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THESIS

Presented by

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Study of Some Elliptic and Parabolic Problems of Dirichlet or Neumann Type in Different Settings

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Dedication

This the

- Profe
- My mother and father
- My sisters and brothers
- My wife and her family
- My profe
- My family and friends.

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List of Publications

The publications that constitute the basis of the PhD thesis can be found in

Published works and works to appear

1. El Ouaarabi, M., Abbassi, A., Allalou, C.: Existence result for a Dirichlet problem governed by nonlinear degenerate elliptic equation in weighted Sobolev spaces. J. Elliptic Parabol Equ. **7**(1), 221–242 (2021). https://doi.org/10.1007/s41808-021-00102-3.

2. El Ouaarabi, M., Allalou, C., Abbassi, A.: On the Dirichlet Problem for some Nonlinear Degenerated Elliptic Equations with Weight. 7^{th} International Conference on Optimization and Applications (ICOA), 1–6 (2021). https://doi.org/10.1109/ICOA51614.2021.9442620.

3. El Ouaarabi, M., Abbassi, A., Allalou, C.: Existence Result for a General Nonlinear Degenerate Elliptic Problems with Measure Datum in Weighted Sobolev Spaces. International Journal On Optimization and Applications. **1**(2), 1–9 (2021).

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6. El Ouaarabi, M., Allalou, C., Melliani, S.: On the existence and uniqueness of solutions for a class of nonlinear degenerate elliptic problems via Browder-Minty theorem. Contrib. Math. **5**, 7–16 (2022). https://doi.org/10.47443/cm.2022.001

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9. M. El Ouaarabi, C. Allalou, S. Melliani, On a class of $p(x)$-Laplacian-like Dirichlet problem depending on three real parameters, Arab. J. Math. **11** (2022), 227–239. https://doi.org/10.1007/s40065-022-00372-2.

10. M. El Ouaarabi, C. Allalou, S. Melliani, Weak solution of a Neumann boundary value problem with $p(x)$-Laplacian-like operator, Analysis (2022). https://doi.org/10.1515/anly-2022-1063.

11. M. El Ouaarabi, C. Allalou, S. Melliani, Existence of weak solution for a class of $p(x)$-Laplacian problems depending on three real parameters with Dirichlet condition, Bol. Soc. Mat. Mex. **28**(2) (2022), 1–16. https://doi.org/10.1007/s40590-022-00427-6.

12. M. El Ouaarabi, C. Allalou, S. Melliani, Existence Result for a Neumann Boundary Value Problem Governed by a Class of $p(x)$-Laplacian-like Equation, Asymptotic Analysis **Preprint** (2022), 1–15. https://doi.org/10.3233/ASY-221791.

13. El Ouaarabi, M., Allalou, C., Melliani, S.: Existence of weak solutions for $p(x)$-Laplacian-like problem with $p(x)$-Laplacian operator under Neumann boundary condition. São Paulo J. Math. Sci. (2022) to appear. https://doi.org/10.1007/s40863-022-00321-z.

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15. M. El Ouaarabi, C. Allalou, S. Melliani, Existence of weak solutions for a double phase variable exponent problem with a gradient dependent reaction term. Miskolc Mathematical Notes, (2022) to appear.

16. M. El Ouaarabi, H. El Hammar, C. Allalou, S. Melliani, A $p(x)$-Kirchhoff type problem involving the $p(x)$-Laplacian-like operators with Dirichlet boundary condition. Studia Universitatis Babes-Bolyai Mathematica, (2022) to appear.

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24. M. El Ouaarabi, C. Allalou, S. Melliani, Existence result for a double phase problem involving the $(p(x), q(x))$-Laplacian operator. Mathematica Slovaca, (2022) to appear.

25. M. El Ouaarabi, C. Allalou, S. Melliani, Neumann Problem Involving the $p(x)$-Kirchhoff-Laplacian-Like Operator in Variable Exponent Sobolev Space, Asia Pac. J. Math. **9** (2022), 18. https://doi.org/10.28924/APJM/9-18.

Submitted works

1. M. El Ouaarabi, C. Allalou, S. Melliani, Existence of weak solutions to a $p(x)$-Kirchhoff type problems involving the $p(x)$-Laplacian-like operators. Ukrainian Mathematical Journal.

2. M. El Ouaarabi, C. Allalou, S. Melliani, On a class of quasilinear elliptic and parabolic problems involving the $p(x)$-Kirchhoff-Laplacian operator. São Paulo Journal of Mathematical Sciences

3. M. El Ouaarabi, C. Allalou, S. Melliani, On a Neumann problem driven by $p(x)$-Laplacian-like operators in variable-exponent Sobolev spaces. Palestine Journal of Mathematics.

4. M. El Ouaarabi, C. Allalou, S. Melliani, Existence of a solution to a new $p(x)$-Kirchhoff type problems driven by $p(x)$-Laplacian-like operators. Nonlinear Studies.

5. M. El Ouaarabi, C. Allalou, S. Melliani, On a new $p(x)$-Kirchhoff type problems with $p(x)$-Laplacian-like operators and Neumann boundary conditions. Acta Universitatis Sapientiae, Mathematica.

6. M. El Ouaarabi, C. Allalou, S. Melliani, Weak solutions for a quasilinear elliptic and parabolic problems involving the $(p(x), q(x))$-Laplacian operator. Journal of Mathematical Sciences (Series A).

Contents

Résumé

Au cours de la dernière décennie, une importante littérature traite de différents aspects des EDP dont la partie principale de l'opérateur a une croissance de type puissance, l'exemple principal étant le p-Laplacien. Il existe un large éventail de directions dans lesquelles le cas de la croissance polynomiale a été développé, notamment les approches à exposant variable, avec poids, double phase à exposant variable et des approches faisant intervenir un opérateur "$p(x)$-Laplacian-like" et un opérateur "$p(x)$-Kirchhoff-Laplacian".

Dans la présente monographie, nous traitons les questions de la théorie de l'existence et de l'unicité aux problèmes elliptiques et paraboliques de type Dirichlet ou Neumann dans différents cadres fonctionnels. L'originalité de ce travail consiste en la présence d'une classe d'opérateurs étudiés permettant de regarder l'importance du cadre fonctionnel, et impliquant des espaces de Lebesgue-Sobolev avec poids et des espaces de Lebesgue-Sobolev à exposant variable. Cette thèse est composée de deux parties principales :

La première partie concerne l'étude de l'existence et de l'unicité de la solution faible de certains problèmes de Dirichlet régis par une équation elliptique non linéaire dégénérée dans le cadre des espaces de Sobolev avec poids où les données sont dans $L^{p'}(\Omega, \omega^{1-p'})$ ou dans $L^{p'}(\Omega, \omega^{1-p'}) + \prod_{j=1}^{n} L^{p'}(\Omega, \omega^{1-p'})$. Notre outil principal, dans cette partie, est basé sur le théorème de Browder-Minty et la théorie des espaces de Sobolev avec poids.

La deuxième partie de cette thèse est consacrée à l'étude de deux classes de problèmes non linéaires, la première classe des problèmes qu'ils discutent dans cette partie sont des problèmes aux limites de Dirichlet ou de Neumann impliquant l'opérateur $p(x)$-Laplacian-like ou l'opérateur $p(x)$-Kirchhoff-Laplacian ou l'opérateur $(p(x), q(x))$-laplacien avec des conditions de croissance non standard. Sous des hypothèses appropriées, ils établissent plusieurs nouveaux résultats concernant l'existence et l'unicité de la solution faible dans le cadre des espaces de Sobolev à exposant variable. Ces résultats sont obtenus par une combinaison de la théorie des espaces de Sobolev à exposant variable et la théorie des degrés topologiques pour une classe d'opérateurs démicontinus de type (S_+) généralisé. La deuxième classe est un problème parabolique associé à l'équation :

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(x, t, \nabla u) = \phi(x, t) + \operatorname{div} \mathcal{B}(x, t, u, \nabla u),$$

ce problème vise à présenter des résultats d'existence d'une solution faible dans l'espace $L^p(0,T;W_0^{1,p}(\Omega,\omega))$ par l'utilisation d'une théorie des degrés topologiques pour les opérateurs du type $\mathcal{T}+\mathcal{S}$, où $\mathcal{S}$ est une application démicontinue bornée de classe (S_+) et $\mathcal{T}$ est une application monotone maximale linéaire définie de manière dense par rapport à un domaine de $\mathcal{T}$.

Mots clés : Problème de Dirichlet, problème de Neumann, problème elliptique, problème parabolique, équation elliptique non linéaire dégénérée, théorie des degrés topologiques, solution faible, espaces de Sobolev avec poids, existence et unicité, espaces de Sobolev à exposant variable, opérateur de type $p(x)$-Kirchhoff-Laplacian, opérateur de type $p(x)$-Laplacian-like, opérateur à double phase avec des exposants variables.

Abstract

Over the last decade, a large literature describes various aspects of PDEs whose main part of the operator has power-type growth with the leading example of the p-Laplacian. There is a wide range of directions in which the polynomial growth case has been developed, including variable exponent, weighted, double phase with variable exponents and approaches involving $p(x)$-Laplacian-like operator and $p(x)$-Kirchhoff-Laplacian operator.

In the following monograph we deal with the questions from existence and uniqueness theory to elliptic and parabolic problems of Dirichlet or Neumann type in different settings. The originality of this work consists of the presence of a class of studied operators allowing to look the importance of the functional framework involves weighted Lebesgue-Sobolev spaces and variable exponent Lebesgue-Sobolev spaces. This thesis covers two main parts :

The first part concerns the study the existence and uniqueness of weak solution to certain Dirichlet problems governed by nonlinear degenerate elliptic equation in the setting of weighted Sobolev spaces with the right-hand side term in $L^{p'}(\Omega, \omega^{1-p'})$ or in $L^{p'}(\Omega, \omega^{1-p'}) + \prod_{j=1}^{n} L^{p'}(\Omega, \omega^{1-p'})$. Our main tool , in this part, is based on the Browder-Minty theorem and the theory of weighted Sobolev spaces.

The second part of this thesis is devoted to study two classes of nonlinear problems, the first classe of problems that we discuss in this part are Dirichlet or Neumann boundary value problems involving the the $p(x)$-Laplacian-like operator or the $p(x)$-Kirchhoff-Laplacian operator or $(p(x), q(x))$-Laplacian operator with nonstandard growth conditions. Under suitable assumptions, we establish new several results concerning the existence and uniqueness of weak solution in the setting of variable exponent Sobolev spaces. These results are obtained by combining the theory of the variable exponent Sobolev spaces and the topological degree theory for a class of demicontinuous operator of generalized (S_+) type. The second classe of problem is a parabolic problem associated with the equation :

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(x, t, \nabla u) = \phi(x, t) + \operatorname{div} \mathcal{B}(x, t, u, \nabla u),$$

this problem aim to present an existence result of a weak solution in the spaces $L^p(0, T; W_0^{1,p}(\Omega, \omega))$ by using a topological degree theory for operators of the type $\mathcal{T} + \mathcal{S}$, where $\mathcal{S}$ is a bounded

demicontinuous map of class (S_+) and $\mathcal{T}$ is a linear densely defined maximal monotone map with respect to a domain of $\mathcal{T}$.

Key words : Dirichlet problem, Neumann problem, elliptic problem, parabolic problem, degenerate nonlinear elliptic equation, topological degree theory, weak solution, weighted Sobolev spaces, existence and uniqueness, variable exponent Sobolev spaces, $p(x)$-Kirchhoff-Laplacian operator, $p(x)$-Laplacian-like operator, double phase operator with variable exponents.

Symbol description

$\forall$	for all
$\exists$	there exists
$\equiv$	equivalent
$\sum$	summation
$\prod$	product
$\mathbb{R}$	set of real numbers
$\mathbb{N}$	set of natural numbers
N	positive integer greater than or equals to 1
$\mathbb{R}^N$	N-dimensional Euclidean space of points $x = (x_1, x_2, \cdots, x_N)$
Ω	open bounded subset of $\mathbb{R}^N$
$\partial\Omega$	boundary of Ω
$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$	an multiindex with $\alpha_i \in \mathbb{N}$
$\lvert\alpha\rvert = \sum\limits_{i=1}^{N} \alpha_i$	the length of the multiindex α
$D^\alpha = \frac{\partial^{\lvert\alpha\rvert}}{\partial x_1^{\alpha_1} . \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}}$	partial derivative of order $\lvert\alpha\rvert$
$\mathcal{C}(\Omega)$	the spaces of continuous functions on Ω
$\mathcal{C}^\infty(\Omega)$	the spaces of infinitely differentiable functions on Ω
$\mathcal{C}_0^\infty(\Omega)$	infinitely differentiable functions with compact support on Ω
∇u	gradient of a function u
a.e	almost everywhere
$\longrightarrow$	strong convergence
$\rightharpoonup$	weak convergence
$\hookrightarrow$	continuous embedding
$\hookrightarrow\hookrightarrow$	compact embedding
X	arbitrary Banach space
X^*	dual space of the Banach space X
$\langle \cdot, \cdot \rangle$	scalar product of $\mathbb{R}^N$, duality between X and X^*

$\overline{\Omega}$	closure of Ω (i.e., Ω plus its boundary)
$[0,T]$	closed interval $0 \leq t \leq T$ in $\mathbb{R}$
Ω_T	the time-space cylinder $\Omega \times (0,T)$ with $T \in (0,\infty)$
$\partial\Omega_T$	boundary of Ω_T
$\|\Omega\|$	measure of the set Ω
x	$(x_1, ..., x_N)$ point in $\mathbb{R}^N$
dx	$dx_1, ..., dx_N$ Lebesgue measure in Ω
$\operatorname{div} f$	$\displaystyle\sum_{i=1}^{N} \frac{\partial f}{\partial x_i}$
p	real number such that $1 \leq p < \infty$
p'	the Hölder conjugate of p
$L^p(\Omega)$	usual Lebesgue space
$L^{p'}(\Omega)$	dual space of $L^p(\Omega)$
$L^\infty(\Omega)$	essentially bounded measurable functions on Ω
$W^{1,p}(\Omega)$	usual Sobolev space
$W_0^{1,p}(\Omega)$	closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ (i.e. w.r.t. the norm $\|\cdot\|_{W^{1,p}(\Omega)}$)
$W^{-1,p'}(\Omega)$	dual space of $W_0^{1,p}(\Omega)$
$L^p(\Omega,\omega)$	weighted Lebesgue Sobolev space
$L^{p'}(\Omega,\omega^{1-p'})$	dual space of $L^p(\Omega,\omega)$
$W^{1,p}(\Omega,\omega)$	weighted Sobolev space
$W_0^{1,p}(\Omega,\omega)$	closure of $\mathcal{C}_0^\infty(\Omega,)$ in $W^{1,p}(\Omega,\omega)$
$W^{-1,p'}(\Omega,\omega^{1-p'})$	dual space of $W_0^{1,p}(\Omega,\omega)$
$p(\cdot)$	measurable function (variable exponent)
p^+	essential sup of $p(\cdot)$
p^-	essential inf of $p(\cdot)$
$p'(\cdot)$	Sobolev conjugate of $p(\cdot)$
$L^{p(\cdot)}(\Omega)$	variable exponent Lebesgue space (generalized Lebesgue space)
$W^{1,p(\cdot)}(\Omega)$	variable exponent Sobolev space (generalized Sobolev space)
$W_0^{1,p(\cdot)}(\Omega)$	closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$
$W^{-1,p'(\cdot)}(\Omega)$	dual space of $W_0^{1,p(\cdot)}(\Omega)$

General introduction

Historical and motivation

Boundary value problems for elliptic and parabolic equations, more precisely, the concept of weak (generalized) solutions, have their background in applications (namely, in the variational approach connected with the critical level of a certain energy functional as well as in numerical methods like FEM etc). This type of approach is closely related to the concept of Sobolev spaces and is well elaborated for both linear and nonlinear equations.

In various applications, we can meet boundary value problems for elliptic and parabolic equations whose ellipticity is "disturbed" in the sense that some degeneration or singularity appears. This "bad" behaviour can be caused by the coefficients of the corresponding differential operator as well as by the solution itself. The so-called p-Laplacian is a prototype of such an operator and its character can be interpreted as a degeneration or as a singularity of the classical (linear) Laplace operator (with $p = 2$). There are several very concrete problems from practice which lead to such differential equations, e.g. from glaceology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction, reaction-diffusion problems, etc.

Let ω be a weight on $\mathbb{R}^N$, i.e., a locally integrable function on $\mathbb{R}^N$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ be open, $1 \leq p < \infty$, and k a nonnegative integer. The weighted Sobolev space $W^{k,p}(\Omega, \omega)$ consists of all functions u with weak derivatives $D^\alpha u$, $|\alpha| \leq k$, satisfying

$$||u||_{W^{k,p}(\Omega,\omega)} = \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p \omega(x)\, dx \right)^{\frac{1}{p}} < \infty.$$

In the case $\omega = 1$, this space is denoted $W^{k,p}(\Omega)$. In general, Sobolev spaces without weights occur as spaces of solutions for elliptic and parabolic partial difierential equations (see [28, 34]). Typically, $2k$ is the order of the equation and the case $p = 2$ corresponds to linear equations. Details can be found in almost any book on partial differential equations. For degenerate partial differential equations, where we have equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces

[87, 88], we mention some works in this direction [7, 11, 12, 130].

The type of a weight depends on the equation type. A class of weights, which is particularly well understood, is the class of A_p weights (or Muckenhoupt class) that was introduced by Muckenhoupt in the early 1970's [113, 114]. This class consists of precisely those weights ω for which the Hardy-Littlewood maximal operator is bounded from $L^p(\mathbb{R}^N, \omega)$ to $L^p(\mathbb{R}^N, \omega)$, when $1 < p < \infty$, and from $L^1(\mathbb{R}^N, \omega)$ to wk-$L^1(\mathbb{R}^N, \omega)$, when $p = 1$. These classes have found many useful applications in harmonic analysis [138]. Another reason for studying A_p weights is the fact that powers of distance to submanifolds of $\mathbb{R}^N$ often belong to A_p [102].

It is well-known that classical potential theory is connected to linear partial differential equations and the Sobolev space $W^{1,2}(\Omega)$. The most striking manifestation of this connection is the Dirichlet principle, which states that the solution to Dirichlet's problem for the Laplace equation in a domain Ω:

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ u = f \quad \text{on } \partial\Omega, \end{cases}$$

where the boundary function f is assumed to belong to $W^{1,2}(\Omega)$, can be obtained by minimizing the energy integral

$$\int_\Omega \nabla u \, dx$$

over all functions $u \in W^{1,2}(\Omega)$ for which $u - f \in W_0^{1,2}(\Omega)$. A corresponding nonlinear potential theory, connected to nonlinear partial differential equations and the space $W^{k,p}$, has been developed. The theory originated in the work by V. G. Maz'ya and J. Serrin. Excellent accounts of this theory and its history are the monographs Adams-Hedberg [6], Maz'ya [111], and Ziemer [155]. A lot of the corresponding weighted theory can be found in Heinonen, Kilpelainen, and Martio [95].

In recent years, partial differential equations with nonlinearities and nonconstant exponents have received a lot of attention (see [119, 120]). The impulse of this topic would come from the new search field that reflects a new type of physical phenomenon is a class of nonlinear problems with variable exponents. Modeling with classic Lebesgue and Sobolev spaces has been demonstrated to be limited for a number of materials with inhomogeneities. In the subject of fluid mechanics, for example, great emphasis has been paid to the study of electrorological fluids, which have the ability to modify their mechanical properties when exposed to an electric field (see [5, 125, 126]). Rajagopal and Rùzicka recently developed a very interesting model for these fluids in [129] (see also [132]), taking into account the delicate interaction between the electric field $E(x)$ and the moving liquid. This type of problem's energy is provided by $\int_{\Omega} |\nabla u|^{p(x)} dx$. This type of energy can also be found in elasticity problems [149]. The natural energy space in which such problems can be studied is the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$. Also, we can find other applications relate to image processing [2, 40], elasticity [150], the flow in porous media [17, 96], and problems in the calculus of variations involving variational integrals with nonstandard growth [1, 4, 21, 22, 109, 122, 121, 123, 150].

For several years, great efforts have been devoted to the study of nonlinear elliptic equations with an operator described by polynomial growth, which is motivated, for example, in the classical Sobolev space, not only by the description of many phenomena appearing in the applied sciences, due to the study of fluid filtration in porous media, constrained heating, elasto-plasticity, optimal control, financial mathematics, and others. Interested readers may refer to [14, 24, 27, 40] and the references therein for more background of applications. But also by the mathematical importance in the theory of this space. In addition, there is a vast literature describing various aspects of PDEs whose main part of the operator has a power-like growth with the preeminent example of the p-Laplacian. There is a wide range of directions in which the polynomial growth case has been developed, including variable exponent, $p(x)$-Laplacian-like operator, $p(x)$-Kirchhoff-Laplacian operator, and double-phase with variable exponent.

The study of various mathematical problems involving double-phase operator has become very attractive in recent decades. Zhikov was the first who studied this type of problem in order to describe models of strongly anisotropic materials by studying the functional

$$u \mapsto \int_{\Omega} \Big(|\nabla u|^p + a(x)|\nabla u|^q \Big) dx \tag{0.0.1}$$

where the integrand switches two different elliptic behaviours. For more results see [151, 152, 153]. Then, several interesting works have been carried out on the double phase problem with a Dirichlet boundary condition. For a deeper comprehension, we refer the reader to [3, 107, 110, 118, 134, 144, 145, 146] and the references therein.

The double phase operator has been used in the modelling of strongly anisotropic materials [150] and in Lavrentiev's phenomenon [151]. In the one hand, we have the physical motivation; since the double phase operator has been used to model the steady-state solutions of reaction-diffusion problems, that arise in biophysic, plasma-physic and in the study of chemical reactions. In the other hand, these operators provide a useful paradigm for describing the behaviour of strongly anisotropic materials, whose hardening properties are linked to the exponent governing the growth of the gradient change radically with the point, where the coefficient $a(\cdot)$ determines the geometry of a composite made of two different materials (see [23, 148] and the references given there).

Moving on to another novel aspect; the double phase problem with variable exponents that few author consider. Ragusa and Tachikawa in [124, 125, 126, 127, 128] and reference therein, are the frst ones who have achieved the regularity theory for minimizers of (0.0.1) with variable exponents (see also [65, 66, 67]). Moreover, in [136] Tachikawa, provides the Hölder continuity up to the boundary of minimizers of so-called double phase functional with variable exponents, under suitable Dirichlet boundary conditions.

In the context of the study of $p(x)$-Laplacian-like problems, arising from capillarity phenomena, Ni and Serrin [115, 116] initiated the study of ground states for equations of the form

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = f(u) \quad \text{in } \mathbb{R}^N, \tag{0.0.2}$$

with very general right hand side f. The operator $-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)$ is usually denoted as the prescribed mean curvature operator. Radial (singular) solutions of the problem (0.0.2) has been studied in the context of the analysis of capillary surfaces for a function f of the form $f(u) = ku$, for $k > 0$ (for more details see [46, 84, 97]).

Capillarity can be briefly explained by considering the effects of two opposing forces: adhesion, i.e. the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, i.e. the attractive force between the molecules of the liquid. The study of capillary phenomenon has gained some attention recently. This increasing interest is motivated not only by fascination in naturally occurring phenomena such as motion of drops, bubbles, and waves but also its importance in applied fields ranging from industrial and biomedical and pharmaceutical to microfluidic systems. Recently, the study of capillarity phenomena has begun to receive more and more attention, for instance [60, 61, 62, 64, 19, 90, 98, 131, 135, 140, 154].

Elliptic boundary value problems involving the mean curvature operator play apivotal role in the mathematical analysis of several physical or geometrical issues, such as capillarity phenomena for incompressible or compressible fluids, mathematical models in physiology or in electrostatics, flux-limited diffusion phenomena, prescribed mean curvature problems for Cartesian surfaces in the Euclidean space: relevant references on these topics include [45, 47, 85, 92].

Let's move on to another innovative aspect; the study of Kirchhoff type problems has already been extended to the case involving the $p(x)$-Laplacian operator. We would like to draw attention to the fact that the $p(x)$-Laplacian operator has more complicated nonlinearity than the p-Laplacian operator. For example, they are non-homogeneous, which prove that the problems involving the $p(x)$-Laplacian operator is more difficult than the problems with p-Laplacian operator.

Kirchhoff [100] has investigated an equation of the form

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{0.0.3}$$

which is called the Kirchhoff equation and which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinguishing feature of the Kirchhoff equation (0.0.3) is that the equation contains a nonlocal coefficient $\left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right)$ which depends on the average $\frac{1}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ of the kinetic energy $\frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2$ on $[0, L]$, and hence the equation is no longer a pointwise identity.

The parameters in (0.0.3) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. Lions [106] has proposed an abstract framework for the Kirchhoff-type equations. After the work by Lions [106], various equations of Kirchhoff-type have been studied extensively, for instance see [15, 18, 20, 38, 39, 42, 49, 50, 51, 52, 73, 76, 78].

Objective

Following the development of nonlinear elliptic and parabolic problems, in this thesis we deal with the existence (and uniqueness) results for some elliptic and parabolic of partial differential equations (PDEs) in different settings.

The first proposals of this thesis is devoted to investigate the existence and uniqueness of the weak solution for some Dirichlet problems governed by nonlinear degenerate elliptic equation in the setting of weighted Sobolev spaces with the right-hand side term in $L^{p'}(\Omega, \omega^{1-p'})$ or in $L^{p'}(\Omega, \omega^{1-p'}) + \prod_{j=1}^{n} L^{p'}(\Omega, \omega^{1-p'})$. The needed results are obtained by means of the Browder-Minty theorem and the theory of weighted Sobolev spaces.

The second part of this thesis is devoted to study two classes of nonlinear problems, the first classe of problems that we discuss in this part are Dirichlet or Neumann boundary value problems involving the $p(x)$-Laplacian-like operator or the $p(x)$-Kirchhoff-Laplacian operator or $(p(x), q(x))$-Laplacian operator with nonstandard growth conditions. Under suitable assumptions, we establish new several results concerning the existence and uniqueness of weak solution in the setting of variable exponent Sobolev spaces. These results are obtained by combining the theory of the variable exponent Sobolev spaces and the topological degree theory for a class of demicontinuous operator of generalized (S_+) type. The second classe of problem is a parabolic problem associated with nonlinear degenerate elliptic equation, this problem aim to present an existence result of weak solution in the space $L^p(0,T;W_0^{1,p}(\Omega,\omega))$ by using the topological degree theory for operators of the type $\mathcal{T}+\mathcal{S}$, where $\mathcal{S}$ is a bounded demicontinuous map of class (S_+) and $\mathcal{T}$ is a linear densely defined maximal monotone map with respect to a domain of $\mathcal{T}$.

Outline

This thesis consists of two parts. Both parts are self-contained and can be studied independently. The parts of this thesis are organized as follows:

Main results of part I

The first part focuses on the study of some nonlinear degenerate elliptic problems in weighted Sobolev spaces with the right-hand side term in $L^{p'}(\Omega,\omega^{1-p'})$ or in $L^{p'}(\Omega,\omega^{1-p'})+\prod_{j=1}^{n} L^{p'}(\Omega,\omega^{1-p'})$.

Let us begin by considering the following nonlinear elliptic of Dirichlet type

$$\begin{cases} \mathcal{L}u(x)=f(x) & \text{in } \Omega,\\ u(x)=0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.4}$$

where $\mathcal{L}$ is the partial differential operator given by $\mathcal{L}u := -\mathrm{div}\Big(\mathcal{A}(x,u,\nabla u)\Big)$ with $\mathcal{A}:\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}^N$ is a Carathéodory function such that

$$|\mathcal{B}(x,\eta,\xi)|\le\gamma(x)+|\eta|^{p-1}+|\xi|^{p-1},\ \ \gamma\in L^{p'}(\Omega), \tag{0.0.5}$$

$$\langle\mathcal{A}(x,\eta,\xi)-\mathcal{A}(x,\eta^{'},\xi^{'}),\xi-\xi^{'}\rangle>0 \ \text{ with } \ \eta\neq\eta^{'} \ \text{ and } \xi\neq\xi^{'}, \tag{0.0.6}$$

$$\langle\mathcal{A}(x,\eta,\xi),\xi\rangle\ge\beta|\xi|^p,\ \ \beta>0, \tag{0.0.7}$$

for all $x\in\Omega$ and whenever $(\eta,\xi),(\eta',\xi')\in\mathbb{R}\times\mathbb{R}^N$. In (0.0.4), the source term f belongs to $W^{-1,p'}(\Omega)$ the dual space of $W_0^{1,p}(\Omega)$. The classical monotone operator methods developed by Minty [112], Browder [36], Brézis [35], Lions [106], Višik [141] and others imply that problem (0.0.4) has at least one weak solution $u\in W_0^{1,p}(\Omega)$.

The Part I is composed of four chapters. First, we begin by a preliminary chapter in which we present all the necessary ingredients thereafter on weight functions, weighted Lebesgue-Sobolev spaces and monotone operators that help us in our analysis. Second, we prove, in Chapter 2, the existence and uniqueness of weak solution u in $W_0^{1,p}(\Omega,\nu_1)$ for a Dirichlet problem associated to (0.0.4) given in the form $\mathcal{L}u := -\mathrm{div}\Big(\nu_1\mathcal{A}(x,\nabla u)+$

$\nu_2\mathcal{B}(x,u,\nabla u)\Big) + \nu_3 g(x,u)$, where ν_1, ν_2 and ν_3 are A_p-weight functions, and $\mathcal{A} : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions allowed to satisfy some conditions similar to (0.0.5), (0.0.6) and (0.0.7) with the right-hand side term f belongs to $L^{p'}(\Omega, \nu_1^{1-p'})$ (cf. [68]). The needed result follows by applying the the Browder-Minty theorem and the theory of the weighted Sobolev spaces.

Thirdly, the aim of the Chapter 3 is to extend the first model given in Chapter 2 to a general form given by $\mathcal{L}u := -\mathrm{div}\Big(\omega_1\mathcal{A}(x,\nabla u) + \omega_2\mathcal{B}(x,u,\nabla u)\Big) + \omega_3 g(x,u) + \omega_4|u|^{p-2}u$ with $1 < p < \infty$. Here $\mathcal{A} : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^N$, $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ are assumed to satisfy some assumptions stated in the sense of (0.0.5), (0.0.6) and (0.0.7), and the source term f belongs to $L^{p'}(\Omega, \omega_1^{1-p'})$ (cf. [56]). We prove that this problem admits a unique weak solution u in $W_0^{1,p}(\Omega, \omega_1, \omega_4)$. The needed result follows also relying on the Browder-Minty theorem and the theory of the weighted Sobolev spaces.

Finally, in Chapter 4, we prove the existence and the uniqueness of weak solution for the problem which has been discussed in the last chapter with the right-hand side term is a measure that decomposes in $L^{p'}(\Omega, \omega_2^{1-p'}) + \prod_{j=1}^{n} L^{p'}(\Omega, \omega_1^{1-p'})$ (cf. [58, 59]) by using the same approach which has been used in the last chapter.

Main results of Part II

The purpose of this part is to study some nonlinear elliptic and parabolic problems in different framework. This part consists of six chapters :

In the first chapter, we present all necessary and relevant ingredients thereafter (defintions, properties, lemmas, theorems ...) about the variable exponent Lebesgue-Sobolev spaces and the topological degree theory.

In the second chapter, we study a Neumann problem with $p(x)$-Laplacian-like operator of the following form

$$\begin{cases} -\mathrm{div}\Big(|\nabla u|^{p(x)-2}\nabla u + \dfrac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big) = \mu|u|^{\alpha(x)-2}u + \lambda f(x,u,\nabla u) & \text{in } \Omega, \\ \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.8}$$

in the setting of the generalized Sobolev spaces $W^{1,p(x)}(\Omega)$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p(\cdot), \alpha(\cdot) \in C_+(\overline{\Omega})$, $\frac{\partial u}{\partial \eta}$ is the exterior normal derivative, μ and λ are two real parameters. Based on the topological degree for a class of demicontinuous operators of generalized (S_+) type, under appropriate assumptions on $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, we obtain a result on the existence of weak solution to the considered problem (cf. [60]).

The third chapter studies an extension of the problem (0.0.8) to a model given in the form

$$\begin{cases} -\Delta^{l}_{p(x)}u + \delta|u|^{\alpha(x)-2}u = \mu g(x,u) + \lambda f(x,u,\nabla u) & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.9}$$

where $\Delta^{l}_{p(x)}$ is the $p(x)$-Laplacian-like operator, δ, μ and λ are three real parameters, $p(\cdot), \alpha(\cdot) \in C_+(\overline{\Omega})$. Under some conditions on the functions $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$, we establish the existence of weak solution for (0.0.9) in variable exponent Sobolev spaces $W_0^{1,p(x)}(\Omega)$ by using also the theory of topological degree and the theory of variable exponent Sobolev spaces (cf. [61]).

In the fourth chapter, we deal with the question of the existence and uniqueness of weak solution for the following Neumann problem involving the $p(x)$-Kirchhoff-Laplacian operator

$$\begin{cases} -M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx\Big)\Big(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - |u|^{p(x)-2}u\Big) = f(x,u,\nabla u) & \text{in } \Omega, \\ \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.0.10}$$

where $\frac{\partial u}{\partial \eta}$ is the exterior normal derivative, $p(x) \in C_+(\overline{\Omega})$, $M(t)$ is a continuous function with $t := \displaystyle\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function. By means of a topological degree of Berkovits for a class of demicontinuous operators of generalized (S_+) type and the theory of the variable exponent Sobolev spaces, under appropriate assumptions on f and M, we obtain a results on the existence and uniqueness of weak solution to the considered problem (cf. [15]). Note that, the problem (0.0.10) is a generalization of the model (0.0.3) introduced by Kirchhoff.

In chapter five, we study the existence of weak solution to a new class of the approximating problems corresponding to a quasilinear elliptic and parabolic equations involving the $(p(x), q(x))$-Laplacian operator, called double phase operator with variable exponents, of the following form

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_{p(x)}u - \Delta_{q(x)}u = \phi(x,t) & \text{in } \Omega_T := \Omega \times (0,T), \\ u(x,t) \ = 0 & \text{on } \partial\Omega_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{0.0.11}$$

and

$$\begin{cases} -\Delta_{p(x)}u - \Delta_{q(x)}u + \omega|u|^{\xi(x)-2}u = \upsilon\,\mathcal{A}(x,u) + \sigma\,\mathcal{B}(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.0.12}$$

where $\phi \in \mathcal{W}^*$ ($\mathcal{W}^*$ denote the dual space of the $\mathcal{W}$, see (5.1.13)), $\mathcal{A} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions that satisfy the assumption of growth, ω, υ

and σ are three real parameters, $T > 0$ is a given final time, and the variables exponents $p, q \in C_+(\overline{\Omega})$ satisfy the assumption (9.0.3). Using the topological degree theory for operators of the type $\mathcal{T} + \mathcal{S}$ (see Subsection 5.2.2), we demonstrate the existence of weak solution $u \in \mathcal{W}$ for (0.0.11), and based on the topological degree theory for a class of demicontinuous operator of generalized (S_+) type (see Subsection 5.2.1), we prove that the Problem (0.0.12) possesses at least one weak solution $u \in W_0^{1,p(x)}(\Omega)$.
In the final chapter, we investigate the parabolic case of (0.0.4), where the partial differential operator $\mathcal{L}$ given by $\mathcal{L}u := -\mathrm{div}\Big(a(x,t,\nabla u) + b(x,t,u,\nabla u)\Big)$ (cf. [69]). The main problem is given in the following form

$$\begin{cases} \frac{\partial u}{\partial t} - \mathrm{div}\ b(x,t,u,\nabla u) = \phi(x,t) + \mathrm{div}\ a(x,t,\nabla u) & \text{in } Q := \Omega \times (0,T), \\ u(x,t) \ = 0 & \text{on } \partial Q, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{0.0.13}$$

where Ω is a bounded open domain of $\mathbb{R}^N$ and $T > 0$. Here ϕ is taken in $L^{p'}(0,T; W^{-1,p'}(\Omega, \omega^{1-p'}))$ the dual space of $L^p(0,T; W_0^{1,p}(\Omega,\omega))$. The problem (0.0.13) aim to present an existence result of weak solution in the space $L^p(0,T; W_0^{1,p}(\Omega,\omega))$ by using a topological degree theory for operators of the type $\mathcal{T} + \mathcal{S}$ (see Subsection 5.2.1).

Part I

Study of some nonlinear degenerate elliptic problems in weighted Sobolev spaces with or without right-hand side measure

Chapter 1

Preliminaries

In this chapter we collect several basic tools on weight functions, weighted Lebesgue-Sobolev spaces and monotone operators, which will be needed throughout this work. The common link between all the results in this chapter is that they are preparatory for the main results, which are contained in the following chapters.

1.1 Weighted Sobolev spaces

The Sobolev spaces $W^{k,p}(\Omega)$ without weights, in general, occur as spaces of solutions for elliptic and parabolic partial difierential equations. For degenerate partial differential equations, where we have equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces [87, 88, 102, 138]. The type of a weight depends on the equation type. This section will be devoted to introduce too the notion of weighted Lebesgue and Sobolev spaces, and some interesting definitions and properties, which are essential to prove some results of existence for weak solutions of the nonlinear elliptic problems studied in this thesis.

1.1.1 Basic results concerning weights

In this section, we review some properties of weights and, in particular, A_p weights, that will be used throughout this thesis. Complete expositions can be found in the monographs by J. Garcia-Cuerva and J. L. Rubio de Prancia [89] and A. Torchinsky [138].

1.1.1.1 General weights

By a weight, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will also be denoted by ω. Thus,

$$\omega(E) = \int_E \omega(x)\,dx \qquad \text{for measurable subset} \quad E \subset \mathbb{R}^n.$$

Definition 1.1.1 *Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $1 < p < \infty$, we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that*

$$||f||_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)\,dx\right)^{\frac{1}{p}} < \infty.$$

We also define wk-$L^1(\Omega, \omega)$, as the set of measurable functions f on Ω satisfying

$$||f||_{wk-L^p(\Omega,\omega)} = \sup_{\lambda>0} \lambda\omega(\{x \in \Omega : |f(x)| > \lambda\}) < \infty.$$

Remark 1.1.2

1. *For $\omega \equiv 1$, we obtain the usual Lebesgue space $L^p(\Omega)$.*
2. *It is a well-known fact that the space $L^p(\Omega, \omega)$ is a Banach space (uniformly convex and hence reflexive if $p > 1$) equipped with the norm $\|\cdot\|_{L^p(\Omega,\omega)}$. We also have that the dual space of $L^p(\Omega, \omega)$, if $p > 1$, is the space $L^{p'}(\Omega, \omega^{1-p'})$ with $\frac{1}{p} + \frac{1}{p'} = 1$.*
3. *If ν is a positive Borel measure on an open set Ω, we shall more generally denote by $L^p(\Omega, \nu)$, $0 < p < \infty$, the set of ν-measurable functions f on Ω for which*

$$\left(\int_\Omega |f(x)|^p d\nu\right)^{\frac{1}{p}} < \infty.$$

We now determine conditions on the weight ω that guarantee that functions in $L^p(\Omega, \omega)$ are locally integrable on Ω.

Proposition 1.1.3 *Let $1 \le p < \infty$ and let ω be a weight such that*

$$\omega^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega) \qquad \text{if} \quad p > 1, \tag{1.1.1}$$

$$\operatorname{ess\,sup}_{x\in B} \frac{1}{\omega(x)} < +\infty \quad \text{if} \quad p = 1, \tag{1.1.2}$$

for every ball $B \subset \Omega$. Then,

$$L^p(\Omega, \omega) \subset L^1_{loc}(\Omega).$$

Proof. Let $1 \le p < \infty$. Suppose that $f \in L^p(\Omega, \omega)$ and let $B \subset \mathbb{R}^n$ be a ball. Then we have the following two cases :

First case : If $p = 1$, then we have

$$\begin{aligned}
\int_B |f(x)|\,dx &= \int_B |f(x)| \frac{\omega(x)}{\omega(x)}\,dx \\
&\le \operatorname{ess\,sup}_{x\in B} \frac{1}{\omega(x)} \int_B |f(x)|\omega(x)\,dx < \infty.
\end{aligned}$$

Hence $f \in L^1_{loc}(\Omega)$.

Second case : If $p > 1$, then we have

$$\begin{aligned}\int_B |f(x)|\,dx &= \int_B |f(x)|(\omega(x))^{\frac{1}{p}}(\omega(x))^{-\frac{1}{p}}\,dx \\ &\leq \left(\int_B |f(x)|(\omega(x))^{\frac{1}{p}}\,dx\right)^{\frac{1}{p}} \left(\int_B \left((\omega(x))^{-\frac{1}{p}}\right)^{p'}\,dx\right)^{\frac{1}{p'}} \\ &= \left(\int_B |f(x)|(\omega(x))^{\frac{1}{p}}\,dx\right)^{\frac{1}{p}} \left(\int_B (\omega(x))^{-\frac{p'}{p}}\,dx\right)^{\frac{1}{p'}}.\end{aligned}$$

Since $p' = \frac{p}{p-1}$, then

$$\int_B |f(x)|\,dx \leq \|f\|_{L^p(\Omega,\omega)} \left(\int_B (\omega(x))^{\frac{-1}{p-1}}\,dx\right)^{\frac{1}{p'}} < \infty.$$

Hence $f \in L^1_{loc}(\Omega)$.

It follows that

$$L^p(\Omega,\omega) \subset L^1_{loc}(\Omega).$$

Remark 1.1.4 1. *As a consequence of Proposition 1.1.3, we have the convergence in $L^p(\Omega,\omega)$ implies local convergence in $L^1(\Omega)$. Moreover, if Ω is bounded, one obtains in the same way that $L^p(\Omega,\omega)$ is continuously embedded in $L^1(\Omega)$.*

2. *Under the assumptions of Proposition 1.1.3, we have $L^p(\Omega,\omega) \subset L^1_{loc}(\Omega)$. Using the usual identification of a regular distribution from $\mathcal{D}'(\Omega)$ with a function from $L^1_{loc}(\Omega)$ we conclude that $L^p(\Omega,\omega) \subset L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$.*
Therefore, every function in $L^p(\Omega,\omega)$ has weak derivatives. It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Note that, in the case $p > 1$, if ω does not satisfy condition (1.1.1) then the injection $L^p(\Omega,\omega) \subset L^1_{loc}(\Omega)$ not hold. This is illustrated by the following example :

Example 1.1.5 *Consider $\Omega = [\frac{-1}{2}, \frac{1}{2}]$ and et $\omega(x) = |x|^{p-1}$ $(p > 1)$.*
We have $(\omega(x))^{\frac{-1}{p-1}} = \left(|x|^{p-1}\right)^{\frac{-1}{p-1}} = |x|^{-1} \notin L^1_{loc}(\Omega)$. Then ω does not satisfy condition (1.1.3).
Now, consider the function f defined by $f(x) = |x|^{-1}|\ln|x||^{\lambda}$ with $\lambda \in [-1, \frac{-1}{p}]$.

We have

$$\begin{aligned}
||f||_{L^p(\Omega,\nu)} &= \int_{\frac{-1}{2}}^{\frac{1}{2}} |x|^{-p}|\ln|x||^{\lambda p}|x|^{p-1}\,dx\\
&= 2\int_0^{\frac{1}{2}} |x|^{-1}|\ln x|^{\lambda p}\,dx\\
&= -2\int_0^{\frac{1}{2}} x^{-1}(\ln x)^{\lambda p}\,dx\\
&= -2\int_{+\infty}^{-\ln(2)} t^{\lambda p}dt\\
&= 2\int_{-\ln(2)}^{+\infty} t^{\lambda p}dt < \infty,\ \text{since } \lambda p < -1.
\end{aligned}$$

Hence $f \in L^p(\Omega,\nu)$.

On another side, we have $f \notin L^1_{loc}(\Omega)$. *In fact, we have*

$$\begin{aligned}
\int_{\frac{-1}{2}}^{\frac{1}{2}} |f(x)|\,dx &= \int_{\frac{-1}{2}}^{\frac{1}{2}} |x|^{-1}|\ln|x||^{\lambda}\,dx\\
&= 2\int_0^{\frac{1}{2}} |x|^{-1}|\ln x|^{\lambda}\,dx\\
&= 2\int_{\frac{1}{2}}^{0} x^{-1}(\ln x)^{\lambda}\,dx\\
&= 2\int_{-\ln(2)}^{+\infty} t^{\lambda}dt = \infty,\ \text{since } \lambda > -1.
\end{aligned}$$

Corollary 1.1.6 *Let* $\varphi \in \mathcal{C}_0^\infty(\Omega)$, ω *be a weight such that* $\omega^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$ *and let a multi-index* $\alpha \in \mathbb{N}^n$ *be fixed. Then the formula*

$$L_\alpha(f) = \int_\Omega f(x)D^\alpha\varphi(x)\,dx \tag{1.1.3}$$

defines a continuous linear functional L_α *on* $L^p(\Omega,\omega)$.

