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Existence and multiplicity results for some parametric anisotropic elliptic problems

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To my dear parents,
for their endless support, encouragement and love.

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Résumé

L'objectif de cette thèse est double : La première partie est consacrée à l'étude de certains problèmes elliptiques anisotropes. Plus précisément, nous examinons deux problèmes elliptiques anisotropes paramétriques avec des conditions aux limites de Dirichlet et de Robin, dans le but d'établir l'existence et la multiplicité de solutions. Un défi majeur se présente lorsque l'on traite de l'existence de solutions, notamment lorsque les non-linéarités dépendent du gradient. Cela implique que des méthodes variationnelles telles que le théorème du Mountain Pass ne peuvent pas être utilisées. Pour surmonter cet obstacle, nous utilisons une méthode topologique basée sur la subjectivité des opérateurs pseudomonotones, comme démontré par Carl et al. [24]. De plus, nous établissons l'unicité de la solution. De plus, nous relâchons la dépendance au gradient des non-linéarités et prouvons un résultat de multiplicité pour trois solutions faibles en utilisant le principe variationnel de Ricceri. Dans la deuxième partie, nous étudions les problèmes de résonance et de non-résonance pour certains systèmes elliptiques paramétriques. Dans les deux cas, les informations disponibles concernant le spectre de Lusternik-Schnirelmann du problème de valeurs propres associé sont limitées. Dans le cas de la résonance, nous nous concentrons sur un système elliptique paramétrique où les non-linéarités impliquent une convection et la convolution de la solution. À l'inverse, dans le cas de la non-résonance, nous examinons la solvabilité à gauche de l'infimum positif de toutes les valeurs propres pour certains problèmes elliptiques quasi-linéaires non-résonants avec des exposants variables. Nous établissons l'existence de la solution en utilisant la même méthode, à savoir la surjectivité des opérateurs pseudomonotones, l'unicité étant prouvée à l'aide d'arguments similaires. Enfin, nous considérons des systèmes de type gradient et prouvons l'existence d'une solution en utilisant une approche variationnelle.

Mots-clefs: Équations elliptiques anisotropes, espaces de Sobolev à exposant variable, existence et multiplicité, problèmes de résonance et de non-résonance, spectre de Lusternik-Schnirelmann.

Abstract

The aim of this thesis is twofold: The first part is devoted to the study of some anisotropic elliptic problems. Specifically, we investigate two parametric anisotropic elliptic problems with Dirichlet and Robin boundary conditions, aiming to establish the existence and multiplicity of solutions. One major challenge arises when dealing with the existence of solutions, particularly when the nonlinearities are gradient-dependent. This implies that variational methods such as the Mountain Pass theorem cannot be employed. To overcome this obstacle, we utilize a topological method based on the subjectivity of pseudomonotone operators, as demonstrated by Carl et al. [24]. Moreover, we establish the uniqueness of the solution. Additionally, we relax the gradient dependence of the nonlinearities and prove a multiplicity result for three weak solutions using Ricceri's variational principle. In the second part, we investigate resonance and non-resonance phenomena for some parametric elliptic systems. In both cases, there is limited available information concerning the Lusternik-Schnirelmann spectrum of the associated eigenvalue problem. In the resonance case, we focus on a parametric elliptic system where the nonlinearities involve convection and the convolution of the solution. Conversely, in the non-resonance case, we investigate the solvability on the left side of the positive infimum of all eigenvalues for specific quasilinear elliptic problems with variable exponents, ensuring they are non-resonant. We establish the existence of the solution by employing the same method, namely the surjectivity of pseudomonotone operators, with uniqueness being proven using similar arguments. Finally, we consider gradient-type systems and prove the existence of a solution using a variational approach.

Keywords: Anisotropic elliptic equations, variable exponent Sobolev spaces, existence and multiplicity, resonance and non-resonance phenomena, Lusternik-Schnirelmann spectrum.

Résumé détaillé

Cette thèse se concentre sur deux aspects principaux : l'étude de problèmes elliptiques anisotropes avec des conditions aux limites de Dirichlet et de Robin, et l'analyse de systèmes elliptiques paramétriques en résonance et non-résonance. Dans les deux parties, l'accent est mis sur l'existence et la multiplicité des solutions, en utilisant des méthodes topologiques et variationnelles adaptées aux non-linéarités dépendant du gradient.

Le manuscrit proposé comporte une introduction générale suivie de cinq chapitres, dont quatre répartis sur deux parties, que l'on peut résumer ainsi :

Le premier chapitre est dédié aux notations, définitions et propriétés des espaces fonctionnels généralisés de Lebesgue et de Sobolev. Nous y présentons également des résultats fondamentaux basés sur la théorie du point fixe et des points critiques.

Le deuxième chapitre traite d'une classe d'équations elliptiques anisotropes à exposants variables avec des termes de convection, soumises à une condition aux limites de Dirichlet :

$$\begin{aligned} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) - \mu \Delta_{\vec{q}(x)}(u) &= \lambda f(x, u, \nabla u) + h(x) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

où l'opérateur

$$\Delta_{\vec{q}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{q_i(x)-2} \partial_{x_i} u \right),$$

est connu sous le nom de pseudo-$\vec{p}(x)$-Laplacien. Dans ce cadre, nous prouvons l'existence d'une solution en utilisant la surjectivité des opérateurs pseudomonotones et, sous certaines conditions supplémentaires, nous montrons l'unicité de cette solution. Ensuite, nous considérons un cas sans dépendance en gradient dans la non-linéarité et prouvons l'existence d'au moins trois solutions faibles en appliquant le principe variationnel de Ricceri.

Le troisième chapitre aborde une autre classe d'équations elliptiques anisotropes à exposants variables, avec des conditions aux limites non linéaires de type Robin, à savoir :

$$-\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{p_M(x)-2}u = \lambda f(x, u, \nabla u) + h(x) \qquad \text{dans } \Omega,$$
$$\sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\nu_i(x) = \mu g(x, u) \qquad \text{sur } \partial\Omega,$$

où nous obtenons des résultats similaires à ceux du deuxième chapitre.

Dans le quatrième chapitre, nous étudions des systèmes quasilinéaires elliptiques à exposants variables dans des domaines non résonants. Nous traitons des problèmes de la forme :

$$\begin{aligned} -\Delta_{p_1(x)} u_1 - \mu_1 \Delta_{q_1(x)} u_1 &= f_1(x, u_1, u_2, \nabla u_1, \nabla u_2) + h_1(x) && \text{dans } \Omega, \\ -\Delta_{p_2(x)} u_2 - \mu_2 \Delta_{q_2(x)} u_2 &= f_2(x, u_1, u_2, \nabla u_1, \nabla u_2) + h_2(x) && \text{dans } \Omega, \\ u_1 = u_2 &= 0 && \text{sur } \partial\Omega, \end{aligned}$$

Nous prouvons l'existence et l'unicité de solutions pour ces systèmes, ainsi que l'existence de solutions pour des systèmes de type gradient non résonants, en utilisant des méthodes variationnelles.

Le cinquième et dernier chapitre est consacré à l'étude de systèmes elliptiques paramétriques dégénérés avec des termes de convection près de la résonance. Nous considérons d'abord un système avec convection proche de la résonance, de la forme :

$$\begin{aligned} -\Delta_{w_1,p_1} u - \mu_1 \Delta_{w_1,q_1} u &= \lambda a_1(x)|u|^{p_1-2}u + f_1(x, u, v, \nabla u, \nabla v) + h_1(x) && \text{dans } \Omega, \\ -\Delta_{w_2,p_2} v - \mu_2 \Delta_{w_2,q_2} v &= \mu a_2(x)|v|^{p_2-2}v + f_2(x, u, v, \nabla u, \nabla v) + h_2(x) && \text{dans } \Omega, \\ u = v &= 0 && \text{sur } \partial\Omega, \end{aligned}$$

puis nous étudions un autre système elliptique à résonance avec termes de convection, sous la forme :

$$\begin{aligned} -\Delta_{w_1,p_1} u - \mu_1 \Delta_{w_1,q_1} u &= \lambda_1 a_1(x)|u|^{p_1-2}u + \lambda_1 w(x)|u|^{\alpha_1-2}u|v|^{\alpha_2} \\ &\quad + f_1(x, u, v, \nabla u, \nabla v) + h_1(x) && \text{dans } \Omega, \\ -\Delta_{w_2,p_2} v - \mu_2 \Delta_{w_2,q_2} v &= \lambda_1 a_2(x)|v|^{p_2-2}v + \lambda_1 w(x)|v|^{\alpha_2-2}v|u|^{\alpha_1} \\ &\quad + f_2(x, u, v, \nabla u, \nabla v) + h_2(x) && \text{dans } \Omega, \\ u = v &= 0 && \text{sur } \partial\Omega. \end{aligned}$$

Dans les deux cas, nous prouvons l'existence d'une solution faible en utilisant la surjectivité des opérateurs pseudomonotones et une approche variationnelle. De plus, nous étudions un système de type gradient à la résonance et prouvons l'existence d'une solution en utilisant une approche variationnelle.

Contents

Introduction

Partial differential equations (PDEs) represent a field of mathematics that is both ancient and profound, offering a blend of precision and beauty. Among these, elliptic PDEs stand out for their clear and elegant structure. Starting from fundamental linear equations like Laplace, Poisson, and Helmholtz, one can quickly advance to deep insights into nonlinear elliptic problems and the nuanced study of their solutions. The application of PDEs to model physical phenomena has been a driving force in their evolution, a tradition that traces back to the innovative ideas of Poincaré. This modern analytical approach has revolutionized our understanding of both linear and nonlinear PDEs. The concept of variable exponent Lebesgue spaces first emerged in Orlicz's work [72] in 1931. Orlicz explored function spaces encompassing all measurable functions $u : \Omega \to \mathbb{R}$ that satisfy

$$\rho(\lambda u) = \int_\Omega \varphi(\lambda |u(x)|)dx,$$

for some $\lambda > 0$ and a function φ meeting certain conditions, where Ω is an open subset of $\mathbb{R}^N$. These spaces, known as Orlicz spaces and denoted by L^φ, did not initially include the case of $|u(x)|^{p(x)}$ with variable exponents. In the 1950s, Nakano [68] furthered the study of modular function spaces, acknowledging variable exponent Lebesgue spaces as a special case within a broader framework. Subsequent research by Polish mathematicians like Musielak [66], and independent developments by Russian researchers such as Tsenov [83] and Sharapudinov [81], expanded the understanding of these spaces. They investigated the minimization of functionals expressed as

$$\int_a^b |u(x) - v(x)|^{p(x)} dx,$$

where u is a given function and v varies within a subspace of $L^{p(x)}([a, b])$. Zhikov [90] initiated a new line of inquiry linking variable exponent spaces to variational integrals with nonstandard growth conditions. Marcellini's work [60] on minimization problems with (p, q)-growth, described by

$$\inf \int_\Omega F(x, |\nabla u|)dx,$$

where $F(x,t)$ satisfies $t^p \leq F(x,t) \leq t^q + 1$ for all $t \geq 0$, further enriched this field. The specific case of variable exponents corresponds to $F(x,t) = t^{p(x)}$, with $p : \Omega \to]1, \infty[$ being a bounded function.

In 1991, Kovacik and Rákosník [54] laid the groundwork for the properties of spaces $L^{p(x)}$ and $W^{1,p(x)}$ with variable exponents. Fan and Zhao [44] extended these findings to the Sobolev spaces $W^{m,p(x)}$. Acerbi and Mingione [1] provided foundational regularity results for functionals with nonstandard growth, while Edmunds and Rákosník [35, 36] contributed to the understanding of smooth function density in $W^{m,p(x)}(\Omega)$ and related embedding properties. The Finnish research group's work on variable exponent spaces and image processing, aiming to explore nonlinear potential theory in variable exponent Sobolev spaces, has been noteworthy. The comprehensive theory of Lebesgue and Sobolev spaces with variable exponents was systematically presented in the monograph by Diening, Harjulehto, Hästö, and Ružička [32].

This thesis delves into two primary areas of interest: exploring anisotropic elliptic problems and investigating resonance and non-resonance phenomena. Anisotropic elliptic problems in PDEs are equations that model physical processes where properties such as diffusivity or conductivity vary with direction. These problems are described by differential operators that take into account the anisotropy of the medium. The operator

$$\Delta_{\vec{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i(x)-2} \, \partial_{x_i} u \right),$$

is known as the pseudo-$\vec{p}(x)$-Laplacian. It generalizes the standard Laplacian to accommodate variable exponents that depend on the spatial variables, reflecting the anisotropic nature of the medium. An elliptic PDE involving this operator might look like

$$-\Delta_{\vec{p}(x)}(u) = f(x,u),$$

where f is a nonlinearity that verifies some growth conditions. Many authors have contributed to this aspect of PDEs, we cite Boureanu and Rădulescu [21] for instance who treated the following problem

$$\begin{aligned} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{p_M(x)-2} u &= f(x,u) && \text{in } \Omega, \\ u &\geq 0 && \text{in } \Omega, \\ -\sum_{i=1}^{N} a_i(x, \partial_{x_i}) v_i &= g(x,u) && \text{on } \partial\Omega. \end{aligned}$$

where a_i, for $i \in \{1, \ldots, N\}$, is a more general operator that verifies conditions $(A_1) - (A_3)$ as mentioned in Chapters 2 and 3.

On the other hand, resonance and non-resonance in the context of elliptic partial differential equations refer to the behavior of solutions in relation to the eigenvalues of the associated differential operator. Resonance occurs when the natural frequencies of the system, represented by the eigenvalues of the differential operator, align with the frequencies of external forces or sources. In the case of elliptic PDEs, this can lead to solutions that exhibit large amplitudes or particular patterns. Mathematically, consider the elliptic operator L acting on a function u in a domain Ω, with boundary conditions on $\partial\Omega$

$$Lu = -\Delta u + V(x)u,$$

where Δ is the Laplacian and $V(x)$ is a potential function. If λ is an eigenvalue of L, then resonance may occur when the external force has a component with frequency λ. Non-resonance, on the flip side, refers to the situation where the external forces do not align with the system's natural frequencies. In this case, the solutions to the elliptic PDEs do not exhibit the amplified behavior seen in resonance. For the same operator L as above, non-resonance would imply that the external force's frequency is not an eigenvalue of L. Consider the boundary value problem for an elliptic PDE

$$\begin{aligned} -\Delta u + v(x)u = f(x) \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega, \end{aligned}$$

where $f(x)$ represents an external force or source term. The resonance condition can be expressed as

$$\int_\Omega f(x)\phi_i(x)dx \neq 0,$$

for some eigenfunction ϕ_i corresponding to the eigenvalue λ_i of L. In contrast, non-resonance would mean

$$\int_\Omega f(x)\phi_i(x)dx = 0,$$

for all eigenfunctions ϕ_i of L. In variable exponent spaces, the concept of resonance and non-resonance can be extended to PDEs with p-Laplacian operators, where the exponent p varies with position

$$-\Delta_{p(x)}u = -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right).$$

Here, the resonance and non-resonance conditions would involve the variable exponent $p(x)$ and the corresponding eigenvalues and eigenfunctions of the Δ_p operator.

This thesis is organized in the following manner: The first chapter is dedicated to notations, the definitions, and properties of functional spaces, as well as the fundamental results upon which we build in subsequent sections. These results are provided without proof, and for more details, we refer to the bibliography.

In the second chapter, we deal with a class of anisotropic elliptic equations with variable exponents and convection terms, and with a Dirichlet boundary condition, that is,

$$\begin{aligned} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) - \mu \Delta_{\vec{q}(x)}(u) &= \lambda f(x, u, \nabla u) + h(x) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where we prove the existence of the solution using the surjectivity result of pseudomonotone operators, and under additional conditions on the data, we show that the solution is unique. Lastly, we drop the gradient dependence on the nonlinearity and establish the existence of at least three weak solutions using the direct Ricceri variational principle. Authored by Ouannasser and El Hachimi [75], this research paper has been submitted for publication.

In the third chapter, we investigate another class of anisotropic elliptic equations with variable exponents, convection terms, but with Robin nonlinear boundary conditions, that is,

$$\begin{aligned} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{p_M(x)-2} u &= \lambda f(x, u, \nabla u) + h(x) \qquad \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\nu_i(x) &= \mu g(x, u) \qquad \text{on } \partial\Omega, \end{aligned}$$

where we prove the same results as in Chapter 2. The difference here lays in the fact that we do not make use of the first eigenvalue related to the following anisotropic eigenvalue problem studied by Mihăilescu and Morosanu [61]

$$\begin{aligned} -\sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \right) &= \lambda |u|^{r(x)-2} u \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Instead, we use the best Sobolev constant in the continuous embeddings $W^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and $W^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$. This work, which is authored by Ouannasser and El Hachimi [74], has been recently accepted in *Filomat*.

In the fourth chapter, we investigate the solvability on the left side of the positive infimum of all eigenvalues for specific quasilinear elliptic problems with variable exponents, ensuring they are non-resonant. In simpler terms, we deal with the following problem

$$\begin{aligned} -\Delta_{p_1(x)}u_1 - \mu_1\Delta_{q_1(x)}u_1 &= f_1\left(x, u_1, u_2, \nabla u_1, \nabla u_2\right) + h_1(x) && \text{in } \Omega, \\ -\Delta_{p_2(x)}u_2 - \mu_2\Delta_{q_2(x)}u_2 &= f_2\left(x, u_1, u_2, \nabla u_1, \nabla u_2\right) + h_2(x) && \text{in } \Omega, \\ u_1 = u_2 &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where we obtain the existence and uniqueness of the solution. Moreover, our focus extends to non-resonant gradient-type systems, where we establish existence using of a variational approach. The recent publication of this work by Ouannasser and El Hachimi [73] appeared in the *Moroccan Journal of Pure and Applied Analysis*.

In the fifth and final chapter, we dedicate the first part to the study of the following degenerate parametric elliptic system with convection terms near resonance

$$\begin{aligned} -\Delta_{w_1,p_1}u - \mu_1\Delta_{w_1,q_1}u &= \lambda a_1(x)|u|^{p_1-2}u + f_1(x,u,v,\nabla u,\nabla v) + h_1(x) && \text{in } \Omega, \\ -\Delta_{w_2,p_2}v - \mu_2\Delta_{w_2,q_2}v &= \mu a_2(x)|v|^{p_2-2}v + f_2(x,u,v,\nabla u,\nabla v) + h_2(x) && \text{in } \Omega, \\ u = v &= 0 && \text{on } \partial\Omega, \end{aligned}$$

while in the second part, we deal with another parametric elliptic system with convection terms at resonance, that is,

$$\begin{aligned} -\Delta_{w_1,p_1}u - \mu_1\Delta_{w_1,q_1}u &= \lambda_1 a_1(x)|u|^{p_1-2}u + \lambda_1 w(x)|u|^{\alpha_1-2}u|v|^{\alpha_2} \\ &\quad + f_1(x,u,v,\nabla u,\nabla v) + h_1(x) && \text{in } \Omega, \\ -\Delta_{w_2,p_2}v - \mu_2\Delta_{w_2,q_2}v &= \lambda_1 a_2(x)|v|^{p_2-2}v + \lambda_1 w(x)|v|^{\alpha_2-2}v|u|^{\alpha_1} \\ &\quad + f_2(x,u,v,\nabla u,\nabla v) + h_2(x) && \text{in } \Omega, \\ u = v &= 0 && \text{on } \partial\Omega. \end{aligned}$$

In both parts, we make use of the surjectivity result of pseudomonotone operators to obtain the existence of at least one weak solution. Furthermore, we study a gradient-type system at resonance and prove the existence of a solution using a variational approach. This work is in progress and is currently being prepared for submission.

Chapter 1

Functional framework

1.1 Classical Lebesgue-Sobolev spaces

The objective of this section is to propose suitable analogs for the Lebesgue spaces L^p and Sobolev spaces $W^{k,p}$. It is evident that a straightforward substitution of p with $p(x)$ in the conventional norm definition for L^p is not viable. However, the Lebesgue spaces can be viewed as specific instances of the Orlicz spaces within a broader family known as modular spaces. This approach allows the establishment of analogous counterparts for the Luxemburg and Orlicz norms in $L^{p(x)}$. When the function p is almost everywhere finite in Ω, then $L^{p(x)}$ becomes a specific case of the Orlicz-Musielak spaces, as treated by Musielak [66].

We extend the definition of $L^{p(x)}$ to include functions p with values in the interval $[1,+\infty[$. Let $\Omega \subset \mathbb{R}^N$ be a measurable subset with a measure $\text{meas}(\Omega) > 0$. The sets $\mathcal{C}(\bar{\Omega})$ and $\mathcal{C}_+(\bar{\Omega})$ are defined as follows

$$\mathcal{C}(\bar{\Omega}) = \left\{u : u \text{ is a continuous function in } \bar{\Omega}\right\},$$
$$\mathcal{C}_+(\bar{\Omega}) = \left\{u \in \mathcal{C}(\bar{\Omega}) : \text{ ess } \inf_{\Omega} u \geq 1\right\}.$$

Let $\mathcal{S}(\Omega)$ denote the set of all measurable real functions defined on Ω. Two functions in $\mathcal{S}(\Omega)$ are considered identical in $\mathcal{S}(\Omega)$ when they are equal almost everywhere. Assuming Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and $p \in \mathcal{C}(\Omega, \mathbb{R})$ with $p(x) > 1$ for any $x \in \Omega$, we denote $p^- = \inf_{x\in\Omega} p(x)$ and $p^+ = \sup_{x\in\Omega} p(x)$.

Additionally, we define the functions $p^*(x)$ and $p^\partial(x)$ as follows

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

and

$$p^{\partial}(x)=\begin{cases}\dfrac{(N-1)p(x)}{N-p(x)} & \text{if } p(x)<N,\\ +\infty & \text{if } p(x)\geq N.\end{cases}$$

It follows that $p^{-}>1$ and $p^{+}<\infty$. Let $\mathcal{M}$ be either Ω or $\partial\Omega$. The variable exponent Lebesgue space $L^{p(x)}(\mathcal{M})$ is defined as the set of measurable functions $u:\mathcal{M}\to\mathbb{R}$ such that $\displaystyle\int_{\mathcal{M}}|u(x)|^{p(x)}\mathrm{d}x<\infty$, equipped with the Luxembourg norm $\|u\|_{p(x)}=\|u\|_{L^{p(x)}(\mathcal{M})}$ given by

$$\|u\|_{p(x)}=\|u\|_{L^{p(x)}(\mathcal{M})}=\inf\left\{\tau>0,\int_{\mathcal{M}}\left|\frac{u(x)}{\tau}\right|^{p(x)}\mathrm{d}x\leq 1\right\}.$$

Proposition 1.1.1 (see Fan and Zhao [44]). *Let* $\rho(u)=\displaystyle\int_{\mathcal{M}}|u(x)|^{p(x)}\mathrm{d}x$. *For* $u,u_k\in L^{p(x)}(\mathcal{M})$ *with* $k\in\{1,2,\ldots\}$, *we have*

1. $\|u\|_{L^{p(x)}(\mathcal{M})}\leq 1\Longrightarrow\|u\|^{p^{+}}_{L^{p(x)}(\mathcal{M})}\leq\rho_{p(\cdot)}(u)\leq\|u\|^{p^{-}}_{L^{p(x)}(\mathcal{M})}$.
2. $\|u\|_{L^{p(x)}(\mathcal{M})}>1\Longrightarrow\|u\|^{p^{-}}_{L^{p(x)}(\mathcal{M})}\leq\rho_{p(\cdot)}(u)\leq\|u\|^{p^{+}}_{L^{p(x)}(\mathcal{M})}$.
3. $\|u_k|_{L^{p(x)}(\mathcal{M})}\longrightarrow 0\Longleftrightarrow\rho_{p(\cdot)}(u_k)\to 0$.
4. $\|u_k\|_{L^{p(x)}(\mathcal{M})}\longrightarrow\infty\Longleftrightarrow\rho_{p(\cdot)}(u_k)\to\infty$.

In addition, if $(u_n)_n\subset L^{p(x)}(\mathcal{M})$, *then*

$$\lim_{n\to+\infty}\|u_n-u\|_{L^{p(x)}(\mathcal{M})}=0\Longleftrightarrow\lim_{n\to+\infty}\rho_{p(\cdot)}(u_n-u)=0$$
$$\Longleftrightarrow (u_n)_n \text{ converges to } u \text{ in measure and } \lim_{n\to+\infty}\rho_{p(\cdot)}(u_n)=\rho_{p(\cdot)}(u).$$

Remark 1.1.2. From Proposition 1.1.1, we have

$$\rho_{p(\cdot)}(u)\leqslant\max\left\{\|u\|^{p^{+}}_{p(x)},\|u\|^{p^{-}}_{p(x)}\right\}\leqslant\|u\|^{p^{+}}_{p(x)}+\|u\|^{p^{-}}_{p(x)}.$$

We define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega):|\nabla u|\in L^{p(x)}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)}=\inf\left\{\tau>0;\int_{\Omega}\left(\left|\frac{\nabla u(x)}{\tau}\right|^{p(x)}+\left|\frac{u(x)}{\tau}\right|^{p(x)}\right)\mathrm{d}x\leq 1\right\}.$$

In general, the smooth functions are not dense in $W^{1,p(x)}(\Omega)$, but if the variable exponent $p \in C_+(\bar{\Omega})$ is logarithmic Hölder continuous, that is

$$|p(x) - p(y)| \leq -\frac{M}{\log(|x-y|)}, \text{ for all } x, y \in \Omega, \text{ such that } |x-y| \leq \frac{1}{2},$$

then the smooth functions are dense in $W^{1,p(x)}(\Omega)$. Now, denote by $W_0^{1,p(x)}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Hereafter, we assume that $p^- > 1$.

Proposition 1.1.3 (see Fan and Zhao [44]). *For any $u \in L^{p(x)}(\mathcal{M})$ and $v \in L^{q(x)}(\mathcal{M})$, the following inequality holds*

$$\int_{\mathcal{M}} |uv| \mathrm{d}x \leq 2\|u\|_{p(x)}\|v\|_{q(x)}, \quad \text{for all } u \in L^{p(x)}(\mathcal{M}) \text{ and } v \in L^{q(x)}(\mathcal{M}).$$

where $\dfrac{1}{p(x)} + \dfrac{1}{q(x)} = 1.$

Proposition 1.1.4 (see Fan and Zhao [44]). *The following Poincaré inequality holds*

$$\|u\|_{L^{p(x)}(\Omega)} \leq c\|\nabla u\|_{L^{p(x)}(\Omega)}, \quad \text{for all } u \in W_0^{1,p(x)}(\Omega),$$

where $c > 0$.

Proposition 1.1.5 (see Fan and Zhao [44]). *The spaces $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 1.1.6 (see Fan and Zhao [44]). *Assume that the boundary of $\Omega \subset \mathbb{R}^N, N \geq 2$ possesses the cone property and $p \in \mathcal{C}_+(\bar{\Omega})$. If $q \in \mathcal{C}(\bar{\Omega})$ and $1 \leq q(x) \leq p^*(x)$ for $x \in \bar{\Omega}$, then there is a compact embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

Proposition 1.1.7 (see Fan and Zhao [44]). *Let $\Omega \subset \mathbb{R}^N, N \geq 2$ be a bounded domain with smooth boundary. Suppose that $p \in \mathcal{C}_+(\bar{\Omega})$ and $r \in \mathcal{C}(\bar{\Omega})$ satisfy the condition $1 \leq r(x) \leq p^\partial(x)$ for $x \in \partial\Omega$, then there is a compact boundary trace embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega).$$

1.2 Classical anisotropic Sobolev spaces

Now, we introduce the anisotropic variable exponent Sobolev space $W^{1,\vec{p}(x)}(\Omega)$, where $\vec{p}:\bar{\Omega}\longrightarrow\mathbb{R}^N$ is a vectorial function defined as

$$\vec{p}(\cdot)=(p_1(\cdot),\ldots,p_N(\cdot)),$$

where for all $i\in\{1,\ldots,N\}, p_i\in\mathcal{C}_+(\bar{\Omega})$, we set

$$p_M(x)=\max\{p_1(x),\ldots,p_N(x)\} \text{ and } p_m(x)=\min\{p_1(x),\ldots,p_N(x)\},$$

for all $x\in\bar{\Omega}$. The anisotropic Sobolev space with variable exponent is introduced by

$$W^{1,\vec{p}(x)}(\Omega)=\left\{u\in L^{p_M(x)}(\Omega):\partial_{x_i}u\in L^{p_i(x)}(\Omega), \text{ for all } i\in\{1,\ldots,N\}\right\}.$$

and it is endowed with the norm

$$\|u\|_{W^{1,\vec{p}(x)}(\Omega)}=|u|_{L^{p_M(x)}(\Omega)}+\sum_{i=1}^{N}|\partial_{x_i}u|_{L^{p_i(x)}(\Omega)}.$$

The space $\left(W^{1,\vec{p}}(\Omega),\|\cdot\|_{W^{1,\vec{p}}(\Omega)}\right)$ forms a reflexive Banach space for any $\vec{p}\in\mathcal{C}\left(\Omega,\mathbb{R}^N\right)$, where each component $p_i^->1$ for all $i\in\{1,\ldots,N\}$. A noteworthy subspace within $W^{1,\vec{p}(x)}(\Omega)$ is $W_0^{1,\vec{p}(x)}(\Omega)$, representing functions that vanish on the boundary. The anisotropic variable exponent Sobolev space $W_0^{1,\vec{p}(x)}(\Omega)$ is defined as the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\vec{p}(x)}=\sum_{i=1}^{N}|\partial_{x_i}u|_{p_i(x)}.$$

When each $p_i\in\mathcal{C}_+(\Omega)$ is a constant function for all $i\in\{1,\ldots,N\}$, the resulting anisotropic Sobolev space is denoted as $W_0^{1,\vec{p}}(\Omega)$, where $\vec{p}$ represents the constant vector $(p_1,\ldots,p_N)$. As indicated in Fan [42], the space $\left(W_0^{1,\vec{p}}(\Omega),\|\cdot\|_{W_0^{1,\vec{p}}(\Omega)}\right)$ forms a reflexive Banach space for any $\vec{p}\in\mathcal{C}\left(\Omega,\mathbb{R}^N\right)$, given that $p_i^->1$ for all $i\in\{1,\ldots,N\}$. Now, denote

$$p_M(x)=\max\{p_1(x),\ldots,p_N(x)\},\quad p_m(x)=\min\{p_1(x),\ldots,p_N(x)\},$$

and

$$p_M^+=\max\left\{p_1^+,\ldots,p_N^+\right\},\quad p_M^-=\max\left\{p_1^-,\ldots,p_N^-\right\},\quad p_m^-=\min\left\{p_1^-,\ldots,p_N^-\right\}.$$

Below we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{1.2.1}$$

and define $p_-^*, p_{-,\infty} \in \mathbb{R}^+$ by

$$p_-^* := \frac{N}{\displaystyle\sum_{i=1}^{N} \frac{1}{p_i^-} - 1} \quad \text{and} \quad p_{-,\infty} := \max\left\{p_M^-, p_-^*\right\}.$$

For any $q \in C_+(\bar{\Omega})$ with $1 < q(x) \le p_{-,\infty}$, for a.e $x \in \bar{\Omega}$, we denote $S_{q,\Omega}$ the best constant in the continuous embedding $W^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, that is,

$$S_{q(x),\Omega} = \inf_{v \in W^{1,\vec{p}(x)}(\Omega)\setminus\{0\}} \frac{\|v\|_{W^{1,\vec{p}(x)}(\Omega)}}{\|v\|_{L^{q(x)}(\Omega)}}.$$

Similarly, for any $q \in C_+(\bar{\Omega})$ verifying $1 \le q(x) \le \min_{x\in\partial\Omega}\{p_1^\partial(x), \ldots, p_N^\partial(x)\}$, where

$$p_i^\partial(x) = \begin{cases} (N-1)p_i(x)/[N-p_i(x)] & \text{if } p_i(x) < N, \\ \infty & \text{if } p_i(x) \ge N, \end{cases}$$

for all $x \in \bar{\Omega}$ and for $i \in \{1, \ldots, N\}$, we denote $S_{q(x),\partial\Omega}$ the best constant in the embedding $W^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$.

Proposition 1.2.1 (see Mihăilescu et al. [62])**.** *Suppose that $\Omega \subset \mathbb{R}^N (N \ge 3)$ is a bounded domain with smooth boundary and relation (1.2.1) is fulfilled. For any $q \in \mathcal{C}(\bar{\Omega})$ verifying $1 < q(x) < p_{-,\infty}$ for all $x \in \bar{\Omega}$, the embedding $W^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.*

Proposition 1.2.2 (see Boureanu and Rădulescu [21])**.** *Suppose that $\vec{p} \in \left(C_+(\bar{\Omega})\right)^N$ and $q \in C_+(\bar{\Omega})$ satisfying $1 < q(x) < \min_{x\in\partial\Omega}\left\{p_1^\partial(x), \ldots, p_N^\partial(x)\right\}$, for all $x \in \partial\Omega$. Then, the embedding $W^{1,\vec{p}(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\partial\Omega)$ is compact.*

Remark 1.2.3. If $p_i(x) \le q_i(x)$ for $i \in \{1, \ldots, N\}$ and any $x \in \bar{\Omega}$. Then, we have

$$L^{q_i(x)}(\Omega) \hookrightarrow L^{p_i(x)}(\Omega).$$

Therefore, there exists $k_1 > 0$ such that

$$\|u\|_{p_i(x)} \le k_1\|u\|_{q_i(x)}, \quad \text{for all } u \in L^{q_i(x)}(\Omega),$$

Remark 1.2.4. If $p_i(x) \le q_i(x)$ for all $i \in \{1, \ldots, N\}$ and for any $x \in \bar{\Omega}$. Then, we have

$$W_0^{1,\vec{q}(x)}(\Omega) \hookrightarrow W_0^{1,\vec{p}(x)}(\Omega).$$

Therefore, there exists $k_2 > 0$ such that

$$\|u\|_{\vec{p}(x)} \leq k_2 \|u\|_{\vec{q}(x)}, \text{ for all } u \in W_0^{1,\vec{q}(x)}(\Omega).$$

Moreover, using Proposition 1.2.1 and Remark 1.2.3, there exists $k_3 > 0$ such that

$$\sum_{i=1}^{N} \rho_{p_i}\left(\partial_{x_i} u\right) \leq k_3 + p_M^+ \sum_{i=1}^{N} \rho_{q_i}\left(\partial_{x_i} u\right).$$

Remark 1.2.5. Using Proposition 1.2.1, we obtain

$$\sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)} \geq \left(\frac{\|u\|_{\vec{p}(x)}^{p_m^-}}{N^{p_m^- - 1}} - N \right), \quad \text{for all } u \in W_0^{1,\vec{p}(x)}(\Omega) \text{ with } \|u\| > 1.$$

1.3 Preliminary results

Lemma 1.3.1. *(see Carl et al. [24]) Let X be a real reflexive Banach space, and assume that $\mathcal{J} : X \to X^*$ is a bounded, pseudomonotone and coercive operator. Then, there exists a solution to the equation $\mathcal{J}(u) = h$, for all $h \in X^*$.*

Theorem 1.3.2. *(see Ricceri [78]) Let X be a reflexive real Banach space and I a real interval. $\mathcal{H} : X \to \mathbb{R}$ a continuously differentiable and sequentially weakly lower semi-continuous functional whose derivative admits a continuous inverse on X^*, $\mathcal{K} : X \to \mathbb{R}$ a continuously differentiable functional whose derivative is compact. Assume that*

(i) $\lim\limits_{\|u\| \to \infty} (\mathcal{H}(u) + \lambda \mathcal{K}(u)) = \infty$*, for all* $\lambda \in I$.

(ii) There exists $\gamma \in \mathbb{R}$ *such that*

$$\sup_{\lambda \in I} \inf_{u \in X} (\mathcal{H}(u) + \lambda(\mathcal{K}(u) + \gamma)) < \inf_{u \in X} \sup_{\lambda \in I} (\mathcal{H}(u) + \lambda(\mathcal{K}(u) + \gamma)).$$

Then, there exist an open interval $\Lambda \subset I$ and a positive real number ρ such that, for each $\lambda \in \Lambda$ and for every $\mathcal{M} : X \to \mathbb{R}$ continuously differentiable, with compact derivative, there exists $\delta > 0$ such that $\forall \mu \in [0, \delta]$, the equation

$$\mathcal{H}'(u) + \mu \mathcal{M}'(u) + \lambda \mathcal{K}'(u) = 0,$$

has at least three solutions in X whose norms are less than ρ.

Part I

On some anisotropic elliptic problems

Chapter 2

Parametric anisotropic elliptic problems with variable exponents and convection terms

In this chapter, we study a class of parametric anisotropic elliptic equations with variable exponents where the nonlinearity may depend on the gradient of the solution. We prove the existence of the solution using the surjectivity result of pseudomonotone operators, and under additional conditions on the data, we show that the solution is unique. Moreover, we establish the existence of at least three weak solutions using the direct Ricceri variational principle when the nonlinearity does not depend on the gradient.

2.1 Introduction

The attention towards anisotropic elliptic problems with variable exponents has grown due to their capability to capture complex physical phenomena that exhibit different behaviors or intensities across various parts of the domain. The variable exponents can arise from the nonlinearity of the underlying physical process or from the heterogeneous nature of the problem domain. Studying these problems is of utmost importance as it allows for a more accurate modeling of real-world situations, including materials with spatially varying properties, porous media flow, and non-Newtonian fluid flows. The investigation of problems with convection terms holds great importance, as they play a vital role in several physical and engineering applications, including fluid dynamics, combustion, heat transfer, and environmental modeling.

In this chapter, we deal with a class of anisotropic elliptic equations with variable exponents and convection terms, and with a Dirichlet boundary condition.

