function, singular term, weak Solutions, local liking, logarithmic non-linearity, infinitely many solutions, Compact Riemannian manifolds, topological degree theory, Calcul of varaitions, Lévy-integrability.

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RÉSUME

Dans cette thèse, notre objectif principal est d'introduire quelques espaces de Sobolev fractionnaires avec un exposant variable et une fonction noyau sur des manifolds Riemanniens compacts $(\mathcal{M}, g)$. En outre, nous prouverons les propriétés fondamentales de ces espaces, telles que la complétude, la réflexivité, séparabilité, l'injection de Sobolev et la densité. Le deuxième objectif de cette thèse est d'étudier quelques problèmes elliptiques non-locaux dans le cadre des espaces de Sobolev généralisés sur des varietes compacts $(\mathcal{M}, g)$.

Nous avons divisé notre thèse en cinq chapitres:

- Dans le premier chapitre, nous présentons l'état de l'art et les outils mathématiques, puis nous incluons les principaux résultats avec un aperçu de la preuve.

- Dans le deuxième chapitre, nous étendons les espaces de Sobolev avec des exposants variables pour couvrir le cas fractionnaire avec des variétés compactes. En utilisant les méthodes variationnelles, nous étudions l'existence d'une solution à un problème elliptique:

$$\begin{cases} (-\Delta_g)^s_{p(z)}\mathrm{u}(z) + \mathcal{V}(z)\,|\mathrm{u}(z)|^{q(z)-2} & = f(z,\mathrm{u}(z)) \quad \text{in} \quad \mathcal{Q}, \\ \mathrm{u} & = 0 \qquad\qquad\quad \text{in} \quad \partial\mathcal{Q}. \end{cases}$$

- L'objectif principal du **troisième chapitre** est de prouver les propriétés fondamentales d'un nouvel espace de Sobolev fractionnaire avec une fonction noyau. Comme application, on utilise les méthodes de degré topologique pour résoudre le problème suivant:

$$\begin{cases} (\mathcal{L}^K_g)_{p(y,\cdot)}w(y) & = \lambda\beta(y)|w(y)|^{r(y)-2}w(y) + f(y,w(y)) \quad \text{in} \quad \mathcal{U}, \\ w & = 0 \qquad\qquad\qquad\qquad\qquad\quad \text{in} \quad \mathcal{M}\backslash\mathcal{U}. \end{cases}$$

- Dans le **quatrième chapitre**, nous discutons de l'existence d'un nombre infini de solutions pour les problèmes fractionnaires à double phase impliquant un terme singulier. Nos traitements dans ce chapitre sont basés sur la théorie de Morse et le calcul du groupe critique.

- Dans le **dernier chapitre**, nous étudions la multiplicité de la solution pour $p(\mathrm{x},\cdot)-$ fractionnaire -Kirchhoff du problème suivant impliquant une non-linéarité logarithmique:

$$\begin{cases} M\left(J_{s,p(\mathtt{x},\cdot)}\mathtt{u}\right)\Delta^{s}_{p(\mathtt{x},\cdot)}\mathtt{u} & = \lambda|\mathtt{u}|^{r(\mathtt{x})-2}\mathtt{u}\log|\mathtt{u}| + \lambda f(\mathtt{x},\mathtt{u}) \quad \text{in} \quad \mathcal{U}, \\ \mathtt{u} & = 0 \quad \text{in} \quad \mathbb{R}^N\backslash\mathcal{U}, \end{cases}$$

L'approche est basée sur le calcul du groupe critique et la théorie de Morse.

Mots clés: Problèmes elliptiques fractionnaires, fonction noyau, espace de Sobolev fractionnaire, théorie de Morse, fonction de Kirchhoff, terme singulier, solutions faibles, vraisemblance locale, non-linéarité logarithmique, nombre infini de solutions, variétés riemanniennes compactes, théorie des degrés topologiques, calcul des variations, intégrabilité de Lévy.

LIST OF SYMBOLS	
$\mathcal{M}$	Compact manifolds
$\mathcal{U}$	open bounded subset of $\mathcal{M}$
$\partial\mathcal{U}$	boudary of $\mathcal{U}$
g	Riemannian metric
$\overline{\mathcal{M}}$	closure of $\mathcal{M}$
$a.e$	almost everywhere
$\rightarrow$	strong convergence
$\rightharpoonup$	weak convergence
$\hookrightarrow$	continuous embedding
$\hookrightarrow\hookrightarrow$	compact embedding
dv_g	Riemannian volume element
X	arbitrary Banach space
X'	dual space of the Banach space X
$supp(u)$	support of function u
$C(\mathcal{U},(1,\infty))$	the set of continuous functions $r:\mathcal{U}\rightarrow(1,\infty)$
$\mathcal{D}(\mathcal{U})$	Test functions in $\mathcal{U}$
$L^p(\mathcal{U})$	$\left\{v:\mathcal{U}\rightarrow\mathbb{R}; v \text{ measurable function }; \int_{\mathcal{U}}\lvert v\rvert^p dx<\infty\right\}$ for all $1<p\leq\infty$
$L^\infty(\mathcal{U})$	$\left\{v:\mathcal{U}\rightarrow\mathbb{R}; v \text{ measurable function; } ess\sup_{x\in\mathcal{U}}\lvert v(x)\rvert<\infty\right\}$
$W^{1,p}(\mathcal{U})$	$\left\{u:\mathcal{U}\rightarrow\mathbb{R}\mid u\in L^p(\mathcal{U}), Du\in\left(L^p(\mathcal{U})\right)^N\right\}$ for $1\leq p\leq+\infty$
$W_0^{1,p}(\mathcal{U})$	Closure of $\mathcal{D}(\mathcal{U})$ in $W^{1,p}(\mathcal{U})$ for $1\leq p<+\infty$
$L^{r(x)}(\mathcal{U})$	Variable exponent Lebesgue space

Introduction

In this thesis, we are interested in a series of results concerning the qualitative properties of the spaces (such as compactness, separability, solutions) of certain classes of PDE's governed by non-local diffusion-reaction operators, which have the following prototypes: defined as follows:

$$(-\Delta)_p^s \mathrm{u} = 2 \lim_{\epsilon \to o^+} \int_{\mathcal{U} \setminus B_\epsilon(x)} \frac{|\mathrm{u}(x) - \mathrm{u}(y)|^{p-2}(\mathrm{u}(x) - \mathrm{u}(y))}{|x-y|^{n+sp}}\, dx,$$

for $s \in (0,1)$, and $1 < p < \infty$. This kind of operator was firstly introduced in [10] in the theory of stochastic partial differentials equations, the kernel corresponding $\mathcal{K}(\mathrm{x},\mathrm{y}) = \frac{1}{|\mathrm{x}-\mathrm{y}|^{s+2N}}$ to $(-\Delta^s)$ determines the probabilities distributions jumps. In [36], the authors introduced the fractional Sobolev space $W^{s,p}$ as follows:

$$W^{s,p}(\mathcal{U}) = \left\{ \mathrm{u} \in L^p(\mathcal{U}) : \frac{|\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y})|}{|\mathrm{x}-\mathrm{y}|^{\frac{n}{p}+s}} \in L^p\left(\mathcal{U} \times \mathcal{U}\right) \right\}.$$

This spaces have been a classical topic in functional and harmonic analysis all along, and some important books, such as [53, 68] treat the topic in detail. Conversely, however, fractional spaces, and the corresponding non-local equations, are now experiencing impressive applications in different subjects, such as, among others, soft thin films [52], conservation laws [52], phase transitions [8, 23, 40, 46, 67], anomalous diffusion [56, 57, 71], stratified materials [28, 8, 65], ultra-relativistic limits of quantum mechanics [41], water waves [30, 31, 32, 33, 34, 35, 44], optimization [72], finance [69], optimal control , game theory [24], gradient potential theory [58], and minimal surfaces [26, 25].
Very recently, a new non-local operators involving a generalized kernel of types $\mathcal{K}(\mathrm{x},\mathrm{y}) = \frac{1}{|\mathrm{x}-\mathrm{y}|^{s+Np(\mathrm{x},\mathrm{y})}}$, and $\mathcal{K}(\mathrm{x},\mathrm{y}) = \frac{1}{d_g(\mathrm{x},\mathrm{y})^{s+Np(\mathrm{x},\mathrm{y})}}$, respectively in the Euclidean and non-Euclidean cases where $sp(\mathrm{x},\mathrm{y}) < N$, and d_g is the distance between x, and y. The approaches for ensuring the existence of weak solutions for a class of nonlocal fractional problems with variable exponents, were addressed in greater depth in [1, 2, 6, 13, 14, 15, 20, 27, 29, 50, 54, 61, 60, 64] and the references therein. In the non-Euclidean case, classical Sobolev spaces on Riemannian manifolds have been investigated for more than seventy years [11, 12, 49]. The theory of these spaces has been applied to isoperimetrical inequalities [49] and the Yamabe

problem [70]. In [45], the authors investigated the theory of generalized Sobolev spaces on compact Riemannian manifolds. Moreover, they proved the compact embeddability of these spaces into the Hölder space. They also studied a PDE problem involving $p(\cdot)$-Laplacian operator.

Now, we give briefly the outline of this thesis and the main results contained in every chapter. This thesis is composed of five chapters:

- In the first chapter, we shall give some fundamental definitions and proprieties about Riemannian manifolds, Sobolev spaces on Riemannian manifolds, fractional Sobolev space with variable exponents, topological degree theory and Morse's theory.

- In the second chapter, we extend the Sobolev spaces with variable exponents to cover the fractional case with complete manifolds. We prove some qualitative properties of this new space. Next, we study the existence of solutions to for the following nonlocal problems involving potentials allowed for vanishing behavior at infinity

$$(\mathcal{P})\begin{cases}(-\Delta_g)^s_{p(z)}\mathrm{u}(z)+\mathcal{V}(z)\,|\mathrm{u}(z)|^{q(z)-2}=f(z,\mathrm{u}(z)) & \text{in } \mathcal{Q}\,,\\ \mathrm{u}\,\big|_{\partial\mathcal{Q}}=0,\end{cases}$$

where $\mathcal{Q}\subset\mathcal{M}$ is an open bounded set with a smooth boundary $\partial\mathcal{Q}$, $s\in(0,1)$, $p\in C(\mathcal{M},(1;\infty))$ with $sp(z)<n$,
$q:\overline{\mathcal{M}}\to(1,\infty)$ satisfying $1<q^-\le q^+<p^-\le p^+<+\infty$, where $q^+=\sup\limits_{z\in\overline{\mathcal{M}}}q(z)$, $q^-=\inf\limits_{z\in\overline{\mathcal{M}}}q(z)$, $p^+=\sup\limits_{z\in\overline{\mathcal{M}}}p(z)$, $p^-=\inf\limits_{z\in\overline{\mathcal{M}}}p(z)$, and

$$(-\Delta_{\mathrm{g}})^s_{p(z)}\mathrm{u}(z)=2\lim_{\epsilon\to o^+}\int_{M\setminus B_\epsilon(z)}\frac{|\mathrm{u}(z)-\mathrm{u}(y)|^{p(z)-2}(\mathrm{u}(z)-\mathrm{u}(y))}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\,dv_{\mathrm{g}}(y)$$

and functions f, $\mathcal{V}$ satisfy some suitable conditions. See section 2.2.
Our motivation on the one hand the work of Fu and Guo [43], who introduced the variable exponent function spaces on Riemannian manifolds in 2012, after that, Gaczkowski and Górka [45] in 2013, examined the above space in the case of compact manifolds, and in 2015 Guo [47] discussed the properties of the Nemytsky operator and obtained the existence of weak solutions for Dirichlet problems of nonhomogeneous $p(m)$-harmonic equations. Finally, in 2016 Gaczkowski, Górka, and Pons [45], studied the variable exponent function spaces on complete non-compact Riemannian manifolds. Furthermore, they proved the continuous embeddings results between Sobolev and Hölder function spaces, using classic assumptions on the geometry. In addition, they established the compact embeddings of $H-$invariant Sobolev spaces, where H is a compact Lie subgroup of the manifold group of isometries, and as an application, they showed the existence of weak solutions to nonhomogeneous $q(z)-$Laplace equations. For further background, we recommend that

readers consult [1, 19] and the references therein. On the other hand, we were also motivated by the work of Guo, Zhang, and Zhang [48] who established the theory of fractional Sobolev spaces on Riemannian manifolds. However, the main difficulty is presented by the fact that the $p(z)-$Laplacian operator has a more complicated non-linearity than the p-laplacian operator. For example, they are non-homogeneous. To the best of our knowledge, there is no known result along this line.

- In the third chapter, we will introduce a new Sobolev space with the kernel function that satisfies Lévy-integrability and coercivity conditions on compact Riemannian manifolds $(\mathcal{M}, g)$. Moreover, we will prove fundamental proprieties of this space, such as, separability, reflexivity, completeness, and embedding compact. In addition, we investigate the existence of weak solutions for the following nonlinear elliptic problem involving $p(\mathrm{y}, \cdot)-$ Laplacian general:

$$\begin{cases} (\mathcal{L}_g^K)_{p(y,\cdot)} w(y) &= \lambda\beta(y)|w(y)|^{r(y)-2}w(y) + f(y, w(y)) \quad \text{in} \quad \mathcal{U}, \\ w &= 0 \quad \text{in} \quad \mathcal{M}\backslash\mathcal{U}. \end{cases}$$

Here, $s \in (0,1)$ is fixed, $r \in C(\mathcal{U}, (1,\infty))$, $\lambda > 0, \mathcal{U} \subset \mathcal{M}$ is an open bounded subset of $\mathcal{M}$, β is a suitable potential function in $\mathbb{R}^+$ with $\beta \in L^\infty(\mathcal{U})$, $p \in \mathcal{C}(\mathcal{U} \times \mathcal{U}, (1,\infty))$ satisfies the following conditions

$$p(z,a) = p(a,z), \text{ for every } (z,a) \in \mathcal{M}^2,$$
$$1 < p^- = \min_{(y,z)\in\mathcal{M}^2} p(y,z) \le p(y,z) < p^+ = \sup_{(y,z)\in\mathcal{M}^2} p(y,z),$$

and $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that:

$(\mathcal{B}_1)$ There exist $\alpha > 0$ and a continuous function $q : \mathcal{M} \to (1, +\infty)$ such that

$$1 < q(y) < p_s^\star(y) = \frac{Np(y,y)}{N - sp(y,y)},$$

and

$$f(y,z) \le \alpha\left(1 + |z|^{q(y)-1}\right), \text{ a.e. } y \in \mathcal{M},\ z \in \mathbb{R},$$
$$r^+ = \sup_{y\in\mathcal{M}} r(y) \le q^- = \min_{y\in\mathcal{M}} q(y) \le p_s^\star(y),$$

and the operator $(\mathcal{L}_g^K)_{p(y,\cdot)}$ is defined by

$$(\mathcal{L}_g^K)_{p(y,\cdot)} w(y) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathcal{M}\backslash\mathfrak{B}_\varepsilon(y)} |w(y) - w(z)|^{p(y,z)-2}(w(y) - w(z))K(y,z)dv_g(z),$$

for every $z \in \mathcal{M}$, where,

$$\mathfrak{B}_\varepsilon(y) = \{z \in \mathcal{M} : d_g(y,z) < \varepsilon,\} \text{ and } dv_g(z) = |g_{ij}|^{\frac{1}{2}}dz.$$

Furthermore, $K : \mathcal{U} \times \mathcal{U} \to (0, +\infty)$ is a symmetric kernel function satisfying the following variant of Lévy-integrability type condition:

$$gK \in L^1\left(\mathcal{M} \times \mathcal{M}, dv_g(y)dv_g(z)\right), \text{ where } g(y,z) = \min\{d_g(y,z), 1\},$$

and the following coercivity condition for some $\alpha_0 > 0$

$$\frac{\alpha_0}{d_g(y,z)^{N+sp(y,z)}} \leq K(y,z), \text{ a.e. } (y,z) \in \mathcal{M}^2, \ y \neq z.$$

Below, some examples of symmetric kernel functions illustrating our results.

Example 0.0.1. *The following functions satisfy satisfy Lévy-integrability and coercivity type conditions.*

- ♣ $K(y,z) = d_g(y,z)^{-N-sp(y,z))}$.
- ♣ $K(y,z) = \frac{\alpha_0}{d_g(y,z)^{N+sp(y,z))}}$, *where* α_0 *is a positive real.*
- ♣ $K(y,z) = \exp\left(\frac{1}{d_g(y,z)^{N+sp(y,z))}}\right)$.
- ♣ $K(y,z) = \exp\left(-\delta d_g(y,z)^2\right)$, *where* δ *is a positive real.*

The work presented in this chapter can be seen as a generalization of [14, 15, 42, 45, 47, 48, 50, 54]. Furthermore, the results are obtained using a topological degree method. However, the main difficulty is presented by the fact that the $p(\cdot,\cdot)$-Laplacian operator has a more complicated nonlinearity than the p-Laplacian operator. For example, it is non-homogeneous. Other complications are due to the non-Euclidean framework of our problem. Also checking for example, the density of the space $C^\infty(\mathcal{M})$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$, because the notion the translation in Riemannian manifolds is not defined.

- In the fourth chapter, we discuss the existence of infinitely many nontrivial solutions of $(p_1(x,.), p_2(x,.))$ double phase fractional Laplacian problem with singular term. More precisely, using Morse's theory and critical groups, we discuss the following fractional double phase problem:

$$\begin{cases} (\mathcal{L}_g^{s_1})_{p_1(x,.)}u(x) + (\mathcal{L}_g^{s_2})_{p_2(x,.)}u(x) &= \frac{f(x,u(x))}{u(x)^{\gamma(x)}} \quad \text{in} \quad \mathcal{U}, \\ u &> 0 \quad \text{in} \quad \mathcal{U}, \\ u &= 0 \quad \text{in} \quad \mathcal{M}\backslash\mathcal{U}. \end{cases} \tag{1}$$

Here, $\mathcal{U} \subset \mathcal{M}$ is an open bounded set of $\mathcal{M}$, $0 < s_1 < s_2 < 1$, $\gamma : \overline{\mathcal{M}} \to (0,1)$ is a continuous function, $p_1, p_1 : \overline{\mathcal{M}} \times \overline{\mathcal{M}} \to (1,\infty)$ are continuous functions satisfy the following conditions: for $i = 1, 2$

$$p_i(z,a) = p_i(a,z) \quad \text{for all } (z,a) \in \mathcal{M}^2,$$

$$1 < p_i^- = \min_{(a,b)\in\mathcal{M}^2} p_i(a,b) \le p_i(a,b) < p_i^+ = \sup_{(a,b)\in\mathcal{M}^2} p_i(a,b), \text{for all } i = 1,2,$$

$$p_2^+ = \sup_{(a,b)\in\mathcal{M}^2} p_2(a,b) \le p_1^+ = \min_{(a,b)\in\mathcal{M}^2} p_1(a,b),$$

$$p_1(x,y) < p_2(x,y) \text{ for all } (x,y) \in \mathcal{M}^2,$$

$(\mathcal{L}_g^{s_i})_{p_i(x,.)}u$ is the $p_i(x,.)-$ Laplacian operator defined by

$$(\mathcal{L}_g^{s_i})_{p_i(x,.)}u(x) = 2\lim_{\varepsilon\to 0^+}\int_{\mathcal{M}\setminus\mathfrak{B}_\varepsilon(x)} \frac{|u(x)-u(y)|^{p_i(x,y)-2}(u(x)-u(y))}{d_g(x,y)^{N+s_ip_i(x,y)}} dv_g(y), \text{ for } i = 1,2,$$

$f : \mathcal{U}\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function that satisfies the following condition
(F_1) There exist $\beta \in L^\infty(\mathcal{M})$, and a continuous function $r : \mathcal{M}\to(1,+\infty)$,
$1 < r(x) < p_{s_1}^\star(x) = \dfrac{Np_1(x,x)}{N - s_1p_1(x,x)}$ satisfies:

$$f(x,y) \le \beta(x)\left(1 + |y|^{r(x)-1}\right), \qquad \text{a.e. } x \in \mathcal{M}, y \in \mathbb{R}.$$

The problem (1) is said of double phase type because of the presence of two different elliptic non standard growths $p_1(x,.)$ and $p_2(x,.)$. This kind of operators was firstly introduced by Zhikov in 1986, [75], to describe models for strongly an-isotropic materials in the context of homogenization and elasticity. In which, He studied the behaviour of the the related energy functional to (1) defined by

$$u \mapsto \int_{\mathcal{U}} \left(|\nabla u|^{p_1} + a(x)|\nabla u|^{p_2}\right) dx. \tag{2}$$

This type of functional with $(p_1,p_2)-$ growths, has been the subject of a great deal of recent work, with applications in many areas of anisotropic physics, for instance see [2, 3, 16, 18].
We highlight here that the main hurdle in our analysis is the energy functional associated with our problem, which is not differentiable due to the presence of a singularity.

- In the last chapter, we analyze the existence results for $p(x,\cdot)-$ fractional Kirchhoff-type elliptic equation involving logarithmic non-linearity. The approach is based on Morse's theory. More precisely, we combine Morse's relation with the computation of critical groups to study the following equation:

$$\begin{cases} M\left(J_{s,p(x,\cdot)}u\right)\Delta^s_{p(x,\cdot)}u &= \lambda|u|^{r(x)-2}u\log|u| + \lambda f(x,u) \quad \text{in} \quad \mathcal{U}, \\ u &= 0 \quad \text{in} \quad \mathbb{R}^N\setminus\mathcal{U}, \end{cases}$$

where λ is a positive parameter, $r \in C(\mathcal{U},(1,\infty))$, $p : \mathcal{U}\times\mathcal{U}\to(1,\infty)$ is a continuous function, $f : \mathcal{U}\times\mathbb{R}\to\mathbb{R}$ is Carathéodory function with $f(x,0) = 0$ and satisfies the suitable conditions see section (5.2).

Furthermore, $M : \mathbb{R}^+\to\mathbb{R}^+$ is a continuous function called Kirchhoff's function that satisfies the following conditions:

$(\mathcal{B}_6)$ There exists $m > 0$, such that

$$m \leq M(t), \text{ for all } t \in \mathbb{R}.$$

$(\mathcal{B}_7)$ There exists $\theta \in (0,1)$ such that

$$\theta t M(t) \leq \widehat{M(t)} \text{ for all } t \in \mathbb{R},$$

where $\widehat{M(t)} = \int_0^t M(s)ds$ is the primitive of function M, and

$$J_{s,p(\mathrm{x},\cdot)}\mathrm{u} = \int_{\mathcal{U}\times\mathcal{U}} \frac{1}{p(\mathrm{x},\mathrm{y})} \frac{|\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{|\mathrm{x}-\mathrm{y}|^{N+sp(\mathrm{x},\mathrm{y})}} d\mathrm{x}d\mathrm{y}, \text{ for all } \mathrm{u} \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}).$$

- We end this thesis with a conclusion and further perspectives.

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1) Aberqi, A., Ouaziz, A.: Morse's theory and local linking for a fractional $(p_1(x,.), p_2(x,.))$: Laplacian problems on compact manifolds. **J. Pseudo-Differ. Oper. Appl**. 14, 41 (2023).

2) Ouaziz, A., Aberqi, A.: Infinitely many solutions to a Kirchhoff-type equation involving logarithmic nonlinearity via Morse's theory. **Boletín de la Sociedad Matemática Mexicana**, 30(2024), p.10.

3) Aberqi, A., Ouaziz, A., Repovš, D.D.: Fractional Sobolev Spaces with Kernel Function on Compact Riemannian Manifolds, **Mediterranean Journal of Mathematics**, 21(1), 6 (2024)

4) Aberqi, A., Benslimane, O., Ouaziz, A., Repovš, D.D.: On a new fractional Sobolev space with variable exponent on complete manifolds. **Boundary Value Problems** 7(1), 1–20 (2022)

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6) Ouaziz, A., Aberqi, A.: Topological Degree Methods for Non-local Fractional $\mathfrak{p}(x,.)$-Laplacian Elliptic Equation. To appear in **Boletim da Sociedade Paranaense de Matemática** xx (yyyy), zzz–zzz

7) Ouaziz, A., Aberqi, A.: Singular fractional double-phase problems with variable exponent via Morse's theory. To appear in **Filomat** xx (yyyy), zzz–zzz

8) Ouaziz, A., Aberqi, A.: Nonlocal Elliptic Problems on Compact Riemannian Manifold. To appear **Boletim da Sociedade Paranaense de Matemática** xx (yyyy), zzz–zzz

SUBMITTED PAPERS

1) Ouaziz, A., Aberqi, A.: Blow-up and global existence of solutions for a new class of parabolic $p(x,)$ Kirchhoff system involving logarithmic nonlinearity.

2) Aberqi, A., Nguyenb, P. D., Ouaziz, A., Ragusa, M. A.: Blow-up and global existence for a new class of parabolic $p(x,)$ Kirchhoff equation involving double phase operator.

3) Knifda, M., Aberqi, A., Ouaziz, A.: Double phase coupled systems in complete manifold.

Chapter 1

Preliminaries and auxilaries results

In this section, we briefly give some basic definitions and properties about Riemanniann manifold. For more details, see [12, 48, 49].

1.1 Riemannian Manifold

Throughout this section, $(\mathcal{M}, g)$ will be a compact Riemannian manifold of dimension N. A local chart on $\mathcal{M}$ is a pair $(\mathcal{U}, \varphi)$, where $\mathcal{U}$ is an open subset of $\mathcal{M}$ and φ is a homeomorphism of $\mathcal{U}$ onto an open subset of $\mathbb{R}^N$. Furthermore, a collection $(\mathcal{U}_k, \varphi_k)_{k\in J}$ of local charts such that $\mathcal{M} = \bigcup_{k\in J} \mathcal{U}_j$, is called an atlas of manifold $\mathcal{M}$. Let $\beta_k : \mathcal{M} \to [0,1]$ be the map. For some atlas $(\mathcal{U}_k, \varphi_k)_{k\in J}$of $\mathcal{M}$, we say that a family $(\mathcal{U}_k, \varphi_k, \beta_k)_{k\in J}$ is a partition of unity subordinate to the covering $(\mathcal{U}_k, \varphi_k)_{k\in J}$ if the following holds:

1) $\sum_{k\in J} \beta_k = 1$,

2) $(\mathcal{U}_k, \varphi_k)_{k\in J}$ is an atlas of $\mathcal{M}$,

3) $\operatorname{supp}(\beta_k) \subset \mathcal{U}_k$, for every $k \in J$.

Definition 1.1.1. (See [49].) Suppose that $w : \mathcal{M} \to \mathbb{R}$ is a continuous function with compact support, $(\mathcal{U}_k, \varphi_k)_{k\in J}$ is an atlas of $\mathcal{M}$, and $(\mathcal{U}_k, \varphi_k, \beta_k)_{k\in J}$ is a partition of unity subordinate to $(\mathcal{U}_k, \varphi_k)_{k\in J}$. We define the Riemannian measure of w in $\mathcal{M}$ as follows:

$$\int_{\mathcal{M}} w(y) dv_g(y) = \sum_{k\in J} \int_{\varphi_k(\mathcal{U}_k)} (|g_{ij}|^{\frac{1}{2}} \beta_k w) \circ \varphi_k^{-1}(y) dy,$$

where $dv_g(y) = |g_{ij}|^{\frac{1}{2}} dy$ is the Riemannian volume element on $(\mathcal{M}, g)$, g_{ij} are the components of the metric g in the local chart $(\mathcal{U}_k, \varphi_k)_{k\in J}$, and dy is the Lebesgue volume of $\mathbb{R}^N$.

Definition 1.1.2. (See [12].) Let $\gamma : [a, b] \to \mathcal{M}$ be a differentiable curve in $\mathcal{M}$ such that $\gamma \in C^1([a, b], \mathcal{M})$. Then the length of γ is given by

$$L(\gamma) = \int_a^b (g(\gamma'(t), \gamma'(t)))^{\frac{1}{2}} dt.$$

Definition 1.1.3. (See [12].) For any $(y,z) \in \mathcal{M}^2$, we define the distance $d_g(y,z)$ between y and z as follows

$$d_g(y,z) = \inf\{L(\gamma) : \gamma(a) = y,\ \gamma(b) = z\}.$$

Theorem 1.1.4. ***(Stine's theorem** [12].) For any $(a,b) \in \mathcal{M}^2$, $d_g(a,b)$ defines a distance on $(\mathcal{M},g)$, and the topology determined by $d_g(a,b)$ is equivalent to the topology of $\mathcal{M}$ as a manifold.*

Definition 1.1.5. (See [49].) Let ∇ be the Levi-Civita connection. For $w \in C^\infty(M)$, $\nabla^k w$ denotes the k-th covariant derivative of w. In local coordinates, the pointwise norm of $\nabla^k w$ is given by

$$\left|\nabla^k w\right| = g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k w)_{i_1 i_2 \ldots i_k} (\nabla^k w)_{j_1 j_2 \ldots j_k}.$$

When $k = 1$, the components of ∇w in local coordinates are given by $(\nabla w)_i = \nabla^i w$. By definition, one has that

$$|\nabla w| = \sum_{i,j=1}^{\infty} g^{ij} \nabla^i w \nabla^j w.$$

Definition 1.1.6. (See [12].) We say that the N-manifold $(\mathcal{M},g)$ has property $B_{vol}(\lambda,v)$ if its geometry is bounded in the following sense:

- The Ricci tensor of g noted by $Rc(g)$ verify, $Rc(g) \geq \lambda(N-1)g$ for some λ.
- There exists some $v > 0$ such that $|B_1(z)|_g \geq v$ for all $z \in \mathcal{M}$, where $B_1(z)$ are the balls of radius 1 centred at some point z in terms of the volume of smaller concentric balls.

1.2 Some fundamental results on integration theory

In this section, we recall some lemmas and elementary properties about integration theory results will play a role in this thesis.

Lemma 1.2.1. *(**Fatou's Lemma**, see [21].) Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(\mathcal{U})$ that satisfy*

- *for all n, $g_n \geq 0$ a.e on $\mathcal{U}$,*
- *$sup_n \int_{\mathcal{U}} g_n dx < \infty$.*

For almost all $x \in \mathcal{U}$ we set $g(x) = \liminf_{n\to} g_n(x) \leq \infty$. Then $g \in L^1(\mathcal{U})$ and

$$\int_{\mathcal{U}} g(x)dx \leq \liminf_{n\to\infty} \int_{\mathcal{U}} g_n(x)dx.$$

Theorem 1.2.2. ***(Dominated Convergence Theorem**, see [21].) Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(\mathcal{U})$ that satisfy*

- *$g_n(x) \to g(x)$ a.e. on $\mathcal{U}$,*

- *there is a function $h \in L^1(\mathcal{U})$ such that for all n, $|g_x| \leq h(x)$ a.e. on $\mathcal{U}$.*

Then, $g \in L^1(\mathcal{U})$ and $\|g_n - g\|_{L^1(\mathcal{U})}$.

Theorem 1.2.3. ***(Vitali's convergence theorem**, see [21].) Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $L^p(\mathcal{U})$ with $1 \leq p < \infty$ such that*

- *$g_n \to g$ a.e. in $\mathcal{U}$.*
- *$\{g_n\}_{n\mathbb{N}}$ is equi-integrable.*

Then $g_n \to g$ in $L^p(\mathcal{U})$ as $n \to \infty$.

Theorem 1.2.4. ***(Tonelli- Fubini theorem**, see [21].) Let $G(x,y) : \mathcal{U}_1 \times \mathcal{U}_2 \subset \mathbb{R}^{2N} \to \mathbb{R}$ be a measurable function satisfying*

- *$\int_{\mathcal{U}_1} |G(x,y)|dx < \infty$,*
- *$\int_{\mathcal{U}_2} |G(x,y)|dy < \infty$.*

Then, $G \in L^1(\mathcal{U}_1 \times \mathcal{U}_2, dxdy)$, and

$$\int_{\mathcal{U}_1\times\mathcal{U}_2} G(x,y)dxdy = \int_{\mathcal{U}_1} dx \int_{\mathcal{U}_2} G(x,y)dy = \int_{\mathcal{U}_2} dy \int_{\mathcal{U}_1} G(x,y)dx.$$

1.3 Fractional Sobolev spaces on Riemannian manifolds

Recalling some fundamental definitions and proprieties of Sobolev spaces on Riemannian manifolds, along with proving a few lemmas that will aid in our study in chapter 5, are the main goals of this part.

Let $0 < s < 1$, and $1 < p < \infty$ be real numbers. The fractional Sobolev space $W^{s,p}(\mathcal{M})$ is defined as follows:

$$W^{s,p}(\mathcal{M}) = \left\{ \mathrm{u} \in L^p(\mathcal{M}) : \frac{|\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y})|}{d_g(\mathrm{x},\mathrm{y})^{\frac{n}{p}+s}} \in L^p\left(\mathcal{M}\times\mathcal{M}\right) \right\}.$$

It is endowed with the natural norm

$$\|\mathrm{u}\|_{W^{s,p}(\mathcal{M})} = \left(\int_{\mathcal{M}} |\mathrm{u}(\mathrm{x})|^p \, d\mu_g(\mathrm{x}) + [\mathrm{u}]^p_{W^{s,p}(\mathcal{M})} \right)^{1/p},$$

with

$$[\mathrm{u}]_{W^{s,p}(\mathcal{M})} = \left(\iint_{\mathcal{M}\times\mathcal{M}} \frac{|\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y})|^p}{(d_g(\mathrm{x},\mathrm{y}))^{N+ps}} dv_g(\mathrm{x}) dv_g(\mathrm{x}) \right)^{1/p},$$

where $[\mathrm{u}]_{W^{s,p}}$ is the Gagliardo semi-norm. We will work in the closed linear subspace

$$W^{s,p}_0(\mathcal{M}) = \{ \mathrm{u} \in W^{s,p}(\mathcal{M}) : supp(\mathrm{u}) \text{ is a compact of } \mathcal{M} \},$$

with $supp(\mathrm{u}) = \overline{\{\mathrm{x} \in \mathcal{M} : \mathrm{u}(\mathrm{x}) \neq 0\}}$.

