ROYAUME DU MAROC
Université Mohammed V - RABAT -
جامعة محمد الخامس – الرباط –
Faculté des sciences
كلية العلوم
CENTRE D'ETUDES DOCTORALES - SCIENCES ET TECHNOLOGIES

**Order number**: 3483

# THESIS

*In order to obtain:* ***Doctorate degree***

***Research center****:* Center of Mathematical Research and Applications of Rabat (CeReMAR).
***Research stricture****:* Laboratory of Mathematical Analysis and Applications (LAMA).
***Discipline****: Mathematics.*
***Specialty****: Mathematical analysis.*

***Presented and defended on:*** *03/ 07 /2021* ***by :***

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## *Title: On Some Quasilinear anisotropic Elliptic and Kirchhoff-type Problems.*

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**Academic year: 2020-2021**

*To My Dear Mother Zahraa El Hour.*

# ACKNOWLEDGEMENTS

This Ph.D. thesis was performed within the Laboratory of Mathematical Analysis and Applications (L.A.M.A), Center of Mathematical Research and Applications of Rabat (CeReMAR), Faculty of Sciences of Mohammed V University in Rabat. Sous la direction Mr. Abderrahmane EL HACHIMI.

First of all, I would like to express my sincere gratitude to my supervisor Prof. Abderrahmane EL HACHIMI (PES), Professor of Higher Education at Faculty of Sciences, Mohammed V University in Rabat, for having directed me in the realization of this work with as much rigor and vigilance and for his continuous support. His patience, his pedagogy, his availability, his encouragement and the confidence granted to me, have contributed to a very large extent, has the complete fulfillment of my work and the culmination of this memory thesis. It is for all these beautiful things I thank him very much.

I would like to express my gratitude to president Mr. Hamza BOUJEMAA (PES), Professor of Higher Education at Faculty of Sciences, Mohammed V University in Rabat. It is an honor for me that he chairs the examining board.

I am very grateful to reviewer and examiner Mr. Mohamed OUANNASSER (PES), Professor of Higher Education at Faculty of Sciences, Mohammed V University in Rabat. It is an honor that he accepted to examine this work. I would like to thank him very much for his availability.

I warmly thank to reviewer and examiner Mr. Houssam MAHZOULI (PH), Qualified professor at Faculty of Sciences, Mohammed V University in Rabat, for agreeing to report on this thesis, for his availability and his scientific rigor.

I want to thank reviewer and examiner Mr. Abdelilah ALAOUI LAMRANI (PH), Qualified professor of Regional Center of Education and training professions (CRMEF) in Fes. I express my sincere recognition of his interest in this work by agreeing to be a reviewer and his participation in the jury of this thesis.

I take this opportunity to thank all the professors, doctorate students and staff of the Department of Mathematics. I also want to send a special thank to all my colleagues, not only for their encouragement and collaboration.

And foremost, I would like to pay best regards and gratefulness to my all time support team, my parents , no words can be enough to thank them.

And praise to God in the beginning and in the end for guiding me in achieving this work.

# ABSTRACT

The aim of this thesis is the study of the existence of solutions of: the anisotropic quasilinear elliptic equation with variable exponent and nonlinear Robin boundary conditions, quasilinear elliptic $\vec{p}(x)$-Kirchhoff type problem with weight and nonlinear Robin boundary conditions , perturbed Kirchhoff-type non-homogeneous problems. By using Mountain Pass, Fountain theorems, and three critical point theorem due to Bonanno and Candito, we establish the existence non trivial weak solution and infinite many pairs of weak solutions to these problems and the existence of three distinct weak solutions for perturbed Kirchhoff-type non-homogeneous Neumann problems. And it contains four chapters. The first chapter is devoted to recall some background facts concerning the generalized Lebesgue–Sobolev spaces, anisotropic–Sobolev spaces, anistropic Orlicz-Sobolev spaces and introduce some notations used below . In the second chapter is mainly devoted to the existence and multiplicity of solutions of quasilinear elliptic equations under nonlinear Robin boundary condition, it turns out that the condition $q^- > P_+^+$ plays an important role in the proofs of our main results. In third chapters, we deal with the existence and multiplicity of weak solutions to a class of quasilinear elliptic $\vec{p}(x)$-Kirchhoff type problems with weight and a nonlinear Robin boundary condition. In fact, we are able in the third chapter to deal with more general nonlinearities in the boundary condition than in [42] and with situations where the function $M$ is unbounded. Perturbed Kirchhoff-type non-homogeneous Neumann problems by means of a variational approach and the use of the anistropic Orlicz-Sobolev spaces is studied in the last chapter, this is the first contribution in this direction. The readers may consult the excellent survey article of M.Mihâilescu [45].

**Keywords:** Anisotropic elliptic equations, nonlocal Kirchhoff equation, variable exponents Lebesgue spaces, anisotropic elliptic system, nonlinear Robin boundary conditions, non-standard growth condition, variational method, existence and multiplicity, three distinct weak solutions, anistropic Orlicz-Sobolev spaces.

# RÉSUMÉ

L'objectif de cette thèse est l'étude de l'existence de solutions de l'équation elliptique quasilinéaire anisotrope avec exposant variable et conditions aux limites non-linéaires de Robin, elliptique quasilinéaire Problème de type $\vec{p}(x)$-Kirchhoff avec poids et conditions aux limites non linéaires de Robin, problème perturbé de type Kirchhoff non-homogène. En utilisant les théorèmes de Mountain Pass et de Fountain, et le théorème des trois points critiques de Bonanno et Candito, nous établissons l'existence d'une solution faible non triviale et d'une infinité de paires de solutions faibles à ces problèmes, et l'existence de trois solutions faibles distinctes pour les problèmes de Neumann non homogènes de type Kirchhoff perturbés. Il contient quatre chapitres. Le premier chapitre sera consacré à rappeler quelques faits de fond concernant les espaces généralisés de Lebesgue-Sobolev, les espaces anisotropes-Sobolev et à introduire quelques notions utilisées ci-dessous.
Dans le deuxième chapitre sera principalement consacré à l'existence et à la multiplicité des solutions d'équations elliptiques quasi-linéaires sous condition aux limites non-linéaire de Robin, il s'avère que la condition $q^- > P_+^+$ joue un rôle primordial dans les preuves de nos principaux résultats. Dans le troisième chapitres, nous allons traiter l'existence et la multiplicité de solutions faibles à une classe de problèmes de type quasi-elliptique $\vec{p}(x)$-Kirchhoff avec le poids et une condition aux limites non-linéaire de Robin. En fin, nous serons capables de traiter des non-linéarités plus générales dans la condition aux limites que dans [42] et des situations où la fonction $M$ est non bornée. Le problème de Neumann non homogène de type Kirchhoff perturbés au moyen d'une approche variationnelle et l'utilisation des espaces anistropiques d'Orlicz-Sobolev sont étudiés dans le dernier chapitre, c'est la première contribution dans cette direction. Les lecteurs peuvent consulter l'excellent article de M.Mihâilescu [45]. .

**Mots Clés :** Équations elliptiques anisotropes, équation de Kirchhoff non locale, exposants variables espaces de Lebesgue, conditions aux limites de Robin non linéaires, , conditions de croissance non standard, méthode variationnelle, existence et multiplicité, trois solutions faibles distinctes, espaces anistropiques Orlicz-Sobolev.

# RÉSUMÉ DÉTAILLÉ

> Trouver quelque chose en mathématiques, c'est vaincre une inhibition et une tradition.
>
> *(Laurent Schwartz)*


Les équations aux dérivées partielles représentent un outil mathématique qui permet de décrire un ensemble des phénomènes physiques observés. Les situations dépendant du temps se traduisent plus particulièrement par des équations d'évolution tenant compte d'éventuelles interactions entre objets et événements.

Nous étudions des problèmes elliptiques non-linéaires faisant intervenir l'opérateur de Laplace $\vec{p}(x)$ défini sur des ouverts bornés de $\mathbb{R}^N$ : La non homogénéité de l'opérateur rend l'étude délicate et fait appel à des espaces fonctionnels non classiques. Ces espaces sont des cas particuliers des espaces d'Orlicz dits espaces anisotropiques de Sobolev généralisés à exposant variable
$W^{1,\overrightarrow{p}(.)}(\Omega)$ , dont la topologie est induite par une norme de manipulation ardue, appelée norme du Luxemburg. Les techniques d'approche restent la théorie des points critiques.

Les travaux présentés dans cette thèse concernent quelques équations aux dérivées partielles du type elliptique faisant intervenir l'opérateur $\Delta_{\vec{p}(x)}$ , définit par :

$$\Delta_{\vec{p}(x)} u = \sum_{i=1}^{N} \partial_{x_i} \left( |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \right).$$

L'analyse mathématique de ces équations aux dérivées partielles nécessite un choix approprié des espaces fonctionnels et une définition claire de la notion de solution (l'existence et parfois l'unicité). Ce travail est divisé en trois parties.

Dans la première partie, on s'intéresse à l'existence et multiplicité des solutions des équations du type elliptique anisotrope avec un exposant variable et une conditions aux

limites de Robin non linéaire :

$$\begin{aligned}
&-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\Big)+\sum_{i=1}^{N}|u|^{p_i(x)-2}u+\lambda|u|^{m(x)-2}u\\
&=\gamma g(x,u) \quad \text{in } \Omega, \\
&\qquad \sum_{i=1}^{N}\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}v_i=\mu|u|^{q(x)-2}u \quad \text{on } \partial\Omega,
\end{aligned} \tag{0.1}$$

Où $\Omega$ est un ouvert borné de $\mathbb{R}^N$ ($N \geq 2$). avec frontière lisse $\partial\Omega$ et $v_i$ sont les composantes du vecteur d'unité normale externe et pour $i \in \{1,\ldots,N\}$, $p_i, m \in \mathcal{C}(\bar{\Omega})$, $q \in \mathcal{C}(\partial\Omega)$.

Les fonctions $p_i$ et $g$ sont censés satisfaire à certaines conditions pour être spécifié ci-dessous, tandis que $\lambda, \gamma$, et $\mu$ sont des paramètres réels, avec $\gamma, \mu > 0$. Nous nous intéressons au cas où $q^- > P_+^+$.

Les résultats sur l'existence et la multiplicité des solutions de notre problème dans le cas où $\frac{\partial u}{\partial \nu} = 0$ sur $\partial\Omega$ et $\gamma = 0$ ont été largement étudiés et ils sont bien documentés (voir [7, 28, 34] ).

Dans la deuxième partie on s'intéresse au cas problème de type Kirchhoff $\vec{p}(x)$ elliptique quasilinéaire avec poids et conditions aux limites de Robin non linéaires. On se propose d'étudier l'existence et la multiplicité des solutions de l'équation suivante :

$$\begin{aligned}
&\left(a+bK\left(\sum_{i=1}^{N}\int_{\Omega}\frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}\mathrm{d}x\right)\right)\left(-\Delta_{\vec{p}(x)}u\right)+\sum_{i=1}^{N}V_i(x)\,|u|^{p_i(x)-2}\,u\\
&\qquad\qquad = \theta(x)\,|u|^{m(x)-2}\,u+f(x,u) \text{ in } \Omega,\\
&\qquad \sum_{i=1}^{N}\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}v_i=\eta|u|^{q(x)-2}u \quad \text{on } \partial\Omega,
\end{aligned} \tag{0.2}$$

où $\Omega$ est un ouvert borné de $\mathbb{R}^N$ ($N \geq 3$). avec frontière lisse $\partial\Omega$ , les fonctions $p_i(x)$ $(1 \leq i \leq N)$ sont continues avec : $2 \leq p_i(x) \leq N$, $a$ et $b$ sont des nombres positives , les fonctions $m$, $q$ $f$, $\theta$ et $V_i(1 \leq i \leq N)$ ont soumis à des conditions appropriées qui seront présentées ci-après.

Des résultats intéressants concernant l'existence et la multiplicité des solutions positives sont obtenus par exemple dans [1], [3], [33] et [43] via des méthodes variationnelles. Plus généralement, les problèmes liés au problème (0.2) ont été étudiés récemment par de nombreux auteurs. L'article [42] de D.Liu semble être plus proche de notre travail et lier au problème suivant

$$\begin{aligned} &\left( M\left[\int_\Omega (|\nabla u|^p + \lambda(x)|u|^p)\ \mathrm{d}x\right]\right)^{p-1} \left(-\Delta_p u + \lambda(x)|u|^{p-2}u\right) = f(x,u), \\ &\text{in } \Omega, \\ &|\nabla u|^{p-2}\frac{\partial u}{\partial n} = \eta |u|^{p-2}u; \quad \text{on } \partial\Omega, \end{aligned} \tag{0.3}$$

Nous avons l'intention ici d'étendre certains des résultats de [42] au problème anisotrope avec des exposants variables et des termes non locaux plus généraux. De plus, nous allons traiter des non-linéarités plus générales dans la condition aux limites que dans [42] et des situations où la fonction $M$ dans (0.3) n'est pas bornée d'en haut. Lorsque $\lambda(x) := 0$ et $p$ est constant dans (0.3), nos résultats s'appliquent par exemple à la fonction $M(t) = (\frac{bk_1}{p}t + a)^{\frac{1}{p-1}}$ , où $k_1$ est une constante réelle positive qui sera définie dans l'hypothèse (K0) ci-dessous. Bien que cette fonction ne satisfasse pas la condition (m1) du théorème 1.1 dans ([42]), l'existence d'une solution faible est cependant garantie par nos résultats.

Dans la troisième partie, nous nous intéressons au problème de Neumann non homogène de type Kirchhoff perturbés de la forme :

$$\begin{aligned} &-K\left(\sum_{i=1}^N \int_\Omega \phi_i(\left|\frac{\partial u}{\partial x_i}\right|) + \phi_i(|u|)\mathrm{d}x\right) \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\alpha_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right)\frac{\partial u}{\partial x_i} + \alpha_i(|u|)\,u\right) \\ &= \Theta(x)|u|^{m(x)}u + \lambda f(x,u) \text{ in } \Omega, \\ &\sum_{i=1}^N \alpha_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right)\frac{\partial u}{\partial x_i}v_i = \mu g(x,u) \quad \text{on } \partial\Omega. \end{aligned} \tag{0.4}$$

tel que $\Omega \subset \mathbb{R}^n$ est un domaine borné où $n \geq 2$, avec la frontière régulière $\partial\Omega$ et $v_i$ sont les composants du vecteur d'unité normale externe, $m \in C(\bar{\Omega})$ et pour $i \in \{1, \ldots, N\}$, $\alpha_i : (0,\infty) \longrightarrow \mathbb{R}$ la fonction donnée pour que la fonction $\varphi_i : \mathbb{R} \longrightarrow \mathbb{R}$ définie par :

$$\varphi_i(t) = \begin{cases} \alpha_i(t)t & \text{for } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases} \tag{0.5}$$

soit impaire , l'homéomorphisme strictement croissant de $\mathbb{R}$ à $\mathbb{R}$. Pour la fonction précédent $\varphi_i$, on définit

$$\phi_i(t) = \int_0^t \varphi_i(s)\mathrm{d}s \text{ for all } t \in \mathbb{R}.$$

Les fonctions $f$ et $g$ sont censées remplir certaines conditions pour être spécifié ci-dessous. L'outil principalement utilisé tout au long de ce travail demeure une variante du théorème des trois points critiques de Bonanno et Candito.

# CONTENTS

# CHAPTER 1

# PRELIMINARIES AND DEFINITIONS

## 1.1 Classical Lebesgue-Sobolev spaces

The aim of this part is to suggest appropriate analogues of the Lebesgue spaces $L^p$ and of the Sobolev spaces $W^{k,p}$. It is clear that we cannot simply replace $p$ by $p(x)$ in the usual definition of the norm in $L^p$. However, the Lebesgue spaces can be considered as particular cases of the Orlicz spaces belonging to a larger family of so called modular spaces. This approach enables to define corresponding counterparts of the Luxemburg and Orlicz norms in $L^{p(x)}$. If the function $p$ is finite a.e. in $\Omega$, then $L^{p(x)}$ is a particular case of the so called Orlicz-Musielak spaces treated by J. Musieiak.

We extend the definition of $L^{p(x)}$ for functions $p$ taking the values from $[1, +\infty[$.
Let $\Omega \subset \mathbb{R}^N$ be a measurable subset with $\text{meas}(\Omega) > 0$. We write

$$\mathcal{C}(\overline{\Omega}) = \{u : u \text{ is a continuous function in } \overline{\Omega}\},$$
$$\mathcal{C}_+(\overline{\Omega}) = \{u \in \mathcal{C}(\overline{\Omega}) : \operatorname{ess\,inf}_\Omega u \geq 1\}.$$

Denote by $\mathcal{S}(\Omega)$ the set of all measurable real functions defined on $\Omega$. Two functions in $\mathcal{S}(\Omega)$ are considered as the same element of $\mathcal{S}(\Omega)$ when they are equal almost everywhere.
Suppose that $\Omega$ is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and $p \in \mathcal{C}(\overline{\Omega}, \mathbb{R})$ with $p(x) > 1$, for any $x \in \Omega$.
Denote $p^- = \inf_{x\in\Omega} p(x)$ , $p^+ = \sup_{x\in\Omega} p(x)$ .

In addition, we denote

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

and

$$p^\partial(x) = \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

then we have, $p^- > 1$ and $p^+ < \infty$. Denote by $\mathcal{M}$ be either $\Omega$ or $\partial\Omega$. Define the variable exponent Lebesgue space

$$L^{p(x)}(\mathcal{M}) = \{u \mid u : \mathcal{M} \to \mathbb{R} \text{ is measurable and } \int_{\mathcal{M}} |u(x)|^{p(x)}\, dx < \infty\},$$

endowed with the Luxembourg norm

$$|u|_{p(x)} = |u|_{L^{p(x)}(\mathcal{M})} = \inf\Big\{\tau > 0; \int_{\mathcal{M}} \Big|\frac{u(x)}{\tau}\Big|^{p(x)}\, dx \leq 1\Big\}.$$

**Proposition 1.1.1** ([20]). *Let $\rho(u) = \int_{\mathcal{M}} |u(x)|^{p(x)}\, dx$. For $u, u_k \in L^{p(x)}(\mathcal{M})(k = 1, 2, \dots)$, we have:*

1. $|u|_{L^{p(x)}(\mathcal{M})} \leq 1 \Rightarrow |u|^{p^+}_{L^{p(x)}(\mathcal{M})} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}(\mathcal{M})}$.
2. $|u|_{L^{p(x)}(\mathcal{M})} > 1 \Rightarrow |u|^{p^-}_{L^{p(x)}(\mathcal{M})} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(\mathcal{M})}$.
3. $|u_k|_{L^{p(x)}(\mathcal{M})} \to 0 \Leftrightarrow \rho(u_k) \to 0$.
4. $|u_k|_{L^{p(x)}(\mathcal{M})} \to \infty \Leftrightarrow \rho(u_k) \to \infty$.

If, in addition, $(u_n)_n \subset L^{p(.)}(\mathcal{M})$ , then

$$\lim_{n\to+\infty} \|u_n - u\|_{L^{p(.)}(\mathcal{M})} = 0 \Leftrightarrow \lim_{n\to+\infty} \rho(u_n - u) = 0$$
$$\Leftrightarrow (u_n)_n \text{ converges to } u \text{ in measure and } \lim_{n\to+\infty} \rho(u_n) = \rho(u) \tag{1.1}$$

We define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

endowed with the norm

$$\|u\| = \inf\Big\{\tau > 0; \int_{\Omega} \Big(\Big|\frac{\nabla u(x)}{\tau}\Big|^{p(x)} + \Big|\frac{u(x)}{\tau}\Big|^{p(x)}\Big)\, dx \leq 1\Big\}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Hereafter, we always assume that $p^- > 1$.

**Proposition 1.1.2** (See [25]). *The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

**Proposition 1.1.3** (See [25]). *The Hölder inequality holds, namely*

$$\int_{\mathcal{M}} |uv|\, dx \leq 2|u|_{p(x)}|v|_{q(x)}; \quad \forall u \in L^{p(x)}(\mathcal{M}),\ \forall v \in L^{q(x)}(\mathcal{M}),$$

*where* $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

**Proposition 1.1.4** (See [25]). *In $W_0^{1,p(x)}(\Omega)$ the Poincaré inequality holds, that is, there exists a positive constant $c$ such that*

$$|u|_{L^{p(x)}(\Omega)} \leq c|\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

*So $|\nabla u|_{L^{p(x)}(\Omega)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$.*

**Proposition 1.1.5** (See [20]). *Let $\rho(u) = \int_\Omega (|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)})\, dx$. For $u, u_k \in W^{1,p(x)}(\Omega)(k = 1, 2, \dots)$, we have*

1. $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$.
2. $\|u\| > 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$.
3. $\|u_k\| \to 0 \Leftrightarrow \rho(u_k) \to 0$.
4. $\|u_k\| \to \infty \Leftrightarrow \rho(u_k) \to \infty$.

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Hereafter, we always assume that $p^- > 1$.

**Proposition 1.1.6** (See [25]). *The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

**Proposition 1.1.7** (See [25]). *In $W_0^{1,p(x)}(\Omega)$ the Poincaré inequality holds, that is, there exists a positive constant $c$ such that*

$$|u|_{L^{p(x)}(\Omega)} \leq c|\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

*So $|\nabla u|_{L^{p(x)}(\Omega)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$.*

**Proposition 1.1.8** (See [25]). *Assume that the boundary of $\Omega \subset \mathbb{R}^N$, $N \geq 2$ possesses the cone property and $p \in C_+(\overline{\Omega})$. If $q \in C(\overline{\Omega})$ and $1 \leq q(x) \leq p^*(x)$ for $x \in \overline{\Omega}$, then there is a compact embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

**Proposition 1.1.9** (See [25]). *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded domain with smooth boundary. Suppose that $p \in \mathcal{C}_+(\overline{\Omega})$ and $r \in \mathcal{C}(\overline{\Omega})$ satisfy the condition*

$$1 \leq r(x) \leq p^{\partial}(x), \quad \forall x \in \partial\Omega$$

*Then there is a compact boundary trace embedding*

$$W^{1,p(.)}(\Omega) \hookrightarrow L^{r(.)}(\partial\Omega).$$

Two functionals that have interesting properties are $\psi, \varphi : W^{1,p(.)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\psi(u) = \int_{\Omega} \frac{1}{q(x)}|u(x)|^{q(x)}\,\mathrm{d}x, \qquad \varphi(u) = \int_{\partial\Omega} \frac{1}{r(x)}|u(x)|^{r(x)}\,\mathrm{d}x \tag{1.2}$$

**Proposition 1.1.10.** *The functionals $\varphi, \psi : W^{1,p(.)}(\Omega) \longrightarrow \mathbb{R}$ are weakly-strongly continuous, that is, $u_n \rightharpoonup u$ (weakly) implies $\psi(u_n) \longrightarrow \psi(u)$, respectively, $\varphi(u_n) \longrightarrow \varphi(u)$, as $n \longrightarrow \infty$.*

*Proof.* Let $(u_n)_n \subset W^{1,p(.)}(\Omega)$ be such that $u_n \rightharpoonup u$ (weakly) in $W^{1,p(.)}(\Omega)$. By proposition (1.1.8) and proposition (1.1.9), we have

$$W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega) \text{ compactly.}$$

and

$$W^{1,p(.)}(\Omega) \hookrightarrow L^{r(.)}(\partial\Omega) \text{ compactly.}$$

Therefore, passing eventually to a subsequence, we get the strong convergence of $(u_n)_n$ to $u$ in $L^{q(.)}(\Omega)$ and $L^{r(.)}(\partial\Omega)$. This only means that

$$\|u_n - u\|_{L^{q(.)}(\Omega)} \longrightarrow 0, \text{ respectively, } \|u_n - u\|_{L^{r(.)}(\partial\Omega)} \longrightarrow 0.$$

as $n \longrightarrow \infty$. Using (1.1) we have arrived to

$$\lim_{n\to+\infty} \rho_{\Omega,q(.)}(u_n) = \rho_{\Omega,q(.)}(u) \text{ , respectively } \lim_{n\to+\infty} \rho_{\Omega,r(.)}(u_n) = \rho_{\partial\Omega,r(.)}(u)$$

Since

$$|\psi(u_n) - \psi(u)| \leq \frac{1}{q^-}|\rho_{\Omega,q(.)}(u_n) - \rho_{\Omega,q(.)}(u)| \text{ and}$$

$$|\varphi(u_n) - \varphi(u)| \leq \frac{1}{r^-}|\rho_{\partial\Omega,q(.)}(u_n) - \rho_{\partial\Omega,q(.)}(u)|,$$

our proof is complete. □

Let us now consider the weighted variable exponent Lebesgue space.
Let $a \in \mathcal{S}(\Omega)$ and $a(x) > 0$ for $x \in \Omega$. Define

$$L^{p(x)}_{a(x)}(\Omega) = \{u \in \mathcal{S}(\Omega) : \int_{\Omega} a(x)|u(x)|^{p(x)}\,\mathrm{d}x < \infty\},$$

with the norm

$$|u|_{L^{p(x)}_{a(x)}(\Omega)} = |u|_{(p(x),a(x))} = \inf\Big\{\tau > 0; \int_\Omega a(x)|\frac{u(x)}{\tau}|^{p(x)}\,\mathrm{d}x \le 1\Big\}.$$

then $L_{(p(x),a(x))}(\Omega)$ is a Banach space. The following proposition follows easily from the definition of $|u|_{L^{p(x)}_{a(x)}(\Omega)}$

**Proposition 1.1.11** ([23]). *Let $\rho(u) = \int_\Omega a(x)|u(x)|^{p(x)}\,\mathrm{d}x$. For $u, u_k \in L^{p(x)}_{a(x)}(\Omega)$ $(k = 1,2,\dots)$, we have:*

1. $|u|_{(p(x),a(x))} \le 1 \Rightarrow |u|^{p^+}_{(p(x),a(x))} \le \rho(u) \le |u|^{p^-}_{(p(x),a(x))}$.
2. $|u|_{(p(x),a(x))} > 1 \Rightarrow |u|^{p^-}_{(p(x),a(x))} \le \rho(u) \le |u|^{p^+}_{(p(x),a(x))}$.
3. $|u_k|_{(p(x),a(x))} \to 0 \Leftrightarrow \rho(u_k) \to 0$.
4. $|u_k|_{(p(x),a(x))} \to \infty \Leftrightarrow \rho(u_k) \to \infty$.