In other words, we study the following problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) - \mu \Delta_{\vec{q}(x)}(u) = \lambda f(x, u, \nabla u) + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 2$, $\lambda, \mu > 0$ are real numbers, $h \in \left(W_0^{1,\vec{p}(x)}(\Omega)\right)^\star$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a function of Carathéodory verifying some growth conditions that will be stated later.

In the last few years, a new operator captured the attention of many authors, and this is thanks to the development of the theory of anisotropic Sobolev spaces. This operator is the pseudo-$\vec{p}$-Laplacian, that is,

$$\Delta_{\vec{p}}(u) = \sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i - 2} \partial_{x_i} u \right). \tag{2.1.2}$$

The same operator is generalized to ultimately become the pseudo-$\vec{p}(x)$-Laplacian, that is,

$$\Delta_{\vec{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i(x) - 2} \partial_{x_i} u \right). \tag{2.1.3}$$

In the current chapter, we deal with a problem that involves a more general type of operator, that is

$$\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u), \tag{2.1.4}$$

where, for all $i \in \{1, \dots, N\}$, $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory-type functions verifying

(A_1) $A_i(x, 0) = 0$ and $A_i(x, -t) = A_i(x, t)$, for all $x \in \Omega$ and $t \in \mathbb{R}$, where

$$A_i(x, t) = \int_0^t a_i(x, s) ds,$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

(A_2) $a_i(x, \cdot)$ is strictly monotone in $\mathbb{R}$, for all $x \in \Omega$, that is

$$\left(a_i(x, s) - a_i(x, t)\right)(s - t) > 0,$$

for all $(s,t) \in \mathbb{R}^2$, with $s \neq t$.

(A_3) There exist positive constants σ_i and ρ_i such that

$$\sigma_i |t|^{p_i(x)} \leq a_i(x,t)t \quad \text{and} \quad |a_i(x,t)| \leq \rho_i |t|^{p_i(x)-1},$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

Note that by hypotheses (A_1) and (A_3), we have

$$\sigma_i |t|^{p_i(x)} \leq p_i(x) A_i(x,t) \leq \rho_i |t|^{p_i(x)}.$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

Many works have discussed problems in which the term on the right-hand side depends on the gradient of the solution in the context of anisotropic elliptic equations. For example, Akdim and Salmani [6] and Di Nardo and Feo [31] investigated this type of problem in the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$. Our aim here is on one hand to deal with a gradient-dependent nonlinearity in the anisotropic variable exponent Sobolev space in the context of parametric elliptic equations. Such class of problems brings many difficulties to overcome, such as the fact that the use of variational methods is no longer possible because of the gradient dependence on the right-hand side. The difficulty also lies in the fact that both $p(\cdot)$-Laplacian and $\vec{p}(\cdot)$-Laplacian operators are nonhomogeneous, unlike the p-Laplacian operator. Recall that the existence of solutions for isotropic elliptic equations has also been obtained by Wang et al. [84].

On the other hand, we are also interested in proving the existence of multiple solutions for nonlinearities that do not depend on the gradient. Recall that variational methods have been employed in several papers to establish the existence of solutions for quasilinear anisotropic elliptic equations, characterized by constant or variable exponents, particularly when the second term lacks dependence on the gradient. We quote Boureanu et al. [20, 22], Ellahyani and El Hachimi [41], Fan [42] and the references therein.

In Section 2.3, we shall also prove the uniqueness of the solution under specific hypotheses on the data for our problem. Now, we situate our work in relation to some publications already produced on this subject.

Example I. The isotropic case $a_i(x,t) = |t|^{p(x)-2}t$, with $w(x) \equiv 1$, $\lambda = 1$ and $h \equiv 0$ has been treated by Wang et al. [84]. They obtained existence and uniqueness results for the $p(x)$-Laplacian equation of the form

$$\begin{cases} -\Delta_{p(x)} u = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1.5}$$

Example II. Considering the general isotropic case related to our problem with $h(x) \equiv 0$, $\mu = 0$, and when dropping the gradient dependence of the nonlinearity f (i.e. $f(x,u,\nabla u) \equiv f(x,u)$), Liu and Yu [57], studied the following problem

$$\begin{cases} -\operatorname{div} A\,(x, \nabla u) = \lambda f(x,u) & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1.6}$$

where, under adequate conditions on the data that are close to ours, the authors showed existence of three weak solutions.

Example III. In the case where $a_i(x,t) = |t|^{p_i(x)-2}t$, $h(x) \equiv 0$, $\mu = 0$, $\lambda = 1$ and also when dropping the gradient dependence of f, Boureanu et al. [20] investigated the multiplicity of solutions for the following class of anisotropic elliptic equations

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i} u|^{p_i(x)-2}\,\partial_{x_i} u\right) = f(x,u) & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1.7}$$

Example IV. El Amrouss and El Mahraoui [37] dealt with the following anisotropic elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{p_M^+-2}u = \lambda f(x,u) & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1.8}$$

where the a_i satisfy the same hypothesis as in this work. They obtained existence of three weak solutions in the case where the primitive F of f is superlinear in some sense. Here, we are conversly interested in the case where F is sublinear and the existence of three weak solutions is obtained.

Example V. Zhou and Ge [87] showed the existence of three solutions for the following problem

$$\begin{cases} -\operatorname{div}\,(A(x, |\nabla u|)\nabla u) = \lambda f(x,u) & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1.9}$$

where

$$A(x,\xi) = \left(1 + \frac{|\xi|^{p(x)}}{\sqrt{1+|\xi|^{2p(x)}}}\right)|\xi|^{p(x)-2}.$$

Our result, in Section 2.3, includes the anisotropic version of their problem which corresponds to the case where

$$a_i(x,\eta)=|\eta|^{p_i(x)-2}\eta+\frac{|\eta|^{2p_i(x)-2}}{\sqrt{1+|\eta|^{2p_i(x)}}}\eta,$$

for $\eta\in\mathbb{R}$, $1\le i\le N$, satisfiy hypotheses $(A_1)-(A_3)$.

On the other hand, many works concerning anisotropic elliptic problems with Neumann boundary conditions have been treated in recent years. We quote the papers of Boureanu and Rădulescu [21], Ellahyani and El Hachimi [41] and Ourraoui and Ragusa [76]. In a forthcoming paper, we shall deal with existence, uniqueness and multiplicity for general parametric anisotropic elliptic problems with nonhomogeneous Neumann boundary conditions.

The remainder of this chapter is organized as follows. In Section 2.2, using the surjectivity result of pseudomonotone operators, we prove the existence of a solution for problem (2.1.1). Moreover, under some suitable hypotheses, we show that the solution is unique. In Section 2.3, we establish the existence of at least three weak solutions using a recent Ricceri variational principle in the case where there is no dependance of the nonlinearity f on the gradient. Finally, in Section 2.4, we give a few concluding examples that verify our theorems.

2.2 Main results

All along this chapter, we shall use the following hypothesis

(A_0) $1<\sum_{i=1}^{N}\frac{1}{p_i^-}<N+1.$

Definition 2.2.1. We say that $u\in W_0^{1,\vec{p}(x)}(\Omega)$ is a weak solution of problem (2.1.1) if

$$\begin{aligned}\int_\Omega\sum_{i=1}^{N}a_i\left(x,\partial_{x_i}u\right)\partial_{x_i}v\,dx+\mu\sum_{i=1}^{N}\int_\Omega|\partial_{x_i}u|^{q_i(x)-2}\partial_{x_i}u\partial_{x_i}v\,dx\\=\lambda\int_\Omega f(x,u,\nabla u)v\,dx+\langle h,v\rangle,\end{aligned}\tag{2.2.1}$$

for all $v\in W_0^{1,\vec{p}(x)}(\Omega)$, where $\langle h,v\rangle$ is the duality pairing between $W_0^{1,\vec{p}(x)}(\Omega)$ and its dual space.

Definition 2.2.2. Let X be a reflexive Banach space, X^* its dual space and denote by $\langle\cdot,\cdot\rangle$ its duality pairing. Consider an application $\mathcal{J}:X\to X^*$. Then, $\mathcal{J}$ is called

(a) to verify the (S^+)-property if $u_n\rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle\mathcal{J}u_n,u_n-u\rangle\le 0$ imply $u_n\to u$ in X.

(b) pseudomonotone if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} \langle \mathcal{J}u_n, u_n - u\rangle \leq 0$ imply $\mathcal{J}u_n \rightharpoonup \mathcal{J}u$ and $\langle \mathcal{J}u_n, u_n\rangle \to \langle \mathcal{J}u, u\rangle$.

(c) coercive if

$$\lim_{\|u\|_X\to\infty} \frac{\langle \mathcal{J}u, u\rangle}{\|u\|_X} = \infty.$$

Before proving the existence and uniqueness of the solution, we first recall the following anisotropic eigenvalue problem studied by Mihăilescu and Morosanu [61]

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\,\partial_{x_i}u\right) = \lambda|u|^{r(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2.2}$$

One can also see the works of Chung [26] and Stancut [82] for some results on eigenvalue problems with weights and variable exponents. The basic assumptions on the functions p_i, r involved in problem (2.2.2) will be the following

(P_1) There exists $j \in \{1,\dots,N\}$ such that $r(x) = r\,(x_1,\dots,\, x_{j-1}, x_{j+1},\dots,x_N)$ (i.e. r is independent of x_j) and $p_j(x) = r(x)$ for all $x \in \bar{\Omega}$,

(P_2) There exists $k \in \{1,\dots,N\}$ ($k \neq j$ with j given in (A_1)) such that

$$\max_{x\in\bar{\Omega}} r(x) < \min_{x\in\bar{\Omega}} p_k(x).$$

We define the Rayleigh type quotients λ_0 and λ_1 associated with problem (2.2.2) as

$$\lambda_0 = \inf_{u\in W_0^{1,\vec{p}(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \sum_{i=1}^{N} |\partial_{x_i}u|^{p_i(x)}\,dx}{\displaystyle\int_\Omega |u|^{r(x)}dx}, \quad \lambda_1 = \inf_{u\in W_0^{1,\vec{p}(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \sum_{i=1}^{N} \frac{1}{p_i(x)}\,|\partial_{x_i}u|^{p_i(x)}\,dx}{\displaystyle\int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx}.$$

The main result in Mihăilescu and Morosanu [61] reads as follows.

Theorem 2.2.3. *(see Mihăilescu and Morosanu [61]) Under assumptions* (P_1) *and* (P_2)*, we have* $0 < \lambda_0 \leq \lambda_1$ *and every* $\lambda \in \,]\lambda_1, \infty[$ *is an eigenvalue of problem* (2.2.2)*, while no* $\lambda \in \,]0, \lambda_0[$ *can be an eigenvalue of problem* (2.2.2).

Now, in order to prove the existence of a solution for problem (2.1.1), we assume the following assumptions on f

(F_1) There exists $k \in L^{r'(x)}(\Omega)$ and $c_1, c_2 > 0$ such that

$$|f(x,t,\xi)| \leq k(x) + c_1|t|^{r(x)-1} + c_2 \sum_{i=1}^{N} |\xi_i|^{p_i(x)\frac{r(x)-1}{r(x)}},$$

for a.e. $x \in \Omega$ and all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

(F_2) There exists $\theta \in L^1(\Omega)$ and $c_3, c_4 \geq 0$ with $|\lambda| \left(c_4 + c_3\lambda_0^{-1}\right) < \sigma_0 + \dfrac{\mu}{p_M^+}$ such that

$$f(x,t,\xi)t \leq \theta(x) + c_3|t|^{r(x)} + c_4 \sum_{i=1}^{N} |\xi_i|^{p_i(x)},$$

for a.e. $x \in \Omega$ and all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

Proposition 2.2.4. *(see Fan and Zhang [43]) We define* $X = W_0^{1,\vec{p}(x)}(\Omega)$ *and*

$$\langle \mathcal{J}(u), v\rangle = \int_\Omega \Big(\sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u\right) \partial_{x_i} v dx + \mu \sum_{i=1}^{N} |\partial_{x_i} u|^{q_i(x)-2} \partial_{x_i} u \partial_{x_i} v \Big) dx,$$

for all $u, v \in X$*. Then*

i) $\mathcal{J} : X \to X^\star$ *is a continuous, bounded and strictly monotone operator.*

ii) $\mathcal{J}$ *is of type* (S^+).

iii) $\mathcal{J}$ *is a homeomorphism.*

The following theorem is the subject of a third publication with my supervisor Professor El Hachimi, which has been submitted for publication.

Theorem 2.2.5. *(see Ouannasser and El Hachimi [75, Theorem 3.4.]) Let us suppose that* $1 < r(x) < p_{-,\infty}$*, for all* $x \in \bar{\Omega}$ *and* $1 < p_i(x) \leq q_i(x)$*, for any* $i \in \{1, \ldots, N\}$*. Assume that hypotheses* $(A_0) - (A_3)$*,* (P_1)*,* (P_2)*,* (F_1) *and* (F_2) *are satisfied. Moreover, suppose that we have* $|\lambda| \left(c_4 + c_3\lambda_0^{-1}\right) < \sigma_0 + \dfrac{\mu}{p_M^+}$*, where* $\sigma_0 = \min_{1\leq i\leq N} \sigma_i$*. Then, for any* $h \in \left(W_0^{1,\vec{p}(x)}(\Omega)\right)^\star$*, problem* (2.1.1) *admits a weak solution in* $W_0^{1,\vec{p}(x)}(\Omega)$*.*

Proof. We define the Nemytskii operator $\bar{N}_f : X \subseteq L^{r(x)}(\Omega) \to L^{r'(x)}(\Omega)$ by $\left(\bar{N}_f u\right)(x) = f(x, u(x), \nabla u(x))$. Furthermore, let $i^* : L^{r'(x)}(\Omega) \to X^*$ be the adjoint operator for the embedding of $i : X \to L^{r(x)}(\Omega)$. We then define $N_f = i^* \circ \bar{N}_f : X \to X^*$, which is well-defined by assumption (F_1). We set

$$\mathbb{A}(u) = \mathcal{J}(u) - N_f(u) - h.$$

Our aim is to apply Lemma 1.3.1. Hence, we need to show that $\mathbb{A}$ is bounded, pseudomonotone and coercive.

- $\mathbb{A}$ ***is bounded:***

Thanks to growth conditions stated in (F_1) on f and the boundedness of $\mathcal{J}$, we obtain the boundedness of $\mathbb{A}$.

- $\mathbb{A}$ ***is pseudomonotone:***

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

$$u_n \rightharpoonup u \text{ in } X \text{ and } \limsup_{n\to+\infty} \langle \mathbb{A}(u_n), u_n - u\rangle \leq 0.$$

Thanks to the compact embeddings $X \hookrightarrow L^{r(x)}(\Omega)$, we get

$$u_n \to u \text{ in } L^{r(x)}(\Omega). \tag{2.2.3}$$

Thus, using (F_1) alongside Hölder's inequality and the boundedness of $(u_n)_{n\in\mathbb{N}}$, we obtain

$$\begin{aligned}
\left|\int_\Omega f(x, u_n, \nabla u_n) \cdot (u_n - u)dx\right| &\leq \Bigg[\int_\Omega k(x)|u_n - u|dx + c_1 \int_\Omega |u_n|^{r(x)-1}|u_n - u|dx \\
&\quad + c_2 \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)\frac{r(x)-1}{r(x)}} |u_n - u|dx\Bigg] \\
&\leq \Bigg[2\|k\|_{r'(x)}\|u_n - u\|_{r(x)} \\
&\quad + 2c_1 \left\||u_n|^{r(x)-1}\right\|_{r'(x)} \|u_n - u\|_{r(x)} \\
&\quad + 2c_2 \sum_{i=1}^N \left\||\partial_{x_i} u_n|^{p_i(x)\frac{r(x)-1}{r(x)}}\right\|_{r'(x)} \|u_n - u\|_{r(x)}\Bigg].
\end{aligned}$$

Therefore, we obtain

$$\left|\int_\Omega f(x,u_n,\nabla u_n)\cdot(u_n-u)dx\right| \leq \Bigg[2\|k\|_{r'(x)}\|u_n-u\|_{r(x)}$$
$$+ 2c_1\left(\left\||u_n|^{r^+-1}\right\|_{r(x)} + \left\||u_n|^{r^--1}\right\|_{r(x)}\right)\|u_n-u\|_{r(x)}$$
$$+ 2c_2\sum_{i=1}^N\left(\left\||\partial_{x_i}u_n|^{p_i^+(\frac{r^+-1}{r^+})}\right\|_{p_i(x)}\right.$$
$$\left.+\left\||\partial_{x_i}u_n|^{p_i^-(\frac{r^--1}{r^-})}\right\|_{p_i(x)}\right)\|u_n-u\|_{r(x)}\Bigg].$$

This, in combination with (2.2.3), leads to the conclusion that

$$\lim_{n\to+\infty}\int_\Omega f(x,u_n,\nabla u_n)\cdot(u_n-u)dx = 0.$$

Passing to the limit in (2.2.1), and replacing u with u_n and v with u_n-u, we obtain

$$\limsup_{n\to+\infty}\langle\mathbb{A}(u),u_n-u\rangle = \limsup_{n\to+\infty}\langle\mathcal{J}(u),u_n-u\rangle \leq 0.$$

Hence, it follows that $u_n\to u$ because $\mathcal{J}$ is of type (S^+). Furthermore, due to the continuity of $\mathbb{A}$, then we get $\mathbb{A}(u_n)\to\mathbb{A}(u)$ in X^* which demonstrates that $\mathbb{A}$ is pseudomonotone.

• $\mathbb{A}$ ***is coercive:***

Hereafter, we shall denote $\|u\| = \|u\|_{\vec{p}(x)}$. We need to show that

$$\lim_{\|u\|\to+\infty}\frac{\langle\mathbb{A}(u),u\rangle}{\|u\|} = +\infty. \tag{2.2.4}$$

Denote the dual norm in X^* by $\|.\|_*$. For any $u\in X$, using (F_2) and Proposition 1.1.1 alongside Remark 1.2.4, we obtain

$$
\begin{aligned}
\langle \mathbb{A}(u), u\rangle &= \int_\Omega \Big(\sum_{i=1}^N a_i\left(x, \partial_{x_i} u\right) \partial_{x_i} u + \mu \sum_{i=1}^N |\partial_{x_i} u|^{q_i(x)} \Big) dx - \int_\Omega \lambda f(x,u,\nabla u) u dx \\
&- \langle h, u\rangle \\
&\geq \sigma_0 \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)}\, dx + \frac{\mu}{p_M^+} \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} - \frac{k_2}{p_M^+} \\
&- |\lambda| \Big(\int_\Omega |\theta(x)| dx - c_3 \int_\Omega |u|^{r(x)} dx - c_4 \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)}\, dx \Big) - \|h\|_* \|u\| \\
&\geq \left(\int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)}\, dx \right) (\sigma_0 + \frac{\mu}{p_M^+} - c_4|\lambda| - c_3 |\lambda| \lambda_0^{-1}) - |\lambda| \|\theta\|_1 \\
&- \|h\|_* \|u\| \\
&\geq \left(\frac{1}{N^{p_m^- - 1}} \|u\|^{p_m^-} - N \right) (\sigma_0 + \frac{\mu}{p_M^+} - c_4|\lambda| - c_3 |\lambda| \lambda_0^{-1}) - |\lambda| \|\theta\|_1 \\
&- \|h\|_* \|u\|.
\end{aligned}
$$

Now, since $\sigma_0 + \dfrac{\mu}{p_M^+} > |\lambda| \left(c_4 + c_3 \lambda_0^{-1}\right)$ and $p_m^- > 1$, then (2.2.4) is satisfied. Hence, $\mathbb{A}$ is coercive.

Consequently, all the conditions outlined in Lemma 1.3.1 are met. Hence, there exists a solution $u \in X$ such that $\mathbb{A}(u) = h$. With this, the proof is concluded. □

Following that, we investigate the uniqueness of the solution for problem (2.1.1) under the following additional hypotheses

(F_3) For all $\xi \in \mathbb{R}^N$ and for a.e $x \in \Omega$, $s \to f(x, s, \xi)$ is decreasing.

(F_4) For all $s \in \mathbb{R}$ and for a.e $x \in \Omega$, we have $f(x, s, \xi) = f(x, s, |\xi|)$ and $|\xi| \to f(x, s, |\xi|)$ is decreasing.

(F_5) $\lambda > 0$ and $q_i(x) \geq 2$, for all $i \in \{1, \ldots, N\}$ and for a.e $x \in \Omega$.

The following theorem is the subject of a third publication with my supervisor Professor El Hachimi, which has been submitted for publication.

Theorem 2.2.6. *(see Ouannasser and El Hachimi [75, Theorem 3.5.]) Suppose the hypotheses outlined in Theorem 3.2.4 alongside hypotheses* $(F_3)-(F_5)$ *are met. Then, problem* (2.1.1) *admits a unique solution.*

Proof. Let u_1 and u_2 be two weak solutions to problem (2.1.1). Using the weak formulation of u_1 and u_2 and choosing $\phi = (u_1 - u_2)_+$ as a test function, we find

$$\int_\Omega \sum_{i=1}^N \left(a_i\left(x,\partial_{x_i}u_1\right)-a_i\left(x,\partial_{x_i}u_2\right)\right)\left(\partial_{x_i}u_1-\partial_{x_i}u_2\right)_+dx$$
$$+\mu\int_\Omega \left(|\partial_{x_i}u_1|^{q_i(x)-2}\partial_{x_i}u_1-|\partial_{x_i}u_1|^{q_i(x)-2}\partial_{x_i}u_2\right)\left(\partial_{x_i}u_1-\partial_{x_i}u_2\right)_+dx$$
$$=\lambda\int_\Omega (f(x,u_1,\nabla u_1)-f(x,u_2,\nabla u_2))\left(u_1-u_2\right)_+dx.$$

By hypotheses (A_2) and (F_5), we obtain

$$0\leq \mu\int_\Omega \sum_{i=1}^N \left|\partial_{x_i}\left(u_1-u_2\right)_+\right|^{q_i(x)}dx$$
$$\leq \int_\Omega \sum_{i=1}^N \left(a_i\left(x,\partial_{x_i}u_1\right)-a_i\left(x,\partial_{x_i}u_2\right)\right)\left(\partial_{x_i}u_1-\partial_{x_i}u_2\right)_+dx$$
$$+\mu\int_\Omega \left(|\partial_{x_i}u_1|^{q_i(x)-2}\partial_{x_i}u_1-|\partial_{x_i}u_1|^{q_i(x)-2}\partial_{x_i}u_2\right)\left(\partial_{x_i}u_1-\partial_{x_i}u_2\right)_+dx$$
$$\leq \lambda\int_\Omega (f(x,u_1,|\nabla u_1|)-f(x,u_2,|\nabla u_2|)\left(u_1-u_2\right)_+dx.$$

Denote

$$\int_\Omega (f(x,u_1,|\nabla u_1|)-f(x,u_2,|\nabla u_2|)\left(u_1-u_2\right)_+dx=\mathcal{I}_1+\mathcal{I}_2,$$

where

$$\mathcal{I}_1=\int_\Omega \left(f\left(x,u_1,|\nabla u_1|\right)-f\left(x,u_2,|\nabla u_1|\right)\right)\cdot\left(u_1-u_2\right)_+dx,$$
$$\mathcal{I}_2=\int_\Omega \left(f\left(x,u_2,|\nabla u_1|\right)-f\left(x,u_2,|\nabla u_2|\right)\right)\cdot\left(u_1-u_2\right)_+dx.$$

According to hypothesis (F_1), we have $\mathcal{I}_1\leq 0$. On the other hand, we get

$$\mathcal{I}_2=$$
$$\int_{\Omega_2}\left(f\left(x,u_2,|\nabla u_1|\right)-f\left(x,u_2,|\nabla u_2|\right)\right)\cdot\left(|\nabla u_1|-|\nabla u_2|\right)_+\frac{\left(u_1-u_2\right)_+}{\left(|\nabla u_1|-|\nabla u_2|\right)_+}dx,$$

where $\Omega_2=\left\{x\in\Omega,\left(|\nabla u_1|-|\nabla u_2|\right)_+(x)\neq 0\right\}$. Thus, according to hypothesis (F_2), we have $\mathcal{I}_2\leq 0$. Therefore, we obtain

$$\lambda\int_\Omega \left(f(x,u_1,\nabla u_1)-f(x,u_2,\nabla u_2)\right)\cdot\left(u_1-u_2\right)_+dx\leq 0,$$

which implies that

$$\int_\Omega \sum_{i=1}^N \left|\partial_{x_i}\left(u_1-u_2\right)_+\right|^{q_i(x)}dx=0.$$

From this, we find $\nabla u_1(x) = \nabla u_2(x)$ for a.e x on $\mathcal{U} = \{x \in \Omega : u_1(x) > u_2(x)\}$. Now, for $x \in \mathcal{U}$, we have $(u_1 - u_2)_+ (x) = (u_1 - u_2)(x)$ and $\nabla (u_1 - u_2)_+ (x) = \nabla u_1(x) - \nabla u_2(x) = 0$, while for $x \in \Omega \backslash \mathcal{U}$, we have $(u_1 - u_2)_+ (x) = 0$ and $\nabla (u_1 - u_2)_+ (x) = 0$.

Consequently, we have $\nabla (u_1 - u_2)_+ (x) = 0$, a.e in Ω and then $(u_1 - u_2)_+ = 0$ because $(u_1 - u_2)_+ \in W_0^{1,\vec{q}(x)}(\Omega)$. Then, we obtain $u_1 \leq u_2 = 0$, a.e in Ω. Correspondingly, we obtain $u_1 \geq u_2 = 0$, a.e in Ω. Hence, $u_1 = u_2$ and the solution is unique. $\square$

2.3 Three weak solutions

In this section, we establish the existence of at least three weak solutions using a Ricceri's recent variational principle to the following problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) - \mu \Delta_{\vec{q}(x)}(u) = \lambda f(x,u) + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3.1}$$

where $h \in L^{(p_M^+)'}(\Omega)$ where $\frac{1}{p_M^+} + \frac{1}{(p_M^+)'} = 1$, $1 < r^- \leq r^+ < p_m^- \leq p_M^+ < q_m^-$, λ is a real number, and $\mu > 0$. For $x_0 \in \Omega$ and $e_0 > 0$, denote

$$\mathcal{B}_1(x_0, D) = \left\{ x \in \mathbb{R}^N : \|x - x_0\|_1 = \sum_{i=1}^{N} |x_i - x_{0i}| \leq D \right\} \subset \Omega.$$

First, we state the following assumptions on f to be used in order to obtain our existence theorem.

(F_6) There exists $k \in L^{\frac{r(x)}{s(x)-1}}(\Omega)$ and $c_6 > 0$ such that

$$|f(x,t)| \leq k(x) + c_6 |t|^{r(x)-1},$$

for a.e. $x \in \Omega$ and all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

(F_7) $\lim_{|t| \to 0} \frac{F(x,t)}{|t|^{p_m^-}} = 0$, uniformly with respect to $x \in \Omega$.

(F_8) $\limsup_{|t| \to +\infty} r(x) \frac{F(x,t)}{|t|^{r(x)}} < \lambda_1$, uniformly with respect to $x \in \Omega$.

(F_9) There exist $e_0 > 0$ and $x_0 \in \Omega$, such that

$$F(x, e_0) + e_0 h(x) \geq 0, \text{ for a.e } x \in \mathcal{B}_1(x_0, D), \tag{2.3.2}$$

and there exist $0 < \alpha_0 < 1$ such that

$$F(x, e_0) + e_0 h(x) > 0, \quad \text{for a.e } x \in \mathcal{B}_1(x_0, \alpha_0 D), \tag{2.3.3}$$

(F_{10}) There exist $\gamma_0 \in \mathbb{R}$ such that

$$G(x, y) := F(x, y) + y h(x) \geq \gamma_0, \qquad \forall (x, y) \in \mathcal{B}_1(x_0, D) \times]-e_0, e_0[. \tag{2.3.4}$$

Remark 2.3.1. Note that, under the hypotheses (F_7) and (F_8), the following hypothesis (F_{10}) is satisfied.

(F_{11}) For all $\epsilon > 0$, there exist an ϵ- uniformly integrable $M_\epsilon \in L^1(\Omega)$ such that for all $t \in \mathbb{R}$ for a.e. $x \in \Omega$,

$$F(x, t) \leq \epsilon |t|^{p_m^-} + \frac{(\lambda_1 + \epsilon)}{r(x)} |t|^{r(x)} + M_\epsilon(x), \, \forall t \in \mathbb{R}.$$

Indeed, let $\epsilon > 0$. By using (F_7), there exists $\eta_\epsilon > 0$ such that

$$|F(x, t)| \leq \epsilon |t|^{p_m^-}, \quad \text{for all } |t| \leq \eta_\epsilon. \tag{2.3.5}$$

Moreover, using (F_8), there exists an ϵ-uniformly integrable function $L_\epsilon \in L^1(\Omega)$ and $\nu_\epsilon > 0$ such that

$$|F(x, t)| \leq \frac{(\lambda_1 + \epsilon)}{r(x)} |t|^{r(x)} + L_\epsilon(x), \quad \forall |t| \geq \nu_\epsilon. \tag{2.3.6}$$

Now, by using condition (F_6) as well as (2.3.5) and (2.3.6), there exists $k_0 > 0$ depending on r , k and c_6 such that

$$F(x, t) \leq \epsilon |t|^{p_m^-} + \frac{(\lambda_1 + \epsilon)}{r(x)} |t|^{r(x)} + M_\epsilon(x) + k_0, \, \forall t \in \mathbb{R}.$$

Then, take $L_\epsilon(x) = M_\epsilon(x) + k_0$. Now, we will state the main result of this section. **The following theorem is the subject of a third publication with my supervisor Professor El Hachimi, which has been submitted for publication.**

Theorem 2.3.2. *(see Ouannasser and El Hachimi [75, Theorem 4.3.]) Assume that hypotheses* $(A_0)-(A_3)$, (P_1), (P_2), $(F_6)-(F_9)$ *are verified and suppose that* $h \in \left(L^{p_M^+}(\Omega)\right)'$ *with* $1 < r^- \leq r^+ < p_m^- \leq p_M^+ < q_m^-$ *and* $N < p_M^+$. *In addition, suppose that* $|\lambda| < \sigma_0$ *where* $\sigma_0 := \min_{1\leq i\leq N} \sigma_i$. *Then, there exist an open interval* $\Lambda \subset]-\sigma_0, \sigma_0[$, *two positive constants* ρ *and* δ *such that for any* $\lambda \in \Lambda$ *and any* $\mu \in [0, \delta]$, *problem* (2.3.1) *has at least three weak solutions in* $W_0^{1,\vec{p}(x)}(\Omega)$, *whose norms are less than* ρ.

Proof. In order to apply Ricceri's result, we define the functionals $\mathcal{H}, \mathcal{M}, \mathcal{K} : W_0^{1,\vec{p}(x)}(\Omega) \to \mathbb{R}$ by

$$\begin{cases} \mathcal{H}(u) = \displaystyle\int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right) dx, \\ \mathcal{M}(u) = \displaystyle\int_\Omega \sum_{i=1}^N \frac{1}{q_i(x)} |\partial_{x_i} u|^{q_i(x)} dx, \\ \mathcal{K}(u) = -\displaystyle\int_\Omega F(x,u)dx - \int_\Omega u(x)h(x)dx. \end{cases}$$

One can see that $\mathcal{H}, \mathcal{K}, \mathcal{M} \in C^1(W_0^{1,\vec{p}(x)}(\Omega), \mathbb{R})$ just by using similar arguments as in Kone et al. [53, Lemma 3.4.] (see also Boureanu [18, Lemma 1.]), with their respective derivatives given by

$$\begin{cases} \langle \mathcal{H}'(u), v\rangle = \displaystyle\int_\Omega \sum_{i=1}^N a_i\left(x, \partial_{x_i} u\right) \partial_{x_i} v dx, \\ \langle \mathcal{M}'(u), v\rangle = \displaystyle\sum_{i=1}^N \int_\Omega |\partial_{x_i} u|^{q_i(x)-2} \partial_{x_i} u \partial_{x_i} v dx, \\ \langle \mathcal{K}'(u), v\rangle = -\displaystyle\int_\Omega f(x,u) v dx - \int_\Omega h v dx, \end{cases}$$

for any $u, v \in W_0^{1,\vec{p}(x)}(\Omega)$. Hence, if there exists a critical point u of the operator $\mathcal{H} + \mu\mathcal{M} + \lambda\mathcal{K}$, we conclude that $u \in W_0^{1,\vec{p}(x)}(\Omega)$ is a weak solution of equation (2.3.1). Then, we can apply Theorem 1.3.2 to look for the weak solutions of problem (2.3.1).

First, by standard arguments (Minty's argument or Lions method), the functional $\mathcal{M}'$ is compact. Denote $\Phi(u) = \mathcal{H}(u) + \lambda\mathcal{K}(u)$. According to the assumption $|\lambda| < \sigma_0$, there exists $0 < \epsilon_1 < \sigma_0$ such that $|\lambda| < \sigma_0 - \epsilon_1$. Take $\epsilon_0 = \dfrac{\lambda_1 \epsilon_1}{\sigma_0 - \epsilon_1}$. Then, we get $|\lambda| < \dfrac{\sigma_0 \lambda_1}{\lambda_1 + \epsilon_0}$. Now, for $u \in W_0^{1,\vec{p}(x)}(\Omega)$ with $\|u\| > 1$, and using (F_8) we obtain

$$\begin{aligned}
\Phi(u) &= \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right) dx - \lambda\left(\int_\Omega F(x,u)dx + \int_\Omega h(x)u(x)dx\right) \\
&\geq \sigma_0 \sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)} \left|\partial_{x_i} u\right|^{p_i(x)} dx - (\lambda_1+\epsilon_0)|\lambda| \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx - |\lambda|\|u\|\|h\|_* \\
&\quad - |\lambda|\|M_{\epsilon_0}\|_1 \\
&\geq \left(\sigma_0 - \frac{(\lambda_1+\epsilon_0)|\lambda|}{\lambda_1}\right) \sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)} \left|\partial_{x_i} u\right|^{p_i(x)} dx - |\lambda|\|u\|\|h\|_* - |\lambda|\|M_{\epsilon_0}\|_1 \\
&\geq \frac{1}{p_M^+}\left(\sigma_0 - \frac{(\lambda_1+\epsilon_0)|\lambda|}{\lambda_1}\right)\left(\frac{\|u\|^{p_m^-}}{N^{p_m^- -1}} - N\right) - |\lambda|\|u\|\|h\|_* - |\lambda|\|M_{\epsilon_0}\|_1 \\
&\geq \frac{1}{p_M^+ N^{p_m^- -1}}\left(\sigma_0 - \frac{(\lambda_1+\epsilon_0)|\lambda|}{\lambda_1}\right)\|u\|^{p_m^-} - |\lambda|\|u_n\|\|h\|_* - |\lambda|\|M_{\epsilon_0}\|_1.
\end{aligned}$$

As $p_m^- > 1$ and $|\lambda| < \dfrac{\sigma_0\lambda_1}{(\lambda_1+\epsilon_0)}$, we then deduce that $\Phi(u) \to \infty$ as $\|u\| \to \infty$, which means that Φ is coercive for any $\lambda \in]-\sigma_0, \sigma_0[$. Therefore, (i) of Theorem 1.3.2 is verified.

Now, in order to establish (ii) of Theorem 1.3.2, by Bonanno and Candito [17, Proposition 1.3], it suffices to show that there exists $w \in X$ and $r > 0$ such that

(B_1) $\mathcal{H}(w) > \rho$,

(B_2) $\sup\limits_{\mathcal{H}(u)<r} \mathcal{K}(u) < \rho \dfrac{\mathcal{K}(w)}{\mathcal{H}(w)}$.

Let us prove (B_1). Consider $x_0 \in \Omega$, $D > 0$ and e_0 given by hypothesis (F_9). For α such that $0 < \alpha < 1$, define u_α such that

$$u_\alpha(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus \mathcal{B}_1(x_0, D), \\ \dfrac{e_0}{D(1-\alpha)}(D - \|x - x_0\|_1) & \text{if } x \in \mathcal{B}_1(x_0, D) \setminus \mathcal{B}_1(x_0, \alpha D), \\ e_0 & \text{if } x \in \mathcal{B}_1(x_0, \alpha D). \end{cases}$$

Then, straightforward calculations give that $u_\alpha \in X$. Moreover, for α less and close to 1, we have

$$\begin{aligned}
\sum_{i=1}^N \int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i(x)} dx &= \sum_{i=1}^N \int_\Omega \left(\frac{e_0}{D(1-\alpha)}\right)^{p_i(x)} dx \\
&\geq N\left(\frac{|e_0|}{D(1-\alpha)}\right)^{p_m^-} meas(T_\alpha),
\end{aligned}$$

where $T_\alpha = \mathcal{B}_1(x_0, D)\backslash\mathcal{B}_1(x_0, \alpha D)$. Therefore, we get

$$\mathcal{H}(u_\alpha) \geq \frac{\sigma_0}{p_m^-}\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u_\alpha}{\partial x_i}\right|^{p_i(x)} dx > \frac{N\sigma_0}{2p_M^+}\left(\frac{e_0}{D(1-\alpha)}\right)^{p_m^-} meas(T_\alpha).$$

Denote

$$r_\alpha = \frac{N\sigma_0}{2p_M^+}\left(\frac{e_0}{D(1-\alpha)}\right)^{p_m^-} meas(T_\alpha).$$

It is well known that $meas(\mathcal{B}_1(x_0, D)) = D^N \frac{2^N}{N!}$. Therefore, we have

$$r_\alpha = \frac{N\sigma_0}{2p_M^+}\left(\frac{e_0}{D(1-\alpha)}\right)^{p_m^-} \frac{(2D)^N}{N!}(1-\alpha^N).$$

As $1 < p_m^-$, we deduce that

$$\lim_{\alpha\to 1^-} r_\alpha = +\infty.$$

In order to complete the proof of (B_1), it suffices hence to choose $w = u_\alpha$ and $\rho = r_\alpha$.