Remark 1.3.1. The space $W_0^{s,p}(\mathcal{M})$ is the closure of $C_0^\infty(\Omega)$ in $W^{s,p}(\mathcal{M})$.

Theorem 1.3.2. *(See [48].) Let $s \in (0,1)$, $p \in (1,\infty)$, and $(\mathcal{M}, d_g)$ be a compact Riemannian manifold. Then the space $W^{s,p}(\mathcal{M})$ is separable and reflexive Banach space.*

Theorem 1.3.3. *(See [48].) Let $s \in (0,1)$, $p \in (1,\infty)$, and $(\mathcal{M}, d_g)$ be a compact Riemannian manifold. Then, there exists a constant $C = C(N,s,p,q,\mathcal{M})$ such that for every $\mathrm{u} \in W_0^{s,p}(\mathcal{M})$ it holds that*

$$\|\mathrm{u}\|_{L^q(\mathcal{M})} \leq C\|\mathrm{u}\|_{W^{s,p}(\mathcal{M})}, \text{ for all } 1 \leq q \leq p_s^*,$$

with

$$p_s^* = \begin{cases} \frac{Np}{N-sp}, & \text{if } sp < N, \\ \infty, & \text{if } sp \geq N, \end{cases} .$$

That is, the space $W^{s,p}(\mathcal{M})$ is continuously embedded in $L^q(\mathcal{M})$ for any $q \in (1, p_s^*)$. Moreover, this embedding is compact.

1.4 Fractional Sobolev spaces with variable exponents

This section contains results that will be used throughout the document concerning fractional Sobolev spaces with variable exponents and the generalized Lebesgue space.

For a bounded domain with Lipschitz boundary $\mathcal{U} \subset \mathbb{R}^N$, we consider two variable exponents, that is, we let $q : \overline{\mathcal{U}} \to (1,+\infty)$ and $p : \overline{\mathcal{U}} \times \overline{\mathcal{U}} \to (1,+\infty)$ be two continuous functions. We assume that p is symmetric, $p(\mathrm{x},\mathrm{y}) = p(\mathrm{y},\mathrm{x})$, and that both p and q are bounded away from 1 and $+\infty$, that is, there exist $1 < q^- < q^+ < +\infty$ and $1 < p^- < p^+ < +\infty$ such that $1 < q^- < q(\mathrm{x}) < q^+$ and $1 < p^- < p(\mathrm{x},\mathrm{y}) < p^+$ for every $(\mathrm{x},\mathrm{y}) \in \overline{\mathcal{U}} \times \overline{\mathcal{U}}$, with $q^- = \min_{\mathrm{x}\in\overline{\mathcal{U}}} q(\mathrm{x})$, $\quad q^+ = \max_{\mathrm{x}\in\overline{\mathcal{U}}} q(\mathrm{x})$, $\quad p^- = \min_{(\mathrm{x},\mathrm{y})\in\overline{\mathcal{U}}\times\overline{\mathcal{U}}} p(\mathrm{x},\mathrm{y})$, and $p^+ = \max_{(\mathrm{x},\mathrm{y})\in\overline{\mathcal{U}}\times\overline{\mathcal{U}}} p(\mathrm{x},\mathrm{y})$.

We define the generalized Lebesgue space $L^{q(\mathrm{x})}(\mathcal{U})$ as usual:

$$L^{q(\mathrm{x})}(\mathcal{U}) = \left\{ u : \mathcal{U} \to \mathbb{R} \text{ is a measurable function } : \exists \lambda > 0 : \int_{\mathcal{U}} |\frac{u(\mathrm{x})}{\lambda}|^{q(\mathrm{x})} d\mathrm{x} < \infty \right\},$$

with its natural norm

$$|\mathrm{u}|_{L^{q(\mathrm{x})}(\mathcal{U})} = \inf \left\{ \xi > 0 : \int_{\mathcal{U}} |\frac{\mathrm{u}(\mathrm{x})}{\xi}|^{q(\mathrm{x})} d\mathrm{x} \leq 1 \right\}.$$

Lemma 1.4.1. ***(Hölder's inequality**, see [39].) Let $\mathcal{U} \subset \mathbb{R}^N$ be a Lipschitz-bounded domain, and $q : \overline{\mathcal{U}} \to (1,+\infty)$ is a continuous function and bounded away from 1 and $+\infty$. Then, the following inequality holds:*

$$|\int_{\mathbb{R}^N} v(\mathrm{x}) w(\mathrm{x}) d\mathrm{x}| \leq (\frac{1}{q^-} + \frac{1}{q'^-}) |v|_{L^{q(\mathrm{x})}(\mathbb{R}^N)} |w|_{L^{q'(x)}(\mathbb{R}^N)},$$

for all $(v,w) \in L^{q(\mathrm{x})}(\mathbb{R}^N) \times L^{q'(x)}(\mathbb{R}^N)$, where $\frac{1}{q(\mathrm{x})} + \frac{1}{q'(x)} = 1$.

Proposition 1.4.2. *(See [39].) Let* u *and* $\{u_n\}_{n\in\mathbb{N}} \in L^{q(x)}(\mathcal{U})$*, then the following statements are equivalent:*

i) $\lim\limits_{n\longrightarrow+\infty} |u_n - u|_{L^{q(x)}(\mathcal{U})} = 0,$

ii) $\lim\limits_{n\longrightarrow+\infty} \rho_{q(x)}(u_n - u) = 0,$

iii) $u_n \to u$ *on measure in* $\mathcal{U}$ *and* $\lim\limits_{n\longrightarrow+\infty} \rho_{q(x)}(u_n) - \rho_{q(x)}u) = 0,$

where, $\rho_{q(x)} : L^{q(x)(\mathcal{U})} \longrightarrow \mathbb{R}$ *is the mapping defined as follows:*

$$\rho_{q(x)}(u) = \int_{\mathcal{U}} |u(x)|^{q(x)} dx.$$

Lemma 1.4.3. *(See [39].) For every* $u \in L^{q(x)}(\mathcal{U})$*, the following properties hold:*

i) If $\|u\|_{L^{q(x)}(\mathcal{U})} < 1$, *then* $\|u\|^{q^-}_{L^{q(x)}(\mathcal{U})} \leq \rho_{q(x)}(u) \leq \|u\|^{q^+}_{L^{q(x)}(\mathcal{U})}.$

ii) If $\|u\|_{L^{q(x)}(\mathcal{U})} > 1$, *then* $\|u\|^{q^+}_{L^{q(x)}(\mathcal{U})} \leq \rho_{q(x)}(u) \leq \|u\|^{q^-}_{L^{u(y)}(\mathcal{U})}.$

iii) $\|u\|_{L^{q(x)}(\mathcal{U})} < 1, = 1, > 1$ *if only if* $\rho_{q(x)}(u) < 1, = 1, > 1.$

Lemma 1.4.4. *(See [39].) Let* $\mathcal{U} \subset \mathbb{R}^N$ *be a Lipschitz-bounded domain, and* $q : \overline{\mathcal{U}} \to (1, +\infty)$ *is a continuous function and bounded away from 1 and* $+\infty$*. Then, we have the following statements:*

i) the space $\left(L^{q(x)}\left(\mathbb{R}^N\right), |.|_{L^{q(x)}(\mathbb{R}^N)}\right)$ *is a separable, reflexive, and Banach space,*

ii) the space $C^\infty(\mathcal{U})$ *is dense in the space* $\left(L^{q(x)}(\mathcal{U}), |.|_{L^{q(x)}(\mathcal{U})}\right).$

Now for $0 < s < 1$, we introduce the variable exponent Sobolev fractional space as follows:

$$W^{s,q(x),p(x,y)}(\mathcal{U}) = \left\{ w \in L^{q(x)}(\mathcal{U}) : \frac{w(x) - w(y)}{\beta |x - y|^{s + \frac{N}{p(x,y)}}} \in L^{p(x,y)}(\mathcal{U} \times \mathcal{U}) \text{ for some } \beta > 0 \right\}. \tag{1.1}$$

Let $[w]^{s,p(x,y)} = \inf\left\{ \beta > 0 : \int_{\mathcal{U}\times\mathcal{U}} \frac{|w(x) - w(y)|^{p(x,y)}}{\beta^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy < 1 \right\}$, be the corresponding variable exponent Gagliardo semi-norm. We equip the space $W^{s,q(x),p(x,y)}(\mathcal{U})$ with the norm

$$\|w\|_{W^{s,q(x),p(x,y)}(\mathcal{U})} = [w]^{s,p(x,y)} + |w|_{q(x)},$$

where $(L^{q(x)}(\mathcal{U}), |.|_{q(x)}$ is the generalized Lebesgue space.

Lemma 1.4.5. *(See [15].) Let* $\mathcal{U} \subset \mathbb{R}^N$ *be a Lipschitz-bounded domain, and* $q : \overline{\mathcal{U}} \to (1, +\infty)$ *and* $p : \overline{\mathcal{U}} \times \overline{\mathcal{U}} \to (1, +\infty)$ *be two continuous functions. Then* $W^{s,q(x),p(x,y}(\mathcal{U})$ *is a separable and reflexive Banach space.*

Theorem 1.4.6. *(See[15].) Let $\mathcal{U} \subset \mathbb{R}^N$ be a Lipschitz-bounded domain, $q : \overline{\mathcal{U}} \to (1, +\infty)$ and $p : \overline{\mathcal{U}} \times \overline{\mathcal{U}} \to (1, +\infty)$ be two continuous functions, and*

$$sp(x, y) < N, \ p(x, x) < q(x), \ \text{for all } (x, y) \in \mathcal{U}^2,$$

and $\ell : \overline{\mathcal{U}} \to (1, +\infty)$ is a continuous variable exponent such that

$$p_s^*(x) = \frac{Np(x, x)}{N - sp(x, x)} > \ell(x) \geq \ell^- = \min_{x \in \overline{\mathcal{U}}} \ell(x) > 1.$$

Then, the space $W^{s,q(x),p(x,y)}(\mathcal{U})$ is continuously embedded in $L^{\ell(y)}(\mathcal{U})$ and there exists a positive constant $C = C(N, s, p, q, \mathcal{U})$ such that

$$|w|_{L^{l(x)}(\mathcal{U})} \leq C\|w\|_{W^{s,q(x),p(x,y)}(\mathcal{U})}, \text{ for all } \quad w \in W^{s,q(x),p(x,y)}(\mathcal{U}).$$

Moreover, this embedding is compact.

In the following, we present several methods used to prove our existence results: the variational method, Morse's method, and the topological degree method.

1.5 Variational methods

Definition 1.5.1. (See [37].) Let X and Y be normed vector spaces, and $X_1 \subset X$, be an open subset of X.

1) A function $f : X_1 \to Y$ is called Fréchet differentiable at $x \in X_1$ if there exists a bounded linear operator $A : X \to Y$ such that

$$\lim_{\|h\|_X \to 0} \frac{\|f(x + h) - f(x) - Ah\|_Y}{\|h\|_X}$$

2) The function f is class C^1 if $A : X_1 \to \mathcal{L}(X, Y)$ is continuous, where $\mathcal{L}(X, Y)$ is the space of all bounded linear operators from X to Y.

Definition 1.5.2. (See [37].) X and Y are Banach spaces, $U \subseteq X$ is open, and $f : U \to Y$. The Gateaux differential $df(u; v)$ of f at $u \in U$ in the direction $v \in X$ is defined as

$$df(u; v) = \lim_{\tau \to 0} \frac{f(u + \tau v) - f(u)}{\tau} = \frac{d}{d\tau} f(u + \tau v)\Big|_{\tau=0}$$

If the limit exists for all $v \in X$, then one says that f is Gateaux differentiable at u.

Definition 1.5.3. (See [62].) Let X be a Banach space and $f \in C^1(X)$. We say that $u \in X$ is a critical point of f, if $df(u) = 0$.

Remark 1.5.4. The critical point of of f are the solutions of the equation $df(u) = 0$ i.e for all $v \in X : df(u) \cdot v = 0$. This corresponds to solving the EDP in the weak solution.

Definition 1.5.5. (See [62].) Let X be a Banach space and X^* be its topological dual. Suppose that $\psi \in C^1(X)$.

(a) We say that ψ satisfies the Palais–Smale condition at the level $c \in \mathbb{R}$ if the following is true:

" every sequence $\{u_n\}_{n\geq 1} \subset X$ such that $\psi(u_n) \to c$ and $\psi'(u_n) \to 0$ in X^* admits a strongly convergent sub-sequence".

(b) We say that ψ satisfies the Cerami condition at the level $c \in \mathbb{R}$ if the following is true:

" every sequence $\{u_n\}_{n\geq 1} \subset X$ such that $\psi(u_n) \to c$ and $(1 + \|u_n\|)\,\psi'(u_n) \to 0$ in X^* admits a strongly convergent sub-sequence".

Theorem 1.5.6. *(**Ambrosetti-Rabinowitz** 1973, see [9].) Let X be a real Banach space and $f \in C^1(X, \mathbb{R})$. Suppose that f satisfies the following conditions:*

- $f(0) = 0$.
- *f satisfies the Palais-Smalle condition.*
- *There exists $v \in X$ with $\|v\|_X > \rho$ such that $f(v) < 0$.*
- *There exists $\rho > 0$ and $\alpha > 0$ such that $f(u) \geq \alpha$ for all $u \in \mathcal{S}^1(0,\rho) = \{u \in X : \|u\|_X = \rho\}$.*

Considering the set $\mathcal{B} := \{\gamma \in C([0,1], X); \gamma(0) = 0,\ \gamma(1) = v\}$. Then,

$$c := \inf_{\gamma\in\mathcal{B}} \max_{t\in[0,1]} f(\gamma(t))) \geq \alpha$$

is a critical value of f.

1.6 Topological Degree Theory

Let E be a real separable Banach space and E^* its dual. Given a non-empty set $U \subset E$, denote by $\bar{U}$ and by ∂U its closure and boundary, respectively.

Definition 1.6.1. (See [21].) Let $f : U \subset E \to E^*$ be an operator.

1) We say that f satisfies condition if for $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$, we have

$$z_n \overset{weakly}{\rightharpoonup} z \text{ weakly and } \limsup_{n\to\infty} \langle fz_n, z_n - z\rangle \leq 0 \;\Rightarrow\; z_n \to z.$$

2) We say that f is a quasi-monotone operator if for every $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$, we have

$$z_n \overset{weakly}{\rightharpoonup} z \Rightarrow \limsup_{n\to\infty} \langle fz_n, z_n - z\rangle \geq 0.$$

Definition 1.6.2. (Condition $(S_+)_{\mathfrak{B}}$, see [51].) Suppose that $U_1 \subset E$ is such that $U \subset U_1$, $\mathfrak{B} : U_1 \to E^*$ is a bounded operator, and $f : U \subset E \to E$ is an operator.

1) We say that f satisfies condition $(S_+)_{\mathfrak{B}}$ if for every $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$, the following combined properties

$$\begin{cases} z_n \overset{weakly}{\rightharpoonup} z \\ a_n = \mathfrak{B}(z_n) \overset{weakly}{\rightharpoonup} a \\ \limsup\limits_{n\to+\infty} \langle fz_n, a_n - a\rangle \geq 0 \end{cases}$$

imply $z_n \to z$.

2) (Property $(QM)_{\mathfrak{B}}$). We say that f satisfies condition $(QM)_{\mathfrak{B}}$ if for every $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$, we have

$$z_n \overset{weakly}{\rightharpoonup} z \text{ and } a_n = \mathfrak{B}z_n \overset{weakly}{\rightharpoonup} a \Rightarrow \limsup_{n\to\infty} \langle fz_n, a - a_n\rangle \geq 0.$$

We consider the following sets

$$\mathcal{F}_0^\star(U) = \{g : U \subset E \to E^* \mid \text{ g is demi-continuous, of type } (S_+), \text{ and is bounded } \}.$$

$$\mathcal{F}_{\mathfrak{B},1}(U) = \{g : U \subset E \to E \mid g \text{ is demi-continuous, bounded, of type } (S_+)_{\mathfrak{B}}\}.$$

$$\mathcal{F}_{\mathfrak{B}}(U) = \{f : U \subset E \to E \mid f \text{ is demi-continuous and satisfies condition } (S_+)_{\mathfrak{B}}\}.$$

Let $U \subset D_f$ and $\mathfrak{B} \in \mathcal{F}_0^\star(U)$, where D_f denotes the domain of f. We denote by $\mathcal{N}$ the collection of all bounded open sets in E. The following operators will be considered

$$\begin{aligned} \mathcal{F}_{S_+}(E) &= \{f \in \mathcal{F}_0^\star(\overline{\omega}) : \omega \in \mathcal{N}\}, \\ \mathcal{F}_B(E) &= \{f \in \mathcal{F}_{\mathfrak{B},1}(\overline{\omega}) : \omega \in \mathcal{N}, \mathfrak{B} \in \mathcal{F}_0^\star(\overline{\omega})\}, \\ \mathcal{F}(E) &= \{f \in \mathcal{F}_{\mathfrak{B}}(\overline{\omega}) : \omega \in \mathcal{N}, \mathfrak{B} \in \mathcal{F}_0^\star(\overline{\omega})\}. \end{aligned}$$

Lemma 1.6.3. *(See [51].) Let ω be a bounded open set in uniformly convex Banach space E, $\mathfrak{B} : \overline{\omega} \to E^*$ a bounded operator, and $f : \overline{\omega} \to E$. Then we have*

1) *If f is locally bounded and satisfies condition $(S_+)_{\mathfrak{B}}$ and $\mathfrak{B}$ is continuous, then f has the property $(QM)_{\mathfrak{B}}$.*

2) *The operator f has the property $(QM)_{\mathfrak{B}}$, if for all $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$*

$$z_n \rightharpoonup z \text{ and } \quad a_n = \mathfrak{B}z_n \rightharpoonup a \ \Rightarrow \liminf \langle fz_n, a_n - a\rangle \geq 0.$$

3) *If operators $f_1, f_2 : \overline{\omega} \to E$ satisfy $(QM)_{\mathfrak{B}}$ condition, then $f_1 + f_2$ and αf_1 also satisfy $(QM)_{\mathfrak{B}}$ condition, for every positive numbers α.*

4) *Let $f_1 : \overline{\omega} \to E$ be an operator of the type $(S_+)_{\mathfrak{B}}$ and $f_2 : \overline{\omega} \to E$ an operator satisfying the property $(QM)_{\mathfrak{B}}$. Then $f_1 + f_2$ satisfies condition $(S_+)_{\mathfrak{B}}$.*

Lemma 1.6.4. *(See [20].) Let B be a bounded open set in E, $\mathfrak{B} \in \mathcal{F}_0^{\star}(\overline{B})$ continuous, and $g : D_g \subset E^* \to E$ a demi-continuous operator such that $\mathfrak{B}(\bar{B}) \subset D_g$. Then the following properties hold:*

a) *If g is quasi-monotone operator, then $I + g \circ \mathfrak{B} \in \mathcal{F}_{\mathfrak{B}}(\bar{B})$, where I denotes the identity operator.*

b) *If g is an operator of type (S_+), then $g \circ \mathfrak{B} \in \mathcal{F}_{\mathfrak{B}}(\overline{B})$.*

Definition 1.6.5. (See [22].) Let $B \subset E$ be a bounded open set, $\mathfrak{B} \in \mathcal{F}_0^{\star}(\overline{B})$ continuous, and $f, g \in \mathcal{F}_{\mathfrak{B}}(\overline{E})$. Then the map $H : [0,1] \times \overline{E} \to E$ given by

$$H(s,w) = (1-s)fw + sgw, \text{ for every } (s,w) \in [0,1] \times \overline{B},$$

is called an admissible affine homotopy.

Lemma 1.6.6. *(See [20].) Let $B \subset E$ be a bounded open set, $\mathfrak{B} \in \mathcal{F}_0^{\star}(\overline{B})$ continuous, and $f, g \in \mathcal{F}_{\mathfrak{B}}(\overline{E})$. Then the homotopy $H(s,\cdot)$ satisfies condition $(S_+)_{\overline{B}}$.*

Theorem 1.6.7. *(See [20].) There exists a unique degree function*

$$d : \left\{(f,F,a) : F \in \mathcal{N}, \mathfrak{B} \in \mathcal{F}_0^{\star}(\overline{B}), f \in \mathcal{F}_{\mathfrak{B},1}(\overline{E}), a \notin f(\partial E)\right\} \to \mathbb{Z}$$

satisfying the following properties:

1) *If $a \in F$, then $d(I,F,a) = 1$.*

2) *If $G : [0,1] \times \overline{B} \to F$ is a bounded admissible affine homotopy with a common continuous essential inner map and $b : [0,1] \to F$ is a continuous mapping in E, then $d(G(x,\cdot),F,b(x))$ is constant for every $x \in [0,1]$ and $b(x) \notin G(t,\partial F)$.*

3) *If F_1 and F_2 are disjoint open subsets of E with $a \notin f\left(\overline{F} \setminus (E_1 \cup E_2)\right)$, then*

$$d(f,F,a) = d(f,F_1,a) + d(f,F_2,a).$$

4) *If $d(f,F,a) \neq 0$, then $fu = a$, has a solution in F.*

Lemma 1.6.8. ***(Simon's inequality**, see [66].) For every $y, z \in \mathbb{R}^N$, the following holds:*

$$\begin{cases} |y-z|^n \leq c_n \left(|y|^{n-2}y - |z|^{n-2}z\right) \cdot (y-z), \ n \geq 2, \\ |y-z|^n \leq C_n \left[\left(|y|^{n-2}y - |z|^{n-2}z\right) \cdot (y-z)\right]^{\frac{n}{2}} \left(|y|^n + |z|^n\right)^{\frac{2-n}{2}}, \ 1 < n < 2, \end{cases}$$

where, $c_n = \left(\frac{1}{2}\right)^{-n}$ and $C_n = \frac{1}{n-1}$.

1.7 Morse's theory and local linking

Morse's theory and the computation of critical groups are useful tools for studying the multiplicity and existence of solutions to nonlinear problems. As far as we know, this method is rarely used in the study of problems of differential equations. For this, we'll provide a brief overview of the technique.

Let W be a real Banach space, $\phi \in C^1(W, \mathbb{R})$, satisfies the Palais Smalle condition, and $c \in \mathbb{R}$. We consider the following sets:

$$\phi^c = \{u \in W : \phi(u) \leq c\} \text{ and } K_\phi = \left\{u \in W : \phi^{'}(u) = 0\right\}.$$

The critical groups of ϕ at u are defined by

$$C_k(\phi, u) = H_k(\phi^c \cap U, \phi^c \cap U \backslash \{u\}),$$

where $k \in \mathbb{N}$, U is a neighborhood of u such that $K_\phi \cap U = \{u\}$, and H_k is the singular relative homology with coefficient in an Abelian group G, (see [62]), for more details. Moreover, the authors in [17], introduced the critical groups of ϕ at infinity by

$$C_k(\phi, \infty) = H_k(W, \phi^a),$$

where a is less than all critical values and $k \in \mathbb{N}$. Concerning the connection between critical groups and critical points of ϕ.

Theorem 1.7.1. *(See [62].) Let $\phi \in C^1(Y, \mathbb{R})$ satisfy the Palais-Smale condition, and let a be a regular value of ϕ. Then, $H_*(Y, \phi^a) \neq 0$, implies that $K_\phi \cap \phi^a \neq \emptyset$.*

Definition 1.7.2. (See [63].) Given Y is a Banach space, $\phi \in C(Y, \mathbb{R})$, and 0 is an isolated critical point of ϕ such that $\phi(0) = 0$. Let $m, n \in \mathbb{N}$. We say that ϕ has a local $(m, n)-$ linking near the origin if there exist a neighbourhood U of 0 and non-empty sets F_0, $F \subset U$, and $D \subset Y$ such that $0 \notin F_0 \subset F$, $F \cap D = \emptyset$ and

1) $\phi|_F \leq 0 < \phi|_{U \cap D \backslash \{0\}}$,

2) 0 is the only critical point of ϕ in $\phi^0 \cap U$, where $\phi^0 = \{u \in Y : \phi(u) = 0\}$

3) $\text{Dim}\, im\,(i^*) - \text{Dim}\, im\,(j^*) \geq n$, where

$$i^* : H_{m-1}(F_0) \to H_{m-1}(Y \backslash D) \text{ and } j^* : H_{m-1}(F_0) \to H_{m-1}(F)$$

are the homomorphisms induced by the inclusion maps $i : F_0 \to Y \backslash D$ and $j : F_0 \to F$.

Theorem 1.7.3. *(see [63]) Given Y is a Banach space, $\phi \in C(Y, \mathbb{R})$. If ψ has a local linking at 0 with respect to Y. Then, we get $C_k(\psi, 0) \neq 0$.*

Lemma 1.7.4. *(Morse relation, see [62].) If Y is a Banach space, $\phi \in C^1(Y, \mathbb{R})$, $a, b \in \mathbb{R} \backslash \phi\left(K_\phi\right)$, $a < b$, $\phi^{-1}((a,b))$ contains a finite number of critical points $\{u_i\}_{i=1}^n$ and ϕ satisfies the Palais-Smale condition, then*

1) *for all $k \in \mathbb{N}_0$, we have $\sum_{i=1}^n \operatorname{rank} C_k(\phi, u_i) \geqslant \operatorname{rank} H_k\left(\phi^b, \phi^a\right)$,*

2) *if the Morse-type numbers $\sum_{i=1}^n \operatorname{rank} C_k(\phi, u_i)$ are finite for all $k \in \mathbb{N}_0$ and vanish for all large $k \in \mathbb{N}_0$, then so do the Betti numbers $\operatorname{rank} H_k\left(\phi^b, \phi^a\right)$ and we have*

$$\sum_{k \geqslant 0} \sum_{i=1}^n \operatorname{rank} C_k(\phi, u_i)\, t^k = \sum_{k \geqslant 0} \operatorname{rank} H_k\left(\phi^b, \phi^a\right) t^k + (1+t) Q(t) \text{ for all } t \in \mathbb{R},$$

where $Q(t)$ is a polynomial in $t \in \mathbb{R}$ with non-negative integer coefficients.

Chapter 2

On a new fractional Sobolev space with variable exponent on complete manifolds

In this chapter, We present the theory of a new fractional Sobolev space in complete manifolds with variable exponent. As a result, we investigate some of our new space's qualitative properties, such as completeness, reflexivity, separability, and density. We also show that continuous and compact embedding results are valid. We apply the conclusions of this study to the variational analysis of a class of fractional $p(z)-$Laplacian problems involving potentials with vanishing behavior at infinity as an application.

2.1 Fractional Sobolev space with variable exponent on complete manifolds

Let $(\mathcal{M}, g)$ be the complete manifold, $\mathcal{Q} \subset \mathcal{M}$, is an open bounded set with a smooth boundary $\partial\mathcal{Q}$, $s \in (0,1)$, $p \in C(\mathcal{M}, (1;\infty))$ with $sp(z) < n$, $p^+ = \sup_{z\in\overline{\mathcal{M}}} p(z)$, $p^- = \inf_{z\in\overline{\mathcal{M}}} p(z)$, and $q : \overline{\mathcal{M}} \to (1,\infty)$ satisfying $1 < q^- \leq q^+ < p^- \leq p^+ < +\infty$, where $q^+ = \sup\limits_{z\in\overline{\mathcal{M}}} q(z)$, $q^- = \inf\limits_{z\in\overline{\mathcal{M}}} q(z)$, We introduce in this section a new fractional Sobolev space with variable exponent and we state our mains results.

Definition 2.1.1. Let $p : \overline{\mathcal{M}} \to (1;\infty)$ be a continuous variable exponent and let $s \in (0,1)$. We define the modular

$$\varrho_{p(\cdot)}(\mathrm{u}) = \int_{\mathcal{M}\times\mathcal{M}} \frac{|\mathrm{u}(z) - \mathrm{u}(y)|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y),$$

where d_g is the distance between z and y, and dv_g is Riemmanian volume, see definitions 1.1.3 and 1.1.1.

We introduce the variable exponent Sobolev fractional space in complete manifolds as follows:

$$W^{s,p(z)}(\mathcal{M}) = \left\{ \mathrm{u} : \mathcal{M} \to \mathbb{R};\ \mathrm{u} \in L^{p(z)}(\mathcal{M}) \text{ such as } \varrho_{p(\cdot)}(\mathrm{u}) < \infty \right\}.$$

Consequently,

$$||u||_{\varrho_{p(\cdot)}} = \inf\left\{\lambda > 0;\ \varrho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\} = [u]_{W^{s,p(z)}(\mathcal{M})}.$$

The modular $\varrho_{p(\cdot)}$ has the following properties.

2.1.1 Fundamental proprieties of a new space

In this part, we will go through some of our new fractional space's qualitative lemmas.

Lemma 2.1.2. *Let $p \in C(\mathcal{M}, (1;\infty))$ be a continuous variable exponent. Then for any $u \in W^{s,p(z)}(\mathcal{M})$, we get*

(1) $[u]_{W^{s,p(z)}(\mathcal{M})} \geq 1 \Longrightarrow [u]^{p^-}_{W^{s,p(z)}(\mathcal{M})} \leq \varrho_{p(\cdot)}(u) \leq [u]^{p^+}_{W^{s,p(z)}(\mathcal{M})}$

(2) $[u]_{W^{s,p(z)}(\mathcal{M})} \leq 1 \Longrightarrow [u]^{p^+}_{W^{s,p(z)}(\mathcal{M})} \leq \varrho_{p(\cdot)}(u) \leq [u]^{p^-}_{W^{s,p(z)}(\mathcal{M})}$

Proof. (1) For all $\theta \in (0,1)$, we have

$$\theta^{p^+}\varrho_{p(\cdot)}(u) \leq \varrho_{p(\cdot)}(\theta u) \leq \theta^{p^-}\varrho_{p(\cdot)}(u).$$

So, if $[u]_{W^{s,p(z)}(\mathcal{M})} > 1$, then $0 < \dfrac{1}{[u]_{W^{s,p(z)}(\mathcal{M})}} < 1$, so we have

$$\frac{\varrho_{p(\cdot)}(u)}{[u]^{p^+}_{W^{s,p(z)}(\mathcal{M})}} \leq \varrho_{p(\cdot)}\left(\frac{u}{[u]_{W^{s,p(z)}(\mathcal{M})}}\right) \leq \frac{\varrho_{p(\cdot)}(u)}{[u]^{p^-}_{W^{s,p(z)}(\mathcal{M})}},$$

and since, $\varrho_{p(\cdot)}\left(\dfrac{u}{[u]_{W^{s,p(z)}(\mathcal{M})}}\right) = 1$, we obtain our result. We proceed in the same way for (2). □

Remark 2.1.3. It is important to note that $\varrho_{p(\cdot)}$ fulfils the results of Proposition 1.4.2.