Proof. Let $f \in L^p(\Omega,\omega)$ and $\varphi \in \mathcal{C}_0^\infty(\Omega)$. If we denote $K = supp(\varphi)$, then by Hölder inequality we obtain

$$\begin{aligned}
|L_\alpha(f)| &\le \int_\Omega |f(x)||D^\alpha\varphi(x)|\,dx\\
&= \int_\Omega |f(x)|\,(\omega(x))^{\frac{1}{p}}\,|D^\alpha\varphi(x)|\,(\omega(x))^{\frac{-1}{p}}\,dx\\
&\le ||f||_{L^p(\Omega,\omega)}\left(\int_\Omega |D^\alpha\varphi(x)|^{\frac{p}{p-1}}\,(\omega(x))^{\frac{-1}{p-1}}\,dx\right)^{\frac{p-1}{p}}\\
&\le ||f||_{L^p(\Omega,\omega)}\left(\int_K |D^\alpha\varphi(x)|^{\frac{p}{p-1}}\,(\omega(x))^{\frac{-1}{p-1}}\,dx\right)^{\frac{p-1}{p}}\\
&\le ||f||_{L^p(\Omega,\omega)}\max_{x\in K}|D^\alpha\varphi(x)|\left(\int_K (\omega(x))^{\frac{-1}{p-1}}\,dx\right)^{\frac{p-1}{p}},
\end{aligned}$$

here, the last integral is finite in view of $\omega^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$.
We have the following result.

Theorem 1.1.7 *Let $\omega \in A_p$, $1 \leq p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_n \longrightarrow u$ in $L^p(\Omega, \omega)$, then there exist a subsequence (u_{n_m}) and $\psi \in L^p(\Omega, \omega)$ such that*

(i) $u_{n_m}(x) \longrightarrow u(x)$, $n_m \longrightarrow \infty$, *a.e. on* Ω.

(ii) $|u_{n_m}(x)| \leq \psi(x)$, *a.e. on* Ω.

Proof. The proof of this theorem follows the lines of [88, Theorem 2.8.1].

Theorem 1.1.8 *(Weighted Sobolev embedding theorem)[80, Theorem 1.2] Given $1 < p < \infty$ and $\omega \in A_p$. Then there exist constants C and $\delta > 0$ such that for all balls B_R, all $u \in C_0^\infty(B_R)$, and all numbers k satisfying $1 \leq k \leq \frac{n}{n-1} + \delta$,*

$$\left(\frac{1}{\omega(B_R)}\int_{B_R} |u|^{kp}\omega \, dx\right)^{\frac{1}{kp}} \leq C\,R\left(\frac{1}{\omega(B_R)}\int_{B_R} |\nabla u|^p\omega \, dx\right)^{\frac{1}{p}}.$$

An immediate consequence of Theorem 1.1.8 is the following theorem :

Theorem 1.1.9 *[80, Theorem 1.3] Let Ω be an open bounded set in $\mathbb{R}^n$. Take $1 < p < \infty$ and a function $\omega \in A_p$. Then there exist positive constants C_Ω and δ such that for all $u \in \mathcal{C}_0^\infty(\Omega)$ and all k satisfying $1 \leq k \leq \frac{n}{n-1} + \delta$,*

$$||u||_{L^{kp}(\Omega,\omega)} \leq C_\Omega ||\nabla u||_{L^p(\Omega,\omega)},$$

where C_Ω depends only on n, p, the A_p constant of ω and the diameter of Ω.

We now turn our attention to the weighted Poincaré inequality.

Theorem 1.1.10 *[80, Theorem 1.5] Let $1 < p < \infty$ and $\omega \in A_p$. Then there are positive constants C and δ such that for all Lipschitz continuous functions u defined on $\overline{B_R}$ and for all $1 \leq k \leq n/(n-1) + \delta$,*

$$\left(\frac{1}{\omega(B_R)}\int_{B_R} |u - u_{B_R}|^{kp}\omega \, dx\right)^{1/kp} \leq C\,R\left(\frac{1}{\omega(B_R)}\int_{B_R} |\nabla u|^p\omega \, dx\right)^{1/p}$$

where $u_{B_R} = \dfrac{1}{\omega(B_R)}\displaystyle\int_{B_R} u(x)\omega(x) \, dx$ *or* $u_{B_R} = \dfrac{1}{|B_R|}\displaystyle\int_{B_R} u(x)\, dx.$

1.1.1.2 A_p weights

The class of A_p weights was introduced by B. Muckenhoupt in [113], where he showed that the A_p weights are precisely those weights ω for which the Hardy-Littlewood maximal operator is bounded from $L^p(\mathbb{R}^n, \omega)$ to $L^p(\mathbb{R}^n, \omega)$, when $1 < p < \infty$, and from $L^1(\mathbb{R}^n, \omega)$ to wk-$L^1(\mathbb{R}^n, \omega)$, when $p = 1$. Here, we define the Hardy-Littlewood maximal function, M_f, for a locally integrable function f on $\mathbb{R}^n$ by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy.$$

The corresponding operator, which takes f to M_f, is denoted by M.

We begin by defining the class of A_p weights. These classes have found many useful applications in harmonic analysis [138].

Definition 1.1.11 *Let $1 \leq p < \infty$. A weight ω is said to be an A_p weight, if there exists a positive constant A such that, for every ball $B \subset \mathbb{R}^n$,*

$$\left(\frac{1}{|B|} \int_B \omega(x)\, dx \right) \left(\frac{1}{|B|} \int_B (\omega(x))^{\frac{-1}{p-1}}\, dx \right)^{p-1} \leq A \qquad \textit{if} \quad p > 1, \tag{1.1.4}$$

$$\left(\frac{1}{|B|} \int_B \omega(x)\, dx \right) \operatorname{ess\,sup}_{x \in B} \frac{1}{\omega(x)} \leq A \qquad \textit{if} \quad p = 1. \tag{1.1.5}$$

The infimum over all such constants A is called the A_p constant of ω. We denote by A_p, $1 \leq p < \infty$, the set of all A_p weights.

We will refer to (1.1.4) and (1.1.5) as the A_p and the A_1 condition, respectively. Muckenhoupt's theorem is now the following [113, p. 209-222].

Theorem 1.1.12 *Suppose that $\omega \in A_p$, where $1 < p < \infty$. Then the Hardy-Littlewood maximal operator M is bounded on $L^p(\mathbb{R}^n, \omega)$, that is, there exists a positive constant C such that*

$$\int_{\mathbb{R}^n} (Mf)^p \omega\, dx \leq C \int_{\mathbb{R}^n} |f|^p \omega\, dx, \tag{1.1.6}$$

for every $f \in L^p(\mathbb{R}^n, \omega)$. The constant C depends only on n, p and the A_p constant of ω. If $\omega \in A_1$ then M is bounded from $L^1(\mathbb{R}^n, \omega)$ to wk-$L^1(\mathbb{R}^n, \omega)$. In other words,

$$\omega(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| \omega\, dx, \tag{1.1.7}$$

for every $f \in L^1(\mathbb{R}^n, \omega)$ and every $\lambda > 0$, with a constant C that only depends on n and the A_1 constant of ω. Conversely, if (1.1.6) holds for every $f \in L^p(\mathbb{R}^n, \omega)$, then $\omega \in A_p$, and if (1.1.7) holds for every $f \in L^1(\mathbb{R}^n, \omega)$, then $\omega \in A_1$.

Remark 1.1.13 *Below we list some simple, but useful properties of A_p weights (see [95, 102, 139] for more informations about A_p weights).*

1. *If $\omega \in A_p$, $1 \leq p < \infty$, then since $\omega^{\frac{-1}{p-1}}$ is locally integrable, when $1 < p < \infty$, and $\frac{1}{\omega}$ is locally bounded, when $p = 1$, we have $L^p(\Omega, \omega) \subset L^1_{loc}(\Omega)$ for every domain Ω. Moreover, if A is the A_p constant of ω, then by the A_p condition, the right-hand sides of (1.1.1) and (1.1.2) do not exceed*
$$A^{\frac{1}{p}}|B|\left(\frac{1}{\omega(B)}\int_B |f|^p \omega \, dx\right)^{\frac{1}{p}}.$$

2. *Note that if ω is a weight, then, by writing $1 = \omega^{\frac{1}{p}}\omega^{\frac{-1}{p}}$, Hölder's inequality implies that, for every ball B,*
$$1 \leq \left(\frac{1}{|B|}\int_B \omega \, dx\right)\left(\frac{1}{|B|}\int_B \omega^{\frac{-1}{p-1}} \, dx\right),$$
when $p > 1$, and similarly for the expression that gives the A_1 condition. It follows that if $\omega \in A_p$, then the A_p constant of ω is ≥ 1.

3. *It also follows from Hölder's inequality that if $1 \leq p < q < \infty$, then $A_p \subset A_q$ and the A_q constant of a weight ω equals the A_p constant of ω.*

4. *If $\omega \in A_p$, where $1 < p < \infty$, then $\omega^{\frac{-1}{p-1}} \in A_{p'}$, and conversely. When p is fixed, we shall sometimes denote the weight $\omega^{\frac{-1}{p-1}}$ by ω'.*

5. *The A_p condition is invariant under translations and dilations, i.e., if $\omega \in A_p$, then the weights $x \mapsto \omega(x+a)$ and $x \mapsto \omega(\delta x)$, where $a \in \mathbb{R}^n$ and $\delta > 0$ are fixed, both belong to A_p with the same A_p constants as ω.*

6. *As it sometimes is more convenient to work with cubes than balls, it is useful to notice that if one replaces the balls in the definition of A_p with cubes, one gets the same class of weights and the different "A_p constants" are comparable.*

7. *It is not so difficult to see that a weight ω belongs to A_1 if and only if $M\omega(x) \leq A\omega(x)$ a.e.*

Example 1.1.14 *(Examples of A_p weights)*

1. *If ω is a weight and there exist two positive constants C and D such that $C \leq \omega(x) \leq D$ for a.e. $x \in \mathbb{R}^n$, then obviously $\omega \in A_p$ for $1 \leq p < \infty$.*

2. *Suppose that $\omega(x) = |x|^\eta$, $y \in \mathbb{R}^n$. Then $\omega \in A_p$ if and only if $-n < \eta < n(p-1)$ for $1 \leq p < \infty$ (see [138, p. 229-236]).*

3. *Let Ω be an open subset of $\mathbb{R}^n$. Then $\omega(x) = e^{\lambda v(x)} \in A_2$, with $v \in W^{1,n}(\Omega)$ and λ is sufficiently small (see Corollary 2.18 in [114]).*

4. *There is a connection between A_p and BMO, the class of functions with bounded mean oscillation. In fact, if ω is a weight, then $\log \omega \in BMO$ if and only if $\omega^\eta \in A_2$ for some $\eta > 0$ (see [138, p. 240]).*

1.1.1.3 Doubling weights

We will often use the fact that A_p weights are doubling.

Definition 1.1.15 *A weight ω is said to be doubling, if there exists a positive constant C such that*

$$\omega(2B) \leq C\omega(B), \tag{1.1.8}$$

for every ball $B \subset \mathbb{R}^n$. The infimum over all constants C, for which (1.1.8) holds, is called the doubling constant of ω.

It follows directly from the A_p condition and Hölder inequality that an A_p weight has the following strong doubling property.

Corollary 1.1.16 *(Strong doubling of A_p weight) Let $\omega \in A_p$ with $1 \leq p < \infty$ and let E be a measurable subset of a ball $B \subset \mathbb{R}^n$. Then*

$$\omega(B) \leq A \left(\frac{|B|}{|E|}\right)^p \omega(E),$$

where A is the A_p constant of ω.

Proof. Let $\omega \in A_p$, $B \subset \mathbb{R}^n$ be a ball and E be a measurable subset of B. Then, by Hölder inequality, we obtain

$$\begin{aligned}
|E| = \int_E dx \;\; &\leq \int_E \omega^{\frac{-1}{p}} \omega^{\frac{1}{p}}\, dx \\
&\leq \left(\int_E \omega\, dx\right)^{\frac{1}{p}} \left(\int_E \omega^{\frac{-p'}{p}}\, dx\right)^{\frac{1}{p'}} \\
&= (\omega(E))^{\frac{1}{p}} \left(\int_E \omega^{\frac{-1}{p-1}}\, dx\right)^{\frac{p-1}{p}} \\
&= (\omega(E))^{\frac{1}{p}} |B|^{\frac{p-1}{p}} \left(\frac{1}{|B|}\int_B \omega^{\frac{-1}{p-1}}\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Since $\nu \in A_p$, then

$$\left(\frac{1}{|B|}\int_B \omega^{\frac{-1}{p-1}}\, dx\right)^{p-1} \leq A \left(\frac{1}{|B|}\int_B \omega\, dx\right)^{-1}.$$

Hence

$$\begin{aligned} |E| &\leq (\omega(E))^{\frac{1}{p}} |B|^{\frac{p-1}{p}} A^{\frac{1}{p}} \left(\frac{1}{|B|} \int_B \nu \, dx \right)^{\frac{-1}{p}} \\ &= A^{\frac{1}{p}} (\omega(E))^{\frac{1}{p}} |B|^{\frac{p-1}{p}} \left(\frac{\omega(B)}{|B|} \right)^{\frac{-1}{p}} \\ &= A^{\frac{1}{p}} (\omega(E))^{\frac{1}{p}} (\omega(B))^{\frac{-1}{p}} |B|, \end{aligned}$$

and consequently

$$\omega(B) \leq A \left(\frac{|B|}{|E|} \right)^p \omega(E).$$

Remark 1.1.17 *1. If $\omega \in A_p$, then ω is doubling (see [43, Corollary 15.7]).*

2. *If $\omega(E) = 0$ then $|E| = 0$. The measure ω and the Lebesgue measure $|\cdot|$ are mutually absolutely continuous, that is they have the same zero sets $\big(\omega(E) = 0$ if and only if $|E| = 0\big)$; so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..*

Lemma 1.1.18 *[43] If $\omega \in A_p$, then there are $0 < q < 1$ and $C > 0$, depending only on n, p, and A, such that*

$$\frac{\omega(E)}{\omega(B)} \leq C \left(\frac{|E|}{|B|} \right)^q,$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B.

We have the following reverse Hölder inequality.

Lemma 1.1.19 *[43] If $\omega \in A_p$, then there are numbers $r > 1$ and $C_r \geq 1$, depending only on n, p, and A, such that*

$$\left(\frac{1}{|B|} \int_B \omega^r \, dx \right)^{1/r} \leq C_r \left(\frac{1}{|B|} \int_B \omega \; dx \right),$$

for all balls B.

We have also the following open-end property of A_p.

Lemma 1.1.20 *[43] Suppose that $\omega \in A_p$ for some p, $1 < p < \infty$. Then there exists a number q, $1 < q < p$, such that $\omega \in A_q$.*

1.1.1.4 A_∞ weights

Another important class of weights is the class of A_∞ weights, introduced by C. Fefferman. The following definition of A_∞, just one of several equivalent ones, suits our purposes best.

Definition 1.1.21 *We say that a weight ω is an A_∞ weight, if there exist two positive constants C and δ such that*

$$\omega(Q) \geq C\left(\frac{|Q|}{|E|}\right)^{\delta}\omega(E)$$

for every cube Q and every measurable subset E of Q. The constants C and δ are called A_∞ constants of ω and the set of A_∞ weights is (of course) denoted A_∞.

The relationship between A_p and A_∞ is clarified by the two theorems below, due to Muckenhoupt [113, p. 214] and [114, p. 104]. Together they show that

$$A_\infty = \bigcup_{1\leq p<\infty} A_p.$$

Theorem 1.1.22 *If $\omega \in A_p$, $1 \leq p < \infty$, then $\omega \in A_\infty$ with A_∞ constants of ω that only depend on n and the A_p constant of ω.*

Theorem 1.1.23 *If $\omega \in A_\infty$, then $\omega \in A_p$ for some p, $1 < p < \infty$, and the A_p constant of ω is majorized by a constant that only depends on n and the A_∞ constants of ω.*

Remark 1.1.24 *A consequence of Theorem 1.1.22 and the defining condition for A_∞ is the fact that $\int_{\mathbb{R}^n}\omega\,dx = \infty$ for every weight $\omega \in A_p$.*

1.1.1.5 p-admissible weights

Let ω be a weight and $1 < p < \infty$. We say that ω is p-admissible if the following four conditions are satisfied :

(I) $0 < \omega(x) < \infty$ a.e. $x \in \mathbb{R}^n$ and ω is doubling.

(II) If Ω is an open set and $\varphi_k \in C^\infty(\Omega)$ is a sequence of functions such that $\int_\Omega |\varphi_k|^p\omega\,dx \to 0$ and $\int_\Omega |\nabla\varphi_k - \vartheta|^p\omega\,dx \to 0$ as $k \to \infty$, then $\vartheta = 0$.

(III) There are constants $k > 1$ and $C_{III} > 0$ such that

$$\left(\frac{1}{\omega(B)}\int_B |\varphi|^{kp}\omega\,dx\right)^{1/kp} \leq C_{III}R\left(\frac{1}{\omega(B)}\int_B |\nabla\varphi|^p\omega\,dx\right)^{1/p},$$

whenever $B = B(x_0, R)$ is a ball in $\mathbb{R}^n$ and $\varphi \in C_0^\infty(B)$.

(IV) There is a constant $C_{IV} > 0$ such that

$$\int_B |\varphi - \varphi_B|^p \omega \, dx \leq C_{IV} R^p \int_B |\nabla \varphi|^p \omega \, dx,$$

whenever $B = B(x_0, R)$ is a ball in $\mathbb{R}^n$ and $\varphi \in C^\infty(B)$ is bounded. Here

$$\varphi_B = \frac{1}{\omega(B)} \int_B \varphi \omega \, dx.$$

Let us make some remarks on conditions (I)-(IV). It follows immediately from condition (I) that the measure ω and Lebesgue measure dx are mutually absolutely continuous. Moreover, it easily follows from the doubling property that $\omega(\mathbb{R}^n) = \infty$.

Condition (II) guarantees that the gradient of a Sobolev function is well defined, a conclusion that cannot be expected in general (Fabes et al. [80, pp. 91-92]).

Condition (III) is the weighted Sobolev embedding theorem or the weighted Sobolev inequality and condition (IV) is the weighted Poincaré inequality.

Example 1.1.25 *(Examples of* p*-admissible weights)*

1. *If* $\omega \in A_p (1 < p < \infty)$ *then* ω *is a* p*-admissible weight.*

2. $\omega(x) = |x|^\alpha,\ x \in \mathbb{R}^n,\ \alpha > -n$, *is a* p*-admissible weight for all* $p > 1$.

3. *If* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a* K *-quasiconformal mapping and* $J_f(x)$ *is the determinant of its jacobian matrix, then* $\omega(x) = J_f(x)^{1-p/n}$ *is* p*-admissible for* $1 < p < n$.

4. *See [32] for non-*A_p *examples of* p*-admissible weights.*

Remark 1.1.26 *P. Hajlasz and P. Koskela [93] showed that conditions (I)-(IV) can be reduced to only two :* ω *is a* p*-admissible weights* $(1 < p < \infty)$ *if and only if* ω *is doubling and there are constants* $C > 0$ *and* $\lambda \geq 1$ *such that*

$$\frac{1}{\omega(B)} \int_B |\varphi - \varphi_B| \omega \, dx \leq C\, R \Big(\frac{1}{\omega(\lambda B)} \int_{\lambda B} |\nabla \varphi|^p \omega \, dx \Big)^{1/p}.$$

Theorem 1.1.27 *Suppose that* ω *is a* p*-admissible weight and* $q > p$*. Then* ω *is* q*-admissible.*

Proof. The steps of the proof follow along the exact lines of [95, Theorem 1.8].

1.1.2 Weighted Sobolev spaces

This subsection explores weighted Sobolev spaces and some there properties. We will consider two types of weighted Sobolev spaces, namely the spaces $W^{1,p}(\Omega,\omega)$ and $W^{1,p}(\Omega,\omega,\nu)$ where $\omega,\nu\in A_p$.

We begin by defining the weighted Sobolev space $W^{1,p}(\Omega,\omega)$. Recall that if $\omega\in A_p$, then $L^p(\Omega,\omega)\subset L^1_{loc}(\Omega)\subset \mathcal{D}'(\Omega)$ for every open set Ω (see Remark 1.1.4). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 1.1.28 *Let $\Omega\subset\mathbb{R}^n$ be a bounded open, $1\le p<\infty$, k be a nonnegative integer and $\omega\in A_p$. We define the weighted Sobolev space $W^{k,p}(\Omega,\omega)$ as the set of functions $u\in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u\in L^p(\Omega,\omega)$ for $|\alpha|\le k$. The norm of u in $W^{1,p}(\Omega,\omega)$ is given by*

$$\|u\|_{W^{k,p}(\Omega,\omega)}=\left(\int_\Omega |u|^p\omega(x)\,dx+\sum_{1\le|\alpha|\le k}\int_\Omega |D^\alpha u|^p\omega(x)\,dx\right)^{\frac{1}{p}}. \tag{1.1.9}$$

We also define $W^{k,p}_0(\Omega,\omega)$ as the closure of $\mathcal{C}^\infty_0(\Omega)$ in $W^{k,p}(\Omega,\omega)$ with respect to the norm (1.1.9). The norm of u in $W^{1,p}_0(\Omega,\omega)$ is given by

$$\|u\|_{W^{1,p}_0(\Omega,\omega)}=\left(\sum_{1\le|\alpha|\le k}\int_\Omega |D^\alpha u|^p\omega(x)\,dx\right)^{\frac{1}{p}}. \tag{1.1.10}$$

Remark 1.1.29
1. *When $\omega=1$, the spaces $W^{1,p}(\Omega,\omega)$ and $W^{1,p}_0(\Omega,\omega)$ will be denoted $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$, respectively.*
2. *The space $W^{1,p}_0(\Omega,\omega)$ is a closed subspace of the space $W^{1,p}(\Omega,\omega)$.*
3. *The dual of space $W^{1,p}_0(\Omega,\omega)$ is the space $\left[W^{1,p}_0(\Omega,\omega)\right]^*=W^{-1,p'}_0(\Omega,\omega^{1-p'})$ given by*

$$W^{-1,p'}_0(\Omega,\omega^{1-p'})=\left\{T=f_0-\operatorname{div}(F):F=(f_1,...,f_n),\ \frac{f_j}{\omega}\in L^{p'}(\Omega,\omega),\ j=0,...,n\right\}.$$

Theorem 1.1.30 *The spaces $(W^{1,p}(\Omega,\omega),\|\cdot\|_{W^{1,p}(\Omega,\omega)})$ and $(W^{1,p}_0(\Omega,\omega),\|\cdot\|_{W^{1,p}(\Omega,\omega)})$ are reflevixe Banach spaces.*

Proof. This theorem is proved exactly the same way as in the case $\omega=1$. Using the completeness of $L^p(\Omega,\omega)$ and the fact that $L^p(\Omega,\omega)\subset L^1_{loc}(\Omega)$ when $\omega\in A_p$ (see [139, Proposition 2.1.2] and [103, p. 540-541]).

Remark 1.1.31 *It is evident that a weight function ω which satisfies $0<c_1\le\omega\le c_2$ for $x\in\Omega$ (where c_1 and c_2 are constants), give nothing new (the space $W^{1,p}_0(\Omega,\omega)$ is then identical with the*

classical Sobolev space $W_0^{1,p}(\Omega)$ since $W_0^{1,p}(\Omega,\omega)$ is isomorphic to the Sobolev space $W_0^{1,p}(\Omega)$). Consequently, we shall interested above all in such weight functions ω which either vanish somewhere in $\Omega\cup\partial\Omega$ or is not bounded (or both).

Another useful consequence of the inclusions $L^p(\Omega,\omega)\subset L^1_{loc}(\Omega)$ for general Ω and $L^p(\Omega,\omega)\subset L^1(\Omega)$ for bounded Ω is the next proposition.

Proposition 1.1.32 *Let $\Omega\subset\mathbb{R}^n$ be open, $1\leq p<\infty$, and k a nonnegative integer. Suppose that $\omega\in A_p$. Then*

$$W^{k,p}(\Omega,\omega)\subset W^{k,1}_{loc}(\Omega),$$

and, if Ω is hounded,

$$W^{k,p}(\Omega,\omega)\subset W^{k,1}(\Omega).$$

Here, $W^{k,1}_{loc}(\Omega)$ denotes the set of functions $u\in L^1_{loc}(\Omega)$ with weak derivatives $D^\alpha u\in L^1_{loc}(\Omega)$ for $|\alpha|\leq k$.

Proof. See [139, Proposition 2.1.3].

Remark 1.1.33 *If $\Omega\subset\mathbb{R}^n$ be open, $m\geq 1$, $1\leq p<\infty$, and $\omega\in A_p$, then $\mathcal{C}^\infty(\Omega)$ is dense in $W^{k,p}(\Omega,\omega)$ (see [139, Corollary 2.1.6]).*

Proposition 1.1.34 *[139, Proposition 2.1.7] Let $\Omega\subset\mathbb{R}^n$ be open, $1\leq p<\infty$, $\omega\in A_p$ and $u\in W^{1,p}(\Omega,\omega)$. Suppose that $F\in\mathcal{C}^1(\mathbb{R})$ with $F'\in L^\infty(\mathbb{R})$. If $F\circ u\in L^p(\Omega,\omega)$, then $F\circ u\in W^{1,p}(\Omega,\omega)$ with $D_i(F\circ u)=F'(u)D_iu,\ \ i=1,...,n$.*

Corollary 1.1.35 *[139, Corollary 2.1.8] Suppose that $\Omega\subset\mathbb{R}^n$ is open, and let $u\in W^{1,p}(\Omega,\omega)$, where $1\leq p<\infty$ and $\omega\in A_p$. Set $u^+=\max\{u,0\}$ and $u^-=\min\{u,0\}$. Then u^+, u^- and $|u|$ belongs to $W^{1,p}(\Omega,\omega)$, and, for $i=1,...,n$,*

$$D_iu^+=\begin{cases}D_iu & \text{if } u>0,\\ 0 & \text{if } u\leq 0,\end{cases}\quad ,\ D_iu^-=\begin{cases}D_iu & \text{if } u<0,\\ 0 & \text{if } u\geq 0,\end{cases}\quad \text{and } D_i|u|=\begin{cases}D_iu & \text{if } u>0,\\ 0 & \text{if } u=0,\\ -D_iu & \text{if } u<0.\end{cases}$$

We now turn our attention to the weighted Sobolev inequality.

Theorem 1.1.36 *Let Ω be an open bounded set in $\mathbb{R}^n$. Take $1<p<\infty$ and $\omega\in A_p$. Then there exist positive constants C_Ω and δ such that for all $u\in W_0^{1,p}(\Omega,\omega)$ and all k satisfying $1\leq k\leq \frac{n}{n-1}+\delta$,*

$$||u||_{L^{kp}(\Omega,\omega)}\leq C_\Omega||\nabla u||_{L^p(\Omega,\omega)},\tag{1.1.11}$$

where C_Ω depends only on n, p, the A_p constant of ω and the diameter of Ω.

Proof. Since $W_0^{1,p}(\Omega,\omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{W^{1,p}(\Omega,\omega)}$, then to show the result its suffices to prove the inequality for functions $u \in \mathcal{C}_0^\infty(\Omega)$ (see Theorem 1.1.9). To extend the estimates (1.1.11) to arbitrary $u \in W_0^{1,p}(\Omega,\omega)$, we let (u_m) be a sequence of $\mathcal{C}_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega,\omega)$. Applying the estimates (1.1.11) to differences $u_p - u_q$, we see that (u_m) will be a Cauchy sequence in $L^p(\Omega,\omega)$ (which is a Banach space). Consequently (u_m) converges to u in $L^p(\Omega,\omega)$, moreover $u \in L^p(\Omega,\omega)$ and u satisfy (1.1.11).

The weighted Sobolev space $W^{1,p}(\Omega,\omega,\nu)$ is defined as follows.

Definition 1.1.37 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open, $1 \leq p < \infty$, k be a nonnegative integer and $\omega, \nu \in A_p$. We define the weighted Sobolev space $W^{k,p}(\Omega,\omega,\nu)$ as the set of functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\nu)$ for $|\alpha| \leq k$. The norm of u in $W^{1,p}(\Omega,\omega,\nu)$ is given by*

$$||u||_{W^{k,p}(\Omega,\omega,\nu)} = \left(\int_\Omega |u|^p \omega(x)\, dx + \sum_{1\leq|\alpha|\leq k} \int_\Omega |D^\alpha u|^p \nu(x)\, dx \right)^{\frac{1}{p}}. \tag{1.1.12}$$

We also define $W_0^{k,p}(\Omega,\omega,\nu)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{k,p}(\Omega,\omega,\nu)$ with respect to the norm (1.1.12).

Equipped by the norm (1.1.12), the spaces $W^{1,p}(\Omega,\omega,\nu)$ and $W_0^{1,p}(\Omega,\omega,\nu)$ are separable and reflexive Banach spaces (see [103, Proposition 2.1.2.] and see [102, 113] for more informations about the spaces $W^{1,p}(\Omega,\omega,\nu)$). The dual of space $W_0^{1,p}(\Omega,\omega,\nu)$ is the space defined by

$$\left[W_0^{1,p}(\Omega,\omega,\nu)\right]^* = \left\{ f - \sum_{i=1}^n D_i f_i : \frac{f}{\omega} \in L^{p'}(\Omega,\omega), \frac{f_i}{\nu} \in L^{p'}(\Omega,\nu), i = 1, ..., n \right\}.$$

Remark 1.1.38 *Let $\omega, \nu \in A_p$. Then,*

(i) *If $\omega = \nu$, then $\mathcal{C}_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega,\omega) = W_0^{1,p}(\Omega,\omega,\omega)$.*

(ii) *If $\varphi \in W_0^{1,p}(\Omega,\omega,\nu)$, then by Theorem 1.1.36 (with $k = 1$), it holds that*

$$||\varphi||_{L^p(\Omega,\omega)} \leq C_\Omega ||\nabla\varphi||_{L^p(\Omega,\omega)} \leq C_\Omega ||\varphi||_{W_0^{1,p}(\Omega,\omega,\nu)}.$$

Hence, $W_0^{1,p}(\Omega,\omega,\nu) \subset W_0^{1,p}(\Omega,\omega)$.

(iii) *If $\nu \leq \omega$, then $W_0^{1,p}(\Omega,\omega) \subset W_0^{1,p}(\Omega,\omega,\nu) \subset W_0^{1,p}(\Omega,\nu)$.*

In this thesis, we consider also the following space

$$\mathcal{X} := L^p(0,T; W_0^{1,p}(\Omega,\omega)), \text{ and } T > 0.$$

In this space, we defined the norm

$$|u|_{\mathcal{X}} = \Big(\int_0^T \|u\|^p_{W^{1,p}(\Omega,\omega)} dt \Big)^{1/p}.$$

Thanks to Poincaré inequality, the expression

$$\|u\|_{\mathcal{X}} = \Big(\int_0^T \|u\|^p_{W_0^{1,p}(\Omega,\omega)} dt \Big)^{1/p},$$

is a norm defined on $\mathcal{X}$ and is equivalent to the norm $|\cdot|_{\mathcal{X}}$.
Note that $\Big(\mathcal{X}, \|\cdot\|_{\mathcal{X}}\Big)$ is a separable and reflexive Banach space.

1.2 Basic results for monotone operators

In this section, we present some basic definitions and results on monotone operators.

Definition 1.2.1 *Let X be a Banach space and let $A : X \longrightarrow X^*$ be an operator where X^* denotes the dual space of X. Then :*

1. *A is called monotone iff*

$$\langle Au - Av, u - v \rangle \geqslant 0$$

for all $u, v \in X$ where $\langle f, u \rangle$ denotes the value of the linear functional $f \in X^*$ at point $u \in X$.

2. *A is called strictly monotone iff*

$$\langle Au - Av, u - v \rangle > 0 \ \textit{ for all } \ u, v \in X \ \textit{ with } \ u \neq v.$$

3. *A is called strongly monotone iff there is a $C > 0$ telle que*

$$\langle Au - Av, u - v \rangle \geqslant C\|u - v\|^2; \quad \textit{for all} \quad u, v \in X.$$

4. *A is called coercive iff*

$$\lim_{\|u\| \longrightarrow \infty} \frac{\langle Au, u \rangle}{\|u\|} = +\infty.$$

5. *A is said to be hemicontinuous iff the real function*

$$t \longmapsto \langle A(u + tv), w \rangle$$

is continuous on $[0, 1]$ for all $u, v, w \in X$.

6. *A is said to be strongly continuous iff*

$$u_n \rightharpoonup u \Rightarrow Au_n \longrightarrow Au.$$

Remark 1.2.2 *Obviously, we have the following implications :*

$$A \text{ is strongly monotone } \Rightarrow A \text{ is strictly monotone } \Rightarrow A \text{ is monotone }.$$

For more informations about monotone, coercive and hemicontinuous operators see [143].

Proposition 1.2.3 *Let $A : X \longrightarrow X^*$ be an operator on the real Banach space X. We set*

$$f(t) = \langle A(u + tv), v\rangle \quad \text{for all} \quad t \in \mathbb{R}.$$

Then the following statements are equivalent.

(a) The operator A is monotone.

(b) The function $f : [0, 1] \longrightarrow \mathbb{R}$ is monotone increasing for all $u, v \in X$.

Proof. If A is monotone, then for $0 \leqslant s < t \leqslant 1$, we have

$$f(t) - f(s) = (t - s)^{-1}\langle A(u + tv) - A(u + sv), (t - s)v\rangle \geqslant 0.$$

Hence f is monotone increasing on $[0, 1]$.
Conversely, if f is monotone increasing on $[0, 1]$, then for all $u, v \in X$, we have

$$\langle A(u + v) - Au, v\rangle = f(1) - f(0) \geqslant 0.$$

Hence A is monotone.

Now we will state the main theorem on monotone operators.

Theorem 1.2.4 ***(Browder(1963), Minty(1963))*** *Let $A : X \longrightarrow X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then the following assertions hold:*

(a) For each $T \in X^*$, the equation $Au = T$ has a solution $u \in X$.

(b) If the operator A is strictly monotone, then equation $Au = T$ has a unique solution $u \in X$ for all $T \in X^*$.

Proof. See [143, Theorem 26.A].

1.3 Some important technical propositions and lemmas

In this section, we introduce some technical propositions lemmas that are used in this thesis.

Proposition 1.3.1 *[43] Let* $1 < p < \infty$.

(i) *There exists a positive constant* C_p *such that for all* $\eta, \xi \in \mathbb{R}^n$*, we have*

$$\left||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right| \leq C_p|\xi - \eta|\Big(|\xi| + |\eta|\Big)^{p-2}.$$

(ii) *There exist two positive constants* β_p *and* γ_p *such that for every* $x, y \in \mathbb{R}^n$*, it holds that*

$$\beta_p\Big(|x| + |y|\Big)^{p-2}|x-y|^2 \leq \Big\langle |x|^{p-2}x - |y|^{p-2}y, x-y\Big\rangle \leq \gamma_p\Big(|x| + |y|\Big)^{p-2}|x-y|^2.$$

Proposition 1.3.2 *[142](Principle of convergence in Banach spaces). A sequence* (x_n) *in a Banach space* X *has the following convergence properties.*

(1) *Strong convergence. Let* x *be a fixed element of* X*. If every subsequence of* (x_n) *has, in turn, a subsequence which converges strongly to* x*, then the original sequence converges strongly to* x*.*

(2) *Weak convergence. Let* x *be a fixed element of* X*. If every subsequence of* (x_n) *has, in turn, a subsequence which converges weakly to* x*, then the original sequence converges weakly to* x*.*

Lemma 1.3.3 *[13] Let* $1 < p < \infty$, $(f_n)_n \subset L^p(\Omega, \omega)$ *and* $f \in L^p(\Omega, \omega)$ *such that* $\|f_n\|_{L^p(\Omega,\omega)} \leq C$*. If* $f_n(y) \to f(y)$ *a.e. in* Ω*, then* $f_n \rightharpoonup f$ *in* $L^p(\Omega, \omega)$*, where* ω *is a weight function on* Ω*.*

Chapter 2

On the existence and uniqueness of solution for a class of nonlinear degenerate elliptic problems via Browder-Minty theorem

The purpose of this chapter is to investigate the existence and uniqueness of weak solution for a class of nonlinear degenerate elliptic problem, under Dirichlet condition, of the form :

$$-\operatorname{div}\Big[\nu_1 a(y,\nabla\varphi) + \nu_2 b(y,\varphi,\nabla\varphi)\Big] + \nu_3 g(y,\varphi) = \phi(y), \tag{2.0.1}$$

where Ω is a bounded open set in $\mathbb{R}^N$, ν_1, ν_2 and ν_3 are A_p-weight functions, and the functions $b : \Omega\times\mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$, $a : \Omega\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ and $g : \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions that satisfy some assumptions with the right-hand side term $\phi \in L^{p'}(\Omega, \nu_1^{1-p'})$.

2.1 Hypotheses and the concept of weak solution

2.1.1 Hypotheses

Now let us present the hypothesis on the problem (2.0.1). Assuming that the following assumptions are true: $\Omega \subset \mathbb{R}^N (N \geq 2)$, ν_1, ν_2 and ν_3 are A_p-weight functions, $a_m : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $b_m : \Omega\times\mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}$ $(m = 1, ..., N)$, with $a(y,\delta) = \Big(a_1(y,\delta), ..., a_N(y,\delta)\Big)$ and $b(y,\mu,\delta) = \Big(b_1(y,\mu,\delta), ..., b_N(y,\mu,\delta)\Big)$ and $g : \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ such that

(H1) a_m, b_m and g are Carathéodory functions.

(H2) There are h_1, h_2, h_3, $h_4 \in L^\infty(\Omega)$ and $f_1 \in L^{p'}(\Omega,\nu_1)$, $f_2 \in L^{q'}(\Omega,\nu_2)$ and $f_3 \in L^{s'}(\Omega,\nu_3)$ such that

$$|a(y,\delta)| \leq f_1(y) + h_1(y)|\delta|^{p-1},$$

$$|b(y,\mu,\delta)| \leq f_2(y) + h_2(y)|\mu|^{q-1} + h_3(y)|\delta|^{q-1},$$

$$|g(y,\mu)| \leq f_3(y) + h_4(y)|\mu|^{s-1},$$

where $(\mu, \delta) \in \mathbb{R} \times \mathbb{R}^n$.

(H3) $\exists\, \lambda > 0$ such that

$$\langle a(y,\delta) - a(y,\delta'), \delta - \delta' \rangle \geqslant \lambda |\delta - \delta'|^p,$$

$$\langle b(y,\mu,\delta) - b(y,\mu',\delta'), \delta - \delta' \rangle \geqslant 0,$$

$$\Big(g(y,\mu) - g(y,\mu')\Big)\Big(\mu - \mu'\Big) \geqslant 0,$$

where $\mu, \mu' \in \mathbb{R}$ and $\delta, \delta' \in \mathbb{R}^n$ with $\mu \neq \mu'$ and $\delta \neq \delta'$.

(H4) $\exists\, \kappa_1,\ \kappa_2,\ \kappa_3 > 0$ such that

$$\langle a(y,\delta), \delta \rangle \geqslant \kappa_1 |\delta|^p,$$

$$\langle b(y,\mu,\delta), \delta \rangle \geqslant \kappa_2 |\delta|^q + \kappa_3 |\mu|^q,$$

$$g(y,\mu)\mu \geqslant 0.$$

2.1.2 The concept of weak solution

The definition of weak solution to (2.0.1) is stated as follows:

Definition 2.1.1 *We say that a function $\varphi \in W_0^{1,p}(\Omega, \nu_1)$ is a weak solution of (2.0.1), if for any $\nu \in W_0^{1,p}(\Omega, \nu_1)$, it satisfies the following:*

$$\int_\Omega \langle a(y, \nabla\varphi), \nabla \nu \rangle \nu_1 dy + \int_\Omega \langle b(y, \varphi, \nabla\varphi), \nabla \nu \rangle \nu_2 dy + \int_\Omega g(y,\varphi) \nu \nu_3 dy = \int_\Omega \phi \nu dy.$$

Remark 2.1.2 *For all $\nu_1,\ \nu_2,\ \nu_3 \in A_p$ we have*

(i) *If $1 < q < p < \infty$ and $\frac{\nu_2}{\nu_1} \in L^{k_1}(\Omega, \nu_1)$ where $k_1 = \frac{p}{p-q}$, then*

$$\|\varphi\|_{L^q(\Omega,\nu_2)} \leqslant \vartheta_{p,q} \|\varphi\|_{L^p(\Omega,\nu_1)},$$

where $\vartheta_{p,q} = \|\frac{\nu_2}{\nu_1}\|^{1/q}_{L^{k_1}(\Omega,\nu_1)}$.

(ii) *Analogously, if $1 < s < p < \infty$ and $\frac{\nu_3}{\nu_1} \in L^{k_2}(\Omega, \nu_1)$ where $k_2 = \frac{p}{p-s}$, then*

$$\|\varphi\|_{L^s(\Omega,\nu_3)} \leqslant \vartheta_{p,s} \|\varphi\|_{L^p(\Omega,\nu_1)},$$

where $\vartheta_{p,s} = \|\frac{\nu_3}{\nu_1}\|^{1/s}_{L^{k_2}(\Omega,\nu_1)}$.

2.2 Main result

Our main result of this chapter can be stated as follows.

Theorem 2.2.1 *If **(H1)**—**(H4)** hold, then the problem* (2.0.1) *admits a unique solution* u *in* $W_0^{1,p}(\Omega, \nu_1)$.