Next, we prove (B_2). Let $u \in X$ such that $\mathcal{H}(u) < r_\alpha$. We have

$$\begin{aligned}\mathcal{H}(u) &\geq \frac{\sigma_0}{p_M^+}\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \\ &\geq \frac{\sigma_0}{p_M^+}\left(\frac{\|u\|^{p_m^-}}{N^{p_m^- -1}} - N\right).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|u\| &\leq \left[\left(\frac{p_M^+}{\sigma_0} r_\alpha + N\right) N^{p_m^- -1}\right]^{\frac{1}{p_m^-}} \\ &\leq \Lambda r_\alpha^{\frac{1}{p_m^-}},\end{aligned}$$

for some $\Lambda > 0$. In order to obtain (B_2), we shall prove the following lemma.

Lemma 2.3.3. *Suppose that hypothesis (F_9) is satisfied and that*

$$F(x, e_0) + e_0 h(x) \geq 0, \ \textit{for a.e } x \in \mathcal{B}(x_0, D), \tag{2.3.7}$$

and there exist $0 < \alpha_0 < 1$ such that

$$F(x, e_0) + e_0 h(x) > 0, \ \textit{for a.e } x \in \mathcal{B}(x_0, \alpha_0 D), \tag{2.3.8}$$

and $\gamma_0 \in \mathbb{R}$ *such that*

$$G(x,y) := F(x,y) + yh(x) \geq \gamma_0, \quad \forall (x,y) \in \mathcal{B}(x_0, D) \times B_{\mathbb{R}}(o, e_0). \tag{2.3.9}$$

Then, we have

$$\lim_{\alpha \to 1^-} \frac{\sup \{\mathcal{K}(u)/\mathcal{H}(u) < r_\alpha\}}{r_\alpha} = 0. \tag{2.3.10}$$

Let us first point out that $\mathcal{K}(u_\alpha)$ is positive for α close to 1. Straightforward calculations give

$$\mathcal{K}(u_\alpha) = I_\alpha + J_\alpha,$$

where

$$I_\alpha = \int_{\mathcal{B}(x_0, \alpha D)} (F(x, e_0) + e_0 h(x))\, dx,$$

and

$$J_\alpha = \int_{T_\alpha} \left(F(x, \frac{e_0}{(1-\alpha)D}\, (D - \|x - x_0\|_1) + \frac{e_0}{(1-\alpha)D} h(x)(D - \|x - x_0\|_1) \right) dx,$$

where $T_\alpha = \mathcal{B}(x_0, D) \setminus \mathcal{B}(x_0, \alpha D)$. Choosing $0 < \alpha < 1$ very close to 1, and using hypothesis (2.3.7) and (2.3.8) we have $I_\alpha > I_{\alpha_0} > 0$, for any $\alpha_0 < \alpha < 1$. On the other hand, thanks to (2.3.9), J_α can be made sufficiently small so that we can have $\mathcal{K}(u_\alpha) > 0$ (Note that $\lim\limits_{\alpha \to 1^-} meas(T_\alpha) = 0$). On the other hand, it is clear that $\mathcal{H}(u_\alpha)$ is positive. Now, if (2.3.10) is satisfied, then for $0 < \epsilon < \dfrac{\mathcal{K}(u_{\alpha_0})}{\mathcal{H}(u_{\alpha_0})}$, there exists $0 < \eta < 1$ such that, for any α satisfying $0 < 1 - \eta < \alpha < 1$, we have

$$\sup \{\mathcal{K}(u)/\mathcal{H}(u) < r_\alpha\} < \epsilon r_\alpha < r_\alpha \frac{\mathcal{K}(u_{\alpha_0})}{\mathcal{H}(u_{\alpha_0})},$$

which gives (B_2). Choosing $0 < \alpha < 1$ very close to 1, Then, by hypothesis (F_9), we have $I_\alpha > 0$; and also, thanks to (F_{10}), J_α can be made sufficiently small so that we can have $\mathcal{K}(u_\alpha) > 0$. On the other hand, it is clear that $\mathcal{H}(u_\alpha)$ is positive. Now, if (2.3.10) is satisfied, then for $0 < \epsilon < \dfrac{\mathcal{K}(u_\alpha)}{\mathcal{H}(u_\alpha)}$, there exists $0 < \eta < 1$ such that, for any α satisfying $0 < 1 - \eta < \alpha < 1$, we have

$$\sup \{\mathcal{K}(u)/\mathcal{H}(u) < r_\alpha\} < \epsilon r_\alpha < r_\alpha \frac{\mathcal{K}(u_\alpha)}{\mathcal{H}(u_\alpha)},$$

which gives (B_2).

Proof of Lemma 2.3.3. Let $\epsilon > 0$. By hypothesis (F_7) and (F_8) , (F_{10}) is satisfied and then, we obtain

$$F(x,t) \leq \epsilon |t|^{p_m^-} + \frac{(\lambda_1 + \epsilon)}{r(x)} |t|^{r(x)} dx + M_\epsilon(x), \forall t \in \mathbb{R}.$$

Therefore, for $u \in X$ and a positive constant c_8, we have

$$\mathcal{K}(u) \leq \epsilon \|u\|^{p_m^-} + c_8 \left(\|u\|^{r^-} + \|u\|^{r^+} \right) + \|M_\epsilon\|_{L^1(\Omega)} + \|h\|_* \|u\|. \tag{2.3.11}$$

Moreover, for $u \in X$ such that $\mathcal{H}(u) \leq r_\alpha$, we have $\|u\| < \Lambda r_\alpha^{\frac{1}{p_m^-}}$. Then, thanks to (2.3.11), there exist positive constants K_1, K_2 and K_3 such that

$$\mathcal{K}(u) \leq \epsilon \Lambda r_\alpha + K_1 \left(\Lambda^{r^-} r_\alpha^{\frac{r^+}{p_m^-}} + \Lambda^{r^-} r_\alpha^{\frac{r^-}{p_m^-}} \right) + K_2 + K_3 \Lambda r_\alpha^{\frac{1}{p_m^-}}.$$

Consequently, we obtain

$$\frac{\mathcal{K}(u)}{r_\alpha} \leq \epsilon \Lambda + K_1 \left(\Lambda^{r^-} r_\alpha^{\frac{r^-}{p_m^-}-1} + \Lambda^{r^+} r_\alpha^{\frac{r^+}{p_m^-}-1} \right) + \frac{K_2}{r_\alpha} + \Lambda K_3 r_\alpha^{\frac{1}{p_m^-}-1}. \tag{2.3.12}$$

As $\dfrac{1}{p_m^-} < 1$, $\dfrac{r^-}{p_m^-} < 1$ and $\dfrac{r^+}{p_m^-} < 1$, then by letting α tend to 1 and ϵ to 0, we can make the second term in (2.3.12) as small as we want. Hence, the proof of the lemma is complete.

Now, we shall end the proof of theorem. Take $0 < \alpha_0 < 1$ such that $\mathcal{K}(u_{\alpha_0}) > 0$ and choose ϵ such that $0 < \epsilon < \epsilon_0 := \dfrac{1}{2} \dfrac{\mathcal{K}(u_{\alpha_0})}{\mathcal{H}(u_{\alpha_0})}$. Using Lemma 2.3.3, there exists η_0 such that, for any $1 - \eta_0 < \alpha < 1$, we have

$$\frac{\sup \{\mathcal{K}(u) / \mathcal{H}(u) < r_\alpha\}}{r_\alpha} < \epsilon_0. \tag{2.3.13}$$

Choose $\alpha_0 > 1 - \eta_0$ and take

$$\xi_0 := \frac{\sup \{\mathcal{K}(u) / \mathcal{H}(u) < r_{\alpha_0}\}}{r_{\alpha_0}},$$

$\delta = \dfrac{b}{2\epsilon_0 - \xi_0}$ with $b > 1$. By Bonanno and Candito [17, Proposition 1.3], for a suitable $\beta > 0$, we have

$$\sup_{\lambda \in \mathbb{R}} \inf_{u \in X} (\mathcal{H}(u) + \lambda(\beta - \mathcal{K}(u))) = \inf_{u \in X} \sup_{\lambda \in [0,\delta]} (\mathcal{H}(u) + \lambda(\beta - \mathcal{K}(u))).$$

Then, using Ricceri [78, Theorem 1], there exist a non empty set $U \subset \,]-\sigma_0, \sigma_0[$ and $\rho > 0$ such that for any $\lambda \in U$, there exists $\nu > 0$, such that for any $\mu \in [0, \nu]$, the equation

$$\mathcal{H}'(u) + \lambda \mathcal{K}'(u) + \mu \mathcal{M}'(u) = 0,$$

has at least three solutions in X whose norms are less than ρ. This complete the proof of Theorem 2.3.2. □

2.4 Concluding examples

Example 2.4.1. Let the function f be defined by

$$f\left(x,s,\xi\right)=-a(x)s-\frac{2}{\pi}\left(\arctan s+\frac{s}{1+s^2}\right)\left(1+d(x)\frac{\left|\xi\right|^2}{1+\left|\xi\right|^2}\right),$$

where a and d are positive functions in $L^\infty(\Omega)$, subject to the condition that $\|a\|_\infty<\lambda_1$. We have

$$\begin{aligned}|f(x,s,\xi)|&\leq\|a\|_\infty|s|+\frac{2}{\pi}\left(\frac{\pi}{2}+\frac{|s|}{1+s^2}\right)(1+\|d(x)\|_\infty)\\&\leq\lambda_1|s|+\left(1+\frac{2}{\pi}\right)(1+\|d(x)\|_\infty)\,.\end{aligned}$$

Then, hypothesis (F_1) is satisfied for $r(x)=2$, for a.e. $x\in\Omega$. Moreover, we have

$$\begin{aligned}f(x,s,\xi)s&=-a(x)s^2-\frac{2}{\pi}\left(\arctan(s)s+\frac{s^2}{1+s^2}\right)\left(1+d(x)\frac{|\xi|^2}{1+|\xi|^2}\right)\\&\leq\|a\|_\infty s^2+\frac{2}{\pi}\left(\frac{\pi}{2}s+1\right)(1+\|d(x)\|_\infty)\\&\leq\lambda_1 s^2+\left(s+\frac{2}{\pi}\right)(1+\|d\|_\infty)\\&\leq c_1s^2+\frac{2}{\pi}(1+\|d\|_\infty)\,.\end{aligned}$$

Therefore, hypothesis (F_2) is satisfied. Furthermore, hypotheses (F_3) and (F_4) can easily be verified. Consequently, f fulfills all the assumptions of Theorem 2.2.6.

Example 2.4.2. Let $D>0$ and consider

$$\begin{gathered}\Omega=\mathcal{B}_1(x_0,D)\text{ and }h\in\left(L^{p_M^+}(\Omega)\right)'\text{ such that }h(x)\geq 0,\text{ for all }x\in\Omega,\\ f(u)=\sum_{i=1}^{N}\beta_i|u|^{\beta_i-1},\text{ with }p_m^-<\beta_i<q_M^+,\text{ for all }i\in\{1,\ldots,N\}.\end{gathered}$$

One can see that hypothesis (F_6) is easily verified. In what follows, set $F(u)=\int_0^u f(v)dv$ and take $e_0>0$ such that $F(e_0)+e_0h(x_0)>0$. Then, hypothesis (F_9) is satisfied. Moreover, we have

$$\lim_{|u|\to 0}\frac{F(u)}{|u|^{p_m^-}}=0\quad\text{ and }\quad\lim_{|u|\to+\infty}\frac{F(u)}{|u|^{q_M^+}}=0.$$

Therefore, hypotheses (F_7) and (F_8) are also satisfied with $r(x)=q_M^+$, for all $x\in\Omega$. So, Theorem 2.3.2 can be applied.

Chapter 3

Existence and multiplicity results for a class of anisotropic Robin elliptic problems

The aim of this chapter is to study a class of anisotropic Robin elliptic equations with variable exponents where the nonlinearity may depend on the gradient of the solution. First, we demonstrate the existence of at least one weak solution using the surjectivity result of pseudomonotone operators. Moreover, under additional conditions on the data, we show that the solution is unique. Furthermore, we prove the existence of at least three weak solutions using the direct Ricceri variational principle when the nonlinearity does not depend on the gradient.

3.1 Introduction

Through recent years, Neumann and Robin elliptic problems have sparked immense interest for their ability to unravel the complexities of physical phenomena that vary in intensity across different areas within a domain. These intriguing problems, characterized by variable exponents, often emerge from the nonlinear dynamics inherent in physical processes or the diverse nature of the domain itself. Understanding these nuances is crucial, as it allows us to create more accurate models of real-world scenarios, including materials with spatially varying properties, porous media flow, and non-Newtonian fluid flows. Henceforth, investigating Robin problems, especially those that may involve convection terms, is pivotal as they play starring roles in fields such as fluid dynamics, combustion, heat transfer, and the intricate world of environmental modeling.

In this chapter, we deal with a class of anisotropic elliptic equations with variable exponents, convection terms, and Robin nonlinear boundary conditions.

In simpler terms, we investigate the following problem

$$\begin{cases} \mathcal{A}(u) + b(x)|u|^{p_M(x)-2}u = \lambda f(x,u,\nabla u) + h(x) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\nu_i(x) = \mu g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{3.1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open bounded domain with smooth boundary, $\mathcal{A}(u) := -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)$ is an anisotropic elliptic operator defined by the functions a_i subjected to the conditions (A_i) below and associated to the continuous variable exponents p_i with $\inf_{x\in\bar{\Omega}} p_i(x) > 1$, and where $p_M(\cdot) := \max_{1\le i\le N} p_i(\cdot)$, while $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and $h \in L^{p'_M(x)}(\Omega)$ with $p'_M(x)$ such that $\dfrac{1}{p_M(x)} + \dfrac{1}{p'_M(x)} = 1$. Finally, $b(x) \in L^\infty(\Omega)$ with $ess \inf_{x\in\Omega} b(x) = b_0 > 0$, $\lambda > 0$ and $\mu \ge 0$ are real parameters, and ν_i constitute the elements comprising the outer normal unit vector for all $i \in \{1, \dots, N\}$.

This chapter addresses a problem featuring the forementioned class of operators, associated with the operaor

$$\mathcal{A}(u) := -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u), \tag{3.1.2}$$

where, for all $i \in \{1, \dots, N\}$, $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ are applications of Carathéodory verifying

(A_1) $A_i(x, 0) = 0$, for all $x \in \Omega$ and $t \in \mathbb{R}$, where A_i is such that

$$a_i(x,t) = \frac{\partial A_i}{\partial t}(x,t),$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

(A_2) $a_i(x, \cdot)$ is strictly monotone in $\mathbb{R}$, that is

$$\left(a_i(x,s) - a_i(x,t)\right)(s-t) > 0,$$

for all $(s,t) \in \mathbb{R}^2$ with $s \neq t$ and $x \in \Omega$.

(A_3) There exist positive constants σ_i and ρ_i with $\sigma_i \le \rho_i$ such that

$$\sigma_i |t|^{p_i(x)} \le a_i(x,t)t \quad \text{and} \quad |a_i(x,t)| \le \rho_i \left(1 + |t|^{p_i(x)-1}\right),$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

Numerous studies have addressed questions regarding the dependence of the right-hand side on the gradient of the solution in the context of anisotropic elliptic equations. For instance, Benboubker et al. [13] (see the references cited in this article) mentioned this category of problems in the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$. Our focus here is twofold: First, to tackle gradient-dependent nonlinearity in variable exponent Sobolev space under anisotropic conditions in the context of parametric elliptic equations. This class of problems presents many challenges, including the limitation of variational methods owing to the dependence of the right-hand side on the gradient. The complexity is further aggravated by the nonhomogeneous nature of the $p(\cdot)$-Laplacian and $\vec{p}(\cdot)$-Laplacian operators, which is distinct from the homogeneous p-Laplacian operator.

Second, our focus extends to establishing the existence of multiple solutions for nonlinearities that do not depend on the gradient. Notably, previous papers employing variational methods have addressed the existence of solutions concerning quasi-linear anisotropic elliptic equations with constant or variable exponents, specifically, when the right-hand side term remains independent of the gradient. Furthermore, our problem adds another layer of difficulty when dealing with nonlinear Robin conditions, as well as an additional term $h(x)$.

Now, we situate our work in relation to some publications already published on this subject. Colasuonno et al. [27] studied the following isotropic Robin boundary type problem

$$\begin{cases} -div(\vec{a}(x,\nabla u)) = \lambda\left(b(x)|u|^{p(x)-2}u + f(x,u)\right) & \text{in } \Omega, \\ \vec{a}(x,\nabla u)\cdot\nu = -a(x)|u|^{p(x)-2}u + \lambda\mu g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{3.1.3}$$

where $p(x) = p$ and the operator $-div(\vec{a}(x,\nabla u))$ is driven by the p-Laplacian. In the case where $b(x) \equiv 0$, they showed the existence of either two nontrivial solutions or only trivial solutions under specific growth conditions verified by functions f and g.

In parallel, in the case where $\lambda = \mu = 1$ and $h(x) \equiv 0$, the nonparametric isotropic problem

$$\begin{cases} \mathcal{A}(u) + b(x)|u|^{p_M(x)-2}u = f(x,u) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,\partial_{x_i}u)\nu_i(x) = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{3.1.4}$$

was treated by Boureanu and Rădulescu [21]. The authors proved the existence of at least a nonnegative solution using the standard minimization result for a weakly lower semicontinuous and coercive functional.

On the other hand, the authors in [19, 28] dealt with problem (3.1.3) in the case where $a(x) \equiv 0$ and obtained the existence of infinitely many solutions. Interesting multiplicity results of solutions for some problems related to problem (3.1.1) have been obtained by several authors, especially Ahmed and Elemine Vall [5], Aydin and Unal [12], Khademloo et al. [51], Kim and Park [52], and Marano and Motreanu [59]. Furthermore, Ourraoui and Ragusa [76] studied a slightly different problem than (3.1.1) when $\mu = 0$ and obtained results regarding the existence of solutions without requiring Ambrosetti-Rabinowitz-type conditions. Ellahyani and El Hachimi [41] also studied a similar problem to (3.1.1) with a Robin boundary condition and established existence and multiplicity results. All these authors established the existence of infinitely many solutions using different variational methods.

Finally, let us point out that the paper by Kim and Park [52] seems to be closer to our present work. These authors obtained the existence of at least three solutions for problem (3.1.3) in the isotropic case and functions f that satisfy the following hypotheses

(KP_1) $|f(x,t)| \leq k_1(x) + \sigma_1(x)|t|^{r_1(x)-1}$, with $r_1^+ < p^-$, for all $(x,t) \in \Omega \times \mathbb{R}$,

(KP_2) $\limsup_{s\to 0} \left(ess \sup_{x\in\Omega} \frac{\left|\int_0^s f(x,t)dt\right|}{|s|^{q(x)}} \right) < +\infty$, with $q \in C_+(\bar{\Omega})$, $p^+ < q_- \leq q < p^*$, where $C_+(\bar{\Omega})$ and the parameters p^-, p^+ and p^* are defined in the next section.

Note that our hypotheses (H_1), (H_2) and (H_3) or (H_3') in Section 3.4 differ from these assumptions. For example, the function given in Example 3.4.2 in the isotropic case, does not verify the conditions (KP_1) and (KP_2) for $\beta(x) < p^+$, for a.e. $x \in \Omega$. However, it satisfies our hypotheses (H) and the existence of three solutions is assured.

The remainder of this chapter is organized as follows. In Section 3.2, using the surjectivity result for pseudomonotone operators, we demonstrate the existence of a solution concerning problem (3.1.1). Moreover, under suitable hypotheses, we show that the solution is unique. In Section 3.3, we demonstrate the existence of a minimum of three weak solutions by applying a recent Ricceri variational principle in the case where there is no dependence of nonlinearity f on the gradient. Finally, in Section 3.4, we give a few concluding examples that verify our theorems.

3.2 Main results

Over the course of this chapter, we suppose that

(A_0) $1 < \sum_{i=1}^{N} \frac{1}{p_i^-} < N+1.$

Definition 3.2.1. An element $u \in W^{1,\vec{p}(x)}(\Omega)$ is termed a weak solution of problem (3.1.1) if

$$\begin{aligned}\int_\Omega \sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u\right) \partial_{x_i} v dx + \int_\Omega b(x)|u|^{p_M(x)-2} uv dx = \lambda \int_\Omega f(x,u,\nabla u) v dx \\ + \mu \int_{\partial\Omega} g(x,u) v d\sigma + \langle h, v\rangle,\end{aligned} \tag{3.2.1}$$

for all $v \in W^{1,\vec{p}(x)}(\Omega)$, where $\langle h, v\rangle$ is the duality pairing between $W^{1,\vec{p}(x)}(\Omega)$ and its dual space.

Definition 3.2.2. Let X be a reflexive Banach space, X^* its dual space and denote by $\langle \cdot, \cdot \rangle$ the duality pairing. Consider an application $\mathcal{J} : X \to X^*$. Then, $\mathcal{J}$ is called

(a) to verify the $\left(S^+\right)$-property if

$$u_n \rightharpoonup u \text{ in } X \text{ and } \limsup_{n\to\infty} \langle \mathcal{J} u_n, u_n - u\rangle \leq 0 \text{ imply } u_n \to u \text{ in } X.$$

(b) pseudomonotone if

$$u_n \rightharpoonup u \text{ in } X \text{ and } \limsup_{n\to\infty} \langle \mathcal{J} u_n, u_n - u\rangle \leq 0 \text{ imply } \mathcal{J} u_n \rightharpoonup \mathcal{J} u \text{ and } \langle \mathcal{J} u_n, u_n\rangle \to \langle \mathcal{J} u, u\rangle.$$

(c) coercive if

$$\lim_{\|u\|_X\to\infty} \frac{\langle \mathcal{J} u, u\rangle}{\|u\|_X} = \infty.$$

Now, in order to state the main existence result for problem (3.1.1), we assume the following assumptions on the functions f and g.

(F_1) There exists $r_1, r_2 \in C_+(\bar{\Omega})$ with $1 < r_1(x) < \max\left\{(\bar{p})^*, p_M(x)\right\}$ for all $x \in \Omega$, and $1 < r_2(x) < \min_{x\in\partial\Omega} \left\{p_1^\partial(x), \ldots, p_N^\partial(x)\right\}$ for all $x \in \partial\Omega$, $k_1 \in L^{r_1'(x)}(\Omega)$, $k_2 \in L^{r_2'(x)}(\partial\Omega)$ and $c_1, c_2, c_1' > 0$ such that

$$|f(x,t,\xi)| \leq k_1(x) + c_1|t|^{r_1(x)-1} + c_2 \sum_{i=1}^{N} |\xi_i|^{p_i(x)\frac{r_1(x)-1}{r_1(x)}}, \text{ for a.e. } x \in \Omega \text{ and all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N,$$

and

$$|g(x,t)| \leq k_2(x) + c_1'|t|^{r_2(x)-1}, \text{ for a.e. } x \in \partial\Omega \text{ and all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N.$$

(F_2) There exists $\theta_1 \in L^1(\Omega)$, $\theta_2 \in L^1(\partial\Omega)$ and $c_3, c_4, c_2' \geq 0$ such that

$$f(x,t,\xi)t \leq \theta_1(x) + c_3|t|^{r_1(x)} + c_4 \sum_{i=1}^{N} |\xi_i|^{p_i(x)},$$
$$g(x,t)t \leq \theta_2(x) + c_2'|t|^{r_2(x)},$$

for a.e. $x \in \Omega$ and all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

Proposition 3.2.3. *We define $X = W^{1,\vec{p}(x)}(\Omega)$ and*

$$\langle \mathcal{J}(u), v \rangle = \int_\Omega \Big(\sum_{i=1}^{N} a_i\left(x, \partial_{x_i} u\right) \partial_{x_i} v dx + b(x)|u|^{p_M(x)-2} uv \Big) dx,$$

for all $u, v \in X$. Then

i) $\mathcal{J} : X \to X^\star$ is bounded and coercive.

ii) $\mathcal{J}$ is of type (S^+).

Proof. *i)* The boundedeness of $\mathcal{J}$ is a consequence of hypothesis (A_3) and the coerciveness is obtained in the proof of Theorem 3.2.4 below.

ii) The proof is similar to that of Boureanu [18, Lemma 2] and is omitted here. □

The following theorem is the subject of a second publication with my supervisor Professor Abderrahmane El Hachimi accepted in Filomat.

Theorem 3.2.4. *(see Ouannasser and El Hachimi [74, Theorem 3.3.]) Assume that hypotheses $(A_0) - (A_3)$, (F_1) and (F_2) are satisfied. Furthermore, suppose that $1 < r_i^- \leq r_i^+ < p_m^-$, for all $i \in \{1,2\}$. Then, for any $\lambda \in \left]-\frac{\sigma_0}{c_4}, \frac{\sigma_0}{c_4}\right[$ and any $h \in \left(W^{1,\vec{p}(x)}(\Omega)\right)^\star$, problem (3.1.1) admits a weak solution in $W^{1,\vec{p}(x)}(\Omega)$.*

Proof. We define the Nemytskii operators $\bar{N}_f : X \subseteq L^{r_1(x)}(\Omega) \to L^{r_1'(x)}(\Omega)$ and $\bar{N}_g : L^{r_2(x)}(\partial\Omega) \to L^{r_2'(x)}(\partial\Omega)$ by $\left(\bar{N}_f u\right)(x) = f(x, u(x), \nabla u(x))$ and $\left(\bar{N}_g u\right)(x) = g(x, u(x))$ respectively. Furthermore, denote $i^* : L^{r_1'(x)}(\Omega) \to X^*$ the adjoint operator for the embedding $i : X \to L^{r_1(x)}(\Omega)$ and $j^* : L^{r_2'(x)}(\partial\Omega) \to X^*$ the adjoint operator for the embedding $j : X \to L^{r_2(x)}(\partial\Omega)$. Subsequently, define $N_f = i^* \circ \bar{N}_f : X \to X^*$ and $N_g = j^* \circ \bar{N}_g \circ j : X \to X^*$, which are well-defined, bounded and continuous operators by assumption (F_1). Now, define the operator $\mathbb{A} : X \to X^*$ as follows

$$\mathbb{A}(u) = \mathcal{J}(u) - \lambda N_f(u) - \mu N_g(u) - h.$$

Our aim is to apply Lemma 1.3.1. Hence, we need to show that $\mathbb{A}$ is bounded, pseudomonotone and coercive.

• $\mathbb{A}$ ***is bounded:***

Thanks to growth conditions on functions f and g stated in (F_1) and the boundedness of $\mathcal{J}$, we obtain the boundedness of $\mathbb{A}$.

• $\mathbb{A}$ ***is pseudomonotone:***

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

$$\begin{cases} u_n \rightharpoonup u \text{ in } X, \\ \limsup\limits_{n\to+\infty} \langle \mathbb{A}(u_n), u_n - u\rangle \leq 0. \end{cases}$$

Thanks to the compact embeddings $X \hookrightarrow\hookrightarrow L^{r_1(x)}(\Omega)$ for $1 < r_1(x) < \max\{\bar{p}^*(x), p_M(x)\}$ for all $x \in \bar{\Omega}$, and $X \hookrightarrow\hookrightarrow L^{r_2(x)}(\partial\Omega)$ for $1 \leq r_2(x) < \min\limits_{x\in\partial\Omega}\{p_1^\partial(x), \dots, p_N^\partial(x)\}$ for all $x \in \partial\Omega$, we obtain

$$u_n \to u \text{ in } L^{r_1(x)}(\Omega) \text{ and } u_n \to u \text{ in } L^{r_2(x)}(\partial\Omega) \tag{3.2.2}$$

Thus, using (F_1) alongside Hölder's inequality and the boundedness of $(u_n)_{n\in\mathbb{N}}$, we obtain

$$\begin{aligned}
\left|\int_\Omega f(x, u_n, \nabla u_n)\cdot(u_n - u)dx\right| &\leq \int_\Omega k_1(x)|u_n - u|dx + c_1\int_\Omega |u_n|^{r_1(x)-1}|u_n - u|dx \\
&+ c_2\int_\Omega \sum_{i=1}^N |\partial_{x_i}u_n|^{p_i(x)\frac{r_1(x)-1}{r_1(x)}}|u_n - u|dx \\
&\leq 2\|k_1\|_{L^{r_1'(x)}(\Omega)}\|u_n - u\|_{L^{r_1(x)}(\Omega)} \\
&+ 2c_1\left\||u_n|^{r_1(x)-1}\right\|_{L^{r_1'(x)}(\Omega)}\|u_n - u\|_{L^{r_1(x)}(\Omega)} \\
&+ 2c_2\sum_{i=1}^N \left\||\partial_{x_i}u_n|^{p_i(x)\frac{r_1(x)-1}{r_1(x)}}\right\|_{L^{r_1'(x)}(\Omega)}\|u_n - u\|_{L^{r_1(x)}(\Omega)},
\end{aligned}$$

and

$$\left|\int_{\partial\Omega} g(x,u_n)\cdot(u_n-u)dx\right| \leq \int_{\partial\Omega} k_2(x)|u_n-u|dx + c_1'\int_{\partial\Omega}|u_n|^{r_2(x)-1}|u_n-u|dx$$
$$\leq 2\|k_2\|_{L^{r_2'(x)}(\partial\Omega)}\|u_n-u\|_{L^{r_2(x)}(\partial\Omega)}$$
$$+ 2c_1'\left\||u_n|^{r_2(x)-1}\right\|_{L^{r_2'(x)}(\partial\Omega)}\|u_n-u\|_{L^{r_2(x)}(\partial\Omega)}.$$

Therefore, we obtain

$$\left|\int_{\Omega} f(x,u_n,\nabla u_n)\cdot(u_n-u)dx\right| \leq \Bigg[2\|k_1\|_{L^{r_1'(x)}(\Omega)}\|u_n-u\|_{L^{r_1(x)}(\Omega)}$$
$$+ 2c_1\left(\left\||u_n|^{r_1^+-1}\right\|_{L^{r_1(x)}(\Omega)} + \left\||u_n|^{r_1^--1}\right\|_{L^{r_1(x)}(\Omega)}\right)$$
$$\cdot\|u_n-u\|_{L^{r_1(x)}(\Omega)} + 2c_2\sum_{i=1}^{N}\left(\left\||\partial_{x_i}u_n|^{p_i^+(\frac{r_1^+-1}{r_1^+})}\right\|_{L^{p_i(x)}(\Omega)}\right.$$
$$\left.+\left\||\partial_{x_i}u_n|^{p_i^-(\frac{r_1^--1}{r_1^-})}\right\|_{L^{p_i(x)}(\Omega)}\right)\|u_n-u\|_{L^{r_1(x)}(\Omega)}\Bigg],$$

and

$$\left|\int_{\partial\Omega} g(x,u_n)\cdot(u_n-u)dx\right| \leq 2\|k\|_{L^{r_2'(x)}(\partial\Omega)}\|u_n-u\|_{L^{r_2(x)}(\partial\Omega)}$$
$$+ 2c_1\left(\left\||u_n|^{r_2^+-1}\right\|_{L^{r_2(x)}(\partial\Omega)} + \left\||u_n|^{r_2^--1}\right\|_{L^{r_2(x)}(\partial\Omega)}\right)$$
$$\cdot\|u_n-u\|_{L^{r_2'(x)}(\partial\Omega)}.$$

This, in combination with (3.2.2), leads to the conclusion that

$$\lim_{n\to+\infty}\int_{\Omega} f(x,u_n,\nabla u_n)\cdot(u_n-u)dx = 0,$$

and

$$\lim_{n\to+\infty}\int_{\partial\Omega} g(x,u_n)\cdot(u_n-u)dx = 0.$$

Taking the limit in (3.2.1), and substituting u with u_n and v with u_n-u, yields

$$\limsup_{n\to+\infty}\langle\mathbb{A}(u),u_n-u\rangle = \limsup_{n\to+\infty}\langle\mathcal{J}(u),u_n-u\rangle \leq 0.$$

Therefore, $u_n \to u$ follows from $\mathcal{J}$ being of type (S^+). Moreover, considering the continuity of $\mathbb{A}$, we deduce $\mathbb{A}(u_n) \to \mathbb{A}(u)$ in X^*, establishing the pseudomonotonicity of $\mathbb{A}$.

• $\mathbb{A}$ ***is coercive:***

We need to show that

$$\lim_{\|u\| \to +\infty} \frac{\langle \mathbb{A}(u), u \rangle}{\|u\|} = +\infty. \tag{3.2.3}$$

Initially, let us give the following notations

$$\mathcal{L}_1 = \left\{ i \in \{1, \dots, N\} : \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)} \leq 1 \right\},$$
$$\mathcal{L}_2 = \left\{ i \in \{1, \dots, N\} : \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)} > 1 \right\}.$$

Then, we get

$$\begin{aligned}
\sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx &= \sum_{i \in \mathcal{L}_1} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx + \sum_{i \in \mathcal{L}_2} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx \\
&\geq \sum_{i \in \mathcal{L}_1} \|\partial_{x_i} u_n\|^{p_M^+}_{L^{p_i(x)}(\Omega)} + \sum_{i \in \mathcal{L}_2} \|\partial_{x_i} u_n\|^{p_m^-}_{L^{p_i(x)}(\Omega)} \\
&\geq \sum_{i=1}^N \|\partial_{x_i} u_n\|^{p_m^-}_{L^{p_i(x)}(\Omega)} - \sum_{i \in \mathcal{L}_1} \|\partial_{x_i} u_n\|^{p_m^-}_{L^{p_i(x)}(\Omega)} \\
&\geq \sum_{i=1}^N \|\partial_{x_i} u_n\|^{p_m^-}_{L^{p_i(x)}(\Omega)} - N \\
&\geq \frac{1}{N^{p_m^- - 1}} \left(\sum_{i=1}^N \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)} \right)^{p_m^-} - N.
\end{aligned}$$

Case 1: If $\|u\|_{L^{p_M(x)}(\Omega)} \geq 1$, we have

$$\begin{aligned}
\langle \mathcal{J}(u), u \rangle &\geq \min\{\sigma_0, b_0\} \left[\frac{1}{N^{p_m^- - 1}} \left(\sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)} \right)^{p_m^-} - N + \|u\|^{p_m^-}_{L^{p_M(x)}(\Omega)} \right] \\
&\geq \frac{\min\{\sigma_0, b_0\}}{(2N)^{p_m^- - 1}} \left[\sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)} + \|u\|_{L^{p_M(x)}(\Omega)} \right]^{p_m^-} - N \min\{\sigma_0, b_0\} \\
&\geq \frac{\min\{\sigma_0, b_0\}}{(2N)^{p_m^- - 1}} \|u\|^{p_m^-} - N \min\{\sigma_0, b_0\}.
\end{aligned}$$

Case 2: If $\|u\|_{L^{p_M(x)}(\Omega)} < 1$, we have

$$
\begin{aligned}
\langle \mathcal{J}(u), u\rangle &\geq \min\{\sigma_0, b_0\}\left[\frac{1}{N^{p_m^- - 1}}\left(\sum_{i=1}^{N}\|\partial_{x_i}u\|_{L^{p_i(x)}(\Omega)}\right)^{p_m^-} + \|u\|^{p_m^-}_{L^{p_M(x)}(\Omega)} - 1 - N\right]\\
&\geq \min\{\sigma_0, b_0\}\left[\frac{1}{N^{p_m^- - 1}}\left(\sum_{i=1}^{N}\|\partial_{x_i}u\|_{L^{p_i(x)}(\Omega)}\right)^{p_m^-} + \|u\|^{p_m^-}_{L^{p_M(x)}(\Omega)}\right]\\
&- (N+1)\min\{\sigma_0, b_0\}\\
&\geq \frac{\min\{\sigma_0, b_0\}}{(2N)^{p_m^- - 1}}\|u\|^{p_m^-} - (N+1)\min\{\sigma_0, b_0\}.
\end{aligned}
$$

Therefore, in both cases we obtain

$$
\langle \mathcal{J}(u), u\rangle \geq d_0\|u\|^{p_m^-} - d_1, \tag{3.2.4}
$$

where

$$
d_0 = \frac{\min\{\sigma_0, b_0\}}{(2N)^{p_m^- - 1}},
$$

and

$$
d_1 = \begin{cases} N\min\{\sigma_0, b_0\} & \text{if } \|u\|_{L^{p_M(x)}(\Omega)} \geq 1,\\ (N+1)\min\{\sigma_0, b_0\} & \text{if } \|u\|_{L^{p_M(x)}(\Omega)} < 1. \end{cases}
$$

Now, denote the dual norm in X^* by $\|\cdot\|_*$. For any $u \in X$, using (F_2) and Proposition 1.1.1, we obtain

$$
\begin{aligned}
\langle \mathbb{A}(u), u\rangle &\geq \sigma_0\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u_n|^{p_i(x)}\,dx + b_0\int_\Omega |u|^{p_M(x)}udx - \lambda\int_\Omega f(x,u,\nabla u)udx\\
&- \mu\int_{\partial\Omega} g(x,u)ud\sigma - \langle h, u\rangle\\
&\geq \sigma_0\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u_n|^{p_i(x)}\,dx + b_0\int_\Omega |u|^{p_M(x)}udx - |\lambda|\Bigg(\int_\Omega |\theta_1(x)|dx\\
&+ c_3\int_\Omega |u|^{r_1(x)} + c_4\int_\Omega\sum_{i=1}^{N}|\partial_{x_i}u|^{p_i(x)}\,dx\Bigg) - |\mu|\Bigg(\int_{\partial\Omega}|\theta_2(x)|d\sigma\\
&+ c_2'\int_{\partial\Omega}|u|^{r_2(x)}d\sigma\Bigg) - \|u\|\|h\|_\star,
\end{aligned}
$$

which gives us

$$\langle \mathbb{A}(u), u\rangle \geq \min\{\sigma_0 - |\lambda|c_4, b_0\}\left(\int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)}\, dx + \int_\Omega |u|^{p_M(x)} dx\right)$$
$$- |\lambda|c_3 \max\left\{\|u\|^{r_1^+}_{L^{r_1(x)}(\Omega)}, \|u\|^{r_1^-}_{L^{r_1(x)}(\Omega)}\right\}$$
$$- |\mu|c_2' \max\left\{\|u\|^{r_2^+}_{L^{r_2(x)}(\partial\Omega)}, \|u\|^{r_2^-}_{L^{r_2(x)}(\partial\Omega)}\right\} - |\lambda|\|\theta_1\|_{L^1(\Omega)}$$
$$- |\mu|\|\theta_2\|_{L^1(\partial\Omega)} - \|u\|\|h\|_\star$$

Now using either Case 1 or Case 2, it follows

$$\langle \mathbb{A}(u), u\rangle \geq \min\{\sigma_0 - \lambda c_4, b_0\}\frac{1}{(2N)^{p_m^- - 1}}\|u\|^{p_m^-} - \lambda c_3 \max\left\{\frac{\|u\|^{r_1^+}}{S^{r_1^+}_{r_1(x),\Omega}}, \frac{\|u\|^{r_1^-}}{S^{r_1^-}_{r_1(x),\Omega}}\right\}$$
$$- \mu c_2' \max\left\{\frac{\|u\|^{r_2^+}}{S^{r_2^+}_{r_2(x),\partial\Omega}}, \frac{\|u\|^{r_2^-}}{S^{r_2^-}_{r_2(x),\partial\Omega}}\right\} - \lambda\|\theta_1\|_{L^1(\Omega)} - \mu\|\theta_2\|_{L^1(\partial\Omega)} - \|u\|\|h\|_\star.$$

Since $1 < r_i^- \leq r_i^+ < p_m^-$, for all $i \in \{1, 2\}$, then (3.2.3) is satisfied. Hence, $\mathbb{A}$ is coercive.