Lemma 2.1.4. *If $(\mathcal{M}, g)$ be a smooth complete compact Riemannian n-manifold, then $W^{s,p(z)}(\mathcal{M})$ is a Banach space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in $W^{s,p(z)}(\mathcal{M})$, since

$$p(z) < p_s^*(z) = \begin{cases} \frac{np_s(z)}{n-p_s(z)} & \text{if } p(z) < n\,, \\ +\infty & \text{if not}\,, \end{cases}$$

for any $z \in \overline{\mathcal{M}}$, it follows that for any $\eta > 0$, there exists μ_η such that, if ℓ, $m \geq \mu_\eta$,

$$||u_\ell - u_m||_{L^{p(z)}(\mathcal{M})} \leq ||u_\ell - u_m||_{W^{s,p(z)}(\mathcal{M})} \leq \eta. \tag{2.1}$$

Since $L^{p(z)}(\mathcal{M})$ is complete (see [47, Lemma 2.5]), there exists $u \in L^{p(z)}(\mathcal{M})$ such that $u_\ell \to u$ strongly in $L^{p(z)}(\mathcal{M})$ as $\ell \to +\infty$. By consequence, we may find subsequences

$\{u_{\ell_t}\}$ of $\{u_\ell\}$ in $W^{s,p(z)}(\mathcal{M})$, such that $u_{\ell_t} \to u$ a.e on $\mathcal{M}$.
Consequently, by the Fatou's Lemma and (2.1) with $\eta = 1$, we obtain

$$\begin{aligned}
\varrho_{p(z)}(u) &= \int_{\mathcal{M}\times\mathcal{M}} \frac{|u(z)-u(y)|^{p(z)}}{(d_g(z,y))^{n+sp(z)}}\, dv_g(z)\, dv_g(y)\\
&\le \liminf_{t\to+\infty} \int_{\mathcal{M}\times\mathcal{M}} \frac{|u_{\ell_t}(z)-u_{\ell_t}(y)|^{p(z)}}{(d_g(z,y))^{n+sp(z)}}\, dv_g(z)\, dv_g(y)\\
&\le 2^{p^+-1} \liminf_{t\to+\infty} \Bigg[\int_{\mathcal{M}\times\mathcal{M}} \frac{|(u_{\ell_t}(z)-u_{\mu_1}(y))-(u_{\ell_t}(y)-u_{\mu_1}(z))|^{p(z)}}{(d_g(z,y))^{n+sp(z)}}\, dv_g(z)\, dv_g(y)\\
&\quad + \int_{\mathcal{M}\times\mathcal{M}} \frac{|u_{\mu_1}(z)-u_{\mu_1}(y)|^{p(x)}}{(d_g(z,y))^{n+sp(z)}}\, dv_g(z)\, dv_g(y) \Bigg]\\
&\le 2^{p^+-1} \big[\liminf_{t\to+\infty} \varrho_{p(\cdot)}(u_{\ell_t}-u_{\mu_1}) + \varrho_{p(\cdot)}(u_{\mu_1}) \big]\\
&\le 2^{p^+-1} \Bigg[\liminf_{t\to+\infty} \Big(||u_{\ell_t}-u_{\mu_1}||^{p^+}_{W^{s,p(z)}(\mathcal{M})}\\
&+ ||u_{\ell_t}-u_{\mu_1}||^{p^-}_{W^{s,p(z)}(\mathcal{M})} \Big) + \Big(||u_{\mu_1}||^{p^+}_{W^{s,p(z)}(\mathcal{M})} + ||u_{\mu_1}||^{p^-}_{W^{s,p(z)}(\mathcal{M})} \Big) \Bigg]\\
&\le 2^{p^+-1} \big(2 + ||u_{\mu_1}||^{p^+}_{W^{s,p(z)}(\mathcal{M})} + ||u_{\mu_1}||^{p^-}_{W^{s,p(z)}(\mathcal{M})}\big) < +\infty.
\end{aligned}$$

Hence, $u \in W^{s,p(z)}(\mathcal{M})$. On the other hand, let $\ell \ge \mu_\eta$, and according to (2.1) and Fatou's Lemma, we get

$$\varrho_{p(\cdot)}(u_\ell - u) \le \liminf_{t\to+\infty} \varrho_{p(\cdot)}(u_\ell - u_{\ell_t}) \le \frac{\eta^{p^+}+\eta^{p^-}}{2} = \eta^*.$$

Thus $\lim_{\ell\to+\infty} \varrho_{p(\cdot)}(u_\ell - u) = 0$. Thanks to Remark 2.1.3, $\lim_{\ell\to+\infty} ||u_\ell - u||_{W^{s,p(z)}(\mathcal{M})} = 0$. That is, $u_\ell \to u$ strongly on $W^{s,p(z)}(\mathcal{M})$ as $\ell \to +\infty$. □

Lemma 2.1.5. *Let $(\mathcal{M}, g)$ be a compact Riemannian n-manifold, and $p(z) \in C(\mathcal{M}, (1,\infty))$ with $sp(z) < n$, for $z \in \overline{\mathcal{M}}$. Then $W^{s,p(z)}(\mathcal{M})$ is a separable and reflexive space.*

Proof. Given $u, v \in W_0^{s,p(z)}(\mathcal{M})$ satisfy $\|u\|_{W_0^{s,p(z)}(\mathcal{M})} = \|v\|_{W_0^{s,p(z)}(\mathcal{M})} = 1$ and $\|u - v\|_{W_0^{s,p(z)}(\mathcal{M})} \ge \varepsilon$, where $\varepsilon \in (0,2)$.
For $p(z) \ge 2$: By [7, inequality (28)], we have that

$$\begin{aligned}
&\left\|\frac{u+v}{2}\right\|^{p(z)}_{W_0^{s,p(z)}(\mathcal{M})} + \left\|\frac{u-v}{2}\right\|^{p(z)}_{W_0^{s,p(z)}(\mathcal{M})} \le \left(\frac{1}{2}\right)^{\frac{p^+}{p^-}} \int_{\mathcal{M}\times\mathcal{M}} \frac{|u(z)-u(y)|^{p(z)}}{\big(d_g(z,y)\big)^{n+sp(z)}} dv_g(z) dv_g(y)\\
&+ \left(\frac{1}{2}\right)^{\frac{p^+}{p^-}} \int_{\mathcal{M}\times\mathcal{M}} \frac{|v(z)-v(y)|^{p(z)}}{\big(d_g(z,y)\big)^{n+sp(z)}} dv_g(z) dv_g(y)\\
&< \frac{1}{2}\left[\int_{\mathcal{M}\times\mathcal{M}} \frac{|u(z)-u(y)|^{p(z)}}{\big(d_g(z,y)\big)^{n+sp(z)}} dv_g(z) dv_g(y) + \int_{\mathcal{M}\times\mathcal{M}} \frac{|v(z)-v(y)|^{p(z)}}{\big(d_g(z,y)\big)^{n+sp(z)}} dv_g(z) dv_g(y) \right]\\
&= \frac{1}{2}\|u\|^{p(z)}_{W_0^{s,p(z)}(\mathcal{M})} + \frac{1}{2}\|v\|^{p(z)}_{W_0^{s,p(z)}(\mathcal{M})}\\
&= 1.
\end{aligned}$$

So, $\left\|\frac{u+v}{2}\right\|^{p(z)}_{W_0^{s,p(z)}(\mathcal{M})} \leq 1-(\varepsilon/2)^{p(z)}$. Taking $\delta=\delta(\varepsilon)$ such as $1-(\varepsilon/2)^{p(z)}=(1-\delta)^{p(z)}$, we obtain $\left\|\frac{u+v}{2}\right\|_{W_0^{s,p(z)}(\mathcal{M})} \leq (1-\delta)$.

For $1<p(z)<2$: Let $p'(z)=\frac{p(z)}{(p(z)-1)}$ we have

$$\|u\|^{p'(z)}_{W_0^{s,p(z)}(\mathcal{M})} = \left[\int_{\mathcal{M}\times\mathcal{M}} \left(\left(\frac{|u(z)-u(y)|}{\left(d_g(z,y)\right)^{\frac{n}{p(z)}+s}}\right)^{p'(z)}\right)^{p(z)-1} dv_g(z)dv_g(y)\right]^{\frac{1}{p(z)-1}}.$$

As a result of Minkowski inequality (see [21, Theorem 2.13] and [7, inequality (27)]), we obtain that

$$\begin{aligned}
&\left\|\frac{u+v}{2}\right\|^{p'(z)}_{W_0^{s,p(z)}(\mathcal{M})} + \left\|\frac{u-v}{2}\right\|^{p'(z)}_{W_0^{s,p(z)}(\mathcal{M})} \\
&\leq \left\{\int\int_{\mathcal{M}\times\mathcal{M}} \left[\left(\left|\frac{(u(z)-u(y))+(v(z)-v(y))}{2\left(d_g(z,y)\right)^{\frac{n}{p(z)}+s}}\right|\right)^{p'(z)}\right.\right. \\
&\left.\left.+\left(\left|\frac{(u(z)-u(y))-(v(z)-v(y))}{2\left(d_g(z,y)\right)^{\frac{n}{p(z)}+s}}\right|\right)^{p'(z)}\right]^{p(z)-1} dv_g(z)dv_g(y)\right\}^{\frac{1}{p(z)-1}} \\
&\leq \left(\frac{1}{2}\|u\|^{p(z)}_{W_0^{s,p(z)}(\mathcal{M})} + \frac{1}{2}\|v\|^{p}_{W_0^{s,p(z)}(\mathcal{M}))}\right)^{p'(z)-1} = 1.
\end{aligned}$$

Hence,

$$\left\|\frac{u+v}{2}\right\|^{p'(z)}_{W_0^{s,p(z)}(M)} \leq 1-\frac{\varepsilon^{p'(z)}}{2^{p'(z)}}$$

Taking $\delta=\delta(\varepsilon)$ such as $1-(\varepsilon/2)^{p'(z)}=(1-\delta)^{p'(z)}$. By the the Milman–Pettis theorem, $W_0^{s,p(z)}(\mathcal{M})$ is reflexive.

Now, we show that $W_0^{s,p(z)}(\mathcal{M})$ is a separable space. Defining the operator

$$\begin{aligned}
T : W^{s,p(z)}(\mathcal{M}) &\longrightarrow L^{p(z)}(\mathcal{M})\times L^{p(z)}(\mathcal{M}\times\mathcal{M}) \\
u &\longmapsto T(u)=\left(u(z), \frac{u(z)-u(y)}{d_g(z,y)^{\frac{n}{p(z)}+s}}\right).
\end{aligned}$$

- T is well defined.
- T is an isometry.

Indeed, for $u\in W_0^{s,p(z)}(\mathcal{M})$, we obtain

$$\|T(u)\|_{L^{p(z)}(\mathcal{M})\times L^{p(z)}(\mathcal{M}\times\mathcal{M})} = \|u\|_{L^{p(z)}(\mathcal{M})} + \left\|\frac{u(z)-u(y)}{d_g(z,y)^{\frac{n}{p(z)}+s}}\right\|_{L^{p(z)}(\mathcal{M}\times\mathcal{M})} = \|u\|_{W_0^{s,p(z)}(\mathcal{M})}.$$

So, $T(W_0^{s,p(z)}(\mathcal{M}))$ is a closed subspace of $L^{p(z)}(\mathcal{M})\times L^{p(z)}(\mathcal{M}\times\mathcal{M})$. Thanks to [21, Proposition 3.17], $T(W_0^{s,p(z)}(\mathcal{M}))$ is separable, therefore $W_0^{s,p(z)}(\mathcal{M})$ is also separable. □

2.1.2 Embedding compact and density

Lemma 2.1.6. *Suppose that $(\mathcal{M}, \mathrm{g})$ satisfies property $B_{vol}(\lambda, v)$ with finite volume. Then $C_0^\infty(\mathcal{M})$ is dense in $W^{s,p(z)}(\mathcal{M})$.*

Proof. Consider the following real-valued function

$$f(t) = \begin{cases} 1 & \text{if } t \leq 0\,, \\ 1-t & \text{if } 0 \leq t \leq 1\,, \\ 0 & \text{if } t \geq 1\,. \end{cases}$$

Let $\varphi \in C^\infty(\mathcal{M}) \cap W^{s,p(z)}(\mathcal{M})$, and y a fixed point of $\mathcal{M}$ such as $\varphi_\nu(\alpha) = \varphi(\alpha) f(d_{\mathrm{g}}(y,\alpha)))$, where d_{g} is Riemannian distance associated to g and $\nu \in \mathbb{N}$. We can easily see that $\varphi_\nu(\alpha) \in W^{s,p(z)}(\mathcal{M})$ for $\nu \in \mathbb{N}$. Then, since $\mathcal{M}$ is a compact Riemannian n-manifolds, we can be covered by a finite number of charts $(\mathcal{Q}_k, \phi_k)_{k=1,\dots,m}$. Let η_k be a smooth partition of unity subordinate to the covering $\mathcal{Q}_k$. We can see that $h = \eta_k \varphi_\nu \circ \phi_k^{-1} \in W^{s,p(z)}(\phi_k(\mathcal{Q}_k))$. So, by [15, Lemma 3.2], we can extract a subsequence $h_t \in C^\infty(\mathbb{R})$ such that $h_t \to h$ strongly in $W^{s,p(z)}(\phi_k(\mathcal{Q}_k))$ as $t \to \infty$. So, $h_t \circ \phi_k \in C^\infty(\mathcal{M})$ and $h_t \circ \phi_k$ converge strongly to $\eta_k \varphi_\nu$ in $W^{s,p(z)}(\mathcal{M})$ as $t \to \infty$. □

Remark 2.1.7. We can also prove the previous lemma by using the following method: For $\mathrm{u} \in C_0^\infty(\mathcal{M})$, we need to prove that

$$\int\int_{\mathcal{M}\times\mathcal{M}} \frac{|\mathrm{u}(z) - \mathrm{u}(y)|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y) < \infty.$$

Notice that, $\forall (z,y) \in \mathcal{M} \times \mathcal{M}$, we have

$$|\mathrm{u}(z) - \mathrm{u}(y)| \leq ||D\mathrm{u}||_{L^\infty(\mathcal{M})}\,.\, d_{\mathrm{g}}(z,y), \;\; |\mathrm{u}(z) - \mathrm{u}(y)| \leq 2\,.\,||\mathrm{u}||_{L^\infty(\mathcal{M})}.$$

Thus,

$$|\mathrm{u}(z) - \mathrm{u}(y)|^{p(z)} \leq ||D\mathrm{u}||^{p(z)}_{L^\infty(\mathcal{M})}\,.\,(d_{\mathrm{g}}(z,y))^{p(z)}, \text{ for all } (z,y) \in \mathcal{M} \times \mathcal{M},$$

and

$$|\mathrm{u}(z) - \mathrm{u}(y)|^{p(z)} \leq 2^{p(z)} ||\mathrm{u}||^{p(z)}_{L^\infty(\mathcal{M})}, \text{ for all } (z,y) \in \mathcal{M} \times \mathcal{M}.$$

Hence,

$$|\mathrm{u}(z) - \mathrm{u}(y)|^{p(z)} \leq \left(||D\mathrm{u}||^{p^+}_{L^\infty(\mathcal{M})} + ||D\mathrm{u}||^{p^-}_{L^\infty(\mathcal{M})}\right)\,.\,(d_{\mathrm{g}}(z,y))^{p(z)}, \text{ for all } (z,y) \in \mathcal{M} \times \mathcal{M},$$

and

$$|\mathrm{u}(z) - \mathrm{u}(y)|^{p(z)} \leq 2^{p^+}\left(||\mathrm{u}||^{p^+}_{L^\infty(\mathcal{M})} + ||\mathrm{u}||^{p^-}_{L^\infty(\mathcal{M})}\right), \text{ for all } (z,y) \in \mathcal{M} \times \mathcal{M}.$$

Hence

$$|\mathrm{u}(z) - \mathrm{u}(y)|^{p(z)} \leq 2^{p^+-1}\left(||\mathrm{u}||^{p^+}_{C^1(\mathcal{M})} + ||\mathrm{u}||^{p^-}_{C^1(\mathcal{M})}\right)\,.\,\min\{1, (d_{\mathrm{g}}(z,y))^{p(z)}\}.$$

Therefore, according to [49], we obtain

$$\begin{aligned}
&\int_{\mathcal{M}\times\mathcal{M}} \frac{|(\eta_s \mathrm{u})(z)-(\eta_s \mathrm{u})(y)|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y)\\
&\leq \mathrm{vol}\,.\,2^{p^+-1}\big(||\mathrm{u}||^{p^+}_{C^1(\mathcal{M})}+||\mathrm{u}||^{p^-}_{C^1(\mathcal{M})}\big)\int_{\mathcal{M}\times\mathcal{M}} \frac{\min\{1,\,(d_{\mathrm{g}}(z,y))^{p(z)}\}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y)\\
&< \infty,
\end{aligned}$$

where, (η_s) is a smooth partition of unity subordinate of the covering $B_{z_k}(r)$ for any k, and $B_{z_k}(r)$ denotes the Euclidean ball of $\mathbb{R}^n$ of centre z_k and radius r. Then we deduce that for $\mathrm{u}\in C_0^\infty(\mathcal{M})$, we have

$$\int_{\mathcal{M}\times\mathcal{M}} \frac{|\mathrm{u}(z)-\mathrm{u}(y)|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y)<\infty.$$

Thus $\mathrm{u}\in W^{s,p(z)}(\mathcal{M})$, and according to the fact that $C_0^\infty(\mathcal{M})$ is dense in $L^{p(z)}(\mathcal{M})$ (see [49]), we conclude that C_0^∞ is dense also in $W^{s,p(z)}(\mathcal{M})$.

Now, we will extend an embedding result between $W^{1,p(z)}(\mathcal{M})$ and $W^{s,p(z)}(\mathcal{M})$ to manifolds.

Lemma 2.1.8. *Suppose that the compact Riemannian n-manifold $(\mathcal{M},\mathrm{g})$ has property $B_{vol}(\lambda,v)$ for some (λ,v), $p\in C(\mathcal{M},(1,+\infty))$ and $s\in(0,1)$. Then $||\mathrm{u}||_{W^{s,p(z)}(\mathcal{M})}\leq C\,||\mathrm{u}||_{W^{1,p(z)}(\mathcal{M})}$, where $C=C(n,s,\lambda,v,p^+,p^-)>0$. In particular, $W^{1,p(z)}(\mathcal{M})\subset W^{s,p(z)}(\mathcal{M})$.*

Proof. As in [48], let $z,y\in\mathcal{M}$ considering the minimizing geodesic map $\varphi\in(0,1)\to\mathcal{M}$ from z and y. Then for $\mathrm{u}\in W^{s,p(z)}(\mathcal{M})$, we have

$$\begin{aligned}
&\int_{\mathcal{M}}\int_{\mathcal{M}\cap\{d_{\mathrm{g}}(z,y)<1\}} \frac{|\mathrm{u}(z)-\mathrm{u}(y)|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_g(y) \qquad (2.2)\\
&\leq \int_{\mathcal{M}}\int_{\mathcal{M}\cap\{d_{\mathrm{g}}(z,y)<1\}}\int_0^1 \frac{|D\mathrm{u}(\varphi(t))|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y)\, dt\\
&\leq \int_{\mathcal{M}} \frac{||D\mathrm{u}||_{L^{p(z)}(\mathcal{M})}}{(d_{\mathrm{g}}(z,y))^{n+(s-1)p(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y)\\
&\leq C(n,s,\lambda,v,p^+,p^-)\,.\,||D\mathrm{u}||_{L^{p(z)}(\mathcal{M})}, \qquad (2.3)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathcal{M}}\int_{\mathcal{M}\cap\{d_{\mathrm{g}}(z,y)\geq 1\}} \frac{|\mathrm{u}(z)-\mathrm{u}(y)|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y)\\
&\leq 2^{p^+-1}\int_{\mathcal{M}}\int_{\mathcal{M}\cap\{d_{\mathrm{g}}(z,y)\geq 1\}} \frac{\big(|\mathrm{u}(z)|^{p(z)}-|\mathrm{u}(y)|^{p(z)}\big)\,\min\{1,\,(d_{\mathrm{g}}(z,y))^{p(z)}\}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y)\\
&\leq C(n,s,\lambda,v,p^+)\,||\mathrm{u}||_{L^{p(z)}(\mathcal{M})}. \qquad (2.4)
\end{aligned}$$

According to (2.2) and (2.4), we get that

$$||\mathrm{u}||_{W^{s,p(z)}(\mathcal{M})} \leq C(n,s,\lambda,v,p^+,p^-)\,.\,||\mathrm{u}||_{W^{1,p(z)}(\mathcal{M})},$$

which completes the proof. □

Theorem 2.1.9. *Given $\mathcal{M}$ a compact Riemannian manifold, $p \in C(\mathcal{M},\,(1;\infty))$, $s \in (0,1)$ with $sp(z) < n$ and $q \in C(\mathcal{M},\,(1;\infty))$. Assume that*

$$1 < q^- = \min_{z\in\mathcal{M}} q(z) \leq q(z) < \frac{n\,p(z)}{n-p(z)} \;\; \textit{for all} \;\; x \in \overline{\mathcal{M}},$$

then $W^{s,p(z)}(\mathcal{M}) \hookrightarrow L^{q(z)}(\mathcal{M})$ is a continuous and compact embedding.

Proof. The demonstration of this theorem is based to an idea introduced in [39, 47, 48]. Let $\varphi : V \subset \mathcal{M} \to \mathbb{R}^n$ be an arbitrary local chart on $\mathcal{M}$, and $G \subset \mathcal{M}$ an open set with closure compact and contrained in V. Taking $\{G_l\}_{l=1,\cdots,k}$ of $\mathcal{M}$ a finite subcovering, such as G_l is homeomorphic to the open unit ball $B_0(1)$ of $\mathbb{R}^n$ and for any l the components g^l_{ij} of g in $(\mathrm{g}_l,\,V_l)$ satisfy

$$\frac{1}{\alpha\,\delta_{ij}} < \mathrm{g}^l_{ij} < \alpha\,\delta_{ij},$$

like bilinear forms, for some constant $\alpha > 1$. Let $\{\pi_l\}_{l=1,\cdots,k}$ be a smooth partition of unity subordinate to the finite covering $\{G_l\}_{l=1,\cdots,k}$. It is clear that if $\mathrm{u} \in W^{s,p(z)}(\mathcal{M})$, then $\pi_l\,\mathrm{u} \in W^{s,p(z)}(G_l)$ and $(\varphi_l^{-1}) * (\pi_l \mathrm{u}) \in W^{s,p(\varphi_l^{-1}(z))}(B_0(1))$ with $\mathrm{u} = \sum_{l=1}^{k} \pi_l \mathrm{u}$. According to Lemma 2.1.8, the Sobolev embedding theorem [39, 47], we get the continuous and compact embedding

$$W^{s,p(z)}(G_l) \hookrightarrow L^{p(z)}(G_l) \;\; \text{for any} \;\; l = 1,\cdots,k.$$

Thus, we can conclude that $W^{s,p(z)}(\mathcal{M}) \subset L^{p(z)}(\mathcal{M})$ and the embedding is continuous and compact. □

2.2 Application: Existence results via variational method

In this part, as an application, we give an existence result to the following problem

$$(\mathcal{P}) \begin{cases} (-\Delta_g)^s_{p(z)}\mathrm{u}(z) + \mathcal{V}(z)\,|\mathrm{u}(z)|^{q(z)-2} = h(z,\mathrm{u}(z)) & \text{in } \mathcal{Q}\,, \\ \mathrm{u}\,|_{\partial\mathcal{Q}} = 0, \end{cases}$$

where $s \in (0,1)$, $p \in C(\mathcal{M},\,(1;\infty))$ with $sp(z) < n$, $q : \overline{\mathcal{Q}} \to (1,\infty)$ satisfy the following condition

$$1 < q^- \leq q^+ < p^- \leq p^+ < \frac{n\,p(z)}{n-p(z)}, \tag{2.5}$$

$(\mathcal{M},\mathrm{g})$ is a compact Riemannian n-manifold, $\mathcal{Q} \subset \mathcal{M}$ is an open bounded set with smooth boundary $\partial\mathcal{Q}$, $h : \mathcal{Q}\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following assumptions:

(h_1) (AR- condition): There exist $\beta > p^+$ and some $I > 0$ such as for each $|\alpha| > I$ we have

$$0 < \int_{\mathcal{M}} H(z,\alpha)\ dv_{\mathrm{g}}(z) \leq \int_{\mathcal{M}} h(z,\alpha)\,.\,\frac{\alpha}{\beta}\ dv_{\mathrm{g}}(z) \text{ a.e } z \in \mathcal{M},$$

where $H(z,\alpha) = \int_0^{\alpha} h(z,v)\ dv$ being the primitive of $h(z,\alpha)$.

(h_2) $h(z,0) = 0$,

(h_3) $\lim_{|\alpha|\to\infty} \frac{h(z,\alpha)}{|\alpha|^{p(z)-1}} = 0$ uniformly a.e $z \in \mathcal{M}$.

and $(-\Delta_{\mathrm{g}})^s_{p(z)}\mathrm{u}(z)$ is the fractional $p(z)-$Laplace operator with (up to normalization factors) may be defined as

$$(-\Delta_{\mathrm{g}})^s_{p(z)}\mathrm{u}(z) = 2 \lim_{\epsilon\to o^+} \int_{M\setminus B_\epsilon(z)} \frac{|\mathrm{u}(z)-\mathrm{u}(y)|^{p(z)-2}(\mathrm{u}(z)-\mathrm{u}(y))}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\ dv_{\mathrm{g}}(z),$$

for $x \in \mathcal{M}$, where $B_\epsilon(z)$ denotes the geodesic ball of $\mathcal{M}$ of centre z and radius ϵ and $d_{\mathrm{g}}(z,y)$ defined a distance on $\mathcal{M}$ whose topology coincides with the original one $\mathcal{M}$. The vanishing potential satisfy the following assumptions:

(i) $\mathcal{V} : \mathcal{M} \to \mathbb{R}$ be a continuous function, and there exists $\theta > 0$, $\gamma > 0$ such as $\mathcal{V}(z) > \theta > 0$ for all $z \in \mathcal{M}$, and

$$\int_{\mathcal{M}} \mathcal{V}(z)\,|\mathrm{u}(z)|^{q(z)}\ dv_{\mathrm{g}}(z) \leq \gamma\,||\mathrm{u}||^{q(z)}_{W^{s,p(z)}(\mathcal{M})},$$

for all $\mathrm{u} \in W_0^{s,p(z)}(\mathcal{M})$.

(ii) $\mathcal{V}(z) \to +\infty$ as $|z| \to +\infty$.

Definition 2.2.1. A measurable function $\mathrm{u} \in W_0^{s,p(z)}(\mathcal{M})$ is said to be a weak solution of $(\mathcal{P})$ if

$$\begin{aligned}&\int_{M\times M} \frac{|\mathrm{u}(z)-\mathrm{u}(y)|^{p(z)-2}(\mathrm{u}(z)-\mathrm{u}(y))\,(\varphi(z)-\varphi(y))}{(d_g(z,y))^{n+sp(z)}}\ dv_g(z)\,dv_g(y)\\ &+\int_M \mathcal{V}(z)|\mathrm{u}(z)|^{q(z)-2}u(x)\varphi(z)\ dv_g(z) = \int_M h(z,\mathrm{u}(z))\,\varphi(z)\ dv_g(z),\end{aligned}$$

for all $\varphi \in W_0^{s,p(z)}(\mathcal{M})$.

Theorem 2.2.2. *Under assumptions* $(h_1)-(h_3)$, (2.5) *and* $(i)-(ii)$, *if* (M,g) *satisfies the property* $B_{vol}(\lambda,v)$. *Then problem* $(\mathcal{P})$ *possesses at least one weak solution.*

Consider the functional $E : W_0^{s,p(z)}(\mathcal{M}) \to \mathbb{R}$ defined by

$$E(\mathrm{u}) = A(\mathrm{u}) - B(\mathrm{u}),$$

where

$$A(\mathrm{u}) = \int_{\mathcal{M}\times\mathcal{M}} \frac{1}{p(z)} \cdot \frac{|\mathrm{u}(z)-\mathrm{u}(y)|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y) + \int_{\mathcal{M}} \frac{\mathcal{V}(z)}{q(z)} |\mathrm{u}(z)|^{q(z)}\, dv_{\mathrm{g}}(z),$$

and

$$B(\mathrm{u}) = \int_{\mathcal{M}} H(z,\mathrm{u}(z))\, dv_{\mathrm{g}}(z).$$

Lemma 2.2.3. *Assume that the assumptions $(i)-(iii)$ hold. Then $A \in C^1(W_0^{s,p(z)}(\mathcal{M}))$ and*

$$\begin{aligned}\langle A'(\mathrm{u}),\, \varphi\rangle = &\int_{\mathcal{M}\times\mathcal{M}} \frac{|\mathrm{u}(z)-\mathrm{u}(y)|^{p(z)-2}(\mathrm{u}(z)-\mathrm{u}(y))(\varphi(z)-\varphi(y))}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y)\\ &+ \int_{\mathcal{M}} \mathcal{V}(z)|\mathrm{u}(z)|^{q(z)-2}\mathrm{u}(z)\varphi(z)\, dv_{\mathrm{g}}(z), \end{aligned} \tag{2.6}$$

for all $\mathrm{u},\, \varphi \in W_0^{s,p(z)}(\mathcal{M})$.

Proof. For $\mathrm{u} \in W_0^{s,p(z)}(\mathcal{M})$. Using Lemma 2.1.2 and (i), we have

$$\begin{aligned} A(\mathrm{u}) &= \int_{\mathcal{M}\times\mathcal{M}} \frac{1}{p(z)} \cdot \frac{|\mathrm{u}(z)-\mathrm{u}(y)|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y) + \int_{\mathcal{M}} \frac{\mathcal{V}(z)}{q(z)}|\mathrm{u}(z)|^{q(z)}\, dv_{\mathrm{g}}(z)\\ &\le \frac{1}{p^-}\int_{\mathcal{M}\times\mathcal{M}} \frac{|\mathrm{u}(z)-\mathrm{u}(y)|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y) + \frac{1}{q^-}\int_{\mathcal{M}} \mathcal{V}(z)\,|\mathrm{u}(z)|^{q(z)}\, dv_{\mathrm{g}}(z)\\ &\le \frac{1}{p^-}\left(||\mathrm{u}||^{p^+}_{W_0^{s,p(z)}(\mathcal{M})} + ||\mathrm{u}||^{p^-}_{W_0^{s,p(z)}(\mathcal{M})}\right) + \frac{\gamma}{q^+}\left(||\mathrm{u}||^{q^+}_{W_0^{s,p(z)}(\mathcal{M})} + ||\mathrm{u}||^{q^-}_{W_0^{s,p(z)}(\mathcal{M})}\right)\\ &< +\infty. \end{aligned}$$

Hence, A is well defined.
To prove that $A \in C^1(W_0^{s,p(z)}(\mathcal{M}))$, we consider $\{\mathrm{u}_t\} \subset W_0^{s,p(z)}(\mathcal{M})$ such as $\mathrm{u}_t \to \mathrm{u}$ strongly in $W_0^{s,p(z)}(\mathcal{M})$ as $t \to +\infty$. Without losing generality, we further assume that $\mathrm{u}_t \longrightarrow \mathrm{u}$ a.e in $\mathcal{M}$ as $t \to +\infty$. So, we get

$$\left\{ \frac{|\mathrm{u}_t(z)-\mathrm{u}_t(y)|^{p(z)}\,(\mathrm{u}_t(z)-\mathrm{u}_t(y))}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}} \right\}_t',$$

is bounded in $L^{p(z)}(\mathcal{M})$ and according to Brezis Lieb Lemma in [42] we obtain

$$\lim_{t\to+\infty}\int_{\mathcal{M}\times\mathcal{M}} \left(\frac{|\mathrm{u}_t(z)-\mathrm{u}_t(y)|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}} - \frac{|\mathrm{u}(z)-\mathrm{u}(y)|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}} \right) dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y) = 0.$$

Similarly,

$$\lim_{t\to+\infty}\int_{\mathcal{M}} \mathcal{V}(z) \left| |\mathrm{u}_t(z)|^{p(z)-2}\mathrm{u}_t(z) - |\mathrm{u}(z)|^{p(z)-2}\mathrm{u}(z) \right|^{p(z)} dv_{\mathrm{g}}(z) = 0.$$

Hölder's inequality, gives

$$||A'(\mathrm{u}_t) - A'(\mathrm{u})||_{(W_0^{s,p(z)}(\mathcal{M}))'} = \sup_{\varphi\in W_0^{s,p(z)}(\mathcal{M});\, ||\varphi||_{W_0^{s,p(z)}(\mathcal{M})}=1} |\langle A'(\mathrm{u}_t) - A'(\mathrm{u}),\, \varphi\rangle| \to 0$$

as $t \to +\infty$. Hence, $A \in C^1(W_0^{s,p(z)}(\mathcal{M}))$. Finally, we can easily verify that the functional A is Gâteaux differentiable on $W_0^{s,p(z)}(\mathcal{M})$ and (2.6) holds for all u, $\varphi \in W_0^{s,p(z)}(\mathcal{M})$. □

Lemma 2.2.4. *The functional E satisfies the Mountain pass geometry in the sense that*

(i) $E(0) = 0.$,

(ii) There exist ζ, $\mu > 0$ *such that* $E(\mathrm{u}) \geq \mu$ *if* $||\mathrm{u}||_{W_0^{s,p(z)}(\mathcal{M})} > \zeta$,

(iii) There exists u, $||\mathrm{u}||_{W_0^{s,p(z)}(\mathcal{M})} > \zeta$ *such that* $E(\mathrm{u}) < 0$.

