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Khalil LAMSAF

On poly-analytic polynomials: a systematic study of the bivariate poly-analytic Hermite polynomials and the two-dimensional (p,q) -heat polynomials of Gould-Hopper type.

Before the JURY:

Zine El Abidine ABDELALI	PES	Faculty of Sciences, Mohammed V University, Rabat.	President
Fouzia EL WASSOULI	PES	Ain Chock Faculty of Sciences A, Hassan II University, Casablanca.	Examinator/Reviewer
Hamid EZZAHRAOUI	MCH	Faculty of Sciences, Mohammed V University, Rabat.	Examinator/ Reviewer
Ahmed HAJJI	MCH	Faculty of Sciences, Mohammed V University, Rabat.	Examinator/ Reviewer
Ali HAFUOD	MCH	Regional Center for Education and Training Professions, Kenitra.	Examinator
Allal GHANMI	PES	Faculty of Sciences, Mohammed V University, Rabat.	Thesis supervisor

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Dedication

To my parents,

Your love, support, and constant encouragement have been essential throughout this journey. You have given me the means to realize my dreams and have always inspired me to persevere. Thank you for everything you have done for me.

To my brothers and sisters,

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Abstract

This thesis explores advanced poly-analytic polynomials, specifically extending the classical Gould-Hopper polynomials to two complex variables and introducing bivariate poly-analytic Hermite polynomials.

Firstly, we extend the classical Gould-Hopper polynomials to encompass two complex variables, incorporating the 1-D and 2-D holomorphic and polyanalytic Itô-Hermite polynomials as particular cases. This study delves into their operational representations, generating functions, and recurrence relations. We establish numerous special identities, including multiplication formulas, Runge type addition formulas, and Nielson type formulas. Higher-order partial differential equations are analyzed, revealing connections to Gould-Hopper polynomials and hypergeometric functions.

Secondly, we introduce a new class of bivariate poly-analytic Hermite polynomials. These are shown to be realizable as the Fourier-Wigner transform of univariate complex Hermite functions, forming a significant orthogonal basis in the classical Hilbert space over two-complex space with respect to the Gaussian measure. We explore their fundamental properties, including three-term recurrence relations, operational realizations, and the differential equations they satisfy. Additionally, we derive various generating functions and integral and exponential operational representations.

keywords: Orthogonal polynomials, (p,q) -heat polynomials, Bi-variate Hermite polynomials, Heat equation, Fourier-Wigner transform.

Résumé

Cette thèse explore des polynômes poly-analytiques avancés, en étendant spécifiquement les polynômes classiques de Gould-Hopper à deux variables complexes et en introduisant des polynômes de Hermite bivariés poly-analytiques.

Premièrement, nous étendons les polynômes classiques de Gould-Hopper pour inclure deux variables complexes, incorporant les polynômes de Itô-Hermite holomorphes et polyanalytiques en 1-D et 2-D comme cas particuliers. Cette étude examine leurs représentations opérationnelles, fonctions génératrices et relations de récurrence. Nous établissons de nombreuses identités spéciales, y compris des formules de multiplication, des formules d'addition de type Runge et des formules de type Nielson. Les équations différentielles partielles d'ordre supérieur sont analysées, révélant des connexions avec les polynômes de Gould-Hopper et les fonctions hypergéométriques.

Deuxièmement, nous introduisons une nouvelle classe de polynômes de Hermite bivariés poly-analytiques. Ceux-ci se révèlent réalisables sous la forme de la transformée de Fourier-Wigner des fonctions de Hermite complexes univariées, formant une base orthogonale significative dans l'espace de Hilbert classique sur un espace à deux variables complexes par rapport à la mesure gaussienne. Nous explorons leurs propriétés fondamentales, y compris les relations de récurrence à trois termes, les réalisations opérationnelles et les équations différentielles (propriété de Bochner) qu'ils satisfont. De plus, nous dérivons diverses fonctions génératrices ainsi que des représentations opérationnelles intégrales et exponentielles.

Mot-clés Polynômes orthogonaux, Polynômes de chaleur (p,q) , Polynômes de Hermite bivariés, Équation de la chaleur, Transformée de Fourier-Wigner

Résumé étendu

Le théorème de Stone-Weierstrass, un pilier de l'analyse fonctionnelle, affirme que toute fonction continue définie sur un intervalle compact de la ligne réelle peut être uniformément approximée par des polynômes. Ce théorème est essentiel pour approcher des fonctions complexes avec des formes polynomiales plus simples, ce qui est avantageux tant pour les méthodes computationnelles que pour l'analyse théorique. La simplicité inhérente et les propriétés analytiques des polynômes en font le choix privilégié pour la construction de bases hilbertiennes. Leur importance dans l'étude des espaces de Hilbert réside dans leur capacité à fournir des représentations efficaces des fonctions, à faciliter l'approximation et la solution de certaines équations différentielles, et à offrir des aperçus sur les propriétés spectrales des opérateurs et des systèmes physiques.

Cette importance est particulièrement évidente lorsqu'on examine des sous-espaces fermés de $L^2(\mathbb{R})$ et $L^2(\mathbb{C})$, constitués de toutes les fonctions mesurables et intégrables au carré sur la ligne réelle et le plan complexe, respectivement, par rapport à la mesure de Lebesgue. Ces sous-espaces comprennent des fonctions ayant une énergie finie lorsqu'elles sont mises au carré et intégrées sur \mathbb{R} ou \mathbb{C} .

La structure mathématique riche et la large applicabilité de certaines classes de polynômes motivent leur étude dans divers domaines des mathématiques et de la physique. Par exemple, dans l'analyse fonctionnelle, l'espace $L^2(\mathbb{C}^2)$ est un espace de Hilbert équipé d'un produit scalaire qui induit une norme, le rendant un espace métrique complet. Cette structure permet d'explorer les propriétés de convergence, de continuité et d'orthogonalité des fonctions au sein de $L^2(\mathbb{C}^2)$.

En mécanique quantique, les éléments de $L^2(\mathbb{C}^2)$ sont interprétés comme des fonctions d'onde décrivant les états de deux particules quantiques, encodant des informations probabilistes sur leurs positions ou leurs moments. De plus, des classes spéciales de fonctions au sein de $L^2(\mathbb{C}^2)$ apparaissent souvent comme solutions à des équations différentielles partielles (EDP) spécifiques, et analyser leurs propriétés aide à comprendre le comporte-

ment de ces solutions dans différentes conditions.

En théorie des opérateurs, les opérateurs agissant sur $L^2(\mathbb{C}^2)$ jouent un rôle central, représentant diverses transformations telles que la différentiation, l'intégration ou les transformations linéaires. Ces opérateurs sont étudiés pour comprendre leurs propriétés spectrales, leurs fonctions propres et le calcul fonctionnel associé. De plus, dans le contexte des transformations intégrales, l'étude des fonctions génératrices pour des classes spéciales de fonctions dans $L^2(\mathbb{C}^2)$ fournit des outils puissants pour résoudre des équations différentielles, analyser des distributions de probabilité et étudier des structures combinatoires. Ces techniques impliquent souvent l'expression des fonctions en termes de systèmes orthogonaux et complets au sein de $L^2(\mathbb{C}^2)$, conduisant à des méthodes computationnelles efficaces et à des aperçus théoriques.

Motivés par ces considérations, ce travail propose d'étudier de nouvelles classes de polynômes en deux variables complexes, notamment les polynômes de chaleur (p, q) $H_{n,m}^{(p,q)}(z, w|\gamma)$ et les polynômes d'Hermite bivariés $H_{m,n,m',n'}(z, w)$.

L'intérêt pour ces polynômes réside dans leur capacité à décrire l'espace de Hilbert sous-jacent $L^2(\mathbb{C}^2)$, connu pour sa structure mathématique riche et ses applications dans divers domaines des mathématiques et de la physique.

La première classe, définie comme

$$H_{n,m}^{(p,q)}(z, w|\gamma) = n!m! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor \wedge \lfloor \frac{m}{q} \rfloor} \frac{\gamma^k}{k!} \frac{z^{n-pk}}{(n-pk)!} \frac{w^{m-qk}}{(m-qk)!}$$

étend diverses catégories de polynômes de type Hermite, y compris les polynômes d'Hermite réels, les polynômes de Gould-Hopper, les polynômes itô-hermitiens polyanalytiques, et les polynômes d'Hermite holomorphes. Cette extension permet un traitement algébrique et analytique unifié de ces polynômes, en explorant leurs propriétés fondamentales, leurs relations de récurrence, leurs formules additives, leurs fonctions génératrices, et les opérateurs différentiels associés.

La seconde classe, les polynômes d'Hermite bivariés, définie comme

$$H_{m,n,m',n'}(z, w) := H_{m,n}(z + iw, \bar{z} - i\bar{w})H_{m',n'}(\bar{z} + i\bar{w}, z - iw),$$

forme une base hilbertienne orthogonale pour l'espace de Hilbert des fonctions gaussiennes sur \mathbb{C}^2 , servant de point de départ fondamental pour la construction d'une base hilbertienne non triviale pour $L^2(\mathbb{C}^2)$. Ces polynômes présentent un comportement analogue à leurs homologues sous les opérateurs dérivés standard et sont examinés à travers divers opérateurs différentiels, représentations intégrales et fonctions génératrices.

Le document est organisé en trois chapitres, complétés par une introduction générale :

- **Chapitre 1** : Un aperçu des outils mathématiques nécessaires tirés de la théorie des fonctions spéciales, y compris les fonctions hypergéométriques, et les polynômes d'Hermite réels et complexes.
- **Chapitre 2** : Une étude systématique des polynômes de chaleur (p, q) , explorant leurs propriétés fondamentales, relations de récurrence, fonctions génératrices, et équations différentielles.
- **Chapitre 3** : Un examen détaillé des polynômes d'Hermite poly-analytiques bivariés, incluant des formules de type Rodrigues, des relations de récurrence, des équations différentielles, et des fonctions génératrices, avec des applications discutées.

En explorant les propriétés des polynômes orthogonaux construits sur \mathbb{C}^2 , cette étude vise à investiguer des applications telles que la résolution des EDP, l'analyse des transformations intégrales, et la dérivation des propriétés des opérateurs dans $L^2(\mathbb{C}^2)$. Ces applications incluent la résolution de problèmes de valeur aux limites, l'étude des systèmes physiques décrits par des EDP, l'analyse des réseaux complexes ou des structures de données, et le développement d'algorithmes numériques efficaces pour divers problèmes scientifiques et d'ingénierie. Dans l'ensemble, cette exploration offre un riche voyage interdisciplinaire dans les domaines de l'analyse fonctionnelle, la théorie des opérateurs, les EDP, et les mathématiques appliquées.

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General introduction

The setting

The Stone-Weierstrass theorem is one of the fundamental results in functional analysis. It states that any continuous function defined on a compact interval in the real line can be uniformly approximated by polynomials. Such theorem is of immense importance as it provides a way to approximate more complex functions by simpler ones, facilitating computational methods and theoretical analysis. The simplicity and the analytical properties of polynomials, with their broad applicability, justify they are considered as the natural and the most practical choice for constructing Hilbertian bases. In fact, their importance in studying Hilbert spaces lies in their ability to provide efficient representations of functions, facilitate approximation and solution of special differential equations. Moreover, they offer insights into the spectral properties of operators and physical systems. This is the case when dealing with particular closed subspaces of $L^2(\mathbb{R})$ and $L^2(\mathbb{C})$ consisting of all square-integrable measurable functions on the real line and complex plane, respectively, with respect to the Lebesgue measure. In other words, the set of functions having finite energy when squared and integrated over \mathbb{R} or \mathbb{C} .

Motivated by these facts, we propose to study, in the present work, new classes of polynomials in two complex variables, each crafted through distinct methodologies. Namely, the so-called (p, q) -heat polynomials $H_{n,m}^{(p,q)}(z, w|\gamma)$ as well as the bivariate Hermite polynomials $H_{m,n,m',n'}(z, w)$. Our interest in the considered polynomials stems from their well description of the underlying Hilbert space $L^2(\mathbb{C}^2)$ on the two-dimensional complex space whose rich mathematical structure and applications in various areas of mathematics and physics is well known. Hereafter, we state some of them:

- Withing the framework of functional analysis, the space $L^2(\mathbb{C}^2)$ is a Hilbert space equipped with an inner product that induces a norm, making it a complete metric

space. This structure allows the study the different notions and properties of convergence, continuity, orthogonality of functions in $L^2(\mathbb{C}^2)$.

- In quantum mechanics, the elements belonging to $L^2(\mathbb{C}^2)$ are even seen as wave functions describing the state of a system of two quantum particles. Such functions encode probabilistic informations about the positions or the momenta of these particles (the probability density of finding them at specific locations).
- Special classes of functions in $L^2(\mathbb{C}^2)$ often arise as solutions to certain specific types of partial differential equations (PDEs). Studying the basic properties of functions in $L^2(\mathbb{C}^2)$ allows analyzing and investigating the behavior of such solutions under different conditions.
- In operator theory, operators acting on $L^2(\mathbb{C}^2)$ play a central role and represent various transforms, including differentiation, integration, or linear transformations. which they are studied to describe their spectral properties, eigenfunctions, and associated functional calculus.
- Within the setting of integral transforms, the study of generating functions, for special classes of functions in $L^2(\mathbb{C}^2)$, provide powerful tools for solving differential equations, analyzing probability distributions, and studying combinatorial structures. These techniques often involve expressing functions in terms of certain orthogonal and complete systems in $L^2(\mathbb{C}^2)$. This leads to efficient computational methods and theoretical insights.

Studied polynomials

The first class is defined by

$$H_{n,m}^{(p,q)}(z,w|\gamma) = n!m! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor \wedge \lfloor \frac{m}{q} \rfloor} \frac{\gamma^k}{k!} \frac{z^{n-pk}}{(n-pk)!} \frac{w^{m-qk}}{(m-qk)!},$$

and serves as an interesting extension of various categories of polynomials of Hermite type, including the real Hermite polynomials, Gould-Hopper polynomials, polyanalytic Itô-Hermite polynomials, and holomorphic Hermite polynomials. Such polynomials allows an unified algebraic and analytic treatment of the aforementioned ones. This innovative extension delineates additional classes by manipulating different parameters (p, q) . Mainly, we explore their fundamental properties such as specific instances and values, recurrence relations contingent upon indices (n, m) , as well as the parameters (p, q) . We

also delve into additive formulas including Runge and Neilson formulas, and we provide closed expressions for their generating functions. These polynomials are elucidated in terms of hypergeometric functions. We also examine the associated differential operator and the generalized heat equation satisfied by these polynomials. These properties, which were the subject of a paper recently published in *“Journal of mathematical Analysis, 530 (2024)”*, are discussed in this document in some details with eventual applications and interesting comments.

The second studied category of polynomials, termed the bivariate Hermite polynomials, is crafted through a fusion of tensorial product and variable and special linear transformations, and delineated by the following definition:

$$H_{m,n,m',n'}(z, w) := H_{m,n}(z + iw, \bar{z} - i\bar{w})H_{m',n'}(\bar{z} + i\bar{w}, z - iw),$$

leading to an orthogonal Hilbertian basis of the Hilbert space of Gaussian functions on the two-dimensional complex space \mathbb{C}^2 , which serves clearly as a foundational step towards constructing a nontrivial Hilbertian basis for $L^2(\mathbb{C}^2)$. A pivotal outcome in this setting, facilitating our study, is the key observation of realizing $L^2(\mathbb{C}^2)$ as the image of $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ through the Fourier-Wigner transform

$$\mathcal{V}_d(f, g)(p, q) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\langle y, q \rangle} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)} dy \quad (0.0.1)$$

The considered polynomials exhibit analogous behavior to their counterparts under standard derivative operators. However, under alternative operators, their characteristics diverge. This will be the task of a detailed discussion in the last chapter of this work, which resumes the work published in *“Results in Mathematics. 76 (2021)”*. The bivariate Hermite polynomials can alternatively be defined via a Rodriguez type formula

$$H_{m,n,m',n'}(z, w) = \left(-\frac{1}{2}\right)^{m+n+m'+n'} e^{|z|^2+|w|^2} \mathfrak{D}_{z,w}^{m,n,m',n'} \left(e^{-|z|^2-|w|^2}\right),$$

with

$$\mathfrak{D}_{z,w}^{m,n,m',n'} := \frac{\partial^m}{\partial(\bar{z} + i\bar{w})^m} \frac{\partial^n}{\partial(z + iw)^n} \frac{\partial^{m'}}{\partial(\bar{z} - i\bar{w})^{m'}} \frac{\partial^{n'}}{\partial(z - iw)^{n'}}.$$

They are also shown to be special eigenfunctions of certain partial differential operators,

including the Landau Hamiltonian

$$L = -\frac{1}{4} \left(\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial w \partial \bar{w}} + i \left(\frac{\partial^2}{\partial z \partial \bar{w}} - \frac{\partial^2}{\partial w \partial \bar{z}} \right) - 2 \left(z \frac{\partial}{\partial z} + w \frac{\partial}{\partial w} \right) - 2i \left(z \frac{\partial}{\partial w} - w \frac{\partial}{\partial z} \right) \right).$$

Additionally, we provide an integral representation of these polynomials, alternative generating functions, and further elucidate their properties with more comprehensive details.

Description of the content

The present document contains three chapters in addition to a general introduction to the studied subjects.

Chapter 1 deals with an extensive overview of different needed mathematical tools from the theory of special functions, like hypergeometric functions, and real and complex Hermite polynomials.

Chapter 2 is concerned with a systematic study of (p, q) -heat polynomials. Thus, by expanding the standard Gould-Hopper to two complex variables, we present a new family of holomorphic polynomials. The 1-D and 2-D holomorphic and polyanalytic Itô-Hermite polynomials are among the considered polynomials as special examples. We focus on investigating their recurrence relations, different generating functions, and operational representation. Additionally, we define several unique identities, such as formulas for Nielson types, Runge addition, and multiplication. Analysis is done on higher-order partial differential equations, and the relationship with hypergeometric functions and Gould-Hopper polynomials is looked into.

In Chapter 3, we provide a nontrivial class of bivariate poly-analytic Hermite polynomials in this study. Certain fundamental characteristics, such as a Rodrigues-type formula and three-term recurrence formulas, are demonstrated, and these polynomials are found to satisfy certain particular differential equations. Various exponential-type generating functions are found. Operational representations that are exponential and integral are obtained. Additionally, a few applications are covered.

Discussion & comments

By exploring properties of the the constructed orthogonal polynomials over \mathbb{C}^2 , we aim to investigate some immediate applications such as solving partial differential equations, analyzing integral transforms, and deriving properties of operators in $L^2(\mathbb{C}^2)$. Such applications could include solving boundary value problems, studying the behavior of physical

systems described by PDEs, analyzing complex networks or data structures, and developing efficient numerical algorithms for solving problems in various scientific and engineering domains. Overall, our exploration of $L^2(\mathbb{C}^2)$ and its associated bases and applications offers a rich and interdisciplinary journey into the realm of functional analysis, operator theory, PDEs, and applied mathematics.

Related published papers

- Two-dimensional (p, q) -heat polynomials of Gould-Hopper type. *J. Math. Anal. Appl.* 530 (2024), no. 1, Paper No. 127682, 18 pp.
- Bivariate poly-analytic Hermite polynomials. *Results Math.* 76 (2021), no. 1, Paper No. 3, 21 pp.

Preliminaries and Background

1.1 Needed tools

1.1.1 Gamma function

The Gamma function $\Gamma(z)$ is a fundamental special function that extends the factorial function to complex numbers. It is defined by the following integral for $\text{Re}(z) > 0$ [16, 48]:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt. \quad (1.1.1)$$

By a change of variable, the gamma function (for $\text{Re}(z) > 0$) can also be written as:

$$\Gamma(z) = 2 \int_0^{+\infty} u^{2z-1} e^{-u^2} du \quad \text{and} \quad \Gamma(z) = \int_0^1 (-\ln s)^{z-1} ds.$$

Fundamental properties of the Gamma function

The Gamma function has several key properties:

- **Recurrence relation:**

$$\Gamma(z + 1) = z\Gamma(z). \quad (1.1.2)$$

- **Reflection formula:**

$$\Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}. \quad (1.1.3)$$

- **Value at positive integers:**

$$\Gamma(n) = (n - 1)! \quad \text{for } n \in \mathbb{N}. \quad (1.1.4)$$

- **Euler's reflection formula:**

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \quad (1.1.5)$$

- **Duplication formula**

The duplication formula, also known as Legendre's duplication formula, is:

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z). \quad (1.1.6)$$

This is a useful identity in various applications of the Gamma function [1]. as a particular case we have

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi} \quad \text{for } n \in \mathbb{N} \quad (1.1.7)$$

- **Multiplication formula:**

The multiplication formula is a generalization of the duplication formula [43],

$$\Gamma(z)\Gamma\left(z + \frac{1}{m}\right)\Gamma\left(z + \frac{2}{m}\right)\cdots\Gamma\left(z + \frac{m-1}{m}\right) = (2\pi)^{(m-1)/2}m^{1/2-mz}\Gamma(mz) \quad (1.1.8)$$

- **Weierstrass formulation**

Weierstrass formulation of the Gamma function is:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}, \quad (1.1.9)$$

where γ is the Euler-Mascheroni constant. This formulation represents the Gamma function as an infinite product [51].

These properties are well-documented in various mathematical references [1, 51]. The incomplete Gamma functions are generalizations of the Gamma function. They are defined as follows:

Lower incomplete Gamma function

The lower incomplete Gamma function $\gamma(s, x)$ is defined by [48]:

$$\gamma(s, x) = \int_0^x t^{s-1}e^{-t} dt. \quad (1.1.10)$$

Upper incomplete Gamma function

The upper incomplete Gamma function $\Gamma(s, x)$ is defined by:

$$\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt. \quad (1.1.11)$$

Properties of incomplete Gamma functions

Some important properties of the incomplete Gamma functions include:

- **Relationship to Gamma function:**

$$\Gamma(s) = \gamma(s, x) + \Gamma(s, x). \quad (1.1.12)$$

- **Differentiation:**

$$\frac{\partial}{\partial x} \gamma(s, x) = x^{s-1} e^{-x}, \quad \frac{\partial}{\partial x} \Gamma(s, x) = -x^{s-1} e^{-x}. \quad (1.1.13)$$

The Pochhammer symbol (rising factorial) $(a)_n$ is defined by [10, 48]:

$$(a)_n = \begin{cases} 1 & \text{if } n = 0, \\ a(a+1) \dots (a+n-1) & \text{if } n > 0, \\ \frac{1}{(a-1)(a-2) \dots (a-|n|)}, \quad a \neq 1, 2, \dots, |n| & \text{if } n < 0, \end{cases} \quad (1.1.14)$$

Properties of the Pochhammer symbol

Key properties of the Pochhammer symbol include [10]:

- **Recursive property:**

$$a(a+1)_n = (a)_{n+1} \quad (1.1.15)$$

- **Addition formula**

$$(a)_{m+n} = (a)_m (a+m)_n \quad (1.1.16)$$

- **Reflection formula**

$$(a)_{-n} = \frac{(-1)^n}{(1-a)_n} \quad (1.1.17)$$

- **Duplication formula**

$$(2a)_{2n} = 2^{2n} (a)_n \quad (1.1.18)$$

- **Relation to Gamma function:**

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \quad (1.1.19)$$

The Gamma function can be expressed as a limit involving the Pochhammer symbol. The formula is given by[16]:

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{(z)_n}, \quad (1.1.20)$$

This limit provides a connection between the Gamma function and the Pochhammer symbol.

We conclude this section with the Bohr-Mollerup Theorem, which states:

Theorem (Bohr-Mollerup): If a function $f(x)$ satisfies the following three conditions, then it is identical to the Gamma function within its domain of definition:

1. $f(x + 1) = xf(x)$.
2. The domain of $f(x)$ includes all $x > 0$ and is log-convex for these x .
3. $f(1) = 1$.

For more details, we can see [16].

In conclusion, The Gamma function, incomplete Gamma functions, and Pochhammer symbols are essential tools in mathematical analysis, with numerous applications in various fields of mathematics and physics. Their fundamental properties and interrelationships make them important subjects of study in special functions.

1.1.2 Hypergeometric funtions

Hypergeometric functions are a class of special functions defined by hypergeometric series. These functions appear in various areas of mathematics and physics, including solutions to differential equations, combinatorial identities, and integrals [1].

The general hypergeometric function is defined as [51]:

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n z^n}{(b_1)_n (b_2)_n \cdots (b_q)_n n!}, \quad (1.1.21)$$

where $(a)_n$ is the Pochhammer symbol.

Conditions of existence: The series converges if [55]:

- $p \leq q$: for all finite z .
- $p = q + 1$: for $|z| < 1$, and also for $|z| = 1$ if $\Re(\sum_{i=1}^q b_i - \sum_{i=1}^p a_i) > 0$.
- $p > q + 1$: it diverges for any non-zero z .

Gauss hypergeometric function: The most well-known hypergeometric function is the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$:

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (1.1.22)$$

where it converges for $|z| < 1$ and can be analytically continued to other values of z [51].

Transformations: Several transformations exist for hypergeometric functions, including Euler's transformation [1]:

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z). \quad (1.1.23)$$

Integral representations: Hypergeometric functions can be represented as integrals. For example:

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \quad (1.1.24)$$

for $\Re(c) > \Re(b) > 0$ [55].

Differential equations: Hypergeometric functions satisfy a second-order linear differential equation. For ${}_2F_1(a, b; c; z)$, the equation is:

$$z(1-z) \frac{d^2 w}{dz^2} + [c - (a+b+1)z] \frac{dw}{dz} - abw = 0, \quad (1.1.25)$$

which is known as the hypergeometric differential equation [51].

