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*“Second Bargmann, fractional Hankel and Cauchy transforms  
on slice hyperholomorphic spaces.”*

## JURY

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# Motivation and introduction

The classical Segal-Bargmann transform  $\mathcal{B}$  is well-known in the literature [17, 47, 107]. It was introduced in Bargmann's seminal paper [17]. And it made the quantum mechanical configuration space  $L^2(\mathbb{R}, \mathbb{C})$  unitarily isomorphic to the phase space of all holomorphic  $\mathbb{C}$ -valued functions on the complex plane that are  $e^{-\nu|z|^2} dx dy$ -square integrable.

The same Bargmann's paper reveals an elegant way to define power of Fourier transform, known nowadays as Fractional Fourier transform (FrFT). In fact, associated to the classical Segal-Bargmann transform  $\mathcal{B}$ , mapping  $L^2(\mathbb{R}, \mathbb{C})$  onto the Fock-Bargmann space, one considers  $\mathcal{R}_\theta := \mathcal{B}^{-1} \circ T_\theta \circ \mathcal{B}$ , with  $T_\theta f(z) := f(\theta z)$ , which defines a unitary homeomorphism transform on  $L^2(\mathbb{C})$  when  $\theta = e^{i\alpha}$ ;  $\alpha \in \mathbb{R}$ , and satisfies the eigenfunction problem

$$\mathcal{R}_{e^{i\alpha}} h_n(x) = e^{in\alpha} h_n(x).$$

For details, see also [19, 56, 96]. Such fractional Fourier transform (FrFT), which is special generalization of the Fourier integral transform, is a powerful tool in many fields of research including mathematics, physics and engineering sciences [5, 86, 73]. Its introduction goes back to 1929 when was considered implicitly in Wiener's work [99], discussing the extension of certain results of H. Weyl and leading later to Fourier developments of fractional order. Mainly, Wiener sets out to find a one-parameter family of unitary integral operators

$$\mathcal{F}_\alpha \varphi(x) := \int_{-\infty}^{+\infty} K_\alpha(x, v) \varphi(v) dv$$

on  $L^2(\mathbb{R}, \mathbb{C})$ , for which the  $n$ -th Hermite function  $h_n(x) = H_n(x) e^{-x^2/2}$  is an eigenfunction with  $e^{in\alpha}$  as corresponding eigenvalue. The explicit Wiener formula for the kernel function  $K_\alpha$  is a limiting case of Mehler's formula for the Hermite functions. This fact was rediscovered sixty years later in quantum mechanics by Namias [84], and showed earlier by Hörmander [64].

Added to this transform, V. Bargmann has introduced in [17, p.203] the integral operator

$$\mathcal{A}_{hol}^\alpha \varphi(z) = (1-z)^{-\alpha-1} \int_0^{+\infty} \varphi(t) \exp\left(\frac{tz}{z-1}\right) \frac{t^\alpha e^{-t}}{\Gamma(\alpha+1)} dt \quad (0.0.1)$$

mapping isometrically the Hilbert space

$$L^{2,\alpha}(\mathbb{R}^+, \mathbb{C}) := L^2\left(\mathbb{R}^+; \frac{t^\alpha e^{-t}}{\Gamma(\alpha+1)} dt\right); \quad \alpha > 0,$$

onto the classical holomorphic Bergman space

$$A_{hol}^{2,\alpha}(D, \mathbb{C}) = Hol(D, \mathbb{C}) \cap L^{2,\alpha}(D, \mathbb{C})$$

consisting of all  $\mathbb{C}$ -valued holomorphic functions on the unit disk  $D = \{z \in \mathbb{C}; |z| < 1\}$  that are square integrable with respect to the hyperbolic measure

$$d\lambda_\alpha(z) := (1 - |z|^2)^{\alpha-1} \frac{\alpha}{\pi} dx dy. \quad (0.0.2)$$

The transform in (0.0.1) is realized as a coherent state transform associated to the lower hyperbolic Landau level of a special magnetic Schrödinger operator on the Poincaré disk [42]. In fact, the involved kernel function is related to the generating function of the generalized Laguerre polynomials  $L_n^{(\alpha)}$ .

The theory of slice regularity initiated by Gentili and Struppa in [54] is quaternionic counterpart of the classical theory of holomorphic functions on  $\mathbb{C}$  and extends in an appropriate way the holomorphic setting on  $\mathbb{C}$  to  $\mathbb{H}$ -valued functions of one quaternionic variable. It was extensively studied and has found many interesting applications in operator theory, quantum physics and Schur analysis [29, 8, 9, 53, 6, 10]. In [7], Alpay *et al.* have considered the slice hyperholomorphic Bargmann-Fock space

$$\mathcal{F}_{slice}^{2,\nu}(\mathbb{H}) = \mathcal{SR}(\mathbb{H}) \cap L^{2,\nu}(\mathbb{C}_I, \mathbb{H}), \quad (0.0.3)$$

where  $\mathcal{SR}(\mathbb{H})$  denotes the space of (left) slice regular  $\mathbb{H}$ -valued functions on quaternion and  $L^{2,\nu}(\mathbb{C}_I, \mathbb{H})$ ,  $\nu > 0$ , is the Hilbert space of  $\mathbb{H}$ -valued functions that are square integrable with respect to the Gaussian measure on an arbitrary slice  $\mathbb{C}_I = \mathbb{R} + \mathbb{R}I$ . The corresponding Segal-Bargmann transform is considered in [35] and maps isometrically the  $L^2$ -Hilbert space  $L^2(\mathbb{R}, \mathbb{H})$  onto the slice hyperholomorphic Bargmann-Fock space  $\mathcal{F}_{slice}^{2,\nu}(\mathbb{H})$ . A quaternionic analogue of the Bergman theory in the setting of the slice regular functions on the open unit ball (centered at the origin) has been introduced by Colombo *et al.* in [24] (see also [25]). Thus, the slice hyperholomorphic Bergman space  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$ , for arbitrary radius, is defined to be

$$A_{slice}^{2,\alpha}(\mathbb{B}_R) := \mathcal{SR}(\mathbb{B}_R) \cap L^{2,\alpha}(\mathbb{B}_{R,I}, \mathbb{H}), \quad \alpha > 0, \quad (0.0.4)$$

where  $L^{2,\alpha}(\mathbb{B}_{R,I}, \mathbb{H})$  is the  $L^2$ -Hilbert space on  $\mathbb{B}_{R,I} = \mathbb{B}_R \cap \mathbb{C}_I$  of  $\mathbb{H}$ -valued functions  $f$  subject to the norm boundedness

$$\|f\|_{\alpha, slice}^2 = \int_{\mathbb{B}_{R,I}} \overline{f(z)} g(z) \left(1 - \frac{|z|^2}{R^2}\right)^{\alpha-1} \frac{\alpha}{\pi R^2} dx dy < +\infty. \quad (0.0.5)$$

The parameter  $\alpha$  corresponds to  $\alpha + 1$  in [24, 25] and the measure in (0.0.5) is the volume measure associated to the quaternionic hyperbolic geometry on  $\mathbb{B}_R$  induced from the scaled Poincaré-type differential metric

$$ds_{\mathbb{B}_R}^2 = R^4(R^2 - |q|^2)^{-2}|d_I q|^2.$$

The metric  $ds_{\mathbb{B}_R}^2$  (with  $R = 1$ ) was defined in [20] by developing a variation of an approach adopted by Ahlfors [2].

Motivated by these recent investigations in the theory of slice regularity and the geometrical fact that the euclidean limit of the balls  $\mathbb{B}_R$  (hyperbolic case) gives rise to the quaternionic space  $\mathbb{H}$  (flat case), as the radius  $R$  goes to  $+\infty$ , quite natural questions arise of whether the analogue of the transform  $\mathcal{A}_{hol}^\alpha$  can be constructed and whether the two theories on  $\mathcal{F}_{slice}^{2,\nu}(\mathbb{H})$  and  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  can be connected. This is Our main purpose in the present paper is to answer these two questions. Namely, we establish a quaternionic analogue of (0.0.1) for the slice hyperholomorphic Bergman space (Theorem 2.3.21). Moreover, related basic properties are studied and the explicit expression of its inverse is obtained (Theorem 2.3.23). We also exhibit an orthonormal basis (Proposition 2.3.1) and give in Theorem 2.3.10 the closed expression of its reproducing kernel generalizing the one obtained in [25, Theorem 4.1-Proposition 4.3] for  $\alpha = 1$ . We also provide an integral representation (Theorem 2.3.16) of this reproducing kernel. This integral representation involves the kernel function of the quaternionic analogue of the second Bargmann transform for which we present in Theorem 2.3.18 a closed form of its explicit formula. The study will be done on the quaternionic ball of radius  $R$  centred at the origin so that the asymptotic behavior as  $R$  goes to infinity can be discussed. We show in Theorem 2.3.24 that the pointwise limit of the weighted Bergman kernel of the slice hyperholomorphic Bergman space  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$ , for the specific  $\alpha = \nu R^2$ , is exactly the reproducing kernel of the slice hyperholomorphic Bargmann-Fock space  $\mathcal{F}_{slice}^{2,\nu}(\mathbb{H})$ . This is to say that one can move from the Bergman universe to the Bargmann universe by taking the "euclidean limit". These ideas can be extended to the weighted hyperholomorphic Bergman spaces. This will be the subject of Chapter 1.

In Chapter 3, we provide to Bargmann a general abstract formalism for constructing fractional transform associated to given special invertible integral transform  $\mathcal{S}_{X,Y} : \mathcal{H}_X \longrightarrow \mathcal{H}_Y$ ,

$$\mathcal{S}_{X,Y}\varphi(y) = \int_X R(x,y)\varphi(x)\omega_X(x)d\lambda(x),$$

on an arbitrary infinite separable functional Hilbert space  $\mathcal{H}_X = L^2(X; \omega_X(x)dx)$ . Namely, we deal with integral transforms of the form  $\mathcal{S}_{X,Y}^{-1} \circ T_\theta \circ \mathcal{S}_{X,Y}$ , where  $T_\theta$  is an appropriate action of a group  $G$ . We show that the performed fractional integral transform inherits numerous properties from the ones of  $\mathcal{S}_{X,Y}$ . The explicit computation shows that the kernel function of  $\mathcal{S}_{X,Y}^{-1} \circ T_\theta \circ \mathcal{S}_{X,Y}$  can be expressed explicitly

in terms of the kernel function  $R(x, y)$  (see (3.3.3) below). As concrete application, we deal with a special quaternionic fractional Fourier transform (QFrFT) acting on the right quaternionic Hilbert space

$$L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+) = L_{\mathbb{H}}^2(\mathbb{R}^+, x^\alpha e^{-x} dx), \quad \alpha > 0,$$

and associated to the second Bargmann transform for hyperholomorphic Bergman space of second kind [39]. More precisely, they are the family of (left) integral transforms

$$\mathcal{L}_\theta^\alpha \varphi(y) := \int_0^\infty K_\theta^\alpha(x, y) \varphi(x) dx, \quad (0.0.6)$$

whose kernel function can be shown to be given in terms of the modified Bessel function  $I_\alpha$ . It verifies  $\mathcal{L}_\theta^\alpha(\varphi_n^\alpha(x)) = \theta^n \varphi_n^\alpha(x)$ . We also prove that  $\mathcal{L}_\theta^\alpha$  is continuous, interpolates continuously the identity operator to the Fourier-Bessel transform and satisfies the index law (semi-group property)  $\mathcal{L}_\theta^\alpha \circ \mathcal{L}_\eta^\alpha = \mathcal{L}_{\theta\eta}^\alpha$ , so that the inverse of  $\mathcal{L}_\theta^\alpha$  reads simply  $\mathcal{L}_{1/\theta}^\alpha$ . When  $|\theta| = 1$ , the constructed family of QFrFT for  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  appears embedded in a strongly continuous one-parameter group of unitary operators and coincides with the fractional Hankel transform [83, 71] with quaternionic parameter (QFrHT). The exposition of these ideas in the quaternionic setting add some technical difficulties which we overcome using tools from the theory of slice regular functions.

As another application of the presented formalism, we will consider the one-parameter (left) integral transforms  $S_y^\alpha$ ;  $\alpha > -1$ , defined in [55], on  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  as the dual transform at  $y \in (0, +\infty)$  of the quaternionic fractional Hankel transform  $\mathcal{L}_\theta^\alpha$  defined in [44]. More precisely, we deal with

$$S_y^\alpha \varphi(q) := \frac{1}{1-q} \int_0^\infty \exp\left(-\frac{x+\theta y}{1-q}\right) I_\alpha\left(\frac{2\sqrt{qxy}}{1-q}\right) \varphi(x) dx, \quad (0.0.7)$$

where  $I_\alpha$  stands for the modified Bessel function [15, p. 222, (4.12.2)]

$$I_\alpha(\xi) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(\alpha + n + 1)} \left(\frac{\xi}{2}\right)^{2n+\alpha}.$$

The motivation of considering  $S_y^\alpha$  lies on the observation that the limiting case  $y = 0$  gives rise to the hyperholomorphic second Bargmann transform in [39]. Accordingly, the study of these new operators is required in order to generate hyperholomorphic-like Bergman spaces.

A concrete description of the spectral properties of  $S_y^\alpha$  was completely described in [55]. Namely, the identification of their null space and the range for arbitrary  $y \geq 0$ . We show that the range of  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  by the transform  $S_y^\alpha$  is contained in a reproducing kernel weighted slice hyperholomorphic with suitable weight function  $\omega$  on the unit ball  $\mathbb{B}$ , extending  $A_{slice}^{2,\alpha}$ .

Afterwards, we consider  $L^2_\mu(\mathbb{H}) := L^2_{\mathbb{H}}(\mathbb{H}; d\mu)$  denotes the right Hilbert space of quaternionic-valued square integrable functions on the quaternions  $\mathbb{H}$  with respect to the Gaussian density  $d\mu = e^{-|q|^2} d\lambda$ , where  $d\lambda$  is the standard Lebesgue measure on  $\mathbb{H} \simeq \mathbb{R}^4$ . The associated norm denoted by  $\|\cdot\|_{\mathbb{H}}$  is the one induced from the scalar product

$$\langle f, g \rangle_{\mathbb{H}} = \int_{\mathbb{H}} \overline{f(q)} g(q) d\mu(q). \quad (0.0.8)$$

We introduce the new weighted quaternionic Cauchy transform (WQCT), with respect to the measure  $\mu$ , defined as a singular integral operator on  $L^2_\mu(\mathbb{H})$  by considering

$$\mathcal{C}_{\mathbb{H}} f(q) := \frac{1}{\pi} \int_{\mathbb{H}} \mathcal{N}(q, p) f(p) d\mu(p) \quad (0.0.9)$$

for  $q \in \mathbb{H}$ , where  $\mathcal{N}$  denotes the Cauchy kernel function given by

$$\mathcal{N}(p, q) := \left( (p - \bar{q})^{-1} p ((p - \bar{q})) - q \right)^{-1}. \quad (0.0.10)$$

This weighted Cauchy transform is in fact a variant form of the one considered in [40]. Here, we have taking into account the linearity of such transform when acting on right vector spaces and the left slice regularity of the images when  $\mathcal{C}_{\mathbb{H}}$  acts on left slice regular functions

Our main purpose is to investigate some spectral properties of the WQCT in (0.0.9). Mainly, we show that  $\mathcal{C}_{\mathbb{H}}$  is a bounded compact operator in  $L^2_\mu(\mathbb{H})$  and belong to  $k$ -Schatten class for every  $k > 2$ . We also give its explicit action on the so-called  $n$ -th S-polyregular Bargmann space of second kind  $\mathcal{A}^2_{n,slice}$  realized as the space of  $L^2$ -eigenfunctions of a slice differential operator [18]. This was possible using the quaternionic Itô–Hermite polynomials that constitute a complete orthogonal system in  $\mathcal{A}^2_{n,slice}$ . Moreover, we study the operator  $P_n \mathcal{C}_{\mathbb{H}}$  defined as the  $n$ -th Bergman projection of  $\mathcal{C}_{\mathbb{H}}$ ,  $P_n$  being the  $n$ -Bergman projector onto  $\mathcal{A}^2_{n,slice}$  given by

$$P_n f(q) = \frac{1}{\pi} \int_{\mathbb{H}} e_*^{[\bar{q}, p]} \star_{sp}^q L_n(|q - p|_{\star_{sp}^q}^2) f(p) d\mu(p), \quad (0.0.11)$$

where the involved quantities are specified in the next section. Thus, we explicit its Schwartz kernel function, we identify its range and determinate its singular values, i.e the nonzero eigenvalues of  $|P_n \mathcal{C}_{\mathbb{H}}| := ((P_n \mathcal{C}_{\mathbb{H}})^* P_n \mathcal{C}_{\mathbb{H}})^{1/2}$ . The obtained results generalize in somehow those elaborated in [38, ?] for the weighted complex Cauchy transform on  $L^2_\mu(\mathbb{C})$  defined by

$$\mathcal{C}_{\mathbb{C}} f(z) := \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\xi)}{z - \bar{\xi}} e^{-|z|^2} dx dy; \quad \bar{\xi} = x + iy, z \in \mathbb{C}. \quad (0.0.12)$$

# Chapter 1

## Preliminaries on slice hyperholomorphic functions

### 1.1 The algebra of quaternions $\mathbb{H}$

We introduce the algebra of quaternions and discuss their main algebraic properties.

**Definition 1.1.1.** *The algebra of quaternions  $\mathbb{H}$  is defined as the 4-dimensional real vector space with basis  $1, e_1, e_2$  and  $e_3$ , that is,*

$$\mathbb{H} = \{x_0 + x_1e_1 + x_2e_2 + x_3e_3 : x_i \in \mathbb{R}\},$$

*endowed with the associative  $\mathbb{R}$ -bilinear product with unity 1 that satisfies*

$$e_1^2 = e_2^2 = e_3^2 = -1$$

$$e_1e_2 = e_3 = -e_2e_1, e_2e_3 = e_1 = -e_3e_2 \quad \text{and} \quad e_3e_1 = e_2 = -e_1e_3. \quad (1.1.1)$$

*Sometimes, when it is more convenient, we will write  $e_0$  instead of 1. Moreover, note that (1.1.1) is equivalent to*

$$e_1e_2e_3 = -1.$$

*As in the complex case, we will identify the subalgebra  $\text{span}\{e_0\}$  with the field of real numbers  $\mathbb{R}$  (see [30]). Moreover, we will identify  $\mathbb{R}^3$  with  $\text{span}\{e_1, e_2, e_3\}$ . The following definitions are formulated in analogy to the case of complex numbers.*

**Definition 1.1.2.** *Let  $x = x_0 + \sum_{i=1}^3 x_i e_i \in \mathbb{H}$ .*

- 1) *We call  $\text{Re}(x) = x_0$  the real part of the quaternion  $x$  and  $\text{Im}(x) = \underline{x} = \sum_{i=1}^3 x_i e_i$  the imaginary or vector part of the quaternion. We call a quaternion  $x$  real, if  $\text{Im}(x) = 0$  and we call it (purely) imaginary if  $\text{Re}(x) = 0$ .*

2) We call  $\bar{x} = x_0 - \sum_{i=1}^3 x_i e_i$  the conjugate of  $x$ .

3) The norm or absolute value of  $x$  is defined as  $|x| = \sqrt{\sum_{i=0}^3 x_i^2}$ .

**Lemma 1.1.3.**

i) The quaternionic conjugation is an  $\mathbb{R}$ -linear involutive antiautomorphism, that is, for all  $x, y \in \mathbb{H}$  and all  $\lambda \in \mathbb{R}$ , we have

$$\overline{x+y} = \bar{x} + \bar{y}, \quad \overline{\lambda x} = \lambda \bar{x}, \quad \overline{\bar{x}} = x, \quad \text{and} \quad \overline{xy} = \bar{y}\bar{x}.$$

Moreover,  $\bar{x} = x$  if and only if  $x \in \mathbb{R}$  and  $\bar{x} = -x$  if and only if  $x$  is purely imaginary.

ii) Let  $x, y \in \mathbb{H}$ . Similar to the complex case, the following identities hold true:

- $Re(x) = \frac{1}{2}(x + \bar{x})$  and  $Im(x) = \frac{1}{2}(x - \bar{x})$ ,
- $\bar{x}x = x\bar{x} = |x|^2$ ,
- $|xy| = |x||y|$ .

*Proof.* From the definition, it is clear that the quaternionic conjugation is  $\mathbb{R}$ -linear and an involution. Moreover, we have

$$\overline{1e_i} = \bar{e}_i = \overline{e_i 1} = \bar{e}_i \bar{1}, i = 1, \dots, 3,$$

and

$$\overline{e_1 e_2} = \bar{e}_3 = -e_3 = e_2 e_1 = (-e_2)(-e_1) = \bar{e}_2 \bar{e}_1.$$

Similarly, we get  $\overline{e_2 e_3} = \bar{e}_3 \bar{e}_2$  and  $\overline{e_1 e_3} = \bar{e}_3 \bar{e}_1$ . Thus,  $\overline{xy} = \bar{y}\bar{x}$  holds if  $x$  and  $y$  are elements of the basis of  $\mathbb{H}$ . Hence, it holds for any  $x, y \in \mathbb{H}$  because of the  $\mathbb{R}$ -bilinearity of the quaternionic product.

It is also clear that  $\bar{x} = x$  if and only if  $Im(x) = 0$ , that is, if and only if  $x$  is real, and that  $\bar{x} = -x$  if and only if  $Re(x) = 0$ , that is, if and only if  $x$  is purely imaginary. Therefore, i) holds true. The identities in ii) are also easy to check. We have

$$x + \bar{x} = Re(x) + Im(x) + Re(x) - Im(x) = 2Re(x)$$

and

$$x - \bar{x} = Re(x) + Im(x) - (Re(x) - Im(x)) = 2Im(x).$$

Since  $e_i e_j = -e_j e_i$  for  $i \neq j \in 1, 2, 3$ , we get

$$\begin{aligned} \bar{x}x &= (x_0 - \sum_{i=1}^3 x_i e_i)(x_0 + \sum_{j=1}^3 x_j e_j) = x_0^2 - \sum_{i=1}^3 x_i x_0 e_i + \sum_{j=1}^3 x_0 x_j e_j - \sum_{i,j=1}^3 x_i x_j e_i e_j \\ &= x_0^2 - \sum_{i,j=1}^3 \sum_{j>i} (x_i x_j - x_j x_i) e_i e_j - \sum_{i=1}^3 x_i^2 e_i^2 = x_0^2 + \sum_{i=1}^3 x_i^2 = |x|^2 \end{aligned}$$

Similarly, we obtain  $x\bar{x} = |x|^2$ . Finally,

$$|xy|^2 = xy\overline{xy} = xy\bar{y}\bar{x} = |y|^2 x\bar{x} = |y|^2 |x|^2$$

Hence  $|xy| = |x||y|$ . □

**Proposition 1.1.4.** *Every quaternion  $x \in \mathbb{H} \setminus \{0\}$  has an multiplicative inverse, namely*

$$x^{-1} = \frac{1}{|x|^2} \bar{x}.$$

*In particular, the quaternions form a skew field. Although the quaternionic multiplication is not commutative.*

*If  $x$  or  $y$  is real, then  $xy = yx$ . As the next Lemma shows, reals are the only quaternions that commute with any other quaternion. We will specify this result later.*

**Lemma 1.1.5.** *A quaternion commutes with every other quaternion if and only if it is real. That is, the center of  $\mathbb{H}$  is the real line  $\mathbb{R}$ .*

*Proof.* Since 1 is the multiplicative neutral element of  $\mathbb{H}$ , it commutes with every quaternion. Moreover, as the multiplication is  $\mathbb{R}$ -bilinear, any  $x \in \mathbb{R} = \text{span}\{1\}$  commutes with every other quaternion, too.

Now, let  $x = x_0 + \sum_{i=1}^3 x_i e_i \in \mathbb{H}$ . be such that  $xy = yx$  for all  $y \in \mathbb{H}$ . In particular,  $xe_1 = e_1x$ . But since

$$e_1x = x_0e_1 + x_1e_1^2 + x_2e_1e_2 + x_3e_1e_3 = x_0e_1 - x_1 + x_2e_3 - x_3e_2$$

and

$$xe_1 = x_0e_1 + x_1e_1^2 + x_2e_2e_1 + x_3e_3e_1 = x_0e_1 - x_1 - x_2e_3 + x_3e_2;$$

this implies  $x_2 = 0$  and  $x_3 = 0$ , Similarly  $xe_2 = e_2x$  together with

$$xe_2 = x_0e_2 + x_1e_1e_2 = x_0e_2 + x_1e_3$$

And

$$e_2x = x_0e_2 + x_1e_2e_1 = x_0e_2 - x_1e_3$$

yields  $x_1 = 0$ . Thus,  $x$  is real. □

**Lemma 1.1.6.** *Let  $x, y \in \mathbb{H} \setminus \{0\}$ . Then  $x$  and  $y$  satisfy  $xy = -yx$  if and only if  $\text{Re}(x) = \text{Re}(y) = 0$  and  $x$  and  $y$  are orthogonal as vectors in  $\mathbb{R}^3$ .*

**Definition 1.1.7.** *A purely imaginary quaternion with absolute value 1 is called an imaginary unit. We denote the set of all imaginary units by  $\mathbb{S}^2$ , that is,*

$$\mathbb{S}^2 = \left\{ \sum_{i=1}^3 x_i e_i \in \mathbb{H} : \sum_{i=1}^3 x_i^2 = 1 \right\}.$$

The name imaginary unit is justified by the fact that, for any  $I \in \mathbb{S}^2$ , we have  $\bar{I} = -I$  and therefore

$$I^2 = -\bar{I}I = -|I|^2 = -1.$$

Thus

$$\mathbb{S}^2 = \{I \in \mathbb{H} : I^2 = -1\}.$$

**Corollary 1.1.8.** For  $I \in \mathbb{S}^2$ , the plane  $\mathbb{C}_I = \{x_0 + x_1I : x_0, x_1 \in \mathbb{R}\}$  is isomorphic to the field of complex numbers  $\mathbb{C}$ .

**Remark 1.1.9.** For any non-real quaternion  $q \in \mathbb{H} \setminus \mathbb{R}$ , there exist, and are unique,  $x, y \in \mathbb{R}$  with  $y > 0$ , and  $I \in \mathbb{S}^2$  such that  $q = x + yI$ . Indeed

$$x = \operatorname{Re} q, \quad I = \frac{q}{|q|}, \quad \text{and } y = |q|.$$

## 1.2 Slice regular functions

In this section, we now introduce a new notion which called C-regular function, was given by Cullen in [31].

**Definition 1.2.1.** Let  $\Omega$  be a domain in  $\mathbb{H}$ . A real differentiable function  $f : \Omega \rightarrow \mathbb{H}$  is said to be C-regular if, for every  $I \in \mathbb{S}^2$ , its restriction  $f_I$  to the complex line  $\mathbb{C}_I = \mathbb{R} + \mathbb{R}I$  passing through the origin and containing 1 and  $I$  is holomorphic on  $\Omega \cap \mathbb{C}_I$ .

Throughout the project, since no confusion can arise, we will refer to C-regular functions as regular functions tout court.

**Remark 1.2.2.** The requirement that  $f : \Omega \rightarrow \mathbb{H}$  is regular is equivalent to require that, for every  $I \in \mathbb{S}^2$ ,

$$\bar{\partial}_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0,$$

on  $\Omega \cap \mathbb{C}_I$ .

We can define a notion of I-derivative as follows:

**Definition 1.2.3.** Let  $\Omega$  be a domain in  $\mathbb{H}$  and let  $f : \Omega \rightarrow \mathbb{H}$  be a real differentiable function. For any  $I \in \mathbb{S}^2$  and any point  $q = x + yI$  in  $\Omega$  ( $x$  and  $y$  are real numbers here) we define the I-derivative of  $f$  in  $q$  by

$$\partial_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f_I(x + yI).$$

Because of the noncommutativity we also have another version of this notion us follows:

**Remark 1.2.4.** *we say that  $f$  is a right slice regular function if, for every  $I \in \mathbb{S}^2$ , we have*

$$\frac{1}{2} \left( \frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy)I \right) = 0.$$

*The first important consequence of our definition of regularity is that, for regular functions, we can introduce a notion of derivative.*

**Definition 1.2.5.** *Let  $\Omega$  be a domain in  $\mathbb{H}$ , and let  $f : \Omega \rightarrow \mathbb{H}$  be a regular function. The Slice derivative of  $f$ ,  $\partial_C f$ , is defined as follows:*

$$\partial_C(f)(q) = \begin{cases} \partial_I(f)(q) & \text{if } q = x + yI \text{ with } y \neq 0 \\ \frac{\partial f}{\partial x}(x) & \text{if } q = x \text{ is real,} \end{cases}$$

*This definition of derivative is well posed because it is applied only to regular functions. In fact, the value of the derivative at a real point  $x$  can be computed using different imaginary units, and a priori there is no reason why the values which one obtains should coincide. However, if a function  $f$  is regular, its derivative in the point  $x$  is immediately shown to be equal to  $\frac{\partial f}{\partial x}(x)$ . It is easy to construct examples which manifest this problem if  $f$  is not regular. Note that this phenomenon is peculiar of the quaternionic case, and does not appear in the complex case. The reason for this is that the unit sphere of imaginary numbers has dimension 2 in the case of quaternions, but it is only made of two points,  $\{i, -i\}$ , in the complex case. Let  $f$  be a regular function. Since for every  $I \in \mathbb{S}^2$  it is  $\bar{\partial}_I(\partial_C(f)) = \partial_C(\bar{\partial}_I(f)) = 0$  we obtain that the Cullen derivative of a regular function is still regular.*

### 1.3 Power series and series expansions for regular functions

In order to study polynomials and power series in  $q$ , we first note that the basic polynomial  $q^n a_n$ , with  $a_n$  a quaternion, is regular according to Definition (1.3.9), however  $aq^n$  is not. Since the sum of regular functions is regular, we immediately have that polynomials with quaternionic coefficients on the right are regular. In order to consider power series  $\sum_{i=0}^{\infty} q^n a_n$ , we will endow the space of regular functions with the natural uniform convergence on compact sets. The same arguments which hold for complex power series, see e.g., allow to obtain the analog of the Abels theorem.

**Theorem 1.3.1.** *For every power series  $\sum_{i=0}^{\infty} q^n a_n$  there exists a number  $R$ ,  $0 \leq R \leq \infty$ , called the radius of convergence, such that the series converges absolutely for every  $q$  with  $|q| < R$  and uniformly for every  $q$  with  $|q| \leq \rho < R$ . Moreover if  $|q| > R$ , the series is divergent.*

Since convergence of power series is uniform on compact sets, it turns out that power series are regular in their domain of convergence. Note also that every power series is also real analytic.

The derivative of a power series can be done term by term because of the uniform convergence, so that

$$\partial_C \left( \sum_{i=0}^{\infty} q^i a_i \right) = \sum_{i=1}^{\infty} q^{i-1} i a_i.$$

This new series has the same radius of convergence as of the original series. In what follows, we will always restrict our attention to functions which are regular on a ball  $B(0, R)$  centered in the origin and of radius  $R$ . In order for us to study regular functions, we will need a simple representation of the restriction of a regular function as a pair of holomorphic functions. To do so, we need a few simple preliminary results on the set  $\mathbb{S}^2$ .

**Corollary 1.3.2.** *For  $a, b \in \mathbb{H}$ , we have*

$$\sum_{n=0}^{\infty} \frac{a^n b^n}{n!} = (\cos(yb) + I_a \sin(yb)) \exp(xb).$$

*Proof.* Let  $a = x + I_a y, b \in \mathbb{H}$ . Then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{a^n b^n}{n!} &= \sum_{n=0}^{\infty} \frac{\sum_{k=0}^n C_n^k (I_a y)^k x^{n-k} b^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{C_n^k (I_a y)^k x^{n-k} b^n}{n!} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{C_{n+k}^k (I_a y)^k x^n b^{n+k}}{(n+k)!} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(I_a y)^k x^n b^{n+k}}{n! k!} \\ &= \sum_{k=0}^{\infty} \frac{(I_a y)^k b^k}{k!} \sum_{n=0}^{\infty} \frac{x^n b^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(I_a y)^k b^k}{k!} \sum_{n=0}^{\infty} \frac{(xb)^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(I_a y)^k b^k}{k!} \exp(xb) \\ &= \left( \sum_{k=0}^{\infty} \frac{(I_a y)^{2k} b^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(I_a y)^{2k+1} b^{2k+1}}{(2k+1)!} \right) \exp(xb) \end{aligned}$$

$$\begin{aligned}
&= \left( \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k} b^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{I_a (-1)^k y^{2k+1} b^{2k+1}}{(2k+1)!} \right) \exp(xb) \\
&= (\cos(yb) + I_a \sin(yb)) \exp(xb).
\end{aligned}$$

Witch complete the proof. □

**Proposition 1.3.3.** *Let  $I = iI_1 + jI_2 + kI_3$  and  $J = iJ_1 + jJ_2 + kJ_3$  be two elements in  $\mathbb{S}^2$ , let  $-\langle I, J \rangle = I_1J_1 + I_2J_2 + I_3J_3 \in \mathbb{R}$  denote the Euclidean scalar product of their coordinates, and let  $I \times J = i(I_2J_3 - I_3J_2) + j(I_3J_1 - I_1J_3) + k(I_1J_2 - I_2J_1) \in \mathbb{R} \cdot \mathbb{S}^2$  be their natural vector product. Then the quaternionic product  $IJ$  can be computed through the following formula:*

$$IJ = \langle I, J \rangle + I \times J.$$

*Proof.* The result follows immediately from the direct computation of the product

$$IJ = (iI_1 + jI_2 + kI_3)(iJ_1 + jJ_2 + kJ_3).$$

□

Note that the previous computation shows, in particular, that the product of two orthogonal elements of  $\mathbb{S}^2$  lies in  $\mathbb{S}^2$  as well. We will use this simple fact to build orthogonal bases in  $\mathbb{S}^2$ .

**Theorem 1.3.4.** *Let  $I$  and  $J$  be two orthogonal elements in  $\mathbb{S}^2$ , and let  $K = IJ$ . Then:*

- $K = IJ = -JI$  is an element of  $\mathbb{S}^2$ ,
- $K$  is orthogonal to both  $I$  and  $J$ ,
- $JK = I = -KJ$  and  $KI = J = -IK$ .

We will prove the three statements independently.

1. This follows immediately from the previous proposition, noting that  $I$  and  $J$  are orthogonal, and that clearly  $I \times J = -J \times I$ .

2. This again is a consequence of the previous proposition, of the orthogonality of  $I$  and  $J$ , and of the fact that  $I \times J$  is always orthogonal to both  $I$  and  $J$ . These three facts imply that  $\langle K, I \rangle = \langle IJ, I \rangle = \langle I \times J, I \rangle = 0$ .

3. Here we apply repeatedly the previous proposition to obtain the sequence of equalities

$$\begin{aligned}
JK &= J(IJ) = J(\langle I, J \rangle + I \times J) \\
&= \langle J, \langle I, J \rangle + I \times J \rangle + J \times (\langle I, J \rangle + I \times J).
\end{aligned}$$

Using the orthogonality of  $I$  and  $J$ , we obtain

$$JK = \langle J, I \times J \rangle + J \langle I \times J, I \times J \rangle.$$

Now note that  $\langle J, I \times J \rangle = 0$  because  $I \times J$  is orthogonal to  $J$ , and that, by the same reason,  $IJ = I \times J$ . Thus to conclude the proof we only need to show that  $J \times K = I$ . This can be obtained by a direct computation. Indeed, we have

**Remark 1.3.5.** *The result we have just proved is simple, but it shows that we can use  $I, J$ , and  $K$  as a basis for  $\mathbb{S}^2$ ; moreover, given any element  $I$  in  $\mathbb{S}^2$ , we can always construct such a basis (though not in a unique way, as the basis will ultimately depend on the choice of  $J$  among vectors which are orthogonal to  $I$ ).*

The following lemma (we will often refer to it as the splitting lemma) is simple to prove but is essential for all the results in this project.

**Lemma 1.3.6.** *(Splitting lemma). If  $f$  is a regular function on  $B = B(0, R)$ , then for every  $I \in \mathbb{S}^2$ , and every  $J$  in  $\mathbb{S}^2$ , perpendicular to  $I$ , there are two holomorphic functions  $F, G : B \cap \mathbb{C}_I \rightarrow \mathbb{C}_I$  such that for any  $z = x + yI$ , it is*

$$f_I(z) = F(z) + G(z)J.$$

*Proof.* Given any pair of orthogonal vectors  $I$  and  $J$  in  $\mathbb{S}^2$ , consider the third element  $K$  of the orthogonal basis  $I, J, K$ , and write  $f_I(x + yI) = f(x + yI)$  as  $f = f_0 + If_1 + Jf_2 + Kf_3$ . Since  $f$  is regular, we know that  $(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y})f_I(x + yI) = 0$ , i.e.

$$\frac{\partial f_0}{\partial x} + I\frac{\partial f_1}{\partial x} + J\frac{\partial f_2}{\partial x} + K\frac{\partial f_3}{\partial x} + I\left(\frac{\partial f_0}{\partial y} + I\frac{\partial f_1}{\partial y} + J\frac{\partial f_2}{\partial y} + K\frac{\partial f_3}{\partial y}\right) = 0.$$

The expression above can be rewritten (taking advantage of the properties of the imaginary units) as

$$\frac{\partial f_0}{\partial x} - \frac{\partial f_1}{\partial y} + I\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_0}{\partial y}\right) + J\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_3}{\partial y}\right) + K\left(\frac{\partial f_3}{\partial x} + \frac{\partial f_2}{\partial y}\right) = 0.$$

This implies that the functions  $f_0 + If_1$  and  $f_2 + If_3$  satisfy the standard CauchyRiemann system and therefore they are both holomorphic. In particular, if we set  $f_0 + If_1 = F$ , and  $f_2 + If_3 = G$ , we obtain that  $f_I(x + yI) = F(x + yI) + G(x + yI)J$ , and so the lemma is demonstrated once we set  $z = x + yI$ .  $\square$

**Remark 1.3.7.** *Given that the functions  $F$  and  $G$  are holomorphic on the plane  $\mathbb{R} + \mathbb{R}I$ , it is not surprising (we will show it in a second) that  $f$  admits, on that plane, a series expansion in powers of  $z$ . What is more surprising is the fact that such an expansion can be used to provide a series expansion for  $f$  in powers of  $q$ . This is a crucial result for this theory, and its proof requires one more preliminary step.*

**Proposition 1.3.8.** *Let  $f : B \rightarrow \mathbb{H}$  be a regular function. Then, for any  $n \in \mathbb{N}$ , its Slice derivative  $\partial_{\mathbb{C}}^n f : B \rightarrow \mathbb{H}$  is regular and it is  $\partial_{\mathbb{C}}^n f(x + yI) = \frac{\partial^n f}{\partial x^n}(x + yI)$ .*

*Proof.* The fact that  $\partial_C^n f$  is well defined has already been established. To prove the equality  $\partial_C^n f(x + yI) = \frac{\partial^n f}{\partial x^n}(x + yI)$  we proceed by induction. First we note that the equality is trivial for  $n = 1$ , since

$$\begin{aligned}\partial_C f(x + yI) &= \frac{1}{2} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) f(x + yI) \\ &= \frac{1}{2} \left( \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) (x + yI) \\ &= \frac{\partial f}{\partial x}(x + yI).\end{aligned}$$

To prove the induction step note that since  $f$  is regular, then

$$\left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) \frac{\partial^n f}{\partial x^n} = \frac{\partial^{n+1} f}{\partial x^{n+1}} + I \frac{\partial^{n+1} f}{\partial x^n \partial y} = \frac{\partial^n}{\partial x^n} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f = 0.$$

Thus we have that

$$\frac{\partial^{n+1} f}{\partial x^{n+1}} = -I \frac{\partial^{n+1} f}{\partial x^n \partial y}$$

and therefore (by the induction hypothesis)

$$\begin{aligned}\partial_C^{n+1} f &= \partial_C(\partial_C^n f) = \partial_C \left( \frac{\partial^n f}{\partial x^n} \right) \\ &= \frac{1}{2} \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) \left( \frac{\partial^n f}{\partial x^n} \right) \\ &= \frac{1}{2} \left( \frac{\partial^{n+1} f}{\partial x^{n+1}} - I \frac{\partial^{n+1} f}{\partial x^n \partial y} \right) \\ &= \frac{\partial^{n+1} f}{\partial x^{n+1}}.\end{aligned}$$

This concludes the proof. □

Now, we can deduce the following important result.