Proof. We will reduce the problem (2.0.1) to a new one governed by an operator problem $\Psi\varphi = \Upsilon$, and we will apply the Theorem 1.2.4.
We define

$$\Phi : W_0^{1,p}(\Omega, \nu_1) \times W_0^{1,p}(\Omega, \nu_1) \longrightarrow \mathbb{R}$$

and

$$\Upsilon : W_0^{1,p}(\Omega, \nu_1) \longrightarrow \mathbb{R},$$

with Φ and Υ are specified in the following paragraphs. Hence

$$\varphi \in W_0^{1,p}(\Omega, \nu_1) \text{ is a weak solution of } (2.0.1) \Leftrightarrow \Phi(\varphi, v) = \Upsilon(v), \qquad \text{for all } v \in W_0^{1,p}(\Omega, \nu_1).$$

The Theorem 2.2.1 is proved in four steps.
Step 1.
We utilize some tools and the condition **(H2)** to show the existence of the operator Ψ and that the problem (2.0.1) is identical to the operator equation $\Psi\varphi = \Upsilon$. By employing the Hölder inequality and Theorem 1.1.36, we get

$$\begin{aligned}
|\Upsilon(\varphi)| &\leq \int_\Omega \frac{|\phi|}{\nu_1} |\varphi| \nu_1 \, dy \\
&\leq \|\phi/\nu_1\|_{L^{p'}(\Omega,\nu_1)} \|\varphi\|_{L^p(\Omega,\nu_1)} \\
&\leq C_\Omega \|\phi/\nu_1\|_{L^{p'}(\Omega,\nu_1)} \|\varphi\|_{W_0^{1,p}(\Omega,\nu_1)}.
\end{aligned}$$

Since $\phi \in L^{p'}(\Omega, \nu_1^{1-p'})$, then $\Upsilon \in W_0^{-1,p'}(\Omega, \nu_1^{1-p'})$.
The operator Φ can be written as

$$\Phi(\varphi, v) = \Phi_1(\varphi, v) + \Phi_2(\varphi, v) + \Phi_3(\varphi, v),$$

where

$$\begin{aligned}
&\Phi_1 : W_0^{1,p}(\Omega, \nu_1) \times W_0^{1,p}(\Omega, \nu_1) \longrightarrow \mathbb{R} \\
&\Phi_1(\varphi, v) = \int_\Omega \langle a(y, \nabla\varphi), \nabla v \rangle \nu_1 dy, \\
&\Phi_2 : W_0^{1,p}(\Omega, \nu_1) \times W_0^{1,p}(\Omega, \nu_1) \longrightarrow \mathbb{R} \\
&\Phi_2(\varphi, v) = \int_\Omega \langle b(y, \varphi, \nabla\varphi), \nabla v \rangle \nu_2 dy, \\
&\Phi_3 : W_0^{1,p}(\Omega, \nu_1) \times W_0^{1,p}(\Omega, \nu_1) \longrightarrow \mathbb{R} \\
&\Phi_3(\varphi, v) = \int_\Omega g(y, \varphi) v \nu_3 dy.
\end{aligned}$$

Then, we have

$$|\Phi(\varphi, v)| \quad \leq \quad |\Phi_1(\varphi, v)| + |\Phi_2(\varphi, v)| + |\Phi_3(\varphi, v)|. \tag{2.2.1}$$

Also, by utilizing Hölder inequality, Remark 2.1.2 (i), **(H2)** and Theorem 1.1.36, we have

$$\begin{aligned}
|\Phi_1(\varphi, v)| &\leq \int_\Omega |a(y, \nabla\varphi)||\nabla v|\nu_1 dy \\
&\leq \int_\Omega \left(f_1 + h_1|\nabla\varphi|^{p-1}\right)|\nabla v|\nu_1 dy \\
&\leq \int_\Omega f_1 \nu_1^{\frac{1}{p'}}|\nabla v|\nu_1^{\frac{1}{p}} dy + \int_\Omega h_1|\nabla\varphi|^{p-1}\nu_1^{\frac{1}{p'}}|\nabla v|\nu_1^{\frac{1}{p}} dy \\
&\leq \|f_1\|_{L^{p'}(\Omega,\nu_1)}\|\nabla v\|_{L^p(\Omega,\nu_1)} + \|h_1\|_{L^\infty(\Omega)}\|\nabla\varphi\|^{p-1}_{L^p(\Omega,\nu_1)}\|\nabla v\|_{L^p(\Omega,\nu_1)} \\
&\leq \left(\|f_1\|_{L^{p'}(\Omega,\nu_1)} + \|h_1\|_{L^\infty(\Omega)}\|\varphi\|^{p-1}_{W_0^{1,p}(\Omega,\nu_1)}\right)\|v\|_{W_0^{1,p}(\Omega,\nu_1)},
\end{aligned}$$

and

$$\begin{aligned}
|\Phi_2(\varphi, v)| &\leq \int_\Omega |b(y, \varphi, \nabla\varphi)||\nabla v|\nu_2 dy \\
&\leq \int_\Omega \left(f_2 + h_2|\varphi|^{q-1} + h_3|\nabla\varphi|^{q-1}\right)|\nabla v|\nu_2 dy \\
&\leq \int_\Omega f_2\nu_2^{\frac{1}{q'}}|\nabla v|\nu_2^{\frac{1}{q}} dy + \int_\Omega h_2|\varphi|^{q-1}\nu_2^{\frac{1}{q'}}|\nabla v|\nu_2^{\frac{1}{q}} dy + \int_\Omega h_3|\nabla\varphi|^{q-1}\nu_2^{\frac{1}{q'}}|\nabla v|\nu_2^{\frac{1}{q}} dy \\
&\qquad + \|h_3\|_{L^\infty(\Omega)}\int_\Omega |\nabla\varphi|^{q-1}\nu_2^{\frac{1}{q'}}|\nabla v|\nu_2^{\frac{1}{q}} dy \\
&\leq \|f_2\|_{L^{q'}(\Omega,\nu_2)}\|\nabla v\|_{L^q(\Omega,\nu_2)} + \|h_2\|_{L^\infty(\Omega)}\|\varphi\|^{q-1}_{L^q(\Omega,\nu_2)}\|\nabla v\|_{L^q(\Omega,\nu_2)} \\
&\qquad + \|h_3\|_{L^\infty(\Omega)}\|\nabla\varphi\|^{q-1}_{L^q(\Omega,\nu_2)}\|\nabla v\|_{L^q(\Omega,\nu_2)} \\
&\leq \|f_2\|_{L^{q'}(\Omega,\nu_2)}\vartheta_{p,q}\|\nabla v\|_{L^p(\Omega,\nu_1)} + \|h_2\|_{L^\infty(\Omega)}C^{q-1}_{p,q}\|\varphi\|^{q-1}_{L^p(\Omega,\nu_1)}\vartheta_{p,q}\|\nabla v\|_{L^p(\Omega,\nu_1)} \\
&\qquad + \|h_3\|_{L^\infty(\Omega)}\vartheta^{q-1}_{p,q}\|\nabla\varphi\|^{q-1}_{L^p(\Omega,\nu_1)}\vartheta_{p,q}\|\nabla v\|_{L^p(\Omega,\nu_1)} \\
&\leq \left[\vartheta_{p,q}\|f_2\|_{L^{q'}(\Omega,\nu_2)} + \vartheta^q_{p,q}\left(C^{q-1}_\Omega\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)}\right)\|\varphi\|^{q-1}_{W_0^{1,p}(\Omega,\nu_1)}\right]\|v\|_{W_0^{1,p}(\Omega,\nu_1)}.
\end{aligned}$$

Similarly, by using Hölder inequality, Theorem 1.1.36, **(H2)** and Remark 2.1.2, we get

$$\begin{aligned}
|\Phi_3(\varphi, v)| &\leq \int_\Omega |g(y, \varphi)||v|\nu_3 dy \\
&\leq \left[C_\Omega\vartheta_{p,s}\|f_3\|_{L^{s'}(\Omega,\nu_3)} + \vartheta^s_{p,s}C^s_\Omega\|h_4\|_{L^\infty(\Omega)}\|\varphi\|^{s-1}_{W_0^{1,p}(\Omega,\nu_1)}\right]\|v\|_{W_0^{1,p}(\Omega,\nu_1)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|\Phi(\varphi, v)| &\leq \Big[\|f_1\|_{L^{p'}(\Omega,\nu_1)} + \|h_1\|_{L^\infty(\Omega)}\|\varphi\|^{p-1}_{W_0^{1,p}(\Omega,\nu_1)} + C_\Omega\vartheta_{p,s}\|f_3\|_{L^{s'}(\Omega,\nu_3)} + \vartheta^q_{p,q}\Big(C^{q-1}_\Omega\|h_2\|_{L^\infty(\Omega)} \\
&+\|h_3\|_{L^\infty(\Omega)}\Big)\|\varphi\|^{q-1}_{W_0^{1,p}(\Omega,\nu_1)} + \vartheta^s_{p,s}C^s_\Omega\|h_4\|_{L^\infty(\Omega)}\|\varphi\|^{s-1}_{W_0^{1,p}(\Omega,\nu_1)} + \vartheta_{p,q}\|f_2\|_{L^{q'}(\Omega,\nu_2)}\Big]\|v\|_{W_0^{1,p}(\Omega,\nu_1)}.
\end{aligned}$$

Thus, $\Phi(\varphi, .)$ is linear and continuous for every $\varphi \in W_0^{1,p}(\Omega, \nu_1)$. As a result, there is a linear and continuous operator on $W_0^{1,p}(\Omega, \nu_1)$ labeled by Ψ that provides

$$\langle \Psi\varphi, v\rangle = \Phi(\varphi, v) \ \text{ for all } \ \varphi, v \in W_0^{1,p}(\Omega, \nu_1).$$

We also have

$$\begin{aligned}\|\Psi\varphi\|_* \;\; &\leq \;\; \|f_1\|_{L^{p'}(\Omega,\nu_1)} + \|h_1\|_{L^\infty(\Omega)}\|\varphi\|^{p-1}_{W_0^{1,p}(\Omega,\nu_1)} + C_\Omega\vartheta_{p,s}\|f_3\|_{L^{s'}(\Omega,\nu_3)} + \vartheta_{p,q}\|f_2\|_{L^{q'}(\Omega,\nu_2)} \\ &\quad + \vartheta^s_{p,s}C^s_\Omega\|h_4\|_{L^\infty(\Omega)}\|\varphi\|^{s-1}_{W_0^{1,p}(\Omega,\nu_1)} + \vartheta^q_{p,q}\left(C^{q-1}_\Omega\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)}\right)\|\varphi\|^{q-1}_{W_0^{1,p}(\Omega,\nu_1)},\end{aligned}$$

where

$$\|\Psi\varphi\|_* := \sup\left\{|\langle\Psi\varphi, v\rangle| = |\Phi(\varphi, v)| : v \in W_0^{1,p}(\Omega,\nu_1), \|v\|_{W_0^{1,p}(\Omega,\nu_1)} = 1\right\},$$

is the norm in $W_0^{-1,p'}(\Omega,\nu_1^{1-p'})$. Therefore, we get the operator

$$\begin{aligned}\Psi : W_0^{1,p}(\Omega,\nu_1) &\longrightarrow W_0^{-1,p'}(\Omega,\nu_1^{1-p'}) \\ \varphi &\longmapsto \Psi\varphi.\end{aligned}$$

Therefore, the problem (2.0.1) is equivalent to the operator equation

$$\Psi\varphi = \Upsilon, \quad \varphi \in W_0^{1,p}(\Omega,\nu_1).$$

Step 2.

In this step, we demonstrate that Ψ is strictly monotone. For all $\varphi_1, \varphi_2 \in W_0^{1,p}(\Omega,\nu_1)$ with $\varphi_1 \neq \varphi_2$, we have

$$\begin{aligned}\langle\Psi\varphi_1 - \Psi\varphi_2, \varphi_1 - \varphi_2\rangle &= \Phi(\varphi_1, \varphi_1 - \varphi_2) - \Phi(\varphi_2, \varphi_1 - \varphi_2) \\ &= \int_\Omega \langle a(y,\nabla\varphi_1), \nabla(\varphi_1 - \varphi_2)\rangle\nu_1 dy - \int_\Omega \langle a(y,\nabla\varphi_2), \nabla(\varphi_1 - \varphi_2)\rangle\nu_1 dy \\ &+ \int_\Omega \langle b(y,\varphi_1,\nabla\varphi_1), \nabla(\varphi_1 - \varphi_2)\rangle\nu_2 dy - \int_\Omega \langle b(y,\varphi_2,\nabla\varphi_2), \nabla(\varphi_1 - \varphi_2)\rangle\nu_2 dy \\ &+ \int_\Omega g(y,\varphi_1)(\varphi_1 - \varphi_2)\nu_3 dy - \int_\Omega g(y,\varphi_2)(\varphi_1 - \varphi_2)\nu_3 dy \\ &= \int_\Omega \langle a(y,\nabla\varphi_1) - a(y,\nabla\varphi_2), \nabla(\varphi_1 - \varphi_2)\rangle\nu_1 dy \\ &+ \int_\Omega \Big(g(y,\varphi_1) - g(y,\varphi_2)\Big)\Big(\varphi_1 - \varphi_2\Big)\nu_3 dy \\ &+ \textstyle\int_\Omega \langle b(y,\varphi_1,\nabla\varphi_1) - b(y,\varphi_2,\nabla\varphi_2), \nabla(\varphi_1 - \varphi_2)\rangle\nu_2 dy.\end{aligned}$$

By using **(H3)**, we obtain

$$\langle\Psi\varphi_1 - \Psi\varphi_2, \varphi_1 - \varphi_2\rangle \;\geq\; \int_\Omega \lambda|\nabla(\varphi_1 - \varphi_2)|^p\nu_1 dy \;\geq\; \lambda\|\nabla(\varphi_1 - \varphi_2)\|^p_{L^p(\Omega,\nu_1)},$$

and by Theorem 1.1.36, we conclude that

$$\langle\Psi\varphi_1 - \Psi\varphi_2, \varphi_1 - \varphi_2\rangle \;\geq\; \frac{\lambda}{(C^p_\Omega + 1)}\|\varphi_1 - \varphi_2\|^p_{W_0^{1,p}(\Omega,\nu_1)},$$

which implies that Ψ is strictly monotone.

Step 3.

This step establishes the coerciveness of the operator Ψ. For all $\varphi \in W_0^{1,p}(\Omega,\nu_1)$, we get

$$\begin{aligned}\langle\Psi\varphi, \varphi\rangle &= \Phi(\varphi,\varphi) \\ &= \int_\Omega \langle a(y,\nabla\varphi), \nabla\varphi\rangle\nu_1 dy + \int_\Omega \langle b(y,\varphi,\nabla\varphi), \nabla\varphi\rangle\nu_2 dy + \int_\Omega g(y,\varphi)u\;\nu_3 dy.\end{aligned}$$

From Theorem 1.1.36 and **(H4)**, it follows that

$$\begin{aligned}
\langle \Psi\varphi, \varphi \rangle &\geq \kappa_1 \int_\Omega |\nabla\varphi|^p \nu_1 dy + \kappa_2 \int_\Omega |\nabla\varphi|^q \nu_2 dy + \kappa_3 \int_\Omega |\varphi|^q \nu_2 dy \\
&\geq \kappa_1 \int_\Omega |\nabla\varphi|^p \nu_1 dy + \min(\kappa_2, \kappa_3) \left[\int_\Omega |\nabla\varphi|^q \nu_2 dy + \int_\Omega |\varphi|^q \nu_2 dy \right] \\
&= \kappa_1 \|\nabla\varphi\|^p_{L^p(\Omega,\nu_1)} + \min(\kappa_2, \kappa_3) \|\varphi\|^q_{W_0^{1,q}(\Omega,\nu_2)} \\
&\geq \kappa_1 \|\nabla\varphi\|^p_{L^p(\Omega,\nu_1)} \\
&\geq \frac{\kappa_1}{(C_\Omega^p + 1)} \|\varphi\|^p_{W_0^{1,p}(\Omega,\nu_1)}.
\end{aligned}$$

Hence, we obtain

$$\frac{\langle \Psi\varphi, \varphi \rangle}{\|\varphi\|_{W_0^{1,p}(\Omega,\nu_1)}} \geq \frac{\kappa_1}{(C_\Omega^p + 1)} \|\varphi\|^{p-1}_{W_0^{1,p}(\Omega,\nu_1)}.$$

Therefore, as $p > 1$, we obtain

$$\frac{\langle \Psi\varphi, \varphi \rangle}{\|\varphi\|_{W_0^{1,p}(\Omega,\nu_1)}} \longrightarrow +\infty \text{ as } \|\varphi\|_{W_0^{1,p}(\Omega,\nu_1)} \longrightarrow +\infty,$$

which means that Ψ is coercive.

Step 4.

In this step, we show that Ψ is continuous. To do this, consider $\varphi_k \longrightarrow \varphi$ in $W_0^{1,p}(\Omega, \nu_1)$ as $k \longrightarrow \infty$. Then $\varphi_k \longrightarrow \varphi$ in $L^p(\Omega, \nu_1)$ et $\nabla\varphi_k \longrightarrow \nabla\varphi$ in $(L^p(\Omega, \nu_1))^n$. Therefore, according to Theorem 1.1.7, there exist (φ_{k_i}), $\psi_1 \in L^p(\Omega, \nu_1)$ and $\psi_2 \in L^p(\Omega, \nu_1)$ in such a way that

$$\begin{array}{ll}
\varphi_{k_i}(y) \longrightarrow \varphi(y), \ \text{as} \ k_i \longrightarrow \infty, & \text{in } \Omega \\
|\varphi_{k_i}(y)| \leq \psi_1(y), & \text{in } \Omega \\
\nabla\varphi_{k_i}(y) \longrightarrow \nabla\varphi(y), \ \text{as} \ k_i \longrightarrow \infty, & \text{in } \Omega \\
|\nabla\varphi_{k_i}(y)| \leq \psi_2(y), & \text{in } \Omega.
\end{array} \tag{2.2.2}$$

We are going to establish that $\Psi\varphi_k \longrightarrow \Psi\varphi$ in $W_0^{-1,p'}(\Omega, \nu_1^{1-p'})$. It is proved in three steps.

Step 4.1.

Let us define the operator

$$\begin{aligned}
&B_j : W_0^{1,p}(\Omega, \nu_1) \longrightarrow L^{p'}(\Omega, \nu_1) \\
&(B_j\varphi)(y) = a_j(y, \nabla\varphi(y)).
\end{aligned}$$

We now show that

$$B_j\varphi_k \longrightarrow B_j\varphi \quad \text{in} \quad L^{p'}(\Omega, \nu_1).$$

(i) For all $\varphi \in W_0^{1,p}(\Omega,\nu_1)$, by Theorem 1.1.36 and **(H2)**, we have

$$\begin{aligned}
\|B_j\varphi\|^{p'}_{L^{p'}(\Omega,\nu_1)} &= \int_\Omega |B_j\varphi(y)|^{p'}\nu_1 dy = \int_\Omega |a_j(y,\nabla\varphi)|^{p'}\nu_1 dy\\
&\leq \int_\Omega \left(f_1 + h_1|\nabla\varphi|^{p-1}\right)^{p'}\nu_1 dy\\
&\leq C_p\int_\Omega \left(f_1^{p'} + h_1^{p'}|\nabla\varphi|^p\right)\nu_1 dy\\
&\leq C_p\left[\|f_1\|^{p'}_{L^{p'}(\Omega,\nu_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|\nabla\varphi\|^p_{L^p(\Omega,\nu_1)}\right]\\
&\leq C_p\left[\|f_1\|^{p'}_{L^{p'}(\Omega,\nu_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{W_0^{1,p}(\Omega,\nu_1)}\right].
\end{aligned}$$

(ii) By **(H2)** and (2.2.2), we obtain

$$\begin{aligned}
\|B_j\varphi_{k_i} - B_j\varphi\|^{p'}_{L^{p'}(\Omega,\nu_1)} &= \int_\Omega |B_j\varphi_{k_i}(y) - B_j\varphi(y)|^{p'}\nu_1 dy\\
&\leq \int_\Omega \Big(|a_j(y,\nabla\varphi_{k_i})| + |a_j(y,\nabla\varphi)|\Big)^{p'}\nu_1 dy\\
&\leq C_p\int_\Omega \Big(|a_j(y,\nabla\varphi_{k_i})|^{p'} + |a_j(y,\nabla\varphi)|^{p'}\Big)\nu_1 dy\\
&\leq C_p\int_\Omega \left[\left(f_1 + h_1|\nabla\varphi_{k_i}|^{p-1}\right)^{p'} + \left(f_1 + h_1|\nabla\varphi|^{p-1}\right)^{p'}\right]\nu_1 dy\\
&\leq C_p\int_\Omega \left[\left(f_1 + h_1\psi_2^{p-1}\right)^{p'} + \left(f_1 + h_1\psi_2^{p-1}\right)^{p'}\right]\nu_1 dy\\
&\leq 2C_pC_p'\int_\Omega \left(f_1^{p'} + h_1^{p'}\psi_2^p\right)\nu_1 dy\\
&\leq 2C_pC_p'\left[\|f_1\|^{p'}_{L^{p'}(\Omega,\nu_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|\psi_2\|^p_{L^p(\Omega,\nu_1)}\right].
\end{aligned}$$

As $k_i \longrightarrow \infty$, by using **(H1)**, we get

$$B_j\varphi_{k_i}(y) = a_j(y,\nabla\varphi_{k_i}(y)) \longrightarrow a_j(y,\nabla\varphi(y)) = B_j\varphi(y), \qquad \text{for almost all } x \in \Omega.$$

Consequently, by Lebesgue's theorem, we have

$$\|B_j\varphi_{k_i} - B_j\varphi\|_{L^{p'}(\Omega,\nu_1)} \longrightarrow 0 \Leftrightarrow B_j\varphi_{k_i} \longrightarrow B_j\varphi \quad \text{in} \quad L^{p'}(\Omega,\nu_1).$$

Finally, considering the principle of convergence in Banach spaces, we conclude

$$B_j\varphi_k \longrightarrow B_j\varphi \quad \text{in} \quad L^{p'}(\Omega,\nu_1). \tag{2.2.3}$$

Step 4.2.
Define

$$\begin{aligned}
&G_j : W_0^{1,p}(\Omega,\nu_1) \longrightarrow L^{q'}(\Omega,\nu_2)\\
&(G_j\varphi)(y) = b_j(y,\varphi(y),\nabla\varphi(y)).
\end{aligned}$$

We also have that

$$G_j\varphi_k \longrightarrow G_j\varphi \quad \text{in} \quad L^{q'}(\Omega,\nu_2).$$

(i) For all $\varphi \in W_0^{1,p}(\Omega,\nu_1)$, by Remark 2.1.2, **(i)** **(H2)** and Theorem 1.1.36, we get

$$\begin{aligned}
\|G_j\varphi\|^{q'}_{L^{q'}(\Omega,\nu_2)} &= \int_\Omega |b_j(y,\varphi,\nabla\varphi)|^{q'}\nu_2 dy \\
&\leq \int_\Omega \left(f_2 + h_2|\varphi|^{q-1} + h_3|\nabla\varphi|^{q-1}\right)^{q'}\nu_2 dy \\
&\leq C_q \int_\Omega \left[f_2^{q'} + h_2^{q'}|\varphi|^q + h_3^{q'}|\nabla\varphi|^q\right]\nu_2 dy \\
&= C_q \left[\int_\Omega f_2^{q'}\nu_2 dy + \int_\Omega h_2^{q'}|\varphi|^q\nu_2 dy + \int_\Omega h_3^{q'}|\nabla\varphi|^q\nu_2 dy\right] \\
&\leq C_q \left[\int_\Omega f_2^{q'}\nu_2 dy + \|h_2\|^{q'}_{L^\infty(\Omega)}\int_\Omega |\varphi|^q\nu_2 dy + \|h_3\|^{q'}_{L^\infty(\Omega)}\int_\Omega |\nabla\varphi|^q\nu_2 dy\right] \\
&\leq C_q \left[\|f_2\|^{q'}_{L^{q'}(\Omega,\nu_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)}\|\varphi\|^q_{L^q(\Omega,\nu_2)} + \|h_3\|^{q'}_{L^\infty(\Omega)}\|\nabla\varphi\|^q_{L^q(\Omega,\nu_2)}\right] \\
&\leq C_q \left[\|f_2\|^{q'}_{L^{q'}(\Omega,\nu_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|\varphi\|^q_{L^p(\Omega,\nu_1)} + \|h_3\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|\nabla\varphi\|^q_{L^p(\Omega,\nu_1)}\right] \\
&\leq C_q \left[\|f_2\|^{q'}_{L^{q'}(\Omega,\nu_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|\varphi\|^q_{W_0^{1,p}(\Omega,\nu_1)} + \|h_3\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|u\|^q_{W_0^{1,p}(\Omega,\nu_1)}\right] \\
&\leq C_q \left[\|f_2\|^{q'}_{L^{q'}(\Omega,\nu_2)} + C^q_{p,q}\left(C^q_\Omega\|h_2\|^{q'}_{L^\infty(\Omega)} + \|h_3\|^{q'}_{L^\infty(\Omega)}\right)\|u\|^q_{W_0^{1,p}(\Omega,\nu_1)}\right].
\end{aligned}$$

(ii) By usin Remark 2.1.2 (i), **(H2)** and the similar reasoning employed in Step 4.1 **(ii)**, we get

$$G_j\varphi_k \longrightarrow G_j\varphi \quad \text{in} \quad L^{q'}(\Omega,\nu_2). \tag{2.2.4}$$

Step 4.3.

We define the operator

$$\begin{aligned}
&H : W_0^{1,p}(\Omega,\nu_1) \longrightarrow L^{s'}(\Omega,\nu_3) \\
&(H\varphi)(y) = g(y,\varphi(y)).
\end{aligned}$$

In this step, we show that

$$H\varphi_k \longrightarrow H\varphi \quad \text{in} \quad L^{s'}(\Omega,\nu_3).$$

(i) For all $\varphi \in W_0^{1,p}(\Omega,\nu_1)$, by Remark 2.1.2 (ii) and **(H2)**, we get

$$\begin{aligned}
\|H\varphi\|^{s'}_{L^{s'}(\Omega,\nu_3)} &= \int_\Omega |g(y,\varphi)|^{s'}\nu_3 dy \\
&\leq C_s \int_\Omega \left(f_3^{s'} + h_4^{s'}|\varphi|^s\right)\nu_3 dy \\
&\leq C_s \left[\|f_3\|^{s'}_{L^{s'}(\Omega,\nu_3)} + \|h_4\|^{p'}_{L^\infty(\Omega)}\|\varphi\|^s_{L^s(\Omega,\nu_3)}\right] \\
&\leq C_s \left[\|f_3\|^{s'}_{L^{s'}(\Omega,\nu_3)} + C^s_{p,s}\|h_4\|^{p'}_{L^\infty(\Omega)}\|\varphi\|^s_{L^p(\Omega,\nu_1)}\right] \\
&\leq C_s \left[\|f_3\|_{L^{s'}(\Omega,\nu_1)} + C^s_{p,s}C^s_\Omega\|h_4\|^{s'}_{L^\infty(\Omega)}\|\varphi\|^s_{W_0^{1,p}(\Omega,\nu_1)}\right].
\end{aligned}$$

(ii) From Remark 2.1.2 (ii) and **(H2)**, we have

$$\begin{aligned}
\|H\varphi_{k_i} - H\varphi\|^{s'}_{L^{s'}(\Omega,\nu_3)} &= \int_\Omega |H\varphi_{k_i}(y) - H\varphi(y)|^{p'}\nu_3 dy \\
&\le \int_\Omega \Big(|g(y,\varphi_{k_i})| + |g(y,\varphi)|\Big)^{s'} \nu_3 dy \\
&\le C_s \int_\Omega \Big(|g(y,\varphi_{k_i})|^{s'} + |g(y,\varphi)|^{s'}\Big)\nu_3 dy \\
&\le C_s \int_\Omega \Big[\Big(f_3 + h_4|\varphi_{k_i}|^{s-1}\Big)^{s'} + \Big(f_3 + h_4|\varphi|^{s-1}\Big)^{s'}\Big]\nu_3 dy \\
&\le C_s \int_\Omega \Big[\Big(f_3 + h_4|\psi_1|^{s-1}\Big)^{s'} + \Big(f_3 + h_4\psi_1^{s-1}\Big)^{s'}\Big]\nu_3 dy \\
&\le 2C_s C'_s \int_\Omega \Big(f_3^{s'} + h_4^{p'}\psi_1^{s}\Big)\nu_3 dy \\
&\le 2C_s C'_s \Big[\|f_3\|^{s'}_{L^{s'}(\Omega,\nu_3)} + \|h_4\|^{s'}_{L^\infty(\Omega)}\|\psi_1\|^{s}_{L^s(\Omega,\nu_3)}\Big] \\
&\le 2C_s C'_s \Big[\|f_3\|^{s'}_{L^{s'}(\Omega,\nu_3)} + \vartheta^s_{p,s}\|h_4\|^{s'}_{L^\infty(\Omega)}\|\psi_1\|^{s}_{L^p(\Omega,\nu_1)}\Big].
\end{aligned}$$

As $k_i \longrightarrow \infty$, by using **(H1)**, we obtain

$$H\varphi_{k_i}(y) = g(y,\varphi_{k_i}(y)) \longrightarrow g(y,u(y)) = H\varphi(y), \qquad \text{a.e. } x \in \Omega.$$

Consequently, by means of Lebesgue's theorem, we have

$$\|H\varphi_{k_i} - H\varphi\|_{L^{s'}(\Omega,\nu_3)} \longrightarrow 0,$$

that is,

$$H\varphi_{k_i} \longrightarrow H\varphi \quad \text{in} \quad L^{s'}(\Omega,\nu_3).$$

Finally, considering the principle of convergence in Banach spaces, we conclude that

$$H\varphi_k \longrightarrow H\varphi \quad \text{in} \quad L^{s'}(\Omega,\nu_3). \tag{2.2.5}$$

At last, by considering $v \in W^{1,p}_0(\Omega,\nu_1)$ and with the help of Theorem 1.1.36, Hölder inequality and Remark 2.1.2, we arrive at

$$\begin{aligned}
|\Phi_1(\varphi_k,v) - \Phi_1(\varphi,v)| &= \Big|\int_\Omega \langle a(y,\nabla\varphi_k) - a(y,\nabla\varphi), \nabla v\rangle \nu_1 dy\Big| \\
&\le \sum_{j=1}^n \int_\Omega |a_j(y,\nabla\varphi_k) - a_j(y,\nabla\varphi)||D_j v|\nu_1 dy \\
&= \sum_{j=1}^n \int_\Omega |B_j\varphi_k - B_j\varphi||D_j v|\nu_1 dy \\
&\le \sum_{j=1}^n \|B_j\varphi_k - B_j\varphi\|_{L^{p'}(\Omega,\nu_1)}\|D_j v\|_{L^p(\Omega,\nu_1)} \\
&\le \left(\sum_{j=1}^n \|B_j\varphi_k - B_j\varphi\|_{L^{p'}(\Omega,\nu_1)}\right)\|v\|_{W^{1,p}_0(\Omega,\nu_1)},
\end{aligned}$$

$$
\begin{aligned}
|\Phi_2(\varphi_k,v)-\Phi_2(\varphi,v)| &= |\int_\Omega \langle b(y,\varphi_k,\nabla\varphi_k)-b(y,\varphi,\nabla\varphi),\nabla v\rangle v_2 dy| \\
&\leq \sum_{j=1}^n \int_\Omega |b_j(y,\varphi_k,\nabla\varphi_k)-b_j(y,\varphi,\nabla\varphi)||D_j v|v_2 dy \\
&= \sum_{j=1}^n \int_\Omega |G_j\varphi_k-G_j\varphi||D_j v|v_2 dy \\
&\leq \left(\sum_{j=1}^n \|G_j\varphi_k-G_j\varphi\|_{L^{q'}(\Omega,v_2)}\right)\|\nabla v\|_{L^q(\Omega,v_2)} \\
&\leq \vartheta_{p,q}\left(\sum_{j=1}^n \|G_j\varphi_k-G_j\varphi\|_{L^{q'}(\Omega,v_2)}\right)\|\nabla v\|_{L^p(\Omega,v_1)} \\
&\leq \vartheta_{p,q}\left(\sum_{j=1}^n \|G_j\varphi_k-G_j\varphi\|_{L^{q'}(\Omega,v_2)}\right)\|v\|_{W_0^{1,p}(\Omega,v_1)},
\end{aligned}
$$

$$
\begin{aligned}
|\Phi_3(\varphi_k,v)-\Phi_3(\varphi,v)| &\leq \int_\Omega |g(y,\varphi_k)-g(y,\varphi)||v|v_3 dy \\
&= \int_\Omega |H\varphi_k-H\varphi||v|v_3 dy \\
&\leq \|H\varphi_k-H\varphi\|_{L^{s'}(\Omega,v_3)}\|v\|_{L^s(\Omega,v_3)} \\
&\leq \vartheta_{p,s}\|H\varphi_k-H\varphi\|_{L^{s'}(\Omega,v_3)}\|v\|_{L^p(\Omega,v_1)} \\
&\leq \vartheta_{p,s}C_\Omega\|H\varphi_k-H\varphi\|_{L^{s'}(\Omega,v_3)}\|v\|_{W_0^{1,p}(\Omega,v_1)}.
\end{aligned}
$$

Hence, for all $v \in W_0^{1,p}(\Omega,v_1)$, we have

$$
\begin{aligned}
|\Phi(\varphi_k,v)-\Phi(\varphi,v)| &\leq |\Phi_1(\varphi_k,v)-\Phi_1(\varphi,v)|+|\Phi_2(\varphi_k,v)-\Phi_2(\varphi,v)|+|\Phi_3(\varphi_k,v)-\Phi_3(\varphi,v)| \\
&\leq \Bigg[\sum_{j=1}^n\left(\|B_j\varphi_k-B_j\varphi\|_{L^{p'}(\Omega,v_1)}+\vartheta_{p,q}\|G_j\varphi_k-G_j\varphi\|_{L^{q'}(\Omega,v_2)}\right) \\
&\qquad +\vartheta_{p,s}C_\Omega\|H\varphi_k-H\varphi\|_{L^{s'}(\Omega,v_3)}\Bigg]\|v\|_{W_0^{1,p}(\Omega,v_1)},
\end{aligned}
$$

and consequently, we get

$$
\|\Psi\varphi_k-\Psi\varphi\|_* \leq \sum_{j=1}^n\left(\|B_j\varphi_k-B_j\varphi\|_{L^{p'}(\Omega,v_1)}+\vartheta_{p,q}\|G_j\varphi_k-G_j\varphi\|_{L^{q'}(\Omega,v_2)}\right)+\vartheta_{p,s}C_\Omega\|H\varphi_k-H\varphi\|_{L^{s'}(\Omega,v_3)}.
$$

Combining (2.2.3), (2.2.4) and (2.2.5), we deduce that

$$
\|\Psi\varphi_k-\Psi\varphi\|_* \longrightarrow 0 \text{ as } m \longrightarrow \infty,
$$

that is, $\Psi\varphi_k \longrightarrow \Psi\varphi$ in $W_0^{-1,p'}(\Omega,v_1^{1-p'})$. Which implies that Ψ is continuous.

We have now proved that Ψ is strictly monotone, coercive and hemicontinuous, and $\Upsilon \in W_0^{-1,p'}(\Omega,v_1^{1-p'})$. Thus, we have verified all the conditions of Theorem 1.2.4. As a result, from Theorem 1.2.4, it follows that the operator equation $\Psi\varphi = \Upsilon$ admits the unique weak solution $\varphi \in W_0^{1,p}(\Omega,v_1)$ and it also follows that u is the unique weak solution for (2.0.1). This completes the proof of Theorem 2.2.1.

2.3 Example

Set $\Omega = \{(y,z) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and let $\nu_1(y,z) = \left(y^2+z^2\right)^{-1/2}$, $\nu_2(y,z) = \left(y^2+z^2\right)^{-1/3}$ and $\nu_3(y,z) = \left(y^2+z^2\right)^{-1}$ $\Big($note that $\nu_1, \nu_2, \nu_3 \in A_4$, $p = 4$, $q = 3$ and $s = 2\Big)$, and we define $b : \Omega \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $a : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$a\Big((y,z),\delta\Big) = h_1(y,z)|\delta|^3 sgn(\delta),$$

$$b\Big((y,z),\mu,\delta\Big) = |\delta|^2 sgn(\delta),$$

$$g\Big((y,z),\mu\Big) = h_4(y,z)|\mu| sgn(\mu),$$

with $h_1(y,z) = 2e^{(y^2+z^2)}$ and $h_4(y,z) = 2 - \cos^2(yz)$. Let us look at the problem

$$\begin{cases} \mathcal{A}\varphi(y,z) = \cos(y+z) & \text{in } \Omega, \\ \varphi(y,z) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3.1}$$

where,

$$\mathcal{A}\varphi(y,z) = -\text{div}\Big[\nu_1 a\Big((y,z),\nabla\varphi(y,z)\Big) + \nu_2 b\Big((y,z),\varphi(y,z),\nabla\varphi(y,z)\Big)\Big] + \nu_3 g\Big((y,z),\varphi(y,z)\Big).$$

From Theorem 2.2.1, it follows that the problem (2.3.1) admits the unique weak solution in $W_0^{1,4}(\Omega,\nu_1)$.

Chapter 3

Existence result for a Dirichlet problem governed by nonlinear degenerate elliptic equation in weighted Sobolev spaces

Our aim in this chapter is to prove the existence and uniqueness of weak solution in the weighted Sobolev space $W_0^{1,p}(\Omega, \omega_1, \omega_4)$ for a Dirichlet boundary value problem for the following nonlinear degenerate elliptic equation

$$\begin{cases} -\mathrm{div}\Big[\omega_1 \mathcal{A}(x, \nabla u) + \omega_2 \mathcal{B}(x, u, \nabla u)\Big] + \omega_3 b(x, u) + \omega_4 |u|^{p-2} u = f & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.0.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$, ω_1, ω_2, ω_3 and ω_4 are A_p-weight functions, $\mathcal{A} : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $b : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions that satisfy the assumptions of growth, ellipticity and monotonicity, and the right-hand side term f belongs to $L^{p'}(\Omega, \omega_1^{1-p'})$.

3.1 Main result

3.1.1 Basic assumptions

We assume that the following assumptions: Ω be a bounded open subset of $\mathbb{R}^n$($n \geq 2$), $1 < q, s < p < \infty$, let ω_1, ω_2, ω_3 and ω_4 are a weights functions, and let $\mathcal{A} : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, with $\mathcal{B}(x, \eta, \xi) = \Big(\mathcal{B}_1(x, \eta, \xi), ..., \mathcal{B}_n(x, \eta, \xi)\Big)$ and $\mathcal{A}(x, \xi) = \Big(\mathcal{A}_1(x, \xi), ..., \mathcal{A}_n(x, \xi)\Big)$ and $b : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the following assumptions:

(A1) For $k = 1, ..., n$, $\mathcal{B}_k$, $\mathcal{A}_k$ and b are Carathéodory functions.

(A2) There are positive functions h_1, h_2, h_3, $h_4 \in L^\infty(\Omega)$ and $\gamma_1 \in L^{p'}(\Omega, \omega_1)$, $\gamma_2 \in L^{q'}(\Omega, \omega_2)$ and $\gamma_3 \in L^{s'}(\Omega, \omega_3)$ such that :

$$|\mathcal{A}(x, \xi)| \leq \gamma_1(x) + h_1(x)|\xi|^{p-1},$$

$$|\mathcal{B}(x,\eta,\xi)| \leq \gamma_2(x) + h_2(x)|\eta|^{q-1} + h_3(x)|\xi|^{q-1},$$

and

$$|b(x,\eta)| \leq \gamma_3(x) + h_4(x)|\eta|^{s-1}.$$

(A3) There exists a constant $\alpha > 0$ such that :

$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\xi'), \xi - \xi' \rangle \geq \alpha|\xi - \xi'|^p,$$

$$\langle \mathcal{B}(x,\eta,\xi) - \mathcal{B}(x,\eta',\xi'), \xi - \xi' \rangle \geq 0,$$

and

$$\Big(b(x,\eta) - b(x,\eta')\Big)\Big(\eta - \eta'\Big) \geq 0,$$

whenever $(\eta,\xi), (\eta',\xi') \in \mathbb{R} \times \mathbb{R}^n$ with $\eta \neq \eta'$ and $\xi \neq \xi'$.

(A4) There are constants $\beta_1,\ \beta_2,\ \beta_3 > 0$ such that :

$$\langle \mathcal{A}(x,\xi), \xi \rangle \geq \beta_1|\xi|^p,$$

$$\langle \mathcal{B}(x,\eta,\xi), \xi \rangle \geq \beta_2|\xi|^q + \beta_3|\eta|^q,$$

and

$$b(x,\eta)\eta \geq 0.$$

3.1.2 Notion of solution

The definition of a weak solution for problem (3.0.1) can be stated as follows.

Definition 3.1.1 *One says $u \in W_0^{1,p}(\Omega,\omega_1,\omega_4)$ is a weak solution to problem (3.0.1), provided that*

$$\int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla v \rangle\, \omega_1\, dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla v \rangle\, \omega_2\, dx + \int_\Omega b(x,u)\, v\, \omega_3\, dx + \int_\Omega |u|^{p-2} u\, v\, \omega_4\, dx = \int_\Omega f v dx,$$

for all $v \in W_0^{1,p}(\Omega,\omega_1,\omega_4)$.

Remark 3.1.2 *We notice, for all $\omega_1,\ \omega_2,\ \omega_3 \in A_p$, that*

(i) *If $\frac{\omega_2}{\omega_1} \in L^{r_1}(\Omega,\omega_1)$ where $r_1 = \frac{p}{p-q}$ and $1 < q < p < \infty$, then, by Hölder inequality we obtain*

$$||u||_{L^q(\Omega,\omega_2)} \leq C_{p,q}||u||_{L^p(\Omega,\omega_1)},$$

where $C_{p,q} = ||\frac{\omega_2}{\omega_1}||^{1/q}_{L^{r_1}(\Omega,\omega_1)}$.

(ii) *Analogously, if $\frac{\omega_3}{\omega_1} \in L^{r_2}(\Omega,\omega_1)$ where $r_2 = \frac{p}{p-s}$ and $1 < s < p < \infty$, then*

$$||u||_{L^s(\Omega,\omega_3)} \leq C_{p,s}||u||_{L^p(\Omega,\omega_1)},$$

where $C_{p,s} = ||\frac{\omega_3}{\omega_1}||^{1/s}_{L^{r_2}(\Omega,\omega_1)}$.

3.1.3 Existence and uniqueness result

We shall prove the following existence and uniqueness theorem.

Theorem 3.1.3 *Let $\omega_i \in A_p (i = 1, 2, 3, 4)$, $1 < q, s < p < \infty$ and assume that the assumptions **(A1)** – **(A4)** hold. If $f \in L^{p'}(\Omega, \omega_1^{1-p})$, $\frac{\omega_2}{\omega_1} \in L^{p/(p-q)}(\Omega, \omega_1)$ and $\frac{\omega_3}{\omega_1} \in L^{p/(p-s)}(\Omega, \omega_1)$, then the problem (3.0.1) has exactly one solution $u \in W_0^{1,p}(\Omega, \omega_1, \omega_4)$.*

Proof. The essential one of our proof is to reduce the (3.0.1) to an operator problem $\mathbf{A}u = \mathbf{G}$ and apply the Theorem 1.2.4.