Proof. i/ $E(0) = 0$ is obvious.

ii/ According to (h_1), (i), Lemmas 2.1.2, 1.4.2, and Theorem 2.1.9 we have

$$\begin{aligned} E(\mathrm{u}) &= \int_{\mathcal{M}\times\mathcal{M}} \frac{1}{p(z)} \frac{|\mathrm{u}(z)-\mathrm{u}(y)|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y) + \int_{\mathcal{M}} \frac{\mathcal{V}(z)}{q(z)} |\mathrm{u}(z)|^{q(z)}\, dv_{\mathrm{g}}(z) \\ &\quad - \int_{\mathcal{M}} H(z,\mathrm{u}(z))\, dv_{\mathrm{g}}(z) \\ &\geq \frac{1}{p^+}\varrho_{p(\cdot)}(\mathrm{u}) + \frac{\theta}{q^+}||\mathrm{u}||^{q^+}_{L^{q(z)}(\mathcal{M})} - \frac{1}{\beta}\,||\mathrm{u}||^{q^-}_{W_0^{s,p(z)}(\mathcal{M})} \\ &\geq \frac{1}{p^+}||\mathrm{u}||^{p^+}_{W_0^{s,p(z)}(\mathcal{M})} + \frac{\theta\, C^{q^+}}{q^+}||\mathrm{u}||^{q^+}_{W_0^{s,p(z)}(\mathcal{M})} - \frac{1}{p^+}\,||\mathrm{u}||^{q^-}_{W_0^{s,p(z)}(\mathcal{M})}. \end{aligned}$$

Since $p^+ > q^+ \geq q^-$, we deduce that $E(\mathrm{u}) > 0$.

iii/ Using there exists $c > 0$ such that

$$H(x,t) > c_2, \quad \text{for all } |t| > c \text{ and } c_2 > 0.$$

According to Lemma 2.1.2, (h_1), (h_3) and (i) we have for $t > 1$ and $\mathrm{u} \neq 0$,

$$\begin{aligned} E(t\mathrm{u}) &= \int_{\mathcal{M}\times\mathcal{M}} \frac{t^{p(z)}}{p(z)} \cdot \frac{|\mathrm{u}(z)-\mathrm{u}(y)|^{p(z)}}{(d_{\mathrm{g}}(z,y))^{n+sp(z)}}\, dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y) \\ &\quad + \int_{\mathcal{M}} \frac{\mathcal{V}(z)}{q(z)} t^{q(z)} |\mathrm{u}(z)|^{q(z)}\, dv_{\mathrm{g}}(z) - \int_{\mathcal{M}} H(z,t\mathrm{u}(z))\, dv_{\mathrm{g}}(z) \\ &\leq \frac{t^{p^+}}{p^-}\varrho_{p(\cdot)}(\mathrm{u}) + \frac{\gamma\, t^{q^-}}{q^-}||\mathrm{u}||^{q^-}_{W_0^{s,p(z)}(\mathcal{M})} - \int_{\{|tu|>c\}} H(z,t\mathrm{u}(z))\, dv_{\mathrm{g}}(z) \\ &\quad - \int_{\{|tu|\leq c\}} H(z,t\mathrm{u}(z))\, dv_{\mathrm{g}}(z) \\ &\leq \frac{t^{p^+}}{p^-}\varrho_{p(\cdot)}(\mathrm{u}) + \frac{\gamma\, t^{q^-}}{q^-}||\mathrm{u}||^{q^-}_{W_0^{s,p(z)}(\mathcal{M})} - c_2 t^{p+1} \int_{\mathcal{M}} |u|^{p^+ +1} dv_{\mathrm{g}}(z) \end{aligned}$$

since $q^- < p^-$, for t small enough, we infer that $E(t\mathrm{u}) < 0$. This completes the demonstration of the lemma.

□

Lemma 2.2.5. *Let $(\mathcal{M}, \mathrm{g})$ be a smooth complete compact Riemannian n-manifolds and $p(\cdot) \in C_+(\mathcal{M})$. Assume that assumptions $(h_1)-(h_3)$ and $(i)-(ii)$ hold. Then the functional E satisfies the Palais-Smale condition.*

Proof. Let $(\mathrm{u}_m)_{m\in\mathbb{N}} \subset W_0^{s,p(z)}(\mathcal{M})$ such that $E(\mathrm{u}_m)$ is bounded and $E'(\mathrm{u}_m) \to 0$ as $m \to \infty$, there exists $C > 0$ such that $dE(\mathrm{u}_m, \mathrm{u}_m) \leq C||\mathrm{u}_m||_{W_0^{s,p(z)}}$ and $|E(\mathrm{u}_m)| < C$. We prove that $(\mathrm{u}_m)_{m\in\mathbb{N}}$ is bounded in $W_0^{s,p(z)}(\mathcal{M})$.
We prove the statement by using the method of contradiction. Indeed, if the assertion were not true, up to a subsequence, $||\mathrm{u}_m||_{W_0^{s,p(z)}(\mathcal{M})} \to \infty$ as $m \longrightarrow \infty$. From (h_1), we have

$$C + C||\mathrm{u}_m||_{W_0^{s,p(z)}(\mathcal{M})} \geq E(\mathrm{u}_m) - \frac{1}{\beta}\langle E'(\mathrm{u}_m),\, \mathrm{u}_m\rangle \geq \left(\frac{1}{p^+} - \frac{1}{\beta}\right) ||\mathrm{u}_m||^{p^+}_{W_0^{s,p(z)}(\mathcal{M})}.$$

This is a contradiction as $||\mathrm{u}_m||_{W_0^{s,p(z)}(\mathcal{M})} \to \infty$. Since $W_0^{s,p(z)}(\mathcal{M})$ is reflexive Banach space, there exists $\mathrm{u} \in W_0^{s,p(z)}(\mathcal{M})$ such that up to a subsequence,

$$\begin{cases} \mathrm{u}_m \rightharpoonup \mathrm{u} & \text{weakly in} \quad W_0^{s,p(z)}(\mathcal{M}), \\ \mathrm{u}_m \longrightarrow \mathrm{u} & \text{a.e in} \quad \mathcal{M}, \\ \mathrm{u}_m \to \mathrm{u} & \text{strongly in} \quad L^{\gamma(z)}(\mathcal{M}) \ \text{ for } 1 \leq \gamma(z) < p_s^*(z). \end{cases}$$

Since $\mathrm{u}_m \longrightarrow \mathrm{u}$ a.e in $\mathcal{M}$, we have that

$$\frac{|\mathrm{u}_m(z) - \mathrm{u}_m(y)|^{p(z)-2}(\mathrm{u}_m(z) - \mathrm{u}_m(y))}{(d_{\mathrm{g}}(z,y))^{\left(\frac{n}{p(z)}+s\right)(p(z)-1)}} \longrightarrow \frac{|\mathrm{u}(z) - \mathrm{u}(y)|^{p(z)-2}(\mathrm{u}(z) - \mathrm{u}(y))}{(d_{\mathrm{g}}(z,y))^{\left(\frac{n}{p(z)}+s\right)(p(z)-1)}}$$

a.e. $(z, y) \in \mathcal{M} \times \mathcal{M}$. On the other hand, $\|\mathrm{u}_m\|_{W_0^{s,p(z)}(\mathcal{M})}$ is uniformly bounded implies that there exists a constant $C > 0$ such that

$$\int_{\mathcal{M}\times\mathcal{M}} \left(\frac{|\mathrm{u}_m(z) - \mathrm{u}_m(y)|}{d_{\mathrm{g}}(z,y)^{\frac{n}{p(z)}+s}}\right)^{p(z)} dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y) \leq C \quad \text{for all} \quad m \geq 1$$

that is,

$$\int_{\mathcal{M}\times\mathcal{M}} \left|\frac{|\mathrm{u}_m(z) - \mathrm{u}_m(y)|^{p(z)-2}(\mathrm{u}_m(z) - \mathrm{u}_m(y))}{d_{\mathrm{g}}(z,y)^{\left(\frac{n}{p(z)}+s\right)(p(z)-1)}}\right|^{\frac{p(z)}{p(z)-1}} dv_{\mathrm{g}}(z)\, dv_{\mathrm{g}}(y) \leq C.$$

Therefore,

$$\frac{|\mathrm{u}_m(z) - \mathrm{u}_m(y)|^{p(z)-2}(\mathrm{u}_m(z) - \mathrm{u}_m(y))}{d_{\mathrm{g}}(z,y)^{\left(\frac{n}{p(z)}+s\right)(p(z)-1)}} \rightharpoonup \frac{|\mathrm{u}(z) - \mathrm{u}(y)|^{p(z)-2}(\mathrm{u}(z) - \mathrm{u}(y))}{d_{\mathrm{g}}(z,y)^{\left(\frac{n}{p(z)}+s\right)(p(z)-1)}}$$

weakly in $L^{p'(z)}(\mathcal{M}\times\mathcal{M})$ with $p'(z)=\frac{p(z)}{p(z)-1}$. If $v\in W_0^{s,p(z)}(\mathcal{M})$, it follows $\frac{v(z)-v(y)}{d_g(z,y)^{\frac{n}{p(z)}+s}}\in L^{p(z)}(\mathcal{M}\times\mathcal{M})$. As a result,

$$\int_{\mathcal{M}\times\mathcal{M}}\frac{|u_m(z)-u_m(y)|^{p(z)-2}\,(u_m(z)-u_m(y))\,(v(z)-v(y))}{d_g(z,y)^{\left(\frac{n}{p(z)}+s\right)(p(z)-1)}d_g(z,y)^{\frac{n}{p(z)}+s}}dv_g(z)\,dv_g(y)$$
$$\longrightarrow\int_{\mathcal{M}\times\mathcal{M}}\frac{|u(z)-u(y)|^{p(z)-2}(u(z)-u(y))(v(z)-v(y))}{d_g(z,y)^{n+sp(z)}}dv_g(z)\,dv_g(y).$$

By Vitali's theorem, we have $E'(u)=0$. We define the sequence $w_n=u_m-u$. Since $(u_m)_{m\in\mathbb{N}}$ is uniformaly bounded and $u_m\longrightarrow u$ a.e in $\mathcal{M}$. Thanks to Brezis-Lieb Lemma in [42], we obtain that

$$\begin{aligned}\int_{\mathcal{M}\times\mathcal{M}}\frac{|u_m(z)-u_m(y)|^{p(z)}}{d_g(z,y)^{n+sp(z)}}dv_g(z)\,dv_g(y)&=\int_{\mathcal{M}\times\mathcal{M}}\frac{|w_m(z)-w_m(y)|^{p(z)}}{d_g(z,y)^{n+sp(z)}}dv_g(z)\,dv_g(y)\\&+\int_{\mathcal{M}\times\mathcal{M}}\frac{|u(z)-u(y)|^{p(z)}}{d_g(z,y)^{n+sp(z)}}dv_g(z)\,dv_g(y)\\&+o(1).\end{aligned}$$

Therefore, a straightforward computation yields,

$$\begin{aligned}\lim_{m\longrightarrow\infty}E(u_m)=\lim_{m\longrightarrow\infty}\int_{\mathcal{M}\times\mathcal{M}}\frac{1}{p(z)}\frac{|w_m(z)-w_m(y)|^{p(z)}}{d_g(z,y)^{n+sp(z)}}dv_g(z)\,dv_g(y)-\int_{\mathcal{M}}H(z,u(z))\;dv_g(z)\\+\int_{\mathcal{M}}\frac{\mathcal{V}(z)}{q(z)}|u(z)|^{q(z)}\;dv_g(z)\\+\int_{\mathcal{M}\times\mathcal{M}}\frac{1}{p(z)}\frac{|u(z)-u(y)|^{p(z)}}{d_g(z,y)^{n+sp(z)}}dv_g(z)\,dv_g(y).\end{aligned}$$

Using $E'(u_m)\longrightarrow 0$ as $m\longrightarrow\infty$, we have

$$\begin{aligned}\int_{\mathcal{M}\times\mathcal{M}}\frac{1}{p(z)}\frac{|w_m(z)-w_m(y)|^{p(z)}}{d_g(z,y)^{n+sp(z)}}dv_g(z)\,dv_g(y)\longrightarrow\int_{\mathcal{M}\times\mathcal{M}}\frac{1}{p(z)}\frac{|u_m(z)-u_m(y)|^{p(z)}}{d_g(z,y)^{n+sp(z)}}dv_g(z)\,dv_g(y)\\+\int_{\mathcal{M}}\mathcal{V}(z)\,|u(z)|^{q(z)}\,dv_g(z)\\-\int_{\mathcal{M}}h(z,u(z))\,.\,u(z)dv_g(z).\end{aligned}$$

By $E'(u)=0$, we have

$$\lim_{m\longrightarrow\infty}\int_{\mathcal{M}\times\mathcal{M}}\frac{1}{p(z)}\frac{|w_m(z)-w_m(y)|^{p(z)}}{d_g(z,y)^{n+sp(z)}}dv_g(z)\,dv_g(y)=0.$$

So, finally, from Lemmas 2.2.4 and 2.2.5, we conclude that our problem $(\mathcal{P})$ possesses a weak solution. □

Conclusion

In this chapter, we have introduced a new fractional Sobolev space whose kernel is $\mathcal{K}(z,y) = \frac{1}{d_g(z,y)^{N+sp(z)}}$ on Riemannian manifolds $(\mathcal{M}, g)$, for which we prove some fundamental properties. Then, we study an existence result in these spaces for an elliptic problem. In the next chapter, we propose a generalization of this space with a symmetrical kernel satisfying Lévy integrability and coercivity conditions.

Chapter 3

Fractional Sobolev Spaces with Kernel Function on Compact Riemannian Manifolds

Let $(\mathcal{M}, g)$ be a compact Riemannian manifold of dimension N. The purpose of this chapter is to present fundamental properties of a new class of Sobolev spaces with a kernel function. Additionally, using the topological degree methods for solving the following equation

$$\begin{cases} (\mathcal{L}_g^K)_{p(y,\cdot)} w(y) &= \lambda\beta(y)|w(y)|^{r(y)-2}w(y) + f(y, w(y)) \quad \text{in} \quad \mathcal{U}, \\ w &= 0 \quad \text{in} \quad \mathcal{M}\backslash\mathcal{U}. \end{cases}$$

Here, $s \in (0,1)$ is fixed, $r \in C(\mathcal{U}, (1,\infty))$, $\lambda > 0$, $\mathcal{U} \subset \mathcal{M}$ is an open bounded subset of $\mathcal{M}$, β is a suitable potential function in $\mathbb{R}^+$ with $\beta \in L^\infty(\mathcal{U})$, $p \in \mathcal{C}(\mathcal{U} \times \mathcal{U}, (1,\infty))$ satisfies the following conditions

$$p(z,a) = p(a,z), \text{ for every } (z,a) \in \mathcal{M}^2, \tag{3.1}$$

$$1 < p^- = \min_{(y,z)\in\mathcal{M}^2} p(y,z) \leq p(y,z) < p^+ = \sup_{(y,z)\in\mathcal{M}^2} p(y,z), \tag{3.2}$$

and $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that:

$(\mathcal{B}_1)$ There exist $\alpha > 0$ and a continuous function $q : \mathcal{M} \to (1, +\infty)$ such that

$$1 < q(y) < p_s^\star(y) = \frac{Np(y,y)}{N - sp(y,y)}$$

and

$$f(y,z) \leq \alpha\left(1 + |z|^{q(y)-1}\right), \text{ a.e. } y \in \mathcal{M},\ z \in \mathbb{R},$$

$$r^+ = \sup_{y\in\mathcal{M}} r(y) \leq q^- = \min_{y\in\mathcal{M}} q(y) \leq p_s^\star(y)$$

and the operator $(\mathcal{L}_g^K)_{p(y,\cdot)}$ is defined by

$$(\mathcal{L}_g^K)_{p(y,\cdot)}w(y) = 2\lim_{\varepsilon\to 0^+}\int_{\mathcal{M}\setminus\mathfrak{B}_\varepsilon(y)} |w(y)-w(z)|^{p(y,z)-2}(w(y)-w(z))K(y,z)dv_g(z),$$

for every $z \in \mathcal{M}$, where,

$$\mathfrak{B}_\varepsilon(y) = \{z \in \mathcal{M} : d_g(y,z) < \varepsilon\} \text{ and } dv_g(z) = |g_{ij}|^{\frac{1}{2}}dz.$$

Furthermore, $K : \mathcal{U}\times\mathcal{U} \to (0,+\infty)$ is a symmetric kernel function satisfying the following variant of Lévy-integrability type condition

$$gK \in L^1\left(\mathcal{M}\times\mathcal{M}, dv_g(y)dv_g(z)\right), \text{ where } g(y,z) = \min\{d_g(y,z),1\} \tag{3.3}$$

and the following coercivity condition for some $\alpha_0 > 0$

$$\frac{\alpha_0}{d_g(y,z)^{N+sp(y,z)}} \le K(y,z), \text{ a.e. } (y,z) \in \mathcal{M}^2,\ y \neq z. \tag{3.4}$$

The results of this chapter are based on the paper [5].

3.1 Fractional Sobolev Spaces with Kernel Function

In this section, we shall introduce fractional Sobolev spaces with a general kernel and prove several qualitative lemmas.

Definition 3.1.1. Suppose that $(\mathcal{M},g)$ is a Riemannian compact manifold of dimension N, $\mathcal{U}$ is a smooth open subset of $\mathcal{M}$, $K : \mathcal{U}\times\mathcal{U} \to (0,+\infty)$ is a symmetric function satisfying Lévy-integrability and coercivity conditions, $q \in C^+(\mathcal{M})$, and $p : \mathcal{U}\times\mathcal{U} \to (1,+\infty)$ satisfying conditions (3.1)-(3.2). We define fractional Sobolev space $W_K^{q(y),p(y,z)}(\mathcal{U})$ with general kernel $K(y,z)$ on compact manifold $\mathcal{M}$ as the set of all measurable functions $w \in L^{q(y)}(\mathcal{U})$ such that $\int_{\mathcal{M}^2}\frac{|w(y)-w(z)|^{p(y,z)}}{\lambda^{p(y,z)}}K(y,z)dv_g(y)dv_g(z) < \infty$, for some $\lambda > 0$ and endow it with the natural norm

$$\|w\|_K^{q(y),p(y,z)}(\mathcal{U}) = [w]_{K,p(y,z)} + |w|_{q(y)},$$

where

$$[w]_{K,p(y,z)} = \inf\left\{\lambda > 0 : \int_{\mathcal{M}^2}\frac{|w(y)-w(z)|^{p(y,z)}}{\lambda^{p(y,z)}}K(y,z)dv_g(y)dv_g(z) < 1\right\},$$

is the Gagliardo seminorm of u and $(L^{q(y)}(\mathcal{U}),|.|_{q(y)})$ is a variable exponent Lebesgue space.

3.1.1 Basic proprieties of the new space

Lemma 3.1.2. *Suppose that $(\mathcal{M}, g)$ is a Riemannian compact manifold of dimension N, $\mathcal{U}$ is a smooth open subset of $\mathcal{M}$, $K : \mathcal{U} \times \mathcal{U} \to (0, +\infty)$ is a symmetric function satisfying Lévy-integrability and coercivity conditions, $q \in C^+(\mathcal{M})$, and $p : \mathcal{U} \times \mathcal{U} \to (1, +\infty)$ satisfies conditions (3.1)-(3.2). Then $(W_K^{q(y),p(y,z)}(\mathcal{U}), ||\cdot||)$ is a Banach space.*

Proof. Let $\{w_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $W_K^{q(y),p(y,z)}(\mathcal{U})$. For any $\varepsilon > 0$, there exists $N_\varepsilon \geq 0$, such that for every $n, m \in \mathbb{N}$, $n, m \geq N_\varepsilon$,

$$|w_n - w_m|_{q(y)} \leq \|w_n - w_m\|_{W_K^{q(y),p(y,z)}(\mathcal{U})} \leq \varepsilon. \tag{3.5}$$

Since $(L^{q(y)}(\mathcal{U}), |.|_{q(y)})$ is a Banach space, there exists $w \in L^{q(y)}(\mathcal{U})$ such that $w_n \to w$ strongly in $L^{q(y)}(\mathcal{U})$ as $n \to +\infty$. Thanks to the converse of the Dominated Convergence Theorem, it follows that for a subsequence still denoted $\{w_n\}$, we have that $w_n \to w$ as $n \to +\infty$ a.e on $\mathcal{U}$.

Let ℓ be an integer such that $\ell \geq N_\varepsilon$ and $w_\ell \in W_K^{q(y),p(y,z)}(\mathcal{U})$. We use Fatou's lemma and inequality (3.5) to get

$$\begin{aligned}
&\int_{\mathcal{U}\times\mathcal{U}} |w(y) - w(z)|^{p(y,z)} K(y,z) dv_g(y) dv_g(z) \\
&\leq \liminf_{n\to+\infty} \int_{\mathcal{U}\times\mathcal{U}} |w_n(y) - w_n(z)|^{p(y,z)} K(y,z) dv_g(y) dv_g(z) \\
&\leq 2^{p^+-1} \liminf_{n\to+\infty} (|w_k(y) - w_\ell(y) - (w_k(z) - w_\ell(z)|^{p(y,z)} K(y,z) \\
&+ |w_\ell(y) - w_\ell(z)|^{p(y,z)} K(y,z)) dv_g(y) dv_g(z) \\
&\leq 2^{p^+} \liminf_{n\to+\infty} (\|w_\ell - w_k\|^{p^+}_{W_K^{q(y),p(y,z)}(\mathcal{U})} + \|w_\ell - w_k\|^{p^-}_{W_K^{q(y),p(y,z)}(\mathcal{U})} \\
&+ \|w_\ell\|^{p^-}_{W_K^{q(y),p(y,z)}(\mathcal{U})} + \|w_\ell\|^{p^+}_{W_K^{q(y),p(y,z)}(\mathcal{U}))} < +\infty.
\end{aligned}$$

Thus, $w \in W_K^{q(y),p(y,z)}(\mathcal{U})$. We combine with Fatou's lemma and inequality (3.5), and obtain

$$\|w_n - w\|^{p^-}_{W_K^{q(y),p(y,z)}(\mathcal{U})} \leq \liminf_{k\to+\infty} \|w_n - w_k\|^{p^-}_{W_K^{q(y),p(y,z)}}(\mathcal{U}) \leq \varepsilon.$$

That is, $w_n \to w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$ as $n \to +\infty$. □

Lemma 3.1.3. *Suppose that $(\mathcal{M}, g)$ is a compact Riemannian manifold with $\dim \mathcal{M} = N$, $\mathcal{U}$ is a smooth open subset of $\mathcal{M}$, $K : \mathcal{U} \times \mathcal{U} \to (0, +\infty)$ is a symmetric function satisfying Lévy-integrability and coercivity conditions, $q \in C^+(\mathcal{M})$, and $p : \mathcal{U} \times \mathcal{U} \to (1, +\infty)$ satisfies conditions* (3.1)-(3.2)*. Then $(W_K^{q(y),p(y,z)}(\mathcal{U}), ||\cdot||)$ is a uniformly convex space.*

Proof. Let $w, v \in W_K^{q(y),p(y,z)}(\mathcal{U})$, and $\eta \in (0, 2)$, such that

$$1 = \|v\|_{W_K^{q(y),p(y,z)}(\mathcal{U})} = \|w\|_{W_K^{q(y),p(y,z)}(\mathcal{U})} \text{ and } \|w - v\|_{W_K^{q(y),p(y,z)}(\mathcal{U})} \geq \eta.$$

Case 1: $p^- \geq 2$. Thanks to [7, Inequality 28], we get

$$\begin{aligned} &\|\frac{w-v}{2}\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})} + \|\frac{w+v}{2}\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})} \\ &\leq \frac{1}{2}\left(\|w\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})} + \|v\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})}\right). \end{aligned} \tag{3.6}$$

Thanks to (3.6), it follows that

$$\|\frac{w+v}{2}\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})} \leq 1 - (\frac{1}{\eta})^{p^+}.$$

We take $\delta = \delta(\eta)$ such that

$$1 - (\frac{\eta}{2})^{p(y,z)} = (1-\delta)^{p(y,z)}$$

and get

$$\|\frac{w+v}{2}\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})} \leq 1 - \eta.$$

Case 2: $1 < p(y,z) < 2$. Note that

$$\|w\|^{p'(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})} = \left[\int_{\mathcal{M}^2} (|w(y) - w(z)|K(y,z))^{p'(y,z)(p(y,z)-1)} dv_g(y) dv_g(z)\right]^{\frac{1}{p(y,z)-1}},$$

with, $\frac{1}{p(y,z)} + \frac{1}{p'(y,z)} = 1$. By the reverse Minkowski's inequality [7, Theorem 2.13] and [7, Inequality 27], we have that

$$\begin{aligned} &\|\frac{w+v}{2}\|^{p'(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})} + \|\frac{w-v}{2}\|^{p'(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})} \\ &\leq \Big\{ \int_{\mathcal{M}^2} \Big[|w(y) - w(z) + (v(y) - v(z))|^{p'(y,z)} + |w(y) - w(z) + (v(y) - v(z))|^{p'(y,z)}\Big] \\ &\times \left(\frac{1}{2}K(y,z)\right)^{p'(y,z)(p(y,z)-1)} dv_g(y) dv_g(z)\Big\}^{\frac{1}{p(y,z)-1}}\Big\} \\ &\leq \frac{1}{2}\|w\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})}) + \frac{1}{2}\|v\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})})^{p'(y,z)-1} = 1, \end{aligned}$$

therefore

$$\|\frac{w+v}{2}\|^{p'(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})} \leq 1 - (\frac{1}{\varepsilon})^{p'(y,z)}.$$

To complete the argument, choose $\delta = \delta(\eta)$ such that $1 - (\frac{\eta}{2})^{p'(y,z)} = (1-\delta)^{p'(y,z)}$. □

Remark 3.1.4. According to the Milman-Petits theorem [7], $W_K^{q(y),p(y,z)}(\mathcal{U})$ is a reflexive space.

Lemma 3.1.5. *Suppose that $(\mathcal{M}, g)$ is a compact Riemannian manifold with $\dim \mathcal{M} = N$, $\mathcal{U}$ is a smooth open subset of $\mathcal{M}$, $K : \mathcal{U} \times \mathcal{U} \to (0, +\infty)$ is a symetric function satisfying conditions Lévy-integrability and coercivity conditions, $q \in C^+(\mathcal{M})$, and $p : \mathcal{U} \times \mathcal{U} \to (1, +\infty)$ satisfies conditions* (3.1)-(3.2)*. Then $(W_K^{q(y),p(y,z)}(\mathcal{U}), ||\cdot||)$ is a separable space.*

Proof. Let $L : W_K^{q(y),p(y,z)}(\mathcal{U}) \to L^{q(y)}(\mathcal{M}) \times L^{p(y,z)}(\mathcal{M} \times \mathcal{M})$ be the operator defined by

$$L(w) = \big(w(y), (w(y) - w(z))K^{\frac{1}{p(y,z)}}(y,z)\big).$$

L is clearly well-defined and is an isometry. By [21, Proposition 3.17], the space $(W_K^{q(y),p(y,z)}(\mathcal{U}), ||\cdot||)$ is indeed separable. □

3.1.2 Embedding Compact

Theorem 3.1.6. *Suppose that $(\mathcal{M}, g)$ is a compact N-dimensional Riemannian manifold, $\mathcal{U}$ is a smooth open subset of $\mathcal{M}$, $K : \mathcal{U} \times \mathcal{U} \to (0, +\infty)$ is a symmetric function satisfying Lévy-integrability and coercivity conditions, $q \in C^+(\mathcal{M})$, $p : \mathcal{U} \times \mathcal{U} \to (1, +\infty)$ satisfies conditions* (3.1)-(3.2) *and*

$$sp(y,z) < N,\ p(y,y) < q(y),\ \text{for every } (y,z) \in \mathcal{U}^2,$$

and $\ell : \overline{\mathcal{M}} \to (1, +\infty)$ is a continuous variable exponent such that

$$p_s^*(y) = \frac{Np(y,y)}{N - sp(y,y)} > \ell(y) \geq \ell^- = \min_{y \in \overline{\mathcal{M}}} \ell(y) > 1.$$

Then the space $W_K^{q(y),p(y,z)}(\mathcal{U})$ is continuously embeddable in $L^{\ell(y)}(\mathcal{U})$ and there exists a positive constant $C = C(N, s, p, q, \mathcal{U})$ such that

$$|w|_{L^{l(y)}(\mathcal{U})} \leq C||w||_{W_K^{q(y),p(y,z)}(\mathcal{U})}, \text{ for every } \quad w \in W_K^{q(y),p(y,z)}(\mathcal{U}).$$

Moreover, this embedding is compact.

Proof. Let p, q, and ℓ be continuous functions and $\mathcal{U}$ an open subset of $\mathcal{M}$. There exist two positive constants α_1 and α_2 such that

$$q(y) \geq p(y,y) + \alpha_1 > 0 \tag{3.7}$$

and

$$\frac{Np(y,y)}{N - sp(y,y)} \geq \ell(y) + \alpha_2 > 0, \tag{3.8}$$

for every $y \in \overline{\mathcal{U}}$. Let $t \in (0, s)$. We use the continuity of p, q, ℓ, (3.7), and (3.8) to find a constant $\varepsilon = \varepsilon(p, q, \ell, \alpha_1, \alpha_2)$ and a finite family of disjoint Lipschitz sets $\mathcal{U}_j$ such that $\mathcal{U} = \bigcup_{j=1}^N \mathcal{U}_j$ and $\operatorname{diam}(\mathcal{U}_j) < \varepsilon$,

$$\frac{Np(m,z)}{N - tp(m,z)} \geq \ell(y) + \frac{\alpha_2}{2} > 0,\ q(y) \geq p(m,z) + \frac{\alpha_1}{2} > 0,\ \forall (y,z,m) \in \mathcal{U}_j^3.$$

Put $p_j = \inf_{(y,z)\in\mathcal{U}_j\times\mathcal{U}_j}\{p(y,z)-\delta\}$. By the continuity of the involved exponents, we can choose $\delta = \delta(\alpha_2)$, with $p^- - 1 > \delta > 0$ such that

$$\frac{Np_j}{N-tp_j} \geq \ell(y) + \frac{\alpha_2}{3}, \text{ for every } y \in \mathcal{U}_j.$$

So, we have the following

$$\frac{Np_j}{N-tp_j} \geq \ell(y) + \frac{\alpha_2}{3}, \text{ for every } y \in \mathcal{U}_j. \tag{3.9}$$

$$p_j + \frac{\alpha_1}{2} \leq q(y), \text{ for every } y \in \mathcal{U}_j. \tag{3.10}$$

By [48, lemma 2.4], there exists a constant $C = C(N,t,\varepsilon,p_j,\mathcal{U}_j)$ such that (see [48] for more details)

$$||w||_{L^{p_j^*}(\mathcal{U}_j)} \leq C(||w||_{L^{p_j}(\mathcal{U}_j)} + [w]^{t,p_j}(\mathcal{U}_j)) \text{ for every } w \in W^{s,p_j}(\mathcal{U}_j). \tag{3.11}$$

Now, we shall prove the following three inequalities.

a) There exists a constant $c_1 > 0$, such as: $c_1|w|_{L^{\ell(y)}(\mathcal{U})} \leq \sum_{j=0}^{N} ||w||_{L^{p_j^*}(\mathcal{U}_j)}$.

b) There exists a constant $c_2 > 0$, such as: $\sum_{j=0}^{N} ||w||_{L^{p_j}(\mathcal{U}_j)} \leq c_2|w|_{L^{q(y)}(\mathcal{U})}$.

c) There exists a constant $c_3 > 0$, such as: $\sum_{j=0}^{N}[w]^{t,p_j}(\mathcal{U}_j)) \leq c_3[w]^{s,p(y,z)}(\mathcal{U})$.

We shall first prove (a). We have that

$$w(y) = \sum_{j=0}^{N} |w(y)|\chi_{\mathcal{U}_j},$$

where $\chi_{\mathcal{U}_j}$ is a characteristic function. Hence, we have

$$|w|_{L^{\ell(y)}(\mathcal{U})} \leq \sum_{j=0}^{N} |w|_{L^{\ell(y)}(\mathcal{U}_j)}.$$

Combining the statement (3.9) with the Hölder inequality, we obtain

$$|w|_{L^{\ell(y)}(\mathcal{U}_j)} \leq C||w||_{L^{p_j^*}(\mathcal{U}_j)}|1|_{L^{a_j(y)}(\mathcal{U}_j)} \leq C(\mathcal{U}_j,a_j)||w||_{L^{p_j^*}(\mathcal{U}_j)},$$

$$\text{where } \frac{1}{\ell(y)} + \frac{1}{p_j^*} = \frac{1}{a_j(y)}, \text{ for every } y \in \mathcal{U}_j. \tag{3.12}$$

Similarly, by using the fact that $q(y) > p_j$ for every $y \in \mathcal{U}_j$, we get (b).