Thus, all the assumptions of Lemma 1.3.1 are satisfied. Therefore, there exists $u \in X$ such that $\mathbb{A}(u) = h$. This completes the proof. □

In what follows, we consider the uniqueness of the solution for problem (3.1.1) under the following additional hypotheses.

- (F_3) For all $\xi \in \mathbb{R}^N$ and for a.e $x \in \Omega$, we have $s \to f(x, s, \xi)$ and $s \to g(x, s)$ are decreasing.
- (F_4) For $s \in \mathbb{R}$, a.e $x \in \Omega$ and all $\xi \in \mathbb{R}^N$, we have $f(x, s, \xi) = f(x, s, |\xi|)$ and $|\xi| \to f(x, s, |\xi|)$ is decreasing.
- (F_5) $\lambda > 0$ and $p_M(x) \geq 2$, for a.e $x \in \Omega$.

The following theorem is the subject of a second publication with my supervisor Professor Abderrahmane El Hachimi accepted in Filomat.

Theorem 3.2.5. *(see Ouannasser and El Hachimi [74, Theorem 3.4.]) Let's consider that the assumptions of Theorem 3.2.4, in addition to hypotheses $(F_3)-(F_5)$, are upheld. Then, problem (3.1.1) admits a unique solution.*

Proof. Consider u_1 and u_2 as two weak solutions to problem (3.1.1). By using the weak formulation of u_1 and u_2, and selecting $\phi = (u_1 - u_2)_+$ as a test function, we derive

$$\int_\Omega \sum_{i=1}^N \left(a_i\left(x,\partial_{x_i}u_1\right)-a_i\left(x,\partial_{x_i}u_2\right)\right)\left(\partial_{x_i}u_1-\partial_{x_i}u_2\right)_+dx$$
$$+\int_\Omega b(x)\left(|u_1|^{p_M(x)-2}u_1-|u_2|^{p_M(x)-2}u_2\right)(u_1-u_2)_+\,dx$$
$$=\lambda\int_\Omega (f(x,u_1,\nabla u_1)-f(x,u_2,\nabla u_2))\,(u_1-u_2)_+\,dx$$
$$+\mu\int_{\partial\Omega}(g(x,u_1)-g(x,u_2))\,(u_1-u_2)_+\,dx.$$

By hypothesis (F_5), we obtain

$$0\le b_0\int_\Omega \left|(u_1-u_2)_+\right|^{p_M(x)}dx$$
$$\le \int_\Omega \sum_{i=1}^N \left(a_i\left(x,\partial_{x_i}u_1\right)-a_i\left(x,\partial_{x_i}u_2\right)\right)\left(\partial_{x_i}u_1-\partial_{x_i}u_2\right)_+dx$$
$$+\int_\Omega b(x)\left(|u_1|^{p_M(x)-2}u_1-|u_2|^{p_M(x)-2}u_2\right)(u_1-u_2)_+\,dx$$
$$\le\lambda\int_\Omega (f(x,u_1,|\nabla u_1|)-f(x,u_2,|\nabla u_2|)\,(u_1-u_2)_+\,dx$$
$$+\mu\int_{\partial\Omega}(g(x,u_1)-g(x,u_2))\,(u_1-u_2)_+\,dx.$$

On the other hand,

$$\lambda\int_\Omega (f(x,u_1,|\nabla u_1|)-f(x,u_2,|\nabla u_2|)\,(u_1-u_2)_+\,dx+\mu\int_{\partial\Omega}(g(x,u_1)-g(x,u_2))\,(u_1-u_2)_+\,dx=\mathcal{I}_1+\mathcal{I}_2,$$

where

$$\mathcal{I}_1=\lambda\int_\Omega\left(f\left(x,u_1,|\nabla u_1|\right)-f\left(x,u_2,|\nabla u_1|\right)\right)\cdot(u_1-u_2)_+\,dx$$
$$+\mu\int_{\partial\Omega}(g(x,u_1)-g(x,u_2))\,(u_1-u_2)_+\,dx,$$
$$\mathcal{I}_2=\lambda\int_\Omega\left(f\left(x,u_2,|\nabla u_1|\right)-f\left(x,u_2,|\nabla u_2|\right)\right)\cdot(u_1-u_2)_+\,dx.$$

According to hypothesis (F_1), we have $\mathcal{I}_1\le 0$. On the other hand, we get

$$\mathcal{I}_2=$$
$$\lambda\int_{\Omega_2}\left(f\left(x,u_2,|\nabla u_1|\right)-f\left(x,u_2,|\nabla u_2|\right)\right)\cdot(|\nabla u_1|-|\nabla u_2|)_+\frac{(u_1-u_2)_+}{(|\nabla u_1|-|\nabla u_2|)_+}dx,$$

where $\Omega_2 = \left\{x, (|\nabla u_1| - |\nabla u_2|)_+ (x) \neq 0\right\}$. Thus, according to hypothesis (F_2), we have $\mathcal{I}_2 \leq 0$. Therefore, we obtain

$$\lambda \int_\Omega (f(x, u_1, \nabla u_1) - f(x, u_2, \nabla u_2)) \cdot (u_1 - u_2)_+ \, dx \leq 0,$$

which implies that

$$b_0 \int_\Omega \left|(u_1 - u_2)_+\right|^{p_M(x)} dx = 0.$$

From this, we find $u_1(x) = u_2(x)$ for a.e x on $\mathcal{U} = \{x \in \Omega : u_1(x) > u_2(x)\}$; while for $x \in \Omega \backslash \mathcal{U}$, we have $(u_1 - u_2)_+ (x) = 0$. Consequently, we have $u_1 \leq u_2$ on Ω. Similarly, we find $u_1 \geq u_2$, in Ω. Hence, $u_1 = u_2$ and the solution is unique. $\square$

3.3 Three weak solutions

In this section, we investigate and obtain the existence of three weak solutions for the following problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{p_M(x)-2}u = \lambda f(x, u) + h(x) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\nu_i(x) = \mu g(x, u) & \text{on } \partial\Omega, \end{cases} \tag{3.3.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open bounded domain with smooth boundary, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function, $h \in L^{(p_M^+)'}(\Omega)$ with $\dfrac{1}{p_M^+} + \dfrac{1}{(p_M^+)'} = 1$, $1 < r_i^- \leq r_i^+ < p_m^- \leq p_M^+$, $b \in L^\infty(\Omega)$ verifying $ess \inf\limits_{x\in\Omega} b(x) = b_0 > 0$, $\lambda > 0$ and $\mu \geq 0$ are real parameters, and ν_i constitute the elements comprising the outer normal unit vector for all $i \in \{1, \ldots, N\}$.

Definition 3.3.1. An element $u \in W^{1,\vec{p}(x)}(\Omega)$ is termed a weak solution of problem (3.3.1) if

$$\begin{aligned} \int_\Omega \sum_{i=1}^{N} a_i \left(x, \partial_{x_i} u\right) \partial_{x_i} v dx + \int_\Omega b(x)|u|^{p_M(x)-2} uv dx = \lambda \int_\Omega f(x, u) v dx \\ + \mu \int_{\partial\Omega} g(x, u) v d\sigma + \langle h, v\rangle, \end{aligned} \tag{3.3.2}$$

for all $v \in W^{1,\vec{p}(x)}(\Omega)$, where $\langle h, v\rangle$ is the duality pairing between $W^{1,\vec{p}(x)}(\Omega)$ and its dual space.

Define

$$F(x,u) := \int_0^u f(x,s)ds, \text{ for all } (x,u) \in \Omega \times \mathbb{R}, \text{ and } G(x,u) := \int_0^u g(x,s)ds,$$

for all $(x,u) \in \partial\Omega \times \mathbb{R}$.

First, we make the following assumptions regarding f and g to obtain the desired result.

(H_1) There exists $q, r_1, r_2 \in C_+(\bar{\Omega})$ with $1 < q(x) < p_m^-$, $1 < q, r_1(x) < \max\{(\bar{p})^*, p_M(x)\}$, for all $x \in \Omega$, $1 < r_2(x) < \min_{x\in\partial\Omega}\{p_1^\partial(x), \dots, p_N^\partial(x)\}$ for all $x \in \partial\Omega$, $k_1 \in L^{r_1'(x)}(\Omega)$, $k_2 \in L^{r_2'(x)}(\partial\Omega)$ and $c_6, c_7 > 0$ such that

$$|f(x,t)| \le k_1(x) + c_6|t|^{r_1(x)-1}, \text{ for all } (x,u) \in \Omega \times \mathbb{R},$$
$$|g(x,t)| \le k_2(x) + c_7|t|^{r_2(x)-1}, \text{ for all } (x,u) \in \partial\Omega \times \mathbb{R}.$$

(H_2) $\limsup_{|t|\to 0} \dfrac{F(x,t)}{|t|^{p_m^-}} = 0$, uniformly with respect to $x \in \Omega$, and

(H_3) $\limsup_{|t|\to+\infty} q(x)\dfrac{F(x,t)}{|t|^{q(x)}} = A(x)$, uniformly with respect to $x \in \Omega$ with A such that $A_0 := \|A\|_\infty > 0$, or

(H_3') $\limsup_{|t|\to+\infty} p_M(x)\dfrac{F(x,t)}{|t|^{p_M(x)}} = 0$, uniformly with respect to $x \in \Omega$.

Remark 3.3.2. Note that, under hypotheses (H_2) and (H_3) (resp. (H_3')), for all $\epsilon > 0$, there exist $M_\epsilon \in L^1(\Omega)$ such that, for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, we have

$$F(x,t) \le \epsilon|t|^{p_m^-} + \frac{(A_0+\epsilon)}{q(x)}|t|^{q(x)} + M_\epsilon(x), \tag{3.3.3}$$

(resp.

$$F(x,t) \le \epsilon\left(|t|^{p_m^-} + \frac{1}{p_M(x)}|t|^{p_M(x)}\right) + M_\epsilon(x)). \tag{3.3.4}$$

Let $\epsilon > 0$. Using (H_3), there exists $\eta_\epsilon > 0$ such that

$$F(x,t) \le \epsilon|t|^{q(x)}, \text{ for all } |t| \le \eta_\epsilon, \text{ uniformly for a.e } x \in \Omega. \tag{3.3.5}$$

Moreover, using (H_2), there exists an ϵ-uniformly integrable function $L_\epsilon \in L^1(\Omega)$ and $\nu_\epsilon > 0$ such that

$$F(x,t) \le \frac{(A(x)+\epsilon)}{q(x)}|t|^{q(x)} + L_\epsilon(x), \text{ for all } |t| \ge \nu_\epsilon, \tag{3.3.6}$$

uniformly for a.e $x \in \Omega$. Now using hypothesis (H_1), (3.3.5) and (3.3.6), there exists $k_0(\epsilon) > 0$ that depends on ϵ, q, k_1 and c_6 such that

$$F(x,t) \leq \epsilon|t|^{p_m^-} + \frac{(A_0+\epsilon)}{q(x)}|t|^{q(x)} + L_\epsilon(x) + k_0(\epsilon), \text{ for all } t \in \mathbb{R},$$

uniformly for a.e $x \in \Omega$. Then, it suffices to take $M_\epsilon(x) = L_\epsilon(x) + k_0(\epsilon)$ and $A_0 = \|A\|_\infty$. Note that the same remark stands for relation (3.3.4), starting from hypotheses (H_2) and (H_3').

For the last hypotheses (H), we suppose that there exists a real $e_0 > 0$ such that

(H_4) $F(x, e_0) + e_0 h(x) \geq 0$, for a.e $x \in \mathcal{B}(x_0, D)$,

(H_5) There exists $0 < \alpha_0 < 1$ such that $F(x, e_0) + e_0 h(x) > 0$, for a.e $x \in \mathcal{B}(x_0, \alpha_0 D)$,

(H_6) There exists $\gamma_0 \in \mathbb{R}$ such that $G(x,y) := F(x,y) + yh(x) \geq \gamma_0$, $\forall (x,y) \in \mathcal{B}(x_0, D)\times]-e_0, e_0[$, where $x_0 \in \Omega$ and $D > 0$ are such that

$$\mathcal{B}(x_0, D) := \left\{ x \in \mathbb{R}^N : \|x - x_0\|_1 = \sum_{i=1}^{N} |x_i - x_{0i}| \leq D \right\} \subset \Omega.$$

Put $\theta_0 := \dfrac{b_0}{A_0}$. The main result of this section is the following.

The following theorem is the subject of a second publication with my supervisor Professor Abderrahmane El Hachimi accepted in Filomat.

Theorem 3.3.3. *(see Ouannasser and El Hachimi [74, Theorem 4.4.]) Assume that hypotheses* $(A_0)-(A_4)$*,* (H_1)*,* (H_2)*,* (H_3) *(resp.*(H_3')*), and* $(H_4)-(H_6)$ *are verified and suppose that* $h \in \left(L^{p_M^+}(\Omega)\right)'$*. Additionally, suppose that* $|\lambda| < \theta_0$ *(resp.* $\lambda \in]-\infty, +\infty[$*). Then, there exist an open interval* $\Lambda \subset]-\theta_0, \theta_0[$ *(resp.* $\subset]-\infty, +\infty[$*), two positive constants* ρ *and* δ *such that for any* $\lambda \in \Lambda$ *and any* $\mu \in [0, \delta]$*, problem* (3.3.1) *has at least three weak solutions in* $W^{1,\vec{p}(x)}(\Omega)$*, whose norms are less than* ρ*.*

Proof. In order to apply Ricceri's result [78], we define the functionals $\mathcal{H}, \mathcal{K}, \mathcal{M} : W^{1,\vec{p}(x)}(\Omega) \to \mathbb{R}$ by

$$\begin{cases} \mathcal{H}(u) = \displaystyle\int_\Omega \sum_{i=1}^{N} A_i(x, \partial_{x_i} u)\, dx + \int_\Omega \frac{b(x)}{p_M(x)} |u|^{p_M(x)} dx, \\ \mathcal{K}(u) = -\displaystyle\int_\Omega F(x,u) dx - \int_\Omega u(x) h(x) dx, \\ \mathcal{M}(u) = -\displaystyle\int_{\partial\Omega} G(x,u) d\sigma. \end{cases}$$

One can see that $\mathcal{H}, \mathcal{K}, \mathcal{M} \in C^1(W^{1,\vec{p}(x)}(\Omega), \mathbb{R})$ just by drawing on similar reasoning as demonstrated in the proof of [53, Lemma 3.4.], with their respective derivatives given by

$$\begin{cases} \langle \mathcal{H}'(u), v\rangle = \displaystyle\int_\Omega \sum_{i=1}^N a_i\left(x, \partial_{x_i} u\right) \partial_{x_i} v dx + \int_\Omega b_i(x)|u|^{p_M(x)-2} uv dx, \\ \langle \mathcal{K}'(u), v\rangle = -\displaystyle\int_\Omega f(x,u) v dx - \int_\Omega hv dx, \\ \langle \mathcal{M}'(u), v\rangle = -\displaystyle\int_{\partial\Omega} g(x,u) v d\sigma, \end{cases}$$

for any $u, v \in W^{1,\vec{p}(x)}(\Omega)$. Hence, if there exists a critical point u of the operator $\mathcal{H} + \lambda\mathcal{K} + \mu\mathcal{M}$, we conclude that $u \in W^{1,\vec{p}(x)}(\Omega)$ is a weak solution to equation (3.3.1). Then, we can apply Theorem 1.3.2 to look for weak solutions to problem (3.3.1). First, the fact that $\mathcal{M}'$ is compact can be shown easily by adapting to the case of $\partial\Omega$ (instead of Ω) the proof of Colasuonno et al. [27, Lemma 3.2.]. Next, we prove (i) in Proposition 1.3.2.

Case 1: Suppose hypothesis (H_3) is satisfied. We denote $\Phi(u) = \mathcal{H}(u) + \lambda\mathcal{K}(u)$. Let $\epsilon_0 > 0$ be fixed such that $|\lambda| < \dfrac{b_0}{\epsilon_0}$ and $u \in W^{1,\vec{p}(x)}(\Omega)$ with $\|u\| > \max\{1, \eta_{\epsilon_0}\}$ (where η_{ϵ_0} is given in (3.3.5)). By using (3.3.6), it follows that

$$\begin{aligned} \Phi(u) &= \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right) dx + \int_\Omega \frac{b(x)}{p_M(x)} |u|^{p_M(x)} dx - \lambda \int_\Omega F(x,u) dx \\ &- \lambda \int_\Omega u(x) h(x) dx \\ &\geq \sigma_0 \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx + b_0 \int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)} dx - |\lambda| \int_\Omega \left(\frac{(A_0 + \epsilon)}{q(x)} |u(x)|^{q(x)} \right. \\ &+ \left. |M_{\epsilon_0}(x)| \right) dx - |\lambda| \|u\| \|h\|_*. \end{aligned}$$

Now, we have

$$\int_\Omega \frac{1}{q(x)} |u(x)|^{q(x)} dx \leq \frac{1}{q^-} \int_\Omega |u(x)|^{q(x)} dx \leq c\left(\|u\|^{q^-} + \|u\|^{q^+}\right).$$

Then, we get

$$\begin{aligned}
\Phi(u) &\geq \min\{\sigma_0, b_0\}\left(\int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}dx + \int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)}dx\right) \\
&- |\lambda|(A_0+\epsilon_0)\int_\Omega \frac{1}{q(x)}|u(x)|^{q(x)}dx - |\lambda|\|M_{\epsilon_0}\|_1 - \|u\|\|h\|_* \\
&\geq \min\{\sigma_0, b_0\}\left(\int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}dx + \int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)}dx\right) - c_\lambda\left(\|u\|^{q^-} + \|u\|^{q^+}\right) \\
&- |\lambda|\|M_{\epsilon_0}\|_1 - \|u\|\|h\|_* \\
&\geq \frac{d_0}{p_M^+}\|u\|^{p_m^-} - c_\lambda\left(\|u\|^{q^-} + \|u\|^{q^+}\right) - |\lambda|\|M_{\epsilon_0}\|_1 - |\lambda|\|u\|\|h\|_* - \frac{d_1}{p_M^+},
\end{aligned}$$

where $c_\lambda := c|\lambda|(A_0+\epsilon_0)$. For $1 < q^- < q^+ < p_m^-$, we deduce that $\Phi(u) \to \infty$ is $\|u\| \to \infty$, which means that Φ is coercive for any $\lambda \in \mathbb{R}$. Therefore, (i) in Proposition 1.3.2 is verified.

Case 2: Suppose hypothesis (H_3') is satisfied. Let $\epsilon_0 > 0$ be fixed such that $|\lambda| < \dfrac{b_0}{\epsilon_0}$ and let $u \in W^{1,\vec{p}(x)}(\Omega)$ with $\|u\| > \max\{1, \eta_{\epsilon_0}\}$ (where η_{ϵ_0} is mentioned in (2.3.5)). By using (3.3.6), it follows that

$$\begin{aligned}
\Phi(u) &= \int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i} u\right)dx + \int_\Omega \frac{b(x)}{p_M(x)}|u|^{p_M(x)}dx - \lambda\int_\Omega F(x,u)dx \\
&- \lambda\int_\Omega u(x)h(x)dx \\
&\geq \sigma_0\int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}dx + b_0\int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)}dx - |\lambda|\int_\Omega \left(\frac{\epsilon_0}{p_M(x)}|u(x)|^{p_M(x)}\right. \\
&\left. + |M_{\epsilon_0}(x)|\right)dx - |\lambda|\|u\|\|h\|_*.
\end{aligned}$$

Then, by (3.2.4), we get

$$\begin{aligned}
\Phi(u) &\geq \sigma_0\int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}dx + (b_0 - |\lambda|\epsilon_0)\int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)}dx \\
&- |\lambda|\|M_{\epsilon_0}\|_1 - |\lambda|\|u\|\|h\|_* \\
&\geq \min\{\sigma_0, b_0 - |\lambda|\epsilon_0\}\left(\int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}dx + \int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)}dx\right) \\
&- |\lambda|\|M_{\epsilon_0}\|_1 - |\lambda|\|u\|\|h\|_* \\
&\geq \frac{d_0\min\{\sigma_0, b_0 - |\lambda|\epsilon_0\}}{p_M^+}\|u\|^{p_m^-} - |\lambda|\|M_{\epsilon_0}\|_1 - |\lambda|\|u\|\|h\|_* - \frac{d_1}{p_M^+}.
\end{aligned}$$

Because $p_m^- > 1$, we deduce that $\Phi(u) \to \infty$ as $\|u\| \to \infty$, which means that Φ is coercive for any $\lambda \in \Big] - \infty, +\infty\Big[$. Therefore, (i) in Proposition 1.3.2 is verified.

Now, in order to establish (ii) of Proposition 1.3.2, by Bonanno and Candito [17, Proposition 1.3], it suffices to show that there exists $w \in X$ and $r > 0$ such that

(B_1) $\mathcal{H}(w) > \rho$,

(B_2) $\sup\limits_{\mathcal{H}(u)<r} \mathcal{K}(u) < \rho \dfrac{\mathcal{K}(w)}{\mathcal{H}(w)}$.

We now prove (B_1). Consider $x_0 \in \Omega$, e_0 and $D > 0$ as defined in hypothesis (H_6). For α verifying $0 < \alpha < 1$, define u_α such that

$$u_\alpha(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus \mathcal{B}(x_0, D), \\ \dfrac{e_0}{D(1-\alpha)}(D - \|x - x_0\|_1) & \text{if } x \in \mathcal{B}(x_0, D) \setminus \mathcal{B}(x_0, \alpha D), \\ e_0 & \text{if } x \in \mathcal{B}(x_0, \alpha D). \end{cases}$$

Straightforward calculations show that $u_\alpha \in X$. Moreover, for α less and close to 1, we have

$$\begin{aligned} \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_\alpha|^{p_i(x)}\, dx + \int_\Omega |u_\alpha|^{p_M(x)} dx &> \int_{T_\alpha} \sum_{i=1}^N \left(\frac{e_0}{D(1-\alpha)}\right)^{p_i(x)} dx \\ &> N \left(\frac{e_0}{D(1-\alpha)}\right)^{p_m^-} meas(T_\alpha), \end{aligned}$$

where $T_\alpha = \mathcal{B}(x_0, D) \backslash \mathcal{B}(x_0, \alpha D)$. Therefore, we get

$$\begin{aligned} \mathcal{H}(u_\alpha) &\geq \frac{\sigma_0}{p_M^+} \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_\alpha|^{p_i(x)}\, dx + \frac{b_0}{p_M^+} \int_\Omega |u_\alpha|^{p_M(x)} dx \\ &> \frac{\sigma_0 N}{p_M^+} \left(\frac{e_0}{D(1-\alpha)}\right)^{p_m^-} meas(T_\alpha), \end{aligned}$$

Denote

$$r_\alpha = \frac{\sigma_0 N}{p_M^+} \left(\frac{e_0}{D(1-\alpha)}\right)^{p_m^-} meas(T_\alpha),$$

and recall that

$$meas(\mathcal{B}(x_0, \tau D)) = \frac{(2\tau D)^N}{N!},$$

for any $\tau > 0$. Therefore, we have

$$\lim_{\alpha \to 1^-} r_\alpha = +\infty.$$

To complete the proof of (B_1), it suffices to consider $w = u_\alpha$ and $\rho = r_\alpha$, with $0 < \alpha < 1$. Next, we prove (B_2). Let $u \in X$ be such that $\mathcal{H}(u) < r_\alpha$.
Case 1: Suppose that hypothesis (H_3) is satisfied. We have

$$\begin{aligned}
\mathcal{H}(u) &\geq \frac{\sigma_0}{p_M^+} \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)}\, dx + \frac{b_0}{p_M^+} \int_\Omega |u|^{p_M(x)} dx \\
&\geq \frac{\min\{\sigma_0, b_0\}}{p_M^+} \left(\int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)}\, dx + \int_\Omega |u|^{p_M(x)} dx \right) \\
&\geq \frac{1}{p_M^+} \left(d_0 \|u\|^{p_m^-} - d_1 \right).
\end{aligned}$$

Therefore, we obtain

$$\|u\| \leq \left(\frac{p_M^+}{d_0} \left(r_\alpha + \frac{d_1}{p_M^+} \right) \right)^{\frac{1}{p_m^-}}.$$

That is,

$$\|u\| \leq \Lambda r_\alpha^{\frac{1}{p_m^-}}. \tag{3.3.7}$$

for certain $\Lambda > 0$.
Case 2: Suppose that hypothesis (H_3') is satisfied. We have

$$\begin{aligned}
\mathcal{H}(u) &\geq \frac{\sigma_0}{p_M^+} \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)}\, dx + \frac{b_0}{p_M^+} \int_\Omega |u|^{p_M(x)} dx \\
&\geq \frac{b_0}{p_M^+} \int_\Omega |u|^{p_M(x)} dx \\
&= \frac{b_0}{p_M^+} \rho_{p_M(x)}(u).
\end{aligned}$$

It gives

$$\rho_{p_M(x)}(u) \leq \frac{p_M^+}{b_0} r_\alpha. \tag{3.3.8}$$

To obtain (B_2), we prove the following lemma.

Lemma 3.3.4. *Suppose that hypotheses (H_1), (H_2), (H_3) (resp. (H_3')) and $(H_4)-(H_6)$ are satisfied. Then, we have*

$$\lim_{\alpha \to 1^-} \frac{\sup \{\mathcal{K}(u) / \mathcal{H}(u) < r_\alpha\}}{r_\alpha} = 0. \tag{3.3.9}$$

Let us first point out that $\mathcal{K}(u_\alpha)$ is positive for α close to 1. Straightforward calculations give

$$\mathcal{K}(u_\alpha) = I_\alpha + J_\alpha,$$

where

$$I_\alpha = \int_{\mathcal{B}(x_0,\alpha D)} (F(x, e_0) + e_0 h(x))\, dx,$$

and

$$J_\alpha = \int_{T_\alpha} \left(F(x, \frac{e_0}{(1-\alpha)D}(D - \|x - x_0\|_1) + \frac{e_0}{(1-\alpha)D} h(x)(D - \|x - x_0\|_1) \right) dx,$$

where $T_\alpha = \mathcal{B}(x_0, D) \setminus \mathcal{B}(x_0, \alpha D)$. Choosing $0 < \alpha < 1$ very close to 1, and using hypotheses (H_4) and (H_5), we have $I_\alpha > I_{\alpha_0} > 0$, for any $\alpha_0 < \alpha < 1$. Similarly, based on hypothesis (H_6), J_α can be made sufficiently small such that we can obtain $\mathcal{K}(u_\alpha) > 0$ (note that $\lim\limits_{\alpha \to 1^-} meas(T_\alpha) = 0$). On the other hand, it is evident that $\mathcal{H}(u_\alpha)$ is positive. Now, if (3.3.9) is satisfied, then for $0 < \epsilon < \dfrac{\mathcal{K}(u_{\alpha_0})}{\mathcal{H}(u_{\alpha_0})}$, there exists $0 < \eta < 1$ such that, for any α satisfying $0 < 1 - \eta < \alpha < 1$, we have

$$\sup\{\mathcal{K}(u)/\mathcal{H}(u) < r_\alpha\} < \epsilon r_\alpha < r_\alpha \frac{\mathcal{K}(u_{\alpha_0})}{\mathcal{H}(u_{\alpha_0})},$$

which yields (B_2).

Proof of Lemma 3.3.4.

Case 1: Suppose hypothesis (H_3) is satisfied. Let $\epsilon > 0$. By using (3.3.3) we derive that, for some positive constant c_8 that may depend on ϵ, we have

$$\mathcal{K}(u) \leq \epsilon \|u\|^{p_m^-} + c_8 \left(\|u\|^{q^-} + \|u\|^{q^+} \right) + \|M_\epsilon\|_{L^1(\Omega)} + \|h\|_* \|u\|, \qquad (3.3.10)$$

for all $u \in X$. Therefore, for $u \in X$ with $\mathcal{H}(u) < r_\alpha$, thanks to (3.3.7) and (3.3.10), there exist positive constants K_1, K_2 and K_3 (K_2 depending on ϵ) such that

$$\mathcal{K}(u) \leq \epsilon \Lambda^{p_m^-} r_\alpha + K_1 \left(r_\alpha^{\frac{q^-}{p_m^-}} + r_\alpha^{\frac{q^+}{p_m^-}} \right) + K_2 + K_3 r_\alpha^{\frac{1}{p_m^-}}.$$

Consequently, we obtain

$$\frac{\mathcal{K}(u)}{r_\alpha} \leq \epsilon \Lambda^{p_m^-} + K_1 \left(r_\alpha^{\frac{q^-}{p_m^-} - 1} + r_\alpha^{\frac{q^+}{p_m^-} - 1} \right) + \frac{K_2}{r_\alpha} + K_3 r_\alpha^{\frac{1}{p_m^-} - 1}. \qquad (3.3.11)$$

Because $p_m^- > \max\{1, q^-, q^+\}$, by letting α tend to 1 and ϵ tend to 0, we can make the second term in (3.3.11) as small as desired.

Case 2: Suppose that hypothesis (H_3') is satisfied. Let $\epsilon > 0$. By using (3.3.4), we derive

$$\mathcal{K}(u) \leq \epsilon \|u\|^{p_m^-} + \frac{\epsilon}{p_M^-}\rho_{p_M(x)} + \|M_\epsilon\|_{L^1(\Omega)} + \|h\|_* \|u\|, \tag{3.3.12}$$

for all $u \in X$. Then, for $u \in X$ with $\mathcal{H}(u) < r_\alpha$, thanks to (3.3.8) and (3.3.12), there exist positive constants K_1, K_2 and K_3 (K_2 depending on ϵ) such that

$$\mathcal{K}(u) \leq \epsilon \left(\Lambda^{p_m^-} + \frac{p_M^+}{b_0 p_M^-}\right) r_\alpha + K_2 + K_3 r_\alpha^{\frac{1}{p_m^-}}.$$

Consequently, we obtain

$$\frac{\mathcal{K}(u)}{r_\alpha} \leq \epsilon \left(\Lambda^{p_m^-} + \frac{p_M^+}{b_0 p_M^-}\right) + \frac{K_2}{r_\alpha} + K_3 r_\alpha^{\frac{1}{p_m^-}-1}. \tag{3.3.13}$$

Because $p_m^- > 1$, by letting α tend to 1 and ϵ to 0, we can make the second term in (3.3.13) as small as desired. Hence, the proof of Lemma 3.3.4 is complete.

Now, we conclude the proof of Theorem 1.3.2. Let $0 < \alpha_1 < 1$ such that $\mathcal{K}(u_{\alpha_1}) > 0$ and choose ϵ such that $0 < \epsilon < \epsilon_0 := \dfrac{1}{2}\dfrac{\mathcal{K}(u_{\alpha_1})}{\mathcal{H}(u_{\alpha_1})}$. Using Lemma 3.3.4, there exists η_0 such that, for any $1 - \eta_0 < \alpha < 1$, we have

$$\frac{\sup\{\mathcal{K}(u)/\mathcal{H}(u) < r_\alpha\}}{r_\alpha} < \epsilon_0. \tag{3.3.14}$$

By choosing $\alpha_1 > 1 - \eta_0$ and taking

$$\xi_0 := \frac{\sup\{\mathcal{K}(u)/\mathcal{H}(u) < r_{\alpha_1}\}}{r_{\alpha_1}} \quad \text{and} \quad \delta = \frac{b}{2\epsilon_0 - \xi_0},$$

with $b > 1$ and applying the result by Bonanno and Candito [17, Proposition 1.3], we deduce that

$$\sup_{\lambda\in\mathbb{R}} \inf_{u\in X} (\mathcal{H}(u) + \lambda(\beta - \mathcal{K}(u))) = \inf_{u\in X} \sup_{\lambda\in[0,\delta]} (\mathcal{H}(u) + \lambda(\beta - \mathcal{K}(u))),$$

for a suitable $\beta > 0$. Then, by using Ricceri [78, Theorem 1], there exist a non-empty set $U \subset]-\sigma_0, \sigma_0[$ (where $\sigma_0 := \theta_0$ if (H_3) is satisfied and $\sigma_0 := +\infty$ when (H_3') is supposed) and $\rho > 0$ such that for any $\lambda \in U$, there exists $\delta > 0$, such that the equation

$$\mathcal{H}'(u) + \lambda\mathcal{K}'(u) + \mu\mathcal{M}'(u) = 0,$$

has at least three solutions in X whose norms are less than ρ. This completes the proof of Theorem 3.3.3. □

3.4 Concluding examples

Example 3.4.1. Let the functions f and g be defined by

$$f(x,s,\xi) = -a(x)s - \frac{2}{\pi}\left(\arctan s + \frac{s}{1+s^2}\right)\left(1 + d(x)\frac{|\xi|^2}{1+|\xi|^2}\right),$$
$$g(x,s) = -b(x)s - \frac{2}{\pi}\left(\arctan s + \frac{s}{1+s^2}\right),$$

where a, b, and d represent positive functions in $L^\infty(\Omega)$. We have

$$\begin{aligned}|f(x,s,\xi)| &\leq \|a\|_\infty |s| + \frac{2}{\pi}\left(\frac{\pi}{2} + \frac{|s|}{1+s^2}\right)(1 + \|d(x)\|_\infty)\\ &\leq \lambda_1|s| + \left(1 + \frac{2}{\pi}\right)(1 + \|d(x)\|_\infty).\end{aligned}$$

In addition, we have

$$\begin{aligned}|g(x,s)| &\leq \|b\|_\infty |s| + \frac{2}{\pi}\left(\frac{\pi}{2} + \frac{|s|}{1+s^2}\right)\\ &\leq \|b\|_\infty|s| + 1 + \frac{2}{\pi}.\end{aligned}$$

Then, hypothesis (F_1) is satisfied for $r_i(x) = 2$, for $i \in \{1,2\}$ and for a.e. $x \in \Omega$. Moreover, we have

$$\begin{aligned}f(x,s,\xi)s &= -a(x)s^2 - \frac{2}{\pi}\left(\arctan(s)s + \frac{s^2}{1+s^2}\right)\left(1 + d(x)\frac{|\xi|^2}{1+|\xi|^2}\right)\\ &\leq \|a\|_\infty s^2 + \frac{2}{\pi}\left(\frac{\pi}{2}s + 1\right)(1 + \|d(x)\|_\infty)\\ &\leq \lambda_1 s^2 + \left(s + \frac{2}{\pi}\right)(1 + \|d\|_\infty)\\ &\leq c_1 s^2 + \frac{2}{\pi}(1 + \|d\|_\infty).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}g(x,s)s &= -b(x)s^2 - \frac{2}{\pi}\left(\arctan(s)s + \frac{s^2}{1+s^2}\right)\\ &\leq \|b\|_\infty s^2 + \frac{2}{\pi}\left(\frac{\pi}{2}s + 1\right)\\ &\leq \|b\|_\infty s^2 + \left(s + \frac{2}{\pi}\right)\\ &\leq c_2 s^2 + \frac{2}{\pi}.\end{aligned}$$

Therefore, hypothesis (F_2) is satisfied. Furthermore, hypotheses (F_3) and (F_4) can easily be verified. Hence, f and g satisfy assumptions of Theorem 3.2.5.