We define

$$\mathbf{F} : W_0^{1,p}(\Omega, \omega_1, \omega_4) \times W_0^{1,p}(\Omega, \omega_1, \omega_4) \longrightarrow \mathbb{R}$$

and

$$\mathbf{G} : W_0^{1,p}(\Omega, \omega_1, \omega_4) \longrightarrow \mathbb{R},$$

where $\mathbf{F}$ and $\mathbf{G}$ are defined below.
Then $u \in W_0^{1,p}(\Omega, \omega_1, \omega_4)$ is a weak solution of (3.0.1) if and only if

$$\mathbf{F}(u, v) = \mathbf{G}(v), \qquad \text{for all } \ v \in W_0^{1,p}(\Omega, \omega_1, \omega_4).$$

The proof of Theorem 3.1.3 is divided into four steps.
Step 1: equivalent operator equation:
In this step, we prove that the problem (3.0.1) is equivalent to an operator equation $\mathbf{A}u = \mathbf{G}$.
Using Hölder inequality, Theorem 1.1.36 and Remark 3.1.2 (ii), we obtain

$$\begin{aligned} |\mathbf{G}(v)| &\leq \int_\Omega \frac{|f|}{\omega_1} |v|\omega_1 \, dx \\ &\leq \|f/\omega_1\|_{L^{p'}(\Omega,\omega_1)} \|v\|_{L^p(\Omega,\omega_1)} \\ &\leq C_\Omega \|f/\omega_1\|_{L^{p'}(\Omega,\omega_1)} \|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}. \end{aligned}$$

Since $f/\omega_1 \in L^{p'}(\Omega, \omega_1)$, then $\mathbf{G} \in \left[W_0^{1,p}(\Omega, \omega_1, \omega_4)\right]^*$.
The operator $\mathbf{F}$ is broken down into the from

$$\mathbf{F}(u, v) = \mathbf{F}_1(u, v) + \mathbf{F}_2(u, v) + \mathbf{F}_3(u, v) + \mathbf{F}_4(u, v),$$

where $\mathbf{F}_i : W_0^{1,p}(\Omega, \omega_1, \omega_4) \times W_0^{1,p}(\Omega, \omega_1, \omega_4) \longrightarrow \mathbb{R}$, for $i = 1, 2, 3, 4$, are defined as

$$\mathbf{F}_1(u, v) = \int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla v \rangle \omega_1 dx, \quad \mathbf{F}_2(u, v) = \int_\Omega \langle \mathcal{B}(x, u, \nabla u), \nabla v \rangle \omega_2 dx,$$

$$\mathbf{F}_3(u, v) = \int_\Omega b(x, u) v \, \omega_3 dx, \qquad \text{and} \quad \mathbf{F}_4(u, v) = \int_\Omega |u|^{p-2} u \, v \, \omega_4 \, dx.$$

Then, we have

$$|\mathbf{F}(u,v)| \leq |\mathbf{F}_1(u,v)| + |\mathbf{F}_2(u,v)| + |\mathbf{F}_3(u,v)| + |\mathbf{F}_4(u,v)|. \tag{3.1.1}$$

On the other hand, we get by using **(A2)**, Hölder inequality, Remark 3.1.2 (i) and Theorem 1.1.36,

$$\begin{aligned}
|\mathbf{F}_1(u,v)| &\leq \int_\Omega |\mathcal{A}(x,\nabla u)||\nabla v|\omega_1 dx \\
&\leq \int_\Omega \left(\gamma_1 + h_1|\nabla u|^{p-1}\right)|\nabla v|\omega_1 dx \\
&= \int_\Omega \gamma_1 \omega_1^{\frac{1}{p'}}|\nabla v|\omega_1^{\frac{1}{p}} dx + \int_\Omega h_1|\nabla u|^{p-1}\omega_1^{\frac{1}{p'}}|\nabla v|\omega_1^{\frac{1}{p}} dx \\
&\leq \|\gamma_1\|_{L^{p'}(\Omega,\omega_1)}\|\nabla v\|_{L^p(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\|\nabla u\|^{p-1}_{L^p(\Omega,\omega_1)}\|\nabla v\|_{L^p(\Omega,\omega_1)} \\
&\leq \left(\|\gamma_1\|_{L^{p'}(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}\right)\|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_4)},
\end{aligned}$$

and

$$\begin{aligned}
|\mathbf{F}_2(u,v)| &\leq \int_\Omega |\mathcal{B}(x,u,\nabla u)||\nabla v|\omega_2 dx \\
&\leq \int_\Omega \left(\gamma_2 + h_2|u|^{q-1} + h_3|\nabla u|^{q-1}\right)|\nabla v|\omega_2 dx \\
&= \int_\Omega \gamma_2\omega_2^{\frac{1}{q'}}|\nabla v|\omega_2^{\frac{1}{q}} dx + \int_\Omega h_2|u|^{q-1}\omega_2^{\frac{1}{q'}}|\nabla v|\omega_2^{\frac{1}{q}} dx + \int_\Omega h_3|\nabla u|^{q-1}\omega_2^{\frac{1}{q'}}|\nabla v|\omega_2^{\frac{1}{q}} dx \\
&\leq \|\gamma_2\|_{L^{q'}(\Omega,\omega_2)}\|\nabla v\|_{L^q(\Omega,\omega_2)} + \|h_2\|_{L^\infty(\Omega)}\|u\|^{q-1}_{L^q(\Omega,\omega_2)}\|\nabla v\|_{L^q(\Omega,\omega_2)} \\
&\qquad + \|h_3\|_{L^\infty(\Omega)}\|\nabla u\|^{q-1}_{L^q(\Omega,\omega_2)}\|\nabla v\|_{L^q(\Omega,\omega_2)} \\
&\leq \|\gamma_2\|_{L^{q'}(\Omega,\omega_2)}C_{p,q}\|\nabla v\|_{L^p(\Omega,\omega_1)} + \|h_2\|_{L^\infty(\Omega)}C_{p,q}^{q-1}\|u\|^{q-1}_{L^p(\Omega,\omega_1)}C_{p,q}\|\nabla v\|_{L^p(\Omega,\omega_1)} \\
&\qquad + \|h_3\|_{L^\infty(\Omega)}C_{p,q}^{q-1}\|\nabla u\|^{q-1}_{L^p(\Omega,\omega_1)}C_{p,q}\|\nabla v\|_{L^p(\Omega,\omega_1)} \\
&\leq \left[C_{p,q}^q\left(C_\Omega^{q-1}\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)}\right)\|u\|^{q-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_4)} + C_{p,q}\|\gamma_2\|_{L^{q'}(\Omega,\omega_2)}\right]\|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}.
\end{aligned}$$

Analogously, using **(A2)**, Hölder inequality, Remark 3.1.2 (ii) and Theorem 1.1.36, we obtain

$$\begin{aligned}
|\mathbf{F}_3(u,v)| &\leq \int_\Omega |b(x,u)||v|\omega_3 dx \\
&\leq \left[C_\Omega C_{p,s}\|\gamma_3\|_{L^{s'}(\Omega,\omega_3)} + C_{p,s}^s C_\Omega^s\|h_4\|_{L^\infty(\Omega)}\|u\|^{s-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}\right]\|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}.
\end{aligned}$$

Next, by applying Hölder inequality and Remark 3.1.2 (ii), we get

$$\begin{aligned}
|\mathbf{F}_4(u,v)| &\leq \int_\Omega |u|^{p-1}|v|\omega_4 dx \\
&\leq \left(\int_\Omega |u|^p\omega_4 dx\right)^{1/p'}\left(\int_\Omega |v|^p\omega_4 dx\right)^{1/p} \\
&= \|u\|^{p-1}_{L^p(\Omega,\omega_4)}\|v\|_{L^p(\Omega,\omega_4)} \\
&\leq C_\Omega\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}\|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}.
\end{aligned}$$

Hence, in (3.1.1) we obtain, for all $u, v \in W_0^{1,p}(\Omega,\omega_1,\omega_4)$

$$\begin{aligned}
|\mathbf{F}(u,v)| \leq \Big[&\|\gamma_1\|_{L^{p'}(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_4)} + C_\Omega C_{p,s}\|\gamma_3\|_{L^{s'}(\Omega,\omega_3)} \\
&+ C_{p,q}\|\gamma_2\|_{L^{q'}(\Omega,\omega_2)} + C_{p,q}^q\left(C_\Omega^{q-1}\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)}\right)\|u\|^{q-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_4)} \\
&+ C_{p,s}^s C_\Omega^s\|h_4\|_{L^\infty(\Omega)}\|u\|^{s-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_4)} + C_\Omega\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}\Big]\|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}.
\end{aligned}$$

Then $\mathbf{F}(u,.)$ is linear and continuous for each $u \in W_0^{1,p}(\Omega,\omega_1,\omega_4)$. Thus, there exists a linear and continuous operator on $W_0^{1,p}(\Omega,\omega_1,\omega_4)$ denoted by $\mathbf{A}$ such that

$$\langle \mathbf{A}u, v\rangle = \mathbf{F}(u,v), \quad \text{for all } u, v \in W_0^{1,p}(\Omega,\omega_1,\omega_4).$$

Moreover, we have

$$\begin{aligned}
\|\mathbf{A}u\|_* \;&\leq \|\gamma_1\|_{L^{p'}(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_4)} + C_\Omega C_{p,s}\|\gamma_3\|_{L^{s'}(\Omega,\omega_3)} \\
&\quad + \; C_{p,q}\|\gamma_2\|_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q}\left(C_\Omega^{q-1}\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)}\right)\|u\|^{q-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_4)} \\
&\quad + \; C^s_{p,s}C^s_\Omega\|h_4\|_{L^\infty(\Omega)}\|u\|^{s-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_4)} + \; C_\Omega\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_4)},
\end{aligned}$$

where

$$\|\mathbf{A}u\|_* := \sup\left\{|\langle \mathbf{A}u, v\rangle| = |\mathbf{F}(u,v)| : v \in W_0^{1,p}(\Omega,\omega_1,\omega_4), \|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_4)} = 1\right\},$$

is the norm in $\left[W_0^{1,p}(\Omega,\omega_1,\omega_4)\right]^*$. Hence, we obtain the operator

$$\begin{aligned}
\mathbf{A} : W_0^{1,p}(\Omega,\omega_1,\omega_4) &\longrightarrow \left[W_0^{1,p}(\Omega,\omega_1,\omega_4)\right]^* \\
u &\longmapsto \mathbf{A}u.
\end{aligned}$$

However, the problem (3.0.1) is equivalent to the operator equation

$$\mathbf{A}u = \mathbf{G}, \quad u \in W_0^{1,p}(\Omega,\omega_1,\omega_4).$$

Step 2: monotonicity of the operator A:

The operator $\mathbf{A}$ is strictly monotone. In fact. Let $v_1, v_2 \in W_0^{1,p}(\Omega,\omega_1,\omega_4)$ with $v_1 \neq v_2$. We have

$$\begin{aligned}
\langle \mathbf{A}v_1 - \mathbf{A}v_2, v_1 - v_2\rangle \;&= \mathbf{F}(v_1, v_1 - v_2) - \mathbf{F}(v_2, v_1 - v_2) \\
&= \int_\Omega \langle \mathcal{A}(x,\nabla v_1), \nabla(v_1 - v_2)\rangle \omega_1 dx - \int_\Omega \langle \mathcal{A}(x,\nabla v_2), \nabla(v_1 - v_2)\rangle \omega_1 dx \\
&\quad + \int_\Omega \langle \mathcal{B}(x, v_1, \nabla v_1), \nabla(v_1 - v_2)\rangle \omega_2 dx - \int_\Omega \langle \mathcal{B}(x, v_2, \nabla v_2), \nabla(v_1 - v_2)\rangle \omega_2 dx \\
&\quad + \int_\Omega b(x, v_1)(v_1 - v_2)\omega_3 dx - \int_\Omega b(x, v_2)(v_1 - v_2)\omega_3 dx \\
&\quad + \int_\Omega |v_1|^{p-2}v_1(v_1 - v_2)\omega_4 dx - \int_\Omega |v_2|^{p-2}v_2(v_1 - v_2)\omega_4 dx \\
&= \int_\Omega \langle \mathcal{A}(x,\nabla v_1) - \mathcal{A}(x,\nabla v_2), \nabla(v_1 - v_2)\rangle \omega_1 dx \\
&\quad + \int_\Omega \langle \mathcal{B}(x, v_1, \nabla v_1) - \mathcal{B}(x, v_2, \nabla v_2), \nabla(v_1 - v_2)\rangle \omega_2 dx \\
&\quad + \int_\Omega \Big(b(x, v_1) - b(x, v_2)\Big)\Big(v_1 - v_2\Big)\omega_3 dx \\
&\quad + \int_\Omega \Big(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\Big)\Big(v_1 - v_2\Big)\omega_4 dx.
\end{aligned}$$

Thanks to **(A3)** and Proposition 1.3.1 (ii), we obtain

$$\begin{aligned}\langle \mathbf{A}v_1 - \mathbf{A}v_2, v_1 - v_2\rangle &\geq \alpha \int_\Omega |\nabla(v_1 - v_2)|^p \ \omega_1 \ dx + \ \beta_p \int_\Omega \Big(|v_1| + |v_2|\Big)^{p-2} |v_1 - v_2|^2 \ \omega_4 \ dx \\ &\geq \alpha \int_\Omega |\nabla(v_1 - v_2)|^p \omega_1 dx \\ &\geq \alpha \|\nabla(v_1 - v_2)\|^p_{L^p(\Omega,\omega_1)}.\end{aligned}$$

Therefore, the operator **A** is strictly monotone.

Step 3: coercivity of the operator A:

In this step, we prove that the operator **A** is coercive. To this purpose let $u \in W^{1,p}_0(\Omega, \omega_1, \omega_4)$. Then, we have

$$\begin{aligned}\langle \mathbf{A}u, u\rangle &= \mathbf{F}(u, u) \\ &= \mathbf{F}_1(u, u) + \mathbf{F}_2(u, u) + \mathbf{F}_3(u, u) + \mathbf{F}_4(u, u) \\ &= \int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla u\rangle \omega_1 dx + \int_\Omega \langle \mathcal{B}(x, u, \nabla u), \nabla u\rangle \omega_2 dx + \int_\Omega b(x, u)u \ \omega_3 dx + \int_\Omega |u|^p \omega_4 dx.\end{aligned}$$

Moreover, from **(A4)** and Theorem 1.1.36, we obtain

$$\begin{aligned}\langle \mathbf{A}u, u\rangle &\geq \beta_1 \int_\Omega |\nabla u|^p \omega_1 dx + \beta_2 \int_\Omega |\nabla u|^q \omega_2 dx + \beta_3 \int_\Omega |u|^q \omega_2 dx + \int_\Omega |u|^p \omega_4 dx \\ &\geq \min(\beta_1, 1) \left[\int_\Omega |\nabla u|^p \omega_1 dx + \int_\Omega |u|^p \omega_4 dx\right] + \min(\beta_2, \beta_3) \left[\int_\Omega |\nabla u|^q \omega_2 dx + \int_\Omega |u|^q \omega_2 dx\right] \\ &= \min(\beta_1, 1) \|u\|^p_{W^{1,p}_0(\Omega,\omega_1,\omega_4)} + \min(\beta_2, \beta_3) \|u\|^q_{L^q(\Omega,\omega_2)} \\ &\geq \min(\beta_1, 1) \|u\|^p_{W^{1,p}_0(\Omega,\omega_1,\omega_4)}.\end{aligned}$$

Hence, we obtain

$$\frac{\langle \mathbf{A}u, u\rangle}{\|u\|_{W^{1,p}_0(\Omega,\omega_1,\omega_4)}} \geq \min(\beta_1, 1) \|u\|^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_4)}.$$

Therefore, since $p > 1$, we have

$$\frac{\langle \mathbf{A}u, u\rangle}{\|u\|_{W^{1,p}_0(\Omega,\omega_1,\omega_4)}} \longrightarrow +\infty \text{ as } \|u\|_{W^{1,p}_0(\Omega,\omega_1,\omega_4)} \longrightarrow +\infty,$$

that is, **A** is coercive.

Step 4: continuity of the operator A:

We need to show that the operator **A** is continuous. To do this, let $u_i \longrightarrow u$ in $W^{1,p}_0(\Omega, \omega_1, \omega_4)$ as $i \longrightarrow \infty$. Then $\nabla u_i \longrightarrow \nabla u$ in $(L^p(\Omega, \omega_1))^n$. Hence, thanks to Theorem 1.1.7, there exist a subsequence (u_{i_j}) and $\psi \in L^p(\Omega, \omega_1)$ such that

$$\begin{aligned}&\nabla u_{i_j}(x) \longrightarrow \nabla u(x), \ \text{ as } \ i_j \longrightarrow \infty, \quad \text{a.e. in } \Omega \\ &|\nabla u_{i_j}(x)| \leq \psi(x), \quad \text{a.e. in } \Omega.\end{aligned} \tag{3.1.2}$$

We will show that $\mathbf{A}u_i \longrightarrow \mathbf{A}u$ in $\left[W_0^{1,p}(\Omega,\omega_1,\omega_4)\right]^*$. In order to prove this convergence we proceed in four steps.

Step 4.1. For $k = 1, ..., n$, we define the operator

$$\begin{aligned}&B_k : W_0^{1,p}(\Omega,\omega_1,\omega_4) \longrightarrow L^{p'}(\Omega,\omega_1)\\&(B_k u)(x) = \mathcal{A}_k(x,\nabla u(x)).\end{aligned}$$

We need to show that $B_k u_i \longrightarrow B_k u$ in $L^{p'}(\Omega,\omega_1)$. We will apply the Lebesgue's theorem and the convergence principle in Banach spaces.

(i) Let $u \in W_0^{1,p}(\Omega,\omega_1,\omega_4)$. Using **(A2)** and Theorem 1.1.36, we obtain

$$\begin{aligned}\|B_k u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |B_k u(x)|^{p'}\omega_1 dx = \int_\Omega |\mathcal{A}_k(x,\nabla u)|^{p'}\omega_1 dx\\&\leq \int_\Omega \left(\gamma_1 + h_1|\nabla u|^{p-1}\right)^{p'}\omega_1 dx\\&\leq C_p \int_\Omega \left(\gamma_1^{p'} + h_1^{p'}|\nabla u|^p\right)\omega_1 dx\\&\leq C_p\left[\|\gamma_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|\nabla u\|^p_{L^p(\Omega,\omega_1)}\right]\\&\leq C_p\left[\|\gamma_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}\right],\end{aligned}$$

where the constant C_p depends only on p.

(ii) Let $u_i \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_4)$ as $i \longrightarrow \infty$. By **(A2)** and (3.1.2), we obtain

$$\begin{aligned}\|B_k u_{i_j} - B_k u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &\leq \int_\Omega \Big(|\mathcal{A}_k(x,\nabla u_{i_j})| + |\mathcal{A}_k(x,\nabla u)|\Big)^{p'}\omega_1 dx\\&\leq C_p\int_\Omega \left(|\mathcal{A}_k(x,\nabla u_{i_j})|^{p'} + |\mathcal{A}_k(x,\nabla u)|^{p'}\right)\omega_1 dx\\&\leq C_p\int_\Omega \left[\left(\gamma_1 + h_1|\nabla u_{i_j}|^{p-1}\right)^{p'} + \left(\gamma_1 + h_1|\nabla u|^{p-1}\right)^{p'}\right]\omega_1 dx\\&\leq C_p\int_\Omega \left[\left(\gamma_1 + h_1\psi^{p-1}\right)^{p'} + \left(\gamma_1 + h_1\psi^{p-1}\right)^{p'}\right]\omega_1 dx\\&\leq 2C_pC_p'\int_\Omega \left(\gamma_1^{p'} + h_1^{p'}\psi^p\right)\omega_1 dx\\&\leq 2C_pC_p'\left[\|\gamma_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|\psi\|^p_{L^p(\Omega,\omega_1)}\right].\end{aligned}$$

Hence, thanks to **(A1)**, we get, as $i_j \longrightarrow \infty$

$$B_k u_{i_j}(x) = \mathcal{A}_k(x,\nabla u_{i_j}(x)) \longrightarrow \mathcal{A}_k(x,\nabla u(x)) = B_k u(x), \quad \text{a.e. } x \in \Omega.$$

Therefore, by Lebesgue's theorem, we obtain

$$\|B_k u_{i_j} - B_k u\|_{L^{p'}(\Omega,\omega_1)} \longrightarrow 0,$$

that is,

$$B_k u_{i_j} \longrightarrow B_k u \quad \text{in} \quad L^{p'}(\Omega,\omega_1).$$

Finally, in view to convergence principle in Banach spaces, we have

$$B_k u_i \longrightarrow B_k u \quad \text{in} \quad L^{p'}(\Omega, \omega_1). \tag{3.1.3}$$

Step 4.2. For $k = 1, ..., n$, we define the operator

$$\begin{aligned} &M_k : W_0^{1,p}(\Omega, \omega_1, \omega_4) \longrightarrow L^{q'}(\Omega, \omega_2) \\ &(M_k u)(x) = \mathcal{B}_k(x, u(x), \nabla u(x)). \end{aligned}$$

We will prove that $M_k u_i \longrightarrow M_k u$ in $L^{q'}(\Omega, \omega_2)$.

(i) Let $u \in W_0^{1,p}(\Omega, \omega_1, \omega_4)$. Using **(A2)**, Remark 3.1.2 **(i)** and Theorem 1.1.36, we obtain

$$\begin{aligned} \|M_k u\|^{q'}_{L^{q'}(\Omega,\omega_2)} &= \int_\Omega |\mathcal{B}_k(x, u, \nabla u)|^{q'} \omega_2 dx \\ &\leq \int_\Omega \left(\gamma_2 + h_2|u|^{q-1} + h_3|\nabla u|^{q-1}\right)^{q'} \omega_2 dx \\ &\leq C_q \int_\Omega \left[\gamma_2^{q'} + h_2^{q'}|u|^q + h_3^{q'}|\nabla u|^q\right] \omega_2 dx \\ &= C_q \left[\int_\Omega \gamma_2^{q'} \omega_2 dx + \int_\Omega h_2^{q'}|u|^q \omega_2 dx + \int_\Omega h_3^{q'}|\nabla u|^q \omega_2 dx\right] \\ &\leq C_q \left[\int_\Omega \gamma_2^{q'} \omega_2 dx + \|h_2\|^{q'}_{L^\infty(\Omega)} \int_\Omega |u|^q \omega_2 dx + \|h_3\|^{q'}_{L^\infty(\Omega)} \int_\Omega |\nabla u|^q \omega_2 dx\right] \\ &\leq C_q \left[\|\gamma_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)} \|u\|^q_{L^q(\Omega,\omega_2)} + \|h_3\|^{q'}_{L^\infty(\Omega)} \|\nabla u\|^q_{L^q(\Omega,\omega_2)}\right] \\ &\leq C_q \left[\|\gamma_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)} C^q_{p,q} \|u\|^q_{L^p(\Omega,\omega_1)} + \|h_3\|^{q'}_{L^\infty(\Omega)} C^q_{p,q} \|\nabla u\|^q_{L^p(\Omega,\omega_1)}\right] \\ &\leq C_q \left[\|\gamma_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q}\left(C^q_\Omega \|h_2\|^{q'}_{L^\infty(\Omega)} + \|h_3\|^{q'}_{L^\infty(\Omega)}\right) \|u\|^q_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}\right], \end{aligned}$$

where the constant C_q depends only on q.

(ii) Let $u_i \longrightarrow u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_4)$ as $i \longrightarrow \infty$. According to **(A2)**, Remark 3.1.2 (i) and the same arguments used in Step 4.1 **(ii)**, we obtain analogously,

$$M_k u_i \longrightarrow M_k u \quad \text{in} \quad L^{q'}(\Omega, \omega_2). \tag{3.1.4}$$

Step 4.3. We define the operator

$$\begin{aligned} &H : W_0^{1,p}(\Omega, \omega_1, \omega_4) \longrightarrow L^{s'}(\Omega, \omega_3) \\ &(Hu)(x) = b(x, u(x)). \end{aligned}$$

In this step, we will show that $Hu_i \longrightarrow Hu$ in $L^{s'}(\Omega, \omega_3)$.

(i) Let $u \in W_0^{1,p}(\Omega, \omega_1, \omega_4)$. Using **(A2)** and Remark 3.1.2 (ii), we obtain

$$\begin{aligned}
\|Hu\|^{s'}_{L^{s'}(\Omega,\omega_3)} &= \int_\Omega |b(x,u)|^{s'}\omega_3 dx \\
&\le \int_\Omega \left(\gamma_3 + h_4|u|^{s-1}\right)^{s'} \omega_3 dx \\
&\le C_s \int_\Omega \left(\gamma_3^{s'} + h_4^{s'}|u|^s\right) \omega_3 dx \\
&\le C_s \left[\|\gamma_3\|^{s'}_{L^{s'}(\Omega,\omega_3)} + \|h_4\|^{p'}_{L^\infty(\Omega)}\|u\|^s_{L^s(\Omega,\omega_3)}\right] \\
&\le C_s \left[\|\gamma_3\|^{s'}_{L^{s'}(\Omega,\omega_3)} + C^s_{p,s}\|h_4\|^{p'}_{L^\infty(\Omega)}\|u\|^s_{L^p(\Omega,\omega_1)}\right] \\
&\le C_s \left[\|\gamma_3\|_{L^{s'}(\Omega,\omega_1)} + C^s_{p,s}C^s_\Omega\|h_4\|^{s'}_{L^\infty(\Omega)}\|u\|^s_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}\right],
\end{aligned}$$

where the constant C_s depends only on s.

(ii) Let $u_i \longrightarrow u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_4)$ as $i \longrightarrow \infty$. By **(A2)** and Remark 3.1.2 (ii), we get

$$\begin{aligned}
\|Hu_{i_j} - Hu\|^{s'}_{L^{s'}(\Omega,\omega_3)} &= \int_\Omega \left|Hu_{i_j}(x) - Hu(x)\right|^{p'} \omega_3 dx \\
&\le \int_\Omega \left(|b(x,u_{i_j})| + |b(x,u)|\right)^{s'} \omega_3 dx \\
&\le C_s \int_\Omega \left(|b(x,u_{i_j})|^{s'} + |b(x,u)|^{s'}\right)\omega_3 dx \\
&\le C_s \int_\Omega \left[\left(\gamma_3 + h_4|u_{i_j}|^{s-1}\right)^{s'} + \left(\gamma_3 + h_4|u|^{s-1}\right)^{s'}\right]\omega_3 dx \\
&\le C_s \int_\Omega \left[\left(\gamma_3 + h_4|\psi|^{s-1}\right)^{s'} + \left(\gamma_3 + h_4\psi^{s-1}\right)^{s'}\right]\omega_3 dx \\
&\le 2C_sC'_s \int_\Omega \left(\gamma_3^{s'} + h_4^{p'}\psi^s\right) \omega_3 dx \\
&\le 2C_sC'_s \left[\|\gamma_3\|^{s'}_{L^{s'}(\Omega,\omega_3)} + \|h_4\|^{s'}_{L^\infty(\Omega)}\|\psi\|^s_{L^s(\Omega,\omega_3)}\right] \\
&\le 2C_sC'_s \left[\|\gamma_3\|^{s'}_{L^{s'}(\Omega,\omega_3)} + C^s_{p,s}\|h_4\|^{s'}_{L^\infty(\Omega)}\|\psi\|^s_{L^p(\Omega,\omega_1)}\right],
\end{aligned}$$

next, using condition **(A1)**, we deduce, as $i_j \longrightarrow \infty$

$$Hu_{i_j}(x) = b(x, u_{i_j}(x)) \longrightarrow b(x, u(x)) = Hu(x), \quad \text{a.e. } x \in \Omega.$$

Therefore, by the Lebesgue's theorem, we obtain

$$\|Hu_{i_j} - Hu\|_{L^{s'}(\Omega,\omega_3)} \longrightarrow 0,$$

that is,

$$Hu_{i_j} \longrightarrow Hu \quad \text{in} \quad L^{s'}(\Omega, \omega_3).$$

We conclude, from the convergence principle in Banach spaces, that

$$Hu_i \longrightarrow Hu \quad \text{in} \quad L^{s'}(\Omega, \omega_3). \tag{3.1.5}$$

Step 4.4. We define the operator $J : W_0^{1,p}(\Omega, \omega_1, \omega_4) \longrightarrow L^{p'}(\Omega, \omega_4)$ by $(Ju)(x) = |u(x)|^{p-2}u(x)$. In this step, we will demonstrate that $Ju_i \longrightarrow Ju$ in $L^{p'}(\Omega, \omega_4)$.

(i) Let $u \in W_0^{1,p}(\Omega, \omega_1, \omega_4)$. We have

$$\begin{aligned} \|Ju\|_{L^{p'}(\Omega,\omega_4)}^{p'} &= \int_\Omega |Ju|^{p'} \omega_4 dx \\ &= \int_\Omega |u|^{(p-1)p'} \omega_4 dx \\ &= \int_\Omega |u|^p \omega_4 dx \\ &= \|u\|_{L^p(\Omega,\omega_4)}^p \\ &\leq \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}^p. \end{aligned}$$

(ii) Let $u_i \longrightarrow u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_4)$ as $i \longrightarrow \infty$. Then $u_i \longrightarrow u$ in $L^p(\Omega, \omega_4)$. Hence, thanks to Theorem 1.1.7, there exist a subsequence (u_{i_j}) and $\varphi \in L^p(\Omega, \omega_4)$ such that

$$\begin{aligned} &u_{i_j}(x) \longrightarrow u(x), \ \text{ as } \ i_j \longrightarrow \infty, && \text{a.e. in } \Omega \\ &|u_{i_j}(x)| \leq \varphi(x), && \text{a.e. in } \Omega. \end{aligned}$$

Next, we get

$$\begin{aligned} \|Ju_{i_j} - Ju\|_{L^{p'}(\Omega,\omega_4)}^{p'} &= \int_\Omega \Big|Ju_{i_j}(x) - Ju(x)\Big|^{p'} \omega_4 dx \\ &\leq \int_\Omega \Big(|Ju_{i_j}(x)| + |Ju(x)|\Big)^{p'} \omega_4 dx \\ &\leq C_p \int_\Omega \Big(|Ju_{i_j}(x)|^{p'} + |Ju(x)|^{p'}\Big) \omega_4 dx \\ &\leq C_p \int_\Omega \Big(||u_{i_j}|^{p-2} u_{i_j}|^{p'} + ||u|^{p-2} u|^{p'}\Big) \omega_4 dx \\ &\leq C_p \int_\Omega \Big(|u_{i_j}|^{(p-1)p'} + |u|^{(p-1)p'}\Big) \omega_4 dx \\ &\leq C_p \int_\Omega \Big(|u_{i_j}|^p + |u|^p\Big) \omega_4 dx \\ &\leq C_p \int_\Omega \Big(|\varphi|^p + |\varphi|^p\Big) \omega_4 dx \\ &\leq 2C_p \int_\Omega |\varphi|^p \omega_4 dx \\ &\leq 2C_p \|\varphi\|_{L^p(\Omega,\omega_4)}^p. \end{aligned}$$

Therefore, by Lebesgue's theorem, we obtain

$$\|Ju_{i_j} - Ju\|_{L^{p'}(\Omega,\omega_4)} \longrightarrow 0, \ \text{ as } \ i_j \longrightarrow \infty,$$

that is,

$$Ju_{i_j} \longrightarrow Ju \quad \text{in} \quad L^{p'}(\Omega, \omega_4).$$

We conclude, in view to convergence principle in Banach spaces, that

$$Ju_i \longrightarrow Ju \quad \text{in} \quad L^{p'}(\Omega, \omega_4). \tag{3.1.6}$$

Finally, let $v \in W_0^{1,p}(\Omega, \omega_1, \omega_4)$ and using Hölder inequality, we obtain

$$\begin{aligned}
|\mathbf{F}_1(u_i, v) - \mathbf{F}_1(u, v)| &= \left|\int_\Omega \langle \mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u), \nabla v\rangle \omega_1 dx\right| \\
&\leq \sum_{k=1}^n \int_\Omega |\mathcal{A}_k(x, \nabla u_i) - \mathcal{A}_k(x, \nabla u)||D_k v|\omega_1 dx \\
&= \sum_{k=1}^n \int_\Omega |B_k u_i - B_k u||D_k v|\omega_1 dx \\
&\leq \sum_{k=1}^n \|B_k u_i - B_k u\|_{L^{p'}(\Omega,\omega_1)} \|D_k v\|_{L^p(\Omega,\omega_1)} \\
&\leq \left(\sum_{k=1}^n \|B_k u_i - B_k u\|_{L^{p'}(\Omega,\omega_1)}\right) \|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_4)},
\end{aligned}$$

and by Remark 3.1.2 (i), we get

$$\begin{aligned}
|\mathbf{F}_2(u_i, v) - \mathbf{F}_2(u, v)| &= \left|\int_\Omega \langle \mathcal{B}(x, u_i, \nabla u_i) - \mathcal{B}(x, u, \nabla u), \nabla v\rangle \omega_2 dx\right| \\
&\leq \sum_{k=1}^n \int_\Omega |\mathcal{B}_k(x, u_i, \nabla u_i) - \mathcal{B}_k(x, u, \nabla u)||D_k v|\omega_2 dx \\
&= \sum_{k=1}^n \int_\Omega |M_k u_i - M_k u||D_k v|\omega_2 dx \\
&\leq \left(\sum_{k=1}^n \|M_k u_i - M_k u\|_{L^{q'}(\Omega,\omega_2)}\right) \|\nabla v\|_{L^q(\Omega,\omega_2)} \\
&\leq C_{p,q} \left(\sum_{k=1}^n \|M_k u_i - M_k u\|_{L^{q'}(\Omega,\omega_2)}\right) \|\nabla v\|_{L^p(\Omega,\omega_1)} \\
&\leq C_{p,q} \left(\sum_{k=1}^n \|M_k u_i - M_k u\|_{L^{q'}(\Omega,\omega_2)}\right) \|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_4)},
\end{aligned}$$

and by Remark 3.1.2 (ii), we get

$$\begin{aligned}
|\mathbf{F}_3(u_i, v) - \mathbf{F}_3(u, v)| &\leq \int_\Omega |g(x, u_i) - g(x, u)||v|\omega_3 dx \\
&= \int_\Omega |Hu_i - Hu||v|\omega_3 dx \\
&\leq \|Hu_i - Hu\|_{L^{s'}(\Omega,\omega_3)} \|v\|_{L^s(\Omega,\omega_3)} \\
&\leq C_{p,s} \|Hu_i - Hu\|_{L^{s'}(\Omega,\omega_3)} \|v\|_{L^p(\Omega,\omega_1)} \\
&\leq C_{p,s} C_\Omega \|Hu_i - Hu\|_{L^{s'}(\Omega,\omega_3)} \|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_4)},
\end{aligned}$$

and by Step 4.4, we obtain

$$\begin{aligned}
|\mathbf{F}_4(u_i, v) - \mathbf{F}_4(u, v)| &\leq \int_\Omega \left||u_i|^{p-2}u_i - |u|^{p-2}u\right||v|\omega_4 dx \\
&= \int_\Omega |Ju_i - Ju||v|\omega_4 dx \\
&\leq \|Ju_i - Ju\|_{L^{p'}(\Omega,\omega_4)} \|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}.
\end{aligned}$$

Hence, for all $v \in W_0^{1,p}(\Omega, \omega_1, \omega_4)$, we have

$$\begin{aligned}|\mathbf{F}(u_i, v) - \mathbf{F}(u, v)| &\leq \sum_{j=1}^{4} \Big|\mathbf{F}_j(u_i, v) - \mathbf{F}_j(u, v)\Big| \\ &\leq \Big[\sum_{k=1}^{n} \Big(\|B_k u_i - B_k u\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|M_k u_i - M_k u\|_{L^{q'}(\Omega,\omega_2)}\Big) \\ &\quad + C_{p,s}C_\Omega\|Hu_i - Hu\|_{L^{s'}(\Omega,\omega_3)} + \|Ju_i - Ju\|_{L^{p'}(\Omega,\omega_4)}\Big] \|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_4)}.\end{aligned}$$

Then, we get

$$\begin{aligned}\|\mathbf{A}u_i - \mathbf{A}u\|_* &\leq \sum_{k=1}^{n} \Big(\|B_k u_i - B_k u\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|M_k u_i - M_k u\|_{L^{q'}(\Omega,\omega_2)}\Big) \\ &\quad + C_{p,s}C_\Omega\|Hu_i - Hu\|_{L^{s'}(\Omega,\omega_3)} + \|Ju_i - Ju\|_{L^{p'}(\Omega,\omega_4)}.\end{aligned}$$

Combining (3.1.3), (3.1.4), (3.1.5) and (3.1.6), we deduce that

$$\|\mathbf{A}u_i - \mathbf{A}u\|_* \longrightarrow 0 \text{ as } i \longrightarrow \infty,$$

that is, $\mathbf{A}u_i \longrightarrow \mathbf{A}u$ in $\Big[W_0^{1,p}(\Omega, \omega_1, \omega_4)\Big]^*$. Hence, $\mathbf{A}$ is continuous and this implies that $\mathbf{A}$ is hemicontinuous.

Therefore, by Theorem 1.2.4, the operator equation $\mathbf{A}u = \mathbf{G}$ has exactly one solution $u \in W_0^{1,p}(\Omega, \omega_1, \omega_4)$ and it is the unique solution for the problem (3.0.1). With this last step the proof of Theorem 3.1.3 is completed.

3.2 Example

Take $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and consider the weight functions $\omega_1(x, y) = (x^2 + y^2)^{-1/2}$, $\omega_2(x, y) = (x^2 + y^2)^{-1/3}$, $\omega_3(x, y) = (x^2 + y^2)^{-1}$ and $\omega_4(x, y) = (x^2 + y^2)^{-3/2}$ $\Big($we have that ω_1, ω_2, ω_3, and ω_4 are A_4-weight, $p = 4$, $q = 3$ and $s = 2\Big)$, and the functions $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\mathcal{A} : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $b : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\mathcal{A}\Big((x, y), \xi\Big) = h_1(x, y)|\xi|^2\xi,$$

where $h_1(x, y) = 4e^{(x^2+y^2)}$, and

$$\mathcal{B}\Big((x, y), \eta, \xi\Big) = h_3(x, y)|\xi|\xi,$$

where $h_3(x, y) = 1 + \cos^2(xy)$, and

$$b\Big((x, y), \eta\Big) = h_4(x, y)\eta,$$

where $h_4(x, y) = 2 - \cos^2(xy)$.

Let us consider the operator

$$\begin{aligned} \mathbf{L}u(x,y) &= -\mathrm{div}\Big[\omega_1(x,y)\mathcal{A}\Big((x,y),\nabla u\Big) + \omega_2(x,y)\mathcal{B}\Big((x,y),u,\nabla u(x,y)\Big)\Big] \\ &\quad + \omega_3(x,y)b\Big((x,y),u\Big) + \omega_4(x,y)|u|^{p-2}u. \end{aligned}$$

Therefore, by Theorem 3.1.3, the problem

$$\begin{cases} \mathbf{L}u(x,y) = \dfrac{\cos(xy)}{x^2+y^2} & \text{in } \Omega, \\ u(x,y) = 0 & \text{on } \partial\Omega, \end{cases}$$

has exactly one solution $u \in W_0^{1,4}(\Omega, \omega_1, \omega_4)$.

Chapter 4

Existence and uniqueness of weak solution in weighted Sobolev spaces for a class of nonlinear degenerate elliptic problems with measure data

In this chapter, we discuss the existence and uniqueness of weak solution of a nonlinear degenerate elliptic equation of the form

$$\begin{cases} -\mathrm{div}\Big[\omega_1\mathcal{A}(x,\nabla u)+\nu_2\mathcal{B}(x,u,\nabla u)\Big]+\nu_1\mathcal{C}(x,u)+\omega_2|u|^{p-2}u=f-\mathrm{div}F & \text{in } \Omega,\\ u(x)=0 & \text{on } \partial\Omega, \end{cases} \tag{4.0.1}$$

in the setting of weighted Sobolev spaces $W_0^{1,p}(\Omega,\omega_1,\omega_2)$, where Ω is a bounded open set in $\mathbb{R}^n$, $1<p<\infty$, ω_1, ν_2, ν_1 and ω_2 are A_p-weight functions, $\mathcal{A}:\Omega\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$ and $\mathcal{B}:\Omega\times\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$, $\mathcal{C}:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ are Carathéodory functions that satisfy some conditions and the right-hand side term $f-\mathrm{div}F$ belongs to $L^{p'}(\Omega,\omega_2^{1-p'})+\prod_{j=1}^{n}L^{p'}(\Omega,\omega_1^{1-p'})$.

4.1 Existence Result

4.1.1 Hypotheses

Let us now give the precise hypotheses on the problem (4.0.1), we assume that the following assumptions: Ω be a bounded open subset of $\mathbb{R}^n$($n\geq 2$), $1<q,s<p<\infty$, let ω_1, ν_2, ν_1 and ω_2 are A_p-weight functions, and let $\mathcal{A}:\ \Omega\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$, $\mathcal{B}:\ \Omega\times\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$, with $\mathcal{A}(x,\xi)=\Big(\mathcal{A}_1(x,\xi),...,\mathcal{A}_n(x,\xi)\Big)$ and $\mathcal{B}(x,\eta,\xi)=\Big(\mathcal{B}_1(x,\eta,\xi),...,\mathcal{B}_n(x,\eta,\xi)\Big)$ and $\mathcal{C}:$ $\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfying the following assumptions:

(A1) For $k=1,...,n$, $\mathcal{A}_k$, $\mathcal{B}_k$ and $\mathcal{C}$ are Carathéodory functions.

(A2) There are positive functions $h_1,\ h_2,\ h_3,\ h_4 \in L^\infty(\Omega)$ and $\gamma_1 \in L^{p'}(\Omega,\omega_1)$, $\gamma_2 \in L^{q'}(\Omega,\nu_2)$ and $\gamma_3 \in L^{s'}(\Omega,\nu_1)$ such that

$$|\mathcal{A}(x,\xi)| \le \gamma_1(x) + h_1(x)|\xi|^{p-1},$$

$$|\mathcal{B}(x,\eta,\xi)| \le \gamma_2(x) + h_2(x)|\eta|^{q-1} + h_3(x)|\xi|^{q-1},$$

and

$$|\mathcal{C}(x,\eta)| \le \gamma_3(x) + h_4(x)|\eta|^{s-1},$$

where $(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n$.

(A3) There exists a constant $\alpha > 0$ such that

$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\xi^{'}), \xi-\xi^{'}\rangle \geqslant \alpha|\xi-\xi^{'}|^p,$$

$$\langle \mathcal{B}(x,\eta,\xi) - \mathcal{B}(x,\eta^{'},\xi^{'}), \xi-\xi^{'}\rangle \geqslant 0,$$

and

$$\Big(\mathcal{C}(x,\eta) - \mathcal{C}(x,\eta^{'})\Big)\Big(\eta-\eta^{'}\Big) \geqslant 0,$$

whenever $\eta,\eta' \in \mathbb{R}$ and $\xi,\xi' \in \mathbb{R}^n$ with $\eta \neq \eta^{'}$ and $\xi \neq \xi^{'}$.

(A4) There are constants $\beta_1,\ \beta_2,\ \beta_3 > 0$ such that

$$\langle \mathcal{A}(x,\xi),\xi\rangle \geqslant \beta_1|\xi|^p,$$

$$\langle \mathcal{B}(x,\eta,\xi),\xi\rangle \geqslant \beta_2|\xi|^q + \beta_3|\eta|^q,$$

and

$$\mathcal{C}(x,\eta)\eta \geqslant 0,$$

for all $(\eta,\xi) \in \mathbb{R}\times\mathbb{R}^n$.

4.1.2 Main result

First let us introduce the definition of a weak solution for problem (4.0.1).

Definition 4.1.1 *One says $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ is a weak solution to problem (4.0.1), provided that*

$$\int_\Omega \langle \mathcal{A}(x,\nabla u),\nabla v\rangle\,\omega_1\,dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u),\nabla v\rangle\,\nu_2\,dx + \int_\Omega \mathcal{C}(x,u)\,v\,\nu_1\,dx$$
$$+ \int_\Omega |u|^{p-2}u\,v\,\omega_2\,dx \ = \int_\Omega fv dx + \sum_{j=1}^n \int_\Omega f_j D_j v dx,$$

for all $v \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

Remark 4.1.2 *Let ω_1, ν_2, $\nu_1 \in A_p$, then*

(i) *If $\frac{\nu_2}{\omega_1} \in L^{r_1}(\Omega, \omega_1)$ where $r_1 = \frac{p}{p-q}$ and $1 < q < p < \infty$, then, by Hölder inequality we obtain*

$$||u||_{L^q(\Omega,\nu_2)} \leqslant C_{p,q}||u||_{L^p(\Omega,\omega_1)},$$

where $C_{p,q} = ||\frac{\nu_2}{\omega_1}||^{1/q}_{L^{r_1}(\Omega,\omega_1)}$.

(ii) *Analogously, if $\frac{\nu_1}{\omega_1} \in L^{r_2}(\Omega, \omega_1)$ where $r_2 = \frac{p}{p-s}$ and $1 < s < p < \infty$, then*

$$||u||_{L^s(\Omega,\nu_1)} \leqslant C_{p,s}||u||_{L^p(\Omega,\omega_1)},$$

where $C_{p,s} = ||\frac{\nu_1}{\omega_1}||^{1/s}_{L^{r_2}(\Omega,\omega_1)}$.

The principal result of this chapter reads as follows:

Theorem 4.1.3 *Let $\omega_i, \nu_i \in A_p (i = 1, 2)$, $1 < q, s < p < \infty$ and assume that the assumptions **(A1)** − **(A4)** hold. If*

1. *$f \in L^{p'}(\Omega, \omega_2^{1-p'})$ and $f_j \in L^{p'}(\Omega, \omega_1^{1-p'})$ for $j = 1, ..., n$,*
2. *$\frac{\nu_2}{\omega_1} \in L^{p/(p-q)}(\Omega, \omega_1)$ and $\frac{\nu_1}{\omega_1} \in L^{p/(p-s)}(\Omega, \omega_1)$,*

then the problem (4.0.1) has a unique solution $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$.

Proof. Our proof is based on the transform of problem (4.0.1) to a new one governed by an operator equation of the form $\mathcal{L}u = \mathcal{T}$, in order to apply the Browder-Minty theorem (Theorem 1.2.4). We define the operators $\mathcal{N} : W_0^{1,p}(\Omega, \omega_1, \omega_2) \times W_0^{1,p}(\Omega, \omega_1, \omega_2) \longrightarrow \mathbb{R}$ and $\mathcal{T} : W_0^{1,p}(\Omega, \omega_1, \omega_2) \longrightarrow \mathbb{R}$ by

$$\mathcal{N}(u, v) := \int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla v\rangle\, \omega_1\, dx + \int_\Omega \langle \mathcal{B}(x, u, \nabla u), \nabla v\rangle\, \nu_2\, dx + \int_\Omega \mathcal{C}(x, u)\, v\, \nu_1\, dx + \int_\Omega |u|^{p-2}u\, v\, \omega_2\, dx,$$

and

$$\mathcal{T}(v) := \int_\Omega f v dx + \sum_{j=1}^{n} \int_\Omega f_j D_j v dx.$$

Then $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ is a weak solution of (4.0.1) if and only if

$$\mathcal{N}(u, v) = \mathcal{T}(v), \qquad \text{for all } \; v \in W_0^{1,p}(\Omega, \omega_1, \omega_2).$$

The proof of Theorem 4.1.3 is divided into four steps.

Step 1.

In this step, we prove that the problem (4.0.1) is equivalent to an operator equation $\mathcal{L}u = \mathcal{T}$.

Let us show that $\mathcal{T} \in \left[W_0^{1,p}(\Omega,\omega_1,\omega_2)\right]^*$ and $\mathcal{N}(u,.)$ is linear and continuous, for each $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$.

