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كلية العلوم ظهر المهراز فاس
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BOUKHRIJ Mohamed

On some class of degenerate elliptic probems in p(x) and anisotropic Sobolev spaces

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Pr. Abdelfattah TOUZANI	***FSDM FES***	***Président***
Pr. Zakaria El ALLALI	***FPN NADOR***	***Rapporteur***
Pr. Abdelkader BOUAJAJA	***FSJES SETTAT***	***Rapporteur***
Pr. Ahmed YOUSSFI	***ENSA FES***	***Rapporteur***
Pr. Ahmed ABERQI	***ENSA FES***	***Examinateur***
Pr. Mounir MEKKOUR	***FSDM FES***	***Examinateur***
Pr. Jaouad BENNOUNA	***FSDM FES***	***Directeur de thèse***

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Notations

Ω	open domain in $\mathbb{R}^N$ for $N \geq 1$
$\partial\Omega$	topological boundary of Ω
$x = (x_1, \ldots, x_N)$	point in $\mathbb{R}^N$ for $N \geq 1$
$dx = dx_1 \ldots dx_N$	Lebesgue measure in Ω
∇u	gradient of function u
$\mathrm{supp} u$	support of function u
$\mathscr{C}(\Omega)$	The function space of continuous functions in Ω
$\mathscr{C}^k(\Omega)$	for $1 \leq k \leq \infty$ the function space of k times continuously differentiable functions in Ω
$\mathscr{D}(\Omega)$	$\mathscr{C}^\infty(\Omega)$ functions in Ω, with compact support
$L^p(\Omega)$	$\left\{f : \Omega \to \mathbb{R} / \ f \text{ measurable}, \int_\Omega \lvert f(x)\rvert^p dx < \infty\right\}$ for $1 \leq p < \infty$
$L^\infty(\Omega)$	$\left\{f : \Omega \to \mathbb{R} / \ f \text{ measurable}, \ \mathrm{esssup}_{x\in\Omega} \lvert f(x)\rvert < \infty\right\}$
$W^{1,p}(\Omega)$	$\left\{f : \Omega \to \mathbb{R} / \ f \text{ measurable}, \ f \in L^p(\Omega), \text{and } \frac{\partial f}{\partial x_i} \in L^p(\Omega)\right\}$ for $1 \leq p \leq \infty$
$W_0^{1,p}(\Omega)$	Closure of $\mathscr{D}(\Omega)$ in $W^{1,p}(\Omega)$ for $1 \leq p \leq \infty$

Résumé

[1]

L'objectif de ce travail est l'étude de divers problèmes d'équations aux dérivées partielles non linéaires de type elliptique dans les espaces de Lebesgue Anisotropiques, et les espaces de Sobolev à exposants variables, faisant intervenir des opérateurs du type Leray-Lions avec des données peu régulières . Cette thèse, est composée de deux parties chaque partie contient trois chapitres :

Première partie :

Chapitre I : ce chapitre est totalement consacré à définir le côté fonctionnel des espaces anisotropique, où les différentes dérivées partielles faibles de la solution u sont intégrables avec différents exposants, collectées dans le vecteur $\vec{p} = (p_1, ..., p_N)$ tel que $1 < p_i < \infty,\ \ i = 1, ..., N$.

Chapitre II : dans ce chapitre nous avons montré l'existence des solutions faibles du problème anisotrope suivant :

$$\begin{cases} -\partial_{x_i} a_i(x,u,\nabla u) + \sum_{i=1}^{N} H^i(x,u,\nabla u) = f - \partial_{x_i} g_i & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases} \tag{1}$$

où Ω est un sous ensemble ouvert borné de $\mathbb{R}^N$, $(N \geq 2)$, pour $i = 1, ..., N$, $a_i(x,u,\nabla u)$ est une fonction Carathéodory avec ellipticité dégénérée et croissance par rapport à u , et le terme d'ordre inférieur $H_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ est une fonction carathéodory qui ne satisfait qu'une condition de croissance par rapport au gradient ∇u

Chapitre III : Notre objectif dans ce chapitre est de prouver l'existence et l'unicité de solutions entropie pour le problème unilatéral associé à l'équation elliptique anisotrope quasi-linéaire suivante

$$\begin{cases} -\sum_{1}^{N} D^i a_i(x,u,\nabla u) + g(x,u) = f & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases} \tag{2}$$

1. *Mots clés et phrases* : Problèmes Elliptique, Espace de Sobolev, Solutions Renormalisées, Solutions Entropique, problèmes Unilateral, problèmes noncoercive, Estimation L^∞, Méthode de Réarrangement.

où $f \in L^1(\Omega)$, les fonctions Carathéodory $a_i(x,s,x_i)$ vérifient une condition de coercivité non standard, et le terme d'ordre inférieur $g(x,s)$ est une fonction Carathéodory satisfaisant seulement une condition de croissance.

La deuxième partie : On commence cette partie par un chapitre contenant un rappel sur le cadre fonctionnel des espaces de Sobolev à exposant variable, on termine ce chapitre par quelques propriétés de réarrangement relative

Chapitre V : dans ce chapitre nous avons etudié le problème elliptique dégénéré non linéaire suivant :

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases} \tag{3}$$

où Ω est un sous-ensemble ouvert borné de $\mathbb{R}^N$ ($N \geq 2$). De plus, nous supposons que $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ est une fonction vectorielle de Carathéodory satisfaisant une condition de coercivité dégénéré.

Le premier résultat dans ce chapitre sert à montrer en utilisons la notion de réarrangement l'estimation L^∞ et l'existence des solutions faible, et le deuxième résultat sera consacré à l'étude de la solution renormalisée.

Chapitre VI : Notre objectif dans ce chapitre est d'étudier dans le cadre des espaces de Sobolev à exposant variable, la classe des problèmes elliptiques dégénéré non linéaire avec des coefficients en $L^1(\Omega)$ de la forme :

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases} \tag{4}$$

Nous prouvons l'existence des solutions faibles et entropiques sous quelques hypothèses sur a. Nous étudions également le même problème avec un terme d'ordre inférieur $H(x,s,\xi)$ qui satisfie seulement la condition de croissance.

Abstract

[2] This work focuses on the results of existence and sometimes uniqueness of solutions for some class of degenerate elliptic problems in $p(x)$ and Anisotropic Sobolev spaces. The thesis is divided into two parts :
First part : We give some preliminary about the anisotropic spaces in the first chapter , and then in the second chapter we study the following non-linear anisotropic degenerate elliptic problem :

$$\begin{cases} -\partial_{x_i} a_i(x,u,\nabla u) + \sum_{i=1}^{N} H^i(x,u,\nabla u) = f - \partial_{x_i} g_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$, for $i = 1,...,N$, $a_i(x,u,\nabla u)$ is allowed to degenerate with respect to the unknown u, and $H_i(x,u,\nabla u)$ is a nonlinear term without a sign condition. Under suitable conditions on a_i and H_i, we prove the existence of weak solutions.

In the last chapter of this part we are also interested in the unilateral problem associated to some quasilinear elliptic equation of the form

$$\begin{cases} -\sum_{1}^{N} D^i a_i(x,u,\nabla u) + g(x,u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$ with boundary $\partial\Omega$, and $g(x,s)$ verify the growth condition, where the data f is assumed to be in $L^1(\Omega)$, we prove existence of entropy solutions for this problem. Moreover, the uniqueness solution was proved under some additional assumptions.
Second part : We start this part with a chapter containing a preliminary on the functional framework of Sobolev spaces with a variable exponent and some rearrangement property. In the second chapter we study the nonlinear degenerate elliptic problem with $L^1(\Omega)$ co-

2. *Key words and phrases* : Elliptic problems, Sobolev space, Renormalized Solutions, Entropy solutions, Unilateral problem, noncoercive problems, L^∞ estimate, Rearrangement method.

efficients

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$, $a(x,u,\nabla u)$ is allowed to be degenerate with the unknown u, and the data f are assumed to be in $L^m(\Omega)$, $m \geq 1$, we prove the L^∞-estimation, and the existence of weak solutions if $m > \max\{1, \frac{N}{p}\}$ and the existence of Renormalized solutions if $m = 1$

We end this part with a chapter that would be dedicated to the study a class of non-linear degenerate elliptic problem with coefficients in $L^1(\Omega)$ of the form

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$, $a(x,u,\nabla u)$ is allowed to be degenerate with the unknown u. We prove the existence of weak and entropy solutions under some hypothesis on f. We also study the same problem with a lower order term.

Introduction

Notre compréhension des phénomènes du monde réel et notre technologie sont aujourd'hui en grande partie basées sur les équations aux dérivées partielles, en effet elles permettent d'aborder d'un point de vue mathématique des phénomènes observés. Ces équations ont été formulées pour la première fois lors de la naissance de la mécanique rationnelle au cours du XVIIème siècle (Newton, Leibniz,...). La liste des types d'EDP s'est enrichie au fur et à mesure du développement des sciences de la physique et en particulier avec les travaux de Newton, d'Euler, de d'Alembert, de Lagrange et de Laplace etc... voir [38]. Les modèles traités ici sont issus principalement de la mécanique des fluides, de la géophysique, des biomathématiques, la théorie électromagnétique, la mécanique quantique, la relativité générale, la lumière électrique, le mouvement des planètes...., La préoccupation première des mathématiciens confronté à une équation aux dérivées partielles est de lui donner un sens dans des espaces fonctionnels appropriés et d'y démontrer l'existence et l'unicité de la solution. Lors de cette démarche, diverses diffcultés peuvent apparaitre, diffcultés liées au type (elliptique, parabolique, hyperbolique, dégénéré, ...) de l'équation, à la régularité des données et aussi à la définition des espaces appropriés.

L'objectif de ce travail est l'étude de divers problèmes d'équations aux dérivées partielles non linéaires de type elliptique dans les espaces de Lebesgue anisotropiques, et les espaces de Sobolev à exposants variables, faisant intervenir des opérateurs du type Leray-Lions avec des données peu régulières . Cette thèse, est composée de deux parties chaque partie contient trois chapitres. Nous allons présenter ici d'une manière détaillée le contenu de chacun d'eux

Première partie :

Chapitre I : ce chapitre est totalement consacré à définir le côté fonctionnel des espaces Anisotropique, où les différentes dérivées partielles faibles de la solution u sont intégrables avec différents exposants, collectées dans le vecteur $\vec{p} = (p_1, ..., p_N)$ tel que $1 < p_i < \infty,\ \ i = 1, ..., N$.

Chapitre II : qui est l'objectif d'un article publié [34], nous avons montré l'existence des

solutions faibles du problème Anisotrope suivant :

$$\begin{cases} -\partial_{x_i} a_i(x,u,\nabla u) + \sum_{i=1}^{N} H^i(x,u,\nabla u) = f - \partial_{x_i} g_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

où Ω est un sous ensemble ouvert borné de $\mathbb{R}^N$, $(N \geq 2)$, pour $i = 1,...,N$, $a_i(x,u,\nabla u)$ est une fonction Carathéodory avec ellipticité dégénérée et croissance par rapport à u (voir (2.3)), et le terme d'ordre inférieur $H_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ est une fonction carathéodory qui ne satisfait qu'une condition de croissance par rapport au gradient ∇u (voir (2.7)).
La solvabilité des problèmes de type (5) a été bien étudié. Dans le cas isotrope, en particulier, il existe une vaste littérature pour l'analyse du cas impliquant l'opérateur p-laplacien et les problèmes énoncés dans le cadre de l'espace de Lebesgue, dans le cadre elliptique la fondation de la branche a été posée par Boccardo et al. [27], Dall'Aglio [39] et Murat [66].
Pour les équations elliptiques anisotropes, nous avons commencé par les travaux de Bendahmane généralisant le travail de H. Brésis et F. Browder [37], à l'espace Anisotrope $W^{1,\vec{p}}(\Omega)$ en utilisant les approximations de type Hedberg. Pour plus de travaux sur les sapces Anisotropes, nous renvoyons le lecteur à ([8], [26], [33], [40], [41], [47] et [83]).
Chapitre III : Notre objectif dans ce chapitre est de prouver l'existence et l'unicité de solutions entropie pour le problème unilatéral associé à l'équation elliptique anisotrope quasi-linéaire suivant

$$\begin{cases} -\sum_{1}^{N} D^i a_i(x,u,\nabla u) + g(x,u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

où $f \in L^1(\Omega)$, les fonctions Carathéodory $a_i(x,s,\ xi)$ vérifient une condition de coercivité non standard, et le terme d'ordre inférieur $g(x,s)$ est une fonction carathéodory satisfaisant seulement une condition de croissance.
Soit Ω un ouvert borné de $\mathbb{R}^N$, $(N \geq 2)$, dans [24], Boccardo et al. ont étudié le problème elliptique quasi-linéaire suivant

$$\begin{cases} -\text{div}\,(A(x,u)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

où la donnée f supposées être dans $L^m(\Omega)$ avec $m \geq 1$. Ils ont prouvé l'existence et quelques résultats de régularité, voir [17, 25, 58], ainsi que [23] pour le problème elliptique unilatéral. Dans le cadre des espaces Orlicz-Sobolev, Aharouch et Bennouna [2] ont traité l'équa-

tion elliptique quasi-linéaire associée au problème unilatéral

$$\begin{cases} -\text{div}\,(a(x,\nabla u)) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{8}$$

Où $f \in L^1(\Omega)$. Ils ont prouvé l'existence et l'unicité de la solution entropie dans $W_0^1 L_M(\Omega)$ en supposant seulement la condition Δ_2 sur le conjugué de N−fonction $M(\cdot)$, on se réfère aussi à [10, 18, 44] pour plus de détails.
Dans le cadre des espaces de Sobolev Anisotropes , Li Feng-Quan a considéré dans [59] l'équation elliptique

$$\begin{cases} -\text{div}\, a(x,u,Du) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

où $f \in L^m(\Omega)$, il a prouvé l'existence de solutions au sens des distributions. Dans [41], Di-Nardo, Feo et Guibé ont étudié la classe suivante de problème elliptique anisotrope non linéaire

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}(a_i(x,u,\nabla u)) + \sum_{i=1}^{N} H_i(x,\nabla u) = f - \sum_{i=1}^{N} \partial_{x_i}(g_i) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{9}$$

avec $f \in L^{p'_\infty}(\Omega)$, $g_i \in L^{p'_i}(\Omega)$. Ils ont prouvé l'existence des solutions faibles pour le problème (9). Ils ont également prouvé l'unicité de la solution sous certaines hypothèses supplémentaires.
Le contenu de ce chatitre est l'objet d'un article soumis [3]

La deuxième partie :
On commence cette partie par un chapitre contenant un rappel sur le cadre fonctionnel des espaces de Sobolev à exposant variable qui sont apparus pour la première fois dans la littérature en 1931 dans un article de W. Orlicz [69], dont Orlicz a également examiné l'espace de fonctions à exposant variable $L^{p(\cdot)}$ sur la droite réelle et a démontré l'inégalité de Hölder dans ce contexte. Ensuite les espaces modulaires ont attiré l'intérêt de plusieurs mathématiciens. On cite par exemple les groupes de Sapporo (Japan), Voronezh (U.S.S.R.) and Leiden (the Netherlands), les chercheurs russes qu'ont développés indépendamment ces espaces. En effet en 1961 I. Sharapunov dans [78], introduit la norme de Lexemberg pour l'espace de Lebesgue et a démontré que cette espace est reflexive si $1 < p^- \leq p^+ < +\infty$ avec p^- et p^+ sont respectivement l'inf et le sup essentiel de l'exposant $p(\cdot)$, parsuite Zhikov en [86] a étudié l'espace à exposant variable pour les modèles variationels avec la condition de croissance non standard. Aussi les travaux de H. Hudzik [52, 53], A. Kamińska [**? ?**] et J. Musielak [**?**], (voir le monographe [42] pour plus de détail).
On termine ce chapitre par quelques propriétés de réarrangement relative introduite initialement par Mossino et Temam dans [65], et développé ensuite par d'autre auteurs, voir

par exemple [64, 74, 72].

Chapitre V : dans ce chapitre nous avons étudié le problème elliptique dégénéré non linéaire suivant :

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

où Ω est un sous-ensemble ouvert borné de $\mathbb{R}^N$ ($N \geq 2$). De plus, nous supposons que $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ est une fonction vectorielle de Carathéodory satisfaisant : il existe une fonction continue $g : \mathbb{R}^+ \mapsto \mathbb{R}^+_*$, tel que $g(0) = 0$, $g(s) > 0$ $\forall s > 0$ et $a(x,s,\xi)\xi \geq \alpha(x) g(|s|)|\xi|^p$, $\forall s \in \mathbb{R}$, a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^N$, où la fonction $\alpha(x)$ satisfait, pour $\lambda > 0$, $\alpha \in L^1(\Omega)$ et $\alpha(x) \geq \lambda$. Le premier résultat dans ce chapitre sert à montrer en utilisons la notion de réarrangement l'estimation L^∞ et l'existence des solutions du problème suivant : Cherchons $u \in V_0$ tel que $a(x,u,\nabla u)\alpha(x)^{\frac{1}{p'}-1} \in (L^{p'}(\Omega))^N$ et

$$\int_\Omega a(x,u,\nabla u)\nabla v \, dx = \int_\Omega f v \, dx \; \forall v \in C_0^1(\Omega), \text{ vrai}$$

avec

$$X_0^p(\Omega) = \{\varphi \in W_0^{1,p}(\Omega) \text{ tel que } \int_\Omega \alpha(x)|\nabla\varphi|^p < \infty\},$$

et

$$V_0 = \{v \text{ est une fonction mesurable tel que } \eta(v) \in W_0^{1,p}(\Omega), \eta(s) = \int_0^s g^{\frac{1}{p-1}}(|\tau|)d\tau\}.$$

Et le deuxième résultat sera consacré à l'étude de la solution renormalisée :
Cherchons $u : \Omega \to \mathbb{R}$ tel que

$$T_k(\eta(u)) \in X_0^p(\Omega) \; \forall \; k \geq 0,$$

$$\lim_{m\to\infty} \int_{m \leq |\eta(u)| \leq m+1} a(x,u,\nabla u)\nabla\eta(u) \, dx = 0,$$

pour toute $h \in W_0^{1.\infty}(\Omega)$ avec un support compact et $\varphi \in W_0^p(\Omega) \cap L^\infty(\Omega)$;

$$\int_\Omega a(x,u,\nabla u)\nabla(h(\eta(u))\varphi) \, dx = \int_\Omega f h(\eta(u))\varphi \, dx.$$

Un exemple modéle de 10

$$\begin{cases} -\operatorname{div}(\alpha(x)|\nabla u|^{p-2}\nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

où Ω est un sous-ensemble ouvert borné de $\mathbb{R}^N$ ($N > 2$). La version linéaire du problème (11) ($p = 2$), avec des coefficients généraux $L^1(\Omega)$, a été abordée pour la première fois par

Trudinger dans [81], en utilisant la méthode de dualité entre les espaces de Sobolev avec poids. Pour des résultats plus récents, dans le cas linéaire, est considéré dans [21] sans le cadre de Sobolev avec poids mais avec hypothèses plus fortes sur la sommabilité de $\alpha(x)$,

Dans le cas $\alpha(\cdot) \equiv 1$, il y a de nombreux résultats concernant l'existence de solutions au problème , nous mentionnons les travaux suivants [20, 46, 72, 76], où l'existence de solutions faibles bornées et de solutions renormalisées est prouvée.

Dans le cas de Sobolev avec poids pour $\alpha = \{\alpha_i(x),\ 0 \le i \le N\}$ soient des fonctions poids satisfaisant certaines conditions d'intégrabilité et $f \in W^{-1,p'}(\Omega, \alpha^*)$. Les auteurs de [6] ont prouvé l'existence du résultat des solutions pour () en utilisant la version L^1 du lemme de Minty.

Signalons que ce chapitre est l'objet d'un article soumis [35]

Chapitre VI : Notre objectif dans ce chapitre est d'étudier dans le cadre des espaces de Sobolev à exposant variable, la classe des problèmes elliptiques dégénéré non linéaire avec des coefficients en $L^1(\Omega)$ de la forme :

$$-\operatorname{div} a(x, u, \nabla u) = f, \text{ in } \Omega.$$

Nous prouvons l'existence des solutions faibles et entropiques sous les hypothèses suivantes : $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ est une fonction Carathéodory tel que quelques soient s dans $\mathbb{R}$, ξ, ξ' dans $\mathbb{R}^N$ ($\xi \neq \xi'$), et p.p $x \in \Omega$,

$$|a(x, s, \xi)| \le \gamma\alpha(x)(1 + g(s))|\xi|^{p(x)-1},$$

$$a(x, s, \xi)\xi \ge \alpha(x)g(s)|\xi|^{p(x)},$$

$$[a(x, s, \xi) - a(x, s, \xi')][\xi - \xi'] > 0,$$

où γ est une constante positive .

La fonction α satisfait, pour $\nu > 0$

$$\alpha \in L^1(\Omega), \quad \alpha(x) > \nu,$$

Nous étudions également le même problème avec un terme d'ordre inférieur $H(x, s, \xi)$ qui satisfait seulement la condition de croissance suivant :

$$| H(x, s, \xi) | \le \beta(x)h(s) | \xi |^{p(x)},$$

pour tout p.p $x \in \Omega$, pour tout $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $\gamma > 0$, avec la fonction $\beta(.)$ satisfait, pour $\theta > 0$

$$\beta \in L^1(\Omega), \quad \beta(x) \ge 0 \text{ et } \beta(x) \le \theta\alpha(x) \text{ p.p. dans } \Omega, \text{ et}$$

$h:\ \mathbb{R}\mapsto\mathbb{R}^+$, est une fonction continue positive.

Soit Ω un sous-ensemble borné de $\mathbb{R}^N$, $N\geq 2$. Porretta et Segura de León [70] ont étudié le problème de Dirichlet pour certaines équations elliptiques non linéaires dont le modéle exemple est :

$$\begin{cases} -\mathrm{div}(g(u)|\nabla u|^{p-2}\nabla u) = h(u)|\nabla u|^p + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

avec $1 < p < \infty$, g est une fonction positive continue. Toujours sous l'hypothèse supplémentaire

$$\lim_{s\to\infty}\frac{h(s)}{g(s)} = 0,$$

ils obtiennent L^∞ - estimation. Ce type de problèmes ont été largement étudié, voir les références classiques [56] et [60].

le contenu de ce chatitre est l'objet d'un article accepté [36].

Dans cette thèse Trois articles sont publiés et deux autres sont soumis pour publication.

Part I

Existence Results For Some Nonlinear Degenerate Problems In The Anisotropic Spaces

Chapter 1

Prelemineries of part 1

1.1 Anisotropic Sobolev spaces and Useful Lemmas

1.1.1 Anisotropic Sobolev spaces

Let Ω be an open bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with boundary $\partial\Omega$.
Let $p_1, \ldots, p_N$ be N real constants numbers, with $1 < p_i < \infty$ for $i = 1, \ldots, N$.
We denote

$$\vec{p} = (1, p_1, \ldots, p_N), \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1, \ldots, N,$$

and we set

$$\underline{p} = \min\{p_1, p_2, \ldots, p_N\} \qquad \text{and} \qquad p_0 = \max\{p_1, p_2, \ldots, p_N\}.$$

We define the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ as follows :

$$W^{1,\vec{p}}(\Omega) = \left\{ u \in W^{1,1}(\Omega) \quad \text{such that} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N \right\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{1.1}$$

The space $\left(W^{1,\vec{p}}(\Omega), \|u\|_{1,\vec{p}}\right)$ is a separable and reflexive Banach space [63].
We define also $W_0^{1,\vec{p}}(\Omega)$ as the closure of $\mathscr{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ with respect to the norm (1.1).
Let us recall the Poincaré and Sobolev type inequalities in the anisotropic Sobolev space.

1.1.2 Some Technical lemmas and Proposition

Proposition 1.1.1 *Let $u \in W_0^{1,\vec{p}}(\Omega)$, we have*
(i) *Poincaré inequality : there exists a constant $C_p > 0$, such that*

$$\|u\|_{L^{p_i}(\Omega)} \leq C_p \|D^i u\|_{L^{p_i}(\Omega)} \qquad \text{for any} \quad i = 1, \ldots, N.$$

(ii) *Sobolev inequality : there exists an other constant $C_s > 0$, such that*

$$\|u\|_{L^q(\Omega)} \leq \frac{C_s}{N} \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)},$$

where

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \qquad \text{and} \qquad \begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N - \overline{p}} & \text{if} \quad \overline{p} < N \\ q \in [1, +\infty[& \text{if} \quad \overline{p} \geq N. \end{cases}$$

Lemma 1.1.1 *Let Ω be a bounded open set in $\mathbb{R}^N$ ($N \geq 2$), we set*

$$p_\infty = \max(q, \max_{1 \leq i \leq N} p_i),$$

then, we have the following embedding :

- *if $\overline{p} < N$ then the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, s[$,*
- *if $\overline{p} = N$ then the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, +\infty[$,*
- *if $\overline{p} > N$ then the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$ is compact.*

If we note by $p_\infty = max\{\overline{p}^*, p^+\}$, we have the continuous embedding
$\forall\, q \in [1, p_\infty]$ $W_0^{1,\vec{p}}(\Omega) \subset L^q(\Omega)$.

The proof of this lemma follows from the Proposition 1.1.1.

Proposition 1.1.2 *We denote the dual of the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ by $W^{-1,\vec{p}'}(\Omega)$, where $\vec{p}' = (p'_\infty, p'_1, \ldots, p'_N)$ with $p_\infty = \max(q, p_0)$.*
For each $F \in W^{-1,\vec{p}'}(\Omega)$, there exists $F_0 \in L^{s'}(\Omega)$ and $F_i \in L^{p'_i}(\Omega)$ for $i = 1, \ldots, N$, such that $F = F_0 - \sum_{i=1}^{N} D^i F_i$. Moreover, for all $u \in W_0^{1,\vec{p}}(\Omega)$ we have

$$\langle F, u \rangle = \sum_{i=0}^{N} \int_\Omega F_i \, D^i u \, dx.$$

We define a norm on the dual space by

$$\|F\|_{-1,\vec{p}'} = \inf \Big\{ \sum_{i=0}^{N} \|F_i\|_{p'_i} \quad / \quad F = F_0 - \sum_{i=1}^{N} D^i F_i \quad \text{with} \quad F_0 \in L^{s'}(\Omega) \quad \text{and} \quad F_i \in L^{p'_i}(\Omega) \Big\}.$$

Let $a_1, ..., a_N$ be positive numbers, we have

$$\prod_{i=1}^{N} a_i^{\frac{1}{N}} \leq \frac{1}{N} \sum_{i=1}^{N} a_i.$$

Definition 1.1.1 *Let $k > 0$, we consider the truncation function $T_k(\cdot) : \mathbb{R} \longmapsto \mathbb{R}$, given by*

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if } |s| > k, \end{cases}$$

and we define

$$\mathcal{T}_0^{1,\vec{p}}(\Omega) := \{u : \Omega \mapsto \mathbb{R} \text{ measurable, such that } T_k(u) \in W_0^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

Proposition 1.1.3 *Let $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$. For any $i \in \{1, \ldots, N\}$, there exists a unique measurable function $v_i : \Omega \mapsto \mathbb{R}$ such that*

$$\forall k > 0 \qquad D^i T_k(u) = v_i \cdot \chi_{\{|u|<k\}} \quad \text{a.e. } x \in \Omega,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivative of u, that is, $v_i = D^i u$.

The proof of the Proposition 1.1.3 follows the usual techniques developed in [17] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [8, 13, 40, 41].

Lemma 1.1.2 *(see [51], Theorem 13.47) Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that*

(i) *$u_n \to u$ a.e. in Ω,*

(ii) *$u_n \geq 0$ and $u \geq 0$ a.e. in Ω,*

(iii) *$\int_\Omega u_n \, dx \to \int_\Omega u \, dx$,*

then $u_n \to u$ in $L^1(\Omega)$.

Chapter 2

Existence of weak solution for nonlinear degenerate elliptic equations in the Anisotropic spaces

Our goal in this chapter is to prove the existence of solutions for the following nonlinear anisotropic degenerate elliptic problem :

$$-\partial_{x_i} a_i(x,u,\nabla u) + \sum_{i=1}^{N} \mathrm{H}_i(x,u,\nabla u) = f - \partial_{x_i} g_i \quad \text{in } \Omega, \tag{2.1}$$

where for $i = 1,...,\mathrm{N}$, $a_i(x,u,\nabla u)$ is allowed to degenerate with respect to the unknown u, and $\mathrm{H}_i(x,u,\nabla u)$ is a nonlinear term without a sign condition. Under suitable conditions on a_i and H_i, we prove the existence of weak solutions.

2.1 Introduction

Let Ω be a bounded subset of $\mathbb{R}^{\mathrm{N}}$ ($\mathrm{N} \geq 2$) with smooth boundary $\partial\Omega$. For the vectorial exponent $\vec{p} = (p_1,...,p_{\mathrm{N}})$ we assume that for $i = 1,...,\mathrm{N}, 1 < p_i < \infty$.

Our aim is to prove the existence of weak solutions to the following anisotropic degenerate elliptic equations

$$\begin{cases} -\partial_{x_i} a_i(x,u,\nabla u) + \sum_{i=1}^{N} \mathrm{H}^i(x,u,\nabla u) = f - \partial_{x_i} g_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where the source data f and $g = (g_1,...,g_{\mathrm{N}})$ belonging a suitable Lebesgue spaces (see assumptions (2.9)).

We assume that for i=1,...,N, the function $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$ is a Carathéodory function

with degenerate ellipticity and growth with respect to u (see (2.3)), and the lower order term $H_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfy only a growth condition with respect to the gradient ∇u (see (2.7)).
Existence to problems like (2.1) is very well understood, in the isotropic case, in particular, there is vast literature for analysis of the case involving the p-Laplacian operator and problems stated in the Lebesgue space setting, in the elliptic setting the foundation of the branch where laid by Boccardo et al.[27], Dall'Aglio [39] and Murat [66].
For the anisotropic elliptic equations, we started by the work of Bendahmane generalizing the work H. Brésis and F. Browder [37], to the anisotropic space $W^{1,\vec{p}}(\Omega)$ using the Hedberg-type approximations. For more works in the anisotropic sapces we refer the reader to ([8], [26], [33], [40], [41], [47] and [83]).
Our main contribution is to prove the existence of weak solutions of the problems2.1, which are in general non coercive, in order to overcome this difficulties, we split the domain Ω in q finite number of small domain Ω_i (see lemma2.3.1), this method used by Bottaro in [29] ; and we avoid the smallness of $\| b_i \|_{L^r}$ (see (2.8)) to get the operator that govern the equation coercive.
A major difficulties if we work with the terms $a_i(x,u,\nabla u)$ and $H_i(x,u,\nabla u)$, we encounter problems because of the degeneracy, (at the level the problems that are well defined, secondly at the level of the a priori estimates of H) and in passing to the limit, we can not apply the classical methods (Young, Hölder, and Gagliardo inequalities..), to overcomes this difficulties we use the positively homogeneous terms $\bar{a}_i(x,u,\nabla u)$ and $\widehat{H}_i(x,u,\nabla u)$ of degree $(p_i - 1)$, with respect to ∇u, instead of the terms $a_i(x,u,\nabla u)$ and $H_i(x,u,\nabla u)$, see hypotheses (2.3) and (2.7).

2.2 Basic assumptions

We assume that $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $H_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions such that

$$\sum_{i=1}^{N} a_i(x,s,\xi)\xi_i \geq \sum_{i=1}^{N} \nu(|s|)|\xi|^{p_i} \quad \forall s \in \mathbb{R}, \xi \in \mathbb{R}^N \text{and a.e in } \Omega, \tag{2.3}$$

with ν is a bounded continuous function such that $\nu > 0$, and

$$\int_0^{+\infty} \nu(t)^{\frac{1}{p_i - 1}} dt = +\infty, \text{ for } i = 1, ..., N. \tag{2.4}$$

There is a Carathéodory vector function $\overline{a}$ such that for a.e.$x \in \Omega$, $\forall s \in \mathbb{R}$, $\forall \xi$, $\xi' \in \mathbb{R}^N$:

$$a_i(x,s,\xi) = \nu(|s|)\bar{a}_i(x,s,\xi),$$

$$|\bar{a}_i(x,s,\xi)| \leq \gamma[|s|^{\frac{s}{p_i'}} + |\xi_i|^{p_i - 1}], \quad \gamma \text{ positive constant}, \tag{2.5}$$

$$[\bar{a}_i(x,s,\xi)-\bar{a}_i(x,s,\xi')][\xi_i-\xi_i']>0 \ \text{ for } \ \xi_i\neq\xi_i'. \tag{2.6}$$

There is a Carathéodory vector function $\widehat{\mathrm{H}}$ such that for a.e.$x\in\Omega,\ \forall s\in\mathbb{R},\ \forall\xi\in\mathbb{R}^{\mathrm{N}}$:

$$\mathrm{H}_i(x,s,\xi)=\nu(|s|)\widehat{\mathrm{H}}_i(x,\xi),$$

$$|\widehat{\mathrm{H}}_i(x,\xi)|\leq b_i(x)|\xi_i|^{p_i-1}, \tag{2.7}$$

where b_i belong to the space $\mathrm{L}^{r_i}(\Omega)$ with $\dfrac{1}{r_i}=\dfrac{1}{p_i}-\dfrac{1}{p_\infty}$ for $i=1,...,\mathrm{N}$.
The function $\bar{a}_i$ and $\widehat{\mathrm{H}}_i$ are positively homogeneous of degree (p_i-1) with respect to the variable ξ, i.e

$$\bar{a}_i(x,s,t\xi)=t^{p_i-1}\bar{a}_i(x,s,\xi),\ \ \widehat{\mathrm{H}}_i(x,s,t\xi)=t^{p_i-1}\widehat{\mathrm{H}}_i(x,s,\xi),\ \ \forall\, t\geq 0. \tag{2.8}$$

$$f\in\mathrm{L}^{p_\infty'}(\Omega)\ \ and\ \ g_i\in\mathrm{L}^{p_i'}(\Omega)\ \text{ for }\ i=1,...,\mathrm{N}. \tag{2.9}$$

2.3 Principal results

We denote by $\tilde{v}$ the function

$$\tilde{v}(s)=\int_0^s \nu(|t|)^{\frac{1}{p_i-1}}\,dt$$

and

$$\mathrm{V}=\{u\ \text{ is measurable function in }\ \Omega:\tilde{v}(u)\in\mathrm{W}_0^{1,\vec{p}}(\Omega)\}.$$

2.3.1 Definition of the weak solution

Definition 2.3.1 *A function u in V is a weak solution to problem2.2 if*
$a_i(.,u,\nabla u)\in\mathrm{L}^{p_i'}(\Omega),\mathrm{H}_i(.,u,\nabla u)\in\mathrm{L}^{p_\infty'}(\Omega)$, *and*

$$\sum_{i=1}^{\mathrm{N}}\int_\Omega [a_i(x,u,\nabla u)\partial_{x_i}\varphi+\mathrm{H}_i(x,u,\nabla u)\varphi]=\int_\Omega [f\varphi+\sum_{i=1}^{\mathrm{N}} g_i\partial_{x_i}\varphi],\ \ \forall\varphi\in\mathrm{W}_0^{1,\vec{p}}(\Omega).$$

To avoid the smallness of the norm of b_i, spliting the domain Ω in a finite number of small domains Ω_s, by adopting the technique introduced in [29] for the linear case, and [82] for the nonlinear case.

Proposition 2.3.1 *Let $\mathrm{A}\in\mathbb{R}^+$ and $u\in\mathrm{V}$, (i.e. $\tilde{v}(u)\in\mathrm{W}_0^{1,\vec{p}}(\Omega)$).*
Then there exists t measurable subsets $\Omega_1,...,\Omega_t$ of Ω and t functions $\tilde{v}(u)_1,...,\tilde{v}(u)_t$ such that $\Omega_i\cap\Omega_j=\emptyset$, for $i\neq j,|\Omega_t|\leq\mathrm{A}$ and $|\Omega_s|=\mathrm{A}$ for $s\in\{1,...,t-1\}$,

$$\{x\in\Omega:|\partial_{x_i}\tilde{v}(u)_s|\neq 0\ \ for\ \ i=1,...,\mathrm{N}\}\subset\Omega_s,\partial_{x_i}\tilde{v}(u)=\partial_{x_i}\tilde{v}(u)_s\ \ a.e.\ in\ \ \Omega_s, \tag{2.10}$$

$$\partial_{x_i}([\tilde{v}(u)]_1 + ... + [\tilde{v}(u)]_s)[\tilde{v}(u)]_s = (\partial_{x_i}\tilde{v}(u))[\tilde{v}(u)]_s, [\tilde{v}(u)]_1 + ... + [\tilde{v}(u)]_s = \tilde{v}(u) \qquad (2.11)$$

and $sign(\tilde{v}(u)) = sign([\tilde{v}(u)]_s)$ *if* $[\tilde{v}(u)]_s \neq 0$ *for* $s \in \{1, ..., t\}$ *and* $i \in \{1, ..., \mathrm{N}\}$.

Proof. Let $0 \leq k < h \leq +\infty$, define $\mathrm{S}_{h,k}(s) = \mathrm{T}_k(s) - \mathrm{T}_h(s)$, where $\mathrm{T}_k(s) = sign(s)(min(|s|, k))$. We put $\Omega_{h,k} = \{x \in \Omega : |\partial_{x_i}\mathrm{S}_{h,k}(\tilde{v}(u))| \neq 0, \text{ for } i = 1, ..; \mathrm{N}\}$, we have $\partial_{x_i}\mathrm{S}_{h,k}(\tilde{v}(u)) = \partial_{x_i}\tilde{v}(u)$ a.e. in $\Omega(h,k)$ for $i = 1, ..., \mathrm{N}$.
Construction of the subset Ω_s and the function $(\tilde{v}(u))_s$: the idea of our approach is inspired from the paper of Del Vecchio et al.(see appendix, [82]).
Let $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ a decreasing sequence constructed in the following way :
If $|\Omega| \leq \mathrm{A}$ then $k_1 = 0$, else k_1 is chosen such that $|\{x \in \Omega : |\tilde{v}(u(x))| \geq k_1| = \mathrm{A}$.
For $j > 2$, if $|\{x \in \Omega : 0 \leq |\tilde{v}(u(x))| \geq k_{j-1}| \leq \mathrm{A}$ then $k_j = 0$ else k_j is chosen such that $0 < k_j < k_{j-1}$ and $|\{x \in \Omega : k_j \leq |\tilde{v}(u(x))| \geq k_{j-1}| = \mathrm{A}$. Let t be the first index such that $k_t = 0$, then we put $\Omega_1 = \Omega(k_1, +\infty)$; $\Omega_s = \Omega(k_s, k_{s-1})$ for $s = 2, .., t$ and $u_1 = \mathrm{S}_{k_1,\infty}(u)$, $u_s = \mathrm{S}_{k_s,k_{s-1}}(u)$ for $s = 2, ..., t$.