Example 3.4.2. Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $p_i \in C^+(\bar{\Omega})$ with $p_i(x) \geq 2$ for all $i \in \{1, \ldots, N\}$ and $(r_1, r_2) \in \left(C^+(\bar{\Omega})\right)^2$ with $2 \leq r_1(x) < \max\{(\bar{p})^*, p_M(x)\}$, for all $x \in \Omega$ and $1 < r_2(x) < \min_{x \in \partial\Omega}\left\{p_1^\partial(x), \ldots, p_N^\partial(x)\right\}$. Consider β and q in $C^+(\bar{\Omega})$ such that $1 < q(x) < p_m^- < \beta(x)$ and $1 + q(x) \leq r_1(x) \leq \beta(x)$ for all $x \in \Omega$. Take $h \in L^{p'_M(x)}(\Omega)$ and $A \in L^\infty(\Omega)$ with

$$h(x) := \left(\frac{1}{1+x^2}\right)^{\frac{1}{p'_M(x)}} \text{ and } A(x) := \frac{\beta(x)}{q(x)}\left(\frac{\pi}{2}\right)^{\beta(x)-q(x)},$$

for all $x \in \Omega$. Note that $\|A\|_\infty = \dfrac{\beta^+}{q^-}\left(\dfrac{\pi}{2}\right)^{\beta^+-q^-}$. Now, define

$$f(x,u) := \begin{cases} \beta(x)|u|^{\beta(x)-2}u & \text{if } |u| \leq \frac{\pi}{2}, \\ q(x)A(x)|u|^{q(x)-2}u\sin u + A(x)|u|^{q(x)}\cos u & \text{if } |u| \geq \frac{\pi}{2}, \end{cases}$$

for all $(x,u) \in \Omega \times \mathbb{R}$, and

$$g(x,u) = \frac{1}{1+x^2}|u|^{\sigma(x)}\cos u, \quad \forall (x,u) \in \partial\Omega \times \mathbb{R},$$

with $\sigma \in C^+(\bar{\Omega})$ such that $1 < \sigma(x) < r_2(x) - 1, \quad \forall x \in \partial\Omega$. Straightforward calculations give that

$$|f(x,u)| \leq c_6|u|^{r_1(x)-1}, \forall (x,u) \in \Omega \times \mathbb{R}, \text{ (resp. } |g(x,u)| \leq |u|^{r_2(x)-1}, \forall (x,u) \in \partial\Omega \times \mathbb{R}),$$

with $c_6 := \max\left\{\beta^+, \left(q^+ + \dfrac{2}{\pi}\right)\|A\|_\infty\right\}$ and

$$F(x,u) := \int_0^u f(x,s)ds = \begin{cases} |u|^{\beta(x)} & \text{if } |u| \leq \frac{\pi}{2}, \\ A(x)|u|^{q(x)} & \text{if } |u| \geq \frac{\pi}{2}, \end{cases}$$

for all $(x,u) \in \Omega \times \mathbb{R}$. Moreover, we have

$$F(x,y) + yh(x) > 0, \quad \forall (x,y) \in \Omega \times \mathcal{B}_R(O, e_0),$$

with $0 < e_0 < \dfrac{\pi}{2}$, and

$$\limsup_{|u|\to 0} \frac{F(x,u)}{|u|^{p_m^-}} = 0 \quad \text{and} \quad \limsup_{|u|\to+\infty} \frac{F(x,u)}{|u|^{q(x)}} = A(x),$$

uniformly for a.e $x \in \Omega$. Then, all hypotheses of Theorem 3.3.3 are satisfied and we obtain the existence of at least three solutions for the following problem

$$\begin{cases} -\sum_{i=1}^{N} |\partial_{x_i} u(x)|^{p_i(x)-2} \partial_{x_i} u + |u|^{p_M(x)-2} u = \lambda f(x,u) + h(x) & \text{in } \Omega, \\ \sum_{i=1}^{N} |\partial_{x_i} u(x)|^{p_i(x)-2} \partial_{x_i} u(x) \nu_i(x) = \mu g(x,u) & \text{on } \partial\Omega. \end{cases}$$

Part II

Resonance and non-resonance phenomena

Chapter 4

Solvability of parametric elliptic systems with variable exponents

In this chapter, we study the solvability to the left of the positive infimum of all eigenvalues for some non-resonant quasilinear elliptic problems involving variable exponents. We first prove the existence of at least a weak solution for some non-variational systems by using a surjectivity result for pseudomonotone operators. Furthermore, under additional conditions, we show that the solution is unique and provide examples. Second, we deal with non-resonant gradient-type systems and obtain existence by using a variational approach.

4.1 Introduction

In recent years, the $p(x)$-Laplacian has gained increasing attention in various fields of science and engineering. This operator, defined as $\Delta_{p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$, is a generalization of the standard Laplacian operator and is commonly used in mathematics, physics, and engineering to describe physical phenomena such as heat flow, fluid dynamics, and electrostatics. One of the main advantages of the $p(x)$-Laplacian is its ability to provide a more flexible description of nonlinear diffusion processes, which are often encountered in real-world applications. In addition, the $p(x)$-Laplacian can also model non-Newtonian fluids, which exhibit nonlinear behavior due to the dependence of viscosity on the applied stress.

Furthermore, it's worth noting that there are other applications of the $p(x)$-Laplacian, such as its use in electrorheological fluids (studied by Acerbi and Mingione [1] and Ružička [79]), image restoration (employed by Chen et al. [25]), and thermorheological fluids (explored by Antontsev et al. [10, 11]).

Given a bounded smooth domain $\Omega \subseteq \mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, this chapter is concerned with the existence of a solution $(u_1, u_2) \in W_0^{1,p_1(x)}(\Omega) \times$

$W_0^{1,p_2(x)}(\Omega)$ for the following elliptic system

$$\begin{cases} -\Delta_{p_1(x)}u_1 - \mu_1\Delta_{q_1(x)}u_1 = f_1\left(x, u_1, u_2, \nabla u_1, \nabla u_2\right) + h_1(x) & \text{in } \Omega, \\ -\Delta_{p_2(x)}u_2 - \mu_2\Delta_{q_2(x)}u_2 = f_2\left(x, u_1, u_2, \nabla u_1, \nabla u_2\right) + h_2(x) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{PS}$$

where for $i \in \{1, 2\}$, $1 < q_i(x) \leq p_i(x) < N$ are variable exponents of class $C^1(\bar{\Omega})$, $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, corresponding essentially to some perturbation below the positive infimum of all eigenvalues of a related eigenvalue problem that we defined in (2.1.2) below and satisfying some structure conditions, (see hypotheses (H_4) and (H_5) in Section 4.2), μ_i are positive constants and $h_i \in W^{-1,p_i'(x)}(\Omega)$ with $\frac{1}{p_i'(x)} + \frac{1}{p_i(x)} = 1$, for all $x \in \Omega$.

Many works have been dedicated to existence results for such quasilinear elliptic systems with constant or variable exponents. We quote for example Alves and Moussaoui [8], Boccardo and De Figueiredo [16], El Hachimi and De Thélin [38], Felmer et al. [46], Faria et al. [45], Hsu [48], Motreanu et al. [64], Nabab and Vélin [67] and the references therein. Recall that there are few works dealing with eigenvalue problems for elliptic systems. One can see De Thélin [30], El Khalil et al. [40], Kandilakis et al. [50], Moussaoui and Vélin [65], Zographopoulos [89] and the references therein. We refer especially to the paper of Moussaoui and Vélin [65], where the authors are concerned with the positive infimum $\lambda_c(p)$, with $p = (p_1, p_2)$, of all eigenvalues for a nonlinear gradient-type elliptic system with weight and involving variable exponents such as

$$\begin{cases} -\Delta_{p_1(x)}u = \lambda c(x)\alpha_1(x) \left|u_1\right|^{\alpha_1(x)-2} u_1 \left|u_2\right|^{\alpha_2(x)} & \text{in } \Omega, \\ -\Delta_{p_2(x)}v = \lambda c(x)\alpha_2(x) \left|u_1\right|^{\alpha_1(x)} u_2 \left|u_2\right|^{\alpha_2(x)-2} & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{VP$_c$}$$

Let be associated to (PS) the eigenvalue problem

$$\begin{cases} -\Delta_{p_1(x)}v_1 = \lambda\alpha_1(x) \left|v_1\right|^{\alpha_1(x)-2} v_1 \left|v_2\right|^{\alpha_2(x)} & \text{in } \Omega, \\ -\Delta_{p_2(x)}v_2 = \lambda\alpha_2(x) \left|v_2\right|^{\alpha_2(x)-2} v_2 \left|v_1\right|^{\alpha_1(x)} & \text{in } \Omega, \\ v_1 = v_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{VP$_1$}$$

where for $i \in \{1, 2\}$, α_i verify

$$\alpha_1(x), \alpha_2(x) > 0 \text{ and } \frac{\alpha_1(x)}{p_1(x)} + \frac{\alpha_2(x)}{p_2(x)} = 1, \text{ for all } x \in \Omega.$$

On one hand, we were motivated by the paper of Motreanu et al. [64] where the authors studied quasilinear parametric elliptic systems, and proved existence and uniqueness of solutions to the following elliptic system with constant exponents p_i, q_i

$$\begin{cases} -\Delta_{p_1} u_1 - \mu_1 \Delta_{q_1} u_1 = f_1\left(x, u_1, u_2, \nabla u_1, \nabla u_2\right) & \text{in } \Omega, \\ -\Delta_{p_2} u_2 - \mu_2 \Delta_{q_2} u_2 = f_2\left(x, u_1, u_2, \nabla u_1, \nabla u_2\right) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{CPS}$$

Their results are obtained for functions

$$F^* : (u_1, u_2) \to f_1\left(x, u_1, u_2, \nabla u_1, \nabla u_2\right) u_1 + f_2\left(x, u_1, u_2, \nabla u_1, \nabla u_2\right) u_2,$$

that are, in some sense, below the first eigenvalues of the related p_i-Laplacian operators. Their principal condition reads

$$\begin{aligned} f_1\left(x, s_1, s_2, \xi_1, \xi_2\right) s_1 + f_2\left(x, s_1, s_2, \xi_1, \xi_2\right) s_2 \leq \omega(x) + c\left(|s_1|^{p_1} + |s_2|^{p_2}\right) \\ + d\left(|\xi_1|^{p_1} + |\xi_2|^{p_2}\right), \end{aligned} \tag{4.1.1}$$

for a.e $x \in \Omega$, all $(s_1, s_2) \in \mathbb{R}^2$ and all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{2N}$, with $\omega \in L^1(\Omega)$ and positive constants c and d such that

$$\min\left(\lambda_{1,p_1}, \lambda_{1,p_2}\right) < \frac{1-c}{d},$$

where λ_{1,p_i} denotes the first eigenvalue of the p_i-Laplacian operator for $i \in \{1, 2\}$. This condition ensures the coerciveness of the operator associated with the problem as defined in Motreanu et al. [64]. On the other hand, the motivation of the present work was to extend to the case of a non-gradient-type system with variable exponents. The earlier results by El Hachimi and De Thélin [38] for the existence of solutions to the problem

$$\begin{cases} -\Delta_{p_i} u_i = \dfrac{\partial F}{\partial u_i}(x, u) + h_i(x) & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{GS}$$

is obtained when the function F satisfies a non-resonance condition near the first eigenvalue of an associated eigenvalue system. Unlike the variational case used in Section 4.3, where solutions are obtained as critical points of C^1-functionals, we shall use in Section 4.2, a topological method based on a fixed point theorem for bounded, pseudomonotone and coercive operators.

Our aim is to establish the existence of a solution for (PS) based on the findings of Moussaoui and Vélin [65]. Broadly speaking, if the condition $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \exists M_\epsilon \in L^1(\Omega), \forall s = (s_1, s_2) \in \mathbb{R}^2$ with $|s|_\infty > \delta_\epsilon, \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^{2N}, \forall x \in \Omega,$

$$\begin{aligned} & f_1\left(x, s_1, s_2, \xi_1, \xi_2\right) s_1 + f_2\left(x, s_1, s_2, \xi_1, \xi_2\right) s_2 \\ & \leq \left(\lambda_1(p) + \epsilon\right)\left(|s_1|^{\alpha_1(x)} |s_2|^{\alpha_2(x)}\right) + c\left(|\xi_1|^{q_1(x)} + |\xi_2|^{q_2(x)}\right) + M_\epsilon(x), \end{aligned} \tag{4.1.2}$$

is satisfied, where c is a positive constant and $\lambda_1(p)$ is the positive infimum of all eigenvalues of (VP_1), then we prove the existence of at least one weak solution for problem (PS). Note that (4.1.2) is verified if one has for example

$$f_\infty(x) \equiv \limsup_{|s|_\infty \longrightarrow +\infty} \frac{f_1(x,s_1,s_2,\xi_1,\xi_2)\,s_1 + f_2(x,s_1,s_2,\xi_1,\xi_2)\,s_2}{|s_1|^{\alpha_1(x)}\,|s_2|^{\alpha_2(x)}} < \lambda_1(p), \quad (4.1.3)$$

uniformly with respect to $\xi = (\xi_1, \xi_2)$. Moreover, under additional conditions on the functions f_1 and f_2 and the variable exponents, we show that the solution is unique.

It should be noted that, even in the case of constant exponents and functions f_i that are not depending on the gradients of the unknown functions, little is known about resonance and non-resonance issues, when considering the interaction with the variational spectrum of the (p_1, p_2)-Laplacians eigenvalue system (VP_1) of the function

$$(s_1, s_2) \longrightarrow \frac{f_1(x,s_1,s_2)\,s_1 + f_2(x,s_1,s_2)\,s_2}{|s_1|^{\alpha_1}\,|s_2|^{\alpha_2}}.$$

The same remark is also valid in the case of gradient-type systems and functions f_i not depending on ∇u_i for the function

$$(s_1, s_2) \longrightarrow \frac{F(x,s_1,s_2)}{|s_1|^{\alpha_1}\,|s_2|^{\alpha_2}},$$

where $\dfrac{\partial F}{\partial s_i}(x,s_1,s_2) = f_i(x,s_1,s_2)$ for $i \in \{1,2\}$. Let us recall that the positive infimum $\lambda_1(p)$ of all eigenvalues can be defined by the expression

$$\lambda_1(p) = \inf_{(u_1,u_2)\in\mathcal{V}\setminus\{0\}} \frac{\mathcal{A}_p(u_1,u_2)}{\mathcal{B}_p(u_1,u_2)}, \quad (4.1.4)$$

where

$$\mathcal{A}_p(u_1,u_2) = \int_\Omega \left(\frac{1}{p_1(x)}|\nabla u_1|^{p_1(x)} + \frac{1}{p_2(x)}|\nabla u_2|^{p_2(x)}\right) dx \text{ and } \mathcal{B}_p(u_1,u_2) = \int_\Omega |u_1|^{\alpha_1(x)}\,|u_2|^{\alpha_2(x)}\,dx.$$

It is worth noting that condition (4.1.1) implies (4.1.2) which clearly implies that $\forall\epsilon > 0, \exists\delta_\epsilon > 0, \exists M_\epsilon \in L^1(\Omega), \forall s = (s_1,s_2) \in \mathbb{R}^2, \forall\xi = (\xi_1,\xi_2) \in \mathbb{R}^{2N}, \forall x \in \Omega,$

$$\begin{aligned} &f_1(x,s_1,s_2,\xi_1,\xi_2)\,s_1 + f_2(x,s_1,s_2,\xi_1,\xi_2)\,s_2 \\ &\le (\lambda_1(p)+\epsilon)\left(|s_1|^{\alpha_1(x)}\,|s_2|^{\alpha_2(x)}\right) + c\left(|\xi_1|^{p_1(x)} + |\xi_2|^{p_2(x)}\right) + M_\epsilon(x). \end{aligned} \quad (4.1.5)$$

Now, according to a recent result by Bozorgnia et al. [23, Theorem 3.3] obtained in the case of constant exponents, the following situation may occur

$$\min(\lambda_{1,p_1}, \lambda_{1,p_2}) < \lambda_1(p).$$

Therefore, in the case of constant exponents, condition (4.1.3) is weaker than that in condition (4.1.1).

The remainder of this chapter is organized as follows. In Section 4.2, we present and prove our main existence result for the problem (PS) and provide conditions to obtain the uniqueness of the solution in Section 4.3. We conclude this section by providing examples of applications. In Section 4.4, we treat the case of a gradient-type system and demonstrate an existence result using a variational approach, similar to that employed in El Hachimi and de Thélin [38]. Finally, in Section 4.5, we give a few concluding examples that verify our theorems.

4.2 Main result

4.2.1 General hypotheses

(H_1) Ω is a bounded open of $\mathbb{R}^N$, with boundary $\partial\Omega$ of class $C^{2,\delta}$, for some $0 < \delta < 1$.

(H_2) μ_1 and μ_2 are two positive constants and p_i and q_i, for $i \in \{1, 2\}$, are two functions in $C^1(\bar{\Omega})$ satisfying $p_i(x) < p_i^*(x)$ and $q_i(x) < q_i^*(x)$, respectively, for all $x \in \bar{\Omega}$, with

$$1 < p_i^- = \inf_{x\in\Omega} p_i(x) \le p_i(x) \le p_i^+ = \sup_{x\in\Omega} p_i(x) < \infty,$$
$$1 < q_i^- = \inf_{x\in\Omega} q_i(x) \le q_i(x) \le q_i^+ = \sup_{x\in\Omega} q_i(x) < \infty.$$

(H_3) For $i \in \{1, 2\}$, $\alpha_i : \bar{\Omega} \to]0, +\infty[$ are two continuous functions satisfying

$$0 < \alpha_i^- = \inf_{x\in\Omega} \alpha_i(x) \le \alpha_i(x) \le \alpha_i^+ = \sup_{x\in\Omega} \alpha_i(x) < \infty,$$

with

$$\frac{\alpha_1(x)}{p_1(x)} + \frac{\alpha_2(x)}{p_2(x)} = 1, \qquad \text{for all } x \in \bar{\Omega}.$$

(H_4) For $i \in \{1, 2\}$, $f_i : \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are Carathéodory functions such that there exist three positive continuous functions α_i^*, q_i^* and δ_i, and a nonnegative function $e_i \in L^{\frac{\gamma_i(x)}{\gamma_i(x)-1}}(\Omega)$ with $1 < \gamma_i(x) < p_i^*(x)$, satisfying the following conditions

$$0<\alpha_i^*(x)<p_i^*(x)-1, \quad 0<\delta_i(x)<\frac{q_i(x)}{\left(q_i^*(x)\right)'},$$
$$\frac{\alpha_1^*(x)}{\gamma_1(x)}+\frac{\alpha_2^*(x)}{\gamma_2(x)}=\frac{\gamma_1(x)-1}{\gamma_1(x)},$$

and nonnegative constants a_i, b_i, such that for a.e. $x \in \Omega$ and all $s_1, s_2 \in \mathbb{R}$, we have

$$\begin{aligned}\left|f_1\left(x, s_1, s_2, \xi_1, \xi_2\right)\right| \leq & a_1\left|s_1\right|^{\alpha_1^*(x)}\left|s_2\right|^{\alpha_2^*(x)} \\ & +b_1\left(\left|\xi_1\right|^{\delta_1(x)}+\left|\xi_2\right|^{\frac{\delta_1(x) q_2(x)}{q_1(x)}}\right)+e_1(x),\end{aligned}$$

and

$$\begin{aligned}\left|f_2\left(x, s_1, s_2, \xi_1, \xi_2\right)\right| \leq & a_2\left|s_1\right|^{\alpha_1^*(x)}\left|s_2\right|^{\alpha_2^*(x)} \\ & +b_2\left(\left|\xi_1\right|^{\frac{\delta_2(x) q_1(x)}{q_2(x)}}+\left|\xi_2\right|^{\delta_2(x)}\right)+e_2(x).\end{aligned}$$

(H_5) $\forall \epsilon>0, \exists \delta_\epsilon>0, \exists M_\epsilon \in L^1(\Omega), \forall s=\left(s_1, s_2\right) \in \mathbb{R}^2$ with $|s|_{\infty}>\delta_\epsilon, \forall \xi=\left(\xi_1, \xi_2\right) \in \mathbb{R}^{2N}, \forall x \in \Omega$

$$\begin{aligned}& f_1\left(x, s_1, s_2, \xi_1, \xi_2\right) s_1+f_2\left(x, s_1, s_2, \xi_1, \xi_2\right) s_2 \\ & \leq\left(\lambda_1(p)+\epsilon\right)|s|^{\alpha(x)}+c\left(\left|\xi_1\right|^{q_1(x)}+\left|\xi_2\right|^{q_2(x)}\right)+M_\epsilon(x).\end{aligned}$$

4.2.2 Preliminary results

Denote $\mathcal{V}=W_0^{1, p_1(x)}(\Omega) \times W_0^{1, p_2(x)}(\Omega), \mathcal{V}'=W^{-1, p_1'(x)}(\Omega) \times W^{-1, p_2'(x)}(\Omega)$ and for $u=\left(u_1, u_2\right) \in \mathcal{V}$ and $\varphi=\left(\varphi_1, \varphi_2\right) \in \mathcal{V}'$

$$\langle u, \varphi\rangle=\left\langle u_1, \varphi_1\right\rangle_{W_0^{1, p_1(x)}(\Omega), W^{-1, p_1'(x)}(\Omega)}+\left\langle u_2, \varphi_2\right\rangle_{W_0^{1, p_2(x)}(\Omega), W^{-1, p_2'(x)}(\Omega)}.$$

The norm considered on $\mathcal{V}$ is given by

$$\|u\|_{\mathcal{V}}=\max \left\{\left\|u_1\right\|_{W_0^{1, p_1(x)}(\Omega)},\left\|u_2\right\|_{W_0^{1, p_2(x)}(\Omega)}\right\}, \quad \forall u=\left(u_1, u_2\right) \in \mathcal{V}.$$

For $u=\left(u_1, u_2\right) \in \mathcal{V}$, define the functionals $\mathcal{A}_p$ and $\mathcal{B}_p$ on $\mathcal{V}$ by

$$\begin{cases}\mathcal{A}_p(u)=\displaystyle\int_{\Omega}\left(\frac{1}{p_1(x)}\left|\nabla u_1\right|^{p_1(x)}+\frac{1}{p_2(x)}\left|\nabla u_2\right|^{p_2(x)}\right) d x, \\ \mathcal{B}_p(u)=\displaystyle\int_{\Omega}\left|u_1\right|^{\alpha_1(x)}\left|u_2\right|^{\alpha_2(x)} d x.\end{cases}$$

For fixed $R > 0$, set $\Omega_R = \{u \in \mathcal{V}, \mathcal{B}_p(u) = R\}$. It is clear that Ω_R is not empty, see Moussaoui and Vélin [65]. Define the Rayleigh quotients as follows

$$\begin{cases} \lambda_R(p) = \inf\limits_{u \in \Omega_R} \dfrac{\mathcal{A}_p(u)}{\mathcal{B}_p(u)}, \\ \lambda_1(p) = \inf\limits_{u \in \mathcal{V}\backslash\{0\}} \dfrac{\mathcal{A}_p(u)}{\mathcal{B}_p(u)}. \end{cases}$$

Now, we recall the following result by Moussaoui and Vélin.

Theorem 4.2.1. *(see Moussaoui and Vélin [65, Theorem 2.2]) Assume that* $(H_1) - (H_3)$ *hold. Then, system* (VP_1) *has a one-parameter family of nontrivial solutions* $((\hat{u}_{1,R}, \hat{u}_{2,R}), \lambda_R^*)$ *for all* $R \in]0, +\infty[$*. Moreover, if one of the following conditions holds*

(P_1) *There are two vectors* $l_1, l_2 \in \mathbb{R}^N\backslash\{0\}$ *such that for all* $x \in \Omega, \theta_1(t_1) = p_1(x + t_1 l_1)$ *and* $\theta_2(t_2) = p_2(x + t_2 l_2)$ *are monotone for* $t_i \in I_{i,x} = \{t_i, x + t_i l_i \in \Omega\}$*, for* $i \in \{1, 2\}$*.*

(P_2) *There are* x_1 *and* $x_2 \notin \Omega$ *such that for all* $w_1, w_2 \in \mathbb{R}\backslash\{0\}$ *with* $\|w_1\|, \|w_2\| = 1$*, the functions* $\theta_1(t_1) = p_1(x_0 + t_1 w_1)$ *and* $\theta_2(t_2) = p_2(x_2 + t_2 w_2)$ *are monotone for* $t_i \in I_{x_i,w_i} = \{t_i \in \mathbb{R}, x_i + t_i w_i \in \Omega\}$*, for* $i \in \{1, 2\}$*.*

Subsequently, $\lambda_1(p) = \inf\limits_{R>0} \lambda_R(p) > 0$ *is the positive infimum eigenvalue of problem* (VP_1)*.*

Definition 4.2.2. A couple $(u_1, u_2) \in \mathcal{V}$ is called a weak solution of problem (PS) if

$$\int_\Omega \left(\mathcal{F}_{p_1(x)}(\nabla u_1) + \mu_1 \mathcal{F}_{q_1(x)}(\nabla u_1)\right) \cdot \nabla\varphi_1 + \left(\mathcal{F}_{p_2(x)}(\nabla u_1) + \mu_2 \mathcal{F}_{q_2(x)}(\nabla u_1)\right) \cdot \nabla\varphi_2 dx = \int_\Omega \mathbf{f}(x, u, \nabla u_1, \nabla u_2) \cdot \varphi dx + \langle h, \varphi\rangle,$$

for any couple $\varphi = (\varphi_1, \varphi_2) \in \mathcal{V}$, where $\mathcal{F}_{r(x)}(t) = |t|^{r(x)-2}t$, for $t \in \mathbb{R}^N$ and $\langle h, \varphi\rangle = \langle h_1, \varphi_1\rangle + \langle h_2, \varphi_2\rangle$ is the duality pairing between $\mathcal{V}$ and its dual space, where we denote

$$\mathbf{f}(x, s, \xi_1, \xi_2).s = f_1(x, s_1, s_2, \xi_1, \xi_2) s_1 + f_2(x, s_1, s_2, \xi_1, \xi_2) s_2.$$

Definition 4.2.3. Let X be a reflexive Banach space, X^* its dual space and denote by $\langle\cdot,\cdot\rangle$ its duality pairing. Let $\mathcal{I} : X \to X^*$, then $\mathcal{I}$ is called

(a) to satisfy the (S_+)-property if $u_n \to u$ in X and $\limsup\limits_{n\to\infty} \langle \mathcal{I}u_n, u_n - u\rangle \leq 0$ imply $u_n \to u$ in X,

(b) pseudomonotone if $u_n \to u$ in X and $\limsup_{n\to\infty} \langle \mathcal{I}u_n, u_n - u\rangle \le 0$ imply $\mathcal{I}u_n \to u$ and $\langle \mathcal{I}u_n, u_n\rangle \to \langle \mathcal{I}u, u\rangle$,

(c) coercive if

$$\lim_{\|u\|_X\to\infty} \frac{\langle \mathcal{I}(u), u\rangle}{\|u\|_X} = \infty.$$

Theorem 4.2.4. *(see Motreanu et al. [64]) Let X be a real, reflexive Banach space, $\mathcal{I} : X \to X^*$ be a bounded, pseudomonotone, and coercive operator, and $h \in X^*$. Then, a solution to the equation $\mathcal{I}u = h$ exists.*

Now, for all $u = (u_1, u_2) \in \mathcal{V}$, we define

$$\begin{aligned}\tilde{\mathcal{A}}(u) &= \big(-\Delta_{p_1(x)}u_1 - \mu_1\Delta_{q_1(x)}u_1, -\Delta_{p_2(x)}u_1 - \mu_2\Delta_{q_2(x)}u_2\big)\\ &= \tilde{\mathcal{A}}_p(u) + \tilde{\mathcal{A}}_q(u),\end{aligned} \tag{4.2.1}$$

where

$$\tilde{\mathcal{A}}_p(u) = \big(-\Delta_{p_1(x)}u_1, -\Delta_{p_2(x)}u_2\big) \quad \text{and} \quad \tilde{\mathcal{A}}_q(u) = \big(-\mu_1\Delta_{q_1(x)}u_1, -\mu_2\Delta_{q_2(x)}u_2\big).$$

The following result summarizes the properties of the operator $\tilde{\mathcal{A}}$.

Lemma 4.2.5. *The operator $\tilde{\mathcal{A}} : \mathcal{V} \to \mathcal{V}'$, defined in (4.2.1), is bounded, continuous, monotone (hence maximal monotone), and of type (S_+).*

Proof. The proof is similar to that obtained by Fan and Zhang [43, Theorem 3.1] for a single equation and $\mu_i = 0$ for $i \in \{1, 2\}$, and is omitted here. □

Next, we provide and prove our first main result.
The following theorem is the subject of a first publication with my supervisor Professor El Hachimi published in the Moroccan Journal of Pure and Applied Analysis.

Theorem 4.2.6. *(see Ouannasser and El Hachimi [73, Theorem 3.1.]) Suppose that the hypotheses (P_1) or (P_2) and $(H_1) - (H_5)$ are verified. Moreover, suppose that $c < \frac{1}{\mu_0\theta_q}$, where $\theta_q := \max\left\{\frac{1}{q_1^-}, \frac{1}{q_2^-}\right\} < 1$ and $\mu_0 = \min\{\mu_1, \mu_2\}$. Then, for all $h \in \mathcal{V}'$, problem (PS) admits at least one solution, $u = (u_1, u_2) \in \mathcal{V}$.*

Proof. In order to apply Theorem 4.2.4, we need to show that the operator $\mathcal{I}$ associated with (PS) , which is defined on $\mathcal{V}$ by

$$\mathcal{I}(u) := \tilde{\mathcal{A}}(u) - N(u) - h,$$

where $N(u) = (N_1(u), N_2(u))$, and for $i \in \{1,2\}$, $N_i : \mathcal{V} \to \mathcal{V}'$ denotes the Nemytskii operator associated with f_i, that is, $N_i(u) = f_i(x, u_1, u_2, \nabla u_1, \nabla u_2)$.

• ***$\mathcal{I}$ is bounded:***

From (H_4), it is easy to see that the Nemytskii operators N_1 and N_2 are well-defined, continuous and bounded. Because $\tilde{\mathcal{A}} : \mathcal{V} \to \mathcal{V}'$ is bounded, we conclude that $\mathcal{I}$ is bounded.

• ***$\mathcal{I}$ is pseudomonotone:***

Let $\{u_n = (u_{n,1}, u_{n,2})\}_{n\in\mathbb{N}} \subset \mathcal{V}$ be a sequence such that

$$u_n \rightharpoonup u \text{ in } \mathcal{V} \quad \text{and} \quad \limsup_{n\to\infty} \langle \mathcal{I}(u_n), (u_n - u)\rangle \leq 0. \tag{4.2.2}$$

Taking the compact embedding $W_0^{1,p_i(x)}(\Omega) \hookrightarrow L^{\alpha_i^*(x)}(\Omega)$ into consideration gives

$$u_{n,1} \to u_1 \text{ in } L^{\alpha_1^*(x)}(\Omega) \quad \text{and} \quad u_{n,2} \to u_2 \text{ in } L^{\alpha_2^*(x)}(\Omega), \tag{4.2.3}$$

since $\alpha_1^*(x) < p_1^*(x)$ and $\alpha_2^*(x) < p_2^*(x)$, respectively. We need to show that

$$\lim_{n\to\infty} \int_\Omega f_1(x, u_{n,1}, u_{n,2}, \nabla u_{n,1}, \nabla u_{n,2})(u_{n,1} - u_1)\, dx = 0 \tag{4.2.4}$$

and

$$\lim_{n\to\infty} \int_\Omega f_2(x, u_{n,1}, u_{n,2}, \nabla u_{n,1}, \nabla u_{n,2})(u_{n,2} - u_2)\, dx = 0. \tag{4.2.5}$$

We proceed as described in Motreanu et al. [64]. In order to establish (4.2.4), we use hypothesis (H_4), Hölder's inequality and compactness embedding relation (4.2.3). We have

$$\left| \int_\Omega f_1(x, u_{n,1}, u_{n,2}, \nabla u_{n,1}, \nabla u_{n,2})(u_{n,1} - u_1)\, dx \right| \leq K_{n,1} + K_{n,2} + K_{n,3} + K_{n,4},$$

where, by using the relation $\dfrac{\alpha_1^*(x)}{\gamma_1(x)} + \dfrac{\alpha_2^*(x)}{\gamma_2(x)} + \dfrac{1}{\gamma_1(x)} = 1$ with $\rho = \dfrac{(\alpha_1^*)^-}{\gamma_1^+} + \dfrac{(\alpha_2^*)^-}{\gamma_2^+} + \dfrac{1}{\gamma_1+}$ and applying Remark 1.1.2, we have

$$\begin{aligned} K_{n,1} &= \int_\Omega a_1 |u_{n,1}|^{\alpha_1^*(x)} |u_{n,2}|^{\alpha_2^*(x)} |u_{n,1} - u_1|\, dx \\ &\leq \rho a_1 \left(\|u_{n,1}\|_{L^{\gamma_1(\cdot)}(\Omega)}^{(\alpha_1^*)^+} + \|u_{n,1}\|_{L^{\gamma_1(\cdot)}(\Omega)}^{(\alpha_1^*)^-} \right) \left(\|u_{n,1}\|_{L^{\gamma_2(\cdot)}(\Omega)}^{(\alpha_2^*)^+} + \|u_{n,1}\|_{L^{\gamma_2(\cdot)}(\Omega)}^{(\alpha_2^*)^-} \right) \\ &\quad \cdot \|u_{n,1} - u_1\|_{L^{\gamma_1(\cdot)}(\Omega)} \to 0, \end{aligned}$$

$$K_{n,2} = \int_\Omega b_1 |\nabla u_{n,2}|^{\delta_1(x)} |u_{n,1} - u_1| \, dx \leq 2 \left\| |\nabla u_{n,2}|^{q_1(x)} \right\|_{L^{\frac{q_1(x)}{\delta_1(x)}}(\Omega)}$$
$$\cdot \|u_{n,1} - u_1\|_{L^{\frac{q_1(x)}{q_1(x)-\delta_1(x)}}(\Omega)} \to 0,$$
$$K_{n,3} = \int_\Omega b_1 |\nabla u_{n,2}|^{\frac{\delta_1(x) q_2(x)}{q_1(x)}} |u_{n,1} - u_1| \, dx \leq 2 \left\| |\nabla u_{n,2}|^{q_2(x)} \right\|_{L^{\frac{q_1(x)}{\delta_1(x)}}(\Omega)}$$
$$\cdot \|u_{n,1} - u_1\|_{L^{\frac{q_1(x)}{q_1(x)-\delta_1(x)}}(\Omega)} \to 0,$$
$$K_{n,4} = \int_\Omega e_1(x) |u_{n,1} - u_1| \, dx \leq 2 \|e_1\|_{L^{\gamma_1'(x)}(\Omega)} \|u_{n,1} - u_1\|_{L^{\gamma_1(x)}(\Omega)} \to 0.$$

Therefore, (4.2.4) is fulfilled. Applying similar calculations yields (4.2.5). Taking the weak formulation in Definition 5.4.1, replacing u_1 by $u_{n,1}, u_2$ by $v_{n,2}, \varphi_1$ by $u_{n,1} - u_1$ and φ_2 by $u_{n,2} - u_2$ and using (4.2.2) as well as (4.2.4) and (4.2.5) lead to

$$\limsup_{n\to\infty} \left\langle \tilde{\mathcal{A}}(u_n), (u_n - u) \right\rangle = \limsup_{n\to\infty} \langle \mathcal{I}(u_n), (u_n - u) \rangle \leq 0. \tag{4.2.6}$$

Since $\tilde{\mathcal{A}}$ satisfies the (S_+)-property, we deduce from (4.2.2) and (4.2.6) that $u_n \to u$ in $\mathcal{V}$. Lastly, because $\mathcal{I}$ is continuous, we conclude that $\mathcal{I}(u_n) \to \mathcal{I}(u)$ in $\mathcal{V}'$, which proves that $\mathcal{I}$ is pseudomonotone.