Relations with classical orthogonal polynomials:

Hypergeometric functions are related to various classical orthogonal polynomials.

- **Laguerre Polynomials:** Laguerre polynomials $L_n^{(\alpha)}(x)$ can be expressed using hypergeometric functions:

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1(-n; \alpha+1; x), \quad (1.1.26)$$

where ${}_1F_1$ is the confluent hypergeometric function [1]. We recall the expression of $L_n^{(\alpha)}(x)$ is [51]:

$$L_n^{(\alpha)}(x) = (1+\alpha)_n \sum_{k=0}^n \frac{(-1)^k x^k}{k!(n-k)!(1+\alpha)_k}, \quad (1.1.27)$$

- **Jacobi polynomials:** Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ are related to hypergeometric func-

tions:

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} {}_2F_1\left(-n, 1 + \alpha + \beta + n; \alpha + 1; \frac{1-x}{2}\right), \quad (1.1.28)$$

which can be derived from the general properties of hypergeometric functions [55].

- **Chebyshev polynomials:** Chebyshev polynomials of the first kind $T_n(x)$ and second kind $U_n(x)$ can be written in terms of hypergeometric functions:

$$T_n(x) = {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right), \quad (1.1.29)$$

$$U_n(x) = {}_2F_1\left(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\right), \quad (1.1.30)$$

which follows from their definitions and generating functions [1].

In conclusion, Hypergeometric functions are a fundamental part of mathematical analysis, with numerous applications in physics and other sciences. Their connections to classical orthogonal polynomials further highlight their importance and versatility.

1.1.3 Useful summation techniques

The techniques outlined in this section is partially based on the reordering of terms in double (or multiple) summations. Such rearrangements in iterated infinite series can be justified in a straightforward manner when, for example, the series in question are absolutely convergent. we have the following identities, for any positive integer p ,

•

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{(k,n)} = \sum_{n=0}^{\infty} \sum_{k=0}^n a_{(k,n-k)} \quad (1.1.31)$$

•

$$\sum_{n=0}^{\infty} \sum_{k=0}^n a_{(k,n)} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{(k,n+k)}. \quad (1.1.32)$$

•

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{(k,n)} = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} a_{(k,n-pk)} \quad (1.1.33)$$

•

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} a_{(k,n)} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{(k,n+pk)}. \quad (1.1.34)$$

For more details see [43] page 100.

1.1.4 Exponential of an operator

The exponential of an operator A , denoted as e^A , is defined by the power series:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

where A^n is the n -th power of the operator A and A^0 is the identity operator.

The exponential of an operator is a powerful tool in mathematics calculus, for example the Taylor series for an analytic function $f(x)$ is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a),$$

where the series converges to corresponding values of f in a neighborhood of x . We can reformulate as following:

$$f(x+a) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} x^k = f(a+x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} a^k = \sum_{k=0}^{\infty} \frac{\left(\frac{d}{dx}\right)^{(k)}}{k!} a^k f(x).$$

Therefore, we get

$$f(x+a) = e^{a \frac{d}{dx}} f(x). \quad (1.1.35)$$

1.2 The Hermite polynomials

This chapter provides a comprehensive overview of various Hermite polynomials, including their Rodrigues formula, explicit expression, differential operators, generating functions, recursion relations, differential equations, expression in terms of hypergeometric functions, orthogonality, integral representation, addition formulas, complex Hermite functions, Mehler formul and other identities.

1.2.1 The classical Hermite polynomials

Hermite polynomials $H_n(x)$ are a classical orthogonal polynomial sequence. They play a significant role in probability, combinatorics, and mathematical physics, especially in the context of the quantum harmonic oscillator.

Rodriguez formula

The Hermite polynomials can be defined by the Rodriguez formula [56]:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \quad (1.2.1)$$

Operational formula

The Hermite polynomials can also be defined by the following operational formula:

$$H_n(x) = e^{-\partial_x} ((2x)^n). \quad (1.2.2)$$

Explicit expression

Hermite polynomials can also be expressed explicitly as [56]:

$$H_n(x) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!(n-2k)!} (2x)^{n-2k}. \quad (1.2.3)$$

Graph of some examples:

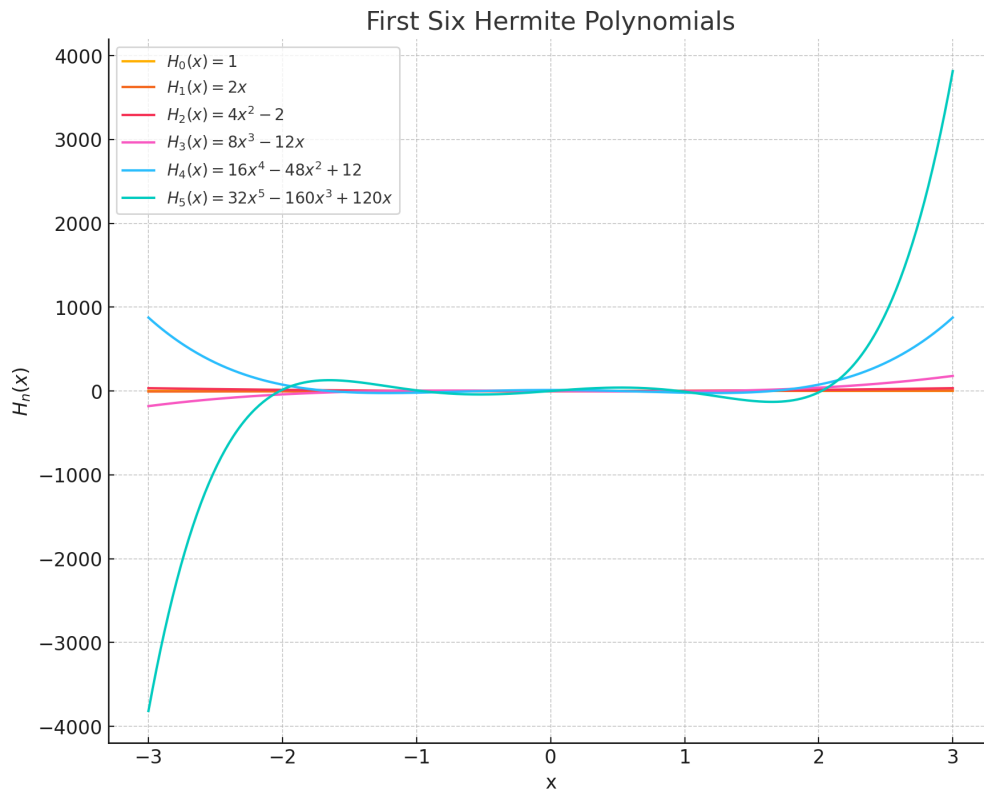


Figure 1.1: The first six Hermite polynomials: $H_0(x)$, $H_1(x)$, $H_2(x)$, $H_3(x)$, $H_4(x)$, and $H_5(x)$.

Generating functions

The generating function for Hermite polynomials is given by [56]:

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2}. \quad (1.2.4)$$

If we translate the polynomials by k (in indexes), we get the following one:

$$\sum_{n=0}^{\infty} H_{n+k}(x) \frac{t^n}{n!} = H_k(x-t)e^{2xt-t^2}. \quad (1.2.5)$$

The generating function for the even-indexed Hermite polynomials is [18]:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{2n}(x) = \frac{1}{\sqrt{1+4t}} \exp\left(\frac{4t}{1+4t}x^2\right). \quad (1.2.6)$$

The generating function for the odd-indexed Hermite polynomials is [18]:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{2n+1}(x) = \frac{1}{(\sqrt{1+4t})^{l+1}} H_l\left(\sqrt{\frac{1}{4t+1}}x\right) \exp\left(\frac{4t}{1+4t}x^2\right). \quad (1.2.7)$$

Another interesting generating function for Hermite polynomials is [56]:

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{[n/2]!} t^n = \frac{1+2xt+4t^2}{(1+4t^2)^{3/2}} \exp\left(\frac{4x^2t^2}{1+4t^2}\right), \quad (1.2.8)$$

The generating function of Hermite polynomials with Pochhammer factor is given by [?]:

$$\sum_{n=0}^{\infty} \frac{(c)_n}{n!} t^n H_n(x) = (1-2xt)^{-c} {}_2F_0\left(\begin{matrix} c/2, (c+1)/2 \\ - \end{matrix} \middle| \frac{-4t^2}{(1-2xt)^2}\right). \quad (1.2.9)$$

Recursion relations

Hermite polynomials satisfy the following recursion relations [56]:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad (1.2.10)$$

with initial conditions

$$H_0(x) = 1 \quad \text{and} \quad H_1(x) = 2x. \quad (1.2.11)$$

Burchnall formula

The Burchnall formula [9] for Hermite polynomials is given by:

$$(D - 2x)^n y = \sum_{r=0}^n (-1)^{n-r} \binom{n}{r} H_{n-r}(x) D^r y. \quad (1.2.12)$$

An interesting result that can be derived from the Burchnall formula is:

$$H_{m+n}(x) = m!n! \sum_{r=0}^{\min(m,n)} \frac{(-1)^r 2^r}{(m-r)!(n-r)!r!} H_{m-r}(x) H_{n-r}(x). \quad (1.2.13)$$

Differential equations

Hermite polynomials satisfy the differential equation [56]:

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0. \quad (1.2.14)$$

Realization by differential operator

The Hermite polynomials are eigenfunctions of the differential operator [56]:

$$\mathcal{H} = -\frac{d^2}{dx^2} + x^2, \quad (1.2.15)$$

such that:

$$\mathcal{H}H_n(x) = (2n + 1)H_n(x). \quad (1.2.16)$$

Expression in terms of hypergeometric functions

Hermite polynomials can be expressed in terms of the confluent hypergeometric function [56]:

$$H_n(x) = 2^n {}_1F_1 \left(-\frac{n}{2}; \frac{1}{2}; x^2 \right). \quad (1.2.17)$$

Integral representation

The integral representation of Hermite polynomials is:

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x + it)^n e^{-t^2} dt. \quad (1.2.18)$$

Orthogonality

Hermite polynomials are orthogonal with respect to the weight function e^{-x^2} [56]:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = \sqrt{\pi} 2^n n! \delta_{mn}. \quad (1.2.19)$$

As a particular case, we have:

$$\|H_n\|_{L^2(\mathbb{R}, e^{-x^2} dx)} = \sqrt{\pi} 2^n n!. \quad (1.2.20)$$

Some integrals involving Hermite polynomials

We have the values [56]:

$$\int_{-\infty}^{\infty} H_n(x) e^{-x^2} dx = 0 \quad \text{for odd } n. \quad (1.2.21)$$

And

$$\int_{-\infty}^{\infty} H_{2n}(x) e^{-x^2} dx = \frac{2^{2n} \sqrt{\pi}}{(2n)!} (2n-1)!!. \quad (1.2.22)$$

We have the following transformation [39]:

$$\int_{-\infty}^{\infty} H_n(2t) e^{-t^2 - 2tx} dt = (-1)^n e^{x^2} H_n(x). \quad (1.2.23)$$

More general, we have [?]:

$$\int_{-\infty}^{\infty} H_n(x) e^{-x^2 + bx} dx = \sqrt{\pi} (1 - b^2)^{n/2} H_n\left(\frac{b}{\sqrt{1 - b^2}}\right). \quad \text{for } b \notin \{-1, 1\}. \quad (1.2.24)$$

Real Hermite functions

The real Hermite functions are defined as [56]:

$$\psi_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x), \quad (1.2.25)$$

and form an orthonormal basis for $L^2(\mathbb{R})$, and they satisfy the differential equation:

$$\psi_n''(x) + (2n + 1 - x^2) \psi_n(x) = 0. \quad (1.2.26)$$

The Hermite functions $\psi_n(x)$ are a set of eigenfunctions of the continuous Fourier transform \mathcal{F} .

Graph of some examples:

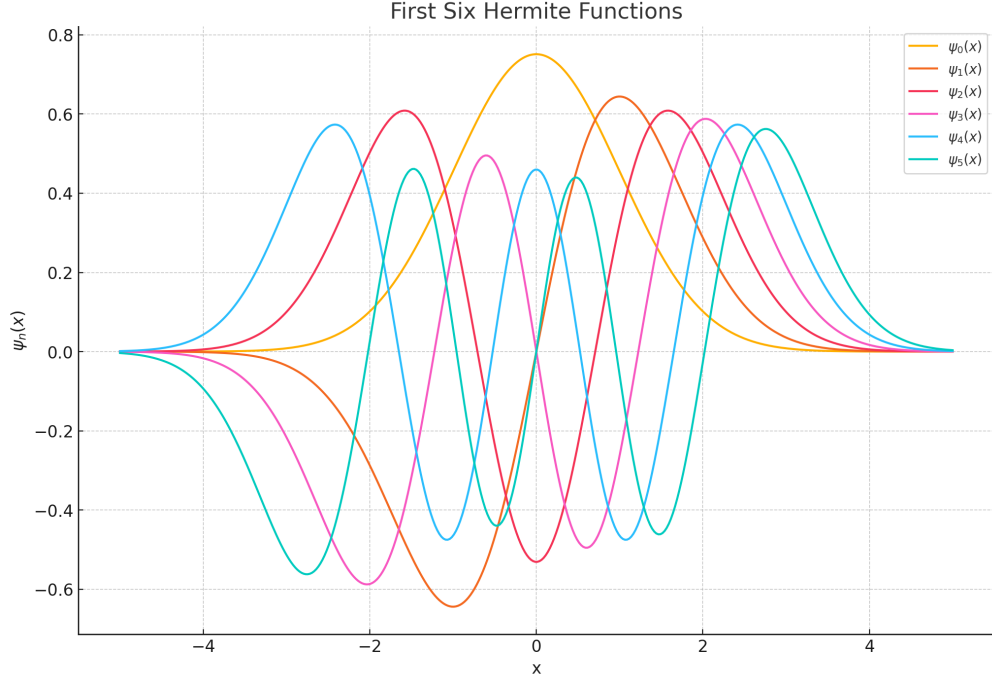


Figure 1.2: The first six Hermite functions: $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, $\psi_3(x)$, $\psi_4(x)$, and $\psi_5(x)$.

Mehler formula

Mehler formula provides a generating function for Hermite polynomials [56]:

$$\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{t^n}{2^n n!} = \frac{1}{\sqrt{1-t^2}} \exp\left(\frac{2txy - (x^2 + y^2)t^2}{1-t^2}\right). \quad (1.2.27)$$

Nielsen formula

The Nielsen formula provides an expansion of $H_{n+m}(x)$ in terms of Hermite polynomials [56]:

$$H_{n+m}(x) = 2^m \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} \frac{(2k)!}{k!} (2x)^{m-2k} H_n(x). \quad (1.2.28)$$

Addition formula

The addition formula for Hermite polynomials [56], called also Runge formula, is:

$$H_n(x+y) = \sum_{k=0}^n \binom{n}{k} H_k(x)H_{n-k}(y). \quad (1.2.29)$$

Homogeneity formula The homogeneity formula or the multiplication formula of the

polynomials H_n is given by [34] :

$$H_n(cx) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!(-1)^k}{k!(n-2k)!} (1-c^2)^k c^{n-2k} H_{n-2k}(x) \quad (1.2.30)$$

Linearization formula

The linearization formula for Hermite polynomials is [56]:

$$H_n(x)H_m(x) = \sum_{k=0}^{\min(n,m)} 2^k k! \binom{n}{k} \binom{m}{k} H_{n+m-2k}(x). \quad (1.2.31)$$

Relation with Laguerre polynomials The Hermite polynomials can be entirely recovered by Laguerre polynomials with the parameters $\alpha = \pm \frac{1}{2}$, we have [56] :

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{(-\frac{1}{2})}(x^2), \quad H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{(\frac{1}{2})}(x^2). \quad (1.2.32)$$

Relation with Jacobi polynomials We can express the Hermite polynomials as a limit of Jacobi ones, more precisely we have:

$$H_n(x) = \lim_{\lambda \rightarrow \infty} \lambda^{-n/2} n! P_n^{(\lambda)}(\lambda^{-\frac{1}{2}}x) \quad (1.2.33)$$

1.2.2 The Gould-Hopper polynomials

Definition

Gould and Hopper introduced the following polynomials in their work [20]:

The Gould-Hopper polynomial is defined as:

$$H_n^{(p)}(x|\gamma) = n! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \frac{\gamma^k}{k!} \frac{x^{n-pk}}{(n-pk)!} \quad (1.2.34)$$

An operational form of the Gould-Hopper polynomial is given by:

$$H_n^{(p)}(x|\gamma) = e^{\gamma \partial_x^p} (x^n) \quad (1.2.35)$$

Generating functions

Generating functions play a crucial role in studying the properties of polynomials. For Gould-Hopper polynomials, the generating function for the polynomials $H_n^{(p)}$ is given by:

$$\sum_{n=0}^{+\infty} H_n^{(p)}(x|\gamma) \frac{u^n}{n!} = e^{\gamma u^p + ux} \quad (1.2.36)$$

which will be useful in deriving additional properties.

We also define the following polynomials

$$P_k^n(z) = \sum_{j=0}^{k-1} (-1)^{(k-j)} (-n)_{(k-j)} \binom{k}{j} z^j \quad (1.2.37)$$

For any $j, k, p, q \in \mathbb{N}$ and $z, w, u, v, \gamma \in \mathbb{C}$, we have the following generating function:

$$\sum_{n=0}^{\infty} (n)_j H_n^{(p)}(x|\gamma) \frac{u^n}{n!} = ux \left((ux)^{j-1} + P_{j-1}^j(ux) \right) e^{xu + \gamma u^p} \quad (1.2.38)$$

Relation in terms of hypergeometric functions

The Gould-Hopper polynomials can be expressed in terms of hypergeometric functions as following :

$$H_n^{(p)}(x|\gamma) = x^n {}_pF_0 \left(\begin{matrix} \frac{-n}{p}, \frac{-n+1}{p}, \dots, \frac{-n+p-1}{p} \\ - \end{matrix} \middle| \frac{(-p)^p \gamma}{x^p} \right) \quad (1.2.39)$$

Using hypergeometric functions, we derive the following summation:

$$\sum_{n=0}^{\infty} (a)_n H_n^{(p)}(x|\gamma) \frac{u^n}{n!} = (1 - uz)^{-a} {}_pF_0 \left(\begin{matrix} \frac{a}{p}, \frac{a+1}{p}, \dots, \frac{a+p-1}{p} \\ - \end{matrix} \middle| \frac{p^p \gamma u}{(1 - uz)^p} \right) \quad (1.2.40)$$

Derivation and differential equations

The derivatives of the Gould-Hopper polynomials have significant properties, they are given by:

$$\partial_x H_n^{(p)}(x|\gamma) = n H_{n-1}^{(p)}(x|\gamma) \quad (1.2.41)$$

$$\partial_\gamma H_n^{(p)}(x|\gamma) = \frac{n!}{(n-p)!} H_{n-p}^{(p)}(x|\gamma) \quad (1.2.42)$$

The polynomials $H_n^{(p)}(x|\gamma_1, \dots, \gamma_p)$ satisfy the following generalized heat equation:

$$\partial_\gamma H_n^{(p)}(x|\gamma) = \partial_x H_n^{(p)}(x|\gamma) \quad (1.2.43)$$

We also have the following operational formula:

$$x^n = n! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \frac{(-\gamma)^k H_{n-pk}^{(p)}(x|\gamma)}{k! (n-pk)!} \quad (1.2.44)$$

This can be expressed as:

$$x^n = e^{-\gamma \partial_x^p} \left(H_n^{(p)}(z|\gamma) \right) \quad (1.2.45)$$

Recursion relations

The recursion relations are vital for iterative computation of the polynomials. The polynomials $H_{n,m}^{(p,q)}$ obey the recurrence relation:

$$H_{n+1}^{(p)}(x|\gamma) = x H_{n,m}^{(p)}(x|\gamma) + \gamma p! \binom{n}{p-1} H_{n+1-p}^{(p)}(x|\gamma) \quad (1.2.46)$$

An alternative form of the recurrence relation is:

$$H_{n+1}^{(p)}(x|\gamma) = (x + p\gamma \partial_x^{p-1}) H_n^{(p)}(x|\gamma) \quad (1.2.47)$$

For any $n, p = 0, 1, 2, \dots$, we have:

$$H_n^{(p)}(z, w|\gamma) = \begin{cases} e^{\gamma z^n}, & \text{if } p = 0, \\ (z + p\gamma \partial_z^{p-1})^n (1), & \text{if } p \geq 1 \text{ and } q = 0, \end{cases} \quad (1.2.48)$$

We have the following recursive formula:

$$H_n^{(p+1)}(x|\gamma) = n! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \sum_{j=0}^k 0^k \binom{k}{j} \gamma^k (-1)^{k-j} \frac{H_{n-j-pk}^{(p)}(x|\gamma)}{(n-j-pk)!} \quad (1.2.49)$$

This can be rewritten in an operational form as:

$$H_n^{(p+1)}(x|\gamma) = e^{\gamma(\partial_x - 1)\partial_x^p} H_n^{(p)}(x|\gamma) \quad (1.2.50)$$

For any positive integer q , we have the general formula:

$$H_n^{(p+q)}(x|\gamma) = e^{\gamma(\partial_x^q - 1)\partial_x^p} H_n^{(p)}(x|\gamma) \quad (1.2.51)$$

Homogeneity property

The homogeneity property of the polynomials is expressed as follows:

For any scalar a , we have:

$$a^n H_n^{(p)}(x|\gamma) = H_{n,m}^{(p,q)}(ax|\gamma a^p) \quad (1.2.52)$$

Taking the limit as t approaches 0, we find:

$$\lim_{t \rightarrow 0} \left(\frac{t}{2}\right)^n H_n^{(p)}\left(\frac{x}{t}\right) = x^n \quad (1.2.53)$$

We also have the following modification formula:

$$H_n^{(p)}(x|c\gamma) = n! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \frac{(c-1)^k \gamma^k}{k!} \frac{H_{n-pk}^{(p)}(x|\gamma)}{(n-pk)!} \quad (1.2.54)$$

In general, for any constants a and c , we have:

$$H_n^{(p)}(az|c\gamma) = n! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \frac{(c-a^p)^k \gamma^k}{k!} \frac{a^{n-pk}}{(n-pk)!} H_{n-pk}^{(p)}(z|\gamma) \quad (1.2.55)$$

Runge formulas

Runge formulas provide a way to combine polynomials, for the Gould-Hopper polynomials we have:

$$H_n^{(p)}(x+y|\gamma+\gamma') = \sum_{k=0}^n \binom{n}{k} H_k^{(p)}(x|\gamma) H_{n-k}^{(p)}(y|\gamma') \quad (1.2.56)$$

Specifically, we have:

$$H_n^{(p)}(x|\gamma) = \sum_{k=0}^n \binom{n}{k} H_k^{(p)}(z|2^{p-1}\gamma) H_{n-k}^{(p)}(z|2^{p-1}\gamma) \quad (1.2.57)$$

Another form of the Runge formula is:

$$x^n = \sum_{k=0}^n \binom{n}{k} H_k^{(p)}\left(\frac{x}{2}|\gamma\right) H_{n-k}^{(p)}\left(\frac{x}{2}|\gamma\right) \quad (1.2.58)$$

Neilson Identity

Neilson identity provides another useful relationship for these polynomials. We have:

$$H_{n+n'}^{(p)}(z|\gamma) = \sum_{i=0}^n \sum_{j=0}^{n'} \binom{n}{i} \binom{n'}{j} (z-z')^{i+j} H_{n+n'-i-j}^{(p)}(z'|\gamma) \quad (1.2.59)$$

Similarly, we find:

$$H_n^{(p)}(x+y|\gamma) = \sum_{i=0}^n \binom{n}{i} x^i w^j H_{n-i, m-j}^{(p)}(y|\gamma) \quad (1.2.60)$$

Moreover, we have:

$$H_n^{(p)}(x|\gamma) = 2^{n+m} \sum_{i=0}^n \binom{n}{i} z^i H_{n-i, m-j}^{(p)}(x|2^{p-1}\gamma) \quad (1.2.61)$$

Integral representation

Haimo and Markett have studied, in [28, 29], a class of polynomials that have nearly the same form as the Gould-Hopper polynomials:

$$p_{n,q}(x, t) = n! \sum_{k=0}^{\lfloor \frac{n}{2q} \rfloor} (-1)^{(q+1)k} \frac{x^{n-2qk}}{(n-2qk)!} \frac{t^k}{k!} \quad (1.2.62)$$

We have the relation:

$$P_{n,q}(x, t) = (-i)^n H_n^{(2q)}(ix|(-1)^q t) \quad (1.2.63)$$

where $t > 0$, and they have proved that these polynomials have the following integral representation:

$$p_{n,q}(x, (-1)^q t) = \int_{-\infty}^{\infty} G_q(y + ix, t) (iy)^n dy \quad (1.2.64)$$

where G_q is defined by:

$$G_q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ts^{2q} + ixs} ds \quad (1.2.65)$$

Then we could deduce that the polynomials $H_n^{(2q)}(ix|(-1)^q t)$ can be expressed as an integral as follows:

$$H_n^{(2q)}(ix|(-1)^q \gamma) = (-1)^n \int_{-\infty}^{\infty} G_q(u - x, t) u^n du \quad (1.2.66)$$

For the polynomials H_{2q+1} , we are working on it in independent work.