2.3.2 Some a priori estimates

We consider a sequence of regularized problems :

$$(\mathrm{P}_n)\begin{cases} -\partial_{x_i} a_i(x, u_n, \nabla u_n) + \sum_{i=1}^{\mathrm{N}} \mathrm{H}_n^i(x, u_n, \nabla u_n) = f - \partial_{x_i} g_i & \text{in} \quad \Omega \\ u_n = 0 & \text{on} \quad \partial\Omega, \end{cases} \qquad (2.12)$$

where $\mathrm{H}_n^i(x, u, \nabla u) = \mathrm{T}_n(\mathrm{H}_i(x, u, \nabla u))$.
It is well known, for the functions b_i with norms small enough, the problems2.12 has at least a weak solution $u_n \in \mathrm{W}_0^{1,\vec{p}}(\Omega)$, we refer to ([60]).

Proposition 2.3.2 *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* V *such that* $\tilde{v}(u_n) \rightharpoonup \tilde{v}(u)$ *in* $\mathrm{W}_0^{1,\vec{p}}(\Omega)$ *and*

$$\sum_{i=1}^{\mathrm{N}} \int_\Omega [\bar{a}_i(x, u_n, \nabla\tilde{v}(u_n)) - \bar{a}_i(x, u_n, \nabla\tilde{v}(u))](\partial_{x_i}\tilde{v}(u_n) - \partial_{x_i}\tilde{v}(u_n)) \to 0,$$

then $\partial_{x_i}\tilde{v}(u_n) \to \partial_{x_i}\tilde{v}(u)$ *a.e. in* Ω *and for* $i = 1, ..., \mathrm{N}$.

Proof. Let $\mathrm{D}_n^i = [\bar{a}_i(x, u_n, \nabla\tilde{v}(u_n)) - \bar{a}_i(x, u_n, \nabla\tilde{v}(u))](\partial_{x_i}\tilde{v}(u_n) - \partial_{x_i}\tilde{v}(u_n))$, $\mathrm{D}_n^i \geq 0$ and $\mathrm{D}_n^i \to 0$ in $\mathrm{L}^1(\Omega)$. Extracting a subsequence $\tilde{v}(u_n)$, by lemma1.1.1 we have $u_n \to u$ *a.e in* Ω, $\mathrm{D}_n^i \to 0$ *a.e in* Ω, which implies that $u_n \to u$ a.e. in Ω.
Let $\mathrm{A} \subset \Omega$, such that $|\mathrm{A}| = 0$, we have $u_n \to u$ and $\mathrm{D}_n^i \to 0$ pointwise in $\Omega\backslash\mathrm{A}$. For all $x \in \Omega\backslash\mathrm{A}$, denoting $\tilde{\xi}_n = \partial_{x_i}\tilde{v}(u_n(x))$, $\tilde{\xi} = \partial_{x_i}\tilde{v}(u(x))$, $\xi_n = \partial_{x_i}u_n(x)$ and $\xi = \partial_{x_i}u(x)$.

Then,

$$\begin{aligned}
\mathrm{D}_n^i = \quad & \bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))\partial_{x_i}\tilde{v}(u_n)+\bar{a}_i(x,u_n,\nabla\tilde{v}(u))\partial_{x_i}\tilde{v}(u)\\
& -\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))\partial_{x_i}\tilde{v}(u)-\bar{a}_i(x,u_n,\nabla\tilde{v}(u))\partial_{x_i}\tilde{v}(u_n)\\
& = a_i(x,u_n,\nabla u_n)\nu(|u_n|)^{\frac{1}{p_i-1}}\partial_{x_i}u_n+a_i(x,u_n,\nabla u)\nu(|u_n|)^{\frac{1}{p_i-1}}\partial_{x_i}u\\
& -\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))\partial_{x_i}\tilde{v}(u_n)-\bar{a}_i(x,u_n,\nabla\tilde{v}(u))\partial_{x_i}\tilde{v}(u_n)\\
& \geq \nu(|u_n|)^{\frac{p_i}{p_i-1}}|\xi_n^i|^{p_i}+\nu(|u|)^{\frac{p_i}{p_i-1}}|\xi^i|^{p_i}-\gamma[|u_n|^{\frac{p_\infty}{p_i'}}+|\tilde{\xi}^i|^{p_i-1}]|\tilde{\xi}_n^i|\\
& -\gamma[|u_n|^{\frac{p_\infty}{p_i'}}+|\tilde{\xi}_n^i|^{p_i-1}]|\tilde{\xi}^i|,
\end{aligned} \tag{2.13}$$

by the fact that ν is real positive bounded function we conclude that

$$\mathrm{D}_n^i \geq c_1\big(|\tilde{\xi}_n^i|^{p_i}-c_x(1+|\tilde{\xi}_n^i|^{p_i-1}+|\tilde{\xi}_n^i|)\big). \tag{2.14}$$

Since u_n is bounded in $\Omega\backslash\mathrm{A}$. Then, $|\xi_n^i|$ is bounded uniformly with respect to n, indeed (2.14) becomes

$$\mathrm{D}_n^i \geq c_2|\tilde{\xi}_n^i|^{p_i}\big(1-\frac{c_x}{|\tilde{\xi}_n^i|^{p_i}}-\frac{c_x}{|\tilde{\xi}_n^i|^{p_i-1}}-\frac{c_x}{|\tilde{\xi}_n^i|}\big)\big).$$

If $\tilde{\xi}_n^i\to\infty$, (for a subsequence) implies that $\mathrm{D}_n^i(x)\to\infty$, which gives a contradiction. Denoting by $\tilde{\xi}^*$ the limit of subsequence of $\tilde{\xi}_n^i$, for i=1,...,N. Applying the continuity of $\bar{a}$ with respect to the two last variables we obtain

$$(\bar{a}_i(x,u,\tilde{\xi}^*)-\bar{a}_i(x,u,\tilde{\xi}))(\tilde{\xi}_i^*-\tilde{\xi}_i)=0.$$

By using (2.6) we get $\tilde{\xi}_n^i\to\tilde{\xi}^i$, a.e. in Ω for $i=1,...,\mathrm{N}$.

Proposition 2.3.3 *Let $u_n\in\mathrm{V}$ be a solution of the approximate problem2.12. Then we have*

$$\sum_{i=1}^{\mathrm{N}}\int_\Omega|\partial_{x_i}\tilde{v}(u_n)|^{p_i}\leq\mathrm{C}, \tag{2.15}$$

where $\mathrm{C}=\mathrm{C}\big(\mathrm{N},\Omega,\gamma,\|\beta\|_\infty,\|f\|_{p_\infty'},\|g_i\|_{p_i'}\big)>0,\quad$ *for* $i=1,...,\mathrm{N}$.

Proof. Let $u_n\in\mathrm{V}$, then $\tilde{v}(u_n)\in\mathrm{W}_0^{1,\vec{p}}(\Omega)$, and $[\tilde{v}(u_n)]_s$ defined as in lemma2.3.1, we have

$$\begin{aligned}
\int_\Omega a_i(x,u_n,\nabla u_n)\partial_{x_i}[\tilde{v}(u_n)]_s \quad & = \int_{\Omega_s}a_i(x,u_n,\nabla u_n)\partial_{x_i}u_n\nu(|u_n|)^{\frac{1}{p_i-1}}\\
& \geq \int_{\Omega_s}\nu(|u_n|)^{\frac{p_i}{p_i-1}}|\partial_{x_i}u_n|^{p_i}\\
& \geq \int_\Omega|\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i}.
\end{aligned} \tag{2.16}$$

We choose $[\tilde{v}(u_n)]_s$ as test function in the approximate problems2.12. By (2.16), Young, Hölder inequalities, and embedding results in lemma1.1.1, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i} &\leq c_1\Big(\|f\|_{p'_\infty}d_s^{\frac{1}{N}} \\ &\quad+\sum_{i=1}^{N}\int_{\Omega}|\mathrm{H}_i^n(x,u_n,\nabla u_n)||[\tilde{v}(u_n)]_s|+\sum_{i=1}^{N}\|g_i\|_{p'_i}^{p'_i}\Big),\end{aligned} \tag{2.17}$$

where $d_s=\prod_{j=1}^{N}\Big(\int_{\Omega}|\partial_{x_j}\tilde{v}(u_n)_s|^{p_j}\Big)^{\frac{1}{p_j}}$.

Here and in what follows, the constants depend on the data but not on u. Using conditions (2.7), (2.10), (2.11) Young, Hölder inequalities, the embedding $\mathrm{W}_0^{1,\vec{p}}(\Omega)\subset\mathrm{L}^{p_\infty}(\Omega)$, and $\mathrm{H}^i(x,u_n,\nabla u_n)=\widehat{\mathrm{H}}^i(x,u_n,\nabla\tilde{v}(u_n))$ we get

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}|\mathrm{H}^i(x,u_n,\nabla u_n)||[\tilde{v}(u_n)]_s|\leq\sum_{i=1}^{N}\int_{\Omega}|b_i(x)||\partial_{x_i}\tilde{v}(u_n)|^{p_i-1}|[\tilde{v}(u_n)]_s| \\ &\leq\sum_{i=1}^{N}\sum_{\sigma=1}^{s}\int_{\Omega_\sigma}|b_i(x)||\partial_{x_i}[\tilde{v}(u_n)]_\sigma|^{p_i-1}|[\tilde{v}(u_n)]_s| \\ &\leq\sum_{i=1}^{N}\|b_i\|_{\mathrm{L}^{r_i}(\Omega')}\sum_{\sigma=1}^{s}\Big[\int_{\Omega_\sigma}|\partial_{x_i}[\tilde{v}(u_n)]_s|^{(p_i-1)r'_i}|[\tilde{v}(u_n)]_s|^{r'_i}\Big]^{\frac{1}{r'_i}} \\ &\leq\sum_{i=1}^{N}\|b_i\|_{\mathrm{L}^{r_i}(\Omega')}\sum_{\sigma=1}^{s}\Big(\int_{\Omega_\sigma}|\partial_{x_i}[\tilde{v}(u_n)]_s|^{(p_i-1)r'_it_i}\Big)^{\frac{1}{r'_it_i}}\Big(\int_{\Omega_\sigma}|[\tilde{v}(u_n)]_s|^{r'_it'_i}\Big)^{\frac{1}{r'_it'_i}} \\ &\leq c_2\sum_{i=1}^{N}\|b_i\|_{\mathrm{L}^{r_i}(\Omega')}\sum_{\sigma=1}^{s}\int_{\Omega_\sigma}\Big[|\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i}+d_s^{\frac{p_i}{N}}\Big] \\ &\leq c_2\sum_{i=1}^{N}\|b_i\|_{\mathrm{L}^{r_i}(\Omega')}\Big[\int_{\Omega_s}|\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i}+\sum_{\sigma=1}^{s-1}\int_{\Omega_\sigma}|\partial_{x_i}[\tilde{v}(u_n)]_\sigma|^{p_i}+d_s^{\frac{p_i}{N}}\Big],\end{aligned} \tag{2.18}$$

where Ω' and t_i is such that

$$\begin{cases}\|b_i\|_{\mathrm{L}^{r_i}(\Omega')} &= \max\{\|b_i\|_{\mathrm{L}^{r_i}(\Omega_\sigma)};\ \sigma=1,..,s\},\\ (p_i-1)r'_it_i &= p_i,\\ r'_it'_i &= p_\infty.\end{cases}$$

Replacing the inequality (2.18) in (2.17) we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i}\leq c_1\Big\{\|f\|_{p'_\infty}d_s^{\frac{1}{N}}+\sum_{i=1}^{N}\|g_i\|_{p'_i}^{p'_i} \\ &+\sum_{i=1}^{N}\|b_i\|_{\mathrm{L}^{r_i}(\Omega')}\Big[\int_{\Omega_s}|\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i}+\sum_{\sigma=1}^{s-1}\int_{\Omega_\sigma}|\partial_{x_i}[\tilde{v}(u_n)]_\sigma|^{p_i}+\sum_{i=1}^{N}d_s^{\frac{p_i}{N}}\Big]\Big\}.\end{aligned} \tag{2.19}$$

Choosing Ω' such that

$$1-c_1\sum_{i=1}^{N}\|b_i\|_{\mathrm{L}^{r_i}(\Omega')}>0, \tag{2.20}$$

(2.19) becomes

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i} &\leq c_2\{\|f\|_{p'_\infty}d_s^{\frac{1}{N}}+\sum_{i=1}^{N}\|g_i\|_{p'_i}^{p'_i}\\ &+(\sum_{i=1}^{N}A^{\frac{1}{p_i}-\frac{1}{p_\infty}})(\sum_{\sigma=1}^{s-1}\sum_{j=1}^{N}\int_{\Omega_\sigma}|\partial_{x_j}[\tilde{v}(u_n)]_\sigma|^{p_j}) \ .\\ &+\sum_{i=1}^{N}\|b_i\|_{L^{r_i}(\Omega')}d_s^{\frac{p_i}{N}}\},\end{aligned}$$

for some constant $c_3>0$. For $s=1$ we get

$$\begin{aligned}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_1|^{p_i} &\leq \sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_1|^{p_i}\\ &\leq c_2\Big[\|f\|_{p'_\infty}d_1^{\frac{1}{N}}+\sum_{i=1}^{N}\|g_i\|_{p'_i}^{p'_i}+\sum_{i=1}^{N}\|b_i\|_{L^{r_i}(\Omega')}d_1^{\frac{p_i}{N}}\Big].\end{aligned}\tag{2.21}$$

Thanks to Proposition 4.3 in [40], by choosing Ω' such that (2.20) and $1-c_2\sum_{i=1}^{N}\|b_i\|_{L^{r_i}(\Omega')}>0$. We get

$$d_1\leq c_3\Big[\big(\|f\|_{p'_\infty}^{\frac{N}{\bar{p}}}+\|\gamma\|_{p'_\infty}^{\frac{N}{\bar{p}}}\big)d_1^{\frac{1}{\bar{p}}}+\sum_{i=1}^{N}\|g_i\|_{p'_i}^{p'_i}\Big].$$

Then there exists a constant $c_4>0$ such that $d_1\leq c_4$, and by (2.21) we obtain

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_1|^{p_i}\leq c_5,\tag{2.22}$$

for some constant $c_5>0$. Moreover, using (2.22) in (2.21) and iterating on s, we have

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i}\leq c_3\Big[\|f\|_{p'_\infty}d_s^{\frac{1}{N}}+\sum_{i=1}^{N}\|g_i\|_{p'_i}^{p'_i}+1+\sum_{i=1}^{N}\|b_i\|_{L^{r_i}(\Omega_\sigma)}d_1^{\frac{p_i}{N}}\Big],\tag{2.23}$$

finally by (2.23), we get

$$\begin{aligned}\|\tilde{v}(u_n)\|_{W_0^{1,\vec{p}}} &= \sum_{i=1}^{N}\Big(\int_{\Omega}|\partial_{x_i}\tilde{v}(u_n)|^{p_i}\Big)^{\frac{1}{p_i}}\\ &\leq \sum_{i=1}^{N}\Big(\int_{\Omega}\Big(\sum_{s=1}^{t}|\partial_{x_i}[\tilde{v}(u_n)]_s|\Big)^{p_i}\Big)^{\frac{1}{p_i}} \ .\\ &\leq k\sum_{i=1}^{N}\Big(\sum_{s=1}^{t}\int_{\Omega}|\partial_{x_i}[\tilde{v}(u_n)]_s|^{p_i}\Big)^{\frac{1}{p_i}}\\ &\leq C.\end{aligned}$$

2.3.3 Existence Theorem

Theorem 2.3.1 *We suppose that the conditions* (2.3)-(2.8) *holds true, then the problem2.1 has at least one weak solution.*

Proof. By proposition2.3.3, we conclude that $\partial_{x_i}\tilde{v}(u_n)$ is bounded in $L^{p_i}(\Omega)$, which gives a weakly convergence of $\partial_{x_i}\tilde{v}(u_n)$ to $\partial_{x_i}\tilde{v}(u)$ in the space $L^{p_i}(\Omega)$, for $i = 1,...,N$, and consequently by embedding theorem we get the strongly convergence of $\tilde{v}(u_n)$ to $\tilde{v}(u)$ in $L^{\bar{p}}(\Omega)$, for some u and some subsequence, still denote by u_n.

1. **Almost everywhere convergence of the gradient**
   According to the proposition 2.3.2, it is enough to show that

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}[\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))-\bar{a}_i(x,u_n,\nabla\tilde{v}(u))][\partial_{x_i}\tilde{v}(u_n)-\partial_{x_i}\tilde{v}(u)]=0. \tag{2.24}$$

   Indeed, we can write the integral of 2.3.2 as follows

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}[\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n)) & -\bar{a}_i(x,u_n,\nabla\tilde{v}(u))][\partial_{x_i}\tilde{v}(u_n)-\partial_{x_i}\tilde{v}(u)]\\ &=\sum_{i=1}^{N}\int_{\Omega}[\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))][\partial_{x_i}\tilde{v}(u_n)-\partial_{x_i}\tilde{v}(u)]dx\\ &-\sum_{i=1}^{N}\int_{\Omega}[\bar{a}_i(x,u_n,\nabla\tilde{v}(u))][\partial_{x_i}\tilde{v}(u_n)-\partial_{x_i}\tilde{v}(u)]dx\\ &=A_n-B_n.\end{aligned}$$

   By assumption (2.3) and (2.8), we have $\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))=a_i(x,u_n,\nabla u_n)$. Thus, we can rewrite A_n as

$$A_n=\sum_{i=1}^{N}\int_{\Omega}[a_i(x,u_n,\nabla u_n)][\partial_{x_i}\tilde{v}(u_n)-\partial_{x_i}\tilde{v}(u)]dx.$$

   We claim that A_n goes to zero as n tend to infinity. Indeed, taking $v=\tilde{v}(u_n)-\tilde{v}(u)$ as test function in the approximate problem we get

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}a_i(x,u_n,\nabla u_n)[\partial_{x_i}\tilde{v}(u_n)-\partial_{x_i}\tilde{v}(u)]dx\\ &+\sum_{i=1}^{N}\int_{\Omega}H^i(x,u_n,\nabla u_n)(\tilde{v}(u_n)-\tilde{v}(u))dx\\ &=\int_{\Omega}f(\tilde{v}(u_n)-\tilde{v}(u))dx+\sum_{i=1}^{N}\int_{\Omega}g_i(\partial_{x_i}\tilde{v}(u_n)-\partial_{x_i}\tilde{v}(u))dx.\end{aligned} \tag{2.25}$$

   Since $\partial_{x_i}\tilde{v}(u_n)-\partial_{x_i}\tilde{v}(u)$ is bounded uniformly in $(L^{p_i}(\Omega))^N$, and $\tilde{v}(u_n)$ converge

strongly to $\tilde{v}(u)$ in $L^{p_i}(\Omega)$, g_i, f belong to $L^{p_i'}(\Omega)$, we conclude that

$$\lim_{n\to\infty}\int_\Omega g_i(\partial_{x_i}\tilde{v}(u_n)-\partial_{x_i}\tilde{v}(u))dx=0 \text{ and } \lim_{n\to\infty}\int_\Omega f(\tilde{v}(u_n)-\tilde{v}(u))dx=0. \quad (2.26)$$

By assumption (2.5), we can show that $H^i(x,u_n,\nabla u_n)\rightharpoonup \eta$ in $L^{p_\infty'}(\Omega)$ and since $\tilde{v}(u_n)$ converge strongly to $\tilde{v}(u)$ in $L^{d_i}(\Omega)$ for all $d_i < p_\infty'$, we get

$$\lim_{n\to\infty}\int_\Omega H^i(x,u_n,\nabla u_n)(\tilde{v}(u_n)-\tilde{v}(u))dx=0. \quad (2.27)$$

By 2.26 and 2.27, we conclude that $\lim_{n\to\infty} A_n = 0$.
Let E be subset of Ω, we have

$$\int_E |\bar{a}_i(x,u_n,\nabla\tilde{v}(u))|^{q_i}dx \le c_1\int_E |u_n|^{\frac{p_\infty q_i}{p_i'}}dx + c_2\int_E |\partial_{x_i}\tilde{v}(u_)|^{p_i}dx, \forall\, q_i < p_{i'}. \quad (2.28)$$

By the strongly convergence $\tilde{v}(u_n)$ to $\tilde{v}(u)$ in $L^{\frac{p_\infty q_i}{p_i'}}(\Omega)$ and since $\tilde{v}$ is bijective (i.e u_n tend to u strongly in $L^{q_i\frac{p_\infty}{p_i'}}(\Omega)$) we have the terms in the right hand side in 2.28 goes to zero as |E| tend to zero, and by almost everywhere convergence of $\bar{a}_i(x,u_n,\nabla\tilde{v}(u))$ to $\bar{a}_i(x,u,\nabla\tilde{v}(u))$, we conclude by Vitali's Theorem that, $\lim_{n\to\infty} B_n = 0$,
according to lemma2.3.2, we conclude that

$$\partial_{x_i}\tilde{v}(u_n) \longrightarrow \partial_{x_i}\tilde{v}(u), \text{ a.e in } \Omega. \quad (2.29)$$

2. **Passage to the limit**
By using the growth condition (2.4), we get

$$|\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))|^{p_i'}dx \le c[|u_n|^{p_\infty}+|\partial_{x_i}\tilde{v}(u_n)|^{p_i}], \quad (2.30)$$

by the continuous embedding in anisotropic space $W_0^{1,\vec{p}}(\Omega)$ into $L^{p_\infty}(\Omega)$, Proposition2.3.3, (2.30) and (2.29) we conclude that

$$\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n))\rightharpoonup \bar{a}_i(x,u,\nabla\tilde{v}(u)) \text{ weakly in } L^{p_i'}(\Omega),$$

and since $\bar{a}_i(x,u_n,\nabla\tilde{v}(u_n)) = a_i(x,u_n,\nabla u_n)$, and $\bar{a}_i(x,u,\nabla\tilde{v}(u)) = a_i(x,u,\nabla u)$ we have

$$a_i(x,u_n,\nabla u_n)\rightharpoonup a_i(x,u,\nabla u) \text{ weakly in } L^{p_i'}(\Omega). \quad (2.31)$$

On the other hand, we have

$$\begin{aligned} |\widehat{\mathrm{H}}^i(x,\nabla\tilde{v}(u_n))|^{p'_\infty} &\leq |b_i(x)|^{p'_\infty}|\partial\tilde{v}(u_n)|^{(p_i-1)p'_\infty} \\ &\leq c_3|b_i(x)|^{r_i} + c_4|\partial_{x_i}\tilde{v}(u_n)|^{(p_i-1)p'_\infty(\frac{r_i}{p'_\infty})'} \quad . \\ &\leq c_3|b_i(x)|^{r_i} + c_4|\partial_{x_i}\tilde{v}(u_n)|^{p_i}, \end{aligned} \tag{2.32}$$

with c_3 and c_4 are the positive constant. (2.32), and (2.29) gives

$$\widehat{\mathrm{H}_i}(x,\nabla\tilde{v}(u_n)) \rightharpoonup \widehat{\mathrm{H}_i}(x,\nabla\tilde{v}(u)) \text{ weakly in } \mathrm{L}^{p'_\infty}(\Omega). \tag{2.33}$$

And since
$\widehat{\mathrm{H}_i}(x,\nabla\tilde{v}(u_n)) = \mathrm{H}_i(x,u_n,\nabla u_n)$ and $\widehat{\mathrm{H}_i}(x,\nabla\tilde{v}(u)) = \mathrm{H}_i(x,u,\nabla u)$ we get

$$\mathrm{H}_i(x,u_n,\nabla u_n) \rightharpoonup \mathrm{H}_i(x,u,\nabla u) \text{ weakly in } \mathrm{L}^{p'_\infty}(\Omega). \tag{2.34}$$

Passing to the limit in the approximate problems (2.12), and using (2.31) and (2.34) we conclude that the problem2.1 has at least a weak solution in the sense of definition (2.3.1).

Chapter 3

Existence and uniqueness of entropy solutions for some anisotropic unilateral elliptic equation

In this chapter we study a unilateral problem associated to some quasilinear elliptic equation of the forme $-\sum_{1}^{N} D^i a_i(x,u,\nabla u) + g(x,u) = f$ in Ω, where Ω is a bounded open domain in $\mathbb{R}^N$ ($N \geq 2$) with boundary $\partial\Omega$, and $g(x,s)$ verify the growth condition, where the data f is assumed to be in $L^1(\Omega)$, we prove existence of entropy solutions for this problem. Moreover, the uniqueness solution was proved under some additional assumptions.

3.1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$). In [24], Boccardo et al. have studied the following quasilinear elliptic problem

$$\begin{cases} -\text{div}\,(A(x,u)\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

where the data f in assumed to be in $L^m(\Omega)$ with $m \geq 1$. They have proved the existence and some regularity results, we refer the reader to [17, 22, 25, 58], and also to [23] for the unilateral elliptic problem.

Bendahmane and Wittbold have studied in [16] the existence and uniqueness of solution for the quasilinear elliptic equation

$$\begin{cases} -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.2}$$

in the Sobolev spaces with variable exponents, where the right-hand side $f \in L^1(\Omega)$ and the exponent $p(\cdot) : \overline{\Omega} \mapsto (1, +\infty)$ is continuous, for more related results we refer to [9, 11, 77]. In the framework of Orlicz-Sobolev spaces, Aharouch and Bennouna [2] have treated the quasilinear elliptic equation associated to the unilateral problem

$$\begin{cases} -\text{div}\,(a(x, \nabla u)) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.3}$$

where $f \in L^1(\Omega)$. They have proved the existence and uniqueness of entropy solution in $W_0^1 L_M(\Omega)$ by assuming only the Δ_2-condition on the conjugate of N−function $M(\cdot)$, we refer also to [10, 18, 44] for more details.

In the recent years, the anisotropic Sobolev spaces have attract the attention of many scientist and researcher, these studies are motivated by their applications to the mathematical modeling of physical in flows of electro-rheological fluids, thermistor problem and the processes of image restoration, (see also [12, 55, 63]).

In the framework of anisotropic Sobolev spaces, Li Feng-Quan has considered in [59] the elliptic equation

$$\begin{cases} -\text{div}\, a(x, u, Du) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $f \in L^m(\Omega)$, he has proved the existence of solutions in the sense of the distributions. In [41], Di-Nardo, Feo and Guibé have studied the following class of nonlinear anisotropic elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}(a_i(x, u, \nabla u)) + \sum_{i=1}^{N} H_i(x, \nabla u) = f - \sum_{i=1}^{N} \partial_{x_i}(g_i) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.4}$$

with $f \in L^{p'_\infty}(\Omega)$, $g_i \in L^{p'_i}(\Omega)$. They have proved the existence of weak solutions for the problem (3.4). Also they have proved the uniqueness of solution under some additional assumptions.

Our aim in this work is to prove the existence and uniqueness of entropy solutions for the unilateral problem associated to the quasilinear anisotropic elliptic equation

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, u, \nabla u) + g(x, u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.5}$$

where $f \in L^1(\Omega)$, the Carathéodory functions $a_i(x, s, \xi)$ verify some non-standard coercivity condition, and the the lower order term $g(x, s)$ is a Carathéodory satisfier only some growth condition.

This chapter is divided into two sections. We introduce in the section 1 some assumptions for which our problem has at least one solutions. The section 2 will be devoted to prove the existence of entropy solutions for our quasilinear elliptic problem (3.5). We conclude this chapter by proving the uniqueness of entropy solution under some additional assumptions.

3.2 Essential assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$, we assume that the vector $\vec{p} = (1, p_1, \ldots, p_N)$ satisfying the requirements that $1 < p_i < \infty$ for any $i = 1, \ldots, N$.
Taking ψ as a measurable function on Ω with values in $\overline{\mathbb{R}}$, such that

$$\psi^+ \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega).$$

We set

$$\underline{p} = \min\{p_1, p_2, \ldots, p_N\} \qquad \text{and} \qquad p_0 = \max\{p_1, p_2, \ldots, p_N\}.$$

We define the convex subset K_ψ by

$$K_\psi = \{v \in W_0^{1,\vec{p}}(\Omega) \ \text{ such that } \ v \geq \psi \ \text{ a.e. in } \ \Omega\}.$$

We consider a Leray-Lions operator $A : W_0^{1,\vec{p}}(\Omega) \longmapsto W^{-1,\vec{p'}}(\Omega)$ given by

$$Au = -\sum_{i=1}^{N} D^i a_i(x, u, \nabla u)$$

where $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}$ are Carathéodory functions, for $i = 1, \ldots, N$, (measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω), which satisfy the following conditions :

$$|a_i(x, s, \xi)| \leq \beta(K_i(x) + |s|^{\tau_i} + |\xi_i|^{p_i - 1}) \qquad \text{for} \quad i = 1, \ldots, N, \tag{3.6}$$

with $0 \leq \tau_i < p_i - 1$.

$$\sum_{i=1}^{N} (a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0 \quad \text{for any} \quad \xi \neq \xi', \tag{3.7}$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $K_i(x)$ is a nonnegative function lying in $L^{p_i'}(\Omega)$ and $\beta > 0$.
We assume that the Carathéodory functions $a_i(x, s, \xi)$ verifying the non-standard coerci-

vity condition

$$a_i(x,s,\xi)\xi_i \geq b(|s|)|\xi_i|^{p_i} \qquad \text{for} \quad i=1,\dots,N, \tag{3.8}$$

such that $b(|\cdot|):\mathbb{R}^+ \mapsto \mathbb{R}^+$ is a decreasing function that belongs to $L^1(R)\cap L^\infty(R)$, and there exists a positive constant b_0 which verifying

$$\frac{b_0}{(1+|s|)^\lambda} \leq b(|s|) \qquad \text{for any} \quad s\in\mathbb{R}. \tag{3.9}$$

where $0<\lambda<min(1,p_0-1,\frac{1}{p_0-1})$. The lower order $g(x,s)$ is a Carathéodory function which satisfies only the growth condition :

$$|g(x,s)| \leq |f_0(x)| + c(x)|s|^\delta \qquad \text{for any} \quad s\in\mathbb{R}, \tag{3.10}$$

where $f_0(x)\in L^1(\Omega)$, the exponent $0<\delta<p_0-\lambda-1$, and the measurable function $c(x)$ is assumed to be in $L^\varrho(\Omega)$ for some $\varrho > \dfrac{p_0-\lambda-1}{p_0-\lambda-\delta-1}$.

We consider the anisotropic quasilinear unilateral elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + g(x,u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.11}$$

where $f\in L^1(\Omega)$.

Now, we recall some important Lemma useful to prove our main result.

Lemma 3.2.1 *(see. [12]) Assuming that* (3.6) − (3.8) *hold, and let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $W_0^{1,\vec{p}}(\Omega)$ *such that* $u_n \rightharpoonup u$ *in* $W_0^{1,\vec{p}}(\Omega)$ *and*

$$\begin{aligned} &\sum_{i=1}^{N}\int_\Omega \Big(a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u)\Big)(D^i u_n - D^i u)\,dx \\ &+\sum_{i=1}^{N}\int_\Omega \Big(|u_n|^{\underline{p}-2}u_n - |u|^{\underline{p}-2}u)\Big)(u_n-u)\,dx \longrightarrow 0 \qquad as \quad n\to\infty, \end{aligned} \tag{3.12}$$

then $u_n \to u$ *in* $W_0^{1,\vec{p}}(\Omega)$ *for a subsequence.*

3.3 Main results : Existence of entropy solutions

Firstly, we introduce the definition of entropy solutions for our quasilinear elliptic problem.

Definition 3.3.1 *A measurable function u is called entropy solution of the unilateral problem associated to the anisotropic quasilinear elliptic equation* (3.11) *if* $T_k(u)\in K_\psi$ *and*

$g(x,u) \in L^1(\Omega)$, *such that* u *verify the following inequality*

$$\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) D^i T_k(u-v)\, dx + \int_\Omega g(x,u) T_k(u-v)\, dx \leq \int_\Omega f T_k(u-v)\, dx, \qquad (3.13)$$

for any $v \in K_\psi \cap L^\infty(\Omega)$.

Our objective is to prove the following existence theorem :

Theorem 3.3.1 *Assuming that the conditions* (3.6) − (3.9) *and* (3.10) *hold true, then the problem* (3.11) *has at least one entropy solution. Moreover, we have*

$$\lim_{h\to\infty} \frac{1}{h} \sum_{i=1}^{N} \int_{\{|u|<h\}} a_i(x,u,\nabla u) D^i u\, dx = 0. \qquad (3.14)$$

Proof of Theorem 3.3.1

Step 1 : Approximate problems.

We set $(f_n)_{n\in\mathbb{N}}$ a sequence of functions in $W^{-1,\vec{p}'}(\Omega) \cap L^1(\Omega)$ which verify : $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \leq |f|$ (for example $f_n = T_n(f)$). We consider the approximate problem :

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, T_n(u_n), \nabla u_n) + g_n(x,u_n) = f_n(x) & \text{in} \quad \Omega, \\ u_n = 0 & \text{on} \quad \partial\Omega, \end{cases} \qquad (3.15)$$

where $g_n(x,s) = T_n(g(x,s))$. We define the operator A_n and $G_n : W_0^{1,\vec{p}}(\Omega) \mapsto W^{-1,\vec{p}'}(\Omega)$ by

$$\langle A_n u, v\rangle = \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i v\, dx \quad \text{and} \quad \langle G_n u, v\rangle = \int_\Omega g_n(x,u) v\, dx \;\forall\, u, v \in W_0^{1,\vec{p}}(\Omega),$$

Lemma 3.3.1 *The operator* $B_n = A_n + G_n$ *acted from* $W_0^{1,\vec{p}}(\Omega)$ *into* $W^{-1,\vec{p}'}(\Omega)$ *is pseudo-monotone. Moreover,* B_n *is coercive in the following sense : there exists* $v_0 \in K_\psi$ *such that*

$$\frac{\langle B_n v, v - v_0\rangle}{\|v\|_{1,\vec{p}(\cdot)}} \longrightarrow \infty \qquad \text{as} \quad \|v\|_{1,\vec{p}(\cdot)} \to \infty \quad \text{for any} \quad v \in K_\psi.$$

Proof of Lemma 3.3.1

Using the Hölder's inequality and the growth condition (3.6), we can show that the operator A_n is bounded, and since $|G_n(v)| = |T_n(g(x,v)| \leq n$ for any $v \in K_\psi$, we conclude that

B_n is bounded.
For the coercivity, let $v_0 \in K_\psi$, for any $v \in K_\psi$ we have

$$\begin{aligned}
\langle B_n v, v_0\rangle &= \langle A_n v, v_0\rangle + \langle G_n v, v_0\rangle \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(v), \nabla v) D^i v_0 \, dx + \int_\Omega g_n(x, v) v_0 \, dx \\
&\le \sum_{i=1}^{N} \Big(\int_\Omega |a_i(x, T_n(v), \nabla v)|^{p_i'} \, dx \Big)^{\frac{1}{p_i'}} \|D^i v_0\|_{p_i} + n \int_\Omega |v_0| \, dx \\
&\le 3^{p_i'-1} \beta \sum_{i=1}^{N} \Big(\int_\Omega (K_i(x)^{p_i'} + n^{p_i} + |D^i v|^{p_i}) \, dx \Big)^{\frac{1}{p_i'}} \|v_0\|_{1,\vec{p}} + n \cdot \|v_0\|_1,
\end{aligned}$$

and

$$\begin{aligned}
\langle B_n v, v\rangle &= \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(v), \nabla v) D^i v \, dx + \int_\Omega g_n(x, v) v \, dx \\
&\ge \sum_{i=0}^{N} \int_\Omega b(|T_n(v)|) |D^i v|^{p_i} \, dx - n \cdot \|v\|_1 \\
&\ge \frac{b_0}{(1+n)^\lambda} \sum_{i=0}^{N} \int_\Omega |D^i v|^{p_i} \, dx - n \cdot \|v\|_1.
\end{aligned}$$

In view of Poincaré inequality, we conclude that

$$\begin{aligned}
\frac{\langle B_n v, v - v_0\rangle}{\|v\|_{1,\vec{p}(\cdot)}} &\ge \frac{b_0}{(1+n)^\lambda \|v\|_{1,\vec{p}}} \sum_{i=1}^{N} \int_\Omega |D^i v|^{p_i} \, dx - \frac{n \cdot (\|v\|_1 + \|v_0\|_1)}{\|v\|_{1,\vec{p}}} \\
&\quad - \frac{3^{p_i'-1} \beta \|v_0\|_{1,\vec{p}}}{\|v\|_{1,\vec{p}}} \sum_{i=1}^{N} \Big(\int_\Omega K_i(x)^{p_i} + n^{p_i} + |D^i v|^{p_i} \, dx \Big)^{\frac{1}{p_i'}} \longrightarrow \infty \\
&\quad \text{as} \quad \|v\|_{1,\vec{p}(\cdot)} \to \infty.
\end{aligned} \tag{3.16}$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases}
u_k \rightharpoonup u & \text{in } W_0^{1,\vec{p}}(\Omega), \\
B_n u_k \rightharpoonup \chi_n & \text{in } W^{-1,\vec{p}'}(\Omega), \\
\limsup\limits_{k\to\infty} \langle B_n u_k, u_k\rangle \le \langle \chi_n, u\rangle.
\end{cases} \tag{3.17}$$

We will prove that

$$\chi_n = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \ \text{ as } \ k \to +\infty.$$

Thanks to Lemma 1.1.1, we have $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$ with $\underline{p} > 1$, then

$$u_k \to u \quad \text{in } L^{\underline{p}}(\Omega) \quad \text{and a.e. in } \Omega, \quad \text{for a subsequence denoted again } (u_k)_{k\in\mathbb{N}}.$$

As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,\vec{p}}(\Omega)$, and by the growth condition, the sequence $(a_i(x, T_n(u_k), \nabla u_k))_{k\in\mathbb{N}}$ is bounded in $L^{p_i'}(\Omega)$. Therefore, there exists a measurable func-

tion $\varphi_i \in L^{p_i'}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \ \text{ in } \quad L^{p_i'}(\Omega) \ \text{ as } \ k \to \infty, \tag{3.18}$$

Also, we have $(g_n(x,u_k))_{k\in\mathbb{N}}$ is uniformly bounded in $L^{\underline{p}'}(\Omega)$, and $g_n(x,u_k) \to g_n(x,u)$ almost everywhere in Ω. In view of Lebesgue dominated convergence theorem, we deduce that

$$g_n(x,u_k) \longrightarrow g_n(x,u) \ \text{ in } \quad L^{\underline{p}'}(\Omega) \ \text{ as } \ k \to \infty. \tag{3.19}$$

On the one hand, we have for any $v \in W_0^{1,\vec{p}}(\Omega)$

$$\begin{aligned} \langle \chi_n, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i v \, dx + \lim_{k\to\infty} \int_\Omega g_n(x,u_k)\ v \, dx \\ &= \sum_{i=1}^{N} \int_\Omega \varphi_i \, D^i v \, dx + \int_\Omega g_n(x,u)\ v \, dx. \end{aligned} \tag{3.20}$$

From relations (3.17) and (3.20), we obtain

$$\begin{aligned} \limsup_{k\to\infty} \langle B_n(u_k), u_k\rangle &= \limsup_{k\to\infty} \Big(\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx + \int_\Omega g_n(x,u_k) u_k \, dx \Big) \\ &\le \sum_{i=1}^{N} \int_\Omega \varphi_i \, D^i u \, dx + \int_\Omega g_n(x,u)\ u \, dx. \end{aligned} \tag{3.21}$$

Thanks to (3.19), and since u_k converges strongly to u in $L^{\underline{p}}(\Omega)$, then

$$\int_\Omega g_n(x,u_k) u_k \, dx \longrightarrow \int_\Omega g_n(x,u)\ u \, dx \quad \text{ as } \quad k \to \infty. \tag{3.22}$$

Therefore

$$\limsup_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx \le \sum_{i=1}^{N} \int_\Omega \varphi_i \, D^i u \, dx. \tag{3.23}$$

On the other hand, in view of (3.8) we have

$$\sum_{i=1}^{N} \int_\Omega (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u) \, dx \ge 0, \tag{3.24}$$

then

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx &\ge \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u \, dx \\ &\quad + \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u)(D^i u_k - D^i u) \, dx. \end{aligned}$$

In view of Lebesgue dominated convergence theorem, we have $T_n(u_k) \to T_n(u)$ in $L^{p_i}(\Omega)$ then $a_i(x, T_n(u_k), \nabla u) \to a_i(x, T_n(u), \nabla u)$ strongly in $L^{p_i'}(\Omega)$, and by (3.18) we get

$$\liminf_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx \geq \sum_{i=1}^{N} \int_\Omega \varphi_i \, D^i u \, dx.$$

Having in mind (3.23), we conclude that

$$\lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx = \sum_{i=1}^{N} \int_\Omega \varphi_i \, D^i u \, dx. \tag{3.25}$$

According to (3.20), (3.22) and (3.25), we obtain

$$\langle B_n u_k, u_k \rangle \longrightarrow \langle \chi_n, u \rangle \text{ as } k \to +\infty.$$

Now, by (3.25) we can prove that

$$\lim_{k\to+\infty} \sum_{i=1}^{N} \int_\Omega (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u) \, dx = 0.$$

Since $u_n \to u$ in $L^{\underline{p}}(\Omega)$ and by virtue of Lemma 3.2.1, we get

$$u_k \longrightarrow u \text{ in } W_0^{1,\vec{p}}(\Omega) \quad \text{and} \quad D^i u_k \longrightarrow D^i u \text{ a.e. in } \Omega,$$

it follows that $a_i(x, T_n(u_k), \nabla u_k) \to a_i(x, T_n(u), \nabla u)$ almost everywhere in Ω, and in view of (3.6) we conclude that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup a_i(x, T_n(u), \nabla u) \qquad \text{in } L^{p_i'}(\Omega) \quad \text{for} \quad i = 1, \ldots, N,$$

and thanks to (3.19) we conclude that $\chi_n = B_n u$, which completes the proof of lemma 3.3.1.