**Proposition 1.1.12** (See [23]). *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$ be a bounded domain with smooth boundary. Suppose that $p \in \mathcal{C}(\overline{\Omega})$. Suppose that $a \in Ł^{r(x)}(\Omega)$, $a(x) > 0$ for $x \in \Omega$, $r \in \mathcal{C}(\overline{\Omega})$ and $r^- > 0$ if $q \in \mathcal{C}(\overline{\Omega})$ and*

$$1 \le q(x) \le \frac{r(x)-1}{r(x)}p^*(x), \quad \forall x \in \overline{\Omega} \tag{1.3}$$

*Then there is a compact boundary trace embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{a(x)}(\Omega).$$

*Proof.* Let $u \in W^{1,p(x)}(\Omega)$. Set $h(x) = r^0(x)r(x) = \frac{r(x)}{r(x)-1}q(x)$. Then (1.3) implies $h(x) < p^*(x)$ and hence there is a compact embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{h(x)}(\Omega).$$

For $u \in W^{1,p(x)}(\Omega)$ we have $|\ u(x)\ |^{q(x)} \in L^{r^0(x)}(\Omega)$ and, by Proposition (1.1.3)

$$\int_\Omega |u(x)|^{q(x)}\,\mathrm{d}x \le 2|a|_{r(x)}|\ |\ u(x)\ |^{q(x)}\ |_{r^0(x)} < \infty.$$

This shows $W^{1,p(x)}(\Omega) \subset L^{p(x)}_{a(x)}(\Omega)$. Now let $\{u_n\} \subset W^{1,p(x)}(\Omega)$ and $u_n \rightharpoonup 0$ (weakly) in $W^{1,p(x)}(\Omega)$. Then $u_n \longrightarrow 0$ (strongly) in $L^{h(x)}(\Omega)$ and from this it follows that $|\ |\ u_n(x)\ |^{q(x)}\ |_{r^0(x)} \longrightarrow 0$. Thus we have

$$\int_\otimes |u_n(x)|^{q(x)}\,\mathrm{d}x \le 2|a|_{r(x)}|\ |\ u_n(x)\ |^{q(x)}\ |_{r^0(x)} \longrightarrow 0.$$

which implies $|u_n|(q(x), a(x)) \longrightarrow 0$. This shows that the embedding

$$W^{1,p(x)}(\Omega) \longrightarrow L^{q(x)}_{a(x)}(\Omega)$$

is compact. The proof is complete □

**Proposition 1.1.13** (See [23]). *Assume that $0 \in \overline{\Omega}$ and the boundary of $\Omega$ possesses the cone property.*
*Suppose that $p, s, q \in \mathcal{C}(\overline{\Omega})$ , $0 \leq s(x) < N$ for $x \in \overline{\Omega}$. If $q$ satisfies the condition*

$$1 \leq q(x) < \frac{N - s(x)}{N} p^*(x), \quad \forall x \in \overline{\Omega} \tag{1.4}$$

*then there is a compact embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{|x|^{-s(x)}}(\Omega).$$

*Proof.* By the compactness of $\overline{\Omega}$ we can find a positive constant $\epsilon$ small enough such that

$$s(x) < N - \epsilon, \quad q(x) < \frac{N - \epsilon - s(x)}{N - \epsilon} p^*(x), \quad \forall x \in \overline{\Omega}$$

Applying proposition (1.1.12) to the case that $a(x) =| x |^{-s(x)}$ and $r(x) = \frac{N-\epsilon}{s(x)}$, we obtain the proposition. □

## 1.2 Classical anisotropic Sobolev space

Now we can present the anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(.)}(\Omega)$ where $\overrightarrow{p} : \overline{\Omega} \longrightarrow \mathbb{R}^N$ is the vectorial function

$$\overrightarrow{p}(.) = (p_1(.), ..., p_N(.)),$$

where for all $i \in \{1, ..., N\}$, $p_i \in \mathcal{C}_+(\overline{\Omega})$, we set

$$p_M(x) = \max\{p_1(x), ..., p_N(x)\} \text{ and } p_m(x) = \min\{p_1(x), ..., p_N(x)\}.$$

for all $x \in \overline{\Omega}$. The anisotropic Sobolev space with variable exponent is introduced by

$$W^{1,\overrightarrow{p}(.)}(\Omega) = \left\{ u \in Ł^{p_M(.)}(\Omega) : \quad \partial_{x_i} u \in L^{p_i(.)}(\Omega) \text{ for all } i \in \{1, ..., N\} \right\}$$

$$\begin{aligned} W^{1,\overrightarrow{p}(.)}(\Omega) &= \left\{ u \in Ł^{p_M(.)}(\Omega) : \quad \partial_{x_i} u \in L^{p_i(.)}(\Omega) \text{ for all } i \in \{1, ..., N\} \right\} \\ &= \left\{ u \in L^1_{loc}(\Omega) : \quad u \in L^{p_i(.)}(\Omega), \partial_{x_i} u \in L^{p_i(.)}(\Omega) \text{ for all } i \in \{1, ..., N\} \right\}. \end{aligned}$$

and it is endowed with the norm

$$\|u\|_{W^{1,\overrightarrow{p}(.)}(\Omega)} = | u |_{L^{p_M(.)}(\Omega)} + \sum_{i=1}^{N} |\partial_{x_i} u|_{L^{p_i(.)}(\Omega)}.$$

the space $(W^{1,\overrightarrow{p}}(\Omega), \|.\|_{W^{1,\overrightarrow{p}}(\Omega)})$ a reflexive Banach space for any $\overrightarrow{p} \in \mathcal{C}(\Omega, \mathbb{R}^N)$ with $p_i^- > 1$ forall $i \in \{1, ..., N\}$. An important subspace of $W^{1,\overrightarrow{p}(.)}(\Omega)$ is $W_0^{1,\overrightarrow{p}(.)}(\Omega)$, that is, the subspace of the functions that are vanishing on the boundary. The anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ as the closure of $\mathcal{C}_0^{\infty}(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}(.)} = \sum_{i=1}^{N} | \partial_{x_i} u |_{p_i(.)} .$$

In the case when $p_i \in \mathcal{C}_+(\Omega)$ are constant functions for any $i \in \{1, ..., N\}$ the resulting anisotropic Sobolev space is denoted by $W_0^{1,\overrightarrow{p}}(\Omega)$, where $p$ is the constant vector $(p_1, ..., p_N)$. According to [28], the space $(W_0^{1,\overrightarrow{p}}(\Omega), \|.\|_{W_0^{1,\overrightarrow{p}}(\Omega)})$ a reflexive Banach space for any $\overrightarrow{p} \in \mathcal{C}(\Omega, \mathbb{R}^N)$ with $p_i^- > 1$ forall $i \in \{1, ..., N\}$.

We denote by $\overrightarrow{P}_+, \overrightarrow{P}_- \in \mathbb{R}^N$ the vectors

$$\overrightarrow{P}_+ = (p_1^+, ..., p_N^+), \quad \overrightarrow{P}_- = (p_1^-, ..., p_N^-),$$

and by $P_+^+, P_-^+, P_-^- \in \mathbb{R}^+$ the following:

$$P_+^+ = \max\{p_1^+, ..., p_N^+\}, \quad P_-^+ = \max\{p_1^-, ..., p_N^-\}, \quad P_-^- = \min\{p_1^-, ..., p_N^-\}.$$

Below we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1 \tag{1.5}$$

and define $P_-^\star, P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^\star := \frac{N}{\sum_{i=1}^N \frac{1}{p_i^-} - 1} \quad \text{and } P_{-,\infty} := \max\{P_-^+, P_-^\star\}$$

**Proposition 1.2.1** ([43], Theorem 1). *Suppose that $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary and relation (1.5) is fulfilled. For any $q \in \mathcal{C}(\overline{\Omega})$ verifying*

$$1 < q(x) < P_{-,\infty} \quad \forall x \in \overline{\Omega}, \tag{1.6}$$

*the embedding*

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega).$$

*is continuous and compact.*

*Proof.* Clearly $L^{p_i(.)}(\Omega)$ is continuously embedded in $L^{p^-}(\Omega)$ for any $i \in \{1, ..., N\}$, since $p_i^- \leq p_i(x)$ for all $x \in \overline{\Omega}$. Thus, for each $i \in \{1, ..., N\}$ there exists a positive constant $C_i > 0$ such that

$$|\varphi|_{p_i^-} \leq C_i |\varphi|_{p_i(.)} \text{ for all } \varphi \in L^{p_i(.)}(\Omega).$$

If $u \in W^{1,\overrightarrow{p}(.)}(\Omega)$, then $\partial_{x_i} u \in L^{p_i(.)}(\Omega)$ for each $i \in \{1, ..., N\}$. The above inequalities imply

$$\|u\|_{\overrightarrow{P}_-} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i^-} \leq C \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(.)} = C\|u\|_{\overrightarrow{P}}(.),$$

where $C = \max\{C_1, ..., C_N\}$. Thus, we deduce that $W^{1,\overrightarrow{p}(.)}(\Omega)$ is continuously embedded in $W^{1,\overrightarrow{p}_-}(\Omega)$. On the other hand, since relation (**??**) holds true, we infer that $m^+ < P_{-,\infty}$. This fact combined with the result of Theorem 1 in [29] implies that $W^{1,\overrightarrow{p}_-}(\Omega)$ is compactly embedded in $L^{m^+}(\Omega)$. Finally, since $m(x) \leq m^+$ we deduce that $L^{m^+}(\Omega)$ is continuously embedded in $L^{m(.)}(\Omega)$. The above piece of information yields to the conclusion that $W^{1,\overrightarrow{p}(.)}(\Omega)$ is compactly embedded in $L^{m(.)}(\Omega)$. The proof of proposition **??** is complete. □

**Proposition 1.2.2** (See [9]). *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded domain with smooth boundary and let $q \in \mathcal{C}(\overline{\Omega})$ satisfy the condition*

$$1 \leq q(x) < \min_{x \in \partial\Omega} \{p_1^\partial(x), ..., p_N^\partial(x)\}, \forall x \in \partial\Omega.$$

*Then, there is a compact boundary trace embedding*

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega).$$

Next, we introduce the notions of he Orlicz–Sobolev spaces $W^1 L_{\phi_i}(\Omega)$ and the anisotropic Orlicz–Sobolev space $W^1 L_{\vec{\phi}}(\Omega)$, with variable exponent.
Assume that $\varphi_i : \mathbb{R} \longrightarrow \mathbb{R}$, $i \in \{1, .., N\}$, are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. Define

$$\phi_i(t) = \int_0^t \varphi_i(s) ds, \quad (\phi_i)^*(t) = \int_0^t (\varphi_i)^{-1}(s) ds \text{ for all } t \in \mathbb{R}, i \in \{1, ..., N\}.$$

We observe that $\phi_i$, $i \in \{1, ..., N\}$, are Young functions; i.e., $\phi_i(0) = 0$, $\phi_i$ are convex, and $\lim_{x\to\infty} \phi_i(x) = +\infty$. Furthermore, since $\phi_i(x) = 0$ if and only if $x = 0$, $\lim_{x\to 0} \frac{\phi_i(x)}{x} = 0$, and $\lim_{x\to\infty} \frac{\phi_i(x)}{x} = \infty$, then $\phi_i$ are called N-functions. The functions $(\phi_i)^*$, $i \in \{1, ..., N\}$, are called the complementary functions of $\phi_i$, $i \in \{1, ..., N\}$, and they satisfy

$$(\phi_i)^*(t) = \sup\{st - \phi_i(s); s \geq 0\}, \quad \forall t \geq 0.$$

We also observe that$(\phi_i)^*$, $i \in \{1, ..., N\}$ are also N-functions, and Young's inequality holds true:

$$st \leq \phi_i(s) + (\phi_i)^*(t), \text{ forall } s, t \geq 0.$$

The Orlicz spaces $L_{\phi_i}(\Omega)$, $i \in \{1, ..., N\}$, defined by the N-functions $\phi_i$ (see [46]) are the spaces of measurable $u : \Omega \longrightarrow \mathbb{R}$ such that

$$||u||_{L_{\phi_i}} := \sup \left\{ \int_\Omega u(x)v(x); \int_\Omega (\phi_i)^*(|v(x)|)dx \leq 1 \right\} < \infty$$

Then $(L_{\phi_i(\Omega)}, ||u||_{L_{\phi_i}})$, $i \in \{1, ..., N\}$ , are Banach spaces whose norms are equivalent to the Luxemburg norms

$$||u||_{\phi_i} := \inf \left\{ k > 0; \int_\Omega \phi_i \left( \frac{u(x)}{k} \right) dx \leq 1 \right\}$$

For Orlicz spaces, Hölder's inequality reads as follows (see [46]):

$$\int_\Omega uv dx \leq 2||u||_{L_{\phi_i}} ||v||_{L_{(\phi_i)^*}}$$

We denote by $W^1L_{\phi_i}(\Omega)$ the Orlicz–Sobolev spaces defined by

$$W^1L_{\phi_i}(\Omega) = \{u \in L_{\phi_i}(\Omega) : \frac{\partial u}{\partial x_j} \in L_{\phi_i}, j = 1, ..., N\},$$

These are Banach spaces with respect to the norms

$$||u||_{1,\phi_i} := ||u||_{\phi_i} + || \mid \nabla u \mid ||_{\phi_i}, \quad \forall i \in \{1, ..., N\}$$

We also define the Orlicz–Sobolev spaces $W_0^1L_{\phi_i}(\Omega)$ as the closure of $C_0^1(\Omega)$ in $W^1L_{\phi_i}(\Omega)$. By [46], we obtain that on $W_0^1L_{\phi_i}(\Omega)$, $i \in \{1, ..., N\}$, we may consider the equivalent norm

$$||u|| := || \mid \nabla u \mid ||_{\phi_i}.$$

Moreover, it can be proved that the above norm is equivalent to the following norm:

$$||u||_{i,1} := \sum_{i=1}^{N} ||\partial_j u||_{\phi_i}.$$

For an easier manipulation of Orlicz–Sobolev spaces, we define

$$(p_i)_0 := \inf_{t>0} \frac{t\varphi_i(t)}{\phi_i(t)} \text{ et } (p_i)^0 := \sup_{t>0} \frac{t\varphi_i(t)}{\phi_i(t)}, \quad i \in \{1, ..., N\}.$$

The above relation implies that each $\phi_i$ satisfies the $\Delta_2$-condition; i.e.,

$$\phi_i(2t) \leq C\phi_i(t), \quad \forall t \geq 0, \tag{1.7}$$

where $C$ is a positive constant (see [46], Proposition 2.3).
Furthermore, in this paper we assume that for each $i \in \{1, ..., N\}$ the function $\phi_i$ satisfies the following condition:

$$\text{the function } t \in [0, \infty) \longrightarrow \phi_i(\sqrt{t}) \text{ is convex .} \tag{1.8}$$

Conditions (1.7) and (1.8) ensure that for each $i \in \{1, ..., N\}$ the Orlicz spaces $L_{\phi_i}(\Omega)$ are uniformly convex spaces, and thus reflexive Banach spaces (see [[46], Proposition 2.2]). That fact implies that the Orlicz–Sobolev spaces $W_0^1 L_{\phi_i}(\Omega)$, $i \in \{1, ..., N\}$ are also reflexive Banach spaces.
Finally, we introduce a natural generalization of the Orlicz–Sobolev spaces $W^1 L_{\phi_i}(\Omega)$ that will enable us to study problem (4.1), with sufficient accuracy. For this purpose, let us denote by $\vec{\phi} : \Omega \longrightarrow \mathbb{R}^N$ the vectorial function $\vec{\phi} = (\phi_1, ..., \phi_N)$. We define $W^1 L_{\vec{\phi}}(\Omega)$ , the anisotropic Orlicz–Sobolev space with respect to the norm

$$||u||_{\vec{\phi}} = \sum_{i=1}^{N} ||\partial_i u||_{\phi_i}.$$

By $E_i(i \in \{1, ..., N\}$ we mean the Orlicz-Sobolev spaces defined by

$$E_i = \left\{ u \in L_{\phi_i}(\Omega) : \int_\Omega \left( \phi_i \left( \left| \frac{\partial u}{\partial x_i} \right| \right) + \phi_i(|u|) \right) \mathrm{d}x < \infty \right\}$$

These are Banach spaces with respect to the norms,

$$\|u\|_i = \inf \left\{ k > 0; \int_\Omega \left( \phi_i \left( \frac{1}{k} \left| \frac{\partial u}{\partial x_i} \right| \right) + \phi_i \left( \frac{|u|}{k} \right) \right) \mathrm{d}x \leq 1 \right\}.$$

and (see [46]) the anisotropic Orlicz-Sobolev space $W^1 L_{\vec{\phi}}(\Omega)$ can be defined as the closure of $\mathcal{C}_0^1(\Omega)$ with respect to the norm

$$\|u\|_{1,\vec{\phi}} = \sum_{i=1}^{N} \|u\|_i.$$

We have that $W^1 L_{\vec{\phi}}(\Omega)$ is compactly embedded in $\mathcal{C}^0(\bar{\Omega})$ and there exists a constant $c > 0$ such that

$$\|u\|_\infty \leq c \|u\|_{1,\vec{\phi}}, \quad \text{for all } u \in W^1 L_{\vec{\phi}}(\Omega) \tag{1.9}$$

where $\|u\|_\infty := \sup_{x \in \bar{\Omega}} |u(x)|$.

**Proposition 1.2.3** (See [45]). *Let $u \in W^1 L_{\vec{\phi}}(\Omega)$ and assume that*

$$\int_\Omega \phi_i \left( \left| \frac{\partial u}{\partial x_i} \right| \right) + \phi_i(|u|) \mathrm{d}x \leq r_i, \text{ for all } i \in \{1, ..., N\}$$

*with $\forall i \in \{1, ..., N\}$, $r_i > 0$ and $\sum_{i=1}^{n} r_i < 1$. Then $\|u\|_{1,\vec{\phi}} < 1$.*

On the other hand, in order to facilitate the manipulation of the space $W^1 L_{\vec{\phi}}(\Omega)$, we introduce $\overrightarrow{P^0}, \overrightarrow{P_0} \in \mathbb{R}^N$ as

$$\overrightarrow{P^0} = \left( (p_1)^0, ..., (p_N)^0 \right), \quad \overrightarrow{P_0} = ((p_1)_0, ..., (p_N)_0)$$

and $(P^0)^+, (P_0)^+, (P_0)^- \in \mathbb{R}^+$ as

$$(P^0)^+ = \max\left\{(p_1)^0, ..., (p_N)^0\right\}, \quad (P_0)^+ = \max\left\{(p_1)_0, ..., (p_N)_0\right\},$$

$$(P_0)^- = \min\left\{(p_1)_0, ..., (p_N)_0\right\}.$$

Throughout this paper we assume that

$$\sum_{i=1}^{N} \frac{1}{(p_i)_0} > 1, \tag{1.10}$$

and define $(P_0)^* \in \mathbb{R}^+$ and $P_{0,\infty} \in \mathbb{R}^+$ by

$$(P_0)^* = \frac{N}{\sum_{i=1}^{N} \frac{1}{(p_i)_0} - 1}, \quad P_{0,\infty} = \max\{(P_0)_+, (P_0)^*\},$$

and for all $i \in \{1, ..., N\}$; $\quad p_i^\partial = \begin{cases} \dfrac{(N-1)(p_i)_0}{N-(p_i)_0} & \text{if } (p_i)_0 < \ N, \\ +\infty & \text{if } (p_i)_0 \geq \ N. \end{cases}$

**Proposition 1.2.4** (See [45]). *Let $u \in W^1 L_{\vec{\phi}}(\Omega)$. Then*

1. $\frac{1}{2^{(P^0)^+} N^{(P^0)^+ - 1}} \|u\|_{1,\vec{\phi}}^{(P^0)^+} \leq \sum_{i=1}^{N} \int_\Omega \phi_i(\left|\frac{\partial u}{\partial x_i}\right|) + \phi_i(|u|) \mathrm{d}x$, *if* $\|u\|_{1,\vec{\phi}} \leq 1$.
2. $\frac{1}{2^{(P^0)^-} N^{(P^0)^+ - 1}} \|u\|_{1,\vec{\phi}}^{(P^0)^-} - 1 \leq \sum_{i=1}^{N} \int_\Omega \phi_i(\left|\frac{\partial u}{\partial x_i}\right|) + \phi_i(|u|) \mathrm{d}x$, *if* $\|u\|_{1,\vec{\phi}} \geq 1$.

**Proposition 1.2.5** (See [45]). *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded domain with smooth boundary. Assume that relation (3.6) is fulfilled. Then, for any $m \in C_+(\overline{\Omega})$ satisfying $1 \leq m(x) \leq P_{0,\infty}$ for all $x \in \overline{\Omega}$, the embedding*

$$W^1 L_{\vec{\phi}}(\Omega) \hookrightarrow L^{m(.)}(\Omega)$$

*is continuous and compact.*

**Proposition 1.2.6** (See [46]). *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded domain with smooth boundary and let $q \in C(\overline{\Omega})$ satisfy the condition*

$$1 \leq q(x) < \min\{p_1^\partial, ..., p_N^\partial\}, \forall x \in \partial\Omega.$$

*Then, there is a compact boundary trace embedding*

$$W^1 L_{\vec{\phi}}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega).$$

## 1.3 Laplace operator

### 1.3.1 Properties of $p(x)$-Laplace operator

In this section, we discuss the $p(x)$-Laplace operator :

$$-\Delta_{p(x)}u := -div(\mid \nabla u \mid^{p(x)-2} \nabla u).$$

Consider the following functional:

$$J(u) = \int_\Omega \frac{1}{p(x)} \mid \nabla u \mid^{p(x)} \,\mathrm{d}x, \quad u \in X := W_0^{1,p(x)}(\Omega).$$

We know that (see [1]), $J \in \mathcal{C}^1(X, \mathbb{R})$, and the $p(x)$-Laplace operator is the derivative operator of $J$ in the weak sense. We denote $L = J' : X \longrightarrow X^*$, then

$$(L(u), v) = \int_\Omega \mid \nabla u \mid^{p(x)-2} \nabla u \nabla v \,\mathrm{d}x \quad \forall u, v \in X.$$

**Theorem 1.3.1.** *i) $L : X \longrightarrow X^*$ is a continuous; bounded and strictly monotone operator;*

*(ii) $L$ is a mapping of type $(S_+)$, i.e. if $u_n \rightharpoonup u$ in $X$ and* $\limsup_{n\to+\infty}(L(u_n) - L(u), u_n - u) \leq 0$, *then $u_n \longrightarrow u$ in $X$;*

*(iii) $L : X \longrightarrow X^*$ is a homeomorphism.*

*Proof.* (i) It is obvious that $L$ is continuous and bounded. For any $\xi, \eta \in \mathbb{R}^N$, we have the following inequalities (see [40]) from which we can get the strictly monotonicity of $L$:

$$\left(\mid \eta \mid^{p-2} \eta - \mid \xi \mid^{p-2} \xi\right).(\eta - \xi)(|\eta|^p + |\xi|^p)^{\frac{2-p}{p}} \geq (p-1)|\eta - \xi|^p, \eta, \xi \in \mathbb{R}^N, \quad (1.11)$$

for $1 < p < 2$.

$$\left(\mid \eta \mid^{p-2} \eta - \mid \xi \mid^{p-2} \xi\right).(\eta - \xi) \geq 2^{-p}|\eta - \xi|^p, \quad \eta, \xi \in \mathbb{R}^N, \quad p \geq 2 \quad (1.12)$$

(ii) From (i), if $u_n \rightharpoonup u$ in $X$ and $\limsup_{n\to+\infty}(L(u_n) - L(u), u_n - u) \leq 0$. In view of (1.11) and (1.12), $\nabla u_n$ converges in measure to $\nabla u$ in $\Omega$, so we get a subsequence (which we still denote by $(\nabla u_n)$ ) satisfying $\nabla u_n(x) \longrightarrow \nabla u(x)$, a.e. $x \in \Omega$. By Fatou Lemma we get

$$\limsup_{n\to+\infty} \int_\Omega \frac{1}{p(x)} \mid \nabla u_n \mid^{p(x)} \,\mathrm{d}x \geq \int_\Omega \frac{1}{p(x)} \mid \nabla u \mid^{p(x)} \,\mathrm{d}x. \quad (1.13)$$

From $u_n \rightharpoonup u$ we have

$$\lim_{n\to+\infty}(L(u_n), u_n - u) = \lim_{n\to+\infty}(L(u_n) - L(u), u_n - u) = 0. \quad (1.14)$$

We also have

$$\begin{aligned}(L(u_n),u_n-u) &= \int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x - \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla u\,\mathrm{d}x \\ &\geq \int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x - \int_\Omega |\nabla u_n|^{p(x)-1}|\nabla u|\,\mathrm{d}x \\ &\geq \int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x - \int_\Omega \left(\frac{p(x)-1}{p(x)}|\nabla u_n|^{p(x)} + \frac{1}{p(x)}|\nabla u|^{p(x)}\right)\mathrm{d}x \\ &\geq \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x - \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x.\end{aligned} \tag{1.15}$$

According to (1.15)–(1.14) we obtain

$$\lim_{n\to+\infty}\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,\mathrm{d}x = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x \tag{1.16}$$

From (1.16) it follows that the integrals of the functions family $\{\frac{1}{p(x)}|\nabla u_n|^{p(x)}\}$ possess absolutely equicontinuity on $\Omega$ (see [48], Chapter 6, Section 3). Since

$$\frac{1}{p(x)}|\nabla u_n(x)-\nabla u(x)|^{p(x)} \leq C\left(\frac{1}{p(x)}|\nabla u_n|^{p(x)} + \frac{1}{p(x)}|\nabla u|^{p(x)}\right), \tag{1.17}$$

the integrals of the family $\{(\frac{1}{p(x)})|\nabla u_n-\nabla u|^{p(x)}\}$ are also absolutely equicontinuous on $\Omega$ (see [48]) and therefore

$$\lim_{n\to+\infty}\int_\Omega \frac{1}{p(x)}|\nabla u_n(x)-\nabla u(x)|^{p(x)}\,\mathrm{d}x = 0. \tag{1.18}$$

By (1.18)

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n(x)-\nabla u(x)|^{p(x)}\,\mathrm{d}x = 0. \tag{1.19}$$

From Proposition (1.19) $u_n \longrightarrow u$, i.e. $L$ is of type $(S_+)$.
(iii) By the strictly monotonicity, $L$ is an injection. Since

$$\lim_{\|u\|\to+\infty}\frac{(Lu,u)}{\|u\|} = \lim_{\|u\|\to+\infty}\frac{\int_\Omega |\nabla u(x)|^{p(x)}\,\mathrm{d}x}{\|u\|} = +\infty,$$

$L$ is coercive, thus $L$ is a surjection in view of Minty-Browder Theorem (see [51]). Hence $L$ has an inverse mapping $L^{-1}: X^* \longrightarrow X$. Therefore, the continuity of $L^{-1}$ is sufficient to ensure $L$ to be a homeomorphism.
If $f_n, f \in X^*, f_n \longrightarrow f$, let $u_n = L^{-1}(f_n), u = L^{-1}(f)$, then $L(u_n) = f_n, L(u) = f$.
So $\{u_n\}$ is bounded in $X$ . Without loss of generality, we can assume that $u_n \rightharpoonup u$.
Since $f_n \longrightarrow f$, then

$$\lim_{n\to+\infty}(L(u_n)-L(u_0),u_n-u_0) = \lim_{n\to+\infty}(f_n,u_n-u_0) = 0.$$

Since $L$ is of type $(S_+)$ , $u_n \longrightarrow u_0$, we conclude that $u_n \longrightarrow u$, so $L^{-1}$ is continuous. □

### 1.3.2 $\vec{p}(x)$-Laplace operator

A new operator takes its place in the mathematical literature, namely

$$\Delta_{\vec{p}(x)}u = \sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\right).$$

see also [9]. This operator will be referred to as the anisotropic variable exponent $\vec{p}(x)$-Laplace operator.

Consider the following functional:

$$J(u) = \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} \mathrm{d}x, \quad u \in X := W^{1,\vec{p}(x)}(\Omega).$$

We know that (see [9]), $J \in C^1(X,\mathbb{R})$, and the $\overrightarrow{p}(x)$-Laplace operator is the derivative operator of $J$ in the weak sense. We denote $L = J' : X \longrightarrow X^*$, then

$$(L(u),v) = \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \mathrm{d}x \quad \forall u,v \in X.$$

**Theorem 1.3.2.** *i) $L : X \longrightarrow X^*$ is a continuous; bounded and strictly monotone operator;*

*(ii) $L$ is a mapping of type $(S_+)$, i.e. if $u_n \rightharpoonup u$ in $X$ and $\limsup_{n\to+\infty}(L(u_n) - L(u), u_n - u) \leq 0$, then $u_n \longrightarrow u$ in $X$;*

*(iii) $L : X \longrightarrow X^*$ is a homeomorphism.*

## 1.4 Some Technical Tools.

**Theorem 1.4.1** ([41], Theorem 6.2.1.). *Let $X$ be a reflexive Banach space, and let $f : M \subseteq X \longrightarrow \mathbb{R}$ be Gâteaux differentiable over the closed, convex set $M$. Then the following conditions are equivalent:*

*i) $f$ is convex over $M$.*

*(ii) We have*

$$f(u) - f(v) \geq (f'(v), u - v)_{X^*\times X} \quad \forall u,v \in M,$$

*where $X^*$ denotes the dual of the space $X$.*

*(iii) The first Gâteaux derivative is monotone, that is*

$$(f'(u) - f'(v), u - v)_{X^*\times X} \geq 0 \quad \forall u,v \in M,$$

*(iv) The second Gâteaux derivative of $f$ exists and it is positive, that is*

$$(f''(u)\circ v, v)_{X^*\times X} \geq 0 \quad \forall u, v \in M.$$

**Theorem 1.4.2** ([41], Theorem 6.2.1.). *Suppose $X$ is a reflexive Banach space with norm $\|.\|_X$ and let $M \subset X$ be a weakly closed subset of $X$. Suppose $\phi : M \longrightarrow \mathbb{R}\cup\{\infty\}$ is coercive and (sequentially) weakly lower semi-continuous on $M$ with respect to $X$, that is, suppose the following conditions are fulfilled:*

*i) $\phi(u) \longrightarrow \infty$ as $\|u\|_X \longrightarrow \infty$, $u \in M$.*

*(ii) For any $u \in M$, and any subsequence $(u_m)m$ in $M$ such that $u_m \rightharpoonup u$ weakly in $X$, it holds that*

$$\phi(u) \leq \lim \inf_{m\longrightarrow+\infty} \phi(u_m).$$

*Then $\phi$ is bounded from below on $M$ and attains its infimum in $M$.*

**Theorem 1.4.3.** *Let $(X, \|.\|_X)$ be a Banach space. Assume that $\phi \in \mathcal{C}^1(X, \mathbb{R})$ satisfies the Palais-Smale condition; that is, any sequence $(u_n)_n \subset X$ such that $(\phi(u_n))_n$ is bounded and $\phi(u_n)\rightarrow 0$ in $X^*$ as $n \longrightarrow \infty$, contains a subsequence converging to a critical point of $\phi$. Also, assume that $\phi$ has a mountain pass geometry; that is,*

*i) there exist two constants $\tau > 0$ and $\rho \in \mathbb{R}$ such that $\phi(u) \geq \rho$ if $\|u\|_X = \tau$;*

*(ii) $\phi(0) < \rho$ and there exists $e \in X$ such that $\|e\|_X > \tau$ and $\phi(e) < \rho$.*

*Then $\phi$ has a critical point $u_0 \in X\backslash\{0, e\}$ with critical value*

$$\phi(u_0) = \inf_{\gamma\in\mathcal{P}} \sup_{u\in\gamma} \phi(u) \geq \rho > 0,$$

*where $\mathcal{P}$ denotes the class of the paths $\gamma \in \mathcal{C}([0,1], X)$ joining 0 to $e$.*

# CHAPTER 2

# EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR ANISOTROPIC ELLIPTIC PROBLEMS WITH VARIABLE EXPONENT AND NONLINEAR ROBIN BOUNDARY CONDITIONS.

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

*(David Hilbert.)*

This article presents sufficient conditions for the existence of solutions of the anisotropic quasilinear elliptic equation with variable exponent and nonlinear Robin boundary conditions,

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\Big)+\sum_{i=1}^{N}|u|^{p_i(x)-2}u+\lambda|u|^{m(x)-2}u=\gamma g(x,u)$$
in $\Omega$,
$$\sum_{i=1}^{N}\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\nu_i=\mu|u|^{q(x)-2}u \quad \text{on } \partial\Omega.$$

Under appropriate assumptions on the data, we prove some existence and multiplicity results. The methods are based on Mountain Pass and Fountain theorems.