1.2.3 The complex Hermite polynomials

Definitions : The complex Hermite polynomials, or the poly-analytic Hermite polynomials, had introduced By Itô in [?] by the following explicit expression:

$$H_{n,m}(z, \bar{z}) = n!m! \sum_{k=0}^{n \wedge m} \frac{(-1)^k}{k!} \frac{z^{n-k}}{(n-k)!} \frac{\bar{z}^{m-k}}{(m-k)!} \quad (1.2.67)$$

Where $n \wedge m = \min(n, m)$. as particular cases we have:

$$H_{n,0}(z, \bar{z}) = z^n \quad (1.2.68)$$

$$H_{0,m}(z, \bar{z}) = \bar{z}^m \quad (1.2.69)$$

an equivalent definition by operational operator is :

$$H_{n,m}(z, \bar{z}) = e^{-\partial_z \partial_{\bar{z}}} \{z^n \bar{z}^m\} \quad (1.2.70)$$

The Rodriguez formula of the polynomials $H_{n,m}(z, \bar{z})$ is obtained as following:

$$H_{n,m}(z, \bar{z}) = e^{|z|} \partial_z^m \partial_{\bar{z}}^n e^{-|z|} \quad (1.2.71)$$

Expression in terms of hypergeometric functions:

Complex Hermite polynomials can be expressed in terms of hypergeometric functions:

$$H_{n,m}(z, \bar{z}) = (-1)^m m! \binom{n}{m} z^{n-m} {}_1F_1(-m, n-m+1, |z|^2).$$

Generating functions the partial generating functions of $H_{n,m}(z, \bar{z})$ (see [22]) is given by

$$\sum_{n=0}^{+\infty} H_{n,m}(z, \bar{z}) \frac{u^n}{n!} = (\bar{z} - u)^m e^{uz} \quad (1.2.72)$$

$$\sum_{m=0}^{+\infty} H_{n,m}(z, \bar{z}) \frac{u^n v^m}{n! m!} = (z - v)^n e^{v\bar{z}} \quad (1.2.73)$$

Then it's easy to deduce the generating function of the polynomials $H_{n,m}(z, \bar{z})$, we have:

$$\sum_{n,m=0}^{+\infty} H_{n,m}(z, \bar{z}) \frac{u^n v^m}{n! m!} = e^{-uv+uz+v\bar{z}} \quad (1.2.74)$$

by noting that

$$e^{|z|} e^{-v\partial_z - u\partial_{\bar{z}}} e^{-|z|} = e^{-uv+uz+v\bar{z}} \quad (1.2.75)$$

The generating function of translated complex polynomials $H_{m+j, n+k}(z, \bar{z})$ is [36]:

$$\sum_{n,m=0}^{\infty} H_{n+j, m+k}(z, \bar{z}) \frac{u^n v^m}{n! m!} = (-1)^{j+k} e^{uz+v\bar{z}-uv} H_{j,k}(z-v, \bar{z}-\bar{v}). \quad (1.2.76)$$

And the generating function of $H_{m+j, n+k}(z, \bar{z})$ with Pochhammer coefficients is given by

[36]:

$$\sum_{n,m=0}^{\infty} \frac{(a)_n (b)_m}{n!m!} u^n v^m H_{n,m}(z, \bar{z}) = (1 - uz)^{-a} (1 - v\bar{z})^{-b} {}_2F_0 \left(\begin{matrix} a, b \\ - \end{matrix} \middle| \frac{uv}{(1 - uz)(1 - v\bar{z})} \right). \quad (1.2.77)$$

Recursion relations and PDEs:

The poly-analytic polynomials $H_{n,m}(z, \bar{z})$ satisfy the following recursion relations

$$H_{n+1,m}(z, \bar{z}) = zH_{n,m}(z, \bar{z}) - mH_{n,m-1}(z, \bar{z}) \quad (1.2.78)$$

and

$$H_{n,m+1}(z, \bar{z}) = \bar{z}H_{n,m}(z, \bar{z}) - nH_{n-1,m}(z, \bar{z}) \quad (1.2.79)$$

The first relation can be proved by using the generating function 1.2.74 (see [?]) or by using operational formulas (see [22]), and for the second one it's useful to apply the following symmetry property :

$$H_{n,m}(z, \bar{z}) = H_{m,n}(\bar{z}, z) \quad (1.2.80)$$

The partial derivatives of $H_{n,m}(z, \bar{z})$ can be found easily by using its generating function 1.2.74, we have:

$$\partial_z H_{n,m}(z, \bar{z}) = nH_{n-1,m}(z, \bar{z}) \quad (1.2.81)$$

$$\partial_{\bar{z}} H_{n,m}(z, \bar{z}) = mH_{n,m-1}(z, \bar{z}) \quad (1.2.82)$$

The recursion relations 1.2.78 and 1.2.79 can be expressed as

$$H_{n+1,m}(z, \bar{z}) = (z - \partial_{\bar{z}})H_{n,m}(z, \bar{z}) \quad (1.2.83)$$

and

$$H_{n,m+1}(z, \bar{z}) = (\bar{z} - \partial_z)H_{n,m}(z, \bar{z}) \quad (1.2.84)$$

and by iterating

$$H_{n+1,m}(z, \bar{z}) = (z - \partial_{\bar{z}})^n (z^m) \quad (1.2.85)$$

and

$$H_{n,m+1}(z, \bar{z}) = (\bar{z} - \partial_z)^m (\bar{z}^n) \quad (1.2.86)$$

Then the Complex Hermite polynomials can be realized using differential operators as

follows:

$$H_{n,m}(z, \bar{z}) = (\bar{z} - \frac{\partial}{\partial z})^n (z - \frac{\partial}{\partial \bar{z}})^m \cdot 1. \quad (1.2.87)$$

Now by combining the last equations and the recursion relations 1.2.78 and 1.2.79, the polynomials $H_{n,m}(z, \bar{z})$ satisfy the following partial derivative equation:

$$\partial_z \partial_{\bar{z}} H_{n,m}(z, \bar{z}) - z H_{n,m}(z, \bar{z}) - n H_{n,m}(z, \bar{z}) = 0 \quad (1.2.88)$$

and

$$\partial_z \partial_{\bar{z}} H_{n,m}(z, \bar{z}) - \bar{z} H_{n,m}(z, \bar{z}) - m H_{n,m}(z, \bar{z}) = 0 \quad (1.2.89)$$

Burchnall formula:

Similarly to Hermite polynomials $H_n(x)$, the complex Hermite polynomials $H_{n,m}(z, \bar{z})$ have a Burchnall formula, more precisely in [22], for a given function f in two variables z and \bar{z} we have:

$$e^{z\bar{z}} \partial_z^n \partial_{\bar{z}}^m e^{-z\bar{z}} f(z, \bar{z}) = (-1)^{n+m} n! m! \sum_{i=0, k=0}^{n,m} \frac{(-1)^{i+k}}{i! k!} \frac{H_{n-i, m-k}(z, \bar{z})}{(n-i)! (m-k)!} \partial_z^i \partial_{\bar{z}}^k f(z, \bar{z}) \quad (1.2.90)$$

And in the same article, we have also the following identity:

$$e^{z\bar{z}} \partial_z^n e^{-z\bar{z}} f(z, \bar{z}) = (z - \partial_z)^n (f) \quad (1.2.91)$$

Now by combining (1.2.90) and (1.2.91), and by considering the symmetry property we get the suitable Burchnall formula :

$$(z - \partial_z)^n (\bar{z} - \partial_{\bar{z}})^m (f) = (-1)^{n+m} n! m! \sum_{i=0, k=0}^{n,m} \frac{(-1)^{i+k}}{i! k!} \frac{H_{n-i, m-k}(z, \bar{z})}{(n-i)! (m-k)!} \partial_z^i \partial_{\bar{z}}^k f(z, \bar{z}) \quad (1.2.92)$$

An interest particular case is when $f = 1$, the complex Hermite polynomials can be realized by:

$$H_{n,m}(z, \bar{z}) = (z - \partial_z)^n (\bar{z} - \partial_{\bar{z}})^m (1) \quad (1.2.93)$$

Noting that we could find the last expression of $H_{n,m}(z, \bar{z})$ by applying the recursion relations (1.2.83) and (1.2.84).

Rung formula:

The Runge formula or the addition formula, see [22] of the polynomials $H_{n,m}(z, \bar{z})$ is given

by :

$$H_{n,m} \left(\frac{z+w}{\sqrt{2}}, \frac{\bar{z}+\bar{w}}{\sqrt{2}} \right) = n!m! \left(\frac{1}{\sqrt{2}} \right)^{n+m} \sum_{i=0, k=0}^{n,m} \frac{H_{i,k}(z, \bar{z})}{i!k!} \frac{H_{n-i, m-k}(w, \bar{w})}{(n-i)!(m-k)!} \quad (1.2.94)$$

The Runge formula can be proved by many ways, we can use the generating function.

Neilson formula:

The Neilson formula is a generalization of the recursion relations, for the complex Hermite polynomials we have [23]:

$$H_{n+p, m+q}(z, \bar{z}) = n!m!p!q! \sum_{j=0}^{\min(n,q)} \sum_{k=0}^{\min(m,p)} \frac{(-1)^{j+k}}{j!k!} \frac{H_{n-j, m-k}(z, \bar{z})}{(n-j)!(m-k)!} \frac{H_{p-k, q-j}(z, \bar{z})}{(p-k)!(q-j)!}. \quad (1.2.95)$$

Linearization of product:

We can linear the product of two complex Hermite polynomials as follow [36]:

$$H_{n_1, m_1}(z, \bar{z}) H_{n_2, m_2}(z, \bar{z}) = \sum_{j,k} \frac{m_1!m_2! H_{n_1+n_2-j-k, m_1+m_2-j-k}(z, \bar{z})}{j!k! (n_1-j)! (m_1-k)! (n_2-k)! (m_2-j)!} \quad (1.2.96)$$

Homogeneity identity:

The polynomials $H_{m,n}(cz, \bar{c}\bar{z})$ satisfy the following property:

$$H_{n,m}(cz, \bar{c}\bar{z}) = \sum_{j=0}^{n \wedge m} H_{n-j, m-j}(z, \bar{z}) \frac{n!m! c^{n-j} \bar{c}^{m-j}}{j!(n-j)!(m-j)!} (c\bar{c} - 1)^j. \quad (1.2.97)$$

Integral representation:

The complex Hermite polynomials can be realized by he following integral [36]:

$$H_{n,m}(iz, i\bar{z}) = \frac{i^{n+m}}{\pi} \int_{\mathbb{R}^2} (r+is)^n (r-is)^m \exp \left(-(r-x)^2 - (s-y)^2 \right) drds. \quad (1.2.98)$$

Orthogonality properties:

The complex Hermite polynomials satisfy the following orthogonality property [32]:

$$\frac{1}{\pi} \int_{\mathbb{R}^2} H_{n,m}(x+iy, x-iy) \overline{H_{p,q}(x+iy, x-iy)} e^{-x^2-y^2} dx dy = n!m! \delta_{n,p} \delta_{m,q}. \quad (1.2.99)$$

and they form a orthogonal basis of $L^2(\mathbb{C}, e^{-x^2-y^2} dx dy)$.

Complex Hermite functions:

The complex Hermite functions are defined as:

$$\psi_{n,m}(z, \bar{z}) = \frac{H_{n,m}(z, \bar{z})}{\sqrt{\pi n! m!}} e^{-|z|^2/2}. \quad (1.2.100)$$

and they form an orthonormal basis of the space of $L^2(\mathbb{C})$.

Mehler formula:

The Mehler formula is corresponding to a reproducing kernel of some Hilbert spaces, For the complex Hermite polynomials we cite two of its variants. In [23], Ghanmi has establish the Mehler formulas for the univariate Hermite polynomials $H_{n,m}^\mu(z, \bar{z})$ then for $\mu = 1$ we find those of $H_{n,m}(z, \bar{z})$. More precisely, For every $u, z \in \mathbb{C}$ such that $|u| < 1$, we have the first one is giving by:

$$\sum_{n,m=0}^{+\infty} \frac{u^n H_{n,m}(z, \bar{z}) \overline{H_{n,m}(w, \bar{w})}}{n! m!} = \frac{e^{\langle w, z \rangle}}{(1-u)} \exp\left(\frac{-u|z-w|^2}{1-u}\right) \quad (1.2.101)$$

And the second :

$$\sum_{n,m=0}^{+\infty} \frac{u^n v^m H_{n,m}(z, \bar{z}) \overline{H_{n,m}(w, \bar{w})}}{n! m!} = \frac{1}{1-uv} \exp\left(-\frac{[(|z|^2 + |w|^2) uv - uz\bar{w} - v\bar{z}w]}{1-uv}\right) \quad (1.2.102)$$

Relation to the generalized Laguerre polynomials:

The complex Hermite polynomials $z = |z|e^{i \arg(z)}$ can be expressed in terms of the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, by using the exponential form of $z = |z|e^{i \arg(z)}$, see [32, 22], as following:

$$H_{n,m}(z, \bar{z}) = (-1)^{\min(n,m)} (\min(n, m))! |z|^{n-m} e^{i(n-m) \arg(z)} L_{\min(n,m)}^{(|n-m|)}(|z|^2) \quad (1.2.103)$$

we have, as special case for $q = p + k$, the following expression :

$$H_{n,m+k}(z, \bar{z}) = (-1)^n n! \bar{z}^k L_p^{(k)}(|z|^2) \quad (1.2.104)$$

Two-dimensional (p, q) -heat polynomials of Gould–Hopper type

2.1 Introduction

The classical Hermite polynomials are defined by

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2x)^{n-2k}}{k! (n-2k)!}$$

and are well studied in the literature. Different generalizations to the multivariate setting have interesting applications in many branches of mathematics, physics, and engineering and have been widely studied in the literature [30, 35, 36, 47, 56, 34]. Notice for instance that the tensor product $H_m(x)H_n(y)$ as well as the holomorphic Hermite polynomials $H_n(z)$ are specific 2D generalizations of $H_n(x)$ to the complex plane \mathbb{C} . The first class is an orthogonal basis of $L^2(\mathbb{R}^2; e^{-x^2-y^2} dx dy)$, while the functions $e^{-z^2/2} H_m(z)$ form an orthogonal basis of a Bargmann–Fock-like space [61, 7]. For their analytic and combinatoric properties, one can refer to [56, 35].

The polyanalytic analogs are the complex Itô–Hermite polynomials

$$H_{n,m}(z, \bar{z}) := (-1)^{m+n} e^{|z|^2} \partial_{\bar{z}}^n \partial_z^m \left(e^{-|z|^2} \right) = n! m! \sum_{j=0}^{n \wedge m} \frac{(-1)^j}{j!} \frac{z^{n-j} \bar{z}^{m-j}}{(n-j)! (m-j)!} \quad (2.1.1)$$

where $n \wedge m = \min\{n, m\}$, are solutions of the iterated Cauchy–Riemann equation

$$2^{n+1} \partial_{\bar{z}}^{n+1} f = (\partial_x + i\partial_y)^{n+1} f = 0.$$

These polynomials, introduced by Itô [31] in the context of the complex Markov process, are basic tools in the nonlinear analysis of traveling wave tube amplifiers [5]. More specifically, they appear in the calculation of the effects of non-linearity on broadband radio frequencies in communication systems. For their properties and applications, one can refer to [21, 22, 35, 15, 23]. Their holomorphic counterparts $H_{n,m}(z, w)$ and $H_{n,m}(z, w, x)$ were introduced and studied recently in [27, 35, 41], respectively.

Another class of generalized Hermite polynomials is referred to as Gould–Hopper [20] and defined by

$$H_n^{(p)}(z|\gamma) = n! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \frac{\gamma^k}{k!} \frac{z^{n-pk}}{(n-pk)!}. \quad (2.1.2)$$

These polynomials enter into the study of the Novikov–Veselov equation [11]. The specific ones $H_n^{(2p)}(x|(-1)^{p+1}t)$ are solutions of the higher-order heat equation associated with $(-1)^{q+1}\partial^{2q}/\partial x^{2q}$ (see e.g., [?, 20, 42]).

Analogously to (2.1.1), a natural extension of $H_n^{(p)}(z|\gamma)$ in (2.1.2) to two complex variables is suggested by the following

$$H_{n,m}^{(p,q)}(z, w|\gamma) = n!m! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor \wedge \lfloor \frac{m}{q} \rfloor} \frac{\gamma^k}{k!} \frac{z^{n-pk}}{(n-pk)!} \frac{w^{m-qk}}{(m-qk)!}. \quad (2.1.3)$$

We call them here two-dimensional complex (p, q) -heat polynomials of Gould–Hopper type. Obviously, the particular cases of the considered polynomials reduce to the classes mentioned above, the real, Gould–Hopper, as well as the 1-D and 2-D holomorphic, and Itô–Hermite polyanalytic complex polynomials. More precisely, we have

1. $H_{n,0}^{(2,0)}(2z, 1| -1) = H_n(z)$, the holomorphic Hermite polynomials.
2. $H_{n,0}^{(p,0)}(z, 1|\gamma) = H_n^{(p)}(z|\gamma)$ and $H_{n,m}^{(p,0)}(z, w|\gamma) = H_{n,m}^{(0,p)}(z, w|\gamma) = w^m H_n^{(p)}(z|\gamma)$, where $H_n^{(p)}(z|\gamma)$ denotes the Gould–Hopper polynomials in (2.1.2).
3. $H_{n,m}^{(1,1)}(z, w| -1) = H_{n,m}(z, w)$ and $H_{n,m}^{(1,1)}(z, \bar{z}| -1) = H_{n,m}(z, \bar{z})$, the 2-D holomorphic Hermite polynomials and their restriction to the non-analytic surface $w = \bar{z}$, respectively.
4. $H_{n,m}^{(1,1)}(z, w|\gamma) = H_{n,m}(z, w, \gamma)$, the Hermite polynomials considered in [41].

Even if this extension is natural, they give rise to new classes of polynomials. Their algebraic properties seem to be derived in a standard way. However, there is no evidence to

suggest exact formulas for their generating functions or the partial differential equations they obey. Furthermore, their analytical properties, including orthogonality, description of associated functional spaces, and integral transforms, are not an easy task. The study of these polynomials $H_{n,m}^{(p,q)}(z, w|\gamma)$ will contribute to providing eventual representations for the solutions of the partial differential equation in the (z, w) -plane

$$c_{p,q} \frac{\partial^{p+q}}{\partial z^p \partial w^q} u(z, w; \gamma) = \frac{\partial}{\partial \gamma} u(z, w; \gamma) \quad (2.1.4)$$

with initial data. Other motivations for considering $H_{n,m}^{(p,q)}(x, y|\gamma)$ include potential applications in quantum mechanics, combinatorics, or applied mathematics. Mainly, they can be used in evaluating transition matrix elements, studying the root dynamics of the σ -flows associated with (2.1.4) similarly to [59], or computing the higher-order moments of a given distribution.

In the present paper, we provide a complete unified description of the basic properties of $H_{n,m}^{(p,q)}(z, w|\gamma)$ in (2.1.3). More precisely, in Section 2 we are concerned with their operational and hypergeometric representations (Subsection 2.2.1), generating functions (Subsection 2.2.2), multiplication formulas (Subsection 2.2.3), and their connection to Gould–Hopper polynomials (Subsection 2.2.4). Different types of recurrence relations are discussed in Section 2.3. The analog of the magnetic Laplacians in connection with $H_{n,m}^{(p,q)}(z, w|\gamma)$ is given in Section ??.

2.2 Complex (p, q) -Heat Polynomials of Gould–Hopper Type

2.2.1 Equivalent Definitions

The two-dimensional complex (p, q) -heat polynomials of Gould–Hopper type we deal with are defined by (2.1.3) with the convention that $H_{0,0}^{(0,0)}(z, w|\gamma) = e^\gamma$ and $\lfloor \frac{j}{k} \rfloor = +\infty$ when $k = 0$. Notice for instance that the monomials $z^n w^m$ can be recovered by taking $\gamma = 0$ or $n < p$ and $m < q$, $H_{n,m}^{(p,q)}(z, w|\gamma) = z^n w^m$, while for $n = p$ and $m = q$, we have $H_{p,q}^{(p,q)}(z, w|0) = z^p w^q + \gamma p! q!$.

Notice also that for $z = w = 0$ we get

$$H_{n,m}^{(p,q)}(0, 0|\gamma) = \frac{n!}{\lfloor \frac{n}{p} \rfloor!} \gamma^{\lfloor \frac{n}{p} \rfloor} \delta_{n-p \lfloor \frac{n}{p} \rfloor, m-q \lfloor \frac{m}{q} \rfloor}$$

when $p \mid n$ and $q \mid m$. Otherwise, we have $H_{n,m}^{(p,q)}(0, w|\gamma) = 0$ if $p \nmid n$ and $H_{n,m}^{(p,q)}(z, 0|\gamma) = 0$

if $q \nmid m$. The case $p = 0$ gives rise to

$$H_{0,qm}^{(0,q)}(0,0|\gamma) = \frac{(qm)!}{m!} \gamma^m$$

and $H_{0,qm+j}^{(0,q)}(0,0|\gamma) = 0$ when $j = 1, \dots, q-1$. This recoups the well-known identity for the real Hermite polynomials $H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$ and $H_{2n+1}(0) = 0$.

Furthermore, it should be mentioned here that the polynomials $H_{n,m}^{(p,q)}(z,w|\gamma)$ satisfy the symmetry property

$$H_{n,m}^{(p,q)}(z,w|\gamma) = H_{m,n}^{(q,p)}(w,z|\gamma). \quad (2.2.1)$$

An equivalent definition of $H_{p,q}^{(p,q)}(z,w|\gamma)$, which offers a broader direction by generalizing e^{∂_z} to several variables and to higher order in the exponent, is given by the following operational formula.

Proposition 2.2.1. *We have*

$$H_{n,m}^{(p,q)}(z,w|\gamma) = e^{\gamma \partial_z^p \partial_w^q} (z^n w^m). \quad (2.2.2)$$

Proof. The result is clear for $p = q = 0$. The particular case of $p = 0$ and $q \geq 1$ immediately follows from the one corresponding to $p \geq 1$ and $q = 0$ by the symmetry property (2.2.1). For the latter one ($p \geq 1$ and $q = 0$), we use [14, Eq. (6)] to get

$$e^{\gamma \partial_z^p} (z^n w^m) = w^m e^{\gamma \partial_z^p} \{z^n\} = w^m H_n^{(p)}(z|\gamma) = H_{n,m}^{(p,0)}(z,w|\gamma).$$

The result for arbitrary $p \geq 1$ and $q \geq 1$ follows by making use of the fact that

$$\partial_z^{pk} (z^n) = \frac{n! z^{n-pk}}{(n-pk)!}$$

for $k \leq \left\lfloor \frac{n}{p} \right\rfloor$ and vanishes otherwise, so that one obtains

$$\begin{aligned} e^{\gamma \partial_z^p \partial_w^q} (z^n w^m) &= \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} (\partial_z^p \partial_w^q)^k \{z^n w^m\} \\ &= n! m! \sum_{k=0}^{\left\lfloor \frac{n}{p} \right\rfloor \wedge \left\lfloor \frac{m}{q} \right\rfloor} \frac{\gamma^k}{k!} \frac{z^{n-pk}}{(n-pk)!} \frac{w^{m-qk}}{(m-qk)!} \\ &= H_{n,m}^{(p,q)}(z,w|\gamma). \end{aligned}$$

■

The next result gives the representation of $H_{n,m}^{(p,q)}(z, w|\gamma)$ in terms of the hypergeometric function

$${}_rF_s \left(\begin{matrix} a_1, a_2, \dots, a_r \\ c_1, c_2, \dots, c_s \end{matrix} \middle| x \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{(c_1)_k (c_2)_k \cdots (c_s)_k} \frac{x^k}{k!}.$$

Proposition 2.2.2. *We have*

$$H_{n,m}^{(p,q)}(z, w|\gamma) = z^n w^m {}_{p+q}F_0 \left(\begin{matrix} \frac{-n}{p}, \frac{-n+1}{p}, \dots, \frac{-n+p-1}{p}, \frac{-m}{q}, \frac{-m+1}{q}, \dots, \frac{-m+q-1}{q} \\ - \end{matrix} \middle| \frac{(-p)^p (-q)^q \gamma}{z^p w^q} \right). \quad (2.2.3)$$

Proof. Starting from (2.1.3), we can rewrite the explicit expression of $H_{n,m}^{(p,q)}$ in the following form

$$H_{n,m}^{(p,q)}(z, w|\gamma) = n!m!z^n w^m \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor \wedge \lfloor \frac{m}{q} \rfloor} \frac{\left(\frac{\gamma}{z^p w^q}\right)^k}{k!(n-pk)!(m-qk)!}.$$

Making appeal to the identity [?, Eq. (21), p. 21] as well as the Gauss multiplication theorem [?, Eq. (26), p. 23],

$$(n-pk)! = \frac{(-1)^{pk} n!}{(-p)^{pk} \prod_{j=1}^p \left(\frac{j-1-n}{p}\right)_k},$$

we can rewrite the involved factorial $(n-pk)!$ as

$$(n-pk)! = \frac{(-1)^{pk} n!}{(-p)^{pk} \prod_{j=1}^p \left(\frac{j-1-n}{p}\right)_k}.$$

Therefore, one obtains

$$H_{n,m}^{(p,q)}(z, w|\gamma) = z^n w^m \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor \wedge \lfloor \frac{m}{q} \rfloor} \prod_{j=0}^{p-1} \left(\frac{-n+j}{p}\right)_k \prod_{j=0}^{q-1} \left(\frac{-m+j}{q}\right)_k \frac{\left(\frac{(-1)^{p+q} p^p q^q \gamma}{z^p w^q}\right)^k}{k!}.$$

This is exactly the desired result (2.2.3). ■

Remark 2.2.3. *By means of the fact*

$${}_2F_0 \left(\begin{matrix} -n, -m \\ - \end{matrix} \middle| \frac{-1}{z} \right) = \frac{z^{-(n \wedge m)} (n \vee m)!}{(|n-m|)!} {}_1F_1 \left(\begin{matrix} -(n \wedge m) \\ |n-m|+1 \end{matrix} \middle| z \right), \quad (2.2.4)$$

where $n \vee m = \max(n, m)$, we can express the classical classes of Hermite polynomials described in the introductory section in terms of the confluent hypergeometric function ${}_1F_1$.