Using Hölder inequality, Theorem 1.1.36 and Remark 4.1.2 (ii), we obtain

$$\begin{aligned}
|\mathcal{T}(v)| &\leq \int_\Omega |f| v dx + \sum_{j=1}^n \int_\Omega |f_j| D_j v dx \\
&= \int_\Omega \frac{|f|}{\omega_2} |v|\omega_2 \, dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega_1} |D_j v| \, \omega_1 dx \\
&\leq \|f/\omega_2\|_{L^{p'}(\Omega,\omega_2)} \|v\|_{L^p(\Omega,\omega_2)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)} \|D_j v\|_{L^p(\Omega,\omega_1)} \\
&\leq \left(C_\Omega \|f/\omega_2\|_{L^{p'}(\Omega,\omega_2)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)} \right) \|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

According to $f \in L^{p'}(\Omega,\omega_2^{1-p'})$ and $f_j \in L^{p'}(\Omega,\omega_1^{1-p'})$ for $j = 1, ..., n$, we deduce that $\mathcal{T} \in \left[W_0^{1,p}(\Omega,\omega_1,\omega_2)\right]^*$.
The operator $\mathcal{N}$ can be written as $\mathcal{N}(u,v) = \mathcal{N}_1(u,v) + \mathcal{N}_2(u,v) + \mathcal{N}_3(u,v) + \mathcal{N}_4(u,v)$, where $\mathcal{N}_i : W_0^{1,p}(\Omega,\omega_1,\omega_2) \times W_0^{1,p}(\Omega,\omega_1,\omega_2) \longrightarrow \mathbb{R}$, for $i = 1,2,3,4$, are defined as

$$\mathcal{N}_1(u,v) = \int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla v \rangle \omega_1 dx, \quad \mathcal{N}_2(u,v) = \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla v \rangle v_2 dx,$$

$$\mathcal{N}_3(u,v) = \int_\Omega \mathcal{C}(x,u) v \, v_1 dx, \quad \text{and} \quad \mathcal{N}_4(u,v) = \int_\Omega |u|^{p-2} u \, v \, \omega_2 \, dx.$$

Then, we have

$$|\mathcal{N}(u,v)| \leq |\mathcal{N}_1(u,v)| + |\mathcal{N}_2(u,v)| + |\mathcal{N}_3(u,v)| + |\mathcal{N}_4(u,v)|. \tag{4.1.1}$$

Also, by utilizing **(A2)**, Hölder inequality, Remark 4.1.2 (i) and Theorem 1.1.36, we have

$$\begin{aligned}
|\mathcal{N}_1(u,v)| &\leq \int_\Omega |\mathcal{A}(x,\nabla u)| |\nabla v| \omega_1 dx \\
&\leq \int_\Omega \left(\gamma_1 + h_1 |\nabla u|^{p-1}\right) |\nabla v| \omega_1 dx \\
&= \int_\Omega \gamma_1 \omega_1^{\frac{1}{p'}} |\nabla v| \omega_1^{\frac{1}{p}} dx + \int_\Omega h_1 |\nabla u|^{p-1} \omega_1^{\frac{1}{p'}} |\nabla v| \omega_1^{\frac{1}{p}} dx \\
&\leq \|\gamma_1\|_{L^{p'}(\Omega,\omega_1)} \|\nabla v\|_{L^p(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)} \|\nabla u\|^{p-1}_{L^p(\Omega,\omega_1)} \|\nabla v\|_{L^p(\Omega,\omega_1)} \\
&\leq \left(\|\gamma_1\|_{L^{p'}(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)} \|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \right) \|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{N}_2(u,v)| &\leq \int_\Omega |\mathcal{B}(x,u,\nabla u)||\nabla v|\nu_2 dx \\
&\leq \int_\Omega \left(\gamma_2 + h_2|u|^{q-1} + h_3|\nabla u|^{q-1}\right)|\nabla v|\nu_2 dx \\
&= \int_\Omega \gamma_2 \nu_2^{\frac{1}{q'}}|\nabla v|\nu_2^{\frac{1}{q}} dx + \int_\Omega h_2|u|^{q-1}\nu_2^{\frac{1}{q'}}|\nabla v|\nu_2^{\frac{1}{q}} dx + \int_\Omega h_3|\nabla u|^{q-1}\nu_2^{\frac{1}{q'}}|\nabla v|\nu_2^{\frac{1}{q}} dx \\
&\leq ||\gamma_2||_{L^{q'}(\Omega,\nu_2)}||\nabla v||_{L^q(\Omega,\nu_2)} + ||h_2||_{L^\infty(\Omega)}||u||^{q-1}_{L^q(\Omega,\nu_2)}||\nabla v||_{L^q(\Omega,\nu_2)} \\
&\qquad + ||h_3||_{L^\infty(\Omega)}||\nabla u||^{q-1}_{L^q(\Omega,\nu_2)}||\nabla v||_{L^q(\Omega,\nu_2)} \\
&\leq ||\gamma_2||_{L^{q'}(\Omega,\nu_2)}C_{p,q}||\nabla v||_{L^p(\Omega,\omega_1)} + ||h_2||_{L^\infty(\Omega)}C^{q-1}_{p,q}||u||^{q-1}_{L^p(\Omega,\omega_1)}C_{p,q}||\nabla v||_{L^p(\Omega,\omega_1)} \\
&\qquad + ||h_3||_{L^\infty(\Omega)}C^{q-1}_{p,q}||\nabla u||^{q-1}_{L^p(\Omega,\omega_1)}C_{p,q}||\nabla v||_{L^p(\Omega,\omega_1)} \\
&\leq \left[C^q_{p,q}\left(C^{q-1}_\Omega||h_2||_{L^\infty(\Omega)} + ||h_3||_{L^\infty(\Omega)}\right)||u||^{q-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} + C_{p,q}||\gamma_2||_{L^{q'}(\Omega,\nu_2)}\right]||v||_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

Similarly, by using **(A2)**, Hölder inequality, Remark 4.1.2 (ii) and Theorem 1.1.36, we get

$$\begin{aligned}
|\mathcal{N}_3(u,v)| &\leq \int_\Omega |\mathcal{C}(x,u)||v|\nu_1 dx \\
&\leq \left[C_\Omega C_{p,s}||\gamma_3||_{L^{s'}(\Omega,\nu_1)} + C^s_{p,s}C^s_\Omega||h_4||_{L^\infty(\Omega)}||u||^{s-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\right]||v||_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

Next, by applying Hölder inequality and Remark 4.1.2 (ii), we get

$$\begin{aligned}
|\mathcal{N}_4(u,v)| &\leq \int_\Omega |u|^{p-1}|v|\omega_2 dx \\
&\leq \left(\int_\Omega |u|^p\omega_2 dx\right)^{1/p'}\left(\int_\Omega |v|^p\omega_2 dx\right)^{1/p} \\
&= ||u||^{p-1}_{L^p(\Omega,\omega_2)}||v||_{L^p(\Omega,\omega_2)} \\
&\leq C_\Omega||u||^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}||v||_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

Hence, in (4.1.1), we obtain

$$\begin{aligned}
|\mathcal{N}(u,v)| &\leq \Big[||\gamma_1||_{L^{p'}(\Omega,\omega_1)} + ||h_1||_{L^\infty(\Omega)}||u||^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} + C_\Omega C_{p,s}||\gamma_3||_{L^{s'}(\Omega,\nu_1)} \\
&\quad + C_{p,q}||\gamma_2||_{L^{q'}(\Omega,\nu_2)} + C^q_{p,q}\left(C^{q-1}_\Omega||h_2||_{L^\infty(\Omega)} + ||h_3||_{L^\infty(\Omega)}\right)||u||^{q-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} \\
&\quad + C^s_{p,s}C^s_\Omega||h_4||_{L^\infty(\Omega)}||u||^{s-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} + C_\Omega||u||^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)}\Big]||v||_{W^{1,p}_0(\Omega,\omega_1,\omega_2)},
\end{aligned}$$

for all $u, v \in W^{1,p}_0(\Omega,\omega_1,\omega_2)$. Therefore, the operator $\mathcal{N}(u,.)$ is linear and continuous for every $u \in W^{1,p}_0(\Omega,\omega_1,\omega_2)$. As a result, there is a linear and continuous operator on $W^{1,p}_0(\Omega,\omega_1,\omega_2)$ labeled by $\mathcal{L}$ that provides $\langle \mathcal{L}u, v\rangle = \mathcal{N}(u,v)$ for all $u, v \in W^{1,p}_0(\Omega,\omega_1,\omega_2)$. We also have

$$\begin{aligned}
||\mathcal{L}u||_* &\leq ||\gamma_1||_{L^{p'}(\Omega,\omega_1)} + ||h_1||_{L^\infty(\Omega)}||u||^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} + C_\Omega C_{p,s}||\gamma_3||_{L^{s'}(\Omega,\nu_1)} \\
&\quad + C_{p,q}||\gamma_2||_{L^{q'}(\Omega,\nu_2)} + C^q_{p,q}\left(C^{q-1}_\Omega||h_2||_{L^\infty(\Omega)} + ||h_3||_{L^\infty(\Omega)}\right)||u||^{q-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} \\
&\quad + C^s_{p,s}C^s_\Omega||h_4||_{L^\infty(\Omega)}||u||^{s-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)} + C_\Omega||u||^{p-1}_{W^{1,p}_0(\Omega,\omega_1,\omega_2)},
\end{aligned}$$

where

$$\|\mathcal{L}u\|_* := \sup\left\{|\langle \mathcal{L}u, v\rangle| = |\mathcal{N}(u,v)| : v \in W_0^{1,p}(\Omega,\omega_1,\omega_2), \|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} = 1\right\},$$

is the norm in $\left[W_0^{1,p}(\Omega,\omega_1,\omega_2)\right]^*$. Therefore, we get the operator

$$\begin{aligned} \mathcal{L} : W_0^{1,p}(\Omega,\omega_1,\omega_2) &\longrightarrow \left[W_0^{1,p}(\Omega,\omega_1,\omega_2)\right]^* \\ u &\longmapsto \mathcal{L}u. \end{aligned}$$

Hence, the problem (4.0.1) is equivalent to the operator equation

$$\mathcal{L}u = \mathcal{T}, \quad u \in W_0^{1,p}(\Omega,\omega_1,\omega_2).$$

Step 2.

In this step, we demonstrate that **A** is strictly monotone. Let $v_1, v_2 \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$ with $v_1 \neq v_2$, then we have

$$\begin{aligned} \langle \mathcal{L}v_1 - \mathcal{L}v_2, v_1 - v_2\rangle &= \mathcal{N}(v_1, v_1 - v_2) - \mathcal{N}(v_2, v_1 - v_2) \\ &= \int_\Omega \langle \mathcal{A}(x,\nabla v_1), \nabla(v_1 - v_2)\rangle \omega_1 dx - \int_\Omega \langle \mathcal{A}(x,\nabla v_2), \nabla(v_1 - v_2)\rangle \omega_1 dx \\ &\quad + \int_\Omega \langle \mathcal{B}(x, v_1, \nabla v_1), \nabla(v_1 - v_2)\rangle \nu_2 dx - \int_\Omega \langle \mathcal{B}(x, v_2, \nabla v_2), \nabla(v_1 - v_2)\rangle \nu_2 dx \\ &\quad + \int_\Omega \mathcal{C}(x, v_1)(v_1 - v_2)\nu_1 dx - \int_\Omega \mathcal{C}(x, v_2)(v_1 - v_2)\nu_1 dx \\ &\quad + \int_\Omega |v_1|^{p-2} v_1 (v_1 - v_2)\omega_2 dx - \int_\Omega |v_2|^{p-2} v_2 (v_1 - v_2)\omega_2 dx \\ &= \int_\Omega \langle \mathcal{A}(x,\nabla v_1) - \mathcal{A}(x,\nabla v_2), \nabla(v_1 - v_2)\rangle \omega_1 dx \\ &\quad + \int_\Omega \langle \mathcal{B}(x, v_1, \nabla v_1) - \mathcal{B}(x, v_2, \nabla v_2), \nabla(v_1 - v_2)\rangle \nu_2 dx \\ &\quad + \int_\Omega \Big(\mathcal{C}(x, v_1) - \mathcal{C}(x, v_2)\Big)\Big(v_1 - v_2\Big)\nu_1 dx \\ &\quad + \int_\Omega \Big(|v_1|^{p-2} v_1 - |v_2|^{p-2} v_2\Big)\Big(v_1 - v_2\Big)\omega_2 dx. \end{aligned}$$

By using **(A3)** and Proposition 1.3.1 (ii), we obtain

$$\begin{aligned} \langle \mathcal{L}v_1 - \mathcal{L}v_2, v_1 - v_2\rangle &\geq \alpha \int_\Omega |\nabla(v_1 - v_2)|^p \; \omega_1 \; dx + \; \beta_p \int_\Omega \Big(|v_1| + |v_2|\Big)^{p-2} |v_1 - v_2|^2 \; \omega_2 \; dx \\ &\geq \alpha \int_\Omega |\nabla(v_1 - v_2)|^p \omega_1 dx \\ &\geq \alpha \|\nabla(v_1 - v_2)\|^p_{L^p(\Omega,\omega_1)}. \end{aligned}$$

Therefore, the operator $\mathcal{L}$ is strictly monotone.

Step 3.

This step establishes the coerciveness of the operator $\mathcal{L}$. For all $u \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$, we have

$$\begin{aligned}\langle \mathcal{L}u, u\rangle &= \mathcal{N}(u,u)\\ &= \mathcal{N}_1(u,u)+\mathcal{N}_2(u,u)+\mathcal{N}_3(u,u)+\mathcal{N}_4(u,u)\\ &= \int_\Omega \langle \mathcal{A}(x,\nabla u),\nabla u\rangle \omega_1 dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u),\nabla u\rangle \nu_2 dx + \int_\Omega \mathcal{C}(x,u)u\, \nu_1 dx + \int_\Omega |u|^p \omega_2 dx.\end{aligned}$$

From **(A4)** and Theorem 1.1.36, it follows that

$$\begin{aligned}\langle \mathcal{L}u, u\rangle &\geq \beta_1 \int_\Omega |\nabla u|^p \omega_1 dx + \beta_2 \int_\Omega |\nabla u|^q \nu_2 dx + \beta_3 \int_\Omega |u|^q \nu_2 dx + \int_\Omega |u|^p \omega_2 dx\\ &\geq \min(\beta_1, 1)\left[\int_\Omega |\nabla u|^p \omega_1 dx + \int_\Omega |u|^p \omega_2 dx\right] + \min(\beta_2,\beta_3)\left[\int_\Omega |\nabla u|^q \nu_2 dx + \int_\Omega |u|^q \nu_2 dx\right]\\ &\geq \min(\beta_1,1)\|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.\end{aligned}$$

Hence, we obtain

$$\frac{\langle \mathcal{L}u, u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}} \geq \min(\beta_1,1)\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.$$

Therefore, as $p > 1$, we get

$$\frac{\langle \mathcal{L}u, u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}} \longrightarrow +\infty \text{ as } \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)} \longrightarrow +\infty,$$

which means that $\mathcal{L}$ is coercive.

Step 4.

In this step, we show that $\mathcal{L}$ is continuous. To do this, consider $u_i \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1,\omega_2)$ as $i \longrightarrow \infty$. Then $\nabla u_i \longrightarrow \nabla u$ in $(L^p(\Omega,\omega_1))^n$. Therefore, according to Theorem 1.1.7, there exist a subsequence (u_{i_j}) and $\psi \in L^p(\Omega,\omega_1)$ in such a way that

$$\begin{aligned} &\nabla u_{i_j}(x) \longrightarrow \nabla u(x), \text{ as } i_j \longrightarrow \infty, \quad \text{a.e. in } \Omega\\ &|\nabla u_{i_j}(x)| \leq \psi(x), \quad \text{a.e. in } \Omega.\end{aligned} \tag{4.1.2}$$

We are going to establish that $\mathcal{L}u_i \longrightarrow \mathcal{L}u$ in $\left[W_0^{1,p}(\Omega,\omega_1,\omega_2)\right]^*$. It is proved in four steps.

Step 4.1.

We define the operator $B_k : W_0^{1,p}(\Omega,\omega_1,\omega_2) \longrightarrow L^{p'}(\Omega,\omega_1)$ by $(B_k u)(x) = \mathcal{A}_k(x,\nabla u(x))$ for $k = 1,...,n$. We need to show that $B_k u_i \longrightarrow B_k u$ in $L^{p'}(\Omega,\omega_1)$. We will apply the Lebesgue's theorem and the convergence principle in Banach spaces.

(i) Let $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$. Using **(A2)** and Theorem 1.1.36, we obtain

$$\begin{aligned}
\|B_k u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |B_k u(x)|^{p'} \omega_1 dx = \int_\Omega |\mathcal{A}_k(x, \nabla u)|^{p'} \omega_1 dx \\
&\leq \int_\Omega \left(\gamma_1 + h_1 |\nabla u|^{p-1}\right)^{p'} \omega_1 dx \\
&\leq C_p \int_\Omega \left(\gamma_1^{p'} + h_1^{p'} |\nabla u|^p\right) \omega_1 dx \\
&\leq C_p \left[\|\gamma_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)} \|\nabla u\|^p_{L^p(\Omega,\omega_1)}\right] \\
&\leq C_p \left[\|\gamma_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)} \|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}\right],
\end{aligned}$$

where the constant C_p depends only on p.

(ii) Let $u_i \longrightarrow u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ as $i \longrightarrow \infty$. By **(A2)** and (4.1.2), we obtain

$$\begin{aligned}
\|B_k u_{i_j} - B_k u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |B_k u_{i_j}(x) - B_k u(x)|^{p'} \omega_1 dx \\
&\leq \int_\Omega \Big(|\mathcal{A}_k(x, \nabla u_{i_j})| + |\mathcal{A}_k(x, \nabla u)|\Big)^{p'} \omega_1 dx \\
&\leq C_p \int_\Omega \left(|\mathcal{A}_k(x, \nabla u_{i_j})|^{p'} + |\mathcal{A}_k(x, \nabla u)|^{p'}\right) \omega_1 dx \\
&\leq C_p \int_\Omega \left[\left(\gamma_1 + h_1 |\nabla u_{i_j}|^{p-1}\right)^{p'} + \left(\gamma_1 + h_1 |\nabla u|^{p-1}\right)^{p'}\right] \omega_1 dx \\
&\leq C_p \int_\Omega \left[\left(\gamma_1 + h_1 \psi^{p-1}\right)^{p'} + \left(\gamma_1 + h_1 \psi^{p-1}\right)^{p'}\right] \omega_1 dx \\
&= 2C_p \int_\Omega \left(\gamma_1 + h_1 \psi^{\frac{p}{p'}}\right)^{p'} \omega_1 dx \\
&\leq 2C_p C'_p \int_\Omega \left(\gamma_1^{p'} + h_1^{p'} \psi^p\right) \omega_1 dx \\
&\leq 2C_p C'_p \left[\|\gamma_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)} \|\psi\|^p_{L^p(\Omega,\omega_1)}\right].
\end{aligned}$$

Hence, thanks to **(A1)**, we get, as $i_j \longrightarrow \infty$

$$B_k u_{i_j}(x) = \mathcal{A}_k(x, \nabla u_{i_j}(x)) \longrightarrow \mathcal{A}_k(x, \nabla u(x)) = B_k u(x), \qquad \text{a.e. } x \in \Omega.$$

Therefore, by Lebesgue's theorem, we obtain

$$\|B_k u_{i_j} - B_k u\|_{L^{p'}(\Omega,\omega_1)} \longrightarrow 0,$$

that is,

$$B_k u_{i_j} \longrightarrow B_k u \quad \text{in} \quad L^{p'}(\Omega, \omega_1).$$

Finally, in view to convergence principle in Banach spaces, we have

$$B_k u_i \longrightarrow B_k u \quad \text{in} \quad L^{p'}(\Omega, \omega_1). \tag{4.1.3}$$

Step 4.2.

Let us define the operator $M_k : W_0^{1,p}(\Omega, \omega_1, \omega_2) \longrightarrow L^{q'}(\Omega, \nu_2)$ by $(M_k u)(x) = \mathcal{B}_k(x, u(x), \nabla u(x))$ for $k = 1, ..., n$, We will prove that $M_k u_i \longrightarrow M_k u$ in $L^{q'}(\Omega, \nu_2)$.

(i) Let $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$. Using **(A2)**, Remark 4.1.2 **(i)** and Theorem 1.1.36, we obtain

$$\begin{aligned}
\|M_k u\|_{L^{q'}(\Omega,\nu_2)}^{q'} &= \int_\Omega |\mathcal{B}_k(x,u,\nabla u)|^{q'} \nu_2 dx \\
&\leq \int_\Omega \left(\gamma_2 + h_2|u|^{q-1} + h_3|\nabla u|^{q-1}\right)^{q'} \nu_2 dx \\
&\leq C_q \int_\Omega \left[\gamma_2^{q'} + h_2^{q'}|u|^q + h_3^{q'}|\nabla u|^q\right] \nu_2 dx \\
&= C_q \left[\int_\Omega \gamma_2^{q'} \nu_2 dx + \int_\Omega h_2^{q'}|u|^q \nu_2 dx + \int_\Omega h_3^{q'}|\nabla u|^q \nu_2 dx\right] \\
&\leq C_q \left[\int_\Omega \gamma_2^{q'} \nu_2 dx + \|h_2\|_{L^\infty(\Omega)}^{q'} \int_\Omega |u|^q \nu_2 dx + \|h_3\|_{L^\infty(\Omega)}^{q'} \int_\Omega |\nabla u|^q \nu_2 dx\right] \\
&\leq C_q \left[\|\gamma_2\|_{L^{q'}(\Omega,\nu_2)}^{q'} + \|h_2\|_{L^\infty(\Omega)}^{q'} \|u\|_{L^q(\Omega,\nu_2)}^q + \|h_3\|_{L^\infty(\Omega)}^{q'} \|\nabla u\|_{L^q(\Omega,\nu_2)}^q\right] \\
&\leq C_q \left[\|\gamma_2\|_{L^{q'}(\Omega,\nu_2)}^{q'} + \|h_2\|_{L^\infty(\Omega)}^{q'} C_{p,q}^q \|u\|_{L^p(\Omega,\omega_1)}^q + \|h_3\|_{L^\infty(\Omega)}^{q'} C_{p,q}^q \|\nabla u\|_{L^p(\Omega,\omega_1)}^q\right] \\
&\leq C_q \left[\|\gamma_2\|_{L^{q'}(\Omega,\nu_2)}^{q'} + C_{p,q}^q \left(C_\Omega^q \|h_2\|_{L^\infty(\Omega)}^{q'} + \|h_3\|_{L^\infty(\Omega)}^{q'}\right) \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}^q\right],
\end{aligned}$$

where the constant C_q depends only on q.

(ii) Let $u_i \longrightarrow u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ as $i \longrightarrow \infty$. According to **(A2)**, Remark 4.1.2 (i) and the same arguments used in Step 1 **(ii)**, we obtain analogously,

$$M_k u_i \longrightarrow M_k u \quad \text{in} \quad L^{q'}(\Omega, \nu_2). \tag{4.1.4}$$

Step 4.3.

We define the operator $H : W_0^{1,p}(\Omega, \omega_1, \omega_2) \longrightarrow L^{s'}(\Omega, \nu_1)$ by $(Hu)(x) = \mathcal{C}(x, u(x))$. In this step, we will show that $Hu_i \longrightarrow Hu$ in $L^{s'}(\Omega, \nu_1)$.

(i) Let $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$. Using **(A2)** and Remark 4.1.2 (ii), we obtain

$$\begin{aligned}
\|Hu\|_{L^{s'}(\Omega,\nu_1)}^{s'} &= \int_\Omega |\mathcal{C}(x,u)|^{s'} \nu_1 dx \\
&\leq \int_\Omega \left(\gamma_3 + h_4|u|^{s-1}\right)^{s'} \nu_1 dx \\
&\leq C_s \int_\Omega \left(\gamma_3^{s'} + h_4^{s'}|u|^s\right) \nu_1 dx \\
&\leq C_s \left[\|\gamma_3\|_{L^{s'}(\Omega,\nu_1)}^{s'} + \|h_4\|_{L^\infty(\Omega)}^{p'} \|u\|_{L^s(\Omega,\nu_1)}^s\right] \\
&\leq C_s \left[\|\gamma_3\|_{L^{s'}(\Omega,\nu_1)}^{s'} + C_{p,s}^s \|h_4\|_{L^\infty(\Omega)}^{p'} \|u\|_{L^p(\Omega,\omega_1)}^s\right] \\
&\leq C_s \left[\|\gamma_3\|_{L^{s'}(\Omega,\omega_1)} + C_{p,s}^s C_\Omega^s \|h_4\|_{L^\infty(\Omega)}^{s'} \|u\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}^s\right],
\end{aligned}$$

where the constant C_s depends only on s.

(ii) By **(A2)** and Remark 4.1.2 (ii), we get

$$\begin{aligned}
\|Hu_{i_j} - Hu\|^{s'}_{L^{s'}(\Omega,\nu_1)} &= \int_\Omega \left|Hu_{i_j}(x) - Hu(x)\right|^{s'} \nu_1 dx \\
&\leq \int_\Omega \left(|\mathcal{C}(x,u_{i_j})| + |\mathcal{C}(x,u)|\right)^{s'} \nu_1 dx \\
&\leq C_s \int_\Omega \left(|\mathcal{C}(x,u_{i_j})|^{s'} + |\mathcal{C}(x,u)|^{s'}\right)\nu_1 dx \\
&\leq C_s \int_\Omega \left[\left(\gamma_3 + h_4|u_{i_j}|^{s-1}\right)^{s'} + \left(\gamma_3 + h_4|u|^{s-1}\right)^{s'}\right]\nu_1 dx \\
&\leq C_s \int_\Omega \left[\left(\gamma_3 + h_4|\psi|^{s-1}\right)^{s'} + \left(\gamma_3 + h_4\psi^{s-1}\right)^{s'}\right]\nu_1 dx \\
&\leq 2C_sC'_s \int_\Omega \left(\gamma_3^{s'} + h_4^{p'}\psi^s\right) \nu_1 dx \\
&\leq 2C_sC'_s \left[\|\gamma_3\|^{s'}_{L^{s'}(\Omega,\nu_1)} + \|h_4\|^{s'}_{L^\infty(\Omega)}\|\psi\|^s_{L^s(\Omega,\nu_1)}\right] \\
&\leq 2C_sC'_s \left[\|\gamma_3\|^{s'}_{L^{s'}(\Omega,\nu_1)} + C^s_{p,s}\|h_4\|^{s'}_{L^\infty(\Omega)}\|\psi\|^s_{L^p(\Omega,\omega_1)}\right],
\end{aligned}$$

next, using condition **(A1)**, we deduce, as $i_j \longrightarrow \infty$

$$Hu_{i_j}(x) = \mathcal{C}(x, u_{i_j}(x)) \longrightarrow \mathcal{C}(x, u(x)) = Hu(x), \quad \text{a.e. } x \in \Omega.$$

Therefore, by the Lebesgue's theorem, we obtain

$$\|Hu_{i_j} - Hu\|_{L^{s'}(\Omega,\nu_1)} \longrightarrow 0,$$

that is,

$$Hu_{i_j} \longrightarrow Hu \quad \text{in} \quad L^{s'}(\Omega, \nu_1).$$

We conclude, from the convergence principle in Banach spaces, that

$$Hu_i \longrightarrow Hu \quad \text{in} \quad L^{s'}(\Omega, \nu_1). \tag{4.1.5}$$

Step 4.4.

We define the operator $J : W_0^{1,p}(\Omega, \omega_1, \omega_2) \longrightarrow L^{p'}(\Omega, \omega_2)$ by $(Ju)(x) = |u(x)|^{p-2}u(x)$. In this step, we will demonstrate that $Ju_i \longrightarrow Ju$ in $L^{p'}(\Omega, \omega_2)$.

(i) Let $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$. We have

$$\begin{aligned}
\|Ju\|^{p'}_{L^{p'}(\Omega,\omega_2)} &= \int_\Omega |Ju|^{p'} \omega_2 dx \\
&= \int_\Omega |u|^{(p-1)p'} \omega_2 dx \\
&= \int_\Omega |u|^p \omega_2 dx \\
&\leq \|u\|^p_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

(ii) Let $u_i \longrightarrow u$ in $W_0^{1,p}(\Omega, \omega_1, \omega_2)$ as $i \longrightarrow \infty$. Then $u_i \longrightarrow u$ in $L^p(\Omega, \omega_2)$. Hence, thanks to Theorem 1.1.7, there exist a subsequence (u_{i_j}) and $\varphi \in L^p(\Omega, \omega_2)$ such that

$$\begin{aligned} u_{i_j}(x) &\longrightarrow u(x), \text{ as } i_j \longrightarrow \infty, && \text{a.e. in } \Omega \\ |u_{i_j}(x)| &\leq \varphi(x), && \text{a.e. in } \Omega. \end{aligned}$$

Next, we get

$$\begin{aligned} \|Ju_{i_j} - Ju\|^{p'}_{L^{p'}(\Omega,\omega_2)} &= \int_\Omega \left|Ju_{i_j}(x) - Ju(x)\right|^{p'} \omega_2 dx \\ &\leq \int_\Omega \Big(|Ju_{i_j}(x)| + |Ju(x)|\Big)^{p'} \omega_2 dx \\ &\leq C_p \int_\Omega \Big(|Ju_{i_j}(x)|^{p'} + |Ju(x)|^{p'}\Big) \omega_2 dx \\ &\leq C_p \int_\Omega \Big(||u_{i_j}|^{p-2}u_{i_j}|^{p'} + ||u|^{p-2}u|^{p'}\Big) \omega_2 dx \\ &\leq C_p \int_\Omega \Big(|u_{i_j}|^{(p-1)p'} + |u|^{(p-1)p'}\Big) \omega_2 dx \\ &\leq C_p \int_\Omega \Big(|u_{i_j}|^{p} + |u|^{p}\Big) \omega_2 dx \\ &\leq C_p \int_\Omega \Big(|\varphi|^{p} + |\varphi|^{p}\Big) \omega_2 dx \\ &\leq 2C_p \int_\Omega |\varphi|^p \omega_2 dx \\ &\leq 2C_p \|\varphi\|^p_{L^p(\Omega,\omega_2)}. \end{aligned}$$

Therefore, by Lebesgue's theorem, we obtain

$$\|Ju_{i_j} - Ju\|_{L^{p'}(\Omega,\omega_2)} \longrightarrow 0,$$

that is,

$$Ju_{i_j} \longrightarrow Ju \quad \text{in} \quad L^{p'}(\Omega, \omega_2).$$

We conclude, in view to convergence principle in Banach spaces, that

$$Ju_i \longrightarrow Ju \quad \text{in} \quad L^{p'}(\Omega, \omega_2). \tag{4.1.6}$$

Finally, let $v \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ and using Hölder inequality, we obtain

$$\begin{aligned} |\mathcal{N}_1(u_i, v) - \mathcal{N}_1(u, v)| &= \left|\int_\Omega \langle \mathcal{A}(x, \nabla u_i) - \mathcal{A}(x, \nabla u), \nabla v\rangle \omega_1 dx\right| \\ &\leq \sum_{k=1}^n \int_\Omega |\mathcal{A}_k(x, \nabla u_i) - \mathcal{A}_k(x, \nabla u)||D_k v|\omega_1 dx \\ &= \sum_{k=1}^n \int_\Omega |B_k u_i - B_k u||D_k v|\omega_1 dx \\ &\leq \sum_{k=1}^n \|B_k u_i - B_k u\|_{L^{p'}(\Omega,\omega_1)} \|D_k v\|_{L^p(\Omega,\omega_1)} \\ &\leq \left(\sum_{k=1}^n \|B_k u_i - B_k u\|_{L^{p'}(\Omega,\omega_1)}\right) \|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}, \end{aligned}$$

and by Remark 4.1.2 (i), we get

$$\begin{aligned}
|\mathcal{N}_2(u_i,v)-\mathcal{N}_2(u,v)| &= |\int_\Omega \langle \mathcal{B}(x,u_i,\nabla u_i)-\mathcal{B}(x,u,\nabla u),\nabla v\rangle v_2 dx| \\
&\leq \sum_{k=1}^n \int_\Omega |\mathcal{B}_k(x,u_i,\nabla u_i)-\mathcal{B}_k(x,u,\nabla u)||D_k v|v_2 dx \\
&= \sum_{k=1}^n \int_\Omega |M_k u_i - M_k u||D_k v|v_2 dx \\
&\leq \left(\sum_{k=1}^n \|M_k u_i - M_k u\|_{L^{q'}(\Omega,v_2)}\right)\|\nabla v\|_{L^q(\Omega,v_2)} \\
&\leq C_{p,q}\left(\sum_{k=1}^n \|M_k u_i - M_k u\|_{L^{q'}(\Omega,v_2)}\right)\|\nabla v\|_{L^p(\Omega,\omega_1)} \\
&\leq C_{p,q}\left(\sum_{k=1}^n \|M_k u_i - M_k u\|_{L^{q'}(\Omega,v_2)}\right)\|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)},
\end{aligned}$$

and by Remark 4.1.2 (ii), we obtain

$$\begin{aligned}
|\mathcal{N}_3(u_i,v)-\mathcal{N}_3(u,v)| &\leq \int_\Omega |g(x,u_i)-g(x,u)||v|v_1 dx \\
&= \int_\Omega |Hu_i - Hu||v|v_1 dx \\
&\leq \|Hu_i - Hu\|_{L^{s'}(\Omega,v_1)}\|v\|_{L^s(\Omega,v_1)} \\
&\leq C_{p,s}\|Hu_i - Hu\|_{L^{s'}(\Omega,v_1)}\|v\|_{L^p(\Omega,\omega_1)} \\
&\leq C_{p,s}C_\Omega\|Hu_i - Hu\|_{L^{s'}(\Omega,v_1)}\|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

and by Step 4.4, we have

$$\begin{aligned}
|\mathcal{N}_4(u_i,v)-\mathcal{N}_4(u,v)| &\leq \int_\Omega \left||u_i|^{p-2}u_i - |u|^{p-2}u\right||v|\omega_2 dx \\
&= \int_\Omega |Ju_i - Ju||v|\omega_2 dx \\
&\leq \|Ju_i - Ju\|_{L^{p'}(\Omega,\omega_2)}\|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

Hence, for all $v \in W_0^{1,p}(\Omega,\omega_1,\omega_2)$, we have

$$\begin{aligned}
|\mathcal{N}(u_i,v)-\mathcal{N}(u,v)| &\leq \sum_{j=1}^4 \left|\mathcal{N}_j(u_i,v)-\mathcal{N}_j(u,v)\right| \\
&\leq \Bigg[\sum_{k=1}^n \left(\|B_k u_i - B_k u\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|M_k u_i - M_k u\|_{L^{q'}(\Omega,v_2)}\right) \\
&\quad + C_{p,s}C_\Omega\|Hu_i - Hu\|_{L^{s'}(\Omega,v_1)} + \|Ju_i - Ju\|_{L^{p'}(\Omega,\omega_2)}\Bigg]\|v\|_{W_0^{1,p}(\Omega,\omega_1,\omega_2)}.
\end{aligned}$$

Then, we get

$$\begin{aligned}
\|\mathcal{L}u_i - \mathcal{L}u\|_* &\leq \sum_{k=1}^n \left(\|B_k u_i - B_k u\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|M_k u_i - M_k u\|_{L^{q'}(\Omega,v_2)}\right) \\
&\quad + C_{p,s}C_\Omega\|Hu_i - Hu\|_{L^{s'}(\Omega,v_1)} + \|Ju_i - Ju\|_{L^{p'}(\Omega,\omega_2)}.
\end{aligned}$$

Combining (4.1.3), (4.1.4), (4.1.5) and (4.1.6), we deduce that

$$\|\mathcal{L}u_i - \mathcal{L}u\|_* \longrightarrow 0 \text{ as } i \longrightarrow \infty,$$

that is, $\mathcal{L}u_i \longrightarrow \mathcal{L}u$ in $\left[W_0^{1,p}(\Omega, \omega_1, \omega_2)\right]^*$. Hence, $\mathcal{L}$ is continuous and this implies that $\mathcal{L}$ is hemicontinuous.

Therefore, by Theorem 1.2.4, the operator equation $\mathcal{L}u = \mathcal{T}$ has exactly one solution $u \in W_0^{1,p}(\Omega, \omega_1, \omega_2)$ and it is the unique solution for the problem (4.0.1).

4.2 Example

Take $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and consider the weight functions $\omega_1(x, y) = (x^2 + y^2)^{-1/2}$, $\nu_2(x, y) = (x^2 + y^2)^{-1/3}$, $\nu_1(x, y) = (x^2 + y^2)^{-1}$ and $\omega_2(x, y) = (x^2 + y^2)^{-3/2}$ $\Big($we have that ω_1, ν_2, ν_1, and ω_2 are A_4–weight, $p = 4$, $q = 3$ and $s = 2\Big)$, and the functions $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\mathcal{A} : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $\mathcal{C} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\mathcal{A}\Big((x, y), \xi\Big) = h_1(x, y)|\xi|^2\xi,$$

where $h_1(x, y) = 4e^{(x^2+y^2)}$, and

$$\mathcal{B}\Big((x, y), \eta, \xi\Big) = h_3(x, y)|\xi|\xi,$$

where $h_3(x, y) = 1 + \cos^2(xy)$, and

$$\mathcal{C}\Big((x, y), \eta\Big) = h_4(x, y)\eta,$$

where $h_4(x, y) = 2 - \cos^2(xy)$.

Let us consider the operator

$$\begin{aligned} \mathbf{L}u(x, y) &= -\mathrm{div}\Big[\omega_1(x, y)\mathcal{A}\Big((x, y), \nabla u\Big) + \nu_2(x, y)\mathcal{B}\Big((x, y), u, \nabla u(x, y)\Big)\Big] \\ &\quad + \nu_1(x, y)\mathcal{C}\Big((x, y), u\Big) + \omega_2(x, y)|u|^{p-2}u \end{aligned}$$

Therefore, by Theorem 4.1.3, the problem

$$\begin{cases} \mathbf{L}u(x, y) = \frac{\cos(x+y)}{(x^2+y^2)} - \frac{\partial}{\partial x}\left(\frac{\sin(x+y)}{(x^2+y^2)}\right) - \frac{\partial}{\partial y}\left(\frac{\sin(x+y)}{(x^2+y^2)}\right) & \text{in } \Omega, \\ u(x, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

has exactly one solution $u \in W_0^{1,4}(\Omega, \omega_1, \omega_2)$.

Part II

Some elliptic and parabolic problems of Dirichlet or Neumann type via the theory of topological degrees in functional spaces

Chapter 5

Preliminaries

In the present chapter we introduce the notations and present all necessary and relevant properties about variable exponent Lebesgue-Sobolev spaces and topological degree theory.

5.1 Preliminaries about the functional framework

In recent years, the nonlinear problems with variable exponential growth is a new research field that drew the interest of many mathematical researcher. The principal interest of these problems come mainly from their applications, such in image processing (remove noise, edge detection and image restoration) and in the modelisation of the movement for a electrorheological fluids. This section will be devoted to introduce too the notion of variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, and some interesting definitions and properties, which are essential to prove some results of existence for solutions of the nonlinear elliptic problems studied in this thesis. For more details on these spaces, we refer the reader to [82, 101].

Let $\Omega \subset \mathbb{R}^N (N > 1)$ be an open with a Lipschitz boundary denoted by $\partial\Omega$. Denote

$$C_+(\overline{\Omega}) = \Big\{ p : \overline{\Omega} \longrightarrow [1, +\infty[\text{ continous such that } p(x) > 1 \Big\}.$$

We define

$$p^+ := \max \Big\{ p(x),\ x \in \overline{\Omega} \Big\} \text{ and } p^- := \min \Big\{ p(x),\ x \in \overline{\Omega} \Big\} \text{ for every } p \in C_+(\overline{\Omega}).$$

We define the Lebesgue space with a variable exponent $p \in C_+(\overline{\Omega})$ by

$$L^{p(x)}(\Omega) = \Big\{ f : \Omega \to \mathbb{R} \text{ is measurable such that } \int_\Omega |f(x)|^{p(x)} dx < +\infty \Big\}.$$

$L^{p(x)}(\Omega)$ is endowed with the following Luxembourg-type norm

$$|f|_{p(x)} = \inf \Big\{ \lambda > 0 : \rho_{p(x)}\Big(\frac{f}{\lambda}\Big) \leq 1 \Big\},$$

with

$$\rho_{p(x)}(f) = \int_\Omega |f(x)|^{p(x)} dx \ \text{ for all } \ f \in L^{p(x)}(\Omega).$$

Proposition 5.1.1 *[101] For any sequence (f_n) and all $f \in L^{p(x)}(\Omega)$, we have*

$$|f|_{p(x)} < 1(\text{resp.} = 1; > 1) \ \Leftrightarrow \ \rho_{p(x)}(f) < 1(\text{resp.} = 1; > 1), \tag{5.1.1}$$

$$|f|_{p(x)} > 1 \ \Rightarrow \ |f|^{p^-}_{p(x)} \leq \rho_{p(x)}(f) \leq |f|^{p^+}_{p(x)}, \tag{5.1.2}$$

$$|f|_{p(x)} < 1 \ \Rightarrow \ |f|^{p^+}_{p(x)} \leq \rho_{p(x)}(f) \leq |u|^{p^-}_{p(x)}, \tag{5.1.3}$$

$$\lim_{n\to\infty} |f_n - f|_{p(x)} = 0 \ \Leftrightarrow \ \lim_{n\to\infty} \rho_{p(x)}(f_n - f) = 0. \tag{5.1.4}$$

Remark 5.1.2 *From (5.1.2) and (5.1.3), we can infer that*

$$|f|_{p(x)} \leq \rho_{p(x)}(f) + 1, \tag{5.1.5}$$

$$\rho_{p(x)}(f) \leq |f|^{p^-}_{p(x)} + |f|^{p^+}_{p(x)}. \tag{5.1.6}$$

Proposition 5.1.3 *[101] The space $\left(L^{p(x)}(\Omega), |\cdot|_{p(x)}\right)$ is a separable and reflexive Banach space.*

Proposition 5.1.4 *[101] Let $f \in L^{p(x)}(\Omega)$ and $g \in L^{p'(x)}(\Omega)$. Then, we have the following Hölder-type inequality*

$$\Big| \int_\Omega fg \, dx \Big| \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big) |f|_{p(x)} |g|_{p'(x)} \leq 2|f|_{p(x)}|g|_{p'(x)}. \tag{5.1.7}$$

Remark 5.1.5 *If $p, \ q \in C_+(\overline{\Omega})$ with $p(x) \leq q(x)$ then $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.*

Now, we define the Sobolev space with a variable exponent $p \in C_+(\overline{\Omega})$ by

$$W^{1,p(x)}(\Omega) = \left\{ f \in L^{p(x)}(\Omega) \ : \ |\nabla f| \in (L^{p(x)}(\Omega))^N \right\},$$

and it is a Banach space under the norm

$$||f||_{1,p(x)} = |f|_{p(x)} + |\nabla f|_{p(x)}.$$

Furthermore, we have the compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$(see [101]).
We also define $W_0^{1,p(x)}(\Omega)$ as the subspace of $W^{1,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $|| \cdot ||_{1,p(x)}$.

Proposition 5.1.6 *[101] If the exponent $p(x)$ satisfy the log-Hölder continuity condition, i.e. there is a constant $a > 0$ such that for every $x,\ y \in \Omega,\ x \neq y$ with $|x-y| \leq \frac{1}{2}$ one has*

$$|p(x) - p(y)| \leq \frac{a}{-\log|x-y|}, \tag{5.1.8}$$

then, there exists $C > 0$ depending only on Ω and the function p such that

$$|f|_{p(x)} \leq C|\nabla f|_{p(x)} \text{ for all } f \in W_0^{1,p(x)}(\Omega). \tag{5.1.9}$$

In this thesis, we shall use the following norm on $W_0^{1,p(x)}(\Omega)$

$$|f|_{1,p(x)} = |\nabla f|_{p(x)},$$

and is equivalent to the norm $\|\cdot\|_{1,p(x)}$ (thanks Poincaré inequality (5.1.9)).

Proposition 5.1.7 *[101] The spaces $\left(W^{1,p(x)}(\Omega), \|\cdot\|_{1,p(x)}\right)$ and $\left(W_0^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\right)$ are separable and reflexive Banach spaces.*

Remark 5.1.8 *The dual space of $W_0^{1,p(x)}(\Omega)$ is the space $W^{-1,p'(x)}(\Omega)$ defined by*

$$W^{-1,p'(x)}(\Omega) := \left\{ f = f_0 - \sum_{i=1}^{N} D_i f_i \text{ with } (f_0, f_1, \ldots, f_N) \in (L^{p'(x)}(\Omega))^N \right\},$$

equipped with the norm

$$|f|_{-1,p'(x)} = \inf\left\{ |f_0|_{p'(x)} + \sum_{i=1}^{N} |f_i|_{p'(x)} \right\}.$$

Remark 5.1.9 *Note that for all $f \in W^{1,p(x)}(\Omega)$, we have*

$$|f|_{p(x)} \leq \|f\|_{1,p(x)} \text{ and } |\nabla f|_{p(x)} \leq \|f\|_{1,p(x)}.$$

Next, for all $f \in W^{1,p(x)}(\Omega)$, we introduce the following notation

$$\rho_{1,p(x)}(f) = \rho_{p(x)}(f) + \rho_{p(x)}(\nabla f).$$

Then, from [82, Theorem 1.3], we have the following result.

Proposition 5.1.10 *If $f \in W^{1,p(x)}(\Omega)$, then the following properties hold true*

$$\|f\|_{1,p(x)} < 1 (\text{resp.} = 1; > 1) \Leftrightarrow \rho_{1,p(x)}(f) < 1 (\text{resp.} = 1; > 1), \tag{5.1.10}$$

$$\|f\|_{1,p(x)} > 1 \Rightarrow \|f\|_{1,p(x)}^{p^-} \leq \rho_{1,p(x)}(f) \leq \|f\|_{1,p(x)}^{p^+}, \tag{5.1.11}$$

$$\|f\|_{1,p(x)} < 1 \Rightarrow \|f\|_{1,p(x)}^{p^+} \leq \rho_{1,p(x)}(f) \leq \|f\|_{1,p(x)}^{p^-}. \tag{5.1.12}$$

We may also consider the generalized Lebesgue space

$$L^{p(x)}(\Omega_T) = \Big\{ f : \Omega_T \to \mathbb{R} \text{ is measurable with } \int_0^T \int_\Omega |f(x,t)|^{p(x)} dxdt < \infty \Big\},$$

endowed with the norm

$$|f|_{L^{p(x)}(\Omega_T)} = \inf \Big\{ \lambda > 0 : \int_0^T \rho_{p(x)}\Big(\frac{f}{\lambda}\Big) dt \leq 1 \Big\},$$

which, of course, shares the same type of properties as $L^{p(x)}(\Omega)$.

As in [25], we introduce the functional space

$$\mathcal{W} := \Big\{ f \in L^{p^-}(0,T; W_0^{1,p(x)}(\Omega)) \ : \ |\nabla f| \in L^{p(x)}(\Omega_T)^N \Big\}, \tag{5.1.13}$$

which is a separable and reflexive Banach space endowed with the norm

$$|f|_{\mathcal{W}} := |f|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} + |\nabla f|_{L^{p(x)}(\Omega_T)}.$$

Thanks to Poincaré inequality (5.1.9), the expression

$$|f|_{\mathcal{W}} := |\nabla f|_{L^{p(x)}(\Omega_T)},$$

is a norm defined on $\mathcal{W}$ and is equivalent to the norm $|\cdot|_{\mathcal{W}}$.

Some interesting properties of the space $\mathcal{W}$ are stated in the following lemma.

Lemma 5.1.11 *[25] Let $\mathcal{W}$ be the space defined as above and $\mathcal{W}^*$ denote its dual space, then:*

1. *We have the following continuous dense embedding*

$$L^{p^+}(0,T; W_0^{1,p(x)}(\Omega)) \hookrightarrow \mathcal{W} \hookrightarrow L^{p^-}(0,T; W_0^{1,p(x)}(\Omega)). \tag{5.1.14}$$

2. *In particular, since $C_0^\infty(\Omega_T)$ is dense in $L^{p^+}(0,T; W_0^{1,p(x)}(\Omega))$, it is dense in $\mathcal{W}$ and for the corresponding dual spaces we have*

$$L^{(p^-)'}(0,\ T; W^{-1,p'(\cdot)}(\Omega)) \hookrightarrow \mathcal{W}^* \hookrightarrow L^{(p^+)'}(0,\ T; W^{-1,p'(\cdot)}(\Omega)). \tag{5.1.15}$$

3. *Under the assumption (9.0.3), we have*

$$|f|^{q^-}_{L^{q(\cdot)}(\Omega_T)} - 1 \leq \int_{\Omega_T} |f|^{q(x)} dxdt \leq |f|^{q^+}_{L^{q(\cdot)}(\Omega_T)} + 1 \leq |f|^{p^-}_{L^{p(x)}(\Omega_T)} - 1 \leq \int_{\Omega_T} |f|^{p(x)} dxdt \leq |f|^{p^+}_{L^{p(x)}(\Omega_T)} + 1. \tag{5.1.16}$$

5.2 Topological degree theory

Now, we give some results and properties from the theory of topological degree. The readers can find more information about the history of this theory in [9, 10, 30, 31, 99].