**Theorem 1.3.9.** *A function  $f : B \rightarrow \mathbb{H}$  is regular if, and only if, it has a series expansion of the form*

$$f(q) = \sum_{i=0}^{\infty} \frac{q^i}{i!} \frac{\partial^i f}{\partial x^i}(0)$$

*converging on  $B$ . In particular if  $f$  is regular then it is  $C^\infty$  on  $B$ .*

*Proof.* Let  $\Delta_I$  The disc centered in origin and with radius  $a > 0$  in the complex numbers  $\mathbb{C}_I$ , where  $a < R$ . Then we can use the representation from the splitting Lemma 1.3.6 to find an integral representation for  $f_I$  inside  $\Delta_I$ . Specifically, using the fact

that both  $F$  and  $G$  are holomorphic in the domain  $B \cap \mathbb{C}_I$  of the complex plane  $\mathbb{C}_I$ , with values in the same complex plane  $\mathbb{C}_I$ , we obtain  $(\zeta - z)^{-1}F(z) = F(z)(\zeta - z)^{-1}$  and  $(\zeta - z)^{-1}G(z) = G(z)(\zeta - z)^{-1}$ , for any  $\zeta \neq z \in B \cap \mathbb{C}_I$ . Therefore for any  $z$  in  $\Delta_I$  we have:

$$f_I(z) = \frac{1}{2\pi I} \int_{\partial\Delta_I} \frac{F(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi I} \left( \int_{\partial\Delta_I} \frac{G(\zeta)}{\zeta - z} d\zeta \right) J.$$

Each of these two integrals may now be transformed into a power series as in classical complex analysis. For example (and the same process can be applied to the integral containing  $G$ ) one has, for any  $z$  in  $\Delta_I$ ,

$$\int_{\partial\Delta_I} \frac{1}{1 - \frac{z}{\zeta}} \frac{F(\zeta)}{\zeta} d\zeta = \int_{\partial\Delta_I} \sum_{i=0}^{\infty} \left( \frac{z}{\zeta} \right)^i \frac{F(\zeta)}{\zeta} d\zeta = \sum_{i=0}^{\infty} z^i \left( \int_{\partial\Delta_I} \frac{F(\zeta)}{\zeta^{i+1}} d\zeta \right) \quad (1.3.1)$$

Notice that in the above formula we have chosen to put  $\left( \frac{z}{\zeta} \right)^i$  on the left (instead of on the right) of  $\frac{F(\zeta)}{\zeta}$  so that the power series will have its coefficients on the right, and will be regular in its domain of convergence. Equality (1.3.1) immediately yields that

$$f_I(z) = \sum_{i=0}^{\infty} z^i \frac{1}{i!} \frac{\partial^i f}{\partial z^i}(0).$$

Now, because of the last proposition, we can transform this equation as follows:

$$f_I(z) = \sum_{i=0}^{\infty} z^i \frac{1}{i!} \frac{\partial^i f}{\partial z^i}(0) = \sum_{i=0}^{\infty} z^i \left( \frac{1}{2} \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) \right)^i f(0) = \sum_{i=0}^{\infty} z^i \frac{1}{i!} \frac{\partial^i f}{\partial x^i}(0).$$

In particular this shows that  $f_I(z)$  can be given a series representation in  $z^n$  with coefficients  $a_n = \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0)$  which do not depend at all on the choice of  $I$ . Therefore the representation we have found holds for any  $I \in \mathbb{S}^2$ , and this concludes the proof  $\square$

**Corollary 1.3.10.** *Let  $f : B \rightarrow \mathbb{H}$  be regular. If there exists  $I \in \mathbb{S}^2$  such that  $f(\mathbb{C}_I) \subseteq \mathbb{C}_I$ , then the series expansion of  $f$*

$$f(q) = \sum_{i=0}^{\infty} \frac{q^i}{i!} \frac{\partial^i f}{\partial x^i}(0)$$

*has all its coefficients in  $\mathbb{C}_I$ .*

*Proof.* If  $I \in \mathbb{S}^2$  is such that  $f(\mathbb{C}_I) \subseteq \mathbb{C}_I$ , then for any real number  $x$  we have  $f(x) = f_I(x) \in \mathbb{C}_I$ . Therefore  $\frac{\partial^n f}{\partial x^n} \in \mathbb{C}_I$  for any  $n \in \mathbb{N}$ ,  $x \in \mathbb{R}$ , and the conclusion follows  $\square$

## 1.4 Cauchy integral formulas

The power series expansion which we have proved for regular functions in the last section is the key ingredient in proving the analog, for regular functions, of many well-known results from the theory of holomorphic functions in one variable, such as the identity principle, the maximum modulus, the Cauchy representation and estimates, and the Liouville and Morera theorems (see [62]). This short section is dedicated to the proofs of such results.

A basic result in the theory of regular functions that is valid is the following version of the identity principle (see: [54]):

**Theorem 1.4.1** (Identity principle). *Let  $f : B \rightarrow \mathbb{H}$  be a regular function. Denote by  $Z_f = \{q \in B : f(q) = 0\}$  the zero set of  $f$ . If there exists  $I \in \mathbb{S}^2$  such that  $\mathbb{C}_I \cap Z_f$  has an accumulation point, then  $f \equiv 0$  on  $B$ .*

*Proof.* On  $\mathbb{C}_I \cap B$  we can write

$$f(x + yI) = F(x + yI) + G(x + yI)J$$

with  $F$  and  $G$  holomorphic functions on  $\mathbb{C}_I$ . Now, under the assumption that  $\mathbb{C}_I \cap Z_f$  has an accumulation point, we deduce that both  $F$  and  $G$  are identically zero on  $\mathbb{C}_I \cap B$ . This implies, in particular, that  $\frac{\partial^n f}{\partial x^n}(0) = 0$  for all values of  $n$ . Since these derivatives are the coefficients of the power series expansion of  $f$ , this implies that  $f \equiv 0$  on  $B$ .  $\square$

**Remark 1.4.2.** *Let  $f$  and  $g$  be regular functions on the ball  $B$ . If there exists  $I \in \mathbb{S}^2$  such that  $f \equiv g$  on a subset of  $\mathbb{C}_I \cap B$  having an accumulation point in  $\mathbb{C}_I \cap B$ , then  $f \equiv g$  everywhere on  $B$ .*

Before proving the maximum principle, we need a preliminary result on the mean value property.

**Proposition 1.4.3** ([62]). *If  $f : B \rightarrow \mathbb{H}$  is a regular function, and if  $I \in \mathbb{S}^2$ , then  $f_I : \mathbb{C}_I \cap B \rightarrow \mathbb{H}$  has the mean value property.*

*Proof.* We know, from Lemma 1.3.6, that we can write  $f_I(x + yI) = F(x + yI) + G(x + yI)J$ . Therefore, for all points  $a$  in  $\mathbb{C}_I \cap B$ , and all positive numbers  $r$  such that  $\overline{\Delta(a; r)} \subset \mathbb{C}_I \cap B$ , we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} f_I(a + re^{I\theta}) d\theta &= \frac{1}{2\pi} \int_0^{2\pi} (F(a + re^{I\theta}) + G(a + re^{I\theta})J) d\theta \\ &= F(a) + G(a)J = f_I(a). \end{aligned}$$

This concludes the proof.  $\square$

We are now in a position to prove the maximum modulus principle for regular functions.

**Theorem 1.4.4.** *Let  $f : B \rightarrow \mathbb{H}$  be a regular function. If  $|f|$  has a relative maximum at a point  $a \in B$ , then  $f$  is constant on  $B$ .*

*Proof.* If  $f(a) = 0$  the result is trivial. We will assume therefore that  $f(a) \neq 0$ . via a this transformation  $\left( f(z) \leftarrow \frac{\overline{f(a)}}{|f(a)|} f(z) \right)$ , we can reduce the theorem to the case in which  $f(a) > 0$ . Let then  $I$  be the normalized imaginary part of  $a = x_0 + y_0 I$ , where  $x_0, y_0 \in \mathbb{R}$ , and  $I \in \mathbb{S}^2$ .

For  $r > 0$  sufficiently small, we set that

$$M(r) = \sup_{\theta \in \mathbb{R}} \{|f(a + re^{I\theta})|\}.$$

By hypothesis, we have that  $f(a) \geq M(r)$  when  $r$  is sufficiently small. On the other hand, since  $f_I$  satisfies the mean value property from the previous proposition, we simply obtain that  $f_I(a) = M(r)$ . Let  $z = x + yI$ , for sufficiently small  $r = |z - a|$ , the function  $g(z) = \operatorname{Re}(f_I(a) - f_I(z))$  is non-negative. In fact, we have that  $g(z) = 0$  if and only if  $f_I(z) = f_I(a)$ . By the mean value property

$$f_I(a) = \frac{1}{2\pi} \int_0^{2\pi} f_I(a + re^{I\theta}) d\theta$$

Since the real part of a holomorphic map also satisfies the mean value property, we obtain

$$g(a) = \frac{1}{2\pi} \int_0^{2\pi} g(a + re^{I\theta}) d\theta = 0$$

The function  $g$  is continuous and non-negative on  $\partial \Delta(a, r)$ , and so we obtain that  $g(a + re^{I\theta}) = 0$  for all  $\theta \in \mathbb{R}$ . As a consequence,  $g(z) = 0$  for all  $z$  in the closed disc, and therefore  $f_I(z) = f_I(a)$  for all points  $z$  in  $\Delta(a, r)$ . Since this last set clearly has an accumulation point in  $\mathbb{C}_I \cap B$ , we use the identity principle to conclude the proof  $\square$

In the futur, we will adopt the following notation. If  $q \in B$ , we set

$$I_q = \begin{cases} \frac{\operatorname{Im}(q)}{|\operatorname{Im}(q)|} \in \mathbb{S}^2 & \text{if } \operatorname{Im}(q) \neq 0 \\ \text{any element of } \mathbb{S}^2 & \text{otherwise,} \end{cases}$$

**Remark 1.4.5.** *For any  $\zeta \in \mathbb{C}_{I_q}$ ,  $\zeta \neq q$  the equality*

$$(\zeta - q)^{-1} d\zeta = d\zeta (\zeta - q)^{-1}$$

**Theorem 1.4.6.** Let  $f : B \rightarrow \mathbb{H}$  be a regular function, and let  $q \in B$ . Then

$$f(q) = \frac{1}{2\pi I_q} \int_{\partial \Delta_q(0,r)} \frac{d\zeta}{(\zeta - q)} f(\zeta)$$

where  $\zeta \in \mathbb{C}_{I_q} \cap B$ , and where  $r > 0$  is such that

$$\Delta_q(\bar{0}, r) = \{x + yI_q : x^2 + y^2 \leq r^2\}$$

is contained in  $B$  and contains  $q$ .

*Proof.* The result follows immediately from the splitting lemma, as indicated by the following equalities.

$$\frac{1}{2\pi I_q} \int_{\partial \Delta_q(0,r)} \frac{d\zeta}{(\zeta - q)} f(\zeta) = \frac{1}{2\pi I_q} \int_{\partial \Delta_q(0,r)} \frac{d\zeta}{(\zeta - q)} f I_q(\zeta)$$

□

As a consequence we obtain the Cauchy estimates:

**Theorem 1.4.7.** Let  $f : B(0, R) \rightarrow \mathbb{H}$  be a regular function, and let  $r < R, I \in \mathbb{S}^2$ . Then If  $M_I = \max\{|f(q)| : q \in \partial \Delta_q(0, r)\}$  and if  $M = \inf\{M_I : I \in \mathbb{S}^2\}$ , then

$$\left| \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0) \right| \leq \frac{M}{r^n}, \quad n \geq 0.$$

*Proof.* The result follows the same ideas as in the case of holomorphic functions of a complex variable. Specifically, the proof of the series representation for a regular function shows that, for any  $I \in \mathbb{S}^2$ , we can write

$$\frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0) = \frac{1}{2\pi I} \int_{\partial \Delta_I(0,r)} \frac{d\zeta}{\zeta^{n+1}} f(\zeta)$$

Then we can write

$$\left| \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0) \right| \leq \frac{1}{2\pi} \int_{\partial \Delta_I(0,r)} \frac{|f(\zeta)|}{r^{n+1}} d\zeta \leq \frac{1}{2\pi} \int_{\partial \Delta_I(0,r)} \frac{M_I}{r^{n+1}} d\zeta \leq \frac{M_I}{r^n}.$$

By taking the infimum, for  $I \in \mathbb{S}^2$ , of the right-hand side of the inequality we prove the assertion. □

We now have all the instruments needed to prove the analog of the Liouville theorem.

**Theorem 1.4.8.** Let  $f : \mathbb{H} \rightarrow \mathbb{H}$  be an entire regular map (i.e. a regular map defined and regular everywhere on  $\mathbb{H}$ ). If  $f$  is bounded, i.e. there exists a positive number  $M$  such that  $|f(q)| \leq M$  on all of  $\mathbb{H}$ , then  $f$  is constant.

*Proof.* The Cauchy estimates yield that, for any  $r \in \mathbb{R}$ ,

$$\left| \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0) \right| \leq \frac{M_I}{r^n}.$$

By letting  $r$  go to infinity, we obtain that  $\frac{\partial^n f}{\partial x^n}(0) = 0$  for any positive  $n$ , and this implies that  $f \equiv f(0)$ . Indeed, all the coefficients of the power series representing  $f$  must be zero, with the possible exception of the first one.  $\square$

We close this section with a version of Moreras theorem.

**Theorem 1.4.9.** *Let  $f : B \rightarrow \mathbb{H}$  be a differentiable function. If, for every  $I \in \mathbb{S}^2$ , the differential form  $f(z)dz, z = x + yI, x, y \in \mathbb{R}$  defined on  $\mathbb{C}_I \cap B$  is closed, then the function  $f$  is regular.*

*Proof.* The hypotheses imply, by the classical Morera theorem, that each  $r$  function  $f_I : \mathbb{C}_I \cap B \rightarrow \mathbb{H}$  is holomorphic. This concludes the proof, in view of our definition of regularity.  $\square$

# Chapter 2

## Second Bargmann transform for the slice hyperholomorphic Bergman space and asymptotic

Abstract:

We complete the study of basic properties of the slice hyperholomorphic Bergman space, including the explicit closed formula for its reproducing kernel function. Next, we give the exact statements and the proofs of our main results concerning the quaternionic analogue of the second Bargmann transform, giving rise to an integral representation of slice hyperholomorphic functions in the Bergman space. The last section discusses the asymptotic behavior, as  $R$  goes to  $+\infty$ , of some elements in the slice hyperholomorphic Bergman theory, like measure, basis and the reproducing kernel function. It will be seen that they give rise to their analogues in the setting of slice hyperholomorphic Bargmann-Fock space.

### 2.1 The slice hyperholomorphic Bergman space of the first kind revised

In this section, we will work with the Bergman theory in quaternionic variable of the first kind. We start by recalling some notations and next discuss the so called  $C$ -property (see:[27]).

**Definition 2.1.1.** *Let us denote the complex conjugation on  $\mathbb{C}$  by  $Z_{\mathbb{C}}(z) = \bar{z}$ , for all  $z \in \mathbb{C}$ . We will say that a domain  $\Omega \subset \mathbb{C}$  is  $Z$ -invariant if and only if  $Z_{\mathbb{C}}(\Omega) = \Omega$ .*

In this section, the domains  $\Omega$  we consider are supposed to be  $Z$ -invariant. Note that using the Cauchy-Riemann equations, one obtains that  $f \in Hol(\Omega)$  if and only if  $Z_{\mathbb{C}} \circ f \circ Z_{\mathbb{C}} \in Hol(\Omega)$ .

**Remark 2.1.2.** *In the sequel, the quaternionic conjugation will be denoted by  $Z_{\mathbb{H}}(q) = \bar{q}$ .*

**Definition 2.1.3.** Let  $\Lambda \subseteq \mathbb{H}$ , we say that  $\Lambda$  is axially symmetric if whenever  $q = x + Iqy$  belongs to  $\Lambda$  all the elements  $x + Iy$  belong to  $\Lambda$  for all  $I \in \mathbb{S}^2$ . We say that  $\Lambda$  is a slice domain, or  $s$ -domain for short, if it is a domain intersecting the real axis and such that  $\Lambda \cap \mathbb{C}_I$  is connected for all  $I \in \mathbb{S}^2$ .

In this section,  $I, J \in \mathbb{S}^2$  are mutually orthogonal vectors, and  $\Lambda \subset \mathbb{H}$  will be an axially symmetric  $s$ -domain. Therefore for any  $I$ , the domain  $\Lambda_I$  is  $Z$ -invariant in the complex plane  $\mathbb{C}_I$ .

**Theorem 2.1.4.** (Representation Formula). Let  $f$  be a slice regular function on an axially symmetric  $s$ -domain  $\Lambda \subseteq \mathbb{H}$ . Choose any  $J \in \mathbb{S}^2$ . Then the following equality holds for all  $q = x + Iy \in \Lambda$  :

$$f(x + yI) = \frac{1}{2}(1 - IJ)f(x + yJ) + \frac{1}{2}(1 + IJ)f(x - yJ).$$

The Splitting Lemma and the Representation Formula, imply the good definition of the following operators, which relate the slice regular space with the space of pairs of holomorphic functions on  $\Lambda_I$ :

$$\begin{aligned} Q_I : SR(\Lambda) &\longrightarrow Hol(\Lambda_I) + Hol(\Lambda_I)J \\ f &\longmapsto f|_{\Lambda_I}, \end{aligned}$$

and

$$P_I : Hol(\Lambda_I) + Hol(\Lambda_I)J \longrightarrow SR(\Lambda),$$

defined for any  $f \in Hol(\Lambda_I) + Hol(\Lambda_I)J$  by

$$P_I[f](q) = P_I[f](x + yIq) = \frac{1}{2} [(1 + IqI)f(x - yI) + (1 - IqI)f(x + yI)].$$

Moreover, we have that

$$P_I \circ Q_I = Id_{SR(\Lambda)} \text{ and } Q_I \circ P_I = Id_{Hol(\Lambda_I) + Hol(\Lambda_I)J}$$

where  $Id$  denotes the identity operator.

We now apply the results of the Appendix to deduce some properties of the regular Bergman spaces (see:[50]).

**Definition 2.1.5.** Let  $\Lambda \subset \mathbb{H}$  be a bounded axially symmetric slice domain. We denote by  $A_\alpha^2(\Lambda_I)$  the holomorphic Bergman space associated to  $\Lambda_I$ , and by  $A_\alpha(\Lambda)$  the slice regular Bergman space associated to  $\Lambda$  i.e.

$$A_\alpha(\Lambda) = \left\{ f \in SR(\Lambda) \mid \int_\Lambda |f|^2 dA_\alpha < \infty \right\}.$$

Where  $dA_\alpha = (\alpha + 1) \left(1 - \frac{|q|^2}{R^2}\right)^\alpha dA$ , and  $dA$  denotes the area element in  $\mathbb{H}$ . The set  $A_\alpha(\Lambda)$  is equipped with the norm inherited from the  $L^2(\Lambda, \mathbb{H})$ -space.

In general, as the complex conjugation  $Z_C$  preserves the norm of the complex numbers and the differential element of area, then we have  $f \in A_\alpha^2(\Omega)$  if and only if  $Z_C \circ f \circ Z_C \in A_\alpha^2(\Omega)$ .  
Moreover, we have

$$\int_{\Omega} |f(z)|^2 dA_\alpha(z) = \int_{\Omega} |Z_C \circ f \circ Z_C(z)|^2 dA_\alpha(z).$$

**Proposition 2.1.6.** (Evaluation functional for  $\Lambda_I$ ) Let  $\Lambda$  be a bounded axially symmetric  $s$ -domain, let  $I \in \mathbb{S}^2$  and let  $K \subset \Lambda_I$  be a compact set. Then there exists a constant  $c_K > 0$  such that

$$\sup \{|f(q)|; q \in K\} \leq c_K \|f\|_{A_\alpha(\Lambda)}, \forall f \in A_\alpha(\Lambda).$$

*Proof.* From the [Theorem 2.1,p40] in [108], one can find a constant  $c_K > 0$  depending of  $K$ , such that

$$\sup \{|g(q)|; q \in K\} \leq c_K \left[ \int_{\Lambda_I} |g|^2 dA_\alpha(\xi) \right]^{\frac{1}{2}}, \forall g \in A_\alpha^2(\Lambda_I).$$

Since every  $f \in A_\alpha(\Lambda)$  is of the form  $f|_{\Lambda_I} = f_1 + f_2J$ , where the unit vector  $J$  is orthogonal to  $I$ , then

$$\begin{aligned} \sup \{|f(q)|; q \in K\} &= \sup \{|f|_{\Lambda_I}(q)|, q \in K\} \\ &\leq \sup \{|f_1(q)|, q \in K\} + \sup \{|f_2(q)|, q \in K\} \\ &\leq c_K \left[ \int_{\Lambda_I} |f_1|^2 dA_\alpha(\xi) \right]^{\frac{1}{2}} + c_K \left[ \int_{\Lambda_I} |f_2|^2 dA_\alpha(\xi) \right]^{\frac{1}{2}} \\ &\leq 2c_K \left[ \int_{\Lambda_I} |f|_{\Lambda_I}|^2 dA_\alpha(\xi) \right]^{\frac{1}{2}}. \end{aligned} \tag{2.1.1}$$

□

**Proposition 2.1.7.** (Evaluation functional for  $\Lambda$ ) Let  $\Lambda \subset \mathbb{H}$  be a bounded axially symmetric  $s$ -domain. For any compact set  $K \subset \Lambda$  there exists a constant  $c_K > 0$  such that

$$\sup \{|f(q)|; q \in K\} \leq c_K \|f\|_{A_\alpha(\Lambda)}, \forall f \in A_\alpha(\Lambda).$$

*Proof.* Let  $J \in \mathbb{S}^2$  and let  $K_J := K \cap C_J$ . Note that  $K_J$  is a compact subset of  $\Lambda_J$ , and Proposition 2.1.6 gives us a constant  $c_{K_J} > 0$  such that

$$\sup \{|f(q)|; q \in K_J\} \leq c_{K_J} \|f\|_{A_\alpha(\Lambda)}, \forall f \in A_\alpha(\Lambda).$$

Applying the inequality

$$|f(x + yI)| \leq |f(x + yJ)| + |f(x - yJ)|, \forall x + yI \in K, y \geq 0,$$

which is a direct consequence of Representation Formula

$$f(q) = \frac{1}{2} [(1 + I_q I)f(x - yI) + (1 - I_q I)f(x + yI)].$$

We obtain that

$$\sup \{ |f(q)| ; q \in K_J \} \leq c_{K_J} \|f\|_{A_\alpha(\Lambda)}, \forall f \in A_\alpha(\Lambda).$$

Setting  $c_K = 2c_{K_J}$  the statement follows.  $\square$

**Theorem 2.1.8.** *Let  $\Lambda \subset \mathbb{H}$  be a bounded axially symmetric  $s$ -domain. The space  $(A_\alpha(\Lambda), \|\cdot\|_{A_\alpha(\Lambda)})$  is complete.*

*Proof.* Let  $\{f_n\}_{n \in \mathbb{N}} \subset A_\alpha(\Lambda)$  be a Cauchy sequence, then there exists  $\hat{f} \in L^2(\Lambda, \mathbb{H})$  such that

$$\lim_{n \rightarrow \infty} \| \hat{f} - f_n \|_{L^2(\Lambda, \mathbb{H})} = 0.$$

Moreover, Proposition 2.1.6 implies the following:

- i) There exists the function  $\hat{f} : \Lambda \rightarrow \mathbb{H}$  given by  $\hat{f}(q) := \lim_{n \rightarrow \infty} f_n(q)$ , for any  $q \in \Lambda$ .
- ii) The sequence  $\{f_n\}$  converges uniformly to  $\hat{f}$  on compact sets. Therefore  $\hat{f}$  is a slice regular function on  $\Lambda$ .

Now, for any compact set  $C \subset \Lambda$  there holds that

$$\begin{aligned} 0 &\leq \int_C |\hat{f} - \hat{f}|^2 dA_\alpha \leq \int_C |\hat{f} - f_n|^2 dA_\alpha + \int_C |\hat{f} - f_n|^2 dA_\alpha \\ &\leq \int_C |\hat{f} - f_n|^2 dA_\alpha + \| \hat{f} - f_n \|_{L^2(\Lambda, \mathbb{H})}^2. \end{aligned} \quad (2.1.2)$$

Then the previous inequality tends to zero when  $n \rightarrow \infty$ ,  $\hat{f} = \hat{f} \in A_\alpha(\Lambda)$ .  $\square$

## 2.2 The Slice Regular Bergman Kernel of the First Kind

By theorem 2.1.8 we have  $A_\alpha$  is a quaternionic right-linear Hilbert space. And for any  $q \in \Lambda$ , the evaluation functional  $\Phi_q : A_\alpha(\Lambda) \rightarrow \mathbb{H}$  defined by

$$\Phi_q(f) = f(q), \forall f \in A_\alpha(\Lambda)$$

is a bounded quaternionic right-linear functional on  $A_\alpha(\Lambda)$ . the classical Riesz representation theory in functional analysis shows that each  $q \in \Lambda$ , there exists unique function  $B_q^\alpha$  in  $A_\alpha$  such that

$$f(q) = \langle B_q^\alpha, f \rangle = \int_\Lambda \overline{B_q^\alpha(\xi)} f(\xi) dA_\alpha(\xi) \quad f \in A_\alpha$$

Denoting  $B(q, \cdot) = \overline{B_q^\alpha}$ , then

$$f(q) = \int_{\Lambda} B(q, \cdot) f dA_\alpha \quad f \in A_\alpha.$$

For the future, we want to work on  $\mathbb{B}_R$  the ball quaternionic centred in 0 of the radius  $R$ .

**Definition 2.2.1.** *The function  $B(\cdot, \cdot) : \mathbb{B}_R \times \mathbb{B}_R \longrightarrow \mathbb{H}$  will be called the Bergman kernel of the first kind associated with  $\mathbb{B}_R$ .*

We now recall some properties of the slice regular Bergman kernel of the first kind, then we have the following proposition

**Proposition 2.2.2.** 1) *The slice regular Bergman kernel of the first kind is hermitian;*

$$B(q, r) = \overline{B(r, q)}, \quad r, q \in \Lambda.$$

2) *The function  $B(\cdot, \cdot)$  is slice regular in its first coordinate and it is slice right anti-regular in its second coordinate.*

*Proof.* 1)  $B(q, r) = \overline{B_q(r)} = \overline{\int_{\Lambda} \overline{B_r} B_q dA_\alpha} = \int_{\Lambda} \overline{B_q} B_r dA_\alpha = B_r(q) = \overline{B(r, q)}$ .

2) We have  $B_q \in A_\alpha(\Lambda)$ , then

$$\left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) B_{q|_{\Lambda_I}}(x + yI) = 0, \quad \text{on } \Lambda_I,$$

or equivalently

$$\left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) B_{|_{\Lambda_I}}(x + yI, q) = 0,$$

which means that  $B$  is slice regular in its first variable. Finally, applying the quaternionic conjugation we obtain that

$$B_{|_{\Lambda_I}}(x + yI, q) \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) = 0,$$

which means that  $B$  is slice right anti-regular in its second variable. □

**Proposition 2.2.3.** 1) *The slice regular Bergman kernel of the first kind is the unique function which satisfies the properties of last proposition.*

2) *Let  $\{\phi_n\}_{n \in \mathbb{N}}$  be an orthonormal total family of functions in  $A_\alpha(\Lambda)$ , and let  $K$  be a compact subset of  $\Lambda$ . Then the series*

$$\sum_{n \in \mathbb{N}} \phi_n(q) \overline{\phi_n(r)},$$

*converges uniformly on  $K \times K$  to the Bergman kernel  $B(q, r)$ .*

*Proof.* 1) Suppose that the function  $H$  satisfies the same properties of  $B$ , then  $H(r, q) = \overline{H(q, r)} = \overline{\int_{\Lambda} B(q, \cdot) H(\cdot, r) dA_{\alpha}} = \int_{\Lambda} H(r, \cdot) B(\cdot, q) dA_{\alpha} = B(r, q)$ .

2) Since  $B(\cdot, r) \in A_{\alpha}(\Lambda)$ , there exists a sequence of quaternions  $\{v_n(r)\}_{n \in \mathbb{N}}$ , such that

$$B_{\mathbb{B}_R}(q, r) = \sum_{n \geq 0} \phi_n(q) v_n(r).$$

Note that for any natural number  $m$  we have:

$$\begin{aligned} \overline{\phi_n(r)} &= \overline{\int_{\Lambda} B(r, q) \phi_n(q) dA_{\alpha}(q)} = \int_{\Lambda} \phi_n(q) B(q, r) dA_{\alpha}(q) \\ &= \int_{\Lambda} \phi_n(q) \sum_{n \geq 0} \phi_n(q) v_n(r) dA_{\alpha}(q). \end{aligned} \quad (2.2.1)$$

From the uniform convergence on the compact set and from the fact that  $\phi_n(q)$  are orthonormal, we get:

$$\overline{\phi_n(r)} = v_n(r).$$

Therefore

$$B_{\mathbb{B}_R}(q, r) = \sum_{n \geq 0} \phi_n(q) \overline{\phi_n(r)}.$$

□

**Theorem 2.2.4.** For  $m, n \in \mathbb{N}$ , we have

$$\int_{\mathbb{B}_R} \bar{q}^n q^m dA_{\alpha}(q) \in \mathbb{R}$$

*Proof.* Let  $m, n \in \mathbb{N}$ , then

$$\begin{aligned} \overline{\int_{\mathbb{B}_R} \bar{q}^n q^m dA_{\alpha}(q)} &= \int_{\mathbb{B}_R} \overline{\bar{q}^n q^m} dA_{\alpha}(q) \\ &= \int_{\mathbb{B}_R} \bar{q}^m q^n dA_{\alpha}(q). \end{aligned}$$

Since, the Quaternionic ball centred in 0 of radius  $R$  is  $Z$ -invariant, and also we have  $dA_{\alpha}(q) = dA_{\alpha}(\bar{q})$  thus

$$\begin{aligned} \overline{\int_{\mathbb{B}_R} \bar{q}^n q^m dA_{\alpha}(q)} &= \int_{\mathbb{B}_R} q^m \bar{q}^n dA_{\alpha}(\bar{q}). \\ &= \int_{\mathbb{B}_R} q^m \bar{q}^n dA_{\alpha}(q). \end{aligned}$$

$q$  and  $\bar{q}$  are commute, so

$$\overline{\int_{\mathbb{B}_R} \bar{q}^n q^m dA_{\alpha}(q)} = \int_{\mathbb{B}_R} \bar{q}^n q^m dA_{\alpha}(q) \in \mathbb{R}$$

□

The family of functions  $\{q^n\}_{n \in \mathbb{N}}$  is not an orthonormal family. Nevertheless, the theorem 1.3.9 tells us that this family generates the set of slice regular functions. Thus, the family  $\{q^n\}_{n \in \mathbb{N}}$  is contained in  $A_\alpha(\mathbb{B}_R)$  and generates any element of  $A_\alpha(\mathbb{B}_R)$ . Then we have the following calculation are obtained passing to the spherical coordinates.

For  $m, n \in \mathbb{N}$  such that  $m \geq n$ , we have

$$\int_{\mathbb{B}_R} q^m \bar{q}^n dA_\alpha(q) = \int_{\mathbb{B}_R} q^{m-n} |q|^{2n} dA_\alpha(q)$$

In spherical coordinates, we get

$$q = \sum_{l=0}^3 x_l e_l = x + I_q y$$

where

$$\left\{ \begin{array}{l} x_0 = r \cos \varphi_1, \quad r \in [0, R], \varphi_1 \in [0, \pi[ \\ x_1 = r \sin \varphi_1 \cos \varphi_2, \quad \varphi_2 \in [0, \pi[ \\ x_2 = r \sin \varphi_1 \sin \varphi_2 \cos \varphi_3, \quad \varphi_3 \in [0, 2\pi[ \\ x_3 = r \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \\ x = x_0 \\ y = r \sin \varphi_1 \\ I_q = \frac{\sum_{l=1}^3 x_l e_l}{\sqrt{\sum_{l=1}^3 x_l^2}} \\ dA_\alpha(q) = (\alpha + 1) \left(1 - \frac{r^2}{R^2}\right)^\alpha \frac{r^3 \sin^2 \varphi_1 \sin \varphi_2}{\pi^2 R^4} dr d\varphi_1 d\varphi_2 d\varphi_3 \end{array} \right.$$

So we obtain

$$\begin{aligned} \int_{\mathbb{B}_R} q^m \bar{q}^n dA_\alpha(q) &= \int_0^R \int_0^{2\pi} \int_0^\pi \int_0^\pi \left( r e^{I_q \varphi_1} \right)^{m-n} r^{2n} dA_\alpha(q) \\ &= \int_0^R \int_0^{2\pi} \int_0^\pi \int_0^\pi r^{m-n} [\cos((m-n)\varphi_1) + I_q \sin((m-n)\varphi_1)] r^{2n} dA_\alpha(q) \end{aligned} \quad (2.2.2)$$

An application of the previous theorem, we now get

$$\int_0^R \int_0^{2\pi} \int_0^\pi \int_0^\pi r^{m-n} I_q \sin((m-n)\varphi_1) r^{2n} dA_\alpha(q) = 0$$

So, the integral formula will be as the following

$$\begin{aligned} \int_{\mathbb{B}_R} q^m \bar{q}^n dA_\alpha(q) &= 2\pi \int_0^R (\alpha + 1) \left(1 - \frac{r^2}{R^2}\right)^\alpha \frac{r^{n+m+3}}{\pi^2 R^4} dr \\ &\quad \times \int_0^\pi \cos((m-n)\varphi_1) \sin^2(\varphi_1) d\varphi_1 \int_0^\pi \sin(\varphi_2) d\varphi_2 \\ &= \frac{4}{\pi} \int_0^R (\alpha + 1) \left(1 - \frac{r^2}{R^2}\right)^\alpha \frac{r^{n+m+3}}{R^4} dr \int_0^\pi \cos((m-n)\varphi_1) \sin^2(\varphi_1) d\varphi_1 \end{aligned}$$

The first integral will be as the following formula

$$\begin{aligned}
\int_0^R (\alpha + 1) \left(1 - \frac{r^2}{R^2}\right)^\alpha \frac{r^{n+m+3}}{R^4} dr &= \frac{R^{m+n}}{2} \int_0^1 (\alpha + 1) (1 - r^2)^\alpha r^{n+m+3} dr \\
&= \frac{R^{m+n}}{2} \int_0^1 (\alpha + 1) (1 - r)^\alpha r^{\frac{n+m+2}{2}} dr \\
&= \frac{R^{m+n}}{2} (\alpha + 1) \mathcal{B}\left(\alpha + 1, \frac{n+m+4}{2}\right) \\
&= \frac{R^{m+n}}{2} (\alpha + 1) \frac{\Gamma(\alpha + 1) \Gamma\left(\frac{n+m+4}{2}\right)}{\Gamma\left(\alpha + \frac{n+m+6}{2}\right)} \\
&= \frac{R^{m+n}}{2} \frac{\Gamma(\alpha + 2) \Gamma\left(\frac{n+m}{2} + 2\right)}{\Gamma\left(\alpha + \frac{n+m}{2} + 3\right)}.
\end{aligned}$$

In the second integral we get

$$\begin{aligned}
\int_0^\pi \cos((m-n)\varphi_1) \sin^2(\varphi_1) d\varphi_1 &= \int_0^\pi \frac{\cos((m-n)\varphi_1)}{2} \\
&\quad - \frac{\cos((m-n+2)\varphi_1) + \cos((m-n-2)\varphi_1)}{4} d\varphi_1
\end{aligned}$$

After computation, we obtain

$$\int_0^\pi \frac{\cos((m-n)\varphi_1)}{2} \sin^2(\varphi_1) d\varphi_1 = \begin{cases} \pi/2 & \text{if } m = n \\ -\pi/4 & \text{if } m = n + 2 \\ 0 & \text{if } m > n \text{ and } m \neq n + 2 \end{cases}$$

Therefore

$$\int_{\mathbb{B}_R} q^m \bar{q}^n dA_\alpha(q) = \begin{cases} R^{2n} \frac{\Gamma(\alpha+2)\Gamma(n+2)}{\Gamma(\alpha+n+3)} & \text{if } m = n \\ -\frac{R^{2n+2}}{2} \frac{\Gamma(\alpha+2)\Gamma(n+3)}{\Gamma(\alpha+n+4)} & \text{if } m = n + 2 \\ 0 & \text{if } m > n \text{ and } m \neq n + 2 \end{cases}$$

Finally,  $\{q^n\}_{n \in \mathbb{N}}$  does not an orthogonal system in  $SR(\mathbb{B}_R)$ .