In view of Lemma 3.3.1 (see [60], Theorem 8.2), there exists at least one weak solution $u_n \in W_0^{1,\vec{p}}(\Omega)$ of the problem (3.15), i.e.

$$\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i(u_n - v) \, dx + \int_\Omega g_n(x, u_n) \, (u_n - v) \, dx \leq \int_\Omega f_n(x) \, (u_n - v) \, dx, \tag{3.26}$$

for any $v \in K_\psi$.

Step 2 : A priori estimates.

Let $k \geq \max(1, \|\psi^+\|_\infty)$ and $M = \|\psi^+\|_\infty + k$. Choosing $0 < \varepsilon < \underline{p} - \lambda - \delta - 1$ and we set

$$H(r) = \int_0^r \frac{ds}{(1+|s|)^{1+\varepsilon}} \qquad \text{for any} \quad r \in \mathbb{R}_+.$$

By taking $v = u_n - \eta T_k(u_n - \psi^+)(1+|T_M(u_n)|)^\lambda e^{H(|u_n|)} \in W_0^{1,\vec{p}}(\Omega)$, we have $v \in W_0^{1,\vec{p}}(\Omega)$ and $v \geq \psi$ for $\eta > 0$ small enough, then $v \in K_\Psi$ is an admissible test function for the approximate problem (3.26), and we obtain

$$\begin{aligned}
&\eta \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i (T_k(u_n - \psi^+)(1+|T_M(u_n)|)^\lambda e^{H(|u_n|)})\, dx \\
&+\eta \int_\Omega g_n(x, u_n)\, T_k(u_n - \psi^+)(1+|T_M(u_n)|)^\lambda e^{H(|u_n|)}\, dx \\
&\leq \eta \int_\Omega f_n(x)\, T_k(u_n - \psi^+)(1+|T_M(u_n)|)^\lambda e^{H(|u_n|)}\, dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \psi^+)(1+|T_M(u_n)|)^\lambda e^{H(|u_n|)}\, dx \\
&\quad +\lambda \sum_{i=1}^N \int_{\{|u_n| \leq M\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n |T_k(u_n - \psi^+)|(1+|T_M(u_n)|)^{\lambda-1} e^{H(|u_n|)}\, dx \\
&\quad + \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i u_n\, |T_k(u_n - \psi^+)| \frac{(1+|T_M(u_n)|)^\lambda}{(1+|u_n|)^{1+\varepsilon}} e^{H(|u_n|)}\, dx \\
&\leq \int_\Omega |f_n(x)|\, |T_k(u_n - \psi^+)|\, (1+|T_M(u_n)|)^\lambda\, e^{H(|u_n|)}\, dx \\
&\quad + \int_\Omega |g_n(x, u_n)|\, |T_k(u_n - \psi^+)|\, (1+|T_M(u_n)|)^\lambda e^{H(|u_n|)}\, dx.
\end{aligned}$$

Thanks to (3.8) and (3.10), we conclude that

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} b(|u_n|)|D^i u_n|^{p_i} (1+|T_M(u_n)|)^\lambda\, dx \\
&\quad +\lambda \sum_{i=1}^N \int_{\{|u_n| \leq M\}} b(|u_n|)|D^i u_n|^{p_i} |T_k(u_n - \psi^+)|(1+|T_M(u_n)|)^{\lambda-1} e^{H(|u_n|)}\, dx \\
&\quad + \sum_{i=1}^N \int_\Omega b(|u_n|)|D^i u_n|^{p_i} \frac{(1+|T_M(u_n)|)^\lambda}{(1+|u_n|)^{1+\varepsilon}}\, |T_k(u_n - \psi^+)| e^{H(|u_n|)}\, dx \\
&\leq \int_\Omega (|f_n(x)| + |f_0(x)|)\, |T_k(u_n - \psi^+)|\, (1+|T_M(u_n)|)^\lambda e^{H(|u_n|)}\, dx \\
&\quad + \int_\Omega c(x)|u_n|^\delta\, |T_k(u_n - \psi^+)|\, (1+|T_M(u_n)|)^\lambda e^{H(|u_n|)}\, dx \\
&\quad + \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} |a_i(x, T_n(u_n), \nabla u_n)|(1+|T_M(u_n)|)^\lambda\, |D^i \psi^+| e^{H(|u_n|)}\, dx.
\end{aligned} \tag{3.27}$$

In view of Poincaré inequality, we have for any $i = 1, \dots, \mathrm{N}$:

$$\begin{aligned}
\int_{\{|u_n-\psi^+|\le k\}} |u_n|^{p_i}\,dx \le\ & 2^{p_i-1}\int_{\{|u_n-\psi^+|\le k\}} |u_n-\psi^+|^{p_i}\,dx \\
& +2^{p_i-1}\int_{\{|u_n-\psi^+|\le k\}} |\psi^+|^{p_i}\,dx \\
\le\ & 2^{p_i-1}\mathrm{C}_p^{p_i}\int_{\{|u_n-\psi^+|\le k\}} |\mathrm{D}^i u_n-\mathrm{D}^i\psi^+|^{p_i}\,dx \\
& +2^{p_i-1}\int_{\{|u_n-\psi^+|\le k\}} |\psi^+|^{p_i}\,dx \\
\le\ & 2^{2p_i-2}\mathrm{C}_p^{p_i}\int_{\{|u_n-\psi^+|\le k\}} |\mathrm{D}^i u_n|^{p_i}\,dx+\mathrm{C}_0
\end{aligned} \tag{3.28}$$

For the second term on the left-hand side of (3.27), Let $\sigma > 0$, in view of Young's and Poincaré inequalities, and using the growth condition (3.6) and (3.28) we have

$$\begin{aligned}
&\sum_{i=1}^{\mathrm{N}}\int_{\{|u_n-\psi^+|\le k\}} |a_i(x,\mathrm{T}_n(u_n),\nabla u_n)|(1+|\mathrm{T}_\mathrm{M}(u_n)|)^{\lambda}\,|\mathrm{D}^i\psi^+|e^{\mathrm{H}(|u_n|)}\,dx \\
&\le \beta e^{\mathrm{H}(\infty)}\sum_{i=1}^{\mathrm{N}}\int_{\{|u_n-\psi^+|\le k\}} (\mathrm{K}_i(x)+|u_n|^{r_i}+|\mathrm{D}^i u_n|^{p_i-1})(1+|\mathrm{T}_\mathrm{M}(u_n)|)^{\lambda}\,|\mathrm{D}^i\psi^+|\,dx \\
&\le \beta e^{\mathrm{H}(\infty)}\sum_{i=1}^{\mathrm{N}}\int_{\Omega} |\mathrm{K}_i(x)|^{p_i'}\,dx+\sigma\sum_{i=1}^{\mathrm{N}}\int_{\{|u_n-\psi^+|\le k\}} |u_n|^{p_i}\,dx+\sigma\sum_{i=1}^{\mathrm{N}}\int_{\{|u_n-\psi^+|\le k\}} |\mathrm{D}^i u_n|^{p_i}\,dx \\
&\quad +\mathrm{C}_1\sum_{i=1}^{\mathrm{N}}\int_{\{|u_n-\psi^+|\le k\}} (1+|\mathrm{T}_\mathrm{M}(u_n)|)^{\lambda p_i}\,|\mathrm{D}^i\psi^+|^{p_i}\,dx \\
&\le \mathrm{C}_2+\sigma\sum_{i=1}^{\mathrm{N}}(2^{2p_i-2}\mathrm{C}_p^{p_i}+1)\int_{\{|u_n-\psi^+|\le k\}} |\mathrm{D}^i u_n|^{p_i}\,dx+\mathrm{C}_3\sum_{i=1}^{\mathrm{N}}(1+k)^{\lambda p_i}.
\end{aligned}$$

it follows that

$$\begin{aligned}
&\sum_{i=1}^{\mathrm{N}}\int_{\{|u_n-\psi^+|\le k\}} b(|u_n|)(1+|u_n|)^{\lambda}|\mathrm{D}^i u_n|^{p_i}e^{\mathrm{H}(|u_n|)}\,dx \\
&\quad +\lambda\sum_{i=1}^{\mathrm{N}}\int_{\{|u_n|\le \mathrm{M}\}} \frac{b(|u_n|)(1+|u_n|)^{\lambda}}{(1+|u_n|)}|\mathrm{D}^i u_n|^{p_i}|\mathrm{T}_k(u_n-\psi^+)|e^{\mathrm{H}(|u_n|)}\,dx \\
&\quad +\sum_{i=1}^{\mathrm{N}}\int_{\Omega} \frac{b(|u_n|)(1+|\mathrm{T}_\mathrm{M}(u_n)|)^{\lambda}}{(1+|u_n|)^{1+\varepsilon}}|\mathrm{D}^i u_n|^{p_i}\,|\mathrm{T}_k(u_n-\psi^+)|e^{\mathrm{H}(|u_n|)}\,dx \\
&\le e^{\mathrm{H}(\infty)}\int_{\Omega}(|f_n(x)|+|f_0(x)|)\,|\mathrm{T}_k(u_n-\psi^+)|\,(1+|\mathrm{T}_\mathrm{M}(u_n)|)^{\lambda}\,dx \\
&\quad +e^{\mathrm{H}(\infty)}\int_{\Omega} c(x)|u_n|^{\delta}\,|\mathrm{T}_k(u_n-\psi^+)|\,(1+|\mathrm{T}_\mathrm{M}(u_n)|)^{\lambda}\,dx \\
&\quad +\mathrm{C}_2+\sigma\sum_{i=1}^{\mathrm{N}}(2^{2p_i-2}\mathrm{C}_p^{p_i}+1)\int_{\{|u_n-\psi^+|\le k\}} |\mathrm{D}^i u_n|^{p_i}\,dx+\mathrm{C}_3\sum_{i=1}^{\mathrm{N}}(1+k)^{\lambda p_i}.
\end{aligned} \tag{3.29}$$

In view of (3.10), and by taking σ small enough (for example $\sigma \max_{1\le i\le N}(2^{2p_i-2}C_p^{p_i}+1)\le \frac{b_0}{2}$), we conclude that

$$\begin{aligned}
&\frac{b_0}{2}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^i u_n|^{p_i}\,dx+\lambda b_0\sum_{i=1}^{N}\int_{\{|u_n|\le M\}}\frac{|D^i u_n|^{p_i}}{(1+|u_n|)}|T_k(u_n-\psi^+)|\,dx\\
&\quad+b_0\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}\frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{1+\varepsilon}}\,|T_k(u_n-\psi^+)|\,dx\\
&\quad+b_0k(1+M)^{\lambda}\sum_{i=1}^{N}\int_{\{|u_n|>M\}}\frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\lambda+1+\varepsilon}}\,dx\\
&\le C_3\sum_{i=1}^{N}(1+k)^{\lambda p_i}+C_4k^{\lambda+1}\Big(\|f(x)\|_{L^1(\Omega)}+\|f_0(x)\|_{L^1(\Omega)}\Big)+C_4k^{\lambda+1}\int_{\Omega}c(x)|u_n|^{\delta}\,dx.
\end{aligned}\tag{3.30}$$

For the last term on the left-hand side of (3.30), in view of Poincaré inequality we have

$$\begin{aligned}
b_0k(1+k)^{\lambda}\sum_{i=1}^{N}\int_{\{|u_n|>M\}}&\frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\lambda+1+\varepsilon}}\,dx\\
&\ge b_0k(1+k)^{\lambda}\sum_{i=1}^{N}\int_{\Omega}\Big|D^i\int_{|T_M(u_n)|}^{|u_n|}\frac{ds}{(1+s)^{\frac{\lambda+1+\varepsilon}{p_i}}}\Big|^{p_i}dx\\
&\ge b_0k^{\lambda+1}\sum_{i=1}^{N}\frac{1}{C_p^{p_i}}\int_{\Omega}\Big|\int_{|T_M(u_n)|}^{|u_n|}\frac{ds}{(1+s)^{\frac{\lambda+1+\varepsilon}{p_i}}}\Big|^{p_i}dx\\
&\ge b_0k^{\lambda+1}\sum_{i=1}^{N}\frac{1}{C_p^{p_i}}\int_{\Omega}\frac{(|u_n|-|T_M(u_n)|)^{p_i}}{(1+|u_n|)^{\lambda+1+\varepsilon}}\,dx\\
&\ge b_0k^{\lambda+1}\sum_{i=1}^{N}\frac{1}{2^{p_i-1}C_p^{p_i}}\int_{\{|u_n|>M\}}\frac{|u_n|^{p_i}}{(1+|u_n|)^{\lambda+1+\varepsilon}}\,dx\\
&\quad-b_0k^{\lambda+1}\sum_{i=1}^{N}\frac{1}{C_p^{p_i}}\int_{\{|u_n|>M\}}\frac{|T_M(u_n)|^{p_i}}{(1+|u_n|)^{\lambda+1+\varepsilon}}\,dx\\
&\ge C_5k^{\lambda+1}\sum_{i=1}^{N}\int_{\{|u_n|>M\}}|u_n|^{p_i-\lambda-1-\varepsilon}\,dx\\
&\quad-C_6k^{\lambda+1}\sum_{i=1}^{N}k^{p_i-\lambda-1-\varepsilon}
\end{aligned}\tag{3.31}$$

Concerning the second term on the right-hand side of (3.30), in view of Young's inequality we have

$$\begin{aligned}
C_4k^{\lambda+1}\int_{\Omega}c(x)|u_n|^{\delta}\,dx&\le C_7k^{\lambda+1}\int_{\Omega}|c(x)|^{\frac{p_0-\varepsilon-\lambda-1}{p_0-\varepsilon-\lambda-\delta-1}}\,dx+\frac{C_5}{2}k^{\lambda+1}\int_{\Omega}|u_n|^{p_0-\varepsilon-\lambda-1}\,dx\\
&\le C_7k^{\lambda+1}\|c(x)\|^{\frac{p_0-\varepsilon-\lambda-1}{p_0-\varepsilon-\lambda-\delta-1}}_{L^{\frac{p_0-\varepsilon-\lambda-1}{p_0-\varepsilon-\lambda-\delta-1}}(\Omega)}+\frac{C_5}{2}k^{\lambda+1}\int_{\{|u_n|\le M\}}|u_n|^{p_0-\varepsilon-\lambda-1}\,dx\\
&\quad+\frac{C_5}{2}k^{\lambda+1}\int_{\{|u_n|>M\}}|u_n|^{p_0-\varepsilon-\lambda-1}\,dx.
\end{aligned}\tag{3.32}$$

By combining (3.30), (3.31) and (3.32) we conclude that

$$\begin{aligned}
&\frac{b_1}{2}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^i u_n|^{p_i}\,dx+\lambda b_0\sum_{i=1}^{N}\int_{\{|u_n|\le M\}}\frac{|D^i u_n|^{p_i}}{(1+|u_n|)}|T_k(u_n-\psi^+)|\,dx\\
&\quad+b_0\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}\frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{1+\varepsilon}}\,|T_k(u_n-\psi^+)|\,dx\\
&\quad+\frac{C_5}{2}k^{\lambda+1}\sum_{i=1}^{N}\int_{\{|u_n|>M\}}|u_n|^{p_i-\lambda-1-\varepsilon}\,dx\\
&\le C_7k^{\lambda+1}+C_6Nk^{p_0-\varepsilon}+\frac{C_5}{2}k^{\lambda+1}\int_{\{|u_n|\le M\}}|u_n|^{p_0-\varepsilon-\lambda-1}\,dx\\
&\le C_8k^{p_0-\varepsilon}.
\end{aligned}\tag{3.33}$$

Having in mind that $\{|u_n|\le k\}\subset\{|u_n-\psi^+|\le k+\|\psi^+\|_\infty\}$, and thanks to (3.33) we conclude that

$$\begin{aligned}
\sum_{i=1}^{N}\int_\Omega|D^iT_k(u_n)|^{p_i}\,dx&=\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}|D^iu_n|^{p_i}\,dx\\
&\le\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k+\|\psi^+\|_\infty\}}|D^iu_n|^{p_i}\,dx\\
&\le C_9(k+\|\psi^+\|_\infty)^{p_0-\varepsilon}\\
&\le C_{10}k^{p_0-\varepsilon}\qquad\text{with}\quad k\ge\max(1,\|\psi^+\|_\infty).
\end{aligned}\tag{3.34}$$

We conclude that $(T_k(u_n))_n$ is bounded uniformly in $W_0^{1,\vec{p}}(\Omega)$, then there exists a subsequence still denoted by $(T_k(u_n))_n$ and a measurable function $\vartheta_k\in W_0^{1,\vec{p}}(\Omega)$, such that

$$\begin{cases}T_k(u_n)\rightharpoonup\vartheta_k\text{ weakly in }W_0^{1,\vec{p}}(\Omega)\\ T_k(u_n)\to\vartheta_k\text{ strongly in }L^{p_0}(\Omega)\text{ and a.e. in }\Omega.\end{cases}\tag{3.35}$$

Now, we will show that $(u_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in measure.
Indeed, for any $\tau>0$ we have

$$\text{meas}\{|u_n-u_m|>\tau\}\le\text{meas}\{|u_n|>k\}+\text{meas}\{|u_m|>k\}+\text{meas}\{|T_k(u_n)-T_k(u_m)|>\tau\}.$$

Thanks to Poincaré inequality and (3.34), we have

$$\begin{aligned} k^{p_0}\text{meas}\{|u_n|>k\} &= \int_{\{|u_n|>k\}} |\mathrm{T}_k(u_n)|^{p_0}\,dx \le \int_\Omega |\mathrm{T}_k(u_n)|^{p_0}\,dx \\ &\le \mathrm{C}_p^{p_0}\int_\Omega |\mathrm{D}^i\mathrm{T}_k(u_n)|^{p_0}\,dx \\ &= \mathrm{C}_p^{p_0}\int_{\{|u_n|\le k\}} |\mathrm{D}^i u_n|^{p_0}\,dx \\ &\le \mathrm{C}_p^{p_0}\sum_{i=1}^{\mathrm{N}}\int_{\{|u_n|\le k\}} |\mathrm{D}^i u_n|^{p_i}\,dx \\ &\le \mathrm{C}_{11}k^{p_0-\varepsilon}, \end{aligned} \tag{3.36}$$

it follows that

$$\text{meas}\{|u_n|>k\} \le \frac{\mathrm{C}_{11}k^{p_0-\varepsilon}}{k^{p_0}} \longrightarrow 0 \qquad \text{as} \quad k\to\infty. \tag{3.37}$$

Thus, for any $\mu>0$, there exists $k=k(\mu)$ large enough such that

$$\text{meas}\{|u_n|>k\} \le \frac{\mu}{3} \qquad \text{and} \qquad \text{meas}\{|u_m|>k\} \le \frac{\mu}{3}. \tag{3.38}$$

On the other hand, thanks to (3.35), we have $\mathrm{T}_k(u_n)\to\vartheta_k$ in $\mathrm{L}^1(\Omega)$ and a.e. in Ω. then, we can assume that $(\mathrm{T}_k(u_n))_{n\in\mathbb{N}^*}$ is a Cauchy sequence in measure, i.e. there exists $n_0(\tau,\mu,k)$ large enough such that

$$\text{meas}\{|\mathrm{T}_k(u_n)-\mathrm{T}_k(u_m)|>\tau\} \le \frac{\mu}{3} \qquad \text{for all } m,n\ge n_0(\tau,\mu,k). \tag{3.39}$$

Finally, by combining (3.38) and (3.39), we conclude that

$$\forall\tau,\mu>0 \;\text{ there exists }\; n_0=n_0(\tau,\mu) \;\text{ such that }\; \text{meas}\{|u_n-u_m|>\tau\}\le\mu \quad \forall n,m\ge n_0(\tau,\mu),$$

which proves that the sequence $(u_n)_n$ is a Cauchy sequence in measure.
We conclude that the sequence of weak solutions $(u_n)_n$ converges almost everywhere to some measurable function u, and thanks to (3.35) we obtain

$$\begin{cases} \mathrm{T}_k(u_n)\rightharpoonup \mathrm{T}_k(u) \text{ weakly in } \mathrm{W}_0^{1,\vec{p}}(\Omega) \\ \mathrm{T}_k(u_n)\to \mathrm{T}_k(u) \text{ strongly in } \mathrm{L}^{p_0}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{3.40}$$

Step 3 : Equi-integrability of the term $(g_n(x,u_n))_n$.

Now, we will show that

$$g_n(x,u_n)\to g(x,u) \qquad \text{strongly in} \quad \mathrm{L}^1(\Omega). \tag{3.41}$$

We have

$$g_n(x,u_n)\to g(x,u) \qquad \text{a.e. in} \quad \Omega,$$

using Vitali's theorem, it is sufficient to prove that $(g_n(x,u_n))_n$ is uniformly equi-integrable. Let $k \geq \max(1, \|\psi^+\|_\infty)$, and let $0 < \varepsilon < p_0 - \lambda - \delta - 1$ such that $\dfrac{p_0 - \varepsilon - \lambda - 1}{p_0 - \varepsilon - \lambda - \delta - 1} \leq \varrho$. We set

$$H(r) = \int_0^r \frac{ds}{(1+|s|)^{1+\varepsilon}} \qquad \text{for any} \quad r \in \mathbb{R}_+.$$

By taking $v = u_n - \eta T_1(u_n - T_k(u_n)) e^{H(|u_n|)}$, we have $v \in W_0^{1,\vec{p}}(\Omega)$ and $v \geq \psi$ for $\eta > 0$ small enough, then $v \in K_\psi$ is an admissible test function for the approximate problem (3.26), and we obtain

$$\begin{aligned}
&\eta \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i (T_1(u_n - T_k(u_n)) e^{H(|u_n|)})\, dx \\
&\quad + \eta \int_\Omega g_n(x,u_n)\, T_1(u_n - T_k(u_n)) e^{H(|u_n|)}\, dx \\
&\leq \eta \int_\Omega f_n(x)\, T_1(u_n - T_k(u_n)) e^{H(|u_n|)}\, dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{k \leq |u_n| \leq k+1\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n e^{H(|u_n|)}\, dx \\
&\quad + \sum_{i=1}^N \int_{\{k \leq |u_n|\}} \frac{a_i(x, T_n(u_n), \nabla u_n) D^i u_n}{(1+|u_n|)^{1+\varepsilon}} |T_1(u_n - T_k(u_n))| e^{H(|u_n|)}\, dx \\
&\leq \int_\Omega |f_n(x)|\, |T_1(u_n - T_k(u_n))|\, e^{H(|u_n|)}\, dx + \int_\Omega |g_n(x,u_n)|\, |T_1(u_n - T_k(u_n))|\, e^{H(|u_n|)}\, dx \\
&\leq e^{H(\infty)} \int_{\{k \leq |u_n|\}} (|f_n(x)| + |f_0(x)|)\, dx + e^{H(\infty)} \int_{\{k \leq |u_n|\}} c(x) |u_n|^\delta\, dx.
\end{aligned}$$

Having in mind (3.8) and (3.10), we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{k \leq |u_n|\}} \frac{b(|u_n|) |D^i u_n|^{p_i}}{(1+|u_n|)^{1+\varepsilon}}\, dx \\
&\leq \sum_{i=1}^N \int_{\{k \leq |u_n| \leq k+1\}} b(|u_n|) |D^i u_n|^{p_i}\, dx + \sum_{i=1}^N \int_{\{k+1 \leq |u_n|\}} \frac{b(|u_n|) |D^i u_n|^{p_i}}{(1+|u_n|)^{1+\varepsilon}}\, dx \\
&\leq e^{H(\infty)} \int_{\{k \leq |u_n|\}} (|f_n(x)| + |f_0(x)|)\, dx + C_0 e^{H(\infty)} \int_{\{k \leq |u_n|\}} |c(x)|^{\frac{p_0 - \varepsilon - \lambda - 1}{p_0 - \varepsilon - \lambda - \delta - 1}}\, dx \\
&+ \sigma e^{H(\infty)} \int_{\{k \leq |u_n|\}} |u_n|^{p_0 - \varepsilon - \lambda - 1}\, dx.
\end{aligned} \tag{3.42}$$

For the last term on the right-hand side of (3.42), we have

$$\begin{aligned}\int_{\{k\le|u_n|\}}|u_n|^{p_0-\varepsilon-\lambda-1}\,dx &\le 2^{\lambda+\varepsilon+1}\int_{\{k\le|u_n|\}}\frac{|u_n|^{p_0}}{(1+|u_n|)^{\lambda+\varepsilon+1}}\,dx\\ &\le \frac{2^{\lambda+\varepsilon+1}}{b_0}\int_{\{k\le|u_n|\}}\frac{b(|u_n|)|u_n|^{p_0}}{(1+|u_n|)^{\varepsilon+1}}\,dx\\ &\le \frac{2^{\lambda+\varepsilon+1}}{b_0}\int_\Omega\Big|\int_{|\mathrm{T}_k(u_n)|}^{|u_n|}\frac{b(|s|)^{\frac{1}{p_0}}}{(1+|s|)^{\frac{\varepsilon+1}{p_0}}}\,ds\Big|^{p_0}\,dx\\ &\le \mathrm{C}_p^{p_0}\frac{2^{\lambda+\varepsilon+1}}{b_0}\int_\Omega\Big|\mathrm{D}^i\int_{|\mathrm{T}_k(u_n)|}^{|u_n|}\frac{b(|s|)^{\frac{1}{p_0}}}{(1+|s|)^{\frac{\varepsilon+1}{p_0}}}\,ds\Big|^{p_0}\,dx\\ &= \mathrm{C}_p^{p_0}\frac{2^{\lambda+\varepsilon+1}}{b_0}\int_{\{k\le|u_n|\}}\frac{b(|u_n|)|\mathrm{D}^i u_n|^{p_0}}{(1+|u_n|)^{\varepsilon+1}}\,dx.\end{aligned}$$

and by choosing $\sigma>0$ small enough (for example $\sigma\le\dfrac{b_0}{2^{\lambda+\varepsilon+2}\mathrm{C}_p^{p_0}e^{\mathrm{H}(\infty)}}$), we obtain

$$\sigma e^{\mathrm{H}(\infty)}\int_{\{k\le|u_n|\}}|u_n|^{p_0-\varepsilon-\lambda-1}\,dx\le\frac{1}{2}\sum_{i=1}^{\mathrm{N}}\int_{\{k\le|u_n|\}}\frac{b(|u_n|)|\mathrm{D}^i u_n|^{p_i}}{(1+|u_n|)^{1+\varepsilon}}\,dx. \tag{3.43}$$

It follows that

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{\mathrm{N}}\int_{\{k\le|u_n|\}}\frac{b(|u_n|)|\mathrm{D}^i u_n|^{p_i}}{(1+|u_n|)^{1+\varepsilon}}\,dx\\ &\le e^{\mathrm{H}(\infty)}\int_{\{k\le|u_n|\}}(|f_n(x)|+|f_0(x)|)\,dx+\mathrm{C}_0e^{\mathrm{H}(\infty)}\int_{\{k\le|u_n|\}}|c(x)|^{\frac{p_0-\varepsilon-\lambda-1}{p_0-\varepsilon-\lambda-\delta-1}}\,dx.\end{aligned} \tag{3.44}$$

Since $c(x)\in\mathrm{L}^{\varrho}(\Omega)$, and thanks to (3.43) and (3.44), by letting k tends to infinity we deduce that

$$\lim_{k\to\infty}\int_{\{k\le|u_n|\}}|u_n|^{p_0-\varepsilon-\lambda-1}\,dx=0. \tag{3.45}$$

In view of Young's inequality we conclude that

$$\begin{aligned}\int_{\{|u_n|>k\}}|g_n(x,u_n)|\,dx &\le \int_{\{|u_n|>k\}}|f_n(x)|\,dx+\int_{\{|u_n|>k\}}c(x)|u_n|^{\delta}\,dx\\ &\le \int_{\{|u_n|>k\}}|f_n(x)|+|c(x)|^{\frac{p_0-\varepsilon-\lambda-1}{p_0-\varepsilon-\lambda-\delta-1}}\,dx\\ &\quad+\int_{\{k\le|u_n|\}}|u_n|^{p_0-\varepsilon-\lambda-1}\,dx\longrightarrow 0\qquad\text{as}\quad k\to\infty.\end{aligned} \tag{3.46}$$

Let $\eta>0$, we deduce that : there exists $h(\eta)>0$ such that

$$\int_{\{|u_n|\ge h(\eta)\}}|g_n(x,u_n)|\,dx\le\frac{\eta}{2}. \tag{3.47}$$

On the other hand, for any measurable subset E of Ω, we have

$$\int_E |g_n(x,u_n)|\,dx \le \int_E |g_n(x,T_{h(\eta)}(u_n))|\,dx + \int_{\{|u_n|\ge h(\eta)\}} |g_n(x,u_n)|\,dx, \tag{3.48}$$

it's clear that, there exists $\mu(\eta)>0$, such that for all $E\subset\Omega$, we have

$$\int_E |g_n(x,T_{h(\eta)}(u_n))|\,dx \le \frac{\eta}{2} \qquad \text{for} \quad meas(E)\le \mu(\eta). \tag{3.49}$$

By combining the inequalities (3.47) and (3.49), we conclude that : For any $\eta>0$, there exists $\mu(\eta)>0$ such that

$$\int_E |g_n(x,u_n)|\,dx \le \eta, \ \text{ for any } E\subset\Omega, \ \text{ such that } meas(E)\le\mu(\eta). \tag{3.50}$$

Thus, the sequence $(g_n(x,u_n))_n$ is equi-integrability, which conclude the prove of (3.41).

Step 4 : Some regularity results

This step is devoted to show that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,dx = 0.$$

Let $h>k>\|\psi^+\|_\infty$ and $M=k+\|\psi^+\|_\infty$, we consider $v_n = u_n - \eta\cdot\dfrac{T_h(u_n-\psi^+)}{h}e^{B(|u_n|)}\in W_0^{1,\vec{p}}(\Omega)$ for η small enough, we have $v_n\in K_\Psi$, thus v_n is an admissible test function for the approximate problem (3.26), and we obtain

$$\begin{aligned}
&\eta\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i\Big(\frac{T_h(u_n-\psi^+)}{h}e^{B(|u_n|)}\Big)\,dx\\
&+\eta\int_\Omega g_n(x,u_n)\frac{T_h(u_n-\psi^+)}{h}e^{B(|u_n|)}\,dx \le \eta\int_\Omega f_n(x)\frac{T_h(u_n-\psi^+)}{h}e^{B(|u_n|)}\,dx,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n e^{H(|u_n|)}\,dx\\
&\quad+\frac{1}{h}\sum_{i=1}^{N}\int_\Omega \frac{a_i(x,T_n(u_n),\nabla u_n)D^i u_n}{(1+|u_n|)^{1+\varepsilon}}\,|T_h(u_n-\psi^+)|\,e^{H(|u_n|)}\,dx\\
&\le e^{H(\infty)}\int_\Omega (|f_n(x)|+|g_n(x,u_n)|)\frac{|T_h(u_n-\psi^+)|}{h}\,dx\\
&\quad+\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^i\psi^+ e^{H(|u_n|)}\,dx.
\end{aligned}$$

Let $\sigma > 0$, by using (3.8) and (3.10) we conclude that

$$\begin{aligned}
&\frac{1}{2h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,dx+\frac{1}{2h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} b(|u_n|)|D^i u_n|^{p_i}\,dx\\
&+\frac{1}{h}\sum_{i=1}^{N}\int_{\Omega}\frac{b(|u_n|)|D^i u_n|^{p_i}}{(1+|u_n|)^{1+\varepsilon}}|T_h(u_n-\psi^+)|\,e^{H(|u_n|)}\,dx\\
&\le e^{H(\infty)}\int_{\Omega}(|f_n(x)|+|g_n(x,u_n)|)\frac{|T_h(u_n-\psi^+)|}{h}\,dx+\frac{C_1}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}}\frac{|D^i\psi^+|^{p_i}}{b(|u_n|)^{\frac{p_i}{p_i'}}}\,dx\\
&\quad+\sum_{i=1}^{N}\frac{\sigma}{h}\int_{\{|u_n-\psi^+|\le h\}}(|K_i(x)|^{p_i'}+|u_n|^{r_i p_i'}+|D^i u_n|^{p_i})b(|u_n|)\,dx\\
&\le e^{H(\infty)}\int_{\Omega}(|f(x)|+|g_n(x,u_n)|)\frac{|T_h(u_n-\psi^+)|}{h}\,dx+\sum_{i=1}^{N}\frac{\sigma}{h}\int_{\{|u_n-\psi^+|\le h\}}|K_i(x)|^{p_i'}b(|u_n|)\,dx\\
&\quad+\frac{\sigma}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}}b(|u_n|)|u_n|^{r_i p_i'}\,dx+\frac{\sigma}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}}b(|u_n|)|D^i u_n|^{p_i}\,dx\\
&\quad+\frac{C_1}{hb_0^{\frac{p_i}{p_i'}}}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}}|D^i\psi^+|^{p_i}(1+|u_n|)^{\frac{\lambda p_i}{p_i'}}\,dx.
\end{aligned}\tag{3.51}$$

For the first term on the right-hand side of (3.51), we have meas $\{|u_n| > h\} \to 0$ as h tends to infinity, then $\dfrac{|T_h(u_n-\psi^+)|}{h} \to 0$ weak$-*$ in $L^\infty(\Omega)$. Thanks to (3.41) we have $g_n(x,u_n) \to g(x,u)$ strongly in $L^1(\Omega)$, and since $|f(x)| \in L^1(\Omega)$ it follows that

$$\varepsilon_1(h) = e^{H(\infty)}\int_{\Omega}(|f(x)|+|g_n(x,u_n)|)\frac{|T_h(u_n-\psi^+)|}{h}\,dx \longrightarrow 0 \qquad \text{as} \quad h\to\infty. \tag{3.52}$$

Concerning the second term on the right-hand side of (3.51), we have $K_i(x) \in L^{p_i'}(\Omega)$ for any $i = 1,\dots,N$, then

$$\varepsilon_2(h) = \sum_{i=1}^{N}\frac{1}{h}\int_{\{|u_n-\psi^+|\le h\}}|K_i(x)|^{p_i'}b(|u_n|)\,dx \le \sum_{i=1}^{N}\frac{\|b(\cdot)\|_{L^\infty(\Omega)}}{h}\int_{\Omega}|K_i(x)|^{p_i'}\,dx \to 0 \text{ as } h\to\infty, \tag{3.53}$$

and since $0 < \lambda < \dfrac{1}{p_0-1}$ then $\dfrac{\lambda p_i}{p_i'} < 1$ for any $i = 1,\dots,N$, it follows that

$$\begin{aligned}
\varepsilon_3(h) &= \frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}}|D^i\psi^+|^{p_i}(1+|u_n|)^{\frac{\lambda p_i}{p_i'}}\,dx\\
&\le \frac{2h^{\frac{\lambda p_i}{p_i'}}}{h}\sum_{i=1}^{N}\int_{\Omega}|D^i\psi^+|^{p_i}\,dx \to 0 \qquad \text{as} \quad h\to\infty.
\end{aligned}\tag{3.54}$$

and since $\{|u_n-\psi^+|\le h\}\subseteq\{|u_n|\le \mathrm{M}\}$, it follows that

$$\begin{aligned}\frac{\sigma}{h}\int_{\{|u_n-\psi^+|\le h\}}|u_n|^{r_i p_i'}b(|u_n|)\,dx &\le \frac{\sigma}{h}\int_{\{|u_n|\le \mathrm{M}\}}|u_n|^{p_i}b(|u_n|)\,dx\\ &\le \frac{\sigma}{h}\int_\Omega\Big|\int_0^{|\mathrm{T_M}(u_n)|}|b(s)|^{\frac{1}{p_i}}\,ds\Big|^{p_i}dx\\ &\le \frac{\sigma \mathrm{C}_p^{p_i}}{h}\int_\Omega\Big|\mathrm{D}^i\int_0^{|\mathrm{T_M}(u_n)|}|b(s)|^{\frac{1}{p_i}}\,ds\Big|^{p_i}dx\\ &= \frac{\sigma \mathrm{C}_p^{p_i}}{h}\int_{\{|u_n|\le \mathrm{M}\}}|\mathrm{D}^i u_n|^{p_i}b(|u_n|)\,dx.\end{aligned}\tag{3.55}$$

By taking $\sigma>0$ small enough (for example $\sigma\le\min(\frac{1}{8\mathrm{C}_p^{p_i}},\frac{1}{8})$) it follows that

$$\begin{aligned}&\frac{1}{2h}\sum_{i=1}^{\mathrm{N}}\int_{\{|u_n-\psi^+|\le h\}}a_i(x,\mathrm{T}_n(u_n),\nabla u_n)\mathrm{D}^i u_n\,dx+\frac{1}{4h}\sum_{i=1}^{\mathrm{N}}\int_{\{|u_n-\psi^+|\le h\}}b(|u_n|)|\mathrm{D}^i u_n|^{p_i}\,dx\\ &+\frac{1}{h}\sum_{i=1}^{\mathrm{N}}\int_\Omega\frac{b(|u_n|)|\mathrm{D}^i u_n|^{p_i}}{(1+|u_n|)^{1+\varepsilon}}|\mathrm{T}_h(u_n-\psi^+)|\,e^{\mathrm{H}(|u_n|)}\,dx\le\varepsilon_4(h).\end{aligned}\tag{3.56}$$