Now, we show (c). Put

$$G(y,z)=\frac{|w(y)-w(z)|}{d_g(y,z)^s}.$$

We use the Hölder inequality and the definition of p_j, to get

$$\begin{aligned}
[w]^{t,p_j}(\mathcal{U}_j)&=\left(\int_{\mathcal{U}_j\times\mathcal{U}_j}\frac{|w(y)-w(z)|^{p_j}}{d_g(y,z)^{N+tp_j}}dv_g(y)dv_g(z)\right)^{\frac{1}{p_j}}\\
&=\left(\int_{\mathcal{U}_j\times\mathcal{U}_j}(\frac{|w(y)-w(z)|}{d_g(y,z)^s})^{p_j}\frac{dv_g(y)dv_g(z)}{d_g(y,z)^{N+(t-s)p_j}}\right)^{\frac{1}{p_j}}\\
&\leq C|G|_{L^{p(y,z)}(\mu_g,\mathcal{U}_j\times\mathcal{U}_j}|1|_{L^{\mathfrak{B}_j(y,z)}(\mu_g,\mathcal{U}_j\times\mathcal{U}_j)}\\
&=C(\mathcal{U}_j,\mathfrak{B}_j)|G|_{L^{p(y,z)}(\mu_g,\mathcal{U}_j\times\mathcal{U}_j)},
\end{aligned}$$

where

$$\frac{1}{p_j}=\frac{1}{p(y,z)}+\frac{1}{\mathfrak{B}_j(y,z)}\ \text{ and } d\mu_g(y,z)=\frac{dv_g(y)dv_g(z)}{d_g(y,z)^{N+(t-s)p_j}}.$$

Next, we prove that $|G|_{L^{p(y,z)}}\leq C[w]^{s,p(y,z)}(\mathcal{U}_j)$. Let $\lambda>0$ be such that

$$\int_{\mathcal{U}_j\times\mathcal{U}_j}\frac{|w(y)-w(z)|^{p(y,z)}}{\lambda^{p(y,z)}d_g(y,z)^{N+sp(y,z)}}dv_g(y)dv_g(z)<1.$$

Put $k=\sup\{1,\ \sup_{(y,z)\in\mathcal{U}\times\mathcal{U}}d_g(y,z)^{s-t}\}$ and $\breve{\lambda}=\lambda k$. Then we have

$$\int_{\mathcal{U}_j^2}\frac{|w(y)-w(z)|^{p(y,z)}}{((d_g(y,z)^s\breve{\lambda})^{p(y,z)}}\frac{dv_g(y)dv_g(z)}{d_g(y,z)^{N+(t-s)p_j}}\leq\int_{\mathcal{U}_j^2}\frac{|w(y)-w(z)|^{p(y,z)}}{d_g(y,z)^{N+sp(y,z)}\lambda^{p(y,z)}}<1,\tag{3.13}$$

therefore

$$|G|_{L^{p(y,z)}(\mu_g(\mathcal{U}_j\times\mathcal{U}_j)}\leq\lambda k.$$

It now follows from (3.4), inequalities (a), (b), (c), and [48, lemma 2.4] that

$$\begin{aligned}
|w|_{L^{\ell(y)}(\mathcal{U})}&\leq c\sum_{j=0}^{N}||u||_{L^{p_j^*}(\mathcal{U})}\\
&\leq c\sum_{j=0}^{N}\left(||w||_{L^{p_j}(\mathcal{U}_j)}+[w]^{t,p_j}(\mathcal{U}_j)\right)\\
&\leq c\left(|w|_{L^{q(y)}(\mathcal{U})}+[w]^{s,p(y,z)}(\mathcal{U})\right)\\
&\leq c\alpha_0\left(|w|_{L^{q(y)}(\mathcal{U})}+[w]^{K,p(y,z)}(\mathcal{U})\right)\\
&=c\alpha_0||w||_{W_K^{q(y),p(y,z)}(\mathcal{U})}.
\end{aligned}$$

We show that this embedding is compact. Let $\{w_n\}$ be a bounded sequence in $W_K^{q(y),p(y,z)}(\mathcal{U})$, we need to prove that there exists $w\in L^{\ell(y)}(\mathcal{M})$ such that for every $\ell(y)\in(1,p_s^*)$,

$$w_n\to w\text{ in }L^{\ell(y)}(\mathcal{M})\quad\text{as}\quad n\to+\infty.$$

Since $\mathcal{M}$ is a compact Riemannian N-manifold, we can cover $\mathcal{M}$ by a finite number of charts $(\mathcal{U}_j, \varphi_j)_{j=1,\dots,m}$ satisfying

$$\frac{1}{Q}\delta_{ij} \leq g_{ij}^s \leq Q\delta_{ij},$$

where g_{ij}^s are bilinear forms and $Q > 1$. Let η_j be a smooth partition of unity subordinate to the chart $(\mathcal{U}_j, \varphi_j)_{j=1,\dots,m}$. Let $w_n \in W_K^{q(y),p(y,z)}(\mathcal{M})$. Then

$$\eta_j w_n \in W_K^{q(y),p(y,z)}(\mathcal{M}) \text{ and } (\eta_j w_n) \circ \varphi_j^{-1} \in W_K^{q(y),p(y,z)}(B_0(1)),$$

where $B_0(1)$ is an open unit ball of $\mathbb{R}^N$. By [50, Theorem 1.1], there exists $w_j \in L^{\ell(y)}(\varphi_j(\mathcal{U}_j))$ such that

$$(\eta_j w_n) \circ \varphi_j^{-1} \to w_j \text{ strongly in } L^{\ell(y)}(\varphi_j(\mathcal{U}_j)) \text{ as } n \to +\infty.$$

Hence

$$\eta_j w_n \to w_j \circ \varphi_j = a_j \text{ strongly in } L^{\ell(y)}(\varphi_j(\mathcal{U}_j)) \text{ as } n \to +\infty.$$

Finally, we put

$$w = \sum_{j=1}^{m} a_j = \sum_{j=1}^{m} w_j \circ \varphi_j \in L^{\ell(y)}(\mathcal{M}).$$

□

3.2 Application: Existence result via topological degree method

In this part, as an application, we give an existence result to the following problem:

$$\begin{cases} (\mathcal{L}_g^K)_{p(y,\cdot)} w(y) = \lambda\beta(y)|w(y)|^{r(y)-2}w(y) + f(y, w(y)) & \text{in } \mathcal{U}, \\ w = 0 & \text{in } \mathcal{M}\backslash\mathcal{U}. \end{cases} \tag{3.14}$$

Definition 3.2.1. We say that $w \in W_K^{q(y),p(y,z)}(\mathcal{U})$ is a weak solution of problem (3.14) if for every $h \in (W_K^{q(y),p(y,z)}(\mathcal{U}))^*$ we have,

$$\begin{aligned} &\int_{\mathcal{M}\times\mathcal{M}} |w(y) - w(z)|^{p(y,z)-2}(w(y) - w(z))(h(y) - h(z))K(y,z)dv_g(y)dv_g(z) \\ &= \lambda \int_{\mathcal{M}} \beta(y)|w(y)|^{r(y)-2}w(y)h(y)dv_g(y) + \int_{\mathcal{M}} f(y, w(y))h(y)dv_g(y). \end{aligned}$$

We consider the functional $\zeta : W_K^{q(y),p(y,z)}(\mathcal{U}) \to \mathbb{R}$ defined by

$$\begin{aligned} \zeta(w) = &\int_{\mathcal{M}\times\mathcal{M}} \frac{1}{p(y,z)} |w(y) - w(z)|^{p(y,z)} K(y,z)dv_g(y)dv_g(z) \\ &- \lambda \int_{\mathcal{M}} \frac{1}{r(y)} \beta(y)|w(y)|^{r(y)}dv_g(y) - \int_{\mathcal{M}} F(y, w(y))dv_g(y). \end{aligned}$$

Then it follows from [15, 50] that $\zeta \in C^1(W_K^{q(y),p(y,z)}(\mathcal{U}), \mathbb{R})$ and

$$\begin{aligned}
&\langle \zeta^{'}(w), h\rangle \\
&= \int_{\mathcal{M}^2} |w(y) - w(z)|^{p(y,z)-2}(w(y) - w(z))(h(y) - h(z))K(y,z)dv_g(y)dv_g(z) \\
&- \lambda \int_{\mathcal{M}} \beta(y)|w(y)|^{r(y)-2}w(y)h(y)dv_g(y) - \int_{\mathcal{M}} f(y, w(y))h(y)dv_g(y) \\
&=< L(w), h > - < S_1(w), h > - < S_2(w), h > .
\end{aligned}$$

Lemma 3.2.2. *Suppose that $(\mathcal{M}, g)$ is a compact N-dimensional Riemannian manifold, $\mathcal{U}$ is a smooth open subset of $\mathcal{M}$, $K : \mathcal{U} \times \mathcal{U} \to (0, +\infty)$ is a symmetric function satisfying Lévy-integrability and coercivity conditions. Assume that assumption $(\mathcal{B}_1)$ holds. Then the operator $L : W_K^{q(y),p(y,z)}(\mathcal{U}) \to (W_K^{q(y),p(y,z)}(\mathcal{U}))^*$ is continuous, bounded and strictly monotone, and*

i) L is an operator of type (S_+),

ii) $L : W_K^{q(y),p(y,z)}(\mathcal{U}) \to (W_K^{q(y),p(y,z)}(\mathcal{U}))^*$ is a homeomorphism.

Proof. It is obvious that L is bounded. We show that L is continuous. Assume that $w_n \to w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$ and we show that $L(w_n) \to L(w)$ in $(W_K^{q(y),p(y,z)}(\mathcal{U}))^*$. Indeed,

$$\begin{aligned}
&\langle Lw_n - Lw, \varphi\rangle = \int_{\mathcal{M}^2} \Big[(|w_n(y) - w_n(z)|^{p(y,z)-2}(w_n(y) - w_n(z)) \\
&- |w(y) - w(z)|^{p(y,z)-2}(w(y) - w(z))\Big] \times K(y,z)(\varphi(y) - \varphi(z))dv_g(y)dv_g(z) \\
&= \int_{\mathcal{M}^2} \Big[|w_n(y) - w_n(z)|^{p(y,z)-2}(w_n(y) - w_n(z))\, K(y,z)^{\frac{p(y,z)-1}{p(y,z)}} \\
&- |w(y) - w(z)|^{p(y,z)-2}(w(y) - w(z))K(y,z)^{\frac{p(y,z)-1}{p(y,z)}}\Big] \\
&\times (\varphi(y) - \varphi(z))K(y,z)^{\frac{1}{p(y,z)}}dv_g(y)dv_g(z).
\end{aligned}$$

Put

$$G_n(y,z) = |w_n(y) - w_n(z)|^{p(y,z)-2}(w_n(y) - w_n(z))K(y,z)^{\frac{p(y,z)-1}{p(y,z)}} \in L^{p^{'}(y,z)}(\mathcal{U} \times \mathcal{U}),$$

$$G(y,z) = |w(y) - w(z)|^{p(y,z)-2}(w(y) - w(z))K(y,z)^{\frac{p(y,z)-1}{p(y,z)}} \in L^{p^{'}(y,z)}(\mathcal{U} \times \mathcal{U}),$$

and

$$F(y,z) = (\varphi(y) - \varphi(z))K(y,z)^{\frac{1}{p(y,z)}} \in L^{p(y,z)}(\mathcal{U} \times \mathcal{U}),$$

where, $\dfrac{1}{p(y,z)} + \dfrac{1}{p^{'}(y,z)} = 1$. Thanks to the Hölder inequality, we have

$$\langle L(w_n) - L(w), \varphi\rangle \leq 2|G_n(y,z) - G(y,z)|_{L^{p^{'}(y,z)}(\mathcal{U}\times\mathcal{U})}|F|_{L^{p(y,z)}(\mathcal{U}\times\mathcal{U}}.$$

Thus,

$$||L(w_n) - L(w)||_{(W_K^{q(y),p(y,z)}(\mathcal{U}))^*} \leq 2|G_n(y,z) - G(y,z)|_{L^{p^{'}(y,z)}(\mathcal{U}\times\mathcal{U})}.$$

Let

$$V_n(y,z) = (w_n(y) - w_n(z))K(y,z)^{\frac{1}{p(y,z)}} \in L^{p(y,z)}(\mathcal{U}\times\mathcal{U}),$$

and

$$V(y,z) = (w(y) - w(z))K(y,z)^{\frac{1}{p(y,z)}} \in L^{p(y,z)}(\mathcal{U}\times\mathcal{U}).$$

Since $w_n \to w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$, we have $V_n \to V$ in $L^{p(y,z)}(\mathcal{U}\times\mathcal{U})$. So, there exist a subsequence of $\{V_n\}_{n\in\mathbb{N}}$ and $h(y,z) \in L^{p(y,z)}(\mathcal{U}\times\mathcal{U})$ such that $V_n \to V$ a.e in $\mathcal{U}\times\mathcal{U}$ and $|V_n| \leq h(y,z)$. Therefore we have $G_n \to G$ a.e in $\mathcal{U}\times\mathcal{U}$ and

$$|G_n(y,z)| = |V_n(y,z)|^{p(y,z)-1} \leq h(y,z)^{p(y,z)-1}.$$

We use the Dominated Convergence Theorem to get

$$G_n \to G \text{ in } L^{p'(y,z)}(\mathcal{U}\times\mathcal{U}).$$

By Lemma 1.6.8, L is strictly monotone. Now, we show that L is mapping of type (S_+). Let $\{w_n\}_{n\in\mathbb{N}} \subset W_K^{q(y),p(y,z)}(\mathcal{U})$ be a sequence with $w_n \rightharpoonup w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$ and $\limsup_{n\to+\infty} \langle L(w_n) - L(w), w_n - w\rangle \leq 0$. Using (i), we get

$$0 = \lim_{n\to+\infty} \langle Lw_n - Lw, w_n - w\rangle \cdot \tag{3.15}$$

By Theorem 3.1.6, we have that $w_n(y) \to w(y)$ a.e. in $\mathcal{U}$. This, in combination with Fatou's lemma, gives us

$$\begin{aligned}&\liminf_{n\to+\infty} \int_{\mathcal{M}^2} |w_n(y) - w_n(z)|^{p(y,z)} K(y,z)dv_g(y)dv_g(z)\\ &\quad\geq \int_{\mathcal{M}^2} |w(y) - w(z)|^{p(y,z)}K(y,z)dv_g(y)dv_g(z).\end{aligned} \tag{3.16}$$

On the other hand, we have

$$\lim_{n\to+\infty} \langle L(w_n), w_n - w\rangle = \lim_{n\to+\infty} \langle L(w_n) - L(w), w_n - w\rangle = 0. \tag{3.17}$$

Using Young's inequality, we can see there is a positive constant $c > 0$ such that

$$\begin{aligned}&\langle L(w_n), w_n - w\rangle\\ &= \int_{\mathcal{M}\times\mathcal{M}} |w_n(y) - w_n(z)|^{p(y,z)} K(y,z)dv_g(y)dv_g(z)\\ &- \int_{\mathcal{M}^2} |w_n(y) - w_n(z)|^{p(y,z)-2} (w_n(y) - w_n(z)) (w(y) - w(z))K(y,z)dv_g(y)dv_g(z)\\ &\geq \int_{\mathcal{M}^2} |w_n(y) - w_n(z)|^{p(y,z)} K(y,z)dv_g(y)dv_g(z)\\ &- \int_{\mathcal{M}^2} |w_n(y) - w_n(z)|^{p(y,z)-1} |w(y) - w(z)|K(y,z)dv_g(y)dv_g(z)\\ &\geq c\int_{\mathcal{M}^2} |w_n(y) - w_n(z)|^{p(y,z)} K(y,z)dv_g(y)dv_g(z)\\ &- c\int_{\mathcal{M}^2} |w(y) - w(z)|^{p(y,z)}K(y,z)dv_g(y)dv_g(z).\end{aligned}$$

According to (3.15) - (3.17), we get

$$\lim_{n\to+\infty}\int_{\mathcal{M}^2}|w_n(y)-w_n(z)|^{p(y,z)}K(y,z)dv_g(y)dv_g(z)$$
$$=\int_{\mathcal{M}^2}|w(y)-w(z)|^{p(y,z)}K(y,z)dv_g(y)dv_g(z)\cdot \tag{3.18}$$

As a consequence of the Brezis-Lieb Lemma [21], (3.15), and (3.18), L is of type $(S_+)\cdot$

We show that L is a homeomorphism. It is easy to see that L is coercive and injective. Thanks to the Minty-Browder Theorem [73, Theorem 26 A], L is surjective. So, L is a bijection. There exists a map
$G : (W_K^{q(y),p(y,z)}(\mathcal{U}))^* \to W_K^{q(y),p(y,z)}(\mathcal{U})$ such that $G\circ L = id_{W_K^{q(y),p(y,z)}(\mathcal{U})}$ and $L\circ G = id_{(W_K^{q(y),p(y,z)}(\mathcal{U}))^*}$.
We show that G is continuous. Let $g_n, g \in W_K^{q(y),p(y,z)}(\mathcal{U})$ be such that $g_n \to g$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$. Let $t_n = G(g_n), w = G(g)$. Then $L(t_n) = g_n$ and $L(w) = g$. Since $\{t_n\}_{n\in\mathbb{N}}$ is bounded in $W_K^{q(y),p(y,z)}(\mathcal{U})$, we have $t_n \rightharpoonup w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$. It follows that

$$\lim_{n\to+\infty} < L(t_n) - L(w), t_n - w >= \lim_{n\to+\infty} < g_n, t_n - w >= 0.$$

Since L is of type (S_+), we get $t_n \to w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$. This completes the proof. □

Lemma 3.2.3. *If f satisfies $(\mathcal{B}_1)$, then the operator $S : W_K^{q(y),p(y,z)}(\mathcal{U}) \to (W_K^{q(y),p(y,z)}(\mathcal{U}))^*$ defined by*

$$\langle Sw,\varphi\rangle = -\lambda\int_{\mathcal{M}}\beta(y)|w(y)|^{r(y)-2}w\varphi dv_g(y) - \int_{\mathcal{M}} f(y,w(y))\varphi dv_g(y),$$

for every $\varphi \in \left(W_K^{q(y),p(y,z)}(\mathcal{U})\right)^*$ is compact.

Proof. Let

$$S_1 : W_K^{q(y),p(y,z)}(\mathcal{U}) \to L^{q'(y)}(\mathcal{U}) \qquad S_2 : W_K^{q(y),p(y,z)}(\mathcal{U}) \to L^{q'(y)}(\mathcal{U})$$
$$w \longmapsto S_1w = -\lambda\beta(y)|w(y)|^{r(y)-2}w(y) \qquad w \longmapsto S_2w = -f(y,w(y)).$$

We shall show that S_1 and S_2 are both bounded and continuous. For every $w \in W_K^{q(y),p(y,z)}(\mathcal{U})$,

$$|S_1w|_{q'(y)} = \lambda\int_{\mathcal{M}}|\beta(y)|w(y)|^{r(y)-2}w(y)|^{q'(y)}dv_g(y)$$
$$\leq \lambda||\beta||_\infty\int_{\mathcal{M}}||w(y)|^{r(y)-1}|^{q'(y)}dv_g(y)$$
$$\leq \lambda C||\beta||_\infty\int_{\mathcal{M}}||w(y)|^{q(y)-1}|^{q'(y}dv_g(y)$$
$$\leq \lambda C||\beta||_\infty\int_{\mathcal{M}}|w(y)|^{q(y)}dv_g(y).$$

This implies that S_1 is bounded in $W_K^{q(y),p(y,z)}(\mathcal{U})$. By condition $(\mathcal{B}_1)$, there exists $\alpha > 0$ such that

$$|f(y, w(y))| \leq \alpha(1 + |w(y)|^{q(y)-1}).$$

Therefore

$$\begin{aligned}
|S_2 w|^{q^{'}(y)}_{q^{'}(y)} &= \int_{\mathcal{M}} |f(y, w(y)|^{q^{'}(y)} dv_g(y) \\
&\leq \int_{\mathcal{M}} \alpha(1 + |w(y)|^{q(y)-1})^{q^{'}(y)} dv_g(y) \\
&\leq 2^{q^{'}+}(|\mathcal{M}| + \int_{\mathcal{M}} |w(y)|^{(q(y)-1)q^{'}(y)} dv_g(y) \\
&\leq \alpha c^{'}(\mathcal{M}, q(y)) \int_{\mathcal{M}} |w(y)|^{(q(y)-1)q^{'}(y)} dv_g(y)) \\
&\leq \alpha c^{'}(\mathcal{U}, q(y)) |w(y)|_{L^{q^{'}(y)}},
\end{aligned}$$

Hence S_2 is bounded in $W_K^{q(y),p(y,z)}(\mathcal{U})$. Next, we show that S_2 is continuous. Let $w_n \in W_K^{q(y),p(y,z)}(\mathcal{U})$ such that $w_n \to w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$. Then $w_n \to w$ in $L^{q(y)}(\mathcal{U})$. Hence there exists a subsequence, still denoted by w_n, and a measurable function g in $L^{q(y)}(\mathcal{U})$ such that $w_n(y) \to w$ and $|w_n(y)| \leq g(y)$, a.e in $\mathcal{U}$.

Since f is a Carathéodory function, we have

$$f(y, w_n) \to f(y, w(y)) \text{ a.e. in } \mathcal{U}. \tag{3.19}$$

According to condition $(\mathcal{B}_1)$, we have that

$$|f(y, w_n(y)| \leq \alpha(1 + g(y))^{q(y)-1} \in L^{q^{'}(y)}(\mathcal{U}).$$

Using Dominated Convergence Theorem, we obtain

$$\int_{\mathcal{M}} |f(y, w_n(y)) - f(y, w(y))|^{q^{'}(y)} dv_g(y) \to 0 \text{ as } n \to \infty.$$

The Dominated Convergence Theorem implies that $S_2 w_n \to S_2 w$ in $L^{q^{'}(y)}(\mathcal{U})$, so S_2 is continuous in $W_K^{q(y),p(y,z)}(\mathcal{U})$. Because the canonical embedding $i : W_K^{q(y),p(y,z)}(\mathcal{U}) \hookrightarrow L^{q(y)}(\mathcal{U})$ is compact, its adjoint operator $i^* : L^{q^{'}(y)}(\mathcal{U}) \to (W_K^{q(y),p(y,z)}(\mathcal{U}))^*$ is also compact. As a result, compositions $i^* \circ S_2$ and $S_2 \circ i^*$ are compact, so we come to the conclusion that the operator S is compact and this completes the proof. □

Theorem 3.2.4. *Suppose that $(\mathcal{M}, g)$ is a compact N-dimensional Riemannian manifold, $\mathcal{U}$ is a smooth open subset of $\mathcal{M}$, $K : \mathcal{U} \times \mathcal{U} \to (0, +\infty)$ is a symmetric function satisfying Lévy-integrability and coercivity conditions. Assume that assumption $(\mathcal{B}_1)$ holds. Then problem* (3.14) *has at least one weak solution $w \in W_K^{q(y),p(y,z)}(\mathcal{U})$.*

Proof. Let $w \in W_K^{q(y),p(y,z)}(\mathcal{U})$. Then w is a weak solution of problem (3.14) if and only if

$$Lw + Sw = 0, \tag{3.20}$$

where L, S are the operators defined in Lemma 5.24 and Lemma 3.2.3. Since S is bounded, continuous, and quasi-monotone (see Lemma 5.24) and L is strictly monotone, thanks to the Minty-Browder Theorem [73, Theorem 26 A], we have that $L^{-1} = G$ is bounded continuous of type (S_+).

Equation (3.20) is equivalent to

$$w = Gh \quad \text{and} \quad h + S \circ Gh = 0. \tag{3.21}$$

To solve (3.21), we shall use the Berkovits topological degree introduced in Section 1.6. To this end, we first show that the set

$$D = \left\{ h \in (W_K^{q(y),p(y,z)}(\mathcal{U}))^* : h + tS \circ Gh = 0 \text{ for some } t \in [0,1] \right\},$$

is bounded. Let $h \in D$ and take $w = Gh$. Using the growth condition $(\mathcal{B}_1)$, the Hölder inequality, the Young inequality, and continuous embedding $W_K^{q(y),p(y,z)}(\mathcal{U}) \hookrightarrow L^{q(y)}(\mathcal{U})$, we get

$$\begin{aligned}
&||Gh||_{W_K^{q(y),p(y,z)}(\mathcal{U})} \leq \int_{\mathcal{M}\times\mathcal{M}} |w(y) - w(z)|^{p(y,z)} K(y,z) dv_g(y) dv_g(z) \\
&= \langle Lw, w \rangle = \langle h, Gh \rangle \\
&\leq |t| \langle S \circ Gh, Gh \rangle \\
&\leq \lambda \int_{\mathcal{M}} \beta(y) |w(y)|^{r(y)} dv_g(y) + \int_{\mathcal{M}} f(y, w(y)) w(y) dv_g(y) \\
&\leq \lambda ||\beta||_\infty C_1 ||w||^{r^+}_{W_K^{q(y),p(y,z)}(\mathcal{U})} + C_2 \left(\int_{\mathcal{M}} |f(y,w)|^{q'(y)} dv_g(y) \right)^{\frac{1}{q'(y)}} \\
&+ C_3 \left(\int_{\mathcal{M}} |w(y)|^{q(y)} dv_g(y) \right)^{\frac{1}{q(y)}} \\
&\leq \lambda ||\beta||_\infty C_1 ||w||^{r^+}_{W_K^{q(y),p(y,z)}(\mathcal{U})} + C_2 ||\beta||_\infty \int_{\mathcal{M}} \left((1 + |w(y)|^{(q(y)-1)q'(y)}) \right)^{\frac{1}{q'(y)}} \\
&+ C_3 \left(\int_{\mathcal{M}} |w(y)|^{q(y)} dv_g(y) \right)^{\frac{1}{q(y)}} \\
&\leq \lambda ||\beta||_\infty C_1 ||w||^{r^+}_{W_K^{q(y),p(y,z)}(\mathcal{U})} + 2^{q^+} C_4 ||\beta||_\infty ||w||_{W_K^{q(y),p(y,z)}(\mathcal{U})} \\
&+ C_4 ||w||_{W_K^{q(y),p(y,z)}(\mathcal{U})}.
\end{aligned}$$

Since S is bounded, it follows that D is bounded in $(W_K^{q(y),p(y,z)}(\mathcal{U}))^*$. As a result, there exists a positive constant $\eta > 0$ such that

$$||h||_{W_K^{q(y),p(y,z)}(\mathcal{U})^*} < \eta, \quad \text{for every } \ h \in D.$$

Furthermore, $h + tS \circ Gh \neq 0$ for every $(h,t) \in \partial B_\eta(0) \times [0,1]$. Using Lemma 1.6.3, and $i + S \circ G \in \mathcal{F}_{\mathfrak{B}}(\overline{B_\eta(0)})$ and $i = L \circ G \in \mathcal{F}_{\mathfrak{B}}(\overline{B_\eta(0)})$ are present. Next, $i + S \circ G$ is also bounded because the operators i, S and G are all bounded. We come to the conclusion that

$$i + S \circ G \in \mathcal{F}_{\mathfrak{B},1}(\overline{B_\eta(0)}) \text{ and } i \in \mathcal{F}_{\mathfrak{B},1}(\overline{B_\eta(0)}).$$

We consider the map $H : [0,1] \times \overline{B_\eta(0)} \to (W_K^{q(y),p(y,z)}(\mathcal{U}))^*$ given by

$$H(t,w) = w + tS \circ Lw.$$

By the statements (1)-(2) in Theorem 1.6.7, we can deduce

$$d(i + S \circ G, B_\eta(0), 0) = d(i, B_\eta(0), 0) = 1,$$

by applying the homotopy invariance and normalization properties of the degree d from Theorem 1.6.7. Therefore there exists $w \in B_\eta(0)$ such that $h + S \circ Gh = 0$. We can now deduce that $w = Gh$ is a weak solution to problem (3.14) in $W_K^{q(y),p(y,z)}(\mathcal{U})$. This completes the proof. □

Conclusion: We have treated the Sobolev spaces with a general kernel, and we have illustrated an existence result via topological degree theory. There are still questions raised about the qualities of these spaces, such as the density of $D(\mathcal{U})$ spaces, which makes approximation techniques very hard. In the next chapter, we'll look at another type of fractional problem called fractional double phase operators.

Chapter 4

Existence and infinitely many solutions to double phase problem involving a singular term

In this chapter, by computing critical groups, we establish the existence of solutions for a class of fractional $(p_1(x,.), p_2(x,.))-$ Laplacian problems involving singular terms and some vanishing potential. More precisely, we use Morse's relation to prove that our problem has infinitely many solutions. The results of this chapter are based on the work [3].

4.1 Analysis of double phase fractional problems with singularity

Let $(\mathcal{M}, g)$ be a compact Riemannian manifold of dimension N. In this chapter, we discuss the existence of multiple nontrivial solutions of $(p_1(x,.), p_2(x,.))$ fractional Laplacian problem with singular term. More precisely, we using Morse's theory to study the following problem:

$$\begin{cases} (\mathcal{L}_g^{s_1})_{p_1(x,.)}u(x) + (\mathcal{L}_g^{s_2})_{p_2(x,.)}u(x) &= \frac{f(x,u(x))}{u(x)^{\gamma(x)}} \quad \text{in} \quad \mathcal{U}, \\ u &> 0 \quad \text{in} \quad \mathcal{U}, \\ u &= 0 \quad \text{in} \quad \mathcal{M}\backslash\mathcal{U}. \end{cases} \tag{4.1}$$

Here $\mathcal{U} \subset \mathcal{M}$ is an open bounded set of $\mathcal{M}$, $0 < s_1 < s_2 < 1$, $\gamma : \overline{\mathcal{M}} \to (0,1)$ is a continuous function, $q_1, q_2 : \overline{\mathcal{M}} \to (1,\infty)$ are continuous functions such that $q_1(x) < q_2(x)$ for all $x \in \mathcal{M}$, $p_1, p_2 : \overline{\mathcal{M}} \times \overline{\mathcal{M}} \to (1,\infty)$ are continuous functions satisfy the following conditions: for $i = 1, 2$

$$p_i(z,a) = p_i(a,z) \quad \text{for all } (z,a) \in \mathcal{M}^2, \tag{4.2}$$

$$1 < p_i^- = \min_{(a,b)\in\mathcal{M}^2} p_i(a,b) \le p_i(a,b) < p_i^+ = \sup_{(a,b)\in\mathcal{M}^2} p_i(a,b), \text{for all } i = 1,2, \tag{4.3}$$

$$p_1^+ = \sup_{(a,b)\in\mathcal{M}^2} p_2(a,b) \leq p_2^+ = \min_{(a,b)\in\mathcal{M}^2} p_1(a,b), \tag{4.4}$$

$$p_1(x,y) < p_2(x,y) \text{ for all } (x,y) \in \mathcal{M}^2, \tag{4.5}$$

$f : \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies the following condition
(F_1) There exist $\beta \in L^\infty(\mathcal{M})$, and a continuous function $r : \mathcal{M} \to (1, +\infty)$,
$1 < r(x) < p_{s_1}^\star(x) = \dfrac{Np_1(x,x)}{N - s_1 p_1(x,x)}$ satisfies:

$$f(x,y) \leq \beta(x)\left(1 + |y|^{r(x)-1}\right), \qquad \text{a.e. } x \in \mathcal{M}, y \in \mathbb{R},$$

and $(\mathcal{L}_g^{s_i})_{p_i(x,.)}$ is defined by

$$(\mathcal{L}_g^{s_i})_{p_i(x,.)}u(x) = 2\lim_{\varepsilon\to 0^+} \int_{\mathcal{M}\setminus\mathfrak{B}_\varepsilon(x)} \frac{|u(x)-u(y)|^{p_i(x,y)-2}(u(x)-u(y))}{d_g(x,y)^{N+s_i p_i(x,y)}} dv_g(y),$$

for all $x \in \mathcal{M}$, where $\mathfrak{B}_\varepsilon(x) := \{y \in \mathcal{M} : d_g(x,y) < \epsilon\}$, $dv_g(y) := |g_{ij}|^{\frac{1}{2}}dy$, see section (1.1) for more details.

Since the energy functional associated with our problem is not differentiable because of the presence of a singular term, we introduce an approximate problem sequence

4.2 The Approximated Problem

We introduce an approximate problem sequence as follows:

$$\begin{cases} (\mathcal{L}_g^{s_1})_{p_1(x,.)}u_n(x) + (\mathcal{L}_g^{s_2})_{p_2(x,.)}u_n(x) = f_n(x,u_n(x))\frac{1}{(u_n(x)+\frac{1}{n})^{\gamma(x)}} & \text{in } \mathcal{U}, \\ u_n > 0 & \text{in } \mathcal{U}, \\ u_n = 0 & \text{in } \mathcal{M}\setminus\mathcal{U}, \end{cases} \tag{4.6}$$

where $f_n(x,t)) = \min(n, f(x,t))$, $F_n(x,t) = \displaystyle\int_0^t \frac{f_n(x,s)}{(s+\frac{1}{n})^{\gamma(x)}} ds$, and f_n satisfies the following conditions: (F_2) There exist $\theta > p_1^+$ and $r > 0$ such that for a.e $x \in \mathcal{M}$ and $|x| \geq r$,

$$0 < F_n(x,t) \leq \frac{tf_n(x,t)}{(t+\frac{1}{n})^{\gamma(x)}}.$$

(F_3) It holds

$$\lim_{t\to+\infty} \frac{f_n(x,t)}{t^{p_1(x,y)}} = l_1 \text{ uniformly for a.e } x \in \mathcal{M},$$

(F_4) There exist $a_3 > 0$ and $l : \mathcal{M} \times \mathcal{M} \to (1, p_1^-)$ such that

$$f_n(x,t)t - p_{s_1}^* F_n(x,t) \geq -a_3|t|^{l(x,y)}$$

for all $x \in \mathcal{M}$ and $t \in \mathbb{R}$.

Notation: For the sake of simplicity, we note $W := W^{s,q(x),p(x,y)}(\mathcal{M})$, $W_1 := W^{s_1,q_1(x),p_1(x,y)}(\mathcal{M})$, and $W_2 := W^{s_2,q_2(x),p_2(x,y)}(\mathcal{M})$. See (1.1).

Definition 4.2.1. We say that $\{u_n\}_{n\in\mathbb{N}}$ to be a weak solution of (4.6) if

$$\int_{\mathcal{M}\times\mathcal{M}} \frac{|u_n(x)-u_n(y)|^{p_1(x,y)-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{d_g(x,y)^{N+s_1p_1(x,y)}}dv_g(x)dv_g(y)+$$
$$\int_{\mathcal{M}\times\mathcal{M}} \frac{|u_n(x)-u_n(y)|^{p_2(x,y)-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{d_g(x,y)^{N+s_2p_2(x,y)}}dv_g(x)dv_g(y) =$$
$$\int_{\mathcal{M}} f_n(x,u_n(x))\frac{1}{(u_n(x)+\frac{1}{n})^{\gamma(x)}}\varphi(x)dv_g(x), \text{ for all } \varphi\in W_1^*.$$

Now, we will use Morse's theory and the critical groups, which are effective tools for studying the multiplicity and existence of solutions to the approximative problems (4.6).