• ***$\mathcal{I}$ is coercive:***

For $u = (u_1, u_2) \in \mathcal{V}$, we have

$$\begin{aligned}
\langle \tilde{\mathcal{A}}(u), u \rangle &= \left(\int_\Omega |\nabla u_1|^{p_1(x)} + \mu_1 |\nabla u_1|^{q_1(x)} \, dx + \int_\Omega |\nabla u_2|^{p_2(x)} + \mu_2 |\nabla u_2|^{q_2(x)} \, dx \right) \\
&\geq \|u_1\|^{\tau_1}_{W_0^{1,p_1(x)}(\Omega)} + \|u_2\|^{\tau_2}_{W_0^{1,p_2(x)}(\Omega)} + \mu_1 \|u_1\|^{\nu_1}_{W_0^{1,q_1(x)}(\Omega)} + \mu_2 \|u_2\|^{\nu_2}_{W_0^{1,q_2(x)}(\Omega)} \\
&\geq \|u\|^\tau_{\mathcal{V}} + \mu_0 \|u\|^\nu_{\mathcal{V}},
\end{aligned}$$

where for $i \in \{1, 2\}$, we have

$$\begin{cases} \tau_i = p_i^-, \text{ if } \displaystyle\int_\Omega |\nabla u_i|^{p_i(x)} \, dx \geq 1, \\ \tau_i = p_i^+, \text{ if } \displaystyle\int_\Omega |\nabla u_i|^{p_i(x)} \, dx \leq 1, \end{cases}$$

and

$$\begin{cases} \nu_i = q_i^-, \text{ if } \displaystyle\int_\Omega |\nabla u_i|^{q_i(x)} \, dx \geq 1, \\ \nu_i = q_i^+, \text{ if } \displaystyle\int_\Omega |\nabla u_i|^{q_i(x)} \, dx \leq 1, \end{cases}$$

with $\tau = \min\{\tau_1, \tau_2\} > 1$ and $\nu = \min\{\nu_1, \nu_2\} > 1$. Using (H_4) and (H_5), for any $\epsilon > 0$, there exists $M_\epsilon \in L^1(\Omega)$ such that for all $s = (s_1, s_2) \in \mathbb{R}^2$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{2N}$ for a.e. $x \in \Omega$, we have

$$\mathbf{f}(x, s, \xi).s \leq (\lambda_1(p) + \epsilon)\left(|s_1|^{\alpha_1(x)}|s_2|^{\alpha_2(x)}\right) + c\left(|\xi_1|^{q_1(x)} + |\xi_2|^{q_2(x)}\right) + M_\epsilon(x).$$

Consequently, we obtain

$$\begin{aligned}\langle \mathcal{I}(u), u\rangle &= \langle \tilde{\mathcal{A}}(u), u\rangle - \int_\Omega \mathbf{f}(x, u, \nabla u)\cdot u dx - \langle h, u\rangle \\ &\geq \left\langle \tilde{\mathcal{A}}_p(u), u\right\rangle - (\lambda_1(p) + \epsilon)\int_\Omega |u_1|^{\alpha_1(x)}|u_2|^{\alpha_2(x)} dx + \left\langle \tilde{\mathcal{A}}_q(u), u\right\rangle \\ &- c\int_\Omega \left(|\nabla u_1|^{q_1(x)} + |\nabla u_2|^{q_2(x)}\right) dx \\ &- \int_\Omega M_\epsilon(x)dx - \|h\|_{\mathcal{V}'}\|u\|_{\mathcal{V}}.\end{aligned}$$

On the other hand, we have

$$\mathcal{A}_q(u) \leq \theta_q\left\langle \tilde{\mathcal{A}}_q(u), u\right\rangle \quad \text{and} \quad \lambda_1(p)\mathcal{B}_p(u) \leq \mathcal{A}_p(u) \leq \theta_p\left\langle \tilde{\mathcal{A}}_p(u), u\right\rangle,$$

with

$$\theta_p := \max\left\{\frac{1}{p_1^-}, \frac{1}{p_2^-}\right\} < 1 \quad \text{and} \quad \theta_q := \max\left\{\frac{1}{q_1^-}, \frac{1}{q_2^-}\right\} < 1.$$

Hence,

$$\begin{aligned}\langle \mathcal{I}(u), u\rangle &\geq \left(1 - \theta_p\frac{\lambda_1(p) + \epsilon}{\lambda_1(p)}\right)\left\langle \tilde{\mathcal{A}}_p(u), u\right\rangle + (1 - c\mu_0\theta_q)\left\langle \tilde{\mathcal{A}}_q(u), u\right\rangle \\ &- \|M_\epsilon\|_{L^1(\Omega)} - \frac{\alpha_0}{\tau}\|u\|_{\mathcal{V}}^\tau - \frac{1}{\alpha_0\tau'}\|h\|_{\mathcal{V}}^{\tau'},\end{aligned}$$

where $0 < \alpha_0 < \tau$, with $\tau' = \dfrac{\tau}{\tau - 1}$. Therefore, we get

$$\begin{aligned}\frac{\langle \mathcal{I}(u), u\rangle}{\|u\|_{\mathcal{V}}} &\geq \gamma\frac{\langle \tilde{\mathcal{A}}(u), u\rangle}{\|u\|_{\mathcal{V}}} - \frac{\|M_\epsilon\|_{L^1(\Omega)}}{\|u\|_{\mathcal{V}}} - \frac{\alpha_0}{\tau}\|u\|_{\mathcal{V}}^{\tau-1} - \frac{1}{\alpha_0\tau'}\frac{\|h\|_{\mathcal{V}}^{\tau'}}{\|u\|_{\mathcal{V}}}, \\ &\geq \left(\gamma - \frac{\alpha_0}{\tau}\right)\|u\|_{\mathcal{V}}^{\tau-1} - \frac{\|M_\epsilon\|_{L^1(\Omega)}}{\|u\|_{\mathcal{V}}} - \frac{1}{\alpha_0\tau'}\frac{\|h\|_{\mathcal{V}}^{\tau'}}{\|u\|_{\mathcal{V}}},\end{aligned}$$

where $\gamma = \min\left\{1 - \theta_p\dfrac{\lambda_1(p) + \epsilon}{\lambda_1(p)}, 1 - c\mu_0\theta_q\right\}$. Now choosing c, ϵ and α_0 such that

$$\begin{cases} c < \dfrac{1}{\mu_0 \theta_q}, \\ 0 < \epsilon < \dfrac{(1-\theta_p)\,\lambda_1(p)}{\theta_p}, \\ 0 < \alpha_0 < \tau\gamma, \end{cases}$$

we obtain $\lim\limits_{\|u\|_{\mathcal{V}} \to \infty} \dfrac{\langle \mathcal{I}(u), u \rangle}{\|u\|_{\mathcal{V}}} = \infty$. Thus, $\mathcal{I}$ is coercive.

Thus, all the assumptions of Theorem 4.2.4 are satisfied. Therefore, there exists $u_0 \in X$ such that $\mathcal{I}(u_0) = 0$. This completes the proof. □

4.3 Uniqueness result

Next, we consider the uniqueness of the solutions of system (PS). We consider the following hypotheses

(F_1) For all $s = (s_1, s_2)$ and $t = (t_1, t_2) \in \mathbb{R}^2$, and for all $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2) \in \mathbb{R}^N$, we have

$$\begin{aligned} &(\mathbf{f}(x, s, \xi_1, \xi_2) - \mathbf{f}(x, t, \eta_1, \eta_2)) \cdot (s - t) \\ &\leq \rho_1 |s_1 - t_1| \, |s_2 - t_2| + \rho_2 \left(|\xi_1 - \eta_1|^2 + |\xi_2 - \eta_2|^2 \right), \end{aligned}$$

for a.e $x \in \Omega$,

(F_2) We have $\rho_2 < 1$ and $\rho_1 < 2(1 - \rho_2)\lambda_1(2, 2)$,

(F_3) For all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^N$ and for a.e $x \in \Omega$, we have $s_j \longrightarrow \mathbf{f}_i(x, s_1, s_2, \xi_1, \xi_2)$ is decreasing, for $i, j \in \{1, 2\}$,

(F_4) For $i, j \in \{1, 2\}$, for all $s = (s_1, s_2) \in \mathbb{R}^2$ and for a.e $x \in \Omega$, we have $\mathbf{f}(x, s_1, s_2, \xi_1, \xi_2) = \mathbf{f}(x, s_1, s_2, |\xi_1|, |\xi_2|)$ and $\eta_j \longrightarrow \mathbf{f}_i(x, s_1, s_2, \eta_1, \eta_2)$ is decreasing,

(F_5) For $i \in \{1, 2\}$, we have $q_i = p_i$.

The following theorem is the subject of a first publication with my supervisor Professor El Hachimi published in the Moroccan Journal of Pure and Applied Analysis.

Theorem 4.3.1. *(see Ouannasser and El Hachimi [73, Theorem 4.1.]) Suppose that the hypotheses of Theorem 4.2.6 alongside* (F_1) *and* (F_2) *are verified. In addition, if* $p_i(x) = 2$ *and* $\alpha_i(x) = 1$*, for all* $x \in \bar{\Omega}$ *and for* $i \in \{1, 2\}$*, then problem* (PS) *has a unique solution.*

Proof. Let $u=(u_1,u_2)$ and $v=(v_1,v_2)$ be two weak solutions to (PS). Considering the weak formulation of (u_1,u_2) and (v_1,v_2), by choosing $\phi = u_1 - v_1$ and $\psi = u_2 - v_2$ as test functions, and adding the obtained equations to each other, we get

$$\begin{aligned}
&\int_\Omega |\nabla (u_1(x)-v_1(x))|^2\,dx + \mu_1 \int_\Omega \big(\mathcal{F}_{q_1(x)}(\nabla u_1) - \mathcal{F}_{q_1(x)}(\nabla v_1)\big)\cdot(\nabla u_1 - \nabla v_1)\,dx \\
&+ \int_\Omega |\nabla (u_2(x)-v_2(x))|^2\,dx + \mu_2 \int_\Omega \big(\mathcal{F}_{q_2(x)}(\nabla u_2) - \mathcal{F}_{q_2(x)}(\nabla v_2)\big)\cdot(\nabla u_2 - \nabla v_2)\,dx \\
&= \int_\Omega (\mathbf{f}(x,u,\nabla u) - \mathbf{f}(x,v,\nabla v))\cdot(u-v)dx.
\end{aligned}$$

Moreover, for $r(x)>1$ for all $x\in\bar{\Omega}$ and $w_1, w_2 \in W_0^{1,r(x)}(\Omega)$, we have

$$0 \le \big(\mathcal{F}_{r(x)}(\nabla w_1) - \mathcal{F}_{r(x)}(\nabla w_2)\big)\cdot \nabla (w_1 - w_2)\,dx.$$

Therefore, using assumption (F_1) leads to

$$\begin{aligned}
&\int_\Omega |\nabla (u_1(x)-v_1(x))|^2 + |\nabla (u_2(x)-v_2(x))|^2\,dx \\
&\le \rho_1 \int_\Omega |u_1 - v_1|\,|u_2 - v_2| + \rho_2\Big(|\nabla (u_1(x)-v_1(x))|^2 + |\nabla (u_2(x)-v_2(x))|^2\Big)dx.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&(1-\rho_2)\left(\int_\Omega |\nabla (u_1(x)-v_1(x))|^2\,dx + \int_\Omega |\nabla (u_2(x)-v_2(x))|^2\,dx\right) \\
&\le \rho_1 \int_\Omega |u_1 - v_1|\,|u_2 - v_2|\,dx.
\end{aligned}$$

Thus, we obtain

$$2(1-\rho_2)\,\mathcal{A}_{(2,2)}(u-v) \le \rho_1 \mathcal{B}_{(2,2)}(u-v). \tag{4.3.1}$$

On the other hand, by using the definition of $\lambda_1(2,2)$, we get

$$\mathcal{B}_{(2,2)}(u-v) \le \frac{1}{\lambda_1(2,2)}\mathcal{A}_{(2,2)}(u-v).$$

By using (4.3.1), we obtain

$$\mathcal{A}_{(2,2)}(u-v) \le \frac{\rho_1}{2(1-\rho_2)\,\lambda_1(2,2)}\mathcal{A}_{(2,2)}(u-v).$$

If $\mathcal{A}_{(2,2)}(u-v)\neq 0$, we obtain $\rho_1 \ge 2(1-\rho_2)\,\lambda_1(2,2)$, which contradicts hypothesis (F_2).

Therefore, we have $\mathcal{A}_{(2,2)}(u-v)=0$, which means that $u=v$, since $(u,v)\in\mathcal{V}$. This completes the proof. □

The following theorem is the subject of a first publication with my supervisor Professor El Hachimi published in the Moroccan Journal of Pure and Applied Analysis.

Theorem 4.3.2. *(see Ouannasser and El Hachimi [73, Theorem 4.2.]) Suppose that the hypotheses of Theorem 4.2.6 and* $(F_3)-(F_5)$ *are verified, then problem* (PS) *has a unique solution.*

Proof. Let $u=(u_1,u_2)$ and $v=(v_1,v_2)$ be two weak solutions to problem (PS). Considering the weak formulation of (u_1,u_2) and (v_1,v_2), by choosing $\phi=(u_1-v_1)_+$and $\psi=(u_2-v_2)_+$as test functions, and adding the obtained equations to each other, we have as in the proof of Theorem 4.3.1

$$\begin{aligned}
0 &\leq \int_{\{u_1(x)>v_1(x)\}} \left(\mathcal{F}_{p_1(x)}\left(\nabla(u_1-v_2)\right)\right)\cdot\nabla\left(u_1-v_1\right)dx\\
&+\int_{\{u_2(x)>v_2(x)\}} \left(\mathcal{F}_{p_2(x)}\left(\nabla(u_2-v_2)\right)\right)\cdot\nabla\left(u_2-v_2\right)dx\\
&=\int_\Omega \left(\mathcal{F}_{p_1(x)}\left(\nabla u_1\right)-\mathcal{F}_{p_1(x)}\left(\nabla v_1\right)\right)\cdot\nabla\left(u_1-v_1\right)_+dx\\
&+\int_\Omega \left(\mathcal{F}_{p_2(x)}\left(\nabla u_2\right)-\mathcal{F}_{p_2(x)}\left(\nabla v_2\right)\right)\cdot\nabla\left(u_2-v_2\right)_+dx\\
&=\int_\Omega \left(\mathbf{f}_1(x,u,\nabla u)-\mathbf{f}_1(x,v,\nabla v)\right)\cdot\left(u_1-v_1\right)_+dx\\
&+\int_\Omega \left(\mathbf{f}_2(x,u,\nabla u)-\mathbf{f}_2(x,v,\nabla v)\right)\cdot\left(u_2-v_2\right)_+dx.
\end{aligned}$$

Then,

$$\int_\Omega \left(\mathbf{f}_1(x,u,\nabla u)-\mathbf{f}_1(x,v,\nabla v)\right)\cdot\left(u_1-v_1\right)_+dx=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4,$$

where

$$\begin{aligned}
\mathcal{I}_1&=\int_\Omega \left(\mathbf{f}\left(x,u_1,u_2,\left|\nabla u_1\right|,\left|\nabla u_2\right|\right)-\mathbf{f}\left(x,v_1,u_2,\left|\nabla u_1\right|,\left|\nabla u_2\right|\right)\right)\cdot\left(u_1-v_1\right)_+dx,\\
\mathcal{I}_2&=\int_\Omega \left(\mathbf{f}\left(x,v_1,u_2,\left|\nabla u_1\right|,\left|\nabla u_2\right|\right)-\mathbf{f}\left(x,v_1,v_2,\left|\nabla v_1\right|,\left|\nabla u_2\right|\right)\right)\cdot\left(u_1-v_1\right)_+dx,\\
\mathcal{I}_3&=\int_\Omega \left(\mathbf{f}\left(x,u_1,v_2,\left|\nabla u_1\right|,\left|\nabla u_2\right|\right)-\mathbf{f}\left(x,v_1,v_2,\left|\nabla u_1\right|,\left|\nabla v_2\right|\right)\right)\cdot\left(u_1-v_1\right)_+dx,\\
\mathcal{I}_4&=\int_\Omega \left(\mathbf{f}\left(x,v_1,v_2,\left|\nabla v_1\right|,\left|\nabla u_2\right|\right)-\mathbf{f}\left(x,v_1,v_2,\left|\nabla v_1\right|,\left|\nabla v_2\right|\right)\right)\cdot\left(u_1-v_1\right)_+dx.
\end{aligned}$$

According to the hypothesis (F_3), we have $\mathcal{I}_1\leq 0$. Similarly, we have

$$\mathcal{I}_2 = \int_{\Omega_2} \left(\mathbf{f}\left(x, v_1, u_2, \left|\nabla u_1\right|, \left|\nabla u_2\right|\right) - \mathbf{f}\left(x, v_1, v_2, \left|\nabla u_1\right|, \left|\nabla u_2\right|\right)\right) \cdot \\ \left(u_2 - v_2\right)_+ \frac{\left(u_1 - v_1\right)_+}{\left(u_2 - v_2\right)_+} dx,$$

where $\Omega_2 = \left\{x, \left(u_2 - v_2\right)_+ (x) \neq 0\right\}$. Using hypothesis (F_3), we have $\mathcal{I}_2 \leq 0$. On the other hand, we get

$$\mathcal{I}_3 = \int_{\Omega_3} \left(\mathbf{f}\left(x, v_1, v_2, \left|\nabla u_1\right|, \left|\nabla u_2\right|\right) - \mathbf{f}\left(x, v_1, v_2, \left|\nabla v_1\right|, \left|\nabla u_2\right|\right)\right) \cdot \\ \left(\left|\nabla u_1\right| - \left|\nabla v_1\right|\right)_+ \frac{\left(u_1 - v_1\right)_+}{\left(\left|\nabla u_1\right| - \left|\nabla v_1\right|\right)_+} dx,$$

where $\Omega_3 = \left\{x, \left(\left|\nabla u_1\right| - \left|\nabla v_1\right|\right)_+ (x) \neq 0\right\}$. According to the hypothesis (F_3), we have $\mathcal{I}_3 \leq 0$. On the other hand, we get

$$\mathcal{I}_4 = \int_{\Omega_4} \left(\mathbf{f}\left(x, v_1, v_2, \left|\nabla v_1\right|, \left|\nabla u_2\right|\right) - \mathbf{f}\left(x, v_1, v_2, \left|\nabla v_1\right|, \left|\nabla v_2\right|\right)\right) \cdot \\ \left(\left|\nabla u_2\right| - \left|\nabla v_2\right|\right)_+ \frac{\left(u_1 - v_1\right)_+}{\left(\left|\nabla u_2\right| - \left|\nabla v_2\right|\right)_+} dx,$$

where $\Omega_4 = \left\{x, \left(\left|\nabla u_2\right| - \left|\nabla v_2\right|\right)_+ (x) \neq 0\right\}$. According to the hypothesis (F_3), we have $\mathcal{I}_4 \leq 0$. Therefore, we get

$$\int_{\Omega} \left(\mathbf{f}_1(x, u, \nabla u) - \mathbf{f}_1(x, v, \nabla v)\right) \cdot \left(u_1 - v_1\right)_+ dx \leq 0.$$

Using the same arguments, we obtain

$$\int_{\Omega} \left(\mathbf{f}_2(x, u, \nabla u) - \mathbf{f}_2(x, v, \nabla v)\right) \cdot \left(u_2 - v_2\right)_+ dx \leq 0,$$

which implies that

$$\int_{\{u_1(x) > v_1(x)\}} \left(\mathcal{F}_{p_1(x)}\left(\nabla u_1\right) - \mathcal{F}_{p_1(x)}\left(\nabla v_1\right)\right) \cdot \nabla\left(u_1 - v_1\right) dx = 0,$$

and

$$\int_{\{u_2(x) > v_2(x)\}} \left(\mathcal{F}_{p_2(x)}\left(\nabla u_2\right) - \mathcal{F}_{p_2(x)}\left(\nabla v_2\right)\right) \cdot \nabla\left(u_2 - v_2\right) dx = 0.$$

respectively. From this, it follows $\nabla u_1(x) = \nabla v_1(x)$ for a.e x on $\mathcal{U}_1 = \{x \in \Omega : u_1(x) > v_1(x)\}$ and $\nabla u_2(x) = \nabla v_2(x)$ for a.e x on $\mathcal{U}_2 = \{x \in \Omega : u_1(x) > v_1(x)\}$. Now, for $x \in \mathcal{U}_1$, we have $\left(u_1 - v_1\right)_+ (x) = \left(u_1 - v_1\right)(x)$ and $\nabla\left(u_1 - v_1\right)_+ (x) = \nabla u_1(x) - \nabla v_1(x) = 0$, while for $x \in \Omega \backslash \mathcal{U}_1$, we have $\left(u_1 - v_1\right)_+ (x) = 0$ and $\nabla\left(u_1 - v_1\right)_+ (x) = 0$.

Consequently, we have $\nabla (u_1 - v_1)_+ (x) = 0$, a.e in Ω and then $(u_1 - v_1)_+ = 0$ because $(u_1 - v_1)_+ \in W_0^{1,p_1(x)}(\Omega)$. Then, we get $u_1 \leq v_1 = 0$, for a.e x in Ω. Similarly, we obtain $u_2 \leq v_2 = 0$ for a.e x in Ω. Similarly, we obtain $u_1 \geq v_1 = 0$, for a.e x in Ω and $u_2 \geq v_2 = 0$, for a.e x in Ω. Therefore, $u = v$ and the solution is unique. $\square$

4.4 Case of a gradient-type system

In this section, we focus on gradient-type systems. Hence, we assume that $f_i(x,u) = \dfrac{\partial F}{\partial u_i}(x,u)$ for $i \in \{1,2\}$. The system (PS) becomes

$$\begin{cases} -\Delta_{p_i(x)} u_i - \mu_i \Delta_{q_i(x)} u_i = \dfrac{\partial F}{\partial u_i}(x,u) + h_i(x) & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{GS}$$

For $u = (u_1, u_2) \in \mathcal{V}$, define $\mathcal{A}_0(u_1,u_2)$ and $\mathcal{B}(u_1,u_2)$ as in Section 4.2 by

$$\begin{cases} \mathcal{A}_0(u) = \displaystyle\int_\Omega \frac{1}{p_1(x)} |\nabla u_1|^{p_1(x)}\, dx + \int_\Omega \frac{1}{p_2(x)} |\nabla u_2|^{p_2(x)}\, dx, \\ \mathcal{A}(u) = \mathcal{A}_0(u) + \displaystyle\int_\Omega \frac{\mu_1}{q_1(x)} |\nabla u_1|^{q_1(x)}\, dx + \int_\Omega \frac{\mu_2}{q_2(x)} |\nabla u_2|^{q_2(x)}\, dx, \\ \mathcal{B}(u) = \displaystyle\int_\Omega |u_1|^{\alpha_1(x)} |u_2|^{\alpha_2(x)}\, dx = \int_\Omega |u|^{\alpha(x)} dx, \end{cases}$$

and denote

$$F_\infty(x) \equiv \limsup_{|s|_\infty \longrightarrow +\infty} \frac{F(x,s)}{|s|^{\alpha(x)}}.$$

Suppose that function F satisfies the following hypotheses

(h_1) $\dfrac{\partial F}{\partial s_1}$ and $\dfrac{\partial F}{\partial s_2} : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions such that there exist positive continuous functions α_i^*, nonnegative functions $e_i \in L^1(\Omega)$ with $1 < \alpha_i^*(x) < p_i^*(x)$, and nonnegative constants a_i for $i \in \{1,2\}$, such that for a.e. $x \in \Omega$ and all $(s_1, s_2) \in \mathbb{R}^2$, we have

$$\left| \frac{\partial F}{\partial s_1}(x, s_1, s_2) \right| \leq a_1 \alpha_1(x) |s_1|^{\alpha_1^*(x)-1} |s_2|^{\alpha_2^*(x)} + e_1(x),$$

and

$$\left|\frac{\partial F}{\partial s_2}(x, s_1, s_2)\right| \leq a_2\alpha_2(x)\,|s_1|^{\alpha_1^*(x)}\,|s_2|^{\alpha_2^*(x)-1} + e_2(x).$$

(h_2) There exist a nonnegative constant A_1 and a nonnegative function $A_2 \in L^1(\Omega)$ such that, for all $x \in \Omega$ we have

$$|F(x,s)| \leq A_1|s|^{\alpha(x)} + A_2(x).$$

(h_3) $F_\infty(x) < \lambda_1$ uniformly for a.e $x \in \Omega$.

In this section, we state and prove a non-resonance result below the positive infimum λ_1. Define

$$\mathcal{I}(u) = \mathcal{A}(u) - \int_\Omega F(x, u(x))dx - \langle h, u\rangle,$$

where $\langle u, \varphi\rangle = \langle u, \varphi\rangle_{\mathcal{V},\mathcal{V}'}$ denotes the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$.
The following theorem is the subject of a first publication with my supervisor Professor El Hachimi published in the Moroccan Journal of Pure and Applied Analysis.

Theorem 4.4.1. *(see Ouannasser and El Hachimi [73, Theorem 5.1.]) Under hypotheses* (P_1) *or* (P_2) *and* $(h_1)-(h_3)$*, problem* (GS) *admits at least one solution* $v \in \mathcal{V}$*, for any* $h \in \mathcal{V}'$*. Moreover, suppose that* (F_1) *and* (F_2) *are verified and* $p_i(x) \geq 2$*, for all* $x \in \Omega$ *and* $i \in \{1, 2\}$*, then the solution is unique.*

Proof. We want to minimize the following energy functional $\mathcal{I}$ associated with problem (GS)

$$\mathcal{I}(u) \equiv \mathcal{A}(u) - \int_\Omega F(x, u(x))dx - \langle h, u\rangle, \forall u \in \mathcal{V}.$$

To this end, we show that $\mathcal{I}$ is weakly lower semicontinuous and coercive.

• $\mathcal{I}$ ***is weakly lower semicontinuous:***

Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{V}$ be a sequence such that

$$u_n \rightharpoonup u \text{ in } \mathcal{V}. \tag{4.4.1}$$

Then, taking the compact embedding $W_0^{1,p_i(x)}(\Omega) \hookrightarrow L^{p_i(x)}(\Omega)$ into consideration gives

$$u_{i,n} \to u \text{ in } L^{p_i(x)}(\Omega). \tag{4.4.2}$$

Moreover, there exist $\theta \in L^{p_i(x)}(\Omega)$ such that

$$|u_{i,n}(x)| \leq \theta_i(x), \text{ for a.e } x \in \Omega.$$

Therefore, by hypothesis (h_2), we have

$$\begin{aligned} F\left(x, u_n(x)\right) &\leq A_1\left|u_n\right|^{\alpha(x)}+A_2(x) \\ &\leq A_1|\theta(x)|^{\alpha(x)}+A_2(x). \end{aligned}$$

Using the fact that $F\left(x, u_n(x)\right) \rightarrow F(x, u(x))$, for a.e $x \in \Omega$ and applying Fatou's lemma, we obtain

$$\limsup_{n \longrightarrow+\infty} \int_{\Omega} F\left(x, u_n(x)\right) d x \leq \int_{\Omega} F(x, u(x)) d x.$$

Hence

$$\begin{aligned} \mathcal{I}(u) &= \mathcal{A}(u)-\int_{\Omega} F(x, u(x)) d x-\langle u, h\rangle \\ &\leq \liminf_{n \rightarrow+\infty} \mathcal{A}\left(u_n\right)-\limsup_{n \rightarrow+\infty} \int_{\Omega} F\left(x, u_n(x)\right) d x-\lim_{n \rightarrow+\infty}\left\langle h, u_n\right\rangle \\ &\leq \liminf_{n \rightarrow+\infty}\left\{\mathcal{A}\left(u_n\right)-\int_{\Omega} F\left(x, u_n(x)\right) d x-\left\langle h, u_n\right\rangle\right\} \\ &\leq \liminf_{n \rightarrow+\infty} \mathcal{I}\left(u_n\right). \end{aligned}$$

Consequently, $\mathcal{I}$ is weakly lower semicontinuous.

• ***$\mathcal{I}$ is coercive:***

Using Remark 1.1.2 for $u=(u_1, u_2)$, we have

$$\begin{aligned} \mathcal{A}(u) \geq \mathcal{A}_0(u) &\geq \eta\left(\|u_1\|_{\mathcal{V}_1}^{\sigma_1}+\|u_2\|_{\mathcal{V}_2}^{\sigma_2}\right) \\ &\geq \eta\|u\|_{\mathcal{V}}^{\sigma}, \end{aligned}$$

where $\eta=\min \left\{\frac{1}{p_1^{+}}, \frac{1}{p_2^{+}}\right\}$, and for $i \in\{1,2\}$, we have

$$\begin{cases}\sigma_i=p_i^{-} & \text{if } \displaystyle\int_{\Omega}\left|\nabla u_i\right|^{p_i(x)} d x \geq 1, \\ \sigma_i=p_i^{+} & \text{if } \displaystyle\int_{\Omega}\left|\nabla u_i\right|^{p_i(x)} d x \leq 1,\end{cases}$$

where $\sigma=\min \left(\sigma_1, \sigma_2\right)>1$. By (h_3), for any $\epsilon>0$ there exist $\delta_\epsilon>0$ and $M_\epsilon \in L^1(\Omega)$ such that

$$F(x, s) \leq\left(F_{\infty}(x)+\epsilon\right)|s|^{\alpha(x)}+M_\epsilon(x), \quad \forall s:|s|>\delta_\epsilon, \text{ for a.e } x \in \Omega.$$

By the embedding theorem, there exist $\gamma_0, \gamma_1>0$ such that

$$\int_{\Omega} \frac{\mu_1}{q_1(x)}\left|\nabla u_1\right|^{q_1(x)} d x+\int_{\Omega} \frac{\mu_2}{q_2(x)}\left|\nabla u_2\right|^{q_2(x)} d x \geq \gamma_0 \mathcal{A}_0(u)-\gamma_1.$$

Now, using hypothesis (h_1), we have

$$F(x,s) \leq (F_\infty(x)+\epsilon)\,|s|^{\alpha(x)} + M_\epsilon(x), \quad \forall s \in \mathbb{R}^2, \text{ for a.e } x \in \Omega.$$

Let $\epsilon > 0$ be fixed. We then have

$$\begin{aligned}\mathcal{I}(u) &= \mathcal{A}(u) - \int_\Omega F(x,u)dx - \langle u,h\rangle \\ &\geq (1+\gamma_0)\,\mathcal{A}_0(u) - (\lambda_1+\epsilon)\int_\Omega |u|^{\alpha(x)}dx - \int_\Omega M_\epsilon(x) - \|h\|_{\mathcal{V}'}\|u\|_{\mathcal{V}} - \gamma_1.\end{aligned}$$

On the other hand, we have

$$\mathcal{B}(u) = \int_\Omega |u|^{\alpha(x)}dx \leq \frac{1}{\lambda_1}\mathcal{A}_0(u).$$

Hence

$$\mathcal{I}(u) \geq \left(1+\gamma_0 - \frac{\lambda_1+\epsilon}{\lambda_1}\right)\mathcal{A}_0(u) - \|M_\epsilon\|_{L^1(\Omega)} - \frac{\gamma}{p^-}\|u\|_{\mathcal{V}}^{p^-} - \frac{1}{\gamma\,(p^-)'}\|h\|_{\mathcal{V}'}^{(p^-)'} - \gamma_1,$$

where $0<\gamma<p^-$ and $\left(p^-\right)' = \dfrac{p^-}{p^- - 1}$. Now, we have $\mathcal{A}_0(u) \geq \eta\|u\|_{\mathcal{V}}^{p^-}$, then

$$\lim_{\|u\|_{\mathcal{V}}\to\infty} \frac{\mathcal{A}_0(u)}{\|u\|_{\mathcal{V}}} = +\infty,$$

and

$$\mathcal{I}(u) \geq \left(\gamma_0 - \frac{\epsilon}{\lambda_1} - \frac{\gamma}{\eta p^-}\right)\mathcal{A}_0(u) - \|M_\epsilon\|_{L^1(\Omega)} - \frac{1}{\gamma p^{-'}}\|h\|_{\mathcal{V}'}^{p'}.$$

By choosing γ and ϵ such that $\gamma_0 > \dfrac{\epsilon}{\lambda_1} + \dfrac{\gamma}{\eta p^-}$, we obtain

$$\lim_{\|u\|_{\mathcal{V}}\to\infty} \frac{\mathcal{I}(u)}{\|u\|_{\mathcal{V}}} = +\infty.$$

Hence the coerciveness of $\mathcal{I}$. Now, $\mathcal{I}$ being lower semicontinuous and coercive, there exists $u \in \mathcal{V}$ such that $\mathcal{I}(u) = \inf_{v\in\mathcal{V}} \mathcal{I}(v)$. Owing to hypothesis (h_1), $\mathcal{I}$ is a C^1-functional and $u \in \mathcal{V}$ is a weak solution of problem (GS) if and only if it is a critical point of $\mathcal{V}$.

The proof of uniqueness of the solution is similar to that of Theorem 4.3.1. This completes the proof. $\square$

4.5 Concluding examples

Now, we provide some examples that verify the hypotheses of the existence and uniqueness theorems.

Example 4.5.1. Let the function $f = (f_1, f_2)$ be defined by

$$f_1(x, s_1, s_2, \xi_1, \xi_2) = |s_2|^{\alpha_2(x)} |s_1|^{\alpha_1(x)-2} s_1 \left(a_1(x) + \frac{d_1(x)}{1 + |s|^{\alpha(x)}} |\xi_1|^{q_1(x)} \right),$$

and

$$f_2(x, s_1, s_2, \xi_1, \xi_2) = |s_1|^{\alpha_1(x)} |s_2|^{\alpha_1(x)-2} s_2 \left(a_2(x) + \frac{d_2(x)}{1 + |s|^{\alpha(x)}} |\xi_2|^{q_2(x)} \right),$$

where the functions $\alpha_1(x)$ and $\alpha_2(x)$ are defined in the first section and $a_1, a_2, d_1,$ and d_2 are in $L^\infty(\Omega)$ where they satisfy the condition $\|a_1 + a_2\|_\infty < \lambda_1(p)$. We have

$$\begin{aligned} |f_1(x, s_1, s_2, \xi_1, \xi_2)| &\leq \|a_1\|_\infty |s_1|^{\alpha_1(x)-1} |s_2|^{\alpha_2(x)} + \frac{|s_1|^{\alpha_1(x)-1} |s_2|^{\alpha_2(x)} d_1(x) |\xi_1|^{q_1(x)}}{1 + |s|^{\alpha(x)}} \\ &\leq \|a_1\|_\infty |s_1|^{\alpha_1(x)-1} |s_2|^{\alpha_2(x)} + \|d_1\|_\infty |\xi_1|^{q_1(x)}. \end{aligned}$$

Similarly for f_2, we obtain

$$|f_2(x, s_1, s_2, \xi_1, \xi_2)| \leq \|a_2\|_\infty |s_1|^{\alpha_1(x)} |s_2|^{\alpha_2(x)-1} + \|d_2\|_\infty |\xi_2|^{q_2(x)},$$

which implies that hypothesis (H_4) is verified for $\alpha_1^*(x) = \alpha_1(x) - 1$, $\alpha_2^*(x) = \alpha_2(x)$, $\delta_1(x) = q_1(x)$ and $\delta_2(x) = q_2(x)$, for a.e. $x \in \Omega$. Furthermore, we have

$$\begin{aligned} \mathbf{f}(x, s, \xi) \cdot s &= (a_1(x) + a_2(x)) |s|^{\alpha(x)} + \frac{|s|^{\alpha(x)}}{1 + |s|^{\alpha(x)}} \left(d_1(x) |\xi_1|^{q_1(x)} + d_2(x) |\xi_2|^{q_2(x)} \right) \\ &\leq \lambda_1(p) |s|^{\alpha(x)} + \|d_1\|_\infty |\xi_1|^{q_1(x)} + \|d_2\|_\infty |\xi_2|^{q_2(x)}, \end{aligned}$$

which implies that f satisfies hypothesis (H_5) and hence all hypotheses of Theorem 4.2.6 are satisfied.

Example 4.5.2. Let the function $f = (f_1, f_2)$ be defined by

$$f_1(x, s_1, s_2, \xi_1, \xi_2) = a_1(x) \sin s_2 + d_1(x) \xi_1,$$

and

$$f_2(x, s_1, s_2, \xi_1, \xi_2) = a_2(x) \sin s_1 + d_2(x) \xi_2,$$

where the functions $\alpha_1(x)$ and $\alpha_2(x)$ are defined in the first section and a_1, a_2, d_1 and d_2 are in $L^\infty(\Omega)$ where they satisfy the condition $\|a_1\|_\infty + \|a_2\|_\infty < \lambda_1(p)$. One can easily check that f_1 and f_2 satisfy hypothesis (F_4). On the other hand, we have

$$\begin{aligned}\mathbf{f}(x,s,\xi)\cdot s &= a_1(x)s_1\sin s_2 + a_2(x)s_2\sin s_1 + d_1(x)s_1\xi_1 + d_2(x)s_2\xi_2\\ &\le (\|a_1\|_\infty + \|a_2\|_\infty)\,|s| + \frac{\|d_1\|_\infty}{\lambda_1(2,2)}|\xi_1|^2 + \frac{\|d_2\|_\infty}{\lambda_1(2,2)}|\xi_2|^2\\ &\le \lambda_1|s| + c\left(|\xi_1|^2 + |\xi_2|^2\right)\end{aligned}$$

which implies that f_1 and f_2 satisfy hypothesis (F_5). Furthermore, we have

$$\begin{aligned}(\mathbf{f}(x,s,\xi_1,\xi_2) - \mathbf{f}(x,t,\eta_1,\eta_2))\cdot(s-t) &= a_1(x)s_1\sin(s_2) + d_1(x)s_1\xi_1 - a_1(x)s_1\sin(t_2)\\ &\quad - d_1(x)s_1\eta_1 - a_1(x)t_1\sin(s_2) - d_1(x)t_1\xi_1\\ &\quad + a_1(x)t_1\sin(t_2) + d_1(x)t_1\eta_1 + a_2(x)s_2\sin(s_1)\\ &\quad + d_2(x)s_2\xi_2 - a_2(x)s_2\sin(t_1) + d_2(x)s_2\eta_2\\ &\quad - a_2(x)t_2\sin(s_1) - d_2(x)t_2\xi_2 + a_2(x)t_2\sin(t_1)\\ &\quad + d_2(x)t_2\eta_2\end{aligned}$$

which implies

$$\begin{aligned}&\le a_1(x)s_1(\sin(s_2) - \sin(t_2)) - a_1(x)t_1(\sin(s_2) - \sin(t_2))\\ &+ a_2(x)s_2(\sin(s_1) - \sin(t_1)) - a_2(x)t_2(\sin(s_1) - \sin(t_1))\\ &+ d_1(x)(s_1-t_1)(\xi_1-\eta_1) + d_2(x)(s_2-t_2)(\xi_2-\eta_2)\\ &\le a_1(x)(s_1-t_1)(s_2-t_2) + a_2(x)(s_2-t_2)(s_1-t_1) + \frac{d_1(x)}{\lambda_1(2,2)}(\xi_1-\eta_1)^2\\ &+ \frac{d_2(x)}{\lambda_1(2,2)}(\xi_2-\eta_2)^2\\ &\le (\|a_1\|_\infty + \|a_2\|_\infty)|s_1-t_1||s_2-t_2|\\ &+ \frac{(\|d_1\|_\infty + \|d_2\|_\infty)}{\lambda_1(2,2)}\left(|\xi_1-\eta_1|^2 + |\xi_2-\eta_2|^2\right)\\ &\le \lambda_1|s_1-t_1||s_2-t_2| + \frac{(\|d_1\|_\infty + \|d_2\|_\infty)}{\lambda_1(2,2)}\left(|\xi_1-\eta_1|^2 + |\xi_2-\eta_2|^2\right)\end{aligned}$$

Then, f satisfies all hypotheses of Theorem 4.3.1.