By letting h tends to infinity in (3.56) we conclude that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{\mathrm{N}}\int_{\{|u_n|\le h\}}a(x,\mathrm{T}_n(u_n),\nabla u_n)\mathrm{D}^i u_n\,dx=0.\tag{3.57}$$

Moreover, we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{\mathrm{N}}\int_{\{|u_n|\le h\}}b(|u_n|)|\mathrm{D}^i u_n|^{p_i}\,dx=0,\tag{3.58}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{\mathrm{N}}\int_{\{|u_n|>h\}}\frac{b(|u_n|)|\mathrm{D}^i u_n|^{p_i}}{(1+|u_n|)^{1+\varepsilon}}\,dx=0.\tag{3.59}$$

Step 4 : Strong convergence of truncations

Let $h>k\ge\max\{1,\|\psi^+\|\}$, and we set $\psi_h(u_n)=(1-\frac{|\mathrm{T}_h(u_n-\mathrm{T}_h(u_n))|}{h})$.
By taking η small enough, we have $v=u_n-\eta(\mathrm{T}_k(u_n)-\mathrm{T}_k(u))\psi_h(u_n)\in\mathrm{K}_\psi$ is an admissible

test function for the approximate problem (3.26), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,\mathrm{T}_n(u_n),\nabla u_n)(\mathrm{D}^i\mathrm{T}_k(u_n)-\mathrm{D}^i\mathrm{T}_k(u))\psi_h(u_n)\,dx\\
&-\frac{1}{h}\sum_{i=1}^{N}\int_{\{h\le|u_n|\le 2h\}} a_i(x,\mathrm{T}_n(u_n),\nabla u_n)\mathrm{D}^i u_n|\mathrm{T}_k(u_n)-\mathrm{T}_k(u)|\,dx\\
&+\int_{\Omega} g_n(x,u_n)(\mathrm{T}_k(u_n)-\mathrm{T}_k(u))\psi_h(u_n)\,dx\\
&=\int_{\Omega} f_n(x)(\mathrm{T}_k(u_n)-\mathrm{T}_k(u))\psi_h(u_n)\,dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,\mathrm{T}_n(u_n),\nabla u_n)(\mathrm{D}^i\mathrm{T}_k(u_n)-\mathrm{D}^i\mathrm{T}_k(u))\psi_h(u_n)\,dx\\
&\le\int_{\Omega}|f(x)|\,|\mathrm{T}_k(u_n)-\mathrm{T}_k(u)|\,dx+\int_{\Omega}|g_n(x,u_n)|\,|\mathrm{T}_k(u_n)-\mathrm{T}_k(u)|\,dx\\
&\quad+\frac{1}{h}\sum_{i=1}^{N}\int_{\{h\le|u_n|\le 2h\}} a_i(x,\mathrm{T}_n(u_n),\nabla u_n)\mathrm{D}^i u_n|\mathrm{T}_k(u_n)-\mathrm{T}_k(u)|\,dx.
\end{aligned}\tag{3.60}$$

For the first and second terms on the right-hand side of (3.60), we have $\mathrm{T}_k(u_n)\rightharpoonup\mathrm{T}_k(u)$ weak$-\star$ in $\mathrm{L}^\infty(\Omega)$, and since $|g_n(x,u_n)|\to|g(x,u)|$ strongly in $\mathrm{L}^1(\Omega)$, with $f(x)\in\mathrm{L}^1(\Omega)$, it follows that

$$\varepsilon_6(n)=\int_{\Omega}|g_n(x,u_n)|\,|\mathrm{T}_k(u_n)-\mathrm{T}_k(u)|\,dx\longrightarrow 0\qquad\text{as}\quad n\to\infty.\tag{3.61}$$

and

$$\varepsilon_5(n)=\int_{\Omega}|f(x)|\,|\mathrm{T}_k(u_n)-\mathrm{T}_k(u)|\,dx\longrightarrow 0\qquad\text{as}\quad n\to\infty,\tag{3.62}$$

On the other hand, according to (3.57) we have

$$\begin{aligned}
\varepsilon_7(h)\quad&=\frac{1}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\le 2h\}} a_i(x,\mathrm{T}_n(u_n),\nabla u_n)\mathrm{D}^i u_n|\mathrm{T}_k(u_n)-\mathrm{T}_k(u)|\,dx\\
&\le\frac{2k}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\le 2h\}} a_i(x,\mathrm{T}_n(u_n),\nabla u_n)\mathrm{D}^i u_n\,dx\longrightarrow 0\qquad\text{as}\quad h\to\infty.
\end{aligned}\tag{3.63}$$

By combining (3.60) and (3.61) – (3.63) we conclude that

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,\mathrm{T}_n(u_n),\nabla u_n)(\mathrm{D}^i\mathrm{T}_k(u_n)-\mathrm{D}^i\mathrm{T}_k(u))\psi_h(u_n)\,dx\le\varepsilon_8(n,h).\tag{3.64}$$

For the term on the left-hand side of (3.64), since $a_i(x,s,0)=0$, it follows that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,\mathrm{T}_n(u_n),\nabla u_n)(\mathrm{D}^i\mathrm{T}_k(u_n)-\mathrm{D}^i\mathrm{T}_k(u))\psi_h(u_n)\,dx\\
&=\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,\mathrm{T}_k(u_n),\nabla \mathrm{T}_k(u_n))(\mathrm{D}^i\mathrm{T}_k(u_n)-\mathrm{D}^i\mathrm{T}_k(u))\,dx\\
&\quad-\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} a_i(x,\mathrm{T}_{h+1}(u_n),\nabla \mathrm{T}_{h+1}(u_n))\mathrm{D}^i\mathrm{T}_k(u)\,\psi_h(u_n)\,dx\\
&=\sum_{i=1}^{N}\int_\Omega \Big(a_i(x,\mathrm{T}_k(u_n),\nabla \mathrm{T}_k(u_n))-a_i(x,\mathrm{T}_k(u_n),\nabla \mathrm{T}_k(u))\Big)(\mathrm{D}^i\mathrm{T}_k(u_n)-\mathrm{D}^i\mathrm{T}_k(u))\,dx\\
&\quad+\sum_{i=1}^{N}\int_\Omega a_i(x,\mathrm{T}_k(u_n),\nabla \mathrm{T}_k(u))(\mathrm{D}^i\mathrm{T}_k(u_n)-\mathrm{D}^i\mathrm{T}_k(u))\,dx\\
&\quad-\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} a_i(x,\mathrm{T}_{h+1}(u_n),\nabla \mathrm{T}_{h+1}(u_n))\mathrm{D}^i\mathrm{T}_k(u)\,\psi_h(u_n)\,dx.
\end{aligned}\tag{3.65}$$

For the second term on the right-hand side of (3.65), we have $\mathrm{T}_k(u_n)\to\mathrm{T}_k(u)$ in $\mathrm{L}^{p_i}(\Omega)$, then, $a_i(x,\mathrm{T}_k(u_n),\nabla\mathrm{T}_k(u))\to a_i(x,\mathrm{T}_k(u),\nabla\mathrm{T}_k(u))$ strongly in $\mathrm{L}^{p_i'}(\Omega)$, and since $\mathrm{D}^i\mathrm{T}_k(u_n)$ converges to $\mathrm{D}^i\mathrm{T}_k(u)$ weakly in $\mathrm{L}^{p_i}(\Omega)$, we obtain

$$\varepsilon_9(n)=\sum_{i=1}^{N}\int_\Omega a_i(x,\mathrm{T}_k(u_n),\nabla\mathrm{T}_k(u))(\mathrm{D}^i\mathrm{T}_k(u_n)-\mathrm{D}^i\mathrm{T}_k(u))\,dx\longrightarrow 0\qquad\text{as}\quad n\to\infty.\tag{3.66}$$

Concerning the third term on the right-hand side of (3.65), we have $(|a_i(x,\mathrm{T}_{2h}(u_n),\nabla\mathrm{T}_{2h}(u_n))|)_n$ is bounded in $\mathrm{L}^{p_i'}(\Omega)$, then there exists $\nu_i\in\mathrm{L}^{p_i'}(\Omega)$ such that $|a_i(x,\mathrm{T}_{2h}(u_n),\nabla\mathrm{T}_{2h}(u_n))|\rightharpoonup\nu_i$ weakly in $\mathrm{L}^{p_i'}(\Omega)$ for any $i=1,\dots,N$, therefore,

$$\begin{aligned}
\varepsilon_{10}(n)&\le\Big|\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} a_i(x,\mathrm{T}_{2h}(u_n),\nabla\mathrm{T}_{2h}(u_n))\mathrm{D}^i\mathrm{T}_k(u)\,\psi_h(u_n)\,dx\Big|\\
&\le\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}}|a_i(x,\mathrm{T}_{2h}(u_n),\nabla\mathrm{T}_{2h}(u_n))|\,|\mathrm{D}^i\mathrm{T}_k(u)|\,dx\\
&\longrightarrow\sum_{i=1}^{N}\int_{\{k<|u|\le 2h\}}\nu_i\,|\mathrm{D}^i\mathrm{T}_k(u)|\,dx=0\qquad\text{as}\quad n\to\infty.
\end{aligned}\tag{3.67}$$

By combining (3.64) – (3.67), we conclude that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega \Big(a_i(x,\mathrm{T}_k(u_n),\nabla \mathrm{T}_k(u_n))-a_i(x,\mathrm{T}_k(u_n),\nabla \mathrm{T}_k(u))\Big)(\mathrm{D}^i\mathrm{T}_k(u_n)-\mathrm{D}^i\mathrm{T}_k(u))dx\\
&\qquad\qquad\le\varepsilon_{11}(n,h).
\end{aligned}\tag{3.68}$$

Having in mind that $T_k(u_n) \to T_k(u)$ strongly in $L^{p_0}(\Omega)$. Thus, by letting n then h tend to infinity we deduce that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))\,dx\\&+\int_{\Omega}(|T_k(u_n)|^{p_0-2}T_k(u_n)-|T_k(u)|^{p_0-2}T_k(u))(T_k(u_n)-T_k(u))\,dx\longrightarrow 0\quad\text{as}\quad n\to\infty.\end{aligned}\tag{3.69}$$

In view of Lemma 3.2.1, we conclude that

$$\begin{cases}T_k(u_n)\to T_k(u) & \text{strongly in}\quad W_0^{1,\vec{p}}(\Omega),\\ D^iu_n\to D^iu & \text{a.e. in}\quad\Omega\quad\text{for}\quad i=1,\dots,N.\end{cases}\tag{3.70}$$

Moreover, we have $a_i(x,T_n(u_n),\nabla u_n)D^iu_n$ tends to $a_i(x,u,\nabla u)D^iu$ almost everywhere in Ω, and in view of Fatou's lemma and (3.57), we conclude that

$$\begin{aligned}&\lim_{h\to\infty}\sum_{i=1}^{N}\frac{1}{h}\int_{\{h<|u|<2h\}}a_i(x,u,\nabla u)D^iu\,dx\\&\le\lim_{h\to\infty}\liminf_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|<2h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx\\&\le\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|<2h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx=0,\end{aligned}\tag{3.71}$$

which conclude the prove of (3.14). Moreover, we have

$$\lim_{h\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u|\le h\}}b(|u|)|D^iu|^{p_i}\,dx\le\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}}b(|u_n|)|D^iu_n|^{p_i}\,dx=0,\tag{3.72}$$

and

$$\lim_{h\to\infty}\sum_{i=1}^{N}\int_{\{|u|>h\}}\frac{b(|u|)|D^iu|^{p_i}}{(1+|u|)^{1+\varepsilon}}\,dx\le\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{\{|u_n|>h\}}\frac{b(|u_n|)|D^iu_n|^{p_i}}{(1+|u_n|)^{1+\varepsilon}}\,dx=0.\tag{3.73}$$

Step 5 : Passage to the limit

In this step, we will passe to the limit in our approximate problem.
Let $\varphi\in K_\psi\cap L^\infty(\Omega)$ and $M=k+\|\varphi\|_\infty$. By taking $v=u_n-\eta T_k(u_n-\varphi)$ as a test function in the approximate problem (3.26), we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx+\int_{\Omega}g_n(x,u_n)T_k(u_n-\varphi)\,dx\\\le\int_{\Omega}f_n(x)T_k(u_n-\varphi)\,dx.\end{aligned}\tag{3.74}$$

Firstly, we have $\{|u_n-\varphi|\le k\}\subseteq\{|u_n|\le \mathrm{M}\}$, then the first term on the left-hand side of (3.74) can be written as

$$\begin{aligned}
&\sum_{i=1}^{\mathrm{N}}\int_\Omega a_i(x,\mathrm{T}_n(u_n),\nabla u_n)\mathrm{D}^i\mathrm{T}_k(u_n-\varphi)\,dx\\
&=\sum_{i=1}^{\mathrm{N}}\int_\Omega a_i(x,\mathrm{T}_{\mathrm{M}}(u_n),\nabla\mathrm{T}_{\mathrm{M}}(u_n))(\mathrm{D}^i\mathrm{T}_{\mathrm{M}}(u_n)-\mathrm{D}^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\
&=\sum_{i=1}^{\mathrm{N}}\int_\Omega (a_i(x,\mathrm{T}_{\mathrm{M}}(u_n),\nabla\mathrm{T}_{\mathrm{M}}(u_n))-a_i(x,\mathrm{T}_{\mathrm{M}}(u_n),\nabla\varphi))(\mathrm{D}^i\mathrm{T}_{\mathrm{M}}(u_n)-\mathrm{D}^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\
&\quad+\sum_{i=1}^{\mathrm{N}}\int_\Omega a_i(x,\mathrm{T}_{\mathrm{M}}(u_n),\nabla\varphi)(\mathrm{D}^i\mathrm{T}_{\mathrm{M}}(u_n)-\mathrm{D}^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx.
\end{aligned}\tag{3.75}$$

In view of Fatou's Lemma, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty}\sum_{i=1}^{\mathrm{N}}\int_\Omega a_i(x,\mathrm{T}_n(u_n),\nabla u_n)\mathrm{D}^i\mathrm{T}_k(u_n-\varphi)\,dx\\
&\ge\sum_{i=1}^{\mathrm{N}}\int_\Omega (a_i(x,\mathrm{T}_{\mathrm{M}}(u),\nabla\mathrm{T}_{\mathrm{M}}(u))-a_i(x,\mathrm{T}_{\mathrm{M}}(u),\nabla\varphi))(\mathrm{D}^i\mathrm{T}_{\mathrm{M}}(u)-\mathrm{D}^i\varphi)\chi_{\{|u-\varphi|\le k\}}\,dx\\
&\quad+\lim_{n\to\infty}\sum_{i=1}^{\mathrm{N}}\int_\Omega a_i(x,\mathrm{T}_{\mathrm{M}}(u_n),\nabla\varphi)(\mathrm{D}^i\mathrm{T}_{\mathrm{M}}(u_n)-\mathrm{D}^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\
&=\sum_{i=1}^{\mathrm{N}}\int_\Omega a_i(x,\mathrm{T}_{\mathrm{M}}(u),\nabla\mathrm{T}_{\mathrm{M}}(u))(\mathrm{D}^i\mathrm{T}_{\mathrm{M}}(u)-\mathrm{D}^i\varphi)\chi_{\{|u-\varphi|\le k\}}\,dx\\
&=\sum_{i=1}^{\mathrm{N}}\int_\Omega a_i(x,u,\nabla u)\mathrm{D}^i\mathrm{T}_k(u-\varphi)\,dx.
\end{aligned}\tag{3.76}$$

Secondly, we have $\mathrm{T}_k(u_n-\varphi)\rightharpoonup\mathrm{T}_k(u-\varphi)$ weak–$\star$ in $\mathrm{L}^\infty(\Omega)$, Having in mind (3.41) and that $f_n(x)\to f(x)$ in $\mathrm{L}^1(\Omega)$ we conclude that

$$\int_\Omega f_n(x)\mathrm{T}_k(u_n-\varphi)\,dx\longrightarrow\int_\Omega f(x)\mathrm{T}_k(u-\varphi)\,dx,\tag{3.77}$$

and

$$\int_\Omega g_n(x,u_n)\mathrm{T}_k(u_n-\varphi)\,dx\longrightarrow\int_\Omega g(x,u)\mathrm{T}_k(u-\varphi)\,dx.\tag{3.78}$$

Consequently, by combining (3.76) – (3.78) we deduce that

$$\sum_{i=1}^{\mathrm{N}}\int_\Omega a_i(x,u,\nabla u)\mathrm{D}^i\mathrm{T}_k(u-\varphi)\,dx+\int_\Omega g(x,u)\mathrm{T}_k(u-\varphi)\,dx\le\int_\Omega f(x)\mathrm{T}_k(u-\varphi)\,dx,\tag{3.79}$$

for all $\varphi\in\mathrm{K}_\psi\cap\mathrm{L}^\infty(\Omega)$, which completes the proof of the theorem 3.3.1.

3.4 Uniqueness of entropy solution

In this section, taking $1 < p_i \le 2$ for $i = 1, \dots, N$, we will show the uniqueness of solution for our quasilinear elliptic unilateral problem (3.11).

(A1) We consider the Leray-Lions operator $Au = -\sum_{i=1}^{N} D^i(a_i(x,u)|D^i u|^{p_i-2}D^i u)$ where $a_i : \Omega \times \mathbb{R} \longmapsto \mathbb{R}$ are Carathéodory functions which verify the assumption :

$$b(|s|) \le |a_i(x,s)| \le \beta \qquad \text{for} \quad i = 1, \dots, N, \tag{3.80}$$

where $\beta > 0$ and $b(|\cdot|) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a decreasing function that belongs to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ verifying the condition (3.9), moreover, we assume that $0 \le \lambda < \dfrac{1}{\underline{p}}$.

(A2) Let $\varphi(\cdot)$ be a real valued function belonging to $C^1(\mathbb{R})$, such that

$$\varphi(0) = 0 \qquad \text{and} \qquad \varphi'(\cdot) \ge 1, \tag{3.81}$$

and there exists $\theta > \dfrac{1}{2}$ such that

$$|s|^p \frac{\varphi'(s)}{(1+|\varphi(s)|)^{2\theta}} \le L_0 \qquad \text{for any} \qquad s \in \mathbb{R}. \tag{3.82}$$

Let $K_0 > 0$ small enough and $0 < K \le K_0$, we assume that : for any $s, r \ge 1$ such that $|\varphi(s) - \varphi(r)| \le K$, we assume that

$$\begin{aligned} &\frac{1}{L_1} \le \frac{\varphi'(s)}{\varphi'(r)} \le L_1 \quad \text{and} \\ &\Big|\frac{a(x,s)}{\varphi'(s)^{p-1}} - \frac{a(x,r)}{\varphi'(r)^{p-1}}\Big| \le \Big(\frac{b(s)}{\varphi'(s)^{p-1}} + \frac{b(r)}{\varphi'(r)^{p-1}}\Big)\frac{L_2 K}{(1+\varphi(s)+\varphi(r))^\theta}, \end{aligned} \tag{3.83}$$

where L_1 and L_2 two strictly positive constants.

(A3) The lower order term $g(x,s)$ is a Carathéodory function that verifying (3.10), and we assume some additional monotone condition, i.e.

$$(g(x,s) - g(x,r))(s-r) \ge 0 \qquad \text{for any} \quad r, s \in \mathbb{R}. \tag{3.84}$$

We consider the unilateral problem associated to the anisotropic quasilinear elliptic equation

$$\begin{cases} -\sum_{i=1}^{N} D^i(a_i(x,u)|D^i u|^{p_i-2}D^i u) + g(x,u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.85}$$

where $f \in L^1(\Omega)$.

Theorem 3.4.1 *Assuming that the conditions* (A1), (A2) *and* (A3) *hold true, then the problem* (3.85) *has a unique entropy solution.*

Proof of Theorem 3.4.1

The existence of solution : In view of the assumptions (A1), (A2) and (A3), the hypothesis of the Theorem 3.4.1 are verified, thus the problem (3.85) has at least one entropy solution.

The uniqueness of solution : In this part, we assume that there exist two entropy solutions u and v of the problem (3.85), and we will show that $u = v$ a. e. in Ω.

Note that the proof of the uniqueness of solution is based essentially on the technics used by the author in [48], (see also [41, 40, 49]).

Let $0 < \mathrm{K} < \mathrm{K}_0$ and we set $\mathrm{K} < h$. We consider u as an entropy solution of the elliptic problem (3.85), and by taking $\psi = \mathrm{T}_h(u) - \mathrm{T}_\mathrm{K}(\varphi(u) - \varphi(v))$ we have

$$\begin{aligned}
&\sum_{i=1}^{\mathrm{N}} \int_\Omega a_i(x,u)|\mathrm{D}^i u|^{p_i-2}\mathrm{D}^i u \mathrm{D}^i \mathrm{T}_\mathrm{K}\big(u - \mathrm{T}_h(u) + \mathrm{T}_\mathrm{K}(\varphi(u) - \varphi(v))\big)\, dx \\
&\quad + \int_\Omega g(x,u)\mathrm{T}_\mathrm{K}\big(u - \mathrm{T}_h(u) + \mathrm{T}_\mathrm{K}(\varphi(u) - \varphi(v))\big)\, dx \\
&\leq \int_\Omega f \mathrm{T}_\mathrm{K}\big(u - \mathrm{T}_h(u) + \mathrm{T}_\mathrm{K}(\varphi(u) - \varphi(v))\big)\, dx.
\end{aligned} \tag{3.86}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{\mathrm{N}} \int_{\{|u-\psi|\leq \mathrm{K}\}} a_i(x,u)|\mathrm{D}^i u|^{p_i-2}\mathrm{D}^i u \mathrm{D}^i \mathrm{T}_\mathrm{K}(\varphi(u) - \varphi(v))\, dx + \\
&\sum_{i=1}^{\mathrm{N}} \int_{\{|u-\psi|\leq \mathrm{K}\}\cap\{|u|>h\}} a_i(x,u)|\mathrm{D}^i u|^{p_i}\, dx \quad + \int_{\{|u-\psi|\leq \mathrm{K}\}} g(x,u)\mathrm{T}_\mathrm{K}(\varphi(u) - \varphi(v))\, dx \\
&+ \int_{\{|u-\psi|\leq \mathrm{K}\}\cap\{|u|>h\}} g(x,u)(u - \mathrm{T}_h(u))\, dx \quad + \int_{\{|u-\psi|>\mathrm{K}\}} g(x,u)\cdot \mathrm{K}\cdot \mathrm{sign}(u-\psi)\, dx \\
&\leq \int_{\{|u-\psi|\leq \mathrm{K}\}} f \mathrm{T}_\mathrm{K}(\varphi(u) - \varphi(v))\, dx + \int_{\{|u-\psi|\leq \mathrm{K}\}\cap\{|u|>h\}} f(u - \mathrm{T}_h(u))\, dx \\
&\quad + \int_{\{|u-\psi|>\mathrm{K}\}} f\cdot \mathrm{K}\cdot \mathrm{sign}(u-\varphi)\, dx.
\end{aligned} \tag{3.87}$$

Concerning the third and last terms on the left-hand side of (3.87), we have $g(x,u)$ belong to $\mathrm{L}^1(\Omega)$, and since $\{|u-\psi| > \mathrm{K}\} \subseteq \{|u| > h\}$, with $\mathrm{meas}\{|u| > h\} \to 0$ as h tends to infinity,

then

$$\begin{aligned}\varepsilon_0(h) &= \Big| \int_{\{|u-\psi|\le K\}\cap\{|u|>h\}} g(x,u)(u-T_h(u))\,dx + \int_{\{|u-\psi|>K\}} g(x,u)\cdot K\cdot \operatorname{sign}(u-\varphi)\,dx \Big| \\ &\le K\int_{\{h<|u|<h+K\}} |g(x,u)|\,dx + K\int_{\{|u-\psi|>K\}} |g(x,u)|\,dx \longrightarrow 0 \qquad \text{as} \quad h\to 0.\end{aligned} \tag{3.88}$$

Similarly, we have $f\in L^1(\Omega)$ then

$$\begin{aligned}\varepsilon_0(h) &= \Big| \int_{\{|u-\psi|\le K\}\cap\{|u|>h\}} f\cdot(u-T_h(u))\,dx + \int_{\{|u-\psi|>K\}} f\cdot K\cdot \operatorname{sign}(u-\varphi)\,dx \Big| \\ &\le K\int_{\{h<|u|<h+K\}} |f|\,dx + K\int_{\{|u-\psi|>K\}} |f|\,dx \longrightarrow 0 \qquad \text{as} \quad h\to 0.\end{aligned} \tag{3.89}$$

By combining (3.87) – (3.89), we conclude that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|u-\psi|\le K\}} a_i(x,u)|D^i u|^{p_i-2}D^i u D^i T_K(\varphi(u)-\varphi(v))\,dx \\ &\quad+\int_{\{|u-\psi|\le K\}} g(x,u)T_K(\varphi(u)-\varphi(v))\,dx \le \int_{\{|u-\psi|\le K\}} fT_K(\varphi(u)-\varphi(v))\,dx+\varepsilon_2(h).\end{aligned} \tag{3.90}$$

It's clear that the set $\{|u-\psi|\le K\}$ tends to Ω as h goes to infinity, thus we conclude that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} a_i(x,u)|D^i u|^{p_i-2}D^i u D^i T_K(\varphi(u)-\varphi(v))\,dx \\ &\quad+\int_{\Omega} g(x,u)T_K(\varphi(u)-\varphi(v))\,dx \le \int_{\Omega} fT_K(\varphi(u)-\varphi(v))\,dx.\end{aligned} \tag{3.91}$$

Similarly, taking v as an entropy solution of the elliptic problem (3.85), and using $\psi = T_h(v) - T_K(\varphi(v)-\varphi(u))$. By follows the same steps we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} a_i(x,v)|D^i v|^{p_i-2}D^i v D^i T_K(\varphi(v)-\varphi(u))\,dx \\ &\quad+\int_{\Omega} g(x,v)T_K(\varphi(v)-\varphi(u))\,dx \le \int_{\Omega} fT_K(\varphi(v)-\varphi(u))\,dx.\end{aligned} \tag{3.92}$$

By adding the two previous inequality, and in view of the monotonicity of the Carathéodory function $g(x,\cdot)$ we conclude that

$$\sum_{i=1}^{N}\int_{\{0<|\varphi(u)-\varphi(v)|<K\}} \Big(a_i(x,u)|D^i u|^{p_i-2}D^i u - a_i(x,v)|D^i v|^{p_i-2}D^i v\Big)(D^i\varphi(u)-D^i\varphi(v))\,dx \le 0. \tag{3.93}$$

We set $\mathrm{U_K} = \{0 < |\varphi(u) - \varphi(v)| < \mathrm{K}\}$, it's clear that

$$\begin{aligned}
&\sum_{i=1}^{\mathrm{N}} \int_{\mathrm{U_K}} (a_i(x,u)|\mathrm{D}^i u|^{p_i-2}\mathrm{D}^i u - a_i(x,v)|\mathrm{D}^i v|^{p_i-2}\mathrm{D}^i v)(\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v))\,dx \\
&= \frac{1}{2}\sum_{i=1}^{\mathrm{N}} \int_{\mathrm{U_K}} \Big(\frac{a_i(x,u)}{\varphi'(u)^{p_i-1}} + \frac{a_i(x,v)}{\varphi'(v)^{p_i-1}}\Big)(|\mathrm{D}^i\varphi(u)|^{p_i-2}\mathrm{D}^i\varphi(u) - |\mathrm{D}^i\varphi(v)|^{p_i-2}\mathrm{D}^i\varphi(v)) \\
&\qquad\qquad \times(\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v))\,dx \\
&\quad + \frac{1}{2}\sum_{i=1}^{\mathrm{N}} \int_{\mathrm{U_K}} \Big(\frac{a_i(x,u)}{\varphi'(u)^{p_i-1}} - \frac{a_i(x,v)}{\varphi'(v)^{p_i-1}}\Big)(|\mathrm{D}^i\varphi(u)|^{p_i-2}\mathrm{D}^i\varphi(u) + |\mathrm{D}^i\varphi(v)|^{p_i-2}\mathrm{D}^i\varphi(v)) \\
&\qquad\qquad \times(\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v))\,dx \\
&\le 0.
\end{aligned} \tag{3.94}$$

Thanks (3.80) we have

$$\begin{aligned}
\mathrm{A1} \quad &= \frac{1}{2}\sum_{i=1}^{\mathrm{N}} \int_{\mathrm{U_K}} \Big(\frac{a_i(x,u)}{\varphi'(u)^{p_i-1}} + \frac{a_i(x,v)}{\varphi'(v)^{p_i-1}}\Big)(|\mathrm{D}^i\varphi(u)|^{p_i-2}\mathrm{D}^i\varphi(u) - |\mathrm{D}^i\varphi(v)|^{p_i-2}\mathrm{D}^i\varphi(v)) \\
&\qquad\qquad \times(\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v))\,dx \\
&\ge \frac{1}{2}\sum_{i=1}^{\mathrm{N}} \int_{\mathrm{U_K}} \Big(\frac{b(u)}{\varphi'(u)^{p_i-1}} + \frac{b(v)}{\varphi'(v)^{p_i-1}}\Big)\frac{|\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v)|^2}{(|\mathrm{D}^i\varphi(u) + |\mathrm{D}^i\varphi(v)|)^{2-p_i}}\,dx.
\end{aligned} \tag{3.95}$$

By using (3.83) and Young's inequality, we have

$$
\begin{aligned}
|\mathrm{A2}| = \frac{1}{2}\Big| \sum_{i=1}^{\mathrm{N}} \int_{\mathrm{U_K}} \Big(\frac{a_i(x,u)}{\varphi'(u)^{p_i-1}} - \frac{a_i(x,v)}{\varphi'(v)^{p_i-1}}\Big)\big(|\mathrm{D}^i\varphi(u)|^{p_i-2}\mathrm{D}^i\varphi(u) + |\mathrm{D}^i\varphi(v)|^{p_i-2}\mathrm{D}^i\varphi(v)\big) \\
\cdot(\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v))\,dx\Big| \\
\leq\ & \frac{\mathrm{L_2K}}{2}\sum_{i=1}^{\mathrm{N}}\int_{\mathrm{U_K}}\Big(\frac{b(u)}{\varphi'(u)^{p_i-1}} + \frac{b(v)}{\varphi'(v)^{p_i-1}}\Big)\frac{(|\mathrm{D}^i\varphi(u)|^{p_i-1} + |\mathrm{D}^i\varphi(v)|^{p_i-1})|\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v)|}{(1+|\varphi(u)|+|\varphi(v)|)^{\theta}}\,dx \\
\leq\ & \frac{1}{4}\sum_{i=1}^{\mathrm{N}}\int_{\mathrm{U_K}}\Big(\frac{b(u)}{\varphi'(u)^{p_i-1}} + \frac{b(v)}{\varphi'(v)^{p_i-1}}\Big)\frac{|\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v)|^2}{(|\mathrm{D}^i\varphi(u)| + |\mathrm{D}^i\varphi(v)|)^{2-p_i}}\,dx \\
& + \frac{\mathrm{L_2^2K^2}}{4}\sum_{i=1}^{\mathrm{N}}\int_{\mathrm{U_K}}\Big(\frac{b(u)}{\varphi'(u)^{p_i-1}} + \frac{b(v)}{\varphi'(v)^{p_i-1}}\Big) \\
& \qquad \times \frac{(|\mathrm{D}^i\varphi(u)|^{p_i-1} + |\mathrm{D}^i\varphi(v)|^{p_i-1})^2(|\mathrm{D}^i\varphi(u)| + |\mathrm{D}^i\varphi(v)|)^{2-p_i}}{(1+|\varphi(u)|+|\varphi(v)|)^{2\theta}}\,dx \\
\leq\ & \frac{1}{4}\sum_{i=1}^{\mathrm{N}}\int_{\mathrm{U_K}}\Big(\frac{b(u)}{\varphi'(u)^{p_i-1}} + \frac{b(v)}{\varphi'(v)^{p_i-1}}\Big)\frac{|\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v)|^2}{(|\mathrm{D}^i\varphi(u)| + |\mathrm{D}^i\varphi(v)|)^{2-p_i}}\,dx \\
& + \frac{\mathrm{C_0L_2^2K^2}}{4}\sum_{i=1}^{\mathrm{N}}\int_{\mathrm{U_K}}\Big(\frac{b(u)}{\varphi'(u)^{p_i-1}} + \frac{b(v)}{\varphi'(v)^{p_i-1}}\Big)\frac{|\mathrm{D}^i\varphi(u)|^{p_i} + |\mathrm{D}^i\varphi(v)|^{p_i}}{(1+|\varphi(u)|+|\varphi(v)|)^{2\theta}}\,dx.
\end{aligned}
\tag{3.96}
$$

Since $\mathrm{A1} \leq |\mathrm{A2}|$, it follows that

$$
\begin{aligned}
& \frac{1}{2}\sum_{i=1}^{\mathrm{N}}\int_{\mathrm{U_K}}\Big(\frac{b(u)}{\varphi'(u)^{p_i-1}} + \frac{b(v)}{\varphi'(v)^{p_i-1}}\Big)\frac{|\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v)|^2}{(|\mathrm{D}^i\varphi(u)| + |\mathrm{D}^i\varphi(v)|)^{2-p_i}}\,dx \\
\leq\ & \frac{1}{4}\sum_{i=1}^{\mathrm{N}}\int_{\mathrm{U_K}}\Big(\frac{b(u)}{\varphi'(u)^{p_i-1}} + \frac{b(v)}{\varphi'(v)^{p_i-1}}\Big)\frac{|\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v)|^2}{(|\mathrm{D}^i\varphi(u)| + |\mathrm{D}^i\varphi(v)|)^{2-p_i}}\,dx \\
& + \frac{\mathrm{C_0L_2^2K^2}}{4}\sum_{i=1}^{\mathrm{N}}\int_{\mathrm{U_K}}\Big(\frac{b(u)}{\varphi'(u)^{p_i-1}} + \frac{b(v)}{\varphi'(v)^{p_i-1}}\Big)\frac{|\mathrm{D}^i\varphi(u)|^{p_i} + |\mathrm{D}^i\varphi(v)|^{p_i}}{(1+|\varphi(u)|+|\varphi(v)|)^{2\theta}}\,dx.
\end{aligned}
\tag{3.97}
$$

Thanks to (3.83), we conclude that

$$
\begin{aligned}
& \frac{1}{4\mathrm{K}^2}\sum_{i=1}^{\mathrm{N}}\int_{\mathrm{U_K}}\Big(\frac{b(u)}{\varphi'(u)^{p_i-1}} + \frac{b(v)}{\varphi'(v)^{p_i-1}}\Big)\frac{|\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v)|^2}{(|\mathrm{D}^i\varphi(u) + |\mathrm{D}^i\varphi(v)|)^{2-p_i}}\,dx \\
\leq\ & \frac{\mathrm{C_0L_2^2}}{4}\sum_{i=1}^{\mathrm{N}}\int_{\mathrm{U_K}}\Big(\frac{b(u)}{\varphi'(u)^{p_i-1}} + \frac{b(v)}{\varphi'(v)^{p_i-1}}\Big)\frac{|\mathrm{D}^i\varphi(u)|^{p_i} + |\mathrm{D}^i\varphi(v)|^{p_i}}{(1+|\varphi(u)|+|\varphi(v)|)^{2\theta}}\,dx \\
\leq\ & \frac{\mathrm{C_1L_2^2}}{4}\big(1 + \|b(\cdot)\|_{\infty}\mathrm{L_1^{p_i-1}}\big)\sum_{i=1}^{\mathrm{N}}\int_{\mathrm{U_K}}\frac{b(u)|\mathrm{D}^i\varphi(u)|^{p_i}}{\varphi'(u)^{p_i-1}(1+|\varphi(u)|)^{2\theta}}\,dx \\
& + \frac{\mathrm{C_1L_2^2}}{4}\big(1 + \|b(\cdot)\|_{\infty}\mathrm{L_1^{p_i-1}}\big)\sum_{i=1}^{\mathrm{N}}\int_{\mathrm{U_K}}\frac{b(v)|\mathrm{D}^i\varphi(v)|^{p_i}}{\varphi'(v)^{p_i-1}(1+|\varphi(v)|)^{2\theta}}\,dx.
\end{aligned}
\tag{3.98}
$$

Since $2\theta > 1$ and $\varphi'(\cdot) \geq 1$, having in mind (3.73), we conclude that

$$\frac{b(u)|\mathrm{D}^i\varphi(u)|^{p_i}}{\varphi'(u)^{p_i-1}(1+|\varphi(u)|)^{2\theta}} \in \mathrm{L}^1(\Omega) \qquad \text{and} \qquad \frac{b(v)|\mathrm{D}^i\varphi(v)|^{p_i}}{\varphi'(v)^{p_i-1}(1+|\varphi(v)|)^{2\theta}} \in \mathrm{L}^1(\Omega). \tag{3.99}$$

Therefore, by letting K goes to 0, the $\mathrm{meas}\mathrm{U}_\mathrm{K}$ tends to zero, it follows that

$$\lim_{\mathrm{K}\to 0} \frac{1}{4\mathrm{K}^2} \sum_{i=1}^{\mathrm{N}} \int_{\mathrm{U}_\mathrm{K}} \Big(\frac{b(u)}{\varphi'(u)^{p_i-1}} + \frac{b(v)}{\varphi'(v)^{p_i-1}}\Big) \frac{|\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v)|^2}{(|\mathrm{D}^i\varphi(u) + |\mathrm{D}^i\varphi(v)|)^{2-p_i}}\, dx = 0. \tag{3.100}$$

Now, we set $\mathrm{P}_h(s) = 1 - \dfrac{|\mathrm{T}_{2h}(s) - \mathrm{T}_h(s)|}{h}$ and $\mathrm{B}(s) = \displaystyle\int_0^s b(\tau)\, d\tau$. Thus, in view of Poincaré inequality, we conclude that

$$\begin{aligned}
&\int_\Omega \frac{\mathrm{P}_h(\mathrm{B}(u))|\mathrm{T}_\mathrm{K}(\varphi(u) - \varphi(v))|}{\mathrm{K}}\, dx \leq \mathrm{C}_p \sum_{i=1}^{\mathrm{N}} \int_\Omega \Big|\mathrm{D}^i\Big(\frac{\mathrm{P}_h(\mathrm{B}(u))|\mathrm{T}_\mathrm{K}(\varphi(u) - \varphi(v))|}{\mathrm{K}}\Big)\Big|\, dx \\
&\leq \mathrm{C}_p \sum_{i=1}^{\mathrm{N}} \int_\Omega \mathrm{P}_h(\mathrm{B}(u)) \frac{|\mathrm{D}^i\mathrm{T}_\mathrm{K}(\varphi(u) - \varphi(v))|}{\mathrm{K}}\, dx \\
&+ \frac{\mathrm{C}_p}{h} \int_{\{h\leq|\mathrm{B}(u)|\leq 2h\}} b(|u|)|\mathrm{D}^i u| \frac{|\mathrm{T}_\mathrm{K}(\varphi(u) - \varphi(v))|}{\mathrm{K}}\, dx \\
&\leq \frac{\mathrm{C}_p}{\mathrm{K}} \sum_{i=1}^{\mathrm{N}} \int_{\mathrm{U}_\mathrm{K}\cap\{|\mathrm{B}(u)|\leq 2h\}} |\mathrm{D}^i\varphi(u) - \mathrm{D}^i\varphi(v)|\, dx + \frac{\mathrm{C}_p}{h} \sum_{i=1}^{\mathrm{N}} \int_{\{h\leq|\mathrm{B}(u)|\leq 2h\}} b(|u|)|\mathrm{D}^i u|\, dx.
\end{aligned} \tag{3.101}$$

For the first term on the right-hand side of (3.101), it's clear that

$$b_0((1+|u|)^{1-\lambda} - 1) \leq \mathrm{B}(|u|) \leq \|b(\cdot)\|_\infty |s|,$$

it follows that

$$\{h \leq |\mathrm{B}(u)| \leq 2h\} \subseteq \Big\{|u| \leq \mathrm{C}_1 h^{\frac{1}{1-\lambda}}\Big\},$$

with $\mathrm{C}_0 = \dfrac{1}{\|b(\cdot)\|_\infty}$ and $\mathrm{C}_1 = 2\Big(\dfrac{2}{b_0}\Big)^{\frac{1}{1-\lambda}}$ for h large enough.