5.2.1 Topological degree theory for operators of the type $\mathcal{T}+\mathcal{S}$

In what follows, let Y is a real reflexive and separable Banach space with dual Y^* and continuous pairing $\langle .,. \rangle$, and given a nonempty subset $\mathcal{D}$ of Y, $\partial\mathcal{D}$ and $\overline{\mathcal{D}}$ represent the boundary and the closure of $\mathcal{D}$ in Y, respectively.

Definition 5.2.1 *We consider a mapping $\mathcal{T}$ defined from* Y *to* Y^* *and its graph is given by*

$$G(\mathcal{T}) = \Big\{(u,v) \in Y \times Y^* : v \in \mathcal{T}(u)\Big\}.$$

1. *$\mathcal{T}$ is said to be monotone if for all (u_1,v_1), (u_2,v_2) in $G(\mathcal{T})$, we get that $\langle v_1 - v_2, u_1 - u_2\rangle \geq 0$.*
2. *$\mathcal{T}$ is said to be maximal monotone if it is monotone and maximal in the sense of graph inclusion among monotone mappings from Y to Y^*, or for any $(u_0,v_0) \in Y \times Y^*$ for which $\langle v_0 - v, u_0 - u\rangle \geq 0$, for all $(u,v) \in G(\mathcal{T})$, we have $(u_0,v_0) \in G(\mathcal{T})$.*

Definition 5.2.2 *Let Z be a real Banach space. A operator $\mathcal{T} : \mathcal{D} \subset Y \to Z$ is said to be*

1. *bounded, if it takes any bounded set into a bounded set.*
2. *demicontinuous, if for any sequence $(u_n) \subset \mathcal{D}$, $u_n \to u$ implies that $\mathcal{T}(u_n) \rightharpoonup \mathcal{T}(u)$.*
3. *compact, if it is continuous and the image of any bounded set is relatively compact.*

Definition 5.2.3 *A mapping $\mathcal{S} : D(\mathcal{S}) \subset Y \to Y^*$ is said to be*

1. *of type (S_+), if for any $(u_n) \subset D(\mathcal{S})$ with $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty}\langle \mathcal{S}u_n, u_n - u\rangle \leq 0$, it follows that $u_n \to u$.*
2. *quasimonotone, if for any sequence $(u_n) \subset D(\mathcal{S})$ with $u_n \rightharpoonup u$, we have $\limsup\limits_{n\to\infty}\langle \mathcal{S}u_n,\ u_n - u\rangle \geq 0$.*

In the sequel, let $\mathcal{L}$ be a linear maximal monotone map from $D(\mathcal{L}) \subset Y$ to Y^*, and we consider the following classes of operators for each open and bounded subset G on Y:

$$\begin{aligned}\mathcal{F}_G := \Big\{&\mathcal{L}+\mathcal{S} : \overline{G} \cap D(\mathcal{L}) \to Y^* : \mathcal{S} \text{ is bounded, demicontinuous} \\ &\qquad\qquad \text{map of type } (S_+) \text{ with respect to } D(\mathcal{L}) \text{ from } \overline{G} \text{ to } Y^*\Big\},\\ \mathcal{H}_G := \Big\{&\mathcal{L}+\mathcal{S}(t) : \overline{G} \cap D(\mathcal{L}) \to Y^* : \mathcal{S}(t) \text{ is a bounded homotopy of type} \\ &\qquad\qquad \text{map of type } (S_+) \text{ with respect to } D(\mathcal{L}) \text{ from } \overline{G} \text{ to } Y^*\Big\}.\end{aligned}$$

Definition 5.2.4 *Let E be a bounded open subset of a real reflexive Banach space Y, $\mathcal{T} \in \mathcal{F}_1(\overline{E})$ be continuous and let $F, \mathcal{S} \in \mathcal{F}_{\mathcal{T}}(\overline{E})$. The affine homotopy $\Pi : [0,1] \times \overline{E} \to Y$ defined by*

$$\Pi(t,u) := (1-t)Fu + t\mathcal{S}u, \quad \textit{for all} \qquad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map $\mathcal{T}$.

Remark 5.2.5 *Note that the class $\mathcal{H}_G$ includes all affine homotopies*

$$\mathcal{L} + (1-t)\mathcal{S}_1 + t\mathcal{S}_2, \textit{ with } (\mathcal{L} + \mathcal{S}_i) \in \mathcal{F}_G, \; i = 1, 2.$$

Now, we introduce the Berkovits and Mustonen topological degree for the class $\mathcal{F}_G$, and see [31, 30] for more informations.

Theorem 5.2.6 *Let $\mathcal{L}$ a linear maximal monotone densely defined map from $D(\mathcal{L}) \subset Y$ to Y^*, and let*

$$\mathcal{E} = \Big\{(F, G, \phi) : F \in \mathcal{F}_G, \; G \textit{ an open bounded subset in } Y, \; \phi \notin F(\partial G \cap D(\mathcal{L}))\Big\}.$$

Then, there exists a topological degree function $d : \mathcal{E} \to \mathbb{Z}$ satisfying the following properties:

1. *(Existence) if $d(F, G, \phi) \neq 0$, then the equation $Fu = \phi$ has a solution in $G \cap D(\mathcal{L})$.*

2. *(Additivity) If G_1 and G_2 are two disjoint open subsets of G such that $\phi \notin F[(\overline{G} \backslash (G_1 \cup G_2)) \cap D(\mathcal{L})]$, then we have*

$$d(F, G, \phi) = d(F, G_1, \phi) + d(F, G_2, \phi).$$

3. *(Homotopy invariance) If $F(t) \in \mathcal{H}_G$ and $f(t) \notin F(t)(\partial G \cap D(\mathcal{L}))$ for all $t \in [0,1]$, where $f(t)$ is a continuous curve in Y^*, then*

$$d(F(t), G, f(t)) = C, \; \forall t \in [0,1].$$

4. *(Normalization) $\mathcal{L} + \mathcal{J}$ is a normalising map, where $\mathcal{J}$ is the duality mapping of Y into Y^*, that is,*

$$d(\mathcal{L} + \mathcal{J}, G, \phi) = 1, \textit{ for all } \phi \in (\mathcal{L} + \mathcal{J})(G \cap D(\mathcal{L})).$$

The following theorem plays an important role in the proof of the existence results in the next chapters.

Theorem 5.2.7 *Let $\mathcal{L} + \mathcal{S} \in \mathcal{F}_Y$ and $\phi \in Y^*$ and assume that there exists a radius $r > 0$ such that*

$$\langle \mathcal{L}u + \mathcal{S}u - \phi, u \rangle > 0, \tag{5.2.1}$$

for all $u \in \partial B_r(0) \cap D(\mathcal{L})$. Then the equation $\mathcal{L}u + \mathcal{S}u = \phi$ has a solution u in $D(\mathcal{L})$.

Proof. To show this theorem, it suffices to prove that $(\mathcal{L}+\mathcal{S})(D(\mathcal{L})) = Y^*$.

Let $F_\omega(t,\ u) = \mathcal{L}u + (1-t)\mathcal{J}u + t(\mathcal{S}u + \omega\mathcal{J}u - \phi)$, for all $\omega > 0$ and $t \in [0,1]$.

From (5.2.1) and since $0 \in \mathcal{L}(0)$, we obtain

$$\begin{aligned}
\langle F_\omega(t,u), u\rangle &= \langle t(\mathcal{L}u + \mathcal{S}u - \phi, u\rangle + \langle (1-t)\mathcal{L}u + (1-t+\omega)\mathcal{J}u, u\rangle \\
&\geq \langle (1-t)\mathcal{L}u + (1-t+\omega)\mathcal{J}u, u\rangle \\
&= (1-t)\langle \mathcal{L}u,\ u\rangle + (1-t+\omega)\langle \mathcal{J}u, u\rangle \\
&\geq (1-t+\omega)|u|^2 \\
&= (1-t+\omega)r^2 > 0.
\end{aligned}$$

Which implies that $0 \notin F_\omega(t,\ u)$.

Since $\mathcal{J}$ and $\mathcal{S} + \omega\mathcal{J}$ are continuous, bounded and of type (S_+), then $\{F_\omega(t,\ \cdot)\}_{t\in[0,1]}$ is an admissible homotopy. Therefore, applying the homotopy invariance and normalisation property of the degree d stated in Theorem 5.2.6, we obtain

$$d(F_\omega(t,\cdot), B_r(0), 0) = d(\mathcal{L}+\mathcal{J}, B_r(0), 0) = 1 \neq 0.$$

Consequently, by existence property of the degree d there exists a point $u_\omega \in D(\mathcal{L})$ such that $0 \in F_\omega(t,\ \cdot)$. In particular, by setting $\omega \to 0^+$ and $t = 1$, we get $\phi \in (\mathcal{L}+\mathcal{S})(D(\mathcal{L}))$ for some $u \in D(\mathcal{L})$ and that for all $\phi \in Y^*$(ϕ is arbitrary). Which implies that $(\mathcal{L}+\mathcal{S})(D(\mathcal{L})) = Y^*$.

5.2.2 Topological degree theory for a class of demicontinuous operators of generalized (S_+)

We start by defining some classes of mappings. In what follows, let X be a real separable reflexive Banach space and X^* be its dual space with dual pairing $\langle\cdot\,,\cdot\rangle$.

Definition 5.2.8 *Let Y be another real Banach space. A operator $F : \mathcal{D} \subset X \to Y$ is said to be*

1. *bounded, if it takes any bounded set into a bounded set.*
2. *demicontinuous, if for any sequence $(u_n) \subset \mathcal{D}$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.*
3. *compact, if it is continuous and the image of any bounded set is relatively compact.*

Definition 5.2.9 *A mapping $F : \mathcal{D} \subset X \to X^*$ is said to be*

1. *of type (S_+), if for any sequence $(u_n) \subset \mathcal{D}$ with $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$.*

2. *quasimonotone, if for any sequence* $(u_n) \subset \mathcal{D}$ *with* $u_n \rightharpoonup u$*, we have* $\limsup_{n\to\infty}\langle Fu_n,\ u_n - u\rangle \geq 0$.

Definition 5.2.10 *Let* $T : \mathcal{D}_1 \subset X \to X^*$ *be a bounded operator such that* $\mathcal{D} \subset \mathcal{D}_1$*. For any operator* $F : \mathcal{D} \subset X \to X$*, we say that*

1. F *is of type* $(S_+)_T$*, if for any sequence* $(u_n) \subset \mathcal{D}$ *with* $u_n \rightharpoonup u$*,* $y_n := Tu_n \rightharpoonup y$ *and* $\limsup_{n\to\infty}\langle Fu_n,\ y_n - y\rangle \leq 0$*, we have* $u_n \to u$*.*
2. F *has the property* $(QM)_T$*, if for any sequence* $(u_n) \subset \mathcal{D}$ *with* $u_n \rightharpoonup u$*,* $y_n := Tu_n \rightharpoonup y$*, we have* $\limsup_{n\to\infty}\langle Fu_n, y - y_n\rangle \geq 0$.

Consider the different types of operators as follows:

$$\begin{aligned}
\mathcal{F}_1(\mathcal{D}) &:= \Big\{F : \mathcal{D} \to X^* : F \text{ is bounded, demicontinuous and of type } (S_+)\Big\},\\
\mathcal{F}_T(\mathcal{D}) &:= \Big\{F : \mathcal{D} \to X : F \text{ is demicontinuous and of type } (S_+)_T\Big\},\\
\mathcal{F}_{T,B}(\mathcal{D}) &:= \Big\{F \in \mathcal{F}_T(\mathcal{D}) : F \text{ is bounded }\Big\},
\end{aligned}$$

for any $\mathcal{D} \subset D(F)$, where $D(F)$ denotes the domain of F, and any $T \in \mathcal{F}_1(\mathcal{D})$.
Now, let $\mathcal{O}$ be the collection of all bounded open sets in X and we define

$$\mathcal{F}(X) := \Big\{F \in \mathcal{F}_T(\overline{E}) : E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E})\Big\},$$

where, $T \in \mathcal{F}_1(\overline{E})$ is called an essential inner map to F.

Lemma 5.2.11 *[99, Lemma 2.3] Let* $T \in \mathcal{F}_1(\overline{E})$ *be continuous and* $S : D(S) \subset X^* \to X$ *be demicontinuous such that* $T(\overline{E}) \subset D(S)$*, where* E *is a bounded open set in a real reflexive Banach space* X*. Then the following statements are true :*

1. *If* S *is quasimonotone, then* $I + S \circ T \in \mathcal{F}_T(\overline{E})$*, where* I *denotes the identity operator.*
2. *If* S *is of type* (S_+)*, then* $S \circ T \in \mathcal{F}_T(\overline{E})$*.*

Definition 5.2.12 *Suppose that* E *is bounded open subset of a real reflexive Banach space* X*,* $T \in \mathcal{F}_1(\overline{E})$ *is continuous and* $F, S \in \mathcal{F}_T(\overline{E})$*. Then the affine homotopy* $\Lambda : [0,1] \times \overline{E} \to X$ *defined by*

$$\Lambda(t,u) := (1-t)Fu + tSu, \quad \textit{for} \quad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T*.*

Remark 5.2.13 *[99, Lemma 2.5]* The above affine homotopy is of type $(S_+)_T$.

Now, we give the topological degree for the class $\mathcal{F}(X)$ (see [99]).

Theorem 5.2.14 *Let*

$$M = \Big\{(F, E, h) : E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E}),\ F \in \mathcal{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\Big\}.$$

Then, there exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ that satisfy the following properties:

1. *(*Normalization*) For any $h \in F(E)$, we have*

$$d(I, E, h) = 1.$$

2. *(*Additivity*) Let $F \in \mathcal{F}_{T,B}(\overline{E})$. If E_1 and E_2 are two disjoint open subsets of E such that $h \notin F(\overline{E}\backslash(E_1 \cup E_2))$, then we have*

$$d(F, E, h) = d(F, E_1, h) + d(F, E_2, h).$$

3. *(*Homotopy invariance*) If $\Lambda : [0, 1] \times \overline{E} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and h: $[0, 1] \to X$ is a continuous path in X such that $h(t) \notin \Lambda(t, \partial E)$ for all $t \in [0, 1]$, then*

$$d(\Lambda(t, \cdot), E, h(t)) = C \textit{ for all } t \in [0, 1].$$

4. *(*Existence*) If $d(F, E, h) \neq 0$, then the equation $Fu = h$ has a solution in E.*

5. *(* Boundary dependence*) If $F, S \in \mathcal{F}_T(\overline{E})$, $F = S$ on ∂E, and $h \notin F(\partial E)$, then*

$$d(F, E, h) = d(S, E, h).$$

Definition 5.2.15 *[99, Definition 3.3] The above degree is defined as follows:*

$$d(F, E, h) \ := d_B(F|_{\overline{E}_0}, E_0, h),$$

where d_B is the Berkovits degree [29] and E_0 is any open subset of E with $F^{-1}(h) \subset E_0$ and F is bounded on $\overline{E}_0$.

Chapter 6

Existence result for Neumann problem with $p(x)$-Laplacian-like operators in generalized Sobolev space

This chapter studies the existence of a weak solutions for Neumann problem with $p(x)$-Laplacian-like operators, originated from a capillary phenomena, of the following form

$$\begin{cases} -\operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u + \dfrac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big) = \mu|u|^{\alpha(x)-2}u + \lambda f(x,u,\nabla u) & \text{in } \Omega, \\ \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.0.1}$$

in the setting of the generalized Sobolev spaces $W^{1,p(x)}(\Omega)$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p(\cdot), \alpha(\cdot) \in C_+(\overline{\Omega})$, $\frac{\partial u}{\partial \eta}$ is the exterior normal derivative, μ and λ are two real parameters. Based on the topological degree for a class of demicontinuous operators of generalized (S_+) type, under appropriate assumptions on f, we obtain a result on the existence of weak solutions to the considered problem.

6.1 Assumptions and notion of weak solution

We assume that $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (5.1.8), $\alpha \in C_+(\overline{\Omega})$ with $2 \leq \alpha^- \leq \alpha(x) \leq \alpha^+ < p^- \leq p(x) \leq p^+ < \infty$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a function such that:

(A_1) f is a Carathéodory condition.

(A_2) There exists $C_1 > 0$ and $\gamma \in L^{p'(x)}(\Omega)$ such that

$$|f(x,\zeta,\xi)| \leq C_1(\gamma(x) + |\zeta|^{q(x)-1} + |\xi|^{q(x)-1})$$

for a.e. $x \in \Omega$ and all $(\zeta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $q \in C_+(\overline{\Omega})$ with $2 \leq q^- \leq q(x) \leq q^+ < p^-$. The definition of a weak solutions for problem (6.0.1) can be stated as follows.

Definition 6.1.1 *We call that $u \in W^{1,p(x)}(\Omega)$ is a weak solution of (6.0.1) if*

$$\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla \varphi \, dx = \int_\Omega \Big(\mu |u|^{\alpha(x)-2}u + \lambda f(x,u,\nabla u)\Big)\varphi \, dx,$$

for all $\varphi \in W^{1,p(x)}(\Omega)$.

Remark 6.1.2

- *Note that* $\displaystyle\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla \varphi \, dx$ *is well defined (see [131]).*
- $\mu |u|^{\alpha(x)-2}u \in L^{p'(x)}(\Omega)$ *and* $\lambda f(x,u,\nabla u) \in L^{p'(x)}(\Omega)$ *under* $u \in W^{1,p(x)}(\Omega)$ *and the given hypotheses about the exponents* p, α *and* q *and* (A_2) *because:* $\gamma \in L^{p'(x)}(\Omega)$, $r(x) = (q(x)-1)p'(x) \in C_+(\overline{\Omega})$ *with* $r(x) < p(x)$ *and* $\beta(x) = (\alpha(x)-1)p'(x) \in C_+(\overline{\Omega})$ *with* $\beta(x) < p(x)$. *Then, by Remark 5.1.5 we can conclude that* $L^{p(x)} \hookrightarrow L^{r(x)}$ *and* $L^{p(x)} \hookrightarrow L^{\beta(x)}$.

  Hence, since $\varphi \in L^{p(x)}(\Omega)$, *we have* $\Big(\mu |u|^{\alpha(x)-2}u + \lambda f(x,u,\nabla u)\Big)\varphi \in L^1(\Omega)$. *This implies that, the integral* $\displaystyle\int_\Omega \Big(\mu |u|^{\alpha(x)-2}u + \lambda f(x,u,\nabla u)\Big)\varphi \, dx$ *exist.*

6.2 Main result

We are now in the position to get the existence result of weak solutions for (6.0.1).

Theorem 6.2.1 *If the assumptions $(A_1) - (A_2)$ hold, then the problem (6.0.1) possesses at least one weak solutions u in $W^{1,p(x)}(\Omega)$.*

Proof. First, we give several lemmas that will be used later.

Let us consider the following functional :

$$\mathcal{J}(u) := \int_\Omega \frac{1}{p(x)}\Big(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\Big)dx.$$

From [131], it is obvious that $\mathcal{J}$ is a continuously Gâteaux differentiable and $\mathcal{T} := \mathcal{J}'(u) \in W^{-1,p'(x)}(\Omega)$ such that

$$\langle \mathcal{T}u, \varphi\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla \varphi \, dx,$$

for all $u, \varphi \in W^{1,p(x)}(\Omega)$ where $\langle \cdot, \cdot \rangle$ means the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. In addition, the following lemma summarizes the properties of the operator $\mathcal{T}$ (see [131, Proposition 3.1.]).

Lemma 6.2.2 *The mapping*

$$\begin{aligned}&\mathcal{T}: W^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)\\ &\langle \mathcal{T}u, \varphi\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla \varphi \, dx,\end{aligned}$$

is a continuous, bounded, strictly monotone operator and is a mapping of class (S_+).

Lemma 6.2.3 *Assume that the assumptions* $(A_1)-(A_2)$ *hold, then the operator* $\mathcal{S}: W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *defined by*

$$\langle \mathcal{S}u, \varphi\rangle = -\int_\Omega \Big(\mu|u|^{\alpha(x)-2}u + \lambda f(x,u,\nabla u)\Big)\varphi \, dx, \quad \textit{for all} \ \ u, \varphi \in W^{1,p(x)}(\Omega),$$

is compact.

Proof. In order to prove this lemma, we proceed in three steps.
First step: Let us define the operator $\Psi : W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi u(x) := -\mu|u(x)|^{\alpha(x)-2}u(x).$$

In this step, we will prove that Ψ is bounded and continuous. It is clear that Ψ is continuous. Next we show that Ψ is bounded. Let $u \in W^{1,p(x)}(\Omega)$, and using (5.1.5) and (5.1.6) we obtain

$$\begin{aligned}|\Psi u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi u) + 1\\ &= \int_\Omega |\mu|u|^{\alpha(x)-2}u|^{p'(x)} dx + 1\\ &= \int_\Omega |\mu|^{p'(x)}|u|^{(\alpha(x)-1)p'(x)} dx + 1\\ &\leq \Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big)\rho_{\beta(x)}(u) + 1\\ &\leq \Big(|\mu|^{p'^-} + |\mu|^{p'^+}\Big)\Big(|u|^{\beta^-}_{\beta(x)} + |u|^{\beta^+}_{\beta(x)}\Big) + 1.\end{aligned}$$

Hence, we deduce from $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and Remark 5.1.9 that

$$|\Psi u|_{p'(x)} \leq C\Big(\|u\|^{\beta^-}_{1,p(x)} + \|u\|^{\beta^+}_{1,p(x)}\Big) + 1.$$

Consequently, Ψ is bounded on $W^{1,p(x)}(\Omega)$.
Second step: We define the operator $\Phi : W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Phi u(x) := -\lambda f(x,u,\nabla u).$$

We will show that Φ is bounded and continuous.

Let $u \in W^{1,p(x)}(\Omega)$. According to (A_2) and the inequalities (5.1.5) and (5.1.6), we obtain

$$\begin{aligned}
|\Phi u|_{p'(x)} &\leq \rho_{p'(x)}(\Phi u) + 1 \\
&= \int_\Omega |\lambda f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\lambda|^{p'(x)} |f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&\leq \left(|\lambda|^{p'^-} + |\lambda|^{p'^+}\right) \int_\Omega |C_1 \left(\gamma(x) + |u|^{q(x)-1} + |\nabla u|^{q(x)-1}\right)|^{p'(x)} dx + 1 \\
&\leq C\left(|\lambda|^{p'^-} + |\lambda|^{p'^+}\right) \int_\Omega \left(\gamma(x)^{p'(x)} + |u|^{(q(x)-1)p'(x)} + |\nabla u|^{(q(x)-1)p'(x)}\right) dx + 1 \\
&\leq C\left(|\lambda|^{p'^-} + |\lambda|^{p'^+}\right) \left(\rho_{p'(x)}(\gamma) + \rho_{r(x)}(u) + \rho_{r(x)}(\nabla u)\right) + 1 \\
&\leq C\left(|\lambda|^{p'^-} + |\lambda|^{p'^+}\right) \left(|\gamma|_{p(x)}^{p'^+} + |\gamma|_{p(x)}^{p'^-} + |u|_{r(x)}^{r^+} + |u|_{r(x)}^{r^-} + |\nabla u|_{r(x)}^{r^+} + |\nabla u|_{r(x)}^{r^-}\right) + 1.
\end{aligned}$$

Taking into account the continuous embedding $L^{p(x)} \hookrightarrow L^{r(x)}$ and Remark 5.1.9, we have then

$$|\Phi u|_{p'(x)} \leq C\left(|\gamma|_{p(x)}^{p'^+} + |\gamma|_{p(x)}^{p'^-} + \|u\|_{1,p(x)}^{r^+} + \|u\|_{1,p(x)}^{r^-}\right) + 1,$$

and consequently Φ is bounded on $W^{1,p(x)}(\Omega)$.

Let us show that Φ is continuous. To this purpose, let us assume $u_n \to u$ in $W^{1,p(x)}(\Omega)$, we need to show that $\Phi u_n \to \Phi u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.

Note that if $u_n \to u$ in $W^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Consequently, there exist a subsequence (u_k) of (u_n) and ϕ in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \;\text{ and }\; \nabla u_k(x) \to \nabla u(x), \;\text{ as } k \longrightarrow \infty, \tag{6.2.1}$$

$$|u_k(x)| \leq \phi(x) \;\text{ and }\; |\nabla u_k(x)| \leq |\psi(x)|, \;\text{ for all } k \in \mathbb{N}, \tag{6.2.2}$$

and for a.e. $x \in \Omega$.

Hence, from (A_2) and (6.2.2), we have

$$|f(x, u_k(x), \nabla u_k(x))| \leq C_1(\gamma(x) + |\phi(x)|^{q(x)-1} + |\psi(x)|^{q(x)-1}).$$

On the other hand, thanks to (A_1) and (6.2.1), we get, as $k \longrightarrow \infty$

$$f(x, u_k(x), \nabla u_k(x)) \to f(x, u(x), \nabla u(x)) \;\text{ a.e. }\; x \in \Omega.$$

Seeing that

$$\gamma + |\phi|^{q(x)-1} + |\psi(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and

$$\rho_{p'(x)}(\Phi u_k - \Phi u) = \int_\Omega |f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then, from the Lebesgue's theorem and the equivalence (5.1.4), we have

$$\Phi u_k \to \Phi u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and since Φ is single-valued, then

$$\Phi u_n \to \Phi u \ \text{ in } \ L^{p'(x)}(\Omega).$$

Third step: Let $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the natural embedding mapping $I : W^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$. We then define

$$I^* \circ \Psi : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

and

$$I^* \circ \Phi : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

On the other hand, due to the compactness of I, I^* also becomes compact. Thus, the compositions $I^* \circ \Psi$ and $I^* \circ \Phi$ are compact, and consequently, $\mathcal{S} = I^* \circ \Psi + I^* \circ \Phi$ is compact. With this last step the proof of Lemma 6.2.3 is completed.

Now we give the proof of the Theorem 6.2.1. The basic idea of our proof is to reformulate the problem (6.0.1) to an abstract formula governed by a Hammerstein equation, and apply the theory of topological degree introduced in Subsection 5.2.1 to show the existence of weak solution to (6.0.1).

First, for all $u, \varphi \in W^{1,p(x)}(\Omega)$, we define the operators $\mathcal{T}$ and $\mathcal{S}$, as defined in Lemmas 6.2.2 and 6.2.3 respectively,

$$\begin{aligned} &\mathcal{T} : W^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{T}u, \varphi \rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}} \Big) \nabla \varphi \, dx, \end{aligned}$$

$$\begin{aligned} &\mathcal{S} : W^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{S}u, \varphi \rangle = -\int_\Omega \Big(\mu |u|^{\alpha(x)-2}u + \lambda f(x,\ u,\ \nabla u) \Big) \varphi \, dx. \end{aligned}$$

Consequently, the problem (6.0.1) is equivalent to the equation

$$\mathcal{T}u = -\mathcal{S}u, \ \ u \in W^{1,p(x)}(\Omega). \tag{6.2.3}$$

Taking into account that, by Lemma 6.2.2, the operator $\mathcal{T}$ is a continuous, bounded, strictly monotone and of class (S_+), then, by [143, Theorem 26 A], the inverse operator

$$\mathcal{L} := \mathcal{T}^{-1} : W^{-1,p'(x)}(\Omega) \to W^{1,p(x)}(\Omega),$$

is also bounded, continuous, strictly monotone and of class (S_+).
On the other hand, according to Lemma 6.2.3, we have that the operator $\mathcal{S}$ is bounded, continuous and quasimonotone.
Consequently, following Zeidler's terminology [143], the equation (6.2.3) is equivalent to the following abstract Hammerstein equation

$$u = \mathcal{L}\varphi \text{ and } \varphi + \mathcal{S} \circ \mathcal{L}\varphi = 0, \quad u \in W^{1,p(x)}(\Omega) \text{ and } \varphi \in W^{-1,p'(x)}(\Omega). \tag{6.2.4}$$

Due to the equivalence of (6.2.3) and (6.2.4), it will be sufficient to solve (6.2.4). In order to solve (6.2.4), we will apply the Berkovits topological degree introduced in Subsection 5.2.1. First, let us set

$$\mathcal{B} := \Big\{ \varphi \in W^{-1,p'(x)}(\Omega) : \exists\, t \in [0,1] \text{ such that } \varphi + t\mathcal{S} \circ \mathcal{L}\varphi = 0 \Big\}.$$

Next, we show that $\mathcal{B}$ is bounded in $\in W^{-1,p'(x)}(\Omega)$. Let us put $u := \mathcal{L}\varphi$ for all $\varphi \in \mathcal{B}$. Taking into account that $|\mathcal{L}\varphi|_{1,p(x)} = ||u||_{1,p(x)}$, then we have the following two cases:
First case : If $||u||_{1,p(x)} \leq 1$, then $|\mathcal{L}\varphi|_{1,p(x)} \leq 1$, that means $\Big\{ \mathcal{L}\varphi : \varphi \in \mathcal{B} \Big\}$ is bounded.
Second case : If $||u||_{1,p(x)} > 1$, then, we deduce from (5.1.11), (A_2), the inequalities (5.1.6) and (5.1.7), and the Young's inequality that

$$\begin{aligned}
||\mathcal{L}\varphi||^{p^-}_{1,p(x)} &= ||u||^{p^-}_{1,p(x)} \\
&\leq \rho_{1,p(x)}(u) \\
&= \rho_{p(x)}(u) + \rho_{p(x)}(\nabla u) \\
&\leq \langle \mathcal{T}u,\ u\rangle \\
&= \langle \varphi,\ \mathcal{L}\varphi\rangle \\
&= -t\langle \mathcal{S} \circ \mathcal{L}\varphi,\ \mathcal{L}\varphi\rangle \\
&= \ t\int_\Omega \Big(\mu|u|^{\alpha(x)-2}u + \lambda f(x,\ u,\ \nabla u)\Big)u\,dx \\
&\leq t\max(|\mu|, C_1|\lambda|)\Big(\int_\Omega |u|^{\alpha(x)}dx + \int_\Omega |\gamma(x)u(x)|dx + \int_\Omega |u(x)|^{q(x)}dx + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big) \\
&= t\max(|\mu|, C_1|\lambda|)\Big(\rho_{\alpha(x)}(u) + \int_\Omega |\gamma(x)u(x)|dx + \rho_{q(x)}(u) + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big) \\
&\leq C\Big(|u|^{\alpha^-}_{\alpha(x)} + |u|^{\alpha^+}_{\alpha(x)} + |\gamma|_{p'(x)}|u|_{p(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} + \frac{1}{q'^-}\rho_{q(x)}(\nabla u) + \frac{1}{q^-}\rho_{q(x)}(u)\Big) \\
&\leq C\Big(|u|^{\alpha^-}_{\alpha(x)} + |u|^{\alpha^+}_{\alpha(x)} + |u|_{p(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} + |\nabla u|^{q^-}_{q(x)} + |\nabla u|^{q^+}_{q(x)}\Big).
\end{aligned}$$

Then, according to Remark 5.1.9, and the continuous embedding $L^{p(x)} \hookrightarrow L^{\alpha(x)}$ and $L^{p(x)} \hookrightarrow L^{q(x)}$, we get

$$||\mathcal{L}\varphi||^{p^-}_{1,p(x)} \leq C\Big(||\mathcal{L}\varphi||^{\alpha^+}_{1,p(x)} + ||\mathcal{L}\varphi||_{1,p(x)} + ||\mathcal{L}\varphi||^{q^+}_{1,p(x)}\Big),$$

what implies that $\Big\{\mathcal{L}\varphi : \varphi \in \mathcal{B}\Big\}$ is bounded.
On the other hand, we have that the operator is $\mathcal{S}$ is bounded, then $\mathcal{S} \circ \mathcal{L}\varphi$ is bounded. Thus, thanks to (6.2.4), we have that $\mathcal{B}$ is bounded in $W^{-1,p'(x)}(\Omega)$. However, there exists a constant $b > 0$ such that

$$|\varphi|_{-1,p'(x)} < b \;\text{ for all }\; \varphi \in \mathcal{B},$$

which leads to

$$\varphi + t\mathcal{S} \circ \mathcal{L}\varphi \neq 0, \;\; \varphi \in \partial\mathcal{B}_b(0) \;\text{ and }\; t \in [0,1],$$

where $\mathcal{B}_b(0)$ is the ball of center 0 and radius b in $W^{-1,p'(x)}(\Omega)$.
Moreover, by Lemma 5.2.11, we conclude that

$$I + \mathcal{S} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_b(0)}) \;\text{ and }\; I = \mathcal{T} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_b(0)}).$$

On another side, taking into account that I, $\mathcal{S}$ and $\mathcal{L}$ are bounded, then $I + \mathcal{S} \circ \mathcal{L}$ is bounded. Hence, we infer that

$$I + \mathcal{S} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{B}_b(0)}) \;\text{ and }\; I = \mathcal{T} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{B}_b(0)}).$$

Next, we define the homotopy

$$\begin{aligned} \mathcal{H} : [0,1] \times \overline{\mathcal{B}_b(0)} &\to W^{-1,p'(x)}(\Omega) \\ (t,\varphi) &\mapsto \mathcal{H}(t,\varphi) := \varphi + t\mathcal{S} \circ \mathcal{L}\varphi. \end{aligned}$$

Hence, thanks to the properties of the degree d as in Theorem 5.2.14, we obtain

$$d(I + \mathcal{S} \circ \mathcal{L}, \mathcal{B}_b(0), 0) = d(I, \mathcal{B}_b(0),\; 0) = 1 \neq 0,$$

what implies that there exists $\varphi \in \mathcal{B}_b(0)$ which verifies

$$\varphi + \mathcal{S} \circ \mathcal{L}\varphi = 0.$$

Finally, we infer that $u = \mathcal{L}\varphi$ is a weak solutions of (6.0.1). Thus, the proof of Theorem 6.2.1 is completed.

Chapter 7

On a class of $p(x)$-Laplacian-like Dirichlet problem depending on three real parameters

7.1 Position of the problem and hypotheses

This chapter establishes the existence of weak solution for a Dirichlet boundary value problem involving the $p(x)$-Laplacian-like operator depending on three real parameters, originated from a capillary phenomena, of the following form:

$$\begin{cases} -\Delta^l_{p(x)} u + \delta |u|^{\alpha(x)-2} u = \mu g(x,u) + \lambda f(x,u,\nabla u) & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1.1}$$

where $\Delta^l_{p(x)}$ is the $p(x)$-Laplacian-like operator, Ω is a smooth bounded domain in $\mathbb{R}^N$, δ, μ and λ are three real parameters, $p(\cdot), \alpha(\cdot) \in C_+(\overline{\Omega})$ and g, f are Carathéodory functions. Under suitable nonstandard growth conditions on g and f and using the topological degree for a class of demicontinuous operator of generalized (S_+) type and the theory of variable-exponent Sobolev spaces, we establish the existence of weak solution for the above problem.

We assume that $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (5.1.8), $\alpha \in C_+(\overline{\Omega})$ with $2 \leq \alpha^- \leq \alpha(x) \leq \alpha^+ < p^- \leq p(x) \leq p^+ < \infty$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are functions such that:

(A_1) f is a Carathéodory function.

(A_2) There exists $C_1 > 0$ and $\gamma \in L^{p'(x)}(\Omega)$ such that

$$|f(x,\zeta,\xi)| \leq C_1(\gamma(x) + |\zeta|^{q(x)-1} + |\xi|^{q(x)-1}).$$

(A_3) g is a Carathéodory function.

(A_4) There are $C_2 > 0$ and $\nu \in L^{p'(x)}(\Omega)$ such that

$$|g(x,\zeta)| \leq C_2(\nu(x) + |\zeta|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and all $(\zeta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $q, s \in C_+(\overline{\Omega})$ with $2 \leq q^- \leq q(x) \leq q^+ < p^-$ and $2 \leq s^- \leq s(x) \leq s^+ < p^-$.

Remark 7.1.1 *First. Note that, for all* $u, \varphi \in W_0^{1,p(x)}(\Omega)$, $\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\varphi dx$ *is well defined (see [131]). Second, we have* $\delta|u|^{\alpha(x)-2}u \in L^{p'(x)}(\Omega)$, $\mu g(x,u) \in L^{p'(x)}(\Omega)$ *and* $\lambda f(x,u,\nabla u) \in L^{p'(x)}(\Omega)$ *under* $u \in W_0^{1,p(x)}(\Omega)$, *the assumptions* (A_2) *and* (A_4) *and the given hypotheses about the exponents* p, α, q *and* s *because:* $r(x) = (q(x)-1)p'(x) \in C_+(\overline{\Omega})$ *with* $r(x) < p(x)$, $\beta(x) = (\alpha(x)-1)p'(x) \in C_+(\overline{\Omega})$ *with* $\beta(x) < p(x)$ *and* $\kappa(x) = (s(x)-1)p'(x) \in C_+(\overline{\Omega})$ *with* $\kappa(x) < p(x)$.
Then, by Remark 5.1.5 we can conclude that $L^{p(x)} \hookrightarrow L^{r(x)}$, $L^{p(x)} \hookrightarrow L^{\beta(x)}$ *and* $L^{p(x)} \hookrightarrow L^{\kappa(x)}$.
Hence, since $\varphi \in L^{p(x)}(\Omega)$, *we have*

$$\Big(-\delta|u|^{\alpha(x)-2}u + \mu g(x,u) + \lambda f(x,u,\nabla u)\Big)\varphi \in L^1(\Omega).$$

This implies that, the integral

$$\int_\Omega \Big(-\delta|u|^{\alpha(x)-2}u + \mu g(x,u) + \lambda f(x,u,\nabla u)\Big)\varphi dx$$

exists.

Then, we shall use the definition of weak solution for problem (7.1.1) in the following sense:

Definition 7.1.2 *We say that an element* $u \in W_0^{1,p(x)}(\Omega)$ *is a weak solution of* (7.1.1) *iff*

$$\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\varphi dx = \int_\Omega \Big(-\delta|u|^{\alpha(x)-2}u + \mu g(x,u) + \lambda f(x,u,\nabla u)\Big)\varphi dx,$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$.

7.2 Main result

Before giving our main result we first give two lemmas that will be used later.
Let us consider the following functional:

$$\mathcal{J}(u) := \int_\Omega \frac{1}{p(x)}\Big(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\Big)dx.$$

From [131], it is obvious that $\mathcal{J}$ is a continuously Gâteaux differentiable and
$\mathcal{T} := \mathcal{J}'(u) \in W^{-1,p'(x)}(\Omega)$ such that

$$\langle \mathcal{T}u, \varphi\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\varphi dx,$$

for all $u, \varphi \in W_0^{1,p(x)}(\Omega)$ where $\langle \cdot, \cdot \rangle$ means the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. In addition, the following lemma summarizes the properties of the operator $\mathcal{T}$ (see [131, Proposition 3.1.]).

Lemma 7.2.1 *The mapping $\mathcal{T} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)$ defined by*

$$\langle \mathcal{T}u, \varphi \rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}} \Big) \nabla \varphi dx, \textit{ for all } u, \varphi \in W_0^{1,p(x)}(\Omega),$$

is a continuous, bounded, strictly monotone operator, and is of class (S_+).

Lemma 7.2.2 *Assume that the assumptions $(A_1) - (A_4)$ hold. Then the operator $\mathcal{S} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by*

$$\langle \mathcal{S}u, \varphi \rangle = -\int_\Omega \Big(-\delta |u|^{\alpha(x)-2}u + \mu g(x,u) + \lambda f(x,u,\nabla u) \Big) \varphi dx, \textit{ for all } u, \varphi \in W_0^{1,p(x)}(\Omega),$$

is compact.

Proof. In order to prove this lemma, we proceed in four steps.
Step 1 : Let $\Upsilon : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\Upsilon u(x) := -\mu g(x,u).$$

In this step, we prove that the operator Υ is bounded and continuous.
First, let $u \in W_0^{1,p(x)}(\Omega)$, bearing (A_4) in mind and using (5.1.5) and (5.1.6), we infer

$$\begin{aligned}
|\Upsilon u|_{p'(x)} &\le \rho_{p'(x)}(\Upsilon u) + 1 \\
&= \int_\Omega |\mu g(x,u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\mu|^{p'(x)} |g(x,u(x)|^{p'(x)} dx + 1 \\
&\le \Big(|\mu|^{p'^-} + |\mu|^{p'^+} \Big) \int_\Omega |C_2 \Big(\nu(x) + |u|^{s(x)-1} \Big)|^{p'(x)} dx + 1 \\
&\le C\Big(|\mu|^{p'^-} + |\mu|^{p'^+} \Big) \int_\Omega \Big(|\nu(x)|^{p'(x)} + |u|^{\kappa(x)} \Big) dx + 1 \\
&\le C\Big(|\mu|^{p'^-} + |\mu|^{p'^+} \Big) \Big(\rho_{p'(x)}(\nu) + \rho_{\kappa(x)}(u) \Big) + 1 \\
&\le C\Big(|\nu|_{p(x)}^{p'^+} + |u|_{\kappa(x)}^{\kappa^+} + |u|_{\kappa(x)}^{\kappa^-} \Big) + 1.
\end{aligned}$$

Then, we deduce from (5.1.9) and $L^{p(x)} \hookrightarrow L^{\kappa(x)}$, that

$$|\Upsilon u|_{p'(x)} \le C\Big(|\nu|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{\kappa^+} + |u|_{1,p(x)}^{\kappa^-} \Big) + 1,$$

that means Υ is bounded on $W_0^{1,p(x)}(\Omega)$.
Second, we show that the operator Υ is continuous. To this purpose let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.

We need to show that $\Upsilon u_n \to \Upsilon u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.
Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$. Hence there exist a subsequence (u_k) of (u_n) and ϕ in $L^{p(x)}(\Omega)$ such that

$$u_k(x) \to u(x) \ \text{ and } \ |u_k(x)| \leq \phi(x), \tag{7.2.1}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$. Hence, from (A_2) and (7.2.1), we have

$$|g(x, u_k(x))| \leq C_2(\nu(x) + |\phi(x)|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
On the other hand, thanks to (A_3) and (7.2.1), we get, as $k \longrightarrow \infty$

$$g(x, u_k(x)) \to g(x, u(x)) \ \text{ a.e. } \ x \in \Omega.$$

Seeing that

$$\nu + |\phi|^{s(x)-1} \in L^{p'(x)}(\Omega) \ \text{ and } \ \rho_{p'(x)}(\Upsilon u_k - \Upsilon u) = \int_\Omega |g(x, u_k(x)) - g(x, u(x))|^{p'(x)} dx,$$

then, from the Lebesgue's theorem and the equivalence (5.1.4), we have

$$\Upsilon u_k \to \Upsilon u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and consequently

$$\Upsilon u_n \to \Upsilon u \ \text{ in } \ L^{p'(x)}(\Omega),$$

that is, Υ is continuous.
Step 2 : We define the operator $\Psi : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi u(x) := \delta |u(x)|^{\alpha(x)-2} u(x).$$

We will prove that Ψ is bounded and continuous.
It is clear that Ψ is continuous. Next we show that Ψ is bounded.
Let $u \in W_0^{1,p(x)}(\Omega)$ and using (5.1.5) and (5.1.6), we obtain

$$\begin{aligned}
|\Psi u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi u) + 1 \\
&= \int_\Omega |\delta |u|^{\alpha(x)-2} u|^{p'(x)} dx + 1 \\
&= \int_\Omega |\delta|^{p'(x)} |u|^{(\alpha(x)-1)p'(x)} dx + 1 \\
&\leq \left(|\delta|^{p'^-} + |\delta|^{p'^+} \right) \int_\Omega |u|^{\beta(x)} dx + 1 \\
&= \left(|\delta|^{p'^-} + |\delta|^{p'^+} \right) \rho_{\beta(x)}(u) + 1 \\
&\leq \left(|\delta|^{p'^-} + |\delta|^{p'^+} \right) \left(|u|^{\beta^-}_{\beta(x)} + |u|^{\beta^+}_{\beta(x)} \right) + 1.
\end{aligned}$$

Hence, we deduce from $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and (5.1.9) that

$$|\Psi u|_{p'(x)} \leq C\left(|u|_{1,p(x)}^{\beta^-} + |u|_{1,p(x)}^{\beta^+}\right) + 1,$$

and consequently, Ψ is bounded on $W_0^{1,p(x)}(\Omega)$.
Step 3 : Let us define the operator $\Phi : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Phi u(x) := -\lambda f(x, u(x), \nabla u(x)).$$

We will show that Φ is bounded and continuous.
Let $u \in W_0^{1,p(x)}(\Omega)$. According to (A_2) and the inequalities (5.1.5) and (5.1.6), we obtain

$$\begin{aligned}
|\Phi u|_{p'(x)} &\leq \rho_{p'(x)}(\Phi u) + 1 \\
&= \int_\Omega |\lambda f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\lambda|^{p'(x)} |f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&\leq \left(|\lambda|^{p'^-} + |\lambda|^{p'^+}\right) \int_\Omega |C_1\left(\gamma(x) + |u|^{q(x)-1} + |\nabla u|^{q(x)-1}\right)|^{p'(x)} dx + 1 \\
&\leq C\left(|\lambda|^{p'^-} + |\lambda|^{p'^+}\right) \int_\Omega \left(|\gamma(x)|^{p'(x)} + |u|^{r(x)} + |\nabla u|^{r(x)}\right) dx + 1 \\
&\leq C\left(|\lambda|^{p'^-} + |\lambda|^{p'^+}\right)\left(\rho_{p'(x)}(\gamma) + \rho_{r(x)}(u) + \rho_{r(x)}(\nabla u)\right) + 1 \\
&\leq C\left(|\gamma|_{p(x)}^{p'^+} + |u|_{r(x)}^{r^+} + |u|_{r(x)}^{r^-} + |\nabla u|_{r(x)}^{r^+} + |\nabla u|_{r(x)}^{r^-}\right) + 1.
\end{aligned}$$

Taking into account that $L^{p(x)} \hookrightarrow L^{r(x)}$ and (5.1.9), we have then

$$|\Phi u|_{p'(x)} \leq C\left(|\gamma|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{r^+} + |u|_{1,p(x)}^{r^-}\right) + 1,$$

and consequently Φ is bounded on $W_0^{1,p(x)}(\Omega)$.
It remains to show that Φ is continuous. Let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, we need to show that $\Phi u_n \to \Phi u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.
Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Hence, there exist a subsequence (u_k) and ϕ in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \text{ and } \nabla u_k(x) \to \nabla u(x),\ |u_k(x)| \leq \phi(x) \text{ and } |\nabla u_k(x)| \leq |\psi(x)|, \tag{7.2.2}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$. Hence, thanks to (A_1) and (7.2.2), we get, as $k \longrightarrow \infty$

$$f(x, u_k(x), \nabla u_k(x)) \to f(x, u(x), \nabla u(x)) \text{ a.e. } x \in \Omega.$$

On the other hand, from (A_2) and (7.2.2), we can deduce the estimate

$$|f(x, u_k(x), \nabla u_k(x))| \leq C_1(\gamma(x) + |\phi(x)|^{q(x)-1} + |\psi(x)|^{q(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$. Seeing that

$$\gamma + |\phi|^{q(x)-1} + |\psi(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{p'(x)}(\Phi u_k - \Phi u) = \int_\Omega |f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then, we conclude from the Lebesgue's theorem and (5.1.4) that

$$\Phi u_k \to \Phi u \ \text{ in } \ L^{p'(x)}(\Omega)$$

and consequently

$$\Phi u_n \to \Phi u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and then Φ is continuous.