Example 4.5.3. Let the function $f = (f_1, f_2)$ be defined by

$$f_1(x,s_1,s_2,\xi_1,\xi_2) = -a_1(x)s_2 - \frac{2}{\pi}\left(\arctan s_2 + \frac{s_2}{1+s_2^2}\right)\left(1 + d_1(x)\frac{|\xi_2|^2}{1+|\xi_2|^2}\right),$$

and

$$f_2(x, s_1, s_2, \xi_1, \xi_2) = -a_2(x)s_1 - \frac{2}{\pi}\left(\arctan s_1 + \frac{s_1}{1+s_1^2}\right)\left(1 + d_2(x)\frac{|\xi_1|^2}{1+|\xi_1|^2}\right),$$

where a_1, a_2, d_1 and d_2 are positive functions in $L^\infty(\Omega)$ that satisfy the condition $\|a_1\|_\infty + \|a_2\|_\infty < \lambda_1(p)$. One can easily check that f_1 and f_2 satisfy hypotheses (H_4) and (H_5). Furthermore, it is obvious that for $i, j \in \{1, 2\}$, $s_j \longrightarrow \mathbf{f}_i(x, s_1, s_2, \xi_1, \xi_2)$ and $\eta_j \longrightarrow \mathbf{f}_i(x, s_1, s_2, \eta_1, \eta_2)$ are decreasing, whilst $\mathbf{f}(x, s_1, s_2, \xi_1, \xi_2) = \mathbf{f}(x, s_1, s_2, |\xi_1|, |\xi_2|)$ is verified. Then, f satisfies all hypotheses of Theorem 4.3.2.

Chapter 5

On degenerate parametric elliptic systems near and at resonance

In this chapter, we study two degenerate parametric elliptic systems in which the nonlinearities are gradient-dependent. The first part is devoted to investigating a system near resonance, where we establish the existence of a weak solution. In the second part, we deal with a system at resonance and prove the same result as in the first part, without the need for Landesman-Lazer conditions. Our existence results are based on a topological method using fixed point theory. Furthermore, we study a gradient-type system at resonance and prove the existence of a solution using a variational approach.

5.1 Introduction

The study of degenerate parametric elliptic systems near resonance is crucial, offering insights into subtle nonlinearities and their effects across multiple disciplines. In engineering, these insights help in designing structures that avoid catastrophic failures and optimize circuit performance (see Bleck-Neuhaus [15]). In physics, they reveal novel quantum states crucial for quantum computing advancements, as explored in depth in the field of magnonics (see Rezende [77]). In environmental sciences, they improve understanding and mitigation strategies for extreme weather events.

Similarly, the investigation of these systems at resonance has provided deep insights into the nonlinear dynamics and coupling effects in such systems. This research has broad applications, from modeling complex fluid dynamics and enhancing understanding of multiphase flows to aiding in the study of material failures like crack propagation. The study of optical and microwave systems using Fano resonances has further highlighted the widespread applicability and significant impact of resonance phenomena in various fields (see Kamenetskii et al. [49]). Collectively, studies near and at resonance offer a rich field of inquiry with profound implications in understanding and manipulating complex systems in various scientific fields.

The aim of this chapter is twofold: Given a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ with Lipschitz boundary, the first part is concerned with the following degenerate parametric elliptic system with convection terms near resonance

$$\begin{cases} -\Delta_{w_1,p_1}u - \mu_1\Delta_{w_1,q_1}u = \lambda a_1(x)|u|^{p_1-2}u + f_1(x,u,v,\nabla u,\nabla v) + h_1(x) & \text{in } \Omega, \\ -\Delta_{w_2,p_2}v - \mu_2\Delta_{w_2,q_2}v = \mu a_2(x)|v|^{p_2-2}v + f_2(x,u,v,\nabla u,\nabla v) + h_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.1}$$

where for $i \in \{1,2\}$, we have $-\Delta_{w_i,p_i}u = -\operatorname{div}(w_i(x)|\nabla u|^{p_i-2}\nabla u)$, $1 < q_i < p_i < N$, λ and μ are nonnegative parameters, $f_1, f_2 : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are functions of Carathéodory satisfying some conditions that will be specified later, $0 \leq a_1, a_2 \in L^\infty(\Omega)$ are weight functions, μ_i are positive constants and $h_i \in \left(W_0^{1,p_i}(w_i,\Omega)\right)^\star$. We revisit certain details concerning the homogeneous degenerate eigenvalue problem defined, for $i = 1, 2$, by

$$\begin{cases} -\operatorname{div}\left(w_i(x)|\nabla u|^{p_i-2}\nabla u_i\right) = \lambda a_i(x)|u|^{p_i-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N, 1 < p_i < N$ and w_i is a weight function that verifies the conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ given in Section 5.2. In this chapter, we assume that for every $i \in \{1,2\}$, the coefficient function a_i satisfies

$$\operatorname{meas}\{x \in \Omega : a_i(x) > 0\} > 0, \quad a_i \in L^{\frac{r_i}{r_i - p_i}}(\Omega), \quad \text{for some } p_i < r_i < p^*_{i,s_i},$$

where p^*_{i,s_i} is also given in Theorem 5.2.2 of Section 5.2. Lê and Schmitt [56] have proven the existence of a sequence of positive eigenvalues $\{\lambda_{k,i}\}_{k\in\mathbb{N}}$, where each $\lambda_{k,i}$ is determined as follows. Consider

$$\Gamma_i = \left\{ u \in W_0^{1,p_i}(w_i,\Omega) : \int_\Omega a_i(x)|u|^{p_i}dx = 1 \right\},$$
$$I_i(u) = \int_\Omega w_i(x)|\nabla u|^{p_i}dx, \quad u \in W_0^{1,p_1}(w_i,\Omega).$$

We then find

$$\lambda_k(p_i) = \inf_{A_i \in \Sigma_{k,i}} \sup_{u \in A_i} I_i(u),$$

where $\Sigma_{k,i} = \Big\{A_i \subset \Gamma_i :$ there exists a continuous, odd and surjective function $\gamma_i : S^{k-1} \to A_i\Big\}$ and S^{k-1} represents the unit sphere in $\mathbb{R}^k$, $\lambda_k(p_i) \to \infty$ as $k \to \infty$. Drábek et al. [33, Chapter 3] established that the principal eigenvalue $\lambda_1(p_i)$ is identified as simple and isolated, and all corresponding eigenfunctions maintain a consistent sign within Ω. Moreover, it is clear that

$$\lambda_1(p_i) = \inf_{u \in \Gamma_i} I_i(u),$$

which implies that for all $u \in W_0^{1,p_i}(w_i, \Omega)$, we have

$$\int_\Omega w_i(x)|\nabla u|^{p_i} dx \geq \lambda_1(p_i) \int_\Omega a_i(x)|u|^{p_i} dx, \quad \text{for all } u \in W_0^{1,p_i}(w_i, \Omega).$$

Furthermore, the corresponding normalized eigenfunction φ_i is also in $W_0^{1,p_i}(w_i, \Omega)$.

Consider also the set

$$W' = \Big\{w = (u,v) \in W_0^{1,p_1}(w_1, \Omega) \times W_0^{1,p_2}(w_2, \Omega) : \int_\Omega a(x)|\varphi_1|^{p_1-2}\varphi_1 u dx = 0 \text{ and } \int_\Omega b(x)|\varphi_2|^{p_2-2}\varphi_2 v dx = 0\Big\}.$$

It is straightforward to demonstrate that W' is a complementary subspace to $W = \langle\varphi_1\rangle \times \langle\varphi_2\rangle$, resulting in the direct sum

$$H = W \oplus W'.$$

There are many papers that have dealt with problems such as (5.1.1) near resonance, we cite Afrouzi et al. [2, 3, 4], An et al. [9], Zographopoulos [89] and the references therein. Most of these works investigated systems that do not involve gradient-dependent nonlinearities, where many existence and multiplicity results were obtained using several variational tools. In this study, we employ a topological method to obtain the existence result for system (5.1.1).

In the second part of this chapter, we deal with the following parametric elliptic system with convection terms at resonance

$$\begin{cases} -\Delta_{w_1,p_1} u - \mu_1 \Delta_{w_1,q_1} u = \lambda_1 a_1(x)|u|^{p_1-2}u + \lambda_1 w(x)|u|^{\alpha_1-2}u|v|^{\alpha_2} \\ \qquad\qquad + f_1(x,u,v,\nabla u,\nabla v) + h_1(x) & \text{in } \Omega, \\ -\Delta_{w_2,p_2} v - \mu_2 \Delta_{w_2,q_2} v = \lambda_1 a_2(x)|v|^{p_2-2}v + \lambda_1 w(x)|v|^{\alpha_2-2}v|u|^{\alpha_1} \\ \qquad\qquad + f_2(x,u,v,\nabla u,\nabla v) + h_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1.2}$$

where for $i \in \{1,2\}$, we have $-\Delta_{w_i,p_i} u = -\operatorname{div}(w_i(x)|\nabla u|^{p_i-2}\nabla u)$, $1 < q_i < p_i < N$, $f_1, f_2 : \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are functions of Carathéodory satisfying some conditions that will be specified later, $w \in C(\Omega)\cap L^\infty(\Omega)$ satisfying $w^+ = \max\{w(x),0\} \neq 0$, μ_i are positive constants and $h_i \in \big(W_0^{1,p_i}(w_i,\Omega)\big)^\star$. Let be associated to system (5.1.2) the following eigenvalue problem

$$\begin{cases} -\operatorname{div}(w_1(x)|\nabla u|^{p_1-2}\nabla u) = \lambda d_1(x)\,|u|^{p_1-2}\,u + \lambda w(x)\,|u|^{\alpha_1-2}\,u\,|v|^{\alpha_2} & \text{in } \Omega,\\ -\operatorname{div}(w_2(x)|\nabla v|^{p_2-2}\nabla v) = \lambda d_2(x)\,|v|^{p_2-2}\,v + \lambda w(x)\,|v|^{\alpha_2-2}\,v\,|u|^{\alpha_1} & \text{in } \Omega,\\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.3}$$

where for $i = 1,2$, α_i verify

$$\alpha_i > 0 \text{ and } \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = 1, \text{ for all } x \in \Omega.$$

Define the functionals $\phi, \varphi : X = W_0^{1,p_1}(w_1,\Omega) \times W_0^{1,p_2}(w_2,\Omega) \to \mathbb{R}$ as

$$\begin{cases} \phi(u,v) = \dfrac{\alpha_1}{p_1}\displaystyle\int_\Omega w_1(x)|\nabla u|^{p_1}dx + \dfrac{\alpha_2}{p_2}\displaystyle\int_\Omega w_2(x)|\nabla v|^{p_2}dx,\\ \varphi(u,v) = \displaystyle\int_\Omega w(x)|u|^{\alpha_1}|v|^{\alpha_2}dx + \dfrac{\alpha_1}{p_1}\displaystyle\int_\Omega d_1(x)|u|^{p_1}dx + \dfrac{\alpha_2}{p_2}\displaystyle\int_\Omega d_2(x)|v|^{p_2}dx, \end{cases}$$

and the manifold

$$\Sigma = \{(u,v) \in X : \varphi(u,v) = 1\}.$$

One can easily prove that $\phi(u,v)$ and $\varphi(u,v)$ are (p_1,p_2)-homogeneous, i.e.,

$$\phi(t^{1/p_1}u, t^{1/p_2}v) = t\phi(u,v) \quad \text{ and } \quad \varphi(t^{1/p_1}u, t^{1/p_2}v) = t\varphi(u,v),$$

for all $t > 0$ and $(u,v) \in X$. We also have Σ as a symmetric manifold in X. Using the same arguments as in Zographopoulos [88], the eigenvalue problem (5.1.3) admits a sequence of eigenvalues where they can be variationally characterized as follows

$$\lambda_k = \inf_{A\in\Sigma_k}\sup_{(u,v)\in A}\phi(u,v),$$

where $\Sigma_k = \Big\{A \subset \Sigma :$ there exists a continuous, odd and surjective function $\varphi_2 : S^{k-1} \to A\Big\}$, with S^{k-1} is the unit sphere in $\mathbb{R}^k$. We also have

$$0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \ldots, \; \lambda_k \to +\infty \text{ as } k \to +\infty.$$

Now, let us define the following

$$\lambda_1' = \inf_{(u,v)\in\Sigma} \frac{\alpha_1}{p_1}\int_\Omega w_1(x)|\nabla u|^{p_1}dx + \frac{\alpha_2}{p_2}\int_\Omega w_2(x)|\nabla v|^{p_2}dx. \tag{5.1.4}$$

Hence, λ_1' is the first eigenvalue of (5.1.3) as well, and taking the definition of λ_1 into account, one can easily derive that $\lambda_1' = \lambda_1$. Moreover, λ_1 is a simple, positive, and isolated principal eigenvalue of (5.1.3). In view of the fact that $\phi(u,v)$ and $\varphi(u,v)$ are (p_1,p_2)-homogeneous, the eigenfunction space corresponding to the first eigenvalue λ_1 is

$$E_{\lambda_1} = \{(u,v) \in X : \phi(u,v) = \lambda_1\varphi(u,v)\},$$

where $(t^{1/p_1}u, t^{1/p_2}v) \in E_{\lambda_1}$ for all $t \geq 0$ and $(u,v) \in E_{\lambda_1}$. In addition, there exists $t_0 > 0$ such that

$$\|t_0^{1/p_1}u_0\|^{p_1}_{W_0^{1,p_1}(w_1,\Omega)} + \|t_0^{1/p_2}v_0\|^{p_2}_{W_0^{1,p_2}(w_2,\Omega)} = 1,$$

where (u_0,v_0) is a normalized eigenvalue satisfying $u_0 \geq 0$, $v_0 \geq 0$ and $\|u_0\|_{W_0^{1,p_1}(w_1,\Omega)} + \|v_0\|_{W_0^{1,p_2}(w_2,\Omega)} = 1$.

Recently, Ouannasser and El Hachimi [73] studied a similar problem to (5.1.2) for the non-resonance case, but with variable exponents. More specifically, they obtained existence and uniqueness results for the following system

$$\begin{cases} -\Delta_{p_1(x)}u_1 - \mu_1\Delta_{q_1(x)}u_1 = f_1(x,u_1,u_2,\nabla u_1,\nabla u_2) + h_1(x) & \text{in } \Omega, \\ -\Delta_{p_2(x)}u_2 - \mu_2\Delta_{q_2(x)}u_2 = f_2(x,u_1,u_2,\nabla u_1,\nabla u_2) + h_2(x) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1.5}$$

Because of the lack of information on the spectrum in the case of variable exponents , we limit ourselves to the study of problem (5.1.2) for constant exponents. One can see that even in the case of constant exponents, little is known about resonance and non-resonance issues, when there is interaction between the Lusternik-Schnirelmann spectrum of the eigenvalue problem (5.1.3) and the function

$$(s,t) \longmapsto \frac{f_1(x,s,t,\xi,\eta)\,s + f_2(x,s,t,\xi,\eta)\,t}{w(x)\,|s|^{\alpha_1}\,|t|^{\alpha_2}}. \tag{5.1.6}$$

The same observation holds also when considering gradient-type systems, that is when functions f_1 and f_2 do not have a dependency on ∇u and ∇v and when the interaction of the spectrum is with the function

$$(s,t) \longmapsto \frac{F(x,s,t)}{w(x)\,|s|^{\alpha_1}\,|t|^{\alpha_2}},$$

where $\dfrac{\partial F}{\partial s}(x,s,t) = f_1(x,s,t)$ and $\dfrac{\partial F}{\partial t}(x,s,t) = f_2(x,s,t)$. The study of resonance has been of a great deal of importance. In our case, it follows that the function should be, in some sense, greater than the first eigenvalue. There are many works dedicated to the study of quasilinear elliptic problems at resonance, we cite Alves et al. [7], Benouhiba and Belyacine [14], De Nápoli and Mariani [29], Haddaoui et al. [47], Ou [69, 70], Ou and Tang [71], and the references therein. Most of these studies established the existence of weak solutions under Landesman-Lazer-type conditions.

In this work, we extend the aforementioned works to a parametric weighted (p_1, p_2)-Laplacian system, where the nonlinearities depend on the gradient of the solution, and obtain the existence of the solution. Similar to the first part of the chapter, the lack of variational structure in problem (5.1.2), caused by convection terms, prevents the use of variational methods. Instead, and without using Landesman-Lazer-type conditions, we employ a topological method based on the surjectivity result of pseudomonotone operators. To the best of our knowledge, this is the first chapter that deals with parametric weighted elliptic systems near and at resonance where the nonlinearities may depend on the gradient of the solution. Such problems bring about many issues to overcome, such as the interaction of the variational spectrum with the function given in (5.1.6).

Similarly, we address the scenario of a gradient-type system and establish the existence of at least one weak solution, following a standard variational method similar to that used by El Hachimi and de Thélin [38]. More specifically, we deal with the following system at resonance

$$\begin{cases} -\Delta_{w_1,p_1}u - \mu_1\Delta_{w_1,q_1}u = \lambda_1 w(x)|u|^{\alpha_1-2}u|v|^{\alpha_2} + \dfrac{\partial F}{\partial u}(x,u,v) + h_1(x) & \text{in } \Omega, \\ -\Delta_{w_2,p_2}v - \mu_2\Delta_{w_2,q_2}v = \lambda_1 w(x)|v|^{\alpha_2-2}v|u|^{\alpha_1} + \dfrac{\partial F}{\partial v}(x,u,v) + h_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1.7}$$

Many studies have dealt with similar problems to the system (5.1.7), studied their resonance, and obtained the existence of solutions using different methods, such as Zhao and Tang [86]. In this section, we extend the works of Lv and Ou [58], El Hachimi and de Thélin [38], Ouannasser and El Hachimi [73], Ou [69], De Nápoli and Mariani [29], Zhao and Tang [86], and deal with a more general class of systems where we obtain the existence of a solution at resonance.

The remainder of this chapter is organized as follows. In Section 5.2, we present the necessary elements regarding the weighted Sobolev spaces. In Section 5.3, we deal with the degenerate parametric elliptic system near resonance given in (5.1.1) and prove the existence of at least one weak solution using the surjectivity result of pseudomonotone operators. Moreover, we establish the same result for the system (5.1.2) at resonance in Section 5.4. In Section 5.5, we delve into the case of a gradient-type system at resonance, and demonstrate the existence of a solution using a variational approach. Finally, in Section 5.6, we give an illustrative example.

5.2 Functional framework

For $i \in \{1,2\}$, let w_i be a measurable and a.e. finite function in $\mathbb{R}^N$ such that

$(\mathcal{A}_1)$ $w_i \in L^1_{loc}(\Omega)$ and $w_i^{\frac{-1}{p_i-1}} \in L^1_{loc}(\Omega)$,

$(\mathcal{A}_2)$ $w_i^{-s_i} \in L^1(\Omega)$ for $s_i \in \left]\frac{N}{p_i}, +\infty\right[\cap \left[\frac{1}{p_i-1}, +\infty\right[$.

In what follows, let $X = W_0^{1,p_1}(w_1,\Omega) \times W_0^{1,p_2}(w_2,\Omega)$ be the product space endowed with the norm

$$\|(u,v)\|_X = \max\left\{\|u\|_{W_0^{1,p_1}(w_1,\Omega)}, \|v\|_{W_0^{1,p_2}(w_2,\Omega)}\right\},$$

where $\|u\|_{W_0^{1,p_1}(w_1,\Omega)} = \left(\int_\Omega w_1(x)|\nabla u|^{p_1} dx\right)^{\frac{1}{p_1}}$ and $\|v\|_{W_0^{1,p_2}(w_2,\Omega)} = \left(\int_\Omega w_2(x)|\nabla v|^{p_2} dx\right)^{\frac{1}{p_2}}$, for every $(u,v) \in X$. It is important to note that condition (a_1) is mandatory in order for $W_0^{1,p_i}(w_i,\Omega)$ to be a Banach space, for $i \in \{1,2\}$. Furthermore, we recall a few more results in relation to this space such as the continuous and compact embeddings.

Theorem 5.2.1 (see Kufner and Opic [55]). *Assume that $(\mathcal{A}_1)$ is verified. Then, for $i \in \{1,2\}$, the weighted Sobolev space $W_0^{1,p_i}(w_i,\Omega)$ is a Banach space.*

Theorem 5.2.2 (see Drábek et al. [33]). *Let $\Omega \subset \mathbb{R}^N$ be an open set and assume that $(\mathcal{A}_2)$ is verified. Then, for $i \in \{1,2\}$, we have the following continuous and compact embeddings*

i) For any r_i verifying $1 \leq r_i \leq p^*_{i,s_i} = \dfrac{Np_is_i}{N(s_i+1) - p_is_i}$ for $p_is_i < N(s_i+1)$, and $r_i \geq 1$ is arbitrary for $p_is_i \geq N(s_i+1)$, the embedding $W_0^{1,p_i}(w_i,\Omega) \hookrightarrow L^{r_i}(\Omega)$ is continous.

ii) For any r_i verifying $1 \leq r_i < p^*_{i,s_i} = \dfrac{Np_is_i}{N(s_i+1) - p_is_i}$, the embedding $W_0^{1,p_i}(w_i,\Omega) \hookrightarrow\hookrightarrow L^{r_i}(\Omega)$ is compact.

iii) *In particular, if* $s_i > \dfrac{N}{p_i}$ *then* $p^*_{i,s_i} > p_i$. *Consequently, the embedding* $W_0^{1,p_i}(w_i,\Omega) \hookrightarrow\hookrightarrow L^{r_i}(\Omega)$ *is compact.*

Other interesting results and properties of weighted Sobolev spaces can be found in Drábek et al. [33] and Kufner and Opic [55]. Now for $i \in \{1,2\}$, let

$$\bar{N}_{f_i} : W_0^{1,p_1}(w_1,\Omega) \times W_0^{1,p_2}(w_2,\Omega) \subset L^{r_1}(\Omega) \times L^{r_2}(\Omega) \to L^{r_1'}(\Omega) \times L^{r_2'}(\Omega)$$

be the Nemytskij operators associated to f_i . In addition, let

$$j_1^* : L^{r_1'}(\Omega) \times L^{r_2'}(\Omega) \to \left(W_0^{1,p_1}(w_1,\Omega)\right)^* \times \left(W_0^{1,p_2}(w_2,\Omega)\right)^*$$

be the adjoint operators for the embedding

$$j_1 : W_0^{1,p_1}(w_1,\Omega) \times W_0^{1,p_2}(w_2,\Omega) \to L^{r_1}(\Omega) \times L^{r_2}(\Omega).$$

Eventually, we define

$$N_{f_i} := j_1^* \circ \bar{N}_{f_i} : W_0^{1,p_1}(w_1,\Omega) \times W_0^{1,p_2}(w_2,\Omega) \to \left(W_0^{1,p_1}(w_1,\Omega)\right)^* \times \left(W_0^{1,p_2}(w_2,\Omega)\right)^*.$$

Definition 5.2.3. Let X be a reflexive Banach space, X^* its dual space and denote by $\langle\cdot,\cdot\rangle$ the duality pairing. Consider an application $\mathcal{I} : X \to X^*$. Then, $\mathcal{I}$ is said

(a) to verify the $\left(S^+\right)$-property if

$$u_n \rightharpoonup u \text{ in } X \text{ and } \limsup_{n\to\infty} \langle \mathcal{I}u_n, u_n - u\rangle \le 0 \text{ imply } u_n \to u \text{ in } X.$$

(b) pseudomonotone if

$$u_n \rightharpoonup u \text{ in } X \text{ and } \limsup_{n\to\infty} \langle \mathcal{I}u_n, u_n - u\rangle \le 0 \text{ imply } \mathcal{I}u_n \rightharpoonup \mathcal{I}u \text{ and } \langle \mathcal{I}u_n, u_n\rangle \to \langle \mathcal{I}u, u\rangle.$$

(c) coercive if

$$\lim_{\|u\|_X\to\infty} \frac{\langle \mathcal{I}u, u\rangle}{\|u\|_X} = \infty.$$

Theorem 5.2.4 (see Carl et al. [24])**.** *Let X be a real reflexive Banach space, and assume that $\mathcal{I} : X \to X^*$ is a bounded, pseudomonotone and coercive operator. Then, there exists a solution to the equation $\mathcal{I}(u) = v$, where $v \in X^*$.*

5.3 Degenerate parametric elliptic system near resonance

In this section, we investigate the following system given in (5.1.1) and we prove the existence of at least one weak solution

$$\begin{cases} -\Delta_{w_1,p_1}u - \mu_1\Delta_{w_1,q_1}u = \lambda a_1(x)|u|^{p_1-2}u + f_1(x,u,v,\nabla u,\nabla v) + h_1(x) & \text{in } \Omega,\\ -\Delta_{w_2,p_2}v - \mu_2\Delta_{w_2,q_2}v = \mu a_2(x)|v|^{p_2-2}v + f_2(x,u,v,\nabla u,\nabla v) + h_2(x) & \text{in } \Omega,\\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

For this purpose, for all $(\varphi_1,\varphi_2) \in X$, let

$$\begin{aligned} \langle \mathcal{J}_1(u,v),(\varphi_1,\varphi_2)\rangle := {} & \langle \tilde{\mathcal{J}}_1(u,v),(\varphi_1,\varphi_2)\rangle - \lambda\int_\Omega a_1(x)|u|^{p_1-2}u\varphi_1 dx \\ & - \mu\int_\Omega a_2(x)|v|^{p_2-2}v\varphi_2 dx - N_{f_1}(u,v)(\varphi_1,\varphi_2) \\ & - N_{f_2}(u,v)(\varphi_1,\varphi_2) - \langle (h_1,h_2),(\varphi_1,\varphi_2)\rangle, \end{aligned} \tag{5.3.1}$$

where

$$\begin{aligned} \langle \tilde{\mathcal{J}}_1(u,v),(\varphi_1,\varphi_2)\rangle = {} & \int_\Omega w_1(x)|\nabla u|^{p_1-2}\nabla u\nabla\varphi_1 + \mu_1 w_1(x)|\nabla u|^{q_1-2}\nabla u\nabla\varphi_1 dx \\ & + \int_\Omega w_2(x)|\nabla v|^{p_2-2}\nabla v\nabla\varphi_2 + \mu_2 w_2(x)|\nabla v|^{q_2-2}\nabla v\nabla\varphi_2 dx. \end{aligned} \tag{5.3.2}$$

Definition 5.3.1. A couple $(u,v) \in X$ is called a weak solution of the system (5.1.1) if

$$\langle \tilde{\mathcal{J}}_1(u,v),(\varphi_1,\varphi_2)\rangle = \lambda\int_\Omega a_1(x)|u|^{p_1-2}u\varphi_1 dx + \mu\int_\Omega a_2(x)|v|^{p_2-2}v\varphi_2 dx + \int_\Omega f(x,u,v,\nabla u,\nabla v)\varphi_1 + g(x,u,v,\nabla u,\nabla v)\varphi_2 dx + \langle (h_1,h_2),(\varphi_1,\varphi_2)\rangle,$$

for all $(\varphi_1,\varphi_2) \in X$, where $\langle (h_1,h_2),(\varphi_1,\varphi_2)\rangle := \langle h_1,\varphi_1\rangle + \langle h_2,\varphi_2\rangle$.

In order to prove the existence result, we need the following assumptions

(H_1) $f_i : \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are Carathéodory functions such that, for $i \in \{1,2\}$, there exist three positive continuous functions α_i^*, q_i^* and δ_i, and a nonnegative function $e_i \in L^{\frac{\gamma_i}{\gamma_i-1}}(\Omega)$ with $1 < \gamma_i < p^*_{i,s_i}$, satisfying the following conditions

$$0 < \alpha_i^* < p^*_{i,s_i} - 1, \quad 0 < \delta_i < \frac{q_i}{(q_i^*)'}, \quad \frac{\alpha_1^*}{\gamma_1} + \frac{\alpha_2^*}{\gamma_2} = \frac{\gamma_1 - 1}{\gamma_1},$$

and nonnegative constants l_i, b_i, such that for a.e. $x \in \Omega$ and all $s, t \in \mathbb{R}$, we have

$$|f_1(x,s,t,\xi,\eta)| \leq l_1 |s|^{\alpha_1^*} |t|^{\alpha_2^*} + b_1 \left(|\xi|^{\delta_1} + |\eta|^{\frac{\delta_1 q_2}{q_1}} \right) + e_1(x),$$

and

$$|f_2(x,s,t,\xi,\eta)| \leq l_2 |s|^{\alpha_1^*} |t|^{\alpha_2^*} + b_2 \left(|\xi|^{\frac{\delta_2 q_1}{q_2}} + |\eta|^{\delta_2} \right) + e_2(x).$$

(H_2) $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \exists C_\epsilon \in L^1(\Omega), \forall (s,t) \in \mathbb{R}^2$ with $\max\{|s|,|t|\} > \delta_\epsilon, \forall (\xi,\eta) \in \mathbb{R}^{2N}, \forall x \in \Omega$, we have

$$\begin{gathered} f_1(x,s,t,\xi,\eta)\, s + f_2(x,s,t,\xi,\eta)\, t \leq \\ \Gamma\left(a_1(x)|s|^{p_1} + a_2(x)|t|^{p_2}\right) + \Lambda\left(w_1(x)|\xi|^{q_1} + w_2(x)|\eta|^{q_2}\right) + C_\epsilon(x), \end{gathered}$$

where the constant Λ satisfies $0 < \Lambda < \min\{\mu_1, \mu_2\}$. On another note, using the embedding theorem, there exist $\theta_0, \theta_1 > 0$ such that

$$\begin{aligned} &(\min\{\mu_1,\mu_2\} - \Lambda)\left[\int_\Omega \left(w_1(x)|\nabla u|^{q_1} + w_2(x)|\nabla v|^{q_2}\right) dx\right] \\ &\geq \theta_0 \left[\int_\Omega \left(w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2}\right) dx\right] - \theta_1. \end{aligned} \tag{5.3.3}$$

(H_3) $\max\left\{(\lambda+\Gamma)(\lambda_1(p_1))^{-1}, (\mu+\Gamma)(\lambda_1(p_2))^{-1}\right\} < 1 + \theta_0.$

The main tool in this section is based on the surjectivity result for pseudomonotone operators from Carl et al. [24, Theorem 2.99], given in Theorem 5.2.4.

Lemma 5.3.2. *The operator $\tilde{\mathcal{J}}_1 : X \to X^\star$ defined in* (5.3.2)*, is bounded, continuous, monotone (hence maximal monotone), and of type* (S^+).

Proof. The proof bears resemblance to the one given by Fan and Zhang [43, Theorem 3.1] for a single equation when taking $\mu_i = 0$ for $i \in \{1,2\}$, and is omitted here. □

The following theorem is the subject of a fourth publication with my supervisor Professor El Hachimi in preparation.

Theorem 5.3.3. *Suppose that hypotheses (H_1) and (H_2) are verified. Then, for all $(h_1,h_2) \in X^\star$, the system* (5.1.1) *admits at least one weak solution $(u,v) \in X$.*

Proof. First, it is worth mentioning that both Nemytskij operators N_f and N_g are well-defined thanks to hypothesis (H_1). Now, and in order to apply Theorem 5.2.4, we need to show that the operator $\mathcal{J}_1$ associated with problem (5.1.2) is bounded, pseudomonotone and coercive.

• $\mathcal{J}_1$ ***is bounded:***

Thanks to the boundedness of $\tilde{\mathcal{J}}_1$ given in Lemma 5.3.2 and the growth conditions stated in (H_1), then $\mathcal{J}_1$ is bounded.

• $\mathcal{J}_1$ ***is pseudomonotone:***

Let $(u_n, v_n)_{n\in\mathbb{N}} \subset W_0^{1,p_1}(w_1, \Omega) \times W_0^{1,p_2}(w_2, \Omega)$ be a sequence such that

$$(u_n, v_n) \rightharpoonup (u, v) \text{ in } X, \tag{5.3.4}$$

and

$$\limsup_{n\to\infty} \langle \mathcal{J}_1(u_n, v_n), (u_n - u, v_n - v)\rangle \leq 0.$$

For $i \in \{1, 2\}$, taking the compact embeddings $W_0^{1,p_i}(w_i, \Omega) \hookrightarrow\hookrightarrow L^{r_i}(\Omega)$ into consideration gives

$$u_n \to u \text{ in } L^{r_1}(\Omega) \quad \text{ and } \quad v_n \to v \text{ in } L^{r_2}(\Omega). \tag{5.3.5}$$

since $r_i < p_{i,s_i}^*$. For $i = 1, 2$, we want to show that,

$$\lim_{n\to\infty} \int_\Omega f_i(x, u_n, v_n, \nabla u_n, \nabla v_n)(u_n - u)\, dx = 0, \tag{5.3.6}$$

Considering the first term of (5.3.6), and thanks to (H_1), we find

$$\begin{aligned}\int_\Omega f_1(x, u_n, v_n, \nabla u_n, \nabla v_n)(u_n - u)dx \leq & \int_\Omega l_1 |u_n|^{\alpha_1^*} |v_n|^{\alpha_2^*} |u_n - u| dx \\ & + \int_\Omega b_1 \left(|\nabla u_n|^{\delta_1} + |\nabla v_n|^{\delta_1 \frac{q_2}{q_1}} \right) |u_n - u| dx \\ & + \int_\Omega |e_1(x)||u_n - u| dx.\end{aligned} \tag{5.3.7}$$

Taking the first term of (5.3.7), and applying Hölder's inequality with exponents $x, y, z > 0$, where

$$x\alpha_1^* = \gamma_1, \quad y\alpha_2^* = \gamma_2, \quad z = \gamma_1 \quad \text{with} \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1,$$

we obtain

$$\int_\Omega l_1 |u_n|^{\alpha_1^*} |v_n|^{\alpha_2^*} |u_n - u| dx \leq l_1 \|u_n\|_{L^{\alpha_1^* x}(\Omega)}^{\alpha_1^*} \|v_n\|_{L^{\alpha_2^* y}(\Omega)}^{\alpha_2^*} \|u_n - u\|_{L^{\gamma_1}(\Omega)} \to 0.$$

As for the second term of (5.3.7), we apply Hölder's inequality once again to obtain

$$\int_\Omega b_1 \left(|\nabla u_n|^{\delta_1} + |\nabla v_n|^{\delta_1 \frac{q_2}{q_1}}\right) |u_n - u| dx \leq b_1 \Big(\|\nabla u_n\|_{L^{\frac{q_1}{\delta_1}}(\Omega)} \|u_n - u\|_{L^{\frac{q_1}{q_1-\delta_1}}(\Omega)} + \|\nabla v_n\|^{q_2}_{L^{\frac{q_1}{\delta_1}}(\Omega)} \|u_n - u\|_{L^{\frac{q_1}{q_1-\delta_1}}(\Omega)} \Big) \to 0.$$

Finally, for the last term of (5.3.7), similar techniques lead to

$$\int_\Omega |e_1(x)||u_n - u| dx \leq \|e_1\|_{L^{\gamma_1'}(\Omega)} \|u_n - u\|_{L^{\gamma_1}(\Omega)}.$$

Combining everything above gives

$$\lim_{n\to\infty} \int_\Omega f_1 \left(x, u_n, v_n, \nabla u_n, \nabla v_n\right) \left(u_n - u\right) dx = 0.$$

Similarly, applying the same arguments, we have

$$\lim_{n\to\infty} \int_\Omega f_2 \left(x, u_n, v_n, \nabla u_n, \nabla v_n\right) \left(v_n - u\right) dx = 0.$$

Taking the weak formulation in Definition 5.3.1, replacing u by u_n, v by v_n, φ by $u_n - u$ and φ_2 by $v_n - v$ and taking (5.3.4) and (5.3.6) into account lead to

$$\limsup_{n\to\infty} \left\langle \tilde{\mathcal{J}}_1 \left(u_n, v_n\right), \left(u_n - u, v_n - v\right) \right\rangle = \limsup_{n\to\infty} \left\langle \mathcal{J}_1 \left(u_n, v_n\right), \left(u_n - u, v_n - v\right) \right\rangle \leq 0. \tag{5.3.8}$$

Because $\tilde{\mathcal{J}}_1$ satisfies the (S^+)-property as seen in Lemma 5.3.2, and from (5.3.4) and (5.3.8), we obtain

$$\left(u_n, v_n\right) \to (u, v) \text{ in } X.$$

Ultimately, since $\mathcal{J}_1$ is continuous, we find $\mathcal{J}_1 \left(u_n, v_n\right) \to \mathcal{J}_1 \left(u, v\right)$ in X^*. Thus, $\mathcal{J}_1$ is pseudomonotone.