2.2.2 Generating functions.

In this section, we derive several generating functions for the polynomials $H_{n,m}^{(p,q)}$.

Proposition 2.2.4. *We have*

$$\sum_{n=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^n}{n!} = H_m^{(q)}(w|u^p\gamma) e^{zu}, \quad (2.2.5)$$

$$\sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{v^m}{m!} = H_n^{(p)}(z|v^q\gamma) e^{wv}, \quad (2.2.6)$$

and

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^n v^m}{n! m!} = e^{zu+wv+\gamma u^p v^q}. \quad (2.2.7)$$

Proof. The proof of (2.2.5) lies on the operational realization in (2.2.2). Indeed, we have

$$\sum_{n=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^n}{n!} = e^{\gamma \partial_z^p \partial_w^q} (e^{zu} w^m) = e^{\gamma u^p \partial_w^q} (w^m) e^{zu} = H_m^{(q)}(w|u^p\gamma) e^{zu}.$$

The assertion in (2.2.6) can be obtained by the symmetry (2.2.1). Subsequently, it follows from (2.2.5) that

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^n v^m}{n! m!} &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^n}{n!} \right) \frac{v^m}{m!} \\ &= \sum_{m=0}^{\infty} H_m^{(q)}(w|u^p\gamma) \frac{v^m}{m!} e^{zu}. \end{aligned}$$

Therefore, the result in (2.2.7) then follows making use of the generating function for the Gould–Hopper polynomials [13, p.72]

$$\sum_{m=0}^{\infty} H_m^{(q)}(w|\gamma) \frac{v^m}{m!} = e^{wv+\gamma v^q}. \quad (2.2.8)$$

Remark 2.2.5. For $p = q = 1$, we retrieve from (2.2.6) the partial generating functions for the complex Hermite polynomials given through [22, Proposition 3.4], while (2.2.7) generalizes the corresponding formula for the Hermite polynomials defined in the introductory section. ■

As an immediate consequence of the generating function (2.2.7) we establish the following identity needed to prove some coming results.

Corollary 2.2.6. *We have*

$$a^n b^m H_{n,m}^{(p,q)}(z, w|\gamma) = H_{n,m}^{(p,q)}(az, bw|\gamma a^p b^q). \quad (2.2.9)$$

Proof. Identity (2.2.9) follows by identification in

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a^n b^m H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^n v^m}{n! m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(az, bw|\gamma a^p b^q) \frac{u^n v^m}{n! m!},$$

which can be obtained starting from the observation that

$$\exp(auz + bvw + \gamma(au)^p (bv)^q) = \exp(auz + bvw + (\gamma a^p b^q) u^p v^q)$$

combined with (2.2.7). ■

The previous result is the key tool in extending the classical limit identities [61, 27]

$$\lim_{t \rightarrow 0} \left(\frac{t}{2}\right)^n H_n\left(\frac{z}{t}\right) = z^n \quad \text{and} \quad \lim_{t \rightarrow 0} t^{m+n} H_{m,n}\left(\frac{z}{t}, \frac{w}{t}\right) = z^m w^n$$

to the (p, q) complex Hermite polynomials.

Corollary 2.2.7. *For $p + q \geq 0$, we have*

$$\lim_{t \rightarrow 0} t^{n+m} H_{n,m}^{(p,q)}\left(\frac{z}{t}, \frac{w}{t} \middle| \gamma\right) = z^n w^m. \quad (2.2.10)$$

Proof. By specifying $a = b = t$ and $z := \frac{z}{t}, w := \frac{w}{t}$ in (2.2.9) and next tending t to zero, we get

$$\lim_{t \rightarrow 0} t^{n+m} H_{n,m}^{(p,q)}\left(\frac{z}{t}, \frac{w}{t} \middle| \gamma\right) = \lim_{t \rightarrow 0} H_{n,m}^{(p,q)}(z, w|t^{p+q}\gamma) = H_{n,m}^{(p,q)}(z, w|0) = 0. \quad \blacksquare$$

We conclude this section by establishing the closed expression for the next generalized generating functions

$$S_{a,b}^{p,q,\gamma}(z, w|u, v) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (a)_n (b)_m H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^n v^m}{n! m!}$$

and

$$G_{j,k}^{p,q,\gamma}(z, w|u, v) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (n)_j (m)_k H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^n v^m}{n! m!}. \quad (2.2.11)$$

To this purpose, we set

$$P_k^n(z) = \sum_{j=0}^{k-1} (-1)^{(k-j)} (-n)_{(k-j)} \binom{k}{j} z^j.$$

Theorem 2.2.8. *We have*

$$G_{j,k}^{p,q,\gamma}(z, w|u, v) = uvzwe^{zu+vw+\gamma u^p v^q} \left((uz)^{j-1} + P_{j-1}^j(uz) \right) \left((vw)^{k-1} + P_{k-1}^k(vw) \right) \quad (2.2.12)$$

and

$$S_{a,b}^{p,q,\gamma}(z, w|u, v) = (1-uz)^{-a} (1-vw)^{-b} \quad (2.2.13)$$

$$\times {}_{p+q}F_0 \left(\begin{matrix} \frac{a}{p'}, \frac{a+1}{p}, \dots, \frac{a+p-1}{p}, \frac{b}{q'}, \frac{b+1}{q}, \dots, \frac{b+q-1}{q} \\ - \end{matrix} \middle| \frac{p^p q^q \gamma uv}{(1-uz)^p (1-vw)^q} \right).$$

Proof. Using (2.1.3), we get

$$\begin{aligned} G_{j,k}^{p,q,\gamma}(z, w|u, v) &= \sum_{k=0}^{\infty} \frac{(\gamma u^p v^q)^k}{k!} \left(\sum_{n=0}^{\infty} (n)_j \frac{(uz)^n}{n!} \sum_{m=0}^{\infty} (m)_k \frac{(vw)^m}{m!} \right) \\ &= e^{\gamma u^p v^q} uze^{uz} \left((uz)^{j-1} + P_{j-1}^j(uz) \right) vwe^{vw} \left((vw)^{k-1} + P_{k-1}^k(vw) \right) \\ &= uvzwe^{zu+vw+\gamma u^p v^q} \left((uz)^{j-1} + P_{j-1}^j(uz) \right) \left((vw)^{k-1} + P_{k-1}^k(vw) \right). \end{aligned}$$

The second equality results from the use of the generating function [50, Theorem 2.1],

$$\sum_{n=0}^{\infty} (n)_j \frac{z^n}{n!} = ze^z \left(z^{j-1} + P_{j-1}^j(z) \right).$$

Now, using the identity [?, Eq. (5), p. 101] combined with the formula [?, Eq. (20), p. 22], we obtain

$$\begin{aligned} S_{a,b}^{p,q,\gamma}(z, w|u, v) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} (a)_{n+pk} (b)_{m+qk} u^{pk} v^{qk} \frac{(uz)^n}{n!} \frac{(vw)^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} (a)_{pk} (a+pk)_n (b)_{qk} (b+qk)_m u^{pk} v^{qk} \frac{(uz)^n}{n!} \frac{(vw)^m}{m!}. \end{aligned}$$

But, by applying the Gauss multiplication theorem [?, Eq. (26), p. 23] and the binomial theorem ${}_1F_0(a|t) = (1-t)^{-a}$, we get

$$S_{a,b}^{p,q,\gamma}(z,w|u,v) = (1-uz)^{-a}(1-vw)^{-b} \sum_{k=0}^{\infty} \prod_{j=1}^p \left(\frac{a+j-1}{p} \right)_k \prod_{j=1}^q \left(\frac{b+i-1}{q} \right)_k \\ \times \frac{1}{k!} \left(\frac{p^p q^q \gamma u v}{(1-uz)^p (1-vw)^q} \right)^k.$$

which gives the right-hand side of (2.2.13). ■

Remark 2.2.9. *The identity (2.2.12) can be seen as a special generalization of (2.2.7). Notice also that for $p = q = 1, w = z$, and $\gamma = -1$, we retrieve [35, Eq. (4.15)]. When $p = 2, m = q = 0, z = 2x$, and $\gamma = -1$, we get [51, p. 190]*

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} t^n H_n(x) = (1-2xt)^{-a} {}_2F_0 \left(\begin{matrix} \frac{(a)}{2}, \frac{(a+1)}{2} \\ - \end{matrix} \middle| \frac{-4t^2}{(1-2xt)^2} \right)$$

2.2.3 Multiplication formulas.

We begin with the following multiplication formula which will be employed to prove certain recursion relation with respect to parameters p and q .

Proposition 2.2.10. *We have the identity*

$$H_{n,m}^{(p,q)}(z,w|c\gamma) = n!m! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor \wedge \lfloor \frac{m}{q} \rfloor} \frac{(c-1)^k \gamma^k}{k!} \frac{H_{n-pk, m-qk}^{(p,q)}(z,w|\gamma)}{(n-pk)!(m-qk)!}. \quad (2.2.14)$$

Proof. From

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z,w|c\gamma) \frac{u^n v^m}{n!m!} = e^{uz+vw+\gamma u^p v^q} e^{(c-1)\gamma u^p v^q}, \\ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c-1)^k \gamma^k}{k!} H_{n,m}^{(p,q)}(z,w|\gamma) \frac{u^{n+pk} v^{m+qk}}{n!m!}.$$

Then (2.2.14) follows by equating the coefficients of $u^n v^m$ on both sides of the last equation. ■

Another multiplication formula is the following.

Proposition 2.2.11. *We have*

$$H_{n,m}^{(p,q)}(az, bw|c\gamma) = n!m! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor \wedge \lfloor \frac{m}{q} \rfloor} \frac{(c - a^p b^q)^k \gamma^k a^{n-pk} b^{m-qk}}{k!(n-pk)!(m-qk)!} H_{n-pk, m-qk}^{(p,q)}(z, w|\gamma). \quad (2.2.15)$$

Proof. Applying (2.2.9), we get

$$H_{n,m}^{(p,q)}(az, bw|c\gamma) = a^n b^m H_{n,m}^{(p,q)}(z, w|ca^{-p}b^{-q}\gamma).$$

By (2.2.14), we obtain

$$H_{n,m}^{(p,q)}(az, bw|c\gamma) = n!m!a^n b^m \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor \wedge \lfloor \frac{m}{q} \rfloor} \frac{(ca^{-p}b^{-q} - 1)^k \gamma^k H_{n-pk, m-qk}^{(p,q)}(z, w|\gamma)}{k!(n-pk)!(m-qk)!}$$

which reduces to the right-hand side of (2.2.15). ■

Remark 2.2.12. *The Gould-Hopper polynomials satisfy*

$$H_n^{(p)}(az|c\gamma) = n! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \frac{(c - a^p)^k \gamma^k}{k!} \frac{a^{n-pk}}{(n-pk)!} H_{n-pk}^{(p)}(z|\gamma). \quad (2.2.16)$$

We note also that for $p = 2$, $q = 0$, $w = 1$, $c = 1$ and $\gamma = -1$, we recover the multiplication formula for real Hermite polynomials H_n in [34, Eq. (4.6.33)], while for $p = q = 1$, $w = \bar{z}$, $\gamma = -1$, $b = a$ and $c = 1$, we find the multiplication formula for polyanalytic polynomials $H_{n,m}$ proved in [35, Eq. (4.13)].

2.2.4 Connection to Gould-Hopper polynomials.

The main aim here is to express the polynomials $H_{n,m}^{(p,q)}$ in terms of the Gould-Hopper polynomials $H_n^{(p)}$ and vice-versa.

Proposition 2.2.13. *We have*

$$H_n^{(p)}(z|\gamma) = \sum_{k=0}^n \binom{n}{k} H_{n-k,k}^{(p-q,q)}(z - w, w|\gamma) \quad (2.2.17)$$

and

$$H_{n,m}^{(p,q)}(z, w|\gamma) = n!m! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \sum_{j=0}^{\lfloor \frac{m}{q} \rfloor} \sum_{l=0}^{\lfloor \frac{n-pk}{p} \rfloor} \sum_{i=0}^{\lfloor \frac{m-qj}{q} \rfloor} \frac{(-2)^{-l-i} (-\gamma)^{k+j} H_{n-p(l+k)}^{(p)}(z|\gamma) H_{m-q(i+j)}^{(q)}(w|\gamma)}{l!i!(k-l)!(j-i)!(n-p(l+k))!(m-q(i+j))!}. \quad (2.2.18)$$

Proof. The expression of $H_n^{(p)}$ in terms of $H_{n,m}^{(p,q)}$ as given through (2.2.17) is in fact equivalent to the following

$$H_n^{(p+q)}(z+w|\gamma) = \sum_{k=0}^n \binom{n}{k} H_{n-k,k}^{(p,q)}(z, w|\gamma) \quad (2.2.19)$$

which readily follows by identification process. Indeed, by taking $v = u$ in the generating function (2.2.7) and substituting there n by $n - k$, we obtain

$$\sum_{n=0}^{\infty} \sum_{k=0}^n H_{n-k,k}^{(p,q)}(z, w|\gamma) \frac{u^n}{(n-k)!k!} = e^{(z+w)u+\gamma u^{p+q}} = \sum_{n=0}^{\infty} H_n^{(p+q)}(z|\gamma) \frac{u^n}{n!}.$$

The last equality follows by making use of the generating function (2.2.8) for Gould–Hopper polynomials.

Using the generating function (2.2.7) and the fact that $e^{zu+vw+\gamma u^p v^q} = e^{zu+\frac{\gamma}{2}u^p v^q} e^{wv+\frac{\gamma}{2}u^p v^q}$, as well as the identity (2.2.16), it follows

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^n v^m}{n!m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} n! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \frac{(v^q - 1)^k \gamma^k H_{n-pk}^{(p)}(z|\gamma)}{k! (n-pk)!} \frac{u^n}{n!} \left(m! \sum_{j=0}^{\lfloor \frac{m}{q} \rfloor} \frac{(u^p - 1)^k \gamma^j H_{m-qj}^{(q)}(w|\gamma)}{j! (m-qj)!} \frac{v^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \sum_{j=0}^{\lfloor \frac{m}{q} \rfloor} \sum_{l=0}^k \sum_{i=0}^j \binom{k}{l} \binom{j}{i} (-1)^{k+j} (-2)^{-l-i} \gamma^{k+j} \frac{H_{n-pk}^{(p)}(z|\gamma)}{k!j!} \frac{H_{m-qk}^{(q)}(w|\gamma)}{(n-pk)!(m-qk)!} u^{n+lp} v^{m+iq} \end{aligned}$$

Replacing $n + ip$ by n and $m + iq$ by m , we get

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^n v^m}{n!m!} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} n!m! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \sum_{j=0}^{\lfloor \frac{m}{q} \rfloor} \sum_{l=0}^{\lfloor \frac{n-pk}{p} \rfloor} \sum_{i=0}^{\lfloor \frac{m-qj}{q} \rfloor} \frac{(-2)^{-l-i} (-\gamma)^{k+j}}{l!i!(k-l)!(j-i)!} \\ &\quad \times \frac{H_{n-p(l+k)}^{(p)}(z|\gamma) H_{m-q(i+j)}^{(q)}(w|\gamma)}{(n-p(l+k))!(m-q(i+j))!} \frac{u^n v^m}{n!m!}. \end{aligned}$$

This completes the proof of (2.2.18). ■

Remark 2.2.14. *The specification of $p = q = 1$, $\gamma = -1$ and $m = 0$ in (2.2.17) (or (2.2.19)) provides us with a new expression of the holomorphic Hermite polynomials $H_n(z)$ in terms of the polyanalytic Ito–Hermite polynomials,*

$$H_n(z) = \sum_{k=0}^n \binom{n}{k} H_{n-k,k}(2i\Im(z), \bar{z}).$$

2.3 Recurrence Relations

In this section, we establish recurrence relations for the polynomials $H_{n,m}^{(p,q)}$. We start by presenting the behavior of the derivative operators with respect to the variables z , w , and γ .

Proposition 2.3.1. *The partial derivatives of $H_{n,m}^{(p,q)}(z, w|\gamma)$ are given by*

$$\partial_z H_{n,m}^{(p,q)}(z, w|\gamma) = n H_{(n-1),m}^{(p,q)}(z, w|\gamma) \quad (2.3.1)$$

and

$$\partial_w H_{n,m}^{(p,q)}(z, w|\gamma) = m H_{(n,m-1)}^{(p,q)}(z, w|\gamma). \quad (2.3.2)$$

Proof. Using the operational formula (2.2.2) and acknowledging that $e^{\gamma \partial_z^p \partial_w^q}$ and ∂_z commute, we derive (2.3.1). Specifically, we have

$$\partial_z H_{n,m}^{(p,q)}(z, w|\gamma) = e^{\gamma \partial_z^p \partial_w^q} \partial_z (z^n w^m) = n e^{\gamma \partial_z^p \partial_w^q} (z^{n-1} w^m).$$

We can obtain (2.3.2) by utilizing the symmetry in 2.2.1. ■

Remark 2.3.2. *The formula*

$$\partial_z^j \partial_w^k H_{n,m}^{(p,q)}(z, w|\gamma) = \frac{n!}{(n-j)!} \frac{m!}{(m-k)!} H_{n-j,m-k}^{(p,q)}(z, w|\gamma) \quad (2.3.3)$$

is valid for any $j \leq n$ and $k \leq m$, and can be shown through mathematical induction. The left-hand side of (2.3.3) is zero otherwise. Consequently, we derive

$$\partial_\gamma^k H_{n,m}^{(p,q)}(z, w|\gamma) = \frac{n!}{(n-pk)!} \frac{m!}{(m-qk)!} H_{n-pk,m-kq}^{(p,q)}(z, w|\gamma) \quad (2.3.4)$$

if $k \leq \left\lfloor \frac{n}{p} \right\rfloor \wedge \left\lfloor \frac{m}{q} \right\rfloor$, and it vanishes otherwise.

Based on the preceding proposition, we affirm the following result.

Proposition 2.3.3. *We have*

$$z^n w^m = n!m! \sum_{k=0}^{\left\lfloor \frac{n}{p} \right\rfloor \wedge \left\lfloor \frac{m}{q} \right\rfloor} \frac{(-\gamma)^k H_{n-pk, m-qk}^{(p,q)}(z, w|\gamma)}{k! (n-pk)! (m-qk)!}. \quad (2.3.5)$$

Consequently, the operational formula

$$z^n w^m = e^{-\gamma \partial_z^p \partial_w^q} \left(H_{n,m}^{(p,q)}(z, w|\gamma) \right) \quad (2.3.6)$$

is valid.

Proof. By expanding the Taylor series of the polynomials $H_{n,m}^{(p,q)}(z, w|\gamma + h)$, viewed as a function in the third variable, and utilizing (2.3.4), we get

$$\begin{aligned} H_{n,m}^{(p,q)}(z, w|\gamma + h) &= \sum_{k=0}^n \frac{h^k}{k!} \partial_\gamma^k H_{n,m}^{(p,q)}(z, w|\gamma) \\ &= n!m! \sum_{k=0}^{\left\lfloor \frac{n}{p} \right\rfloor \wedge \left\lfloor \frac{m}{q} \right\rfloor} \frac{h^k H_{n-pk, m-qk}^{(p,q)}(z, w|\gamma)}{k! (n-pk)! (m-qk)!}. \end{aligned}$$

By setting $h = -\gamma$ and using the fact that $H_{n,m}^{(p,q)}(z, w|0) = z^n w^m$, we derive (2.3.5). The proof of (2.3.6) follows from operational calculus. Starting from the right-hand side, we use the result from Proposition 2.2.11 with $a = b = 1$ and $c = 0$ to obtain

$$e^{-\gamma \partial_z^p \partial_w^q} \left(H_{n,m}^{(p,q)}(z, w|\gamma) \right) = n!m! \sum_{k=0}^{\left\lfloor \frac{n}{p} \right\rfloor \wedge \left\lfloor \frac{m}{q} \right\rfloor} \frac{(-\gamma)^k H_{n-pk, m-qk}^{(p,q)}(z, w|\gamma)}{k! (n-pk)! (m-qk)!} = z^n w^m. \quad \blacksquare$$

Thus, we can establish the primary three-term recurrence formulas in this section.

Proposition 2.3.4. *The polynomials $H_{n,m}^{(p,q)}$ satisfy the recurrence relations*

$$H_{n+1,m}^{(p,q)}(z, w|\gamma) = z H_{n,m}^{(p,q)}(z, w|\gamma) + \gamma p!q! \binom{n}{p-1} \binom{m}{q} H_{n+1-p, m-q}^{(p,q)}(z, w|\gamma) \quad (2.3.7)$$

and

$$H_{n+1,m}^{(p,q)}(z, w|\gamma) = (z + p\gamma\partial_z^{p-1}\partial_w^q)H_{n,m}^{(p,q)}(z, w|\gamma). \quad (2.3.8)$$

Proof. Using the generating function in (2.2.7), replacing n with $n + p - 1$ and m with $m + q$, we obtain

$$\begin{aligned} (u^{p-1}v^q)e^{zu+wv+\gamma u^p v^q} &= \sum_{n=p-1}^{\infty} \sum_{m=q}^{\infty} H_{n+1-p,m-q}^{(p,q)}(z, w|\gamma) \frac{u^n v^m}{(n+1-p)!(m-q)!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (p-1)!q! \binom{n}{p-1} \binom{m}{q} H_{n+1-p,m-q}^{(p,q)}(z, w|\gamma) \frac{u^n v^m}{n!m!}. \end{aligned}$$

Since $\partial_u e^{zu+wv+\gamma u^p v^q} = (z + p\gamma u^{p-1}v^q)e^{zu+wv+\gamma u^p v^q}$, we have

$$\partial_u e^{zu+wv+\gamma u^p v^q} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(zH_{n,m}^{(p,q)}(z, w|\gamma) + \gamma p!q! \binom{n}{p-1} \binom{m}{q} H_{n+1-p,m-q}^{(p,q)}(z, w|\gamma) \right) \frac{u^n v^m}{n!m!}.$$

Additionally, since $H_{n+1,m}^{(p,q)}(z, w|\gamma) = \partial_u^n \partial_v^m (\partial_u e^{zu+wv+\gamma u^p v^q})|_{u=v=0}$, we deduce that

$$H_{n+1,m}^{(p,q)}(z, w|\gamma) = zH_{n,m}^{(p,q)}(z, w|\gamma) + \gamma p!q! \binom{n}{p-1} \binom{m}{q} H_{n+1-p,m-q}^{(p,q)}.$$

This confirms (2.3.7). For (2.3.8), note that

$$\partial_z^{p-1} \partial_w^q H_{n,m}^{(p,q)}(z, w|\gamma) = (p-1)!q! \binom{n'}{p-1} \binom{m'}{q} H_{n'+1-p,m'-q}^{(p,q)}.$$

Therefore,

$$\begin{aligned} H_{n+1,m}^{(p,q)}(z, w|\gamma) &= zH_{n,m}^{(p,q)}(z, w|\gamma) + \gamma p!q! \binom{n'}{p-1} \binom{m'}{q} H_{n'+1-p,m'-q}^{(p,q)}(z, w|\gamma) \\ &= (z + p\gamma\partial_z^{p-1}\partial_w^q)H_{n,m}^{(p,q)}(z, w|\gamma). \end{aligned}$$

■

Remark 2.3.5. Using the symmetry identity (2.2.1), we can derive the following three-term recurrence relations:

$$H_{n,m+1}^{(p,q)}(z, w|\gamma) = wH_{n,m}^{(p,q)}(z, w|\gamma) + \gamma p!q! \binom{n}{p} \binom{m}{q-1} H_{n-p,m-1-q}^{(p,q)}(z, w|\gamma) \quad (2.3.9)$$

and

$$H_{n,m+1}^{(p,q)}(z, w|\gamma) = (w + q\gamma\partial_z^p\partial_w^{q-1})H_{n,m}^{(p,q)}(z, w|\gamma). \quad (2.3.10)$$

Based on the previously derived recurrence relations, the polynomials $H_{n,m}^{(p,q)}$ can be reformulated using certain creation-like operators, with the monomials z^n and w^m serving as generators. We propose the following:

Proposition 2.3.6. *For any $n, m, p, q = 0, 1, 2, \dots$, we have*

$$H_{n,m}^{(p,q)}(z, w|\gamma) = \begin{cases} e^\gamma z^n w^m, & \text{if } p = q = 0, \\ (z + p\gamma\partial_z^{p-1})^n (w^m), & \text{if } p \geq 1 \text{ and } q = 0, \\ (z + p\gamma\partial_z^{p-1}\partial_w^q)^n (w^m), & \text{if } p \geq 1 \text{ and } q \geq 1. \end{cases} \quad (2.3.11)$$

Proof. The first identity in (2.3.11) is evident, considering the convention that $\left[\frac{j}{k}\right] = +\infty$ when $k = 0$. The second identity, for $q = 0$ and $p \geq 1$, can be derived using $H_n^{(p)}(z|\gamma) = (z + p\gamma\partial_z^{p-1})^n(1)$ from [14, Eq (6), p. 18]. Specifically, we have

$$H_{n,m}^{(p,0)}(z, w|\gamma) = w^m H_n^{(p)}(z|\gamma) = (z + p\gamma\partial_z^{p-1})^n (w^m).$$

The last identity, for $p \geq 1$ and $q \geq 1$, can be shown by induction on n , noting that $H_{0,m}^{(p,q)}(z, w|\gamma) = w^m$. Indeed, we have

$$H_{n,m}^{(p,q)}(z, w|\gamma) = (z + p\gamma\partial_z^{p-1}\partial_w^q)H_{n-1,m}^{(p,q)}(z, w|\gamma) = (z + p\gamma\partial_z^{p-1}\partial_w^q)^n H_{0,m}^{(p,q)}(z, w|\gamma). \quad \blacksquare$$

Remark 2.3.7. *The analogues of the second and third recursion formulas in (2.3.11), with respect to the z variable, are $H_{n,m}^{(p,q)}(z, w|\gamma) = (w + q\gamma\partial_w^{q-1})^m (z^n)$ and $H_{n,m}^{(p,q)}(z, w|\gamma) = (w + q\gamma\partial_z^p\partial_w^{q-1})^m (z^n)$, respectively, due to the symmetry property.*

An immediate consequence of (2.3.11) is the following:

Corollary 2.3.8. *We have*

$$H_{n,m}^{(p,q)}(z, w|\gamma) = (z + p\gamma\partial_z^{p-1}\partial_w^q)^n (w + q\gamma\partial_z^p\partial_w^{q-1})^m(1). \quad (2.3.12)$$

Proof. This follows from $H_{0,0}^{(p,q)}(z, w|\gamma) = 1$ and from (2.3.11):

$$\begin{aligned} H_{n,m}^{(p,q)}(z, w|\gamma) &= (z + p\gamma\partial_z^{p-1}\partial_w^q)^n H_{0,m}^{(p,q)}(z, w|\gamma) \\ &= (z + p\gamma\partial_z^{p-1}\partial_w^q)^n (w + q\gamma\partial_z^p\partial_w^{q-1})^m H_{0,0}^{(p,q)}(z, w|\gamma). \end{aligned}$$

■

Next, we present a recursion relation with respect to the parameters p and q .