Step 4 : Let $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the operator $I : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$. Next, we define

$$I^* o\Upsilon : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

$$I^* o\Psi : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

and

$$I^* o\Phi : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

On another side, taking into account that I is compact, then I^* is compact. Thus, the compositions $I^* o\Upsilon$, $I^* o\Psi$ and $I^* o\Phi$ are compact, that means $\mathcal{S} = I^* o\Upsilon + I^* o\Psi + I^* o\Phi$ is compact. With this last step the proof of Lemma 7.2.2 is completed.

We are now in the position to get the existence result of weak solution for (7.1.1).

Theorem 7.2.3 *Assume that the assumptions $(A_1) - (A_4)$ hold, then the problem (7.1.1) possesses at least one weak solution u in $W_0^{1,p(x)}(\Omega)$.*

Proof. The basic idea of our proof is to reduce the problem (7.1.1) to a new one governed by a Hammerstein equation, and apply the topological degree theory to show the existence of weak solution to the state problem. First, for all $u, \varphi \in W_0^{1,p(x)}(\Omega)$, we define the operators $\mathcal{T} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)$ and $\mathcal{S} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)$ by

$$\langle \mathcal{T}u, \varphi \rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2} \nabla u + \frac{|\nabla u|^{2p(x)-2} \nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}} \Big) \nabla \varphi dx,$$

$$\langle \mathcal{S}u, \varphi \rangle = -\int_\Omega \Big(-\delta |u|^{\alpha(x)-2} u + \mu g(x, u) + \lambda f(x, u, \nabla u) \Big) \varphi dx.$$

Consequently, the problem (7.1.1) is equivalent to the equation

$$\mathcal{T}u = -\mathcal{S}u, \;\; u \in W_0^{1,p(x)}(\Omega). \tag{7.2.3}$$

Taking into account that, by Lemma 7.2.1, the operator $\mathcal{T}$ is a continuous, bounded, strictly monotone and of class (S_+), then, by [143, Theorem 26 A], the inverse operator

$$\mathcal{L} := \mathcal{T}^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega),$$

is also bounded, continuous, strictly monotone and of class (S_+).
On another side, according to Lemma 7.2.2, we have that the operator $\mathcal{S}$ is bounded, continuous and quasimonotone.
Consequently, following Zeidler's terminology [143], the equation (7.2.3) is equivalent to the following abstract Hammerstein equation

$$u = \mathcal{L}\varphi \;\text{ and }\; \varphi + \mathcal{S}o\mathcal{L}\varphi = 0, \;\; u \in W_0^{1,p(x)}(\Omega) \text{ and } \varphi \in W^{-1,p'(x)}(\Omega). \tag{7.2.4}$$

Seeing that (7.2.3) is equivalent to (7.2.4), then to solve (7.2.3) it is thus enough to solve (7.2.4). In order to solve (7.2.4), we will apply the Berkovits topological degree introducing in Subsection 5.2. First, let us set

$$\mathcal{B} := \Big\{\varphi \in W^{-1,p'(x)}(\Omega) : \exists\, t \in [0,1] \;\text{ such that }\; \varphi + t\mathcal{S}o\mathcal{L}\varphi = 0\Big\}.$$

Next, we show that $\mathcal{B}$ is bounded in $\in W^{-1,p'(x)}(\Omega)$.
Let us put $u := \mathcal{L}\varphi$ for all $\varphi \in \mathcal{B}$. Taking into account that $|\mathcal{L}\varphi|_{1,p(x)} = |\nabla u|_{p(x)}$, then we have the following two cases:
First case : If $|\nabla u|_{p(x)} \leq 1$, then $|\mathcal{L}\varphi|_{1,p(x)} \leq 1$, that means $\Big\{\mathcal{L}\varphi : \varphi \in \mathcal{B}\Big\}$ is bounded.
Second case : If $|\nabla u|_{p(x)} > 1$, then, we deduce from (5.1.2), (A_2) and (A_4), the inequalities

(5.1.7) and (5.1.6) and the Young's inequality that

$$\begin{aligned}
|\mathcal{L}\varphi|_{1,p(x)}^{p^-} &= |\nabla u|_{p(x)}^{p-}\\
&\leq \rho_{p(x)}(\nabla u)\\
&\leq \langle \mathcal{T}u,\ u\rangle\\
&= \langle \varphi,\ \mathcal{L}\varphi\rangle\\
&= -t\langle \mathcal{S}o\mathcal{L}\varphi,\ \mathcal{L}\varphi\rangle\\
&= t\int_\Omega \Big(-\delta|u|^{\alpha(x)-2}u+\mu g(x,u)+\lambda f(x,u,\nabla u)\Big)u\,dx\\
&\leq t\max(|\delta|, C_2|\mu|, C_1|\lambda|)\Big(\int_\Omega |u|^{\alpha(x)}dx+\int_\Omega |\nu(x)u(x)|dx+\int_\Omega |u(x)|^{s(x)}dx\\
&\qquad +\int_\Omega |\gamma(x)u(x)|dx+\int_\Omega |u(x)|^{q(x)}dx+\int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big)\\
&= t\max(|\delta|, C_2|\mu|, C_1|\lambda|)\Big(\rho_{\alpha(x)}(u)+\int_\Omega |\nu(x)u(x)|dx+\int_\Omega |\gamma(x)u(x)|dx\\
&\qquad +\rho_{s(x)}(u)+\rho_{q(x)}(u)+\int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big)\\
&\leq C\Big(|u|_{\alpha(x)}^{\alpha^-}+|u|_{\alpha(x)}^{\alpha^+}+|\nu|_{p'(x)}|u|_{p(x)}+|\gamma|_{p'(x)}|u|_{p(x)}+|u|_{s(x)}^{s^+}+|u|_{s(x)}^{s^-}+|u|_{q(x)}^{q^+}+|u|_{q(x)}^{q^-}\\
&\qquad +\frac{1}{q'^-}\rho_{q(x)}(\nabla u)+\frac{1}{q-}\rho_{q(x)}(u)\Big)\\
&\leq C\Big(|u|_{\alpha(x)}^{\alpha^-}+|u|_{\alpha(x)}^{\alpha^+}+|u|_{p(x)}+|u|_{s(x)}^{s^+}+|u|_{s(x)}^{s^-}+|u|_{q(x)}^{q^+}+|u|_{q(x)}^{q^-}+|\nabla u|_{q(x)}^{q^+}\Big),
\end{aligned}$$

then, according to $L^{p(x)} \hookrightarrow L^{\alpha(x)}$, $L^{p(x)} \hookrightarrow L^{s(x)}$ and $L^{p(x)} \hookrightarrow L^{q(x)}$, we get

$$|\mathcal{L}\varphi|_{1,p(x)}^{p^-} \leq C\Big(|\mathcal{L}\varphi|_{1,p(x)}^{\alpha^+}+|\mathcal{L}\varphi|_{1,p(x)}+|\mathcal{L}\varphi|_{1,p(x)}^{s^+}+|\mathcal{L}\varphi|_{1,p(x)}^{q^+}\Big),$$

what implies that $\Big\{\mathcal{L}\varphi : \varphi \in \mathcal{B}\Big\}$ is bounded.

On the other hand, we have that the operator is $\mathcal{S}$ is bounded, then $\mathcal{S}o\mathcal{L}\varphi$ is bounded. Thus, thanks to (7.2.4), we have that $\mathcal{B}$ is bounded in $W^{-1,p'(x)}(\Omega)$.

However, there exists $\tau > 0$ such that

$$|\varphi|_{-1,p'(x)} < \tau \text{ for all } \varphi \in \mathcal{B},$$

which leads to

$$\varphi + t\mathcal{S}o\mathcal{L}\varphi \neq 0,\quad \varphi \in \partial B_\tau(0) \text{ and } t \in [0,1],$$

Moreover, by Lemma 5.2.11, we conclude that

$$I+\mathcal{S}o\mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_\tau(0)}) \text{ and } I = \mathcal{T}o\mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_\tau(0)}).$$

On another side, taking into account that I, $\mathcal{S}$ and $\mathcal{L}$ are bounded, then $I+\mathcal{S}o\mathcal{L}$ is bounded. Hence, we infer that

$$I+\mathcal{S}o\mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{B}_\tau(0)}) \text{ and } I = \mathcal{T}o\mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{B}_\tau(0)}).$$

Next, we define the homotopy

$$\begin{aligned}\mathcal{H} : [0,1] \times \overline{B_\tau(0)} &\longrightarrow W^{-1,p'(x)}(\Omega)\\ (t,\varphi) &\mapsto \mathcal{H}(t,\varphi) := \varphi + t\mathcal{S}o\mathcal{L}\varphi.\end{aligned}$$

Hence, thanks to the properties of the degree d seen in Theorem 5.2.14, we obtain

$$d(I + \mathcal{S}o\mathcal{L}, B_\tau(0), 0) = d(I, B_\tau(0),\ 0) = 1 \neq 0,$$

which implies that there exists $\varphi \in B_\tau(0)$ verifying

$$\varphi + \mathcal{S}o\mathcal{L}\varphi = 0.$$

Finally, we infer that $u = \mathcal{L}\varphi$ is a weak solution of (7.1.1). The proof is completed.

Chapter 8

Existence and uniqueness results for a class of $p(x)$-Kirchhoff-type problems with convection term and Neumann boundary data

We establish an existence and uniqueness results for a homogeneous Neumann boundary value problem involving the $p(x)$-Kirchhoff-Laplacian operator of the following form

$$\begin{cases} -M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx\Big)\Big(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - |u|^{p(x)-2}u\Big) = f(x,u,\nabla u) & \text{in } \Omega, \\ \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.0.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\frac{\partial u}{\partial \eta}$ is the exterior normal derivative, $p(x) \in C_+(\overline{\Omega})$ with $p(x) \geq 2$. By means of a topological degree of Berkovits for a class of demicontinuous operators of generalized (S_+) type and the theory of the variable exponent Sobolev spaces, under appropriate assumptions on f and M, we obtain a results on the existence and uniqueness of weak solution to the considered problem.

8.1 Hypothesis and notion of weak solution

In this chapter we will discuss the existence and uniqueness of weak solution of (8.0.1). For this, we list our assumptions on f and M associated with our problem to show the existence result.

From new on, we always assume that $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\overline{\Omega})$ with $2 \leq p^- \leq p(x) \leq p^+ < \infty$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a function such that:

(A_1) f satisfies the Carathéodory condition.

(A_2) There exists $\beta_1 > 0$ and $\gamma \in L^{p'(x)}(\Omega)$ such that

$$|f(x,\zeta,\xi)| \leq \beta_1\Big(\gamma(x) + |\zeta|^{q(x)-1} + |\xi|^{q(x)-1}\Big)$$

for a.e. $x \in \Omega$ and all $(\zeta,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $2 \leq q^- \leq q(x) \leq q^+ < p^-$.
Furthermore,

(M_0) $M : [0,+\infty) \to (m_0,+\infty)$ is a continuous and increasing function with $m_0 > 0$.

Let us recall that the definition of a weak solution for problem (8.0.1) can be stated as follows.

Definition 8.1.1 *A function* $u \in W^{1,p(x)}(\Omega)$ *is called a weak solution of* (8.0.1) *if*

$$M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\Big)\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u\nabla\varphi + |u|^{p(x)-2}u\ \varphi\Big)\,dx = \int_\Omega f(x,u,\nabla u)\varphi\,dx,$$

for all $\varphi \in W^{1,p(x)}(\Omega)$.

8.2 Existence and uniqueness results

We are now in the position to get the existence result of weak solution for (8.0.1).

Theorem 8.2.1 *If the assumptions* $(A_1)-(A_2)$ *and* (M_0) *hold, then the problem* (8.0.1) *admits at least one weak solution* u *in* $W^{1,p(x)}(\Omega)$.

Proof. First, we give several lemmas that will be used later.

Let us consider the following functional:

$$\Phi(u) := \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx\Big),$$

where $\widehat{M}(s) = \displaystyle\int_0^s M(\tau)d\tau$, such that $M(s)$ satisfies the condition (M_0).
It is obvious that the functional Φ is a continuously Gâteaux differentiable whose Gâteaux derivative at the point $u \in W^{1,p(x)}(\Omega)$ is the functional $\mathcal{T} := \Phi'(u) \in W^{-1,p'(x)}(\Omega)$, given by

$$\langle \mathcal{T}u,\varphi\rangle = M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\Big)\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u\nabla\varphi + |u|^{p(x)-2}u\ \varphi\Big)\,dx,$$

for all $u,\varphi \in W^{1,p(x)}(\Omega)$ where $\langle\cdot,\cdot\rangle$ means the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. Furthermore, the properties of the operator $\mathcal{T}$ are summarized in the following lemma (see [51, Theorem 2.1])

Lemma 8.2.2 *If* (M_0) *holds, then*

1. *$\mathcal{T} : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is a continuous, bounded and strictly monotone operator.*

2. *$\mathcal{T}$ is a mapping of type* (S_+).

Lemma 8.2.3 *Assume that the assumptions* $(A_1)-(A_2)$ *hold, then the operator* $\mathcal{S}: W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *defined by*

$$\langle \mathcal{S}u, \varphi\rangle = -\int_\Omega f(x,u,\nabla u)\varphi\, dx, \ \textit{for all}\ u,\varphi \in W^{1,p(x)}(\Omega)$$

is compact.

Proof. Let $\Phi: W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\Phi u(x) := -f(x,\ u,\ \nabla u)\ \text{ for }\ u \in W^{1,p(x)}(\Omega)\ \text{ and }\ x \in \Omega.$$

Next, we split the proof in several steps.
We first show that Φ is bounded and continuous. By using (A_2), the inequalities (5.1.5) and (5.1.6), we obtain

$$\begin{aligned}
|\Phi u|_{p'(x)} &\leq \rho_{p'(x)}(\Phi u) + 1\\
&= \int_\Omega |f(x,u(x),\nabla u(x))|^{p'(x)}\, dx + 1\\
&\leq C\Big(\rho_{p'(x)}(\gamma) + \rho_{r(x)}(u) + \rho_{r(x)}(\nabla u)\Big) + 1\\
&\leq C\Big(|\gamma|^{p'^+}_{p(x)} + |u|^{r^+}_{r(x)} + |u|^{r^-}_{r(x)} + |\nabla u|^{r^+}_{r(x)} + |\nabla u|^{r^-}_{r(x)}\Big) + 1,
\end{aligned}$$

for all $u \in W^{1,p(x)}(\Omega)$ where $r(x) = (q(x)-1)p'(x) < p(x)$.
Then, by the continuous embedding $L^{p(x)} \hookrightarrow L^{r(x)}$ and Remark 5.1.9, we have

$$|\Phi u|_{p'(x)} \leq C\Big(|\gamma|^{p'^+}_{p(x)} + |u|^{r^+}_{1,p(x)} + |u|^{r^-}_{1,p(x)}\Big) + 1.$$

This implies that Φ is bounded on $W^{1,p(x)}(\Omega)$.
To show that Φ is continuous, let $u_n \to u$ in $W^{1,p(x)}(\Omega)$. Then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Hence there exist a subsequence (u_k) of (u_n) and measurable functions ϕ in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x)\ \text{ and }\ \nabla u_k(x) \to \nabla u(x),$$

$$|u_k(x)| \leq \phi(x)\ \text{ and }\ |\nabla u_k(x)| \leq |\psi(x)|, \tag{8.2.1}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, thanks to (A_1), we get, as $k \longrightarrow \infty$

$$f(x,u_k(x),\nabla u_k(x)) \to f(x,u(x),\nabla u(x))\ \text{ a.e. }\ x \in \Omega.$$

On the other hand, it follows from (A_2) and (8.2.1) that

$$|f(x, u_k(x), \nabla u_k(x))| \leq \beta_1\Big(\gamma(x) + |\phi(x)|^{q(x)-1} + |\psi(x)|^{q(x)-1}\Big),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
Since

$$\gamma + |\phi|^{q(x)-1} + |\psi(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and

$$\rho_{p'(x)}(\Phi u_k - \Phi u) = \int_\Omega \Big|f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))\Big|^{p'(x)} dx,$$

therefore, the Lebesgue's theorem and the equivalence (5.1.4) implies that

$$\Phi u_k \to \Phi u \text{ in } L^{p'(x)}(\Omega).$$

Thus the entire sequence (Φu_n) converges to Φu in $L^{p'(x)}(\Omega)$.
Moreover, let $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator for the embedding of $I : W^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$. Let us define

$$I^* \circ \Phi : W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

which is well-defined by assumption (A_2).
Since the embedding I is compact, it is known that the adjoint operator I^* is also compact. Therefore, $I^* \circ \Phi$ is compact. This completes the proof.
Now we give the proof of the Theorem 8.2.1. For that, we transform this Neumann boundary value problem into a new one governed by a Hammerstein equation, so by using the theory of topological degree introduced in Section 5.2, we show the existence of weak solution to the state problem.

First, for all $u, \varphi \in W^{1,p(x)}(\Omega)$, we define the operators $\mathcal{T}$ and $\mathcal{S}$, as defined in Lemmas 8.2.2 and 8.2.3 respectively,

$$\begin{aligned}&\mathcal{T} : W^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)\\ &\langle \mathcal{T}u, \varphi\rangle = M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\Big) \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla\varphi + |u|^{p(x)-2}u\,\varphi\Big)\, dx,\end{aligned}$$

$$\begin{aligned}&\mathcal{S} : W^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)\\ &\langle \mathcal{S}u, \varphi\rangle = -\int_\Omega f(x, u, \nabla u)\varphi\, dx.\end{aligned}$$

Then $u \in W^{1,p(x)}(\Omega)$ is a weak solution of (8.0.1) if and only if

$$\mathcal{T}u = -\mathcal{S}u. \tag{8.2.2}$$

Thanks to the properties of the operator $\mathcal{T}$ seen in Lemma 8.2.2 and in view of Minty-Browder Theorem (see [143, Theorem 26 A], the inverse operator

$$\mathcal{L} := \mathcal{T}^{-1} : W^{-1,p'(x)}(\Omega) \to W^{1,p(x)}(\Omega),$$

is bounded, continuous and of type (S_+). Moreover, note by Lemma 8.2.3 that the operator $\mathcal{S}$ is bounded, continuous and quasimonotone.
Consequently, Equation (8.2.2) is equivalent to the abstract Hammerstein equation

$$u = \mathcal{L}\varphi \ \text{ and } \ \varphi + \mathcal{S} \circ \mathcal{L}\varphi = 0. \tag{8.2.3}$$

To solve Equation (8.2.3), we will apply the degree theory introduced in Subsection 5.2.2. To do this, set

$$\mathcal{B} := \Big\{ \varphi \in W^{-1,p'(x)}(\Omega) : \exists\, t \in [0,1] \ \text{ such that } \ \varphi + t\mathcal{S} \circ \mathcal{L}\varphi = 0 \Big\}$$

Next,we prove that $\mathcal{B}$ is bounded in $\in W^{-1,p'(x)}(\Omega)$.
Let $\varphi \in \mathcal{B}$ and set $u := \mathcal{L}\varphi$, then $|\mathcal{L}\varphi|_{1,p(x)} = |u|_{1,p(x)}$.
If $|u|_{1,p(x)} \leq 1$, then $|\mathcal{L}\varphi|_{1,p(x)}$ is bounded.
If $|u|_{1,p(x)} > 1$, then by (5.1.11), the growth condition (A_2), the Hölder inequality (5.1.7), the inequality (5.1.6) and the Young inequality, we get

$$\begin{aligned}
|\mathcal{L}\varphi|_{1,p(x)}^{p^-} &= |u|_{1,p(x)}^{p^-} \\
&\leq \rho_{1,p(x)}(u) \\
&= \rho_{p(x)}(u) + \rho_{p(x)}(\nabla u) \\
&\leq \langle \mathcal{T}u, u \rangle \\
&= \langle \varphi, \mathcal{L}\varphi \rangle \\
&= -t \langle \mathcal{S} \circ \mathcal{L}\varphi, \mathcal{L}\varphi \rangle \\
&= t \int_\Omega f(x,u,\nabla u) u \, dx \\
&\leq C\Big(\int_\Omega |\gamma(x) u(x)| \, dx + \rho_{q(x)}(u) + \int_\Omega |\nabla u|^{q(x)-1} |u| \, dx \Big) \\
&\leq C\Big(|\gamma|_{p'(x)} |u|_{p(x)} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + \frac{1}{q'^-} \rho_{q(x)}(\nabla u) + \frac{1}{q^-} \rho_{q(x)}(u) \Big) \\
&\leq C\Big(|u|_{p(x)} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + |\nabla u|_{q(x)}^{q^+} \Big).
\end{aligned}$$

From Remark 5.1.9 and the continuous embedding $L^{p(x)} \hookrightarrow L^{q(x)}$, we conclude that

$$|\mathcal{L}\varphi|_{1,p(x)}^{p^-} \leq C\Big(|\mathcal{L}\varphi|_{1,p(x)} + |\mathcal{L}\varphi|_{1,p(x)}^{q^+} \Big).$$

So, we infer that $\left\{\mathcal{L}\varphi|\varphi \in \mathcal{B}\right\}$ is bounded.
Since the operator $\mathcal{S}$ is bounded, it is obvious from (8.2.3) that $\mathcal{B}$ is bounded in $W^{-1,p'(x)}(\Omega)$.
Consequently, there exists $R > 0$ such that

$$|\varphi|_{-1,p'(x)} < R \text{ for all } \varphi \in \mathcal{B}.$$

Therefore

$$\varphi + t\mathcal{S} \circ \mathcal{L}\varphi \neq 0 \text{ for all } \varphi \in \partial\mathcal{B}_R(0) \text{ and all } t \in [0,1],$$

where $\mathcal{B}_R(0)$ is the ball of center 0 and radius R in $W^{-1,p'(x)}(\Omega)$.
Moreover, from Lemma 5.2.11 we conclude that

$$I + \mathcal{S} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_R(0)}) \text{ and } I = \mathcal{T} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_R(0)}).$$

Next, consider a homotopy $\Lambda : [0,1] \times \overline{\mathcal{B}_R(0)} \to W^{-1,p'(x)}(\Omega)$ given by

$$\Lambda(t,\varphi) := \varphi + t\mathcal{S} \circ \mathcal{L}\varphi \text{ for } (t,\varphi) \in [0,1] \times \overline{\mathcal{B}_R(0)}.$$

Hence, by using the normalization property and the homotopy invariance of degree d seen in Theorem 5.2.14, we obtain

$$d(I + \mathcal{S} \circ \mathcal{L}, \mathcal{B}_R(0), 0) = d(I, \mathcal{B}_R(0),\ 0) = 1 \neq 0.$$

Then, there exists $\varphi \in \mathcal{B}_R(0)$ such that

$$\varphi + \mathcal{S} \circ \mathcal{L}\varphi = 0.$$

Thus, we conclude that $u = \mathcal{L}\varphi$ is a weak solution of (8.0.1). The proof is completed.
Next, we consider the uniqueness of solution of (8.0.1). To this end, we also need the following hypothesis:

(A_3) There exists $C_2 \geq 0$ such that

$$\Big(f(x,t,\xi) - f(x,s,\zeta)\Big)(t-s) \leq C_2|t-s|^{p(x)}$$

for a.e. $x \in \Omega$ and all $t, s \in \mathbb{R}$, $\xi, \zeta \in \mathbb{R}^N$.
We are now in the position to state our uniqueness result.

Theorem 8.2.4 *If the assumptions* $(A_1) - (A_3)$ *and* (M_0) *hold, then the weak solution of* (8.0.1) *is unique provided* $\dfrac{2^{p^+}C_2}{m_0} < 1$.

Proof. Let $u_1, u_2 \in W^{1,p(x)}(\Omega)$ be two weak solutions of (8.0.1). Then, by taking $\varphi = u_1 - u_2$ in the Definition 8.1.1, we get

$$\Psi_1 := M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_1|^{p(x)} + |u_1|^{p(x)})dx\Big)\int_\Omega \Big(|\nabla u_1|^{p(x)-2}\nabla u_1 \nabla(u_1 - u_2) + |u_1|^{p(x)-2}u_1(u_1 - u_2)\Big)\,dx = \int_\Omega f(x, u_1, \nabla u_1)(u_1 - u_2)\,dx,$$

and

$$\Psi_2 := M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_2|^{p(x)} + |u_2|^{p(x)})dx\Big)\int_\Omega \Big(|\nabla u_2|^{p(x)-2}\nabla u_2 \nabla(u_1 - u_2) + |u_2|^{p(x)-2}u_2(u_1 - u_2)\Big)\,dx = \int_\Omega f(x, u_1, \nabla u_1)(u_1 - u_2)\,dx.$$

Subtracting the above two equations, we obtain

$$\Psi_1 - \Psi_2 = \int_\Omega \Big(f(x, u_1, \nabla u_1) - f(x, u_2, \nabla u_2)\Big)(u_1 - u_2)\,dx. \tag{8.2.4}$$

Denote $\rho_{1,p(x)}(u) = \int_\Omega \Big(|\nabla u|^{p(x)} + |u|^{p(x)}\Big)\,dx,\ \forall\, u \in W^{1,p(x)}(\Omega)$.

By the same proof as [82], we can show that the Theorem 1.4 of [82] hold for $\rho_{1,p(x)}(u)$. In particular, if we take $u_k \equiv v$ in Theorem 1.4 of [82], we can easily see that $u = v$ in $W^{1,p(x)}(\Omega)$ if and only if $\rho_{1,p(x)}(u) = \rho_{1,p(x)}(v)$. Hence, for any u, $v \in W^{1,p(x)}(\Omega)$ with $u \neq v$ in $W^{1,p(x)}(\Omega)$, we can see that $\rho_{1,p(x)}(u) \neq \rho_{1,p(x)}(v)$.

Without loss of generality, we may assume that $\rho_{1,p(x)}(u_1) > \rho_{1,p(x)}(u_2)$. It follows that

$$M\Big(\int_\Omega \frac{1}{p(x)}\Big(|\nabla u_1|^{p(x)} + |u_1|^{p(x)}\Big)dx\Big) \geq M\Big(\int_\Omega \frac{1}{p(x)}\Big(|\nabla u_2|^{p(x)} + |u_2|^{p(x)}\Big)dx\Big). \tag{8.2.5}$$

since $M(t)$ is a monotone function.

If $|u_1| > |u_2|$, using (8.2.5) and the assumption (M_0), we obtain

$$\begin{aligned}
\Psi_1 - \Psi_2 &\geq M\Big(\int_\Omega \frac{1}{p(x)}\Big(|\nabla u_2|^{p(x)} + |u_2|^{p(x)})dx\Big)\int_\Omega \Big(|\nabla u_1|^{p(x)-2}\nabla u_1 \nabla(u_1 - u_2) \\
&\qquad + |u_1|^{p(x)-2}u_1(u_1 - u_2)\Big)\,dx - \Psi_2 \\
&\geq M\Big(\int_\Omega \frac{1}{p(x)}\Big(|\nabla u_2|^{p(x)} + |u_2|^{p(x)}\Big)dx\Big)\Big[\int_\Omega \Big[\Big(|\nabla u_1|^{p(x)-2}\nabla u_1 \\
&\qquad - |\nabla u_2|^{p(x)-2}\nabla u_2\Big)\nabla(u_1 - u_2) + \Big(|u_1|^{p(x)-2}u_1 - |u_2|^{p(x)-2}u_2\Big)(u_1 - u_2)\Big]\,dx\Big] \\
&\geq m_0\Big[\int_\Omega \Big[\Big(|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla u_2|^{p(x)-2}\nabla u_2\Big)\nabla(u_1 - u_2) \\
&\qquad + \Big(|u_1|^{p(x)-2}u_1 - |u_2|^{p(x)-2}u_2\Big)(u_1 - u_2)\Big]\,dx\Big].
\end{aligned} \tag{8.2.6}$$

On the other hand, since $p(x) \geq 2$, then we have the following inequalities (see [82]):

$$\Big(|u_1|^{p(x)-2}u_1 - |u_2|^{p(x)-2}u_2\Big)(u_1 - u_2) \geq \Big(\frac{1}{2}\Big)^{p(x)}|u_1 - u_2|^{p(x)}, \tag{8.2.7}$$

$$\Big(|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla u_2|^{p(x)-2}\nabla u_2\Big)\nabla(u_1-u_2) \geq \Big(\frac{1}{2}\Big)^{p(x)}|\nabla u_1 - \nabla u_2|^{p(x)}. \tag{8.2.8}$$

So, by (8.2.7) and (8.2.8), we get

$$\begin{aligned}\Psi_1 - \Psi_2 &\geq m_0\Big(\frac{1}{2}\Big)^{p^+}\int_\Omega \Big(|\nabla u_1 - \nabla u_2|^{p(x)} + |u_1-u_2|^{p(x)}\Big)\,dx \\ &\geq m_0\Big(\frac{1}{2}\Big)^{p^+}\int_\Omega |u_1-u_2|^{p(x)}\,dx.\end{aligned} \tag{8.2.9}$$

Using (8.2.9) in (8.2.4), we obtain

$$m_0\Big(\frac{1}{2}\Big)^{p^+}\int_\Omega |u_1-u_2|^{p(x)}\,dx \leq \int_\Omega \Big(f(x,u_1,\nabla u_1) - f(x,u_2,\nabla u_2)\Big)(u_1-u_2)\,dx. \tag{8.2.10}$$

Then it follows from the assumption (A_3) that

$$m_0\Big(\frac{1}{2}\Big)^{p^+}\int_\Omega |u_1-u_2|^{p(x)}\,dx \leq C_2\int_\Omega |u_1-u_2|^{p(x)}\,dx. \tag{8.2.11}$$

If $|u_2| > |u_1|$, changing the role of u_1 and u_2 in (8.2.4)-(8.2.11), we obtain

$$m_0\Big(\frac{1}{2}\Big)^{p^+}\int_\Omega |u_2-u_1|^{p(x)}\,dx \leq C_2\int_\Omega |u_2-u_1|^{p(x)}\,dx. \tag{8.2.12}$$

Consequently, when $\dfrac{2^{p^+}C_2}{m_0} < 1$, then $u_1 = u_2$ and so the solution of (8.0.1) is unique. This completes the proof.

Chapter 9

Weak solutions for a quasilinear elliptic and parabolic problems involving the $(p(x), q(x))$-Laplacian operator

In this chapter, we study the existence of weak solution to the following quasilinear problems:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_{p(x)}u - \Delta_{q(x)}u = \phi(x,t) & \text{in } \Omega_T := \Omega \times (0,T), \\ \\ u(x,t) \ = 0 & \text{on } \Gamma := \partial\Omega \times (0,T), \\ \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{9.0.1}$$

and

$$\begin{cases} -\Delta_{p(x)}u - \Delta_{q(x)}u + \omega|u|^{\xi(x)-2}u = \upsilon\, \mathcal{A}(x,u) + \sigma\, \mathcal{B}(x,u,\nabla u) & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.0.2}$$

where $\phi \in \mathcal{W}^*$, $u_0 \in L^2(\Omega)$, $\mathcal{A} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions that satisfy the assumption of growth, and the variables exponents $p, q \in C_+(\overline{\Omega})$ are assumed to satisfy the following assumption:

$$1 < q^- \leq q \leq q^+ < p^- \leq p \leq p^+ < +\infty. \tag{9.0.3}$$

9.1 Quasilinear parabolic problem involving the $(p(x), q(x))$-Laplacian operator

In this section, we will prove the existence of weak solution of the problem (9.0.1). First we will state a lemma that will be used later.

Lemma 9.1.1 *The operator $\mathcal{S} := -\Delta_{p(x)}u - \Delta_{q(x)}u$ defined from $\mathcal{W}$ into $\mathcal{W}^*$ by*

$$\langle \mathcal{S}u, v\rangle_{\mathcal{W}^*,\mathcal{W}} = \int_{\Omega_T} \Big(|\nabla u|^{p(x)-2}\nabla u\nabla v + |\nabla u|^{q(x)-2}\nabla u\nabla v\Big)\,dxdt,$$

is bounded, continuous and of type (S_+).

Proof. Let $t \in]\,0,\ T[$ and denote by $\mathcal{A}$ the operator defined from $W_0^{1,p(x)}(\Omega)$ into $W^{-1,p'(x)}(\Omega)$ by

$$\langle \mathcal{A}u(x,t), v(x,t)\rangle := \langle \mathcal{A}_1 u(x,t), v(x,t)\rangle + \langle \mathcal{A}_2 u(x,t), v(x,t)\rangle,$$

where

$$\langle \mathcal{A}_1 u(x,t), v(x,t)\rangle := \int_{\Omega} \Big(|\nabla u(x,t)|^{p(x)-2}\nabla u(x,t)\nabla v(x,t)\Big)\,dx,$$

and

$$\langle \mathcal{A}_2 u(x,t), v(x,t)\rangle := \int_{\Omega} \Big(|\nabla u(x,t)|^{q(x)-2}\nabla u(x,t)\nabla v(x,t)\Big)\,dx,$$

for all $u(\cdot,t), v(\cdot,t) \in W_0^{1,p(x)}(\Omega)$, with $\langle\cdot,\cdot\rangle$ is the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$.
Then, we obtain

$$\langle \mathcal{S}u, v\rangle_{\mathcal{W}^*,\mathcal{W}} = \int_0^T \langle \mathcal{A}u(x,t), v(x,t)\rangle dt, \quad \text{for all } \ u, v \in \mathcal{W},$$

with $\langle\cdot,\cdot\rangle_{\mathcal{W}^*,\mathcal{W}}$ is the duality pairing between $\mathcal{W}^*$ and $\mathcal{W}$.
Next, it follows from [81, Lemma 3.1] that $\mathcal{A}_1$ and $\mathcal{A}_2$ are bounded, continuous and of type (S_+); so the operator $\mathcal{A} := \mathcal{A}_1 + \mathcal{A}_2$ is bounded, continuous and of type (S_+) and consequently the operator $\mathcal{S}$ is bounded, continuous and of type (S_+).
We are now in the position to get existence result of weak solution for (9.0.1).

Theorem 9.1.2 *Let $\phi \in \mathcal{W}^*$ and $u_0 \in L^2(\Omega)$, then the problem (9.0.1) admits at least one weak solution $u \in D(\mathcal{L})$, where $D(\mathcal{L}) = \Big\{u \in \mathcal{W} : \frac{du}{dt} \in \mathcal{W}^* \ \text{and} \ \ u(0) = 0\Big\}$.*

Proof. First, let us define the operator $\mathcal{L} := \dfrac{d}{dt}$ with domain $D(\mathcal{L})$ given by

$$D(\mathcal{L}) = \{u \in \mathcal{W} \ : \ \frac{du}{dt} \in \mathcal{W}^* \ \text{ and } \ u(0) = 0\},$$

where the time derivative $\dfrac{du}{dt}$ is understood in the sense of vector-valued distributions, i.e.,

$$\langle \mathcal{L}u,\ v\rangle_{\mathcal{W}^*,\mathcal{W}} = \int_0^T \langle u'(t), v(t)\rangle dt,\ \forall\, v \in \mathcal{W},$$

with $\langle \cdot,\cdot \rangle_{\mathcal{W}^*,\mathcal{W}}$ means the duality pairing between $\mathcal{W}^*$ and $\mathcal{W}$, and $\langle \cdot,\cdot \rangle$ means the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$.
Second, we define the operator $\mathcal{S} : \mathcal{W} \to \mathcal{W}^*$ as defined in Lemma 9.1.1

$$\langle \mathcal{S}u, v \rangle_{\mathcal{W}^*,\mathcal{W}} = \int_{\Omega_T} \Big(|\nabla u|^{p(x)-2} \nabla u \nabla v + |\nabla u|^{q(x)-2} \nabla u \nabla v \Big) dxdt.$$

Consequently, the weak formulation of the problem (9.0.1) is given by the operator equation

$$u \in D(\mathcal{L}) : \mathcal{L}u + \mathcal{S}u = \phi.$$

Next, it follows from lemma 9.1.1 that $\mathcal{S}$ is bounded, continuous and of type (S_+), and the operator $\mathcal{L}$ is well known to be closed, densely defined, and maximal monotone [143, Theorem 32.L, pp.897-899].
Let $u \in \mathcal{W}$. Using the monotonicity of $\mathcal{L}$ and the inequality (5.1.16), we obtain

$$\begin{aligned}
\langle \mathcal{L}u + \mathcal{S}u, u \rangle &\geq \langle \mathcal{S}u, u \rangle \\
&= \int_{\Omega_T} (|\nabla u|^{p(x)} + |\nabla u|^{q(x)}) dxdt \\
&\geq 2(|\nabla u|^{p^-}_{L^{p(x)}(\Omega_T)} - 1) \\
&\geq 2(|u|^{p^-}_{\mathcal{W}} - 1).
\end{aligned}$$

Because the right-hand side of the previous inequality approximates to ∞ when $|u|_{\mathcal{W}} \to \infty$, then the operator $\mathcal{L} + \mathcal{S}$ is coercive. Thus for each $\phi \in \mathcal{W}^*$ there is a radius $r = r(\phi) > 0$ such that

$$\langle \mathcal{L}u + \mathcal{S}u - \phi, u \rangle > 0, \quad \text{for each} \quad u \in B_r(0) \cap D(\mathcal{L}).$$

So all the conditions of Theorem 5.2.7 are satisfied. Consequently, Theorem 5.2.7 leads us to the conclusion that the equation $\mathcal{L}u + \mathcal{S}u = \phi$ has a weak solution in $D(\mathcal{L})$, which implies that the problem (9.0.1) has a weak solution $u \in D(\mathcal{L})$. This completes the proof.

9.2 Quasilinear elliptic problem involving the $(p(x), q(x))$-Laplacian operator

In this section, we will discuss the existence of weak solution of (9.0.2). In the beginning, let us assume that $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (5.1.8), $\xi \in C_+(\overline{\Omega})$ with $2 \leq \xi^- \leq \xi(x) \leq \xi^+ < p^- \leq p(x) \leq p^+ < \infty$, $\mathcal{A} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are functions such that:

(A_1) $\mathcal{B}$ is a Carathéodory function.

(A_2) There exists $\alpha_1 > 0$ and $f \in L^{p'(x)}(\Omega)$ such that

$$|\mathcal{B}(x,y,z)| \leq \alpha_1 (f(x) + |y|^{k(x)-1} + |z|^{k(x)-1}).$$

(A_3) $\mathcal{A}$ is a Carathéodory function.

(A_4) There are $\alpha_2 > 0$ and $g \in L^{p'(x)}(\Omega)$ such that

$$|\mathcal{A}(x,y)| \leq \alpha_2 (g(x) + |y|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and all $(y,z) \in \mathbb{R} \times \mathbb{R}^N$, where $q, s \in C_+(\overline{\Omega})$ with $2 \leq k^- \leq k(x) \leq k^+ < p^-$ and $2 \leq s^- \leq s(x) \leq s^+ < p^-$.

Remark 9.2.1

1. *Let* $\vartheta \in W_0^{1,p(x)}(\Omega)$, *then*

$$\int_\Omega \Big(|\nabla u|^{p(x)-2} \nabla u \nabla \vartheta + |\nabla u|^{q(x)-2} \nabla u \nabla \vartheta \Big) dx$$

is well defined (see [81]).

2. *Let* $u \in W_0^{1,p(x)}(\Omega)$, *then we have* $\omega |u|^{\xi(x)-2} u \in L^{p'(x)}(\Omega)$, $\upsilon\, \mathcal{A}(x,u) \in L^{p'(x)}(\Omega)$ *and* $\sigma\, \mathcal{B}(x,u,\nabla u) \in L^{p'(x)}(\Omega)$ *under the assumptions* (A_2) *and* (A_4) *and the given hypotheses about the exponents* p, ξ, q *and* s *because:* $f \in L^{p'(x)}(\Omega)$, $g \in L^{p'(x)}(\Omega)$, $r(x) = (k(x) - 1)p'(x) \in C_+(\overline{\Omega})$ *with* $r(x) < p(x)$, *and* $\beta(x) = (\xi(x) - 1)p'(x) \in C_+(\overline{\Omega})$ *with* $\beta(x) < p(x)$ *and* $\kappa(x) = (s(x) - 1)p'(x) \in C_+(\overline{\Omega})$ *with* $\kappa(x) < p(x)$.
Then, using Remark 5.1.5, we conclude that $L^{p(x)} \hookrightarrow L^{r(x)}$, $L^{p(x)} \hookrightarrow L^{\beta(x)}$ *and* $L^{p(x)} \hookrightarrow L^{\kappa(x)}$.
Therefore, with $\vartheta \in L^{p(x)}(\Omega)$, *we have*

$$\Big(-\omega |u|^{\xi(x)-2} u + \upsilon\, \mathcal{A}(x,u) + \sigma\, \mathcal{B}(x,u,\nabla u) \Big) \vartheta \in L^1(\Omega).$$

This means that

$$\int_\Omega \Big(-\omega |u|^{\xi(x)-2} u + \upsilon\, \mathcal{A}(x,u) + \sigma\, \mathcal{B}(x,u,\nabla u) \Big) \vartheta dx < \infty.$$

Then, let us introduce the definition of a weak solution for (9.0.2).

Definition 9.2.2 *We say that a function* $u \in W_0^{1,p(x)}(\Omega)$ *is a weak solution of* (9.0.2), *if for any* $\vartheta \in W_0^{1,p(x)}(\Omega)$, *it satisfy the following:*

$$\int_\Omega \Big(|\nabla u|^{p(x)-2} \nabla u \nabla \vartheta + |\nabla u|^{q(x)-2} \nabla u \nabla \vartheta \Big) dx = \int_\Omega \Big(-\omega |u|^{\xi(x)-2} u + \upsilon\, \mathcal{A}(x,u) + \sigma\, \mathcal{B}(x,u,\nabla u) \Big) \vartheta dx.$$

Let us now give some lemmas that will be used later. First, let us consider the following functional:

$$\mathcal{C}(u) := \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx.$$

From [81], it is clear that the derivative operator of the functional $\mathcal{C}$ in the weak sense at the point $u \in W_0^{1,p(x)}(\Omega)$ is the functional $\mathcal{G}(u) := \mathcal{C}'(u) \in W^{-1,p'(x)}(\Omega)$ given by

$$\langle \mathcal{G}u, \vartheta\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla\vartheta + |\nabla u|^{q(x)-2}\nabla u\nabla\vartheta\Big)dx,$$

for all $u, \vartheta \in W_0^{1,p(x)}(\Omega)$ where $\langle\cdot,\cdot\rangle$ means the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. Furthermore, we have the following properties of the operator $\mathcal{G}$.

Lemma 9.2.3 *[81, Theorem 3.1.]The mapping*

$$\begin{aligned} &\mathcal{G} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)\\ &\langle \mathcal{G}u, \vartheta\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla\vartheta + |\nabla u|^{q(x)-2}\nabla u\nabla\vartheta\Big)dx, \end{aligned} \tag{9.2.1}$$

is a continuous, bounded, strictly monotone operator and is of type (S_+).