By the orthonormalization process gives us the following

$$\begin{aligned}
v_1 &= q; \text{ set } \phi_1 = \frac{v_1}{\|v_1\|_{A_\alpha(\mathbb{B}_R)}} \\
v_2 &= q^2 - \langle q^2, \phi_1 \rangle_{A_\alpha(\mathbb{B}_R)} \phi_1; \text{ set } \phi_2 = \frac{v_2}{\|v_2\|_{A_\alpha(\mathbb{B}_R)}} \\
&\vdots \\
v_n &= q^n - \sum_{j=1}^{n-1} \langle q^n, \phi_j \rangle_{A_\alpha(\mathbb{B}_R)} \phi_j; \text{ set } \phi_n = \frac{v_n}{\|v_n\|_{A_\alpha(\mathbb{B}_R)}}
\end{aligned}$$

Therefore,  $\{\phi_n\}_{n \in \mathbb{N}}$  is an orthonormal total family contained in  $A_\alpha(\mathbb{B}_R)$ .  
As consequence, we now can define our Bergman space kernel was given

**Proposition 2.2.5.** *The slice regular Bergman kernel of the first kind for the unit ball  $\mathbb{B}_R$  is given by*

$$B_{\mathbb{B}_R}(q, r) = \frac{\Gamma(\alpha + 4)}{R^2\Gamma(3)\Gamma(\alpha + 2)}q\bar{r} + \frac{\Gamma(\alpha + 5)}{R^4\Gamma(4)\Gamma(\alpha + 2)}q^2\bar{r}^2 \\ + \sum_{n \geq 3} \lambda_{\alpha, n} \left[ q^n + \frac{nR^2}{2(\alpha + n + 1)}q^{n-2} \right] \left[ \bar{r}^n + \frac{nR^2}{2(\alpha + n + 1)}\bar{r}^{n-2} \right].$$

$$\text{where } \lambda_{\alpha, n} = \frac{\Gamma(\alpha + n + 3)}{\Gamma(\alpha + 2)\Gamma(n + 1)R^{2n}} \left[ n + 1 - \frac{n(n + \alpha + 2)}{4(n + \alpha + 1)} \right]^{-1}$$

*Proof.* Applying the orthonormalization of Gramm-Schmid process on the family  $\{q^n\}_{n \in \mathbb{N}}$ , we check the following orthonormal system  $\{\phi_n\}_{n \in \mathbb{N}}$

$$\phi_1(q) = \sqrt{\frac{\Gamma(\alpha + 4)}{R^2\Gamma(3)\Gamma(\alpha + 2)}}q;$$

$$\phi_2(q) = \sqrt{\frac{\Gamma(\alpha + 5)}{R^4\Gamma(4)\Gamma(\alpha + 2)}}q^2;$$

⋮

$$\phi_n(q) = \sqrt{\frac{\Gamma(\alpha + n + 3)}{n!R^{2n}\Gamma(\alpha + 2)}} \left[ n + 1 - \frac{n(n + \alpha + 2)}{4(n + \alpha + 1)} \right]^{-1/2} \left[ q^n + \frac{nR^2}{2(\alpha + n + 1)}q^{n-2} \right], \quad n > 2.$$

Thus, the Bergman space kernel in first kind given by

$$B_{\mathbb{B}_R}(q, r) = \sum_{n \geq 0} \phi_n(q)\overline{\phi_n(r)}.$$

□

## 2.3 The slice hyperholomorphic Bergman space of the second kind

The slice hyperholomorphic Bergman space on the open unit ball  $\mathbb{B}$  (centered at the origin) was first presented in [24]. This was possible by extending the complex holomorphic functions on the disc to the whole  $\mathbb{B}$  by the representation formula (Theorem 2.1.4). For arbitrary radius  $R$ , the slice hyperholomorphic Bergman space  $A_{slice}^{2, \alpha}(\mathbb{B}_R)$  is defined by (0.0.4). Namely,

**Definition 2.3.1.** For every  $I \in \mathbb{S}^2$ , the slice hyperholomorphic Bergman spaces is defined by

$$A_{slice}^{2,\alpha}(\mathbb{B}_R) := \{f \in \mathcal{SR}(\mathbb{B}_R) \mid \|f\|_{A_\alpha(\mathbb{B}_{R,I})}^2 := \int_{\mathbb{B}_{R,I}} |f|_{\mathbb{B}_{R,I}}^2 dA_{\alpha,I} < \infty\},$$

where  $\mathbb{B}_{R,I} = \mathbb{B}_R \cap \mathbb{C}_I$  and  $dA_{\alpha,I}$  denotes the area measure in the complex plane  $\mathbb{B}_{R,I}$  given by:

$$dA_{\alpha,I}(z) = \left(1 - \frac{|z|^2}{R^2}\right)^{\alpha-1} \frac{\alpha}{\pi R^2} dx dy, \quad z = x + iy.$$

On The Bergman space  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  is quaternionic right linear Hilbert space. So, on  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  we define the scalar product

$$\langle f, g \rangle_{A_{slice}^{2,\alpha}(\mathbb{B}_R)} = \int_{\mathbb{B}_{R,I}} \bar{f}(z)g(z) \left(1 - \frac{|z|^2}{R^2}\right)^{\alpha-1} \frac{\alpha}{\pi R^2} dx dy.$$

Therefore, we have the following result:

**Proposition 2.3.2.** The spaces  $(A_{slice}^{2,\alpha}(\mathbb{B}_R), \|\cdot\|_{A_{slice}^{2,\alpha}(\mathbb{B}_R)})$  contains the monomials  $e_n(q) := q^n$ ,  $n \in \mathbb{N}$  which form an orthogonal basis. The square norm of the  $e_n$  is given by

$$\|e_n\|_{\alpha,slice}^2 = \frac{n!R^{2n}\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}. \quad (2.3.1)$$

*Proof.* The first assertion follows by similar arguments as in the classical case. The norm of  $e_n(q) := q^n = r^n e^{In\theta}$  can be computed easily using the polar coordinates and making use of the appropriate change of variable  $t = r^2/R^2$ . Indeed,

$$\begin{aligned} \langle e_n, e_m \rangle_{\alpha,slice} &= \left(\frac{\alpha}{\pi R^2}\right) \int_{\mathbb{B}_{R,I}} \bar{q}^n q^m \left(1 - \frac{|q|^2}{R^2}\right)^{\alpha-1} dx dy \\ &= \alpha R^{2n} \delta_{m,n} \int_0^1 t^n (1-t)^{\alpha-1} dt \\ &= \frac{n!R^{2n}\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \delta_{m,n}. \end{aligned}$$

The last equality follows making use of  $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$  as well as of the well-known Euler's Beta integral [29, Theorem 7, p. 19]

$$\int_0^1 t^{a-1} (1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

valid for  $\Re(a) > 0$  and  $\Re(b) > 0$ . □

Now, choose  $I \in \mathbb{S}^2$ , and let us  $f$  an entire slice regular function in  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$ . According to the theorem 1.3.9, we have

$$f(q) = \sum_{i=0}^{\infty} \frac{q^n}{n!} \frac{\partial^n f}{\partial x^n}(0),$$

for all  $q$  in  $\mathbb{H}$  and  $a_n$  are a constants in  $\mathbb{H}$ . Therefore, the norm of  $f$  was written by

$$\|f\|_{A_{slice}^{2,\alpha}(\mathbb{B}_R)}^2 = \sum_{n \geq 0} |a_n|^2 \langle q^n, q^n \rangle_{A_{slice}^{2,\alpha}(\mathbb{B}_R)},$$

where

$$\langle q^n, q^n \rangle_{A_{slice}^{2,\alpha}(\mathbb{B}_R)} = \frac{n! R^{2n} \Gamma(\alpha + 2)}{\Gamma(n + \alpha + 2)}.$$

Let now  $J \in \mathbb{S}^2$ , such that  $I \neq J$ . Since  $\langle q^n, q^n \rangle_{A_{slice}^{2,\alpha}(\mathbb{B}_R)}$  is a real, imply that is does not depend to  $I$ , namely

$$\|f\|_{A_{slice}^{2,\alpha}(\mathbb{B}_R)}^2 = \|f\|_{A_{\alpha}(\mathbb{B}_{R,I})}^2.$$

Thus, the norm of  $f$  does not depend to imaginary unit.

When  $R=1$ , the quaternionic Bergman space in setting of the slise regular functions was introduced by Colombo and al. [24]. So, all analytics properties are still holds however much  $R$  be nonnegatif real. Then, we have the following results:

**Theorem 2.3.3.** *The space  $(A_{slice}^{2,\alpha}(\mathbb{B}_R), \|\cdot\|_{A_{slice}^{2,\alpha}(\mathbb{B}_R)})$  is complete.*

*Proof.* Let  $\{f_n\}_n$  be a sequence in  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$ . In particular case in complex plane, we obtain that  $\{f_n|_{\mathbb{B}_{R,I}}\}_n$  converges uniformly on every compact subset of  $\mathbb{B}_{R,I}$  to  $f|_{\mathbb{B}_{R,I}}$ , i.e. that  $\{f_n\}_n \rightarrow f$  in  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$ . Therefore the completeness of  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  holds.  $\square$

**Proposition 2.3.4.** *For any  $q \in \mathbb{B}_{R,I}$ , the evaluation functional  $\varphi_q : A_{slice}^{2,\alpha}(\mathbb{B}_R) \rightarrow \mathbb{H}$ , defined by*

$$\varphi_q(f) := f(q)$$

*is a bounded quaternionic right-linear functional on  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$ .*

*Proof.* With Spilliting lemma, we can write  $f|_{\mathbb{B}_{R,I}} = F + GJ$ , where  $F, G : \mathbb{B}_{R,I} \rightarrow \mathbb{C}_I$ , are holomorphic functions and  $J \perp I$ , then for  $f|_{\mathbb{B}_{R,I}}$  in the set consists of all  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  there holds: two following inequalities

$$\int_{\mathbb{B}_{R,I}} |F|^2 dA_{\alpha,I} \leq \|f|_{\mathbb{B}_{R,I}}\|_{A_{slice}^{2,\alpha}(\mathbb{B}_R)}^2.$$

and

$$\int_{\mathbb{B}_{R,I}} |G|^2 dA_{\alpha,I} \leq \|f|_{\mathbb{B}_{R,I}}\|_{A_{slice}^{2,\alpha}(\mathbb{B}_R)}^2.$$

We can claim that the evaluation functional is a bounded quaternionic right-linear functional on  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$ .  $\square$

The Riesz representation theorem [24] for quaternionic right-linear Hilbert spaces shows the existence of the unique function  $K_q(I) \in A_{slice}^{2,\alpha}(\mathbb{B}_R)$ , such that

$$\varphi_q[f|_{\mathbb{B}_{R,I}}] = \langle K_q(I), f|_{\mathbb{B}_{R,I}} \rangle_{A_{slice}^{2,\alpha}(\mathbb{B}_R)}.$$

Denoting  $K_{R,slice}^\alpha(q, \cdot) := \overline{K_q(I)}$  we have

$$f|_{\mathbb{B}_{R,I}}(q) = \int_{\mathbb{B}_{R,I}} K_{R,slice}^\alpha(q, z) f(z) dA_{\alpha,I}(z), \quad \forall f|_{\mathbb{B}_{R,I}} \in A_{slice}^{2,\alpha}(\mathbb{B}_R),$$

Notice for instance that by the representation formula (Lemma 2.1.4), the involved integral does not depend on the choice of  $I \in \mathbb{S}^2$ . Moreover, it is clear that the restriction  $K_{R,I}^\alpha := K_{R,slice}^\alpha|_{\mathbb{B}_{R,I} \times \mathbb{B}_{R,I}}$  of the Bergman kernel to  $\mathbb{B}_{R,I} \times \mathbb{B}_{R,I}$  coincides with the classical Bergman kernel  $K_{R,hol}^\alpha(z, w)$  on  $\mathbb{B}_{R,I}$  given by

$$K_{R,hol}^\alpha(z, w) = \left(1 - \frac{z\bar{w}}{R^2}\right)^{-\alpha-1} = K_{R,I}^\alpha(z, w). \quad (2.3.2)$$

The answer is contained in the next result.

**Proposition 2.3.5.** *The Bergman kernel  $K_{R,slice}^\alpha$  coincides with the classical Bergman kernel on  $\mathbb{B}_{R,I}$ , for any  $I \in \mathbb{S}^2$ .*

*Proof.* By the Splitting Lemma,  $f|_{\mathbb{B}_{R,I}} = F + GJ$ , where  $F, G \in Hol(\mathbb{B}_{R,I})$ . So, we have

$$\varphi_q[f|_{\mathbb{B}_{R,I}}] = \langle K_q(I), F + GJ \rangle_{A_{slice}^{2,\alpha}(\mathbb{B}_R)} \quad (2.3.3)$$

$$= \langle K_q(I), F \rangle_{A_{slice}^{2,\alpha}(\mathbb{B}_R)} + \langle K_q(I), G \rangle_{A_{slice}^{2,\alpha}(\mathbb{B}_R)} J, \quad \forall f|_{\mathbb{B}_{R,I}} \in A_{slice}^{2,\alpha}(\mathbb{B}_R). \quad (2.3.4)$$

If  $q \in \mathbb{C}_I$ , we get

$$F(q) = \int_{\mathbb{B}_{R,I}} K_{R,slice}^\alpha(q, z) F(z) dA_{\alpha,I}(z),$$

and

$$G(q) = \int_{\mathbb{B}_{R,I}} K_{R,slice}^\alpha(q, z) G(z) dA_{\alpha,I}(z).$$

However, on the complex plane  $\mathbb{C}_I$ , the holomorphic functions  $F, G$  can be written using the classical Bergman kernel in one complex variable. By the uniqueness of the function  $K_{R,slice}^\alpha|_{\mathbb{B}_{R,I} \times \mathbb{B}_{R,I}}(\cdot, \cdot)$ , it coincides with the classical Bergman kernel.  $\square$

**Definition 2.3.6.** *Let  $p$  be a quaternionic in  $\mathbb{H}$ . Let us consider the function defined by*

$$K_{R,slice}^\alpha(x + yI, p) := \frac{1}{2}(1 - II_p)K_{R,I_p}^\alpha(x + yI_p, p) + \frac{1}{2}(1 + II_p)K_{R,I_p}^\alpha(x - yI_p, p) \quad (2.3.5)$$

Both functions defined by the right-hand side of (2.3.5) is slice regular, by construction, and coincide with the Bergman kernel  $K_{\mathbb{B}_{R,I}}(z, w)$  on the complex plane  $\mathbb{C}_I$ . Thus, by the identity principle they coincide on  $\mathbb{B}_R$ .

Then, the following definition is well posed.

**Proposition 2.3.7.** *The function*

$$\begin{aligned} K_{R,slice}^\alpha(x + yI, p) &= \frac{1}{2}(1 - II_p)K_{R,I_p}^\alpha(x + yI_p, p) + \frac{1}{2}(1 + II_p)K_{R,I_p}^\alpha(x - yI_p, p) \\ &= \sum_{n \in \mathbb{N}} \frac{\Gamma(n + \alpha + 1)}{n!R^{2n}\Gamma(\alpha + 1)}(x + yI)^n(\bar{p})^n, \end{aligned} \tag{2.3.6}$$

so-called slice Bergman kernel of the first kind.

Some standard Properties of the slice Bergman kernel of the first kind are summaries below.

**Proposition 2.3.8.** *The slice Bergman kernel of the first kind  $K_{R,slice}^\alpha$  satisfies the properties:*

- (i)  $K_{R,slice}^\alpha(q, p) = \overline{K_{R,slice}^\alpha(p, q)}$ , for  $p, q \in \mathbb{B}_R$ .
- (ii)  $K_{R,slice}^\alpha(\cdot, \cdot)$  is slice right anti-regular with respect to its first variable.

*Proof.* (i) We have the following equality :

$$\begin{aligned} \overline{K_{R,slice}^\alpha(x + yI, p)} &= \overline{\frac{1}{2}(1 - II_p)K_{R,I_p}^\alpha(x + yI_p, p) + \frac{1}{2}(1 + II_p)K_{R,I_p}^\alpha(x - yI_p, p)} \\ &= \overline{K_{R,I_p}^\alpha(x + yI_p, p)}\frac{1}{2}(1 - II_p) + \overline{K_{R,I_p}^\alpha(x - yI_p, p)}\frac{1}{2}(1 + II_p) \\ &= K_{R,I_p}^\alpha(p, x + yI_p)\frac{1}{2}(1 - I_pI) + K_{R,I_p}^\alpha(p, x - yI_p)\frac{1}{2}(1 + I_pI). \end{aligned}$$

Since (2.3) is the right hand side of the representation formula for slice right anti-regular functions, we get:

$$\overline{K_{R,slice}^\alpha(x + yI, p)} = K_{R,slice}^\alpha(p, x + yI),$$

which is equivalent to the formula (i).

- (ii) It follows from representation formula and ant-holomorphic with respect to its first variable of  $K_{\mathbb{B}_{R,I}}(\cdot, \cdot)$ . □

**Remark 2.3.9.** *The integral*

$$f(q) = \int_{\mathbb{B}_{R,I}} K_{R,slice}^\alpha(q, z)f(z)dA_{\alpha,I}(z)$$

does not depend on  $I \in \mathbb{S}^2$  by the representation formula.

The explicit expression of  $K_{R,slice}^\alpha(q, p)$  for  $\alpha = 1$  and  $R = 1$ , is proved in [25, Theorem 4.1-Proposition 4.3] to be given by

$$K_{R,slice}^\alpha(q, p) = (1 - 2\bar{q}\bar{p} + \bar{q}^2\bar{p}^2)(1 - 2\Re(q)\bar{p} + |q|^2\bar{p}^2)^{-2} \quad (2.3.7)$$

$$= (1 - 2q\Re(p) + q^2|p|^2)^{-2}(1 - 2qp + q^2p^2). \quad (2.3.8)$$

A direct computation shows that the two expressions (2.3.7) and (2.3.8) are the same. For general  $\alpha > 0$ , the expression of  $K_{R,slice}^\alpha(q, p)$  can be given in terms of the special function

$$I^a(q, p) := \sum_{n=0}^{\infty} \frac{(a)_n}{n!} q^n p^n, \quad (2.3.9)$$

with real parameter  $a$  and quaternionic variables  $q, p \in \mathbb{H}$ , which is a particular case of the left-sided Gauss hypergeometric function

$${}_2F^*_1 \left( [q, p] \middle| \begin{matrix} a, b \\ c \end{matrix} \right) = \sum_{n=0}^{\infty} \frac{q^n p^n}{n!} \frac{(a)_n (b)_n}{(c)_n} \quad (2.3.10)$$

defined here for real  $c$  and quaternionic  $a, b \in \mathbb{H}$ . Above  $(a)_k$  denotes the Pochhammer symbol  $(a)_k = a(a+1) \cdots (a+k-1)$  with  $(a)_0 = 1$ . The above series converges absolutely and uniformly on  $K \times K'$  for any compact subsets  $K, K' \subset \mathbb{B}_R$ .

**Theorem 2.3.10.** *The weighted Bergman kernel is given by*

$$K_{R,slice}^\alpha(q, p) = I^{-\alpha-1} \left( \frac{\bar{q}}{R'}, \frac{\bar{p}}{R} \right) \left( 1 - 2\frac{\Re(q)\bar{p}}{R^2} + \frac{|q|^2\bar{p}^2}{R^4} \right)^{-\alpha-1} \quad (2.3.11)$$

and

$$K_{R,slice}^\alpha(q, p) = I^{\alpha+1} \left( \frac{\bar{q}}{R'}, \frac{\bar{p}}{R} \right). \quad (2.3.12)$$

*Proof.* □

The explicit expression of  $K_{R,slice}^\alpha(q, p)$  in (2.3.11) in terms of the special function  $I^{-\alpha-1}$  in (2.3.9) can be suggested starting from (2.3.2) and using the extension Lemma 2.1.4. Indeed, fix  $p \in \mathbb{B}_R$  and let  $q = x + Iy \in \mathbb{B}_R$  and take  $z_p$  to be  $z_p = x + I_p y$ . Then, by the representation formula and the explicit expression of the classical weighted Bergman kernel given through (2.3.2), we obtain

$$\begin{aligned} K_{R,slice}^\alpha(q, p) &= K_{R,slice}^\alpha(x + Iy, p) \\ &= \frac{1}{2} (K_{R,slice}^\alpha(z_p, p) + K_{R,slice}^\alpha(\bar{z}_p, p)) + \frac{II_p}{2} (K_{R,slice}^\alpha(\bar{z}_p, p) - K_{R,slice}^\alpha(z_p, p)) \\ &= \frac{1}{2} \left( \frac{\left(1 - \frac{z_p\bar{p}}{R^2}\right)^{\alpha+1} + \left(1 - \frac{\bar{z}_p\bar{p}}{R^2}\right)^{\alpha+1}}{\left(1 - \frac{z_p\bar{p}}{R^2}\right)^{\alpha+1} \left(1 - \frac{\bar{z}_p\bar{p}}{R^2}\right)^{\alpha+1}} \right) + \frac{II_p}{2} \left( \frac{\left(1 - \frac{z_p\bar{p}}{R^2}\right)^{\alpha+1} - \left(1 - \frac{\bar{z}_p\bar{p}}{R^2}\right)^{\alpha+1}}{\left(1 - \frac{z_p\bar{p}}{R^2}\right)^{\alpha+1} \left(1 - \frac{\bar{z}_p\bar{p}}{R^2}\right)^{\alpha+1}} \right). \end{aligned}$$

Straightforward computation shows that

$$\left(1 - \frac{z_p \bar{p}}{R^2}\right)^{\alpha+1} \left(1 - \frac{\bar{z}_p p}{R^2}\right)^{\alpha+1} = \left(1 - 2\frac{\Re(q)\bar{p}}{R^2} + \frac{|q|^2 \bar{p}^2}{R^4}\right)^{\alpha+1}$$

and

$$\left(1 - \frac{z_p \bar{p}}{R^2}\right)^{\alpha+1} \pm \left(1 - \frac{\bar{z}_p p}{R^2}\right)^{\alpha+1} = \sum_{k=0}^{\infty} \frac{(-\alpha-1)_k}{k!} (z_p^k \pm \bar{z}_p^k) \left(\frac{\bar{p}}{R^2}\right)^k.$$

The equality follows using the binomial theorem for real exponent

$$(a-b)^\beta = \sum_{k=0}^{\infty} \frac{(-\beta)_k}{k!} a^{\beta-k} b^k$$

valid for  $|a| > |b|$ . Consequently, the expression of  $K_{R,slice}^\alpha(q, p)$  becomes

$$\begin{aligned} K_{R,slice}^\alpha(q, p) &= \left( \sum_{k=0}^{\infty} \frac{(-\alpha-1)_k}{k!} \frac{1}{2} \left[ (z_p^k + \bar{z}_p^k) + II_p(z_p^k - \bar{z}_p^k) \right] \left(\frac{\bar{p}}{R^2}\right)^k \right) \\ &\quad \times \left( 1 - 2\frac{\Re(q)\bar{p}}{R^2} + \frac{|q|^2 \bar{p}^2}{R^4} \right)^{-\alpha-1}. \end{aligned}$$

Now, making use of the well-established fact

$$\bar{q}^k = (x - Iy)^k = \frac{1}{2} \left( ((x + Jy)^k + (x - Jy)^k) + IJ((x + Jy)^k - (x - Jy)^k) \right),$$

for every nonnegative integer  $k$  and arbitrary  $q \in \mathbb{H}$  and  $J \in \mathbb{S}$ , we conclude easily that

$$K_{R,slice}^\alpha(q, p) = \left( \sum_{k=0}^{\infty} \frac{(-\alpha-1)_k}{k!} \frac{\bar{q}^k \bar{p}^k}{R^{2k}} \right) \left( 1 - 2\frac{\Re(q)\bar{p}}{R^2} + \frac{|q|^2 \bar{p}^2}{R^4} \right)^{-\alpha-1}.$$

The proof presented below is more simpler.

*Proof.* Fix  $I \in \mathbb{S}$  and  $q \in \mathbb{B}_{R,I}$ . The equality (2.3.11) holds trivially for every  $p \in \mathbb{B}_{R,I}$  since both sides of (2.3.11) reduce further to  $K_{R,I}^\alpha(q, p)$  in (2.3.2). The assertion of Theorem 2.3.10 for arbitrary  $p \in \mathbb{B}_R$  immediately follows from the identity principle (Lemma 1.4.1) for right anti-slice regular functions. Indeed, the function

$$p \longmapsto \overline{K_{R,slice}^\alpha(q, p)} - I^{-\alpha-1} \left( \frac{p}{R}, \frac{q}{R} \right) \left( 1 - 2\frac{\Re(q)p}{R^2} + \frac{|q|^2 p^2}{R^4} \right)^{-\alpha-1}$$

vanishes on  $\mathbb{B}_{R,I}$  and is left slice regular for the coefficients in the expansion series of  $I^{-\alpha-1}$  being reals. Thus, it is identically zero on the whole  $\mathbb{B}_R$ . The proof of (2.3.12) can be handled using similar arguments based essentially on the counterpart of the identity principle for right slice regular functions. This completes the proof.  $\square$

**Remark 2.3.11.** For  $\alpha$  being a nonnegative integer the expression (2.3.11) reduces further to the following

$$K_{R,slice}^\alpha(q, p) = P_{\alpha+1} \left( \frac{\bar{q}}{R'}, \frac{\bar{p}}{R} \right) \left( 1 - 2 \frac{\Re(q)\bar{p}}{R^2} + \frac{|q|^2 \bar{p}^2}{R^4} \right)^{-\alpha-1}, \quad (2.3.13)$$

where  $P_{\alpha+1}(\bar{q}, \bar{p})$  is the polynomial of degree  $\alpha + 1$  given by

$$P_{\alpha+1}(\bar{q}, \bar{p}) = \sum_{k=0}^{\alpha+1} \frac{(-\alpha-1)_k}{k!} \bar{q}^k \bar{p}^k.$$

When taking  $\alpha = 1$ , we recover (2.3.7) obtained in [25].

**Corollary 2.3.12.** We have the following identity (for the hypergeometric function  ${}_2F^*_1$  in (4.2.9)),

$$I^{\alpha+1} \left( \frac{q}{R'}, \frac{\bar{p}}{R} \right) = I^{-\alpha-1} \left( \frac{\bar{q}}{R'}, \frac{\bar{p}}{R} \right) \left( 1 - 2 \frac{\Re(q)\bar{p}}{R^2} + \frac{|q|^2 \bar{p}^2}{R^4} \right)^{-\alpha-1}. \quad (2.3.14)$$

In particular, for  $\alpha = 1$ , we have

$$I^2 \left( \frac{q}{R'}, \frac{\bar{p}}{R} \right) = \left( 1 - 2 \frac{q\Re(p)}{R^2} + \frac{q^2|p|^2}{R^4} \right)^{-2} \left( 1 - 2 \frac{qp}{R^2} + \frac{q^2p^2}{R^4} \right) \quad (2.3.15)$$

*Proof.* The identity (2.3.14) follows by equating the right hand sides in both (2.3.11) and (2.3.12). It can also be obtained using similar arguments as in the proof of Theorem 2.3.10. A direct proof outside the framework of slice regular functions seems to be hard to obtain for the lack of commutativity in the quaternions. The second identity (2.3.15) is a particular case keeping in mind the expression of  $K_{R,slice}^\alpha(q, p)$ , for arbitrary  $R$ , given through (2.3.8).  $\square$

**Remark 2.3.13.** The formula (2.3.12) and therefore the identity (2.3.14) can be reproved using Proposition 2.3.1 below, since the  $K_{R,slice}^\alpha(q, p)$  can be realized as

$$K_{R,slice}^\alpha(q, p) = \sum_{n=0}^{\infty} \phi_n(q) \overline{\phi_n(p)} \quad (2.3.16)$$

for any orthonormal total family of functions  $(\phi_n)_n$  in  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$ . The involved series converges uniformly on  $K \times K$  for any compact subset  $K \subset \mathbb{B}_R$ .

As immediate consequence of Proposition 2.3.1, one can easily obtain the following

$$\langle f, g \rangle_{\alpha, slice} = \sum_{n \geq 0} \frac{n! R^{2n} \Gamma(\alpha + 1)}{\Gamma(n + \alpha + 1)} \bar{a}_n b_n$$

for any  $f(q) = \sum_{n=0}^{\infty} q^n a_n$  and  $g(q) = \sum_{n=0}^{\infty} q^n b_n$  in  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$ . In particular, we assert

**Corollary 2.3.14.** A given  $f(q) = \sum_{n=0}^{\infty} q^n a_n$ ;  $a_n \in \mathbb{H}$ , belongs to  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  if and only if the coefficients  $a_n$  satisfies the growth condition

$$\|f\|_{\alpha,slice}^2 = \sum_{n=0}^{\infty} \frac{n! R^{2n} \Gamma(\alpha + 1)}{\Gamma(n + \alpha + 1)} |a_n|^2 < +\infty. \quad (2.3.17)$$

**Remark 2.3.15.** The identity (2.3.17) shows in particular that the quantity  $\|f\|_{\alpha,slice}^2$  is independent of the choice of the purely imaginary unit  $I \in \mathbb{S}$ .

Now, let define  $A_{R,slice}^{\alpha}$  to be

$$A_{R,slice}^{\alpha}(t; x + yI_q) = \frac{1 - I_q J}{2} A_{R,hol}^{\alpha}(t; x + yJ) + \frac{1 + I_q J}{2} A_{R,hol}^{\alpha}(t; x - yJ) \quad (2.3.18)$$

defined on  $\mathbb{R}^+ \times \mathbb{B}_R$ , where

$$A_{R,hol}^{\alpha}(t; z) := \exp\left(\frac{tz}{z - R}\right) \left(1 - \frac{z}{R}\right)^{-\alpha-1} \quad (2.3.19)$$

is the kernel function of the second Bargmann transform (0.0.1). The following result shows that the sliced weighted Bergman kernel  $K_{R,slice}^{\alpha}$  is connected to the kernel function  $A_{R,slice}^{\alpha}$ .

**Theorem 2.3.16.** For every  $q, q' \in \mathbb{B}_R$ , we have

$$\int_0^{+\infty} A_{R,slice}^{\alpha}(t; q) \overline{A_{R,slice}^{\alpha}(t; q')} \frac{t^{\alpha} e^{-t}}{\Gamma(\alpha + 1)} dt = K_{R,slice}^{\alpha}(q, q').$$

*Proof.* Thanks to the identity principle for slice regular functions, we need only to prove the result for a fixed  $I \in \mathbb{S}$ . In fact, the function  $t \mapsto A_{R,hol}^{\alpha}(t; z)$  belongs to  $L^{2,\alpha}(\mathbb{R}^+, \mathbb{C}_I)$  and satisfies

$$\int_0^{+\infty} A_{R,hol}^{\alpha}(t; z) \overline{A_{R,hol}^{\alpha}(t; w)} \frac{t^{\alpha} e^{-t}}{\Gamma(\alpha + 1)} dt = K_{R,I}^{\alpha}(z, w)$$

for every fixed  $z, w \in \mathbb{B}_{R,I}$ . This follows readily using the generating function character of the kernel function  $A_{R,hol}^{\alpha}(t; z)$  in (2.3.19), to wit  $A_{R,hol}^{\alpha}(t; z) = \sum_{n=0}^{\infty} z^n L_n^{(\alpha)}(t)$ , combined with the orthogonality property [29, Eq. (4), p. 205 - Eq. (7), p. 206]

$$\int_0^{+\infty} L_n^{(\alpha)}(t) L_m^{(\alpha)}(t) \frac{t^{\alpha} e^{-t}}{\Gamma(\alpha + 1)} dt = \frac{\Gamma(\alpha + n + 1)}{\Gamma(n + 1)\Gamma(\alpha + 1)} \delta_{n,m}. \quad (2.3.20)$$

Above  $L_n^{(\alpha)}(t)$  denotes the generalized Laguerre polynomials defined by [29, p. 203 and P. 204]

$$L_n^{(\alpha)}(t) = \sum_{k=0}^n \frac{\Gamma(\alpha + n + 1)}{\Gamma(n - k + 1)\Gamma(\alpha + k + 1)} \frac{(-t)^k}{k!} = \frac{t^{-\alpha} e^t}{n!} \frac{d^n}{dt^n} (t^{n+\alpha} e^{-t}). \quad (2.3.21)$$

□

**Corollary 2.3.17.** For every fixed  $q \in \mathbb{B}_R$ , the function  $t \mapsto A_{R,slice}^\alpha(t; q)$  belongs to

$$L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+) := L_{\mathbb{H}}^2\left(\mathbb{R}^+; \frac{t^\alpha e^{-t}}{\Gamma(\alpha+1)} dt\right); \alpha > 0,$$

the right quaternionic Hilbert space of all square integrable  $\mathbb{H}$ -valued functions on the half-real line with respect to the scalar product

$$\langle \phi, \varphi \rangle_{\alpha, \mathbb{R}^+} = \int_0^\infty \phi(t) \overline{\varphi(t)} \frac{t^\alpha e^{-t}}{\Gamma(\alpha+1)} dt.$$

We conclude this section by giving an explicit closed formula of the kernel  $A_{R,slice}^\alpha$  in (2.3.18).

**Theorem 2.3.18.** For every  $t \in \mathbb{R}^+$  and  $q \in \mathbb{B}_R$ , we have

$$A_{R,slice}^\alpha(t; q) = \tilde{A}_{R,slice}^\alpha(t; q) := \left(1 - \frac{q}{R}\right)^{-\alpha-1} \exp\left(\frac{tq}{q-R}\right).$$

*Proof.* Fix  $t \in \mathbb{R}^+$ . It is clear that the restriction  $\tilde{A}_{R,slice}^\alpha(t; \cdot)$  to any  $\mathbb{B}_{R,I}$  is holomorphic and coincides with the kernel function  $A_{R,hol}^\alpha(t; \cdot)$ , given through (2.3.19), of the second Bargmann transform for the classical complex holomorphic Bergman space. On the other hand, the function  $A_{R,slice}^\alpha(t; q)$  is clearly slice regular in  $q$ -variable and coincides with  $A_{R,hol}^\alpha(t; \cdot)$  when restricted to  $\mathbb{B}_{R,I}$ . Thus, by Lemma 1.4.1, we conclude that  $\tilde{A}_{R,slice}^\alpha(t; \cdot) = A_{R,slice}^\alpha(t; \cdot)$  on the whole  $\mathbb{B}_R$ .  $\square$

$A_{slice}^{2,\alpha}(\mathbb{B}_R) := \mathcal{SR}(\mathbb{B}_R) \cap L^{2,\alpha}(\mathbb{B}_{R,I}, \mathbb{H})$ . The extra normalisation factor  $\alpha/\pi R^2$  in (0.0.5) defining the measure is implemented to simplify later formulas and mainly to get the asymptotic behavior when  $R$  goes to infinity. It turns out that  $A_{slice}^{2,\alpha}(\mathbb{B}_R)|_{\mathbb{B}_{R,I}} := \{f_I; f \in A_{slice}^{2,\alpha}(\mathbb{B}_R)\}$  is the usual Bergman space on the disc  $\mathbb{B}_{R,I} = D_I(0, R) \subset \mathbb{C}_I$  with respect to the norm  $\|\cdot\|_{\alpha, slice}$  in (0.0.5). Moreover, it is shown in [24, Theorem 3, p. 50] that  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  is a reproducing kernel Hilbert space with respect to (0.0.5), whose the reproducing kernel  $K_R^\alpha(q, q')$  satisfies

$$f(q) = \int_{\mathbb{B}_{R,I}} K_{R,slice}^\alpha(q, z) f(z) \left(1 - \frac{|z|^2}{R^2}\right)^{\alpha-1} \frac{\alpha}{\pi R^2} dx dy$$

for all  $f \in A_{slice}^{2,\alpha}(\mathbb{B}_R)$  and every  $I \in \mathbb{S}^2$ .

### 2.3.1 An integral transform from $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$ onto $A_{slice}^{2,\alpha}(\mathbb{B}_R)$

In this section, we consider and study a special integral transform from  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  into  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$ . A complete orthonormal system for  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  is given by

$$\phi_n(x) = \left(\frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)}\right)^{1/2} L_n^{(\alpha)}(x), \quad n = 0, 1, \dots, \quad (2.3.22)$$

where  $L_n^{(\alpha)}(x)$  denotes the generalized Laguerre polynomials given by (2.3.21). Recall also that the set of functions

$$f_n(q) = \left( \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)} \right)^{1/2} \left( \frac{q}{R} \right)^n, \quad (2.3.23)$$

is an orthonormal basis of the slice hyperholomorphic Bergman space  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  (see Proposition 2.3.1). Accordingly, by considering the kernel function  $A_{R,slice}^\alpha(t; q)$  defined on  $\mathbb{R}^+ \times \mathbb{B}_R$  by (2.3.18) or equivalently by its explicit expression given in Theorem 2.3.18, we can prove the following.

**Lemma 2.3.19.** *The kernel function  $A_{R,slice}^\alpha$  can be realized as*

$$A_{R,slice}^\alpha(t; q) = \sum_{n=0}^{\infty} \phi_n(t) f_n(q) = \sum_{n=0}^{\infty} \left( \frac{q}{R} \right)^n L_n^{(\alpha)}(t)$$

for every  $t \geq 0$  and  $q \in \mathbb{B}_R$ .