Let $0 < \varepsilon \leq \underline{p}(1-\lambda) - 1$ thus $\dfrac{1}{1-\lambda} \cdot \dfrac{1+\varepsilon}{p_i} \leq 1$, it follows that $\dfrac{(1+|u|)^{\frac{1+\varepsilon}{p_i}}}{h}$ is bounded on the

set $\{|u| \le C_1 h^{\frac{1}{1-\lambda}}\}$, and in view of (3.73) we conclude that

$$\begin{aligned}
&\frac{C_p}{h}\int_{\{h\le|B(u)|\le 2h\}} b(|u|)|D^i u|\frac{|T_K(\varphi(u)-\varphi(v))|}{K}\,dx\\
&\le \frac{C_p}{h}\sum_{i=1}^{N}\int_{\{h\le|B(u)|\le 2h\}} b(|u|)|D^i u|\,dx\\
&\le \frac{C_p}{h}\sum_{i=1}^{N}\int_{\{C_0 h\le|u|\le C_1 h^{\frac{1}{1-\lambda}}\}} \frac{b(|u|)|D^i u|}{(1+|u_n|)^{\frac{1+\varepsilon}{p_i}}}(1+|u_n|)^{\frac{1+\varepsilon}{p_i}}\,dx\\
&\le \frac{C_p}{h}\sum_{i=1}^{N}\Big\|\frac{b(|u|)^{\frac{1}{p_i}}|D^i u|}{(1+|u_n|)^{\frac{1+\varepsilon}{p_i}}}\,dx\Big\|_{L^{p_i}(\{C_0 h\le|u|\})}\cdot\Big\|b(|u|)^{\frac{1}{p_i'}}(1+|u_n|)^{\frac{1+\varepsilon}{p_i}}\Big\|_{L^{p_i}(\{|u|\le C_1 h^{\frac{1}{1-\lambda}}\})}\\
&\le C_p\sum_{i=1}^{N}\|b(\cdot)\|^{\frac{1}{p_i'}}_{L^\infty(\mathbb{R})}\Big(\int_{\{C_0 h\le|u|\}}\frac{b(|u|)|D^i u|^{p_i}}{(1+|u|)^{1+\varepsilon}}\,dx\Big)^{\frac{1}{p_i}}\cdot\Big\|\frac{(1+|u|)^{\frac{1+\varepsilon}{p_i}}}{h}\Big\|_{L^{p_i'}(\{|u|\le C_1 h^{\frac{1}{1-\lambda}}\})}.
\end{aligned}$$

It follows that

$$\lim_{h\to\infty}\limsup_{K\to 0}\frac{C_p}{h}\int_{\{h\le|B(u)|\le 2h\}} b(|u|)|D^i u|\frac{|T_K(\varphi(u)-\varphi(v))|}{K}\,dx=0. \tag{3.102}$$

Concerning the second term on the right-hand side of (3.101), we have

$$\begin{aligned}
&\frac{C_p}{K}\sum_{i=1}^{N}\int_{U_K\cap\{|B(u)|\le 2h\}} P_h(B(u))|D^i\varphi(u)-D^i\varphi(v)|\,dx\\
&\le C_p\sum_{i=1}^{N}\Big(\frac{1}{K^2}\int_{U_K\cap\{|u|\le C_1 h^{\frac{1}{1-\lambda}}\}}\frac{b(u)}{\varphi'(u)^{p_i-1}}\frac{|D^i\varphi(u)-D^i\varphi(v)|^2}{(|D^i\varphi(u)|+|D^i\varphi(v)|)^{2-p_i}}\,dx\Big)^{\frac{1}{2}}\\
&\qquad\times\Big(\int_{U_K\cap\{|u|\le C_1 h^{\frac{1}{1-\lambda}}\}}\frac{\varphi'(u)^{p_i-1}}{b(u)}(|D^i\varphi(u)|+|D^i\varphi(v)|)^{2-p_i}\,dx\Big)^{\frac{1}{2}}\\
&\le C_p\sum_{i=1}^{N}\Big(\frac{1}{K^2}\int_{U_K\cap\{|u|\le C_1 h^{\frac{1}{1-\lambda}}\}}\frac{b(u)}{\varphi'(u)^{p_i-1}}\frac{|D^i\varphi(u)-D^i\varphi(v)|^2}{(|D^i\varphi(u)|+|D^i\varphi(v)|)^{2-p_i}}\,dx\Big)^{\frac{1}{2}}\\
&\qquad\times\Big(\int_{\{|u|\le C_1 h^{\frac{1}{1-\lambda}}\}\cap\{|v|\le C_2 h^{\frac{1}{1-\lambda}}\}}\frac{\varphi'(u)^{p_i-1}}{b_0}(1+|u|)^{\lambda}(|D^i\varphi(u)|+|D^i\varphi(v)|)^{2-p_i}\,dx\Big)^{\frac{1}{2}}\\
&.\quad\le C(h,u,v)\sum_{i=1}^{N}\Big(\frac{1}{K^2}\int_{U_K\cap\{|u|\le C_1 h^{\frac{1}{1-\lambda}}\}}\frac{b(u)}{\varphi'(u)^{p_i-1}}\frac{|D^i\varphi(u)-D^i\varphi(v)|^2}{(|D^i\varphi(u)|+|D^i\varphi(v)|)^{2-p_i}}\,dx\Big)^{\frac{1}{2}}.
\end{aligned}$$

where $C(h,u,v)$ is a positive constant which depends only on u, v, $\varphi(\cdot)$ and h. Thus, in view of (3.100) we conclude that

$$\lim_{K\to 0}\frac{C_p}{K}\sum_{i=1}^{N}\int_{U_K\cap\{|B(u)|\le 2h\}} P_h(B(u))|D^i\varphi(u)-D^i\varphi(v)|\,dx=0. \tag{3.103}$$

Having in mind that $P_h(B(u))\longrightarrow 1$ almost everywhere in Ω and weak$-*$ in $L^\infty(\Omega)$ as h tends to infinity. By combining (3.101) – (3.103), we obtain

$$\int_{\{\varphi(u)\ne\varphi(v)\}} 1\,dx=\lim_{h\to\infty}\lim_{K\to 0}\int_\Omega\frac{P_h(B(u))|T_K(\varphi(u)-\varphi(v))|}{K}\,dx=0. \tag{3.104}$$

We conclude that $\varphi(u) = \varphi(v)$ almost everywhere in Ω, and since $\varphi(\cdot)$ is strictly increasing function we conclude that $u = v$ almost everywhere in Ω.

Part II

Study of Degenerate Elliptic Equations with L^1 Coefficient

Chapter 4

Prelemineries of part 2

4.1 Sobolev Spaces with Variable Exponent and Useful Lemmas

4.1.1 Lebesgue and Sobolev Spaces with Variable Exponent

In this section, we define at first Lebesgue and Sobolev spaces with variable exponent and recall some of their properties. Let Ω be an open bounded set in $\mathbb{R}^N$, $N \geq 2$. The function $p(\cdot)$ satisfies the log-Hölder continuity on Ω if

$$|p(y)-p(x)| \leq \frac{C}{|log|y-x||}, \text{ for all } x,\ y \in \overline{\Omega} \text{ such that } |y-x| \leq \frac{1}{2}, \tag{4.1}$$

with C being a positive constant.

We denote

$$C^+(\overline{\Omega}) = \{\text{log-Hölder continuous function } p : \overline{\Omega} \to \mathbb{R} \text{ with } 1 < p^- \leq p^+ < N\},$$

where

$$p^+ = \max_{x \in \overline{\Omega}} p(x) \quad \text{and} \quad p^- = \min_{x \in \overline{\Omega}} p(x).$$

The variable exponent Lebesgue space is defined as

$$L^{p(\cdot)}(\Omega) = \{u : u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)} dx < \infty\}.$$

We can introduce the norm on $L^{p(\cdot)}(\Omega)$ by

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \quad \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\}.$$

The variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many res-

pects : they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p_- < p_+ < \infty$ and continuous functions are dense in $L^{p(\cdot)}(\Omega)$ if $p_+ < \infty$, see (K*ováčik* and R*ákosník* [73]).

We denote by $L^{p'(.)}(\Omega)$ the dual space of $L^{p(\cdot)}(\Omega)$ where $1/p(\cdot)+1/p'(.) = 1$ (see [68],[60]). For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$, the generalized Hölder inequality

$$\left| \int_\Omega uv \right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) \| u \|_{p(x)} \| v \|_{p'(x)},$$

holds true.

Proposition 4.1.1 *(see[45],[85])*
if we denote

$$\rho(u) = \int_\Omega | u |^{p(x)} \, dx, \forall u \in L^{p(\cdot)}(\Omega),$$

then the following assertions holds :
(i) $\| u \|_{p(x)} < 1 \ \ (resp, = 1, > 1) \Leftrightarrow \rho(u) < 1 \ \ (resp, = 1, > 1)$,
(ii)
$$\begin{cases} \| u \|_{p(x)} > 1 \Rightarrow \| u \|^{p_-}_{p(x)} \leq \rho(u) \leq \| u \|^{p_+}_{p(x)}, \\ \| u \|_{p(x)} < 1 \Rightarrow \| u \|^{p_+}_{p(x)} \leq \rho(u) \leq \| u \|^{p_-}_{p(x)} \end{cases}$$
(iii) $\| u \|_{p(x)} \to 0 \ \Leftrightarrow \ \rho(u) \to 0; \| u \|_{p(x)} \to \infty \ \Leftrightarrow \ \rho(u) \to \infty.$
We define the variable Sobolev space by

$$W^{1.p(\cdot)}(\Omega) = \{ u \in L^{p(\cdot)}(\Omega) \ \ and \ \ | \nabla u | \in L^{p(\cdot)}(\Omega) \}.$$

normed by,

$$\| u \|_{1.p(\cdot)} = \| u \|_{p(\cdot)} + \| \nabla u \|_{p(\cdot)} \quad \forall u \in W^{1.p(\cdot)}(\Omega). \tag{4.2}$$

We denote by $W_0^{1.p(\cdot)}(\Omega)$ *the closure of* $\mathscr{C}_0^\infty(\Omega)$ *in* $W^{1.p(\cdot)}(\Omega)$ *and*

$$p^*(.) = \begin{cases} \dfrac{Np(\cdot)}{N - p(\cdot)} \ \ for \ \ p(\cdot) < N, \\ \infty \ \ for \ \ p(\cdot) \geq N. \end{cases}$$

Proposition 4.1.2 *(see [45])*
(i) Assuming $p_- > 1$, *the spaces* $W^{1.p(\cdot)}(\Omega)$ *and* $W_0^{1.p(\cdot)}(\Omega)$ *are separable and reflexive Banach spaces.*
(ii) if $q \in \mathscr{C}_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *then*

$$W_0^{1.p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p(\cdot)}(\Omega) \tag{4.3}$$

is compact and continuous (for more details we refer to Theorem 8.4.2 [42]).

(iii) Let $p \in \mathscr{C}_+(\overline{\Omega})$. Then, for $u \in W_0^{1.p(\cdot)}(\Omega)$, the $p(\cdot)-$Poincaré inequality

$$\| u \|_{p(\cdot)} \leq C \| \nabla u \|_{p(\cdot)} \text{ holds,}$$

where the positive constant C depends on $p(\cdot)$ and Ω.

Remark 4.1.1 By (iii) of Proposition 2.2, we know that $\| \nabla u \|_{p(\cdot)}$ and $\| u \|_{p(\cdot)}$ are equivalent norms on $W_0^{1.p(\cdot)}(\Omega)$.

4.1.2 Some technical Lemmas

Lemma 4.1.1 *[85] Let $q \in \mathscr{C}_+(\overline{\Omega})$, $g \in L^{q(\cdot)}(\Omega)$ and $(g_n)_n \in L^{q(\cdot)}(\Omega)$ with $\| g_n \|_{q(\cdot)} \leq C$, if $g_n(x) \to g(x)$ a.e in Ω , then $g_n(x) \to g(x)$ in $L^{q(x)}(\Omega)$, where C is a positive constant.*

Lemma 4.1.2 *[85] Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipshitz function, with $F(0) = 0$ and $p \in \mathscr{C}_+(\overline{\Omega})$.*
if $u \in W_0^{1.p(\cdot)}(\Omega)$, then $F(u) \in W_0^{1.p(\cdot)}(\Omega)$. Moreover, if the set D of discontinuity point of F' in finite, then,

$$\frac{\partial (F \circ u)}{\partial x_i} = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & \text{for a.e. in} \quad \{x \in \Omega \backslash u(x) \notin D\}, \\ 0 & \text{for a.e. in} \quad \{x \in \Omega \backslash u(x) \in D\}, \end{cases}$$

Lemma 4.1.3 *[45] Let $u \in W_0^{1.p(\cdot)}(\Omega)$, then, we have $T_k(u) \in W_0^{1.p(\cdot)}(\Omega)$, with $k > 0$. Moreover, we have also $T_k(u) \to u \in W_0^{1.p(\cdot)}(\Omega)$ when $k \to \infty$.*

Lemma 4.1.4 *[12] Let $(u_n)_n$ be a sequence in $W_0^{1.p(\cdot)}(\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1.p(\cdot)}(\Omega)$. then, $T_k(u_n) \rightharpoonup T_k(u)$ in $W_0^{1.p(\cdot)}(\Omega)$.*

4.2 Properties of the relative rearrangement

Let Ω be a bounded open subset of $\mathbb{R}^N$, and let u be a measurable function of from Ω into $\mathbb{R}$, we denote by $|\Omega|$ the Lebesgue measure of Ω and $|E|$ denote more generally the Lebesgue measure of a set E. We also write $|u > t|$ for the Lebesgue measure of $\{x \in \Omega,\ u(x) > t\}$, the definition of $|u \geq t|$, $|u = t|$, are similar. We recall that the distribution function of u the real function

$$t \in \mathbb{R} \longrightarrow \mu(t) = |u > t|,$$

and the decreasing rearrangement of u, denoted by u_*, is the generalized inverse function of μ defined on Ω^*, where $\Omega^* = (0.|\Omega|)$ as follows

$$u_*(s) = \inf\{v \in \mathbb{R},\ |u > t| \leq s\} \quad \forall s \in [0.|\Omega|].$$

We recall that the functions u and u_* are equimeasurable, i.e

$$|u > t| = |u_* > t|; \quad |u \geq t| = |u_* \geq t|; \ \forall t \in \mathbb{R}$$

which implies that for any Borel function f, $f \geq 0$

$$\int_\Omega f(u(x))dx = \int_0^{|\Omega|} f(u_*(s))ds, \tag{4.4}$$

and if $\mathrm{E} \subset \Omega$ a measurable subset, then

$$\int_\mathrm{E} u(x)dx \leq \int_0^{|\mathrm{E}|} u_*(s)ds. \tag{4.5}$$

In particular $f(r) = |r|$ $(1 \leq p < \infty)$, we have

$$\|u\|_{\mathrm{L}^p(\Omega)} = \|u_*\|_{\mathrm{L}^p(\Omega^*)} \ \forall p,\ 1 \leq p \leq \infty$$

and the mapping

$$u \in \mathrm{L}^p(\Omega) \to u_* \in \mathrm{L}^p(\Omega^*)$$

is well defined from $\mathrm{L}^p(\Omega)$ into $\mathrm{L}^p(\Omega^*)$

Definition 4.2.1 *(see[54]) Let now $v \in \mathrm{L}^p(\Omega)$ and u a measurable function; we define a function w in Ω^* by*

$$w(s) = \begin{cases} \displaystyle\int_{u>u_*(s)} v(x)dx \ \text{if} |u = u_*(s)| = 0, \\ \displaystyle\int_{u>u_*(s)} v(x)dx + \int_0^{s-|u>u_*(s)|} (v/\mathrm{P}(s))_*(t)dt \ \text{if not,} \end{cases}$$

where $(v/\mathrm{P}(s))_*$ is the decreasing rearrangement of the restriction of v to $\mathrm{P}(s) = \{u = u_*(s)\}$. The function $\dfrac{dw}{ds}$ is called the rearrangement of v with respect to u and is denoted as v_{*u}.

Lemma 4.2.1 *(see[65]) Let $v \in \mathrm{L}^p(\Omega)$ and u a measurable function from Ω to $\mathbb{R}$, then*

1. $w \in \mathrm{W}^{1.p}(\Omega^*)$.
2. $\|\dfrac{dw}{ds}\|_{\Omega^*} \leq \|v\|_\Omega$.
3. *if $v \geq 0$ a.e. in Ω then $\dfrac{dw}{ds} \geq 0$ a.e. in Ω.*

Lemma 4.2.2 *(see[76]) Let Ω be a bounded open set of $\mathbb{R}^N$ and let us assume that $u \in W_0^{1.1}(\Omega)$; $u > 0$ and $v \in L^1(\Omega)$. Then, for almost every $t \in \mathbb{R}$,*

$$N C_N^{\frac{1}{N}} \mu_u(t)^{1-\frac{1}{N}} \leq -\frac{d}{dt}\int_{u>t} |\nabla u| \, dx \leq (\mu'_u(t))^{\frac{1}{p}} (-\frac{d}{dt}\int_{u>t} |\nabla u|^p \, dx)^{\frac{1}{p}}, \tag{4.6}$$

where C_N denotes the measure of the unit ball in $\mathbb{R}^N$.

Lemma 4.2.3 *(see[76]) If $f_1 \geq 0$ is locally integrable function on Ω^*, then for every s, s', $0 \leq s \leq s' \leq |\Omega|$,*

$$\int_{u_*(s')}^{u_*(s)} f_1(\mu(\theta))(-\mu'(\theta)) \, d\theta \leq \int_s^{s'} f_1(\sigma) \, d\sigma.$$

Definition 4.2.2 *For all $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ can be defined by*

$$T_k(s) = \begin{cases} s & if \ |s| \leq k, \\ k.sign(s) & if \ |s| > k. \end{cases}$$

Let $S_{t,h}(s)$ be a real function defined for $t > 0$; $h > 0$ by

$$S_{t,h}(s) = \begin{cases} 1 & if \ \ s > t + h, \\ \dfrac{s-t}{h} & if \ \ t \leq s \leq t + h, \\ 0 & if \ \ |s| \leq t, \\ \dfrac{s+t}{h} & if \ \ -h - t \leq s \leq -t, \\ -1 & if \ \ s \leq -t - h. \end{cases}$$

Chapter 5

Existence of solution for degenerate elliptic equations with $\mathrm{L}^1(\Omega)$ coefficients

In this chapitre, we study the existence solutions for the nonlinear degenerate elliptic problem with $\mathrm{L}^1(\Omega)$ coefficients

$$-\operatorname{div} a(x,u,\nabla u) = f \text{ in } \Omega,$$

where $a(x,u,\nabla u)$ is allowed to be degenerate with the unknown u, and the data f are assumed to be in $\mathrm{L}^m(\Omega)$, $m \geq 1$.

5.1 Introduction

Consider the following nonlinear degenerate elliptic problem :

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is a bounded open subset of $\mathbb{R}^{\mathrm{N}}$ ($\mathrm{N} \geq 2$).
Moreover we assume that $a : \Omega \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \to \mathbb{R}^{\mathrm{N}}$ is a Carathéodory vector function satisfying the following structural assumption : there exists a continuous function $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$, such that $g(0) = 0$ and $g(s) > 0$ if $s > 0$ and $a(x,s,\xi)\xi \geq \alpha(x) g(|s|)|\xi|^p$, $\forall s \in \mathbb{R}$, a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^{\mathrm{N}}$, where the function $\alpha(x)$ satisfies, for $\lambda > 0$, $\alpha \in \mathrm{L}^1(\Omega)$ and $\alpha(x) \geq \lambda$. So problem(5.1) is degenerate on the subset $\{x \in \Omega,\ \ u(x) = 0\}$. The main objective of this study is to show that, despite the presence of the singular coefficient $\alpha(x)$, the proplem5.1 admits a bounded weak solution for all $f \in \mathrm{L}^m(\Omega), m > \max\{\frac{\mathrm{N}}{p}, 1\}$, and if $m = 1$ we always prove that there is a solution of (5.1) in the renormalized sense (see definition5.3.2).

A model example of this problem

$$\begin{cases} -\mathrm{div}\,(\alpha(x)|\nabla u|^{p-2}\nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ ($N > 2$). The linear version of problem 5.2 ($p = 2$), with general $L^1(\Omega)$ coefficients, has been addressed for the first time by Trudinger in [81], using duality method between weighted Sobolev Spaces. For more recently results, in the linear case, is considered in [21] without the weighted framework but with stronger assumptions on the summability of $\alpha(x)$.

In the case $\alpha(.) \equiv 1$, there are many results concerning the existence of solutions to problem5.1. Without the aim to be complete, we only mention the works of [20, 46, 72, 76], where the existence of bounded weak solutions and renormalized solutions is proved.

Concerning the case that $\alpha = \{\alpha_i(x),\ 0 \le i \le N\}$ be a vector of weight functions (i.e., every component $\alpha_i(x)$ is a measurable function which is strictly positive a.e. in Ω) satisfying some integrability conditions and $f \in W^{-1,p'}(\Omega, \alpha^*)$, where $\alpha^* = \{\alpha_i^* = \alpha_i^{1-p'},\ i = 0...N\}$. The authors in [6] proved the existence result of solutions for problem5.1 by using the L^1-version of Minty's lemma.

The main difficulties in dealing with this problem are in the first the non linearity of the operator and the singularity of the coefficient $\alpha(x)$, plus we can't expect that ∇u belongs to $L^p(\Omega)$, due to the fact that the main operator is degenerate for the subset $\{x \in \Omega,\ \ u(x) = 0\}$ (see [20]).
Secondly, we want to extend our results to the case of $f \in L^1(\Omega)$. Since no growth condition is required for b (see (5.4)), it is not obvious that the term $-\mathrm{div}a(x, u, \nabla u)$ makes sense even as a distribution.
We stress that in our treatment we avoid the use of weighted Sobolev spaces.

5.2 Assumptions and useful lemmas

Lets Ω is a bounded open subset of $\mathbb{R}^N$ ($N \ge 2$).
We assume that $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector function satisfying : there exists a continuous function $g :\ \mathbb{R}^+ \mapsto \mathbb{R}^+$, such that $g(0) = 0$ and $g(s) > 0$ if $s > 0$ and

$$a(x, s, \xi)\xi \ge \alpha(x) g(|s|)|\xi|^p,\ \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N. \tag{5.3}$$

where $\int_0^{+\infty} g(\tau)^{\frac{1}{p-1}} d\tau = +\infty$,
and the function $\alpha(x)$ satisfies, for $\lambda > 0$ $\alpha \in L^1(\Omega)$ and $\alpha(x) \ge \lambda$

There is a Carathéodory vector function $\overline{a}$ such that for a.e.$x \in \Omega,\ \forall \xi,\ \xi' \in \mathbb{R}^N$: (5.4)

1. $a(x,s,\xi) = g(|s|)\overline{a}(x,s,\xi)$,
2. $[\overline{a}(x,s,\xi) - \overline{a}(x,s,\xi')][\xi - \xi'] > 0, \ \xi \neq \xi'$,
3. There is an increasing function $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, and $\gamma > 0$ such that : $|\overline{a}(x,s,\xi)| \leq \gamma\alpha(x)(1 + b(|s|)|\xi|^{p-1}$.
4. The function $\overline{a}$ is a positively homogeneous of degree $(p-1)$ with respect to the variable ξ, i.e. $\overline{a}(x,s,t\xi) = t^{p-1}\overline{a}(x,s,\xi) \ \forall t \geq 0$,

$$f \in \mathrm{L}^m(\Omega) \text{ with } m \geq 1. \tag{5.5}$$

Notations and definitions
We define the spaces of functions

$$\mathrm{X}_0^p(\Omega) = \{\varphi \in \mathrm{W}_0^{1,p}(\Omega) \text{ such that } \int_\Omega \alpha(x)|\nabla\varphi|^p < \infty\}, \tag{5.6}$$

and

$$\mathrm{V}_0 = \{v \text{ is measurable function such that } \eta(v) \in \mathrm{W}_0^{1,p}(\Omega)\}, \tag{5.7}$$

where the function η is defined by

$$\eta(s) = \int_0^s g^{\frac{1}{p-1}}(|\tau|)d\tau. \tag{5.8}$$

Then, for $u \in \mathrm{V}_0$, we can define the field ∇u on Ω through the formula

$$\nabla u = \begin{cases} \dfrac{\nabla\eta(u)}{g^{\frac{1}{p-1}}(|u|)} & if \ \ u \neq 0, \\ 0 \quad if \ \ u = 0. \end{cases} \tag{5.9}$$

Now, we give some lemmas that will be often used in the sequel.

Lemma 5.2.1 *Assume that there is a measurable sequence $(u_n)_n \in \mathrm{L}^\infty(\Omega)$ and a measurable function u satisfying*

$$\eta(u_n) \rightharpoonup \eta(u) \text{ weakly in } \mathrm{W}_0^{1,p}(\Omega), \text{ and } \|u_n\|_{\mathrm{L}^\infty(\Omega)} \leq \infty,$$

and

$$\lim_{n\to\infty}\int_\Omega [\overline{a}_n(x,u_n,\nabla\eta(u_n)) - \overline{a}_n(x,u_n,\nabla\eta(u))][\nabla\eta(u_n) - \nabla\eta(u)]dx = 0, \tag{5.10}$$

where $\overline{a}_n(x,s,\xi)$ is defining as (5.27)*.Then*

$$\nabla\eta(u_n) \to \nabla\eta(u) \text{ a.e. in } \Omega.$$

Proof. Let D_n defined by

$$D_n = [\overline{a}_n(x, u_n, \nabla\eta(u_n)) - \overline{a}_n(x, u_n, \nabla\eta(u))][\nabla\eta(u_n) - \nabla\eta(u)].$$

Then by (5.3), D_n is a positive function and (5.10) implies

$$D_n \longrightarrow 0 \text{ strongly in } L^1(\Omega).$$

Extracting a subsequence (still denoted by u_n), we have

$$\eta(u_n) \longrightarrow \eta(u) \text{ a.e. in } \Omega \text{ and } D_n \longrightarrow 0 \text{ a.e. in } \Omega.$$

Then there exists a subset E of Ω of zero measure, such that for $x \in \Omega\backslash E$

$$|\eta(u(x))| < \infty,\ |\nabla\eta(u(x))| < \infty, \text{ and } \eta(u_n(x)) \longrightarrow \eta(u(x)),\ D_n \longrightarrow 0.$$

Defining $\xi_n = \nabla\eta(u_n(x))$, $\xi = \nabla\eta(u(x))$ we have

$$D_n(x) = [\overline{a}_n(x, u_n, \xi_n)) - \overline{a}_n(x, u_n, \xi)][\xi_n - \xi],$$

and using (5.3)-(5.4) we get

$$\begin{aligned} D_n(x) &\geq \alpha_n|\xi_n|^p + \alpha_n|\xi|^p - \gamma\alpha_n(1 + b(|u_n||))|\xi_n|^{p-1}|\xi| \\ &\quad - \gamma\alpha_n(1 + b(|u_n|))|\xi|^{p-1}|\xi_n| \\ &\geq \lambda(|\xi_n|^p + |\xi|^p - \gamma(1 + b(|u_n|))|\xi_n|^{p-1}|\xi| - \gamma(1 + b(|u_n|))|\xi|^{p-1}|\xi_n|) \\ &\geq C(|\xi_n|^p - C(x)(|\xi_n|^{p-1} + |\xi_n|), \end{aligned}$$

where $C(x)$ is a constant which depends on x but does not depend on n. Thus $|\xi_n|$ is bounded uniformly with respect to n, since $D_n(x) \longrightarrow 0$.
Let ξ^* be a cluster point of ξ_n. We have $|\xi^*| < \infty$ and

$$[\overline{a}(x, u, \xi^*) - \overline{a}(x, u, \xi)][\xi^* - \xi] = 0,$$

because of the continuity of $\overline{a}$ with respect to s and ξ, (5.3) implies that $\xi^* = \xi$. The uniqueness of the cluster point means that for the whole sequence

$$\nabla\eta(u_n(x)) \longrightarrow \nabla\eta(u(x))\ \forall x \in \Omega\backslash E.$$

i.e. $\nabla\eta(u_n) \longrightarrow \nabla\eta(u)$ a.e. in Ω.

Lemma 5.2.2 *Assume that u_n and u are two measurable functions such that for fixed $k > 0$,*

$\widehat{k} > 0$

$$\mathrm{T}_k(\eta(u_n)) \rightharpoonup \mathrm{T}_k(\eta(u)) \text{ weakly in } \mathrm{W}_0^{1.p}(\Omega), \text{ and}$$

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega [\overline{a}_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_k(\eta(u_n))) - \overline{a}_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_k(\eta(u)))] \\ \times[\nabla\mathrm{T}_k(\eta(u_n)) - \nabla\mathrm{T}_k(\eta(u))]\, dx = 0,\end{aligned} \tag{5.11}$$

then $\nabla\mathrm{T}_k(\eta(u_n)) \to \nabla\mathrm{T}_k(\eta(u_n))$ *a.e. in* Ω.

The proof of Lemma 5.2.2 is the same as that of Lemma 5.2.1.

Lemma 5.2.3 *(see [31] Lemma 3.3) Let* σ_n *be a sequence of non negative bounded functions, almost everywhere convergent to some function* σ, *and let* $\rho_n : \mathbb{R}^N \to \mathbb{R}^N$ *be a sequence of functions which is weakly convergent in* $(\mathrm{L}^q(\Omega))^N$ *(*$q > 1$*) to some function* ρ. *If the sequence* $\sigma_n|\rho_n|^q$ *is bounded in* $\mathrm{L}^1(\Omega)$, *then* $\sigma|\rho|^q$ *belongs to* $\mathrm{L}^1(\Omega)$ *and*

$$\int_\Omega \sigma|\rho|^q \le \liminf_{n\to+\infty}\int_\Omega \sigma_n|\rho_n|^q. \tag{5.12}$$

If moreover $\sigma_n \to \sigma$ *strongly in* $\mathrm{L}^1(\Omega)$ *then it holds true also that*

$$\sigma_n^{1/q}\rho_n \rightharpoonup \sigma^{1/q}\rho \text{ weakly in } (\mathrm{L}^q(\Omega))^N.$$

Proof. Let $k > 0$ and $\psi \in (\mathrm{L}^{q'}(\Omega))^N$ we have that

$$\mathrm{T}_k(\sigma_n)^{1/q}\rho_n \to \mathrm{T}_k(\sigma)^{1/q}\rho \text{ weakly in } (\mathrm{L}^q(\Omega))^N.$$

Hence by the lower semi-continuity of the L^q-norm it results

$$\int_\Omega \mathrm{T}_k(\sigma)|\rho|^q \le \liminf_{n\to+\infty}\int_\Omega \mathrm{T}_k(\sigma_n)|\rho_n|^q \le \liminf_{n\to+\infty}\int_\Omega \sigma_n|\rho_n|^q \le \mathrm{C}_0.$$

Then letting k tend to infinity and using the monotone convergence theorem we obtain (5.12). Let now ψ be an arbitrary element of $(\mathrm{L}^{q'}(\Omega))^N$, then

$$\begin{aligned}\int_\Omega \sigma_n^{1/q}\rho_n\psi &= \int_{\{\sigma_n\le k\}} \sigma_n^{1/q}\rho_n\psi + \int_{\{\sigma>k\}} \sigma_n^{1/q}\rho_n\psi \\ &= \varepsilon_n + \int_{\{\sigma\le k\}} \sigma^{1/q}\rho\psi + \int_{\{\sigma_n>k\}} \sigma_n^{1/q}\rho_n\psi.\end{aligned}$$

Moreover

$$\begin{aligned}\int_\Omega \sigma^{1/q}\rho\psi &= \int_{\{\sigma_n\le k\}} \sigma^{1/q}\rho\psi + \int_{\{\sigma_n>k\}} \sigma^{1/q}\rho\psi \\ &= \varepsilon_n + \int_{\{\sigma\le k\}} \sigma^{1/q}\rho\psi + \int_{\{\sigma_n>k\}} \sigma^{1/q}\rho\psi.\end{aligned}$$

It follows that

$$\Big|\int_\Omega [\sigma_n^{1/q}\rho_n - \sigma^{1/q}\rho].\psi\Big| \le \varepsilon_n + \int_{\{\sigma_n>k\}} \sigma_n^{1/q}|\rho_n||\psi| + \int_{\{\sigma_n>k\}} \sigma^{1/q}|\rho||\psi|.$$

Using Hölder inequality we have that

$$\int_{\sigma_n>k} \sigma_n^{1/q}|\rho_n||\psi| \le \Big(\int_\Omega \sigma_n|\rho_n|^q\Big)^{1/q}\Big(\int_\Omega |\psi|^{q'}\Big)^{1/q'},$$

and

$$\int_{\sigma_n>k} \sigma^{1/q}|\rho_n||\psi| \le \Big(\int_\Omega \sigma|\rho|^q\Big)^{1/q}\Big(\int_\Omega |\psi|^{q'}\Big)^{1/q'}.$$

Thanks to the assumption $\sigma_n \longrightarrow \sigma$ in $L^1(\Omega)$ we deduce that the limit with respect to $k \longrightarrow \infty$ of the quantities above is zero uniformly with respect to n. So we can conclude that $\sigma_n^{1/q}\rho_n \longrightarrow \sigma^{1/q}\rho$ weakly in $L^q(\Omega)$.

5.3 Existence results

We first give the definition of a weak and renormalized solution to problem5.1 Then, we will discuss the existence of a bounded weak solution and renormalized solution to problem5.1.

Definition 5.3.1 *A measurable function $u \in V_0$ is called a weak solution to problem5.1 if $a(x,u,\nabla u)\alpha(x)^{\frac{1}{p'}-1} \in (L^{p'}(\Omega))^N$ and*

$$\int_\Omega a(x,u,\nabla u)\nabla v\, dx = \int_\Omega f v\, dx \ \forall v \in C_0^1(\Omega), \text{ hold} \tag{5.13}$$

Definition 5.3.2 *A measurable function $u : \Omega \to \mathbb{R}$ is a renormalized solution of problem5.1 if*

$$T_k(\eta(u)) \in X_0^p(\Omega) \text{ for any } k \ge 0, \tag{5.14}$$

$$\lim_{m\to\infty} \int_{m\le|\eta(u)|\le m+1} a(x,u,\nabla u)\nabla\eta(u)\, dx = 0, \tag{5.15}$$

and for any $h \in W^{1,\infty}(\Omega)$ with compact support and $\varphi \in W_0^p(\Omega)\cap L^\infty(\Omega)$, u satisfies

$$\int_\Omega a(x,u,\nabla u)\nabla(h(\eta(u))\varphi)\, dx = \int_\Omega f h(\eta(u))\varphi\, dx. \tag{5.16}$$

Remark 5.3.1 *By the results of [17], (5.14) allows us to give a sense to $\nabla\eta(u)$ almost everywhere in Ω Therefore, we can give the definition of ∇u as in (5.9).*

The main results of this section are as follows :

Theorem 5.3.1 *Assume that* (5.3)-(5.5) *hold, and* $m > \max\{\frac{N}{p}, 1\}$. *Then any weak solution* u *to problem 5.1 satisfies the following estimates :*

$$\|u\|_{L^\infty(\Omega)} \le M,$$

where M is a constant which depends only on N, p, m, λ, μ, *and* $\|f\|_{L^m(\Omega)}$.

Theorem 5.3.2 *Assume that* (5.3)-(5.5) *hold and* $m > \max\{\frac{N}{p}, 1\}$. *Then there exists a bounded weak solution* $u \in W_0^{1,p}$ *to problem5.1 in the of definition5.3.1.*

If $m = 1$, the solution of problem5.1 may not be bounded. Since no growth condition is required for b (see assumption (5.4), (3)), it is not obvious that the term $-\mathrm{div}a(x,s,\xi)$ makes sense even as a distribution. To overcome this difficulty, here we will use the notion of a renormalized solution introduced by DiPerna and Lions in [43] (see also [28, 62]).