## 2.1 Introduction

Many problems in physics and mechanics can be modeled with sufficient accuracy using classical Lebesgue and Sobolev spaces, $L^p(\Omega)$ and $W^{1,p}(\Omega)$, where $p$ is a fixed constant and $\Omega$ is an appropriate domain. But for the electrorheological fluids (Smart fluids), this is not adequate but rather, the exponent should be able to vary. This leads to study the problem in the frame-work of variable exponent Lebesgue and Sobolev spaces, $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$, where $p(\cdot)$ is a real-valued function; see, e.g. [25, 26].

On the other hand, it has been experimentally shown that the above-mentioned fluids may have their viscosity undergoing a significant change; see, e.g. [7]. Consequently, the mathematical modelling of such fluids requires the introduction of the so-called anisotropic variable spaces.Indeed, there is by now a large number of papers and increasing interest about anisotropic problems. With no hope of being complete, let us mention some pioneering works on anisotropic Sobolev spaces [39, 50] and some more recent regularity results for minimizers of anisotropic functionals [2, 15, 43].

Therefore, in the recent years, the study of various mathematical problems modeled by quasilinear elliptic and parabolic equations with both anisotropic and variable exponent has received considerable attention. Let us mention many works in that direction by Antontsev and Shmarev; see, e.g. [3] and the references therein.

Our paper is mainly devoted to the existence and multiplicity of solutions of quasilinear elliptic equations under nonlinear Robin boundary condition such as

$$\begin{aligned}&-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\Big(\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\Big)+\sum_{i=1}^{N}|u|^{p_i(x)-2}u+\lambda|u|^{m(x)-2}u\\&=\gamma g(x,u) \quad \text{in } \Omega,\\&\sum_{i=1}^{N}\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\nu_i=\mu|u|^{q(x)-2}u \quad \text{on } \partial\Omega,\end{aligned} \tag{2.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with $n\geq 2$, with smooth boundary $\partial\Omega$ and $\nu_i$ are the components of the outer normal unit vector and for $i\in\{1,\ldots,N\}$, $p_i, m\in C(\bar{\Omega})$,

$q \in \mathcal{C}(\partial\Omega)$. The functions $p_i$ and $g$ are supposed to satisfy some conditions to be specified below, while $\lambda, \gamma$, and $\mu$ are real parameters, with $\gamma, \mu > 0$.

We shall give conditions under which problem (4.1) has infinitely many solutions. According to the behaviour of $g$ and to the kind of results we want to prove, variational methods turn out to be more appropriate.

When $\lim_{s\to 0} g(x,s)/|s|^{\sigma} = 0$, $\sigma$ to be made precise later, Mountain Pass theorem provides the existence of at least a solution of (4.1) and, on the other hand, when $g$ is an odd function, Fountain's theorem yields the existence of infinitely many solutions.

A host of publications exist for this type of problems when the boundary condition is replaced by $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ and $\gamma = 0$ ; see, e.g. [34] and the references therein, where the authors obtained existence results by means of standard variational tools. The associated problem with Dirichlet boundary conditions has also been treated by many authors; see, e.g. [7, 28]. Furthermore, existence of positive solutions for nonlinear Robin problem involving the $p(x)$-Laplacian have been studied by S. G. Deng; in [21], by using the sub-super solutions and variational methods. We consider here the case where $\mu$ is positive and $g$ satisfies more hypotheses than in [36], to use the Mountain Passe and Fountain theorems. It turns out that the condition $q^- > P_+^+$ plays an important role in the proofs of our main results.

This article is divided into four sections. In the second section, we introduce some basic properties of the generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ and anisotropic Sobolev spaces $W^{1,\overrightarrow{p}(x)}(\Omega)$ , and state the existence and multiplicity results concerning the problem (4.1). The third section is devoted to the proofs of the main results and finally, in the fourth section we deal with a generalized equation related to our problem (4.1).

## 2.2 Main results

We define

$$J(u) = \int_\Omega \Big( \sum_{i=1}^N \frac{1}{p_i(x)} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)} + \sum_{i=1}^N \frac{1}{p_i(x)} |u|^{p_i(x)} \Big)\, \mathrm{d}x,$$
$$G(x,u) = \int_0^u g(x,s)\mathrm{d}s.$$

We have

$$(J'u, v) = \int_\Omega \Big( \sum_{i=1}^N \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + \sum_{i=1}^N |u|^{p_i(x)-2} uv \Big)\, \mathrm{d}x,$$

for all $v \in X$. In all this paper $C$, $C_i (i = 0, 1, 2, \dots)$ represents different positive real constants.

We make the following assumptions on the functions $q$ and $g$.

(H0) $q \in \mathcal{C}(\partial\Omega)$ satisfies : $1 \le q(x) \le \frac{(N-1)P_-^-}{N-P_-^-}$ for all $x \in \partial\Omega$ and $q^- < P_+^+$.

(H1) $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Caratheodory type function and there exist a constant $C>0$ and a function $\alpha\in\mathcal{C}(\bar{\Omega})$ such that: $1<\alpha(x)<\frac{NP_-^-}{N-P_-^-}$, for all $x\in\bar{\Omega}$ and

$$|g(x,s)|\leq C(1+|s|^{\alpha(x)-1})\quad \text{for all } (x,s)\in\Omega\times\mathbb{R}.$$

(H2) There exists $M>0$ and $\theta_\lambda\geq m^+$ (resp $\theta_\lambda\leq m^-$ ) if $\lambda\geq 0$ (resp $\lambda<0$ ). such that for all $s$ with $|s|\geq M$ and $x\in\Omega$, we have

$$0<\theta_\lambda G(x,s)\leq sg(x,s).$$

(H3) $g(x,s)=\circ(|s|^{P_+^+})$ as $s\to 0$ and uniformly for $x\in\Omega$,

(H4) $g(x,-s)=-g(x,s)$, $x\in\Omega, s\in\mathbb{R}$.

We say that $u\in X$ is a weak solution of (4.1) if

$$\int_\Omega\Big(\sum_{i=1}^N\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}+\sum_{i=1}^N|u|^{p_i(x)-2}uv\Big)\,\mathrm{d}x+\lambda\int_\Omega|u|^{m(x)-2}uv\,\mathrm{d}x$$
$$=\int_\Omega\gamma g(x,u)v\,\mathrm{d}x+\mu\int_{\partial\Omega}|u|^{q(x)-2}uv\,\mathrm{d}x,$$

for all $v\in X$.

The energy functional associated with problem (4.1) is

$$\Phi(u)=\int_\Omega\Big(\sum_{i=1}^N\frac{1}{p_i(x)}\Big|\frac{\partial u}{\partial x_i}\Big|+\sum_{i=1}^N\frac{1}{p_i(x)}|u|^{p_i(x)}\Big)\,\mathrm{d}x$$
$$+\lambda\int_\Omega\frac{1}{m(x)}|u|^{m(x)}\,\mathrm{d}x-\gamma\int_\Omega G(x,u)\,\mathrm{d}x-\mu\int_{\partial\Omega}\frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x. \tag{2.2}$$

**Proposition 2.2.1** (See [27, 24]).

*(1)* $L\equiv J':X\to X^*$ *is a continuous, bounded and strictly monotone operator;*

*(2)* $L$ *is a mapping of type* $(S_+)$, *i.e. if* $u_n\rightharpoonup u$ *in* $X$, *and* $\overline{\lim}_{n\to+\infty}(L(u_n)-L(u),u_n-u)\leq 0$, *then* $u_n\to u$ *in* $X$;

*(3)* $L:X\to X^*$ *is a homeomorphism.*

The following are embedding results on anisotropic generalized Sobolev spaces and will be used later.

**Proposition 2.2.2** (See [43]). *Suppose* $\Omega\subset\mathbb{R}^N$ *is a bounded domain with smooth boundary. For any* $q\in\mathcal{C}_+(\overline{\Omega})$ *satisfying* $q(x)<\frac{NP_-^-}{N-P_-^-}$ *for all* $x\in\overline{\Omega}$, *the embedding*

$$W^{1,\overrightarrow{p}(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$$

*is continuous and compact.*

**Proposition 2.2.3** (See [43]). *Assume that the boundary of $\Omega$ possesses the cone property and $p_i \in \mathcal{C}(\overline{\Omega}), 2 \leq p_i < N$ for all $i \in \{1, 2, \ldots, N\}$. If $q \in \mathcal{C}(\partial\Omega)$ satisfies the hypothesis $1 < q(x) < \frac{(N-1)P_-^-}{N-P_-^-}$ for all $x \in \partial\Omega$, then the embedding*

$$W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$$

*is continuous and compact.*

The main results of this article are as follows:

**Theorem 2.2.4.** *Suppose that* (H0)–(H2), (H4) *hold with* $m^+ < \frac{NP_-^-}{N-P_-^-}$ *and* $P_+^+ < \min(\alpha^-, m^-)$. *Then, for any* $\lambda \in \mathbb{R}$ *and* $\mu, \gamma > 0$, *problem* (4.1) *has at least a nontrivial weak solution.*

**Theorem 2.2.5.** *Suppose that* (H0)–(H2) , (H5) *hold with* $m^+ < \frac{NP_-^-}{N-P_-^-}$ *and* $P_+^+ < \min(\alpha^-, m^-)$. *Then, for any* $\lambda \in \mathbb{R}$ *and* $\mu, \gamma > 0$, *problem* (4.1) *has infinite many pairs of weak solutions.*

## 2.3 Proof of Theorem (2.2.4)

This section is devoted to prove Theorem (2.2.4).To prove Theorem (4.2.3), we shall use the Mountain Pass theorem [53]. We first start with the following lemmas.

**Lemma 2.3.1.** *If* (H0)–(H2) *hold, then for any* $\lambda \in \mathbb{R}$, *the functional* $\Phi$ *satisfies the Palais Smale condition (PS).*

*Proof.* Suppose that $(u_n) \subset X$ is a Palais Smale sequence , ie,

$$\sup |\Phi(u_n)| \leq C, \Phi'(u_n) \to 0, \text{ as } n \to \infty.$$

We shall prove that $(u_n)$ has a convergent subsequence.

Let us show that $(u_n)$ is bounded in $X$. Denote by $\widetilde{m} :\equiv m^+$ if $\lambda > 0$ and $\widetilde{m} :\equiv m^-$ if $\lambda \leq 0$. Since $\Phi(u_n)$ is bounded, then by using (H1), we have for large $n$,

$$\begin{aligned}
C + C\|u_n\| &\geq \Phi(u_n) - \theta_\lambda \Phi'(u_n) \\
&\geq \Big(\frac{1}{P_+^+} - \frac{1}{\theta_\lambda}\Big) \sum_{i=1}^{N} \int_\Omega \Big(|\frac{\partial u}{\partial x_i}|^{p_i(x)} + |u_n|^{p_i(x)}\Big)\, dx \\
&\quad + \lambda\Big(\frac{1}{\widetilde{m}} - \frac{1}{\theta_\lambda}\Big) \int_\Omega |u_n|^{m(x)}\, dx - \gamma \int_\Omega (G(x, u_n) - \theta_\lambda g(x, u_n) u_n)\, dx \\
&\quad - \frac{1}{\theta_\lambda}\langle \Phi'(u_n), u_n\rangle + \mu \int_{\partial\Omega} \Big(\frac{1}{\theta_\lambda} - \frac{1}{q(x)}\Big)|u_n|^{q(x)}\, dx \\
&\geq \Big(\frac{1}{P_+^+} - \frac{1}{\theta_\lambda}\Big) \sum_{i=1}^{N} |\frac{\partial u}{\partial x_i}|_{p_i(x)}^{P_-^-} - \frac{1}{\theta_\lambda}(\Phi'(u_n), u_n) \\
&\quad + \mu\Big(\frac{1}{\theta_\lambda} - \frac{1}{q^-}\Big) \int_{\partial\Omega} |u_n|^{q(x)}\, dx.
\end{aligned}$$

Now, according to [10, page 6], we have

$$\frac{\|u\|^{P_-^-}_{\overrightarrow{p}(\cdot)}}{2^{P_-^- -1}N^{P_-^- -1}} \leq \sum_{i=1}^{N}\Big(|\frac{\partial u}{\partial x_i}|^{P_-^-}_{p_i(\cdot)} + |u|^{P_-^-}_{p_i(\cdot)}\Big)$$
$$\leq \sum_{i=1}^{N}\int_{\Omega}\Big(|\frac{\partial u}{\partial x_i}|^{p_i(x)} + |u|^{p_i(x)}\Big)\,\mathrm{d}x.$$

Then,

$$C + C\|u_n\| \geq \frac{1}{2^{P_-^- -1}N^{P_-^- -1}}\Big(\frac{1}{P_+^+} - \frac{1}{\theta_\lambda}\Big)\|u_n\|^{P_-^-}_{\overrightarrow{p}(\cdot)} - \frac{C_1}{\theta_\lambda}\|u_n\|_{\overrightarrow{p}(\cdot)} - C.$$

Since $\mu > 0$, then by using condition (H2) and the inequality above, we deduce that $u_n$ is bounded in $X$. The proof is complete. □

**Lemma 2.3.2.** *There exist $r_1, C' > 0$ such that $\Phi(u) \geq C'$, for all $u \in X$ such that $\|u\| = r_1$.*

*Proof.* Conditions (H0), (H1) and (H2) ensure that, for any $\epsilon > 0$, we have

$$|G(x,s)| \leq \epsilon|s|^{P_+^+} + C(\epsilon)|s|^{\alpha(x)}, \quad \text{for all } (x,s) \in \Omega \times \mathbb{R}.$$

For $\|u\|$ small enough, we thus obtain

$$\begin{aligned}\Phi(u) \geq& \frac{1}{P_+^+}\sum_{i=1}^{N}\int_{\Omega}\Big(|\frac{\partial u}{\partial x_i}|^{p_i(x)} + |u|^{p_i(x)}\Big)\,\mathrm{d}x + \lambda\int_{\Omega}\frac{1}{m(x)}|u|^{m(x)}\,\mathrm{d}x\\ &- \int_{\Omega}(\epsilon|u|^{P_+^+} + C(\epsilon)|u|^{\alpha(x)})\,\mathrm{d}x - \frac{\mu}{q^-}\int_{\partial\Omega}|u|^{q(x)}\,\mathrm{d}x\\ \geq& \frac{1}{P_+^+ 2^{P_+^+ -1}N^{P_+^+ -1}}\|u\|^{P_+^+}_{\overrightarrow{p}(\cdot)} - \frac{|\lambda|}{m^-}\int_{\Omega}|u|^{m(x)}\,\mathrm{d}x - \int_{\Omega}\epsilon|u|^{P_+^+}\,\mathrm{d}x\\ &- \int_{\Omega}C(\epsilon)|u|^{\alpha(x)}\,\mathrm{d}x - \frac{\mu}{q^-}\int_{\partial\Omega}|u|^{q(x)}\,\mathrm{d}x.\end{aligned} \tag{2.3}$$

Since $P_+^+ < \alpha^- \leq \alpha(x) < \frac{NP_-^-}{N-P_-^-}$, for all $x \in \Omega$ and $q(x) < \frac{(N-1)P_-^-}{N-P_-^-}$, for all $x \in \partial\Omega$; then, we have

$$W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{P_+^+}(\Omega) \quad \text{and} \quad W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega),$$

with continuous and compact embeddings. Consequently, there exist two constants $C_1' > 0$ and $C_2' > 0$ such that

$$|u|_{L^{P_+^+}(\Omega)} \leq C_2'\|u\|, \quad |u|_{L^{q(x)}(\Omega)} \leq C_1'\|u\|, \text{ for all } u \in X. \tag{2.4}$$

By using (2.4) for $\|u\|$ small enough, we obtain from (3.7) that

$$\Phi(u) \geq \frac{1}{P_+^+ 2^{P_+^+ -1}N^{P_+^+ -1}}\|u\|^{P_+^+} - \frac{|\lambda|}{m^-}\max\{|u|^{m^+}_{L^{m(x)}(\Omega)}, |u|^{m^-}_{L^{m(x)}(\Omega)}\}$$
$$- \epsilon C_2'\|u\|^{P_+^+} - C(\epsilon)C_3'\|u\|^{\alpha^-} - \frac{\mu}{q^-}C_1'\|u\|^{q^-}.$$

Since $W^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{m^+}(\Omega)$, we have

$$\Phi(u) \geq \frac{1}{P_+^+ 2^{P_+^+-1} N^{P_+^+-1}} \|u\|^{P_+^+} - \frac{|\lambda| C}{m^-} \max\{\|u\|^{m^+}, \|u\|^{m^-}\}$$
$$- \epsilon C_2'^{P_+^+} \|u\|^{P_+^+} - C(\epsilon) C_3' \|u\|^{\alpha^-} - \frac{\mu}{q^-} C_1' \|u\|^{q^-}.$$

Now, let $\epsilon > 0$ be small enough so that:

$$0 < \epsilon C_2'^{P_+^+} \leq \frac{1}{2P_+^+ 2^{P_+^+-1} N^{P_+^+-1}} =: c_0.$$

We have

$$\Phi(u) \geq c_0 \|u\|^{P_+^+} - \frac{|\lambda| C}{m^-} \max\{\|u\|^{m^+}, \|u\|^{m^-}\} - C(\epsilon) \|u\|^{\alpha^-} - \frac{\mu C_1'}{q^-} \|u\|^{q^-}$$
$$\geq \|u\|^{P_+^+} \Big( c_0 - \frac{|\lambda| C}{m^-} \max\{\|u\|^{m^+ - P_+^+}, \|u\|^{m^- - P_+^+}\} \Big)$$
$$- \|u\|^{P_+^+} \Big( C(\epsilon) \|u\|^{\alpha^- - P_+^+} + \frac{\mu C_1'}{q^-} \|u\|^{q^- - P_+^+} \Big).$$

Since $P_+^+ < \min(\alpha^-, m^-, q^-)$, then there exist $r_1 > 0$ and $C' > 0$ such that

$$\Phi(u) \geq C' > 0, \quad \text{for any } u \in X.$$

Hence, the proof is complete. □

*Proof of Theorem 4.2.3.* To apply the Mountain Pass theorem ([53]), we have to prove that $\Phi(tu) \to -\infty$ as $t \to +\infty$, for some $u \in X$. From (H2), it follows that

$$G(x,s) \geq C|s|^{\theta_\lambda}, \quad \forall x \in \bar{\Omega}, \forall |s| \geq M.$$

For $u \in X$ and $t > 1$, we have

$$\Phi(tu) \leq \frac{1}{P_-^-} \sum_{i=1}^N \int_\Omega \Big( \Big|\frac{\partial(tu)}{\partial x_i}\Big|^{p_i(x)} + |tu|^{p_i(x)} \Big) \,\mathrm{d}x + \lambda \int_\Omega \frac{1}{m(x)} |tu|^{m(x)} \,\mathrm{d}x$$
$$- \int_\Omega G(x,tu) \,\mathrm{d}x - \frac{\mu}{q^+} \int_{\partial\Omega} |tu|^{q(x)} \,\mathrm{d}x$$
$$\leq \frac{t^{P_+^+}}{P_-^-} \sum_{i=1}^N \int_\Omega \Big( \Big|\frac{\partial(tu)}{\partial x_i}\Big|^{p_i(x)} + |u|^{p_i(x)} \Big) \,\mathrm{d}x + \lambda t^{\widetilde{m}} \int_\Omega \frac{1}{m(x)} |u|^{m(x)} \,\mathrm{d}x$$
$$- C t^{\theta_\lambda} \int_\Omega |u|^{\theta_\lambda} \,\mathrm{d}x - \frac{\mu t^{q^-}}{q^+} \int_{\partial\Omega} |u|^{q(x)} \,\mathrm{d}x,$$

where again $\widetilde{m} = m^+$ if $\lambda > 0$ and $\widetilde{m} = m^-$ if $\lambda \leq 0$.

By (H0) and (H2), it follows that, for any $\lambda \in \mathbb{R}$, $\Phi(tu) \to -\infty$ as $(t \to +\infty)$. Since $\Phi(0) = 0$, it follows that $\Phi$ satisfies the condition of the Mountain Pass lemma, and so $\Phi$ admits at least one nontrivial critical point $u_0 \in X$; which is characterized by

$$\tau = \inf_{h\in\Gamma} \sup_{t\in[0,1]} \Phi(h(t)),$$

where

$$\Gamma = \{h \in C([0,1], X); h(0) = 0 \text{ and } h(1) = e\}.$$

□

## 2.4 Proof of Theorem 2.2.5

Let $X$ be a reflexive and separable Banach space. It is well know (see, e.g. [2]) that there are $\{e_j\}_{j=1}^{\infty} \subset X$ and $\{e_j^*\}_{j=1}^{\infty} \subset X^*$ (where $X^*$ is the topological dual of $X$) such that

$$X = \overline{\text{span}}\{e_j : 1, 2, \ldots\}, \quad X^* = \overline{\text{span}}\{e_j^* : 1, 2, \ldots\},$$

and

$$\langle e_j^*, e_i \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.5}$$

For convenience, we write $X_j = \text{span}\{e_j\}$, $Y_k = \oplus_{j=1}^{k} X_j$ and $Z_k = \oplus_{j=k}^{\infty} X_j$. Denote

$$p^*(x) = \begin{cases} Np(x)/(N - p(x)) & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

**Lemma 2.4.1** (See [20, 23]). *Let $\beta(x) \in C_+(\bar{\Omega})$, with $\beta(x) < p^*(x)$ for $x \in \bar{\Omega}$ and $\alpha_k := \sup\{|u|_{L^{\beta(x)}(\Omega)}; \|u\| = 1, u \in Z_k\}$. Then, we have $\lim_{k\to\infty} \alpha_k = 0$.*

To prove Theorem 4.2.4, we shall use the Fountain theorem (see [53, Theorem 3.6] ). Obviously, $\Phi \in C^1(X, \mathbb{R})$ is an even functional. By using (H0) and (H1) we first prove that if $k$ is large enough, then there exist $\rho_k > \nu_k > 0$ such that

$$b_k := \inf\{\Phi(u)/u \in Z_k, \|u\| = \nu_k\} \to +\infty \quad \text{as } k \to +\infty; \tag{2.6}$$
$$a_k := \max\{\Phi(u)/u \in Y_k, \|u\| = \rho_k\} \to 0 \quad \text{as } k \to +\infty. \tag{2.7}$$

Proof of (3.20): For any $u \in Z_k$, $|u| = r_k > 1$, we have

$$\begin{aligned}\Phi(u) &\geq \frac{1}{P_+^+}\sum_{i=1}^{N}\int_\Omega \left(|\frac{\partial u}{\partial x_i}|^{p_i(x)} + |u|^{p_i(x)}\right) \mathrm{d}x - \frac{|\lambda|}{m^-}\int_\Omega |u|^{m(x)}\,\mathrm{d}x \\ &\quad - C\int_\Omega (1+|u|^{\alpha(x)})\,\mathrm{d}x - \frac{\mu}{q^-}\int_{\partial\Omega}|u|^{q(x)}\,\mathrm{d}x \\ &\geq \frac{1}{P_+^+}\sum_{i=1}^{N}\int_\Omega |\frac{\partial u}{\partial x_i}|^{P_-^-}\,\mathrm{d}x - \frac{|\lambda|}{m^-}\int_\Omega |u|^{m(x)}\,\mathrm{d}x \\ &\quad - C\int_\Omega (1+|u|^{\alpha(x)})\,\mathrm{d}x - \frac{\mu}{q^-}\int_{\partial\Omega}|u|^{q(x)}\,\mathrm{d}x \\ &\geq \frac{1}{P_+^+ 2^{P_-^- -1} N^{P_-^- -1}}\|u\|^{P_-^-} - \frac{C|\lambda|}{m^-}|u|^{m(\xi)}_{L^{m(x)}} \\ &\quad - C_1|u|^{\alpha(\xi)}_{\alpha(x)} - \frac{\mu}{q^-}|u|^{q(\xi)}_{L^{q(x)}(\partial\Omega)} - C_2, \quad \text{for some } \xi\in\Omega.\end{aligned} \tag{2.8}$$

So, for the study of the previous inequality, we only need to consider either the case where $m(x) \geq \alpha(x)$ or the case where $m(x) < \alpha(x)$ for all $x \in \Omega$.

Let us assume that $m(x) \leq \alpha(x)$ for all $x \in \Omega$. Then, we have $L^{\alpha(x)}(\Omega) \subset L^{m(x)}(\Omega)$. Thus, there is a positive constant $C_3 > 0$ such that

$$|u|_{L^{m(x)}(\Omega)} \leq C_3 |u|_{L^{\alpha(x)}(\Omega)} \quad \text{for all } u \in X.$$

So, for any $\xi \in \Omega$, we have

$$|u|^{m(\xi)}_{L^{m(x)}(\Omega)} \leq C^{m(\xi)}|u|^{m(\xi)}_{L^{\alpha(x)}(\Omega)}.$$

Let us denote $e := 1/(2^{P_-^- -1}N^{P_-^- -1})$. Then, for any $\xi \in \Omega$, we have

$$\begin{aligned}&\Phi(u) \\ &\geq \frac{e}{P_+^+}\|u\|^{P_-^-} - C'|u|^{m(\xi)}_{L^{\alpha(x)}} - C_1|u|^{\alpha(\xi)}_{L^{\alpha(x)}(\Omega)} - \frac{\mu}{q^-}|u|^{q(\xi)}_{L^{q(x)}(\partial\Omega)} - C_2 \\ &\geq \frac{e}{P_+^+}\|u\|^{P_-^-} - C\max\{|u|^{\alpha(\xi)}_{L^{\alpha(x)}(\Omega)}, |u|^{m(\xi)}_{L^{\alpha(x)}(\Omega)}\} - \frac{\mu}{q^-}|u|^{q(\xi)}_{L^{q(x)}(\partial\Omega)} - C_2.\end{aligned} \tag{2.9}$$

Denote

$$\begin{aligned} E &= L^{\alpha(x)}(\Omega) \cap L^{q(x)}(\partial\Omega), \\ A &= \{u \in E : |u|_{L^{\alpha(x)}(\Omega)} \leq 1, |u|_{L^{q(x)}(\partial\Omega)} \leq 1\}, \\ B &= \{u \in E : |u|_{L^{\alpha(x)}(\Omega)} > 1, |u|_{L^{q(x)}(\partial\Omega)} \leq 1\}, \\ C &= \{u \in E; |u|_{L^{\alpha(x)}(\Omega)} \leq 1, |u|_{L^{q(x)}(\partial\Omega)} > 1\}, \\ D &= \{u \in E; |u|_{L^{\alpha(x)}(\Omega)} \geq 1, |u|_{L^{q(x)}(\partial\Omega)} > 1\}. \end{aligned}$$

From (2.9), we have

$$\Phi(u) \geq \begin{cases} \frac{e}{P_+^+}\|u\|^{P_-^-} - C_1 & \text{if } u \in A, \\ \frac{e}{P_+^+}\|u\|^{P_-^-} - C_1(\alpha_k|u|)^{\alpha^+} - C_2 & \text{if } u \in B, \\ \frac{e}{P_+^+}\|u\|^{P_-^-} - \frac{\mu}{q^-}(\beta_k|u|)^{q^+} - -C_1 & \text{if } u \in C, \\ \frac{e}{P_+^+}\|u\|^{P_-^-} - C_1(\alpha_k|u|)^{\alpha^+} - \frac{\mu}{q^-}(\beta_k|u|)^{q^+} - C_2, & \text{if } u \in D, \end{cases}$$

where $\beta_k = \sup\{|u|_{L^{q(x)}(\partial\Omega)}; \|u\| = 1, u \in Z_k\}$. It is obvious that $\Phi(u) \to +\infty$ as $\|u\| \to +\infty$ in $A$.