4.2.1 Computation of group criticals

In this section, we will consider the associate energy functional to (4.6), we shall prove that $C^1(W^1,\mathbb{R})$, and we compute the criticals groups at 0 and ∞.
Consider the energy functional $\phi : W_1 \to \mathbb{R}$ defined by

$$\phi(u_n) = \phi_1(u_n) - \phi_2(u_n),$$

where

$$\phi_1(u_n) = \int_{\mathcal{M}\times\mathcal{M}} \frac{1}{p_1(x,y)}\frac{|u_n(x)-u_n(y)|^{p_1(x,y)}}{d_g(x,y)^{N+s_1p_1(x,y)}}dv_g(x)dv_g(y)$$
$$+\int_{\mathcal{M}\times\mathcal{M}} \frac{1}{p_2(x,y)}\frac{|u_n(x)-u_n(y)|^{p_2(x,y)}}{d_g(x,y)^{N+s_2p_2(x,y)}}dv_g(x)dv_g(y),$$

$\phi_2(u_n) = \int_{\mathcal{M}} F_n(x,u_n(x))dv_g(x)$, and $F_n(x,t) = \int_0^t \frac{f_n(x,s)}{(s+\frac{1}{n})^{\gamma(x)}}ds$.

Lemma 4.2.2. *If $\{f_n\}_{n\in\mathbb{N}}$ satisfies condition (F_1), then $\phi_2 \in C^1(W_1,\mathbb{R})$ and*

$$\langle \phi_2'(u_n), v_n\rangle = \int_{\mathcal{M}} \frac{f_n(x,u_n(x)}{(u_n(x)+\frac{1}{n})^{\gamma(x)}}v_n(x)dv_g(x),$$

for all $u_n, v_n \in W_1$.

Proof. (i) ϕ_2 is Gateaux differentiable in W_1.

Let $u_n, v_n \in W_1$, and $0 < t < 1$, we have

$$\frac{1}{t}(F_n(x,u_n+tv_n) - F_n(x,u_n)) = \frac{1}{t}\int_0^{u_n+tv_n}\frac{f_n(x,s)}{(s+\frac{1}{n})^{\gamma(x)}}ds - \frac{1}{t}\int_0^{u_n}\frac{f_n(x,s)}{(s+\frac{1}{n})^{\gamma(x)}}ds$$
$$= \frac{1}{t}\int_{u_n}^{u_n+tv_n}\frac{f_n(x,s)}{(s+\frac{1}{n})^{\gamma(x)}}ds.$$

By the mean value theorem, there exists $0 < \delta < 1$ such that

$$\frac{1}{t}(F_n(\mathrm{x}, \mathrm{u}_n + t\mathrm{v}_n) - F_n(\mathrm{x}, \mathrm{u}_n)) = \frac{f_n(x, \mathrm{u}_n + \delta t\mathrm{v}_n)}{(\mathrm{u}_n + \delta t\mathrm{v}_n + \frac{1}{n})^{\gamma(\mathrm{x})}}\mathrm{v}_n.$$

Combining with (F_1) with Young's inequality, we have

$$\begin{aligned} f_n(\mathrm{x}, \mathrm{u}_n + \delta t\mathrm{v}_n) &\leq f(\mathrm{x}, \mathrm{u}_n + \delta t\mathrm{v}_n) \\ &\leq \beta(|\mathrm{v}_n| + |\mathrm{u}_n + \delta t\mathrm{v}_n|^{r(\mathrm{x})}|\mathrm{v}_n|) \\ &\leq \beta 2^{r^+}(1 + |\mathrm{u}_n|^{r(\mathrm{x})} + |\mathrm{v}_n|^{r(\mathrm{x})}). \end{aligned}$$

Since $r(\mathrm{x}) \in (1, p_{s_1^*})$, we have $\mathrm{u}_n, \mathrm{v}_n \in L^{r(\mathrm{x}}(\mathcal{M})$. Thanks to the Lebesgue's dominated converge Theorem, we get

$$\begin{aligned} \lim_{t\to 0}\frac{1}{t}(F_n(\mathrm{x}, \mathrm{u}_n + t\mathrm{v}_n) - F_n(\mathrm{x}, \mathrm{u}_n)) &= \lim_{t\to 0}\int_{\mathcal{M}} \frac{f_n(\mathrm{x}, \mathrm{u}_n + \delta t\mathrm{v}_n)}{(\mathrm{u}_n + \delta t\mathrm{v}_n + \frac{1}{n})^{\gamma(\mathrm{x})}}\mathrm{v}_n dv_{\mathrm{g}}(\mathrm{x}) \\ &= \int_{\mathcal{M}} lim_{t\to 0}\frac{f_n(\mathrm{x}, \mathrm{u}_n + \delta t\mathrm{v}_n)}{(\mathrm{u}_n + \delta t\mathrm{v}_n + \frac{1}{n})^{\gamma(\mathrm{x})}}\mathrm{v}_n dv_{\mathrm{g}}(\mathrm{x}) \\ &= \int_{\mathcal{M}} \frac{f_n(\mathrm{x}, \mathrm{u}_n)}{(\mathrm{u}_n + \frac{1}{n})^{\gamma(\mathrm{x})}}\mathrm{v}_n dv_{\mathrm{g}}(\mathrm{x}). \end{aligned}$$

□

(ii) The continuity of Gateaux-derivatives. Let $\{\mathrm{u}_{n,k}\}_{k\in\mathbb{N}} \subset W_1, \mathrm{u}_n \in W_1$ such that $\mathrm{u}_{n,k} \to \mathrm{u}_n$ strongly in W_1 as $k \to +\infty$. We use Hölder's inequality and condition (F_1), we have that

$$\begin{aligned} \int_{\mathcal{M}} |\frac{f_n(\mathrm{x}, \mathrm{u}_{n,k})}{(\mathrm{u}_{n,k} + \frac{1}{n})^{\gamma(\mathrm{x})}}|^{r'(\mathrm{x})} dv_{\mathrm{g}}(\mathrm{x}) &\leq \int_{\mathcal{M}} |f_n(\mathrm{x}, \mathrm{u}_{n,k})|^{r'(\mathrm{x})} dv_{\mathrm{g}}(\mathrm{x}) \\ &\leq \int_{\mathcal{M}} |f(\mathrm{x}, \mathrm{u}_{n,k})|^{r'(\mathrm{x})} dv_{\mathrm{g}}(\mathrm{x}) \\ &\leq 2^{\frac{r^+ + 1}{r^+ - 1}} \|\beta\|_\infty^{\frac{r^+ + 1}{r^+ - 1}} \int_{\mathcal{M}} |\mathrm{u}_{n,k}|^{r(\mathrm{x})} dv_{\mathrm{g}}(\mathrm{x}) \\ &\leq C(\beta, r^+) \int_{\mathcal{M}} |\mathrm{u}_{n,k}|^{r(\mathrm{x})} dv_{\mathrm{g}}(\mathrm{x}) \\ &\leq C(\|\beta\|_\infty, r^+) \||\mathrm{u}_{n,k}|\|_{L^{\frac{p_{s_1^*(\mathrm{x})}}{r(\mathrm{x})}}(\mathcal{M})} \|1\|_{L^{\frac{p_{s_1^*(\mathrm{x})}}{p_{s_1^*(\mathrm{x})} - r(\mathrm{x})}}(\mathcal{M})}. \end{aligned}$$

So, the sequence $\{|\frac{f_n(\mathrm{x}, \mathrm{u}_{n,k})}{(\mathrm{u}_{n,k} + \frac{1}{n})^{\gamma(\mathrm{x})}} - \frac{f_n(\mathrm{x}, \mathrm{u}_n)}{(\mathrm{u}_n + \frac{1}{n})^{\gamma(\mathrm{x})}}|^{r(\mathrm{x})}\}_{k\in\mathbb{N}}$ is uniformly bounded and equi-integrable in $L^1(\mathcal{M})$. Thanks to Vitali converge theorem implies

$$\lim_{k\to+\infty}\int_{\mathcal{M}} |\frac{f_n(\mathrm{x}, \mathrm{u}_{n,k})}{(\mathrm{u}_{n,k} + \frac{1}{n})^{\gamma(\mathrm{x})}} - \frac{f_n(\mathrm{x}, \mathrm{u}_n)}{(\mathrm{u}_n + \frac{1}{n})^{\gamma(\mathrm{x})}}|^{r'(\mathrm{x})} dv_{\mathrm{g}}(\mathrm{x}) = 0,$$

where $\frac{1}{r'(\mathrm{x})} + \frac{1}{r(\mathrm{x})} = 1$. Thus, by Theorem 1.4.6 and Hölder's inequality, we have

$$\|\phi_2^{'}(u_{n,k})-\phi_2^{'}(u_n)\|_{W_1^*}=\sup_{v_n\in W_1,\|v_n\|_{W_1}\leq 1}|\langle\phi_2^{'}(u_{n,k})-\phi_2^{'}(u_n),v_n\rangle|$$
$$\leq\|\frac{f_n(x,u_{n,k})}{(u_{n,k}+\frac{1}{n})^{\gamma(x)}}-\frac{f_n(x,u_n)}{(u_n+\frac{1}{n})^{\gamma(x)}}\|_{L^{q_1'(x)}(\mathcal{M})}\|v_n\|_{L^{q_1(x)}(\mathcal{M})}\to 0$$

as $k\to+\infty$, where W_1^* is the dual space of W_1.

Remark 4.2.3. 1) Clearly, $\phi_1\in C^1(W_1,\mathbb{R})$, and

$$\begin{aligned}&\langle\phi_1^{'}(u_{n,k}),v_n\rangle\\&=\int_{\mathcal{M}\times\mathcal{M}}\frac{|u_{n,k}(x)-u_{n,k}(y)|^{p_1(x,y)-2}(u_{n,k}(x)-u_{n,k}(y))(v_n(x)-v_n(y))}{d_g(x,y)^{N+s_1p_1(x,y)}}dv_g(x)dv_g(y)\\&+\int_{\mathcal{M}\times\mathcal{M}}\frac{|u_{n,k}(x)-u_{n,k}(y)|^{p_2(x,y)-2}(u_{n,k}(x)-u_{n,k}(y))(v_n(x)-v_n(y))}{d_g(x,y)^{N+s_2p_2(x,y)}}dv_g(x)dv_g(y)\end{aligned}$$

2) Consequently, critical points of ϕ are the weak solutions to problem (4.6).

Theorem 4.2.4. *The functional ϕ satisfies the Palais-Smalle condition at level $c\in\mathbb{R}$.*

Proof. Let $\{u_{n,k}\}_{k\in\mathbb{N}}\subset W_1$ such that $\phi(u_{n,k})\to c$ as $k\to+\infty$ and $\phi^{'}(u_{n,k})\to 0$ as $k\to+\infty$ in W_1^*.

$\star$ We shall prove that $\{u_{n,k}\}_{k\in\mathbb{N}}$ is bounded in W_1. By contradiction. We assume the claim does not hold, that is up to a sub-sequence still denote by $\{u_{n,k}\}_{k\in\mathbb{N}}$ such that $u_{n,k}\to+\infty$ as $k\to+\infty$ in W_1. Let us $v_{n,k}:=\frac{u_{n,k}}{\|u_{n,k}\|_{W_1}}$. Clearly $\{v_{n,k}\}_{k\in\mathbb{N}}$ is bounded in W_1. Since W_1 is a reflexive Banach space, up to a subsequence still denoted by $\{v_{n,k}\}_{k\in\mathbb{N}}$ such that $v_{n,k}\rightharpoonup v_n$ as $k\to+\infty$ in W_1 and $v_{n,k}\to v_n$ strongly $k\to+\infty$ in $L^{\alpha(x)}(\mathcal{M})$ for $1<\alpha(x)<p_{s_1}^*(x)$. We use the fact $\lim_{k\to+\infty}\frac{1}{\|u_{n,k}\|_{W_1}}=0$, we have that

$$\begin{aligned}&\lim_{k\to+\infty}\frac{1}{p_1^-}\|v_{n,k}\|_{W_1}^{p_1(x,y)}+\frac{1}{p_2^-}\|u_{n,k}\|_{W_1}^{p_2(x,y)-p_1(x,y)}\|v_{n,k}\|_{W_2}^{p_2(x,y)}\\&=\lim_{k\to+\infty}\|u_{n,k}\|_{W_1}^{-p_1(x,y)}\int_{\mathcal{M}}F_n(x,u_{n,k})dv_g(x),\end{aligned}\tag{4.7}$$

and

$$\begin{aligned}&\lim_{k\to+\infty}\|v_{n,k}\|_{W_1}^{p_1(x,y)}+\|u_{n,k}\|_{W_1}^{p_2(x,y)-p_1(x,y)}\|v_{n,k}\|_{W_2}^{p_2(x,y)}\\&=\lim_{k\to+\infty}\|u_{n,k}\|_{W_1}^{-p_1(x,y)}\int_{\mathcal{M}}\frac{f_n(x,u_{n,k})}{(u_{n,k}+\frac{1}{n})^{\gamma(x)}}u_{n,k}(x)dv_g(x).\end{aligned}\tag{4.8}$$

We use (4.8) and (4.7), we have

$$\begin{aligned}&\lim_{k\to+\infty}\left(\frac{p_{s_1}^*}{p_1^-}-1\right)\|v_{n,k}\|_{W_1}+\left(\frac{p_{s_2}^*}{p_2^-}-1\right)\|u_{n,k}\|_{W_1}^{p_2(x,y)-p_1(x,y)}\|v_{n,k}\|_{W_2}^{p_2(x,y)}\\&=\lim_{k\to+\infty}\|u_{n,k}\|_{W_1}^{-p_1(x,y)}p_{s_1}^*\left(\int_{\mathcal{M}}F_n(x,u_{n,k})-\frac{f_n\,(x,u_{n,k})}{\left(u_{n,k}+\frac{1}{n}\right)^{\gamma(x)}}u_{n,k}(x)dv_g(x)\right).\end{aligned}\tag{4.9}$$

Using (F_4), we get

$$\begin{aligned}\lim_{k\to+\infty}\left(\frac{p_{s_1}^*}{p_1^-}-1\right)\|\mathrm{v}_{n,k}\|_{W_1} &= \lim_{k\to+\infty}\Big[(1-\frac{p_{s_2}^*}{p_2^+})\|\mathrm{u}_{n,k}\|_{W_1}^{p_2(\mathrm{x},\mathrm{y})-p_1(\mathrm{x},\mathrm{y})}\|\mathrm{v}_{n,k}\|_{W_2}^{p_2(\mathrm{x},\mathrm{y})}+\\ &\|\mathrm{u}_{n,k}\|_{W_1}^{-p_1(\mathrm{x},\mathrm{y})}p_{s_1}^*\Big(\int_{\mathcal{M}}F_n(\mathrm{x},\mathrm{u}_{n,k})-\frac{f_n(\mathrm{x},\mathrm{u}_{n,k})}{(\mathrm{u}_{n,k}+\frac{1}{n})^{\gamma(\mathrm{x})}}\mathrm{u}_{n,k}(\mathrm{x})dv_{\mathrm{g}}(\mathrm{x})\Big)\Big]\\ &\leq \lim_{k\to+\infty}\Big[\left(1-\frac{p_{s_2}^*}{p_2^+}\right)\|\mathrm{u}_{n,k}\|_{W_1}^{p_2(\mathrm{x},\mathrm{y})-p_1(\mathrm{x},\mathrm{y})}\|\mathrm{v}_{n,k}\|_{W_2}^{p_2(\mathrm{x},\mathrm{y})}\\ &+a_3\|\mathrm{u}_{n,k}\|_{W_1}^{l(\mathrm{x},\mathrm{y})-p_1(\mathrm{x},\mathrm{y})}\|\mathrm{v}_n\|_{L^{l(\mathrm{x},\mathrm{y})}(\mathcal{M})}^{l(\mathrm{x},\mathrm{y})}\Big].\end{aligned}\tag{4.10}$$

This is a contradiction as $\|\mathrm{v}_n\|_{W_1}=1$. We have also $\{\mathrm{u}_{n,k}\}_{k\in\mathbb{N}}$ is bounded in W_2. Since $\mathrm{u}_{n,k}\to\mathrm{u}_n$ a.e. in $\mathcal{M}$ as $k\to+\infty$,

we have that

$$\begin{aligned}&\frac{|\mathrm{u}_{n,k}(\mathrm{x})-\mathrm{u}_{n,k}(\mathrm{y})|^{p_1(\mathrm{x},\mathrm{y})-2}(\mathrm{u}_{n,k}(\mathrm{x})-\mathrm{u}_{n,k}(\mathrm{y}))}{d_{\mathrm{g}}(\mathrm{x},\mathrm{y})^{(\frac{N}{p_1(\mathrm{x},\mathrm{y})}+s_1)(p_1(\mathrm{x},\mathrm{y})-1)}}\\ &\to\frac{|\mathrm{u}_n(\mathrm{x})-\mathrm{u}_n(\mathrm{y})|^{p_1(\mathrm{x},\mathrm{y})-2}(\mathrm{u}_n(\mathrm{x})-\mathrm{u}_n(\mathrm{y}))}{d_{\mathrm{g}}(\mathrm{x},\mathrm{y})^{(\frac{N}{p_1(\mathrm{x},\mathrm{y})}+s_1)(p_1(\mathrm{x},\mathrm{y})-1)}}\end{aligned}$$

a.e $(\mathrm{x},\mathrm{y})\in\mathcal{M}\times\mathcal{M}$ as $k\to+\infty$. Since $\{\mathrm{u}_{n,k}\}_{k\in\mathbb{N}}$ is bounded in W_1, there exist $c>0$ such that

$$\int_{\mathcal{M}\times\mathcal{M}}\Big|\frac{|\mathrm{u}_{n,k}(\mathrm{x})-\mathrm{u}_{n,k}(\mathrm{y})|^{p_1(\mathrm{x},\mathrm{y})-2}(\mathrm{u}_{n,k}(\mathrm{x})-\mathrm{u}_{n,k}(\mathrm{y}))}{d_{\mathrm{g}}(\mathrm{x},\mathrm{y})^{(\frac{N}{p_1(\mathrm{x},\mathrm{y})}+s_1)(p_1(\mathrm{x},\mathrm{y})-1)}}\Big|^{\frac{p_1(\mathrm{x},\mathrm{y})}{p_1(\mathrm{x},\mathrm{y})-1}}dv_{\mathrm{g}}(\mathrm{x})dv_{\mathrm{g}}(\mathrm{y})\leq C.$$

So, we have that

$$\begin{aligned}&\frac{|\mathrm{u}_{n,k}(\mathrm{x})-\mathrm{u}_{n,k}(\mathrm{y})|^{p_1(\mathrm{x},\mathrm{y})-2}(\mathrm{u}_{n,k}(\mathrm{x})-\mathrm{u}_{n,k}(\mathrm{y}))}{d_{\mathrm{g}}(\mathrm{x},\mathrm{y})^{(\frac{N}{p_1(\mathrm{x},\mathrm{y})}+s_1)(p_1(\mathrm{x},\mathrm{y})-1)}}\\ &\rightharpoonup\frac{|\mathrm{u}_n(\mathrm{x})-\mathrm{u}_n(\mathrm{y})|^{p_1(\mathrm{x},\mathrm{y})-2}(\mathrm{u}_n(\mathrm{x})-\mathrm{u}_n(\mathrm{y}))}{d_{\mathrm{g}}(\mathrm{x},\mathrm{y})^{(\frac{N}{p_1(\mathrm{x},\mathrm{y})}+s_1)(p_1(\mathrm{x},\mathrm{y})-1)}}\end{aligned}$$

weakly in $L^{p_1'(\mathrm{x},\mathrm{y})}(\mathcal{M}\times\mathcal{M})$, where $\frac{1}{p_1'(\mathrm{x},\mathrm{y})}+\frac{1}{p_1(\mathrm{x},\mathrm{y})}=1$. Let $\mathrm{w}_n\in W_1$, it is follows that

$$\frac{\mathrm{w}_n(\mathrm{x})-\mathrm{w}_n(\mathrm{y})}{d_{\mathrm{g}}(\mathrm{x},\mathrm{y})^{\frac{N}{p_1(\mathrm{x},\mathrm{y})}+s_1}}\in L^{p_1(\mathrm{x},\mathrm{y}}(\mathcal{M}\times\mathcal{M}),\text{ and }\frac{\mathrm{w}_n(\mathrm{x})-\mathrm{w}_n(\mathrm{y})}{d_{\mathrm{g}}(\mathrm{x},\mathrm{y})^{\frac{N}{p_2(\mathrm{x},\mathrm{y})}+s_2}}\in L^{p_2(\mathrm{x},\mathrm{y}}(\mathcal{M}\times\mathcal{M}).$$

Finally, we get that

$$\int_{\mathcal{M}\times\mathcal{M}} \frac{|u_{n,k}(x)-u_{n,k}(y)|^{p_1(x,y)-2}(u_{n,k}(x)-u_{n,k}(y))}{d_g(x,y)^{(\frac{N}{p_1(x,y)}+s_1)p_1(x,y)}} dv_g(x)dv_g(y)$$
$$\to \int_{\mathcal{M}\times\mathcal{M}} \frac{|u_n(x)-u_n(y)|^{p_1(x,y)-2}(u_n(x)-u_n(y))}{d_g(x,y)^{(\frac{N}{p_1(x,y)}+s_1)p_1(x,y)}} dv_g(x)dv_g(y) \text{ as } k\to\infty$$

and

$$\int_{\mathcal{M}\times\mathcal{M}} \frac{|u_{n,k}(x)-u_{n,k}(y)|^{p_2(x,y)-2}(u_{n,k}(x)-u_{n,k}(y))}{d_g(x,y)^{(\frac{N}{p_1(x,y)}+s_2)p_2(x,y)}} dv_g(x)dv_g(y)$$
$$\to \int_{\mathcal{M}\times\mathcal{M}} \frac{|u_n(x)-u_n(y)|^{p_2(x,y)-2}(u_n(x)-u_n(y))}{d_g(x,y)^{(\frac{N}{p_2(x,y)}+s_2)p_2(x,y)}} dv_g(x)dv_g(y) \text{ as } k\to\infty.$$

$\star$ We shall prove that $u_{n,k}\to u_n$ strongly in W_1 as $k\to+\infty$.
Consider the sequence defined by $v_{n,k}=u_{n,k}-u_{n,k}$. Since $u_{n,k}\to u_n$ a.e in $\mathcal{M}$ and $\{u_{n,k}\}_{k\in\mathbb{N}}$ is uniformly bounded in W_1 and W_2. Thanks to Brezis-Lieb Lemma in [42], we have that

$$\begin{aligned}
&\lim_{k\to+\infty}\int_{\mathcal{M}\times\mathcal{M}} \frac{|u_{n,k}(x)-u_{n,k}(y)|^{p_1(x,y)-2}(u_{n,k}(x)-u_{n,k}(y))}{d_g(x,y)^{(\frac{N}{p_1(x,y)}+s_1)p_1(x,y)}} dv_g(x)dv_g(y)\\
&=\lim_{k\to+\infty}\Big[\int_{\mathcal{M}\times\mathcal{M}} \frac{|v_{n,k}(x)-v_{n,k}(y)|^{p_1(x,y)-2}(v_{n,k}(x)-v_{n,k}(y))}{d_g(x,y)^{(\frac{N}{p_1(x,y)}+s_1)p_1(x,y)}} dv_g(x)dv_g(y)\\
&\quad+\int_{\mathcal{M}\times\mathcal{M}} \frac{|u_n(x)-u_n(y)|^{p_1(x,y)-2}(u_n(x)-u_n(y))}{d_g(x,y)^{(\frac{N}{p_1(x,y)}+s_1)p_1(x,y)}} dv_g(x)dv_g(y)\Big],
\end{aligned} \tag{4.11}$$

thus $\lim_{k\to+\infty}\|u_{n,k}\|_{W_1}^{p_1(x,y)}-\|v_{n,k}\|_{W_1}^{p_1(x,y)}=\|u_n\|_{W_1}^{p_1(x,y)}$.
Similarly, we get $\lim_{k\to+\infty}\|u_{n,k}\|_{W_2(\mathcal{M})}^{p_2(x,y)}-\|v_{n,k}\|_{W_2}^{p_2(x,y)}=\|u_n\|_{W2}^{p_2(x,y)}$. So, we have that

$$\begin{aligned}
c=\lim_{k\to+\infty}\phi(u_{n,k})=\lim_{k\to+\infty}\Big[&\frac{1}{p_1(x,y)}\|v_{n,k}\|_{W_1}^{p_1(x,y)}+\frac{1}{p_2(x,y)}\|v_{n,k}\|_{W_2}^{p_2(x,y)}\\
&+\frac{1}{p_1(x,y)}\|u_n\|_{W_1}^{p_1(x,y)}+\frac{1}{p_2(x,y)}\|u_n\|_{W_2}^{p_2(x,y)}-\int_{\mathcal{M}}F_n(x,u_n(x))dv_g(x)\Big].
\end{aligned} \tag{4.12}$$

On the other hand, using $\phi'(u_{n,k})\to 0$ as $k\to+\infty$, we have that

$$\begin{aligned}
\lim_{k\to+\infty}\|v_{n,k}\|_{W_1}^{p_1(x,y)}+\|v_{n,k}\|_{W_2}^{p_2(x,y)}=&\int_{\mathcal{M}}\frac{f_n(x,u_n)}{(u_n+\frac{1}{n})^{\gamma(x)}}u_n(x)dv_g(x)\\
&-\|u_n\|_{W_1}^{p_1(x,y)}-\|u_n\|_{W_2}^{p_2(x,y)}.
\end{aligned} \tag{4.13}$$

We combine (4.13) with $\phi(u_n)=0$, we obtain that

$$\lim_{k\to+\infty}\Big[\|v_{n,k}\|_{W_1}^{p_1(x,y)}+\|v_{n,k}\|_{W_2}^{p_2(x,y)}\Big]=0.$$

Since $\|v_{n,k}\|_{W_1}^{p_1(x,y)}$ and $\|v_{n,k}\|_{W_2}^{p_2(x,y)}$ are bounded sequences, we can write $\lim_{k\to+\infty}\|v_{n,k}\|_{W_1}^{p_1(x,y)} = a$ and $\lim_{k\to+\infty}\|v_{n,k}\|_{W_2}^{p_2(x,y)} = b$. Since $a, b \geq 0$ and $a + b = 0$, we get that $a = b = 0$. Finally, $u_{n,k} \to u_n$ strongly in W_1 as $k \to +\infty$.

□

Now, we will use the notion of the local $(m, n)-$ linking for computing $\dim C_k(\phi, 0)$.

Theorem 4.2.5. *The functional ϕ has a local $(1,1)-$ linking at the origin.*

Proof. We define $V = \mathbb{R}$. Clearly V is a one dimensional vector space subspace of W_1. We choose $r \in (0,1)$ such that $K_\phi \cap \overline{B_r(0)} = \{0\}$, where $B_r(0) = \{u_n \in W_1 : \|u_n\|_{W_1} < r\}$ and $K_\phi = \{u_n \in W_1 : \phi'(u_n = 0)\}$. We consider the set $E = V \cap \overline{B_r(0)}$ for small enough $r \in (0,1)$. Recall the on a finite dimensional normed space all norms are equivalent. So, by taking $r \in (0,1)$ even smaller as necessary, we obtain that

$$\|u_n\|_{W_1} \leq r \Rightarrow |u_n| \leq \delta \quad \text{for all} \quad u_n \in V = \mathbb{R}.$$

Then, for any $u_n \in V \cap \overline{B_r(0)}$, we have

$$\begin{aligned}
\phi(u_n) &= \int_{\mathcal{M}\times\mathcal{M}} \frac{1}{p_1(x,y)} \frac{|u_n(x) - u_n(y)|^{p_1(x,y)}}{d_g(x,y)^{N+s_1p_1(x,y)}} dv_g(x)dv_g(y) - \int_{\mathcal{M}} F_n(x, u_n(x))dv_g(x) \\
&+ \int_{\mathcal{M}\times\mathcal{M}} \frac{1}{p_2(x,y)} \frac{|u_n(x) - u_n(y)|^{p_2(x,y)}}{d_g(x,y)^{N+s_2p_2(x,y)}} dv_g(x)dv_g(y) \\
&\leq \int_{\mathcal{M}\times\mathcal{M}} \frac{1}{p_1(x,y)} \frac{|u_n(x) - u_n(y)|^{p_1(x,y)}}{d_g(x,y)^{N+s_1p_1(x,y)}} dv_g(x)dv_g(y) - \int_{\mathcal{M}} F_n(x, u_n(x))dv_g(x) \\
&+ \int_{\mathcal{M}\times\mathcal{M}} \frac{1}{p_2(x,y)} \frac{|u_n(x) - u_n(y)|^{p_2(x,y)}}{d_g(x,y)^{N+s_2p_2(x,y)}} dv_g(x)dv_g(y) \\
&\leq \frac{1}{p_1^-} \int_{\mathcal{M}\times\mathcal{M}} \frac{|u_n(x) - u_n(y)|^{p_1(x,y)}}{d_g(x,y)^{N+s_1p_1(x,y)}} dv_g(x)dv_g(y) - \int_{\mathcal{M}} F_n(x, u_n(x))dv_g(x) \\
&+ \frac{1}{p_2^-} \int_{\mathcal{M}\times\mathcal{M}} \frac{|u_n(x) - u_n(y)|^{p_2(x,y)}}{d_g(x,y)^{N+s_2p_2(x,y)}} dv_g(x)dv_g(y) \\
&\leq 0.
\end{aligned}$$

Further, we consider the set

$$D = \left\{ u_n \in W_1 : \min\left(\frac{1}{p_2^+}, \frac{1}{p_1^+}\right) \|u_n\|_{W_1}^{p_1(x,y)} > 2\|\beta\|_\infty \|u_n\|_{W_1}^{l(x)} \right\}.$$

By (F_1), and Lemma 1.4.6, we have that for any $u_n \in D$,

$$\begin{aligned}\phi(u_n) &= \int_{\mathcal{M}\times\mathcal{M}} \frac{1}{p_1(x,y)} \frac{|u(x)-u_n(y)|^{p_1(x,y)}}{d_g(x,y)^{N+s_1p_1(x,y)}} dv_g(x)dv_g(y) - \int_{\mathcal{M}} F_n(x,u_n(x))dv_g(x) \\ &+ \int_{\mathcal{M}\times\mathcal{M}} \frac{1}{p_2(x,y)} \frac{|u_n(x)-u_n(y)|^{p_2(x,y)}}{d_g(x,y)^{N+s_2p_2(x,y)}} dv_g(x)dv_g(y) \\ &\geq \frac{1}{p_1^+}\|u_n\|_{W_1}^{p_1(x,y)} + \frac{1}{p_2^+}\|u_n\|_{W_2}^{p_2(x,y)} - 2\|\beta\|_\infty \|u_n\|_{W_1}^{l(x)} > 0.\end{aligned}$$

Let $U = \overline{B_r(0)}$, $E_0 = V \cap \partial B_r(0)$, $E = V \cap \overline{B_r(0)}$, and D as above, we have that $0 \notin E_0 \subset E \subset U = \overline{B_r(0)}$ and $E_0 \cap D = \emptyset$. Therefore, we arrive the following

$$\phi_{|E} \leq 0 < \phi_{|D\cap\overline{B_r(0)}}.$$

Let Y be the topological complement of V. We have that $W_1 = V \bigoplus Y$. So, every $u_n \in W_1$ can be written in unique way as

$$u_n = v_n + y_n \quad \text{with} \quad v_n \in V, y_n \in Y.$$

We consider the map $h : [0,1] \times W_1 \to W_1 \backslash D$ defined by

$$h(t,u_n) = (1-t)u_n + tr\frac{v_n}{\|v_n\|}.$$

We have $h(0,u_n) = u_n$ and $h(1,u_n) = r\dfrac{v_n}{\|v_n\|} \in V \cap \partial B_r(0) = E_0$. It follows that E_0 is a deformation retract of $W_1 \backslash D$. Hence

$$i^* : H_0(E_0) \to H_0(W_1)$$

i an isomorphism. Note that $E_0 = \{a, -a\}$ for some $a \neq 0$. Therefore, from $\dim H_0(E_0) = 2$, since $H_0(E_0) = \mathbb{R} \bigoplus \mathbb{R}$. Thus,

$$\dim im(i^*) = 2.$$

The set $E = V \cap B_r(0)$ is contractible (it is an interval). By [38, Theorem 11.5], we have that $H_0(E, E_0) = 0$. Thanks to [38, Remark 6.1.26], we get $\dim im(j^*) = 1$. So, finally

$$\dim im(i^*) - \dim im(j^*) = 2 - 1 = 1.$$

Thus, the hypothesis of Definition 1.7.2 are satisfied. Hence, ϕ has a local $(1,1)-$ linking at 0. □

Remark 4.2.6. For all $k \in \mathbb{N}$, $C_k(\phi, 0) \neq 0$.

Proof. Since ϕ has a local $(1,1)-$ linking at the origin. By [55, proposition 2.1], we get that $\dim C_k(\phi, 0) \geq 1$. □

Now, we will compute the group critical of ϕ infinitely.