*Proof.* This follows readily from the generating function [29, Eq. (14), p. 135]

$$\sum_{n=0}^{\infty} \zeta^n L_n^{(\alpha)}(t) = \frac{1}{(1 - \zeta)^{\alpha+1}} \exp\left(\frac{t\zeta}{\zeta - 1}\right).$$

Indeed, we have

$$\sum_{n=0}^{\infty} \phi_n(t) f_n(q) = \sum_{n=0}^{\infty} \left( \frac{q}{R} \right)^n L_n^{(\alpha)}(t) = \exp\left(\frac{tq}{q - R}\right) \left(1 - \frac{q}{R}\right)^{-\alpha-1}.$$

□

Associated to  $A_{R,slice}^\alpha$  we perform the integral operator

$$\mathcal{A}_{R,slice}^\alpha \varphi(q) = \left(1 - \frac{q}{R}\right)^{-\alpha-1} \int_0^{+\infty} \exp\left(\frac{tq}{q - R}\right) \varphi(t) \frac{t^\alpha e^{-t}}{\Gamma(\alpha + 1)} dt. \quad (2.3.24)$$

**Lemma 2.3.20.** *The integral transform  $\mathcal{A}_{R,slice}^\alpha$  is well-defined on  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$ .*

*Proof.* This can be handled easily making use of the Cauchy-Schwarz inequality. Indeed, we have

$$\begin{aligned} |\mathcal{A}_{R,slice}^\alpha \varphi(q)| &\leq \int_0^{\infty} |A_{R,slice}^\alpha(t; q)| |\varphi(t)| \frac{t^\alpha e^{-t}}{\Gamma(\alpha + 1)} dt \\ &\leq \left( \int_0^{\infty} |A_{R,slice}^\alpha(t; q)|^2 \frac{t^\alpha e^{-t}}{\Gamma(\alpha + 1)} dt \right)^{\frac{1}{2}} \left( \int_0^{\infty} |\varphi(t)|^2 \frac{t^\alpha e^{-t}}{\Gamma(\alpha + 1)} dt \right)^{\frac{1}{2}}. \end{aligned}$$

Now, since  $t \mapsto A_{R,slice}^\alpha(t; q)$  belongs to  $L^{2,\alpha}(\mathbb{R}^+, \mathbb{H})$  for every fixed  $q \in \mathbb{H}$  (see Theorem 2.3.16), we deduce

$$|\mathcal{A}_{R,slice}^\alpha \varphi(q)| \leq \|A_{R,slice}^\alpha(\cdot; q)\|_{L^{2,\alpha}(\mathbb{R}^+, \mathbb{H})} \|\varphi\|_{L^{2,\alpha}(\mathbb{R}^+, \mathbb{H})}.$$

□

Moreover, we can prove the following

**Theorem 2.3.21.** *The integral operator  $\mathcal{A}_{R,slice}^\alpha$  defines a unitary isometry transform from  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  onto  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$ . Moreover, we have  $\mathcal{A}_{R,slice}^\alpha \phi_n(q) = f_n(q)$ , where  $\phi_n(x)$  and  $f_n(q)$  are respectively the orthogonal bases of  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  and the slice hyperholomorphic Bergman space  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  given by (2.3.22) and (3.4.3), respectively.*

*Proof.* Fix  $I \in \mathbb{S}$ . The identification of the slice  $\mathbb{C}_I$  with the complex plane  $\mathbb{C}$  and  $\mathbb{B}_{R,I}$  with the disc  $D_R = D(0, R)$  of  $\mathbb{C}$  leads to the consideration of the unitary isometry  $\mathcal{A}_{R,hol}^\alpha$  from  $L^{2,\alpha}(\mathbb{R}^+, \mathbb{C}_I)$  onto  $A_{hol}^{2,\alpha}(\mathbb{B}_{R,I})$ . It is specified by the rescaled version of the formula (0.0.1), to wit

$$\mathcal{A}_{R,hol}^\alpha \varphi(z) = \left(1 - \frac{z}{R}\right)^{-\alpha-1} \int_0^{+\infty} \varphi(t) \exp\left(\frac{tz}{z-R}\right) \frac{t^\alpha e^{-t}}{\Gamma(\alpha+1)} dt$$

for  $z, w \in \mathbb{B}_{R,I}$ . Now, for  $J \in \mathbb{S}$  such that  $J \perp I$ , we split any  $\varphi \in L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  as  $\varphi_I = F + GJ$  for some  $F, G : \mathbb{R}_I^+ \rightarrow \mathbb{C}_I$ . Obviously, we have  $\mathbb{R}_I^+ = \mathbb{R}^+$ ,  $\varphi_I = \varphi$  as well as

$$(\mathcal{A}_{R,slice}^\alpha \varphi)_I = \mathcal{A}_{R,slice}^\alpha F + (\mathcal{A}_{R,slice}^\alpha G)J = \mathcal{A}_{R,hol}^\alpha F + (\mathcal{A}_{R,hol}^\alpha G)J$$

by means of (2.3.24) and therefore  $\mathcal{A}_{R,slice}^\alpha \phi_n(q) = f_n(q)$ . Moreover, it is evident to see that  $F, G \in L^{2,\alpha}(\mathbb{R}^+, \mathbb{C}_I)$  with

$$\begin{aligned} \|\mathcal{A}_{R,slice}^\alpha \varphi\|_{\alpha,slice}^2 &= \|\mathcal{A}_{hol}^\alpha F\|_{\alpha,slice}^2 + \|\mathcal{A}_{hol}^\alpha G\|_{\alpha,slice}^2 \\ &= \|F\|_{L^{2,\alpha}(\mathbb{R}^+, \mathbb{C}_I)}^2 + \|G\|_{L^{2,\alpha}(\mathbb{R}^+, \mathbb{C}_I)}^2 \\ &= \|\varphi\|_{L^{2,\alpha}(\mathbb{R}^+, \mathbb{H})}^2. \end{aligned}$$

Accordingly, the transform  $\mathcal{A}_{R,slice}^\alpha$  from  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  into  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  is injective and an isometry. On the other hand, since  $\mathcal{A}_{R,hol}^\alpha$  is surjective, we see that so is  $\mathcal{A}_{R,slice}^\alpha$ . Therefore,  $\mathcal{A}_{R,slice}^\alpha$  is a unitary isometry from  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  onto  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$ .  $\square$

**Remark 2.3.22.** *The argument of splitting  $\varphi$  as  $\varphi_I = F + GJ$  for some  $F, G : \mathbb{R}_I^+ \rightarrow \mathbb{C}_I$  with  $J \in \mathbb{S}$  and that  $J \perp I$  is used in [89] and is the basic idea in the splitting Lemma 1.3.6. The result of Theorem 2.3.21 can also be proved by using (2.3.18) in order to rewrite the integral transform  $\mathcal{A}_{R,slice}^\alpha$  acting on  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$ , in (2.3.24), as*

$$\begin{aligned} \mathcal{A}_{R,slice}^\alpha \varphi(q) &= \frac{1 - I_q J}{2} \int_0^{+\infty} A_{R,hol}^\alpha(t; z_q) \varphi(t) \frac{t^\alpha e^{-t}}{\Gamma(\alpha+1)} dt \\ &\quad + \frac{1 + I_q J}{2} \int_0^{+\infty} A_{R,hol}^\alpha(t; \bar{z}_q) \varphi(t) \frac{t^\alpha e^{-t}}{\Gamma(\alpha+1)} dt. \end{aligned}$$

The second main result of this section is the following

**Theorem 2.3.23.** *The inverse transform  $[\mathcal{A}_{R,slice}^\alpha]^{-1} : A_{slice}^{2,\alpha}(\mathbb{B}_R) \longrightarrow L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  of  $\mathcal{A}_{R,slice}^\alpha$  is given by*

$$[\mathcal{A}_{R,slice}^\alpha]^{-1}f(t) = \left(\frac{\alpha}{\pi R^2}\right) \int_{\mathbb{B}_{R,I}} \exp\left(\frac{t\bar{q}}{\bar{q}-R}\right) \frac{\left(1-\frac{|q|^2}{R^2}\right)^{\alpha-1}}{\left(1-\frac{\bar{q}}{R}\right)^{\alpha+1}} f(q) dx dy. \quad (2.3.25)$$

*Proof.* The inverse of the unitary isometric transform  $\mathcal{A}_{R,hol}^\alpha$  in (2.3.24) is given by its hermitian conjugate (the adjoint). More explicitly, we claim that

$$[\mathcal{A}_{R,hol}^\alpha]^{-1}F(t) = \left(\frac{\alpha}{\pi R^2}\right) \int_{\mathbb{B}_{R,I}} F(z) \exp\left(\frac{t\bar{z}}{\bar{z}-R}\right) \frac{\left(1-\frac{|z|^2}{R^2}\right)^{\alpha-1}}{\left(1-\frac{\bar{z}}{R}\right)^{\alpha+1}} dx dy \quad (2.3.26)$$

for every  $F \in A_{hol}^{2,\alpha}(\mathbb{B}_{R,I})$ . Consequently, the inverse transform of  $\mathcal{A}_{R,slice}^\alpha$  in (2.3.24), the quaternionic analogue of  $\mathcal{A}_{R,hol}^\alpha$ , is given by (2.3.25). This follows readily from the splitting Lemma 1.3.6 combined with (2.3.26) above.  $\square$

## 2.3.2 Asymptotic behavior: from slice Bergman to slice Bargmann

Intuitively, the space  $\mathbb{H}$  can be viewed as the euclidean limit of balls  $\mathbb{B}_R$  in  $\mathbb{H}$ , as the radius  $R$  goes to  $+\infty$ . This intuitive limit can be justified geometrically. The analogue of the Poincaré (the real hyperbolic) differential metric on the unit open ball  $\mathbb{B}$  in the quaternionic setting is given by  $ds_{\mathbb{B}}^2 = (1-|q|^2)^{-2}|d_I q|^2$ , where  $q = x + Iy$ ;  $I \in \mathbb{S}$ , and  $d_I q = dx + I dy$ . It was proposed by Bisi and Gentili in [20] by developing a variation of an approach adopted by Ahlfors [2]. The quaternionic hyperbolic geometry on  $\mathbb{B}_R$  is described by the scaled Poincaré-type differential metric

$$ds_{\mathbb{B}_R}^2 = R^4(R^2 - |q|^2)^{-2}|d_I q|^2.$$

The associated volume measure is given by

$$d\lambda_{I,R}^\alpha(q = x + Iy) = \left(\frac{\alpha}{\pi R^2}\right) \left(1 - \frac{|q|^2}{R^2}\right)^{\alpha-1} dx dy.$$

Therefore, the sectional curvature of  $(\mathbb{B}_R, ds_{\mathbb{B}_R}^2)$ , given by  $\kappa_R = -4/R^2$ , tends to 0 which corresponds to the curvature of the flat hermitian manifold  $(\mathbb{H}, ds_{\mathbb{H}}^2)$  endowed with the flat metric  $ds_{\mathbb{H}}^2 = |d_I q|^2$ . Moreover, if we parameterize  $\alpha$  as  $\alpha = \nu R^2$  for some fixed  $\nu > 0$ , we see that the sliced measure  $d\lambda_{I,R}^\alpha$  converges pointwisely to the sliced volume measure on  $\mathbb{H}$ ,

$$\lim_{R \rightarrow +\infty} d\lambda_{I,R}^{\nu R^2}(q = x + Iy) = \left(\frac{\nu}{\pi}\right) \lim_{R \rightarrow +\infty} \left(1 - \left|\frac{q}{R}\right|^2\right)^{\nu R^2} dx dy = \left(\frac{\nu}{\pi}\right) e^{-\nu|q|^2} dx dy.$$

With respect to this parametrization, the orthonormal basis of the slice hyperholomorphic Bergman space  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  given by the functions

$$f_n(q) = \left( \frac{\Gamma(n + \nu R^2 + 1)}{n! \Gamma(\nu R^2 + 1)} \right)^{1/2} \left( \frac{q}{R} \right)^n,$$

(see Proposition ??), also gives rise to

$$e_n(q) = \left( \frac{\nu^n}{n!} \right)^{1/2} q^n,$$

pointwisely, when  $R$  goes to infinity. The set of  $e_n(q)$  is in fact an orthonormal basis of the slice hyperholomorphic Bargmann-Fock space

$$\mathcal{F}_{slice}^{2,\nu}(\mathbb{H}) = \mathcal{SR}(\mathbb{H}) \cap L^{2,\nu}(\mathbb{C}_I, \mathbb{H}) \quad (2.3.27)$$

with respect to the sliced Gaussian measure  $\left(\frac{\nu}{\pi}\right) e^{-\nu|q|^2} dx dy$ . This follows readily thanks to the Binet formula [29]

$$\lim_{x \rightarrow +\infty} \frac{\Gamma(x+a)}{x^{a-b} \Gamma(x+b)} = 1.$$

The main result of this section concerns the pointwise convergence of the reproducing kernel function.

**Theorem 2.3.24.** *The pointwise limit of the weighted Bergman kernel  $K_{R,slice}^{\nu R^2}$  of the slice hyperholomorphic Bergman space  $A_{slice}^{2,\nu R^2}(\mathbb{B}_R)$  is exactly the reproducing kernel of the slice hyperholomorphic Bargmann-Fock space  $\mathcal{F}_{slice}^{2,\nu}(\mathbb{H})$  in (2.3.27). More precisely, for every fixed  $(q, p) \in \mathbb{H} \times \mathbb{H}$ , we have*

$$\lim_{R \rightarrow +\infty} K_{R,slice}^{\nu R^2}(q, p) = e_*^{[\nu q, \bar{p}]} := \sum_{n=0}^{+\infty} \frac{\nu^n q^n \bar{p}^n}{n!}.$$

*Proof.* Recall first that for  $\alpha = \nu R^2$  and  $R$  being reals, the expression of the reproducing kernel  $K_{R,slice}^\alpha$  given by (2.3.12) reads

$$K_{R,slice}^\alpha(q, p) = {}_2F^*_1 \left( \begin{matrix} \nu R^2 + 1, 0 \\ 0 \end{matrix} \middle| \left[ \begin{matrix} q & \bar{p} \\ R' & R \end{matrix} \right] \right).$$

Accordingly, what is needed to conclude is an asymptotic behavior of the involved hypergeometric function. Thus, we claim that for every fixed  $q, p \in \mathbb{H}$  and reals  $a, b, c$ , we have (see [57] for a rigorous proof that we can extend to our context by means of the identity principle for slice regular functions)

$$\lim_{\rho \rightarrow +\infty} {}_2F^*_1 \left( \begin{matrix} a + \rho, b \\ c \end{matrix} \middle| \left[ \begin{matrix} q & \bar{p} \\ \rho & \rho \end{matrix} \right] \right) = {}_1F^*_1 \left( \begin{matrix} b \\ c \end{matrix} \middle| [q, \bar{p}] \right). \quad (2.3.28)$$

Moreover, the convergence is uniform on compact sets of  $\mathbb{H} \times \mathbb{H}$ . Therefore, one obtains

$$\lim_{R \rightarrow +\infty} K_{R, \text{slice}}^{vR^2}(q, p) = \sum_{n=0}^{+\infty} \frac{v^n q^n \bar{p}^n}{n!} = e_*^{[vq, \bar{p}]}.$$

□

**Remark 2.3.25.** *According to the uniform convergence of the series in (2.3.28) on compact sets of  $\mathbb{H} \times \mathbb{H}$ , the convergence in Theorem 2.3.24 of the reproducing kernel function is uniform in  $(q, p)$  in any compact set of  $\mathbb{H} \times \mathbb{H}$ .*

## 2.4 Weighted hyperholomorphic left Bergman Hilbert spaces

In order to identify the range of the integral transform  $S_y^\alpha$  in (0.0.7) when acting on  $L_{\mathbb{H}}^{2, \alpha}(\mathbb{R}^+)$ , we begin by examining a class of weighted hyperholomorphic left Bergman Hilbert spaces for which we provide a closed expression of their reproducing kernel in terms of a  $\star$ -regularization of Gauss hypergeometric function. Let  $\omega$  be a given positive measurable mapping on  $(0, 1)$  such that  $\omega(t)dt$  be a finite measure. We extend  $\omega$  to the whole unit ball  $\mathbb{B} \subset \mathbb{H}$  by taking  $\tilde{\omega}(q) := \omega(|q|^2)$ . We define the  $\omega$ -hyperholomorphic left Bergman Hilbert space  $A_{\text{slice}}^{2, \omega} := \mathcal{SR}(\mathbb{B}) \cap L_{\mathbb{H}}^{2, \omega}(\mathbb{B}_I)$  as the space of all slice left slice regular functions  $\varphi$  in  $\mathbb{B}$  belonging to  $L_{\mathbb{H}}^{2, \omega}(\mathbb{B}_I) := L^2(\mathbb{B}_I, \omega(|q|^2)dxdy)$  and endowed with the norm induced from the slice inner product on  $\mathbb{B}_I = \mathbb{B} \cap \mathbb{C}_I$ ,

$$\langle f, g \rangle_\omega = \int_{\mathbb{B}_I} \overline{f(x + Iy)} g(x + Iy) \omega(x^2 + y^2) dx dy.$$

More explicitly, the Hilbert space  $A_{\text{slice}}^{2, \omega}$  consists of all convergent power series  $\varphi(q) = \sum_{n=0}^{\infty} q^n c_n$  on  $\mathbb{B}$  for which the quaternionic sequence  $(c_n)_n$  satisfies the growth condition

$$\sum_{n=0}^{\infty} \gamma_n |c_n|^2 < \infty; \quad \gamma_n := \int_0^1 t^n \omega(t) dt.$$

The specification of the weight function

$$\omega_{\beta, \eta}(t) := t^{\beta-1} (1-t)^{\eta-1}, \quad \eta, \beta > 0,$$

gives rise to the weighted hyperholomorphic Hilbert space

$$A_{\text{slice}}^{2, \beta, \eta} := \left\{ \varphi(q) = \sum_{n=0}^{\infty} q^n c_n; \sum_{n=0}^{\infty} \gamma_n^{\beta, \eta} |c_n|^2 < \infty \right\}, \quad (2.4.1)$$

where

$$\gamma_n^{\beta,\eta} := \frac{\Gamma(\eta)\Gamma(\beta+n)}{\Gamma(\beta+\eta+n)}.$$

It should be mentioned here that the monomials  $e_n(q) = q^n$  form an orthogonal basis of  $A_{slice}^{2,\beta,\eta}$  with square norm given by

$$\|e_n\|_{\beta,\eta}^2 = \pi\gamma_n^{\beta,\eta}.$$

Moreover, appealing to the continuity of the evaluation linear form and the quaternionic version of Riesz representation theorem, we claim that  $A_{slice}^{2,\beta,\eta}$  is a reproducing kernel Hilbert space, whose kernel function is expressible in terms of the quaternionic Gauss hypergeometric function (of first kind)

$${}_2F_1^* \left( \begin{matrix} a, b \\ c \end{matrix} \middle| [p, q] \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{p^k q^k}{k!} \quad (2.4.2)$$

for  $p, q \in \mathbb{B}$  and reals  $a, b$  and  $c$ , where  $(a)_k = a(a+1)\cdots(a+k-1)$  with  $(a)_0 = 1$ . The series in (2.4.2) converges absolutely and uniformly on  $K \times K'$  for any compact subsets  $K, K' \subset \mathbb{B}$ . The function  ${}_2F_1^*$  can be seen as the slice regularization of the classical Gauss hypergeometric function with respect to star product for slice functions, in order to get a left slice regular function in  $p$  and a right slice one in  $q$ . Namely, we assert

**Proposition 2.4.1.** *The reproducing kernel of  $A_{slice}^{2,\beta,\eta}$  is given by*

$$K_{\beta,\eta}(p, q) = \frac{\Gamma(\beta+\eta)}{\pi\Gamma(\eta)\Gamma(\beta)} {}_2F_1^* \left( \begin{matrix} 1, \eta + \beta \\ \beta \end{matrix} \middle| [p, \bar{q}] \right). \quad (2.4.3)$$

*Proof.* The explicit expression of  $K_{\beta,\eta}(p, q)$  follows easily since

$$\begin{aligned} K_{\beta,\eta}(p, q) &= \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{e_n(p) \overline{e_n(q)}}{\gamma_n^{\beta,\eta}} \\ &= \frac{\Gamma(\beta+\eta)}{\pi\Gamma(\eta)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(\beta+\eta)_n}{(\beta)_n} e_n(p) \overline{e_n(q)}. \end{aligned} \quad (2.4.4)$$

□

**Remark 2.4.2.** For  $\beta = 1$ , the space  $A_{slice}^{2,\eta,1}$  is the one described in the introduction,  $A_{slice}^{2,\eta,1} = A_{slice}^{2,\eta}$ . Moreover, the  $K_{\eta,1}(p, q)$  reduces further to the reproducing kernel of classical weighted Bergman space  $A_{slice}^{2,\eta}$  given by [39, Theorem 3.1]

$$K_{\eta,1}(p, q) = \frac{\eta}{\pi} {}_1F_0^* \left( \begin{matrix} -\eta - 1 \\ - \end{matrix} \middle| [p, \bar{q}] \right) \left( 1 - 2\Re(q)\bar{p} + |q|^2\bar{p}^2 \right)^{-\eta-1}. \quad (2.4.5)$$

The restriction of  $K_{\eta,1}$  to  $\mathbb{B}_I$  coincides with the classical Bergman kernel  $K_{\eta,1}(z, w) = (\eta/\pi)(1 - z\bar{w})^{-\eta-1}$ ;  $z, w \in \mathbb{B}_I$ .

# Chapter 3

## Bargmann's Versus for the FrHT and applications

Abstract:

We begin by recalling the definition of FrHT due to Namias that we adapt to the quaternionic setting. In Section 3, we present a general abstract formalism for constructing QFrHT by means of eigenvalue equation involving orthogonal basis of certain quaternionic Hilbert space. Section 4 is devoted to the reconstruction of QFrHT for  $L^2_{\mathbb{H}}(\mathbb{R}^+)$  by Bargmann versus, and show how to derive in a simple way their basic properties such as the Plancherel and inversion formulas.

### 3.1 Hankel transform

The Hankel transform of order  $\alpha$  of a function  $f$  from  $\mathbb{R}^+$  to  $\mathbb{R}$ , such that is given in [87] by

$$\mathcal{H}_\alpha(k) = \int_0^\infty f(r) I_\alpha(kr) r dr,$$

where  $I_\alpha$  is the Bessel function of the first kind of order  $\alpha$  given in by

$$I_\alpha(\xi) = \left(\frac{\xi}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(\alpha + n + 1)} \left(\frac{\xi}{2}\right)^{2n}$$

such that  $\alpha \geq -1/2$ . The inverse Hankel transform of  $\mathcal{H}_\alpha(k)$  is defined as

$$f(r) = \int_0^\infty \mathcal{H}_\alpha(k) I_\alpha(kr) k dk,$$

which can be readily verified using the orthogonality relationship described below.

## 3.2 Quaternionic fractional Hankel transform (à la Namias)

In this section, we review à la Namias the classical approach of constructing fractional Hankel transform that we adopt to give rise to the fractional Hankel transform for the quaternionic right Hilbert space  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$ ;  $\alpha > 0$ , of all quaternionic-valued functions on the half real line  $\mathbb{R}^+$  that are square integrable with respect to the inner product given by

$$\langle \varphi, \psi \rangle_{\alpha} = \int_{\mathbb{R}^+} \overline{\varphi(x)} \psi(x) x^{\alpha} e^{-x} dx.$$

We denote by  $\|\cdot\|_{\alpha}$  the associated norm. A complete orthonormal system in  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  is given by the functions  $\varphi_n^{\alpha}$  in (2.3.22). Accordingly, the series function in Definition 3.3.4 reduces further to Hille–Hardy identity [15, (6.2.25) p. 288]

$$\begin{aligned} R_{\theta}^{\alpha}(x, y) &= \sum_{n=0}^{+\infty} \frac{n!}{\Gamma(\alpha + n + 1)} \theta^n L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) \\ &= \frac{1}{1-\theta} \left( \frac{1}{\theta xy} \right)^{\alpha/2} \exp\left(-\frac{\theta(x+y)}{1-\theta}\right) I_{\alpha}\left(\frac{2\sqrt{\theta}}{1-\theta}\sqrt{xy}\right) \end{aligned} \quad (3.2.1)$$

valid for  $|\theta| < 1$  and nonnegative integer  $\alpha$ , where  $I_{\alpha}(\xi)$  denotes the modified Bessel function [15, p.222]

$$I_{\alpha}(\xi) = \left(\frac{\xi}{2}\right)^{\alpha} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(\alpha + n + 1)} \left(\frac{\xi}{2}\right)^{2n}.$$

Thus, we can rewrite the kernel function  $K_{\theta}^{\alpha}(x, y) = x^{\alpha} e^{-x} R_{\theta}^{\alpha}(x, y)$  as

$$K_{\theta}^{\alpha}(x, y) = \frac{1}{1-\theta} \left( \frac{x}{\theta y} \right)^{\alpha/2} \exp\left(-\frac{x+\theta y}{1-\theta}\right) I_{\alpha}\left(\frac{2\sqrt{\theta}}{1-\theta}\sqrt{xy}\right), \quad (3.2.2)$$

so that the corresponding integral operator is well-defined on  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  by

$$\mathcal{L}_{\theta}^{\alpha}(\varphi)(y) = \int_0^{+\infty} K_{\theta}^{\alpha}(x, y) \varphi(x) dx. \quad (3.2.3)$$

The Laguerre polynomial  $\varphi_n^{\alpha}(x)$  in (2.3.22) is (left) eigenfunction of  $\mathcal{L}_{\theta}^{\alpha}$  with  $\theta^n$  as corresponding (right) eigenvalue,

$$\mathcal{L}_{\theta}^{\alpha}(\varphi_n^{\alpha}(x)) = \varphi_n^{\alpha}(x) \theta^n.$$

This readily follows from the definition of  $\mathcal{L}_{\theta}^{\alpha}$ .

**Remark 3.2.1.** *Such transform is closely connected to the fractional Hankel transform [83, 71]. In fact the last one appears as the limit case of  $\mathcal{L}_{\theta}^{\alpha}$  when restricting  $\theta$  to  $|\theta| = 1$  such that  $\theta \neq 1$ .*

We conclude this section by proving that for  $|\theta| < 1$ , the integral transform  $\mathcal{L}_\theta^\alpha$  defines a continuous  $k$ -contraction from  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  into itself with  $k = (1 - |\theta|^2)^{-1/2}$ .

**Proposition 3.2.2.** *For  $|\theta| < 1$  and every  $\varphi \in L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$ , we have*

$$\|\mathcal{L}_\theta^\alpha(\varphi)\|^2 \leq \left( \frac{1}{1 - |\theta|^2} \right) \|\varphi\|^2.$$

*Proof.* Since  $(\varphi_n^\alpha)_n$  in (2.3.22) is a complete orthonormal system in  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$ , we can expand any  $f \in L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  as  $f(x) = \sum_{n=0}^{+\infty} \varphi_n^\alpha c_n$  for some  $c_n \in \mathbb{H}$ . Hence, using the fact that  $\mathcal{L}_\theta^\alpha(\varphi_n^\alpha)(x) = \varphi_n^\alpha(x)\theta^n$ , we get

$$\mathcal{L}_\theta^\alpha(f)(x) = \sum_{n=0}^{+\infty} \varphi_n^\alpha(x)\theta^n c_n. \quad (3.2.4)$$

Using the orthogonality of  $\varphi_n^\alpha$ , we obtain

$$\begin{aligned} \|\mathcal{L}_\theta^\alpha(\varphi)\|^2 &= \sum_{n=0}^{+\infty} |\theta|^{2n} |c_n|^2 \\ &\leq \left( \sum_{n=0}^{+\infty} |\theta|^{2n} \right) \left( \sum_{n=0}^{+\infty} |c_n|^2 \right) \\ &\leq \left( \frac{1}{1 - |\theta|^2} \right) \|\varphi\|^2 \end{aligned}$$

which requires  $|\theta| < 1$ . □

### 3.3 Bargmann's Versus for fractional integral transform: Abstract formalism

This section is devoted to present a general formalism for constructing like fractional Fourier transform on an infinite functional right quaternionic separable Hilbert space  $\mathcal{H}_X$  on given set  $X$ . For this, we explore Bargmann idea related to Segal-Bargmann transform. It will applied in Section 4 to recover the quaternionic fractional Hankel transform discussed in Section 2. Thus, let  $\mathcal{H}_X$  and  $\mathcal{H}_Y$  be two arbitrary infinite functional right quaternionic separable Hilbert spaces with orthonormal bases  $\{\varphi_n; n \in \mathbb{N}\}$  and  $\{\psi_n; n \in \mathbb{N}\}$  defined on  $X$  and  $Y$ , respectively. The corresponding inner scalar products are given by

$$\langle \varphi, \phi \rangle_{\mathcal{H}_X} = \int_X \bar{\varphi}(x)\phi(x)\omega_X(x)dx$$

and

$$\langle \Psi, \Phi \rangle_{\mathcal{H}_Y} = \int_Y \overline{\Psi}(y) \Phi(y) \omega_Y(y) dy,$$

respectively, for some weight functions  $\omega_X$  and  $\omega_Y$ . Associated to the data  $(X, \mathcal{H}_X, \varphi_n)$  and  $(Y, \mathcal{H}_Y, \psi_n)$ , we consider the integral transform  $T_{XY} : \mathcal{H}_X \rightarrow \mathcal{H}_Y$  of the form

$$T_{XY}(\varphi)(y) = \int_X \overline{R(x, y)} \varphi(x) \omega_X(x) dx.$$

We assume that  $T_{XY}$  is well defined on  $\mathcal{H}_X$  such that  $T_{XY}(\varphi_n) = \psi_n$ . This is equivalent to say that the kernel function  $R(x, y)$  on  $X \times Y$  can be expanded as

$$R(x, y) = \sum_{n=0}^{\infty} \varphi_n(x) \overline{\psi_n(y)}$$

whenever the series in the right-hand side is uniformly and absolutely convergent.

Then we have the following result.

**Proposition 3.3.1.**  $T_{XY}$  is an invertible integral kernel transform, whose inverse is given by

$$T_{XY}^{-1}\psi(x) = \int_Y R(x, y) \psi(y) \omega_Y(y) dy$$

for  $x \in X$  and  $\psi \in \mathcal{H}_Y$ .

*Proof.* With assumption that the series  $R(x, y) = \sum_{n=0}^{\infty} \varphi_n(x) \overline{\psi_n(y)}$  is uniformly converges on all the compact subset of  $X \times Y$ . Then we obtain for all  $n \in \mathbb{N}$   $T_{XY}(\varphi_n) = \psi_n$ , indeed

$$\begin{aligned} T_{XY}(\varphi_n) &= \int_X \left( \sum_{i=0}^{\infty} \psi_i(y) \overline{\varphi_i(x)} \right) \varphi_n(x) \omega_X(x) dx \\ &= \sum_{i \geq 0} \psi_i(y) \left( \int_X \overline{\varphi_i(x)} \varphi_n(x) \omega_X(x) dx \right) \\ &= \psi_n(y) \end{aligned} \tag{3.3.1}$$

Thus,  $T_{XY}$  is an isometry transformation from  $\mathcal{H}_X$  onto  $\mathcal{H}_Y$ . To establish the unitarity of  $T_{XY}$ , it's sufficient to show that its range is dense in  $\mathcal{H}_Y$ .

Since the principal vectors  $\{R(x, \cdot) : x \in X\}$  complete in  $\mathcal{H}_Y$ , i.e their finite linear combinations are dense in  $\mathcal{H}_Y$  (because the only orthogonal to all of them is  $f \equiv 0$ , integral form). We obtain that the range of  $T_{XY}$  is dense in  $\mathcal{H}_Y$ .  $\square$

We then perform the fractional transform associated to  $T_{XY}$  to be the  $\mathcal{F}_g$  the diagrams commutative

$$\begin{array}{ccc} \mathcal{H}_X & \xrightarrow{T_{XY}} & \mathcal{H}_Y \\ \mathcal{F}_g \downarrow & \circlearrowleft & \downarrow \Gamma_g \\ \mathcal{H}_X & \xleftarrow{T_{XY}^{-1}} & \mathcal{H}_Y \end{array} \quad \begin{array}{ccc} Y & \xrightarrow{\psi} & \mathbb{H} \\ \tilde{\Gamma}_g \downarrow & \circlearrowleft & \downarrow \Gamma_g \\ Y & \xrightarrow{\psi} & \mathbb{H} \end{array}$$

for every  $g \in G$ ,  $\psi \in \mathcal{H}_Y$ , where  $\tilde{\Gamma} : G \times Y \longrightarrow Y$ ;  $(g, y) \longmapsto \tilde{\Gamma}(g, y) = \tilde{\Gamma}_g(y) = g(y)$ , is a special action of some group  $G$  on  $Y$  that we have extended to  $\mathcal{H}_Y$  by considering  $\Gamma : G \times \mathcal{H}_Y \longrightarrow \mathcal{H}_Y$  with  $\Gamma(g, \psi)(y) = \Gamma_g(\psi)(y) = \psi(g(y))$  with  $y \in Y$  and  $\psi \in \mathcal{H}_Y$ . Namely,

$$\mathcal{F}_g = T_{XY}^{-1} \circ \Gamma_g \circ T_{XY}; \quad g \in G.$$

Therefore, for every  $\psi \in \mathcal{H}_Y$ , we have

$$\mathcal{F}_g(\varphi)(x) = \int_Y R(x, y) \left( \int_X \overline{R(x', g(y))} \varphi(x') \omega_X(x') dx' \right) \omega_Y(y) dy. \quad (3.3.2)$$

To change the order of the integrals, stronger conditions need to be imposed on the integrand so the requirements of Fubini's theorem are met. This holds true when for example  $\omega_Y(y) dy$  is a finite measure on  $Y$  and the function  $(x', y) \longmapsto R(x, y) \overline{R(x', g(y))}$  belongs to  $L^2(X \times Y, \omega_X(x') \omega_Y(y) dx' dy)$  for every fixed  $x \in X$  and  $g \in G$ . A sufficient condition, when  $\omega_X(x') dx'$  and  $\omega_Y(y) dy$  are finite measures on  $X$  and  $Y$ , respectively, is  $|R(x, y)|^2 \in L^2(Y, \omega_Y(y) dx' dy)$  for every fixed  $x \in X$  and  $|R(x', g(y))|^2 \in L^2(X \times Y, \omega_X(x') \omega_Y(y) dx' dy)$  for every fixed  $g \in G$ . Thus, under such kind of conditions, we get

**Proposition 3.3.2.**

$$\mathcal{F}_g(\varphi)(x) \stackrel{\text{Fubini}}{=} \int_X \widetilde{R}_g(x', x) \varphi(x') \omega_X(x') dx',$$

where  $\widetilde{R}_g(x', x)$  stands for

$$\widetilde{R}_g(x', x) = \langle R(x', g(y)), R(x, y) \rangle_{\mathcal{H}_Y}. \quad (3.3.3)$$

An expansion of  $\widetilde{R}_g(x', x)$ , at least formally, is the following

**Proposition 3.3.3.**

$$\begin{aligned} \widetilde{R}_g(x', x) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varphi_n(x) \langle \psi_n, \Gamma_g \psi_m \rangle_{\mathcal{H}_Y} \overline{\varphi_m(x')} \\ &= \sum_{n=0}^{\infty} \varphi_n(x) \chi_n(g) \overline{\varphi_n(x')} =: R_g(x', x). \end{aligned} \quad (3.3.4)$$

The last equality follows under the additional assumption that

$$\Gamma_g \psi_m(y) = \psi_m(g(y)) = \psi_m(y) \chi_m(g). \quad (3.3.5)$$

According to the above discussion, we reformulate the following definitions.