For $u \in B \cup C$, we have

$$\Phi(u) \geq \frac{e}{P_+^+}\|u\|^{P_-^-} - C_2(\widetilde{\alpha}_k|u|)^{\widetilde{\alpha}^+} - C_3,$$

By taking $\widetilde{\alpha}_k$ to be either $\alpha_k$ or $\beta_k$ and $\widetilde{\alpha} = \alpha^+$ or $q^+$, we obtain

$$\Phi(u) \geq e\Big(\frac{1}{P_+^+} - \frac{1}{\widetilde{\alpha}^+}\Big)\left(\frac{C_2}{e}\widetilde{\alpha}^+\widetilde{\alpha}_k^{\widetilde{\alpha}^+}\right)^{\frac{P_-^-}{P_-^- - \widetilde{\alpha}^+}} - C_3.$$

Since $\widetilde{\alpha}_k \to 0$ as $k \to \infty$ and $\widetilde{\alpha}^+ > P_+^+$, then $\left(\frac{C_2}{e}\widetilde{\alpha}^+\widetilde{\alpha}_k^{\widetilde{\alpha}^+}\right)^{\frac{1}{P_-^- - \widetilde{\alpha}^+}} \to \infty$ as $k \to \infty$. Consequently, we have

$$\Phi(u) \to +\infty \quad \text{as } \|u\| \to +\infty, u \in Z_k.$$

If $u \in D$, then

$$\Phi(u) \geq \frac{e}{P_+^+}\|u\|^{P_-^-} - C(\alpha_k^{\alpha^+}|u|^{\alpha^+}) - \frac{\mu}{q^-}(\beta_k^{q^+}|u|^{q^+}) - C_1.$$

By assuming $\alpha^+ \leq q^+$, we obtain

$$\begin{aligned} \Phi(u) &\geq \frac{e}{P_+^+}\|u\|^{P_-^-} - C(\alpha_k^{\alpha^+}|u|^{q^+}) - \frac{\mu}{q^-}(\beta_k^{q^+}|u|^{q^+}) - C_1 \\ &\geq \frac{e}{P_+^+}\|u\|^{P_-^-} - (C\alpha_k^{\alpha^+} + \frac{\mu}{q^-}\beta_k^{q^+})|u|^{q^+} - C_1 \\ &\geq \frac{e}{P_+^+}\|u\|^{P_-^-} - C_3(\alpha_k^{\alpha^+} + \beta_k^{q^+})|u|^{q^+} - C_1 \\ &\geq e\Big(\frac{1}{P_+^+} - \frac{1}{q^+}\Big)\left[C_3 q^+(\alpha_k^{\alpha^+} + \beta_k^{q^+})\right]^{\frac{P_-^-}{P_-^- - q^+}} - C_1. \end{aligned}$$

Since $q^+ > P_+^+$, we then have $[C_3 q^+(\alpha_k^{\alpha^+} + \beta_k^{q^+})]^{\frac{1}{P_-^- - q^+}} \to \infty$, as $k \to \infty$. Consequently, we obtain $\Phi(u) \to +\infty$ as $\|u\| \to +\infty$, $u \in Z_k$. Now, from condition (H2), we have

$$G(x,s) \geq C_1|s|^{\theta_\lambda} - C_2, \quad \text{for any } (x,s) \in \Omega \times \mathbb{R}.$$

Then there exist constants $C_1', C_3' > 0$ such that

$$\Phi(u) \leq \frac{C_1'}{P_-^-} \sum_{i=1}^{N} \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)} \mathrm{d}x + \frac{\lambda}{\widehat{m}} \int_\Omega |u|^{m(x)} \mathrm{d}x - C_2' \|u\|^{\theta_\lambda} - C_3',$$

where $\widehat{m} = m^-$ if $\lambda > 0$ and $\widehat{m} = m^+$ if $\lambda \leq 0$. Hence, we obtain the inequality

$$\sum_{i=1}^{N} \Big|\frac{\partial u}{\partial x_i}\Big|^{P_+^+}_{L^{p_i(x)}(\Omega)} \leq C\Big(\sum_{i=1}^{N} \Big|\frac{\partial u}{\partial x_i}\Big|_{L^{p_i(x)}(\Omega)}\Big)^{P_+^+},$$

Where $C$ is a positive constant.

In the case $\lambda > 0$, we obtain

$$\Phi(u) \leq \frac{C'}{P_-^-} \|u\|^{P_+^+} + \frac{C_4 \lambda}{\widehat{m}} \|u\|^{m^+} - C_2' \|u\|^{\theta_\lambda} - C_3'.$$

But we have $W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{m(x)}(\Omega)$, and $W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$. Then, as $\theta_\lambda > \max(P_+^+, m^+)$ and $\dim Y_k = k$, it is easy to see that

$$\Phi(u) \to -\infty \quad \text{as } \|u\| \to +\infty \text{ for } u \in Y_k.$$

For the case $\lambda \leq 0$, we have

$$\Phi(u) \leq \frac{C'}{P_-^-} \|u\|^{P^+} - C_2' \|u\|^{\theta_\lambda} - C_3'.$$

Now, as we have $\theta_\lambda > P_+^+$ and $\dim Y_k = k$, it is also easy to see that

$$\Phi(u) \to -\infty \quad \text{as } \|u\| \to +\infty \text{ for } u \in Y_k.$$

## 2.5 A generalized equation

We shall now consider the generalized equation

$$\begin{aligned} &-\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\Big) + \sum_{i=1}^{n} |u|^{p_i(x)-2}u \\ &= \lambda g_1(x,u) + \nu g_2(x,u) \quad \text{in } \Omega, \\ &\sum_{i=1}^{n} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\upsilon_i = \mu f(x,u) \quad \text{on } \partial\Omega, \end{aligned} \tag{2.10}$$

where $\lambda, \nu, \mu > 0$ are real numbers, $p_i(x) \in \mathcal{C}(\bar{\Omega})$ with $2 \leq p_i(x) \leq N$ for all $i \in \{1, 2, \ldots, N\}$, and $g_1, g_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ are two functions of class $\mathcal{C}^1$ with respect to the $\Omega$-variable, and $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is of class $\mathcal{C}^1$ with respect to the $\partial\Omega$-variable. We make the following assumptions on the functions $q, g_1, g_2$ and $f$.

(H5) For $i = 1, 2$, $q_i \in \mathcal{C}(\bar{\Omega})$ satisfies $1 < q_i(x) < \frac{NP_-^-}{N-P_-^-}$ for all $x \in \Omega$.

(H6) (i) For $i = 1, 2$, $g_i : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition, and there exist some positive constant $C_i$ such that

$$|g_i(x,s)| \leq C_1 + C_2|s|^{q_i(x)-1} \quad \text{for } (x,s) \in \Omega \times \mathbb{R}.$$

(ii) There exists $M > 0$, $\sigma > P_+^+$ such that for all $|s| \geq M$ and $x \in \Omega$,

$$0 < \sigma G_2(x,s) \leq g_2(x,s)s.$$

(H7) There exist $\delta_1 > 0, C_3 > 0$ and $q_3 \in \mathcal{C}(\bar{\Omega})$ such that

$$G_1(x,s) \geq C_3|s|^{q_3(x)}, \forall (x,s) \in \Omega \times (0, \delta_1],$$

where $\max(q_3^-, q_1^+) < P_-^- < P_+^+ < q_2^-$.

(H8) $g_2(x,s) = \circ(|s|^{P_+^+-1})$ as $s \to 0$ uniformly for $x \in \Omega$.

(H9) (i) $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition and there exists a constant $C > 0$ such that

$$|f(x,s)| \leq C(1 + |s|^{\beta(x)-1}), \quad \forall (x,s) \in \partial\Omega \times \mathbb{R};$$

where $\beta(x) \in \mathcal{C}(\partial\Omega)$ with $1 < \beta^- \leq \beta^+ < P_-^-$ and $\beta(x) < \frac{(N-1)P_-^-}{N-P_-^-}$ for all $x \in \partial\Omega$.

(ii) There exist $R > 0$, such that for all $|s| \geq R$ and $x \in \partial\Omega$

$$0 < \sigma F(x,s) \leq f(x,s)s.$$

(H10) There exist $\delta_2 > 0$, $C_4 > 0$ and $q_4(x) \in \mathcal{C}(\partial\Omega)$ such that

$$F(x,s) \geq C_4|s|^{q_4(x)}, \quad \forall x \in \partial\Omega, \ \forall |s| \leq \delta_2,$$

where $1 < q_4 < \frac{(N-1)P_-^-}{N-P_-^-}$ and $q_4^+ < P_-^-$ for all $x \in \partial\Omega$.

(H11) For $i = 1, 2$, $g_i(x, -s) = -g_i(x,s)$ for all $(x,s) \in \Omega \times \mathbb{R}$, and $f(x, -s) = -f(x,s)$ for all $(x,s) \in \partial\Omega \times \mathbb{R}$.

We denote

$$g(x,s) = \lambda g_1(x,s) + \gamma g_2(x,s), \quad G_i(x,s) = \int_0^s g_i(x,t)\,\mathrm{d}t \quad (i = 1,2),$$

$$G(x,s) = \int_0^s g(x,t)\,\mathrm{d}t, \quad F(x,s) = \int_0^s f(x,t)\,\mathrm{d}t;$$

and the associated functional

$$\Phi(u) = \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} \Big( \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)} + |u|^{p_i(x)} \Big)\,\mathrm{d}x - \int_\Omega G(x,u)\,\mathrm{d}x - \mu \int_{\partial\Omega} F(x,u)\,\mathrm{d}x.$$

**Proposition 2.5.1.** *If* (H5), (H6) *and* (H9) *hold, then for every* $\lambda, \gamma, \mu \geq 0$ *the functional* $\Phi$ *satisfies the Palais Small condition (PS).*

*Proof.* We use the following inequalities: For $x \in \Omega, s \in \mathbb{R}$

$$\begin{aligned} \sigma G_2(x,s) &\leq g_2(x,s)s + C_3, \\ \sigma G(x,s) - g(x,s)s &\leq [\sigma G_1(x,s) - s g_1(x,s)] + [\sigma G_2(x,s) - s g_2(x,s)] \\ &\leq (C_1 + C_2|s|^{q_1(x)}) + C_3. \end{aligned}$$

Suppose that $(u_n) \subset X$ is a (PS) sequence ; i.e,

$$\sup|\Phi(u_n)| \leq C, \Phi'(u_n) \to 0, \text{ as } n \to \infty.$$

Let us show that $(u_n)$ is bounded in $X$. Since $\Phi(u_n)$ is bounded, then by using hypothesis (H6) and (H9), we have for $n$ large enough

$$\begin{aligned} C + C\|u_n\| &\geq \sigma\Phi(u_n) - \Phi'(u_n) \\ &\geq \Big(\frac{1}{P_+^+} - \frac{1}{\sigma}\Big) \sum_{i=1}^{N} \int_\Omega \Big(|\frac{\partial u}{\partial x_i}|^{p_i(x)} + |u_n|^{p_i(x)}\Big)\, dx \\ &\quad - \int_\Omega (\sigma G(x,u_n) - g(x,u_n)u_n)\, dx \\ &\quad - \mu \int_{\partial\Omega} (\sigma f(x,u_n) - f(x,u_n)u_n)\, dx \\ &\geq \frac{1}{2^{P_-^- -1} N^{P_-^- -1}} \Big(\frac{1}{p_+^+} - \frac{1}{\sigma}\Big) \|u_n\|_{\overrightarrow{p}(\cdot)}^{P_-^-} - C' \int_\Omega |u_n|^{q_1}\, dx \\ &\quad - C' - \int_{\partial\Omega} (\sigma f(x,u_n) - f(x,u_n)u_n)\, dx. \end{aligned} \tag{2.11}$$

Applying (H9) for $\|u_n\|$ large enough, we then get

$$C + C\|u_n\| \geq \frac{1}{2^{P_-^- -1} N^{P_-^- -1}} \Big(\frac{1}{p_+^+} - \frac{1}{\sigma}\Big) \|u_n\|^{P_-^-} - C'\|u_n\|^{q_1^+} - C_3'.$$

Now, as $W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q_1^+}(\Omega)$ is a continuous and compact embedding, from the inequality above, we deduce that $u_n$ is bounded in $X$. The proof is complete. □

**Remark 2.5.2.** It follows from (H6) that

$$G_2(x,s) \geq C_5|s|^{1/\sigma} - C_6, \quad \forall x \in \Omega,\ \forall s \in \mathbb{R}.$$

The main results of this section are as follows:

**Proposition 2.5.3** ([27]). *Assume that* $\psi : X \to \mathbb{R}$ *is weakly-strongly continuous and that* $\psi(0) = 0$. *Let* $\nu > 0$ *be given. Set*

$$\beta_k = \beta_k(\nu) = \sup_{u \in Z_k, \|u\| \leq \nu} |\psi(u)|.$$

*Then* $\beta_k \to 0$ *as* $k \to \infty$.

**Theorem 2.5.4.** *Assume that* (H5), (H6) *and* (H9) *hold.*

*(1) If in addition,* (H10) *holds, then for every* $\gamma, \mu > 0$*, there exists* $r_0(\gamma) > 0$ *such that when* $0 \le \lambda, \mu \le r_0(\gamma)$*, problem* (3.8) *has a nontrivial solution* $u_1$ *such that* $\Phi(u_1) > 0$.

*(2) If in addition,* (H7) *and* (H10) *hold, then for every* $\gamma, \mu > 0$*, there exists* $r_0(\gamma) > 0$ *such that when* $0 \le \lambda, \mu \le r_0(\gamma)$*, problem* (3.8) *has two nontrivial solutions* $u_1, v_1$ *such that* $\Phi(u_1) > 0$ *and* $\Phi(v_1) < 0$.

*(3) If in addition,* (H7), (H10) *and* (H11) *hold, then for every* $\lambda, \gamma, \mu > 0$*, problem* (3.8) *has a sequence of solutions* $\{\pm u_k\}$ *such that* $\Phi(\pm u_k) \to +\infty$ *as* $k \to +\infty$.

*Proof.* (1) We denote

$$\psi_1(u) = \lambda \int_\Omega G_1(x, u(x))\, dx, \quad \psi_2(u) = \gamma \int_\Omega G_2(x, u(x))\, dx.$$

When the assumptions in (1) hold, then for sufficiently small $\|u\|$, we get

$$G_2(x, u) \le \epsilon |u|^{P_+^+} + C(\epsilon)|u|^{q_2}, \quad \forall (x, s) \in \Omega \times \mathbb{R},$$

Then

$$\psi_2(u) \le \gamma\epsilon \int_\Omega |u|^{P_+^+}\, dx + \gamma C(\epsilon) \int_\Omega |u|^{q_2(x)}\, dx.$$

Since $1 < q_2 < \frac{NP_-^-}{N-P_-^-}$, for all $x \in \Omega$, then we have

$$W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{P_+^+}(\Omega), \quad \text{and} \quad W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q_2(x)}(\Omega),$$

with continuous and compact embeddings. This implies the existence of $C_1, C_2 > 0$ such that

$$\psi_2(u) \le \gamma\epsilon C_1 \|u\|^{P_+^+} + \gamma C(\epsilon) C_2 \|u\|^{q_2^-}.$$

Choose $\epsilon > 0$ small enough so that $0 < \gamma\epsilon C_2 < \frac{1}{2^{P_+^+-1} N^{P_+^+-1}}$. Then, we have

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} \Big( |\frac{\partial u}{\partial x_i}|^{p_i(x)} + |u|^{p_i(x)} \Big)\, dx - \psi_2(u) \\ &\ge \frac{1}{2^{P_+^+} N^{P_+^+-1}} \|u\|^{P_+^+} - \gamma C(\epsilon) C_2 \|u\|^{q_2^-}.\end{aligned}$$

Since $q_2^- > P_+^+$, there exist $r_1 > 0$ and $\alpha > 0$ such that

$$\sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} \Big( |\frac{\partial u}{\partial x_i}|^{p_i(x)} + |u|^{p_i(x)} \Big)\, dx - \psi_2(u) \ge \alpha > 0, \quad \text{for } \|u\| = r_1.$$

We can find $r_0(\gamma) > 0$ such that when $\mu, \lambda \le r_0(\gamma)$, we obtain

$$\psi_1(u) \le \frac{\alpha}{2}, \quad \forall u \in S_{r_1} = \{u \in X; \|u\| = r_1\}.$$

Therefore, $\lambda, \mu \leq r_0(\gamma)$. So, we obtain

$$\Phi(u) \geq \frac{\alpha}{2} > 0, \quad \forall u \in S_{r_1}.$$

Let $u \in X$ and $t > 1$, we have

$$\begin{aligned} \Phi(tu) &= \sum_{i=1}^{N} \int_\Omega \frac{t^{p_i(x)}}{p_i(x)} \Big( \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)} + |u|^{p_i(x)} \Big) \,\mathrm{d}x - \lambda \int_\Omega G_1(x, tu) \,\mathrm{d}x \\ &\quad - \gamma \int_\Omega G_2(x, tu) \,\mathrm{d}x - \mu \int_{\partial\Omega} F(x, tu) \,\mathrm{d}x. \end{aligned} \tag{2.12}$$

From Remark (2.5.2), we obtain

$$\begin{aligned} \Phi(tu) &\leq t^{P_+^+} \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} \Big( \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)} + |u|^{p_i(x)} \Big) \,\mathrm{d}x - \lambda \int_\Omega G_1(x, tu) \,\mathrm{d}x \\ &\quad - \gamma C_5 t^{1/\sigma} \int_\Omega |u|^{1/\sigma} \,\mathrm{d}x - \mu \int_{\partial\Omega} F(x, tu) \,\mathrm{d}x. \end{aligned} \tag{2.13}$$

Now, since

$$G_1(x, tu) = o((t|u|)^{q_1^+}), \quad F(x, tu) = o((t|u|)^{\beta^+}) \quad \text{when } t \to +\infty,$$

(because $P_+^+ \leq q_1^+$ and $\beta^+ < P_+^+ < \frac{1}{\sigma}$), we obtain

$$\Phi(tu) \to -\infty, \quad \text{when } t \to +\infty.$$

Hence, It follows that there exist $u_0 \in X$ such that $\|u_0\| > r_1$ and $\Phi(u_0) < 0$. Therefore, By the Mountain Pass theorem, problem (3.8) has a nontrivial solution $u_1$ such that $\Phi(u_1) > 0$.

(2) Under the assumpti9ns in (2) hold, (1), we know that there exist $r_0(\gamma) > 0$ such that when $0 \leq \lambda, \mu \leq r_0(\gamma)$, problem has a nontrivial solution $u_1$ such that $\Phi(u_1) > 0$. For $t \in (0, 1)$ small enough, and $v_0 \in C_0^\infty(\Omega)$ such that $0 \leq v_0(x) \leq \min\{\delta_1, \delta_2\}$, we have

$$\begin{aligned} &\Phi(tv_0) \\ &= \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} \Big( \Big|\frac{\partial (tv_0)}{\partial x_i}\Big|^{p_i(x)} + |tv_0|^{p_i(x)} \Big) \,\mathrm{d}x - \lambda \int_\Omega G_1(x, tv_0) \,\mathrm{d}x \\ &\quad - \gamma \int_\Omega G_2(x, tv_0) \,\mathrm{d}x - \mu \int_{\partial\Omega} F(x, tv_0) \,\mathrm{d}x \\ &\leq t^{P_-^-} \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} \Big( \Big|\frac{\partial v_0}{\partial x_i}\Big|^{p_i(x)} + |v_0|^{p_i(x)} \Big) \,\mathrm{d}x - \lambda C_3 \int_\Omega |tv_0|^{q_3(x)} \,\mathrm{d}x \\ &\quad - \gamma \int_\Omega G_2(x, tv_0) \,\mathrm{d}x - \mu C_4 \int_{\partial\Omega} |tv_0|^{q_4(x)} \,\mathrm{d}x. \end{aligned} \tag{2.14}$$

For $t \in (0, 1)$ small enough, we obtain

$$G_2(x, tv_0) = o(|tv_0|^{P_+^+}), \quad \text{as } t \to \infty$$

So, we have

$$\begin{aligned}&\Phi(tv_0)\\&\le t^{P_-^-}\sum_{i=1}^{N}\int_\Omega \frac{1}{p_i(x)}\Big(|\frac{\partial tv_0}{\partial x_i}|^{p_i(x)}+|tv_0|^{p_i(x)}\Big)\,\mathrm{d}x-\lambda C_3 t^{q_3^+}\int_\Omega |v_0|^{q_3(x)}\,\mathrm{d}x\\&\quad-\gamma M t^{P_+^+}\int_\Omega v_0\,\mathrm{d}x-\mu C_4 t^{q_4^+}\int_{\partial\Omega}|v_0|^{q_4(x)}\,\mathrm{d}x.\end{aligned}\tag{2.15}$$

Since $\min(q_3^+, q_4^+) < P_-^-$, by factoring the right side of (3.13) by $t^{q_3^+}$ if $q_4^+ > q_3^+$, and by $t^{q_4^+}$ if $q_3^+ > q_4^+$, we obtain

$$\lim_{t\to 0}\Phi(tv_0) < 0.$$

Then there exist $w \in X$ such that $\|w\| \le r_1$, and $\Phi(w) < 0$.

(3) $\Phi$ is an even functional. We denote

$$\psi(u)=\lambda\int_\Omega G_1(x,u)\,\mathrm{d}x+\gamma\int_\Omega G_2(x,u)\,\mathrm{d}x+\mu\int_{\partial\Omega}F(x,u)\,\mathrm{d}x$$

As $\beta_k(\nu)$ is defined in Proposition (2.2.2), for each positive integer, there exist a positive integer $k_0$ such that $\beta_k(n) \le 1$ for all $k \ge k_0(n)$. We can assume $k_0(n) < k_0(n+1)$ for each $n$. We define $\{\nu_k : k = 1, 2, \dots\}$ by

$$\nu_k=\begin{cases} n & \text{if } k_0\le k< k_0(n+1),\\ 1 & \text{if } 1\le k<k_0.\end{cases}\tag{2.16}$$

We see that $\nu_k \to \infty$ when $k \to \infty$, then for $u \in Z_k$ *with* $\|u\| = \nu_k$, we obtain

$$\begin{aligned}\Phi(u)&=\sum_{i=1}^{N}\int_\Omega\frac{1}{p_i(x)}\Big(|\frac{\partial u}{\partial x_i}|^{p_i(x)}+|u|^{p_i(x)}\Big)\,\mathrm{d}x-\psi(u)\\&\ge\frac{1}{P_+^+2^{P_-^- -1}N^{P_-^- -1}}(\nu_k)^{P_-^-}-1.\end{aligned}$$

Consequently,

$$\inf_{u\in Z_k,\|u\|=\nu_k}\Phi(u)\to\infty\quad\text{as } k\to\infty.$$

So the hypotheses (3.21) of Fountain theorem are satisfied. Indeed, by (H6), (H9) and Remark (2.5.2), for $\|u\| \ge 1$ we obtain

$$\Phi(u)\le\frac{C}{P_-^-}\|u\|^{P_+^+}+C_1\lambda|u|_{q_1(x)}^{q_1^+}-C_5\gamma|u|_{\frac{1}{\sigma}}^{1/\sigma}+C_6\mu|u|_{\beta(x)}^{\beta^+}+C_7.$$

As the space $Y_k$ has finite dimension i.e all norms are equivalents, we then have

$$\Phi(u)\le\frac{C}{P_-^-}\|u\|^{P_+^+}+C_1'\lambda\|u\|^{q_1^+}-C_5'\gamma\|u\|^{1/\sigma}+C'\mu\|u\|^{\beta^+}+C_7.$$

Since $\min(q_1^+, q_2^+) < P_+^+ < \frac{1}{\sigma}$, we obtain $\Phi(u) \to -\infty$ as $\|u\| \to +\infty$, $u \in Y_k$. Finally, the proof of (3) is complete. □

# CHAPTER 3

# SOLUTIONS FOR A QUASILINEAR ELLIPTIC $\vec{P}(X)$-KIRCHHOFF TYPE PROBLEM WITH WEIGHT AND NONLINEAR ROBIN BOUNDARY CONDITIONS

Mathematics, rightly viewed, possesses not only truth, but supreme beauty.

*( Bertrand Russell)*

This paper deals with the existence and multiplicity of weak solutions to a class of quasilinear elliptic $\vec{p}(x)$-Kirchhoff type problems with weight and a nonlinear Robin boundary condition such as

$$\left(a+bK\left(\sum_{i=1}^{N}\int_{\Omega}\frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}\mathrm{d}x\right)\right)(-\Delta_{\vec{p}(x)}u)+\sum_{i=1}^{N}V_i(x)\,|u|^{p_i(x)-2}\,u$$
$$=\theta(x)\,|u|^{m(x)-2}\,u+f(x,u)\text{ in }\Omega,$$
$$\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}v_i=\eta|u|^{q(x)-2}u\quad\text{on }\partial\Omega,$$

where $\Omega$ is a smooth bounded domain. Under suitable conditions on the data, we show the existence and multiplicity of weak solutions to this problem by means of a variational approach and the use of the anisotropic Sobolev spaces with variable exponents setting.

## 3.1 Introduction and statement of main results

This work is mainly devoted to the existence and multiplicity of weak solutions to the following class of $\vec{p}(x)$-Kirchhoff type quasilinear elliptic prooblems

$$\begin{aligned}&\left(a+bK\left(\sum_{i=1}^{N}\int_{\Omega}\frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}\mathrm{d}x\right)\right)\left(-\Delta_{\vec{p}(x)}u\right)+\sum_{i=1}^{N}V_i(x)\,|u|^{p_i(x)-2}\,u\\&\qquad=\theta(x)\,|u|^{m(x)-2}\,u+f(x,u)\text{ in }\Omega,\\&\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}v_i=\eta|u|^{q(x)-2}u\quad\text{on }\partial\Omega,\end{aligned}\tag{3.1}$$

where $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain ($N\geq 3$), the $p_i(x)$ $(1\leq i\leq N)$ are continuous functions with : $2\leq p_i(x)<N$, $\Delta_{\vec{p}(x)}u=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right)$ is the so-called anisotropic $\vec{p}(x)$-Laplacian operator, $a$ and $b$ are positive reals and the functions $m$, $q$ $f$, $\theta$ and $V_i(1\leq i\leq N)$ are subjected to some appropriate conditions that will be presented hereafter.

Concretely, we are interested in the existence of at least a nontrivial weak solution for problem (3.1) and on existence of infinitely many solutions as well. Actually, by using the Mountain Pass theorem, we prove that (3.1) has at least a nontrivial solution; while Fountain's theorem provides us with the existence of infinitely many solutions when $f$ is odd.

This chapter is motivated by recent works on nonlinear Kirchhoff elliptic equations (see for example[1],[3],[17],[42],[33] and the references therein). When $b=0$, these questions have been treated by the authors in a recent work [22] .The main goal here is to extend the results in [22] to the case of $\vec{p}(x)$ -Kirchhoff type quasilinear elliptic prooblems of the form (3.1).

Let us point out that, in the last decade, many authors studied the following elliptic nonlocal Kirchhoff type problem

$$\begin{aligned} -\left(a+b\int_{\Omega}|\nabla u|^2\right)\Delta u = f(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{3.2}$$

Interesting results concerning the existence and multiplicity of positive solutions are obtained for example in [1],[3], [33] and [43] by means of variational methods.
More generally, problems related to problem 3.1 have been studied recently by many authors. The paper [42] by D.Liu seems to be closer to our work and is related to the following problem

$$\begin{aligned} \left(M\left[\int_{\Omega}(|\nabla u|^p+\lambda(x)|u|^p)\ \mathrm{d}x\right]\right)^{p-1}\left(-\Delta_p u+\lambda(x)|u|^{p-2}u\right) = f(x,u), \\ \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = \eta|u|^{p-2}u; \quad \text{on } \partial\Omega, \end{aligned} \tag{3.3}$$

We intend here to extend some of the results in [42] to the anisotropic problem with variable exponents and more general nonlocal terms. Besides, we are able to deal with more general nonlinearities in the boundary condition than in [42] and with situations where the function $M$ in 4.6 is unbounded from above. When $\lambda(x) := 0$ and $p$ is constant in 4.6, our results apply for example to the function $M(t) = (\frac{bk_1}{p}t+a)^{\frac{1}{p-1}}$ , where $k_1$ is some positive real constant that will be defined in hypothesis (K0) below. Although this function does not satisfy condition (m1) of theorem 1.1 in [42], the existence of a weak solution is however guaranted by our results.

This chapter is organized as follows: In section 2 we recall some basic properties of the Lebesgue-Sobolev space $W^{1,p}(\Omega)$ and the anisotropic Orlicz-Sobolev space $W^{1,\overrightarrow{p}(x)}(\Omega)$, and state our existence and multiplicity results concerning problem 3.1. Section 4 is devoted to the proofs of the main results and finally in section 5, we give some concluding examples.

## 3.2 Preliminaries

Let us first recall the definition of the Cerami condition which was introduced by G. Cerami in [18].