Proposition 2.3.9. *We have*

$$H_{n,m}^{(p+1,q)}(z, w|\gamma) = n!m! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor \wedge \lfloor \frac{m}{q} \rfloor} \sum_{j=0}^k \binom{k}{j} \gamma^k (-1)^{k-j} \frac{H_{n-j-pk, m-qk}^{(p,q)}(z, w|\gamma)}{(n-j-pk)!(m-qk)!}. \quad (2.3.13)$$

Proof. Using the generating function in (2.2.7), we obtain

$$R_\gamma^{p+1,q}(z, w|u, v) = e^{uz+vw+\gamma u^{p+1}v^q} = e^{uz+vw+(\gamma u)u^p v^q} = R_{u\gamma}^{p,q}(z, w|u, v).$$

In view of (2.2.15), we have

$$\begin{aligned} R_\gamma^{p+1,q}(z, w|u, v) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor \wedge \lfloor \frac{m}{q} \rfloor} \gamma^k (u-1)^k H_{n-pk, m-qk}^{(p,q)}(z, w|\gamma) \frac{u^n v^m}{n!m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor \wedge \lfloor \frac{m}{q} \rfloor} \sum_{j=0}^k \binom{k}{j} \gamma^k (-1)^{k-j} \frac{H_{n-j-pk, m-qk}^{(p,q)}(z, w|\gamma)}{(n-j-pk)!(m-qk)!} \frac{u^n v^m}{n!m!}. \end{aligned}$$

Therefore, the result (2.3.13) follows by identification. ■

The previous recursion relation is equivalent to the operational formula

$$H_{n,m}^{(p+1,q)}(z, w|\gamma) = e^{\gamma(\partial_z-1)\partial_z^p\partial_w^q} H_{n,m}^{(p,q)}(z, w|\gamma), \quad (2.3.14)$$

which can be derived from the definition of $H_{n,m}^{(p+1,q)}(z, w|\gamma)$ using (2.3.3). Specifically, we

have

$$\begin{aligned}
H_{n,m}^{(p+1,q)}(z,w|\gamma) &= \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor \wedge \lfloor \frac{m}{q} \rfloor} \gamma^k \sum_{j=0}^k \binom{k}{j} \partial_z^j (-1)^{k-j} \partial_z^{pk} \partial_w^{qk} H_{n,m}^{(p,q)}(z,w|\gamma) \\
&= \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor \wedge \lfloor \frac{m}{q} \rfloor} \gamma^k (\partial_z - 1)^k (\partial_z^p \partial_w^q)^k H_{n,m}^{(p,q)}(z,w|\gamma) \\
&= e^{\gamma(\partial_z - 1)} \partial_z^p \partial_w^q H_{n,m}^{(p,q)}(z,w|\gamma).
\end{aligned}$$

More generally, we have the following results for any positive integers h and k :

Proposition 2.3.10. *For any positive integers h and k , we have:*

$$H_{n,m}^{(p+h,q)}(z,w|\gamma) = e^{\gamma(\partial_z^h - 1)} \partial_z^p \partial_w^q H_{n,m}^{(p,q)}(z,w|\gamma), \quad (2.3.15)$$

$$H_{n,m}^{(p,q+k)}(z,w|\gamma) = e^{\gamma(\partial_w^k - 1)} \partial_z^p \partial_w^q H_{n,m}^{(p,q)}(z,w|\gamma), \quad (2.3.16)$$

and

$$H_{n,m}^{(p+h,q+k)}(z,w|\gamma) = e^{\gamma(\partial_z^h \partial_w^k - 1)} \partial_z^p \partial_w^q H_{n,m}^{(p,q)}(z,w|\gamma). \quad (2.3.17)$$

Proof. First, we note that

$$H_{n,m}^{(p+h,q)}(z,w|\gamma) = H_{n,m}^{(p+(h-1)+1,q)}(z,w|\gamma) = e^{\gamma(\partial_z - 1)} \partial_z^{p+h-1} \partial_w^q H_{n,m}^{(p+h-1,q)}(z,w|\gamma).$$

By successive application ($h - 1$ times) of (2.3.14) and the commutative property of the operators $(\partial_z - 1) \partial_z^{p+i} \partial_w^q$ for $1 \leq i \leq h - 1$, we get

$$\begin{aligned}
H_{n,m}^{(p+h,q)}(z,w|\gamma) &= \prod_{i=1}^{h-1} e^{\gamma(\partial_z - 1)} \partial_z^{p+i} \partial_w^q H_{n,m}^{(p,q)}(z,w|\gamma) \\
&= e^{\sum_{i=1}^{h-1} \gamma(\partial_z - 1)} \partial_z^{p+h-1} \partial_w^q H_{n,m}^{(p,q)}(z,w|\gamma) \\
&= e^{\gamma(\partial_z - 1)} \sum_{i=1}^{h-1} \partial_z^i \partial_z^p \partial_w^q H_{n,m}^{(p,q)}(z,w|\gamma).
\end{aligned}$$

Finally, using the following identity

$$(\partial_z - 1) \sum_{i=1}^{h-1} \partial_z^i = \partial_z^h - 1,$$

we obtain the first relation (2.3.15). Subsequently, the recurrence formula (2.3.16) follows from (2.3.14) by the symmetry property for the polynomials $H_{n,m}^{(p,q)}$. The last identity (2.3.17) follows by combining (2.3.15) and (2.3.16), considering that the operators $(\partial_z^h - 1)\partial_z^p\partial_w^{q+k}$ and $(\partial_w^k - 1)\partial_z^p\partial_w^q$ commute. Indeed, we proceed as follows:

$$\begin{aligned} H_{n,m}^{(p+h,q+k)}(z, w|\gamma) &= e^{\gamma(\partial_z^h - 1)\partial_z^p\partial_w^{q+k}} H_{n,m}^{(p,q+k)}(z, w|\gamma) \\ &= e^{\gamma(\partial_z^h - 1)\partial_z^p\partial_w^{q+k}} e^{\gamma(\partial_w^k - 1)\partial_z^p\partial_w^q} H_{n,m}^{(p,q)}(z, w|\gamma) \\ &= e^{\gamma(\partial_z^h\partial_w^k - \partial_w^k + \partial_w^k - 1)\partial_z^p\partial_w^q} H_{n,m}^{(p,q)}(z, w|\gamma) \\ &= e^{\gamma(\partial_z^h\partial_w^k - 1)\partial_z^p\partial_w^q} H_{n,m}^{(p,q)}(z, w|\gamma). \end{aligned}$$

■

Remark 2.3.11. *These recursion relations are novel even when restricted to the Gould–Hopper polynomials.*

2.4 High-Order Magnetic Laplacian

We extend the classical results applicable to Hermite, Ito-Hermite, and Gould–Hopper polynomials to the (p, q) Gould–Hopper polynomials in a unified manner. These polynomials are likely to be essential in studying the high-order partial differential equation in (2.1.4).

The previously derived recurrence formulas demonstrate that the polynomials $H_{n,m}^{(p,q)}$ satisfy specific partial differential equations, generalizing those obtained for real and complex Hermite-type polynomials. Notably, the complex Itô-Hermite polynomials $H_{n,m}(z, \bar{z})$ form an orthogonal complete system of eigenfunctions of the magnetic Laplacian

$$\tilde{\Delta} = -\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{z} \frac{\partial}{\partial \bar{z}}, \quad (2.4.1)$$

acting on the Hilbert space of Gaussian functions, possessing a purely discrete spectrum (Landau levels) given by an arithmetic sequence of eigenvalues with infinite degeneracy.

This operator is unitarily equivalent to the Schrödinger operator (Landau Hamilto-

nian)

$$\Delta_L = \left(i \frac{\partial}{\partial x} - 2y \right)^2 + \left(i \frac{\partial}{\partial y} + 2x \right)^2, \quad (2.4.2)$$

which describes a non-relativistic particle moving in the complex plane in the presence of a constant homogeneous magnetic field.

The generalization of $\tilde{\Delta}$ in (2.4.1) to higher-order partial differential operators is given by

$$\widetilde{\Delta}_\gamma^{p,q} := w \partial_w + \gamma \partial_z^p \partial_w^q. \quad (2.4.3)$$

For $w = \bar{z}$ and $p = q = -\gamma = 1$, we recover $\tilde{\Delta}$. Similarly, we define the generalized high-order Landau-like Hamiltonian as

$$\Delta_\gamma^{p,q} := z \partial_z + \gamma \partial_z^p \partial_w^q. \quad (2.4.4)$$

Theorem 2.4.1. *The polynomials $H_{n,m}^{(p,q)}$ satisfy the (p, q) -heat differential equation*

$$\partial_\gamma H_{n,m}^{(p,q)}(z, w | \gamma) = \partial_z^p \partial_w^q H_{n,m}^{(p,q)}(z, w | \gamma). \quad (2.4.5)$$

Moreover, the polynomials $H_{n,m}^{(p,q)}$ are eigenfunctions of the operators $\Delta_\gamma^{p,q}$ and $\widetilde{\Delta}_\gamma^{p,q}$, with $\Delta_\gamma^{p,q} H_{n,m}^{(p,q)} = n H_{n,m}^{(p,q)}$ and $\widetilde{\Delta}_\gamma^{p,q} H_{n,m}^{(p,q)} = m H_{n,m}^{(p,q)}$.

Proof. By noting that $\partial_\gamma e^{\gamma \partial_z^p \partial_w^q} = \partial_z^p \partial_w^q e^{\gamma \partial_z^p \partial_w^q}$, it is evident that the polynomials $H_{n,m}^{(p,q)}$ are solutions of the heat equation in (2.4.5).

For the second part, using (2.3.1) and (2.3.7), we find that

$$(z + \gamma \partial_z^{p-1} \partial_w^q) \partial_z H_{n,m}^{(p,q)}(z, w | \gamma) = n H_{n,m}^{(p,q)}(z, w | \gamma)$$

and

$$(w + \gamma \partial_z^p \partial_w^{q-1}) \partial_w H_{n,m}^{(p,q)}(z, w | \gamma) = m H_{n,m}^{(p,q)}(z, w | \gamma).$$

This shows that the polynomials $H_{n,m}^{(p,q)}$ are eigenfunctions of the differential operators $\Delta_\gamma^{p,q}$ and $\widetilde{\Delta}_\gamma^{p,q}$ with eigenvalues n and m , respectively. ■

Remark 2.4.2. *From the previous result, the polynomials $H_{n,m}^{(p,q)}$ are also eigenfunctions of the operator $(z + \gamma \partial_z^{p-1} \partial_w^q)(w + \gamma \partial_z^p \partial_w^{q-1}) \partial_z \partial_w$. Specifically, they satisfy*

$$(z + \gamma \partial_z^{p-1} \partial_w^q)(w + \gamma \partial_z^p \partial_w^{q-1}) \partial_z \partial_w u = n m u. \quad (2.4.6)$$

This holds because the operators ∂_z and $(w + \gamma \partial_z^p \partial_w^{q-1})$ commute.

2.5 Runge and Nielsen formulas

2.5.1 Runge formulas

In this section, we present a Runge-type formula for $H_{n,m}^{(p,q)}$. This formula generalizes the known Runge-type formulas for the real Hermite and the Itô–Hermite polynomials, as found in [53] and [22] respectively.

Proposition 2.5.1. *We have*

$$H_{n,m}^{(p,q)}(z + z', w + w' | \gamma + \gamma') = \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} H_{k,j}^{(p,q)}(z, w | \gamma) H_{n-k, m-j}^{(p,q)}(z', w' | \gamma'). \quad (2.5.1)$$

Proof. Define

$$T_{\gamma, \gamma'}^{p,q}(z, z', w, w' | u, v) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z + z', w + w' | \gamma + \gamma') \frac{u^n v^m}{n! m!}.$$

Using (2.2.7) and rewriting $e^{(z+z')u + (w+w')v + (\gamma + \gamma')u^p v^q}$ as $e^{zu + wv + \gamma u^p v^q} e^{z'u + w'v + \gamma' u^p v^q}$, we get

$$\begin{aligned} T_{\gamma, \gamma'}^{p,q}(z, z', w, w' | u, v) &= \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w | \gamma) \frac{u^n v^m}{n! m!} \right) \left(\sum_{n'=0}^{\infty} \sum_{m'=0}^{\infty} H_{n',m'}^{(p,q)}(z', w' | \gamma') \frac{u^{n'} v^{m'}}{n'! m'!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\sum_{k=0}^n \sum_{j=0}^m \frac{H_{k,j}^{(p,q)}(z, w | \gamma) H_{n-k, m-j}^{(p,q)}(z', w' | \gamma')}{k! j! (n-k)! (m-j)!} \right) u^n v^m. \end{aligned}$$

Equating the coefficients of $u^n v^m$ in the last equation yields (2.5.1). ■

Remark 2.5.2. *A special case of $\gamma = -\gamma'$ in (2.5.1) leads to the identity*

$$\sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} H_{k,j}^{(p,q)}\left(\frac{z}{2}, \frac{w}{2} | \gamma\right) H_{n-k, m-j}^{(p,q)}\left(\frac{z}{2}, \frac{w}{2} | -\gamma\right) = z^n w^m.$$

Remark 2.5.3. *By taking $z = z'$, $w = w'$, and $\gamma = \gamma'$ in (2.5.1) and using (2.2.9), we obtain the identity*

$$H_{n,m}^{(p,q)}(z, w | \gamma) = \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} H_{k,j}^{(p,q)}(z, w | 2^{p+q-1} \gamma) H_{n-k, m-j}^{(p,q)}(z, w | 2^{p+q-1} \gamma).$$

As an immediate consequence of Proposition 2.5.1, we assert the following:

Corollary 2.5.4. *We have*

$$H_{n,m}^{(p,q)} \left(\frac{z+z'}{\sqrt[2p]{2}}, \frac{w+w'}{\sqrt[2q]{2}} \middle| \gamma \right) = 2^{-\left(\frac{n}{2p} + \frac{m}{2q}\right)} \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \binom{m}{j} H_{k,j}^{(p,q)}(z, w | \gamma) H_{n-k, m-j}^{(p,q)}(z', w' | \gamma). \quad (2.5.2)$$

Proof. Equation (2.5.2) follows from the case $\gamma = \gamma'$ in Proposition 2.5.1 and equation (2.2.9) with $a = 2^{1/2p}$ and $b = 2^{1/2q}$. ■

Remark 2.5.5. *For $p = q = 1$ and $w = \bar{z}$, we recover the Runge formula for the Itô–Hermite polynomials in [22, p.9, Eq 3.22]. For $p = 2$ and $m = q = 0$, we recover the formula for the real Hermite polynomials from [53].*

2.5.2 Nielsen Identities

In this section, we prove some summation formulas of Nielsen type for the polynomials $H_{n,m}^{(p,q)}$, which can be used to derive additional summation formulas.

Theorem 2.5.6. *We have the following identities:*

$$H_{n+n',m}^{(p,q)}(z, w | \gamma) = \sum_{i=0}^n \sum_{j=0}^{n'} \binom{n}{i} \binom{n'}{j} (z - z')^{i+j} H_{n+n'-i-j,m}^{(p,q)}(z', w' | \gamma) \quad (2.5.3)$$

and

$$H_{n,m+m'}^{(p,q)}(z, w | \gamma) = \sum_{k=0}^m \sum_{l=0}^{m'} \binom{m}{k} \binom{m'}{l} (w - w')^{k+l} H_{n, m+m'-k-l}^{(p,q)}(z, w' | \gamma). \quad (2.5.4)$$

Proof. We need to prove only the first identity. Using (2.2.5), we have

$$\begin{aligned} H_m^{(q)}(w | (u+t)^p \gamma) e^{z(u+t)} &= \sum_{n=0}^{\infty} H_{n,m}^{(p,q)}(z, w | \gamma) \frac{(u+t)^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} H_{n+n',m}^{(p,q)}(z, w | \gamma) \frac{u^n}{n!} \frac{t^{n'}}{n'!}. \end{aligned} \quad (2.5.5)$$

Applying this fact twice for given z and z' , we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} H_{n+n',m}^{(p,q)}(z,w|\gamma) \frac{u^n t^{n'}}{n! n'!} &= e^{(z-z')(u+t)} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} H_{n+n',m}^{(p,q)}(z',w|\gamma) \frac{u^n t^{n'}}{n! n'!} \\
&= \sum_{k=0}^{\infty} \frac{(z-z')^k (u+t)^k}{k!} \left(\sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} H_{n+n',m}^{(p,q)}(z',w|\gamma) \frac{u^n t^{n'}}{n! n'!} \right) \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^k (z-z')^{k+j} \frac{u^k t^j}{k! j!} \sum_{n,n'=0}^{\infty} H_{n+n',m}^{(p,q)}(z',w|\gamma) \frac{u^n t^{n'}}{n! n'!}.
\end{aligned}$$

The last equality uses [?, Eq. (1), p. 100]. Now, substituting n by $n-k$ and n' by $n'-j$ leads to

$$\sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} H_{n+n',m}^{(p,q)}(z,w|\gamma) \frac{u^n t^{n'}}{n! n'!} = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} n! n'! \sum_{k=0}^n \sum_{j=0}^{n'} \frac{(z-z')^{k+j} H_{n+n'-k-j,m}^{(p,q)}(z',w|\gamma)}{k! j! (n-k)! (n'-j)!} \frac{u^n t^{n'}}{n! n'!}.$$

This proves (2.5.3). ■

A generalization of Theorem 2.5.6 follows from applying (2.5.3) to $H_{n+n',m+m'}^{(p,q)}(z,w|\gamma)$ and (2.5.4). we denote

$$\binom{n, n', m, m'}{i, j, k, l} := \binom{n}{i} \binom{n'}{j} \binom{m}{k} \binom{m'}{l}.$$

Proposition 2.5.7. *We have*

$$\begin{aligned}
H_{n+n',m+m'}^{(p,q)}(z,w|\gamma) &= \sum_{i=0}^n \sum_{j=0}^{n'} \sum_{k=0}^m \sum_{l=0}^{m'} \binom{n, n', m, m'}{i, j, k, l} (z-z')^{i+j} (w-w')^{k+l} \\
&\quad \times H_{n+n'-i-j, m+m'-k-l}^{(p,q)}(z', w'|\gamma).
\end{aligned} \tag{2.5.6}$$

Remark 2.5.8. *For the particular case $m = q = 0$, we recover the result for the Gould-Hopper polynomials as in [38].*

As an immediate consequence, we obtain the following addition formula with respect to the variables z and w .

Corollary 2.5.9. *We have*

$$H_{n,m}^{(p,q)}(z+z', w+w'|\gamma) = \sum_{i=0}^n \sum_{j=0}^m \binom{n}{i} \binom{m}{j} z^i w^j H_{n-i, m-j}^{(p,q)}(z', w'|\gamma). \tag{2.5.7}$$

Proof. This follows directly from (2.5.6) by setting $n' = m' = 0$ and substituting z with $z - z'$ and w with $w - w'$. ■

Corollary 2.5.10. *We have*

$$H_{n,m}^{(p,q)}(z, w|\gamma) = 2^{n+m} \sum_{i=0}^n \sum_{j=0}^m \binom{n}{i} \binom{m}{j} z^i w^j H_{n-i, m-j}^{(p,q)}(z, w|2^{p+q-1}\gamma). \quad (2.5.8)$$

Proof. It suffices to replace z and z' by $\frac{z}{2}$, and w and w' by $\frac{w}{2}$ in (2.5.7), and then apply (2.2.9). ■

The bi-variate Hermite polynomials

3.1 Introduction

The univariate (poly-analytic) complex Hermite polynomials (UHCP), denoted by $H_{m,n}(z, \bar{z})$, serve as an orthogonal basis in the classical Hilbert space on the complex plane with the Gaussian measure $e^{-|z|^2} dx dy$. These polynomials were introduced by Itô in [31] within the context of complex Markov processes and have since found applications in various fields. They are utilized in the analysis of nonlinear phenomena in traveling wave tube amplifiers [5], spectral theory of certain second-order differential operators [23, 44, 58], studies of special integral transforms [8, 32], coherent states theory [4, 3], combinatorial mathematics [37, 36], and signal processing [12, 54]. For comprehensive properties and applications, see [22, 37, 15, 8, 23].

Bivariate complex Hermite-type polynomials can be defined in multiple ways. A natural approach involves considering the tensor product $H_m(z)H_n(w)$ of the univariate holomorphic Hermite polynomials $H_m(z)$, or by substituting \bar{z} in $H_{m,n}(z, \bar{z})$ with the variable w , resulting in the two-variable holomorphic Hermite polynomials $H_{m,n}(z, w)$ studied in [37]. An in-depth examination of their analytic properties is presented in [27], while [65] explores a three-variable variant $H_{m,n}(z, w, u)$. The variable u can be interpreted as a physical parameter representing time or the magnitude of a magnetic field [8, 23]. Another class arises from the tensor product $H_{m,n}(z, \bar{z})H_{m',n'}(w, \bar{w})$, leading to bivariate poly-analytic Hermite polynomials.

This paper introduces a novel class of bivariate (poly-analytic) complex orthogonal polynomials. These are not simple tensor products of UHCP but involve special composition operators. Specifically, following the approach that yields UHCP from real Hermite

polynomials via a binomial-like formula, we define

$$H_{m,n,m',n'}(z,w) := H_{m,n}(z+iw, \bar{z}-i\bar{w})H_{m',n'}(\bar{z}+i\bar{w}, z-iw). \quad (3.1.1)$$

We will explore their fundamental properties, including creation and annihilation operators, three-term recurrence relations, Rodrigues-type formula, and specific differential equations. We will also discuss their connection to UCHP, provide various representations (e.g., exponential operational and integral representations involving monomials), and investigate their realization as the Fourier–Wigner transform of UCHP.

Additionally, we will demonstrate that these polynomials form an orthogonal basis in the Hilbert space $\mathcal{H}^2(\mathbb{C}_{z,w}^2) := L^2(\mathbb{C}^2, e^{-2(|z|^2+|w|^2)}d\lambda)$, where $d\lambda$ is the Lebesgue measure on \mathbb{C}^2 . Summation formulas, including generating functions, will be derived. Applications in integral transforms and L^2 -spectral analysis of special magnetic Laplacians, though beyond the scope of this paper, will be briefly described and will be the focus of a subsequent study.

The foundational topics needed for these developments are compiled in Section 2, which includes a review of the Fourier–Wigner transform and univariate poly-analytic Hermite polynomials. Our main results are presented in Section 3, followed by concluding remarks and applications in the final section.

3.2 Backgrounds

3.2.1 Fourier–Wigner Transform

The Fourier–Wigner transform is a bilinear mapping defined on $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ by

$$\mathcal{V}_d(f,g)(p,q) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle y,q \rangle} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)} dy \quad (3.2.1)$$

for every $(p,q) \in \mathbb{R}^d \times \mathbb{R}^d$. This transform is crucial in various fields such as harmonic analysis, signal processing, engineering, and physical sciences. It is particularly important in studying the Weyl transform [19, 60, 63] and interpreting quantum mechanics as a form of nondeterministic statistical dynamics [46]. The Fourier–Wigner transform \mathcal{V}_d maintains the tensor product property:

$$\mathcal{V}_d(\otimes_{j=0}^d f_j, \otimes_{j=0}^d g_j)(p,q) = \prod_{j=1}^n \mathcal{V}_1(f_j, g_j)(p_j, q_j), \quad (3.2.2)$$

for given $f_j, g_j \in L^2(\mathbb{R})$, where \mathcal{V}_1 denotes the one-dimensional Fourier–Wigner transform. Additionally, it satisfies the Moyal formula:

$$\langle \mathcal{V}_d(f, g), \mathcal{V}_d(\varphi, \psi) \rangle_{L^2(\mathbb{C}^d)} = \langle f, \varphi \rangle_{L^2(\mathbb{R}^d)} \langle \psi, g \rangle_{L^2(\mathbb{R}^d)}. \quad (3.2.3)$$

This property, along with the action of the Fourier–Wigner transform \mathcal{V}_1 on the classical univariate real Hermite functions

$$h_n^{real}(x) = e^{-\frac{x^2}{2}} H_n^{real}(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-x^2}),$$

is fundamental in reproving that the UCHP constitute an orthogonal basis of the Hilbert space $L^2(\mathbb{C}, e^{-|z|^2} dx dy)$ (see [32, ?] for details). In fact, we have [2, Theorem 3.1]:

$$H_{m,n}(z, \bar{z}) = (-1)^n \frac{\sqrt{2}}{\sqrt{2}^{m+n}} e^{\frac{|z|^2}{2}} \mathcal{V}_1(h_m^{real}, h_n^{real})(\sqrt{2}x, \sqrt{2}y). \quad (3.2.4)$$

It is therefore natural to consider the set of functions $\mathcal{V}_2(h_{m,n}, h_{m',n'})$, where

$$h_{m,n}(z, \bar{z}) := e^{-|z|^2/2} H_{m,n}(z, \bar{z})$$

denotes the Hermite functions associated with UCHP, and to investigate their explicit expressions and fundamental properties. This is the subject of Section 3. Below, we collect the basic properties of UCHP needed for the development of this paper.