**Definition 3.3.4.** *If the series in the right-hand side of (3.3.4) converges absolutely and uniformly to  $R_g(x', x)$ , then*

$$\mathcal{F}_g(\varphi)(x) := \int_X R_g(x', x) \varphi(x') \omega_X(x') dx'$$

*defines a like-fractional Fourier transform for the data  $(\mathcal{H}_X, \varphi_n, \chi_n)$ .*

**Remark 3.3.5.** *We have  $\mathcal{F}_g(\varphi_n) = \varphi_n \chi_n(g)$ . This gives an integral representation for  $\varphi_n$ .*

**Definition 3.3.6.** *We call fractional Fourier transform associated to  $T_{XY}$  and  $\Gamma$  the integral transform*

$$\tilde{\mathcal{F}}_{rg}(\varphi)(x) = \int_X \tilde{R}_g(x', x) \varphi(x') \omega_X(x') dx'$$

*with*

$$\tilde{R}_g(x', x) = \langle R(x', g(y)), R(x, y) \rangle_{\mathcal{H}_Y} \quad (3.3.6)$$

*provided that (3.3.6) exists.*

**Remark 3.3.7.** *The equality  $\tilde{\mathcal{F}}_{rg}(\varphi)(x) = T_{XY}^{-1} \Gamma_g T_{XY}(\varphi)(x)$  holds true under further assumptions on the kernel function allowing the application of Fubini's theorem to (3.3.2). In this case, if  $\chi_m$  in (3.3.5) is a character of the group  $G$ , then  $\tilde{\mathcal{F}}_{rg}$  is invertible with inverse given by  $\tilde{\mathcal{F}}_{g^{-1}}$ .*

As immediate consequences

**Remark 3.3.8.** *The formalism presented here is valid for the complex context. Here we have preferred to present it in the quaternionic context to avoid technical problem associated to the non commutatif of the product when passing from  $\mathbb{C}$  to  $\mathbb{H}$ .*

**Remark 3.3.9.** *Possible description of other properties of the considered QFrFT, like its behavior with ordinary derivatives, with fractional derivatives, with fractional integrals, as well as the discussion of its eventual role in the resolution of ordinary and partial differential equations is closely connected to the initial transform  $T_{X,Y}$  and its kernel function.*

## 3.4 The Quaternionic Fractional Fourier Transform for

$$L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+):$$

### (Application 1)

In view of the explicit expression of the kernel function in (3.5.1), we see that we can consider the limit case of the Hille–Hardy formula which corresponds to  $|\theta| = 1$  with  $\theta \neq 1$ . We show below that this can be recovered by the formalism presented in Definition 3.3.6 and specified for  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$ , so that for  $\theta = 1$  the considered transform

reduces further to the identity operator of  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$ . To this end, we begin by recalling that the hyperholomorphic second Bargmann integral transform [39], defined in 2.3.24 which represent the quaternionic analogue of the complex second Bargmann transform introduced by Bargmann himself in [17, p.203]. It establishes a unitary isometry from  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  onto the slice hyperholomorphic Bergman space (of second kind) on the unit ball  $\mathbb{B}$  in  $\mathbb{R}^4$ ,

$$A_{slice}^{2,\alpha}(\mathbb{B}) := \mathcal{SR}(\mathbb{B}) \cap L^{2,\alpha}(\mathbb{B}_I). \quad (3.4.1)$$

Notice for instance that the definition of  $A_{slice}^{2,\alpha}(\mathbb{B})$  is based on the classical one on a given disc  $\mathbb{B}_I$ . This was possible by extending the complex holomorphic functions to the whole  $\mathbb{B}$  by the representation formula (see for example [28]). While the transform  $\mathcal{A}_{R,slice}^\alpha$  in (2.3.24) is associated to the kernel function

$$A_{slice}^\alpha(x; q) := \frac{1}{\sqrt{\pi\Gamma(\alpha)}(1-q)^{\alpha+1}} \exp\left(\frac{xq}{q-1}\right) \quad (3.4.2)$$

on  $\mathbb{R}^+ \times \mathbb{B}$ , and obtained as bilinear generating function involving the functions  $(\varphi_n^\alpha)_n$  in (2.3.22) and the orthonormal basis of  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  given by the functions

$$f_n(q) = \left(\frac{\Gamma(n+\alpha+1)}{\pi\Gamma(\alpha)n!}\right)^{1/2} q^n. \quad (3.4.3)$$

Now, by means of  $\mathcal{A}_{R,slice}^\alpha$ , its inverse  $[\mathcal{A}_{R,slice}^\alpha]^{-1}$  and the angular unitary operator  $\Gamma_\theta(f)(q) = f(q\theta)$ , we perform the transform

$$\widetilde{\mathcal{L}}_\theta^\alpha := [\mathcal{A}_{R,slice}^\alpha]^{-1} \Gamma_\theta \mathcal{A}_{R,slice}^\alpha \quad (3.4.4)$$

on  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$ . Here we consider the  $U_{\mathbb{H}}(1)$ -action  $\theta(q) := q\theta$  of  $G = U_{\mathbb{H}}(1)$  on  $\mathbb{B}$ , that we extend to the hyperholomorphic Bergman space  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  by considering

$$\Gamma_\theta(f)(q) = f_\star(q\theta) := \sum_{n=0}^{\infty} q^n \theta^n c_n \quad (3.4.5)$$

for given  $f(q) = \sum_{n=0}^{\infty} q^n c_n \in A_{slice}^{2,\alpha}(\mathbb{B}_R)$ . The function  $q \mapsto f_\star(q\theta)$  is in fact the slice regularization of  $q \mapsto f(q\theta)$  obtained by making use of the left  $\star_s^L$ -product for left slice regular functions  $f(q) = \sum_{n=0}^{\infty} q^n a_n$  and  $g(q) = \sum_{n=0}^{\infty} q^n b_n$  on  $\mathbb{H}$  defined by [51]

$$(f \star_s^L g)(q) = \sum_{n=0}^{\infty} q^n \left( \sum_{k=0}^n a_k b_{n-k} \right). \quad (3.4.6)$$

In particular, we have

$$(f_n)_\star(q\theta) := f_n(q)\theta^n, \quad (3.4.7)$$

and therefore we may prove the following.

**Theorem 3.4.1.** For  $\theta \in \mathbb{H}$  with  $|\theta| \leq 1$ , the transform  $\widetilde{\mathcal{L}}_\theta^\alpha$  in (3.4.4) defines a continuous integral transform from  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  onto  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  with norm not exceed 1. For  $|\theta| = 1$ , we have

$$\langle \widetilde{\mathcal{L}}_\theta^\alpha \varphi, \widetilde{\mathcal{L}}_\theta^\alpha \psi \rangle = \langle \varphi, \psi \rangle.$$

*Proof.* The operator  $\widetilde{\mathcal{L}}_\theta^\alpha$  in (3.4.4) is well-defined from  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  into itself if and only if the action  $\Gamma_\theta$  leaves the space  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$  invariant, which is clear from the definition of  $\Gamma_\theta$  given through (3.4.5). Moreover, using the fact  $\mathcal{A}_{R,slice}^\alpha \varphi_n^\alpha = f_n$  as well as (3.4.7), we get

$$\widetilde{\mathcal{L}}_\theta^\alpha(\varphi_n^\alpha(y)) = [\mathcal{A}_{R,slice}^\alpha]^{-1}(f_n(\cdot)\theta^n)(y) = \varphi_n^\alpha(y)\theta^n.$$

In addition, under the condition that  $|\theta| = 1$ , it is clear that  $\Gamma_\theta$  preserves the scalar product in  $A_{slice}^{2,\alpha}(\mathbb{B}_R)$ . Indeed, for every  $f = \sum_{n=0}^\infty f_n c_n$  and  $g = \sum_{n=0}^\infty f_n d_n \in A_{slice}^{2,\alpha}(\mathbb{B}_R)$ , we have

$$\begin{aligned} \langle \Gamma_\theta f, \Gamma_\theta g \rangle_{A_{slice}^{2,\alpha}(\mathbb{B}_R)} &= \sum_{n,m=0}^\infty \overline{c_n} \overline{\theta^n} \langle f_n, f_m \rangle_{A_{slice}^{2,\alpha}(\mathbb{B}_R)} \theta^m d_m \\ &= \sum_{n=0}^\infty \overline{c_n} |\theta^n|^2 d_n \\ &= \langle f, g \rangle_{A_{slice}^{2,\alpha}(\mathbb{B}_R)}. \end{aligned}$$

Accordingly, the identity  $\langle \widetilde{\mathcal{L}}_\theta^\alpha \varphi, \widetilde{\mathcal{L}}_\theta^\alpha \psi \rangle = \langle \varphi, \psi \rangle$  follows as composition of operators preserving scalar product.  $\square$

**Remark 3.4.2.** As particular case we have the Plancherel formula  $\|\widetilde{\mathcal{L}}_\theta^\alpha \psi\| = \|\psi\|$  when  $|\theta| = 1$ . This can be recovered directly from the definition of  $\widetilde{\mathcal{L}}_\theta^\alpha$ , since in this case  $\Gamma_\theta$  is an isometry like the Bargmann transform and its inverse.

**Corollary 3.4.3.** If  $|\theta| = 1$ , then the QFrFT in (3.4.4) defines a unitary transform from  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  into  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$ .

**Remark 3.4.4.** The family of one-parameter transforms  $\mathcal{L}_\theta^\alpha$  verifies the semi-group property  $\mathcal{L}_\theta^\alpha \circ \mathcal{L}_\eta^\alpha = \mathcal{L}_{\theta\eta}^\alpha$ , so that its inverse is  $\mathcal{L}_{1/\theta}^\alpha$  when  $\theta \neq 0$ . But we do not have  $\mathcal{L}_\theta^\alpha \circ \mathcal{L}_\eta^\alpha = \mathcal{L}_\eta^\alpha \circ \mathcal{L}_\theta^\alpha$  in general, for lack of commutativity in  $\mathbb{H}$ . However,  $\mathcal{L}_\theta^\alpha \circ \mathcal{L}_\eta^\alpha = \mathcal{L}_{\theta\eta}^\alpha = \mathcal{L}_\eta^\alpha \circ \mathcal{L}_\theta^\alpha$  holds only when  $\theta$  and  $\psi$  belongs to the same slice  $\mathbb{C}_I := \mathbb{R} + I\mathbb{R} \subset \mathbb{H}$ ;  $I^2 = -1$ .

The next result gives the explicit expression of the inverse of  $\widetilde{\mathcal{L}}_\theta^\alpha$ .

**Proposition 3.4.5.** For any quaternionic  $\theta \neq 0$ , the inverse of  $\widetilde{\mathcal{L}}_\theta^\alpha$  is given by

$$(\widetilde{\mathcal{L}}_\theta^\alpha)^{-1} = \mathcal{A}^{-1} \Gamma_\theta^{-1} \mathcal{A} = \widetilde{\mathcal{L}}_{1/\theta}^\alpha.$$

*Proof.* It is immediate from the definition of  $\widetilde{\mathcal{L}}_\theta^\alpha$  and the fact that  $\Gamma_\theta \circ \Gamma_\eta = \Gamma_{\theta\eta}$ .  $\square$

The following result identifies the kernel function given by (3.3.6),

$$\widetilde{R}_\theta^\alpha(x, y) := \langle A_{slice}^\alpha(x; \theta(\cdot)), A_{slice}^\alpha(y; \cdot) \rangle_{L^{2,\alpha}(\mathbb{B}_I)} \quad (3.4.8)$$

of the QFrFT transform

$$[\widetilde{\mathcal{L}}_\theta^\alpha(\varphi)](y) = \left\langle \widetilde{R}_\theta^\alpha(\cdot, y), \varphi \right\rangle_{L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)}$$

**Theorem 3.4.6.** *The kernel function  $\widetilde{R}_\theta^\alpha(x, y)$  is a left slice regular and coincides with the kernel function of the fractional Hankel transform on the quaternionic unit ball. Moreover, the explicit expression of  $\widetilde{\mathcal{L}}_\theta^\alpha$  is given by*

$$\widetilde{\mathcal{L}}_\theta^\alpha \varphi(y) = \frac{e^{\frac{\theta y}{\theta-1}}}{(1-\theta)(\theta y)^{\alpha/2}} \int_0^\infty x^{\alpha/2} I_\alpha \left( \frac{2\sqrt{\theta}}{(1-\theta)} \sqrt{xy} \right) e^{\frac{x}{\theta-1}} \varphi(x) dx \quad (3.4.9)$$

for any  $\theta \in \mathbb{H}$  with  $|\theta| \leq 1$  and  $\theta \neq 1$ .

*Proof.* Notice first that for  $\theta = 1$  there is nothing to prove since in this case, the operator  $\widetilde{\mathcal{L}}_\theta^\alpha$  reduces further to the identity operator of the Hilbert space  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  and the  $R_\theta^\alpha(x, y)$  in (3.4.8) can be considered as the Dirac delta function. To identify the closed expression of the kernel  $\widetilde{R}_\theta^\alpha(x, y)$ , we should notice that the  $\Gamma_\theta$ -action reads

$$\Gamma_\theta(q \mapsto A_{slice}^\alpha(x; q)) = (1 - q\theta)^{-\alpha-1} \star \exp_\star \left( xq\theta, [q\theta - 1]^{-1} \right),$$

where

$$\exp_\star(f(q), g(q)) = \sum_{n=0}^{\infty} \frac{f^{n\star}(q) \star g^{n\star}(q)}{n!}.$$

For  $\theta$  being a non real quaternionic number, there exists a unique imaginary unit  $I_\theta$ ;  $I_\theta^2 = -1$ , such that  $\theta \in \mathbb{C}_{I_\theta} \cap S^3$ . By means of (3.4.8) and the independence of the scalar product  $\langle f, g \rangle_I$  in  $I$  when acting on  $A_{slice}^{2,\alpha}(\mathbb{B})$ , we may write

$$\begin{aligned} \widetilde{R}_\theta^\alpha(x, y) &:= \langle \Gamma_\theta A_{slice}^\alpha(x; \cdot), A_{slice}^\alpha(y; \cdot) \rangle_{L^{2,\alpha}(\mathbb{B}_{I_\theta})} \\ &= \frac{1}{\pi\Gamma(\alpha)} \int_{B_I} \frac{\exp\left(\frac{xz\theta}{z\theta-1}\right) \exp\left(\frac{y\bar{z}}{\bar{z}-1}\right)}{(1-z\theta)^{\alpha+1}(1-\bar{z})^{\alpha+1}} (1-|z|^2)^{\alpha-1} d\lambda_I(z) \end{aligned}$$

in view of the explicit expression of the kernel function  $A_{slice}^\alpha$  in (3.4.2). Using the generating function for generalized Laguerre polynomials [15, p.288]

$$(1-z)^{-\alpha-1} \exp\left(\frac{xz}{z-1}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) z^n,$$

provided  $|z| < 1$ , as well as Fejer's formula [95, Theorem 8.22.1, p. 198], it is not hard to see that the involved  $z$ -function series are uniformly convergent on any compact set contained in unit disk. Therefore, direct computation yields

$$\begin{aligned}\widetilde{R}_\theta^\alpha(x, y) &= \frac{1}{\pi\Gamma(\alpha)} \int_{\mathbb{D}} \left( \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \theta^n z^n \right) \left( \sum_{m=0}^{\infty} L_m^{(\alpha)}(y) \bar{z}^m (1 - |z|^2)^{\alpha-1} \right) d\lambda(z) \\ &= \frac{1}{\pi\Gamma(\alpha)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \theta^n L_n^{(\alpha)}(x) L_m^{(\alpha)}(y) \int_{\mathbb{D}} z^n \bar{z}^m (1 - |z|^2)^{\alpha-1} d\lambda(z) \\ &= \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+1+\alpha)} \theta^n L_n^{(\alpha)}(x) L_n^{(\alpha)}(y)\end{aligned}\quad (3.4.10)$$

for  $|\theta z| < 1$  which holds true when  $|\theta| \leq 1$  and  $|z| < 1$ . This provides the expansion series of the restriction of  $\widetilde{R}_\theta^\alpha(x, y)$  to any  $\mathbb{B}_I$ . For  $|\theta| < 1$ , we recognize the Hille-Hardy identity (3.2.1) for Laguerre polynomials. Thus, we have

$$\widetilde{R}_\theta^\alpha(x, y) = \frac{1}{(1-\theta)} \left( \frac{1}{xy\theta} \right)^{\alpha/2} I_\alpha \left( \frac{2\theta^{1/2}}{1-\theta} \sqrt{xy} \right) \exp \left( -\frac{\theta(x+y)}{1-\theta} \right) \quad (3.4.11)$$

for  $|\theta| < 1$  and  $\theta \notin \mathbb{R}$ . This leads to (3.2.3) by considering the kernel function  $\widetilde{R}_\theta^\alpha(x, y) x^\alpha e^{-x}$ . The right-hand side in (3.4.11) is clearly a slice regular function in  $\theta \in \mathbb{B}$  for  $x, y$  being reals. The extension of (3.4.11) to the whole unit open ball  $\mathbb{B}$  relies on the Identity Principle for left slice regular functions [51], since both sides of (3.4.11) are left slice regular and coincide at least on the upper half unit ball. To conclude, we need only to examine the validity of the closed expression in the right-hand side of (3.4.11) for the expansion of  $\widetilde{R}_\theta^\alpha(x, y)$  which remains valid when  $|\theta| = 1$  with  $\theta \neq 1$ . This can be handled by fixing  $\theta$  and let  $\varepsilon \in (0, 1)$ , so that (3.4.11) holds true for  $|\varepsilon\theta| < 1$ , and next sending  $\varepsilon$  to  $1^-$ , at least formally. This can be rigorously justified making use of test functions and classical argument from the Schwartz theory of distributions.  $\square$

**Remark 3.4.7.** By taking  $\theta = -1$  with  $\sqrt{\theta} = i$  in (3.4.9), we recover the classical Fourier-Bessel transform [83, 71]

$$(\mathcal{H}_\alpha \psi)(y) := \int_0^\infty u J_\alpha(yu) \psi(u) du$$

for  $\psi \in L^2(\mathbb{R}^+)$ , where  $J_\alpha$  is the Bessel function. Indeed, by setting  $\widetilde{\mathcal{L}}^\alpha = \widetilde{\mathcal{L}}_{-1}^\alpha$  and making the change of variable  $u^2 = x$  and the function  $\psi(u) = x^{\alpha/2} e^{-x/2} \varphi(x) = u^\alpha e^{-u^2/2} \varphi(u^2)$  we get

$$\widetilde{\mathcal{L}}^\alpha \varphi(y^2) = \frac{e^{y^2/2}}{i^\alpha y^\alpha} \int_0^\infty u^{\alpha+1} I_\alpha(iyu) e^{-u^2/2} \varphi(u^2) du = \frac{e^{y^2/2}}{y^\alpha} (\mathcal{H}_\alpha \psi)(y).$$

The last equality follows since  $I_\alpha(x) = i^{-\alpha} J_\alpha(ix)$ .

**Remark 3.4.8.** *The considered family of QFrFT on the real half-line appears embedded in a strongly continuous one-parameter group of unitary operators the quaternionic context. Moreover, it is continuous and interpolates continuously the identity operator ( $\theta = 1$ ) to the Hankel transform [15, p. 216] corresponding to  $\theta = -1$ .*

**Remark 3.4.9.** *The considered transform can be used to reintroduce the hyperholomorphic Bergman space  $A_{\text{slice}}^{2,\alpha}(\mathbb{B})$  in (3.4.1) as well as some of their specific generalization in the context of slice regular functions on the unit quaternionic ball by considering the dual transform of  $\theta \mapsto \widetilde{\mathcal{L}}_\theta^\alpha \varphi(y)$ , for fixed  $y \in (0, +\infty)$ . For the limit case of  $y = 0$ , the last transform is nothing than the Bargmann transform in (2.3.24). The concrete study of the spectral properties of these dual transforms is studied in [55].*

### 3.5 The dual transforms $S_y^\alpha$ : Application 2

We begin by noticing that the transform  $S_y^\alpha$  satisfies  $S_y^\alpha \varphi(pq) = S_y^\alpha \circ \mathcal{L}_q^\alpha(\varphi)(p)$  by means of the semi-group property  $\mathcal{L}_p^\alpha \circ \mathcal{L}_q^\alpha = \mathcal{L}_{pq}^\alpha$  for the quaternionic fractional Hankel transform, as well as the eigenvalue equation  $S_y^\alpha(\varphi_n^\alpha) = \varphi_n^\alpha(y)e_n$  since the normalized generalized Laguerre polynomials given in 2.3.22 are solutions of  $\mathcal{L}_q^\alpha(L_n^{(\alpha)}) = q^n L_n^{(\alpha)}$ . Moreover, the kernel function

$$R_q^\alpha(x, y) = \frac{1}{(1-q)\sqrt{qxy}^\alpha} \exp\left(-\frac{q(x+y)}{1-q}\right) I_\alpha\left(\frac{2\sqrt{qxy}}{1-q}\right), \quad (3.5.1)$$

for the transform  $S_y^\alpha$  in (0.0.7), has the expansion series [83, 44],

$$R_p^\alpha(x, y) = \sum_{n=0}^{\infty} e_n(p) \varphi_n^\alpha(x) \varphi_n^\alpha(y) \quad (3.5.2)$$

which follows from the Hille–Hardy formula for the Laguerre polynomials [15, (6.2.25) p. 288]. Such kernel function satisfies the following reproducing property.

**Proposition 3.5.1.** *Let  $K_{\beta,\eta}(p, q)$  be as in (2.4.3). Then, for every  $y \in (0, +\infty)$ , we have*

$$R_q^\alpha(x, y) = \int_{\mathbb{B}_I} \overline{K_{\beta,\eta}(p, q)} R_p^\alpha(x, y) \omega_{\beta,\eta}(|p|^2) d\lambda_I(p) \quad (3.5.3)$$

and

$$R_{|q|^2}^\alpha(y, y) = \int_{\mathbb{R}^+} |R_q^\alpha(x, y)|^2 x^\alpha e^{-x} dx. \quad (3.5.4)$$

*Proof.* Both (3.5.3) and (3.5.4) can be proved, at least formally, using the expansion series of the involved kernels given by (2.4.4) and (3.5.2).  $\square$

Proposition 3.5.1 can be used to reprove the reproducing property satisfied by the functions in the range of  $S_y^\alpha$  by means of the kernel  $K_{\beta,\eta}$ . Indeed, by rewriting  $S_y^\alpha$  as  $S_y^\alpha \varphi(q) = \left\langle \overline{R_q^\alpha(\cdot, y)}, \varphi \right\rangle_{L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)}$ .

for every  $\varphi \in L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$ .

**Proposition 3.5.2.** *See [55]: The transform  $S_y^\alpha$  is well defined on  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$ . In addition,, under the assumption that*

$$\int_{\mathbb{B}_I} R_{|q|^2}^\alpha(y, y) \omega(|q|^2) dudv < +\infty; \quad q = u + Iv, \quad (3.5.5)$$

the transform  $S_y^\alpha$  is a bounded operator from  $L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+)$  into  $L_{\mathbb{H}}^{2,\omega}(\mathbb{B}_I)$ .

For  $y = 0$ , the assumption that (3.5.5) reduces further to

$$\int_0^1 R_t^\alpha(0, 0) \omega(t) dt = \frac{1}{2^\alpha \Gamma(\alpha + 1)} \int_0^1 \frac{\omega(t)}{(1-t)^{\alpha+1}} dt \quad (3.5.6)$$

be finite. The convergence of the integral in (3.5.6) readily holds when  $\eta > \alpha + 1$  for the special case of  $\omega(t) = \omega_{\beta,\eta}(t) := t^{\beta-1}(1-t)^{\eta-1}$  with  $\alpha > -1$  and  $\beta, \eta > 0$ . The next results extends this condition to includes  $y > 0$  qnd it was proved in [ [55] ].

**Proposition 3.5.3.** *Let  $\alpha > -1$ ,  $\beta, \eta > 0$  and  $y \geq 0$ . It in addition  $\eta > \alpha + 1$ , then the integral operator  $S_y^\alpha : L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+) \longrightarrow L_{\mathbb{H}}^{2,\omega_{\beta,\eta}}(\mathbb{B}_I)$  is bounded.*

The next result refines the boundedness condition of  $S_y^\alpha$  provided in the previous assertion. It shows that  $\eta > \alpha + 1$  can be relaxed. To this end, we distinguish two cases  $y = 0$  and  $y > 0$ .

**Proposition 3.5.4.** *Let  $\alpha > -1$ ,  $\beta, \eta > 0$  and  $y \geq 0$ . Then, the integral operator  $S_y^\alpha : L_{\mathbb{H}}^{2,\alpha}(\mathbb{R}^+) \longrightarrow L_{\mathbb{H}}^{2,\omega_{\beta,\eta}}(\mathbb{B}_I)$  is bounded for any  $y > 0$ . The boundedness of  $S_y^\alpha$  at  $y = 0$  holds when  $\eta \geq \alpha$ .*

# Chapter 4

## Cauchy transform on $\mathbb{H}$ .

Abstract: We investigate some spectral properties of the weighted quaternionic Cauchy transform when acting on the right quaternionic Hilbert space of Gaussian integrable functions. We study its boundedness, compactness and memberships to  $k$ -Schatten class, and we identify its range. This is done by means of its restriction to the  $n$ -th S-polyregular Bargmann space of second kind, for which we provide explicit closed expression for its action on the quaternionic Itô–Hermite polynomials constituting an orthogonal basis. We also exhibit an orthogonal basis of eigenfunctions of its  $n$ -Bergman projection leading to the explicit determination of its singular values. The obtained results generalize those given for weighted Cauchy transform on the complex plane to the quaternionic setting.

### 4.1 Introduction

Let  $\mathbb{C}$  denotes the complex plane endowed with  $d\lambda(z) = dx dy; z = x + iy$ , being the usual Lebesgue area measure. We denote by  $L^2(G)$  the space of square integrable complex valued-functions on given domain  $G$  for which the norm

$$\|f\| = \left( \int_G |f(z)|^2 d\lambda(z) \right)^{\frac{1}{2}}$$

is finite. The Cauchy integral operator on  $L^2(G)$ , is

$$\mathcal{C}_G(f)(w) = \frac{1}{\pi} \int_G \frac{f(z)}{z - w} d\lambda(z).$$

If  $G$  be a bounded domain with smooth boundary, the operator  $\mathcal{C}_G$  is compact as an operator from  $L^2(G)$  to  $L^2(G)$ . For more details, we can see the work of J. M. Anderson, D. Khavinson, and V. Lomonosov, [14]. Precisely, the case when the domain  $G$  is the unit disk  $\mathbb{D} = \{z, |z| < 1\}$ , J. M. Anderson and A.Hinkkanen, [13]

determine the precise value of its norm which is given by  $\beta$  the smallest positive zero of the Bessel function  $J_0$  of order zero

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$$

More precisely, we have  $\|\mathcal{C}\| = \frac{2}{\beta}$ , where  $\beta = 2,4048256$ .

In addition, in [14], the authors investigated the spectrale estimates of eigenvalue of Cauchy operator which is related with the product of the Cauchy operator  $\mathcal{C}_G$  and its adjoint  $\mathcal{C}_G^*$ . In [16] are derived a number of results concerning more subtle operator-theoretic properties of operator  $\mathcal{C}_G$ . It is observed that the singulars values of the product of Bergman projection and Cauchy operator decrease as those of  $\mathcal{C}$ , and show that  $\mathcal{C}_G$  belongs to the "weak Schatten ideal".

In the other hand, when  $G$  is the whole complex plane  $\mathbb{C}$ . The Cauchy integral operator is not bounded when  $\mathcal{C}_G$  is considered as an operator on  $L^2(\mathbb{C})$ , the Hilbert space of all complex valued functions in  $\mathbb{C}$  with respect to ordinary Lebesgue measure in  $\mathbb{C}$  to itself as quoted in [13]. In fact, the Cauchy transform action of the characteristic function of any bounded domain whose modulus bahaves like  $|w|^{-1}$  as  $w \rightarrow \infty$ . Hence does not belong to  $L^2(\mathbb{C})$ .

Nonetheless, the boundness and compactness are yeild when Cauchy operator is considered as an operator from  $L^2(\mathbb{C}, d\mu(z))$  to itself, where  $d\mu(z) = e^{-|z|^2} d\lambda(z)$  is the Gaussian density on the complex plane  $\mathbb{C}$ .

The authors in [16] have investigate some numbers spectrale properties of Cauchy transform defined on  $L^2(\mathbb{C})$  by

$$\mathcal{C}_{\mathbb{C}}(f)(w) = \int_{\mathbb{C}} \frac{f(z)}{z-w} d\mu(z).$$

Namely, they studied the asymptotic baheviour of the sequence of singulars values of the Cauchy operator and its relation with those of the product operators of Cauchy operator with orthogonal projection operator from  $L^2(\mathbb{C}, d\mu(z))$  onto  $B_0(\mathbb{C})$  the Bargmann space of entire functions in  $L^2(\mathbb{C}, d\mu(z))$ . But, Abdelkader Intissar and Ahmed Intissar in [66] discussed some spectral properties of the Cauchy transform  $\mathcal{C}_{\mathbb{C}}$  on the orthogonal complement  $B_0(\mathbb{C})^{\perp}$  of Bargmann space  $B_0(\mathbb{C})$  in  $L^2(\mathbb{C}, d\mu(z))$ .

## 4.2 quaternionic Cauchy singular integral

Let  $\mathbb{H}$  denotes the real skew algebra of quaternions defined as the 4-component extended complex numbers  $q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$ , where  $x_0, x_1, x_2, x_3 \in \mathbb{R}$ , endowed with the Hamiltonian computation rules so that the imaginary components  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  satisfy  $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ ;  $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$ . The algebraic conjugate and the modulus are defined by  $x_0 - x_1i - x_2j - x_3k$  and  $|q| = \sqrt{q\bar{q}}$ , respectively. If  $\mathbb{S}$  denotes the set of imaginary units  $I^2 = -1$ , identified to the unit sphere  $S^2$  in

$\Im\mathbb{H} = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ , then we can rewrite any  $q \in \mathbb{H}$  as  $q = re^{I_q\theta}$  (polar representation) or as  $q = x + I_qy$  (slice representation). Here  $r = |q| \geq 0$ ,  $I_q \in \mathbb{S}$ ,  $\theta \in [0, 2\pi[$  and  $x, y \in \mathbb{R}$ . The last representation gives rise to the notion of slice plane  $\mathbb{C}_I := \mathbb{R} + \mathbb{R}I$ , so that  $\bigcap_{I \in \mathbb{S}} \mathbb{C}_I = \mathbb{R}$  and

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I = \bigcup_{I \in \tilde{\mathbb{S}}} \mathbb{C}_I,$$

where  $\tilde{\mathbb{S}}$  denotes the hemisphere of purely imaginary quaternions defined as

$$\tilde{\mathbb{S}} := \{I = \cos(\phi)\mathbf{i} + \sin(\phi)\cos(\psi)\mathbf{j} + \sin(\phi)\sin(\psi)\mathbf{k}; \phi, \psi \in (0, \pi)\},$$

and endowed with surface area measure  $d\sigma(I) = \sin(\phi)d\phi d\psi$ . Thereafter,  $|\tilde{\mathbb{S}}|$  will denote the area of  $\tilde{\mathbb{S}}$ .

In accordance with the slice representation of quaternionic numbers, Gentili and Struppa were able in 2007 to develop the theory of slice regular functions on specific domains of  $\mathbb{H}$  and similar to the one of holomorphic functions on complex domains [54]. A given quaternionic valued real differentiable function  $f$  on  $\mathbb{H} \sim \mathbb{R}^4$  is said to be left slice regular and we write  $f \in \mathcal{SR}(\mathbb{H})$ , if its restriction  $f|_{\mathbb{C}_I}$  to any slice  $\mathbb{C}_I$ ;  $I \in \mathbb{S} = \{q \in \mathbb{H}; q^2 = -1\}$ , satisfies the Cauchy–Riemann equation  $\overline{\partial}_I f = 0$ , where  $\overline{\partial}_I f$  stands for the conjugate of the left slice derivative,

$$(\overline{\partial}_I f)(x + Iy) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f|_{\mathbb{C}_I}(x + yI). \quad (4.2.1)$$

As specific space of these functions is the slice quaternionic Bargmann space defined in [7] as

$$\mathcal{A}_{slice}^2 := \left\{ f = \sum_{n=0}^{\infty} q^n c_n; c_n \in \mathbb{H}; \sum_{n=0}^{\infty} n! |c_n|^2 < +\infty \right\} \quad (4.2.2)$$

and can be seen as closed subspace of all slice regular functions  $f \in \mathcal{SR}(\mathbb{H})$  belonging to the Hilbert space  $L_{\mu_I}^2(\mathbb{C}_I) := L_{\mathbb{H}}^2(\mathbb{C}_I, d\mu_I)$  endowed with the scalar product

$$\langle \varphi, \psi \rangle_{\mathbb{C}_I} = \int_{\mathbb{C}_I} \overline{\varphi(x + Iy)} \psi(x + Iy) d\mu_I(q), \quad (4.2.3)$$

where  $d\mu_I(x + Iy) = e^{-x^2 - y^2} dx dy$ . In [18], the space  $\mathcal{A}_{slice}^2$  has been realized as a specific  $L^2$ -eigenspace of the semi-elliptic (slice) second-order differential operator

$$\square_q = -\partial_s \overline{\partial}_s + \overline{q} \overline{\partial}_s, \quad (4.2.4)$$

where for  $q = x + I_qy \in \mathbb{H}$ , we have

$$\overline{\partial}_s f(q) = \begin{cases} \overline{\partial}_{I_q} f(x + I_qy), & \text{if } y \neq 0, \\ \frac{df}{dx}(x), & \text{if } y = 0. \end{cases} \quad (4.2.5)$$

The generalization to the  $n$ -polyregular functions  $\mathcal{SR}_n(\mathbb{H})$ , i.e., those whose  $(n + 1)$ -th slice derivative  $\bar{\partial}_I^{n+1}$  vanishes identically on  $\mathbb{H}$  for every  $I \in \mathbb{S}$ , is performed by considering the eigenvalue problem  $\square_q f = n f$  for  $f$  in  $L_\mu^2(\mathbb{H})$ . Here, we focus our attention on the one defined as the  $n$ -th  $\mathbb{S}$ -polyregular space of second kind and denoted  $\mathcal{A}_{n,slice}^2$ . More explicitly, the space  $\mathcal{A}_{n,slice}^2$  is characterized as the right  $\mathbb{H}$ -vector space of all convergent series (on the whole  $\mathbb{H}$ ) of the form

$$f(q) = \sum_{m=0}^{+\infty} H_{m,n}^{\mathcal{Q}}(q, \bar{q}) \alpha_m$$

belonging  $L_{\mu_I}^2(\mathbb{C}_I)$ . The occurring quaternionic coefficients  $\alpha_m$  are independent of  $q$  and  $I$ , and satisfy the growth condition (clearly independent of  $I$ )

$$\sum_{m=0}^{+\infty} m! |\alpha_m|^2 < +\infty.$$

Above, the  $H_{m,n}^{\mathcal{Q}}$  stand for the  $(m, n)$ -th quaternionic Itô–Hermite polynomials [41],

$$H_{m,n}^{\mathcal{Q}}(q, \bar{q}) = \sum_{\ell=0}^{m \wedge n} (-1)^\ell \ell! \binom{m}{\ell} \binom{n}{\ell} q^{m-\ell} \bar{q}^{n-\ell}, \quad (4.2.6)$$

where here and elsewhere after  $m \wedge n = \min(m, n)$  and  $m \vee n = \max(m, n)$ . The system  $\varphi_{m,n}^{\mathcal{Q}}(q, \bar{q}) := (\pi m! n! |\tilde{\mathbb{S}}|)^{-1/2} H_{m,n}^{\mathcal{Q}}(q, \bar{q})$  with varying  $m, n$  is orthonormal in  $L_\mu^2(\mathbb{H})$ . For fixed  $n$ , they form a complete orthonormal system in  $\mathcal{A}_{n,slice}^2$ , endowed with the norm induced from (4.2.3). To authorize  $H_{m,n}^{\mathcal{Q}}$  taking negative index, we have to use its hypergeometric representation

$$H_{m,n}^{\mathcal{Q}}(q, \bar{q}) = c_{m,n} \frac{q^m \bar{q}^n}{|q|^{2(m \wedge n)}} {}_1F_1 \left( \begin{matrix} -(m \wedge n) \\ |m - n| + 1 \end{matrix} \middle| |q|^2 \right) \quad (4.2.7)$$

with

$$c_{m,n} := \frac{(-1)^{m \wedge n} (m \vee n)!}{|m - n|!}. \quad (4.2.8)$$

Here,  ${}_1F_1$  denotes the classical Kummer's function [79, Ch. 6, p. 262]

$${}_1F_1 \left( \begin{matrix} a \\ c \end{matrix} \middle| x \right) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{\Gamma(a+j)}{\Gamma(c+j)} \frac{x^j}{j!} \quad (4.2.9)$$

for given  $a, c, x \in \mathbb{R}$  with  $c \neq 0, -1, -2, \dots$ . Thus, we write

$$H_{-1,n}^{\mathcal{Q}}(q, \bar{q}) := -\frac{\bar{q}^{n+1}}{n+1} {}_1F_1 \left( \begin{matrix} 1 \\ n+2 \end{matrix} \middle| |q|^2 \right). \quad (4.2.10)$$

The performed sequence of spaces  $\mathcal{A}_{n,slice}^2$  form an orthogonal slice Hilbertian decomposition for  $L_{\mu_I}^2(\mathbb{C}_I)$  [18],

$$L_{\mu_I}^2(\mathbb{C}_I) = \bigoplus_{n=0}^{+\infty} \mathcal{A}_{n,slice}^2,$$

in the sense that for every  $\varphi \in L_{\mu_I}^2(\mathbb{C}_I)$  there exists a sequence of quaternions  $\alpha_{m,n}^I$  depending in  $I \in \mathbb{S}$ , and satisfying the growth condition

$$\pi \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} m!n! |\alpha_{m,n}^I|^2 < +\infty \quad (4.2.11)$$

such that

$$\varphi(q) = \sum_{n=0}^{\infty} \varphi_n^I(q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} H_{m,n}^Q(q, \bar{q}) \alpha_{m,n}^I$$

with

$$\varphi_n^I(x + Iy) = \sum_{m=0}^{\infty} H_{m,n}^Q(x + Iy, x - Iy) \alpha_{m,n}^I. \quad (4.2.12)$$

The functions  $\varphi_n^I$  can be extended to the whole  $\mathbb{H}$  leading to a S-polyregular of order  $n - 1$  by considering

$$\tilde{\varphi}_n^I(x + Jy) = \sum_{m=0}^{\infty} H_{m,n}^Q(x + Jy, x - Jy) \alpha_{m,n}^I \quad (4.2.13)$$

for any  $J \in \mathbb{S}$ . Clearly, we have  $\tilde{\varphi}_n^I \in \mathcal{A}_{n,slice}^2$ . The situation is different when considering functions  $f$  on  $\mathbb{H}$  which admits a slice decompositions in terms of the quaternionic Itô–Hermite polynomials, i.e. for each  $I \in \mathbb{S}$ , we have

$$f|_{\mathbb{C}_I}(q) = \sum_{m=0}^{\infty} f_n^I(q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} H_{m,n}^Q(q, \bar{q}) \alpha_{m,n}^I.$$

In this case the  $f_n^I$  are S-polyregular but do not belong necessary to  $\mathcal{A}_{n,slice}^2$ . However,  $f_n^I \in L_{\mu}^2(\mathbb{H})$  (resp.  $f \in L_{\mu}^2(\mathbb{H})$ ) if and only if the coefficients satisfy

$$\sum_{m=0}^{+\infty} m!n! \int_{\mathbb{S}} |\alpha_{m,n}^I|^2 < +\infty \quad (\text{resp. } \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} m!n! \int_{\mathbb{S}} |\alpha_{m,n}^I|^2 < +\infty). \quad (4.2.14)$$