• $\mathcal{J}_1$ ***is coercive:***

Using hypothesis (H_2) and (5.3.3), we have

$$
\begin{aligned}
\langle \mathcal{J}_1(u,v),(u,v)\rangle =& \int_\Omega w_1(x)\left(|\nabla u|^{p_1-2}\nabla u + \mu_1|\nabla u|^{q_1-2}\nabla u\right)\nabla u dx\\
&+ \int_\Omega w_2(x)\left(|\nabla v|^{p_2-2}\nabla v + \mu_2|\nabla v|^{q_2-2}\nabla v\right)\nabla v dx\\
&- \lambda\int_\Omega a_1(x)|u|^{p_1}dx - \mu\int_\Omega a_2(x)|v|^{p_2}dx\\
&- \int_\Omega \left(f_1(x,u,v,\nabla u,\nabla v)u + f_2(x,u,v,\nabla u,\nabla v)v\right)dx\\
&- \langle (h_1,h_2),(u,v)\rangle.
\end{aligned}
$$

It follows

$$
\begin{aligned}
\langle \mathcal{J}_1(u,v),(u,v)\rangle \geq& \int_\Omega \left(w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2}\right)dx\\
&+ \left(\min\{\mu_1,\mu_2\} - \Lambda\right)\left[\int_\Omega \left(w_1(x)|\nabla u|^{q_1} + w_2(x)|\nabla v|^{q_2}\right)dx\right]\\
&- \lambda\int_\Omega a_1(x)|u|^{p_1}dx - \mu\int_\Omega a_2(x)|v|^{p_2}dx - \|(u,v)\|_X\|(h_1,h_2)\|_{X^\star}\\
&- \Gamma\left(\int_\Omega a_1(x)|u|^{p_1}dx + \int_\Omega a_2(x)|v|^{p_2}dx\right) - \int_\Omega C_\epsilon(x)dx\\
\geq& \left(1+\theta_0\right)\int_\Omega \left(w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2}\right)dx\\
&- \theta_1 - (\lambda+\Gamma)\left(\lambda_1(p_1)\right)^{-1}\int_\Omega w_1(x)|\nabla u|^{p_1}dx\\
&- (\mu+\Gamma)\left(\lambda_1(p_2)\right)^{-1}\int_\Omega w_2(x)|\nabla v|^{p_2}dx - \|C_\epsilon(x)\|_{L^1(\Omega)}\\
&- \|(u,v)\|_X\|(h_1,h_2)\|_{X^\star}\\
\geq& \left[1+\theta_0 - \max\left\{(\lambda+\Gamma)(\lambda_1(p_1))^{-1}, (\mu+\Gamma)(\lambda_1(p_2)))^{-1}\right\}\right]\cdot\\
&\int_\Omega \left(w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2}\right)dx - \theta_1 - \|C_\epsilon(x)\|_{L^1(\Omega)}\\
&- \|(u,v)\|_X\|(h_1,h_2)\|_{X^\star}\\
\geq& \left[1+\theta_0 - \max\left\{(\lambda+\Gamma)(\lambda_1(p_1))^{-1}, (\mu+\Gamma)(\lambda_1(p_2)))^{-1}\right\}\right]\cdot\\
&\left(\|u\|^{p_1}_{W_0^{1,p_1}(w_1,\Omega)} + \|v\|^{p_2}_{W_0^{1,p_2}(w_2,\Omega)}\right) - \|C_\epsilon(x)\|_{L^1(\Omega)} - \frac{\alpha_0}{\tau}\|(u,v)\|^\tau_X\\
&- \frac{1}{\alpha_0\tau'}\|(h_1,h_2)\|^{\tau'}_{X^\star} - \theta_1,
\end{aligned}
$$

where $0 < \alpha_0 < \tau < \min\{p_1,p_2\}$ is chosen as small as we want with $\tau' = \dfrac{\tau}{\tau-1}$.

Thus, we obtain

$$\langle \mathcal{J}_1(u,v),(u,v)\rangle \geq \Big[1+\theta_0 - \max\left\{(\lambda+\Gamma)(\lambda_1(p_1))^{-1},(\mu+\Gamma)(\lambda_1(p_2))^{-1}\right\} - \frac{\alpha_0}{\tau}\Big]\cdot$$
$$\|(u,v)\|_X^{\tau} - \|C_\epsilon(x)\|_{L^1(\Omega)} - \frac{1}{\alpha_0\tau'}\|(h_1,h_2)\|_{X^\star}^{\tau'} - \theta_1.$$

Using hypothesis (H_3) and choosing appropriate values for α_0 and τ such that

$$1+\theta_0 > \max\left\{(\lambda+\Gamma)(\lambda_1(p_1))^{-1},(\mu+\Gamma)(\lambda_1(p_2))^{-1}\right\} + \frac{\alpha_0}{\tau},$$

we obtain the coerciveness of $\mathcal{J}_1$.

Thus, all assumptions of Theorem 5.2.4 are verified. Therefore, there exists $(u,v)\in X$ such that $\mathcal{J}_1(u,v)=0$. Adhering to the definition of $\mathcal{J}_1$ in (5.3.1), it follows that (u,v) is a weak solution to system (5.1.1). This finishes the proof. □

5.4 Degenerate parametric elliptic system at resonance

In this section, we deal with the following system defined in (5.1.2) and we establish the existence of at least one weak solution

$$\begin{cases} -\Delta_{w_1,p_1}u - \mu_1\Delta_{w_1,q_1}u = \lambda_1 a_1(x)|u|^{p_1-2}u + \lambda_1 w(x)|u|^{\alpha_1-2}u|v|^{\alpha_2} & \\ \qquad\qquad + f_1(x,u,v,\nabla u,\nabla v) + h_1(x) & \text{in } \Omega,\\ -\Delta_{w_2,p_2}v - \mu_2\Delta_{w_2,q_2}v = \lambda_1 a_2(x)|v|^{p_2-2}v + \lambda_1 w(x)|v|^{\alpha_2-2}v|u|^{\alpha_1} & \\ \qquad\qquad + f_2(x,u,v,\nabla u,\nabla v) + h_2(x) & \text{in } \Omega,\\ u=v=0 & \text{on } \partial\Omega. \end{cases}$$

To this end, for all $(\varphi_1,\varphi_2)\in X$, we set forth

$$\begin{aligned}\langle \mathcal{J}_2(u,v),(\varphi_1,\varphi_2)\rangle := {}& \langle \tilde{\mathcal{J}}_2(u,v),(\varphi_1,\varphi_2)\rangle - \lambda_1\int_\Omega a_1(x)|u|^{p_1-2}u\varphi_1 dx\\ & - \lambda_1\int_\Omega a_2(x)|v|^{p_2-2}v\varphi_2 dx - \lambda_1\langle \mathcal{A}(u,v),(\varphi_1,\varphi_2)\rangle\\ & - N_{f_1}(u,v)(\varphi_1,\varphi_2) - N_{f_2}(u,v)(\varphi_1,\varphi_2) - \langle (h_1,h_2),(\varphi_1,\varphi_2)\rangle,\end{aligned} \tag{5.4.1}$$

where

$$\begin{aligned}\langle \tilde{\mathcal{J}}_2(u,v),(\varphi_1,\varphi_2)\rangle = {}& \int_\Omega w_1(x)|\nabla u|^{p_1-2}\nabla u\nabla\varphi_1 + \mu_1 w_1(x)|\nabla u|^{q_1-2}\nabla u\nabla\varphi_1 dx\\ & + \int_\Omega w_2(x)|\nabla v|^{p_2-2}\nabla v\nabla\varphi_2 + \mu_2 w_2(x)|\nabla v|^{q_2-2}\nabla v\nabla\varphi_2 dx,\end{aligned} \tag{5.4.2}$$

and

$$\langle \mathcal{A}(u,v),(\varphi_1,\varphi_2)\rangle = \int_\Omega w(x)\left(|u|^{\alpha_1-2}u|v|^{\alpha_2}\varphi_1 + |v|^{\alpha_2-2}v|u|^{\alpha_1}\varphi_2\right)dx.$$

Definition 5.4.1. A couple $(u,v) \in X$ is called a weak solution of the system (5.1.2) if

$$\begin{aligned}\langle \tilde{\mathcal{J}}_2(u,v),(\varphi_1,\varphi_2)\rangle &= \lambda_1 \int_\Omega a_1(x)|u|^{p_1-2}u\varphi_1 dx + \lambda_1 \int_\Omega a_2(x)|v|^{p_2-2}v\varphi_2 dx \\ &+ \lambda_1\langle \mathcal{A}(u,v),(\varphi_1,\varphi_2)\rangle + N_{f_1}(u,v)(\varphi_1,\varphi_2) + N_{f_2}(u,v)(\varphi_1,\varphi_2) \\ &+ \langle h_1,\varphi_1\rangle + \langle h_2,\varphi_2\rangle,\end{aligned}$$

for all $(\varphi_1,\varphi_2) \in X$, where $\langle (h_1,h_2),(\varphi_1,\varphi_2)\rangle := \langle h_1,\varphi_1\rangle + \langle h_2,\varphi_2\rangle$.

In order to prove the existence result, we need the following assumptions

(H_1') $f_1, f_2 : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions such that, for $i \in \{1,2\}$, there exist three positive continuous functions α_i^*, q_i^* and δ_i, and a nonnegative function $e_i \in L^{\frac{\gamma_i}{\gamma_i-1}}(\Omega)$ with $1 < \gamma_i < p_{i,s_i}^*$, satisfying the following conditions

$$0 < \alpha_i^* < p_{i,s_i}^* - 1, \quad 0 < \delta_i < \frac{q_i}{(q_i^*)'}, \quad \frac{\alpha_1^*}{\gamma_1} + \frac{\alpha_2^*}{\gamma_2} = \frac{\gamma_1 - 1}{\gamma_1},$$

and nonnegative constants l_i, b_i, such that for a.e. $x \in \Omega$ and all $s, t \in \mathbb{R}$, we have

$$|f_1(x,s,t,\xi,\eta)| \le l_1 |s|^{\alpha_1^*} |t|^{\alpha_2^*} + b_1\left(|\xi|^{\delta_1} + |\eta|^{\frac{\delta_1 q_2}{q_1}}\right) + e_1(x),$$

and

$$|f_2(x,s,t,\xi,\eta)| \le l_2 |s|^{\alpha_1^*} |t|^{\alpha_2^*} + b_2\left(|\xi|^{\frac{\delta_2 q_1}{q_2}} + |\eta|^{\delta_2}\right) + e_2(x).$$

(H_2') $\forall \epsilon > 0, \exists \delta_\epsilon > 0, \exists C_\epsilon \in L^1(\Omega), \forall (s,t) \in \mathbb{R}^2$ with $\max\{|s|,|t|\} > \delta_\epsilon, \forall (\xi,\eta) \in \mathbb{R}^{2N}, \forall x \in \Omega$, we have

$$\begin{gathered} f_1(x,s,t,\xi,\eta)\, s + f_2(x,s,t,\xi,\eta)\, t \le \\ (\lambda_2(p) + \epsilon)\, w(x)|s|^{\alpha_1}|t|^{\alpha_2} + c\left(w_1(x)\,|\xi|^{q_1} + w_2(x)\,|\eta|^{q_2}\right) + C_\epsilon(x),\end{gathered}$$

where the constant c satisfies $0 < c < \min\{\mu_1, \mu_2\}$.

Remark 5.4.2. The hypothesis (H_2') is fulfilled, for instance, when the following condition holds

$$\limsup_{|(s,t)|\to\infty} \frac{f(x,s,t,\xi,\eta)s + g(x,s,t,\xi,\eta)t - c\left(w_1(x)|\xi|^{q_1} + w_2(x)|\eta|^{q_2}\right)}{w(x)|s|^{\alpha_1}|t|^{\alpha_2}} < \lambda_2(p).$$

uniformly with respect to x, ξ, η.

The main tool in this section is based on the surjectivity result for pseudomonotone operators from Carl et al. [24, Theorem 2.99], given in Theorem 5.2.4.

Lemma 5.4.3. *The operator $\tilde{\mathcal{J}}_2 : X \to X^\star$ defined in* (5.4.2), *is bounded, continuous, monotone (hence maximal monotone), and of type* (S^+).

Proof. The proof bears resemblance to the one given by Fan and Zhang [43, Theorem 3.1] for a single equation when taking $\mu_i = 0$ for $i \in \{1, 2\}$, and is omitted here. □

The following theorem is the subject of a fourth publication with my supervisor Professor El Hachimi in preparation.

Theorem 5.4.4. *Suppose that hypotheses (H_1') and (H_2') are verified. Then, for all $(h_1, h_2) \in X^\star$, the system* (5.1.2) *admits at least one weak solution $(u, v) \in X$.*

Proof. In order to apply Theorem 5.2.4, we need to show that the operator $\mathcal{J}_2$ associated with problem (5.1.2) is bounded, pseudomonotone and coercive. The proof of the boundedness and pseudomonotonicity of $\mathcal{J}_2$ is identical to the one given in the proof of Theorem 5.3.3. It remains to prove that $\mathcal{J}_2$ is coercive.

First, using the embedding theorem, there exist $\theta_0', \theta_1' > 0$ such that

$$\begin{aligned} &(\min\{\mu_1, \mu_2\} - c)\left[\int_\Omega \left(w_1(x)|\nabla u|^{q_1} + w_2(x)|\nabla v|^{q_2}\right) dx\right] \\ &\geq \theta_0' \left[\int_\Omega \left(w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2}\right) dx\right] - \theta_1'. \end{aligned} \tag{5.4.3}$$

Therefore, using hypothesis (H_2') and (5.4.3), we find

$$
\begin{aligned}
\langle \mathcal{J}_2(u,v),(u,v)\rangle &= \int_\Omega w_1(x)\left(|\nabla u|^{p_1}+\mu_1|\nabla u|^{q_1}\right)+w_2(x)\left(|\nabla v|^{p_2}+\mu_2|\nabla v|^{q_2}\right)dx \\
&- \lambda_1\int_\Omega w(x)|u|^{\alpha_1}|v|^{\alpha_2}dx-\int_\Omega \Big(f_1(x,u,v,\nabla u,\nabla v)u \\
&+ f_2(x,u,v,\nabla u,\nabla v)v\Big)dx-\langle h_1,u\rangle-\langle h_2,v\rangle \\
&\geq \int_\Omega \left(w_1(x)|\nabla u|^{p_1}+w_2(x)|\nabla v|^{p_2}\right)dx+\left(\min\{\mu_1,\mu_2\}-c\right)\cdot \\
&\left[\int_\Omega \left(w_1(x)|\nabla u|^{q_1}+w_2(x)|\nabla v|^{q_2}\right)dx\right]-\lambda_1\int_\Omega w(x)|u|^{\alpha_1}|v|^{\alpha_2}dx \\
&- (\lambda_2+\epsilon)\int_\Omega w(x)|u|^{\alpha_1}|v|^{\alpha_2}dx-\int_\Omega C_\epsilon(x)dx \\
&- \|(u,v)\|_X\|(h_1,h_2)\|_{X^\star} \\
&\geq (1+\theta_0')\int_\Omega \left(w_1(x)|\nabla u|^{p_1}+w_2(x)|\nabla v|^{p_2}\right)dx-\theta_1' \\
&- (\lambda_1+\lambda_2+\epsilon)\int_\Omega w(x)|u|^{\alpha_1}|v|^{\alpha_2}dx \\
&- \|C_\epsilon(x)\|_{L^1(\Omega)}-\|(u,v)\|_X\|(h_1,h_2)\|_{X^\star},
\end{aligned}
$$

which implies

$$
\begin{aligned}
\langle \mathcal{J}_2(u,v),(u,v)\rangle &\geq (1+\theta_0')\int_\Omega \left(w_1(x)|\nabla u|^{p_1}+w_2(x)|\nabla v|^{p_2}\right)dx-\theta_1'-\|C_\epsilon(x)\|_{L^1(\Omega)} \\
&- \|(u,v)\|_X\|(h_1,h_2)\|_{X^\star}-\frac{(\lambda_1+\lambda_2+\epsilon)}{\lambda_1}\Big(\int_\Omega \frac{\alpha_1 w_1(x)}{p_1}|\nabla u|^{p_1} \\
&+ \frac{\alpha_2 w_2(x)}{p_2}|\nabla v|^{p_2}dx\Big) \\
&\geq \left[1+\theta_0'-\frac{(\lambda_1+\lambda_2+\epsilon)\max\{\alpha_1,\alpha_2\}}{\lambda_1\min\{p_1,p_2\}}\right]\Big(\|u\|^{p_1}_{W_0^{1,p_1}(w_1,\Omega)} \\
&+ \|v\|^{p_2}_{W_0^{1,p_2}(w_2,\Omega)}\Big)-\theta_1'-\|C_\epsilon(x)\|_{L^1(\Omega)}-\frac{\alpha_0}{\tau}\|(u,v)\|_X^\tau \\
&- \frac{1}{\alpha_0\tau'}\|(h_1,h_2)\|^{\tau'}_{X^\star},
\end{aligned}
$$

where $0<\alpha_0<\tau<\min\{p_1,p_2\}$ is chosen as small as we want with $\tau'=\dfrac{\tau}{\tau-1}$. Thus, we obtain

$$
\begin{aligned}
\langle \mathcal{J}_2(u,v),(u,v)\rangle \geq &\left[1+\theta_0'-\frac{(\lambda_1+\lambda_2+\epsilon)\max\{\alpha_1,\alpha_2\}}{\lambda_1\min\{p_1,p_2\}}-\frac{\alpha_0}{\tau}\right]\|(u,v)\|_X^\tau-\theta_1' \\
&- \|C_\epsilon(x)\|_{L^1(\Omega)}-\frac{1}{\alpha_0\tau'}\|(h_1,h_2)\|^{\tau'}_{X^\star}.
\end{aligned}
$$

Choosing appropriate values for ϵ and α_0 such that

$$1+\theta_0' > \frac{(\lambda_1+\lambda_2+\epsilon)\max\{\alpha_1,\alpha_2\}}{\lambda_1\min\{p_1,p_2\}} + \frac{\alpha_0}{\tau},$$

with the conditions

$$\lambda_2 < \lambda_1\left[\frac{\min\{p_1,p_2\}}{\max\{\alpha_1,\alpha_2\}}\left(1+\theta_0'-\frac{\alpha_0}{\tau}\right)-1\right] \quad \text{and}$$

$$\frac{\min\{p_1,p_2\}}{\max\{\alpha_1,\alpha_2\}\max\{p_1,p_2\}} > 1,$$

leads to the coerciveness of $\mathcal{J}_2$.

Thus, all assumptions of Theorem 5.2.4 are verified. Therefore, there exists $(u,v)\in X$ such that $\mathcal{J}_2(u,v)=0$. Adhering to the definition of $\mathcal{J}_2$ in (5.4.1), it follows that (u,v) is a weak solution to system (5.1.2). This finishes the proof. □

5.5 Case of a gradient-type system

In this section, we focus our study on a gradient-type system. In other words, we consider the following system

$$\begin{cases} -\Delta_{w_1,p_1}u-\mu_1\Delta_{w_1,q_1}u=\lambda_1 w(x)|u|^{\alpha_1-2}u|v|^{\alpha_2}+\dfrac{\partial F}{\partial u}(x,u,v)+h_1(x) & \text{in } \Omega,\\ -\Delta_{w_2,p_2}v-\mu_2\Delta_{w_2,q_2}v=\lambda_1 w(x)|v|^{\alpha_2-2}v|u|^{\alpha_1}+\dfrac{\partial F}{\partial v}(x,u,v)+h_2(x) & \text{in } \Omega,\\ u=u=0 & \text{on } \partial\Omega. \end{cases} \tag{5.5.1}$$

Similarly, in order to prove the existence of at least one weak solution, we need the following assumptions on F

(F_1) $\dfrac{\partial F}{\partial s}$ and $\dfrac{\partial F}{\partial t}:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions such that there exist positive continuous functions α_i^*, nonnegative functions $e_i\in L^1(\Omega)$ verifying $1<\alpha_i^*<p_{i,s_i}^*$, and nonnegative constants a_i for $i\in\{1,2\}$, such that for all $(s,t)\in\mathbb{R}^2$, we have

$$\left|\frac{\partial F}{\partial s}(x,s,t)\right| \le a_1 w(x)\,|s|^{\alpha_1^*-1}\,|t|^{\alpha_2^*}+e_1(x),$$

and

$$\left|\frac{\partial F}{\partial t}(x,s,t)\right| \le a_2 w(x)\,|s|^{\alpha_1^*}\,|t|^{\alpha_2^*-1}+e_2(x).$$

(F_2) There exist a nonnegative constant C_1 and a nonnegative function $C_2 \in L^1(\Omega)$ such that, for all $x \in \Omega$ we have

$$|F(x,s,t)| \leq C_1 w(x)|s|^{\alpha_1}|t|^{\alpha_2} + C_2(x).$$

(F_3) $F_\infty(x) \equiv \limsup\limits_{|(s,t)|\to+\infty} \dfrac{F(x,s,t)}{w(x)|s|^{\alpha_1}|t|^{\alpha_2}} < \lambda_2$ uniformly for a.e $x \in \Omega$.

Define

$$\mathcal{J}_2(u,v) = \tilde{\mathcal{J}}_2(u,v) - \lambda_1 \mathcal{B}(u,v) - \int_\Omega F(x,u,v)dx - \langle h_1, u\rangle - \langle h_2, v\rangle,$$

where

$$\begin{aligned}\tilde{\mathcal{J}}_2(u,v) = &\int_\Omega \left(\frac{w_1(x)}{p_1}|\nabla u|^{p_1} + \frac{\mu_1 w_1(x)}{q_1}|\nabla u|^{q_1}\right)dx \\ &+ \int_\Omega \left(\frac{w_2(x)}{p_2}|\nabla v|^{p_2} + \frac{\mu_2 w_2(x)}{q_2}|\nabla v|^{q_2}\right)dx,\end{aligned}$$

and

$$\mathcal{B}(u,v) = \int_\Omega w(x)|u|^{\alpha_1}|v|^{\alpha_2}dx.$$

Next, we state and prove the existence result for the system (5.5.1) at resonance. **The following theorem is the subject of a fourth publication with my supervisor Professor El Hachimi in preparation.**

Theorem 5.5.1. *Suppose that hypotheses* $(F_1) - (F_3)$ *are verified. Then, the system* (5.5.1) *admits at least one weak solution.*

Proof. In order to obtain the desired result, we need to minimize the energy functional associated with the system (5.5.1), that is,

$$\mathcal{J}_2(u,v) = \tilde{\mathcal{J}}_2(u,v) - \lambda_1 \mathcal{B}(u,v) - \int_\Omega F(x,u,v)dx - \langle h_1, u\rangle - \langle h_2, v\rangle.$$

To this end, we show that $\mathcal{J}_2$ is weakly lower semicontinuous and coercive.

• ***$\mathcal{J}_2$ is weakly lower semicontinuous:***
Let $(u_n, v_n)_{n\in\mathbb{N}}$ be a sequence such that

$$(u_n, v_n) \rightharpoonup (u,v) \text{ in } X.$$

Taking the compact embedding $W_0^{1,p_i}(w_i,\Omega) \hookrightarrow\hookrightarrow L^{r_i}(\Omega)$ into consideration, for $i \in \{1,2\}$, we obtain

$$(u_n, v_n) \to (u,v) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega),$$

since $r_i \in \left[1, p^*_{i,s_i}\right]$, for $i \in \{1,2\}$. In addition, there exist $\theta_1 \in L^{r_1}(\Omega)$ and $\theta_2 \in L^{r_2}(\Omega)$ such that

$$\begin{cases} |u_n(x)| \leq \theta_1(x), \\ |v_n(x)| \leq \theta_2(x), \end{cases}$$

for a.e. $x \in \Omega$. Hence, using hypothesis (F_2), we find

$$\begin{aligned} F(x, u_n(x), v_n(x)) &\leq C_1 w(x)|u_n|^{\alpha_1}|v_n|^{\alpha_2} + C_2(x) \\ &\leq C_1 w(x)|\theta_1|^{\alpha_1}|\theta_2|^{\alpha_2} + C_2(x). \end{aligned}$$

Using the fact that $F(x, u_n(x), v_n(x)) \to F(x, u(x), v(x))$ for a.e. $x \in \Omega$, and applying Fatou's lemma, we obtain

$$\limsup_{n\to+\infty} \int_\Omega F(x, u_n(x), v_n(x))dx \leq \int_\Omega F(x, u(x), v(x))dx.$$

Thus, we derive

$$\begin{aligned} \mathcal{J}_2(u,v) &\leq \liminf_{n\to+\infty} \tilde{\mathcal{J}}_2(u_n, v_n) - \lambda_1 \liminf_{n\to+\infty} \mathcal{B}(u_n, v_n) - \limsup_{n\to+\infty} \int_\Omega F(x, u_n, v_n)dx \\ &\quad - \lim_{n\to+\infty} \langle h_1, u_n\rangle - \lim_{n\to+\infty} \langle h_2, v_n\rangle \\ &\leq \liminf_{n\to+\infty} \Bigg\{ \tilde{\mathcal{J}}_2(u_n, v_n) - \lambda_1 \mathcal{B}(u_n, v_n) - \int_\Omega F(x, u_n, v_n)dx \\ &\quad - \langle h_1, u_n\rangle - \langle h_2, v_n\rangle \Bigg\} \\ &\leq \liminf_{n\to+\infty} \mathcal{J}_2(u_n, v_n). \end{aligned}$$

Therefore, $\mathcal{J}_2$ is weakly lower semicontinuous.

- ***$\mathcal{J}_2$ is coercive:***

Using hypotheses (F_3), and for all $\epsilon > 0$, there exist $\delta_\epsilon > 0$ and $C_\epsilon \in L^1(\Omega)$ such that

$$F(x, u, v) \leq (F_\infty(x) + \epsilon)w(x)|u|^{\alpha_1}|v|^{\alpha_2} + C_\epsilon(x),$$

for every $(u, v) \in \mathbb{R}^2$ satisfying $|(u, v)|_\infty > \delta_\epsilon$, for a.e. $x \in \Omega$. On the flip side, applying the embedding theorem, there exist $\eta_0, \eta_1 > 0$ such that

$$\begin{aligned} &\min\{\mu_1, \mu_2\} \left(\int_\Omega \frac{w_1(x)}{q_1}|\nabla u|^{q_1} + \frac{w_2(x)}{q_2}|\nabla v|^{q_2} dx \right) \\ &\geq \eta_0 \left(\int_\Omega \frac{w_1(x)}{p_1}|\nabla u|^{p_1} + \frac{w_2(x)}{p_2}|\nabla v|^{p_2} dx \right) - \eta_1. \end{aligned} \tag{5.5.2}$$

Therefore, using (5.5.2) we find

$$
\begin{aligned}
\mathcal{J}_2(u,v) &= \tilde{\mathcal{J}}_2(u,v) - \lambda_1\mathcal{B}(u,v) - \int_\Omega F(x,u,v)dx - \langle h_1, u\rangle - \langle h_2, v\rangle \\
&\geq \int_\Omega \frac{w_1(x)}{p_1}|\nabla u|^{p_1} + \frac{\mu_1 w_1(x)}{q_1}|\nabla u|^{q_1}dx + \int_\Omega \frac{w_2(x)}{p_2}|\nabla v|^{p_2} \\
&+ \frac{\mu_2 w_2(x)}{q_2}|\nabla v|^{q_2}dx - \int_\Omega C_\epsilon(x)dx - \lambda_1\int_\Omega w(x)|u|^{\alpha_1}|v|^{\alpha_2}dx \\
&- \int_\Omega (\lambda_2+\epsilon)w(x)|u|^{\alpha_1}|v|^{\alpha_2}dx - \|(u,v)\|_X\|(h_1,h_2)\|_{X^\star}.
\end{aligned}
$$

Using the definition of the first eigenvalue in (5.1.4), we find

$$
\begin{aligned}
\mathcal{J}_2(u,v) &\geq \frac{(1+\eta_0)}{\max\{p_1,p_2\}}\left(\int_\Omega w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2}dx\right) - \|C_\epsilon(x)\|_{L^1(\Omega)} \\
&- \left(\frac{\lambda_1+\lambda_2+\epsilon}{\lambda_1}\right)\left(\int_\Omega \frac{\alpha_1 w_1(x)}{p_1}|\nabla u|^{p_1} + \frac{\alpha_2 w_2(x)}{p_2}|\nabla v|^{p_2}dx\right) \\
&- \|(u,v)\|_X\|(h_1,h_2)\|_{X^\star} - \eta_1 \\
&\geq \left[\frac{(1+\eta_0)}{\max\{p_1,p_2\}} - \left(\frac{\lambda_1+\lambda_2+\epsilon}{\lambda_1}\right)\frac{\max\{\alpha_1,\alpha_2\}}{\min\{p_1,p_2\}}\right]\left(\int_\Omega w_1(x)|\nabla u|^{p_1}\right. \\
&\left. + w_2(x)|\nabla v|^{p_2}dx\right) - \|C_\epsilon(x)\|_{L^1(\Omega)} - \frac{\beta_0}{\rho}\|(u,v)\|_X^\rho - \frac{1}{\beta_0\rho'}\|(h_1,h_2)\|_{X^\star}^{\rho'} - \eta_1 \\
&\geq \left[\frac{(1+\eta_0)}{\max\{p_1,p_2\}} - \left(\frac{\lambda_1+\lambda_2+\epsilon}{\lambda_1}\right)\frac{\max\{\alpha_1,\alpha_2\}}{\min\{p_1,p_2\}}\right]\left(\|u\|^{p_1}_{W_0^{1,p_1}(w_1,\Omega)}\right. \\
&\left. + \|v\|^{p_2}_{W_0^{1,p_2}(w_2,\Omega)}\right) - \|C_\epsilon(x)\|_{L^1(\Omega)} - \frac{\beta_0}{\rho}\|(u,v)\|_X^\rho - \frac{1}{\beta_0\rho'}\|(h_1,h_2)\|_{X^\star}^{\rho'} - \eta_1,
\end{aligned}
$$

where $0 < \beta_0 < \rho < \min\{p_1,p_2\}$ is chosen as small as we want with $\rho' = \dfrac{\rho}{\rho-1}$. Thus, we obtain

$$
\mathcal{J}_2(u,v) \geq \left[\frac{(1+\eta_0)}{\max\{p_1,p_2\}} - \left(\frac{\lambda_1+\lambda_2+\epsilon}{\lambda_1}\right)\frac{\max\{\alpha_1,\alpha_2\}}{\min\{p_1,p_2\}} - \frac{\beta_0}{\rho}\right]\|(u,v)\|_X^\rho - \\
\|C_\epsilon(x)\|_{L^1(\Omega)} - \frac{1}{\beta_0\rho'}\|(h_1,h_2)\|_{X^\star}^{\rho'} - \eta_1.
$$

Choosing appropriate values for ϵ and β_0 such that

$$
\frac{(1+\eta_0)}{\max\{p_1,p_2\}} > \left(\frac{\lambda_1+\lambda_2+\epsilon}{\lambda_1}\right)\frac{\max\{\alpha_1,\alpha_2\}}{\min\{p_1,p_2\}} + \frac{\beta_0}{\rho},
$$

with the conditions

$$\lambda_2 < \lambda_1 \left[\frac{\min\{p_1, p_2\}}{\max\{\alpha_1, \alpha_2\}} \left(\frac{1+\eta_0}{\max\{p_1, p_2\}} - \frac{\beta_0}{\rho} \right) - 1 \right] \quad \text{and}$$
$$\frac{\min\{p_1, p_2\}}{\max\{\alpha_1, \alpha_2\}\max\{p_1, p_2\}} > 1,$$
leads to the coerciveness of $\mathcal{J}_2$.

Hence, $\mathcal{J}_2$ being weakly lower semicontinuous and coercive, then $\mathcal{J}_2$ admits a minimum point (u, v) in X. Thanks to hypothesis (F_1), $\mathcal{J}_2$ is a C^1-functional. Furthermore, $(u, v) \in X$ is a weak solution for the system (5.5.1) if and only if (u, v) is a critical point. This completes the proof. □

5.6 Concluding examples

Example 5.6.1. Let the functions f_1 and f_2 be defined by
$$f_1\,(x, s, t, \xi, \eta) = |s|^{\alpha_1 - 2}\, s\, |t|^{\alpha_2} \left(a_1(x) + \frac{d_1(x)}{1 + |s|^{\alpha_1}|t|^{\alpha_2}}\, |\xi|^{q_1} \right),$$
and
$$f_2\,(x, s, t, \xi, \eta) = |s|^{\alpha_1}\, |t|^{\alpha_2 - 2}\, t \left(a_2(x) + \frac{d_2(x)}{1 + |s|^{\alpha_1}|t|^{\alpha_2}}\, |\eta|^{q_2} \right),$$
where α_1 and α_2 are defined in Section 5.1, and $a_1, a_2, d_1, d_2 \in L^\infty(\Omega)$ with the condition $\|a_1 + a_2\|_{L^\infty(\Omega)} < \lambda_2$ satisfied. We have
$$\begin{aligned} |f_1(x, s, t, \xi, \eta)| &\le \|a_1\|_{L^\infty(\Omega)}|s|^{\alpha_1 - 1}|t|^{\alpha_2} + \frac{|s|^{\alpha_1 - 1}|t|^{\alpha_2}\|d_1(x)\|_{L^\infty(\Omega)}|\xi|^{q_1}}{1 + |s|^{\alpha_1}|t|^{\alpha_2}} \\ &\le \|a_1\|_{L^\infty(\Omega)}|s|^{\alpha_1 - 1}|t|^{\alpha_2} + \|d_1\|_{L^\infty(\Omega)}|\xi|^{q_1}. \end{aligned}$$
Similarly for f_2, we obtain
$$|f_2(x, s, t, \xi, \eta)| \le \|a_2\|_{L^\infty(\Omega)}|s|^{\alpha_1}|t|^{\alpha_2 - 1} + \|d_2\|_{L^\infty(\Omega)}|\eta|^{q_2},$$
which implies that hypothesis (H_1') is verified for $\alpha_1^* = \alpha_1 - 1$, $\alpha_2^* = \alpha_2$, $\delta_1 = q_1$ and $\delta_2 = q_2$. Furthermore, we have
$$\begin{aligned} f_1(x, s, t, \xi, \eta)s + f_2(x, s, t, \xi, \eta)t &= (a_1(x) + a_2(x))\, |s|^{\alpha_1}|t|^{\alpha_2} \\ &\quad + \frac{|s|^{\alpha_1}|t|^{\alpha_2}}{1 + |s|^{\alpha_1}|t|^{\alpha_2}}\, (d_1(x)\, |\xi|^{q_1} + d_2(x)\, |\eta|^{q_2}) \\ &\le \lambda_2(p)|s|^{\alpha_1}|t|^{\alpha_2} + \|d_1\|_{L^\infty(\Omega)}|\xi|^{q_1} + \|d_2\|_{L^\infty(\Omega)}|\eta|^{q_2}, \end{aligned}$$
which implies that f_1 and f_2 satisfy hypothesis (H_2') and hence all hypotheses of Theorem 5.4.4 are satisfied.

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Résumé

L'objectif de cette thèse est double : La première partie est consacrée à l'étude de certains problèmes elliptiques anisotropes. Plus précisément, nous examinons deux problèmes elliptiques anisotropes paramétriques avec des conditions aux limites de Dirichlet et de Robin, dans le but d'établir l'existence et la multiplicité de solutions. Un défi majeur se présente lorsque l'on traite de l'existence de solutions, notamment lorsque les non-linéarités dépendent du gradient. Cela implique que des méthodes variationnelles telles que le théorème du Mountain Pass ne peuvent pas être utilisées. Pour surmonter cet obstacle, nous utilisons une méthode topologique basée sur la subjectivité des opérateurs pseudomonotones, comme démontré par Carl et al. (*Nonsmooth Variational Problems and Their Inequalities, Springer, 2007*). De plus, nous établissons l'unicité de la solution. De plus, nous relâchons la dépendance au gradient des non-linéarités et prouvons un résultat de multiplicité pour trois solutions faibles en utilisant le principe variationnel de Ricceri. Dans la deuxième partie, nous étudions les problèmes de résonance et de non-résonance pour certains systèmes elliptiques paramétriques. Dans les deux cas, les informations disponibles concernant le spectre de Lusternik-Schnirelmann du problème de valeurs propres associé sont limitées. Dans le cas de la résonance, nous nous concentrons sur un système elliptique paramétrique où les non-linéarités impliquent une convection et la convolution de la solution. À l'inverse, dans le cas de la non-résonance, nous examinons la solvabilité à gauche de l'infimum positif de toutes les valeurs propres pour certains problèmes elliptiques quasi-linéaires non-résonants avec des exposants variables. Nous établissons l'existence de la solution en utilisant la même méthode, à savoir la surjectivité des opérateurs pseudomonotones, l'unicité étant prouvée à l'aide d'arguments similaires. Enfin, nous considérons des systèmes de type gradient et prouvons l'existence d'une solution en utilisant une approche variationnelle.

Mots-clefs (5) : Équations elliptiques anisotropes, espaces de Sobolev à exposant variable, existence et multiplicité, problèmes de résonance et de non-résonance, spectre de Lusternik-Schnirelmann.

Abstract

The aim of this thesis is twofold: The first part is devoted to the study of some anisotropic elliptic problems. Specifically, we investigate two parametric anisotropic elliptic problems with Dirichlet and Robin boundary conditions, aiming to establish the existence and multiplicity of solutions. One major challenge arises when dealing with the existence of solutions, particularly when the nonlinearities are gradient-dependent. This implies that variational methods such as the Mountain Pass theorem cannot be employed. To overcome this obstacle, we utilize a topological method based on the subjectivity of pseudomonotone operators, as demonstrated by Carl et al. (*Nonsmooth Variational Problems and Their Inequalities, Springer, 2007*). Moreover, we establish the uniqueness of the solution. Additionally, we relax the gradient dependence of the nonlinearities and prove a multiplicity result for three weak solutions using Ricceri's variational principle. In the second part, we investigate resonance and non-resonance issues for some parametric elliptic systems. In both cases, there is limited available information concerning the Lusternik-Schnirelmann spectrum of the associated eigenvalue problem. In the resonance case, we focus on a parametric elliptic system where the nonlinearities involve convection and the convolution of the solution. Conversely, in the non-resonance case, we investigate the solvability on the left side of the positive infimum of all eigenvalues for specific quasilinear elliptic problems with variable exponents, ensuring they are non-resonant. We establish the existence of the solution by employing the same method, namely the surjectivity of pseudomonotone operators, with uniqueness being proven using similar arguments. Finally, we consider gradient-type systems and prove the existence of a solution using a variational approach.

Keywords (5) : Anisotropic elliptic equations, variable exponent Sobolev spaces, existence and multiplicity, resonance and non-resonance issues, Lusternik-Schnirelmann spectrum.