3.2.2 The Univariate Poly-analytic Hermite Polynomials

The orthogonal UCHP are defined by the Rodrigues formula:

$$H_{m,n}(z, \bar{z}) = (-1)^{m+n} e^{|z|^2} \frac{\partial^{m+n}}{\partial \bar{z}^m \partial z^n} \left(e^{-|z|^2} \right) \quad (3.2.5)$$

and satisfy the orthogonality relation:

$$\int_{\mathbb{C}} H_{m,n}(z, \bar{z}) H_{j,k}(z, \bar{z}) e^{-|z|^2} d\lambda(z) = \pi m! n! \delta_{m,n}. \quad (3.2.6)$$

Their expression in terms of the generalized Laguerre polynomials is provided in [32, Eq. (2.3)], while their form in terms of the univariate real Hermite polynomials h_m^{real} is given

by [22, 37]:

$$H_{m,n}(z, \bar{z}) = \left(\frac{1}{2}\right)^{m+n} m!n! \sum_{j=0}^m \sum_{k=0}^n \frac{(-1)^k (i)^{j+k} H_{m+n-j-k}^{real}(x) H_{j+k}^{real}(y)}{j!k! (m-j)!(n-k)!} \quad (3.2.7)$$

with $z = x + iy$; $x, y \in \mathbb{R}$. The corresponding exponential operational formula is given by [37, Theorem 2.1]:

$$H_{m,n}(z, \bar{z}) = e^{-\Delta_{\mathbb{C}}} (z^m \bar{z}^n), \quad \Delta_{\mathbb{C}} := \frac{\partial^2}{\partial z \partial \bar{z}}. \quad (3.2.8)$$

In addition to the integral representation (3.2.4) via the Fourier–Wigner transform, these polynomials satisfy [8, Theorem 2.4]:

$$H_{m,n}(z, \bar{z}) = \frac{\mu(-\alpha)^m (\beta)^n}{\pi} e^{|z|^2} \int_{\mathbb{C}} \xi^m \bar{\xi}^n e^{-\gamma|\xi|^2 + \alpha\langle \xi, z \rangle - \beta\overline{\langle \xi, z \rangle}} d\lambda(\xi). \quad (3.2.9)$$

Here α, β are complex numbers such that $\alpha\beta = \gamma > 0$. For instance, by taking $\gamma = 1$ and $\alpha = -\beta = i$, the integral representation (3.2.9) simplifies to the form given by Ismail in [37, Theorem 5.1].

3.2.3 Generating and Bilinear Generating functions

The considered polynomials can also be defined using the generating function:

$$\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{u^m v^n}{m! n!} H_{m,n}(z, \bar{z}) = e^{-uv + zu + \bar{z}v}. \quad (3.2.10)$$

Moreover, the polynomials $H_{m,n}$ satisfy several interesting partial generating functions [8]. We summarize here some bilinear generating functions of Mehler type that generalize the classical Poisson kernel for real Hermite polynomials $H_m^{real}(x)$. The following result [8, Theorem 3.1] holds for every t in the unit circle and $z, w \in \mathbb{C}$:

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} H_{m,n}(z, \bar{z}) H_{n,m'}(w, \bar{w}) = (-t)^{m'} H_{m,m'}(z - tw, \bar{z} - \bar{t}\bar{w}) e^{tw\bar{z}}. \quad (3.2.11)$$

Three broader generalizations of (3.2.11) exist. The first one asserts that the quantity

$$E(u, v|z, w) := \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{u^m v^n}{m!n!} H_{m,n}(z, \bar{z}) H_{m,n}(w, \bar{w})$$

is given by the formula:

$$\frac{1}{1-uv} \exp\left(-\frac{uv(|z|^2 + |w|^2) - uz\bar{w} - v\bar{z}\bar{w}}{1-uv}\right) \quad (3.2.12)$$

for $u, v \in \mathbb{C}$ such that $|uv| < 1$. This is Mehler's formula for $H_{m,n}(z; \bar{z})$ given by Wünsche [64] and recovered by Ismail [37, Theorem 3.3] as a specific case of his Kibble-Slepian formula [37, Theorem 1.1] (see [23, Theorem 4.1] for a special generalization). As a consequence of (3.2.12) and the integral representation (3.2.9), one can derive an interesting self-reciprocity property [23, Theorem 4.2]. The explicit expression of the heat kernel function for a special magnetic Laplacian is given in [23, Theorem 3.3]. Extending to t in the unit circle and $|u| < 1$, we have [8, Theorem 2.4]:

$$E(u, t|z, \bar{w}) = \frac{1}{(1-tu)} \exp\left(\frac{-tu|z-tw|^2}{1-tu}\right) e^{t\bar{w}z} \quad (3.2.13)$$

as well as [8, Theorem 2.3]:

$$\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{u^m t^n}{m!n!} H_{m,n}(z, \bar{z}) H_{n,m'}(w, \bar{w}) = (\bar{w} - t\bar{z} + u)^{m'} e^{t\bar{z}w - ut(w - \bar{z})}. \quad (3.2.14)$$

These identities have significant applications in integral transforms connecting $L^2(\mathbb{C}, e^{-|z|^2} dx dy)$ or the generalized Bargmann–Fock spaces to the two-dimensional Bargmann-Fock space $\mathcal{F}^2(\mathbb{C}^2)$, as presented in [8].

3.3 Bivariate Complex Hermite Polynomials

The polynomials $H_{m,n,m',n'}$ defined in (3.1.1), which depend on the variables z, w , and their complex conjugates, are referred to as bivariate complex poly-analytic Hermite polynomials (BCPHP). Throughout this paper, we will use both the notation $H_{m,n,m',n'}(z, w)$ and the abbreviated form $H_M(Z, \bar{Z})$ to denote the BCPHP. Here, M is a 4-tuple, and $Z = (z, w)$ with $\bar{Z} = (\bar{z}, \bar{w})$. Additionally, we will use multi-index notation. For a given $M = (m_1, m_2, m'_1, m'_2)$ and $N = (n_1, n_2, n'_1, n'_2)$, we define $|M| = m_1 + m_2 + m'_1 + m'_2$ and the Kronecker delta $\delta_{M,N} := \delta_{m_1,n_1} \delta_{m_2,n_2} \delta_{m'_1,n'_1} \delta_{m'_2,n'_2}$, with $M! := m_1! m_2! m'_1! m'_2!$. The binomial coefficient for $N \leq M$ (i.e., $m_\ell \leq n_\ell$ and $m'_\ell \leq n'_\ell$ for $\ell = 1, 2$) is given by

$$\binom{M}{N} = \prod_{\ell=1,2} \binom{m_\ell}{n_\ell} \binom{m'_\ell}{n'_\ell}.$$

From (3.1.1), it is clear that $H_{m,n,m',n'}(z,w)$ are polynomials in $z + iw$ of degree m , in $\bar{z} - i\bar{w}$ of degree n , in $\bar{z} + i\bar{w}$ of degree m' , and in $z - iw$ of degree n' . These polynomials are also functions in z, \bar{z}, w , and \bar{w} of degrees $m + n, m' + n', m + n$, and $m' + n'$, respectively. Specifically, we have $H_{m,n,0,0}(z,w) = H_{m,n}(z + iw, \bar{z} - i\bar{w})$, $H_{m,0,m',0} = (z + iw)^m (\bar{z} + i\bar{w})^{m'}$, and $H_{m,0,0,n'} = (z + iw)^m (z - iw)^{n'}$.

Additionally, these polynomials exhibit the following symmetry properties under complex conjugation:

$$\overline{H_{m,n,m',n'}(z,w)} = H_{n,m,n',m'}(z,w) = H_{m,n,m',n'}(\bar{z}, -\bar{w}) \quad (3.3.1)$$

and

$$H_{m,n,m',n'}(\bar{z}, \bar{w}) = H_{m',n',m,n}(z,w). \quad (3.3.2)$$

To each $(z,w) \in \mathbb{C}^2$, we associate the complex numbers $\xi = z + iw$, $\xi^* = \bar{z} + i\bar{w}$, $\bar{\xi} = \bar{z} - i\bar{w}$, and $\tilde{\xi} = z - iw$. Note that $\bar{\xi}$ and $\tilde{\xi}$ are the complex conjugates of ξ and ξ^* , respectively. The operations $\bar{\cdot}$, \cdot^* , and $\tilde{\cdot}$ are involutions, meaning $\xi^{**} = \tilde{\xi} = \bar{\bar{\xi}} = \xi$. These operations are pairwise commuting, as $\bar{\xi}^* = \bar{\xi} = \tilde{\tilde{\xi}}$, $\tilde{\xi} = \bar{\bar{\xi}} = \xi^*$, and $\tilde{\xi}^* = \tilde{\xi} = \bar{\bar{\xi}}$. Therefore, z and w are real if and only if $\bar{\xi} = \xi^*$ and $\tilde{\xi} = \xi$.

3.3.1 Orthogonality

The main result in this subsection demonstrates that the polynomials H_M are orthogonal in $\mathcal{H}^2(\mathbb{C}_{z,w}^2)$.

Theorem 3.3.1. *The bivariate complex Hermite polynomials H_M form an orthogonal basis for the Hilbert space $\mathcal{H}^2(\mathbb{C}_{z,w}^2)$, and we have*

$$\int_{\mathbb{C}^2} H_M(Z, \bar{Z}) H_N(\bar{Z}, Z) e^{-2|Z|^2} d\lambda(Z) = \frac{\pi^2}{4} M! \delta_{M,N}, \quad (3.3.3)$$

where $|Z| := \sqrt{|z|^2 + |w|^2}$ represents the Euclidean norm of the 2D complex number Z .

Proof. Let $I_{M,N}$ denote the left-hand side of (3.3.3). Using the fact that the Lebesgue measure on $\mathbb{C}^2 = \mathbb{R}^4$ satisfies $4d\lambda(Z) = 4d\lambda(z,w) = d\lambda(\xi, \tilde{\xi})$, and applying Fubini's theorem,

we get

$$\begin{aligned} I_{M,N} &= \frac{1}{4} \int_{\mathbb{C}^2} H_{m_1, m_2}(\zeta, \bar{\zeta}) H_{m'_1, m'_2}(\zeta^*, \bar{\zeta}) \overline{H_{n_1, n_2}(\zeta, \bar{\zeta}) H_{n'_1, n'_2}(\zeta^*, \bar{\zeta})} e^{-|\zeta|^2 - |\bar{\zeta}|^2} d\lambda(\zeta, \bar{\zeta}) \\ &= \frac{1}{4} \langle H_{m_1, m_2}, H_{n_1, n_2} \rangle_{L^2(\mathbb{C}_{\bar{\zeta}}, e^{-|\bar{\zeta}|^2} d\lambda)} \langle H_{m'_1, m'_2}, H_{n'_1, n'_2} \rangle_{L^2(\mathbb{C}_{\zeta}, e^{-|\zeta|^2} d\lambda)}, \end{aligned}$$

where $M = (m_1, m_2, m'_1, m'_2)$ and $N = (n_1, n_2, n'_1, n'_2)$. The result follows from the orthogonality property (3.2.6) for the UCHP.

To complete the proof, we need to show completeness. Suppose $f \in \mathcal{H}^2(\mathbb{C}_{z,w}^2)$ such that

$$\int_{\mathbb{C}^2} f(z, w) H_{m, n, m', n'}(z, w) e^{-2(|z|^2 + |w|^2)} d\lambda(z, w) = 0$$

for every 4-tuple $M = (m, n, m', n')$. This implies

$$\int_{\mathbb{C}} \left(\int_{\mathbb{C}} f(\zeta, \bar{\zeta}) H_{m, n}(\zeta, \bar{\zeta}) e^{-|\zeta|^2} d\lambda(\zeta) \right) H_{m', n'}(\zeta^*, \bar{\zeta}) e^{-|\bar{\zeta}|^2} d\lambda(\bar{\zeta}) = 0.$$

Therefore, the function $\psi : \bar{\zeta} \mapsto \int_{\mathbb{C}} f(\zeta, \bar{\zeta}) H_{m, n}(\zeta, \bar{\zeta}) e^{-|\zeta|^2} d\lambda(\zeta)$ vanishes almost everywhere on \mathbb{C} , as $\psi \in L^2(\mathbb{C}, e^{-|\bar{\zeta}|^2} d\lambda(\bar{\zeta}))$ and the UCHP form a basis of it. By the same reasoning, $f(\zeta, \bar{\zeta}) = 0$ almost everywhere on \mathbb{C}^2 . This completes the proof. \blacksquare

3.3.2 Rodrigues-Type formula

In this context, any function f in $(z, w) \in \mathbb{C}^2$ can also be viewed as a function of $\zeta = z + iw$ and $\bar{\zeta} = z - iw$ (and implicitly in $\bar{\zeta} = \bar{z} - i\bar{w}$ and $\zeta^* = \bar{z} + i\bar{w}$). We introduce the following first-order differential operators:

$$A_{\zeta} := \frac{1}{2} \left(\frac{\partial}{\partial z} - i \frac{\partial}{\partial w} \right) = \frac{\partial}{\partial \zeta}, \quad \text{and} \quad A_{\zeta^*} := \frac{1}{2} \left(\frac{\partial}{\partial \bar{z}} - i \frac{\partial}{\partial \bar{w}} \right) = \frac{\partial}{\partial \zeta^*} \quad (3.3.4)$$

and their conjugate-like operators:

$$A_{\bar{\zeta}} := \frac{1}{2} \left(\frac{\partial}{\partial \bar{z}} + i \frac{\partial}{\partial \bar{w}} \right) = \frac{\partial}{\partial \bar{\zeta}}, \quad \text{and} \quad A_{\bar{\zeta}^*} := \frac{1}{2} \left(\frac{\partial}{\partial z} + i \frac{\partial}{\partial w} \right) = \frac{\partial}{\partial \bar{\zeta}^*}. \quad (3.3.5)$$

The notation on the right-hand sides of (3.3.4) and (3.3.5) is justified by the following lemma, whose proof is straightforward.

Lemma 3.3.2. *The operators $A_{\zeta}, A_{\bar{\zeta}}, A_{\zeta^*}$, and $A_{\bar{\zeta}^*}$ commute pairwise. Moreover, they act as derivation operators with respect to $\zeta, \bar{\zeta}, \zeta^*$, and $\bar{\zeta}^*$ (seen as independent variables), respectively.*

Specifically, we have:

$$\begin{aligned}
A_{\zeta}(\zeta^m \bar{\zeta}^n \zeta^{*j} \tilde{\zeta}^k) &= m \zeta^{m-1} \bar{\zeta}^n \zeta^{*j} \tilde{\zeta}^k, & A_{\bar{\zeta}}(\zeta^m \bar{\zeta}^n \zeta^{*j} \tilde{\zeta}^k) &= n \zeta^m \bar{\zeta}^{n-1} \zeta^{*j} \tilde{\zeta}^k, \\
A_{\zeta^*}(\zeta^m \bar{\zeta}^n \zeta^{*j} \tilde{\zeta}^k) &= j \zeta^m \bar{\zeta}^n \zeta^{*j-1} \tilde{\zeta}^k, & A_{\tilde{\zeta}}(\zeta^m \bar{\zeta}^n \zeta^{*j} \tilde{\zeta}^k) &= k \zeta^m \bar{\zeta}^n \zeta^{*j} \tilde{\zeta}^{k-1}, \\
A_{\zeta}(\bar{\zeta}^n \zeta^{*j} \tilde{\zeta}^k) &= A_{\bar{\zeta}}(\zeta^m \zeta^{*j} \tilde{\zeta}^k) = A_{\zeta^*}(\zeta^m \bar{\zeta}^n \tilde{\zeta}^k) = A_{\tilde{\zeta}}(\zeta^m \bar{\zeta}^n \zeta^{*j}) = 0.
\end{aligned}$$

The following observation is crucial.

Lemma 3.3.3. *We have:*

$$(-1)^m e^{|\zeta|^2} \left(A_{\bar{\zeta}}^m (\bar{\zeta}^n e^{-|\zeta|^2}) \right) = H_{m,n}(\zeta, \bar{\zeta})$$

and

$$(-1)^{m'} \left(A_{\tilde{\zeta}}^{m'} (\tilde{\zeta}^{n'} e^{-|\zeta^*|^2}) \right) = H_{m',n'}(\zeta^*, \tilde{\zeta}).$$

Proof. The result can be proved using Leibniz's rule for the operators $A_{\bar{\zeta}}^m$ and $A_{\tilde{\zeta}}^{m'}$ along with appropriate variable changes. However, the proof we present here relies on the commutation rules and their explicit actions on monomials (Lemma 3.3.2), which can be extended to Gaussian functions. ▀

Consequently, we state the following.

Theorem 3.3.4. *Using the above notation, the bivariate complex Hermite polynomials $H_{m,n,m',n'}$ satisfy:*

$$H_{m,n,m',n'}(z, w) := (-1)^{m+n+m'+n'} e^{|\zeta|^2 + |\zeta^*|^2} A_{\bar{\zeta}}^m A_{\zeta}^n A_{\tilde{\zeta}}^{m'} A_{\zeta^*}^{n'} \left(e^{-|\zeta|^2 - |\zeta^*|^2} \right). \quad (3.3.6)$$

Proof. Starting from (3.1.1) and using Lemma 3.3.3, we obtain:

$$\begin{aligned}
H_{m,n,m',n'}(z, w) &= H_{m,n}(\zeta, \bar{\zeta}) H_{m',n'}(\zeta^*, \tilde{\zeta}) \\
&= (-1)^{m+m'} e^{|\zeta|^2 + |\zeta^*|^2} \left(A_{\bar{\zeta}}^m (\bar{\zeta}^n e^{-|\zeta|^2}) \right) \left(A_{\tilde{\zeta}}^{m'} (\tilde{\zeta}^{n'} e^{-|\zeta^*|^2}) \right) \\
&= (-1)^{m+n+m'+n'} e^{|\zeta|^2 + |\zeta^*|^2} A_{\bar{\zeta}}^m A_{\zeta}^n A_{\tilde{\zeta}}^{m'} A_{\zeta^*}^{n'} \left(e^{-|\zeta|^2 - |\zeta^*|^2} \right).
\end{aligned}$$

This completes the proof of (3.3.6). ▀

3.3.3 Creation and Annihilation Operators

The three-term recurrence relations for the polynomials $H_{m,n,m',n'}$ can be derived from the Rodrigues-type formula (3.3.6) and Lemma 3.3.2. These relations are given by:

$$\begin{aligned} H_{m+1,n,m',n'} &= \zeta H_{m,n,m',n'} - A_{\bar{\zeta}} H_{m,n,m',n'} \\ H_{m,n+1,m',n'} &= \bar{\zeta} H_{m,n,m',n'} - A_{\zeta} H_{m,n+1,m',n'} \\ H_{m,n,m'+1,n'} &= \zeta^* H_{m,n,m',n'} - A_{\bar{\zeta}^*} H_{m,n,m'+1,n'} \\ H_{m,n,m',n'+1} &= \tilde{\zeta} H_{m,n,m',n'} - A_{\zeta^*} H_{m,n,m',n'} \end{aligned}$$

Thus, the operators $\zeta - A_{\bar{\zeta}}$, $\bar{\zeta} - A_{\zeta}$, $\tilde{\zeta} - A_{\zeta^*}$, and $\zeta^* - A_{\bar{\zeta}^*}$ act as raising operators for the polynomials $H_{m,n,m',n'}(z, w)$:

$$(\zeta - A_{\bar{\zeta}})H_{m,n,m',n'} = H_{m+1,n,m',n'} \quad (3.3.7)$$

$$(\bar{\zeta} - A_{\zeta})H_{m,n,m',n'} = H_{m,n+1,m',n'} \quad (3.3.8)$$

$$(\tilde{\zeta} - A_{\zeta^*})H_{m,n,m',n'} = H_{m,n,m'+1,n'} \quad (3.3.9)$$

$$(\zeta^* - A_{\bar{\zeta}^*})H_{m,n,m',n'} = H_{m,n,m',n'+1} \quad (3.3.10)$$

The following representations of $H_{m,n,m',n'}$ are particularly noteworthy.

Proposition 3.3.5. *We have:*

$$H_{m,n,m',n'} = (\zeta^* - A_{\bar{\zeta}})^{m'} (\tilde{\zeta} - A_{\zeta^*})^{n'} H_{m,n,0,0} \quad (3.3.11)$$

$$= (\zeta - A_{\bar{\zeta}})^m (\bar{\zeta} - A_{\zeta})^n H_{0,0,m',n'} \quad (3.3.12)$$

$$= (\zeta - A_{\bar{\zeta}})^m (\bar{\zeta} - A_{\zeta})^n (\zeta^* - A_{\bar{\zeta}})^{m'} (\tilde{\zeta} - A_{\zeta^*})^{n'} \cdot (1) \quad (3.3.13)$$

as well as:

$$H_{m,n,m',n'} = (\zeta - A_{\bar{\zeta}})^m (\zeta^* - A_{\bar{\zeta}})^{m'} (\bar{\zeta}^n \tilde{\zeta}^{n'}) \quad (3.3.14)$$

$$= (\bar{\zeta} - A_{\zeta})^n (\tilde{\zeta} - A_{\zeta^*})^{n'} (\bar{\zeta}^m \zeta^{*m'}) \quad (3.3.15)$$

Proof. The proof of (3.3.13) follows immediately from (3.3.11) and (3.3.12) since $H_{0,0,0,0} = 1$. The proofs of the first and second assertions are similar.

The proof of (3.3.11) follows directly using $H_{m,n,m',n'} = (\zeta - A_{\bar{\zeta}})^m H_{0,n,m',n'}$ and $H_{m,n,m',n'} = (\bar{\zeta} - A_{\zeta})^n H_{m,0,m',n'}$, which can be shown by induction using the three-term recurrence formulas (3.3.7) and (3.3.8).

By similar arguments, we get:

$$H_{m,n,m',n'} = (\zeta - A_{\bar{\zeta}})^m (\zeta^* - A_{\tilde{\zeta}})^{m'} H_{0,n,0,n'}.$$

This leads to the identity (3.3.14) since $H_{0,n,0,n'} = \bar{\zeta}^n \tilde{\zeta}^{n'}$. The proof of (3.3.15) is similar, but it can also be obtained by complex conjugation using (3.3.1) and the identities $\overline{\bar{\zeta}} = \zeta$, $\overline{\zeta^*} = \tilde{\zeta}$, and $\overline{\tilde{\zeta}} = \zeta^*$. ■

Remark 3.3.6. Notice that the operator $\tilde{\zeta} - A_{\tilde{\zeta}^*}$ commutes with $\bar{\zeta} - A_{\zeta}$ and $\zeta - A_{\bar{\zeta}}$ but not with $\zeta^* - A_{\tilde{\zeta}}$. This fact leads to other possible forms of the polynomials $H_{m,n,m',n'}$. For example, we have:

$$H_{m,n,m',n'}(z, w) = (\zeta^* - A_{\tilde{\zeta}})^{m'} (\tilde{\zeta} - A_{\tilde{\zeta}^*})^{n'} (\zeta - A_{\bar{\zeta}})^m (\bar{\zeta} - A_{\zeta})^n \cdot (1) \quad (3.3.16)$$

$$= (\zeta - A_{\bar{\zeta}})^m (\zeta^* - A_{\tilde{\zeta}})^{m'} (\bar{\zeta} - A_{\zeta})^n (\tilde{\zeta} - A_{\tilde{\zeta}^*})^{n'} \cdot (1). \quad (3.3.17)$$

As an immediate consequence of Proposition 3.3.5, we show that A_{ζ} , $A_{\bar{\zeta}}$, A_{ζ^*} , and $A_{\tilde{\zeta}}$ are lowering operators for the $H_{m,n,m',n'}$. Specifically, we have the following.

Proposition 3.3.7. We have:

$$A_{\zeta} H_{m,n,m',n'} = m H_{m-1,n,m',n'} \quad (3.3.18)$$

$$A_{\bar{\zeta}} H_{m,n,m',n'} = n H_{m,n-1,m',n'} \quad (3.3.19)$$

$$A_{\zeta^*} H_{m,n,m',n'} = m' H_{m,n,m'-1,n'} \quad (3.3.20)$$

$$A_{\tilde{\zeta}} H_{m,n,m',n'} = n' H_{m,n,m',n'-1} \quad (3.3.21)$$

Proof. The identity (3.3.19) (resp. (3.3.21)) follows directly from (3.3.15) (resp. (3.3.14)) by applying the operator $A_{\bar{\zeta}}$ (resp. $A_{\tilde{\zeta}}$) and using the identity $A_{\bar{\zeta}}(\bar{\zeta}^m \bar{\zeta}^n) = n \bar{\zeta}^m \bar{\zeta}^{n-1}$ (resp. $A_{\tilde{\zeta}}(\zeta^{*m'} \tilde{\zeta}^{n'}) = n' \zeta^{*m'} \tilde{\zeta}^{n'-1}$). The identities (3.3.18) and (3.3.20) can be obtained by complex conjugation using (3.3.1). ■

3.3.4 Exponential Operational Representation

This result establishes an analog of the operational formula (3.2.8). To this end, we denote by

$$\Delta_{\mathbb{C}^2} := \frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial w \partial \bar{w}}$$

the Laplace-Beltrami operator on \mathbb{C}^2 .