This holds true when for example  $\alpha_{m,n}^I$  are constant with respect to  $I$ . This conditions is fulfilled if we restrict ourself to the space  $\mathcal{H}_{slice}^2(\mathbb{H})$  of all quaternionic valued

functions on  $\mathbb{H}$ ,  $f : \mathbb{H} \rightarrow \mathbb{H}$ , whose restriction  $f|_{\mathbb{C}_I}$ , belongs to  $L^2_{\mathbb{H}}(\mathbb{C}_I, d\mu_I)$  for any  $I \in \tilde{\mathbb{S}}$  and such that

$$\|f\|_{\infty} := \sup_{I \in \tilde{\mathbb{S}}} \|f|_{\mathbb{C}_I}\|_{\mathbb{C}_I} \quad (4.2.15)$$

is finite. We claim that  $\|\cdot\|_{\infty}$  defines a norm in  $\mathcal{H}^2_{slice}(\mathbb{H})$  and that  $(\mathcal{H}^2_{slice}(\mathbb{H}), \|\cdot\|_{\infty})$  is a complet space which is contained in  $L^2_{\mu}(\mathbb{H})$  and we have

$$\|f\|_{\mathbb{H}} \leq |\tilde{\mathbb{S}}| \|f\|_{\infty}.$$

We conclude this section by noticing that  $\mathcal{A}^2_{n,slice} \subset \mathcal{H}^2_{slice}(\mathbb{H})$  and its elements satisfy the estimation

$$|f(q)| \leq \frac{e^{-\frac{|q|^2}{2}}}{\sqrt{\pi}} \|f\|_{\mathbb{C}_I} \quad (4.2.16)$$

for any  $I \in \mathbb{S}$ . This to say that the evaluation map  $\delta_q f = f(q)$ , for every fixed  $q \in \mathbb{H}$ , is a continuous linear form on  $\mathcal{A}^2_{n,slice}$ , and therefore  $\mathcal{A}^2_{n,slice}$  is a reproducing kernel Hilbert space, i.e., there exists some function  $\mathcal{K}_n$  on  $\mathbb{H} \times \mathbb{H}$  such that  $p \mapsto \mathcal{K}_n(q, p) = \overline{\mathcal{K}_n(p, q)}$  belongs to  $\mathcal{A}^2_{n,slice}$  and satisfies the reproducing property  $f(q) = \langle \mathcal{K}_n(\cdot, q), f \rangle_{\mathbb{C}_I}$  for every  $f \in \mathcal{A}^2_{n,slice}$ . This kernel function can be expanded as

$$\mathcal{K}_n(q, p) = \sum_{m=0}^{\infty} \frac{H_{m,n}^Q(q, \bar{q}) \overline{H_{m,n}^Q(p, \bar{p})}}{\pi m! n!}. \quad (4.2.17)$$

Its closed expression is given by [18, Theorem 3.6]

$$\mathcal{K}_n(q, p) = \frac{e_*^{[\bar{q}, p]}}{\pi} \star_{sp}^q L(|q - p|_{\star_{sp}^q}^2), \quad (4.2.18)$$

where  $|q - p|_{\star_{sp}^q}^2$  denotes the regularization of the modulus function  $q \mapsto (p - q)(\bar{p} - \bar{q})$  with respect to the left star product  $\star_{sp}^q$  for S-polyregular functions in  $q$ ,  $L$  the classical Laguerre polynomials and

$$e_*^{[a,b]} = \sum_{k=0}^{\infty} \frac{a^k b^k}{k!}.$$

Moreover, the integral operator

$$P_n f(q) = \int_{\mathbb{C}_I} \mathcal{K}_n(p, q) f(p) d\mu_I(p) \quad (4.2.19)$$

is the orthogonal projection of  $L^2_{\mu_I}(\mathbb{C}_I)$  on  $\mathcal{A}^2_{n,slice}$ . The problem of extending  $P_n$  to a larger Hilbert space can be handled by averaging the coefficients over the hemisphere  $\tilde{\mathbb{S}}$ . Thus, for each nonnegative integer  $n$ , we define the transform  $P_n$  on

$\mathcal{H}_{slice}^2(\mathbb{H})$  by the series

$$P_n f(q) = \frac{1}{|\widetilde{\mathbb{S}}|} \sum_{m=0}^{\infty} H_{m,n}^Q(q, \bar{q}) \int_{\widetilde{\mathbb{S}}} \alpha_{m,n}^I d\sigma(I). \quad (4.2.20)$$

which is absolutely and uniformly convergent on compact sets of  $\mathbb{H}$  thanks to (4.2.14). It belongs to  $\mathcal{A}_{n,slice}^2$  and one recovers the (4.2.19) when  $f \in L_{\mu_1}^2(\mathbb{C}_I)$ . This justifies the use of the same notation. Moreover, using the fact

$$\langle H_{m,n}^Q, f \rangle_{\mathbb{H}} = \pi m! n! \int_{\widetilde{\mathbb{S}}} \alpha_{m,n}^I d\sigma(I)$$

and the orthogonality of  $H_{m,n}^Q$  in  $L_{\mu}^2(\mathbb{H})$ , we can rewrite  $P_n$  in (4.2.20) as

$$P_n f(q) = \frac{1}{|\widetilde{\mathbb{S}}|} \langle \mathcal{K}_n(\cdot, q), f \rangle_{\mathbb{H}}, \quad (4.2.21)$$

Thus, we have proved the following

**Lemma 4.2.1.** *The transform  $P_n$  in (4.2.21) defines an orthogonal projection of  $L_{\mu}^2(\mathbb{H})$  onto the  $S$ -polyregular Bargmann space  $\mathcal{A}_{n,slice}^2$ . Its integral representation is given by (4.2.21),*

$$P_n f(q) = \frac{1}{\pi |\widetilde{\mathbb{S}}|} \int_{\mathbb{H}} L_n(|q - p|_{\star_{sp}^q}^2) \star_{sp}^q e^{[\bar{q}, p]} f(p) d\mu(p). \quad (4.2.22)$$

**Remark 4.2.2.** *For  $n = 0$ ,  $P_0$  is the quaternionic Bergman projector on the slice hyperholomorphic Bargmann space in (4.2.2).*

**Definition 4.2.3.** *We call  $n$ -th Bergman projection the transform  $P_n$ .*

### 4.3 Weighted quaternionic Cauchy singular integral

In order to develop the quaternionic analogue of the Cauchy transform, one can consider the kernel function  $p \mapsto (q - p)^{-1}$ . However, this function is in general neither left nor right slice regular, unless  $q \in \mathbb{R}$ . Instead, one has to consider the one given by the power series representation

$$\mathcal{N}(q, p) = \begin{cases} -\sum_{\ell=0}^{+\infty} q^{\ell} p^{-1-\ell}; & p \in \overline{B_{|q|}}^c \\ \sum_{\ell=0}^{+\infty} q^{-1-\ell} p^{\ell}; & p \in B_{|p|}. \end{cases} \quad (4.3.1)$$

where for given  $q \in \mathbb{H}$ , we have denoted by  $B_{|q|} := \{p \in \mathbb{H}; |p| < |q|\}$  the ball of radius  $|q|$  in  $\mathbb{H} \equiv \mathbb{R}^4$  and centered at origin and by  $\overline{B_{|q|}}^c := \{p \in \mathbb{H}; |p| > |q|\}$

the complement of its closure. It turns out to be an appropriate combination of the  $p$ -left and the  $q$ -right non-commutative Cauchy kernel series in the terminology of Colombo and his collaborators (see e.g. [23, 49]). More precisely, the closed expression is the one given by (5.3.1) or equivalently [23, Theorem 2.10]

$$\mathcal{N}(q, p) = \left( q^2 - 2q\Re(p) + |p|^2 \right)^{-1} (q - \bar{p})$$

for  $q \in \mathbb{H}$  and  $p \in V_q := \{p \in \mathbb{H}, pq - qp \neq 0\}$ . It should be mentioned here that the Cauchy kernel function in (5.3.1) in variable  $p$ , can be identified as the regular inverse  $(q - p)^{-*}$  in the slice regular sense (see [53, 52], which is the unique left slice regular extension of  $\mathbb{C}_q \ni p \mapsto (q - p)^{-1}$ ). The next uniform estimation is needed forthcoming investigation.

**Lemma 4.3.1.** *For every  $0 \leq r < 2$ , we have*

$$\sup_{q \in \mathbb{H}} \int_{\mathbb{H}} |\mathcal{N}(q, p)|^r d\mu(q) \leq \frac{|\tilde{\mathbb{S}}|}{\pi^{r-1}} \Gamma\left(1 - \frac{r}{2}\right), \quad (4.3.2)$$

where  $\Gamma(\cdot)$  is the Euler gamma-function.

*Proof.* By observing that the modulus  $|\mathcal{N}(q, p)|$  is independent of  $I_p$  and  $I_q$  for  $p = x + I_p y$  and  $q = x' + I_q y'$  in  $\mathbb{H}$ , we obtain

$$\begin{aligned} \int_{\mathbb{H}} |\mathcal{N}(q, p)|^r d\mu(p) &= \int_{I \in \tilde{\mathbb{S}}} \int_{\mathbb{C}_I} |\mathcal{N}(q, p)|^r d\mu_I(p) d\sigma(I) \\ &= \int_{I \in \tilde{\mathbb{S}}} \int_{\mathbb{C}_{I_q}} |\mathcal{N}(q, p)|^r d\mu_{I_q}(p) d\sigma(I) \\ &= |\tilde{\mathbb{S}}| \int_{\mathbb{C}_{I_q}} |\mathcal{N}(q, p)|^r d\mu_{I_q}(p). \end{aligned}$$

But since for  $p \in \mathbb{C}_{I_q}$ , we have  $\mathcal{N}(q, p) = (q - p)^{-1}$  and therefore we can conclude for (4.3.2) by appealing to Lemma 3.2 in [66].  $\square$

**Corollary 4.3.2.** *For the specific case of  $r = 1$ , we get*

$$\sup_{q \in \mathbb{H}} \int_{\mathbb{H}} |\mathcal{N}(q, p)| d\mu(p) \leq \sqrt{\pi} |\tilde{\mathbb{S}}|. \quad (4.3.3)$$

Associated to the Schwartz kernel  $\mathcal{N}(p, q)$ , there are several ways to extend the definition of weighted Cauchy transform in (0.0.12) to acts on the right quaternionic Hilbert space  $L^2_{\mu}(\mathbb{H})$ . Notice for instance that  $\mathcal{C}_{\mathbb{H}}^I f(q)$  and  $\mathcal{C}_{\mathbb{H}}^S f(q)$  below are defined by means of slices, i.e., by restricting the occurred integral to a fixed slice

$$\mathcal{C}_{\mathbb{H}}^I f(q) := \frac{1}{\pi} \int_{\mathbb{C}_I} \mathcal{N}(q, p) f|_{\mathbb{C}_I}(p) d\mu_I(p) \quad (4.3.4)$$

for given fixed  $I \in \mathbb{S}$  independently of  $q \in \mathbb{H}$ , or to dynamic slice depending in  $q$ , to wit

$$\mathcal{C}_{\mathbb{H}}^{\mathbb{S}}f(q) := \frac{1}{\pi} \begin{cases} \int_{\mathbb{C}_{Iq}} \mathcal{N}(q, p)f(p)d\mu_I(p); & q \in \mathbb{H} \setminus \mathbb{R}, \\ \int_{\mathbb{R}} f(t)\mathcal{N}(t, x)e^{-t^2} dt; & q = x \in \mathbb{R}. \end{cases} \quad (4.3.5)$$

**Remark 4.3.3.** We notice that the computation with  $\mathcal{C}_{\mathbb{H}}^{\mathbb{S}}$  in (4.3.5) is quite similar to the one considered in the complex setting [38, 66], and the restriction of  $\mathcal{C}_{\mathbb{H}}^{\mathbb{S}}f|_{\mathbb{C}}$  to the complex plane  $\mathbb{C} = \mathbb{C}_i$  with  $f$  being complex valued function on  $\mathbb{H}$  reduces also to (0.0.12).

**Remark 4.3.4.** The emplacement of the kernel function  $\mathcal{N}(q, p)$  in the left hand-side of the testing function  $f$  is required to get the linearity of the constructed integral transform when acting on right vector spaces. Otherwise we have to regularize the product  $f(p)\mathcal{N}(p, q)$  in the integrand to overcome technical difficulties for the lack of commutativity. This follows a general scheme for constructing linear quaternionic analogue of the classical integral transforms.

According to the last remark, we can suggest

$$\mathcal{C}_{\mathbb{H}}^*f(q) := \frac{1}{\pi} \int_{\mathbb{H}} f(p) \star^q \star^p \mathcal{N}(q, p)d\mu(p). \quad (4.3.6)$$

One can also consider the singular integral operator defined by

$$\mathcal{C}_{\mathbb{H}}f(q) := \frac{1}{\pi} \int_{\mathbb{H}} \mathcal{N}(q, p)f(p)d\mu(p). \quad (4.3.7)$$

The following result shows that the Cauchy transform  $\mathcal{C}_{\mathbb{H}}$  can be seen as the averaging operator of the  $I$ -slice Cauchy transform  $\mathcal{C}_{\mathbb{H}}^I$  over the hemisphere  $\tilde{\mathbb{S}}$  of purely imaginary quaternions.

**Lemma 4.3.5.** We have

$$\mathcal{C}_{\mathbb{H}}f(q) = \int_{\tilde{\mathbb{S}}} \mathcal{C}_{\mathbb{H}}^I f(q) d\sigma(I).$$

*Proof.* The identity is immediate by definition of the integral on  $\mathbb{H}$ .  $\square$

Subsequently, the uniform boundedness of the operators  $\mathcal{C}_{\mathbb{H}}^I; I \in \tilde{\mathbb{S}}$ , restricted to  $\mathcal{A}_{n, \text{slice}}^2$ , is a sufficient condition for the boundedness of  $\mathcal{C}_{\mathbb{H}}$  on  $\mathcal{A}_{n, \text{slice}}^2$ . In fact, by applying the Cauchy-Schwartz inequality, we get

$$\|\mathcal{C}_{\mathbb{H}}f\|_{\mathbb{H}}^2 = O\left(\sup_{I \in \tilde{\mathbb{S}}} \|\mathcal{C}_{\mathbb{H}}^I\|^2 \|f\|_{\mathbb{C}_I}^2\right).$$

Now, since  $\|f\|_{\mathbb{C}_I}^2$  is independent of  $I$  when  $f \in \mathcal{A}_{n, \text{slice}}^2$ , we get  $\|f\|_{\mathbb{C}_I}^2 = O(\|f\|_{\mathbb{H}}^2)$  and therefore

$$\|\mathcal{C}_{\mathbb{H}}f\|_{\mathbb{H}}^2 = O\left(\sup_{I \in \tilde{\mathbb{S}}} \|\mathcal{C}_{\mathbb{H}}^I\|^2\right) \|f\|_{\mathbb{H}}^2.$$

Clearly, this argument is also valid for  $\mathcal{C}_{\mathbb{H}}$  acting on  $\mathcal{H}_{slice}^2(\mathbb{H})$ . We note also that the boundedness of  $\mathcal{C}_{\mathbb{H}}$  can also be deduced by applying the Schur sufficiency condition for boundedness of integral operator with Schwartz kernel. Below, we present a variant proof.

**Theorem 4.3.6.** *The weighted Cauchy transform  $\mathcal{C}_{\mathbb{H}}$  is a bounded and compact linear operator from  $L_{\mu}^2(\mathbb{H})$  to  $L_{\mu}^2(\mathbb{H})$  with norm operator satisfying*

$$\|\mathcal{C}_{\mathbb{H}}\| \leq \frac{|\tilde{\mathcal{S}}|}{\sqrt{\pi}}.$$

Moreover, it belongs to the  $k$ -Schatten ideal of bounded operators  $S_k(L_{\mu}^2(\mathbb{H}))$  for every  $k > 2$ .

For the proof, we use the following lemma:

**Lemma 4.3.7** (Russo's Lemma [90]). *An integral operator of the form*

$$Tf(y) = \int_X \mathcal{S}(y, x)f(x)dv(x),$$

for given finite measure  $\nu$  on  $X$ , belongs to the  $k$ -Schatten class as an operator on  $L^2(X, dv)$ , if its kernel function satisfies

$$\int_{\mathbb{H}} \left( \int_{\mathbb{H}} |\mathcal{S}(q, p)|^r dv(p) \right)^{k/r} dv(q) \text{ and } \int_{\mathbb{H}} \left( \int_{\mathbb{H}} |\mathcal{S}(q, p)|^r dv(q) \right)^{k/r} dv(p)$$

are finite for  $1 \leq r < 2$  which is the exponent conjugate of  $k$ ,  $1/k + 1/r = 1$ .

*Proof.* of theorem: The Cauchy-Schwartz inequality and Lemma 4.3.2 imply

$$\begin{aligned} |\mathcal{C}_{\mathbb{H}}(f)(q)|^2 &\leq \frac{1}{\pi^2} \left( \int_{\mathbb{H}} |\mathcal{N}(q, p)| d\mu(p) \right) \left( \int_{\mathbb{H}} |\mathcal{N}(q, p)| |f(p)|^2 d\mu(p) \right) \\ &\leq \frac{|\tilde{\mathcal{S}}|}{\pi^{3/2}} \int_{\mathbb{H}} |\mathcal{N}(q, p)| |f(p)|^2 d\mu(p). \end{aligned}$$

Therefore,

$$\begin{aligned} \|\mathcal{C}_{\mathbb{H}}(f)\|_{\mathbb{H}}^2 &\leq \frac{|\tilde{\mathcal{S}}|}{\pi^{3/2}} \int_{\mathbb{H}} \int_{\mathbb{H}} |\mathcal{N}(q, p)| |f(p)|^2 d\mu(p) d\mu(q) \\ &\leq \frac{|\tilde{\mathcal{S}}|}{\pi^{3/2}} \int_{\mathbb{H}} |f(p)|^2 \left( \int_{\mathbb{H}} |\mathcal{N}(q, p)| d\mu(q) \right) d\mu(p) \\ &\leq \frac{|\tilde{\mathcal{S}}|}{\pi^{3/2}} \left( \sup_{p \in \mathbb{H}} \int_{\mathbb{H}} |\mathcal{N}(q, p)| d\mu(q) \right) \|f\|_{\mathbb{H}}^2. \end{aligned}$$

Finally, by Lemma 4.3.2 again, we get

$$\|\mathcal{C}_{\mathbb{H}}(f)\|_{\mathbb{H}}^2 \leq \frac{(|\tilde{\mathcal{S}}|)^2}{\pi} \|f\|_{\mathbb{H}}^2.$$

The compactness of  $\mathcal{C}_{\mathbb{H}}$  follows in a similar way as in [66] for the complex Cauchy transform (0.0.12). Indeed, using the estimation provided by Lemma 4.3.2 and the fact that  $d\mu$  is a finite measure on  $\mathbb{H}$ , we get

$$\int_{\mathbb{H}} \left( \int_{\mathbb{H}} |\mathcal{N}(q, p)|^r d\mu(p) \right)^{k/r} d\mu(q) \leq \frac{\pi(|\tilde{\mathcal{S}}|)^2}{2} \left( \frac{\Gamma(1 - \frac{r}{2})}{\pi^{r-1}} \right)^{k/r}$$

for every  $1 \leq r < 2$ , where  $k$  is its exponent conjugate  $1/k + 1/r = 1$ . Thus, since  $|\mathcal{N}(q, p)| = |\mathcal{N}(p, q)|$ , the two requirements in Russo's lemma [90], for compactness and membership in  $k$ -Schatten class of integral operator with Schwartz kernel, are satisfied. □

**Remark 4.3.8.** *The compactness of  $\mathcal{C}_{\mathbb{H}}$  can also be studied by performing the operator  $\mathcal{C}_{\mathbb{H}}^* \mathcal{C}_{\mathbb{H}}$  and proceed as in [16], where  $\mathcal{C}_{\mathbb{H}}^*$  denotes the adjoint of  $\mathcal{C}_{\mathbb{H}}$  defined as operator from  $L_{\mu}^2(\mathbb{H})$  into  $L_{\mu}^2(\mathbb{H})$  by*

$$\mathcal{C}_{\mathbb{H}}^*(g)(p) = \frac{1}{\pi} \int_{\mathbb{H}} \overline{\mathcal{N}(q, p)} g(q) d\mu(q) = -\frac{1}{\pi} \int_{\mathbb{H}} \mathcal{N}(\bar{p}, \bar{q}) g(q) d\mu(q).$$

The theorem just proved and the upcoming ones remain valid for the Cauchy transform  $\mathcal{C}_{\mathbb{H}}^*$  in (4.3.6). This readily follows by observing that  $\mathcal{C}_{\mathbb{H}}^*$  coincides with  $\mathcal{C}_{\mathbb{H}}$  in (4.3.7) (see Corollary 4.3.11 below). For the proof, we use the following result giving the explicit expression for  $\mathcal{C}_{\mathbb{H}}$  and  $\mathcal{C}_{\mathbb{H}}^*$  on the generic elements  $e_{m,n}(q, \bar{q}) := q^m \bar{q}^n$ . For exact statement, we let  $\varepsilon_k$  stands for

$$\varepsilon_k = \begin{cases} 1 & \text{if } k \geq 0, \\ 0 & \text{if } k < 0. \end{cases}$$

**Proposition 4.3.9.** *For every nonnegative integers  $m, n$  and  $q \in \mathbb{H}$ , we have  $\mathcal{C}_{\mathbb{H}}^*(e_{m,n}) = \mathcal{C}_{\mathbb{H}}(e_{m,n})$  and*

$$[\mathcal{C}_{\mathbb{H}}^*(e_{m,n})](q) = |\tilde{\mathcal{S}}| \left( q^m H_{-1,n}^{\mathcal{Q}}(q, \bar{q}) e^{-|q|^2} - n! \varepsilon_{m-n-1} q^{m-n-1} \right). \quad (4.3.8)$$

*Proof.* We need only to prove the identity (4.3.8), since the computation of  $\mathcal{C}_{\mathbb{H}}(e_{m,n})$  can be handled in a similar way to give rise to the right hand-side in (4.3.8). To this end, we keep notation of the ball of radius  $B_{|q|}$ , the complement of its closure  $\overline{B_{|q|}}^c$

and the hemisphere  $\tilde{\mathbb{S}}$  as above. Thus, making use of the expansion series (5.3.1) of the Cauchy kernel  $\mathcal{N}$ , we can rewrite (4.3.6) as

$$\begin{aligned}
[\mathcal{C}_{\mathbb{H}}^*(e_{m,n})](q) &= \frac{1}{\pi} \left( \int_{B_{|q|}} p^m \bar{p}^n \star^q \star^p \mathcal{N}(q,p) d\mu(p) + \int_{B_{|q|}^c} p^m \bar{p}^n \star^q \star^p \mathcal{N}(q,p) d\mu(p) \right) \\
&= \frac{1}{\pi} \sum_{\ell=0}^{+\infty} \left( q^{-1-\ell} \int_{B_{|q|}} p^{m+\ell} \bar{p}^n d\mu(p) - q^\ell \int_{B_{|q|}^c} p^{m-1-\ell} \bar{p}^n d\mu(p) \right) \\
&= \frac{1}{\pi} \sum_{\ell=0}^{+\infty} q^{-1-\ell} \int_0^{|q|} \int_{\tilde{\mathbb{S}}} \int_0^{2\pi} r^{m+\ell+n} e^{I(m+\ell-n)\theta} e^{-r^2} r dr d\sigma(I) d\theta \\
&\quad - \frac{1}{\pi} \sum_{\ell=0}^{+\infty} q^\ell \int_{|q|}^\infty \int_{\tilde{\mathbb{S}}} \int_0^{2\pi} r^{m-1-\ell+n} e^{I(m-1-\ell-n)\theta} e^{-r^2} r dr d\sigma(I) d\theta \\
&= |\tilde{\mathbb{S}}| q^{m-n-1} \left( \varepsilon_{n-m} \int_0^{|q|^2} t^n e^{-t} dt - \varepsilon_{m-n-1} \int_{|q|^2}^\infty t^n e^{-t} dt \right), \quad (4.3.9)
\end{aligned}$$

where we have used the polar coordinates  $q = re^{I\theta}$  with  $r > 0$ ,  $\theta \in [0, 2\pi]$  and  $I$  in the hemisphere  $\tilde{\mathbb{S}}$ , and next the change of variable  $t = r^2$ . Above  $dr$  and  $d\theta$  are the Lebesgue measure on positive real line and unit circle, respectively. Therefore, keeping in mind the expression of  $H_{m-1,n}^Q(q, \bar{q})$  in (4.2.10), we can check (4.3.8) by observing that

$$\psi_n(|q|^2) := \int_0^{|q|^2} t^n e^{-t} dt = n! - \int_{|q|^2}^\infty t^n e^{-t} dt$$

and using  $\varepsilon_{n-m} + \varepsilon_{m-n-1} = 1$ , as well as

$$\psi_n(|q|^2) = \frac{|q|^{2(n+1)}}{n+1} {}_1F_1 \left( \begin{matrix} 1 \\ n+2 \end{matrix} \middle| |q|^2 \right) e^{-|q|^2} \quad (4.3.10)$$

which follows thanks to the integral representation of confluent hypergeometric function [79, p. 275]

$${}_1F_1 \left( \begin{matrix} a \\ c \end{matrix} \middle| x \right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} e^z \int_0^1 e^{-zt} t^{c-a-1} (1-t)^{a-1} dt \quad (4.3.11)$$

with  $a = 1$  and  $c = n + 2$ . □

**Remark 4.3.10.** *The result of Proposition (4.3.9) generalizes the one obtained for the standard unweighted complex Cauchy transform on the monomials  $z^m \bar{z}^n$ .*

**Corollary 4.3.11.** *We have  $\mathcal{C}_{\mathbb{H}}^* = \mathcal{C}_{\mathbb{H}}$ .*

*Proof.* This readily follows from Proposition 4.3.9 since the polynomials  $e_{m,n}(q) = q^m \bar{q}^n$  are dense in  $L_\mu^2(\mathbb{H})$ . □

In order to provide concrete description of basic properties of  $\mathcal{C}_{\mathbb{H}}$ , we need to its explicit expression when acting on the quaternionic Itô–Hermite polynomials  $H_{m,n}^{\mathbb{Q}}(q, \bar{q})$ . From now on, we normalize the area measure on the hemisphere  $\tilde{\mathbb{S}}$  such that  $|\tilde{\mathbb{S}}| = 1$ .

**Theorem 4.3.12.** *The explicit action of  $\mathcal{C}_{\mathbb{H}}$  on the orthonormal basis  $\varphi_{m,n}^{\mathbb{Q}} = (\pi m!n!)^{-1/2} H_{m,n}^{\mathbb{Q}}$ , for every  $m > 0$ , is given by*

$$\left[ \mathcal{C}_{\mathbb{H}} \varphi_{m,n}^{\mathbb{Q}} \right] (q) = -\frac{e^{-|q|^2}}{\sqrt{m}} \varphi_{m-1,n}^{\mathbb{Q}}(q, \bar{q}), \quad (4.3.12)$$

while for  $m = 0$ , we have

$$\left[ \mathcal{C}_{\mathbb{H}} \varphi_{0,n}^{\mathbb{Q}} \right] (q) = -\frac{e^{-|q|^2}}{\sqrt{\pi n!}} H_{-1,n}^{\mathbb{Q}}(q, \bar{q}).$$

*Proof.* Notice first that the specific case of  $m = 0$  can be checked easily by taking  $m = 0$  in Proposition 4.3.9. Indeed,

$$\mathcal{C}_{\mathbb{H}}(H_{0,n}^{\mathbb{Q}}) = \mathcal{C}_{\mathbb{H}}(e_{0,n}) = -\frac{\bar{q}^{n+1}}{n+1} {}_1F_1 \left( \begin{matrix} 1 \\ n+2 \end{matrix} \middle| |q|^2 \right) e^{-|q|^2}$$

which is close to the hypergeometric function  $H_{-1,n}^{\mathbb{Q}}(q, \bar{q})$  in (4.2.10).

For  $m > 0$ , we make use of (4.3.9) and the linearity of  $\mathcal{C}_{\mathbb{H}}$  to have

$$\mathcal{C}_{\mathbb{H}} H_{m,n}(q) = q^{m-n-1} \left( \varepsilon_{n-m} I_{m,n}(|q|^2) - \varepsilon_{m-n-1} J_{m,n}(|q|^2) \right),$$

with

$$I_{m,n}(|q|^2) := \frac{(-1)^m n!}{(n-m)!} \int_0^{|q|^2} t^{n-m} {}_1F_1 \left( \begin{matrix} -m \\ n-m+1 \end{matrix} \middle| t \right) e^{-t} dt$$

and

$$J_{m,n}(|q|^2) := \int_{|q|^2}^{\infty} {}_1F_1 \left( \begin{matrix} -n \\ m-n+1 \end{matrix} \middle| t \right) e^{-t} dt.$$

Direct computation making appeal to indefinite integrals for the confluent hypergeometric function  ${}_1F_1$ , see for instance [79, p. 266], or equivalently the differentiation formulas in [22, 4, p. 73] with  $a = 1 - m$ ,  $b = n - m + 2$  and  $c = 1 = k$  for the evaluation of  $I_{m,n}(|q|^2)$  and [22, 6, p. 73] with  $a = -n$ ,  $b = |m - n|$  and  $c = 1 = k$  for  $J_{m,n}(|q|^2)$ , keeping in mind the hypergeometric representation of  $H_{m-1,n}^{\mathbb{Q}}(q, \bar{q})$  given through (4.2.7), we arrive at

$$I_{m,n}(|q|^2) = -q^{n-m+1} H_{m-1,n}^{\mathbb{Q}}(q, \bar{q}) e^{-|q|^2} = -J_{m,n}(|q|^2). \quad (4.3.13)$$

Therefore, the identity (4.3.12) follows since  $\varepsilon_{n-m} + \varepsilon_{m-n-1} = 1$ .  $\square$

**Remark 4.3.13.** Green formula was employed in [16, 33] to get the explicit expression of the unweighted complex Cauchy transform on bounded domains when acting on specific elementary functions. See also [66] for the weighted complex Cauchy transform on the complex Itô–Hermite polynomials of total degree great than 1. Adoption of similar approach for the quaternionic Cauchy transform requires an appropriate quaternionic analogue of this famous Green formula (which can be obtained starting from quaternionic version of Cauchy representation theorem [23, 26]). Here, we have given a direct proof using the expansion series of the kernel function combined with integral formula for special function.

For every nonnegative integers  $m, n$ , we denote by  $\psi_{m,n}$  the Hermite functions defined by

$$\psi_{m,n}(p) := -e^{-|p|^2} H_{m,n-1}^Q(p, \bar{p}) = \overline{\mathcal{C}_{\mathbb{H}}(H_{n,m}^Q)(p)}. \quad (4.3.14)$$

Accordingly, for given nonnegative integer  $j$ , we perform the spaces

$$E_j^+ = \overline{\text{span}\{\psi_{n,n+j}; n = 0, 1, 2, \dots\}}^{L_\mu^2(\mathbb{H})}$$

and

$$E_j^- = \overline{\text{span}\{\psi_{n+j,n}; n = 0, 1, 2, \dots\}}^{L_\mu^2(\mathbb{H})}$$

as well as the spaces  $E_\ell = E_{|\ell|}^+$  if  $\ell \geq 0$  and  $E_\ell = E_{|\ell|}^-$  when  $\ell < 0$ .

**Theorem 4.3.14.** The spaces  $E_\ell$ , for varying integer  $\ell$ , form an orthogonal Hilbertian decomposition of the range of the weighted Cauchy transform  $\mathcal{C}_{\mathbb{H}}$  in  $L_\mu^2(\mathbb{H})$ .

*Proof.* We begin by noticing that  $\psi_{n,m} \in L_\mu^2(\mathbb{H})$ . Indeed, we have

$$\|\psi_{n,m}\|_{\mathbb{H}}^2 = \left\| e^{-|p|^2} H_{m-1,n}^Q \right\|_{\mathbb{H}}^2 \leq |\tilde{\mathcal{S}}| \left\| H_{m-1,n}^Q \right\|_{\mathbb{C}_I}^2 < \infty$$

by means of (v) in [66, Proposition 3.2]. Next, by Theorem 4.3.12, the hypergeometric representation (4.2.7) of  $H_{m,n}^Q$  and Fubini's theorem, we can show that the orthogonality of the system  $(\psi_{n,m})_{m,n}$  in  $L_\mu^2(\mathbb{H})$  is equivalent to the nullity of the angular part in the integral giving  $\langle \psi_{n,m}, \psi_{k,j} \rangle_{\mathbb{H}}$  given by

$$A_{m,n,j,k} := \int_0^{2\pi} e^{I_q(n+j-m-k)\theta} d\theta = 2\pi \delta_{m-j, n-k}.$$

Indeed, direct computation shows that

$$\langle \psi_{n,m}, \psi_{k,j} \rangle_{\mathbb{H}} = \pi c_{m-1,n} c_{j-1,k} \delta_{m-j, n-k} \int_0^\infty \frac{e^{-3t} t^{m+n-1} R_{m,n,j,k}(t)}{t^{\min(m-1,n) + \min(j-1,k)}} dt, \quad (4.3.15)$$

where the constants  $c_{m,n}$  are as in (4.2.8) and  $R_{m,n,j,k}$  is given by

$$R_{m,n,j,k}(t) := {}_1F_1 \left( \begin{matrix} -(m-1) \wedge n \\ |m-1-n|+1 \end{matrix} \middle| t \right) {}_1F_1 \left( \begin{matrix} -(j-1) \wedge k \\ |j-1-k|+1 \end{matrix} \middle| t \right). \quad (4.3.16)$$

Thus, for  $m-j = \ell \neq n-k = \ell'$ , we have  $\langle \psi_{n,m}, \psi_{k,j} \rangle_{\mathbb{H}} = 0$ . This proves in particular that the  $E_\ell; \ell \in \mathbb{Z}$ , form an orthogonal sequence in  $L^2_\mu(\mathbb{H})$ . Moreover, we have

$$\bigoplus_{\ell \in \mathbb{Z}} E_\ell = \mathcal{C}_{\mathbb{H}} \left( \bigoplus_{n=0}^{\infty} \mathcal{A}_{n,slice}^2 \right).$$

□

**Corollary 4.3.15.** *The functions  $\psi_{n,m}$ , for varying  $m = 0, 1, \dots$ , and fixed  $n$ , (resp. for varying  $n = 0, 1, \dots$ , and fixed  $m$ ) constitute an orthogonal system in  $L^2_\mu(\mathbb{H})$  whose square norm is given by*

$$\|\psi_{n,m}\|_{\mathbb{H}}^2 = \frac{\pi}{3^{m+n}} \begin{cases} \frac{4^{m-1}((n)!)^2}{(n-m+1)!} {}_2F_1 \left( \begin{matrix} 1-m, 1-m \\ n-m+2 \end{matrix} \middle| \frac{1}{4} \right); & m \leq n+1, \\ \frac{4^n((m-1)!)^2}{(m-1-n)!} {}_2F_1 \left( \begin{matrix} -n, -n \\ m-n \end{matrix} \middle| \frac{1}{4} \right); & m \geq n+1. \end{cases} \quad (4.3.17)$$

*Proof.* The orthogonality in  $L^2_\mu(\mathbb{H})$  of the systems  $(\psi_{n,m})_m$ , for fixed  $n$ , and  $(\psi_{n,m}^Q)_n$ , for fixed  $m$ , readily follows from (4.3.15). We need only to compute its norm. We have

$$\|\psi_{n,m}\|_{\mathbb{H}}^2 = \pi(c_{m-1,n})^2 \int_0^\infty t^{|m-1-n|} R_{m,n,m,n}(t) e^{-3t} dt.$$

Thus, we get (4.3.17) by appealing to the integral formula [79, p. 293]. □

## 4.4 The $k$ -Bergman projection of $\mathcal{C}_{\mathbb{H}}$

The proposition just proved shows that the functions  $\psi_{n,m} = \mathcal{C}_{\mathbb{H}} H_{m,n}^Q; m; n = 0, 1, 2, \dots$ , belong to  $L^2_{\mu_I}(\mathbb{C}_I)$  which possesses the slice orthogonal decomposition in the sense described in the preliminaries section. Accordingly, the determination of the component function of  $\psi_{n,m} = \mathcal{C}_{\mathbb{H}} H_{m,n}^Q$  in  $\mathcal{A}_{n,slice}^2$  requires the consideration of the orthogonal projection in (4.2.21).

The next assertion gives the closed integral representation of  $P_k \mathcal{C}_{\mathbb{H}}$ , the  $k$ -Bergman projection of the weighted Cauchy singular integral  $\mathcal{C}_{\mathbb{H}}$ , in terms of the special polynomials

$$H_{m,n}^{Q,\star q}(q-p, \bar{q}-\bar{p}) := \sum_{\ell=0}^{m \wedge n} (-1)^\ell \ell! \binom{m}{\ell} \binom{n}{\ell} (q-p)^{\star q(m-\ell)} \star^q (\bar{q}-\bar{p})^{\star \bar{q}(n-\ell)}, \quad (4.4.1)$$

obtained as the unique left slice polyregular extension in  $q$  and right slice polyregular in  $\bar{q}$  to the whole  $\mathbb{H}$  of  $q \mapsto H_{k-1,k}^{\mathcal{Q},\star^q}(q-p, \bar{q}-\bar{p})$  on  $\mathbb{C}_p$ , for fixed  $p$ . It is also the unique right slice polyregular extension in  $p$  and left slice polyregular in  $\bar{p}$  to the whole  $\mathbb{H}$  of  $p \mapsto H_{k-1,k}^{\mathcal{Q},\star^q}(q-p, \bar{q}-\bar{p})$  on  $\mathbb{C}_q$ , for fixed  $q$ .

**Theorem 4.4.1.** *The integral transform  $P_k \mathcal{C}_{\mathbb{H}} : L_{\mu}^2(\mathbb{H}) \longrightarrow \mathcal{A}_{k,slice}^2$  is given by*

$$P_k \mathcal{C}_{\mathbb{H}} f(q) = \int_{\mathbb{H}} \mathcal{R}_k(q, p) f(p) d\mu(p) \quad (4.4.2)$$

where  $\mathcal{R}_k$  stands for

$$\mathcal{R}_k(q, p) := \frac{1}{\pi k!} e^{-|p|^2} e_*^{[q, \bar{p}]} \star_{sp}^q H_{k-1,k}^{\star}(q-p, \bar{q}-\bar{p}). \quad (4.4.3)$$

*Proof.* By means of (4.2.21), the definition of  $\mathcal{C}_{\mathbb{H}}$  and the Fubini's theorem,

$$P_k \mathcal{C}_{\mathbb{H}} f(q) = \int_{\mathbb{H}} \mathcal{K}_k(q, \xi) \mathcal{C}_{\mathbb{H}} f(\xi) d\mu(\xi) = \int_{\mathbb{H}} \mathcal{R}_k(q, p) f(p) d\mu(p)$$

for  $q \in \mathbb{H}$ , where the kernel function is given by

$$\begin{aligned} \mathcal{R}_k(q, p) &= \int_{\mathbb{H}} \mathcal{K}_k(q, \xi) \mathcal{N}(\xi, p) d\mu(\xi) \\ &= \sum_{m=0}^{\infty} \frac{H_{m,k}^{\mathcal{Q}}(q, \bar{q})}{\pi m! k!} \int_{\mathbb{H}} \overline{H_{m,k}^{\mathcal{Q}}(\xi, \bar{\xi})} \mathcal{N}(\xi, p) d\mu(\xi) \\ &= - \sum_{m=0}^{\infty} \frac{H_{m,k}^{\mathcal{Q}}(q, \bar{q})}{\pi m! k!} \mathcal{C}_{\mathbb{H}} H_{k,m}^{\mathcal{Q}}(p) \\ &= \sum_{m=0}^{\infty} \frac{H_{m,k}^{\mathcal{Q}}(q, \bar{q})}{\pi m! k!} H_{k-1,m}^{\mathcal{Q}}(p, \bar{p}) e^{-|p|^2}. \end{aligned} \quad (4.4.4)$$

Above we have made use of the expansion series of  $\mathcal{K}_k$  as given by (4.2.17) and the facts  $\overline{\mathcal{N}(\xi, p)} = -\mathcal{N}(\bar{p}, \bar{\xi})$ ,  $H_{k,m}^{\mathcal{Q}}(p, \bar{p}) = H_{m,k}^{\mathcal{Q}}(p, \bar{p}) = H_{k,m}^{\mathcal{Q}}(\bar{p}, p)$  and  $\mathcal{C}_{\mathbb{H}} H_{k,m}^{\mathcal{Q}}(\bar{p}) = \mathcal{C}_{\mathbb{H}} H_{k,m}^{\mathcal{Q}}(p)$ . The last equality (4.4.4) follows from Theorem 4.3.12.