Lemma 9.2.4 *If* $(A_1) - (A_2)$ *hold, then the operator*

$$\begin{aligned} &\mathcal{N} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)\\ &\langle \mathcal{N}u, \vartheta\rangle = -\int_\Omega \Big(-\omega|u|^{\xi(x)-2}u + \upsilon\,\mathcal{A}(x,u) + \sigma\,\mathcal{B}(x,u,\nabla u)\Big)\vartheta dx, \end{aligned} \tag{9.2.2}$$

is compact.

Proof. We follow four steps to prove this lemma.
Step 1 : Let $\Psi_1 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\Psi_1 u(x) := -\upsilon\mathcal{A}(x,u).$$

We wiil prove that the operator Ψ_1 is bounded and continuous. Let $u \in W_0^{1,p(x)}(\Omega)$, bearing (A_4) in mind and using (5.1.5) and (5.1.6), we infer

$$\begin{aligned} |\Psi_1 u|_{p'(x)} &\le \rho_{p'(x)}(\Psi_1 u) + 1\\ &= \int_\Omega |\upsilon\mathcal{A}(x,u(x))|^{p'(x)}dx + 1\\ &= \int_\Omega |\upsilon|^{p'(x)}|\mathcal{A}(x,u(x)|^{p'(x)}dx + 1\\ &\le \Big(|\upsilon|^{p'^-} + |\upsilon|^{p'^+}\Big)\int_\Omega |\alpha_2\Big(g(x) + |u|^{s(x)-1}\Big)|^{p'(x)}dx + 1\\ &\le C\Big(|\upsilon|^{p'^-} + |\upsilon|^{p'^+}\Big)\int_\Omega \Big(|g(x)|^{p'(x)} + |u|^{\kappa(x)}\Big)dx + 1\\ &\le C\Big(|\upsilon|^{p'^-} + |\upsilon|^{p'^+}\Big)\Big(\rho_{p'(x)}(g) + \rho_{\kappa(x)}(u)\Big) + 1\\ &\le C\Big(|g|_{p(x)}^{p'^+} + |u|_{\kappa(x)}^{\kappa^+} + |u|_{\kappa(x)}^{\kappa^-}\Big) + 1. \end{aligned}$$

Then, we deduce from (5.1.9) and $L^{p(x)} \hookrightarrow L^{\kappa(x)}$, that

$$|\Psi_1 u|_{p'(x)} \leq C\Big(|g|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{\kappa^+} + |u|_{1,p(x)}^{\kappa^-}\Big) + 1,$$

that means Ψ_1 is bounded on $W_0^{1,p(x)}(\Omega)$.
Second, we show that the operator Ψ_1 is continuous. To this purpose let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. We need to show that $\Psi_1 u_n \to \Psi_1 u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.
Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$. Hence there exist a subsequence (u_m) of (u_n) and ϕ in $L^{p(x)}(\Omega)$ such that

$$u_m(x) \to u(x) \text{ and } |u_m(x)| \leq \phi(x), \tag{9.2.3}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, from $(\mathcal{A}_2)$ and (9.2.3), we have

$$|\mathcal{A}(x, u_m(x))| \leq \alpha_2(g(x) + |\phi(x)|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
On the other hand, thanks to $(\mathcal{A}_3)$ and (9.2.3), we get, as $k \longrightarrow \infty$

$$\mathcal{A}(x, u_m(x)) \to \mathcal{A}(x, u(x)) \text{ a.e. } x \in \Omega.$$

Seeing that

$$g + |\phi|^{s(x)-1} \in L^{p'(x)}(\Omega) \text{ and } \rho_{p'(x)}(\Psi_1 u_m - \Psi_1 u) = \int_\Omega |\mathcal{A}(x, u_m(x)) - \mathcal{A}(x, u(x))|^{p'(x)} dx,$$

then, from the Lebesgue's theorem and the equivalence (5.1.4), we have

$$\Psi_1 u_m \to \Psi_1 u \text{ in } L^{p'(x)}(\Omega),$$

and consequently

$$\Psi_1 u_n \to \Psi_1 u \text{ in } L^{p'(x)}(\Omega),$$

that is, Ψ_1 is continuous.
Step 2 : We define the operator $\Psi_2 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi_2 u(x) := \omega |u(x)|^{\xi(x)-2} u(x).$$

We will prove that Ψ_2 is bounded and continuous.
It is clear that Ψ_2 is continuous. Next we show that Ψ_2 is bounded.

Let $u \in W_0^{1,p(x)}(\Omega)$ and using (5.1.5) and (5.1.6), we obtain

$$\begin{aligned}
|\Psi_2 u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi_2 u) + 1 \\
&= \int_\Omega |\omega |u|^{\xi(x)-2} u|^{p'(x)} dx + 1 \\
&= \int_\Omega |\omega|^{p'(x)} |u|^{(\xi(x)-1)p'(x)} dx + 1 \\
&\leq \left(|\omega|^{p'^-} + |\omega|^{p'^+}\right) \int_\Omega |u|^{\beta(x)} dx + 1 \\
&= \left(|\omega|^{p'^-} + |\omega|^{p'^+}\right) \rho_{\beta(x)}(u) + 1 \\
&\leq \left(|\omega|^{p'^-} + |\omega|^{p'^+}\right)\left(|u|^{\beta^-}_{\beta(x)} + |u|^{\beta^+}_{\beta(x)}\right) + 1.
\end{aligned}$$

Hence, we deduce from $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and (5.1.9) that

$$|\Psi_2 u|_{p'(x)} \leq C\left(|u|^{\beta^-}_{1,p(x)} + |u|^{\beta^+}_{1,p(x)}\right) + 1,$$

and consequently, Ψ_2 is bounded on $W_0^{1,p(x)}(\Omega)$.

Step 3 : Let us define the operator $\Psi_3 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi_3 u(x) := -\sigma \mathcal{B}(x, u(x), \nabla u(x)).$$

We will show that Ψ_3 is bounded and continuous.

Let $u \in W_0^{1,p(x)}(\Omega)$. According to (A_2) and the inequalities (5.1.5) and (5.1.6), we obtain

$$\begin{aligned}
|\Psi_3 u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi_3 u) + 1 \\
&= \int_\Omega |\sigma \mathcal{B}(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\sigma|^{p'(x)} |\mathcal{B}(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&\leq \left(|\sigma|^{p'^-} + |\sigma|^{p'^+}\right) \int_\Omega \left|\alpha_1 \left(f(x) + |u|^{k(x)-1} + |\nabla u|^{k(x)-1}\right)\right|^{p'(x)} dx + 1 \\
&\leq C\left(|\sigma|^{p'^-} + |\sigma|^{p'^+}\right) \int_\Omega \left(|f(x)|^{p'(x)} + |u|^{r(x)} + |\nabla u|^{r(x)}\right) dx + 1 \\
&\leq C\left(|\sigma|^{p'^-} + |\sigma|^{p'^+}\right)\left(\rho_{p'(x)}(f) + \rho_{r(x)}(u) + \rho_{r(x)}(\nabla u)\right) + 1 \\
&\leq C\left(|f|^{p'^+}_{p(x)} + |u|^{r^+}_{r(x)} + |u|^{r^-}_{r(x)} + |\nabla u|^{r^+}_{r(x)} + |\nabla u|^{r^-}_{r(x)}\right) + 1.
\end{aligned}$$

Taking into account that $L^{p(x)} \hookrightarrow L^{r(x)}$ and (5.1.9), we have then

$$|\Psi_3 u|_{p'(x)} \leq C\left(|f|^{p'^+}_{p(x)} + |u|^{r^+}_{1,p(x)} + |u|^{r^-}_{1,p(x)}\right) + 1,$$

and consequently Ψ_3 is bounded on $W_0^{1,p(x)}(\Omega)$.

It remains to show that Ψ_3 is continuous. Let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, we need to show that

$\Psi_3 u_n \to \Psi_3 u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.
Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$.
Hence, there exist a subsequence (u_m) and Ψ_3 in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_m(x) \to u(x) \ \text{ and } \ \nabla u_m(x) \to \nabla u(x), \tag{9.2.4}$$

$$|u_m(x)| \leq \phi(x) \ \text{ and } \ |\nabla u_m(x)| \leq |\psi(x)|, \tag{9.2.5}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.
Hence, thanks to (A_1) and (9.2.4), we get, as $k \longrightarrow \infty$

$$\mathcal{B}(x, u_m(x), \nabla u_m(x)) \to \mathcal{B}(x, u(x), \nabla u(x)) \ \text{ a.e. } \ x \in \Omega.$$

On the other hand, from (A_2) and (9.2.5), we can deduce the estimate

$$|\mathcal{B}(x, u_m(x), \nabla u_m(x))| \leq \alpha_1(f(x) + |\phi(x)|^{k(x)-1} + |\psi(x)|^{k(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
Seeing that

$$f + |\phi|^{k(x)-1} + |\psi(x)|^{k(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{p'(x)}(\Psi_3 u_m - \Psi_3 u) = \int_\Omega |\mathcal{B}(x, u_m(x), \nabla u_m(x)) - \mathcal{B}(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then, we conclude from the Lebesgue's theorem and (5.1.4) that

$$\Psi_3 u_m \to \Psi_3 u \ \text{ in } \ L^{p'(x)}(\Omega)$$

and consequently

$$\Psi_3 u_n \to \Psi_3 u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and then Ψ_3 is continuous.
Step 4 : Let $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the operator $I : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$. Hence, we define the operators

$$I^* \circ \Psi_1 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

$$I^* \circ \Psi_2 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

and

$$I^* \circ \Psi_3 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

On another side, taking into account that I is compact, then I^* is compact. Thus, the compositions $I^* \circ \Psi_1$, $I^* \circ \Psi_2$ and $I^* \circ \Psi_3$ are compact, that means $\mathcal{N} = I^* \circ \Psi_1 + I^* \circ \Psi_2 + I^* \circ \Psi_3$ is compact. With this last step the proof of Lemma 9.2.4 is completed.
We are now in the position to get the existence result of weak solution for (9.0.2).

Theorem 9.2.5 *Assume that the assumptions* $(A_1) - (A_4)$ *hold, then the problem* (9.0.2) *possesses at least one weak solution* u *in* $W_0^{1,p(x)}(\Omega)$.

Proof. The basic idea of our proof is to reduce the problem (9.0.2) to a new one governed by a Hammerstein equation, and apply the theory of topological degree introduced in Subsection 5.2.2 to show the existence of a weak solution to the state problem.
First, for all $u, \vartheta \in W_0^{1,p(x)}(\Omega)$, we define the operators $\mathcal{G}$ and $\mathcal{N}$ by

$$\begin{aligned} &\mathcal{G} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{G}u, \vartheta \rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla \vartheta + |\nabla u|^{q(x)-2}\nabla u \nabla \vartheta \Big) dx, \end{aligned}$$

$$\begin{aligned} &\mathcal{N} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{N}u, \vartheta \rangle = -\int_\Omega \Big(-\omega |u|^{\xi(x)-2}u + \upsilon \mathcal{A}(x,u) + \sigma \mathcal{B}(x,u,\nabla u) \Big) \vartheta dx. \end{aligned}$$

Consequently, the problem (9.0.2) is equivalent to the equation

$$\mathcal{G}u = -\mathcal{N}u, \ \ u \in W_0^{1,p(x)}(\Omega). \tag{9.2.6}$$

Taking into account that, by Lemma 9.2.3, the operator $\mathcal{G}$ is a continuous, bounded, strictly monotone and of type (S_+), then, by [143, Theorem 26 A], the inverse operator

$$\mathcal{M} := \mathcal{G}^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega),$$

is also bounded, continuous, strictly monotone and of type (S_+).
On another side, according to Lemma 9.2.4, we have that the operator $\mathcal{N}$ is bounded, continuous and quasimonotone.
Consequently, following Zeidler's terminology [143], the equation (9.2.6) is equivalent to the following abstract Hammerstein equation

$$u = \mathcal{M}\vartheta \ \text{ and } \ \vartheta + \mathcal{N} \circ \mathcal{M}\vartheta = 0, \ \ u \in W_0^{1,p(x)}(\Omega) \text{ and } \vartheta \in W^{-1,p'(x)}(\Omega). \tag{9.2.7}$$

Seeing that (9.2.6) is equivalent to (9.2.7), then to solve (9.2.6) it is thus enough to solve (9.2.7). In order to solve (9.2.7), we will apply the Berkovits topological degree introduced in Subsection 5.2.2.
First, let us set

$$\mathcal{R} := \Big\{ \vartheta \in W^{-1,p'(x)}(\Omega) \ \text{ such that there exists } \ t \in [0,1] \ \text{ such that } \ \vartheta + t\mathcal{N} \circ \mathcal{M}\vartheta = 0 \Big\}.$$

Next, we show that $\mathcal{R}$ is bounded in $W^{-1,p'(x)}(\Omega)$.
Let us put $u := \mathcal{M}\vartheta$ for all $\vartheta \in \mathcal{R}$. Taking into account that $|\mathcal{M}\vartheta|_{1,p(x)} = |\nabla u|_{p(x)}$, then we

have the following two cases:

First case : If $|\nabla u|_{p(x)} \leq 1$.

Then $|\mathcal{M}\vartheta|_{1,p(x)} \leq 1$, that means $\left\{\mathcal{M}\vartheta : \vartheta \in \mathcal{R}\right\}$ is bounded.

Second case : If $|\nabla u|_{p(x)} > 1$.

Then, we deduce from (5.1.2), (A_2) and (A_4), the inequalities (5.1.7) and (5.1.6) and the Young's inequality that

$$
\begin{aligned}
|\mathcal{M}\vartheta|^{p^-}_{1,p(x)} &= |\nabla u|^{p^-}_{p(x)} \\
&\leq \rho_{p(x)}(\nabla u) \\
&\leq \langle \mathcal{G}u,\ u\rangle \\
&= \langle \vartheta,\ \mathcal{M}\vartheta\rangle \\
&= -t\langle \mathcal{N}\circ\mathcal{M}\vartheta,\ \mathcal{M}\vartheta\rangle \\
&= t\int_\Omega \Big(-\omega|u|^{\xi(x)-2}u + \upsilon\,\mathcal{A}(x,u) + \sigma\,\mathcal{B}(x,u,\nabla u)\Big)u\,dx \\
&\leq t\max(|\omega|, \alpha_2|\upsilon|, \alpha_1|\sigma|)\Big(\int_\Omega |u|^{\xi(x)}dx + \int_\Omega |g(x)u(x)|dx + \int_\Omega |u(x)|^{s(x)}dx \\
&\qquad + \int_\Omega |f(x)u(x)|dx + \int_\Omega |u(x)|^{k(x)}dx + \int_\Omega |\nabla u|^{k(x)-1}|u|dx\Big) \\
&= t\max(|\omega|, \alpha_2|\upsilon|, \alpha_1|\sigma|)\Big(\rho_{\xi(x)}(u) + \int_\Omega |g(x)u(x)|dx + \int_\Omega |f(x)u(x)|dx \\
&\qquad + \rho_{s(x)}(u) + \rho_{k(x)}(u) + \int_\Omega |\nabla u|^{k(x)-1}|u|dx\Big) \\
&\leq C\Big(|u|^{\xi^-}_{\xi(x)} + |u|^{\xi^+}_{\xi(x)} + |g|_{p'(x)}|u|_{p(x)} + |f|_{p'(x)}|u|_{p(x)} + |u|^{s^+}_{s(x)} + |u|^{s^-}_{s(x)} + |u|^{k^+}_{k(x)} + |u|^{k^-}_{k(x)} \\
&\qquad + \frac{1}{k'^-}\rho_{k(x)}(\nabla u) + \frac{1}{k^-}\rho_{k(x)}(u)\Big) \\
&\leq C\Big(|u|^{\xi^-}_{\xi(x)} + |u|^{\xi^+}_{\xi(x)} + |u|_{p(x)} + |u|^{s^+}_{s(x)} + |u|^{s^-}_{s(x)} + |u|^{k^+}_{k(x)} + |u|^{k^-}_{k(x)} + |\nabla u|^{k^+}_{k(x)}\Big).
\end{aligned}
$$

Then, according to $L^{p(x)} \hookrightarrow L^{\xi(x)}$, $L^{p(x)} \hookrightarrow L^{s(x)}$ and $L^{p(x)} \hookrightarrow L^{k(x)}$, we get

$$
|\mathcal{M}\vartheta|^{p^-}_{1,p(x)} \leq C\Big(|\mathcal{M}\vartheta|^{\xi^+}_{1,p(x)} + |\mathcal{M}\vartheta|_{1,p(x)} + |\mathcal{M}\vartheta|^{s^+}_{1,p(x)} + |\mathcal{M}\vartheta|^{k^+}_{1,p(x)}\Big),
$$

what implies that $\left\{\mathcal{M}\vartheta : \vartheta \in \mathcal{R}\right\}$ is bounded.

On the other hand, we have that the operator is $\mathcal{N}$ is bounded, then $\mathcal{N}\circ\mathcal{M}\vartheta$ is bounded. Thus, thanks to (9.2.7), we have that $\mathcal{R}$ is bounded in $W^{-1,p'(x)}(\Omega)$.

However, there exists $r > 0$ such that

$$
|\vartheta|_{-1,p'(x)} < r \ \text{ for all } \ \vartheta \in \mathcal{R},
$$

which leads to

$$
\vartheta + t\mathcal{N}\circ\mathcal{M}\vartheta \neq 0, \ \ \vartheta \in \partial\mathcal{R}_r(0) \ \text{ and } \ t \in [0,1],
$$

where $\mathcal{R}_r(0)$ is the ball of center 0 and radius r in $W^{-1,p'(x)}(\Omega)$.
Moreover, by Lemma 5.2.11, we conclude that

$$I + \mathcal{N} \circ \mathcal{M} \in \mathcal{F}_{\mathcal{M}}(\overline{\mathcal{R}_r(0)}) \text{ and } I = \mathcal{G} \circ \mathcal{M} \in \mathcal{F}_{\mathcal{M}}(\overline{\mathcal{R}_r(0)}).$$

On another side, taking into account that I, $\mathcal{N}$ and $\mathcal{M}$ are bounded, then $I + \mathcal{N} \circ \mathcal{M}$ is bounded. Hence, we infer that

$$I + \mathcal{N} \circ \mathcal{M} \in \mathcal{F}_{\mathcal{M},B}(\overline{\mathcal{R}_r(0)}) \text{ and } I = \mathcal{G} \circ \mathcal{M} \in \mathcal{F}_{\mathcal{M},B}(\overline{\mathcal{R}_r(0)}).$$

Next, we define the homotopy

$$\begin{aligned} \mathcal{H} : [0,1] \times \overline{\mathcal{R}_r(0)} &\to W^{-1,p'(x)}(\Omega) \\ (t,\vartheta) &\mapsto \mathcal{H}(t,\vartheta) := \vartheta + t\mathcal{N} \circ \mathcal{M}\vartheta. \end{aligned}$$

Hence, thanks to the properties of the degree d seen in Theorem 5.2.14, we obtain

$$d(I + \mathcal{N} \circ \mathcal{M}, \mathcal{R}_r(0), 0) = d(I, \mathcal{R}_r(0),\ 0) = 1 \neq 0,$$

what implies that there exists $\vartheta \in \mathcal{R}_r(0)$ which verifies

$$\vartheta + \mathcal{N} \circ \mathcal{M}\vartheta = 0.$$

Finally, we infer that $u = \mathcal{M}\vartheta$ is a weak solution of (9.0.2). The proof is completed.

Chapter 10

Existence of weak solutions to a class of nonlinear degenerate parabolic equations in weighted Sobolev spaces by Topological degree methods

In this chapter, we prove the existence of a weak solutions for the initial boundary value problem associated with the nonlinear degenerate parabolic equations

$$\frac{\partial u}{\partial t} - \operatorname{div} b(x,t,u,\nabla u) = \phi(x,t) + \operatorname{div} a(x,t,\nabla u). \tag{10.0.1}$$

We will use the topological degree theory for operators of the type $\mathcal{T} + \mathcal{S}$ to study this problem in the space $L^p(0,T;W_0^{1,p}(\Omega,\omega))$, where Ω is a bounded domain in $\mathbb{R}^N(N \geq 2)$, $p \geq 2$ and ω is a vector of weight functions.

10.1 Hypotheses and technical lemmas

we focus our attention on the basic assumptions and the operators associated with our problem to prove the existence results, and we introduce some useful technical lemmas to prove existence results.

Throughout this paper, we assume that the operators $a : \Omega_T \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and $b : \Omega_T \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ are Carathéodory's functions satisfying the following assumptions:

(A_1) There exists c_1, c_2 positive consts and $k_1, k_2 \in L^q(\Omega_T)$ such that

$$|a_i(x,t,\zeta)| \leq c_1 \omega_i^{1/p}\Big(k_1(x,t) + \sum_{i=1}^{N} \omega_i^{1/q}|\zeta_i|^{p-1}\Big),$$

$$|b_i(x,t,\eta,\zeta)| \leq c_2 \omega_i^{1/p}\Big(k_2(x,t) + \sum_{i=1}^{N} \omega_i^{1/q}|\zeta_i|^{p-1}\Big),$$

for all $i \in \{1,\cdots,N\}$.

$(A_2)(a(x,t,\zeta) - a(x,t,\zeta'))(\zeta - \zeta') > 0$, $(b(x,t,\eta,\zeta) - b(x,t,\eta,\zeta'))(\zeta - \zeta') > 0$.

(A_3) There exists $\alpha_1,\ \alpha_2$ positive constants such that
$$\sum_{i=1}^{N} a_i(x,t,\zeta)\zeta_i \geq \alpha_1 \sum_{i=1}^{N} \omega_i|\zeta_i|^p,\ \sum_{i=1}^{N} b_i(x,t,\eta,\zeta)\zeta_i \geq \alpha_2 \sum_{i=1}^{N} \omega_i|\zeta_i|^p,$$

for all $(x,\ t) \in \Omega_T$, $\eta \in \mathbb{R}$ and $(\zeta',\zeta) \in \mathbb{R}^N \times \mathbb{R}^N$ with $\zeta' \neq \zeta$.

Now, we give the property of the related operator which will be used later.

Lemma 10.1.1 *Assume that the assumptions* $(A_1)-(A_3)$ *hold. Then the operator* $\mathcal{S}$ *defined from* $\mathcal{X}$ *to* $\mathcal{X}^*$ *by*
$$\langle \mathcal{S}u, v\rangle = \sum_{i=1}^{N}\int_{\Omega_T}\Big(a_i(x,t,\nabla u) + b_i(x,t,u,\nabla u)\Big)\partial_i v dx dt,\ \ u,v \in \mathcal{X}$$
is bounded, continuous and of class (S_+).

Proof. Firstly, let us show that the operator $\mathcal{S}$ is bounded.

Let $u, v \in \mathcal{X}$. By using the Hölder's inequality, we get

$$\begin{aligned}
&|\langle \mathcal{S}u, v\rangle| \\
&\leq \int_0^T \Big[\sum_{i=1}^{N}\int_\Omega |a_i(x,t,\nabla u) + b_i(x,t,u,\nabla u)|\omega_i^{-1/p}|\partial_i v|\omega_i^{1/p}dx\Big]dt \\
&\leq \int_0^T \Big[\sum_{i=1}^{N}\int_\Omega |a_i(x,t,\nabla u)|\omega_i^{-1/p}|\partial_i v|\omega_i^{1/p}dx\Big]dt \\
&\quad + \int_0^T \Big[\sum_{i=1}^{N}\int_\Omega |b_i(x,t,u,\nabla u)|\omega_i^{-1/p}|\partial_i v|\omega_i^{1/p}dx\Big]dt \\
&\leq \int_0^T \Big[\sum_{i=1}^{N}\Big(\int_\Omega \big|a_i(x,t,\nabla u)\omega_i^{-1/p}\big|^q dx\Big)^{1/q}\Big(\int_\Omega |\partial_i v|^p\omega_i dx\Big)^{1/p}\Big]dt \\
&\quad + \int_0^T \Big[\sum_{i=1}^{N}\Big(\int_\Omega \big|b_i(x,t,u,\nabla u)\omega_i^{-1/p}\big|^q dx\Big)^{1/q}\Big(\int_\Omega |\partial_i v|^p\omega_i dx\Big)^{1/p}\Big]dt \\
&\leq \int_0^T \Big[\Big(\sum_{i=1}^{N}\int_\Omega \big|a_i(x,t,\nabla u)\omega_i^{-1/p}\big|^q dx\Big)^{1/q}\Big(\sum_{i=1}^{N}\int_\Omega |\partial_i v|^p\omega_i dx\Big)^{1/p}\Big]dt \\
&\quad + \int_0^T \Big[\Big(\sum_{i=1}^{N}\int_\Omega \big|b_i(x,t,u,\nabla u)\omega_i^{-1/p}\big|^q dx\Big)^{1/q}\Big(\sum_{i=1}^{N}\int_\Omega |\partial_i v|^p\omega_i dx\Big)^{1/p}\Big]dt \\
&= \int_0^T \Big[\Big(\sum_{i=1}^{N}\int_\Omega |a_i(x,t,\nabla u)|^q\omega_i^{1-q}dx\Big)^{1/q}\Big(\sum_{i=1}^{N}\int_\Omega |\partial_i v|^p\omega_i dx\Big)^{1/p}\Big]dt \\
&\quad + \int_0^T \Big[\Big(\sum_{i=1}^{N}\int_\Omega |b_i(x,t,u,\nabla u)|^q\omega_i^{1-q}dx\Big)^{1/q}\Big(\sum_{i=1}^{N}\int_\Omega |\partial_i v|^p\omega_i dx\Big)^{1/p}\Big]dt \\
&= \int_0^T \sum_{i=1}^{N}\Big[\|a_i(x,t,\nabla u)\|_{L^q(\Omega,\omega_i^{1-q})} + \|b_i(x,t,u,\nabla u)\|_{L^q(\Omega,\omega_i^{1-q})}\Big]\|v\|dt.
\end{aligned}$$

Thanks to (A_1) and for all $i \in \{1, ..., N\}$, we can easily prove that $\|a_i(x, t, \nabla u)\|_{L^q(\Omega,\omega_i^{1-q})}$ and $\|b_i(x, t, u, \nabla u)\|_{L^q(\Omega,\omega_i^{1-q})}$ are bounded for all $u \in W_0^{1,p}(\Omega, \omega)$. Therefore

$$|\langle \mathcal{S}u, v\rangle| \leq \text{const} \int_0^T \|v\| dt = \text{const}\|v\|_{L^1(0,T;W_0^{1,p}(\Omega,\omega))}.$$

From the continuous embedding $\mathcal{X} \hookrightarrow L^1(0, T; W_0^{1,p}(\Omega, \omega))$, we concludes that

$$|\langle \mathcal{S}u, v\rangle| \leq \text{const}\|v\|_{\mathcal{X}}.$$

Hence, the operator $\mathcal{S}$ is bounded.

Secondly, we show that $\mathcal{S}$ is continuous. Let $u_n \to u$ in $\mathcal{X}$. We need to show that $\mathcal{S}u_n \to \mathcal{S}u$. By using the Hölder's inequality, we have for all $v \in \mathcal{X}$

$$\begin{aligned}
|\langle \mathcal{S}u_n - \mathcal{S}u, v\rangle| \leq & \int_0^T \Big(\int_\Omega |a(x, t, \nabla u_n) - a(x, t, \nabla u)|\omega^{-1/p} \cdot |\nabla v|\omega^{1/p} dx\Big) dt \\
& + \int_0^T \Big(\int_\Omega |b(x, t, u, \nabla u_n) - b(x, t, u, \nabla u)|\omega^{-1/p} \cdot |\nabla v|\omega^{1/p} dx\Big) dt \\
& \leq \int_0^T \|a(x, t, \nabla u_n) - a(x, t, \nabla u)\|_{L^q(\Omega,\omega^{1-q})}\|\nabla v\|_{L^p(\Omega,\omega)} dt \\
& + \int_0^T \|b(x, t, u_n, \nabla u_n) - b(x, t, u, \nabla u)\|_{L^q(\Omega,\omega^{1-q})}\|\nabla v\|_{L^p(\Omega,\omega)} dt \\
& \leq \Big[\|a(x, t, \nabla u_n) - a(x, t, \nabla u)\|_{L^q(\Omega_T,\omega^{1-q})} \\
& \qquad + \|b(x, t, u_n, \nabla u_n) - b(x, t, u, \nabla u)\|_{L^q(\Omega_T,\omega^{1-q})}\Big] \|v\|_{\mathcal{X}},
\end{aligned}$$

so, we need to show that

$$\|a(x, t, \nabla u_n) - a(x, t, \nabla u)\|_{L^q(\Omega_T,\omega^{1-q})} \to 0,$$

and

$$\|b(x, t, u_n, \nabla u_n) - b(x, t, u, \nabla u)\|_{L^q(\Omega_T,\omega^{1-q})} \to 0.$$

On the other hand, note that if $u_n \to u$ in $\mathcal{X}$, then $\nabla u_n \to \nabla u$ in $\prod_{i=1}^N L^p(\Omega_T, \omega_i)$. Hence, by Theorem 1.1.7, there exist a subsequence (u_k) and functions φ in $L^p(\Omega_T, \omega_0)$ and ψ in $\prod_{i=1}^N L^p(\Omega_T, \omega_i)$ such that

$$u_k \to u \text{ and } \nabla u_k \to \nabla u,$$

$$|u_k(x, t)| \leq \varphi(x, t) \text{ and } |\nabla u_k(x, t)| \leq |\psi(x, t)|, \tag{10.1.1}$$

for a.e. $(x, t) \in \Omega_T$ and all $k \in \mathbb{N}$.

Then, in the light of the operators a and b are Carathéodory functions, we deduce that

$$a(x, t, \nabla u_k(x, t)) \to a(x, t, \nabla u(x, t)) \text{ a.e. } (x, t) \in \Omega_T, \tag{10.1.2}$$

$$b(x,t,u_k,\nabla u_k(x,t)) \to b(x,t,u,\nabla u(x,t)) \text{ a.e. } (x,t) \in \Omega_T. \tag{10.1.3}$$

On another side, in view of (A_1), we get for all $i = 1, \cdots, N$

$$|a_i(x,t,\zeta)| \leq c_1 \omega_i^{1/p}\Big(k_1(x,t) + \sum_{i=1}^{N} \omega_i^{1/q}|\psi_i(x,t)|^{p-1}\Big),$$

$$|b_i(x,t,\eta,\zeta)| \leq c_2 \omega_i^{1/p}\Big(k_2(x,t) + \sum_{i=1}^{N} \omega_i^{1/q}|\psi_i(x,t)|^{p-1}\Big),$$

for a.e. $(x,\ t) \in \Omega_T$.

As

$$c_1 \omega_i^{1/p}\Big(k_1(x,t) + \sum_{i=1}^{N} \omega_i^{1/q}|\psi_i(x,t)|^{p-1}\Big) \in \prod_{i=1}^{N} L^q(\Omega_T, \omega_i^{1-q}),$$

and

$$c_2 \omega_i^{1/p}\Big(k_2(x,t) + \sum_{i=1}^{N} \omega_i^{1/q}|\psi_i(x,t)|^{p-1}\Big) \in \prod_{i=1}^{N} L^q(\Omega_T, \omega_i^{1-q}),$$

therefore, thanks to (10.1.2), (10.1.3) and the dominated convergence theorem, we obtain

$$a(x,t,\nabla u_k(x,t)) \to a(x,t,\nabla u(x,t)) \text{ in } L^q(\Omega_T, \omega^{1-q}),$$

$$b(x,t,u_k,\nabla u_k(x,t)) \to b(x,t,u,\nabla u(x,t)) \text{ in } L^q(\Omega_T, \omega^{1-q}).$$

Thus, in view to convergence principle in Banach spaces, we conclude that

$$a(x,t,\nabla u_n(x,t)) \to a(x,t,\nabla u(x,t)) \text{ in } L^q(\Omega_T, \omega^{1-q}), \tag{10.1.4}$$

$$b(x,t,u_n,\nabla u_n(x,t)) \to b(x,t,u,\nabla u(x,t)) \text{ in } L^q(\Omega_T, \omega^{1-q}). \tag{10.1.5}$$

According to (10.1.4) and (10.1.5), we deduce that

$$\langle \mathcal{S}u_n - \mathcal{S}u, v\rangle \to 0, \text{ for all } v \in \mathcal{X},$$

that means, the operator $\mathcal{S}$ is continuous.

Next, we prove that the operator $\mathcal{S}$ is of class (S_+). Let $(u_n)_n \subset \mathcal{X}$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ in } \mathcal{X}, \\ \limsup\limits_{n\to\infty} \langle \mathcal{S}u_n,\ u_n - u\rangle \leq 0. \end{cases} \tag{10.1.6}$$

We will prove that

$$u_n \to u \text{ in } \mathcal{X}.$$

Since $u_n \rightharpoonup u$ in $\mathcal{X}$, then $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, \omega)$, then there exist a subsequence still denoted by (u_n) such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, \omega)$,

$$u_n \to u \quad \text{in } L^p(\Omega, \omega_0) \quad \text{and a.e in } \Omega.$$

On the other hand, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \mathcal{S}u_n,\ u_n-u\rangle \\
&\qquad= \limsup_{n\to\infty}\langle \mathcal{S}u_n-\mathcal{S}u,\ u_n-u\rangle \\
&\qquad= \limsup_{n\to\infty}\Big[\int_{\Omega_T}\Big(a(x,t,\nabla u_n(x,t))-a(x,t,\nabla u(x,t))\Big).\Big(\nabla u_n-\nabla u\Big)dxdt \\
&\qquad\quad+\int_{\Omega_T}\Big(b(x,t,u_n,\nabla u_n(x,t))-b(x,t,u,\nabla u(x,t))\Big).\Big(\nabla u_n-\nabla u\Big)dxdt\Big] \\
&\qquad\leq 0.
\end{aligned}$$

From (A_2) and (10.1.6), we obtain

$$\lim_{n\to\infty}\langle \mathcal{S}u_n,\ u_n-u\rangle=\lim_{n\to\infty}\langle \mathcal{S}u_n-\mathcal{S}u,\ u_n-u\rangle=0. \tag{10.1.7}$$

Let

$$\Theta_n(x,t)=\Big(a(x,t,\nabla u_n)-a(x,t,\nabla u)\Big).\Big(\nabla u_n-\nabla u\Big),$$

Under (10.1.7), we have

$$\Theta_n\to 0 \text{ in } L^1(\Omega_T) \text{ and a.e. in } \Omega_T,$$

Since $\Theta_n\to 0$ a.e in Ω_T, then there exists a subset B of Ω_T ($mes(B)=0$) such that for all $(x,t)\in\Omega\backslash B$,

$$|u(x,t)|<\infty,\quad |\nabla u(x.t)|<\infty,\quad u_n\to u,\quad \Theta_n\to 0.$$

Thanks to (A_1) and (A_3), if we pose $\zeta_n=\nabla u_n$ and $\zeta=\nabla u$, we get

$$\begin{aligned}
\Theta_n(x,t)&=\Big(a(x,t,\zeta_n)-a(x,t,\zeta)\Big).\Big(\zeta_n-\zeta\Big)\\
&=a(x,t,\zeta_n).\zeta_n+a(x,t,\zeta).\zeta-a(x,t,\zeta_n).\zeta-a(x,t,\zeta).\zeta_n\\
&\geq \alpha_1\sum_{i=1}^N\omega_i|\zeta_n^i|^p+\alpha_1\sum_{i=1}^N\omega_i|\zeta^i|^p\\
&\quad-\sum_{i=1}^N c_1\omega_i^{1/p}\Big(k_1(x,t)+\sum_{j=1}^N\omega_j^{1/q}|\zeta_n^j|^{p-1}\Big)|\zeta_n^i|\\
&\quad-\sum_{i=1}^N c_1\omega_i^{1/p}\Big(k_1(x,t)+\sum_{j=1}^N\omega_j^{1/q}|\zeta_n^j|^{p-1}\Big)|\zeta^i|\\
&\geq \alpha_1\sum_{i=1}^N\omega_i|\zeta_n^i|^p-C\Big(1+\sum_{i=1}^N\omega_i^{1/q}|\zeta_n^i|^{p-1}+\sum_{i=1}^N\omega_i^{1/q}|\zeta_n^i|\Big)
\end{aligned}$$

where C is a const which depends only on x.

Then by a standard argument $(\zeta_n)_n$ is bounded a.e. Ω_T, we deduce that

$$\Theta_n(x,t)\geq\sum_{i=1}^N|\zeta_n^i|^p\Big(\alpha_1\omega_i-\frac{C}{N|\zeta_n^i|^p}-\frac{C\omega_i^{1/q}}{|\zeta_n^i|}-\frac{C\omega_i^{1/q}}{|\zeta_n^i|^{p-1}}\Big).$$

Hence, if $|\zeta_n| \to \infty$, then $\Theta_n \to \infty$; what is contradiction with $\Theta_n \to 0$ in $L^1(\Omega_T)$..

Next, for ζ^* be an adherent point of ζ_n, we have $|\zeta^*| < \infty$ and the continuity of a, with respect to the last two variables, we will obtain

$$\Big(a(x,t,\zeta_n) - a(x,t,\zeta)\Big)\big(\zeta^* - \zeta\big) = 0. \tag{10.1.8}$$

Analogously, if we choose

$$\Lambda_n(x,t) = \Big(b(x,t,u_n,\nabla u_n) - b(x,t,u,\nabla u)\Big).\Big(\nabla u_n - \nabla u\Big),$$

and we take $\zeta_n = \nabla u_n$ and $\zeta = \nabla u$, then, by the same arguments used above, we obtain

$$\Big(b(x,t,\eta,\zeta_n) - b(x,t,\eta,\zeta)\Big)\big(\zeta^* - \zeta\big) = 0. \tag{10.1.9}$$

Then, according to (10.1.8), (10.1.9) and (A_2) we get $\zeta^* = \zeta$. Hence, by the uniqueness of the adherent point, we deduce that

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega_T. \tag{10.1.10}$$

On the other hand, seeing that $a(x,t,\nabla u_n)$ and $b(x,t,u_n,\nabla u_n)$ are bounded in $\prod_{i=1}^{N} L^q(\Omega,\omega^{1-q})$, and

$$a(x,t,\nabla u_n) \longrightarrow a(x,t,\nabla u) \text{ a.e. in } \Omega_T,$$

$$b(x,t,u_n,\nabla u_n) \longrightarrow b(x,t,u,\nabla u) \text{ a.e. in } \Omega_T,$$

then, by Lemma 1.3.3, we have

$$a(x,t,\nabla u_n) \rightharpoonup a(x,t,\nabla u) \quad \text{in } \prod_{i=1}^{N} L^q(\Omega,\omega_i^{1-q}),$$

$$b(x,t,u,\nabla u_n) \rightharpoonup b(x,t,u,\nabla u) \quad \text{in } \prod_{i=1}^{N} L^q(\Omega,\omega_i^{1-q}).$$

If we pose

$$\overline{\rho}_n = \Big(a(x,t,\nabla u_n) + b(x,t,u_n,\nabla u_n)\Big)\cdot\nabla u_n,$$

$$\overline{\rho} = \Big(a(x,t,\nabla u) + b(x,t,u,\nabla u)\Big)\cdot\nabla u,$$

we can write

$$\overline{\rho}_n \to \overline{\rho} \quad \text{in } L^1(\Omega_T).$$

Thanks to (A_3), we obtain

$$\overline{\rho}_n \geq (\alpha_1+\alpha_2)\sum_{i=1}^{N}\omega_i|\partial_i u_n|^p \quad \text{and} \quad \overline{\rho} \geq (\alpha_1+\alpha_2)\sum_{i=1}^{N}\omega_i|\partial_i u|^p.$$

In view of $\tau_n = \sum_{i=1}^{N} \omega_i |\partial_i u_n|^p$, $\tau = \sum_{i=1}^{N} \omega_i |\partial_i u|^p$, $\rho_n = \dfrac{\overline{\rho}_n}{(\alpha_1 + \alpha_2)}$ and $\rho = \dfrac{\overline{\rho}}{(\alpha_1 + \alpha_2)}$, we have

$$\rho_n \geq \tau_n \quad \text{and} \quad \rho \geq \tau.$$

Then by Fatou's lemma, we get

$$\int_{\Omega_T} 2\,\rho\, dxdt \leq \liminf_{n \to \infty} \int_{\Omega_T} \rho + \rho_n - |\tau_n - \tau| dxdt,$$

i.e.,

$$0 \leq - \limsup_{n \to \infty} \int_{\Omega_T} |\tau_n - \tau|\, dxdt.$$

So

$$0 \leq \liminf_{n \to \infty} \int_{\Omega_T} |\tau_n - \tau|\, dxdt \leq \limsup_{n \to \infty} \int_{\Omega_T} |\tau_n - \tau|\, dxdt \leq 0,$$

consequently

$$\nabla u_n \longrightarrow \nabla u \quad \text{in} \quad \prod_{i=1}^{N} L^p(\Omega, \omega_i). \tag{10.1.11}$$

According to (10.1.10) and (10.1.11), we have

$$u_n \longrightarrow u \text{ in } W_0^{1,p}(\Omega, \omega),$$

this implies

$$u_n \to u \text{ in } \mathcal{X},$$

what implies that $\mathcal{S}$ is of type (S_+), which completes the proof.

10.2 Main result

First, let us recall that the definition of a weak solution for problem (10.0.1) can be stated as follows.

Definition 10.2.1 *We say that the function $u \in \mathcal{X}$ is a weak solution of (10.0.1) if*

$$-\int_{\Omega_T} u v_t dxdt + \sum_{i=1}^{N} \int_{\Omega_T} \Big(a_i(x,t,\nabla u) + b_i(x,t,u,\nabla u) \Big) \partial_i v dxdt = \int_{\Omega_T} \phi v dxdt,$$

for all $v \in \mathcal{X}$.

We are now in the position to get existence result of weak solution for (10.0.1).

Theorem 10.2.2 *Let $\phi \in \mathcal{X}^*$, $u_0 \in L^2(\Omega)$ and assume that the assumptions $(A_1) - (A_3)$ hold. Then, the problem (10.0.1) admits at least one weak solution $u \in D(\mathcal{T})$, where $D(\mathcal{T}) = \{v \in \mathcal{X} : v' \in \mathcal{X}^*,\ v(0) = 0\}$.*

Proof. Let $\mathcal{S}$ and $\mathcal{T}$ be the operators defined from $D(\mathcal{T}) \subset \mathcal{X}$ to $\mathcal{X}^*$, by

$$\langle \mathcal{S}u, v\rangle = \sum_{i=1}^{N} \int_{\Omega_T} \Big(a_i(x,t,\nabla u) + b_i(x,t,u,\nabla u)\Big)\partial_i v dxdt,$$

$$\langle \mathcal{T}u,\ v\rangle = -\int_{\Omega_T} u\, v_t \, dx\, dt,$$

for all $u \in D(\mathcal{T}),\ v \in \mathcal{X}$. Then $u \in D(\mathcal{T})$ is a weak solution for (10.0.1) if and only if

$$\mathcal{T}u + \mathcal{S}u = \phi \text{ for all } u \in D(\mathcal{T}).$$

One can verify, as in Zeidler [143], that the operator $\mathcal{T}$ is linear densely defined and maximal monotone [143, Theorem 32.L, pp.897-899].

Next, it follows from Lemma 10.1.1 that $\mathcal{S}$ is bounded, continuous and of class (S_+).

Let $u \in \mathcal{X}$. Using the monotonicity of $\mathcal{T}$ $\big(\langle \mathcal{L}u, u\rangle \geq 0$ for all $u \in D(\mathcal{L})\big)$ and the assumption (A_2), we deduce that

$$\begin{aligned}
\langle \mathcal{T}u + \mathcal{S}u, u\rangle &\geq \langle \mathcal{S}u, u\rangle \\
&= \int_{\Omega_T} \Big(a(x,t,\nabla u) + b(x,t,u,\nabla u)\Big)\cdot \nabla v dxdt \\
&\geq \int_{\Omega_T} \alpha_1 \sum_{i=1}^{N} \omega_i |\nabla u|^p dxdt + \int_{\Omega_T} \alpha_2 \sum_{i=1}^{N} \omega_i |\nabla u|^p dxdt \\
&\geq \min(\alpha_1,\alpha_2) \int_{\Omega_T} \sum_{i=1}^{N} \omega_i |\nabla u|^p dxdt \\
&= \min(\alpha_1,\alpha_2) \int_0^T \|u\|^p dt \\
&= \min(\alpha_1,\alpha_2) \|u\|^p_{\mathcal{X}}.
\end{aligned}$$

Because the right-hand side of the previous inequality approximates to ∞ when $\|u\|_{\mathcal{X}} \to \infty$, then for every $\phi \in \mathcal{X}^*$ there is a radius $r = r(\phi) > 0$ such that

$$\langle \mathcal{T}u + \mathcal{S}u - \phi, u\rangle > 0, \quad \text{for each} \quad u \in B_r(0) \cap D(\mathcal{T}).$$

So, all the conditions of Theorem 5.2.7 are satisfied. Consequently, Theorem 5.2.7 leads us to the conclusion that the equation $\mathcal{T}u + \mathcal{S}u = \phi$ has a weak solution in $D(\mathcal{T})$, which implies that the problem (10.0.1) admits at least one weak solution. This completes the proof.

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