**Definition 3.1.** *Let $X$ be a real Banach space and $\varphi \in C^1(X,\mathbb{R})$. We say that $\varphi$ satisfies the Cerami condition at level $c \in \mathbb{R}$ ($(C)_c$ for short) if any sequence $\{u_n\} \subset X$ along with*

$$\varphi(u_n) \longrightarrow c \quad \textit{and } (1+\|u_n\|)\left\|\varphi'(u_n)\right\| \longrightarrow 0, \quad \textit{as } n \longrightarrow \infty \tag{3.4}$$

*possesses a convergent subsequence. On the other hand, we say that $\varphi$ satisfies the Cerami condition if $\varphi$ satisfies $(C)_c$ for all $c \in \mathbb{R}$.*

Let us mention that the Cerami condition is weaker than the Palais-Smale (PS) condition. We have the following well-known result

**Proposition 3.2.1.** *(See [52]) Let $X$ be a real Banach space, $\varphi \in C^1(X,\mathbb{R})$, $e \in X$ and $r > 0$ be such that $\|e\| > r$ and*

$$b := \inf_{\|u\|=r} \varphi(u) > \max\{\varphi(0), \varphi(e)\}. \tag{3.5}$$

*Let*

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)),$$

*where*

$$\Gamma := \{\gamma \in C([0,1], X) \mid \gamma(0) = 0, \gamma(1) = e\}.$$

*If $\varphi$ satisfies the condition $(C)_c$, then $c$ is a critical value of $\varphi$.*

In order to discuss our problem, we need to state some properties concerning the spaces $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W^{1,\overrightarrow{p}(x)}(\Omega)$, with a real scalar $p > 1$ and a real vector function $\overrightarrow{p}(x) = (p_1(x), p_2(x), ..., p_N(x))$, such that $p_i(x) > 1 (1 \leq i \leq N)$. Let

$$\mathcal{C}_+(\bar{\Omega}) = \{u \in \mathcal{C}(\bar{\Omega}) : \operatorname{ess\,inf}_\Omega u \geq 1\}.$$

Denote by $p^- = \inf_{x\in\Omega} p(x)$ and $p^+ = \sup_{x\in\Omega} p(x)$.We have $p^- > 1$ and $p^+ < \infty$. Let $\mathcal{S}$ be either $\Omega$ or $\partial\Omega$. Define the variable exponent Lebesgue space

$$L^{p(x)}(\mathcal{S}) = \{u \mid u : \mathcal{S} \to \mathbb{R} \text{ is measurable and } \int_{\mathcal{M}} |u(x)|^{p(x)}\, dx < \infty\},$$

endowed with the Luxembourg norm

$$|u|_{p(x)} = |u|_{L^{p(x)}(\mathcal{S})} = \inf\Big\{\tau > 0; \int_{\mathcal{S}} \Big|\frac{u(x)}{\tau}\Big|^{p(x)}\, dx \leq 1\Big\},$$

and define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

endowed with the norm

$$\|u\| = \inf\Big\{\tau > 0; \int_{\Omega}\Big(\Big|\frac{\nabla u(x)}{\tau}\Big|^{p(x)} + \Big|\frac{u(x)}{\tau}\Big|^{p(x)}\Big)\, dx \leq 1\Big\}.$$

Now, we introduce the anisotropic Orlicz-Sobolev space $W^{1,\overrightarrow{p}(x)}(\Omega)$. Set

$$P_+^+ = \max\{p_1^+, p_2^+, \ldots, p_N^+\}, \quad P_-^+ = \max\{p_1^-, p_2^-, \ldots, p_N^-\},$$

$$P_-^- = \min\{p_1^-, p_2^-, \ldots, p_N^-\}.$$

$W^{1,\overrightarrow{p}(x)}(\Omega)$ is defined by

$$W^{1,\overrightarrow{p}(x)}(\Omega) = \{u \in L^{P_+^+}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i(x)}(\Omega), \forall i \in \{1, ..., N\}\},$$

and is endowed with the norm

$$\|u\|_{\overrightarrow{p}(x)} = \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(x)} + \sum_{i=1}^{N} |\, u \,|_{p_i(x)} \,.$$

By $E_i(i \in \{1, ..., N\})$ we mean the Orlicz-Sobolev spaces defined by

$$E_i = \left\{ u \in L^{p_i(x)}(\Omega); \int_\Omega \left( a \left| \frac{\partial u}{\partial x_i}(x) \right|^{p_i(x)} + V_i(x) \, |u(x)|^{p_i(x)} \right) \mathrm{d}x < \infty \right\}.$$

These are Banach spaces with respect to the norms, For $i \in \{1, ..., N\}$

$$\|u\|_i = \inf \left\{ \tau > 0; \int_\Omega ( \frac{a}{\tau} \left| \frac{\partial u}{\partial x_i}(x) \right|^{p_i(x)} + V_i(x) \left| \frac{u(x)}{\tau} \right|^{p_i(x)} ) \, dx \leq 1 \right\},$$

and (see [20]) the anisotropic Orlicz-Sobolev space $X = W^{1,\overrightarrow{p}(x)}(\Omega)$ can be defined as the closure of $\mathcal{C}_0^1(\Omega)$ with respect to the norm

$$\|u\|_* = \sum_{i=1}^{N} \|u\|_i,$$

which is itself equivalent to the norm

$$\|u\|_{\overrightarrow{p}(x)} = \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{p_i(x)}.$$

Throughout this work we shall assume

$$\sum_1^N \frac{1}{p_i^-} > 1. \tag{3.6}$$

We also define

$$P_-^\star := \frac{N}{\sum_1^N \frac{1}{p_i^-} - 1} \quad \text{and } P_{-,\infty} := \max\{P_-^+, P_-^\star\}$$

and for all $i \in \{1, ..., N\}$ $p_i^\partial(x) = \begin{cases} \dfrac{(N-1)p_i(x)}{N - p_i(x)} & \text{if } p_i(x) < N, \\ +\infty & \text{if } p_i(x) \geq N. \end{cases}$

The following are embedding results on anisotropic generalized Sobolev spaces and will be used later.

**Proposition 3.2.2** (See [43]). *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with smooth boundary. Assume that relation (3.6) is fulfilled. Then, for any $m \in C_+(\overline{\Omega})$ satisfying $1 \leq m(x) \leq P_{-,\infty}$ for all $x \in \overline{\Omega}$, the embedding*

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{m(.)}(\Omega)$$

*is continuous and compact.*

**Proposition 3.2.3** (See [9]). *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with smooth boundary and let $q \in C(\overline{\Omega})$ satisfy the condition*

$$1 \leq q(x) < \min_{x\in\partial\Omega}\{p_1^{\partial}(x), ..., p_N^{\partial}(x)\}, \forall x \in \partial\Omega.$$

*Then, there is a compact boundary trace embedding*

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega).$$

## 3.3 Main results

In this section we state the main existence theorems of this work.
Throughout this paper, by weak solutions of problem (3.1) we understand critical points of the following associated energy functional $\varphi$, acting on the anisotropic Orlicz-Sobolev space $X = W^{1,\overrightarrow{p}(x)}(\Omega)$ and defined by

$$\begin{aligned} \varphi(u) &= a\sum_{i=1}^{N}\int_{\Omega}\left|\frac{1}{p_i(x)}\frac{\partial u}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x + b\hat{K}\left(\sum_{i=1}^{N}\int_{\Omega}\frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x\right) \\ &+ \sum_{i=1}^{N}\int_{\Omega}\frac{1}{p_i(x)}V_i(x)\,|u|^{p_i(x)}\,\mathrm{d}x - \int_{\Omega}F(x,u)\,\mathrm{d}x \\ &- \int_{\Omega}\frac{\theta(x)}{m(x)}\mid u\mid^{m(x)}\,\mathrm{d}x - \eta\int_{\partial\Omega}\frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x, \end{aligned} \tag{3.7}$$

where $\hat{K}(t) = \int_0^t K(s)\,\mathrm{d}s$ and $F(x,u) = \int_0^u f(x,t)\,\mathrm{d}t$.
In all this paper $C, C_i(i = 0,1,2,\dots)$ represents different positive real constants .

Now, we introduce the followings assumptions:

(P) $2 \leq P_-^- \leq P_+^+ \leq N$ , $2P_-^- > \frac{3P_+^+ + 2}{2P_+^+ + 1}P_+^+$ and $\sum_1^N \frac{1}{p_i^-} > 1$.

(T) $\theta \in L^\infty(\Omega)$ is non-negative, $m \in C_+(\overline{\Omega})$ verifies $1 \leq m(x) \leq P_{-,\infty}$, for all $x \in \overline{\Omega}$ and $q \in C(\overline{\Omega})$ satisfies the condition

$$1 \leq q(x) < \min_{x\in\partial\Omega}\{p_1^{\partial}(x), ..., p_N^{\partial}(x)\}, \forall x \in \partial\Omega.$$

(V) For all $i \in \{1, ..., N\}$, $V_i \in C(\Omega, \mathbb{R})$ is such that $\inf\limits_{x\in\Omega} V_i(x) \geq a_i$, for some $a_i > 0$ and for each $M_i > 0$, $meas\{x \in \Omega : V_i(x) \leq M_i\} < +\infty$, where meas denotes the Lebesgue measure.

(K0) $K : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous nondecreasing function and there exist two constants $k_0 > 0$ and $k_1 > 0$ with $k_0 < k_1$ such that

$$k_0 \leq K(t) \leq k_1 t, \text{ for all } t > 1.$$

(K1) There exists a postive constant $\eta$ with $\frac{P_+^+ + 1}{2P_+^+ + 1} < \eta < \frac{2P_-^- - P_+^+}{P_+^+}$ such that

$$\hat{K}(t) \geq (1-\eta)K(t)t, \quad \text{for all } t \geq 0.$$

(f0) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist $c > 0$ and $\alpha \in \mathcal{C}(\bar{\Omega})$ such that: $P_+^+ < \alpha(x) < P_{-,\infty}$, for all $x \in \bar{\Omega}$ and

$$|f(x,t)| \leq c(1 + |t|^{\alpha(x)-1}) \quad \text{for all } (x,t) \in \bar{\Omega} \times \mathbb{R}.$$

(f1) $f(x,t) = \circ(|t|^{P_-^- - 1})$ uniformly in $x$ as $t \longrightarrow 0$.

(f2) There exist two constants $\sigma > \max\{m, \frac{P_+^+}{1-\eta}\}$ and $r > 0$ such that

$$\inf_{x\in\Omega, |u|=r} F(x,u) > 0$$

and

$$\sigma F(x,u) \leq f(x,u)u, \quad \text{for all } x \in \Omega \quad \text{and } |u| \geq r;$$

which is an Ambrosetti-Rabinowitz type condition (see [24]).

(f3) There exists a constant $r \geq 1$, such that for any $s \in [0,1]$ and $t \in \mathbb{R}$, we have the inequality

$$r\mathcal{F}(x,t) \geq \mathcal{F}(x,st) \text{ hold for a.e. } x \in \Omega,$$

where $\mathcal{F}$ is defined by

$$\mathcal{F}(x,t) = f(x,t)t - \frac{1}{2P_+^+}F(x,t).$$

(f4) $f(x,-t) = -f(x,t)$, $x \in \Omega, t \in \mathbb{R}$.

Finally, denote by $\lambda_1^+$ and $\lambda_1^-$ the first eigenvalues of the problems $-\Delta_{\vec{p}(x)}u = \lambda |u|^{P_+^+-2} u$ and $-\Delta_{\vec{p}(x)}u = \lambda |u|^{P_-^--2} u$ respectively, that are defined by

$$\lambda_1^+ = \inf_{u\in W^{1,\vec{p}(x)}(\Omega)\setminus\{0\}} \left\{ \frac{\sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx}{\int_\Omega |u|^{P_+^+} dx} \right\}$$

and

$$\lambda_1^- = \inf_{u\in W^{1,\vec{p}(x)}(\Omega)\setminus\{0\}} \left\{ \frac{\sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx}{\int_\Omega |u|^{P_-^-} dx} \right\}.$$

The main results of this article are as follows:

**Theorem 3.3.1.** *Suppose that (P),(T),(V), (K0),(K1), (f0),(f1) and (f2) hold with $q^- > P_+^+$. Then, problem (3.1) has at least a nontrivial weak solution.*

**Theorem 3.3.2.** *Suppose that (P),(T),(V), (K0),(K1), (f0),(f2) and (f4) hold with $q^- > P_+^+$. Then, problem (3.1) has infinite many pairs of weak solutions.*

**Theorem 3.3.3.** *Suppose that (P),(T),(V), (f0),(K0) and (K1) hold with $\min\{m^-, q^-\} \geq \frac{P_+^+}{1-\eta}$. Furthermore, assume that*

*(F1)* $\lim_{|t|\to+\infty} \left( \frac{1}{2P_+^+} f(x,t)t - F(x,t) + \frac{a\lambda_1^-}{2P_+^+}|t|^{P_-^-} \right) = +\infty$*, uniformly for a.e. $x \in \Omega$ ,*

*(F2) there exists $\lambda < \lambda_1^+$ such that $F(x,t) \leq \frac{\lambda}{N^{P_+^+-1}}|t|^{P_+^+}$, for $|t|$ small and a.e. $x \in \Omega$.*

*Then, (3.1) has at least a nontrivial weak solution.*

**Theorem 3.3.4.** *Assume that (P),(T),(V),(K0), (f0) and (f3) hold with $\max\{q^+, m^+\} < P_-^-$ and $a > \lambda_1^+$. Furthermore, assume that*

*(f'1)* $\lim_{|t|\to+\infty} \frac{f(x,t)}{t^{2P_+^+-1}} = \infty$*, uniformly for a.e $x \in \Omega$,*

*(f'2) there exists a nonnegative function $k \in L^\infty(\Omega)$ such that $\|k\|_\infty < \lambda_1^+$ and*

$$\limsup_{|t|\to 0} \frac{f(x,t)}{at} = k(x), \text{ uniformly for a.e } x \in \Omega,$$

*(K3)* $\hat{K}(t) \geq \frac{P_-^-}{2P_+^+}K(t)t, \quad$ *for all $t \geq 0$.*

*Then, problem (3.1) has at least a nontrivial weak solution.*

## 3.4 Proofs of main results

*Proof of Theorem 3.3.1.* First, we show the following lemmas.

**Lemma 3.4.1.** *Under the assumptions (P), (T), (V), (K0), (K1) and (f0) to (f2), $\varphi$ satisfies the condition $(C)_c$, where c is defined in proposition (3.2.1) .*

**Lemma 3.4.2.** *Under the assumptions (P), (T), (V), (K0), (f0) and (f1), there exist two positive reals $r_1$ and $C'$ such that $\varphi(u) \geq C'$, for all $u \in X$ satisfying $\|u\| = r_1$.*

**Proof of lemma 4.1 .** The constant $c$ defined in(3.2.1) proposition is positive. Let us show that $\varphi$ satisfies the condition $(C)_c$. By hypothesis, there exists a sequence $\{u_n\} \subset X$ such that

$$\varphi(u_n) \longrightarrow c, \, (1 + \|u_n\|) \left\|\varphi'(u_n)\right\| \longrightarrow 0. \tag{3.8}$$

Let $\{u_n\} \subset X$ be a Cerami sequence of $\varphi$. It suffices to show that $\{u_n\}$ is bounded. We shall argue by contradiction. Assume that $\|u_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$ and consider $w_n = u_n/\|u_n\|$; then $\|w_n\| = 1, \forall n \geq 1$. Since $X$ is a reflexive Banach space, there exists $w \in X$ such that

$$\begin{aligned} w_n &\rightharpoonup w \quad \text{in } X, \\ w_n &\longrightarrow w \quad \text{in } L^r(\Omega), \\ w_n(x) &\longrightarrow w(x) \quad \text{for a.e. } x \in \Omega, \end{aligned} \tag{3.9}$$

for any $r$ with $1 \leq r < P_{-,\infty}$.
From (3.8) we obtain

$$| < \varphi'(u_n), u_n > | \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

On the other hand, we have

$$\begin{aligned} < \varphi'(u_n), u_n >= {} & a \int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x + \sum_{i=1}^N \int_\Omega V_i(x)|u_n|^{p_i(x)} \mathrm{d}x \\ & + bK\left(\sum_{i=1}^N \frac{1}{p_i(x)} \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x\right) \int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x \\ & - \int_\Omega \theta(x)|u_n|^{m(x)} \mathrm{d}x - \int_\Omega f(x, u_n)u_n \, \mathrm{d}x \\ & -\eta \int_{\partial\Omega} |u_n|^{q(x)} \mathrm{d}x. \end{aligned} \tag{3.10}$$

Therefore, it yields for large $n$ that

$$\left|A - B - \int_\Omega f(x, u_n)u_n \, \mathrm{d}x\right| \leq 1, \tag{3.11}$$

where

$$A := a\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x + \sum_{i=1}^N \int_\Omega V_i(x)|u_n|^{p_i(x)}\,\mathrm{d}x$$
$$+bK\left(\sum_{i=1}^N \frac{1}{p_i(x)}\int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x\right)\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x$$

and

$$B := \int_\Omega \theta(x)|u_n|^{m(x)}\,\mathrm{d}x + \eta\int_{\partial\Omega}|u_n|^{q(x)}\,\mathrm{d}x.$$

Hence

$$\int_\Omega f(x,u_n)u_n\,\mathrm{d}x \le A - B + 1 \le A + 1.$$

So, we obtain

$$\int_\Omega f(x,u_n)u_n\,\mathrm{d}x \le \sum_{i=1}^N \int_\Omega \left(a\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} + V_i(x)|u_n|^{p_i(x)}\right)\mathrm{d}x$$
$$+bK\left(\sum_{i=1}^N \frac{1}{p_i(x)}\int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x\right)\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x.$$

By virtue of hypotheses (K0), we then get

$$\int_\Omega f(x,u_n)u_n\,\mathrm{d}x \le \sum_{i=1}^N \int_\Omega \left(a\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} + V_i(x)|u_n|^{p_i(x)}\right)\mathrm{d}x$$
$$+\frac{bk_1}{P^+}\left(\sum_{i=1}^N \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x\right)^2.$$

We use the following notation

$$\boldsymbol{\ell}_1 = \{i \in \{1,...,N\} : \left|\frac{\partial u}{\partial x_i}\right|_{p_i(x)} \le 1\}, \quad \boldsymbol{\ell}_1' = \{i \in \{1,...,N\} : \|u\|_i \le 1\}$$

and

$$\boldsymbol{\ell}_2 = \{i \in \{1,...,N\} : \left|\frac{\partial u}{\partial x_i}\right|_{p_i(x)} > 1\} \quad \boldsymbol{\ell}_2' = \{i \in \{1,...,N\} : \|u\|_i > 1\}.$$

We get

$$\begin{aligned}\int_\Omega f(x,u_n)u_n\,\mathrm{d}x &\le \sum_{i\in\ell_2'}\int_\Omega\left(a\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)}+V_i(x)|u_n|^{p_i(x)}\right)\,\mathrm{d}x+N\\ &+\frac{bk_1}{P_+^+}\left(\sum_{i\in\ell_2}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)}\,\mathrm{d}x+N\right)^2\\ &\le \sum_{i\in\ell_2'}\|u_n\|_i^{P_+^+}+\frac{bk_1}{P_+^+}\left(\sum_{i\in\ell_2}\left|\frac{\partial u_n}{\partial x_i}\right|_{p_i(x)}^{P_+^+}+N\right)^2+N\\ &\le \left(\sum_{i=1}^N\|u_n\|_i\right)^{P_+^+}+\frac{bk_1}{P_+^+}\left[\left(\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|_{p_i(x)}\right)^{P_+^+}+N\right]^2\\ &+N.\end{aligned}$$

Consequently, there exists $c_1>0$ such that

$$\int_\Omega f(x,u_n)u_n\,\mathrm{d}x\le\|u\|^{P_+^+}+\frac{bk_1}{P_+^+}\left[c_1^{P_+^+}\|u\|^{P_+^+}+N\right]^2+N. \tag{3.12}$$

By (f2) we know that

$$\lim_{n\to+\infty}\int_\Omega\frac{f(x,u_n)u_n\,\mathrm{d}x}{\|u_n\|^{2P_+^++1}}=0. \tag{3.13}$$

Set $\Omega_1=\{x\in\Omega:w(x)\neq0\}$ and suppose $meas(\Omega_1)>0$. We have $|u_n(x)|\longrightarrow\infty$, as $n\longrightarrow\infty$, for a.e. $x\in\Omega_1$. Moreover, by (f2) there exits $C>0$ such that

$$F(x,t)\ge C|t|^\sigma,\forall x\in\Omega,\forall|t|\ge r;$$

which implies that

$$\lim_{t\to+\infty}\frac{F(x,t)}{|t|^{2P^++1}}=\infty. \tag{3.14}$$

Now, it follows from (f1), (f2) and (3.14) that there exist $C_1,C_2>0$ such that

$$f(x,t)t\ge C_1|t|^{2P_+^++1}-C_2|t|^{P_-^-},\quad\forall(x,t)\in\Omega\times\mathbb{R}.$$

Hence

$$\int_\Omega\frac{f(x,u_n)u_n}{\|u_n\|^{2P_+^++1}}\,\mathrm{d}x\ge C_1\int_{\Omega_1}\frac{|u_n|^{2P_+^++1}}{\|u_n\|^{2P_+^++1}}\,\mathrm{d}x-C_2\int_{\Omega_1}\frac{|u_n|^{P_-^-}}{\|u_n\|^{2P_+^++1}}\,\mathrm{d}x.$$

Therefore, we get

$$\int_{\Omega_1}\frac{f(x,u_n)u_n}{\|u_n\|^{2P_+^++1}}\,\mathrm{d}x\ge C_1\int_{\Omega_1}|w_n|^{2P_+^++1}\,\mathrm{d}x-C_2\frac{|w_n|_{P_-^-}^{P_-^-}}{\|u_n\|^{2P_+^+-P_-^-+1}}. \tag{3.15}$$

Thus, by using (3.15) and Fatou's lemma we obtain

$$0 = \lim_{n\to+\infty} \int_{\Omega_1} \frac{f(x,u_n)u_n\,\mathrm{d}x}{\|u_n\|^{2P_+^+ +1}} \geq C_1 \int_{\Omega_1} |w|^{2P_+^+ +1}\,\mathrm{d}x > 0.$$

This is a contradiction. Hence $meas(\Omega_1) = 0$. Therefore, we get $w(x) = 0$, for a.e. $x \in \Omega$.

Now, we have

$$\begin{aligned}\left(\varphi(u_n) - \frac{1}{\sigma} < \varphi'(u_n), u_n >\right) &= \sum_{i=1}^{N} \int_{\Omega} \left(\frac{1}{p_i(x)} - \frac{1}{\sigma}\right) a \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x \\ + \sum_{i=1}^{N} \int_{\Omega} \left(\frac{1}{p_i(x)} - \frac{1}{\sigma}\right) V_i(x) |u_n|^{p_i(x)}\,\mathrm{d}x &+ \int_{\Omega} \theta(x) \left(\frac{1}{\sigma} - \frac{1}{m(x)}\right) |u_n|^{m(x)}\,\mathrm{d}x \\ + \int_{\Omega} \left(\frac{1}{\sigma} f(x,u_n)u_n - F(x,u_n)\right) \mathrm{d}x &+ \eta \int_{\partial\Omega} \left(\frac{1}{\sigma} - \frac{1}{q(x)}\right) |u|^{q(x)}\,\mathrm{d}x,\end{aligned}$$

where

$$\begin{aligned}L := &- \frac{b}{\sigma} K \left(\sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_i(x)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx\right) \left(\sum_{i=1}^{N} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx\right) \\ &+ b\hat{K} \left(\sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_i(x)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} (\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} + V_i(x) |u_n|^{p_i(x)})\,\mathrm{d}x &\geq \sum_{i\in\ell_1'} \|u_n\|_i^{P_+^+} + \sum_{i\in\ell_2'} \|u_n\|_i^{P_-^-} \\ &\geq \sum_{i=1}^{N} \|u_n\|_i^{P_-^-} - \sum_{i\in\ell_1'} (\|u_n\|_i^{P_-^-} - \|u_n\|_i^{P_+^+}) \\ &\geq \sum_{i=1}^{N} \|u_n\|_i^{P_-^-} - N.\end{aligned}$$

Applying the Jensen inequality to the convex function $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $h(t) = t^{P_-^-}$(we have $P_-^- \geq 2$), then yields

$$\frac{1}{2^{P_-^- -1} N^{P^- -1}} \|u_n\|_{\vec{p}(x)}^{P_-^-} - N \leq \sum_{i=1}^{N} \int_{\Omega} \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} + V_i(x) |u_n|^{p_i(x)}\right) \mathrm{d}x. \tag{3.16}$$

Now, by (f0), (f1) and (f2) we know that there exist $C_3 > 0$ such that

$$tf(x,t) - \sigma F(x,t) \geq -C_3 t^{P_-^-}; \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Then, we get

$$\begin{aligned}\varphi(u_n) - \frac{1}{\sigma} < \varphi'(u_n), u_n >\geq \frac{1}{2^{P_-^- -1} N^{P_-^- -1}} \left(\frac{1}{P_+^+} - \frac{1}{\sigma}\right) \|u_n\|^{P_-^-} - C_4 \\ -C_3 \int_\Omega |u_n|^{P_-^-} \mathrm{dx} + \hat{K} \left(\sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x\right) \\ -\frac{P_-^-}{\sigma} K \left(\sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x\right) \left(\sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x\right).\end{aligned}$$

Therfore, for large $n$ one has

$$0 \geq \frac{1}{2^{P_-^- -1} N^{P_-^- -1}} \left(\frac{1}{P_+^+} - \frac{1}{\sigma}\right) - C_3 |w|_{P_-^-}^{P_-^-};$$

which implies that $0 \geq \left(\frac{1}{P_+^+} - \frac{1}{\sigma}\right)$. We obtain a contradiction.
Therefore, $u_n$ is bounded in $X$. This completes the proof.

**Proof of lemma 4.2.** By (f0) and (f1), for each $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that

$$|F(x,t)| \leq \epsilon |t|^{P_-^-} + C(\epsilon)|t|^{\alpha(x)}, \quad \text{for all } t \in \mathbb{R}.$$

For $\|u\|$ small enough, by (K0) we thus obtain

$$\begin{aligned}\varphi(u) \;&\geq \tfrac{1}{P_+^+} \textstyle\sum_{i=1}^N \displaystyle\int_\Omega \left(a \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + V_i(x) \mid u \mid^{p_i(x)}\right) \mathrm{d}x - \int_\Omega \epsilon \mid u \mid^{P_+^+} \mathrm{d}x \\ &+ \int_\Omega C(\epsilon) \mid u \mid^{\alpha(x)} \mathrm{d}x - \frac{\|\theta\|_\infty}{m^-} \int_\Omega \mid u \mid^{m(x)} \mathrm{d}x - \frac{\eta}{q^-} \int_{\partial\Omega} |u|^{q(x)} \mathrm{d}x, \\ &\geq \tfrac{1}{P_+^+ 2^{P_+^+ -1} N^{P_+^+ -1}} \|u\|^{P_+^+} - \int_\Omega \epsilon \mid u \mid^{P_+^+} \mathrm{d}x - \int_\Omega C(\epsilon) \mid u \mid^{\alpha(x)} \mathrm{d}x \\ &- \tfrac{\|\theta\|_\infty}{m^-} \int_{\partial\Omega} \mid u \mid^{m(x)} \mathrm{d}x - \frac{\eta}{q^-} \int_{\partial\Omega} |u|^{q(x)} \mathrm{d}x. \end{aligned} \tag{3.17}$$

Since $m(x) < P_{-,\infty}$ , $q(x) < \min_{x\in\partial\Omega}\{p_1^\partial(x), ..., p_N^\partial(x)\}$ and $P_+^+ < \alpha(x) \leq P_{-,\infty}$, we then have

$$W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{P_+^+}(\Omega) \text{ and } W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega),$$

with continuous and compact embeddings.
Consequently, there exist $C_1' > 0$, $C_2' > 0$ and $C_3' > 0$ such that for all $u \in X$

$$\mid u \mid_{L^{P_+^+}(\Omega)} \leq C_2' \|u\|, \mid u \mid_{L^{m(x)}(\Omega)} \leq C_1' \|u\| \text{ and } |u|_{L^{q(x)}(\partial\Omega)} \leq C_3' \|u\|. \tag{3.18}$$

For $\|u\|$ small enough, we obtain by applying (4.18) in (3.17)

$$\begin{aligned}\varphi(u) \geq & \frac{1}{P_+^+ 2^{P_+^+-1} N^{P_+^+-1}} \|u\|^{P_+^+} - \epsilon C_2' \|u\|^{P_+^+} - C(\epsilon) C_3' \|u\|^{\alpha^-} \\ & - C_1' \frac{\|\theta\|_\infty}{m^-} \max\{|u|^{m^+}_{L^{m(x)}(\Omega)}, |u|^{m^-}_{L^{m(x)}(\Omega)}\} - \frac{\eta C_3'}{q} \|u\|^{q^-}.\end{aligned}$$

Now, let $\epsilon > 0$ be small enough so that: $0 < \epsilon {C_2'}^{P_+^+} \leq \frac{1}{2P_+^+ 2^{P_+^+-1} N^{P_+^+-1}}$.

Denote by $c_0 :\equiv \frac{1}{P_+^+ 2^{P_+^+-1} N^{P_+^+-1}}$. We have

$$\begin{aligned}\varphi(u) \geq & \|u\|^{P_+^+} (c_0 - C(\epsilon) \|u\|^{\alpha^- - P_+^+} - \frac{\|\theta\|_\infty C_1'}{m^-} \max\{\|u\|^{m^+ - P_+^+}, \|u\|^{m^- - P_+^+}\}) \\ & - \|u\|^{P_+^+} \left( \frac{\eta C_3'}{q} \|u\|^{q^- - P_+^+} \right).\end{aligned}$$

Since $P_+^+ < \min(\alpha^-, m^-, q^-)$, there exist therefore two real constants $r_1 > 0$ and $C' > 0$ such that

$$\varphi(u) \geq C' > 0, \text{ for all } u \in X \text{ satisfying } \|u\| = r_1.$$

The proof of lemma 3.4.2 is then complete.