Now, the function

$$\tilde{\mathcal{R}}_k(q, p) := e^{|p|^2} \mathcal{R}_k(q, p) = \sum_{m=0}^{\infty} \frac{H_{m,k}^{\mathcal{Q}}(q, \bar{q})}{\pi m! k!} H_{k-1,m}^{\mathcal{Q}}(p, \bar{p})$$

is clearly left slice polyregular of order  $k+1$  in  $q$  and right slice polyregular of order  $k$  in  $p$ , since it can be rewritten as

$$\tilde{\mathcal{R}}_k(q, p) = \sum_{j=0}^k \bar{q}^j f_j^{(p)}(q) = \sum_{j=0}^{k-1} g_j^{(q)}(p) \bar{p}^j$$

with  $f_j^{(p)}$  (resp.  $g_j^{(q)}$ ) being left (resp. right) slice regular functions in  $q$  (resp. in  $p$ ). Moreover, Proposition 3.6 in [58] shows that the expression of  $\tilde{\mathcal{R}}_k(q, p) := e^{|p|^2} \mathcal{R}_k(q, p)$  reduces to

$$E_k(q, p) = \frac{e^{q\bar{p}}}{\pi k!} H_{k,k-1}(q - p, \bar{q} - \bar{p})$$

on  $\mathbb{C}_{I_p}$  (resp.  $\mathbb{C}_{I_q}$ ) as function in  $q$  (resp.  $p$ ) for fixed  $p$  (resp.  $q$ ). Finally, using Identity Principle for slice regular functions, one can proceed as in [18] to prove that the function in the right hand-side of (4.4.3) (in the  $(q, p)$  variables) is the the unique extension to left slice hyperholomorphic function in  $q$  and right one in  $\bar{q}$  outside of  $\mathbb{C}_p$  and right slice hyperholomorphic in  $p$  and left slice hyperholomorphic in  $\bar{p}$  outside of  $\mathbb{C}_q$ . Thus, we have

$$\tilde{\mathcal{R}}_k(q, p) = \frac{e_*^{[q, \bar{p}]}}{\pi k!} \star_{sp}^q H_{m,n}^{Q, \star q}(q - p, \bar{q} - \bar{p}).$$

This completes the proof.  $\square$

**Proposition 4.4.2.** *For every nonnegative integers  $k, m, n$ , there exists certain nonzero real constant  $d_k^{m,n}$  such that*

$$P_k \mathcal{C}_{\mathbb{H}} \psi_{m,n}(q) = \frac{\varepsilon_{n+k-m} d_k^{m,n}}{\pi(n+k-m)! k!} H_{n+k-m,k}^Q(q, \bar{q}). \quad (4.4.5)$$

In particular, the systems  $P_k \mathcal{C}_{\mathbb{H}}(\psi_{m,n}); n = 0, 1, 2, \dots$ , for fixed  $m$ , and  $P_k \mathcal{C}_{\mathbb{H}}(\psi_{m,n}); m = 0, 1, 2, \dots$ , for fixed  $n$  are orthogonal in  $L_\mu^2(\mathbb{H})$ .

*Proof.* By taking into account the expansion series of the kernel given by (4.4.4), we can rewrite the integral in (4.4.2) in terms of  $\psi_{j,k}$  in (4.3.14), as

$$P_k \mathcal{C}_{\mathbb{H}} f(q) = \sum_{j=0}^{\infty} \frac{H_{j,k}^Q(q, \bar{q})}{\pi j! k!} \int_{\mathbb{H}} \overline{\psi_{k,j}(p)} f(p) d\mu(p),$$

so that for the specific function  $f = \psi_{m,n}$ , we see from (4.3.15) that the only possible nonzero term corresponds to  $j = n + k - m$  under the assumption that  $n + k \geq m$ . Thus, we have

$$P_k \mathcal{C}_{\mathbb{H}} \psi_{m,n}(q) = \frac{\varepsilon_{n+k-m}}{\pi(n+k-m)! k!} d_k^{m,n} H_{n+k-m,k}^Q(q, \bar{q}).$$

The constant  $d_k^{m,n}$  is given by  $d_k^{m,n} := \langle \psi_{k,n+k-m}, \psi_{m,n} \rangle_{\mathbb{H}}$  which is finite and real.  $\square$

**Remark 4.4.3.** *The polynomials  $H_{n,k}^Q$  are eigenfunction of the operator  $P_k \mathcal{C}_{\mathbb{H}} \mathcal{C}_{\mathbb{H}}$ . This clearly follows from the fact*

$$P_k \mathcal{C}_{\mathbb{H}}(\psi_{k,n})(q) = \frac{\left\| \mathcal{C}_{\mathbb{H}}(H_{k,n}^Q) \right\|_{\mathbb{H}}^2}{\pi n! k!} H_{n,k}^Q(q, \bar{q}). \quad (4.4.6)$$

**Remark 4.4.4.** The image of  $E_\ell^+$  by  $P_k\mathcal{C}_\mathbb{H}$  reduces to zero for all  $\ell > k$ . Otherwise, it is one dimensional vector space. While  $P_k\mathcal{C}_\mathbb{H}(E_\ell^-)$  is always a one dimensional vector space.

We conclude by giving the explicit sequence of eigenvalues of the integral operator

$$|P_k\mathcal{C}_\mathbb{H}| := ((P_k\mathcal{C}_\mathbb{H})^*P_k\mathcal{C}_\mathbb{H})^{1/2}.$$

**Proposition 4.4.5.** The functions  $\psi_{k,n}$ ;  $n = 0, 1, 2, \dots$ , are eigenfunctions of  $|P_k\mathcal{C}_\mathbb{H}|^2 = (P_k\mathcal{C}_\mathbb{H})^*P_k\mathcal{C}_\mathbb{H}$  with  $\lambda_{k,n} := \|\psi_{k,n}\|^2 / \pi n!k!$  as associated eigenvalues.

*Proof.* Notice first that the formal adjoint of  $P_k\mathcal{C}_\mathbb{H}$  is given by

$$(P_k\mathcal{C}_\mathbb{H})^*g(p) = \int_{\mathbb{H}} \overline{\mathcal{R}_k(q, p)}g(q)d\mu(q).$$

Hence, from the expansion series of  $\overline{\mathcal{R}_k(q, p)} = \mathcal{R}_k(\bar{p}, \bar{q})$ , we get

$$(P_k\mathcal{C}_\mathbb{H})^*g(p) = \sum_{m=0}^{\infty} \frac{e^{-|p|^2}H_{m,k-1}^{\mathcal{Q}}(p, \bar{p})}{\pi m!k!} \int_{\mathbb{H}} \overline{H_{m,k}^{\mathcal{Q}}(q, \bar{q})}g(q)d\mu(q).$$

so that

$$(P_k\mathcal{C}_\mathbb{H})^*(H_{n,\ell}^{\mathcal{Q}})(p) = e^{-|p|^2}H_{n,k-1}^{\mathcal{Q}}(p, \bar{p})\delta_{k,\ell}. \quad (4.4.7)$$

This combined with (4.4.5) yield

$$\begin{aligned} (P_k\mathcal{C}_\mathbb{H})^*P_k\mathcal{C}_\mathbb{H}\psi_{\ell,n}(p) &= \frac{\varepsilon_{n+k-\ell}d_k^{\ell,n}}{\pi(n+k-\ell)!k!}(P_k\mathcal{C}_\mathbb{H})^*H_{n+k-\ell,k}^{\mathcal{Q}}(p) \\ &= \frac{\varepsilon_{n+k-\ell}d_k^{\ell,n}}{\pi(n+k-\ell)!k!}e^{-|p|^2}H_{n+k-\ell,k-1}^{\mathcal{Q}}(p, \bar{p}) \\ &= \frac{\varepsilon_{n+k-\ell}d_k^{\ell,n}}{\pi(n+k-\ell)!k!}\overline{C_\mathbb{H}H_{k,n+k-\ell}^{\mathcal{Q}}(p)} \end{aligned} \quad (4.4.8)$$

This shows in particular that for  $\ell = k$ , we have

$$|P_k\mathcal{C}_\mathbb{H}|^2\psi_{k,n} = \frac{\|\psi_{k,n}\|_{\mathbb{H}}^2}{\pi n!k!}\psi_{k,n} = \lambda_{k,n}\psi_{k,n}.$$

□

**Corollary 4.4.6.** The singular values of  $P_k\mathcal{C}_\mathbb{H}$  are given by

$$s_n^k = \frac{1}{3^{(n+k)/2}} \begin{cases} \left[ \frac{4^{n-1}(k)!}{n!(k-n+1)!} {}_2F_1 \left( \begin{matrix} 1-n, 1-n \\ k-n+2 \end{matrix} \middle| \frac{1}{4} \right) \right]^{1/2}; & n \leq k+1, \\ \left[ \frac{4^k(n-1)!}{nk!(n-1-k)!} {}_2F_1 \left( \begin{matrix} -k, -k \\ n-k \end{matrix} \middle| \frac{1}{4} \right) \right]^{1/2}; & n \geq k+1. \end{cases}$$

**Remark 4.4.7.** The previous result shows in particular that the product of  $k$ -Bergman projector with the weighted quaternionic Cauchy transform  $\mathcal{C}_{\mathbb{H}}$  and its complex analogue have the same singular values, up to multiplicative constant. Subsequently, these singular values are asymptotically as

$$s_n^k \sim \left( \frac{4^k (n-1)!}{3^{n+k} n! (n-1-k)!} \right)^{1/2}$$

for  $k$  is fixed and  $n$  large enough.

**Remark 4.4.8.** The previous result can be obtained using the integral representation of the operator  $(P_k \mathcal{C}_{\mathbb{H}})^* P_k \mathcal{C}_{\mathbb{H}}$ ,

$$(P_k \mathcal{C}_{\mathbb{H}})^* P_k \mathcal{C}_{\mathbb{H}} f(p) = \int_{\mathbb{H}} \mathcal{S}_k(p, q) f(q) d\mu(q) \quad (4.4.9)$$

where the Schwartz kernel  $\mathcal{S}_k$  is given by

$$\begin{aligned} \mathcal{S}_k(p, q) &:= \int_{\mathbb{H}} \overline{\mathcal{R}_k(\xi, p)} \mathcal{R}_k(\xi, q) d\mu(\xi) \\ &= e^{-|p|^2 - |q|^2} \sum_{m=0}^{\infty} \frac{H_{m, k-1}^{\mathbb{Q}}(p, \bar{p}) H_{m, k-1}^{\mathbb{Q}}(q, \bar{q})}{\pi m! k!}. \end{aligned} \quad (4.4.10)$$

The last equality follows from (4.4.4) and the orthogonality of  $H_{m, k}^{\mathbb{Q}}$ .

**Remark 4.4.9.** For  $k \geq 1$ , the kernel  $\mathcal{S}_k$  is closely connected to the reproducing kernel of  $\mathcal{A}_{k-1, \text{slice}}^2$

$$\mathcal{S}_k(p, q) = \frac{e^{-|p|^2 - |q|^2}}{k} \mathcal{K}_{k-1}(\bar{p}, \bar{q}). \quad (4.4.11)$$

# Chapter 5

## Cauchy transform on the hyperbolic complex and quaternionic disc

### 5.1 Complex Cauchy transform on the unit disc

The main tool in the description of the complex Cauchy transform  $\mathcal{C}$  is the so-called Zernike polynomials. An accurate analysis of the basic analytic properties of  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$ , like recurrence relations, differential equations they obey, generating functions and so on, have been developed in various papers. For a very nice account on these polynomials, one can refer to Koornwinder seminal works [74, 75], as well as [100] for an elegant reintroduction of them. See [1] for useful operational formulas of Burchall type and generating functions. An complete overview of needed properties are given Appendix. We give the explicit action of the Cauchy transform  $\mathcal{C}$  defined as bounded compact operator from  $L^2(D)$  into  $L^2(D)$  by

$$[\mathcal{C}(f)](z) = \frac{1}{\pi} \int_D \frac{f(w)}{w - z} dx dy$$

on the associated Zernike functions  $\mathfrak{z}_{m,n}^\gamma(z, \bar{z}) = (1 - |z|^2)^\gamma \mathcal{Z}_{m,n}^\gamma$ . More precisely, we show the following

$$[\mathcal{C}(\mathfrak{z}_{m,n}^\gamma)](z) = \mathfrak{z}_{m,n-1}^{\gamma+1}(z, \bar{z}); \quad m, n \geq 0, . \quad (5.1.1)$$

Accordingly, special identities for integrals involving the Gauss hypergeometric functions are derived. We also discuss some spectral properties of  $\mathcal{C}$  such as the boundedness and compactness of  $\mathcal{C}$ .

## 5.2 The action of the Cauchy transform on $\mathcal{Z}_{m,n}^\gamma$

We define the Cauchy transform  $C_\gamma$ ;  $\gamma > -1$ , of a given  $f \in L^2(D, d\mu_\gamma)$  by

$$[C_\gamma(f)](z) = \frac{1}{\pi} \int_D \frac{f(w)}{w-z} d\mu_\gamma(w).$$

To provide the explicit expression of the Cauchy transform  $C_\gamma$  on the generalized Zernike polynomials, we proceed by steps. We begin by giving the action of  $C_\gamma$  of the functions

$$e_{m,n}^s(z, \bar{z}) := \bar{z}^m z^n (1 - |z|^2)^s \quad (5.2.1)$$

in terms of the Gauss hypergeometric function. To this end, we need to the following elementary results.

**Lemma 5.2.1.** *For every  $z \in D$  and nonnegative real number  $r$ , we have*

$$\int_0^{2\pi} \frac{e^{i(n-m)\theta}}{re^{i\theta} - z} d\theta = \begin{cases} -\frac{2\pi r^{m-n}}{z^{m-n+1}} & \text{if } r < |z| \\ 0 & \text{if } |z| < r \end{cases} \quad (5.2.2)$$

when  $m \geq n$ , and

$$\int_0^{2\pi} \frac{e^{i(n-m)\theta}}{re^{i\theta} - z} d\theta = \begin{cases} 0 & \text{if } r < |z| \\ \frac{2\pi z^{n-m-1}}{r^{n-m}} & \text{if } |z| < r \end{cases} \quad (5.2.3)$$

otherwise.

*Proof.* The proof is straightforward and can be handled by expanding  $(re^{i\theta} - z)^{-1}$  as power series and next using the fact that  $\int_0^{2\pi} e^{i(k-k')\theta} d\theta = 2\pi\delta_{k,k'}$  for every integers  $k, k'$ .  $\square$

**Lemma 5.2.2.** *The functions  $C_\gamma(e_{m,n}^s)(z)$  admit the integral representation*

$$C_\gamma(e_{m,n}^s)(z) = -\frac{z^n}{z^{m+1}} \int_0^{|z|^2} t^m (1-t)^{\gamma+s} dt \quad (5.2.4)$$

for  $n \leq m$  and

$$C_\gamma(e_{m,n}^s)(z) = \frac{z^n}{z^{m+1}} \int_0^{1-|z|^2} t^{\gamma+s} (1-t)^m dt \quad (5.2.5)$$

when  $n > m$ .

*Proof.* Using the polar coordinates, one can rewrite  $C_\gamma(e_{m,n}^s)$  in the following form

$$C_\gamma(e_{m,n}^s)(z) = \frac{1}{\pi} \int_0^1 r^{m+n} (1-r^2)^{\gamma+s} \left( \int_0^{2\pi} \frac{e^{i(n-m)\theta}}{re^{i\theta} - z} d\theta \right) r dr.$$

In virtue of (5.2.2) for  $m \geq n$  and the change of variable  $t = r^2$ , we get

$$\begin{aligned} C_\gamma(e_{m,n}^s)(z) &= \frac{-2}{z^{m-n+1}} \int_0^{|z|} r^{2m} (1-r^2)^{\gamma+s} r dr \\ &= \frac{-1}{z^{m-n+1}} \int_0^{|z|^2} t^m (1-t)^{\gamma+s} dt. \end{aligned}$$

To prove (5.2.5) for  $n > m$ , we proceed in a similar way making use of (5.2.3). Indeed, we have

$$\begin{aligned} C_\gamma(e_{m,n}^s)(z) &= 2z^{n-m-1} \int_{|z|}^1 r^{2m} (1-r^2)^{\gamma+s} r dr \\ &= z^{n-m-1} \int_0^{1-|z|^2} t^{\gamma+s} (1-t)^m dt. \end{aligned}$$

□

**Lemma 5.2.3.** For  $\Re(b) > 0$  and  $x \in [0;1]$ , we have

$$\int_0^x u^{b-1} (1-u)^{-a} du = \frac{1}{b} x^b (1-x)^{1-a} {}_2F_1 \left( \begin{matrix} 1, b+1-a \\ b+1 \end{matrix} \middle| x \right). \quad (5.2.6)$$

*Proof.* Recall first that the Euler integral representation of the Gauss hypergeometric function [15, Theorem 2.2.1, p. 65] reads

$$\int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| x \right)$$

for  $\Re(c) > \Re(b) > 0$  and  $\arg(z)$ . In virtue of the change of variable  $u = xt$  with  $c = b+1$  and  $\Re(b) > 0$  we get

$$\int_0^x u^{b-1} (1-u)^{-a} du = \frac{x^b}{b} {}_2F_1 \left( \begin{matrix} a, b \\ b+1 \end{matrix} \middle| x \right).$$

from which we deduce (5.2.6) by applying Euler's transformation [29, Theorem 21, p. 60]

$${}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} \middle| x \right) = (1-x)^{c-a-b} {}_2F_1 \left( \begin{matrix} c-a, c-b \\ c \end{matrix} \middle| x \right).$$

□

Accordingly, we can formulate the following.

**Proposition 5.2.4.** *Let  $m, n$  be two given nonnegative integers and  $s > 0$ . Then, for  $m \geq n$ , we have*

$$C_\gamma(e_{m,n}^s)(z) = \frac{-z^n \bar{z}^{m+1}}{m+1} (1-|z|^2)^{\gamma+s+1} {}_2F_1\left(\begin{matrix} 1, \gamma+s+m+2 \\ m+2 \end{matrix} \middle| |z|^2\right). \quad (5.2.7)$$

If  $n > m$ , then

$$C_\gamma(e_{m,n}^s)(z) = \frac{z^n \bar{z}^{m+1}}{(\gamma+s+1)} (1-|z|^2)^{\gamma+s+1} {}_2F_1\left(\begin{matrix} 1, \gamma+s+m+2 \\ \gamma+s+2 \end{matrix} \middle| 1-|z|^2\right). \quad (5.2.8)$$

*Proof.* The proof is immediate using Lemmas 5.2.2 and 5.2.3. Indeed, the case of  $m \geq n$  follows by applying (5.2.6) to the integral in the right hand-side of (5.2.4) with  $b = m+1$ ,  $-a = \gamma+s$  and  $x = |z|^2$ . The identity (5.2.8) corresponding to  $n > m$  follows in a similar way with  $b = \gamma+s+1$ ,  $a = -m$  and  $x = 1-|z|^2$ .  $\square$

Therefore, the action of  $C_\gamma$  on  $\mathcal{Z}_{m,0}^\gamma(z, \bar{z})$  and  $\mathcal{Z}_{0,n}^\gamma(z, \bar{z})$  is given by the following.

**Corollary 5.2.5.** *For  $\gamma > -1$ , If  $m \geq 0$ , we have*

$$C_\gamma(\mathcal{Z}_{m,0}^\gamma)(z) = \frac{-(\gamma+1)_m \bar{z}^{m+1} (1-|z|^2)^\gamma}{(m+1)} {}_2F_1\left(\begin{matrix} 1, \gamma+m+2 \\ m+2 \end{matrix} \middle| |z|^2\right) \quad (5.2.9)$$

and if  $n \geq 1$ , we have

$$C_\gamma(\mathcal{Z}_{0,n}^\gamma)(z) = \frac{z^n \bar{z}}{(\gamma+1)} (1-|z|^2)^{\gamma+1} {}_2F_1\left(\begin{matrix} 1, \gamma+m+2 \\ \gamma+2 \end{matrix} \middle| 1-|z|^2\right). \quad (5.2.10)$$

*Proof.* It suffices to observe that  $\mathcal{Z}_{m,0}^\gamma(z, \bar{z}) = (\gamma+1)_m \bar{z}^m = (\gamma+1)_m e_{m,0}^0(z, \bar{z})$  and  $\mathcal{Z}_{0,n}^\gamma(z, \bar{z}) = (\gamma+1)_n z^n = (\gamma+1)_n e_{0,n}^0(z, \bar{z})$ , and then apply Proposition 5.2.4.  $\square$

The following is the main theorem of this section.

**Theorem 5.2.6.** *For every nonnegative integers  $m \geq n \geq 1$  and real  $\gamma > -1$ , we have*

$$[C(\mathfrak{z}_{m,n}^\gamma)](z, \bar{z}) = \mathfrak{z}_{m,n-1}^{\gamma+1}(z, \bar{z}). \quad (5.2.11)$$

*Proof.* Let  $m \geq n > 0$  and notice that the explicit expression (5.3.4) of the polynomials  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$  can be rewritten as

$$\mathcal{Z}_{m,n}^\gamma(z, \bar{z}) = m!n!(\gamma+1)_{m+n} \sum_{j=0}^n \frac{(-1)^j}{(\gamma+1)_j j!} \frac{e_{m-j, n-j}^j(z, \bar{z})}{(m-j)!(n-j)!}.$$

By Proposition 5.2.4 (Eq. (5.2.7)), the linearity of the Cauchy transform and the fact  $(-1)^j(m+1)m! = (-m+1)_j(m+1-j)!$ , we get

$$\begin{aligned} [C_\gamma(\mathcal{Z}_{m,n}^\gamma)](z) &= -(\gamma+1)_{m+n} \frac{z^n \bar{z}^{m+1}}{m+1} (1-|z|^2)^{\gamma+1} \\ &\quad \times n! \sum_{j=0}^n \frac{(-m+1)_j}{(\gamma+1)_j} \frac{\left(\frac{1-|z|^2}{|z|^2}\right)^j}{j!(n-j)!} {}_2F_1\left(\begin{matrix} 1, \gamma+m+2 \\ m+2-j \end{matrix} \middle| |z|^2\right) \end{aligned}$$

Now, in virtue of [22, p.415]

$$n! \sum_{j=0}^n \frac{(1-c)_j}{(b-c+1)_j} \frac{\left(\frac{1-x}{x}\right)^j}{j!(n-j)!} {}_2F_1\left(\begin{matrix} a, b \\ c-j \end{matrix} \middle| x\right) = \frac{(1-c)_n}{(b-c+1)_n} x^{-n} {}_2F_1\left(\begin{matrix} a-n, b \\ c-n \end{matrix} \middle| x\right)$$

with  $a = 1, b = \gamma + m + 2, c = m + 2$  and  $x = |z|^2$ , we see that the expression pf  $[C_\gamma(\mathcal{Z}_{m,n}^\gamma)](z)$  reduces to

$$-\frac{(\gamma+1)_{m+n}(-m-1)_n}{(\gamma+1)_n(m+1)} \bar{z}^{m+1-n} (1-|z|^2)^{\gamma+1} {}_2F_1\left(\begin{matrix} 1-n, \gamma+m+2 \\ m-n+2 \end{matrix} \middle| |z|^2\right)$$

and moreover to

$$(-1)^{n-1} \frac{(\gamma+n+1)_m m!}{(m-n+1)!} \bar{z}^{m-(n-1)} (1-|z|^2)^{\gamma+1} {}_2F_1\left(\begin{matrix} -n+1, \gamma+1+m+1 \\ m-(n-1)+1 \end{matrix} \middle| |z|^2\right).$$

Thus, we recognize the expression of  $\mathcal{Z}_{m,n-1}^{\gamma+1}(z, \bar{z})$  given through (6.0.6). This completes the proof of Theorem 5.2.6.  $\square$

The counterpart of Theorem 5.2.6 for  $0 < m < n$  is the following.

**Theorem 5.2.7.** *For every positive integers  $0 \leq m < n$  and real  $\gamma > -1$ , we have*

$$[C_\gamma(\mathcal{Z}_{m,n}^\gamma)](z, \bar{z}) = (1-|z|^2)^{\gamma+1} \mathcal{Z}_{m,n-1}^{\gamma+1}(z, \bar{z}). \quad (5.2.12)$$

*Proof.* The proof for  $n > m$  can be handled as follows. By direct computation, using (5.2.8) and the facts  $(\gamma+j+1)(\gamma+1)_j = (\gamma+1)(\gamma+2)_j$  and  $\frac{(-1)^j n!}{(n-j)!} = (-n)_j$ , we arrive at

$$\begin{aligned} [C_\gamma(\mathcal{Z}_{m,n}^\gamma)](z) &= \frac{(\gamma+1)_{m+n}}{(\gamma+1)} z^n \bar{z}^{m+1} (1-|z|^2)^{\gamma+1} \\ &\quad \times \sum_{j=0}^m \binom{n}{j} \frac{(-m)_j}{(\gamma+2)_j} \left(\frac{1-|z|^2}{|z|^2}\right)^j {}_2F_1\left(\begin{matrix} 1, \gamma+m+2 \\ \gamma+2+j \end{matrix} \middle| 1-|z|^2\right) \end{aligned}$$

Now, by setting  $a = 1, b = \gamma + m + 2, c = \gamma + 2$  and  $x = 1 - |z|^2$ , and extending the sum to  $n$  for  $(-m)_j = 0$  when  $j > m$ , the quantity  $[C_\gamma(\mathcal{Z}_{m,n}^\gamma)](z)$  takes the form

$$[C_\gamma(\mathcal{Z}_{m,n}^\gamma)](z) = \frac{(\gamma + 1)_{m+n}}{(\gamma + 1)} z^n \bar{z}^{m+1} (1 - |z|^2)^{\gamma+1} \\ \times \sum_{j=0}^n \binom{n}{j} \frac{(c-b)_j}{(c)_j} \left(\frac{x}{1-x}\right)^j {}_2F_1\left(\begin{matrix} a, b \\ c+j \end{matrix} \middle| x\right)$$

for which we can apply the summation formula [22, Eq. 5., 5.16.1., p. 414] to get

$$[C_\gamma(\mathcal{Z}_{m,n}^\gamma)](z) = \frac{(\gamma + 1)_{m+n}}{(\gamma + 1)} \bar{z}^{m+1-n} (1 - |z|^2)^{\gamma+1} {}_2F_1\left(\begin{matrix} 1-n, \gamma+m+2 \\ \gamma+2 \end{matrix} \middle| 1-|z|^2\right) \\ = \frac{(\gamma + 1)_{m+n}}{(\gamma + 1)} \bar{z}^m z^{n-1} (1 - |z|^2)^{\gamma+1} {}_2F_1\left(\begin{matrix} -m, -n+1 \\ -\gamma-m-n \end{matrix} \middle| \frac{1}{|z|^2}\right).$$

The last equality is immediate from (see: equality: 15.8.6)

$${}_2F_1\left(\begin{matrix} -m, b \\ c \end{matrix} \middle| x\right) = \frac{(b)_m}{(c)_m} (1-x)^m {}_2F_1\left(\begin{matrix} -m, c-b \\ 1-b-m \end{matrix} \middle| \frac{1}{1-x}\right).$$

This completes the proof of (5.3.11) thanks to (6.0.5).  $\square$

In the sequel, we denote by  $E$  the orthogonal complement of the subspace spanned by the set  $\{\mathcal{Z}_{m,0}^\gamma(z, \bar{z}); \text{ for all } m \in \mathbb{N}\}$  in the Hilbert space  $L^2(D, d\mu_\gamma)$ .

**Proposition 5.2.8.** *If  $\gamma > -1/2$ , the Cauchy transform is continuous on  $E$ , with  $\|C_\gamma\| \leq \sqrt{\gamma+1}$ .*

*Proof.* We need only to prove the assertion for  $f$  belonging to  $\{\mathcal{Z}_{m,0}^\gamma(z, \bar{z}) \text{ for all } m \in \mathbb{N}\}^\perp$ . Such function can be expanded as

$$f(z) = \sum_{m=0, n=1}^{\infty} a_{m,n} \mathcal{Z}_{m,n}^\gamma(z, \bar{z}).$$

Thus using Theorem 5.3.11, combined with  $\gamma > -1/2$ , we get

$$\begin{aligned} \left\| [C_\gamma(f)](z) \right\|_\gamma^2 &= \left\| \sum_{m=0, n=1}^{\infty} a_{m,n} [C_\gamma(\mathcal{Z}_{m,n}^\gamma)](z) \right\|_\gamma^2 \\ &= \left\| \sum_{m=0, n=1}^{\infty} a_{m,n} (1 - |z|^2)^{\gamma+1} [\mathcal{Z}_{m,n-1}^{\gamma+1}](z) \right\|_\gamma^2 \\ &\leq \left\| \sum_{m=0, n=1}^{\infty} a_{m,n} (1 - |z|^2)^{1/2} [\mathcal{Z}_{m,n-1}^{\gamma+1}](z) \right\|_\gamma^2 \\ &\leq \int_{\mathbb{D}} \sum_{m=0, n=1}^{\infty} \sum_{k=0, l=1}^{\infty} a_{m,n} \overline{a_{k,l}} [\mathcal{Z}_{m,n-1}^{\gamma+1}](z) \overline{[\mathcal{Z}_{k,l-1}^{\gamma+1}](z)} (1 - |z|^2)^{\gamma+1} d\lambda \\ &\leq \sum_{m=0, n=1}^{\infty} \sum_{k=0, l=1}^{\infty} a_{m,n} \overline{a_{k,l}} \int_{\mathbb{D}} [\mathcal{Z}_{m,n-1}^{\gamma+1}](z) \overline{[\mathcal{Z}_{k,l-1}^{\gamma+1}](z)} (1 - |z|^2)^{\gamma+1} d\lambda. \end{aligned}$$

or  $\mathcal{Z}_{m,n-1}^{\gamma+1}$  is an orthogonale basis in the  $L^2(\mathbb{D}, (1 - |z|^2)^{\gamma+1} d\lambda)$  space, from 6.2.3 we get

$$\begin{aligned}
\| [C_\gamma(f)](z) \|_\gamma^2 &\leq \sum_{m=0, n=1}^{\infty} |a_{m,n}|^2 \| \mathcal{Z}_{m,n-1}^{\gamma+1}(z) \|_{\gamma+1}^2 \\
&\leq \sum_{m=0, n=1}^{\infty} |a_{m,n}|^2 \pi m! (n-1)! (\gamma+2+m)_{n-1} (\gamma+1+n)_m \frac{\Gamma(\gamma+2)^2}{\gamma+1+m+n} \\
&\leq \sum_{m=0, n=1}^{\infty} |a_{m,n}|^2 \frac{(\gamma+1)^2 \pi m! n!}{n(\gamma+1+m)} (\gamma+1+m)_n (\gamma+1+n)_m \frac{\Gamma(\gamma+1)^2}{\gamma+1+m+n} \\
&\leq \sum_{m=0, n=1}^{\infty} |a_{m,n}|^2 \frac{(\gamma+1)^2}{n(\gamma+1+m)} \| [\mathcal{Z}_{m,n}^\gamma](z) \|_\gamma^2 \\
&\leq \sum_{m=0, n=1}^{\infty} |a_{m,n}|^2 \| [\mathcal{Z}_{m,n}^\gamma](z) \|_\gamma^2 \\
&\leq (\gamma+1) \|f\|_\gamma^2.
\end{aligned}$$

this completes the proof.  $\square$

**Corollary 5.2.9.** *The adjoint operator of  $C_\gamma$  on  $E$  is given by*

$$[C_\gamma]^*(f) = -\overline{C_\gamma \bar{f}}$$

*Proof.* We know that  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$  with  $m, n$  are a nonnegatives integers form an orthogonal basis in  $L^2(D, d\mu_\gamma)$  space. So the result follows from: For all  $m, n, p, q \in \mathbb{N}$  with  $n, p > 0$ , we have

$$\begin{aligned}
\langle C_\gamma(\mathcal{Z}_{m,n}^\gamma(z, \bar{z})), \mathcal{Z}_{p,q}^\gamma(z, \bar{z}) \rangle_\gamma &= \langle (1 - |z|^2)^{\gamma+1} \mathcal{Z}_{m,n-1}^{\gamma+1}(z, \bar{z}), \mathcal{Z}_{p,q}^\gamma(z, \bar{z}) \rangle_\gamma \\
&= \int_{\mathbb{D}} (-1)^{m+n-1} \frac{\partial^{m+n-1}}{\partial z^m \partial \bar{z}^{n-1}} \left( (1 - |z|^2)^{\gamma+m+n} \right) \\
&\quad \times (-1)^{p+q} \frac{\partial^{p+q}}{\partial z^q \partial \bar{z}^p} \left( (1 - |z|^2)^{\gamma+p+q} \right) d\mu(z) \\
&= - \int_{\mathbb{D}} (-1)^{m+n} \frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} \left( (1 - |z|^2)^{\gamma+m+n} \right) \\
&\quad \times (-1)^{p+q-1} \frac{\partial^{p+q-1}}{\partial z^q \partial \bar{z}^{p-1}} \left( (1 - |z|^2)^{\gamma+p+q} \right) d\mu(z) \\
&= - \int_{\mathbb{D}} (1 - |z|^2)^\gamma \mathcal{Z}_{m,n}^\gamma(z, \bar{z}) (1 - |z|^2)^{\gamma+1} \mathcal{Z}_{q,p-1}^{\gamma+1}(z, \bar{z}) d\mu(z) \\
&= - \int_{\mathbb{D}} (1 - |z|^2)^\gamma \mathcal{Z}_{m,n}^\gamma(z, \bar{z}) C_\gamma(\mathcal{Z}_{q,p}^\gamma(z, \bar{z})) d\mu(z) \\
&= - \langle \mathcal{Z}_{m,n}^\gamma(z, \bar{z}), C_\gamma(\overline{\mathcal{Z}_{p,q}^\gamma(z, \bar{z})}) \rangle_\gamma
\end{aligned}$$

The result follows from the linearity of Cauchy transform  $C_\gamma$ .  $\square$

**Proposition 5.2.10.** Let  $M_\gamma$  denote the multiplication operator by  $(1 - |z|^2)^\gamma$ . Then, the following first order differential equation

$$\frac{\partial C_\gamma f}{\partial \bar{z}} = -M_\gamma f,$$

holds true for every  $f \in E$ .

*Proof.* We need only to proof the assertion for  $f$  belonging to  $\{\mathcal{Z}_{m,o}^\gamma(z, \bar{z}); \text{ for all } m \in \mathbb{N}\}^\perp$ . Such function can be expanded as

$$f(z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{m,n}^\gamma \mathcal{Z}_{m,n}^\gamma(z, \bar{z}).$$

Thus using the linearity of Cauchy operator  $C$  and Theorem 5.2.6, we get

$$[C_\gamma(f)](z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{m,n}^\gamma [C_\gamma(\mathcal{Z}_{m,n}^\gamma)](z) \quad (5.2.13)$$

$$= (1 - |z|^2)^{\gamma+1} \left( \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{m,n}^\gamma \mathcal{Z}_{m,n-1}^{\gamma+1}(z, \bar{z}) \right). \quad (5.2.14)$$

Thus,

$$\frac{\partial [C_\gamma(f)](z)}{\partial \bar{z}} = - \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{m,n}^\gamma \left( (1 - |z|^2)^\gamma \mathcal{Z}_{m,n}^\gamma(z, \bar{z}) \right) \quad (5.2.15)$$

$$= -M_\gamma f. \quad (5.2.16)$$

This, which is completes the proof. □

### 5.3 The quaternionic Cauchy transform on unit disc

To deal with a quaternionic analogue of the Cauchy transform, one consider again

$$\mathcal{N}(q, p) = \begin{cases} - \sum_{\ell=0}^{+\infty} q^\ell p^{-1-\ell}; & p \in B_1 \cap \overline{B_{|q|}}^c \\ \sum_{\ell=0}^{+\infty} q^{-1-\ell} p^\ell; & p \in B_{|q|}. \end{cases} \quad (5.3.1)$$

where  $B_{|q|} := \{p \in \mathbb{H}; |p| < |q|\}$  the ball of radius  $|q|$  in  $\mathbb{H} \equiv \mathbb{R}^4$  and centered at origin and by  $\overline{B_{|q|}}^c := \{p \in \mathbb{H}; |p| > |q|\}$  the complement of its closure. It is an appropriate combination of the  $p$ -left and the  $q$ -right non-commutative Cauchy

kernel series in (see e.g. [23, 49]). More precisely, the closed expression is the one given by (5.3.1) or equivalently [23, Theorem 2.10]

$$\mathcal{N}(q, p) = \left( q^2 - 2q\Re(p) + |p|^2 \right)^{-1} (q - \bar{p})$$

for  $q \in \mathbb{H}$  and  $p \in V_q := \{p \in \mathbb{H}, pq - qp \neq 0\}$ . It should be mentioned here that the Cauchy kernel function in variable  $p$ , can be identified as the regular inverse  $(q - p)^{-*}$  in the slice regular sense (see [53, 52], which is the unique left slice regular extension of  $\mathbb{C}_q \ni p \mapsto (q - p)^{-1}$ ).