Theorem 5.3.3 *Assume that (5.3)-(5.5) hold and* $m = 1$. *Then problem5.1 admits a renormalized solution* u *in the sense of Definition 5.3.2.*

Proof of Theorem 5.3.1

Taking $v = S_{t,h}(\eta(u))$ as test function in (5.13), it is easy to see that $S_{t,h}(\eta(u)) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, by (5.3) we get the following estimate

$$\begin{aligned} &\lambda \frac{1}{h} \int_{t \le |\eta(u)| \le t+h} |\nabla \eta(u)|^p \, dx \le \frac{1}{h} \int_{t \le |\eta(u)| \le t+h} \alpha(x) |g(|u|)|^{\frac{p}{p-1}} |\nabla u|^p \, dx \\ &\le \int_{t \le |\eta(u)| \le t+h} a(x,u,\nabla u) \frac{\nabla \eta(u)}{h} \, dx \le \int_{|\eta(u)| > t} |f| \, dx, \end{aligned} \tag{5.17}$$

and if we let $h \to 0$ in (5.17), then

$$\lambda \frac{d}{dt} \int_{|\eta(u)| > t} |\nabla \eta(u)|^p \, dx \le \int_{|\eta(u)| > t} |f| \, dx,$$

setting $w = |\eta(u)|$, then it follows from (4.5) that

$$\lambda \frac{d}{dt} \int_{w > t} |\nabla \eta(u)|^p \, dx \le \int_0^{\mu_w(t)} |f|_*(s) \, ds. \tag{5.18}$$

(4.6) and (5.18) imply that

$$\lambda C_N^{\frac{1}{N}} \mu_w(t)^{1-\frac{1}{N}} \le (-\mu_w'(t))^{\frac{1}{p'}} \Big(\int_0^{\mu_w(t)} |f|_*(s) \, ds \Big)^{\frac{1}{p}}, \tag{5.19}$$

it follows by Lemma4.2.3 that

$$(-w_*(s))' \leq \frac{((-w_*(s))')^{\frac{1}{p'}}}{\lambda \mathrm{N}\mathrm{C}_{\mathrm{N}}^{\frac{1}{\mathrm{N}}} s^{1-\frac{1}{\mathrm{N}}}} \Big(\int_0^s |f|_*(\sigma)\, d\sigma \Big)^{\frac{1}{p}}. \tag{5.20}$$

Applying Young's inequality in (5.20), we get

$$(-w_*(s))' \leq \Big(\frac{1}{(\lambda \mathrm{N}\mathrm{C}_{\mathrm{N}}^{\frac{1}{\mathrm{N}}} s^{1-\frac{1}{\mathrm{N}}})^p} \int_0^s |f|_*(\sigma)\, d\sigma \Big)^{\frac{p'}{p}}. \tag{5.21}$$

Since $w_*(|\Omega|) = ess\inf_{\Omega} w = 0$ and $w_*(0) = \|w\|_{\mathrm{L}^\infty(\Omega)}$, a simple integral of (5.21) leads to

$$\|w\|_{\mathrm{L}^\infty(\Omega)} \leq \int_0^{|\Omega|} \Big(\frac{1}{(\lambda \mathrm{N}\mathrm{C}_{\mathrm{N}}^{\frac{1}{\mathrm{N}}} s^{1-\frac{1}{\mathrm{N}}})^p} \int_0^s |f|_*(\sigma)\, d\sigma \Big)^{\frac{p'}{p}}. \tag{5.22}$$

By Hölder's inequality (4.4) and $m > \max\{\frac{\mathrm{N}}{p}, 1\}$ we have

$$\begin{aligned} \|w\|_{\mathrm{L}^\infty(\Omega)} &\leq \frac{(\|f\|_{\mathrm{L}^m(\Omega)})^{\frac{1}{(p-1)}}}{(\lambda \mathrm{N}\mathrm{C}_{\mathrm{N}}^{\frac{1}{\mathrm{N}}})^{p'}} \int_0^{|\Omega|} \frac{1}{s^{(1-\frac{1}{\mathrm{N}})p'}} \times s^{\frac{1-1/m}{p-1}}\, ds \\ &\leq \Big[\frac{(\|f\|_{\mathrm{L}^m(\Omega)})^{\frac{1}{(p-1)}}}{(\lambda \mathrm{N}\mathrm{C}_{\mathrm{N}}^{\frac{1}{\mathrm{N}}})^{p'}} |\Omega|^{\frac{p'}{\mathrm{N}} - 1/m(p-1)} \frac{(p-1)\mathrm{N}m}{pm-\mathrm{N}} \Big] = \mathrm{M}_0, \end{aligned} \tag{5.23}$$

it follows

$$\|\eta(u)\|_{\mathrm{L}^\infty(\Omega)} \leq \mathrm{M}_0. \tag{5.24}$$

The function η is continuous, odd, and strictly increasing, so it is invertible and its inverse η^{-1} is also strictly increasing. Thus, (5.24) is equivalent to

$$\|u\|_{\mathrm{L}^\infty(\Omega)} \leq \mathrm{M}, \text{ where } \mathrm{M} = \eta^{-1}(\mathrm{M}_0). \tag{5.25}$$

Proof of Theorem 5.3.2

Step 1 : Approximation and a priori estimates Let us set for $n \in \mathbb{N}^*$

$$a^n(x,s,\xi) = \frac{1}{n}|\xi|^{p-2}\xi + a_n(x,s,\xi), \tag{5.26}$$

where $a_n(x,s,\xi) = \dfrac{a(x,\mathrm{T}_n(s),\xi)\alpha_n(x)}{\alpha(x)}$, and $\alpha_n(x) = \mathrm{T}_n(\alpha(x))$.
Note that by definition $a_n(x,s,\xi)$ satisfies $a_n(x,s,\xi)\xi \geq \alpha_n(x) g_n(|s|)|\xi|^p$,

$$\begin{aligned}
&(1)\ \ a_n(x,s,\xi) = g_n(|s|)\overline{a}_n(x,s,\xi),\ \overline{a}_n(x,s,\xi) = \frac{\overline{a}(x,\mathrm{T}_n(s),\xi)\alpha_n(x)}{\alpha(x)},\\
&(2)\ \ [\overline{a}_n(x,s,\xi) - \overline{a}_n(x,s,\xi')][\xi - \xi'] > 0,\\
&(3)\ \ |\overline{a}_n(x,s,\xi)| \leq \gamma\alpha_n(x)(1 + b_n(|s|)|\xi|^{p-1},\\
&(4)\ \ \overline{a}_n(x,s,t\xi) = t^{p-1}\overline{a}_n(x,s,\xi),
\end{aligned} \tag{5.27}$$

for almost every $x \in \Omega$, every $s \in \mathbb{R}$ and every $\xi, \xi' \in \mathbb{R}^{\mathrm{N}}$ with $\xi \neq \xi$.

Here, we consider the following approximate problem :

$$\begin{cases} -\mathrm{div}\, a^n(x,u,\nabla u) = f & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.28}$$

Then by the classical method in [60], problem5.28 admits a weak solution $u_n \in$ for any fixed $n >$ M such that

$$\begin{aligned}
\frac{1}{n}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n v\, dx + \int_\Omega a_n(x,u_n,\nabla u_n)\nabla v\, dx\\
= \int_\Omega f v\, dx\ v \in \mathrm{W}_0^{1,p}(\Omega).
\end{aligned} \tag{5.29}$$

where M is the same constant as in (5.25).
Arguing as in the proof of Theorem5.3.1, we get

$$\|u_n\| \leq \mathrm{M},$$

where M is the constant given in (5.25) It is easy to see that $\eta(u_n) \in \mathrm{W}_0^{1,p}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ with $\|\eta(u_n)\| \leq \mathrm{C}_1$, where C_1 is constant, only depends on M Take $v = \eta(u_n)$ as a test function of problem (5.29).Noting that $\mathrm{T}_n(u_n) = u_n$, by (6.2) and Hölder inequality and Young's inequality we obtain

$$\begin{aligned}
&\lambda\int_\Omega |\nabla\eta(u_n)|^p\, dx \leq \int_\Omega \alpha_n(x)|\nabla\eta(u_n)|^p\, dx,\\
&\leq \frac{1}{n}\int_\Omega g(|u_n|)^{\frac{1}{p-1}}|\nabla u_n|^p\, dx + \int_\Omega \alpha(x) g(|u_n|)^{p'}|\nabla u_n|^p\, dx,\\
&\leq \mathrm{C}_1(\|f\|_{\mathrm{L}^1(\Omega)}).
\end{aligned} \tag{5.30}$$

As a consequence, there exists a subsequence (still denoted by $\eta(u_n)$) and a measurable

function $w:\ \Omega \to \mathbb{R}$, which is finite almost everywhere in Ω, such that

$$\begin{aligned} &\eta(u_n) \rightharpoonup w \text{ weakly in } \mathrm{W}_0^{1,p}(\Omega), \\ &\eta(u_n) \to w \text{ a.e. in } \Omega, \\ &\eta(u_n) \to w \text{ weakly* in } \mathrm{L}^{\infty}(\Omega). \end{aligned} \tag{5.31}$$

For almost every x in Ω, let us define a function u by

$$u(x) = \inf\{s \in \mathbb{R},\ w(x) \leq \eta(s)\}. \tag{5.32}$$

Note that u is well defined since η is strictly increasing. Furthermore, it is easy to check that u is measurable on Ω. Thus, (5.32) shows that $\eta(u) = w$ and (5.31) can be rewritten as

$$\begin{aligned} &\eta(u_n) \rightharpoonup \eta(u) \text{ weakly in } \mathrm{W}_0^{1,p}(\Omega), \\ &\eta(u_n) \to \eta(u) \text{ a.e. in } \Omega, \\ &\eta(u_n) \to \eta(u) \text{ weakly* in } \mathrm{L}^{\infty}(\Omega). \end{aligned} \tag{5.33}$$

Since η is strictly increasing, we also have

$$u_n \to u \text{ a.e. in } \Omega, \tag{5.34}$$

and on the other, by Lemma 5.2.3

$$\begin{aligned} \alpha_n(x)^{\frac{1}{p}} \nabla \eta(u_n) \rightharpoonup &\alpha(x)^{\frac{1}{p}} \nabla \eta(u) \text{ weakly in } (\mathrm{L}^p(\Omega)^{\mathrm{N}}, \\ &\alpha(x)|\nabla \eta(u)|^p \in \mathrm{L}^1(\Omega). \end{aligned} \tag{5.35}$$

Step 2 : $\nabla \eta(u_n) \to \nabla \eta(u)$ *a.e. in* Ω
Since $n > \mathrm{M}$, taking $v = u_n$ as a test function for problem (5.29), (5.3) and (5.25) lead to

$$\frac{1}{n} \int_{\Omega} |\nabla u_n|^p \, dx + \int_{\Omega} \alpha_n(x) g(|u_n|) |\nabla u_n|^p \, dx \leq \mathrm{M}(\|f\|_{\mathrm{L}^1(\Omega)}). \tag{5.36}$$

Taking $v = \eta(u_n) - \eta(u)$ as test function in (5.29) and noting $\mathrm{T}_n(u_n) = u_n$ for $n > \mathrm{M}$ we get

$$\begin{aligned} &\frac{1}{n} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n (\nabla \eta(u_n) - \nabla \eta(u)) \, dx \\ &+ \int_{\Omega} a_n(x, u_n, \nabla u_n)(\nabla \eta(u_n) - \nabla \eta(u)) \, dx = \int_{\Omega} f(\eta(u_n) - \eta(u)) \, dx. \end{aligned} \tag{5.37}$$

Hölder's inequality and (5.36) yield

$$\begin{aligned}\frac{1}{n}|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla\eta(u_n)-\nabla\eta(u))\,dx| \\ &\le \frac{1}{n}\int_\Omega |\nabla u_n|^{p-1}|(\nabla\eta(u_n)-\nabla\eta(u))|\,dx \\ &\le \frac{1}{n}\|\nabla\eta(u_n)-\nabla\eta(u)\|_{\mathrm{L}^p(\Omega)}\|\nabla u_n\|^{\frac{p}{p'}}_{\mathrm{L}^p(\Omega)} \\ &\le \mathrm{C}_2(\frac{1}{n})^{\frac{1}{p'}}\|\nabla\eta(u_n)-\nabla\eta(u)\|_{\mathrm{L}^p(\Omega)},\end{aligned} \tag{5.38}$$

where C_2 is a positive constant such that $\mathrm{C}_2 = (\mathrm{M}(\|f\|_{\mathrm{L}^1(\Omega)})^{\frac{1}{p'}}$.
(5.30) and (5.36) imply that $\nabla\eta(u_n)-\nabla\eta(u)$ in bounded uniformly in $(\mathrm{L}^p(\Omega))^{\mathrm{N}}$. Therefore by (5.38) we get

$$\overline{\lim_{n\to\infty}}\frac{1}{n}|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla\eta(u_n)-\nabla\eta(u))\,dx| \le 0,$$

which show that

$$\lim_{n\to\infty}\frac{1}{n}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla\eta(u_n)-\nabla\eta(u))\,dx = 0. \tag{5.39}$$

By (5.33) and $f\in \mathrm{L}^m(\Omega)$ with $m > \max\{\frac{\mathrm{N}}{p}, 1\}$ we have

$$\lim_{n\to\infty}\int_\Omega f(\eta(u_n)-\eta(u))\,dx = 0. \tag{5.40}$$

From (5.37), (5.39) and (5.40) we obtain

$$\lim_{n\to\infty}\int_\Omega a_n(x,u_n,\nabla u_n)(\nabla\eta(u_n)-\nabla\eta(u))\,dx = \lim_{n\to\infty}\mathrm{D}_n = 0. \tag{5.41}$$

By (5.27), it is easy to get that $a_n(x,u_n,\nabla u_n) = \overline{a}_n(x,u_n,\nabla\eta(u_n))$ Thus, we can rewrite D_n as

$$\begin{aligned}\mathrm{D}_n &= \int_\Omega a_n(x,u_n,\nabla u_n)(\nabla\eta(u_n)-\nabla\eta(u))\,dx \\ &= \int_\Omega \overline{a}_n(x,u_n,\nabla\eta(u_n))(\nabla\eta(u_n)-\nabla\eta(u))\,dx\end{aligned} \tag{5.42}$$

$$\begin{aligned}&= \int_\Omega [\overline{a}_n(x,u_n,\nabla\eta(u_n))-\overline{a}_n(x,u_n,\nabla\eta(u))](\nabla\eta(u_n)-\nabla\eta(u))\,dx \\ &+ \int_\Omega \overline{a}_n(x,u_n,\nabla\eta(u))(\nabla\eta(u_n)-\nabla\eta(u))\,dx.\end{aligned}$$

The last term above can be written as

$$\begin{aligned}\int_\Omega \overline{a}_n(x,u_n,\nabla\eta(u))(\nabla\eta(u_n)-\nabla\eta(u))\,dx \\ =\int_\Omega \frac{\overline{a}(x,u_n,\nabla\eta(u))}{\alpha(x)}\alpha_n(x)(\nabla\eta(u_n)-\nabla\eta(u))\,dx.\end{aligned} \tag{5.43}$$

That converges to zero as n diverges thanks to (5.35), the a.e. convergence of u_n. Hence, taking $n\to\infty$ in (5.42), by (5.41) and (5.43) we get

$$\int_\Omega [\overline{a}_n(x,u_n,\nabla\eta(u_n))-\overline{a}_n(x,u_n,\nabla\eta(u))](\nabla\eta(u_n)-\nabla\eta(u))\,dx\to 0, \tag{5.44}$$

and by Lemma 5.2.1 we get

$$\nabla\eta(u_n)\to\nabla\eta(u), \text{ a.e. in } \Omega. \tag{5.45}$$

Step 3 : passage to limit We prove that u is a weak solution to problem (5.29) in the sense of Definition .
(5.25),(5.30), (5.34) and (5.45) imply that the sequence

$$\frac{\overline{a}_n(x,u_n,\nabla\eta(u_n))}{\alpha(x)}\alpha_n(x)^{\frac{1}{p'}} \text{ is bounded in } (\mathrm{L}^{p'}(\Omega))^{\mathrm{N}},$$

and converges almost everywhere to $\overline{a}(x,u,\nabla\eta(u))\alpha(x)^{\frac{1}{p'}-1}$. Hence it also converges weakly in $(\mathrm{L}^{p'}(\Omega))^{\mathrm{N}}$ to its a.e. limit, i.e.

$$\frac{a_n(x,u_n,\nabla u_n)}{\alpha(x)}\alpha_n(x)^{\frac{1}{p'}}\rightharpoonup a(x,u,\nabla u)\alpha(x)^{\frac{1}{p'}-1} \text{ weakly in } (\mathrm{L}^{p'}(\Omega))^{\mathrm{N}}. \tag{5.46}$$

Noticing that for every $v\in \mathrm{X}_0^p(\Omega)$

$$\alpha_n(x)^{\frac{1}{p}}v\rightharpoonup\alpha(x)^{\frac{1}{p}}v \text{ strongly in } (\mathrm{L}^{p}(\Omega))^{\mathrm{N}} \tag{5.47}$$

We deduce from (5.46) and (5.47) that

$$\lim_{n\to\infty}\int_\Omega a_n(x,u_n,\nabla u_n)\nabla v dx=\int_\Omega a(x,u,\nabla u)\nabla v dx. \tag{5.48}$$

Arguing as (5.39), it is easy to get that

$$\lim_{n\to\infty}\frac{1}{n}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla v\,dx=0. \tag{5.49}$$

Let n tend to infinity in (5.29) and note that $\mathrm{T}_n(u_n)=u_n$ for $n>\mathrm{M}$; it follows from (5.48)

and (5.49) that

$$\int_\Omega a(x,u,\nabla u)\nabla v\ dx = \int_\Omega f v\ dx \quad v \in W_0^{1,p}(\Omega).$$

Thus we complete the proof of Theorem 5.3.2.

Proof of Theorem 5.3.3

Step 1 : Approximation and a priori estimates.

For $n > 0$, let f_n be a sequence of $L^\infty(\Omega)$ such that

$$f_n \to f \text{ strongly in } L^1(\Omega). \tag{5.50}$$

We consider the approximate problem :

$$\begin{cases} -\text{div}\, a_n(x,u_n,\nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.51}$$

By the Theorem 5.3.1 for $n > 0$, there exists at least a weak solution u_n satisfying $\eta(u_n) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such

$$\int_\Omega a_n(x,u_n,\nabla u_n)\nabla v\ dx = \int_\Omega f_n v\ dx,\ \forall v \in W_0^{1,p}(\Omega). \tag{5.52}$$

For any given $k > 0$, let us take $T_k(\eta(u_n))$ as the test function in (5.52). So form (5.3) and Hölder's inequality

$$\begin{aligned} \lambda \int_\Omega |\nabla T_k(\eta(u_n))|^p\ dx &\le \int_\Omega \alpha_n(x)|\nabla T_k(\eta(u_n))|^p\ dx \\ &\le \int_\Omega a_n(x,u_n,\nabla u_n)\nabla T_k(\eta(u_n))\ dx \le k\|f\|_{L^1(\Omega)}. \end{aligned} \tag{5.53}$$

It follows that

$$\int_\Omega |\nabla T_k(\eta(u_n))|^p\ dx \le C_3 k. \tag{5.54}$$

where C_3 is a positive constant independent of n.

Let $k \ge 1$, we have $k\text{meas}\{|\eta(u_n)| > k\} = \int_{|\eta(u_n)|>k} |T_k(A(u_n))| dx \le C_3 k^{\frac{1}{p}}$,

which implies that $\text{meas}\{|\eta(u_n)| > k\} \le C_3 \dfrac{1}{k^{1-\frac{1}{p}}} \longrightarrow 0$ as $k \to \infty$.

Reasoning as in the proof of Theorem 6.1 of [17] it follows from (5.54) that. Consequently, we can assume that $(T_k(\eta(u_n)))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Thus, $(T_k(\eta(u_n)))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Hence, there exists a measurable function u and a subsequence (still denoted by $(\eta(u_n))$), such that

$$\eta(u_n) \to \eta(u_n) \text{ a.e. in } \Omega. \tag{5.55}$$

From (5.54) and (5.55), we deduce that

$$\mathrm{T}_k(\eta(u_n)) \to \mathrm{T}_k(\eta(u)) \text{ weaky in } \mathrm{W}_0^{1.p}(\Omega). \tag{5.56}$$

Since η is strictly increasing, we also have

$$u_n \to u \text{ a.e. in } \Omega, \tag{5.57}$$

and on the other, by Lemma 5.2.3

$$\begin{gathered} \alpha(x)|\nabla \mathrm{T}_k(\eta(u))|^p \in \mathrm{L}^1(\Omega), \\ \alpha(x)^{\frac{1}{p}} \nabla \mathrm{T}_k(\eta(u_n)) \to \alpha(x)^{\frac{1}{p}} \nabla \mathrm{T}_k(\eta(u)) \text{ weakly in } (\mathrm{L}^p(\Omega))^{\mathrm{N}}. \end{gathered} \tag{5.58}$$

Step 2 : Strong convergence of truncations.

Let $j > k$ and take $v = \mathrm{T}_{2k}(\eta(u_n) - \mathrm{T}_j(\eta(u_n)) + \mathrm{T}_k(\eta(u_n)) - \mathrm{T}_k(\eta(u)))$ as a test function in (5.52).
Note that $\nabla v = 0$ as $|\eta(u_n)| > 4k + j = \mathrm{J}$, therefore for $n > \mathrm{J}$ we can writ (5.52) as

$$\int_\Omega a_n(x, \mathrm{T}_{\widehat{\mathrm{J}}}(u_n), \nabla \mathrm{T}_{\widehat{\mathrm{J}}}(u_n)) \nabla v \, dx = \int_\Omega f_n v \, dx, \tag{5.59}$$

where $\widehat{\mathrm{J}}$ is a positive constant depending on J such that $\{|\eta(u_n)| > \mathrm{J}\}$ can be equivalently replaced by $\{|u_n| > \widehat{\mathrm{J}}\}$ (recall that η strictly increasing and continuous).
Splitting the integral on the left side of (5.59) on the sets $\{|\eta(u_n)| > k\}$ and $\{|\eta(u_n)| \le k\}$ by (5.3) we obtain

$$\begin{aligned} &\int_\Omega a_n(x, \mathrm{T}_{\widehat{\mathrm{J}}}(u_n), \nabla \mathrm{T}_{\widehat{\mathrm{J}}}(u_n)) \\ &\qquad \times \nabla \mathrm{T}_{2k}(\eta(u_n) - \mathrm{T}_j(\eta(u_n)) + \mathrm{T}_k(\eta(u_n)) - \mathrm{T}_k(\eta(u))) dx \\ &\qquad \ge \int_\Omega a_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_{\widehat{k}}(u_n)) \nabla (\mathrm{T}_k(\eta(u_n)) - \mathrm{T}_k(\eta(u))) \, dx \\ &\qquad - \int_{\{|\eta(u_n)| > k\}} |a_n(x, \mathrm{T}_{\widehat{\mathrm{J}}}(u_n), \nabla \mathrm{T}_{\widehat{\mathrm{J}}}(u_n))| |\nabla \mathrm{T}_k(\eta(u))| \, dx, \end{aligned} \tag{5.60}$$

where $\widehat{k}$ is a constant defined as $\widehat{\jmath}$. From (5.59) and (5.60)

$$\begin{aligned}
&\int_\Omega [a_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_{\widehat{k}}(u_n)) - a_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_{\widehat{k}}(u))] \\
&\quad \times [\nabla \mathrm{T}_k(\eta(u_n)) - \nabla \mathrm{T}_k(\eta(u))]\, dx \\
&\quad \le \int_{\{|\eta(u_n)|>k\}} |a_n(x, \mathrm{T}_{\widehat{\jmath}}(u_n), \nabla \mathrm{T}_{\widehat{\jmath}}(u_n))||\nabla \mathrm{T}_k(\eta(u))|\, dx \\
&\quad - \int_\Omega a_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_{\widehat{k}}(u))[\nabla \mathrm{T}_k(\eta(u_n)) - \nabla \mathrm{T}_k(\eta(u))]\, dx \\
&\quad + \int_\Omega f_n v\, dx.
\end{aligned} \tag{5.61}$$

By (5.4), it is easy to check that $a_n(x,s,\xi) = \overline{a}_n(x,s,g(s)^{\frac{1}{p-1}}\xi)$. Thus, by the definition of η we can rewrite (5.61) as

$$\begin{aligned}
&\int_\Omega [\overline{a}_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_k(\eta(u_n))) - \overline{a}_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_k(\eta(u)))] \\
&\quad \times [\nabla \mathrm{T}_k(\eta(u_n)) - \nabla \mathrm{T}_k(\eta(u))]\, dx \\
&\quad \le \int_{\{|\eta(u_n)|>k\}} |\overline{a}_n(x, \mathrm{T}_{\widehat{\jmath}}(u_n), \nabla \mathrm{T}_j(\eta(u_n)))||\nabla \mathrm{T}_k(\eta(u))|\, dx \\
&\quad + \int_\Omega f_n \mathrm{T}_{2k}(\eta(u_n) - \mathrm{T}_j(\eta(u_n)) + \mathrm{T}_k(\eta(u_n)) - \mathrm{T}_k(\eta(u)))\, dx \\
&\quad - \int_\Omega \overline{a}_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_k(\eta(u))[\nabla \mathrm{T}_k(\eta(u_n)) - \nabla \mathrm{T}_k(\eta(u))]\, dx \\
&\quad = \mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3.
\end{aligned} \tag{5.62}$$

The first term on the right hand side of (5.62) can be written as

$$\begin{aligned}
\mathrm{A}_1 &= \int_{\{|\eta(u_n)|>k\}} |\overline{a}_n(x, \mathrm{T}_{\widehat{\jmath}}(u_n), \nabla \mathrm{T}_j(\eta(u_n)))||\nabla \mathrm{T}_k(\eta(u))|\, dx \\
&= \int_\Omega \Big| \frac{\overline{a}(x, \mathrm{T}_{\widehat{\jmath}}(u_n), \nabla \mathrm{T}_j(\eta(u_n)))}{\alpha(x)} \alpha_n(x) \Big| |\nabla \mathrm{T}_k(\eta(u))| \chi_{\{|\eta(u_n)|>k\}}\, dx.
\end{aligned}$$

For any $j > 0$, (5.4) and (5.58) imply that the sequence

$$\frac{\overline{a}(x, \mathrm{T}_{\widehat{\jmath}}(u_n), \nabla \mathrm{T}_j(\eta(u_n)))}{\alpha(x)} \alpha_n(x)^{\frac{1}{p'}} \text{ is bounded in } \mathrm{L}^{p'}(\Omega)^{\mathrm{N}}. \tag{5.63}$$

While $\alpha_n(x)^{\frac{1}{p}} \nabla \mathrm{T}_k(\eta(u)) \chi_{\{|\eta(u_n)|>k\}}$ converge strongly to zero in $\mathrm{L}^p(\Omega)$
as n tends to infinity. Hence we have

$$\lim_{n\to\infty} \mathrm{A}_1 = \lim_{n\to\infty} \int_\Omega |\overline{a}_n(x, \mathrm{T}_{\widehat{\jmath}}(u_n), \nabla \mathrm{T}_j(\eta(u_n)))||\nabla \mathrm{T}_k(\eta(u))| \chi_{\{|\eta(u_n)|>k\}} dx = 0. \tag{5.64}$$

By (5.55) and (5.57) we can deduce that

$$\begin{aligned}&\mathrm{T}_{2k}(\eta(u_n)-\mathrm{T}_j(\eta(u_n))+\mathrm{T}_k(\eta(u_n))-\mathrm{T}_k(\eta(u)))\ \rightharpoonup\ \mathrm{T}_{2k}(\eta(u_n)-\mathrm{T}_j(\eta(u)))\\&\text{weakly in }\ \mathrm{W}_0^{1.p}(\Omega)\end{aligned}$$

and weakly* in $\mathrm{L}^{\infty}(\Omega)$ as $n\to\infty$. So, from (5.50) we obtain

$$\lim_{n\to\infty}\mathrm{A}_2=\int_{\Omega}f\mathrm{T}_{2k}(\eta(u)-\mathrm{T}_j(\eta(u_n)))\,dx. \tag{5.65}$$

By (5.4), (5.57), (5.58) and (5.63) it gollows that

$$\begin{aligned}&\lim_{n\to\infty}\int_{\Omega}\frac{\overline{a}(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_k(\eta(u))}{\alpha(x)}\alpha_n(x)[\nabla\mathrm{T}_k(\eta(u_n))-\nabla\mathrm{T}_k(\eta(u))]dx.\\&=\lim_{n\to\infty}\mathrm{A}_3=0\end{aligned} \tag{5.66}$$

Let n tend to infinity in (5.62) ; it follows from (5.64)-(5.66) that

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\int_{\Omega}[\overline{a}_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_k(\eta(u_n)))-\overline{a}_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_k(\eta(u)))]\\&\times[\nabla\mathrm{T}_k(\eta(u_n))-\nabla\mathrm{T}_k(\eta(u))]\,dx\le\int_{\Omega}f\mathrm{T}_{2k}(\eta(u)-\mathrm{T}_j(\eta(u_n)))\,dx.\end{aligned} \tag{5.67}$$

Furthermore by (5.55)-(5.57) we have

$$\begin{gathered}\mathrm{T}_{2k}(\eta(u)-\mathrm{T}_j(\eta(u_n)))\to 0\text{ weakly in }\ \mathrm{W}_0^{1.p}(\Omega),\\\text{and weakly* in }\ \mathrm{L}^{\infty}(\Omega)\text{ as }j\to\infty.\end{gathered}$$

Let j tend to infinity in (5.67)

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\int_{\Omega}[\overline{a}_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_k(\eta(u_n)))-\overline{a}_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_k(\eta(u)))]\\&\qquad\times[\nabla\mathrm{T}_k(\eta(u_n))-\nabla\mathrm{T}_k(\eta(u))]\,dx\le 0.\end{aligned}$$

Hence by (2) of (5.4), we obtain

$$\begin{aligned}&\int_{\Omega}[\overline{a}_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_k(\eta(u_n)))-\overline{a}_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_k(\eta(u)))]\\&\times[\nabla\mathrm{T}_k(\eta(u_n))-\nabla\mathrm{T}_k(\eta(u))]\,dx=0\text{ as }n\to\infty.\end{aligned} \tag{5.68}$$

By Lemma 5.3.3 and (5.68), we have

$$\nabla\mathrm{T}_k(\eta(u_n))\to\nabla\mathrm{T}_k(\eta(u))\ \text{ a.e. in }\Omega\text{ as }n\to\infty. \tag{5.69}$$

Furthermore, we have

$$\begin{aligned}
&\int_\Omega \overline{a}_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_k(\eta(u_n)))\nabla \mathrm{T}_k(\eta(u_n)) \\
&\quad = \int_\Omega \overline{a}_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_k(\eta(u_n)))\nabla \mathrm{T}_k(\eta(u))\ dx \\
&\quad + \int_\Omega \overline{a}_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_k(\eta(u)))[\nabla \mathrm{T}_k(\eta(u_n)) - \nabla \mathrm{T}_k(\eta(u))] \\
&\quad + \int_\Omega [\overline{a}_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_k(\eta(u_n))) - \overline{a}_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_k(\eta(u)))] \\
&\quad \times [\nabla \mathrm{T}_k(\eta(u_n)) - \nabla \mathrm{T}_k(\eta(u))]\ dx
\end{aligned} \tag{5.70}$$

(5.4),(5.57) and (5.58) imply that

$$\lim_{n\to\infty} \int_\Omega \overline{a}_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_k(\eta(u)))[\nabla \mathrm{T}_k(\eta(u_n)) - \nabla \mathrm{T}_k(\eta(u))] = 0\ dx. \tag{5.71}$$

By assumption (5.4), (5.57), (5.58) and (5.69), we get

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega \overline{a}_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_k(\eta(u_n)))\nabla \mathrm{T}_k(\eta(u))\ dx \\
&\qquad = \int_\Omega \overline{a}(x, \mathrm{T}_{\widehat{k}}(u), \nabla \mathrm{T}_k(\eta(u)))\nabla \mathrm{T}_k(\eta(u))\ dx.
\end{aligned} \tag{5.72}$$

Now from (5.68), (5.70), (5.71) and (5.72) we deduce

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega \overline{a}_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_k(\eta(u_n)))\nabla \mathrm{T}_k(\eta(u_n))\ dx \\
&\qquad = \int_\Omega \overline{a}(x, \mathrm{T}_{\widehat{k}}(u), \nabla \mathrm{T}_k(\eta(u)))\nabla \mathrm{T}_k(\eta(u))\ dx.
\end{aligned} \tag{5.73}$$

Set

$\mathrm{Z} = \overline{a}(x, \mathrm{T}_{\widehat{k}}(u), \nabla \mathrm{T}_k(\eta(u)))\nabla \mathrm{T}_k(\eta(u)), \mathrm{Z}_n = \overline{a}_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_k(\eta(u_n)))\nabla \mathrm{T}_k(\eta(u_n))$

Then it is easy to see that

$$\mathrm{Z}_n \geq 0\ \mathrm{Z}_n \longrightarrow \mathrm{Z} \text{ a.e. in } \Omega\ \mathrm{Z} \in \mathrm{L}^1(\Omega),\ \int_\Omega \mathrm{Z}_n \longrightarrow \int_\Omega \mathrm{Z}.$$

Note that

$$\int_\Omega |\mathrm{Z}_n - \mathrm{Z}| = 2\int_{0\leq \mathrm{Z}_n \leq \mathrm{Z}} \mathrm{Z} - \mathrm{Z}_n + \int_\Omega \mathrm{Z}_n - \mathrm{Z}$$

Combining this with Lebesgue's dominated convergence theorem, we obtain

$$\mathrm{Z}_n \longrightarrow \mathrm{Z} \text{ strongly in } \mathrm{L}^1(\Omega),$$

i.e.

$$\begin{aligned}&\overline{a}_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_k(\eta(u_n)))\nabla\mathrm{T}_k(\eta(u_n))\longrightarrow\\&\overline{a}(x,\mathrm{T}_{\widehat{k}}(u),\nabla\mathrm{T}_k(\eta(u)))\nabla\mathrm{T}_k(\eta(u))\text{ strongly in }\mathrm{L}^1(\Omega).\end{aligned}\tag{5.74}$$

Now from (5.74) and assumptions (5.3), (5.4) and (5.58) we deduce, using Vitali's theorem,

$$\nabla\mathrm{T}_k(\eta(u_n))\to\nabla\mathrm{T}_k(\eta(u))\ \text{ strongly in }(\mathrm{L}^p(\Omega))^{\mathrm{N}}\text{ as }n\to\infty.\tag{5.75}$$

Furthermore by (5.53),(5.75) and Lemma 5.2.3

$$\begin{aligned}&\alpha(x)|\nabla\mathrm{T}_k(\eta(u))|^p\in\mathrm{L}^1(\Omega),\\&\alpha_n(x)^{\frac{1}{p}}\nabla\mathrm{T}_k(\eta(u_n))\to\alpha(x)^{\frac{1}{p}}\nabla\mathrm{T}_k(\eta(u))\text{ strongly in }(\mathrm{L}^p(\Omega))^{\mathrm{N}}.\end{aligned}\tag{5.76}$$

Step 3 : Renormalized solution.

In this step we will show that u is a renormalized solution to problem.
Let $\varphi\in\mathrm{W}_0^{1.p}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ and $h\in\mathrm{C}_0^1(\Omega)$ with $supp(h)\subset[-k.k]$, taking $v=h(\eta(u_n))\varphi$ as a test function in (5.52)

$$\begin{aligned}\int_\Omega a_n(x,u_n,\nabla u_n)h(\eta(u_n))\nabla\varphi\,dx=&-\int_\Omega a_n(x,u_n,\nabla u_n)\nabla\eta(u_n)h'(\eta(u_n))\varphi\,dx\\&+\int_\Omega f_nh(\eta(u_n))\varphi\,dx=-\mathrm{B}_1+\mathrm{B}_2.\end{aligned}\tag{5.77}$$

Since $a_n(x,u_n,\nabla u_n)=\overline{a}_n(x,u_n,\nabla\eta(u_n))$ and $a(x,u,\nabla u)=\overline{a}(x,u,\nabla\eta(u))$, from assumption (5.4), (5.55), (5.57),(5.75) and (5.76)

$$\begin{gathered}\overline{a}_n(x,u_n,\nabla\eta(u_n))h(\eta(u_n))\alpha_n(x)^{\frac{1}{p'}}\to\overline{a}(x,u,\nabla\eta(u))h(\eta(u))\alpha(x)^{\frac{1}{p'}-1}\\\text{weakly in }(\mathrm{L}^{p'}(\Omega))^{\mathrm{N}},\end{gathered}$$

i.e.

$$\begin{gathered}a_n(x,u_n,\nabla u_n)h(\eta(u_n))\alpha_n(x)^{\frac{1}{p'}}\to a(x,u,\nabla u)h(\eta(u))\alpha(x)^{\frac{1}{p'}-1}\\\text{weakly in }(\mathrm{L}^{p'}(\Omega))^{\mathrm{N}}.\end{gathered}\tag{5.78}$$

Noticing that for every $\varphi\in\mathrm{W}_0^{1.p}(\Omega)$

$$\alpha_n(x)^{\frac{1}{p}}\nabla\varphi\to\alpha(x)^{\frac{1}{p}}\nabla\varphi\text{ strongly in }(\mathrm{L}^p(\Omega))^{\mathrm{N}}.\tag{5.79}$$

(5.78) and (5.79) imply that

$$\lim_{n\to\infty}\int_\Omega a_n(x,u_n,\nabla u_n)h(\eta(u_n))\nabla\varphi dx=\int_\Omega a(x,u,\nabla u)h(\eta(u))\nabla\varphi dx.\tag{5.80}$$

By (5.53), (5.55), (5.75), (5.76) and Vitali's theorem, we obtain

$$\begin{gathered}\alpha_n(x)^{\frac{1}{p}}\nabla\eta(u_n)h'(\eta(u_n))\varphi \to \alpha(x)^{\frac{1}{p}}\nabla\eta(u)h'(\eta(u))\varphi \\ \text{strongly in } (\mathrm{L}^p(\Omega))^{\mathrm{N}},\end{gathered} \tag{5.81}$$

on the other, we have that the sequence

$$\overline{a}_n(x,u_n,\nabla\eta(u_n))\alpha_n(x)^{\frac{1}{p'}} \text{ is bounded in } (\mathrm{L}^{p'}(\Omega))^{\mathrm{N}},$$

and converges almost everywhere to $\overline{a}(x,u,\nabla\eta(u))\alpha(x)^{\frac{1}{p'}-1}$ tanks to (5.53), (5.55) and (5.57). Hence it also converges weakly in $(\mathrm{L}^{p'}(\Omega))^{\mathrm{N}}$ to its a.e. limit. i.e.

$$a_n(x,u_n,\nabla u_n)\alpha_n(x)^{\frac{1}{p'}} \to a(x,u,\nabla u)\alpha(x)^{\frac{1}{p'}-1} \text{ weakly in } (\mathrm{L}^{p'}(\Omega))^{\mathrm{N}}. \tag{5.82}$$

By (5.81) and (5.82), we have

$$\begin{aligned}\lim_{n\to\infty}\mathrm{B}_2 &= \lim_{n\to\infty}\int_\Omega a_n(x,u_n,\nabla u_n)\nabla\eta(u_n)h'(\eta(u_n))\varphi\,dx \\ &= \int_\Omega a(x,u,\nabla u)\nabla\eta(u)h'(\eta(u))\nabla\varphi\,dx.\end{aligned} \tag{5.83}$$

It follows from (5.50), (5.55) and Vitali's theorem that

$$\lim_{n\to\infty}\mathrm{B}_2 = \lim_{n\to\infty}\int_\Omega f_n h(\eta(u_n))\varphi\,dx = \int_\Omega f h(\eta(u))\varphi\,dx. \tag{5.84}$$

Let n tend to infinity in (5.77) ; by (5.80), (5.83) and (5.84) we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla((\eta(u_n))\varphi)\,dx = \int_\Omega f h(\eta(u))\varphi\,dx. \tag{5.85}$$

For any given $m>0$, taking $v=\mathrm{T}_{m+1}(\eta(u_n)-\mathrm{T}_m(\eta(u_n)))$ in (5.52), we have

$$\int_\Omega a_n(x,u_n,\nabla u_n)\nabla\mathrm{T}_{m+1}(\eta(u_n)-\mathrm{T}_m(\eta(u_n))) = \int_\Omega f_n\mathrm{T}_{m+1}(\eta(u_n)-\mathrm{T}_m(\eta(u_n)))\,dx. \tag{5.86}$$

For the term of the left hand side of (5.86), by the condition (4) of (5.4) we have

$$\begin{aligned}&\int_\Omega a_n(x,u_n,\nabla u_n)\nabla\mathrm{T}_{m+1}(\eta(u_n)-\mathrm{T}_m(\eta(u_n))) \\ &= \int_{\{m\le|\eta(u_n)|\le m+1\}} a_n(x,u_n,\nabla u_n)\nabla\eta(u_n)\,dx \\ &= \int_\Omega \overline{a}_n(x,\mathrm{T}_{\widehat{m+1}}(\eta(u_n)),\nabla\mathrm{T}_{m+1}(\eta(u_n)))\nabla\mathrm{T}_{m+1}(\eta(u_n))\chi_{\{m\le|\eta(u_n)|\le m+1\}}dx.\end{aligned} \tag{5.87}$$

Using (5.4), (5.57), and replacing k by $m+1$ in (5.75) and (5.76), we get

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \overline{a}_n(x,\mathrm{T}_{\widehat{m+1}}(\eta(u_n)),\nabla\mathrm{T}_{m+1}(\eta(u_n)))\nabla\mathrm{T}_{m+1}(\eta(u_n))\chi_{\{m\le|\eta(u_n)|\le m+1\}}\,dx\\
&=\lim_{n\to\infty}\int_\Omega \frac{\overline{a}(x,\mathrm{T}_{\widehat{m+1}}(\eta(u_n)),\nabla\mathrm{T}_{m+1}(\eta(u_n)))}{\alpha(x)}\alpha_n(x)\\
&\times\nabla\mathrm{T}_{m+1}(\eta(u_n))\chi_{\{m\le|\eta(u_n)|\le m+1\}}\,dx \qquad (5.88)\\
&=\int_\Omega \overline{a}(x,\mathrm{T}_{\widehat{m+1}}(\eta(u)),\nabla\mathrm{T}_{m+1}(\eta(u)))\nabla\mathrm{T}_{m+1}(\eta(u))\chi_{\{m\le|\eta(u)|\le m+1\}}\,dx\\
&=\int_{\{m\le|\eta(u_n)|\le m+1\}} a(x,u,\nabla u)\nabla\eta(u)\,dx.
\end{aligned}$$

From (5.55) we have $\mathrm{T}_{m+1}(\eta(u_n)-\mathrm{T}_m(\eta(u_n)))\to\mathrm{T}_{m+1}(\eta(u)-\mathrm{T}_m(\eta(u)))$ weakly* in $\mathrm{L}^\infty(\Omega)$ as n tends to infinity. So, from (5.50) we obtain

$$\lim_{n\to\infty}\int_\Omega f_n\mathrm{T}_{m+1}(\eta(u_n)-\mathrm{T}_m(\eta(u_n)))\,dx=\int_\Omega f\mathrm{T}_{m+1}(\eta(u)-\mathrm{T}_m(\eta(u)))\,dx. \qquad (5.89)$$

Let n tend to infinity in (5.86) ; it follows from (5.88) and (5.89) that

$$\int_{\{m\le|\eta(u_n)|\le m+1\}} a(x,u,\nabla u)\nabla\eta(u)\,dx=\int_\Omega f\mathrm{T}_{m+1}(\eta(u)-\mathrm{T}_m(\eta(u)))\,dx. \qquad (5.90)$$

From (5.90), we deduce

$$\lim_{m\to\infty}\int_{\{m\le|\eta(u_n)|\le m+1\}} a(x,u,\nabla u)\nabla\eta(u)\,dx=0. \qquad (5.91)$$

We conclude that u is a renormalized solution to problem5.1 in the sense of Definition 5.3.2.