Now, in order to apply the Mountain Pass theorem ([53]), we have to prove that $\varphi(tu) \to -\infty$ as $t \to +\infty$, for some $u \in X$. It follows from (f2) that

$$F(x,s) \geq C \mid s \mid^\sigma, \forall x \in \Omega, \forall \mid s \mid \geq M.$$

On the other hand, when $t > t_0$ with $t_0$ an arbitrarily positive constant, we can easily obtain from (K1) that

$$\hat{K}(t) \leq \frac{\hat{K}(t_0)}{t_0^{\frac{1}{1-\eta}}} t^{\frac{1}{1-\eta}} = c t^{\frac{1}{1-\eta}}.$$

Therefore, for $u \in X \setminus \{0\}$ and $t > 1$, we have

$$\begin{aligned}\varphi(tu) \leq & \frac{1}{P_-^-} \sum_{i=1}^N \int_\Omega \left( a \left| \frac{\partial tu}{\partial x_i} \right|^{p_i(x)} + V_i(x) \mid u \mid^{p_i(x)} \right) \mathrm{d}x \\ & - \int_\Omega F(x,tu)\, \mathrm{d}x + \hat{K} \left( \sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)} \left| \frac{\partial tu}{\partial x_i} \right|^{p_i(x)} \right) \\ & - \int_\Omega \theta(x) \frac{1}{m(x)} \mid tu \mid^{m(x)}\, \mathrm{d}x - \frac{\eta t^{q^-}}{q^-} \int_{\partial\Omega} |u|^{q(x)}\, \mathrm{d}x\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\varphi(tu) \leq \quad & \frac{t^{P_+^+}}{P_-^-}\sum_{i=1}^N \int_\Omega \left( a\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + V_i(x) \mid u \mid^{p_i(x)}\right) \mathrm{d}x \\ & +ct^{\frac{P_+^+}{1-\eta}} \left(\sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} \mathrm{d}x\right)^{\frac{1}{1-\eta}} - t^\sigma \int_\Omega |u|^\sigma \mathrm{d}x \\ & -\frac{\eta t^{q^-}}{q^-}\int_{\partial\Omega} |u|^{q(x)} \mathrm{d}x.\end{aligned}$$

Now, as $\sigma > \frac{P_+^+}{1-\eta}$, it follows that

$$\varphi(tu) \to -\infty \text{ as } (t \to +\infty).$$

Since $\varphi(0) = 0$, then $\varphi$ satisfies the condition of Mountain Pass lemma and so $\varphi$ admits at least a nontrivial critical point $u_0 \in X$ such that $\varphi(u_0) = \tau$ where $\tau$ is characterized by

$$\tau = \inf_{h\in\Gamma} \sup_{t\in[0,1]} \varphi(h(t)),$$

where

$$\Gamma = \{h \in \mathcal{C}([0,1], X); h(0) = 0 \text{ and } h(1) = e\}.$$

This completes the proof of theorem 3.1. □

*Proof of Theorem 3.3.2.* Let $X$ be a reflexive and separable Banach space. It is well know (see, e.g., [2]) that there are $\{e_j\}_{j=1}^\infty \subset X$ and $\{e_j^*\}_{j=1}^\infty \subset X^*(X^*$ being the topological dual of $X)$ such that

$$X = \overline{span}\{e_j : 1, 2, ...\}, \qquad X^* = \overline{span}\{e_j^* : 1, 2, ...\},$$

and

$$< e_j^*, e_i >= \begin{cases} 1 & \text{if } i \ = j, \\ 0 & \text{if } i \ \neq j. \end{cases}$$

For convenience, we write $X_j = span\{e_j\}$, $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \oplus_{j=k}^\infty X_j$.

**Lemma 3.4.3.** *(see [27]) Let $\beta$ be a real such that $1 < \beta < P_-^\star$.*
*Set $\alpha_k := \sup\{\mid u \mid_{L^\beta(\Omega)}; \|u\|_* = 1, u \in Z_k\}$. Then, we have $\lim\limits_{k\to\infty} \alpha_k = 0$.*

To prove theorem 3.2, we shall use the Fountain theorem ( see [53] theorem 3.6). Obviously, $\varphi \in \mathcal{C}^1(X, \mathbb{R})$ and is an even functional. We first prove that if $k$ is large enough, then there exist reals $\rho_k$ and $\nu_k$ with $\rho_k > \nu_k > 0$ such that

$$b_k := \inf\{\varphi(u)/u \in Z_k, \|u\|_* = \nu_k\} \to +\infty \quad \text{as } k \to +\infty, \tag{3.20}$$
$$a_k := \max\{\varphi(u)/u \in Y_k, \|u\|_* = \rho_k\} \to 0 \quad \text{as } k \to +\infty. \tag{3.21}$$

Proof of (3.20): For any $u \in Z_k$ with $|u| = r_k > 1$, we have

$$\begin{aligned} \varphi(u) \geq & \frac{1}{P_+^+}\sum_{i=1}^{N}\int_\Omega \left(a\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + V_i(x) \mid u \mid^{p_i(x)}\right) \mathrm{d}x \\ & - C\int_\Omega (1+|u|^{\alpha(x)})\,\mathrm{d}x - \frac{\|\theta\|_\infty}{m^-}\int_\Omega |u|^{m(x)}\,\mathrm{d}x - \frac{\eta}{q^-}\int_{\partial\Omega}|u|^{q(x)}\,\mathrm{d}x \\ \geq & \frac{1}{P_+^+ 2^{P_-^- -1}N^{P_-^- -1}}\|u\|^{P_-^-} - C_1|u|_{\alpha(x)}^{\alpha(\xi)} \\ & - \frac{\|\theta\|_\infty}{m^-}|u|^{m(\xi)}_{L^{m(x)}(\Omega)} - \frac{\eta}{q^-}|u|^{q(\xi)}_{L^{q(x)}(\partial\Omega)} - C_2, \quad \text{for some } \xi \in \Omega. \end{aligned} \tag{3.22}$$

So, to prove (3.20) we only need to consider the case where $m(x) \leq \alpha(x)$, for all $x \in \Omega$. The case where $m(x) > \alpha(x)$ for all $x \in \Omega$, is similar.

Let us assume that $m(x) \leq \alpha(x)$ for all $x \in \Omega$. Then, we have $L^{\alpha(x)}(\Omega) \subset L^{m(x)}(\Omega)$. Thus, there is a positive constant $C_3 > 0$ such that

$$|u|_{L^{m(x)}(\Omega)} \leq C_3 |u|_{L^{\alpha(x)}(\Omega)} \quad \text{for all } u \in X.$$

Therefore, for any $\xi \in \Omega$ we have

$$|u|^{m(\xi)}_{L^{m(x)}(\Omega)} \leq C^{m(\xi)}|u|^{m(\xi)}_{L^{\alpha(x)}(\Omega)}.$$

Define $e := 1/(2^{P_-^- -1}N^{P_-^- -1})$. We have

$$\begin{aligned} \varphi(u) \geq \frac{e}{P_+^+}\|u\|^{P_-^-} - C\max\{|u|^{\alpha(\xi)}_{L^{\alpha(x)}(\Omega)}, |u|^{m(\xi)}_{L^{m(x)}(\Omega)}\} \\ - \frac{\eta}{q^-}|u|^{q(\xi)}_{L^{q(x)}(\partial\Omega)} - C_2. \end{aligned}$$

Let us write

$$\begin{aligned} E &= L^{\alpha(x)}(\Omega) \cap L^{q(x)}(\partial\Omega), \\ A &= \{u \in E : |u|_{L^{\alpha(x)}(\Omega)} \leq 1, |u|_{L^{q(x)}(\partial\Omega)} \leq 1\}, \\ B &= \{u \in E : |u|_{L^{\alpha(x)}(\Omega)} > 1, |u|_{L^{q(x)}(\partial\Omega)} \leq 1\}, \\ C &= \{u \in E; |u|_{L^{\alpha(x)}(\Omega)} \leq 1, |u|_{L^{q(x)}(\partial\Omega)} > 1\}, \\ D &= \{u \in E; |u|_{L^{\alpha(x)}(\Omega)} \geq 1, |u|_{L^{q(x)}(\partial\Omega)} > 1\}. \end{aligned}$$

From (3.4), we have

$$\varphi(u) \geq \begin{cases} \frac{e}{P_+^+}\|u\|^{P_-^-} - C_1 & \text{if } u \in A, \\ \frac{e}{P_+^+}\|u\|^{P_-^-} - C_1(\alpha_k|u|)^{\alpha_-} - C_2 & \text{if } u \in B, \\ \frac{e}{P_+^+}\|u\|^{P_-^-} - \frac{\eta}{q_-}(\beta_k|u|)^{q_-} - C_1 & \text{if } u \in C, \\ \frac{e}{P_+^+}\|u\|^{P_-^-} - C_1(\alpha_k|u|)^{\alpha_-} - \frac{\eta}{q}(\beta_k|u|)^{q_-} - C_2, & \text{if } u \in D, \end{cases}$$

where $\beta_k = \sup\{|u|_{L^{q(x)}(\partial\Omega)}; \|u\|_* = 1, \text{with } u \in Z_k\}$.

Using the same steps as in ([22]), we easily obtain that $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$, $u \in Z_k$.

Proof of (3.21): From condition (f2), we have

$$F(x,s) \geq C_1|s|^{\sigma} - C_2, \quad \text{for any } (x,s) \in \Omega \times \mathbb{R}.$$

Since we have

$$\sum_{i=1}^{N} \Big|\frac{\partial u}{\partial x_i}\Big|_{L^{p_i(x)}(\Omega)}^{P_+^+} \leq C\Big(\sum_{i=1}^{N} \Big|\frac{\partial u}{\partial x_i}\Big|_{L^{p_i(x)}(\Omega)}\Big)^{P_+^+},$$

where $C$ is a positive constant, then we get

$$\varphi(u) \leq \frac{1}{P_-^-}\sum_{i=1}^{N}\|u\|_i^{P_+^+} + \frac{c}{(P_-^-)^{\frac{1}{1-\eta}}}\left(\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}\right)^{\frac{1}{1-\eta}}$$
$$- C_3\|u\|^{\sigma} - C_4.$$

Therefore, it yields

$$\varphi(u) \leq \frac{C}{P_-^-}\|u\|^{P_+^+} + \frac{C}{(P_-^-)^{\frac{1}{1-\eta}}}\|u\|^{\frac{P_+^+}{1-\eta}} - C_3\|u\|^{\sigma} - C_4.$$

Consequently, as $\sigma > \frac{P_+^+}{1-\eta}$ and $\dim Y_k = k$, it is easy to see that

$$\varphi(u) \to -\infty \quad \text{as } \|u\| \to +\infty \text{ for } u \in Y_k.$$

This completes the proof. □

*Proof of Theorem 3.3.3.*

**Lemma 3.4.4.** *Assume that (P),(T),(V),(K1), (f0), (F1) and (F2) hold with* $\min\{m^-, q^-\} \geq \frac{P_+^+}{1-\eta}$. *Then, the functional* $\varphi$ *satisfies the Cerami-condition (C).*

**Proof.** Let $c \in \mathbb{R}$ and $\{u_n\} \subset X$ be a Cerami sequence of $\varphi$ at level $c$. Let us show that $\{u_n\}$ is bounded. We argue by contradiction. Suppose that $\{u_n\}$ is unbounded. Then, up to a subsequence we may assume that

$$\begin{aligned}
1+c &\geq \varphi(u_n) - \frac{1}{2P_+^+} < \varphi'(u_n), u_n > \\
&\geq (\frac{1}{P_+^+} - \frac{1}{2P_+^+}) \sum_{i=1}^{N} \int_\Omega \left( a \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} + V_i(x) \mid u_n \mid^{p_i(x)} \right) \mathrm{d}x \\
&+\eta \left( \frac{1}{2P_+^+} - \frac{1}{q^-} \right) \int_{\partial\Omega} |u_n|^{q(x)} \,\mathrm{d}x + \int_\Omega \theta(x) \left( \frac{1}{2P_+^+} - \frac{1}{m(x)} \right) |u_n|^{m(x)} \,\mathrm{d}x \\
&+ \int_\Omega \left( \frac{1}{2P_+^+} f(x,u_n)u - F(x,u_n) \right) \mathrm{d}x. \\
&\geq \frac{a}{2P_+^+} \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} \mathrm{d}x - \frac{a\lambda_1^-}{2P^+} \int_\Omega |u_n|^{P_-^-} \,\mathrm{d}x \\
&+ \int_\Omega \left( \frac{1}{2P_+^+} f(x,u_n)u_n - F(x,u_n) + \frac{a\lambda_1^-}{2P_+^+} |u_n|^{P_-^-} \right) \mathrm{d}x.
\end{aligned}$$

By (F1), there exists $M > 0$ such that

$$\frac{1}{2P_+^+} f(x,t)t - F(x,t) + \frac{a\lambda_1^-}{P_+^+} |t|^{P_-^-} \geq -M. \tag{3.23}$$

Then, we get

$$1+c \geq \frac{a}{P_+^+ 2^{P_-^-} N^{P_-^- -1}} \|u_n\|^{P_-^-} \left( 1 - \frac{\lambda_1^- \int_\Omega |u_n|^{P_-^-} \,\mathrm{d}x}{\sum_{i=1}^N \int_\Omega \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} \mathrm{d}x} \right) - M|\Omega|; \tag{3.24}$$

from which We deduce a contradiction. Therefore, $u_n$ is bounded in $X$; which completes the proof.

Now, let us show that $\varphi$ satisfies the hypotheses of lemma 3.4.2 .

By (F2) it is easy to see that

$$F(x,t) \leq \frac{\lambda}{N^{P_+^+ -1}} |t|^{P_+^+} + c\,|t|^{\alpha_+} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

Consequently, we get

$$\begin{aligned}
\varphi(u) &\geq \frac{a}{P_+^+ 2^{P_+^+} N^{P_+^+ -1}} \|u\|^{P_+^+} \left( 1 - \frac{\lambda \int_\Omega |u|^{P_+^+} \,\mathrm{d}x}{\sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} \mathrm{d}x} \right) \\
&- cC_3' \|u\|^{\alpha^+} - C_{1,m(\xi)}' \frac{\|\theta\|_\infty}{m^-} \|u\|^{m(\xi)} - \frac{\eta C_{3,q(\xi)}'}{q^-} \|u\|^{q(\xi)}, \text{ for some } \xi \in \Omega.
\end{aligned}$$

Since $P_+^+ < \min(\alpha^-, m^-, q^-)$, we conclude that there exist $r_1 > 0$ and $C' > 0$ such that

$$\varphi(u) \geq C' > 0, \quad \text{for all } u \in X \text{ satisfying } \|u\| = r_1 .$$

Hence, the proof is complete.
Now, to apply the Mountain Pass theorem, we need to prove that $\varphi(tu) \to -\infty$ as $t \to +\infty$, for some $u \in X$. We have

$$\begin{aligned}\varphi(tu) \leq \quad & \frac{t^{P_+^+}}{P_-^-}\sum_{i=1}^N \int_\Omega \left( a\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} + V_i(x) \mid u \mid^{p_i(x)} \right) \mathrm{d}x \\ & + ct^{\frac{P_+^+}{1-\eta}} \left( \sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} \right)^{\frac{1}{1-\eta}} - \frac{\eta t^{q^-}}{q^+} \int_{\partial\Omega} |u|^{q(x)} \mathrm{d}x.\end{aligned}$$

Due to the relation $q^- > \frac{P_+^+}{1-\eta}$, it follows that

$$\varphi(tu) \to -\infty \text{ as } (t \to +\infty).$$

□

*Proof of Theorem 3.3.4.* It is similar to that of lemma 3.4.1 and theorem 3.3.1.
We use the same notations therein.

**First step**. We show that $u_n$ is bounded in $X$.
Let $w_n = u_n/\|u_n\|$, where $u_n$ is an unbounded sequence such that $w_n \rightharpoonup w$ in $X$. It suffices to consider the cases $w = 0$, a.e. and $w \neq 0$, separately.
Let us suppose $w = 0$, a.e. By (f0), (3.9) and the Lebesgue dominated convergence theorem, we get

$$\lim_{n \to +\infty} \int_\Omega F(x, w_n)\, \mathrm{d}x = 0. \tag{3.25}$$

Following [35], we choose a sequence $(s_n) \subset \mathbb{R}$ such that

$$\varphi(s_n u_n) := \max_{s \in [0,1]} \varphi(s u_n).$$

For any given positive integre $l \geq 1$, set

$$v_n(x) = \|u_l\| w_n(x), \text{ for all } x \in \Omega \text{ and } n \geq 1.$$

By (3.25), we can easily conclude that

$$\lim_{n \to +\infty} \int_\Omega F(x, v_n)\, \mathrm{d}x = \int_\Omega F(x, 0)\, \mathrm{d}x = 0. \tag{3.26}$$

Note that $\|u_n\| \to \infty$ by assumption, then we can choose a subsequence $\{u_n\}$, still denoted by $\{u_n\}$ such that

$$0 < t_n := \frac{\|u_l\|}{\|u_n\|} \leq 1.$$

By (3.8) and (3.26), we then have for $n$ large enough

$$\begin{aligned}\varphi(s_n u_n) &\geq \varphi(t_n u_n) = \varphi(v_n)\\ &\geq \frac{1}{P_+^+}\sum_{i=1}^{N}\int_\Omega \left(a\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i(x)} + V_i(x)\mid v_n\mid^{p_i(x)}\right) \mathrm{d}x\\ &- \int_\Omega F(x, v_n)\,\mathrm{d}x + \hat{K}\left(\sum_{i=1}^{N}\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i(x)}\right)\\ &- \int_\Omega \frac{\theta(x)}{m(x)}\mid v_n\mid^{m(x)}\ \mathrm{d}x - \frac{\eta}{q^-}\int_{\partial\Omega}|v_n|^{q(x)}\,\mathrm{d}x.\end{aligned}$$

By (K0) , (K1) and (4.18), it follows that

$$\begin{aligned}\varphi(s_n u_n) &\geq \varphi(v_n)\\ &\geq \frac{1}{P_+^+}\sum_{i=1}^{N}\int_\Omega \left(a\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i(x)} + V_i(x)\mid v_n\mid^{p_i(x)}\right) \mathrm{d}x\\ &- \frac{\|\lambda\|_\infty}{m^-}\int_\Omega \mid v_n\mid^{m(x)}\ \mathrm{d}x - \frac{\eta}{q^-}\int_{\partial\Omega}|v_n|^{q(x)}\,\mathrm{d}x.\end{aligned}$$

On the other hand, there exists a positive constant $D$ such that

$$\int_\Omega \mid v_n\mid^{m(x)}\ \mathrm{d}x \leq D\max\{\|v_n\|^{m^+}, \|v_n\|^{m^-}\} \tag{3.27}$$

By using (3.16), we therefore get

$$\begin{aligned}\varphi(s_n u_n) \geq\ & \frac{1}{P_+^+}\left(\frac{1}{N^{P_-^- -1}}\|v_n\|_{\vec{p}(x)}^{P_-^-} - N\right) - \frac{\eta C_3'}{q^-}\max\{\|v_n\|^{q^+}, \|v_n\|^{q^-}\}\\ &- \frac{D\|\theta\|_\infty}{m^-}\max\{\|v_n\|^{m^+}, \|v_n\|^{m^-}\}\\ &\geq \frac{1}{2P_+^+}\left(\frac{1}{N^{P_-^- -1}}\|u_m\|_{\vec{p}(x)}^{P_-^-} - N\right);\end{aligned}$$

which implies that

$$\varphi(s_n u_n) \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \tag{3.28}$$

Now, for $s_n \in (0,1)$ we have

$$< \varphi'(s_n u_n), s_n u_n > \longrightarrow 0;$$

that is, for large $n$

$$\begin{aligned}\int_\Omega f(x,s_nu_n)s_nu_n\,\mathrm{d}x &= a\int_\Omega\sum_{i=1}^N\left|\frac{\partial s_nu_n}{\partial x_i}\right|^{p_i(x)}\mathrm{d}x+\sum_{i=1}^N\int_\Omega V_i(x)|s_nu_n|^{p_i(x)}\,\mathrm{d}x \\ &+bK\left(\sum_{i=1}^N\frac{1}{p_i(x)}\left|\frac{\partial s_nu_n}{\partial x_i}\right|^{p_i(x)}\mathrm{d}x\right)\int_\Omega\sum_{i=1}^N\left|\frac{\partial s_nu_n}{\partial x_i}\right|^{p_i(x)}\mathrm{d}x \\ &-\int_\Omega\theta(x)|s_nu_n|^{m(x)}\,\mathrm{d}x-\eta\int_{\partial\Omega}|s_nu_n|^{q(x)}\,\mathrm{d}x.\end{aligned}\tag{3.29}$$

Hence, for large $n$, the combination of (3.28) and (3.29) implies that

$$\begin{aligned}\int_\Omega\Big(\frac{1}{2P_+^+}f(x,s_nu_n)s_nu_n-F(x,s_nu_n)\Big)\,\mathrm{d}x &\geq \varphi(s_nu_n)-b\mathcal{K}(s_nu_n) \\ &+\frac{a}{2P_+^+}|s_n|^{P_+^+}\sum_{i=1}^N\int_\Omega\left(a\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i(x)}+V_i(x)\mid v_n\mid^{p_i(x)}\right)\mathrm{d}x \\ +\int_\Omega\theta(x)\left(\frac{1}{m(x)}-\frac{1}{2P_+^+}\right)\mid s_nu_n\mid^{m(x)}\,\mathrm{d}x &+\eta\int_{\partial\Omega}\left(\frac{1}{q(x)}-\frac{1}{2P_+^+}\right)\mid s_nu_n\mid^{q(x)}\,\mathrm{d}x \\ &\geq\varphi(s_nu_n)-b\mathcal{K}(s_nu_n),\end{aligned}\tag{3.30}$$

where $\mathcal{K}$ is the function definded by

$$\begin{aligned}\mathcal{K}(u) = -\frac{1}{2P_+^+}K\left(\sum_{i=1}^N\int_\Omega\frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}\right)\left(\sum_{i=1}^N\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}\right) \\ +\hat{K}\left(\sum_{i=1}^N\int_\Omega\frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}\right).\end{aligned}$$

Therefore, for n large enough, by (f3), (3.28) and (3.26) it follows that

$$\begin{aligned}\int_\Omega\left[\frac{1}{2P_+^+}f(x,u_n)u_n-F(x,u_n)\right]\mathrm{d}x &= \int_\Omega\frac{1}{2P_+^+}\mathcal{F}(x,u_n)\,\mathrm{d}x \\ &\geq\int_\Omega\frac{1}{2rP_+^+}\mathcal{F}(x,s_nu_n)\,\mathrm{d}x \\ &=\frac{1}{r}\int_\Omega\left[\frac{1}{2P_+^+}f(x,s_nu_n)s_nu_n-F(x,s_nu_n)\right]\mathrm{d}x \\ &\geq\frac{1}{rP_+^+N^{P_-^--1}}\|u_l\|_{\vec{p}(x)}^{P_-^-}-\frac{b}{r}\mathcal{K}(u_n)-\frac{N}{rP_+^+}. \\ -\frac{D\|\theta\|_\infty}{rm^-}\max\{\|u_n\|^{m^+},\|u_n\|^{m^-}\} &-\frac{\eta C_3'}{rq^-}\max\{\|u_n\|^{q^+},\|u_n\|^{q^-}\}\end{aligned}\tag{3.31}$$

On the other hand, by choosing $n$ sufficiently large and using (3.8), there exists a constant $C_5' > 0$ such that

$$\begin{aligned}\int_\Omega \left[\frac{1}{2P_+^+} f(x,u_n)u_n - F(x,u_n)\right] \mathrm{d}x &= \varphi(u_n) - \frac{1}{2P_+^+} < \varphi'(u_n), u_n > \\ &\quad - b\mathcal{K}(u_n) + \int_\Omega \left(\frac{1}{m(x)} - \frac{1}{2P_+^+}\right)\theta(x) \mid u_n \mid^{m(x)} \mathrm{d}x \\ &\quad + \eta \int_{\partial\Omega} \left(\frac{1}{q(x)} - \frac{1}{2P_+^+}\right) \mid u_n \mid^{q(x)} \mathrm{d}x \\ &\leq C_5' + D\|\theta\|_\infty \left(\frac{1}{m^-} - \frac{1}{2P_+^+}\right) \max\{\|u_n\|^{m^+}, \|u_n\|^{m^-}\} \\ &\quad - b\mathcal{K}(u_n) + C_3' \left(\frac{1}{q^-} - \frac{1}{2P_+^+}\right) \max\{\|u_n\|^{q^+}, \|u_n\|^{q^-}\}. \end{aligned} \tag{3.32}$$

Now, combining (3.31) and (3.32) yields

$$\begin{aligned}\frac{1}{rP_+^+ N^{P_-^- -1}} \|u_m\|_{\vec{p}(x)}^{P_-^-} &\leq C_6' + C_7' \max\{\|u_n\|^{m^+}, \|u_n\|^{m^-}\} \\ &\quad + C_8' \max\{\|u_n\|^{q^+}, \|u_n\|^{q^-}\} + \frac{b}{r}(r-1)\mathcal{K}(u_n),\end{aligned}$$

where $C_6' = C_5' + \frac{N}{rP_+^+}$ and $C_7' = \frac{D\|\theta\|_\infty}{r}\left(\frac{1+r}{rm^-} - \frac{P_+^+}{(P_-^-)^2}\right)$;
which is the desired contradiction due to the fact that $\max\{m^+, q^+\} < P_-^-$ and the arbitrariness of $l$. Consequently, $\{u_n\}$ is bounded in $X$.

**Second step**. The sequence $\{u_n\}$ satisfying (2.1) has a convergent subsequence in $X$, by the reflexivity of $X$. From step 1, there exists a positive constant $B$ such that $\|u_n\| \leq B$. We can therefore assume that there exists $u \in X$ such that

$$u_n \rightharpoonup u \quad \text{in } X, u_n \longrightarrow u \quad \text{in } L^r(\Omega) \text{ and in } L^r(\partial\Omega), u_n(x) \longrightarrow u(x), \quad \text{for a.e. } x \in \Omega,$$

where $1 \leq r < P_{-,\infty}$. Hence, by (f0) there exists $C > 0$ such that, as $n \longrightarrow +\infty$ we have

$$\begin{aligned}\int_\Omega f(x,u_n)(u-u_n)\,\mathrm{d}x &\leq C'|f(x,u_n)|_{\frac{\alpha(x)}{\alpha(x)-1}} |u-u_n|_{\alpha(x)} \\ &\leq C|u-u_n|_{\alpha(x)} \longrightarrow 0,\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \theta(x)|u_n|^{m(x)-1}|u-u_n|\,\mathrm{d}x &\leq C'\|\theta\|_\infty |u_n^{m(x)-1}|_{\frac{m(x)}{m(x)-1}} |u-u_n|_{m(x)} \longrightarrow 0. \\ \int_{\partial\Omega} |u_n|^{q_i(x)-1}|u-u_n|\,\mathrm{d}x &\leq C_2'|u_n^{q(x)-1}|_{L^{\frac{q(x)}{q(x)-1}}(\partial\Omega)} |u-u_n|_{L^{q(x)}(\partial\Omega)} \longrightarrow 0.\end{aligned}$$

$$\int_\Omega V_i(x)|u_n|^{p_i(x)-1}|u-u_n|\,\mathrm{d}x \leq C_1'\|V_i\|_\infty |u_n^{p_i(x)-1}|_{\frac{p_i(x)}{p_i(x)-1}}|u-u_n|_{p_i(x)} \longrightarrow 0,$$

for all $i \in \{1,...,N\}$.
Moreover,

$$\begin{aligned}\left[a+bK\left(\sum_{i=1}^N \frac{1}{p_i(x)}\int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)}\mathrm{d}x\right)\right]\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u_n}{\partial x_i}\frac{\partial(u-u_n)}{\partial x_i}\,\mathrm{d}x \\ +\sum_{i=1}^N\int_\Omega V_i(x)|u_n|^{p_i(x)-2}u_n(u-u_n)\,\mathrm{d}x-\int_\Omega \theta(x)|u_n|^{m(x)-2}u_n(u-u_n)\,\mathrm{d}x \\ -\int_\Omega f(x,u_n)(u-u_n)\,\mathrm{d}x-\eta\int_{\partial\Omega}|u_n|^{q(x)-2}u_n(u-u_n)\,\mathrm{d}x \\ =<\varphi'(u_n),(u-u_n)>\longrightarrow 0.\end{aligned} \tag{3.33}$$

From (k0) and (3.8), it must be

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u_n}{\partial x_i}\left(\frac{\partial u}{\partial x_i}-\frac{\partial u_n}{\partial x_i}\right)\mathrm{d}x \longrightarrow 0. \tag{3.34}$$

Furthermore, since $u_n \rightharpoonup u \quad \text{in } X$, we have

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\left(\frac{\partial u}{\partial x_i}-\frac{\partial u_n}{\partial x_i}\right)\mathrm{d}x \longrightarrow 0. \tag{3.35}$$

From (3.34) and (3.35), we deduce that

$$\int_\Omega \sum_{i=1}^N \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}-\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u_n}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_i}-\frac{\partial u_n}{\partial x_i}\right)\mathrm{d}x \longrightarrow 0. \tag{3.36}$$

Next, we apply the following well-known inequality:

$$\left(|\,\xi_i\,|^{r_i-2}\,\xi_i-|\,\psi_i\,|^{r_i-2}\,\psi_i\right).(\xi_i-\psi_i)\geq 2^{-r_i}|\xi_i-\psi_i|^{r_i}, \quad \xi_i,\psi_i\in\mathbb{R}^N, \tag{3.37}$$

valid for all $r_i \geq 2$. From the relations (3.36) and (3.37), we infer that

$$u_n \longrightarrow u \quad \text{in } X$$

Now, since $\varphi \in C^1(X,\mathbb{R})$, we have

$$\varphi'(u)=0, \text{ and } \varphi(u)=c>0.$$

Therefore, $u$ is a nontrivial critical point of $\varphi$. This completes the proof. □

## 3.5 Concluding examples

In this section we present examples of the application of Theorems ,

**Example 3.5.1.** *Set* $K(t) = t^\alpha$ *and* $f(x,t) = \mu|t|^{\alpha-1}t$, *with* $0 < \alpha < \min\{1, \frac{2P_-^- - P_+^+}{2P_+^+ - 2P_-^-}\}$ *and* $\mu > 0$. *Then, all the hypotheses of theorems* (3.3.1) *and* (3.3.2) *are satisfied by K and f.*

It is easy to see that condition (K0) is satisfied, To check that condition (K1) holds, let

$$\hat{K}(t) = \int_0^t s^\alpha \, ds = \frac{t^{\alpha+1}}{\alpha+1}.$$

In order to have $\hat{K}(t) \geq (1-\eta)K(t)t, \quad$ for all $t \geq 0$, with a postive constant $\eta$ such that $\frac{P_+^+ + 1}{2P_+^+ + 1} < \eta < \frac{2P_-^- - P_+^+}{P_+^+}$, it suffices to suppose that $\frac{1}{\alpha+1} \geq 1 - \eta$ ; that is

$$\alpha \leq \frac{P^+}{2(P_+^+ - P_-^-)} - 1 = \frac{2P_-^- - P_+^+}{2(P_+^+ - P_-^-)}$$

Note that $f(x,t) = \mu|t|^{\alpha-2}t$ for $\mu > 0$, satisfies the conditions (f0), (f1) and (f2).