Theorem 4.2.7. *Under assumption (F_3), there exists a constant $A > 0$ such that ϕ^a is homotopic to $S^1 = \{u_n \in W_1 : \|u_n\|_{W_1} = 1\}$, for all $a < -A$.*

Proof. We use the condition (F_3), we get that

$$\begin{aligned}\phi(tu_n) &= \int_{\mathcal{M}\times\mathcal{M}} \frac{t^{p_1(x,y)}}{p_1(x,y)} \frac{|u_n(x)-u_n(y)|^{p_1(x,y)}}{d_g(x,y)^{N+s_1p_1(x,y)}} dv_g(x)dv_g(y) - \int_{\mathcal{M}} F_n(x,tu_n(x))dv_g(x)\\ &+ \int_{\mathcal{M}\times\mathcal{M}} \frac{t^{p_2(x,y)}}{p_2(x,y)} \frac{|u_n(x)-u_n(y)|^{p_2(x,y)}}{d_g(x,y)^{N+s_2p_2(x,y)}} dv_g(x)dv_g(y)\\ &\leq \frac{1}{p_1^-} t^{p_1^+} + \frac{1}{p_2^-} t^{p_2^+} \|u_n\|_{W_2}^{p_2(x,y)} - \frac{l}{2}\frac{1}{p_{s_1}^{+*}} t^{p_{s_1}^{+*}} \int_{\mathcal{M}} u_n(x)^{p_{s_1}^{+*}} dv_g(x),\end{aligned}$$

where $p_{s_1}^{+*} = \max_{x\in\mathcal{M}} p_{s_1}^*(x)$. Since $p_{s_1}^{+*} > p_1^+ > p_2^+$, we have that $\phi(tu_n) \to -\infty$ as $t \to +\infty$. Let $A \in \mathbb{R}$ there exists $t \in \mathbb{R}$ such that $\|tu_n\|_{W_1} \geq B$, we have that $\phi(tu_n) \leq A$.
Since $u_n \in S^1$, we have that

$$\begin{aligned}\frac{d}{dt}\phi(tu_n) &= \int_{\mathcal{M}\times\mathcal{M}} t^{p_1(x,y)-1} \frac{|u_n(x)-u_n(y)|^{p_1(x,y)}}{d_g(x,y)^{N+s_1p_1(x,y)}} dv_g(x)dv_g(y)\\ &- \int_{\mathcal{M}} u_n(x) \frac{f_n(x,tu_n(x))}{(tu_n(x)+\frac{1}{n})^{\gamma(x)}} dv_g(x)\\ &+ \int_{\mathcal{M}\times\mathcal{M}} t^{p_2(x,y)-1} \frac{|u_n(x)-u_n(y)|^{p_2(x,y)}}{d_g(x,y)^{N+s_2p_2(x,y)}} dv_g(x)dv_g(y)\\ &\leq t^{p_1^+-1} + t^{p_2^+-1}\|u_n\|_{W_2}^{p_2(x,y)} - \int_{\mathcal{M}} u_n(x) \frac{f_n(x,tu_n(x))}{(tu_n(x)+\frac{1}{n})^{\gamma(x)}} dv_g(x)\\ &\leq \frac{p_1^+}{t}\left[A + \int_{\mathcal{M}} F_n(x,tu_n(x))dv_g(x) - \frac{p_1^+}{t}\int_{\mathcal{M}} tu_n(x)\frac{f_n(x,tu_n(x))}{(tu_n(x)+\frac{1}{n})^{\gamma(x)}} dv_g(x)\right]\\ &\leq \frac{p_1^+}{t}\left[A + \left(\frac{1}{\theta} - \frac{1}{p_1^+}\right)\int_{\mathcal{M}} tu_n(x)\frac{f_n(x,tu_n(x))}{(tu_n(x)+\frac{1}{n})^{\gamma(x)}} dv_g(x)\right]\\ &\leq \frac{p_1^+}{t}\left[A + C_1\left(\frac{1}{\theta} - \frac{1}{p_1^+}\right)\right] < 0.\end{aligned}$$

By the implicit function Theorem, there exists an unique $T \in C(S^1, \mathbb{R})$ such that for any $u_n \in S^1$,

$$\phi(T(u_n)u_n) = A.$$

For any $u_n \neq 0$, set $\tau(u_n) = \frac{1}{\|u_n\|} T(\frac{u_n}{\|u_n\|})$. Then $\tau \in C(W_1 \backslash 0, \mathbb{R})$ and for all $u_n \in W_1$, $\phi(u_n \tau(u_n)) = A$. Moreover, if $\phi(u_n) = A$, then $\tau(u_n) = 1$.
Define a function $\tau_1 : W_1 \to \mathbb{R}$ as

$$\tau_1(u_n) := \begin{cases} \tau(u_n), & \text{if } \phi(u_n) \geq A, \\ 1, & \text{if } \phi(u_n) \leq A. \end{cases}$$

Since $\phi(u_n) = A$ implies that $\tau(u_n) = 1$, we conclude that $\tau_1 \in C(W_1 \setminus \{0\}, \mathbb{R})$.
Finally, we set $H : [0,1] \times W_1 \setminus \{0\} \to W_1 \setminus \{0\}$ as

$$H(t, u_n) = (1-t)u_n + t\tau_1(u_n)u_n.$$

We have $H(0, u_n) = u_n$, $H(1, u_n) = \tau_1(u_n)u_n \in \phi^A$, and $H(t, .)_{|\phi^A} = id_{|\phi^A}$ for all $t \in [0,1]$. It follows that

$$\phi^A \text{ is a strong deformation retract of } W_1 \setminus \{0\}. \tag{4.14}$$

We consider the radial retraction $r : W_1 \to \mathbb{R}$ defined by

$$r(u_n) = \frac{u_n}{\|u_n\|} \text{ for all } u_n \in W_1.$$

This map is continuous and $r_{|S^1} = id_{|S^1}$. Therefore, S^1 is a retract of $W_1 \setminus \{0\}$.
We consider the map defined by

$$h(t, u_n) = (1-t)u_n + tr(u_n) \text{ for all } (t, u_n) \in [0,1] \times W_1 \setminus \{0\}.$$

Then, $h(0, u_n) = u_n$, $h(1, u_n) = r(u_n) \in S^1$, and $h(1, .)_{|S^1} = id_{|S^1}$. Hence, we refer that

$$S^1 \text{ is a deformation retract of } W_1 \setminus \{0\}. \tag{4.15}$$

Finally, by (4.14) and (5.23) it follows that ϕ^A and S^1 are homotopie equivalent. □

Remark 4.2.8. $C_k(\phi, \infty) = 0$, for all $k \in \mathbb{N}$.

Proof. Since ϕ^A and S^1 are homotopy equivalent, we have that

$$\begin{aligned} C_k(\phi, \infty) &= H_k(W_1, \phi^A) \\ &= H_k(W_1, S^1). \end{aligned}$$

The space W_1 is infinite dimensional and S^1 is contractible. Hence, from [62, p.147]we have $H_k(W_1, S^1) = 0$ for all $k \in \mathbb{N}$. Therefore, $C_k(\phi, \infty) = 0$. □

Theorem 4.2.9. *Assume that $(F_1) - (F_4)$. Then, the problem (4.6) has nontrivial weak solution in W_1.*

Proof. Since ϕ has a local $(1,1)-$ linking near the origin, then $\dim C_1(\phi, 0) \geq 1$, i,e $C_k(\phi, 0) \neq 0$ for some $k \in \mathbb{N}$. Thanks to [62, Theorem 6.2.42], there exists $u_n \in K_\phi$. □

Theorem 4.2.10. *Assume that $(F_1)-(F_4)$. Then, the problem (4.6) has at least nontrivial weak solution in W_1.*

Proof. Thanks to Theorem 4.2.4 ϕ satisfies the Palais-Smalle condition and is bounded from below and the trivial solution $u_n = 0$ is homological nontrivial and is a minimizer. The conclusion follows from [55, Theorem 2.1]. □

Now, using Morse's relation for proving problem (4.6) has many solutions.

Theorem 4.2.11. *Assume that $(F_1)-(F_4)$. Then, the problem (4.6) has an infinitely nontrivial weak solutions in W_1.*

Proof. Suppose that these are the three nontrivial solutions of problem (4.6) that is, $K_\phi = \{0, u_n, v_n\}$. By Morse relation, we get that

$$C_n(\phi, 0) = \begin{cases} \mathbb{R}, & k = m(0) \\ 0, & \text{otherwise}, \end{cases}$$

where $m(0)$ is a Morse index of 0.
We use the Morse relation, we get that

$$\begin{aligned}\sum_{k\geq 0} \operatorname{rank} C_k(\phi,\infty)X^k + (1+X)Q(X) =& \sum_{k\geq 0} \operatorname{rank} C_k(\phi,0)X^k + \sum_{k\geq 0} \operatorname{rank} C_k(\phi,u_n)X^k \\ &+ \sum_{k\geq 0} \operatorname{rank} C_k(\phi,v_n)X^k \\ =& X^{m(0)} + 2\sum_{k\geq 0}\beta_k X^k.\end{aligned}$$

Since $C_k(\phi,\infty) = 0$, we obtain that

$$(1+X)Q(X) = X^{m(0)} + 2\sum_{k\geq 0}\beta_k X^k,$$

where β_k non-negative integer and Q is a polynomial with non-negative integer coefficient. In particular, for $X = 1$ we have $2a = 1 + 2\sum_{k\geq 0}\beta_k$. Since $\beta_k \in \mathbb{N}$, we have that $\sum_{k\geq 0}\beta_k = +\infty$ leads to a contradiction. Thus, there exist infinitely many solutions to problem (4.6). □

4.2.2 Existence and multiplicity of solutions via Morse's theory

Now, introduce our fundamental result.

Definition 4.2.12. We say that u is a weak solution of (4.6) if

$$\begin{aligned}&\int_{\mathcal{M}\times\mathcal{M}} \frac{|u(x)-u(y)|^{p_1(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{d_g(x,y)^{N+s_1p_1(x,y)}} dv_g(x)dv_g(y) + \\ &\int_{\mathcal{M}\times\mathcal{M}} \frac{|u(x)-u(y)|^{p_2(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{d_g(x,y)^{N+s_2p_2(x,y)}} dv_g(x)dv_g(y) = \\ &\int_{\mathcal{M}} \frac{f(x,u(x))}{(u(x))^{\gamma(x)}}\varphi(x)dv_g(x), \text{ for all } \varphi \in W_1^*.\end{aligned}$$

Theorem 4.2.13. *Assume that conditions $(F_1)-(F_4)$ holds. Then, problem (4.1) admits an infinitely weak solutions in W_1.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset W_1$ be the sequence of solutions to problem (4.6). So, we have that

$$\begin{aligned}
&\int_{\mathcal{M}\times\mathcal{M}} \frac{|u_n(x)-u_n(y)|^{p_1(x,y)-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{d_g(x,y)^{N+s_1p_1(x,y)}}dv_g(x)dv_g(y)\\
&+\int_{\mathcal{M}\times\mathcal{M}} \frac{|u_n(x)-u_n(y)|^{p_2(x,y)-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{d_g(x,y)^{N+s_2p_2(x,y)}}dv_g(x)dv_g(y) \qquad (4.16)\\
&\quad=\int_{\mathcal{M}} f_n(x,u_n(x))\frac{1}{(u_n(x)+\frac{1}{n})^{\gamma(x)}}\varphi(x)dv_g(x),\ \text{for all } \varphi\in W_1^*.
\end{aligned}$$

Choosing u_n as test function in (4.16), we obtain that

$$\begin{aligned}
&\int_{\mathcal{M}\times\mathcal{M}} \frac{|u_n(x)-u_n(y)|^{p_1(x,y)}}{d_g(x,y)^{N+s_1p_1(x,y)}}dv_g(x)dv_g(y)+\int_{\mathcal{M}\times\mathcal{M}} \frac{|u_n(x)-u_n(y)|^{p_2(x,y)}}{d_g(x,y)^{N+s_2p_2(x,y)}}dv_g(x)dv_g(y)\\
&=\int_{\mathcal{M}} f_n(x,u_n(x))\frac{1}{(u_n(x)+\frac{1}{n})^{\gamma(x)}}u_n(x)dv_g(x)\le\int_{\mathcal{M}} f(x,u_n(x))|u_n(x))|^{1-\gamma(x)}dv_g(x)\\
&\le\int_{\mathcal{M}}\beta(x)(1+|u_n(x))|^{r(x)-1)})|u_n(x)|^{1-\gamma(x)}dv_g(x).
\end{aligned}$$

Since $r(x)-1\le r(x)-\gamma(x)$ for all $x\in\mathcal{M}$, we get that

$$\|u_n\|_{W_1}\le\|\beta\|_\infty C\left(r(x),p_1(x,y),q_1(x),\gamma(x),s_1,\mathcal{M}\right).$$

Hence, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in W_1. Since W_1 is a reflexive Banach space, up to a subsequence, still denoted by $\{u_n\}$ such that $u_n\rightharpoonup u$ weakly in W_1, $u_n\to u$ strongly in $L^{a(x)}(\mathcal{M})$ for $1\le a(x)<p_{s_1}^*(x)$, and $u_n\to u$ a.e in $\mathcal{M}$. Similarly, be proof of Theorem 4.2.4, we get that

$$\begin{aligned}
&\lim_{n\to\infty}\Big[\int_{\mathcal{M}\times\mathcal{M}} \frac{|u_n(x)-u_n(y)|^{p_1(x,y)-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{d_g(x,y)^{N+s_1p_1(x,y)}}dv_g(x)dv_g(y)\\
&+\int_{\mathcal{M}\times\mathcal{M}} \frac{|u_n(x)-u_n(y)|^{p_2(x,y)-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{d_g(x,y)^{N+s_2p_2(x,y)}}dv_g(x)dv_g(y)\Big]\\
&=\int_{\mathcal{M}\times\mathcal{M}} \frac{|u(x)-u(y)|^{p_1(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{d_g(x,y)^{N+s_1p_1(x,y)}}dv_g(x)dv_g(y)+\\
&\int_{\mathcal{M}\times\mathcal{M}} \frac{|u(x)-u(y)|^{p_2(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{d_g(x,y)^{N+s_2p_2(x,y)}}dv_g(x)dv_g(y).
\end{aligned} \qquad (4.17)$$

Since $u_n(x)>0$, we get that

$$\left|\frac{f_n(x,u_n(x))\varphi(x)}{(\frac{1}{n}+u_n(x))^{\gamma(x)}}\right|\le|f(x,u(x))\varphi(x)|.$$

By the dominated converge theorem, we conclude that

$$\lim_{n\to+\infty}\int_{\mathcal{M}}\frac{f_n(x,u_n(x))\varphi(x)}{(\frac{1}{n}+u_n(x))^{\gamma(x)}}dv_g(x)=\int_{\mathcal{M}}\frac{f(x,u(x))\varphi(x)}{(u(x))^{\gamma(x)}}dv_g(x).$$

Finally, passing to the limit in (4.16), we conclude that

$$\begin{aligned}
&\int_{\mathcal{M}\times\mathcal{M}} \frac{|u(x)-u(y)|^{p_1(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{d_g(x,y)^{N+s_1p_1(x,y)}}dv_g(x)dv_g(y)\\
&+\int_{\mathcal{M}\times\mathcal{M}} \frac{|u(x)-u(y)|^{p_2(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{d_g(x,y)^{N+s_2p_2(x,y)}}dv_g(x)dv_g(y) \qquad (4.18)\\
&\qquad = \int_{\mathcal{M}} \frac{f(x,u(x))\varphi(x)}{(u(x))^{\gamma(x)}}dv_g(x), \text{ for all } \varphi \in W_1^*,
\end{aligned}$$

namely u is a weak solution to (4.1). □

4.2.3 Remarks and comments

Similarly, we prove the existence of solutions for a general class of fractional non-local problems involving a singular term and vanishing potential of the form:

$$\begin{cases}
(\mathcal{L}_g^{s_1})_{p_1(x,.)}u(x)+(\mathcal{L}_g^{s_2})_{p_2(x,.)}u(x) &= \frac{f(x,u(x))}{u(x)^{\gamma(x)}}+\mathcal{V}(x)|u(x)|^{\sigma(x)-2}u(x) \quad \text{in} \quad \mathcal{U},\\
u &> 0 \quad \text{in} \quad \mathcal{U},\\
u &= 0 \quad \text{in} \quad \mathcal{M}\backslash\mathcal{U},
\end{cases} \qquad (4.19)$$

has infinitely many solutions. Where $\sigma : \mathcal{U} \to (1,\infty)$, is a continuous function, and $\mathcal{V}$ vanishing potential satisfies the following assumptions:

(V) $\mathcal{V} : \mathcal{M} \to \mathbb{R}$ is a continuous function, there exist $\theta_1 > 0$, and $0 < \eta_1 < 1$ such that

$$\mathcal{V}(x) > \theta_1 > 0 \text{ and } \int_{\mathcal{M}} \mathcal{V}(x)|u(x)|^{\sigma(x)}dx \le \eta_1\|u\|_{W_1},$$

for all $x \in \mathcal{M}$, and $u \in W1$.

This results is a content of the accepted paper [4].

Chapter 5

Infinitely many Solutions to a Kirchhoff type equation involving logarithmic nonlinearity via Morse's Theory

This chapter is devoted to the study of a fractional elliptic problems involving the Kirchhoff function. Combining variational methods and Morse's theory for studying the existence of infinitely many solutions. The results of this chapter are based on the paper [59].

5.1 Auxiliary Inequalities

Now, we give some lemmas that are basic in our main result.

Lemma 5.1.1. *For every $a > 0$. Then, we have:*

1) $t^a|\log(t)| \leq \frac{1}{a\exp(1)}$, *for all* $t \in (0,1]$;

2) $\log(t) \leq \frac{t^a}{a\exp(1)}$, *for all* $t > 1$.

Proof. For (1). We consider the function by $g : (0,1] \to \mathbb{R}$ as $g(t) = t^a|\log(t)|$. The function is continous on $(0,1]$, and $\lim_{t\to 0} t^a|\log(t)| = 0$. Using a direct computation, we show that the function g acheies the maximum at $t_0 = \exp(\frac{-1}{a})$. Finally, we have $t^a|\log(t)| \leq \frac{1}{a\exp(1)}$, for all $t \in (0,1]$. Now, we prove (2). We construct the following function:

$$f(t) = \log(t) - \frac{1}{a\exp(1)}t^a, \text{ for all } t \in [1,\infty).$$

Obvious, we prove that the function f achieves the maximum at $t^* = \exp(\frac{1}{a})$, for all $t \in [1,\infty)$. So, we get $f(t) \leq f(t^*)$. □

Lemma 5.1.2. *Let $r : \mathcal{U} \to (1,\infty$ be a continuous function such that $1 < r^- \leq r(\mathrm{x}) \leq r^+ < p_s^*(x)$, for each $x \in \mathcal{U}$. Then, we have the following estimate:*

$$\int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\log|\mathrm{u}(\mathrm{x})|d\mathrm{x} \leq C\max\left\{\|u\|^{r^-}, \|u\|^{r^+}\right\} + \log\|u\|\int_{\mathcal{U}}\frac{1}{r(\mathrm{x})}|u|^{r(\mathrm{x})}dx,$$

for all $u \in W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U})\backslash\{0\}$, where $C = C\left(|\mathcal{U}|, r, p_s^*(x)\right)$ is a suitable constant.

Proof. Let $\mathcal{U}_1 = \{x \in \mathcal{U} : |\mathrm{u}(\mathrm{x})| \leq \|u\|\}$, and $\mathcal{U}_2 = \{x \in \mathcal{U} : |\mathrm{u}(\mathrm{x})| \geq \|u\|\}$. From Lemma 5.1.1 (1) with $a = r^-$, we obtain that

$$\begin{aligned}\int_{\mathcal{U}_1} \frac{1}{r(\mathrm{x})}|u|^{r(\mathrm{x})} \log \frac{|u|}{\|u\|} dx &\leq \frac{1}{r^-} \max\{\|u\|^{r^-}, \|u\|^{r^+} \int_{\mathcal{U}_1} \left(\frac{|u|}{\|u\|}\right)^{r(\mathrm{x})} \left|\log \frac{|u|}{\|u\|}\right| dx \\ &\leq \frac{|\mathcal{U}|}{(r^-)^2 \exp(1)} \max\{\|u\|^{r^-}, \|u\|^{r^+}\}.\end{aligned} \tag{5.1}$$

Estimating the second integral expression. Combining Lemma 3.1.6 (2) with Lemma 5.1.1 such that $a = (p_s^*)^- - \epsilon - r^+$, for some sufficiently small $\epsilon > 0$, we have that

$$\begin{aligned}&\int_{\mathcal{U}_2} \frac{1}{r(\mathrm{x})}|u|^{r(\mathrm{x})} \log \frac{|u|}{\|u\|} dx \leq \frac{1}{r^-} \int_{\mathcal{U}_2} |u|^{r(\mathrm{x})} \log \frac{|u|}{\|u\|} dx \\ &\leq \frac{1}{r^- \exp\left((p_s^*)^- - \epsilon - r^+\right)} \int_{\mathcal{U}_2} |u|^{r(\mathrm{x})} \left(\frac{|u|}{\|u\|}\right)^{(p_s^*)^- - \epsilon - r^+} dx \\ &\leq \frac{1}{\exp\left((p_s^*)^- - \epsilon - r^+\right) r^-} \int_{\mathcal{U}_2} |u|^{r(\mathrm{x})} \left(\frac{|u|}{\|u\|}\right)^{(p_s^*)^- - \epsilon - r(\mathrm{x})} dx \\ &\leq \frac{1}{\min\left(\|u\|^{(p_s^*)^- - \epsilon - r^-}, \|u\|^{(p_s^*)^- - \epsilon - r^+}\right) \exp\left((p_s^*)^- - \epsilon - r^+\right) r^-} \int_{\mathcal{U}_2} |u|^{(p_s^*)^- - \epsilon} dx \\ &\leq \frac{C_{p_s^{*-}-\epsilon}}{\min\left(\|u\|^{(p_s^*)^- - \epsilon - r^-}, \|u\|^{(p_s^*)^- - \epsilon - r^+}\right) \exp\left((p_s^*)^- - \epsilon - r^+\right) r^-} \|u\|^{(p_s^*)^- - \epsilon} \\ &= \frac{C_{p_s^{*-}-\epsilon}}{\exp\left((p_s^*)^- - \epsilon - r^+\right) r^-} \min\left(\|u\|^{r^-}, \|u\|^{r^+}\right) \\ &\leq \frac{C_{p_s^{*-}-\epsilon}}{\exp\left((p_s^*)^- - \epsilon - r^+\right) r^-} \max\left(\|u\|^{r^-}, \|u\|^{r^+}\right),\end{aligned} \tag{5.2}$$

where $C_{p_s^{*-}-\epsilon} > 0$ is constant. From (5.1), and (5.2, we deduce that

$$\int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})} \log|\mathrm{u}(\mathrm{x})| d\mathrm{x} \leq C \max\left\{\|u\|^{r^-}, \|u\|^{r^+}\right\} + \log\|u\| \int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}|u|^{r(\mathrm{x})} dx.$$

□

5.2 Analysis of Kirchhoff fractional problems

Let $\mathcal{U} \subset \mathbb{R}^N$ be an open bounded set ($N \geq 2$). Our objective in this chapter is to discuss the existence of multiple nontrivial solutions of the Kirchhoff-type elliptic equation. More precisely, we combine Morse's relation with the computation of critical groups for studying the following equation:

$$\begin{cases} M\left(J_{s,p(\mathrm{x},\cdot)}(\mathrm{u})\right) \Delta^s_{p(\mathrm{x},\cdot)} \mathrm{u} &= \lambda|\mathrm{u}|^{r(\mathrm{x})-2}\mathrm{u}\log|\mathrm{u}| + \lambda f(\mathrm{x},\mathrm{u}) \quad \text{in} \quad \mathcal{U}, \\ \mathrm{u} &= 0 \quad \text{in} \quad \mathbb{R}^N \backslash \mathcal{U}, \end{cases} \tag{5.3}$$

where λ is a positive parameter, $r \in C(\mathcal{U}, (1,\infty))$, $p : \mathcal{U} \times \mathcal{U} \to (1,\infty)$ is a continuous function that verifies the following conditions:

$$p(y-m, z-m) = p(y,z), \text{ for all } (y,z,m) \in \mathbb{R}^N \times \times \mathbb{R}^N, \tag{5.4}$$

$$p(z,a) = p(a,z), \text{ for all } (z,a) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{5.5}$$

$$1 < p^- = \min_{(y,z)\in\mathbb{R}^N\times\mathbb{R}^N} p(y,z) \le p(y,z) < p^+ = \sup_{(y,z)\in\mathbb{R}^N\times\mathbb{R}^N} p(y,z), \tag{5.6}$$

$f : \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function with $f(\mathrm{x}, 0) = 0$ and satisfies below conditions:

$(\mathcal{B}_1)$ There exist $\alpha > 0$ and a continuous function $q : \mathbb{R}^N \to (1, +\infty)$ such that

$$1 < q(\mathrm{x}) < p_s^\star(\mathrm{x}) = \frac{Np(\mathrm{x},\mathrm{x})}{N - sp(\mathrm{x},\mathrm{x})},$$

and

$$f(\mathrm{x},\mathrm{y}) \le \alpha\left(1 + |\mathrm{y}|^{q(\mathrm{x})-1}\right), \text{ a.e. } \mathrm{x} \in \mathbb{R}^N, \mathrm{y} \in \mathbb{R}, r^+ = \sup_{\mathrm{x}\in\mathbb{R}^N} r(\mathrm{x}) \le q^- < p_s^\star(\mathrm{x}),$$

$(\mathcal{B}_2)$ There exists $R > 0$ such that $\frac{f(\mathrm{x},t)}{|t|^{p(\mathrm{x},\mathrm{y})-2}t}$ is increasing for $t \ge R$ and is decreasing for $t \le -R$ for all $\mathrm{x} \in \mathcal{U}$.

$(\mathcal{B}_3)$ $\lim_{t\to\infty} \frac{F(\mathrm{x},t)}{|t|^{p^+}} = +\infty$, where $F(\mathrm{x},t) = \int_0^t f(\mathrm{x},s)ds$ is the primitive of function f.

$(\mathcal{B}_4)$ There are small constants and R with $0 < r < R$ such that

$$C_2|t|^{\alpha(\mathrm{x})} \le \beta(\mathrm{x})F(\mathrm{x},t) \le C_3|t|^{\beta(\mathrm{x})} \text{ for all } r \le t \le R, \text{ a.e } \mathrm{x} \in \mathcal{U},$$

where C_2, C_3 are positive constants with $0 < C_2 < C_3 < 1$, and α, $\beta \in C(\bar{\mathcal{U}})$ with $1 < \alpha(\mathrm{x}) < \beta(\mathrm{x}) < p_s^\star(\mathrm{x})$.

$(\mathcal{B}_5)$ There exist $\beta > p^+$ and some $I > 0$ such that, for each $|\alpha| > I$, we have

$$0 < \int_{\mathcal{U}} F(\mathrm{y},\mathrm{x})d\mathrm{y} \le \int_{\mathcal{U}} f(\mathrm{y},\mathrm{x})\frac{\alpha}{\beta}d\mathrm{y},$$

$\Delta^s_{p(\mathrm{x},\cdot)}$ is the fractional $p(\mathrm{x},.)$-Laplace operator which (up to normalization factors) may be defined as

$$\Delta^s_{p(\mathrm{x},\cdot)}\mathrm{u}(\mathrm{x}) = 2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus\mathfrak{B}_\varepsilon(\mathrm{x})} \frac{|\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})-2}(\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{N+sp(\mathrm{x},\mathrm{y})}}d\mathrm{x}, \text{ for all } \mathrm{y} \in \mathbb{R}^N, \tag{5.7}$$

where $\mathfrak{B}_\varepsilon(\mathrm{x})$ denotes the bole of center x, and radius ϵ, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function called Kirchhoff's function that satisfies the following conditions:

$(\mathcal{B}_6)$ There exists $m > 0$, such that

$$m \leq M(t), \text{ for all } t \in \mathbb{R}.$$

$(\mathcal{B}_7)$ There exists $\theta \in (0,1)$ such that

$$\theta t M(t) \leq \widehat{M(t)} \text{ for all } t \in \mathbb{R},$$

where $\widehat{M(t)} = \int_0^t M(s)ds$ is the primitive of function M, and

$$J_{s,p(\mathrm{x},\cdot)}(\mathrm{u}) = \int_{\mathcal{U}\times\mathcal{U}} \frac{1}{p(\mathrm{x},\mathrm{y})} \frac{|\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y})|^{p(\mathrm{x},\mathrm{y})}}{|\mathrm{x}-\mathrm{y}|^{N+sp(\mathrm{x},\mathrm{y})}} d\mathrm{x}d\mathrm{y}, \text{ for all } \mathrm{u} \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U}).$$

5.2.1 Computation of group critical

In this section, the major result is following theorem:

Theorem 5.2.1. *Under assumption $(\mathcal{B}_1) - (\mathcal{B}_7)$. Then, the problem (5.3) has a weak solution in $W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$.*

Definition 5.2.2. A measurable function $\mathrm{u} \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$ is said to be a weak solution of (5.3) if

$$\begin{aligned} M\left(J_{s,p(x,\cdot)}(u)\right) \int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y})|^{p(x,\mathrm{y})-2}(\mathrm{u}(\mathrm{x}) - \mathrm{u}(\mathrm{y}))(\mathrm{v}(x) - \mathrm{v}(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{N+sp(x,\mathrm{y})}} d\mathrm{x}d\mathrm{y} \\ = \lambda \int_{\mathcal{U}} |\mathrm{u}|^{r(\mathrm{x})-2} \log|\mathrm{u}| \mathrm{u}\mathrm{v}d\mathrm{x} + \lambda \int_{\mathcal{U}} f(x,\mathrm{u})\mathrm{v}dx, \end{aligned}$$

for all $\mathrm{v} \in W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})$.

We consider the functional $\zeta : W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U}) \to \mathbb{R}$ defined by

$$\begin{aligned} \zeta(u) &:= \widehat{M}\left(J_{s,p(x,\cdot)}(u)\right) - \lambda \int_{\mathcal{U}} \frac{1}{r(\mathrm{x})} \left(|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})} \log|\mathrm{u}(\mathrm{x})| - \frac{1}{r(\mathrm{x})} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})} \right) dx \\ &\quad - \lambda \int_{\mathcal{U}} F(x, \mathrm{u}(\mathrm{x}))dx \\ &= L_1(u) - L_2(u) - L_3(u) \end{aligned} \tag{5.8}$$

Then it follows from [15, 50] that $L_1 - L_3 \in C^1\left(W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U}), \mathbb{R}\right)$ and

$$\begin{aligned} &\left\langle (L_1 - L_3)'(u), \mathrm{v} \right\rangle \\ &= M\left(J_{s,p(x,\cdot)}(u)\right) \int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{u}(\mathrm{x}) - u(\mathrm{y})|^{p(x,\mathrm{y})-2}(\mathrm{u}(\mathrm{x}) - u(\mathrm{y}))(\mathrm{v}(x) - \mathrm{v}(\mathrm{y}))}{|x-\mathrm{y}|^{N+sp(x,x)}} d\mathrm{x}d\mathrm{y} \\ &- \lambda \int_{\mathcal{U}} f(x, \mathrm{u}(\mathrm{x}))\mathrm{v}(x)dx. \end{aligned}$$

Ouar first result is the folowing Lemma.

Lemma 5.2.3. *Let $\mathcal{U} \subset \mathbb{R}^N$ be a Lipschitz-bounded domain, λ be a parametre positive, and $r : \mathcal{U} \to (1, \infty)$ be a continuous function. Then, we have $L_2 \in C^1\left(W^{s,q(x),p(x,y)}(\mathcal{U}), \mathbb{R}\right)$, and*

$$\left\langle L_2^{'}(u), \mathrm{v} \right\rangle = \lambda \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{u}(\mathrm{x}) \log |\mathrm{u}(\mathrm{x})|\mathrm{v}(x)d\mathrm{x},$$

for all $u, \mathrm{v} \in W^{s,q(x),p(x,y)}(\mathcal{U})$.