It should be noted here that

$$A_{\bar{\zeta}}A_{\zeta} = \frac{1}{4}\Delta_{\mathbb{C}^2} + \frac{i}{4}\square_{\mathbb{C}^2}$$

and

$$A_{\zeta^*}A_{\bar{\zeta}} = \frac{1}{4}\Delta_{\mathbb{C}^2} - \frac{i}{4}\square_{\mathbb{C}^2},$$

so that

$$A_{\bar{\zeta}}A_{\zeta} + A_{\zeta^*}A_{\bar{\zeta}} = \frac{1}{2}\Delta_{\mathbb{C}^2}.$$

Above, $\square_{\mathbb{C}^2}$ denotes the second-order differential operator given by

$$\square_{\mathbb{C}^2} := \frac{\partial^2}{\partial z \partial \bar{w}} - \frac{\partial^2}{\partial w \partial \bar{z}}.$$

Theorem 3.3.8. *For every $(z, w) \in \mathbb{C}^2$, we have the following operational formula:*

$$H_{m,m',n,n'}(z, w) = e^{-(A_{\bar{\zeta}}A_{\zeta} + A_{\zeta^*}A_{\bar{\zeta}})} \left(\zeta^m \bar{\zeta}^n \zeta^{*m'} \tilde{\zeta}^{n'} \right). \quad (3.3.22)$$

Proof. The result readily follows using (3.2.8). Indeed, by viewing ζ and ζ^* as complex variables with complex conjugates $\bar{\zeta}$ and $\tilde{\zeta}$, respectively, we can write

$$\begin{aligned} H_{m,m',n,n'}(z, w) &= H_{m,n}(\zeta, \bar{\zeta}) H_{m',n'}(\zeta^*, \tilde{\zeta}) \\ &= e^{-A_{\bar{\zeta}}A_{\zeta}} \left(\zeta^m \bar{\zeta}^n \right) e^{-A_{\zeta^*}A_{\bar{\zeta}}} \left(\zeta^{*m'} \tilde{\zeta}^{n'} \right) \\ &= e^{-(A_{\bar{\zeta}}A_{\zeta} + A_{\zeta^*}A_{\bar{\zeta}})} \left(\zeta^m \bar{\zeta}^n \zeta^{*m'} \tilde{\zeta}^{n'} \right). \end{aligned}$$

Thus, one concludes for (3.3.22) since $A_{\bar{\zeta}}A_{\zeta}$ and $A_{\zeta^*}A_{\bar{\zeta}}$ commute. ■

Remark 3.3.9. *The result of Theorem 3.3.8 can also be derived using Theorem 3.3.13 below.*

3.3.5 Special Second Order Differential Equations

Using the introduced lowering and raising operators, it is easy to see that the polynomials $H_{m,n,m',n'}$ solve some second-order differential equations. Indeed, by applying $\zeta - A_{\bar{\zeta}}$ (resp. $\bar{\zeta} - A_{\zeta}$, $\zeta^* - A_{\tilde{\zeta}}$, and $\tilde{\zeta} - A_{\zeta^*}$) to (3.3.18) (resp. (3.3.19), (3.3.20), and (3.3.21)), it is easy to check the following result.

Lemma 3.3.10. *The polynomials $H_{m,n,m',n'}$ satisfy Bochner's property for being eigenfunctions of the second-order differential operator $L_{\bar{\zeta}} := \zeta A_{\bar{\zeta}} - A_{\bar{\zeta}} A_{\zeta}$ (resp. $L_{\zeta} := \bar{\zeta} A_{\zeta} - A_{\zeta} A_{\bar{\zeta}}$, $L_{\tilde{\zeta}} :=$*

$\zeta^* A_{\zeta^*} - A_{\zeta^*} A_{\bar{\zeta}}$, and $L_{\bar{\zeta}} := \bar{\zeta} A_{\bar{\zeta}} - A_{\zeta^*} A_{\bar{\zeta}}$ with m (resp. n , m' , and n') as the corresponding eigenvalue.

Consequently, the polynomials $H_{m,n,m',n'}$ are solutions of the eigenvalue and the common eigenvalue problems for the second-order differential operators

$$S_{\zeta} := -\frac{1}{4} (\Delta_{\mathbb{C}^2} + i\Box_{\mathbb{C}^2} - 2(E_z + E_w) + 2i(F_{zw} - F_{wz})) \quad (3.3.23)$$

$$S_{\zeta^*} := -\frac{1}{4} (\Delta_{\mathbb{C}^2} - i\Box_{\mathbb{C}^2} - 2(E_{\bar{z}} + E_{\bar{w}}) + 2i(F_{\bar{z}\bar{w}} - F_{\bar{w}\bar{z}})), \quad (3.3.24)$$

and their complex conjugates $S_{\bar{\zeta}}$ and $S_{\bar{\zeta}^*}$. Here, $E_z := z \frac{\partial}{\partial z}$ and its complex conjugate $E_{\bar{z}} := \bar{z} \frac{\partial}{\partial \bar{z}}$ are the usual Euler operators on \mathbb{C} , and F_{zw} , $F_{z\bar{w}}$, $F_{\bar{z}w}$, $F_{\bar{z}\bar{w}}$, F_{wz} , $F_{w\bar{z}}$, $F_{\bar{w}z}$, and $F_{\bar{w}\bar{z}}$ are the coupled Euler operators defined by

$$F_{uv} := u \frac{\partial}{\partial v}.$$

For proving the previous assertion, it suffices to make the following observation.

Lemma 3.3.11. *The operators $S_{\bar{\zeta}}$, $S_{\bar{\zeta}^*}$, and S_{ζ} are exactly those involved in Lemma 3.3.10, i.e., $S_{\bullet} = L_{\bullet}$.*

Moreover, the L^2 -eigenvalue problem associated with the special Landau Hamiltonian $S_{\zeta} = L_{\zeta} := \zeta A_{\zeta} - A_{\zeta} A_{\bar{\zeta}}$ acting on $\mathcal{H}^2(\mathbb{C}_{z,w}^2)$ can be explicitly described. Namely, we prove the following:

Theorem 3.3.12. *The spectrum of the operator S_{ζ} (resp. $S_{\bar{\zeta}}$, $S_{\bar{\zeta}^*}$, and S_{ζ}) acting on $\mathcal{H}^2(\mathbb{C}_{z,w}^2)$ is purely discrete and consists of the eigenvalues $\ell = 0, 1, 2, \dots$.*

Proof. Lemmas 3.3.10 and 3.3.11 show that the polynomials $H_{m,n,m',n'}$ are eigenfunctions of L_{ζ} (resp. $L_{\bar{\zeta}}$, $L_{\bar{\zeta}^*}$, and L_{ζ}). Moreover, they belong to the Hilbert space $\mathcal{H}^2(\mathbb{C}_{z,w}^2)$, thanks to Theorem 3.3.1, with m as the corresponding eigenvalue (Lemma 3.3.10). Therefore, one may conclude in virtue of their completeness proved in Theorem 3.3.1. ■

3.3.6 Connection to the UCHP

As a variant of (3.1.1), the polynomials H_M are closely connected to the univariate complex Hermite polynomials $H_{j,k}$ by means of the so-called (4,2)-binomial operator $\mathcal{B}_M^{(4,2)}$, defined on double function sequences $f = (f_{m,n})_{m,n}$ on the complex plane by

$$\mathcal{B}_M^{(4,2)}(f)(z, w) := \sum_{j=0}^M (-1)^{k+k'} i^{|j|} \binom{M}{j} f_{m+n'-j-k', m'+n-j'-k}(z, \bar{z}) f_{j+k', j'+k}(w, \bar{w}).$$

with $M = (m, n, m', n')$ and $J = (j, k, j', k')$.

Theorem 3.3.13. *Set $H^{\text{complex}} := (H_{m,n})_{m,n}$. Then, we have*

$$H_{m,n,m',n'}(z, w) = \frac{1}{\sqrt{2}^{|M|}} \mathcal{B}_M^{(4,2)}(H^{\text{complex}})(\sqrt{2}z, \sqrt{2}w). \quad (3.3.25)$$

Proof. Direct computation shows that the quantity

$$Q_{m,n,m',n'}^a(z, w) := e^{-a\Delta c^2} \left((z + iw)^m (\bar{z} - i\bar{w})^n (\bar{z} + i\bar{w})^{m'} (z - iw)^{n'} \right),$$

for any positive real number $a > 0$, is equal to

$$Q_{m,n,m',n'}^a(z, w) = \mathcal{B}_M^{(4,2)}(e^{-a\Delta c}(E))(z, w),$$

where E stands for the sequence $(e_{m,n})_{m,n}$ with $e_{m,n}(z, \bar{z}) := z^m \bar{z}^n$. Therefore, using the fact that

$$e^{-a\Delta \zeta}(\zeta^m \bar{\zeta}^n) = \sqrt{a}^{m+n} H_{m,n} \left(\frac{\zeta}{\sqrt{a}}, \frac{\bar{\zeta}}{\sqrt{a}} \right),$$

we get

$$Q_{m,n,m',n'}^a(z, w) = \sqrt{a}^{|M|} \mathcal{B}_M^{(4,2)}(H^{\text{complex}}) \left(\frac{\zeta}{\sqrt{a}}, \frac{\bar{\zeta}}{\sqrt{a}} \right).$$

Finally, the identity (3.3.25) follows by taking $a = 1/2$ and making use of Theorem 3.3.8, since $Q_{m,n,m',n'}^{1/2} = H_{m,n,m',n'}$. ■

Remark 3.3.14. *The result of Theorem 3.3.13 can also be obtained starting from Theorem 3.3.4 and making use of the binomial identity for commuting operators and Rodrigues' formula (3.2.5) for the UCHP.*

Remark 3.3.15. *By taking $m' = n' = 0$ and replacing w by $-iw$ in (3.3.25), we recover the Runge formula for the UCHP [22, Proposition 3.8],*

$$H_{m,n}(z + w, \bar{z} + \bar{w}) = \frac{m!n!}{\sqrt{2}^{m+n}} \sum_{j=0}^m \sum_{k=0}^n \frac{H_{m-j-k',n-k}(\sqrt{2}z, \sqrt{2}\bar{z})}{(m-j)!(n-k)!} \frac{H_{j,k}(\sqrt{2}w, \sqrt{2}\bar{w})}{j!k!}. \quad (3.3.26)$$

A linearization formula for the product $H_{m,n}(z, \bar{z})H_{m',n'}(z, \bar{z})$ is proved by Ismail in [37, Theorem 4.1]. Here, Theorem 3.3.13 furnishes another linearization formula for such a product.

Corollary 3.3.16. *We have*

$$H_{m,n}(z, \bar{z})H_{m',n'}(\bar{z}, z) = \frac{1}{\sqrt{2}^{|M|}} \sum_{J=0; j+k'=j'+k}^M (-1)^{j+k} (j+k')! i^{|J|} \binom{M}{J} \quad (3.3.27)$$

$$\times H_{m+n'-j-k', n+m'-j-k'}(\sqrt{2}z, \sqrt{2}\bar{z}).$$

Proof. It is immediate by taking $w = 0$ in Theorem 3.3.13 and the fact that $H_{m,n,m',n'}(z, 0) = H_{m,n}(z, \bar{z})H_{n',m'}(z, \bar{z})$ as well as $H_{r,s}(0, 0) = (-1)^r r!$ when $r = s$ and $H_{r,s}(0, 0) = 0$ otherwise. ■

3.3.7 Integral Representations

The first integral representation for $H_M(Z, \bar{Z})$ follows immediately from their definition and the integral representation (3.2.9) of $H_{m,n}(z, \bar{z})$. Namely, if for given complex numbers $\alpha, \alpha', \beta, \beta'$ such that $\mu := \alpha\beta > 0$ and $\mu' := \alpha'\beta' > 0$, we let $E_{\alpha, \beta}^{\alpha', \beta'}(u, v|X, Y)$ denote the exponential function

$$E_{\alpha, \beta}^{\alpha', \beta'}(u, v|X, Y) := \exp\left(-\mu|u|^2 - \mu'|v|^2 + \alpha \langle u, X \rangle - \beta \overline{\langle u, X \rangle} + \alpha' \langle v, Y \rangle - \beta' \overline{\langle v, Y \rangle}\right), \quad (3.3.28)$$

then the following holds true

$$H_{m,n,m',n'}(z, w) = \frac{(-1)^{m+m'} \mu \mu' \alpha^m \beta^n (\alpha')^{m'} (\beta')^{n'}}{\pi^2} e^{2(|z|^2 + |w|^2)} \quad (3.3.29)$$

$$\times \int_{\mathbb{C}^2} u^m \bar{u}^n v^{m'} \bar{v}^{n'} E_{\alpha, \beta}^{\alpha', \beta'}(u, v|z + iw, \bar{z} + i\bar{w}) d\lambda(u, v).$$

It should be mentioned here that the independence of the left hand-side in the parameters $\alpha, \beta, \alpha', \beta'$ can be proved making specific change of variables. Thus one can rewrite it in a simple form for specific values of such parameters. However, we preferred to keep this form for later use.

A variant integral representation of (3.3.29), involving the auxiliary variables $\xi_{u,v} := u + iv, \bar{\xi}_{u,v} := \bar{u} - i\bar{v}, \xi_{u,v}^* := \bar{u} + i\bar{v}$ and $\widetilde{\xi}_{u,v} := u - iv$ in the integrand, is the following.

Theorem 3.3.17. For every $\alpha, \beta \in \mathbb{C}$ such that $\mu = \alpha\beta > 0$, we have

$$H_{m,n,m',n'} \left(\frac{z}{\sqrt{2}}, \frac{w}{\sqrt{2}} \right) = \frac{\mu^2 (-\alpha)^{m+n'} \beta^{m'+n}}{\pi^2 \sqrt{2}^{|M|}} e^{|z|^2 + |w|^2} \quad (3.3.30)$$

$$\times \int_{\mathbb{C}^2} (u + iv)^m (\bar{u} - i\bar{v})^n (\bar{u} + i\bar{v})^{m'} (u - iv)^{n'} E_{\alpha,\beta}^{\alpha,\beta}(u, v|z, w) d\lambda(u, v).$$

Proof. We make use of Theorem 3.3.13 and the integral representation of the UCHP given through (3.2.9). Indeed, the quantity $H_{m,n,m',n'} \left(\frac{z}{\sqrt{2}}, \frac{w}{\sqrt{2}} \right)$ can be rewritten as

$$\frac{\mu^2 (-\alpha)^{m+n'} \beta^{m'+n}}{\pi^2 \sqrt{2}^{|M|}} e^{|z|^2 + |w|^2} \int_{\mathbb{C}^2} S_M(u, v) E_{\alpha,\beta}^{\alpha,\beta}(u, v|z, w) d\lambda(u, v),$$

where $S_M(u, v) := \mathcal{B}_M^{(4,2)}(E)(u, v)$. Straightforward computation shows that $S_M(u, v)$ is simply given by

$$S_M(u, v) = (u + iv)^m (\bar{u} - i\bar{v})^n (\bar{u} + i\bar{v})^{m'} (u - iv)^{n'}.$$

This proves (3.3.30). ■

Remark 3.3.18. For the particular case of $\alpha = -\beta = i$, the identity (3.3.30) reads simply

$$H_{m,n,m',n'} \left(\frac{z}{\sqrt{2}}, \frac{w}{\sqrt{2}} \right) = \frac{(-i)^{|M|}}{\pi^2 \sqrt{2}^{|M|}} e^{|z|^2 + |w|^2} \quad (3.3.31)$$

$$\times \int_{\mathbb{C}^2} \zeta_{u,v}^m \bar{\zeta}_{u,v}^n \zeta_{u,v}^{*m'} \bar{\zeta}_{u,v}^{n'} e^{-|u|^2 - |v|^2 + 2i\Re(\langle u, z \rangle + \langle v, w \rangle)} d\lambda(u, v).$$

3.3.8 Realization as Fourier–Wigner transform of the UCHP

The next result provides another integral representation of the bivariate complex Hermite polynomials by means of the UCHP. To this end, we consider the two-dimensional Fourier–Wigner transform \mathcal{V}_2 ($d = 2$) and we set

$$\mathcal{V}(\psi, \varphi)(z; w) := \mathcal{V}_2(\psi, \varphi)((x_1, x_2); (y_1, y_2))$$

for given complex variables $z_\ell = x_\ell + iy_\ell$; $\ell = 1, 2$, with $x_\ell, y_\ell \in \mathbb{R}$, and $\psi, \varphi \in L^2(\mathbb{R}^2)$.

Theorem 3.3.19. Let $h_{m,n}(z, \bar{z}) := e^{-\frac{|z|^2}{2}} H_{m,n}(z, \bar{z})$ denote the complex Hermite functions.

Then, we have

$$H_{m,n,m',n'}(z,w) = 2(-1)^{m'+n} e^{|z|^2+|w|^2} \mathcal{V}(h_{m,n'}, h_{m',n})(2z, 2w). \quad (3.3.32)$$

Proof. Direct computation using (3.2.7) and the product formula (3.2.2), shows that for $M = (m, n, m', n')$ we have

$$\begin{aligned} & \mathcal{V}_2(h_{m,n'}, h_{m',n})(\sqrt{2}z, \sqrt{2}w) \\ &= \frac{1}{2^{|M|}} \sum_{J=0}^M (-1)^{k+k'} i^{|J|} \binom{M}{J} \mathcal{V}_2(h_{m+n'-j-k'}^{real} \otimes h_{j+k'}^{real}, h_{m'+n-j'-k}^{real} \otimes h_{j'+k}^{real})(\sqrt{2}z, \sqrt{2}w) \\ &= \frac{1}{2^{|M|}} \sum_{J=0}^M (-1)^{k+k'} i^{|J|} \binom{M}{J} \mathcal{V}_1(h_{m+n'-j-k'}^{real}, h_{m'+n-j'-k}^{real})(\sqrt{2}z) \mathcal{V}_1(h_{j+k'}^{real}, h_{j'+k}^{real})(\sqrt{2}w). \end{aligned}$$

With $M = (m, m', n, n')$ and

$$V := (V_{r,s})_{r,s} = \mathcal{V}_1(h_r^{real}, h_s^{real}).$$

But, in view of (3.2.4), we arrive at

$$\begin{aligned} \mathcal{V}_2(h_{m,n'}, h_{m',n})(\sqrt{2}z, \sqrt{2}w) &= \frac{1}{2^{|M|}} \frac{(-1)^{m'+n} \sqrt{2}^{|M|}}{2} e^{-\frac{1}{2}(|z|^2+|w|^2)} \sum_{J=0}^M (-1)^{k+k'} i^{|J|} \binom{M}{J} \\ &\quad \times H_{m+n'-j-k', m'+n-j'-k}(z, \bar{z}) H_{j+k', j'+k'}(w, \bar{w}) \\ &= \frac{(-1)^{m'+n}}{2\sqrt{2}^{|M|}} e^{-\frac{1}{2}(|z|^2+|w|^2)} \mathcal{B}_M^{(4,2)}(H^{complex})(z, w) \\ &= \frac{(-1)^{m'+n}}{2} e^{-\frac{1}{2}(|z|^2+|w|^2)} H_M\left(\frac{z}{\sqrt{2}}, \frac{w}{\sqrt{2}}\right). \end{aligned}$$

This establishes (3.3.32) by means of Theorem 3.3.13. ▀

Remark 3.3.20. *Orthogonality and completeness (Theorem 3.3.1) of $H_M(Z, \bar{Z})$ in the Hilbert space $\mathcal{H}^2(\mathbb{C}_{z,w}^2)$ can be recovered via the realization of these polynomials as the Fourier–Wigner transform of the UCHP and using the Moyal identity.*

The previous result implicitly states that the bivariate complex Hermite polynomials are expressible as a finite sum of the Fourier–Wigner transform of the tensor product

$$h_{m,m'}^{real}(t, t') := h_m^{real}(t) h_{m'}^{real}(t'). \quad (3.3.33)$$

The next result gives a direct representation of $H_{m,n,m',n'}$ as the Fourier–Wigner transform

of such a tensor product. Thus, we set

$$V_{m',n'}^{m,n} := \mathcal{V}_2(h_m^{real} \otimes h_{m'}^{real}, h_n^{real} \otimes h_{n'}^{real}). \quad (3.3.34)$$

Theorem 3.3.21. *We have*

$$H_{m,n,m',n'} \left(\frac{z}{\sqrt{2}}, \frac{w}{\sqrt{2}} \right) = \frac{(-1)^{n+n'}}{\sqrt{2}^{|M|-2}} e^{\frac{1}{2}(|z|^2+|w|^2)} V_{m',n'}^{m,n}(z+iw, \bar{z}+i\bar{w}). \quad (3.3.35)$$

Proof. Starting from (3.3.34) and using the product formula (3.2.2) as well as (3.2.4), we get

$$\begin{aligned} V_{m',n'}^{m,n}(\xi, \xi^*) &= \mathcal{V}_1(h_m^{real}, h_n^{real})(\xi) \mathcal{V}_1(h_{m'}^{real}, h_{n'}^{real})(\xi^*) \\ &= (-1)^{n+n'} \frac{\sqrt{2}^{|M|}}{2} e^{-\frac{1}{4}(|\xi|^2+|\xi^*|^2)} H_{m,n} \left(\frac{\xi}{\sqrt{2}}, \frac{\bar{\xi}}{\sqrt{2}} \right) H_{m',n'} \left(\frac{\xi^*}{\sqrt{2}}, \frac{\bar{\xi}^*}{\sqrt{2}} \right). \end{aligned}$$

By taking $\xi = z+iw$, $\bar{\xi} = \bar{z}-i\bar{w}$, $\xi^* = \bar{z}+i\bar{w}$ and $\bar{\xi}^* = z-iw = \tilde{\xi}$ for given $z, w \in \mathbb{C}$, we obtain

$$V_{m',n'}^{m,n}(\xi, \xi^*) = (-1)^{n+n'} \frac{\sqrt{2}^{|M|}}{2} e^{-\frac{1}{4}(|\xi|^2+|\xi^*|^2)} H_{m,n,m',n'} \left(\frac{z}{\sqrt{2}}, \frac{w}{\sqrt{2}} \right).$$

This is exactly (3.3.21). ■

3.3.9 Exponential generating function

In this section, we investigate some basic generating functions. The few first ones follow from Mehler formulas for the UCHP presented in Subsection 2.3.

Proposition 3.3.22. *We have*

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} H_{m,n,n,m'}(z, w) = (-t)^{-m'} H_{m,m'}(\zeta_{t,z,w}, \overline{\zeta_{t,z,w}}) e^{t(\bar{z}^2+\bar{w}^2)}. \quad (3.3.36)$$

where $\zeta_{t,z,w} := (z-t\bar{z}) + i(w-t\bar{w})$.

Proof. From (3.2.11), one deduces

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_{m,n,n,m'}(z, w) &= \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_{m,n}(\xi, \bar{\xi}) H_{n,m'}(\xi^*, \tilde{\xi}) \\ &= (-t)^{m'} H_{m,m'}(\xi - t\xi^*, \bar{\xi} - t\tilde{\xi}) e^{t\xi^*\bar{\xi}}. \end{aligned}$$

This yields (3.3.36) since $\zeta^* \bar{\zeta} = \bar{z}^2 + \bar{w}^2$ and $\zeta - t\bar{\zeta}^* = (z - t\bar{z}) + i(w - t\bar{w}) = \zeta_{t,z,w}$. ■

Remark 3.3.23. The particular case of $t = -1$ reduces further to

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} H_{m,n,n,m'}(z, w) = H_{m,m'}(2\zeta_{-1,z,w}, 2\overline{\zeta_{-1,z,w}}) e^{-(\bar{z}^2 + \bar{w}^2)}.$$

with $\zeta_{-1,z,w} = \Re(z) + i\Re(w)$.

The next one concerns the generating function

$$G_2(u, v|z, w) := \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{u^m v^n}{m!n!} H_{m,n,m,n}(z, w). \quad (3.3.37)$$

Proposition 3.3.24. For any $u, v \in \mathbb{C}$ such $|uv| < 1$, we have

$$G_2(u, v|z, w) = \frac{1}{1-uv} \exp\left(\frac{(u+v-2uv)|z|^2 - (u+v+2uv)|w|^2 + 2i\Re(z\bar{w})}{1-uv}\right).$$

Proof. The proof is straightforward and follows easily making use of (3.2.12). ■

Proposition 3.3.25. For $|u| < 1$ and $|t| = 1$, we have

$$\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{u^m t^n}{m!n!} H_{m,n,n,m}(z, w) = \frac{e^{t(\bar{z}^2 + \bar{w}^2)}}{1-ut} \exp\left(-\frac{ut}{1-ut} |z - t\bar{z} + i(w - t\bar{w})|^2\right). \quad (3.3.38)$$

In particular, we have

$$\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{u^m}{m!n!} H_{m,n,n,m}(z, w) = \frac{e^{\bar{z}^2 + \bar{w}^2}}{1-u} \exp\left(-\frac{4u}{1-u} (\Im(z))^2 + \Im(w)^2\right) \quad (3.3.39)$$

for every $u \in \mathbb{C}$ such that $|u| < 1$.