**Example 3.5.2.** *For small* $\epsilon > 0$, *set* $F(x,t) = |u|^{2P_+^+ + \epsilon} + \epsilon|u|^{2P_+^+} sin^2(\frac{|u|^\epsilon}{\epsilon})$. *Then, f satisfies all conditions of theorem* (3.3.3).

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f_\epsilon(x,t) = \frac{a\lambda_1}{4P_+^+}\left(4P_+^+ t^{2P_+^+ - 1} ln(1 + t^{P_+^+}) + \frac{2P^+ t^{3P_+^+ - 1}}{1 + t^{P_+^+}}\right) + \frac{a\lambda_1}{4P_+^+}\epsilon P^- t^{P^- - 1} cos(t^{P_-^-}).$$

We have $F(x,t) = \displaystyle\int_0^t f(x,u)\, du$.

It is easy to see that condition (F1) is satisfied. Indeed, it follows from the following equality

$$\frac{1}{2P_+^+} f_\epsilon(x,t)t - F(x,t) + \frac{a\lambda_1^-}{2P_+^+}|t|^{P_-^-} = \frac{a\lambda_1^-}{4P_+^+}\left(\frac{t^{3P_+^+}}{1 + t^{P_+^+}} + \frac{\epsilon P_-^- t^{P_-^-} cos(t^{P_-^-})}{2P_+^+}\right) + \frac{a\lambda_1^-}{4P_+^+}\left(2|t|^{P_-^-} - \epsilon sin(t^{P_-^-})\right)$$

that $\displaystyle\lim_{|t|\to+\infty}\left(\frac{1}{2P_+^+} f(x,t)t - F(x,t) + \frac{a\lambda_1^-}{2P_+^+}|t|^{P_-^-}\right) = +\infty$, uniformly in $x \in \Omega$.

To check that condition (F2) holds, we choose $\epsilon$ such that $\frac{a\lambda_1^- \epsilon}{P_+^+} < \frac{\lambda_1^+}{N^{P_+^+ - 1}}$, which implies that $\displaystyle\lim_{|t|\to 0}\frac{F_\epsilon(x,t)}{|t|^{P_-^-}} = \frac{a\lambda_1^- \epsilon}{P_+^+}$.

**Example 3.5.3.** *The function $f(x,u) = k(u^{\beta P_+^+} + u)$ with $\beta = \frac{1}{P_+^+} + 4$ and $|k| < a\lambda_1^+$, satisfies the hypotheses of theorem* (3.3.4).

Let us show that $f$ satisfies the condition (f0). From the definition of $f$, it is clear that there exists $c > 0$ such that

$$|f(x,u)| \leq c(1 + |u|^{\alpha-1}) \quad \text{for all } (x,t) \in \bar{\Omega} \times \mathbb{R},$$

with $\alpha = \beta P_+^+ + 1$ if $|u| \geq 1$ and $\alpha = \beta P_+^+$ if $|u| \leq 1$. We have for all cases $P_+^+ < \alpha < P_-^\star$. To check that condition (f3) holds, we write

$$\begin{aligned} \mathcal{F}(x,u) &= f(x,u)u - \frac{1}{2P_+^+} F(x,u) \\ &= k\left[\left(1 - \frac{1}{2P_+^+(\beta P_+^+ + 1)}\right) u^{\beta P_+^+ + 1} + \left(1 - \frac{1}{4P_+^+}\right) u^2\right]. \end{aligned}$$

Since we have $1 - \frac{1}{2P_+^+(\beta P_+^+ + 1)} > 0$ and $1 - \frac{1}{4P_+^+} > 0$, we can easily conclude that for any $s \in [0,1]$ and $u \in \mathbb{R}$, the following inequality holds for a.e x in $\Omega$

$$1\mathcal{F}(x,u) \geq \mathcal{F}(x,su).$$

The hypotheses (f'1) and (f'2) are also satisfied since we have
$\lim\limits_{|u| \to +\infty} \dfrac{f(x,u)}{u^{2P_+^+ - 1}} = \lim\limits_{|u| \to +\infty} \dfrac{k(u^{\beta P_+^+} + u)}{u^{2P_+^+ - 1}} = \infty$, uniformly uniformly for a.e $x \in \Omega$; and

$$\limsup_{|u| \to 0} \frac{f(x,u)}{au} = \frac{k}{a},$$

respectively.

# CHAPTER 4

# EXISTENCE OF THREE DISTINCT WEAK SOLUTIONS FOR PERTURBED KIRCHHOFF-TYPE NON-HOMOGENEOUS NEUMANN PROBLEMS WITH NONLINEAR ROBIN BOUNDARY CONDITIONS

> Les découvertes mathématiques, petites ou grandes, ne naissent jamais d'une génération spontanée.
>
> *(Henri Poincaré.)*

The paper deals with the existence multiple solutions for perturbed Kirchhoff-type non-homogeneous problems with anistropic Orlicz-Sobolev spaces and nonstandard growth conditions such as

$$-K\left(\sum_{i=1}^{N}\int_{\Omega}\Phi_i(\left|\frac{\partial u}{\partial x_i}\right|)+\Phi_i(|u|)\mathrm{d}x\right)\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\alpha_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right)\frac{\partial u}{\partial x_i}+\alpha_i\left(|u|\right)u\right)$$
$$=\theta(x)|u|^{m(x)}u+\lambda f(x,u) \text{ in } \Omega,$$
$$\sum_{i=1}^{N}\alpha_i\left(|\frac{\partial u}{\partial x_i}|\right)\frac{\partial u}{\partial x_i}v_i=\mu g(x,u) \quad \text{on } \partial\Omega.$$

where $\Omega$ is a smooth bounded domain. Under suitable conditions on the data, We establish the existence of three distinct weak solutions for perturbed Kirchhoff-type non-homogeneous Neumann problems by means of a variational approach and the use of the anistropic Orlicz-Sobolev spaces .

## 4.1 Introduction

Our paper is mainly devoted to the the existence of three distinct weak solutions for perturbed Kirchhoff-type non-homogeneous Neumann problems with nonstandard growth conditions such as

$$\begin{aligned}&-K\left(\sum_{i=1}^{N}\int_{\Omega}\phi_i(\left|\frac{\partial u}{\partial x_i}\right|)+\phi_i(|u|)\mathrm{d}x\right)\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\alpha_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right)\frac{\partial u}{\partial x_i}+\alpha_i\left(|u|\right)u\right)\\&\quad=-\Theta(x)|u|^{m(x)}u+\lambda f(x,u) \text{ in } \Omega,\\&\sum_{i=1}^{N}\alpha_i\left(|\frac{\partial u}{\partial x_i}|\right)\frac{\partial u}{\partial x_i}v_i=\mu g(x,u) \quad \text{on } \partial\Omega.\end{aligned} \tag{4.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with $n\geq 2$, with smooth boundary $\partial\Omega$ and $v_i$ are the components of the outer normal unit vector, $m\in\mathcal{C}(\bar{\Omega})$ and for $i\in\{1,\dots,N\}$, $\alpha_i:(0,\infty)\longrightarrow\mathbb{R}$ be such that the mapping $\varphi_i:\mathbb{R}\longrightarrow\mathbb{R}$ defined by

$$\varphi_i(t)=\begin{cases}\alpha_i(t)t & \text{for } t\neq 0,\\ 0 & \text{if } t=0.\end{cases} \tag{4.2}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. For the function $\varphi_i$ above, let us define

$$\phi_i(t)=\int_0^t\varphi_i(s)\mathrm{d}s \text{ for all } t\in\mathbb{R}.$$

The functions $f$ and $g$ are supposed to satisfy some conditions to be specified below, while $\lambda$ and $\mu$ real parameters, with $\mu>0$. The problem (4.1) is related to the stationary problem of a model introduced by Kirchhoff [4]. More precisely, Kirchhoff introduced a model given by the following equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|\mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2}=0. \tag{4.3}$$

which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations.
Concretely, in the present paper, employing two kinds of three critical points theorems obtained in [12, 13] and on of at least three weak solutions as well. Actually, by using Bonanno and Candito, we prove that (4.1) has at least three weak solutions;

This chapter is motivated by recent works on nonlinear Kirchhoff elliptic equations (see for example ,[11], [14], [30] and the references therein). When $\varphi(t) = |t|^{p-1}t$, $\forall t \in \mathbb{R}$ and $\Theta = 0$, these questions have been treated by the authors in a recent work [31]. In the case $K(t) = 1$, $\forall t \geq 0$ and $\Theta = 0$ , in the Orlicz-Sobolev space $W^1L_\phi(\Omega)$, while Mihãilescu and Repovš in [44] by combining Orlicz-Sobolev spaces theory with adequate variational methods and a variant of Mountain Pass Lemma established the existence of at least two non-negative and non-trivial weak solutions for the problem

$$\begin{aligned} -div(\alpha(|\nabla u(x)|)\nabla u(x)) = \lambda f(x,u(x)), \quad \text{in}\Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{4.4}$$

where $\alpha$ is the same with $\alpha_i$ for $i \in \{1, ..., N\}$ in problem (4.1) , $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function and $\lambda$ is a positive parameter. The main goal here is to extend the results in [31] to the case of $\vec{p}(x)$ -Kirchhoff type quasilinear elliptic prooblems of the form (4.1).
Let us point out that, in the last decade, many authors studied the following kirchhoff-type non-homogeneous neumann problems through orlicz-sobolev spaces

$$\begin{aligned} -K\left(\int_\Omega \phi(|\nabla u|) + \phi(|u|)\mathrm{d}x\right)(-div(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u), \\ = \lambda f(x,u) + \mu g(x,u), \quad \text{in}\Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{4.5}$$

Interesting results concerning the existence of three distinct weak solutions are obtained for example in [12],[13] and [32] by means of variational methods.
More generally, problems related to problem 4.1 have been studied recently by many authors. The paper [31] by Shapour Heidarkhani seems to be closer to our work and is related to the following problem

$$\begin{aligned} M\left(\int_\Omega \frac{1}{p(x)}\left(|\alpha(x)u(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\right)\mathrm{d}x\right)\left(-\Delta_p u + \alpha(x)|u|^{p(x)-2}u\right) \\ = \lambda f(x,u), \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} = \mu g(\gamma(u(x))); \quad \text{on } \partial\Omega, \end{aligned} \tag{4.6}$$

The main goal here to extend some of the results in [30] to the anistropic Orlicz-Sobolev spaces and more general nonlocal terms. Besides, we are able to deal with more general nonlinearities in the boundary condition than in [30] and with situations where the function $M$ in 4.6 is unbounded from above. Although this function does not satisfy condition $m_0 \leq M(t) \leq m_1$ forall $t \geq 0$ in [31] and [32],the existence of at least three weak solutions is however guaranted by our results.

## 4.2 Preliminaries and main results

Our main tools are the following three critical point theorems. In the first one the coercivity of the functional $\Phi - \lambda\Psi$ is required, in the second one a suitable sign hypothesis is assumed

**Theorem 4.2.1** (See [12]). *Let $X$ be a reflexive real Banach space, $J : X \longrightarrow \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semi continuous functional whose Gâteaux derivative admits a continuous inverse on $X^*$, $I : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $J(0) = I(0) = 0$. Assume that there exist $r > 0$ and $\bar{v} \in X$, with $r < J(\bar{v})$ such that*

$$\frac{\sup_{J^{-1}(-\infty,r]} I(u)}{r} < \frac{I(\bar{v})}{J(\bar{v})}. \tag{4.7}$$

$$\text{for each } \lambda \in \Lambda_r = \left]\frac{J(\bar{v})}{I(\bar{v})}, \frac{r}{\sup_{J^{-1}(-\infty,r]}}\right[ \text{ the functional } J - \lambda I \text{ is coercive.} \tag{4.8}$$

*Then, for each $\lambda \in \Lambda$ the functional $J - \lambda I$ has at least three distinct critical points in $X$.*

Our main tool to ensure the existence of three solutions for the problem (4.1) is the following three critical point theorem due to Bonanno and Candito.
Let $X$ be a nonempty set and $J, I : X \longrightarrow \mathbb{R}$ be two functions. For all $r, r_1, r_2 > Inf_X J$, $r_2 > r_1$, $r_3 > 0$, we define

$$\varphi(r) := \inf_{u\in J^{-1}(]-\infty,r[)} \frac{(\sup_{u\in J^{-1}(]-\infty,r[} I(u)) - I(u)}{r - J(u)},$$

$$\beta(r_1, r_2) := \inf_{u\in J^{-1}(]-\infty,r[)} \sup_{v\in J^{-1}([r_1,r_2[)} \frac{I(v) - I(u)}{J(v) - J(u)},$$

$$\gamma(r_2, r_3) := \frac{\sup_{u\in J^{-1}(]-\infty,r[)} I(u)}{r_3},$$

$$\alpha(r_1, r_2, r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\},$$

**Theorem 4.2.2** (See [12]). *Let $X$ be a reflexive real Banach space, $J : X \longrightarrow \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on $X^*$, $I : X \longrightarrow \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

*($a_1$)* $\inf_X J = J(0) = I(0) = 0$;

*($a_2$)* *for every* $u_1, u_2$ *such that* $I(u_1) \geq 0$ *and* $I(u_2) \geq 0$, *one has*

$$\inf_{s\in[0,1]} I(su_1 + (1-s)u_2) \geq 0.$$

*Assume that there are three positive constants* $r_1, r_2, r_3$ *with* $r_1 < r_2$, *such that*

*($a_3$)* $\varphi(r_1) < \beta(r_1, r_2)$;

*($a_4$)* $\varphi(r_2) < \beta(r_1, r_2)$;

*($a_5$)* $\gamma(r_2, r_3) < \beta(r_1, r_2)$;

*Then, for each* $\lambda \in ]\frac{1}{\beta(r_1,r_2)} \frac{1}{\alpha(r_1,r_2,r_2)}, [$ *the functional* $J - \lambda I$ *admits three distinct critical points* $u_1, u_2, u_3$ *such that* $u_1 \in J^{-1}(]-\infty, r_1[)$, $u_2 \in J^{-1}([r_1, r_2[)$ *and* $u_3 \in J^{-1}(]-\infty, r_2 + r_3[)$.

Write $X = W^1 L_{\vec{\phi}}(\Omega)$.
Corresponding to $f$, $g$ and $K$ we introduce the functions $F : \bar{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$, $G : \partial\Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, , respectively, as follows

$$F(x,t) := \int_0^t f(x,s)\,\mathrm{d}s, \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R},$$

$$G(x,t) := \int_0^t g(x,s)\,\mathrm{d}s, \quad \forall (x,t) \in \partial\Omega \times \mathbb{R}.$$

Moreover, we set $G^\theta = \int_{\partial\Omega} \sup_{|t|\leq\theta} G(x,t)\,\mathrm{d}x$ for every $\theta > 0$ and $G_\eta = \inf_{\partial\Omega\times[0,\eta]} G$ for every $\eta > 0$. If $g$ is sign-changing, then $G^\theta \geq 0$ and $G_\eta \leq 0$.
We define

$$J(u) = \hat{K}\left(\sum_{i=1}^N \int_\Omega \phi_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right) + \phi_i\left(|u|\right)\,\mathrm{d}x\right)\left(\sum_{i=1}^N \int_\Omega \phi_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right) + \phi_i\left(|u|\right)\,\mathrm{d}x\right).$$

where $\hat{K}(t) = \displaystyle\int_0^t K(s)\,\mathrm{d}s$. We have

$$(J'(u), \psi) = K\left(\sum_{i=1}^N \int_\Omega \phi_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right) + \phi_i\left(|u|\right)\,\mathrm{d}x\right)$$
$$\times \sum_{i=1}^N \int_\Omega \left(\alpha_i\left(|\frac{\partial u}{\partial x_i}|\right)\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i} + \alpha_i(|u|)u\psi\right)\,\mathrm{d}x$$

. The energy functional associated with problem (4.1) is

$$\begin{aligned}\Phi(u) = \hat{K}&\left(\sum_{i=1}^N \int_\Omega \phi_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right) + \phi_i\left(|u|\right)\,\mathrm{d}x\right)\left(\sum_{i=1}^N \int_\Omega \phi_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right) + \phi_i\left(|u|\right)\,\mathrm{d}x\right)\\ &- \int_\Omega \frac{\Theta(x)}{m(x)}|u|^{m(x)}\,\mathrm{d}x - \lambda\int_\Omega F(x,u)\,\mathrm{d}x - \mu\int_{\partial\Omega} G(x,u)\,\mathrm{d}x.\end{aligned} \tag{4.9}$$

It is easy to verify that $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ and $u$ is a weak solution of problem (4.1) if and only if $u$ is a critical point of $\Phi$.
In all this paper $C$, $C_i (i = 0, 1, 2, \dots)$ represents different positive real constants.

(S) $2(P_0)^- < \min\left\{P_{0,\infty}, p_1^{\partial}, ..., p_N^{\partial}\right\}$.

(K0) $K : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous nondecreasing function and there exist two constants $k_0 > 0$ and $k_1 > 0$ with $k_0 < k_1$ such that

$$k_0 \leq K_i(t) \leq k_1 t, \text{ for all } t > 1.$$

(K1) There exists a postive constant $\zeta < 0$ such that

$$\hat{K}(t) \geq (1-\zeta)K(t)t, \quad \text{for all } t \geq 0.$$

(H0)

$$\limsup_{t \longrightarrow \infty} \frac{\sup_{x\in\bar{\Omega}} F(x,t)}{t^{2(P_0)^-}} \leq 0, \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Finally, denote $\delta_{\theta,\eta}$ by

$$\delta_{\theta,g} = \frac{k_0 \left(\frac{\theta N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+}}{\frac{mes(\Omega)\|\Theta\|_\infty \max\{\theta^{m^+}, \theta^{m^-}\}}{\lambda m^-} + \int_\Omega \sup\limits_{|t|\leq\theta} F(x,t)\, dx}$$

To introduce our first result, fixing two positive constants $\theta$ and $\eta$ such that

$$\frac{mes(\Omega)\left(\sum_{i=1}^N \phi_i(\eta)\right)\hat{K}\left(\sum_{i=1}^N \int_\Omega \phi_i(\eta)\, dx\right)}{\frac{|\eta|^{m^+}}{\lambda m^+}\|\Theta\|_{L^1(\Omega)} + \int_\Omega F(x,\eta)\, dx} < \delta_{\theta,g}.$$

and taking

$$\lambda \in \Lambda_1 = \left]\frac{mes(\Omega)\left(\sum_{i=1}^N \phi_i(\eta)\right)\hat{K}\left(\sum_{i=1}^N \int_\Omega \phi_i(\eta)\, dx\right)}{\frac{|\eta|^{m^+}}{\lambda m^+}\|\Theta\|_{L^1(\Omega)} + \int_\Omega F(x,\eta)\, dx}, \delta_{\theta,g}\right[.$$

Set

$$\sigma_{\theta,g} = \frac{k_0 \left(\frac{\theta N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+} - \lambda \int_\Omega \sup\limits_{|t|\leq\theta} F(x,t)\, dx - \frac{mes(\Omega)\|\Theta\|_\infty \max\{\theta^{m^+}, \theta^{m^-}\}}{m^-}}{G^\theta},$$

$$\sigma_{\theta_2,\theta_3,g} = \frac{k_0\left(\frac{N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+}(\theta_3^{(P_0)^+}-\theta_2^{(P_0)^+}) - \lambda\int_\Omega \sup_{|t|\leq\theta_3} F(x,t)\,dx}{G^{\theta_3}} - \frac{\frac{mes(\Omega)\|\Theta\|_\infty \max\{\theta_3^{m^+},\theta_3^{m^-}\}}{m^-}}{G^{\theta_3}},$$

$$\pi_{\eta,g} = \frac{mes(\Omega)\left(\sum_{i=1}^N \phi_i(\eta)\right)\hat{K}\left(\sum_{i=1}^N\int_\Omega \phi_i(\eta)\,dx\right) - \frac{|\eta|^{m^+}}{m^+}\|\Theta\|_{L^1(\Omega)} - \lambda\int_\Omega F(x,\eta)\,dx}{G_\eta},$$

$$\rho_{\lambda,g} = \min\left\{\sigma_{\theta,g},\pi_{\eta,g}\right\}. \tag{4.10}$$

,

$$\nu_{\lambda,g} = \min\left\{\sigma_{\theta_1,g},\sigma_{\theta_2,g},\sigma_{\theta_2,\theta_3,g}\right\}. \tag{4.11}$$

and

$$\overline{\rho_{\lambda,g}} = \min\left\{\rho_{\lambda,g}, \frac{1}{\max\left\{0, \frac{2^{(P^0)^- -1}N^{(P^0)^+-1}}{k_0}\limsup_{t\longrightarrow\infty}\frac{\sup_{x\in\partial\Omega}G(x,t)}{t^{2(P_0)^-}}\right\}}\right\}. \tag{4.12}$$

$$\overline{\nu_{\lambda,g}} = \min\left\{\nu_{\lambda,g}, \frac{\frac{k_1 mes(\Omega)^2}{2}(\sum_{i=1}^N|\phi_i(\eta)|)^2 - \lambda(\int_\Omega F(x,\eta)\,dx - \int_\Omega F(x,\theta_1)\,dx)}{G_\eta - G^{\theta_1}}\right\}. \tag{4.13}$$

The main results of this article are as follows:

**Theorem 4.2.3.** *Suppose that* (P),(K0)–(K1),(H0)–(H1) *hold, assume that there exist two positive constants $\theta$ and $\eta$ with*

$$\theta < \frac{c}{N^{\frac{1}{(P_0)^+}-1}}\min\left\{1, \left[\left(mes(\Omega)\sum_{i=1}^N\phi_i(\eta)\right)\hat{K}\left(mes(\Omega)\sum_{i=1}^N\phi_i(\eta)\right)\right]^{\frac{1}{(P_0)^+}}\right\}$$

*such that*

$$\frac{mes(\Omega)\left(\sum_{i=1}^N\phi_i(\eta)\right)\hat{K}\left(\sum_{i=1}^N\int_\Omega\phi_i(\eta)\,dx\right)}{\frac{|\eta|^{m^+}}{\lambda m^+}\|\Theta\|_{L^1(\Omega)} + \int_\Omega F(x,\eta)\,dx} < \delta_{\theta,g} \tag{4.14}$$

*Then, for each $\lambda\in\Lambda_1$ and for every $L^1(\partial\Omega)$-Carathéodory function $g:\partial\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfying the condition*

$$\limsup_{|t|\longrightarrow\infty}\frac{\sup_{x\in\partial\Omega}G(x,t)}{t^{2(P_0)^-}} < +\infty.$$

*there exists $\overline{\rho_{\lambda,g}}$ given by* (4.12) *such that, for each $\mu\in[0,\overline{\rho_{\lambda,g}}[$, the problem* (4.1) *possesses at least three distinct weak solutions in $W^1L_{\vec{\phi}}(\Omega)$.*

**Theorem 4.2.4.** *Assume that there exist positive constants* $\eta \in ]0,1[$, $\theta_1 < \eta$, $\theta_2$ *and* $\theta_3$ *with* $\theta_3 > \theta_2$, $\quad \max\{\theta_1, \theta_2\} < \frac{c}{N^{\frac{1}{(P_0)^+}-1}} \min\left\{1, \left(mes(\Omega)\sum_{i=1}^N \mid \phi_i(\eta) \mid\right)^{\frac{1}{(P_0)^+}}\right\}$ *and*

$$\theta_2 > \left(\frac{k_1 mes(\Omega)^2}{2k_0}\right)^{\frac{1}{(P_0)^+}} \left(\sum_{i=1}^N \mid \phi_i(\eta) \mid\right)^{\frac{2}{(P_0)^+}},$$

*(A1)* $f(x,t) \geq 0$ *for each* $(x,t) \in \Omega \times [-\theta_3, \theta_3]$;

*(A2)*

$$\max\left\{\frac{\int_\Omega F(x,\theta_1)\,\mathrm{d}x}{\theta_1^{(P_0)^+}}, \frac{\int_\Omega F(x,\theta_2)\,\mathrm{d}x}{\theta_2^{(P_0)^+}}, \frac{\int_\Omega F(x,\theta_3)\,\mathrm{d}x}{\theta_3^{(P_0)^+} - \theta_2^{(P_0)^+}}\right\}$$
$$+\frac{mes(\Omega)\|\Theta\|_\infty \max\{\theta_1^{m^-}, \theta_3^{m^+}, \theta_3^{m^-}\}}{m^-\theta_1^{(P_0)^+}}$$
$$\leq \frac{2k_0\,(2c)^{(P_0)^+}}{k_1 mes(\Omega)^2\left(N^{\frac{1}{(P_0)^+}-1}\right)^{(P_0)^+}} \frac{\int_\Omega F(x,\eta)\,\mathrm{d}x - \int_\Omega F(x,\theta_1)\,\mathrm{d}x}{\left(\sum_{i=1}^N \mid \phi_i(\eta) \mid\right)^2}$$

*Then, for every* $\lambda \in \mathbb{R}$ *such that:*

$$\max\left\{\frac{\int_\Omega F(x,\theta_1)\,\mathrm{d}x}{\theta_1^{(P_0)^+}}, \frac{\int_\Omega F(x,\theta_2)\,\mathrm{d}x}{\theta_2^{(P_0)^+}}, \frac{\int_\Omega F(x,\theta_3)\,\mathrm{d}x}{\theta_3^{(P_0)^+} - \theta_2^{(P_0)^+}}\right\}$$
$$+\frac{mes(\Omega)\|\Theta\|_\infty \max\{\theta_1^{m^-}, \theta_3^{m^+}, \theta_3^{m^-}\}}{m^-\theta_1^{(P_0)^+}}$$
$$< \frac{1}{\lambda} < \frac{2k_0\,(2c)^{(P_0)^+}}{k_1 mes(\Omega)^2\left(N^{\frac{1}{(P_0)^+}-1}\right)^{(P_0)^+}} \frac{\int_\Omega F(x,\eta)\,\mathrm{d}x - \int_\Omega F(x,\theta_1)\,\mathrm{d}x}{\left(\sum_{i=1}^N \mid \phi_i(\eta) \mid\right)^2}$$

*for every nonnegative continuous function* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *there exists* $\overline{v_{\lambda,g}} > 0$ *given by* (4.13) *such that, for each* $\mu \in (0, \overline{v_{\lambda,g}})$, *the problem* (4.1) *possesses at least three distinct weak solutions* $u_i \in W^1L_{\vec{\phi}}(\Omega)$ *for* $i = 1,2,3$. *such that*

$$\max_{x\in\Omega} |u_i(x)| < \theta_i, \quad \forall i \in \{1,3\}.$$

## 4.3 Proofs of main results

To prove Theorem 4.2.3, we shall use the theorem (4.2.1). We take $X = W^1L_{\vec{\phi}}(\Omega)$ and we introduce the functionals $J, I : X \longrightarrow \mathbb{R}$. For each $u \in X$, as follows

$$J(u) = \hat{K}\left(\sum_{i=1}^{N}\int_{\Omega}\phi_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right) + \phi_i\left(|u|\right)\,\mathrm{d}x\right)\left(\sum_{i=1}^{N}\int_{\Omega}\phi_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right) + \phi_i\left(|u|\right)\,\mathrm{d}x\right).$$

$$I(u) = \frac{1}{\lambda}\int_{\Omega}\frac{\Theta(x)}{m(x)}|u|^{m(x)}\,\mathrm{d}x + \int_{\Omega}F(x,u)\,\mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega}G(x,u)\,\mathrm{d}x.$$