Proof. Let $\mathrm{v}, u \in W^{s,q(x),p(x,y)}(\mathcal{U})$. For each $x \in \mathcal{U}$, and $0 < t < 1$. By the definition of Gâteaux-differentiable, we get

$$\begin{aligned}\left\langle L_2^{'}(u), \mathrm{v} \right\rangle &= \lim_{t\to 0} \frac{L_2(u + t\mathrm{v}) - L_2(u)}{t} \\ &= \lim_{t\to 0} \lambda \int_{\mathcal{U}} \frac{1}{r(\mathrm{x})} \frac{\left(|u + t\mathrm{v}|^{r(\mathrm{x})} \log |u + t\mathrm{v}| - |u|^{r(\mathrm{x})} \log |u| + \frac{1}{r(\mathrm{x})}\left(|u + t\mathrm{v}|^{r(\mathrm{x})} - |u|^{r(\mathrm{x})}\right)\right)}{t} d\mathrm{x}.\end{aligned}$$

We consider the function defined by $K : [0, 1] \to \mathbb{R}$ as

$$K(y) = \frac{|u + yt\mathrm{v}|^{r(\mathrm{x})} \log |u + yt\mathrm{v}|}{r(\mathrm{x})} - \frac{|u + yt\mathrm{v}|^{r(\mathrm{x})}}{r^2(x)}.$$

According to the mean value Theorem, there exists $\theta \in (0, 1)$ such that

$$K^{'}(y)(\theta) = K(1) - K(0).$$

Combining the Lebesgue's dominated converge theorem with a direct computation, we have

$$\left\langle L_2^{'}(u), \mathrm{v} \right\rangle = \lambda \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{u}(\mathrm{x}) \log |\mathrm{u}(\mathrm{x})|\mathrm{v}(x)d\mathrm{x}.$$

Using the same method as appears in paper [48], we can easily prove that $L_2 \in C^1\left(W^{s,q(x),p(x,y)}(\mathcal{U}), \mathbb{R}\right)$. □

Lemma 5.2.4. *We assume that the conditions $(\mathcal{B}_5) - (\mathcal{B}_7)$ are fulfilled. Then, the functional ζ satisfies the Palais- Smalle condition at level $c \in \mathbb{R}$.*

Proof. Let $\{\mathrm{u}_n\}_{n\in\mathbb{N}} \subset W^{s,q(x),p(x,y)}(\mathcal{U})$ with $\zeta(\mathrm{u}_n) \to c$ as $n \to +\infty$ and $\zeta^{'}(\mathrm{u}_n) \to 0$ as $n \to +\infty$ in $W^{s,q(x),p(x,y)}(\mathcal{U})$. Without loss of generality, we assume that $\|\mathrm{u}_n\|_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})} \geq 1$. By contradiction, we prove the sequence $\{\mathrm{u}_n\}_{n\in\mathbb{N}}$ is bounded in $W^{s,q(x),p(x,y)}(\mathcal{U})$. So, there exist $C > 0$ such that

$$\langle \zeta\mathrm{u}_n, \mathrm{u}_n\rangle \leq C\|\mathrm{u}_n\|_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})} \text{ and } \zeta(\mathrm{u}_n) \leq C.$$

We combine condition $(\mathcal{B}_5)$ with condition $(\mathcal{B}_7)$, we have

$$
\begin{aligned}
&C + C\|\mathrm{u}_n\|_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})} \geq \zeta(\mathrm{u}_n) - \frac{1}{r^-}\left\langle \zeta'(\mathrm{u}_n), \mathrm{u}_n \right\rangle \\
&= \widehat{M}\left(J_{s,p(x,\cdot)}(\mathrm{u}_n)\right) - \lambda\int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}\left(|\mathrm{u}_n(x)|^{r(\mathrm{x})}\log|\mathrm{u}_n(x)| - \frac{1}{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\right)d\mathrm{x} \\
&- \lambda\int_{\mathcal{U}} F(x,\mathrm{u}_n(x))d\mathrm{x} \\
&- \frac{1}{r^-}M\left(J_{s,p(x,\cdot)}(\mathrm{u}_n)\right)\int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{u}_n(x)-\mathrm{u}_n(\mathrm{y})|^{p(x,\mathrm{y})}}{|x-\mathrm{y}|^{N+sp(x,x)}}d\mathrm{x}d\mathrm{y} + \frac{\lambda}{r^-}\int_{\mathcal{U}}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\log|\mathrm{u}_n(x)|d\mathrm{x} \\
&+ \frac{\lambda}{r^-}\int_{\mathcal{U}} f(x,\mathrm{u}_n(x))\mathrm{u}_n(x)d\mathrm{x} \\
&\geq (1-\alpha)M\left(J_{s,p(x,\cdot)}(\mathrm{u}_n)\right)J_{s,p(x,\cdot)}(\mathrm{u}_n) - \frac{1}{r^-}M\left(J_{s,p(x,\cdot)}(\mathrm{u}_n)\right)\int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{u}_n(x)-\mathrm{u}_n(\mathrm{y})|^{p(x,\mathrm{y})}}{|x-\mathrm{y}|^{N+sp(x,x)}}d\mathrm{x}d\mathrm{y} \\
&+ \frac{\lambda}{r^-}\int_{\mathcal{U}}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\log|\mathrm{u}_n(x)|d\mathrm{x} - \lambda\int_{\mathcal{U}}\frac{1}{r(\mathrm{x})}|\mathrm{u}_n(x)|^{r(\mathrm{x})}\log|\mathrm{u}_n(x)|d\mathrm{x} \\
&+ \frac{\lambda}{r^-}\int_{\mathcal{U}} f\left(x,\mathrm{u}_n(x)\right)\mathrm{u}_n(x)d\mathrm{x} - \lambda\int_{\mathcal{U}} F(x,\mathrm{u}_n(x))d\mathrm{x} \\
&\geq m\left[\int_{\mathcal{U}\times\mathcal{U}}\frac{1}{p(x,\mathrm{y})}\frac{|\mathrm{u}_n(x)-\mathrm{u}_n(\mathrm{y})|^{p(x,\mathrm{y})}}{|x-\mathrm{y}|^{N+sp(x,x)}}d\mathrm{x}d\mathrm{y} - \frac{1}{r^-}\int_{\mathcal{U}\times\mathcal{U}}\frac{|\mathrm{u}_n(x)-\mathrm{u}_n(\mathrm{y})|^{p(x,\mathrm{y})}}{|x-\mathrm{y}|^{N+sp(x,x)}}d\mathrm{x}d\mathrm{y}\right] \\
&+ \frac{\lambda}{r^-}\int_{\mathcal{U}}|\mathrm{u}_n(x)|^{r(\mathrm{x})}\log|\mathrm{u}_n(x)|d\mathrm{x} - \frac{\lambda}{r^-}\int_{\mathcal{U}}|\mathrm{u}_n(x)|^{r(\mathrm{x})}\log|\mathrm{u}_n(x)|d\mathrm{x} \\
&\geq m\left(\frac{1}{p^-}-\frac{1}{r^-}\right)\|\mathrm{u}_n\|^{p^+}_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})}.
\end{aligned}
$$

This is a contradiction as $\|\mathrm{u}_n\|_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})} \to \infty$. From Lemma 3.1.5, we get that there exists $u \in W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U})$ and a subsequence of u_n still denoted by u_n that satisfies the following inequality:

$$
\begin{cases}
\mathrm{u}_n \to u \text{ a.e in } \mathcal{U}, \\
\mathrm{u}_n \rightharpoonup u \text{ weakly in } W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U}), \\
\mathrm{u}_n \to u \text{ strongly in } L^{\sigma(x)}(\mathcal{U}) \text{ for } 1 \leq \sigma(x) < p_s^*(x).
\end{cases} \tag{5.9}
$$

Since $W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U})$ is a reflexive space, we deduce that

$$\left\langle \zeta'(\mathrm{u}_n), \mathrm{u}_n - u \right\rangle \to 0 \text{ as } n \to \infty.$$

Thus, we get that

$$
\begin{aligned}
\left\langle \zeta'(\mathrm{u}_n), \mathrm{u}_n - u \right\rangle &= -\lambda\int_{\mathcal{U}}|\mathrm{u}_n(x)|^{r(\mathrm{x})-2}\log|\mathrm{u}_n(x)|\mathrm{u}_n(x)\left(\mathrm{u}_n(x)-\mathrm{u}(\mathrm{x})\right)d\mathrm{x} - \lambda\int_{\mathcal{U}} f\left(x,\mathrm{u}_n(x)\right)\left(\mathrm{u}_n(x)-\mathrm{u}(\mathrm{x})\right)d\mathrm{x} \\
&+M\left(J_{s,p(x,\cdot)}(\mathrm{u}_n)\right)\int_{\mathcal{U}\times\mathcal{U}}\frac{|\mathrm{u}_n(x)-\mathrm{u}_n(\mathrm{y})|^{p(x,\mathrm{y})-2}(\mathrm{u}_n(x)-\mathrm{u}_n(\mathrm{y}))(\mathrm{u}_n(x)-\mathrm{u}_n(\mathrm{y})-(\mathrm{u}(\mathrm{x})-u(\mathrm{y}))}{|x-\mathrm{y}|^{N+sp(x,x)}}d\mathrm{x}d\mathrm{y}.
\end{aligned}
$$

as $\mathrm{n} \to \infty$. Now, we show that

$$\lim_{n\to\infty}\int_{\mathcal{U}}|\mathrm{u}_n(x)|^{r(\mathrm{x})-2}\mathrm{u}_n(x)\left(\mathrm{u}_n(x)-\mathrm{u}_n(x)\right)\log|\mathrm{u}_n(x)|\,dx = \lim_{n\to\infty}\int_{\mathcal{U}}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{u}(\mathrm{x})\left(\mathrm{u}_n(x)-\mathrm{u}(\mathrm{x})\right)\log|\mathrm{u}(\mathrm{x})|dx.$$

From (5.9) it easy to see that

$$\lim_{n\to\infty} |\mathrm{u}_n(x)|^{r(\mathrm{x})} \log|\mathrm{u}_n(x)| = |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})} \log|u(x)| \text{ a.e in } \mathcal{U}. \tag{5.10}$$

Let $\gamma \in (0, p_s^{*-} - r^+)$. From lemma 5.1.1, and theorem 3.1.6, we have that:

$$\begin{aligned}\int_{\mathcal{U}} |\mathrm{u}_n(x)|^{r(\mathrm{x})} \log|\mathrm{u}_n(x)\|dx = &\int_{\mathcal{U}\cap\{|\mathrm{u}_n(x)|\le 1\}} |\mathrm{u}_n(x)|^{r(\mathrm{x})} \log|u_n(x)\ |dx \\ &+ \int_{\mathcal{U}\cap\{|\mathrm{u}_n(x)|>1\}} |\mathrm{u}_n(x)|^{r(\mathrm{x})} \log|\mathrm{u}_n(x)||dx\end{aligned} \tag{5.11}$$

where $L = \sup \|\mathrm{u}_n\|^{r^+ +\gamma} < \infty$. So, the sequence $\left\{|\mathrm{u}_n(x)|^{r(\mathrm{x})} \log|\mathrm{u}_n(x)\|\right\}_{n\ge 1}$ is equi-integrale in $L^1(\mathcal{U})$, and uniformaly bounded. Combining (5.11), (5.10) with Vitali's convergence theorem, we have that

$$\lim_{n\to\infty} \int_{\mathcal{U}} |\mathrm{u}_n(x)|^{r(\mathrm{x})} \log|\mathrm{u}_n(x)\|dx = \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})} \log|u(x)\|dx. \tag{5.12}$$

Similarly, we prove

$$\lim_{n\to\infty} \int_{\mathcal{U}} u\,|\mathrm{u}_n(x)|^{r(\mathrm{x})-2}\,\mathrm{u}_n(x)|\log|\mathrm{u}_n(x)\|dx = \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})} \log|u(x)\|dx, \tag{5.13}$$

and

$$\lim_{n\to\infty} \int_{\mathcal{U}} \mathrm{u}_n(x)\,|\mathrm{u}_n(x)|^{r(\mathrm{x})-2} \log|u(x)\|\mathrm{u}(\mathrm{x})dx = \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})} \log|u(x)\|dx. \tag{5.14}$$

From (5.12), (5.13), and (5.14), we have that:

$$\lim_{n\to\infty} \int_{\mathcal{U}} |\mathrm{u}_n(x)|^{r(\mathrm{x})-2}\,\mathrm{u}_n(x)\,(\mathrm{u}_n(x)-\mathrm{u}_n(x))\log|\mathrm{u}_n(x)|\,dx = \lim_{n\to\infty} \int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})-2}\mathrm{u}(\mathrm{x})\,(\mathrm{u}_n(x)-\mathrm{u}(\mathrm{x}))\log|\mathrm{u}(\mathrm{x})|dx. \tag{5.15}$$

Combining (5.15) with the same argument in Lemma 3.1 [2], we get that $\mathrm{u}_n \to u$ strongly in $W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U})$. □

Remark 5.2.5. We assume that the conditions $(\mathcal{B}_1) - (\mathcal{B}_7)$ are fulfilled. Then, the functional ζ satisfies the (C_c) condition.

Proof. We use the same technical in Theorem 4 [62] and from Lemma 5.2.4, we deduce that the functional ζ satisfies the (C_c) condition. □

Nown we compute the criticals groups. From Lemma 5.2.5, it follows that $C_k(\zeta, \infty)$ make sence.

Theorem 5.2.6. *We assume that the functional ζ satisfies the conditions $(\mathcal{B}_1)$, $(\mathcal{B}_7)$. Then, we get $C_k(\zeta, \infty) = 0$.*

Proof. Let $G(x,t) = f(x,t)t - p^+F(x,t)$ and $c_1 = 1 + \sup_{\bar{\mathcal{U}}\times[-R;R]} G(x,t) - \inf_{\bar{\mathcal{U}}\times[-R;R]} G(x,t)$. From the condition (H_5), we get that

$$G(x,s) \leq G(x,t) + c_1 \text{ for all } x \in \bar{\mathcal{U}} \text{ and } 0 \leq s < t \text{ or } t \leq s \leq 0. \tag{5.16}$$

By (5.16), we get

$$G(x,t) \geq -c_1 \text{ when } s = 0 \tag{5.17}$$

Let $u \in \mathcal{S}^1 = \left\{u \in W^{s,q(x),p(x,y)}(\mathcal{U}) : \|u\| = 1\right\}$ and $t \geq 1$. From Fatou's Lemma and condition $(\mathcal{B}_3)$, we get that

$$+\infty = \int_{\mathcal{U}} \lim_{t\to\infty} \frac{F(x,tu)}{|tu|^{r^+}}|u|^{r^+}d\mathrm{x} \leq \lim_{t\to\infty}\int_{\mathcal{U}} \frac{F(x,tu)}{|t|^{r^+}}d\mathrm{x}. \tag{5.18}$$

Using the condition $(\mathcal{B}_7)$, it is easy to see that

$$\widehat{M}(t) \leq c_1^{'}t. \tag{5.19}$$

By (5.19) and (5.18), we have

$$\begin{aligned}
\zeta(tu) &= \widehat{M}\left(J_{s,p(x,\cdot)}(tu)\right) - \lambda\int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}|t|^{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}|\log|tu(x)\|d\mathrm{x} \\
&\quad + \lambda\int_{\mathcal{U}} \frac{1}{r^2(x)}|t|^{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}d\mathrm{x} - \lambda\int_{\mathcal{U}} F(x,t\mathrm{u}(\mathrm{x}))d\mathrm{x} \\
&\leq c_1^{'}\frac{t^{p^+}}{p^+}\int_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{u}(\mathrm{x}) - u(\mathrm{y})|^{p(x,\mathrm{y})}}{|x-\mathrm{y}|^{N+sp(x,\mathrm{y})}}d\mathrm{x}d\mathrm{y} + \frac{t^{r^+}}{(r^-)^2}\lambda\int_{\mathcal{U}}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}d\mathrm{x} - \lambda\int_{\mathcal{U}} F(x,t\mathrm{u}(\mathrm{x}))d\mathrm{x} \\
&\leq t^{r^+}\left(t^{p^+-r^+}\frac{c_1^{'}}{p^+} + \frac{\lambda}{(r^-)^2}\int_{\mathcal{U}}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}d\mathrm{x} - \lambda\int_{\mathcal{U}}\frac{F(x,t\mathrm{u}(\mathrm{x}))}{t^{r^+}}dx\right) \to -\infty \text{ as } t\to+\infty.
\end{aligned}$$

Choosing $a < \min\left\{\inf_{\|u\|\leq 1}\zeta(u); \frac{-\lambda|\mathcal{U}|c_1}{p^+}\right\}$, then for $u \in \mathcal{S}^1$, there exists $t_0 > 1$ such that $\zeta(t_0u) \leq a$. Therefore, if

$$\begin{aligned}
\zeta(tu) &= \widehat{M}\left(J_{s,p(x,\cdot)}(tu)\right) - \lambda\int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}|t|^{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}|\log|tu(x)\|d\mathrm{x} + \lambda\int_{\mathcal{U}} \frac{1}{r^2(x)}|t|^{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}d\mathrm{x} \\
&\quad - \lambda\int_{\mathcal{U}} F(x,t\mathrm{u}(\mathrm{x}))d\mathrm{x} \\
&\leq a.
\end{aligned}$$

Then,

$$\begin{aligned}
&M\left(J_{s,p(x,\cdot)}(tu)\right)\int_{\mathcal{U}\times\mathcal{U}}|t|^{p(x,\mathrm{y})}\frac{|\mathrm{u}(\mathrm{x}) - u(\mathrm{y})|^{p(x,\mathrm{y})}}{|x-\mathrm{y}|^{N+sp(x,\mathrm{y})}}d\mathrm{x}d\mathrm{y} - \lambda\int_{\mathcal{U}}\frac{1}{r(\mathrm{x})}|t|^{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}|\log|tu(x)\|d\mathrm{x} \\
&\leq \frac{p^+}{\theta}\left[a + \lambda\int_{\mathcal{U}} F(x,t\mathrm{u}(\mathrm{x}))d\mathrm{x}\right].
\end{aligned}$$

Using (5.17), we get that

$$\begin{aligned}\frac{d}{dt}\zeta(tu) &= \frac{1}{t}\left\langle \zeta'(tu), tu\right\rangle \\ &= \frac{1}{t}\left(M\left(J_{s,p(x,\cdot)}(tu)\right)\int_{\mathcal{U}\times\mathcal{U}}|t|^{p(x,\mathrm{y})}\frac{|\mathrm{u}(\mathrm{x})-u(\mathrm{y})|^{p(x,\mathrm{y})}}{|x-\mathrm{y}|^{N+sp(x,\mathrm{y})}}d\mathrm{x}d\mathrm{y}\right) \\ &- \frac{1}{t}\int_{\mathcal{U}}|t|^{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}d\mathrm{x} - \frac{1}{t}\lambda\int_{\mathcal{U}}f(x,tu)tu)d\mathrm{x}) \\ &\leq \frac{1}{t}\left(ap^{+} + p^{+}\lambda\int_{\mathcal{U}}F(x,t\mathrm{u}(\mathrm{x}))d\mathrm{x} - \lambda\int_{\mathcal{U}}f(x,tu)tu)d\mathrm{x}\right) \\ &\leq \frac{1}{t}\left(ap^{+} + \lambda c_1|\mathcal{U}|\right) < 0,\end{aligned}$$

where $|\mathcal{U}|$ denote the measure of the domain $\mathcal{U}$. Thanks to the implicite function theorem, there exists an unique $T \in C(\mathcal{S}^1, \mathbb{R})$ such that $\zeta(T(u)u) = a$ for any $u \in \mathcal{S}^1$. We extend T to all of $W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U})$ by

$$T_0(u) = \frac{1}{\|u\|}T\left(\frac{u}{\|u\|}\right) \text{ for all } u \in W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U})\backslash\{0\}.$$

Then $T_0 \in C^1(W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U},\mathbb{R})\backslash\{0\}$, and $\zeta(T_0(u)u) = a$. Also, if $\zeta(u) = a$, then $T_0(u) = 1$. We define a function $\widehat{T_0} : W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U}) \to \mathbb{R}$ as

$$\widehat{T_0}(u) := \begin{cases} T_0(u), & \text{if } \zeta(u) \geq a, \\ 1, & \text{if } \zeta(u) < a. \end{cases}$$

Clearly, $\widehat{T_0} \in C(W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U}),\mathbb{R})\backslash 0$. Let $h : [0,1] \times W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U}) \to W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U})$ be the map defined as

$$h(t,u) = (1-t)u + tu\widehat{T_0}(u) \text{ for all } (t,u) \in [0,1] \times W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U})\backslash 0.$$

Evidently, we have

$$h(0,u) = u, \text{ and } h(1,u) = \widehat{T_0}(u)u \in \zeta^a \tag{5.20}$$

From (5.20), we get

$$h(t,.)_{|\zeta^a} = id_{|\zeta^a} \text{ for all } t \in [0,1].$$

It is follows that

$$\zeta^a \text{ is a strongly deformation of } W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U}). \tag{5.21}$$

We consider the radial retraction $\bar{T} : W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U}) \to \mathbb{R}$ defined by

$$\bar{T}(u) = \frac{u}{\|u\|} \text{ for all } u \in W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U})\backslash\{0\}.$$

This map is continuous and $\bar{T}_{|\mathcal{S}^1} = id_{|\mathcal{S}^1}$. Then $\mathcal{S}^1$ is a retract of $W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U})\backslash\{0\}$. Let $\bar{h} : [0,1] \times W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U}) \to W^{s,q(x),p(x,\mathrm{y})}(\mathcal{U})$ be the map defined as

$$\bar{h}(t,u) = (1-t)u + t\bar{T}(u).$$

Clearly, we have

$$\bar{h}(0,u) = u, \ \ \bar{h}(0,u) \text{ and } \bar{h}(1,.)_{|\mathcal{S}^1} = id_{|\mathcal{S}^1}. \tag{5.22}$$

Hence, we refer that

$$\mathcal{S}^1 \text{ is a deformation retract of } W^{s,q(x),p(x,y)}(\mathcal{U})\backslash\{0\}. \tag{5.23}$$

Combining (5.23) with (5.21), it follows that

$$\zeta^a \text{ and } \mathcal{S}^1 \text{ are homotopie equivalent .}$$

So, we have

$$H_k\left(W^{s,q(x),p(x,y)}(\mathcal{U}), \zeta^a\right) = H_k\left(W^{s,q(x),p(x,y)}(\mathcal{U}), \mathcal{S}^1\right) \text{ for all } k \in \mathbb{N}.$$

We already know that the space $W^{s,q(x),p(x,y)}(\mathcal{U})$ is an infinite dimensional Banach space and $\mathcal{S}^1$ is a contractible space. See Remark 6.1.13 in [62]. So, it follows that

$$C_k(\zeta, \infty) = H_k\left(W^{s,q(x),p(x,y)}(\mathcal{U}), \zeta^a\right) = H_k\left(W^{s,q(x),p(x,y)}(\mathcal{U}), \mathcal{S}^1\right) = 0 \text{ for all } k \in \mathbb{N}. \tag{5.24}$$

□

Theorem 5.2.7. *We assume that the conditions* $(\mathcal{B}_1) - (\mathcal{B}_7)$ *are fulfilled. Then, there exists* $k_0 \in \mathbb{N}$ *such that* $C_{k_0}(\zeta, 0) \neq 0$

Proof. Evidently, the zero function is a critical point of ζ. Since $W^{s,q(x),p(x,y)}(\mathcal{U})$ is a separable and reflexive Banach space, from Theorem 2-3 in [74], there exist $\{e_i\}_{i=1}^{\infty} \subset W^{s,q(x),p(x,y)}(\mathcal{U})$ and $\{f_i\}_{i=1}^{\infty} \subset W^{s,q(x),p(x,y)}(\mathcal{U})^*$ such that

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m, \end{cases}$$

$$W^{s,q(x),p(x,y)}(\mathcal{U}) = \overline{span\{e_j : j = 1, 2, \ldots\}}, \text{ and } W^{s,q(x),p(x,y)}(\mathcal{U})^* = \overline{span\{f_j : j = 1, 2, \ldots\}}$$

For convenience, we write $X_j = span\{e_j\}$, $Y_k = \bigoplus_{j=1}^{k} X_j$, and $Z_k = \bigoplus_{j=k}^{\infty} X_j$. Thus, we have $W^{s,q(x),p(x,y)}(\mathcal{U}) = Y_k \bigoplus Z_k$. Let $u \in Y_k$. Since Y_k is a finite dimensional space, we get that for given $R > 0$, there exists $0 < \rho < 1$ small such that

$$u \in Y_k, |||u|||_{Y_k} < \rho \Rightarrow |\mathrm{u(x)}| < R \text{ for all } x \in \mathcal{U}.$$

Let $0 < r < R$. We consider the following sets: $\mathcal{U}_1 = \{x \in \mathcal{U} : |\mathrm{u(x)}| < r\}$, $\mathcal{U}_2 = \{x \in \mathcal{U} : r < |\mathrm{u(x)}| < R\}$, and $\mathcal{U}_3 = \{x \in \mathcal{U} : |\mathrm{u(x)}| > R\}$. We put $G(x,t) = F(x,t) -$

$\frac{C}{p^-}|u|^{\alpha(x)}$. Obviously, we get that $\mathcal{U}_i \cap \mathcal{U}_j$ and $\mathcal{U} = \cup_{i=1}^{3}\mathcal{U}_i$. We combine condition $(\mathcal{B}_7)$ with condition $(\mathcal{B}_4)$, we obtain that

$$\begin{aligned}
\zeta(u) &= \widehat{M}\left(J_{s,p(x,\cdot)}(u)\right) - \int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}d\mathrm{x} - \lambda\int_{\mathcal{U}} F(x,\mathrm{u}(\mathrm{x}))d\mathrm{x} \\
&\leq C_1 J_{s,p(x,\cdot)}(u) - \frac{1}{r^-}\int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}d\mathrm{x} - \lambda\int_{\mathcal{U}} F(x,\mathrm{u}(\mathrm{x}))d\mathrm{x} \\
&\leq C_1 J_{s,p(x,\cdot)}(u) - \frac{1}{r^-}\int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}d\mathrm{x} - \lambda\int_{\mathcal{U}} \frac{C_2}{p^+}|\mathrm{u}(\mathrm{x})|^{\alpha(x)}d\mathrm{x} - \lambda\int_{\mathcal{U}_1} G(x,\mathrm{u}(\mathrm{x}))d\mathrm{x} \\
&- \lambda\int_{\mathcal{U}_2} G(x,\mathrm{u}(\mathrm{x}))d\mathrm{x} - \lambda\int_{\mathcal{U}_3} G(x,\mathrm{u}(\mathrm{x}))d\mathrm{x}.
\end{aligned}$$

From Theorem 3.1.6, there exists a positive constant such that

$$|u|_{L^{\alpha(x)}}(\mathcal{U}) \leq C\|u\|_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})} \leq 2C\||u|\|_{Y_k} \leq 2C\rho.$$

If $\rho < \frac{1}{2C}$, then $|u|_{L^{\alpha(x)}(\mathcal{U})} \leq 1$. Since $\mathcal{U}$ is compact, there exist a finite sub-covering $\{\mathcal{Q}_j\}_{j=1}^{m}$ such that

$$\overline{\mathcal{U}} = \cup_{i=1}^{m}\mathcal{Q}_i.$$

Notice that $\int_{\mathcal{U}_1} G(x,\mathrm{u}(\mathrm{x}))d\mathrm{x} \to 0$ as $r \to 0$. Therefore, we get

$$\zeta(u) \leq \sum_{j=1}^{m}\left[|u|^{p^+}_{L^{r(\mathrm{x})}(\mathcal{Q}_j)} - \frac{1}{r^-}|u|^{r(\mathrm{x})}_{L^{r(\mathrm{x})}(\mathcal{Q}_j)} - \frac{1}{p^+}|u|^{\alpha(x)}_{L^{\alpha(x)}(\mathcal{Q}_j)}\right] - \int_{\mathcal{U}_1} G(x,\mathrm{u}(\mathrm{x}))d\mathrm{x} \leq 0.$$

Let $u \in Z_k$. Since $q(x), r(\mathrm{x}) < p_s^*(x)$, from Theorem 3.1.6, we deduce that, there exist a constant c_1 and c_1 such that

$$|u|_{L^{r(\mathrm{x})}(\mathcal{U})} \leq c_1|u|_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})} \text{ and } |u|_{L^{\beta(x)}(\mathcal{U})} \leq c_2|u|_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})}. \tag{5.25}$$

Using (5.25), $(\mathcal{B}_4)$, and $(\mathcal{B}_7)$, we deduce that

$$\begin{aligned}
\zeta(u) &= \widehat{M}\left(J_{s,p(x,\cdot)}(u)\right) - \lambda\int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}\left(|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\log|\mathrm{u}(\mathrm{x})| - \frac{1}{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}\right)d\mathrm{x} - \lambda\int_{\mathcal{U}} F(x,\mathrm{u}(\mathrm{x}))d\mathrm{x} \\
&\geq \widehat{M}\left(J_{s,p(x,\cdot)}(u)\right) - \int_{\mathcal{U}} \frac{1}{r(\mathrm{x})}|\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}d\mathrm{x} - \lambda\int_{\mathcal{U}} F(x,\mathrm{u}(\mathrm{x}))d\mathrm{x} \\
&\geq m_1 J_{s,p(x,\cdot)}(u) - \frac{1}{r^-}\int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{r(\mathrm{x})}d\mathrm{x} - \lambda\frac{C_3}{\beta^-}\int_{\mathcal{U}} |\mathrm{u}(\mathrm{x})|^{\beta(x)}d\mathrm{x} \\
&\geq \frac{m_1}{p^+}\|u\|^{p^+}_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})} - \frac{C_1}{r^-}\|u\|^{r^-}_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})} - \frac{C_3}{\beta^-}\|u\|^{\beta^-}_{W^{s,q(\mathrm{x}),p(\mathrm{x},\mathrm{y})}(\mathcal{U})}.
\end{aligned}$$

Since $r^-, \beta^- < p^+$, we deduce that

$$\zeta(u) > 0 \text{ for all } u \in Z_k.$$

Finally, from Theorem 1.7.3, there exists $k_0 \in \mathbb{N}$ such that $C_{k_0}(\zeta, 0) \neq 0$.

Conclusion and proof of theorem 5.2.1

By Theorem 5.2.6 and Theorem 5.2.7, we deduce that our problem admits at least three solution in $W^{s,q(x),p(x,y)}(\mathcal{U})$. □

5.2.2 Fundamental Theorem

Theorem 5.2.8. *Under assumption $(\mathcal{B}_1)-(\mathcal{B}_7)$. Then, the problem (5.3) has an infinitely weak solutions in $W^{s,q(x),p(x,y)}(\mathcal{U})$.*

Proof. We suppose that our problem admits three solutions $W^{s,q(x),p(x,y)}(\mathcal{U})$. That is $K_\zeta = \{0, u, \mathrm{v}\}$. From the Morse's relation, it follows that

$$C_n(\zeta, 0) := \begin{cases} \mathbb{R}, & \text{if } n = m(0), \\ 0, & \text{otherwise}, \end{cases}$$

where $m(0)$ is a Morse index of 0. See [17] for more details. We use Morse's relation, we get that

$$\begin{aligned} \sum_{k\geq 0} C_k(\zeta,\infty)X^k + (1+X)Q(X) = &\sum_{k\geq 0} \operatorname{rank} C_k(\zeta,0)X^k + \sum_{k\geq 0} \operatorname{rank} C_k(\zeta,u)X^k \\ &+ \sum_{k\geq 0} \operatorname{rank} C_k(\zeta,\mathrm{v})X^k \\ =&X^{m(0)} + 2\sum_{k\geq 0}\beta_k X^k. \end{aligned}$$

From (5.24), it follows that

$$(1+X)Q(X) = X^{m(0)} + 2\sum_{k\geq 0}\beta_k X^k,$$

where β_k nonnegative integer and Q is a polynomial with nonnegative integer coefficient. In particular, for $X = 1$ we have $2a = 1 + 2\sum_{k\geq 0}\beta_k$. Since $\beta_k \in \mathbb{N}$, we have that $\sum_{k\geq 0}\beta_k = +\infty$ leads to a contradiction. Thus, there exist infinitly solutions to problem (5.3). □

Conclusion and further perspectives

In this thesis, we focused our attention on the results on the existence of weak solutions to different types of fractional elliptic problems in fractional Sobolev spaces involving a kernel function and exponent variables in both Euclidean and non-Euclidean cases.
Our first goal was the study some non-local problems involving Kirchhoff function or double phase operator with Carathéodory function, term singularity, and logarithmic non-linearity. Using Morse's theory, topological degree methods, and variational methods. Namely by considering the non-local ellpitic PDE's

$$Au = f(x, u), \tag{5.26}$$

where A is an operator non local and f is the Carathéodory function that satisfies Ambrosetti-Rabinowitz condition, convergence uniform, limit uniform, and super-linear growth condition in the first state. In the second state, we use the topological degree methods to prove the existence of solution to problem (5.26) with f satisfying Ambrosetti-Rabinowitz condition, convergence uniform, and limit uniform.
The second goal of this thesis was to introduce a new fractional Sobolev space with a kernel function that satisfied the Lévy integrability and coercivity conditions.
We conclude this thesis with some remarks

- In the second chapter, we have introduced the new fractional Sobolev space with kernel function and we have obtained the existence of solutions for the following problems:

$$\begin{cases} (\mathcal{L}_g^K)_{p(y,\cdot)} w(y) &= \lambda\beta(y)|w(y)|^{r(y)-2}w(y) + f(y, w(y)) \quad \text{in} \quad \mathcal{U}, \\ w &= 0 \quad \text{in} \quad \mathcal{M}\backslash\mathcal{U}, \end{cases} \tag{5.27}$$

  with $p : \mathcal{U} \times \mathcal{U} \to (1, \infty)$ is a continuous function. A natural question is: What happens if p is a measurable function. A second question is: what are the conditions that the function f must fulfill in order for equation (5.27) to have a unique solution using Picone's inequality.

· In the third chapter, we have used Morse's theory to study the existence of infinitely many solutions to the following equation:

$$\begin{cases} (\mathcal{L}_g^{s_1})_{p_1(x,.)}u(x) + (\mathcal{L}_g^{s_2})_{p_2(x,.)}u(x) & = \frac{f(x,u(x))}{u(x)^{\gamma(x)}} & \text{in} & \mathcal{U}, \\ u & > 0 & \text{in} & \mathcal{U}, \\ u & = 0 & \text{in} & \mathcal{M}\backslash\mathcal{U}, \end{cases}$$

under condition $0 < \gamma(x) < 1$. A natural question is: what happens if $\gamma(x) > 1$.

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