The quaternionic Zernike polynomials  $\mathcal{Z}_{m,n}^\gamma(q, \bar{q})$  of the two conjugate quaternionic variables  $z = x + I_q y$  and  $\bar{z} = x - I_q y$ ,  $x, y \in \mathbb{R}$ , in the unit ball  $B_1 = \{q \in \mathbb{C}, |q| < 1\}$ , are defined by the Rodrigues formula (up to a multiplicative constant)

$$\mathcal{Z}_{m,n}^\gamma(q, \bar{q}) = (-1)^{m+n} (1 - |q|^2)^{-\gamma} \frac{\partial^{m+n}}{\partial s^m \partial \bar{s}^n} \left( (1 - |q|^2)^{\gamma+m+n} \right) \quad (5.3.2)$$

With respect to slice derivative

$$\partial_s f(q) = \begin{cases} \partial_{I_q} f(x + I_q y), & \text{if } y \neq 0, \\ \frac{df}{dx}(x), & \text{if } y = 0. \end{cases} \quad (5.3.3)$$

where for  $q = x + I_q y \in \mathbb{H}$ . Explicitly

$$\mathcal{Z}_{m,n}^\gamma(q, \bar{q}) = m!n!(\gamma + 1)_{m+n} \sum_{j=0}^{m \wedge n} \frac{(-1)^j (1 - |z|^2)^j}{(\gamma + 1)_{jj}!} \frac{\bar{q}^{m-j}}{(m-j)!} \frac{q^{n-j}}{(n-j)!}. \quad (5.3.4)$$

So, the following result is referred by Kanjin in [68]. And we have

**Theorem 5.3.1.** *The polynomials  $\mathcal{Z}_{m,n}^\gamma(q, \bar{q})$  form an orthogonal basis of the Hilbert space  $L^2(B_1, (1 - |q|^2)^\gamma)$  with*

$$\int_{B_1} \mathcal{Z}_{m,n}^\gamma(q, \bar{q}) \mathcal{Z}_{n,m}^\gamma(q, \bar{q}) d\mu_\gamma(q) = \frac{|\tilde{\mathbb{S}}| \pi \Gamma(\gamma + 1)^2 m!n!(\gamma + n + 1)_m (\gamma + m + 1)_n}{m + n + \gamma + 1}. \quad (5.3.5)$$

*Proof.* We have

$$\int_{B_1} \mathcal{Z}_{m,n}^\gamma(q, \bar{q}) \mathcal{Z}_{n,m}^\gamma(q, \bar{q}) d\mu_\gamma(q) = \int_{I_q \in \tilde{\mathbb{S}}} \int_{B_1 \cap \mathbb{C}_{I_q}} \mathcal{Z}_{m,n}^\gamma(q, \bar{q}) \mathcal{Z}_{n,m}^\gamma(q, \bar{q}) d\mu_\gamma(q) d\sigma(I_q) \quad (5.3.6)$$

From the theorem 6.2.3 in Appendix, we obtain

$$\int_{B_1} \mathcal{Z}_{m,n}^\gamma(q, \bar{q}) \mathcal{Z}_{n,m}^\gamma(q, \bar{q}) d\mu_\gamma(q) = \frac{|\tilde{\mathbb{S}}| \pi \Gamma(\gamma + 1)^2 m!n!(\gamma + n + 1)_m (\gamma + m + 1)_n}{m + n + \gamma + 1} \quad (5.3.7)$$

Witch complete the proof.  $\square$

Now, we give the explicit expression of the Cauchy transform  $C$  defined from  $L^2(B_1, (1 - |q|^2)^\gamma)$  into  $L^2(B_1, (1 - |q|^2)^\gamma)$  by

$$[C_{\mathbb{B}}(f)](q) = \frac{1}{\pi} \int_{B_1} \mathcal{N}(q, p) f(p) (1 - |p|^2)^\gamma d\lambda(q)$$

on the associated Zernike functions  $\mathcal{Z}_{m,n}^\gamma(q, \bar{q})$ . Now for instance that for the generic elements  $e_{m,n,s}(q, \bar{q}) := q^n \bar{q}^m (1 - |q|^2)^s$ . we have

$$\varepsilon_k = \begin{cases} 1 & \text{if } k \geq 0, \\ 0 & \text{if } k < 0. \end{cases}$$

Then, as analog of Cauchy's actions on the generic elements in quaternionic setting. We have the following result:

**Theorem 5.3.2.** *Let  $m, n$  be two given nonnegative integers and  $s$  such that . Then, for  $m \geq n$ , we have*

$$C_{\mathbb{B}}(e_{m,n,s}(q, \bar{q})) = -\frac{|\tilde{\mathcal{S}}| q^n \bar{q}^{m+1}}{(m+1)} (1 - |q|^2)^{\gamma+s+1} {}_2F_1 \left( \begin{matrix} 1, \gamma + s + m + 2 \\ m + 2 \end{matrix} \middle| |q|^2 \right). (a)$$

And if  $n > m$ , we have

$$C_{\mathbb{B}}(e_{m,n,s}(q, \bar{q})) = \frac{|\tilde{\mathcal{S}}| q^n \bar{q}^{m+1}}{(m+s+1)} (1 - |q|^2)^{\gamma+s+1} {}_2F_1 \left( \begin{matrix} 1, \gamma + s + m + 2 \\ m + s + 2 \end{matrix} \middle| 1 - |q|^2 \right). (b)$$

*Proof.* The proof is immediate using Lemmas 5.2.2 and 5.2.3. Then, we have

$$\begin{aligned} C_{\mathbb{B}}(e_{m,n,s})(q) &= \frac{1}{\pi} \left( \int_{B_{|q|}} \mathcal{N}(q, p) p^m \bar{p}^n d\mu(p) + \int_{B_{|q|}^c} \mathcal{N}(q, p) p^m \bar{p}^n d\mu(p) \right) \\ &= \frac{1}{\pi} \sum_{\ell=0}^{+\infty} \left( q^{-1-\ell} \int_{B_{|q|}} p^{m+\ell} \bar{p}^n (1 - |p|^2)^{\gamma+s} d\lambda(q) \right. \\ &\quad \left. - q^\ell \int_{B_{|q|}^c} p^{m-1-\ell} \bar{p}^n (1 - |p|^2)^{\gamma+s} d\lambda(q) \right) \\ &= \frac{1}{\pi} \sum_{\ell=0}^{+\infty} q^{-1-\ell} \int_0^{|q|} \int_{\tilde{\mathcal{S}}} \int_0^{2\pi} r^{m+\ell+n} e^{I(m+\ell-n)\theta} (1 - r^2)^{\gamma+s} r dr d\sigma(I) d\theta \\ &\quad - \frac{1}{\pi} \sum_{\ell=0}^{+\infty} q^\ell \int_{|q|}^1 \int_{\tilde{\mathcal{S}}} \int_0^{2\pi} r^{m-1-\ell+n} e^{I(m-1-\ell-n)\theta} (1 - r^2)^{\gamma+s} r dr d\sigma(I) d\theta \\ &= |\tilde{\mathcal{S}}| q^{m-n-1} \left( \varepsilon_{n-m} \int_0^{|q|^2} t^n (1-t)^{\gamma+s} dt - \varepsilon_{m-n-1} \int_{|q|^2}^1 t^n (1-t)^{\gamma+s} dt \right) \\ &= |\tilde{\mathcal{S}}| q^{m-n-1} \left( \varepsilon_{n-m} \int_0^{|q|^2} t^n (1-t)^{\gamma+s} dt - \varepsilon_{m-n-1} \int_{|q|^2}^1 t^n (1-t)^{\gamma+s} dt \right), \end{aligned}$$

By taking the case of  $n \geq m$  and applying (5.2.6) to the integral in the right hand-side of (5.2.4) with  $b = m + 1$ ,  $-a = \gamma + s$  and  $x = |q|^2$ .

The identity (5.2.8) corresponding to  $m > n$  follows in a similar way with  $b = \gamma + s + 1$ ,  $a = -m$  and  $x = 1 - |q|^2$ .  $\square$

The next result is another equivalent hypergeometric representation of  $\mathcal{Z}_{m,n}^\gamma(q, \bar{q})$ .

**Proposition 5.3.3.** *For nonnegative integers  $m, n \in \mathbb{N}$ , we have*

$$\mathcal{Z}_{m,n}^\gamma(q, \bar{q}) = \frac{(-1)^{n \wedge m} (|m - n| + 1)_{m \wedge n} (\gamma + 1)_{m+n}^2 (\bar{q})^m q^n}{(\gamma + 1)_n (\gamma + 1)_m (\gamma + m \vee n + 1)_{m \wedge n} |q|^{2 \cdot m \wedge n}} {}_2F_1 \left( \begin{matrix} -n \wedge m, \gamma + m \vee n + 1 \\ |m - n| + 1 \end{matrix} \middle| |q|^2 \right). \quad (5.3.8)$$

*Proof.* The proof is immediate by making use of the previous proposition (6.0.9) in complex setting, and taking  $k = n \wedge m$ ,  $b = -m \vee n$ ,  $c = -(\gamma + m + n)$  and  $x = 1/|q|^2$ , so that

$$\mathcal{Z}_{m,n}^\gamma(q, \bar{q}) = \frac{((\gamma + 1)_{m+n})^2}{(\gamma + 1)_{m \vee n} (\gamma + 1)_n} \frac{(-1)^{n \wedge m} (-m \vee n)_{n \wedge m} (\bar{q})^m q^n}{(-\gamma + m + n)_{n \wedge m} |q|^{2 \cdot m \wedge n}} {}_2F_1 \left( \begin{matrix} -n, \gamma + m + 1 \\ m - n + 1 \end{matrix} \middle| |q|^2 \right).$$

which completes the proof.  $\square$

**Remark 5.3.4.** *The right-hand side of (5.3.9) can be used to extend the definition of  $\mathcal{Z}_{m,n}^\gamma$  to negative integers  $n, m$ . Therefore for  $\gamma > -1$ , if  $m \geq 0$ , we have*

$$C_{\mathbb{B}} \left( \mathcal{Z}_{m,0}^\gamma \right) (q) = -|\tilde{\mathcal{S}}| \frac{(\gamma + 1)_m \bar{q}^{m+1} (1 - |q|^2)^\gamma}{(m + 1)} {}_2F_1 \left( \begin{matrix} 1, \gamma + m + 2 \\ m + 2 \end{matrix} \middle| |q|^2 \right) \quad (5.3.9)$$

If  $n \geq 1$ , we have

$$C_{\mathbb{B}} \left( \mathcal{Z}_{0,n}^\gamma \right) (q) = \frac{|\tilde{\mathcal{S}}| q^n \bar{q}}{(\gamma + 1)} (1 - |q|^2)^{\gamma+1} {}_2F_1 \left( \begin{matrix} 1, \gamma + m + 2 \\ \gamma + 2 \end{matrix} \middle| 1 - |q|^2 \right). \quad (5.3.10)$$

**Theorem 5.3.5.** *For every positive integers  $0 < m, n$  and real  $\gamma > -1$ , we have*

$$[C_{\mathbb{B}}(\mathcal{Z}_{m,n}^\gamma)](q, \bar{q}) = |\tilde{\mathcal{S}}| (1 - |q|^2)^{\gamma+1} \mathcal{Z}_{m,n-1}^{\gamma+1}(q, \bar{q}). \quad (5.3.11)$$

*Proof.* The proof can be handled as follows. By direct computation, using (5.3.2)) and the facts That the polynomials  $\mathcal{Z}_{m,n}^\gamma(q, \bar{q})$  in (5.3.4) can be rewritten as

$$\mathcal{Z}_{m,n}^\gamma(q, \bar{q}) = m!n!(\gamma + 1)_{m+n} \sum_{j=0}^m \frac{(-1)^j}{(\gamma + 1)_j j!} \frac{e_{m-j, n-j}^j(q, \bar{q})}{(m-j)!(n-j)!}.$$

we arrive at

$$\begin{aligned} [C_{\mathbb{B}}(\mathcal{Z}_{m,n}^{\gamma})](q) &= |\tilde{\mathcal{S}}| [C_{\gamma}(\mathcal{Z}_{m,n}^{\gamma}(q, \bar{q}))](q) \\ &= |\tilde{\mathcal{S}}| (1 - |q|^2)^{\gamma+1} \mathcal{Z}_{m,n-1}^{\gamma+1}(q, \bar{q}). \end{aligned}$$

This completes the proof of theorem . □

Now, we denote  $E$  the orthogonal complement of the subspace spanned by the set  $\{\mathcal{Z}_{m,o}^{\gamma}(q, \bar{q}); \text{ for all } m \in \mathbb{N}\}$  in the space  $L^2(\mathbb{B}, d\mu_{\gamma})$ .

**Proposition 5.3.6.** *The adjoint operator of  $C_{\gamma}$  on  $E$  was given by*

$$[C_{\mathbb{B}}]^*(f) = -\overline{C_{\mathbb{B}}f}$$

**Corollary 5.3.7.** *If  $\gamma > -1/2$ , the Cauchy transform is continuous on  $E$  with*

$$\|C_{\mathbb{B}}\| \leq |\tilde{\mathcal{S}}| \sqrt{\gamma + 1}.$$

**Proposition 5.3.8.** *Let  $M_{\gamma}$  denote the multiplication operator by  $(1 - |B|^2)^{\gamma}$ . Then, the following first order differential equation*

$$\frac{\partial C_{\mathbb{B}}f}{\partial \bar{s}} = -|\tilde{\mathcal{S}}| M_{\gamma}f,$$

holds true for every  $f \in \{\mathcal{Z}_{m,o}^{\gamma}(q, \bar{q}); \text{ for all } m \in \mathbb{N}\}^{\perp}$  in the space  $L^2(\mathbb{B}, d\mu_{\gamma})$ .

*Proof.* We need only to proof the assertion for  $f$  belonging to  $\{\mathcal{Z}_{m,o}^{\gamma}(q, \bar{q}); \text{ for all } m \in \mathbb{N}\}^{\perp}$ . Such function can be expanded as

$$f(q) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{m,n}^{\gamma} \mathcal{Z}_{m,n}^{\gamma}(q, \bar{q}).$$

Thus using the linearity of Cauchy operator  $C$  and Theorem 5.2.6, we get

$$\begin{aligned} [C_{\mathbb{B}}(f)](q) &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{m,n}^{\gamma} [C_{\mathbb{B}}(\mathcal{Z}_{m,n}^{\gamma})](q) \\ &= |\tilde{\mathcal{S}}| (1 - |q|^2)^{\gamma+1} \left( \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{m,n}^{\gamma} \mathcal{Z}_{m,n-1}^{\gamma+1}(q, \bar{q}) \right). \end{aligned}$$

So,

$$\begin{aligned} \frac{\partial [C_{\mathbb{B}}(f)](q)}{\partial \bar{s}} &= -|\tilde{\mathcal{S}}| \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{m,n}^{\gamma} \left( (1 - |q|^2)^{\gamma} \mathcal{Z}_{m,n}^{\gamma}(q, \bar{q}) \right) \\ &= -|\tilde{\mathcal{S}}| M_{\gamma}f. \end{aligned}$$

This, which is completes the proof. □

# Chapter 6

## Appendix: Generalized Zernike functions

We review and complete the study of the generalized 2D Zernike polynomials. We provide a hypergeometric representation and we derive a new integral representation involving a modified Blaschke function restricted to the unit circle. To this end, we follow the idea in [72] by Kazantsev and Bukhgeim.

The remaining sections are organized as follows. In Section 2, we review the Schrödinger factorization algebraic method for generating the disk polynomials  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$ . We also establish an integral representation for this class of polynomials and show that they form an orthogonal basis for the Hilbert space  $L^2(D, d\mu_\gamma)$ , with respect to the measure  $d\mu_\gamma(z) = (1 - |z|^2)^\gamma dx dy$ ;  $\gamma > -1$ . Section 3 is devoted to the explicit action of the Cauchy transform  $\mathcal{C}_\gamma$  on the associated functions. Furthermore, it is shown there that the construction of  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$  is intimately related to the factorization of a special magnetic Laplacian.

The disk polynomials  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$  of the two conjugate complex variables  $z = x + iy$  and  $\bar{z} = x - iy$ ,  $x, y \in \mathbb{R}$ , in the unit disk  $D = \{z \in \mathbb{C}, |z| < 1\}$ , are defined by the Rodrigues formula (up to a multiplicative constant)

$$\mathcal{Z}_{m,n}^\gamma(z, \bar{z}) = (-1)^{m+n} (1 - |z|^2)^{-\gamma} \frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} \left( (1 - |z|^2)^{\gamma+m+n} \right) \quad (6.0.1)$$

and are often referred to as generalized Zernike polynomials. Their explicit expression is given by

$$\mathcal{Z}_{m,n}^\gamma(z, \bar{z}) = m!n!(\gamma + 1)_{m+n} \sum_{j=0}^{m \wedge n} \frac{(-1)^j (1 - |z|^2)^j}{(\gamma + 1)_{jj}!} \frac{\bar{z}^{m-j}}{(m-j)!} \frac{z^{n-j}}{(n-j)!}. \quad (6.0.2)$$

The suggested definition of  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$  agrees with the one provided by Koornwinder [74, 75] and Dunkl [34, 36] as well as the one considered by Wünsche [100], being indeed

$$\mathcal{Z}_{m,n}^\gamma(z, \bar{z}) = (\gamma + 1)_{m+n} \overline{P_{m,n}^\gamma(z, \bar{z})} = (\gamma + 1)_{m+n} \overline{R_{m,n}^{(\gamma)}(z)}.$$

The limit case of  $\gamma = -1$  leads to the so-called Scattering polynomials, while for  $\gamma = 0$  and  $m \leq n$ , they turn out to be related to the real Zernike polynomials  $R_k^V(x)$ , introduced by Zernike himself ([104]) in his framework on optical problems involving telescopes and microscopes. More exactly, we have

$$\mathcal{Z}_{m,n}^0(z, \bar{z}) = (m+n)! e^{i(n-m) \arg z} R_{m+n}^{n-m}(\sqrt{z\bar{z}}).$$

The polynomials  $R_k^V(x)$  play an important role in expressing the wavefront data in optical tests and in studying diffraction problems [105]. The polynomials  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$  are closely connected to the real Jacobi polynomials

$$P_n^{(\alpha, \beta)}(x) = (1-x)^{-\alpha} (1+x)^{-\beta} \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \left( (1-x)^{n+\alpha} (1+x)^{n+\beta} \right).$$

Namely, we have ([59, 69])

$$\mathcal{Z}_{m,n}^\gamma(z, \bar{z}) = c_{m,n} |z|^{|m-n|} e^{i[(n-m) \arg z]} P_{m \wedge n}^{(|m-n|, \gamma)}(1 - 2|z|^2). \quad (6.0.3)$$

Where  $c_{m,n} = \Gamma(\gamma+1) (-1)^{m \wedge n} (m \wedge n)! (\gamma + m \wedge n + 1)_{m \vee n}$ .

Moreover, starting from (6.2.3) or (5.3.4), one can express  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$  in terms of the Gauss-hypergeometric function  ${}_2F_1$  reads ([100, p. 137])

$$\mathcal{Z}_{m,n}^\gamma(z, \bar{z}) = (\gamma+1)_{m+n} \bar{z}^m z^n {}_2F_1 \left( \begin{matrix} -m, -n \\ \gamma+1 \end{matrix} \middle| 1 - \frac{1}{|z|^2} \right). \quad (6.0.4)$$

Equivalently ([34, p. 692] or [36, p. 535]), we have

$$\mathcal{Z}_{m,n}^\gamma(z, \bar{z}) = \frac{((\gamma+1)_{m+n})^2}{(\gamma+1)_m (\gamma+1)_n} \bar{z}^m z^n {}_2F_1 \left( \begin{matrix} -m, -n \\ -\gamma - m - n \end{matrix} \middle| \frac{1}{|z|^2} \right) \quad (6.0.5)$$

The next result is another equivalent hypergeometric representation of  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$ .

**Proposition 6.0.9.** *For nonnegative integers  $m, n \in \mathbb{N}$ , we have*

$$\begin{aligned} \mathcal{Z}_{m,n}^\gamma(z, \bar{z}) &= \frac{(-1)^{n \wedge m} (\gamma+1)_{m+n}^2 (|m-n|+1)_{m \wedge n}}{(\gamma+1)_n (\gamma+1)_m (\gamma+m \vee n+1)_{m \wedge n}} \frac{(\bar{z})^m z^n}{|z|^{2 \cdot m \wedge n}} \\ &\quad \times {}_2F_1 \left( \begin{matrix} -n \wedge m, \gamma + m \vee n + 1 \\ |m-n|+1 \end{matrix} \middle| |z|^2 \right). \end{aligned} \quad (6.0.6)$$

*Proof.* This is immediate making use of (6.0.5) combined with [85, Eq. (15.8.6)]

$${}_2F_1 \left( \begin{matrix} -k, b \\ c \end{matrix} \middle| x \right) = \frac{(b)_k}{(c)_k} (-x)^k {}_2F_1 \left( \begin{matrix} -k, 1-c-k \\ 1-b-k \end{matrix} \middle| \frac{1}{x} \right)$$

for  $k = n \wedge m$ ,  $b = -m \vee n$ ,  $c = -(\gamma + m + n)$  and  $x = 1/|z|^2$ , so that

$$\begin{aligned} \mathcal{Z}_{m,n}^\gamma(z, \bar{z}) &= \frac{((\gamma + 1)_{m+n})^2 (-1)^{n \wedge m} (-m \vee n)_{n \wedge m} (\bar{z})^m z^n}{(\gamma + 1)_{m \vee n} (\gamma + 1)_{n \wedge m} (-\gamma + m + n)_{n \wedge m} |z|^{2 \cdot m \wedge n}} \\ &\quad \times {}_2F_1 \left( \begin{matrix} -n \wedge m, \gamma + m \vee n + 1 \\ |m - n| + 1 \end{matrix} \middle| |z|^2 \right). \end{aligned}$$

Therefore, we get (6.0.6) since

$$(-a)_{n \wedge m} = (-1)^{n \wedge m} (a - n \wedge m + 1)_{n \wedge m}$$

and

$$(\gamma + 1)_{m+n} = (\gamma + m \vee n + 1)_{n \wedge m} (\gamma + 1)_{m \vee n}$$

□

**Remark 6.0.10.** The right-hand side of (6.0.6) can be used to extend the definition of  $\mathcal{Z}_{m,n}^\gamma$  to negative integers  $n, m$ .

## 6.1 Integral representation for $\mathcal{Z}_{m,n}^\gamma$

The expressions (6.0.4), (6.0.5) and (6.0.3) can be used to derive integral representations for  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$ , using the classical integral representations of the Gauss hypergeometric functions and the Jacobi polynomials. However, using elementary facts we give a new integral representation which is of independent interest.

**Theorem 6.1.1.** The following integral representation

$$\mathcal{Z}_{m,n}^\gamma(z, \bar{z}) = \frac{(-1)^m (\gamma + m + 1)_n m!}{2\pi i} \left(1 - |z|^2\right)^{-\gamma} \oint_{|t|=1} t^{\gamma+m+n} \frac{(\bar{t} - \bar{z})^{\gamma+m}}{(t - z)^{m+1}} dt \quad (6.1.1)$$

holds for the generalized Zernike polynomials  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$ .

*Proof.* For the proof, we proceed as in [72]. Start from

$$\mathcal{Z}_{m,n}^\gamma(z, \bar{z}) = (-1)^m (\gamma + m + 1)_n \left(1 - |z|^2\right)^{-\gamma} \frac{\partial^m}{\partial z^m} \left( z^n \left(1 - |z|^2\right)^{\gamma+m} \right)$$

and use the ordinary binomial expansion with the factorial function

$$(1 - \xi)^{-a} = \sum_{j=0}^{+\infty} \frac{(a)_j}{j!} \xi^j$$

to get

$$\mathcal{Z}_{m,n}^\gamma(z, \bar{z}) = (-1)^m (\gamma + m + 1)_n (1 - |z|^2)^{-\gamma} \sum_{j=0}^{+\infty} \frac{(-\gamma - m)_j}{j!} \frac{\partial^m}{\partial z^m} (z^{j+n}) \bar{z}^j.$$

Now, since

$$\frac{\partial^m}{\partial z^m} (z^{j+n}) = \frac{m!}{2\pi i} \oint_{|t|=1} \frac{t^{j+n}}{(t-z)^{m+1}} dt,$$

it follows

$$\mathcal{Z}_{m,n}^\gamma(z, \bar{z}) = \frac{(-1)^m (\gamma + m + 1)_n m!}{2\pi i} (1 - |z|^2)^{-\gamma} \oint_{|t|=1} \frac{t^n}{(t-z)^{m+1}} \left( \sum_{j=0}^{+\infty} \frac{(-\gamma - m)_j}{j!} t^j \bar{z}^j \right) dt.$$

This proves (6.1.1).  $\square$

**Corollary 6.1.2.** *For every nonnegative integers  $m$  and  $n$ , real  $\gamma > -1$  and  $z \in D$ , we have the following estimate*

$$|\mathcal{Z}_{m,n}^\gamma(z, \bar{z})| \leq 2^{\gamma-1} |(\gamma + m + 1)_n| m! (1 - |z|^2)^{-\gamma}.$$

*Proof.* From (6.1.1), we obtain

$$\begin{aligned} |\mathcal{Z}_{m,n}^\gamma(z, \bar{z})| &\leq \frac{|(\gamma + m + 1)_n| m!}{2\pi} (1 - |z|^2)^{-\gamma} \int_0^{2\pi} \frac{|1 - e^{i\theta} \bar{z}|^{\gamma+m}}{|1 - e^{-i\theta} z|^{m+1}} d\theta \\ &\leq \frac{|(\gamma + m + 1)_n| m!}{2\pi} (1 - |z|^2)^{-\gamma} \int_0^{2\pi} |1 - e^{-i\theta} z|^{\gamma-1} dt \\ &\leq 2^{\gamma-1} |(\gamma + m + 1)_n| m! (1 - |z|^2)^{-\gamma}. \end{aligned}$$

$\square$

Another immediate consequence, we derive the following.

**Corollary 6.1.3.** *We have*

$$\mathcal{Z}_{m,n}^\gamma(0, 0) = (-1)^m (\gamma + m + 1)_m m! \delta_{m,n}. \quad (6.1.2)$$

*Proof.* The result readily follows from (6.1.1) by specifying there  $z = 0$ . Indeed, we have

$$\begin{aligned} \mathcal{Z}_{m,n}^\gamma(0, 0) &= \frac{(-1)^m (\gamma + m + 1)_n m!}{2\pi i} (1 - |z|^2)^{-\gamma} \oint_{|t|=1} t^{n-(m+1)} dt \\ &= (-1)^m (\gamma + m + 1)_m m! \delta_{m,n}. \end{aligned}$$

$\square$

Moreover, we recover its analogue obtained in [67, Eq (3.13) Theorem 3.3] for the complex Hermite polynomials

$$H_{m,n}(z, \bar{z}) := (-1)^{m+n} e^{|z|^2} \frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} \left( e^{-|z|^2} \right).$$

**Corollary 6.1.4.** *We have*

$$H_{m,n}(0, 0) = (-1)^m m! \delta_{m,n}.$$

*Proof.* Starting from the explicit expression of  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$  and  $H_{m,n}(z, \bar{z})$ , it is not hard to prove the following limit

$$\lim_{\rho \rightarrow +\infty} \frac{\mathcal{Z}_{m,n}^{\rho^2}(z/\rho, \bar{z}/\rho)}{\rho^{m+n}} = H_{m,n}(z, \bar{z}).$$

Thus, using the Binet formula, we get

$$H_{m,n}(0, 0) = \lim_{\rho \rightarrow +\infty} (-1)^m m! \delta_{m,n} \frac{\Gamma(\rho^2 + 2m + 1)}{\rho^{2m} \Gamma(\rho^2 + m + 1)} = (-1)^m m! \delta_{m,n}.$$

□

We conclude this section by proving that the polynomials  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$  constitute an orthogonal basis of the Hilbert space  $L^{2,\gamma}(D) := L^2(D, d\mu_\gamma)$  of all square integrable complex-valued functions on  $D$  with respect the measure

$$d\mu_\gamma(z) = \left(1 - |z|^2\right)^\gamma dx dy, \gamma > -1,$$

$dx dy$  being the Lebesgue measure on  $D$ ; where  $z = x + iy$  and  $x, y \in \mathbb{R}$ .

## 6.2 Spectral realization of generalized Zernike polynomials

The unit disk  $D = \{z \in \mathbb{C}; |z| < 1\}$  is endowed with the standard hyperbolic geometry associated to its Poincar metric

$$ds^2 := \frac{1}{(1 - |z|^2)^2} dz \otimes d\bar{z}.$$

The magnetic Schrödinger operator associated to a given the potential vector  $\theta$  (differential 1-form) is defined by

$$\mathcal{L}_v = (d + ivext(\theta))^*(d + ivext(\theta)),$$

where  $d$  and  $ext(\theta)$  are respectively the differential operator and the exterior multiplication by the differential 1-form  $\theta$ . The adjoint operation is taken with respect to the Hermitian scalar product on compactly supported differential forms

$$(\alpha, \beta) := \int_D \alpha \wedge \star \beta,$$

where  $\star$  is the Hodge star operator canonically associated to the hyperbolic metric  $ds^2$ .

For the particular

$$\theta(z) = (\partial - \bar{\partial}) \log(1 - |z|^2) = \frac{-iv(\bar{z}dz - zd\bar{z})}{1 - |z|^2}; \nu > 0,$$

the associated  $\mathfrak{L}_\nu$  leads to the twisted Laplacian

$$\mathfrak{L}_\nu = -(1 - |z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}} - \nu(1 - |z|^2) \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right) + \nu^2 |z|^2$$

acting on the  $L^2$ -Hilbert space  $L^2(D; (1 - |z|^2)^{-2} dx dy)$ . It is essentially the Laplace-Beltrami operator

$$-(1 - |z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}},$$

perturbed by the rotation operator

$$N = \nu(1 - |z|^2) \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right).$$

The second order differential operator  $\mathfrak{L}_\nu$  is clearly an elliptic self-adjoint and their spectral properties are well-known in the literature (see for example [106, 60] and the references therein). Its discrete  $L^2$ -spectrum is nontrivial if and only if  $\nu > 1/2$ . It is given by the eigenvalues

$$\lambda_{\nu, m} := \nu(2m + 1) - m(m + 1)$$

for varying positive integer  $m$  such that  $0 \leq m < \nu - 1/2$ . Its factorization la Schrödinger involves the first order differential operators  $\nabla_\alpha$  and and its formal adjoint  $\nabla_\alpha^*$ ,

$$\nabla_\alpha = -(1 - |z|^2) \frac{\partial}{\partial z} + \alpha \bar{z}, \quad \nabla_\alpha^* = (1 - |z|^2) \frac{\partial}{\partial \bar{z}} + (\alpha + 1)z.$$

More exactly, we have the following algebraic relationship ([46, 59])

$$\mathfrak{L}_\nu = \nabla_\nu^* \nabla_\nu - \nu = \nabla_{\nu-1} \nabla_{\nu-1}^* + \nu,$$

and therefore

$$\mathfrak{L}_\nu \nabla_{\nu-1} = (\nabla_{\nu-1} \nabla_{\nu-1}^* + \nu) \nabla_{\nu-1} = \nabla_{\nu-1} (\mathfrak{L}_{\nu-1} + (2\nu - 1)).$$

Hence, the first order differential operator  $\nabla_{\nu-1}$  allows one to generate  $L^2$ -eigenfunctions of  $\mathfrak{L}_\nu$  from those of  $\mathfrak{L}_{\nu-1}$ . More generally, if  $\varphi_0$  is a nonzero  $L^2$ -eigenfunction associated to the lowest eigenvalue of  $\mathfrak{L}_{\nu-m}$ , then

$$\nabla_m^\nu \varphi_0 = \nabla_{\nu-1} \circ \nabla_{\nu-2} \circ \cdots \circ \nabla_{\nu-m} \varphi_0$$

is an  $L^2$ -eigenfunction of  $\mathfrak{L}_\nu$ .

**Proposition 6.2.1.** *For every fixed real number  $\nu > \frac{1}{2}$  and nonnegative integer  $m$  such that  $0 \leq m < \nu - \frac{1}{2}$ , the functions*

$$\psi_{m,n}^\nu(z, \bar{z}) := \nabla_m^\nu \left( z^n (1 - |z|^2)^{\nu-m} \right), \quad (6.2.1)$$

varying  $n = 0, 1, 2, \dots$ , constitute an orthogonal basis of the generalized Bergman spaces

$$A_m^{2,\nu}(D) := \{ \varphi \in L^2(D; (1 - |z|^2)^{-2} dx dy); \mathfrak{L}_\nu \varphi = \lambda_{\nu,m} \varphi \}. \quad (6.2.2)$$

*Sketched proof.* From the discussion above, The function  $\psi_{m,n}^\nu$  in (6.2.1) are clearly eigenfunctions of  $\mathfrak{L}_\nu$ . Moreover,  $\square$

Associated to  $\psi_{m,n}^\nu(z, \bar{z})$ , we consider the polynomials

$$\mathcal{Z}_{m,n}^\gamma(z, \bar{z}) := (\gamma + m + 1)_n (1 - |z|^2)^{-\nu+m} \nabla_m^\nu \left( z^n (1 - |z|^2)^{\nu-m} \right).$$

Therefore, one recovers the Rodrigues representation of  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$  given through (6.2.3) with  $\gamma = 2(\nu - m) - 1$ .

**Theorem 6.2.2.** *The disk polynomials  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$  form an orthogonal basis of the Hilbert space  $L^{2,\gamma}(D)$  with*

$$\int_D \mathcal{Z}_{m,n}^\gamma(z, \bar{z}) \mathcal{Z}_{n,m}^\gamma(z, \bar{z}) d\mu_\gamma(z) = \frac{\pi \Gamma(\gamma + 1)^2 m! n! (\gamma + n + 1)_m (\gamma + m + 1)_n}{m + n + \gamma + 1}. \quad (6.2.3)$$

*Proof.* Notice first that under the condition  $\gamma > -1$ , the monomials  $z^m$ , and so  $\bar{z}^n$ ,  $z^m \bar{z}^n$  and  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$ , belong to the Hilbert space  $L^{2,\gamma}(D)$ .

The orthogonal relation follows from writing the integral in polar coordinates and applying the orthogonality of the Laguerre polynomials.

For completeness of  $\mathcal{Z}_{m,n}^\gamma(z, \bar{z})$  in  $L^{2,\gamma}(D)$ , we show that for every function  $h \in L^{2,\gamma}(D)$  such that  $S_{m,n}(h) := \langle h, \mathcal{Z}_{m,n}^\gamma \rangle_{L^{2,\gamma}(D)} = 0$ , for all  $m, n$ , then  $h$  is a vanishing

function. To this, notice that we can rewrite  $S_{m,n}(h)$  as

$$\begin{aligned} S_{m,n}(h) &= \int_D h(z) \overline{\mathcal{Z}_{m,n}^\gamma(z, \bar{z})} (1 - |z|^2)^\gamma dx dy \\ &= \int_0^1 \int_0^{2\pi} h(re^{i\theta}) r^{|m-n|} e^{i(m-n)\theta} P_{m \wedge n}^{|m-n|, \gamma} (1 - 2r^2) (1 - r^2)^\gamma r dr d\theta \\ &= \int_{-1}^1 \left( \int_0^{2\pi} h \left( \sqrt{\frac{1-t}{2}} e^{i\theta} \right) e^{i(m-n)\theta} d\theta \right) P_{m \wedge n}^{|m-n|, \gamma} (t) (1-t)^{|m-n|/2} (1+t)^\gamma dt. \end{aligned}$$

This follows using polar coordinates as well as the explicit representation of  $\mathcal{Z}_{m,n}^\gamma$  in terms of the Jacobi polynomials (See (6.0.3)) and the change of variable  $t = 1 - 2r^2$ . By discussing the signum of the integer  $k = m - n$ , we conclude that  $S_{m,n}(h) = 0$ , for all  $m, n$ , is equivalent to

$$\int_{-1}^1 \left( \int_0^{2\pi} h \left( \sqrt{\frac{1-t}{2}} e^{i\theta} \right) e^{ik\theta} d\theta \right) P_s^{|k|, \gamma} (t) (1-t)^{|k|/2} (1+t)^\gamma dt = 0,$$

for all nonnegative integer  $s (= \min(m, n))$ . Making use of the Cauchy inequality, we can show that involved function

$$t \mapsto (1+t)^{\gamma/2} g_k(t); \quad g_k : t \mapsto \int_0^{2\pi} h \left( \sqrt{\frac{1-t}{2}} e^{i\theta} \right) e^{ik\theta} d\theta$$

belongs  $L^2([-1, 1], dt)$ , for which it is known (see e.g. [45]) that the functions

$$t \mapsto P_s^{|k|, \gamma} (t) (1-t)^{|k|/2} (1+t)^{\gamma/2},$$

for varying  $s$ , form an orthogonal basis. Therefore,  $(1+t)^{\gamma/2} g_k = 0$  and hence  $g_k = 0$  a.e on  $[-1, 1]$ , for every  $k \in \mathbb{Z}$ . Subsequently, for every  $t \in [-1, 1] \setminus N$ , with  $N := \cup_k \{t \in [-1, 1]; g_k \neq 0\}$ , we have

$$[0, 2\pi] \ni \theta \mapsto h_t(\theta) = h \left( \sqrt{\frac{1-t}{2}} e^{i\theta} \right)$$

belongs to  $L^2([0, 2\pi]; d\theta)$ , by means Fubini theorem, and its Fourier transform vanishes at  $k$ ,  $\mathcal{F}(h_t)(-k) = g_k(t) = 0$  for  $t \in [-1, 1] \setminus N$ .

Thus, the function  $h_t = 0$  a.e. on  $[0, 2\pi]$ , and for almost every  $t \in [-1, 1]$ . But, the explicit expression of the norm is given in [68] by:

$$\|\mathcal{Z}_{m,n}^\gamma\| = \left( \frac{\pi \Gamma(\gamma + 1)^2 m! n! (\gamma + n + 1)_m (\gamma + m + 1)_n}{m + n + \gamma + 1} \right)^{1/2}$$

This completes the proof. □

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