Chapter 6

On some class of degenerate elliptic equations with $\mathrm{L}^1(\Omega)$-coefficient in p(x)-Sobolev spaces

In this chapter we study a class of nonlinear degenerate elliptic problem with coefficients in $\mathrm{L}^1(\Omega)$ of the form $-\operatorname{div} a(x,u,\nabla u) = f$, where $a(x,u,\nabla u)$ is allowed to be degenerate with the unknown u. We prove the existence of weak and entropy solutions under some hypothesis on f. We also study the same problem with a lower order term.

6.1 Introduction

Let Ω be a bounded subset of $\mathbb{R}^{\mathrm{N}}, \mathrm{N} \geq 2$. Porretta and Segura de León [70] have studied the Dirichlet problem for some nonlinear elliptic equations whose simplest model is

$$\begin{cases} -\operatorname{div}(g(u)|\nabla u|^{p-2}\nabla u) = h(u)|\nabla u|^p + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $1 < p < \infty$, g is a continuous positive function. Also under the additional hypothesis

$$\lim_{s\to\infty} \frac{h(s)}{g(s)} = 0,$$

they obtain L^∞-estimate. This kind of problems has been widely studied. The classical references are [56] and [60].

In this chapter we study the existence of weak and entropy solutions for the following nonlinear degenerate elliptic problem :

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

in the setting of the Sobolev space with variable exponent $W_0^{1,p(\cdot)}(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$), and

$$f \in L^m(\Omega), \text{ where } m = 1 \text{ or } m = (p^*)'.$$

Moreover we assume that $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory function such that for every s in $\mathbb{R}$, ξ, ξ' in $\mathbb{R}^N$ ($\xi \neq \xi'$), and for almost every $x \in \Omega$,

$$|a(x,s,\xi)| \leq \gamma\alpha(x)(1+g(s))|\xi|^{p(x)-1}, \tag{6.2}$$

$$a(x,s,\xi)\xi \geq \alpha(x)g(s)|\xi|^{p(x)}, \tag{6.3}$$

$$[a(x,s,\xi) - a(x,s,\xi')][\xi - \xi'] > 0, \tag{6.4}$$

where γ is a positive real constant.
The function $\alpha(x)$ satisfies, for $\nu > 0$

$$\alpha \in L^1(\Omega), \quad \alpha(x) > \nu, \tag{6.5}$$

and

$$g: \ \mathbb{R} \mapsto \mathbb{R}^+, \text{ is a positive continuous function, and} \tag{6.6}$$

$$\int_0^{+\infty} g(t)^{\frac{1}{p^{\pm}-1}}(t)\,dt = +\infty. \tag{6.7}$$

Then we study the perturbated problem

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) + H(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.8}$$

where $H(x,s,\xi)$ is a Carathéodory function satisfying

$$|\, H(x,s,\xi)\,| \leq \beta(x)h(s)\,|\,\xi\,|^{p(x)}, \tag{6.9}$$

for almost every $x \in \Omega$, for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, $\gamma > 0$, where the functions $\beta(.)$ satisfies, for $\theta > 0$

$$\beta \in L^1(\Omega), \quad \beta(x) \geq 0 \text{ and } \beta(x) \leq \theta\alpha(x) \text{ a.e. in } \Omega, \text{ and} \tag{6.10}$$

$h: \ \mathbb{R} \mapsto \mathbb{R}^+$, is a positive continuous functions $g > 0$,

$$\frac{h}{g} \in L^1(\mathbb{R}). \tag{6.11}$$

Furthermore, we will study the case when $f \in L^1(\Omega)$ and $f \in L^{(p^*(x))'}(\Omega)$.

To deal with this kind of problem, it is natural to work under the framework of Sobolev spaces with variable exponents. The study of differential equations with variable exponents has been a very active field in recent years, with applications in electro-rheological fluids and image processing, and so on. We refer the readers to [67] and references therein.

Some remarks concerning the difficulties in dealing with this problem are in order. First of all, the nonlinearity of the operator and the singularity of the coefficient will be solved by coupling energy estimates of type $\int_\Omega \alpha(x)|\nabla u|^{p(x)} < \infty$, on the one hand, we suppose no growth limitation on function g so that it can happen $a(x,u,\nabla u) \notin W^{-1,p'(\cdot)}(\Omega)$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$ and, on the other hand, it is not coercive. To deal with that equation we will obtain the solution by approximation, getting a sequence of approximated solutions $(u_n)_n$ which converge to the solution u. To avoid some troubles in this convergence process, we will consider the convergence of another sequence $(A(u_n))_n$, function A being the primitive of $g^{\frac{1}{p^{\pm}-1}}$ such that $A(0) = 0$. The other remark is about the lower order term : it does not satisfy the sign condition, since h is positive. Thus, it appears the problem of getting the a priori estimates. This hindrance is overcome by considering test functions of exponential type.
With a modified version of the classical Minty Lemma (see [30]). We stress that in our treatment we avoid the use of weighted Sobolev spaces.

6.2 Main results

6.2.1 Some definitions and technical lemmas

Definition 6.2.1 *For all $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ can be defined by*

$$T_k(s) = \begin{cases} s & if \quad |s| \le k \\ k \cdot sign(s) & if \quad |s| > k, \end{cases}$$

and we define $G_k(s) = s - T_k(s)$.

We define

$$\eta(s) = \int_0^s \frac{h(\tau)}{g(\tau)} d\tau, \tag{6.12}$$

we moreover define the space of functions

$$X_0^{p(\cdot)}(\Omega) = \{\varphi \in W_0^{1,p(\cdot)}(\Omega) \text{ such that } \int_\Omega \alpha(x)|\nabla\varphi|^{p(x)} < \infty\}, \tag{6.13}$$

and

$$\mathcal{T}_0^{1,p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable such that } T_k(u) \in W_0^{1,p(\cdot)}(\Omega)\},$$

i.e. the set of measurable functions whose truncates belong to $W_0^{1,p(\cdot)}(\Omega)$ (we refer to [17] for more details).
We note

$$A(s) = \int_0^s g(\tau)^{1/(p^+-1)} d\tau \text{ if } g \text{ is unbounded,} \tag{6.14}$$

$$A(s) = \int_0^s g(\tau)^{1/(p^--1)} d\tau \text{ if } g \text{ is bounded.} \tag{6.15}$$

We state and prove a preliminary Lemma (see [31] Lemma 3.3) that will be often used in the sequel.

Lemma 6.2.1 *Let σ_n be a sequence of nonnegative bounded functions, almost everywhere convergent to some function σ, and let $\rho_n : \mathbb{R}^N \to \mathbb{R}^N$ be a sequence of functions which is weakly convergent in $(L^{q(x)}(\Omega))^N$ ($q(x) > 1$) to some function ρ. If the sequence $\sigma_n|\rho_n|^{q(x)}$ is bounded in $L^1(\Omega)$, then $\sigma|\rho|^{q(x)}$ belongs to $L^1(\Omega)$ and*

$$\int_\Omega \sigma|\rho|^{q(x)} \le \lim_{n\to+\infty} \inf \sigma_n|\rho_n|^{q(x)}. \tag{6.16}$$

If moreover $\sigma_n \to \sigma$ strongly in $L^1(\Omega)$ then it holds true also that

$$\sigma_n^{1/q(x)}\rho_n \rightharpoonup \sigma^{1/q(x)}\rho \text{ weakly in } L^{q(x)}(\Omega)^N. \tag{6.17}$$

Proof. Since ρ_n weakly convergent to ρ in $(L^{q(x)}(\Omega))^N$, then there exists a functional $h \in (L^{q(x)}(\Omega))^N$ such that $|\rho_n| \le h$ Let $k > 0$ and $\psi \in (L^{q'(x)}(\Omega))^N$ we have $|T_k(\sigma_n)^{1/q(x)}\rho_n.\psi| \le k^{1/q(\bar{x})}h(x).\psi(x)$ and $T_k(\sigma_n)^{1/q(x)}\rho_n.\psi \to T_k(\sigma)^{1/q(x)}\rho.\psi$ a.e. in Ω, witch give

$$T_k(\sigma_n)^{1/q(x)}\rho_n \rightharpoonup T_k(\sigma)^{1/q(x)}\rho \text{ weakly in } (L^{q(x)}(\Omega))^N. \tag{6.18}$$

Hence by the lower semi-continuity of the $L^{q(x)}$-norm it results

$$\int_\Omega T_k(\sigma)|\rho|^{q(x)} \le \lim_{n\to+\infty} \inf \int_\Omega T_k(\sigma_n)|\rho_n|^{q(x)} \le \lim_{n\to+\infty} \inf \sigma_n|\rho_n|^{q(x)} \le C. \tag{6.19}$$

Then letting k tend to infinity and using the monotone convergence theorem we obtain (6.16). Let now ψ be an arbitrary element of $(L^{q'(x)}(\Omega))^N$, then

$$\begin{aligned}\int_\Omega \sigma_n^{1/q(x)}\rho_n\psi &= \int_{\{\sigma_n\le k\}} \sigma_n^{1/q(x)}\rho_n\psi + \int_{\{\sigma_n>k\}} \sigma_n^{1/q(x)}\rho_n\psi \\ &= \varepsilon_n + \int_{\{\sigma\le k\}} \sigma_n^{1/q(x)}\rho_n\psi + \int_{\{\sigma_n>k\}} \sigma_n^{1/q(x)}\rho_n\psi.\end{aligned}$$

Moreover

$$\int_\Omega \sigma^{1/q(x)}\rho_n\psi = \int_{\{\sigma_n\le k\}} \sigma^{1/q(x)}\rho\psi + \int_{\{\sigma_n>k\}} \sigma^{1/q(x)}\rho\psi$$
$$= \varepsilon_n + \int_{\{\sigma\le k\}} \sigma^{1/q(x)}\rho\psi + \int_{\{\sigma_n>k\}} \sigma^{1/q(x)}\rho\psi.$$

It follows that

$$\Big|\int_\Omega [\sigma_n^{1/q(x)}\rho_n - \sigma^{1/q(x)}\rho].\psi\Big| \le \varepsilon_n + \int_{\{\sigma_n>k\}} \sigma_n^{1/q(x)}|\rho_n||\psi| + \int_{\{\sigma_n>k\}} \sigma^{1/q(x)}|\rho||\psi|.$$

Using Hölder inequality we have that

$$\int_{\{\sigma_n>k\}} \sigma_n^{1/q(x)}\rho_n\psi \le \|\sigma_n^{1/q(x)}\rho_n\|_{q(x)}\|\psi\|_{\mathrm{L}^{q'(x)}(\sigma_n>k)},$$

and

$$\int_{\{\sigma_n>k\}} \sigma^{1/q(x)}\rho\psi \le \|\sigma^{1/q(x)}\rho\|_{q(x)}\|\psi\|_{\mathrm{L}^{q'(x)}(\sigma_n>k)}.$$

Lemma 6.2.2 *[85] Assume that* (6.2)-(6.4) *hold, with* $g \ge \lambda$ *in* (6.3), $\lambda > 0$ *and let* $(u_n)_n$ *be a sequence in* $\mathrm{W}_0^{1.p(\cdot)}(\Omega)$ *such that* $u_n \rightharpoonup u$ *in* $\mathrm{W}_0^{1.p(\cdot)}(\Omega)$ *and*

$$\int_\Omega [a_n(x,u_n,\nabla u_n) - a_n(x,u_n,\nabla u)]\nabla(u_n-u) \to 0.$$

Then, $u_n \to u$ *in* $\mathrm{W}_0^{1.p(\cdot)}(\Omega)$.

6.2.2 Existence results

Theorem 6.2.1 *Assume that (6.2)-(6.7) hold true, then for every* $f \in \mathrm{L}^1(\Omega)$ *there exists* $u \in \mathscr{T}_0^{1,p(\cdot)}(\Omega)$ *entropy solution of (6.1) in the following sense*
$\forall k>0$, $\int_\Omega \alpha(x)g(u)^{p'(x)}|\nabla \mathrm{T}_k(u)|^{p(x)} < \infty$, *and*

$$\int_\Omega a(x,u,\nabla u)\nabla \mathrm{T}_k(u-\phi) = \int_\Omega f\mathrm{T}_k(u-\phi) \quad \forall \phi \in \mathrm{W}_0^{1,p(\cdot)}(\Omega)\cap \mathrm{L}^\infty(\Omega). \tag{6.20}$$

If moreover $f \in \mathrm{L}^{(p^*(.))'}(\Omega)$, *then* $u \in \mathrm{W}_0^{1,p(\cdot)}(\Omega)$
$\int_\Omega \alpha(x)g(u)^{p'(x)}|\nabla u|^{p(x)} < \infty$,*and*

$$\int_\Omega a(x,u,\nabla u)\nabla\phi = \int_\Omega f\phi \quad \forall \phi \in \mathrm{W}_0^{1.p(\cdot)}(\Omega).$$

Theorem 6.2.2 *Assume that (6.2)-(6.7) and (6.9)-(6.11) hold true, then for every* $f \in L^1(\Omega)$ *there exists* $u \in \mathcal{T}_0^{1,p(\cdot)}(\Omega)$ *entropy solution of (6.8) in the following sense*
$\forall k > 0$, $\int_\Omega \alpha(x) g(u)^{p'(x)} |\nabla T_k(u)|^{p(x)} < \infty$, $H(x,u,\nabla u) \in L^1(\Omega)$, *and*

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi) + \int_\Omega H(x,u,\nabla u) T_k(u-\phi) = \int_\Omega f T_k(u-\phi),\ \forall \phi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega).$$

If moreover $f \in L^{(p^*(.))'}(\Omega)$, *then* $u \in W_0^{1,p(\cdot)}(\Omega)$
$\int_\Omega \alpha(x) g(u)^{p'(x)} |\nabla u|^{p(x)} < \infty$, $H(x,u,\nabla u) \in L^1(\Omega)$
and

$$\int_\Omega a(x,u,\nabla u)\nabla \phi + \int_\Omega H(x,u,\nabla u)\phi = \int_\Omega f\phi,\ \forall \phi \in W_0^{1.p(\cdot)}(\Omega) \cap L^\infty(\Omega).$$

With C_i, $i \in \mathbb{N}$, we indicate generic positive constants that may depend on the dimension N, on the domain Ω, and on the other data of the problem, while with ε_n we indicate a generic sequence that goes to zero as n diverges.

Proof of Theorem 6.2.1

Step 1 : Approximation and a priori estimates.
Let $(f_n)_{u\in\mathbb{N}}$ be a sequence of smooth function such $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \le |f|$, we consider the following problem

$$\int_\Omega a_n(x,u_n,\nabla u_n)\nabla \phi dx = \int_\Omega f_n \phi dx, \quad \forall \phi \in W_0^{1.p(\cdot)}(\Omega), \tag{6.21}$$

where

$$a_n(x,s,\xi) = \frac{a(x,T_n(s),\xi)}{\alpha(x)}\alpha_n(x),$$
$$\alpha_n(x) = T_n(\alpha(x)),\ \ g_n(s) = g(T_n(s)).$$

Note that g is continuous, so there exists $\lambda_n \ge 0$ such that $g_n(s) \ge \lambda_n$ and since $\int_0^\infty g_n^{\frac{1}{p^\pm - 1}}(t)dt = \infty$. Then by definition $a_n(x,s,\xi)$ satisfies

$$|a_n(x,s,\xi)| \le \gamma \alpha_n(x)(1+g_n(s))|\xi|^{p(x)-1},$$

$$a_n(x,s,\xi)\xi \ge \alpha_n(x) g_n(s)|\xi|^{p(x)} \ge \lambda_n \alpha_n(x)|\xi|^{p(x)},$$

$$[a_n(x,s,\xi) - a_n(x,s,\xi')][\xi - \xi'] > 0, \text{ for } \xi \ne \xi',$$

we can use the theorem 4.3 of [4] to infer the existence of $u_n \in W_0^{1.p(x)}(\Omega) \cap L^\infty(\Omega)$ solution to problem (6.21).
Case 1. The function g is unbounded, i.e. $\lim_{s\to+\infty} g(s) = +\infty$

Then, there exists $g_0 > 0$ such that $\alpha(s) > 1$ for all $s \geq g_0$ since $\lim_{s\to+\infty} g(s) = +\infty$. For every $k > A(g_0)$, with

$$A(s) = \int_0^s g(\tau)^{1/(p^+-1)} d\tau,$$

taking $T_k(A(u_n))$ as test function in (6.21), there exists $C_0 > 0$, and by assumption (6.3), we have

$$\begin{aligned} C_0 \nu \int_{|A(u)|\leq k} |\nabla T_k(A(u_n))|^{p(x)} &\leq C_0 \int_{|A(u)|\leq k} \alpha_n(x) |\nabla T_k(A(u_n))|^{p(x)} \\ &\leq C_0 \int_{|A(u)|\leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p^+-1}} |\nabla u_n|^{p(x)} \\ &\leq \int_{|A(u)|\leq k} \alpha_n(x) g(u_n)^{\frac{p^+}{p^+-1}} |\nabla u_n|^{p(x)} \\ &\leq \int_{|A(u)|\leq k} a_n(x, u_n, \nabla u_n) \nabla u_n g^{\frac{1}{p^+-1}}(u_n) dx \\ &\leq \int_\Omega |f_n| |T_k(A(u_n))| dx \\ &\leq k \|f\|_{L^1}. \end{aligned} \tag{6.22}$$

Case 2. The function g is bounded, i.e. there exists a constant $M > 0$ such that $g(s) \leq M$, for every $s \in [0, +\infty[$. Taking $T_k(A(u_n))$ as an admissible test function in (6.21), with

$$A(s) = \int_0^s g(\tau)^{1/(p^--1)} d\tau.$$

There exists $C_1 > 0$, and by assumption (6.3), we have

$$\begin{aligned} C_1 \nu \int_{|A(u)|\leq k} |\nabla T_k(A(u_n))|^{p(x)} &\leq C_1 \int_{|A(u)|\leq k} \alpha_n(x) |\nabla T_k(A(u_n))|^{p(x)} \\ &\leq C_1 \int_{|A(u)|\leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p^--1}} |\nabla u_n|^{p(x)} \\ &\leq \int_{|A(u)|\leq k} a_n(x, u_n, \nabla u_n) \nabla u_n g^{\frac{1}{p^--1}}(u_n) dx \\ &\leq \int_\Omega |f_n| |T_k(A(u_n))| dx \\ &\leq k \|f\|_{L^1}. \end{aligned} \tag{6.23}$$

And consequently there exist $C_2,\ C_3 > 0$ such that

$$\int_\Omega |\nabla T_k(A(u_n))|^{p(x)} dx \leq C_2 \int_\Omega \alpha_n(x) |\nabla T_k(A(u_n))|^{p(x)} \leq C_3 k. \tag{6.24}$$

Therefore,

$$\|\nabla T_k(A(u_n))\|_{p(\cdot)}^r dx \leq C_3 k, \tag{6.25}$$

with

$$r = \begin{cases} p^+ & \text{if} \quad \|\nabla \mathrm{T}_k(\mathrm{A}(u_n))\|_{p(\cdot)} < 1, \\ p^- & \text{if} \quad \|\nabla \mathrm{T}_k(\mathrm{A}(u_n))\|_{p(\cdot)} \geq 1. \end{cases}$$

Let $k \geq 1$, we have

$$k\mathrm{meas}\{|\mathrm{A}(u_n)| > k\} = \int_{|\mathrm{A}(u_n)|>k} |\mathrm{T}_k(\mathrm{A}(u_n))| dx \leq \mathrm{C}_3 k^{\frac{1}{r}},$$

which implies that

$$\mathrm{meas}\{|\mathrm{A}(u_n)| > k\} \leq \mathrm{C}_3 \frac{1}{k^{1-\frac{1}{r}}} \longrightarrow 0 \text{ as } \; k \to \infty.$$

By the usual method we get that $\mathrm{T}_k(\mathrm{A}(u_n))$ is bounded in $\mathrm{W}_0^{1.p(\cdot)}(\Omega)$, then there exists a subsequence still denoted $(\mathrm{T}_k(\mathrm{A}(u_n)))_{n\in\mathbb{N}}$ such that

$$\mathrm{T}_k(\mathrm{A}(u_n)) \rightharpoonup \eta_k \text{ weakly in } \mathrm{W}_0^{1.p(\cdot)}(\Omega),$$

and by the Sobolev compact embedding, we have

$$\mathrm{T}_k(\mathrm{A}(u_n)) \to \eta_k \text{ strongly in } \mathrm{L}^{p(\cdot)}(\Omega), \text{ and a.e. in } \Omega.$$

Consequently, we can assume that $(\mathrm{T}_k(\mathrm{A}(u_n)))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Thus,

$$\mathrm{meas}\{|\mathrm{T}_k(\mathrm{A}(u_n)) - \mathrm{T}_k(\mathrm{A}(u_m))| > \delta\} \leq \frac{\varepsilon}{2}, \text{ for all } m, n \geq n_0(\delta, \varepsilon).$$

We conclude that for all $\delta, \varepsilon > 0$ there exists $n_0 = n_0(\delta, \varepsilon)$ such that $meas\{|u_n - u_m| > \delta\} \leq \varepsilon$ for all $\delta, \varepsilon > 0$. It follows that $(\mathrm{T}_k(\mathrm{A}(u_n)))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, then it converges almost everywhere, for a subsequence, to some measurable function u.
Consequently, we have

$$\begin{aligned} &\mathrm{T}_k(\mathrm{A}(u_n)) \rightharpoonup \mathrm{T}_k(\mathrm{A}(u)) \text{ weakly in } \mathrm{W}_0^{1.p(\cdot)}(\Omega), \\ &\mathrm{T}_k(\mathrm{A}(u_n)) \to \mathrm{T}_k(\mathrm{A}(u)) \text{ strongly in } \mathrm{L}^{p(\cdot)}(\Omega), \text{ and a.e. in } \Omega. \end{aligned} \tag{6.26}$$

Moreover by (6.24), (6.26) and Lemma 6.2.1 we get

$$\begin{aligned} &\alpha_n(x)^{\frac{1}{p(x)}} \nabla \mathrm{T}_k(\mathrm{A}(u_n)) \rightharpoonup \alpha(x)^{\frac{1}{p(x)}} \nabla \mathrm{T}_k(\mathrm{A}(u)) \text{ weakly in } \mathrm{L}^{p(\cdot)}(\Omega)^{\mathrm{N}}, \text{and} \\ &\alpha(x)|\nabla \mathrm{T}_k(\mathrm{A}(u))|^{p(x)} \in \mathrm{L}^1(\Omega). \end{aligned} \tag{6.27}$$

Step 2 : Strong convergence of truncations. Let $w_n = \mathrm{T}_{2k}(\mathrm{Z}_n)$ be a test function in (6.21), where $\mathrm{Z}_n = (\mathrm{A}(u_n) - \mathrm{T}_h(\mathrm{A}(u_n)) + \mathrm{T}_k(\mathrm{A}(u_n)) - \mathrm{T}_k(\mathrm{A}(u)))$, for $h > k > 0$. Taking $\mathrm{M} = 4k + h$ we

get that,

$$\int_\Omega a_n(x,u_n,\nabla u_n)\nabla w_n dx \le \int_\Omega |f_n||w_n| dx.$$

We have

$$\begin{aligned}\int_\Omega a_n(x,u_n,\nabla u_n)\nabla w_n dx &= \int_{\{|\mathrm{A}(u_n)|>k\}} a_n(x,u_n,\nabla u_n)\nabla w_n dx \\ &\quad + \int_{\{|\mathrm{A}(u_n)|\le k\}} a_n(x,u_n,\nabla u_n)\nabla w_n dx.\end{aligned} \tag{6.28}$$

Concerning the first term on the right hand side of (6.28), since $\nabla w_n = 0$ on $\{|\mathrm{A}(u_n)| > \mathrm{M}\}$, we have

$$\begin{aligned}\int_{\{|\mathrm{A}(u_n)|>k\}} a_n(x,u_n,\nabla u_n)\nabla w_n dx &= \int_{\{|\mathrm{A}(u_n)|>k\}} a(x,\mathrm{T}_{\widehat{\mathrm{M}}}(u_n),\nabla \mathrm{T}_{\widehat{\mathrm{M}}}(u_n))\nabla z_n \\ &\ge -\int_{\{|\mathrm{A}(u_n)|>k\}} a_n(x,\mathrm{T}_{\widehat{\mathrm{M}}}(u_n),\nabla \mathrm{T}_{\widehat{\mathrm{M}}}(u_n))\nabla \mathrm{T}_k(\mathrm{A}(u))dx \\ &\ge -\varepsilon_0(n).\end{aligned}$$

Where $\widehat{\mathrm{M}}$ is a positive constant depending on M such that $\{|\mathrm{A}(u_n)| > \mathrm{M}\}$, can be equivalently replaced by $\{|u_n| > \widehat{\mathrm{M}}\}$.
For the second term on the right hand side of (6.28) ; we have for $\widehat{k}$ defined as $\widehat{\mathrm{M}}$

$$\begin{aligned}\int_{\{|\mathrm{A}(u_n)|\le k\}} a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla \mathrm{T}_{\widehat{k}}(u_n))(\nabla \mathrm{T}_k(\mathrm{A}(u_n)) - \nabla \mathrm{T}_k(\mathrm{A}(u)))dx \\ \le \int_\Omega |f_n||w_n| dx + \varepsilon_0(n).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\int_{\{|\mathrm{A}(u_n)|\le k\}} a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla \mathrm{T}_{\widehat{k}}(u_n))(\nabla \mathrm{T}_k(\mathrm{A}(u_n)) - \nabla \mathrm{T}_k(\mathrm{A}(u)))dx \\ &\quad = \int_{\{|\mathrm{A}(u_n)|\le k\}} (a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla \mathrm{T}_{\widehat{k}}(u_n)) - a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla \mathrm{T}_{\widehat{k}}(u))) \\ &\qquad\qquad \times (\nabla \mathrm{T}_k(\mathrm{A}(u_n)) - \nabla \mathrm{T}_k(\mathrm{A}(u)))dx \\ &\quad + \int_{\{|\mathrm{A}(u_n)|\le k\}} a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla \mathrm{T}_{\widehat{k}}(u))(\nabla \mathrm{T}_k(\mathrm{A}(u_n)) - \nabla \mathrm{T}_k(\mathrm{A}(u)))dx.\end{aligned}$$

By (6.27) and the definition of $a_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_{\widehat{k}}(u))$ the third terms of the right hand tend to 0, as n tend to infinity. So, we have

$$\begin{aligned}\int_{\{|\mathrm{A}(u_n)|\leq k\}} & (a_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_{\widehat{k}}(u_n)) - a_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_{\widehat{k}}(u))) \\ & \times (\nabla \mathrm{T}_k(\mathrm{A}(u_n)) - \nabla \mathrm{T}_k(\mathrm{A}(u)))dx \\ & = \int_{\{|\mathrm{A}(u_n)|\leq k\}} a_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_{\widehat{k}}(u_n))(\nabla \mathrm{T}_k(\mathrm{A}(u_n)) - \nabla \mathrm{T}_k(\mathrm{A}(u)))dx + \varepsilon_1(n) \\ & \leq \int_\Omega |f_n(x)||w_n|dx + \varepsilon_1(n) + \varepsilon_0(n) \leq \varepsilon_2(n),\end{aligned} \tag{6.29}$$

then

$$\begin{aligned}\int_\Omega [a_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_{\widehat{k}}(u_n)) - a_n(x, \mathrm{T}_{\widehat{k}}(u_n), \nabla \mathrm{T}_{\widehat{k}}(u_n))](\nabla \mathrm{T}_k(\mathrm{A}(u_n)) - \nabla \mathrm{T}_k(\mathrm{A}(u)))dx \\ \leq \varepsilon_2(n),\end{aligned}$$

as $f_n \to f$ strongly in $\mathrm{L}^1(\Omega)$, and $w_n \rightharpoonup 0$ weakly* in $\mathrm{L}^\infty(\Omega)$.
So by the Lemma 6.2.2, we conclude that

$$\mathrm{T}_k(\mathrm{A}(u_n)) \to \mathrm{T}_k(\mathrm{A}(u)) \text{ in } \mathrm{W}_0^{1,p(\cdot)}(\Omega),$$

$$\nabla \mathrm{T}_k(\mathrm{A}(u_n)) \to \nabla \mathrm{T}_k(\mathrm{A}(u)) \text{ almost everywhere in } \Omega$$

.

If g is unbounded, we have

$$\begin{aligned}|\nabla \mathrm{T}_k(\mathrm{A}(u_n))|^{p(x)} &= |\nabla \mathrm{A}(u_n)|^{p(x)}\chi_{|\mathrm{A}(u_n)|\leq k} \\ &= g^{\frac{p(x)}{p^+-1}}(u_n)\chi_{|\mathrm{A}(u_n)|\leq k}|\nabla u_n|^{p(x)}, \\ g^{p(x)/(p^+-1)}(u_n)\chi_{|u_n|\leq \widehat{k}}|\nabla u_n|^{p(x)} &= g^{p(x)/(p^+-1)}(\mathrm{T}_{\widehat{k}}(u_n))|\nabla \mathrm{T}_{\widehat{k}}(u_n)|^{p(x)}, \\ |\nabla \mathrm{T}_{\widehat{k}}(u_n)|^{p(x)} &= \frac{|\nabla \mathrm{T}_k(\mathrm{A}(u_n))|^{p(x)}}{g^{\frac{p(x)}{p^+-1}}(\mathrm{T}_{\widehat{k}}(u_n))},\end{aligned}$$

and if g is bounded we have

$$|g\mathrm{T}_k(\mathrm{A}(u_n))|^{p(x)} = |\nabla \mathrm{A}(u_n)|^{p(x)}\chi_{|\mathrm{A}(u_n)|\leq k} = g^{\frac{p(x)}{p^--1}}(u_n)\chi_{|\mathrm{A}(u_n)|\leq k}|\nabla u_n|^{p(x)},$$

$$g^{\frac{p(x)}{p^--1}}(u_n)\chi_{|u_n|\leq \widehat{k}}|\nabla u_n|^{p(x)} = g^{\frac{p(x)}{p^--1}}(\mathrm{T}_{\widehat{k}}(u_n))|\nabla \mathrm{T}_m(u_n)|^{p(x)},$$

$$|\nabla \mathrm{T}_{\widehat{k}}(u_n)|^{p(x)} = \frac{|\nabla \mathrm{T}_k(\mathrm{A}(u_n))|^{p(x)}}{g^{\frac{p(x)}{p^--1}}(\mathrm{T}_{\widehat{k}}(u_n))},$$

since g continuous we we have $g(\mathrm{T}_{\widehat{k}}(u_n)) \geq \min_{[0,\widehat{k}]}(g(s)) = g_{\widehat{k}}$. Finally, we have

$$|\nabla \mathrm{T}_m(u_n)|^{p(x)} \leq \mathrm{C}|\nabla \mathrm{T}_k(\mathrm{A}(u_n))|^{p(x)}. \tag{6.30}$$

and on the other, by Lemma 6.2.1, that for every $k > 0$

$$\alpha(x)|\nabla \mathrm{T}_k(u)|^{p(x)} \in \mathrm{L}^1(\Omega),$$

and

$$\alpha_n(x)^{\frac{1}{p(x)}} \nabla \mathrm{T}_k(u_n) \rightharpoonup \alpha(x)^{\frac{1}{p(x)}} \nabla \mathrm{T}_k(u) \ \ weakly \ in \ (\mathrm{L}^{p(x)}(\Omega))^{\mathrm{N}}. \tag{6.31}$$

Step 3 : Entropy solutions. Case 1. The function α is unbounded, taking $\mathrm{T}_k(\mathrm{A}(u_n))$ as test function in (6.21), with $\mathrm{A}(s) = \int_0^s g(\tau)^{1/(p^- - 1)} d\tau$, we obtaining

$$\begin{aligned} \mathrm{C}_4 \int_{|\mathrm{A}(u)|\leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} &\leq \int_{|\mathrm{A}(u)|\leq k} a_n(x, u_n, \nabla u_n) \nabla \mathrm{T}_k(u_n) g^{\frac{1}{p^- - 1}}(u_n) dx \\ &\leq \int_\Omega |f_n||\mathrm{T}_k(\mathrm{A}(u_n))| dx \\ &\leq k\|f\|_{\mathrm{L}^1}. \end{aligned} \tag{6.32}$$

Case 2. The function g is bounded, taking $\mathrm{T}_k(\mathrm{A}(u_n))$ as test function in (6.21), with $\mathrm{A}(s) = \int_0^s g(\tau)^{1/(p^+ - 1)} d\tau$, we have

$$\begin{aligned} \mathrm{C}_5 \int_{|\mathrm{A}(u)|\leq k} \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} &\leq \int_{|\mathrm{A}(u)|\leq k} a_n(x, u_n, \nabla u_n) \nabla \mathrm{T}_k(u_n) g^{\frac{1}{p^+ - 1}}(u_n) dx \\ &\leq \int_\Omega |f_n||\mathrm{T}_k(\mathrm{A}(u_n))| dx \\ &\leq k\|f\|_{\mathrm{L}^1}. \end{aligned} \tag{6.33}$$

By means of (6.32) and (6.33), we obtain

$$\int_\Omega \alpha_n(x) g(u_n)^{p'(x)} |\nabla \mathrm{T}_k(u_n)|^{p(x)} \leq \mathrm{C}_6 k\|f\|_{\mathrm{L}^1}. \tag{6.34}$$

Let $\phi \in \mathrm{W}_0^{1,p(\cdot)}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, taking $\mathrm{T}_k(u_n - \phi)$ as a test function in the approximate problem (6.21), we get

$$\int_\Omega a_n(x, u_n, \nabla u_n) \nabla \mathrm{T}_k(u_n - \phi) dx = \int_\Omega f_n \mathrm{T}_k(u_n - \phi) dx.$$

Adding and subtracting in the equation above the term

$$\int_\Omega a_n(x,u_n,\nabla\phi)\nabla \mathrm{T}_k(u_n-\phi)dx,$$

and taking advantage of the monotonicity condition (6.4) we get

$$\int_\Omega a_n(x,u_n,\nabla\phi)\nabla \mathrm{T}_k(u_n-\phi)dx \le \int_\Omega f_n \mathrm{T}_k(u_n-\phi)dx. \tag{6.35}$$

Noticing that $\{|u_n-\phi|\le k\}\subset\{|u_n|\le k+\|\phi\|_{\mathrm{L}^\infty}\}$ and that

$$\int_\Omega a_n(x,u_n,\nabla\phi)\nabla \mathrm{T}_k(u_n-\phi)dx = \int_\Omega \frac{a(x,\mathrm{T}_n(u_n),\nabla\phi)}{\alpha(x)}\alpha_n(x)\nabla \mathrm{T}_k(u_n-\phi)dx,$$

we can pass to the limit in (6.35) using assumption (6.2),(6.30),(6.31) and (6.34). Hence we obtain, for every $k>0$,

$$\int_\Omega a(x,u,\nabla\phi)\nabla \mathrm{T}_k(u-\phi)dx \le \int_\Omega f\mathrm{T}_k(u-\phi)dx. \tag{6.36}$$

To recover (6.20), let h and k be positive real numbers, let $t=\pm 1$, and let ψ be a function in $\mathrm{W}_0^{1.p(x)}(\Omega)\cap\mathrm{L}^\infty(\Omega)$. Choose $\phi=\mathrm{T}_h(u)+t\mathrm{T}_k(u-\psi)$ in the inequality in (6.36) we obtain

$$\begin{aligned}\int_\Omega a(x,u,\nabla \mathrm{T}_h(u)+t\nabla \mathrm{T}_k(u-\psi))\nabla \mathrm{T}_k(\mathrm{G}_h(u)-t\mathrm{T}_k(u-\psi))dx\\ \le \int_\Omega f\mathrm{T}_k(\mathrm{G}_h(u)-t\mathrm{T}_k(u-\psi))dx,\end{aligned} \tag{6.37}$$

and we follow the same steps as the Lemma 7 in [30] we get

$$-t\int_\Omega a(x,u,\nabla u+t\nabla \mathrm{T}_k(u-\psi))\nabla \mathrm{T}_k(u-\psi)dx \le -t\int_\Omega f\mathrm{T}_k(u-\psi)dx. \tag{6.38}$$

for every $\psi\in \mathrm{W}_0^{1.p(\cdot)}(\Omega)\cap\mathrm{L}^\infty(\Omega)$, and for every $k>0$. Choosing $t>0$, dividing by t, and then letting t tend to zero, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla \mathrm{T}_k(u-\psi)dx \ge \int_\Omega f\mathrm{T}_k(u-\psi)dx.$$

While the reverse inequality is obtained choosing $t<0$, dividing by $-t$, and then letting t tend to zero.