Proof. The proof of (3.3.38) is immediate in view of (3.2.13). While the generating function (3.3.39) is in fact a particular case with $t = 1$. ■

Proposition 3.3.26. For $|u| < 1$ and $|t| = 1$, we have

$$\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{u^m v^n}{m!n!} H_{m,n,n,m'}(z, w) = (tu + \zeta_{t,z,w})^{m'} \exp\left(t(\bar{z}^2 + \bar{w}^2) + u\overline{\zeta_{t,z,w}}\right). \quad (3.3.40)$$

Proof. The direct use of (3.2.14), that we rewrite in the form

$$\sum_{m,n=0}^{+\infty} \frac{u^m t^n}{m!n!} H_{m,n,n,m'}(z,w) = (t\tilde{\zeta} - t\bar{\zeta} + tu)^{m'} e^{t\bar{\zeta}\tilde{\zeta}^* + u(\tilde{\zeta} - t\tilde{\zeta}^*)},$$

which yields (3.3.40), since $\tilde{\zeta}^* \bar{\zeta} = \bar{z}^2 + \bar{w}^2$ and $\tilde{\zeta} - t\bar{\zeta} = (z - t\bar{z}) - i(w - t\bar{w}) = \tilde{\zeta}_{t,z,w}$. \blacksquare

The next result concerns the special sum

$$G_4(z,w|u,v,u',v') := \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \sum_{m'=0}^{+\infty} \sum_{n'=0}^{+\infty} \frac{u^m v^n u'^{m'} v'^{n'}}{m! n! m'! n'!} H_{m,m',n,n'}(z,w). \quad (3.3.41)$$

Proposition 3.3.27. *The function $G_4(z,w|u,v,u',v')$ is given by*

$$G_4(z,w|u,v,u',v') = e^{-uv-u'v'} e^{z(u+v)+\bar{z}(v+u')+iw(u-v)+i\bar{w}(u'-v)}. \quad (3.3.42)$$

for every $z,w,u,v,u',v' \in \mathbb{C}$.

Proof. The closed expression of $G_4(z,w|u,v,u',v')$ is immediate from the definition of the BCPHP and using the classical generating function (3.2.10) for the UCHP. \blacksquare

The last results of this section are partial generating functions for H_M .

Proposition 3.3.28. *For any $z,w,u,v,u',v' \in \mathbb{C}$, we have*

$$\sum_{m=0}^{+\infty} \sum_{m'=0}^{+\infty} \frac{u^m u'^{m'}}{m! m'!} H_{m,n,m',n'}(z,w) = (\bar{z} - i\bar{w} - u)^n (z - iw - u')^{n'} e^{u(z+iw)+u'(\bar{z}+i\bar{w})}. \quad (3.3.43)$$

and

$$\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{u^m v^n}{m! n!} H_{m,n,m',n'}(z,w) = e^{uv-u(z+iw)-v(\bar{z}-i\bar{w})} H_{m',n'}(\bar{z} + i\bar{w}, w - iw). \quad (3.3.44)$$

Proof. By rewriting the left hand-side in (3.3.43) as

$$\left(\sum_{m=0}^{+\infty} \frac{u^m}{m!} H_{m,n}(z + iw, \bar{z} - i\bar{w}) \right) \left(\sum_{m'=0}^{+\infty} \frac{u'^{m'}}{m'!} H_{m',n'}(\bar{z} + i\bar{w}, w - iw) \right)$$

and next applying the identity [22, Proposition 3.4, Eq. (3.13)]

$$\sum_{m=0}^{+\infty} \frac{u^m}{m!} H_{m,n}(\zeta, \bar{\zeta}) = (\bar{\zeta} - u) e^{u\zeta},$$

we get the identity (3.3.43). While (3.3.44) is immediate from (3.2.10) since the left hand-side takes the form

$$\left(\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{u^m v^n}{m! n!} H_{m,n}(z + iw, \bar{z} - i\bar{w}) \right) H_{m',n'}(\bar{z} + i\bar{w}, w - iw).$$

■

Remark 3.3.29. *Similar partial generating functions to the ones in Proposition 3.3.28 can be obtained by means of the symmetry properties (3.3.1) and (3.3.2).*

3.4 Concluding Remarks

Many interesting algebraic properties of the introduced polynomials can be derived as immediate consequences of those obtained in Section 3. The simpler ones, like the generating functions in Propositions 3.3.22–3.3.27, follow directly from different Mehler formulas for the UCHP. For instance, a Runge addition type formula for the polynomials H_M can be easily proved using (3.3.26):

$$H_{m,n,m',n'}(z + z', w + w') = \frac{1}{\sqrt{2}^{|M|}} \sum_{J=0}^M \binom{M}{J} H_J(\sqrt{2}z, \sqrt{2}w) H_{M-J}(\sqrt{2}z', \sqrt{2}w'). \quad (3.4.1)$$

Similarly, employing the Nielson identity for the UCHP [22, Proposition 3.2],

$$H_{m+p,n+q}(z, \bar{z}) = m!n!p!q! \sum_{j=0}^{m \wedge q} \sum_{k=0}^{n \wedge p} \frac{(-1)^{j+k}}{j!k!} \frac{H_{m-j,n-k}(z, \bar{z})}{(m-j)!(n-k)!} \frac{H_{p-k,q-j}(z, \bar{z})}{(p-k)!(q-j)!}$$

where $m \wedge q := \min(m, q)$, we can derive the following quadratic recurrence formula:

$$H_{M+N}(z, w) = \sum_{J=0}^{M \wedge N^t} (-1)^{|J|} \binom{M}{J} \binom{N}{J^t} H_{M-J}(z, w) H_{N-J^t}(z, w) \quad (3.4.2)$$

for given $M = (m, n, m', n')$ and $N = (p, q, p', q')$, with $M \wedge N^t := (m \wedge q, n \wedge p, m' \wedge q', n' \wedge p')$. Here, J^t is defined by $J^t = (k, j, k', j')$ for $J = (j, k, j', k')$.

Notice also that the closed formula of the following special generating function for the UCHP:

$$T_M(u, v|z, w) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} \frac{(-1)^k i^j u^j v^{j'} z^k w^{k'}}{j!k!j'!k'!} H_{j'+m', k'+n'}(\bar{z}, z) H_{j+m, k+n}(w, \bar{w}) \quad (3.4.3)$$

readily follows by rewriting $H_{m,n,m',n'}$ in the form

$$H_{m,n,m',n'}(z, w) = H_{m,n}(iw - (-z), -i\bar{w} - (-\bar{z})) H_{m',n'}(\bar{z} - (-i\bar{w}), z - iw)$$

and next applying [37, Theorem 4.10]:

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} H_{j+m, k+n}(z, \bar{z}) \frac{u^j v^k}{j!k!} = (-1)^{m+n} e^{uz+v\bar{z}-uv} H_{m,n}(z-v, \bar{z}-\bar{v}),$$

keeping in mind the fact that $H_{m,n}(iz, -i\bar{z}) = i^{m+n} H_{m,n}(z, \bar{z})$. Hence, we assert:

$$T_M(u, v|z, w) = (-1)^{|M|} i^{m+n'} e^{u\xi+v\bar{\xi}^*} H_{m,n,m',n'}(z, w). \quad (3.4.4)$$

Some analytic aspects of H_M are encoded in Theorems 3.3.1, 3.3.1, 3.3.19, and 3.3.21. For example, Theorem 3.3.1 shows that the Hilbert space $\mathcal{H}^2(\mathbb{C}_{z,w}^2)$ possesses different special L^2 -Hilbertian orthogonal decompositions in terms of new functional poly-analytic Hilbert spaces of Bargmann type spanned by the polynomials $H_{m,n,j,k}(z, w)$. By considering, for example,

$$\mathcal{A}_{n,j,k}^2(\mathbb{C}^2) = \overline{\text{Span}\{H_{m,n,j,k}(z, w); m = 0, 1, \dots\}}^{\mathcal{H}^2(\mathbb{C}_{z,w}^2)},$$

for fixed n, j, k , we claim that $\mathcal{A}_{n,j,k}^2(\mathbb{C}^2)$ is a Hilbert subspace of $\mathcal{H}^2(\mathbb{C}_{z,w}^2)$. The special case $\mathcal{A}_{0,0,0}^2(\mathbb{C}^2)$ is realized as $\mathcal{A}_{0,0,0}^2(\mathbb{C}^2) = \ker(A_{\bar{\xi}}) \cap \ker(A_{\xi^*}) \cap \ker(A_{\tilde{\xi}}) \cap \mathcal{H}^2(\mathbb{C}_{z,w}^2)$ and therefore is contained in the two-dimensional Bargmann–Fock space $\mathcal{F}^2(\mathbb{C}^2)$ of L^2 -holomorphic functions on \mathbb{C}^2 . It coincides with the phase space:

$$\mathcal{A}^2(\mathbb{C}^2) := \left\{ F \in \mathcal{F}^2(\mathbb{C}^2); \left(\frac{\partial}{\partial z} + i \frac{\partial}{\partial w} \right) F = 0 \right\} \quad (3.4.5)$$

which is unitary isomorphic to the configuration space $L^{2,\nu}(\mathbb{R}, \mathbb{C})$ by means of the integral transform:

$$\mathcal{G}^\nu f(z, w) := \left(\frac{c_\nu^2}{\pi} \right)^{\frac{1}{2}} \int_{\mathbb{R}} f(x) e^{-\left(x - \frac{z+iw}{2}\right)^2} dx \quad (3.4.6)$$

obtained as the composition of 1d- and 2d-Segal–Bargmann transforms (see [6]). Another interesting class of functional spaces in $\mathcal{H}^2(\mathbb{C}_{z,w}^2)$ are those defined by

$$\mathcal{A}_m^2(\mathbb{C}^2) = \overline{\text{Span}\{H_{m,n,j,k}(z,w); n,j,k = 0,1,\dots\}}^{\mathcal{H}^2(\mathbb{C}_{z,w}^2)},$$

leading to another orthogonal Hilbertian decomposition of $\mathcal{H}^2(\mathbb{C}_{z,w}^2)$. For every fixed m , the space $\mathcal{A}_m^2(\mathbb{C}^2)$ is closely connected to the concrete spectral analysis of the second-order differential operator S_ξ in (3.3.23) acting on $\mathcal{H}^2(\mathbb{C}_{z,w}^2)$. In fact, one can show that $\mathcal{A}_m^2(\mathbb{C}^2)$ is an L^2 -eigenspace of S_ξ with m as the associated eigenvalue:

$$\mathcal{A}_m^2(\mathbb{C}^2) = \ker(S_\xi - m\text{Id})|_{\mathcal{H}^2(\mathbb{C}_{z,w}^2)}.$$

In addition to $\mathcal{A}_{m,n,j}^2(\mathbb{C}^2)$, $\mathcal{A}_m^2(\mathbb{C}^2)$, and their variants, one can define the spaces $\mathcal{A}_{m,n}^2(\mathbb{C}^2)$ (as well as their variants) similarly. The following local orthogonal Hilbertian decompositions hold true:

$$\mathcal{A}_m^2(\mathbb{C}^2) = \bigoplus_{n=0}^{\infty} \mathcal{A}_{m,n}^2(\mathbb{C}^2)$$

and

$$\mathcal{A}_{m,n}^2(\mathbb{C}^2) = \bigoplus_{j=0}^{\infty} \mathcal{A}_{m,n,j}^2(\mathbb{C}^2).$$

Therefore, the global ones for $\mathcal{H}^2(\mathbb{C}_{z,w}^2)$ are the following:

$$\mathcal{H}^2(\mathbb{C}_{z,w}^2) = \bigoplus_{m=0}^{\infty} \mathcal{A}_m^2(\mathbb{C}^2) = \bigoplus_{m,n=0}^{\infty} \mathcal{A}_{m,n}^2(\mathbb{C}^2) = \bigoplus_{m,n,j=0}^{\infty} \mathcal{A}_{m,n,j}^2(\mathbb{C}^2).$$

The concrete description of these spaces is the subject of a forthcoming work.

Bibliography

- [1] Andrews G.E., Askey R., Roy R. Special functions. Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press: Cambridge; 1999.
- [2] Agorram F., Benkhadra A., El Hamyani A., Ghanmi A., Complex Hermite functions as Fourier–Wigner transform. *Integral Transforms Spec. Funct.* 27 (2016), no. 2, 94–100.
- [3] Ali S.T., Bagarello F., Honnouvo G., Modular structures on trace class operators and applications to Landau levels, *J. Phys. A*, 43, no. 10 (2010) 105202, 17 pp.
- [4] Ali S.T., Bagarello F., Gazeau J-P., Quantizations from reproducing kernel spaces. *Ann. Physics* 332 (2013), 127–142
- [5] Barrett M.J., Nonlinear analysis of travelling wave tube amplifiers using complex Hermite polynomials. Preprin, 1990.
- [6] Benahmedi A., Diki K., Ghanmi A., *On composition of Segal–Bargmann transforms.* *Complex Var. Elliptic Equ.* 64 (2019), no. 6, 950–964.
- [7] Benahmedi A., Ghanmi A., Souid El Ainin M., A special orthogonal complement basis for holomorphic Hermite functions and associated 1d- and 2d-fractional Fourier transforms. *Integral Transforms Spec. Funct.*(2019).
- [8] Benahmedi A., Ghanmi A., Non-trivial 1d and 2d Segal-Bargmann transforms. *Integral Transforms Spec. Funct.* 30 (2019), no. 7, 547–563.
- [9] BURCHNALL, J. L. A note on the polynomials of Hermite. *The Quarterly Journal of Mathematics*, 1941, no 1, p. 9-11.
- [10] CARLSON, Bille Chandler. *Special functions of applied mathematics.* 1977.

- [11] Chan J-H., The Gould-Hopper polynomials in the Novikov-Veselov equation, *Journal of Mathematical Physics*, 52 (9) (2011) 092703.
- [12] Dallinger R., Ruotsalainen H., Wichman R., Rupp M., Adaptive pre-distortion techniques based on orthogonal polynomials. In Conference Record of the 44th Asilomar Conference on Signals, Systems and Computers, IEEE (2010) pp 1945-1950.
- [13] Dattoli G., Chiccoli C., Lorenzutta S., Maino G., Torre , A Theory of generalized Hermite polynomials. *Computers and Mathematics with Applications*, 28 (4) (1994) 71-83.
- [14] Dattoli G., Germano B., Ricci P.E., Hermite polynomials with more than two variables and associated bi-orthogonal functions, *Integral Transforms and Special functions*, 20 (1) (2009) 17-22.
- [15] Dunkl C., Xu Y. Orthogonal polynomials of several variables. *Encyclopedia of Mathematics and its Applications*, 155. 2nd ed. Cambridge: Cambridge University Press; 2014.
- [16] Emil Artin ,The Gamma function, translated by Michael Butler,HOLT, Holt, Rinchart and Winston, Inc. 1964.
- [17] Erdélyi A., Magnus W. , Oberhettinger F. , Tricomi F. G., Higher Transcendental functions, Vol. II, McGraw-Hill, 1953.
- [18] Fan, Hong-Yi, and De-Hui Zhan. "New generating function formulae of even-and odd-Hermite polynomials obtained and applied in the context of quantum optics." *Chinese Physics B* 23.6 (2014): 060301.
- [19] Folland G.B., Harmonic analysis in phase space. *Annals of Mathematics Studies*, 122. Princeton University Press, Princeton, NJ, 1989.. 23, no. 6, p. 060301, 2014.
- [20] Gould, Henry W., and A. T. Hopper. "Operational formulas connected with two generalizations of Hermite polynomials." (1962): 51-63.
- [21] Ghanmi A., A class of generalized complex Hermite polynomials. *J. Math. Anal. Appl.*, 340 (2) (2008) 1395–1406.
- [22] GHANMI, Allal. Operational formulae for the complex Hermite polynomials $H_{p,q}(z, \bar{z})$. *Integral transforms and special functions*, 2013, vol. 24, no 11, p. 884-895.

- [23] GHANMI, Allal. Mehler's formulas for the univariate complex Hermite polynomials and applications. *Mathematical Methods in the Applied Sciences*, 2017, vol. 40, no 18, p. 7540-7545.
- [24] Ghanmi, Allal, and Khalil Lamsaf. "Bivariate poly-analytic hermite polynomials." *Results in Mathematics* 76.1 (2021): 3.
- [25] Ghanmi, Allal, and Khalil Lamsaf. "Two-dimensional (p, q)-heat polynomials of Gould-Hopper type." *Journal of Mathematical Analysis and Applications* 530.1 (2024): 127682.
- [26] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer, 2001.
- [27] Górska K., Horzela A., Szafraniec F.H., Holomorphic Hermite polynomials in two variables *Journal of Mathematical Analysis and Applications*, 470 (2) (2019) 750-769
- [28] HAIMO, Deborah Tepper and MARKETT, Clemens. A representation theory for solutions of a higher order heat equation, I. *Journal of mathematical analysis and applications*, 1992, vol. 168, no 1, p. 89-107.
- [29] HAIMO, Deborah Tepper and MARKETT, Clemens. A representation theory for solutions of a higher order heat equation, II. *Journal of mathematical analysis and applications*, 1992, vol. 168, no 2, p. 289-305.
- [30] Hermite C., Sur un nouveau développement en série des fonctions, *Oeuvres de Charles Hermite*, (1908) 293–308.
- [31] ITÔ, Kiyosi. Complex multiple Wiener integral. In : *Japanese journal of mathematics: transactions and abstracts*. The Mathematical Society of Japan, 1952. p. 63-86.
- [32] INTISSAR, Abdelkader et INTISSAR, Ahmed. Spectral properties of the Cauchy transform on $L^2(\mathbb{C}, e^{-|z|} \lambda(z))$. *Journal of mathematical analysis and applications*, 2006, vol. 313, no 2, p. 400-418.
- [33] Intissar A., Intissar A., Spectral properties of the Cauchy transform on $L^2(\mathbb{C}, e^{-|z|^2} \lambda(z))$, *Journal of mathematical analysis and applications*, 2006, vol. 313 (2) (2006) 400-418.
- [34] ISMAIL, Mourad. *Classical and quantum orthogonal polynomials in one variable*. Cambridge university press, 2005.

- [35] Ismail M.E.H., Analytic properties of complex Hermite polynomials. *Trans. Amer. Math. Soc.* 368 (2) (2016) 1189-1210.
- [36] Ismail M.E.H., Simeonov P., Complex Hermite polynomials: their combinatorics and integral operators, *Proc. Amer. Math. Soc.* 143 (4) (2015) 1397-1410.
- [37] ISMAIL, Mourad. Analytic properties of complex Hermite polynomials. *Transactions of the American Mathematical Society*, 2016, vol. 368, no 2, p. 1189-1210.
- [38] Khan S., Al-Saad, M.W., Summation formulae for Gould–Hopper generalized Hermite polynomials, *Computers and Mathematics with Applications*, 61 (6) (2011) 1536-1541.
- [39] N. N. Lebedev, *Special functions and Their Applications*, Dover Publications, 1972.
- [40] L. Lovász and M. D. Plummer, *Matching Theory*, North-Holland, 1986.
- [41] Liu Z-G., On the complex Hermite polynomials and partial differential equations. *arXiv:1707.08708v2*, 2017.
- [42] Lévêque O., Vignat C., About sum rules for Gould-Hopper polynomials, *arXiv preprint arXiv:1103.5168*, 2011.
- [43] Manocha H.L., Srivastava H., *A treatise on generating functions*, Mathematics and Its Applications, Ellis Horwood Series, 1984.
- [44] Matsumoto H., Quadratic Hamiltonians and associated orthogonal polynomials. *J.Funct. Anal.*, 140 (1996), 218-255.
- [45] Mehler F.G., Ueber die Entwicklung einer function von beliebig vielen Variabeln nach Laplaceschen functionen höherer Ordnung. *J. Reine Angew. Math.* 1866; 66:161–176.
- [46] Moyal J.E., Quantum mechanics as a statistical theory. *Proc. Cambridge Philos. Soc.* 45, (1949). 99–124.
- [47] Nielsen N., *Recherches sur les polynômes de Hermite*. *Mathematisk-fysiske meddelelser*. 79 pages, Det Kgl. Danske Videnskabernes Selskab, 1918, vol. 1.
- [48] Olver, Frank W., et al. *NIST handbook of mathematical functions*. Cambridge university press, 2010.
- [49] Paris, Richard B., Asymptotics of integrals of Hermite polynomials. *Appl. Math. Sci.* 4 (2010) 3043-3056.

- [50] Petojević A., A Note about the Pochhammer Symbol, *Mathematica Moravica*, 12 (1) (2008) 37-42.
- [51] Rainville, Earl D, *Special functions*, Chelsea Publishing Co., Bronx, NY, 1971.
- [52] J. Riordan, *Introduction to Combinatorial Analysis*, Princeton University Press, 1958.
- [53] Runge C., Über eine besondere Art von Integralgleichungen, *Mathematische Annalen*, 75 (1) (1914) 130-132.
- [54] Raich R., Zhou G., Orthogonal polynomials for complex Gaussian processes. *IEEE Trans. Signal Process.*, vol. 52 (2004) no. 10, pp. 2788-2797.
- [55] Slater L. J. , *Generalized Hypergeometric functions*, Cambridge University Press, 1966.
- [56] Szegő G., *Orthogonal polynomials*. Fourth edition, American Mathematical Society, Providence, R.I., 1975.
- [57] R. P. Stanley, *Enumerative Combinatorics, Volume 2*, Cambridge University Press, 1999.
- [58] Shigekawa I., Eigenvalue problems of Schrödinger operator with magnetic field on compact Riemannian manifold. *J. Funct. Anal.*, 75 (1987) 92-127.
- [59] Taimanov I.A., Tsarev S.P., Two-dimensional rational solitons and their blow-up via the Moutard transformation, *Theoret. Math. Phys.* 157 (2) (2008) 1525–1541.
- [60] Thangavelu S., *Harmonic analysis on the Heisenberg group*. Progress in Mathematics, 159. Birkhäuser Boston, Inc., Boston, MA, 1998.
- [61] Van Eijndhoven S.J.L., Meyers J.L.H., New orthogonality relations for the Hermite polynomials and related Hilbert spaces, *Journal of Mathematical Analysis and Applications*, 146 (1) (1990) 89-98.
- [62] N. C. Wormald, *Models of Random Regular Graphs*, in *Surveys in Combinatorics*, Cambridge University Press, 1999.
- [63] Wong M.W., *Weyl transforms*. Universitext. Springer-Verlag, New York, 1998.
- [64] Wünsche A., *Transformations of Laguerre 2D-polynomials and their applications to quasiprobabilities*,
- [65] Zhi-Guo Liu, *On the ternary Hermite polynomials*. arXiv:1707.08708v1, 2017.

Abstract

This thesis explores advanced poly-analytic polynomials, specifically extending the classical Gould-Hopper polynomials to two complex variables and introducing bivariate poly-analytic Hermite polynomials. Firstly, we extend the classical Gould-Hopper polynomials to encompass two complex variables, incorporating the 1-D and 2-D holomorphic and poly-analytic Itô-Hermite polynomials as particular cases. This study delves into their operational representations, generating functions, and recurrence relations. We establish numerous special identities, including multiplication formulas, Runge type addition formulas, and Nielson type formulas. Higher-order partial differential equations are analyzed, revealing connections to Gould-Hopper polynomials and hypergeometric functions. Secondly, we introduce a new class of bivariate poly-analytic Hermite polynomials. These are shown to be realizable as the Fourier-Wigner transform of univariate complex Hermite functions, forming a significant orthogonal basis in the classical Hilbert space over two-complex space with respect to the Gaussian measure. We explore their fundamental properties, including three-term recurrence relations, operational realizations, and the differential equations (Bochner's property) they satisfy. Additionally, we derive various generating functions and integral and exponential operational representations.

Keywords (5): Orthogonal polynomials, (p,q) -heat polynomials, Bi-variate Hermite polynomials, Heat equation, Fourier-Wigner transform.

Résumé

Cette thèse explore des polynômes poly-analytiques avancés, en étendant spécifiquement les polynômes classiques de Gould-Hopper à deux variables complexes et en introduisant des polynômes de Hermite bivariés poly-analytiques. Premièrement, nous étendons les polynômes classiques de Gould-Hopper pour inclure deux variables complexes, incorporant les polynômes de Itô-Hermite holomorphes et poly-analytiques en 1-D et 2-D comme cas particuliers. Cette étude examine leurs représentations opérationnelles, fonctions génératrices et relations de récurrence. Nous établissons de nombreuses identités spéciales, y compris des formules de multiplication, des formules d'addition de type Runge et des formules de type Nielson. Les équations différentielles partielles d'ordre supérieur sont analysées, révélant des connexions avec les polynômes de Gould-Hopper et les fonctions hypergéométriques.

Deuxièmement, nous introduisons une nouvelle classe de polynômes de Hermite bivariés poly-analytiques. Ceux-ci se révèlent réalisables sous la forme de la transformée de Fourier-Wigner des fonctions de Hermite complexes univariées, formant une base orthogonale significative dans l'espace de Hilbert classique sur un espace à deux variables complexes par rapport à la mesure gaussienne. Nous explorons leurs propriétés fondamentales, y compris les relations de récurrence à trois termes, les réalisations opérationnelles et les équations différentielles qu'ils satisfont. De plus, nous dérivons diverses fonctions génératrices ainsi que des représentations opérationnelles intégrales et exponentielles.

Mots-clefs (5) : Polynômes orthogonaux, Polynômes de chaleur (p,q) , Polynômes de Hermite bivariés, Équation de la chaleur, Transformée de Fourier-Wigner.