Let us prove that the functionals $I$ and $J$ satisfy the required conditions in Theorem 4.2.1. It is well known that $I$ is a differentiable functional whose differential at the point $u \in X$ is

$$I'(u)(v) = \frac{1}{\lambda}\int_{\Omega}\Theta(x)|u|^{m(x)-1}uv\,\mathrm{d}x + \int_{\Omega}f(x,u)v\,\mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega}g(x,u)v\,\mathrm{d}x.$$

for every $v \in X$. Furthermore, $I' : X \longrightarrow X^*$ is a compact operator. Moreover, $J$ is continuously differentiable whose differential at the point $u \in X$ is

$$(J'(u), v) = K\left(\sum_{i=1}^{N}\int_{\Omega}\phi_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right) + \phi_i\left(|u|\right)\,\mathrm{d}x\right)$$

$$\times\sum_{i=1}^{N}\int_{\Omega}\left(\alpha_i\left(|\frac{\partial u}{\partial x_i}|\right)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} + \alpha_i(|u|)uv\right)\,\mathrm{d}x.$$

for every $v \in X$, we prove that $J'$ admits a continuous inverse on $X^*$. Assuming $\|u\| > 1$, we have

$$J'(u)(u) \geq k_0\left(\frac{1}{2^{(P^0)^-}N^{(P^0)^+-1}}\|u\|_{1,\vec{\phi}}^{(P^0)^-} - 1\right)^2.$$

and since $(P^0)^- > 1$, it follows that is coercive. Since $J'$ is the Fréchet derivative of $J$, it follows that $J'$ is continuous and bounded. Using the elementary inequality

$$|x-y|^p \leq 2^p(|x|^{p-2}x - |y|^{p-2}y)(x-y) \text{ if } r \geq 2$$

for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $N \geq 1$, we obtain for all $u, v \in X$ such that $u \neq v$

$$\langle J'(u) - J'(v), u - v\rangle > 0,$$

which means that $J'$ is strictly monotone. Thus $J'$ is injective. Consequently, thanks to Minty–Browder theorem [51], the operator $J'$ is an surjection and has an inverse $J'^{-1} : X \longrightarrow X^*$, and one has $J'^{-1}$ is continues. Furthermore, $J$ is sequentially weakly lower semicontinuous. Put $r = k_0 \left( \frac{\theta N^{\frac{1}{(P_0)^+}-1}}{2c} \right)^{(P_0)^+}$ and $w(x) = \eta \in ]0,1[$ for all $x \in \Omega$. We have $w \in X$. Hence

$$\begin{aligned} J(w) &= \hat{K} \left( \sum_{i=1}^{N} \int_{\Omega} \phi_i \left( \left| \frac{\partial w(x)}{\partial x_i} \right| \right) + \phi_i \left( |w(x)| \right) \mathrm{d}x \right) \\ &\times \left( \sum_{i=1}^{N} \int_{\Omega} \phi_i \left( \left| \frac{\partial w(x)}{\partial x_i} \right| \right) + \phi_i \left( |w(x)| \right) \mathrm{d}x \right) \\ &= \hat{K} \left( \sum_{i=1}^{N} \int_{\Omega} \mid \phi_i (\eta) \mid \mathrm{d}x \right) \left( \sum_{i=1}^{N} \int_{\Omega} \mid \phi_i (\eta) \mid \mathrm{d}x \right) \end{aligned}$$

Since $\theta < cN^{1-\frac{1}{(P_0)^+}} \left[ \left( mes(\Omega) \sum_{i=1}^{N} \phi_i (\eta) \right) \hat{K} \left( mes(\Omega) \sum_{i=1}^{N} \phi_i (\eta) \right) \right]^{\frac{1}{(P_0)^+}}$, one has $r < J(w)$. Hence, recalling $\theta < \frac{c}{N^{\frac{1}{(P_0)+}-1}}$. Therefore, $r < k_0$. So, we obtain

$$\begin{aligned} J^{-1}(-\infty, r] &\subseteq \left\{ u \in X; \quad \left( k_0 \sum_{i=1}^{N} \int_{\Omega} \phi_i \left( \left| \frac{\partial u(x)}{\partial x_i} \right| \right) + \phi_i \left( |u(x)| \right) \mathrm{d}x \right)^2 \leq r \right\} \\ &\subseteq \left\{ u \in X; \quad \left( \sum_{i=1}^{N} \int_{\Omega} \phi_i \left( \left| \frac{\partial u(x)}{\partial x_i} \right| \right) + \phi_i \left( |u(x)| \right) \mathrm{d}x \right)^2 \leq \frac{r}{k_0} \right\} \\ &\subseteq \left\{ u \in X; \quad \frac{1}{2^{(P^0)^+} N^{(P^0)^+ - 1}} \|u\|_{1,\vec{\phi}}^{(P^0)^+} \leq \frac{r}{k_0} \right\} \\ &\subseteq \left\{ u \in X; \quad \|u\|_{\infty} \leq 2c \left( \frac{r}{k_0} \right)^{\frac{1}{(P^0)^+}} N^{1-\frac{1}{(P^0)^+}} \right\} \\ &\subseteq \{ u \in X; \quad |u(x)| \leq \theta, \text{ for all } x \in \Omega \} \end{aligned}$$

Therefore, it yields

$$\begin{aligned} \sup_{u \in J^{-1}(-\infty, r]} I(u) &\leq \frac{1}{\lambda} \int_{\Omega} \frac{\Theta(x)}{m(x)} \sup_{|t| \leq \theta} |t|^{m(x)} \mathrm{d}x + \int_{\Omega} \sup_{|t| \leq \theta} F(x,t) \, \mathrm{d}x + \frac{\mu}{\lambda} \int_{\partial\Omega} \sup_{|t| \leq \theta} G(x,t) \, \mathrm{d}x \\ &\leq \frac{mes(\Omega) \|\Theta\|_{\infty} \max\{\theta^{m^+}, \theta^{m^-}\}}{\lambda m^-} + \int_{\Omega} \sup_{|t| \leq \theta} F(x,t) \, \mathrm{d}x + \frac{\mu}{\lambda} G^{\theta} \end{aligned}$$

On the other hand, we have

$$I(w) = \frac{1}{\lambda} \int_{\Omega} \frac{\Theta(x)}{m(x)} |\eta|^{m(x)} \mathrm{d}x + \int_{\Omega} F(x,\eta) \, \mathrm{d}x + \frac{\mu}{\lambda} \int_{\partial\Omega} G(x,\eta) \, \mathrm{d}x.$$

that we have

$$\frac{\sup_{u\in J^{-1}(-\infty,r]} I(u)}{r} \leq \frac{\frac{mes(\Omega)\|\Theta\|_\infty \max\{\theta^{m^+},\theta^{m^-}\}}{\lambda m^-} + \int_\Omega \sup_{|t|\leq\theta} F(x,t)\,\mathrm{d}x + \frac{\mu}{\lambda}G^\theta}{k_0\left(\frac{\theta N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+}}. \tag{4.15}$$

On the other hand, we have

$$\frac{I(w)}{j(w)} = \frac{\frac{1}{\lambda}\int_\Omega \frac{\Theta(x)}{m(x)}|\eta|^{m(x)}\,\mathrm{d}x + \int_\Omega F(x,\eta)\,\mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega} G(x,\eta)\,\mathrm{d}x}{mes(\Omega)\left(\sum_{i=1}^N \phi_i(\eta)\right)\hat{K}\left(\sum_{i=1}^N \int_\Omega \phi_i(\eta)\ \mathrm{d}x\right)}$$

and

$$\begin{aligned}\frac{I(w)}{J(w)} &\geq \frac{\frac{|\eta|^{m^+}}{\lambda m^+}\int_\Omega \frac{\Theta(x)}{m(x)}\,\mathrm{d}x + \int_\Omega F(x,\eta)\,\mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega} G(x,\eta)\,\mathrm{d}x}{J(w)} \\ &\geq \frac{\frac{|\eta|^{m^+}}{\lambda m^+}\|\Theta\|_{L^1(\Omega)} + \int_\Omega F(x,\eta)\,\mathrm{d}x + \frac{\mu}{\lambda}G_\eta}{mes(\Omega)\left(\sum_{i=1}^N \phi_i(\eta)\right)\hat{K}\left(\sum_{i=1}^N \int_\Omega \phi_i(\eta)\ \mathrm{d}x\right)}.\end{aligned} \tag{4.16}$$

Moreover, by virtue of $\mu < \rho_{\lambda,g}$, one clearly has

$$\mu < \frac{k_0\left(\frac{\theta N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+} - \int_\Omega \sup_{|t|\leq\theta} F(x,t)\,\mathrm{d}x - \frac{mes(\Omega)\|\Theta\|_\infty \max\{\theta^{m^+},\theta^{m^-}\}}{\lambda m^-}}{G^\theta}$$

herewith,

$$\frac{\frac{mes(\Omega)\|\Theta\|_\infty \max\{\theta^{m^+},\theta^{m^-}\}}{\lambda m^-} + \int_\Omega \sup_{|t|\leq\theta} F(x,t)\,\mathrm{d}x + \frac{\mu}{\lambda}G^\theta}{k_0\left(\frac{\theta N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+}} < \frac{1}{\lambda}$$

Furthermore,

$$\mu < \frac{mes(\Omega)\left(\sum_{i=1}^N \phi_i(\eta)\right)\hat{K}\left(\sum_{i=1}^N \int_\Omega \phi_i(\eta)\ \mathrm{d}x\right) - \frac{|\eta|^{m^+}}{m^+}\|\Theta\|_{L^1(\Omega)} - \lambda\int_\Omega F(x,\eta)\,\mathrm{d}x}{G_\eta}$$

herewith,

$$\frac{1}{\lambda} < \frac{\frac{|\eta|^{m^+}}{\lambda m^+}\|\Theta\|_{L^1(\Omega)} + \int_\Omega F(x,\eta)\,\mathrm{d}x + \frac{\mu}{\lambda}G_\eta}{mes(\Omega)\left(\sum_{i=1}^N \phi_i(\eta)\right)\hat{K}\left(\sum_{i=1}^N \int_\Omega \phi_i(\eta)\ \mathrm{d}x\right)}$$

So, we obtain

$$\frac{\frac{mes(\Omega)\|\Theta\|_{\infty}\max\{\theta^{m^+},\theta^{m^-}\}}{\lambda m^-}+\int_{\Omega}\sup_{|t|\leq\theta}F(x,t)\,dx+\frac{\mu}{\lambda}G^{\theta}}{k_0\left(\frac{\theta N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+}} < \frac{1}{\lambda} < \frac{\frac{|\eta|^{m^+}}{\lambda m^+}\|\Theta\|_{L^1(\Omega)}+\int_{\Omega}F(x,\eta)\,dx+\frac{\mu}{\lambda}G_{\eta}}{mes(\Omega)\left(\sum_{i=1}^{N}\phi_i(\eta)\right)\hat{K}\left(\sum_{i=1}^{N}\int_{\Omega}\phi_i(\eta)\,dx\right)}. \tag{4.17}$$

Hence, the combination of (4.15), (4.16) and (4.17) implies that the condition (4.2.1) of proposition (4.2.1) is satisfied. Now, in order to apply the proposition (4.2.1), we have to prove that $\lim_{\|u\|\to+\infty}(J(u)-\lambda I(u))=+\infty$. Since $m(x)<P_{-,\infty}$ $,2(P_0)^-<\min\{P_{-,\infty},\min_{x\in\partial\Omega}\{p_1^{\partial}($ we then have

$$W^1L_{\vec{\phi}}(\Omega)\hookrightarrow L^{m(.)}(\Omega),\quad W^1L_{\vec{\phi}}(\Omega)\hookrightarrow L^{2(P_0)^-}(\Omega),\text{ and}$$

$$W^1L_{\vec{\phi}}(\Omega)\hookrightarrow L^{2(P_0)^-}(\partial\Omega).$$

with continuous and compact embeddings.
Consequently, there exist $C_1>0$, $C_2>0$ and $C_3>0$ such that for all $u\in X$

$$|\,u\,|_{L^{2(P_0)^-}(\Omega)}\leq C_2\|u\|,|\,u\,|_{L^{m(x)}(\Omega)}\leq C_1\|u\|\text{ and }|u|_{L^{2(P_0)^-}(\partial\Omega)}\leq C_3\|u\|. \tag{4.18}$$

It follows since $\mu<\overline{\rho_{\lambda,g}}$ that there exist $l>0$ such that

$$\limsup_{|t|\longrightarrow\infty}\frac{\sup_{x\in\partial\Omega}G(x,t)}{t^{2(P_0)^-}}<lC_3.$$

with $\frac{k_0}{2^{(P^0)^--1}N^{(P^0)^+-1}}-\mu lC_3>0$. Therefore, there exists a function $h\in L^1(\partial\Omega)$ such that

$$G(x,t)\leq lt^{2(P_0)^-}+h(x);\quad\forall(x,t)\in\partial\Omega\times\mathbb{R}. \tag{4.19}$$

Now, let $\epsilon>0$ be small enough so that

$$0<\lambda\epsilon C_2<\frac{k_0}{2^{(P^0)^--1}N^{(P^0)^+-1}}-\mu lC_3$$

From (H0), there is a function $h_{\epsilon}\in Ł^1(\Omega)$ such that

$$F(x,t)\leq\epsilon t^{2(P_0)^-}+h_{\epsilon}(x);\quad\forall(x,t)\in\Omega\times\mathbb{R}. \tag{4.20}$$

$$\begin{aligned}
J(u)-\lambda I(u) &= \hat{K}\left(\sum_{i=1}^{N}\int_{\Omega}\phi_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right)+\phi_i\left(|u|\right)\,\mathrm{d}x\right)(\sum_{i=1}^{N}\int_{\Omega}\phi_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right)+\phi_i\left(|u|\right)\,\mathrm{d}x)\\
&\quad-\int_{\Omega}\frac{\Theta(x)}{m(x)}|u|^{m(x)}\,\mathrm{d}x-\lambda\int_{\Omega}F(x,u)\,\mathrm{d}x-\mu\int_{\partial\Omega}G(x,u)\,\mathrm{d}x\\
&\geq k_0\left(\sum_{i=1}^{N}\int_{\Omega}\phi_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right)+\phi_i\left(|u|\right)\,\mathrm{d}x\right)^2-\frac{\|\Theta\|_\infty}{m^-}\int_{\Omega}|u|^{m(x)}\,\mathrm{d}x\\
&\quad-\lambda\epsilon\int_{\Omega}|u|^{2(P_0)^-}\,\mathrm{d}x-\lambda\int_{\Omega}h_\epsilon(x)\,\mathrm{d}x-\mu l\int_{\partial\Omega}|u|^{2(P_0)^-}\,\mathrm{d}x\\
&\quad-\mu\int_{\partial\Omega}h_\epsilon(x)\,\mathrm{d}x\\
&\geq\frac{k_0}{2^{(P^0)^--1}N^{(P^0)^+-1}}\|u\|^{2(P^0)^-}-\frac{C_1\|\Theta\|_\infty}{m^-}\max\{\|u\|^{m^+},\|u\|^{m^-}\}\\
&\quad-\lambda\epsilon C_2\|u\|^{2(P^0)^-}-\lambda\|h_\epsilon\|_{L^1(\Omega)}-\mu l C_3\|u\|^{2(P^0)^-}-\mu\|h\|_{L^1(\Omega)}\\
&\geq\left(\frac{k_0}{2^{(P^0)^--1}N^{(P^0)^+-1}}-\lambda\epsilon C_2-\mu l C_3\right)\|u\|^{2(P^0)^-}\\
&\quad-\frac{C_1\|\Theta\|_\infty}{m^-}\max\{\|u\|^{m^+},\|u\|^{m^-}\}-C.
\end{aligned}\tag{4.21}$$

Since $m^- > 2(P^0)^-$. Then, we have

$$\lim_{\|u\|\to+\infty} I(u)-\lambda J(u)=+\infty.$$

which means the functional $J-\lambda I$ is coercive. The proof of the theorem (4.2.1) is now completed.

*Proof of Theorem 4.2.4.* Our aim is to apply Theorem to our problem. Let $X$ be the anistropic Orlicz-Sobolev spaces $W^1L_{\vec{\phi}}(\Omega)$. We consider the auxiliary problem

$$\begin{aligned}
&-K\left(\sum_{i=1}^{N}\int_{\Omega}\phi_i(\left|\frac{\partial u}{\partial x_i}\right|)+\phi_i(|u|)\mathrm{d}x\right)\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\alpha_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right)\frac{\partial u}{\partial x_i}+\alpha_i\left(|u|\right)u\right)\\
&\quad=\Theta(x)|u|^{m(x)}u+\lambda\widehat{f}(x,u)\text{ in }\Omega,\\
&\qquad\sum_{i=1}^{N}\alpha_i\left(|\frac{\partial u}{\partial x_i}|\right)\frac{\partial u}{\partial x_i}v_i=\mu g(x,u)\quad\text{on }\partial\Omega.
\end{aligned}\tag{4.22}$$

where $\hat{f}:\Omega\times\mathbb{R}\to\mathbb{R}$ is an $L^1$ -Carathéodory function, defined as follows

$$\hat{f}(x,\xi)=\begin{cases} f(x,0), & \text{if}\quad \xi<-\theta_3\\ f(x,\xi), & \text{if}\quad -\theta_3\leq\xi\leq\theta_3\\ f\left(x,\theta_3\right), & \text{if}\quad \xi>\theta_3\end{cases}$$

If any solution of the problem (4.1) satisfies the condition $-\theta_3\leq u(x)\leq\theta_3$ for every $x\in\Omega$, then, any weak solution of the problem (4.22) clearly turns to be also a weak solution of (4.1) Therefore, for our goal, it is enough to show that our conclusion holds for (4.1). in order to apply the theorem 4.2.2, we have $J$ is sequentially weakly lower semicontinuous. Put $r_1 := k_0\left(\frac{\theta_1 N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+}$, $r_2 := k_0\left(\frac{\theta_2 N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+}$, $r_3 := k_0\left(\frac{N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+} \times (\theta_3^{(P_0)^+} - \theta_2^{(P_0)^+})$ and $w(x) = \eta \in ]0,1[$ for all $x \in \Omega$. We have $w \in X$. Hence, we have definitively,

$$k_0 mes(\Omega)(\sum_{i=1}^{N} \mid \phi_i(\eta) \mid) \leq J(w) \leq \frac{k_1 mes(\Omega)^2}{2}(\sum_{i=1}^{N} \mid \phi_i(\eta) \mid)^2.$$

From the conditions, $\theta_1 < \frac{c}{N^{\frac{1}{(P_0)+}-1}} \min\left\{\frac{N^{\frac{1}{(P_0)+}-1}}{c}\eta, 1, \left(mes(\Omega)\sum_{i=1}^{N} \mid \phi_i(\eta) \mid\right)^{\frac{1}{(P_0)^+}}\right\}$, $\theta_3 > \theta_2$, and $\left(\frac{k_1 mes(\Omega)^2}{2k_0}\right)^{\frac{1}{(P_0)^+}} (\sum_{i=1}^{N} \mid \phi_i(\eta) \mid)^{\frac{2}{(P_0)^+}} < \theta_2$, we get $r_3 > 0$ and $r_1 < J(w) < r_2$. From the definition of $r_1$, it follows that

$$J^{-1}(-\infty, r_1]) \subseteq \{u \in X; \quad |u(x)| \leq \theta_1, \text{ for all } x \in \Omega\}$$

By using the assumption (A1), one has

$$\sup_{u \in J^{-1}(-\infty, r_1]} \int_\Omega F(x, u(x))\, dx \leq \int_\Omega \sup_{|t| \leq \theta_1} F(x,t)\, dx \leq \int_\Omega F(x, \theta_1)\, dx$$

In a similar way, we have

$$\sup_{u \in J^{-1}(-\infty, r_2]} \int_\Omega F(x, u(x))\, dx \leq \int_\Omega F(x, \theta_2)\, dx$$

and

$$\sup_{u \in J^{-1}(-\infty, r_2 + r_3]} \int_\Omega F(x, u(x))\, dx \leq \int_\Omega F(x, \theta_3)\, dx$$

Therefore, since $0 \in J^{-1}(-\infty, r_1)$ et $I(0) = J(0) = 0$ it yields

$$\begin{aligned}
\varphi(r_1) &\leq \inf_{u\in J^{-1}(]-\infty,r_1[)} \frac{(\sup_{u\in J^{-1}(]-\infty,r_1[} I(u)) - I(u)}{r_1 - J(u)} \\
&\leq \frac{\sup_{u\in J^{-1}(]-\infty,r_1[} I(u)}{r_1} \\
&= \frac{\inf_{u\in J^{-1}(]-\infty,r_1[)} \left[\frac{1}{\lambda}\int_\Omega \frac{\Theta(x)}{m(x)}|u|^{m(x)}\,\mathrm{d}x + \int_\Omega F(x,u)\,\mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega} G(x,u)\,\mathrm{d}x.\right]}{r_1} \\
&\leq \frac{\frac{mes(\Omega)\|\Theta\|_\infty \theta_1^{m^-}}{\lambda m^-} + \int_\Omega \sup_{|t|\leq\theta_1} F(x,t)\,\mathrm{d}x + \frac{\mu}{\lambda}G^{\theta_1}}{k_0\left(\frac{\theta_1 N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+}} \\
&\leq \frac{\frac{mes(\Omega)\|\Theta\|_\infty \theta_1^{m^-}}{\lambda m^-} + \int_\Omega F(x,\theta_1)\,\mathrm{d}x + \frac{\mu}{\lambda}G^{\theta_1}}{k_0\left(\frac{\theta_1 N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+}}.
\end{aligned}$$

$$\begin{aligned}
\varphi(r_2) &\leq \inf_{u\in J^{-1}(]-\infty,r_2[)} \frac{(\sup_{u\in J^{-1}(]-\infty,r_2[} I(u)) - I(u)}{r_1 - J(u)} \\
&\leq \frac{\sup_{u\in J^{-1}(]-\infty,r_2[} I(u)}{r_2} \\
&= \frac{\inf_{u\in J^{-1}(]-\infty,r_2[)} \left[\frac{1}{\lambda}\int_\Omega \frac{\Theta(x)}{m(x)}|u|^{m(x)}\,\mathrm{d}x + \int_\Omega F(x,u)\,\mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega} G(x,u)\,\mathrm{d}x.\right]}{r_2} \\
&\leq \frac{\frac{mes(\Omega)\|\Theta\|_\infty \max\{\theta_2^{m^+},\theta_2^{m^-}\}}{\lambda m^-} + \int_\Omega F(x,\theta_2)\,\mathrm{d}x + \frac{\mu}{\lambda}G^{\theta_2}}{k_0\left(\frac{\theta_2 N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+}}.
\end{aligned}$$

$$\begin{aligned}
\gamma(r_2, r_3) &:= \frac{\sup_{u\in J^{-1}(]-\infty,r_2+r_3[)} I(u)}{r_3} \\
&\leq \frac{\frac{mes(\Omega)\|\Theta\|_\infty \max\{\theta_3^{m^+},\theta_3^{m^-}\}}{\lambda m^-} + \int_\Omega F(x,\theta_3)\,\mathrm{d}x + \frac{\mu}{\lambda}G^{\theta_3}}{k_0\left(\frac{N^{\frac{1}{(P_0)^+}-1}}{2c}\right)^{(P_0)^+}(\theta_3^{(P_0)^+} - \theta_2^{(P_0)^+})}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned} I(w) &= \frac{1}{\lambda}\int_\Omega \frac{\Theta(x)}{m(x)}|\eta|^{m(x)}\,\mathrm{d}x + \int_\Omega F(x,\eta)\,\mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega} G(x,\eta)\,\mathrm{d}x \\ &\geq \frac{|\eta|^{m^+}}{\lambda m^+}\|\Theta\|_{L^1(\Omega)} + \int_\Omega F(x,\eta)\,\mathrm{d}x + \frac{\mu}{\lambda}G_\eta. \end{aligned}$$

For each $u \in J^{-1}(]-\infty, r_1[)$ one has

$$\begin{aligned} \beta(r_1,r_2) &\geq \frac{\frac{(\eta^{m^+}-\theta_1^{m^+})}{\lambda m^+}\|\Theta\|_{L^1(\Omega)} + \int_\Omega F(x,\eta)\,\mathrm{d}x - \int_\Omega F(x,\theta_1)\,\mathrm{d}x + \frac{\mu}{\lambda}(G_\eta - G^{\theta_1})}{J(w)-J(u)} \\ &\geq \frac{\int_\Omega F(x,\eta)\,\mathrm{d}x - \int_\Omega F(x,\theta_1)\,\mathrm{d}x + \frac{\mu}{\lambda}(G_\eta - G^{\theta_1})}{\frac{k_1 mes(\Omega)^2}{2}(\sum_{i=1}^N \mid \phi_i(\eta)\mid)^2}. \end{aligned}$$

Due to (A2) we get

$$\alpha(r_1,r_2,r_3) < \beta(r_1,r_2).$$

Now, we show that the functional $I$ satisfies the assumption (a2) of Theorem 2.2. Let $u_1$ and $u_2$ be two local minima for $I$. Then $u_1$ and $u_2$ are critical points for $I$, and so, in the same way as for theorem 3.2 in [32] they are nonnegative weak solutions for the problem (4.1). Thus, it follows that $su_1 + (1-s)u_2 \geq 0$ for all $s \in [0,1]$, and that

$$(\lambda f + \mu g)(x, su_1 + (1-s)u_2) \geq 0$$

and consequently, $I(su_1 + (1-s)u_2) \geq 0$, for every $s \in [0,1]$. Hence, Theorem 2.1 implies that for every $\lambda \in \mathbb{R}$ such that:

$$\begin{aligned} &\max\left\{ \frac{\int_\Omega F(x,\theta_1)\,\mathrm{d}x}{\theta_1^{(P_0)^+}}, \frac{\int_\Omega F(x,\theta_2)\,\mathrm{d}x}{\theta_2^{(P_0)^+}}, \frac{\int_\Omega F(x,\theta_3)\,\mathrm{d}x}{\theta_3^{(P_0)^+} - \theta_2^{(P_0)^+}} \right\} \\ &\qquad + \frac{mes(\Omega)\|\Theta\|_\infty \max\{\theta_1^{m^-}, \theta_3^{m^+}, \theta_3^{m^-}\}}{m^-\theta_1^{(P_0)^+}} \\ &< \frac{1}{\lambda} < \frac{2k_0\,(2c)^{(P_0)^+}}{k_1 mes(\Omega)^2\left(N^{\frac{1}{(P_0)^+}-1}\right)^{(P_0)^+}} \frac{\int_\Omega F(x,\eta)\,\mathrm{d}x - \int_\Omega F(x,\theta_1)\,\mathrm{d}x}{(\sum_{i=1}^N \mid \phi_i(\eta)\mid)^2} \end{aligned}$$

□

the functional $J - \lambda I$ has at least three distinct critical points which are the weak solutions of problem (4.1). This completes the proof.

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## *Abstract:*

*The aim of this thesis is the study of the existence of solutions of: the anisotropic quasilinear elliptic equation with variable exponent and nonlinear Robin boundary conditions, quasilinear elliptic $\vec{p}(x)$-Kirchhoff type problem with weight and nonlinear Robin boundary conditions , , perturbed Kirchhoff-type non-homogeneous problems. By using Mountain Pass, Fountain theorems, and three critical point theorem due to Bonanno and Candito. , we establish the existence non trivial weak solution and infinite many pairs of weak solutions to these problems and the existence of three distinct weak solutions for perturbed Kirchhoff-type non-homogeneous Neumann problems . And it contains four chapters. The first chapter is devoted to recall some background facts concerning the generalized Lebesgue–Sobolev spaces, anisotropic–Sobolev spaces and introduce some notations used below . In the second chapter is mainly devoted to the existence and multiplicity of solutions of quasilinear elliptic equations under nonlinear Robin boundary condition,it turns out that the condition $q^- > \boldsymbol{P}_+^+$ plays an important role in the proofs of our main results. In third chapters, we deal with the existence and multiplicity of weak solutions to a class of quasilinear elliptic $\vec{p}(x)$Kirchhoff type problems with weight and a nonlinear Robin boundary condition. In fact, we are able in the third chapter to deal with more general nonlinearities in the boundary condition and with situations where the function M is unbounded. Perturbed Kirchhoff-type non-homogeneous Neumann problems by means of a variational approach and the use of the anistropic Orlicz-Sobolev spaces is studied in the last chapter, this is the first contribution in this direction. The readers may consult the excellent survey article of M.Mihailescu.*


***Keywords:*** *Anisotropic elliptic equations, nonlocal Kirchhoff equation, variable exponents Lebesgue spaces,non-standard growth condition, variational method, existence and multiplicity, three distinct weak solutions, anistropic Orlicz-Sobolev spaces.*


## *Résumé :*

L'objectif de cette thèse est l'étude de l'existence de solutions de l'équation elliptique quasilinéaire anisotrope avec exposant variable et conditions aux limites non-linéaires de Robin, elliptique quasilinéaire Problème de type $\vec{p}(x)$--Kirchhoff avec poids et conditions aux limites non linéaires de Robin, problème perturbé de type Kirchhoff non-homogène. En utilisant les théorèmes de Mountain Pass et de Fountain, et le théorème des trois points critiques de Bonanno and Candito, nous établissons l'existence d'une solution faible non triviale et d'une infinité de paires de solutions faibles à ces problèmes, et l'existence de trois solutions faibles distinctes pour les problèmes de Neumann non homogènes de type Kirchhoff perturbés. Il contient quatre chapitres. Le premier chapitre sera consacré à rappeler quelques faits de fond concernant les espaces généralisés de Lebesgue-Sobolev, les espaces anisotropes-Sobolev et à introduire quelques notions utilisées ci-dessous. Le deuxième chapitre sera principalement consacré à l'existence et à la multiplicité des solutions d'équations elliptiques quasi-linéaires sous condition aux limites non-linéaire de Robin, il s'avère que la condition $q^- > \boldsymbol{P}_+^+$ joue un rôle primordial dans les preuves de nos principaux résultats. Dans le troisième chapitre, nous allons traiter l'existence et la multiplicité de solutions faibles à une classe de problèmes de type quasi-elliptique $\vec{p}(x)$-Kirchhoff avec le poids et une condition aux limites non-linéaire de Robin. En fin, nous serons capables de traiter des non-linéarités plus générales dans la condition aux limites et des situations où la fonction M est non bornée Le problème de Neumann non homogène de type Kirchhoff perturbés au moyen d'une approche variationnelle et l'utilisation des espaces anistropiques d'Orlicz-Sobolev sont étudiés dans le dernier chapitre, c'est la première contribution dans cette direction. Les lecteurs peuvent consulter l'excellent article de *M.Mihailescu.*


***Mots-clés :*** *Équations elliptiques anisotropes, équation de Kirchhoff non locale, exposants variables espaces de Lebesgue,, conditions de croissance non standard, méthode variationnelle, existence et multiplicité, trois solutions faibles distinctes, espaces anistropiques Orlicz-Sobolev.*

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