Step 4 : weak solutions. In this last part of the proof we consider the stronger assumption $f\in \mathrm{L}^{(p^*(\cdot))'}(\Omega)$. Recalling that $u_n\in \mathrm{W}_0^{1.p(\cdot)}(\Omega)\cap\mathrm{L}^\infty(\Omega)$, we can choose $\mathrm{A}(u_n)$ as test function in (6.21), where $\mathrm{A}(s)=\displaystyle\int_0^s g(\tau)^{1/(p^+-1)}d\tau$.

We obtain

$$\begin{aligned}\int_\Omega a_n(x,u_n,\nabla u_n)\nabla u_n g^{\frac{1}{p^+-1}}(u_n)dx &\le \int_\Omega |f||\mathrm{T}_k(\mathrm{A}(u_n))|dx \\ &\le \|\mathrm{A}(u_n)\|_{\mathrm{L}^{p*(x)}}\|f\|_{\mathrm{L}^{(p*(x))'}}.\end{aligned} \tag{6.39}$$

Using Sobolev inequality we get

$$\int_\Omega \alpha_n(x)g(u_n)^{p'(x)}|\nabla u_n|^{p(x)} \le \frac{(\mathrm{C}_7\|f\|_{\mathrm{L}^{(p^*(x))'}})^{p^{+'}}}{\nu^{\frac{1}{1+p^+}}}. \tag{6.40}$$

Thanks to the a.e. convergence of ∇u_n to ∇u and Fatou Lemma,

$$\int_\Omega \alpha(x)g(u)^{p'(x)}|\nabla u|^{p(x)} \le \frac{(\mathrm{C}_7\|f\|_{\mathrm{L}^{(p^*(x))'}})^{p^{+'}}}{\nu^{\frac{1}{1+p^+}}}. \tag{6.41}$$

Thus we have that the sequence

$$\frac{a(x,\mathrm{T}_n(u_n),\nabla u_n)}{\alpha(x)}\alpha_n(x)^{\frac{1}{p'(x)}} \text{ is bounded in } (\mathrm{L}^{p'(\cdot)}(\Omega))^{\mathrm{N}},$$

and converges almost everywhere to $a(x,u,\nabla u)\alpha(x)^{\frac{1}{p'(x)}-1}$. Hence it also converges weakly in $(\mathrm{L}^{p'(\cdot)}(\Omega))^{\mathrm{N}}$ to its a.e.-limit. Noticing that for every $\phi \in \mathrm{X}_0^{p(\cdot)}(\Omega)$

$$\alpha_n(x)^{\frac{1}{p(x)}}\nabla\phi \to \alpha(x)^{\frac{1}{p(x)}}\nabla\phi \text{ strongly in } \mathrm{L}^{p(\cdot)}(\Omega)^{\mathrm{N}},$$

we can pass to the limit in (6.21) and conclude the proof.

Proof of Theorem 6.2.2

Step 1 : Approximation and a priori estimates.
Let $(f_n)_{u\in\mathbb{N}}$ be a sequence of smooth function such $f_n \to f$ in $\mathrm{L}^1(\Omega)$ and $|f_n| \le |f|$, we consider the following problem

$$\begin{cases} -\mathrm{div}\, a_n(x,u_n,\nabla u_n) + \mathrm{H}_n(x,u_n,\nabla u_n) = f_n \text{ in } \Omega, \\ u_n \in \mathrm{W}_0^{1.p(\cdot)}(\Omega). \end{cases} \tag{6.42}$$

Where

$$\mathrm{H}_n(x,s,\xi) = \min[\mathrm{T}_n(\beta_n(x)h_n(s)|\xi|^{p(x)}), \max(-\mathrm{T}_n(\beta_n(x)h_n(s)|\xi|^{p(x)}), \mathrm{H}_n(x,s,\xi))],$$

$$h_n(s) = g_n(s)\frac{h(s)}{g(s)},\ \beta_n(x) = \mathrm{T}_n(\beta(x)),\ \text{and } \beta_n(x) \le \theta\alpha_n(x).$$

Note that the function H_n is bounded and

$$|H_n(x,s,\xi)| \le T_n(\beta_n(x)h_n(s)|\xi|^{p(x)-1}) \le \beta_n(x)h_n(s)|\xi|^{p(x)-1}. \tag{6.43}$$

We also observe that $\dfrac{h_n}{g_n} = \dfrac{h}{g} \in L^1(\mathbb{R})$ and

$$h_n \le \max\{g(s) :| s |\le n\}\frac{h}{g}, \text{ so that } h_n \in L^1(\mathbb{R}).$$

Let $(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}$ be a test function in (6.42), we have

$$\begin{aligned}\int_\Omega a_n(x,u_n,\nabla u_n)\nabla T_k(A(u_n))e^{\theta\eta(u_n)}dx \\ + \int_\Omega H_n(x,u_n,\nabla u_n)(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}dx \\ + \int_\Omega a_n(x,u_n,\nabla u_n)\nabla u_n \frac{\theta h(u_n)}{g(u_n)}(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}dx \\ = \int_\Omega f_n(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}dx.\end{aligned} \tag{6.44}$$

By (6.3) and (6.9), we have

$$\begin{aligned}\int_\Omega H_n(x,u_n,\nabla u_n)(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}dx \\ + \int_\Omega a_n(x,u_n,\nabla u_n)\nabla u_n \frac{\theta h(u_n)}{g(u_n)}(T_k(A(u_n)))^+ e^{\theta\eta(u_n)}dx \ge 0.\end{aligned}$$

It follows that

$$\int_{A(u_n)>0} a_n(x,u_n,\nabla u_n)\nabla(T_k(A(u_n)))e^{\theta\eta(u_n)}dx \le \int_\Omega |f_n|(T_k(A(u_n)))^+ e^{\theta\eta(u_n)},$$

so we get, if g unbounded,

$$\int_{0<A(u_n)\le k} a_n(x,u_n,\nabla u_n)\nabla u_n g^{\frac{1}{p^+-1}}(u_n)e^{\theta\eta(u_n)}dx \le \int_\Omega |f_n||T_k(A(u_n))|e^{\theta\eta(u_n)}dx. \tag{6.45}$$

By the assumption (6.3), there exist $C_8,\ C_9 > 0$ such that

$$\begin{aligned}\nu\int_{0<A(u_n)\le k}|\nabla T_k(A(u_n))|^{p(x)}dx &\le C_8\int_{0<A(u_n)\le k}\alpha_n(x)|\nabla T_k(A(u_n))|^{p(x)} \\ &\le \int_{0<A(u_n)\le k}\alpha_n(x)g(u_n)^{\frac{p(x)}{p^+-1}}|\nabla u_n|^{p(x)} \le C_9 k\|f\|_{L^1},\end{aligned} \tag{6.46}$$

and if g bounded we obtain

$$\int_{0<\mathrm{A}(u_n)\leq k} a_n(x,u_n,\nabla u_n)\nabla u_n g^{\frac{1}{p^- -1}}(u_n)e^{\theta\eta(u_n)}dx \leq \int_\Omega |f_n||\mathrm{T}_k(\mathrm{A}(u_n))|e^{\theta\eta(u_n)}dx. \tag{6.47}$$

Using again the assumption (6.3), there exist $\mathrm{C}_{10},\mathrm{C}_{11}>0$ such that

$$\begin{aligned}\nu\int_{0<\mathrm{A}(u_n)\leq k}|\nabla \mathrm{T}_k(\mathrm{A}(u_n))|^{p(x)}dx &\leq \mathrm{C}_{10}\int_{0<\mathrm{A}(u_n)\leq k}\alpha_n(x)|\nabla \mathrm{T}_k(\mathrm{A}(u_n))|^{p(x)}\\ &\leq \int_{0<\mathrm{A}(u_n)\leq k}\alpha_n(x)g(u_n)^{\frac{p(x)}{p^- -1}}|\nabla u_n|^{p(x)} \leq \mathrm{C}_{11}k\|f\|_{\mathrm{L}^1},\end{aligned} \tag{6.48}$$

By using the test function $-(\mathrm{T}_k(\mathrm{A}(u_n)))^- e^{-\eta(u_n)}$, and reasoning as before we get

$$\begin{aligned}\nu\int_{-k<\mathrm{A}(u_n)\leq 0}|\nabla \mathrm{T}_k(\mathrm{A}(u_n))|^{p(x)}dx &\leq \mathrm{C}_{12}\int_{-k<\mathrm{A}(u_n)\leq 0}\alpha_n(x)|\nabla \mathrm{T}_k(\mathrm{A}(u_n))|^{p(x)}\\ &\leq \mathrm{C}_{13}k\|f\|_{\mathrm{L}^1}.\end{aligned} \tag{6.49}$$

Combining (6.48) and (6.49), we get

$$\nu\int_\Omega |\nabla \mathrm{T}_k(\mathrm{A}(u_n))|^{p(x)} \leq \mathrm{C}_{14}\int_\Omega \alpha_n(x)|\nabla \mathrm{T}_k(\mathrm{A}(u_n))|^{p(x)} \leq \mathrm{C}_{15}k\|f\|_{\mathrm{L}^1}. \tag{6.50}$$

Therefore,

$$\|\nabla \mathrm{T}_k(\mathrm{A}(u_n))\|^{r_1}_{p(\cdot)}dx \leq \mathrm{C}_{15}k, \tag{6.51}$$

$$r_1 = \begin{cases} p^+ & \text{if} \quad \|\nabla \mathrm{T}_k(\mathrm{A}(u_n))\|_{p(\cdot)}<1,\\ p^- & \text{if} \quad \|\nabla \mathrm{T}_k(\mathrm{A}(u_n))\|_{p(\cdot)}\geq 1.\end{cases}$$

That implies that $(\mathrm{T}_k(\mathrm{A}(u_n)))_{n\in\mathbb{N}}$ converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$\begin{aligned}&\mathrm{T}_k(\mathrm{A}(u_n)) \rightharpoonup \mathrm{T}_k(\mathrm{A}(u)) \ \text{ weakly in } \ \mathrm{W}_0^{1.p(\cdot)}(\Omega),\\ &\mathrm{T}_k(\mathrm{A}(u_n)) \to \mathrm{T}_k(\mathrm{A}(u)) \ \text{ strongly in } \ \mathrm{L}^{p(\cdot)}(\Omega), \ \text{ and a.e. in } \ \Omega,\end{aligned}$$

and on the other, by Lemma 6.2.1, that for every $k>0$

$$\begin{aligned}&\alpha_n(x)^{\frac{1}{p(x)}}\nabla \mathrm{T}_k(\mathrm{A}(u_n)) \rightharpoonup \alpha(x)^{\frac{1}{p(x)}}\nabla \mathrm{T}_k(\mathrm{A}(u)) \ \text{ weakly in } \ \mathrm{L}^{p(\cdot)}(\Omega)^{\mathrm{N}}, \text{and}\\ &\alpha(x)|\nabla \mathrm{T}_k(\mathrm{A}(u))|^{p(x)} \in \mathrm{L}^1(\Omega).\end{aligned} \tag{6.52}$$

Step2 : Strong convergence of truncations.
Let $w_n^+ e^{\theta\eta(u_n)}$ be a test function in (6.42), where $w_n = \mathrm{T}_{2k}(\mathrm{Z}_n)$ be a test function in (6.21), where $\mathrm{Z}_n = (\mathrm{A}(u_n)-\mathrm{T}_h(\mathrm{A}(u_n))+\mathrm{T}_k(\mathrm{A}(u_n))-\mathrm{T}_k(\mathrm{A}(u)))$, for $h>k>0$. Taking $\mathrm{M}=4k+h$, we

have

$$\begin{aligned}&\int_{w_n>0} a_n(x,u_n,\nabla u_n)\nabla w_n e^{\theta\eta(u_n)}dx+\int_{w_n>0}\mathrm{H}_n(x,u_n,\nabla u_n)w_n^+e^{\theta\eta(u_n)}dx\\&+\theta\int_{w_n>0} a_n(x,u_n,\nabla u_n)\nabla u_n\frac{h(u_n)}{g(u_n)}w_n^+e^{\theta\eta(u_n)}dx=\int_{w_n>0} f_n w_n^+e^{\theta\eta(u_n)}dx,\end{aligned}\tag{6.53}$$

by (6.3) and (6.9), we have

$$\begin{aligned}&\int_{w_n>0}\mathrm{H}_n(x,u_n,\nabla u_n)w_n^+e^{\theta\eta(u_n)}dx\\&\qquad+\theta\int_{w_n>0} a_n(x,u_n,\nabla u_n)\nabla u_n\frac{h(u_n)}{g(u_n)}w_n^+e^{\theta\eta(u_n)}dx\geq 0.\end{aligned}$$

So we get that,

$$\int_{w_n>0} a_n(x,u_n,\nabla u_n)\nabla w_n e^{\theta\eta(u_n)}dx\leq\int_{w_n>0} f_n w_n^+e^{\Upsilon(u_n)}dx,$$

We have

$$\begin{aligned}\int_{\{w_n>0\}} a_n(x,u_n,\nabla u_n)\nabla w_n e^{\theta\eta(u_n)}dx&=\int_{\{w_n>0\}\cap\{|\mathrm{A}(u_n)|>k\}} a_n(x,u_n,\nabla u_n)\nabla w_n e^{\theta\eta(u_n)}dx\\&+\int_{\{w_n>0\}\cap\{|\mathrm{A}(u_n)|\leq k\}} a_n(x,u_n,\nabla u_n)\nabla w_n e^{\theta\eta(u_n)}dx.\end{aligned}\tag{6.54}$$

Concerning the first term on the right hand side of (6.54) ; since $\nabla w_n=0$ on $\{|\mathrm{A}(u_n)|>\mathrm{M}\}$, we have

$$\begin{aligned}&\int_{\{w_n>0\}\cap\{|\mathrm{A}(u_n)|>k\}} a_n(x,u_n,\nabla u_n)\nabla w_n e^{\theta\eta(u_n)}dx\\&\qquad=\int_{\{w_n>0\}\cap\{|\mathrm{A}(u_n)|>k\}} a_n(x,\mathrm{T}_{\widehat{\mathrm{M}}}(u_n),\nabla\mathrm{T}_{\widehat{\mathrm{M}}}(u_n))\nabla\mathrm{Z}_n e^{\theta\eta(u_n)}\\&\qquad\geq-\int_{\{w_n>0\}\cap\{|\mathrm{A}(u_n)|>k\}} a_n(x,\mathrm{T}_{\widehat{\mathrm{M}}}(u_n),\nabla\mathrm{T}_{\widehat{\mathrm{M}}}(u_n))\nabla\mathrm{T}_k(\mathrm{A}(u))e^{\theta\eta(u_n)}dx\\&\qquad\geq-e^{\theta\eta(\infty)}\int_{\{|\mathrm{A}(u_n)|>k\}} a_n(x,\mathrm{T}_{\widehat{\mathrm{M}}}(u_n),\nabla\mathrm{T}_{\widehat{\mathrm{M}}}(u_n))\nabla\mathrm{T}_k(\mathrm{A}(u))dx\\&\qquad\geq-\varepsilon_3(n).\end{aligned}$$

For the second term on the right hand side of (6.54) ; we have for $\widehat{k}>0$

$$\begin{aligned}&\int_{\{w_n>0\}\cap\{|\mathrm{A}(u_n)|\leq k\}} a(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u_n))(\nabla\mathrm{T}_k(\mathrm{A}(u_n))-\nabla\mathrm{T}_k(\mathrm{A}(u)))e^{\theta\eta(u_n)}dx\\&\qquad\leq e^{\theta\eta(\infty)}\int_{\{w_n>0\}}|f_n||w_n|dx+\varepsilon_3(n).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\{w_n>0\}\cap\{|\mathrm{A}(u_n)|\le k\}} a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u_n))[\nabla\mathrm{T}_k(\mathrm{A}(u_n))-\nabla\mathrm{T}_k(\mathrm{A}(u))]e^{\theta\eta(u_n)}dx\\
&\quad=\int_{\{w_n>0\}\cap\{|\mathrm{A}(u_n)|\le k\}}[a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u_n))-a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u))]\\
&\qquad\qquad\times[\nabla\mathrm{T}_k(\mathrm{A}(u_n))-\nabla\mathrm{T}_k(\mathrm{A}(u))]e^{\theta\eta(u_n)}dx\\
&\qquad+\int_{\{w_n>0\}\cap\{|\mathrm{A}(u_n)|\le k\}} a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u))[\nabla\mathrm{T}_k(\mathrm{A}(u_n))-\nabla\mathrm{T}_k(\mathrm{A}(u))]e^{\theta\eta(u_n)}dx.
\end{aligned}$$

The third terms of the right hand tend to 0, as n tend to infinity. Thanks to step 2 of theorem 6.2.1. So, we have

$$\begin{aligned}
&\int_{\{w_n>0\}\cap\{|\mathrm{A}(u_n)|\le k\}}[a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u_n))\\
&-a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u))][\nabla\mathrm{T}_k(\mathrm{A}(u_n))-\nabla\mathrm{T}_k(\mathrm{A}(u))]e^{\theta\eta(u_n)}dx\\
&=\int_{\{|\mathrm{A}(u_n)|\le k\}} a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u_n))[\nabla\mathrm{T}_k(\mathrm{A}(u_n))-\nabla\mathrm{T}_k(\mathrm{A}(u))]e^{\theta\eta(u_n)}dx+\varepsilon_4(n)\\
&\le e^{\theta\eta(\infty)}\int_{\{w_n>0\}}|f_n(x)||w_n|dx+\varepsilon_4(n)+\varepsilon_3(n)\le\varepsilon_5(n),
\end{aligned}\tag{6.55}$$

as $f_n\to f$ strongly in $\mathrm{L}^1(\Omega)$, and $w_n\rightharpoonup 0$ weakly* in $\mathrm{L}^\infty(\Omega)$.
Let $-(w_n)^-\exp(-\theta\eta(u_n))$ be a test function in the problem (6.42), we obtain

$$\begin{aligned}
&\int_{w_n<0} a_n(x,u_n,\nabla u_n)\nabla w_n e^{-\theta\eta(u_n)}dx+\int_{w_n<0}-\mathrm{H}_n(x,u_n,\nabla u_n)(w_n)^-e^{-\theta\eta(u_n)}dx\\
&+\theta\int_{w_n<0} a_n(x,u_n,\nabla u_n)-\nabla u_n\frac{h(u_n)}{g(u_n)}-(w_n)^-e^{-\theta\eta(u_n)}dx=\int_{w_n<0}-f_n(w_n)^-e^{-\theta\eta(u_n)}dx,
\end{aligned}\tag{6.56}$$

Reasoning as before, we get that

$$\int_{\{w\le 0\}\cap\{|\mathrm{A}(u_n)|>k\}} a(x,u_n,\nabla u_n)\nabla w_n e^{-\theta\eta(u_n)}dx\ge-\varepsilon_6(n),$$

where $\varepsilon_6(n)$ tend to 0 as n tend to infinity.

$$\begin{aligned}
&\int_{w\le 0} a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u_n))(\nabla\mathrm{T}_{\widehat{k}}(u_n)-\nabla\mathrm{T}_{\widehat{k}}(u))e^{-\theta\eta(u_n)}dx\\
&\qquad\qquad\le\int_{w\le 0}|f_n||w_n|e^{-\theta\eta(u_n)}dx+\varepsilon_6(n).
\end{aligned}$$

Since w_n tend to 0 weakly * in $\mathrm{L}^\infty(\Omega)$, and f_n converge strongly to f in $\mathrm{L}^1(\Omega)$ we conclude that

$$\int_{w\le 0} a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u_n))(\nabla\mathrm{T}_k(\mathrm{A}(u_n))-\nabla\mathrm{T}_k(\mathrm{A}(u)))e^{-\theta\eta(u_n)}dx\le\varepsilon_7(n).$$

By adding the term to the last expression, we get

$$\int_{w\leq 0}[a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u_n))-a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u_n))] \times(\nabla\mathrm{T}_k(\mathrm{A}(u_n))-\nabla\mathrm{T}_k(\mathrm{A}(u)))e^{-\theta\eta(u_n)}dx\leq\varepsilon_7(n). \tag{6.57}$$

Combining (6.55) and (6.57), we get

$$\int_\Omega[a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u_n))-a_n(x,\mathrm{T}_{\widehat{k}}(u_n),\nabla\mathrm{T}_{\widehat{k}}(u_n))]\times (\nabla\mathrm{T}_k(\mathrm{A}(u_n))-\nabla\mathrm{T}_k(\mathrm{A}(u)))e^{-\theta\eta(u_n)}dx\leq\varepsilon_8(n),$$

so by the Lemma 6.2.2, we conclude that

$$\mathrm{T}_k(\mathrm{A}(u_n))\to\mathrm{T}_k(\mathrm{A}(u))\text{ in }\mathrm{W}_0^{1,p(\cdot)}(\Omega),\ \ \nabla\mathrm{A}(u_n)\to\nabla\mathrm{A}(u)\ \text{ a.e. in }\Omega,$$

moreover we have

$$|\nabla\mathrm{T}_m(u_n)|^{p(x)}\leq c|\nabla\mathrm{T}_k(\mathrm{A}(u_n))|^{p(x)}, \tag{6.58}$$

thanks to the continuity of g.
By Lemma 6.2.1, for every $k>0$ we have

$$\alpha(x)|\nabla\mathrm{T}_k(u)|\in\mathrm{L}^1(\Omega), \tag{6.59}$$

and

$$\alpha_n(x)^{\frac{1}{p(x)}}\nabla\mathrm{T}_k(u_n)\rightharpoonup\alpha(x)^{\frac{1}{p(x)}}\nabla\mathrm{T}_k(u)\quad\text{weakly in }\ (\mathrm{L}^{p(x)}(\Omega))^{\mathrm{N}}. \tag{6.60}$$

Step 3 : The equi-integrability of $\mathrm{H}_n(x,u_n,\nabla u_n)$
In order to pass to the limit in the approximate problem, we shall show that

$$\mathrm{H}_n(x,u_n,\nabla u_n)\to\mathrm{H}(x,u,\nabla u)\ \text{ in }\ \mathrm{L}^1(\Omega). \tag{6.61}$$

Let E a set of Ω such that meas(E) = 0, and $l>0$. We have

$$\begin{aligned}
&\int_{\mathrm{E}}|\mathrm{H}_n(x,u_n,\nabla u_n)|dx\leq\int_{\mathrm{E}}\beta(x)h(u_n)|\nabla u_n|^{p(x)}dx\\
&=\int_{\mathrm{E}\cap\{|u_n|>l\}}\beta(x)h(u_n)|\nabla u_n|^{p(x)}dx+\int_{\mathrm{E}\cap\{|u_n|\leq l\}}\beta(x)h(u_n)|\nabla u_n|^{p(x)}dx\\
&=\int_{\mathrm{E}\cap\{|u_n|>l\}}\beta(x)h(u_n)|\nabla u_n|^{p(x)}dx+\int_{\mathrm{E}\cap\{|u_n|\leq l\}}\beta(x)h(\mathrm{T}_l(u_n))|\nabla\mathrm{T}_l(u_n)|^{p(x)}dx\\
&\leq\int_{\mathrm{E}\cap\{|u_n|>l\}}\beta(x)h(u_n)|\nabla u_n|^{p(x)}dx+\max_{[0,l]}(h(s))\int_{\mathrm{E}\cap\{|u_n|\leq l\}}\beta(x)|\nabla\mathrm{T}_l(u_n)|^{p(x)}dx.
\end{aligned}$$

From (6.59), we deduce that the second term of the right hand side of the last inequality

equal to 0 as meas(E) = 0. Prove that

$$\int_{\mathrm{E}\cap\{|u_n|>l\}} \beta(x)h(u_n)|\nabla \mathrm{T}_l(u_n)|^{p(x)}dx \to 0.$$

Let $(\mathrm{T}_1(u_n-\mathrm{T}_l(u_n))^+\exp(2\theta\eta(u_n))$ be a test function in the problem (6.42). We have

$$\begin{aligned}
&\int_{l<u_n\le l+1} a_n(x,u_n,\nabla u_n)\nabla \mathrm{T}_1(u_n-\mathrm{T}_l(u_n))\exp(2\theta\eta(u_n))dx \\
&\qquad+\int_{l<u_n} 2\theta a_n(x,u_n,\nabla u_n)\nabla u_n\frac{h(u_n)}{g(u_n)}(\mathrm{T}_1(u_n-\mathrm{T}_l(u_n)))^+\exp(2\theta\eta(u_n))dx \\
&\qquad+\int_{l<u_n} \mathrm{H}_n(x,u_n,\nabla u_n)\mathrm{T}_1(u_n-\mathrm{T}_l(u_n))^+\exp(2\theta\eta(u_n))dx \\
&\qquad=\int_{l<u_n} f_n(\mathrm{T}_1(u_n-\mathrm{T}_l(u_n)))^+\exp(2\eta(u_n))dx.
\end{aligned}$$

By assumption (6.3) and (6.9) we have

$$\int_{l<u_n}|\nabla u_n|^{p(x)}\beta(x)h(u_n)(\mathrm{T}_1(u_n-\mathrm{T}_l(u_n))^+\exp(2\theta\eta(u_n))dx\le \mathrm{C}_{16}\int_{l<u_n}|f|dx. \tag{6.62}$$

Let $-(\mathrm{T}_1(u_n-\mathrm{T}_l(u_n))^-\exp(-2\theta\eta(u_n))$ be a test function in the problem (6.42). Reason as above, we get

$$\int_{u_n<-l}|\nabla u_n|^{p(x)}\beta(x)h(u_n)(\mathrm{T}_1(u_n-\mathrm{T}_l(u_n))^+\exp(2\eta(u_n))dx\le \mathrm{C}_{17}\int_{u_n<-l}|f|dx. \tag{6.63}$$

From (6.62) and (6.63) we conclude that

$$\int_{l<|u_n|}|\nabla u_n|^{p(x)}\beta(x)h(u_n)dx\le \mathrm{C}_{18}\int_{l<|u_n|}|f|dx. \tag{6.64}$$

Let l tend to infinity we get

$$\int_{l<|u_n|}|\nabla u_n|^{p(x)}\beta(x)h(u_n)dx\to 0.$$

Finally we get the equi-integrability of H.
Step 4 : Entropy solutions.

Let $(\mathrm{T}_k(\mathrm{A}(u_n)))^+ e^{\theta\eta(u_n)}$ be a test function in (6.42), we have

$$\begin{aligned}\int_\Omega & a_n(x,u_n,\nabla u_n)\nabla \mathrm{T}_k(\mathrm{A}(u_n))e^{\theta\eta(u_n)}dx \\ &+\int_\Omega \mathrm{H}_n(x,u_n,\nabla u_n)(\mathrm{T}_k(\mathrm{A}(u_n)))^+ e^{\theta\eta(u_n)}dx \\ &+\int_\Omega a_n(x,u_n,\nabla u_n)\nabla u_n \frac{\theta h(u_n)}{g(u_n)}(\mathrm{T}_k(\mathrm{A}(u_n)))^+ e^{\theta\eta(u_n)}dx \\ &=\int_\Omega f_n(\mathrm{T}_k(\mathrm{A}(u_n)))^+ e^{\theta\eta(u_n)}dx.\end{aligned} \tag{6.65}$$

By (6.3) and (6.9), we have

$$\begin{aligned}\int_\Omega & \mathrm{H}_n(x,u_n,\nabla u_n)(\mathrm{T}_k(\mathrm{A}(u_n)))^+ e^{\theta\eta(u_n)}dx \\ &+\int_\Omega a_n(x,u_n,\nabla u_n)\nabla u_n \frac{\theta h(u_n)}{g(u_n)}(\mathrm{T}_k(\mathrm{A}(u_n)))^+ e^{\theta\eta(u_n)}dx \geq 0.\end{aligned}$$

It follows that

$$\int_{\mathrm{A}(u_n)>0} a_n(x,u_n,\nabla u_n)\nabla(\mathrm{T}_k(\mathrm{A}(u_n)))e^{\theta\eta(u_n)}dx \leq \int_\Omega |f_n|(\mathrm{T}_k(\mathrm{A}(u_n)))^+ e^{\theta\eta(u_n)}.$$

If the function g is unbounded, taking $\mathrm{A}(s)=\int_0^s g(\tau)^{1/(p^- -1)}d\tau$, and by (6.3) we have

$$\begin{aligned}\mathrm{C}_{19}\int_{0<\mathrm{A}(u)\leq k} & \alpha_n(x)g_n(u_n)^{\frac{p(x)}{p(x)-1}}|\nabla u_n|^{p(x)}e^{\theta\eta(u_n)} \\ &\leq \int_{0<\mathrm{A}(u)\leq k} a_n(x,u_n,\nabla u_n)\nabla \mathrm{T}_k(u_n)g^{\frac{1}{p^- -1}}(u_n)e^{\theta\eta(u_n)}dx \\ &\leq \int_\Omega |f_n||\mathrm{T}_k(\mathrm{A}(u_n))|e^{\theta\eta(u_n)}dx \\ &\leq \mathrm{C}_{20}k\|f\|_{\mathrm{L}^1}.\end{aligned} \tag{6.66}$$

Else, taking $\mathrm{A}(s)=\int_0^s g(\tau)^{1/(p^+ -1)}d\tau$, and by (6.3) we obtain

$$\begin{aligned}\mathrm{C}_{21}\int_{0<\mathrm{A}(u)\leq k} & \alpha_n(x)g_n(u_n)^{\frac{p(x)}{p(x)-1}}|\nabla u_n|^{p(x)}e^{\theta\eta(u_n)} \\ &\leq \int_{0<\mathrm{A}(u)\leq k} a_n(x,u_n,\nabla u_n)\nabla \mathrm{T}_k(u_n)g^{\frac{1}{p^+ -1}}(u_n)e^{\theta\eta(u_n)}dx \\ &\leq \int_\Omega |f_n||\mathrm{T}_k(\mathrm{A}(u_n))|e^{\theta\eta(u_n)}dx \\ &\leq \mathrm{C}_{22}k\|f\|_{\mathrm{L}^1}.\end{aligned} \tag{6.67}$$

By means of (6.66) and (6.67), we obtain

$$\int_{0<A(u)\le k} \alpha_n(x) g_n(u_n)^{p'(x)} |\nabla u_n|^{p(x)} \le C_{23} k \|f\|_{L^1}. \tag{6.68}$$

Let $(T_k(A(u_n)))^- exp(-\theta\eta(u_n))$ be a test function in the problem (6.42). Reason as above, we get

$$\int_{-k<A(u)\le 0} \alpha_n(x) g_n(u_n)^{p'(x)} |\nabla u_n|^{p(x)} \le C_{24} k \|f\|_{L^1}. \tag{6.69}$$

Combining (6.68) and (6.69), we have

$$\int_{\Omega} \alpha_n(x) g_n(u_n)^{p'(x)} |\nabla u_n|^{p(x)} \le C_{25} k \|f\|_{L^1}. \tag{6.70}$$

Let $\varphi \in X_0^{p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, where $X_0^{p(\cdot)}(\Omega)$ is the space defined in (6.13),
taking $\phi = T_k(u_n - \varphi)$ as a test function in the approximate problem, we get

$$\int_{\Omega} a_n(x, u_n, \nabla u_n) \nabla T_k(u_n - \varphi) dx + \int_{\Omega} H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx = \int_{\Omega} f_n T_k(u_n - \varphi) dx.$$

Adding and subtracting in the equation above the term

$$\int_{\Omega} a_n(x, u_n, \nabla \varphi) \nabla T_k(u_n - \varphi) dx$$

we obtain

$$\begin{aligned} \int_{\Omega} \Big[a_n(x, u_n, \nabla u_n) - a_n(x, u_n, \nabla \varphi) \Big] \nabla T_k(u_n - \varphi) dx + \int_{\Omega} a_n(x, u_n, \nabla \varphi) \nabla T_k(u_n - \varphi) dx \\ + \int_{\Omega} H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx = \int_{\Omega} f_n T_k(u_n - \varphi) dx. \end{aligned}$$

We then have, once again by (6.4)

$$\int_{\Omega} a_n(x, u_n, \nabla \varphi) \nabla T_k(u_n - \varphi) dx + \int_{\Omega} H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx \le \int_{\Omega} f_n T_k(u_n - \varphi) dx. \tag{6.71}$$

Noticing that $\{|u_n - \varphi| \le k\} \subset \{|u_n| \le k + \|\varphi\|_{L^\infty}\}$ and that

$$\int_{\Omega} a_n(x, u_n, \nabla \varphi) \nabla T_k(u_n - \varphi) dx = \int_{\Omega} \frac{a(x, T_n(u_n), \nabla \varphi)}{\alpha(x)} \alpha_n(x) \nabla T_k(u_n - \varphi) dx$$

we can pass to the limit in (6.71) using assumption (6.2), the a.e. convergence of u_n and the weak convergence of $\alpha(x)^{\frac{1}{p(x)}} \nabla T_k(u_n)$ proved in Step 2. Hence we obtain, for every $k > 0$,

$$\int_{\Omega} a(x, u, \nabla \varphi) \nabla T_k(u - \varphi) dx + \int_{\Omega} H(x, u, \nabla u) T_k(u - \varphi) dx \le \int_{\Omega} f T_k(u - \varphi) dx. \tag{6.72}$$

As in the last part of Theorem 6.2.1 we take advantage of Lemma 7 of [30] to infer that, for every $k>0$,

$$\int_\Omega a(x,u,\nabla u)\nabla \mathrm{T}_k(u-\varphi)+\int_\Omega \mathrm{H}(x,u,\nabla u)\mathrm{T}_k(u-\varphi)=\int_\Omega f\mathrm{T}_k(u-\varphi),$$

$\forall \varphi \in \mathrm{W}_0^{1,p(x)}(\Omega)\cap \mathrm{L}^\infty(\Omega)$.

Step 5 : weak solutions. Let $f\in \mathrm{L}^{(p^*(\cdot))'}(\Omega)$, for each $n\in\mathbb{N}$, there exists a weak solution $u_n\in \mathrm{W}_0^{1.p(\cdot)}(\Omega)$, which is an admissible test function in the weak sense (6.42). By Theorem 2.10 in [4], we have $u_n\in \mathrm{L}^\infty(\Omega)$, and so $\mathrm{A}(u_n)\in \mathrm{W}_0^{1.p(\cdot)}(\Omega)\cap \mathrm{L}^\infty(\Omega)$ for all $n\in\mathbb{N}$, where $\mathrm{A}(s)=\int_0^s g(\tau)^{1/(p^--1)}d\tau$,

Let $v=e^{\theta\eta(u_n)}(\mathrm{A}(u_n))^+$ be a test function in (6.42), we obtain

$$\begin{aligned}&\int_{\mathrm{A}(u_n)\geq 0} a_n(x,u_n,\nabla u_n)\nabla \mathrm{A}(u_n)e^{\theta\eta(u_n)}dx+\int_{\mathrm{A}(u_n)\geq 0}\mathrm{H}_n(x,u_n,\nabla u_n)\mathrm{A}(u_n)e^{\theta\eta(u_n)}dx\\ &+\int_{\mathrm{A}(u_n)\geq 0} a_n(x,u_n,\nabla u_n)\mathrm{A}(u_n)\frac{\theta h(u_n)}{g(u_n)}e^{\theta\eta(u_n)}\nabla u_n dx=\int_{\mathrm{A}(u_n)\geq 0} f\mathrm{A}(u_n)e^{\theta\eta(u_n)}dx.\end{aligned}\tag{6.73}$$

On the other hand, by (6.3) and (6.9) we have

$$\begin{aligned}\int_{\mathrm{A}(u_n)\geq 0} a_n(x,u_n,\nabla u_n)\mathrm{A}(u_n)\frac{\theta h(u_n)}{g(u_n)}e^{\theta\eta(u_n)}\nabla u_n dx&\\ +\int_{\mathrm{A}(u_n)\geq 0}\mathrm{H}_n(x,u_n,\nabla u_n)\mathrm{A}(u_n)e^{\theta\eta(u_n)}dx\geq 0&\end{aligned}$$

So (6.73) becomes

$$\int_{\mathrm{A}(u_n)\geq 0} a_n(x,u_n,\nabla u_n)\nabla \mathrm{A}(u_n)e^{\theta\eta(u_n)}dx\leq \int_{\mathrm{A}(u_n)\geq 0} f\mathrm{A}(u_n)e^{\theta\eta(u_n)}dx.\tag{6.74}$$

By the assumption (6.3), there exists $\mathrm{C}_{26}>0$ such that

$$\mathrm{C}_{26}\int_{\mathrm{A}(u_n)\geq 0}\alpha_n(x)g(u_n)^{\frac{p(x)}{p(x)-1}}|\nabla u_n|^{p(x)}e^{\theta\eta(u_n)}\leq \int_{\mathrm{A}(u_n)\geq 0} f\mathrm{A}(u_n)e^{\theta\eta(u_n)}dx.\tag{6.75}$$

Let $v=-e^{-\theta\eta(u_n)}(\mathrm{A}(u_n))^-$ be a test function in (6.42), and reasing as above we get

$$\mathrm{C}_{27}\int_{\mathrm{A}(u_n)\leq 0}\alpha_n(x)g(u_n)^{\frac{p(x)}{p(x)-1}}|\nabla u_n|^{p(x)}e^{\theta\eta(u_n)}\leq \int_{\mathrm{A}(u_n)\leq 0} f\mathrm{A}(u_n)e^{\theta\eta(u_n)}dx\tag{6.76}$$

Adding up (6.75) and (6.76), we conclude that there exists $\mathrm{C}_{28}>0$ such that

$$\int_\Omega \alpha_n(x)g(u_n)^{\frac{p(x)}{p(x)-1}}|\nabla u_n|^{p(x)}\leq \mathrm{C}_{28}\int_\Omega f\mathrm{A}(u_n)dx\leq \mathrm{C}_{28}\|\mathrm{A}(u_n)\|_{\mathrm{L}^{p*(x)}}\|f\|_{\mathrm{L}^{(p*(x))'}}.\tag{6.77}$$

Using Sobolev inequality and, thanks to the a.e. convergence of ∇u_n to ∇u and Fatou

Lemma,

$$\int_\Omega \alpha_n(x) g(u_n)^{\frac{p(x)}{p(x)-1}} |\nabla u_n|^{p(x)} \leq C_{29} (\|f\|_{(p^*(\cdot))'})^{p^{+'}}.$$

Thus we have that the sequence

$$\frac{a(x, T_n(u_n), \nabla u_n)}{\alpha(x)} \alpha_n(x)^{\frac{1}{p'(x)}} \text{ is bounded in } (L^{p'(\cdot)}(\Omega))^N, \tag{6.78}$$

and converges almost everywhere to $a(x,u,\nabla u) g(|u|)^{\frac{1}{p'(x)}-1}$ and weakly in $L^{p'(x)}(\Omega)^N$ to its a.e.-limit. Noticing that for every $\phi \in X_0^{p(\cdot)}(\Omega)$

$$\alpha_n(x)^{\frac{1}{p(x)}} \nabla \phi \to \alpha(x)^{\frac{1}{p(x)}} \nabla \phi \text{ strongly in } L^{p(\cdot)}(\Omega)^N.$$

Let $\varphi \in X_0^{p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, then

$$\int_\Omega a_n(x, u_n, \nabla u_n) \varphi dx + \int_\Omega H_n(x, u_n, \nabla u_n) \varphi dx = \int_\Omega f \varphi dx, \tag{6.79}$$

it follows from (6.61) and (6.78) that we may pass to the limit in (6.79) and conclude the proof.

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