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On a class of singular nonlinear elliptic problems

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Contents

Abstract

This thesis concerns the study of some elliptic, nonlinear singular problems. In the model problems considered in this work, we place ourselves on a bounded domain Ω of $\mathbb{R}^N$, with homogeneous Dirichlet boundary conditions. The singular character of the various problems encountered is then reflected by the presence in the equation of a non-linear term of the form $u^{-\gamma}$, with $0 < \gamma \leq 1$, which tends to infinity at the edge of the Ω domain. This poses a certain number of difficulties, linked to the lack of regularity and therefore compactness of the solutions, which do not allow us to use directly the classical methods of non-linear analysis. Through Chapters 3 to 7, we have shown how these difficulties can be overcome and demonstrated new results concerning the existence, regularity and asymptotic behaviour of weak solutions. The general idea used to overcome these obstacles is to first introduce a class of approximate problems, then by using the Fixed Point Theorem we will prove the existence of approximate solutions and then we will establish some estimates for the solutions by taking appropriate test functions, and finally, we will use compactness results in Sobolev spaces to pass to the limit in approximation problems.

Keywords: Elliptic PDEs, Singular elliptic problem, Coercivity , Schauder fixed point theorem, Sobolev spaces.

Résumé

Cette thèse concerne l'étude de certains problèmes elliptiques, non linéaires singuliers. Dans les problèmes-modèles considérés au cours de ce travail, nous nous plaçons sur un domaine borné Ω de $\mathbb{R}^N$, avec des conditions aux limites de type Dirichlet homogène. Le caractère singulier des différents problèmes rencontrés, se traduit alors par la présence dans l'équation d'un terme non-linéaire de la forme $u^{-\gamma}$, avec $0 < \gamma \leq 1$, qui tend vers l'infini au bord du domaine Ω. Ceci pose un certain nombre de difficultés, liées au manque de régularité et donc de compacité des solutions, qui ne nous permettent pas d'utiliser directement les méthodes classiques de l'analyse non-linéaire. A travers les Chapitres 3 à 7, nous avons montré comment ces difficultés peuvent être surmontées et démontré de nouveaux résultats concernant l'existence, la régularité et le comportement asymptotique des solutions faibles. Pour pallier ces obstacles, nous introduirons d'abord une classe de problèmes approchés, en suite par utilisation de Théorème de point fixe on arrivera à démontrer l'existence des solutions approximées puis nous établirons certaines estimations pour les solutions en prenant des fonctions test appropriées, et enfin, nous utiliserons des résultats de compacité dans les espaces de Sobolev pour passer à la limite dans les problèmes d'approximation.
Mots clefs: EDP non linéaire, problème elliptique singulier, coercivité, théorème du point fixe de Schauder, espaces de Sobolev.

Preface

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Notations

Everywhere in the sequel we use the following notations:

- $\mathbb{N}$: the set of all positive natural numbers.

- $\mathbb{R}^N$: the N-dimensional Euclidean space with the distance $|x| = \left(\sum_{i=1}^N x_i^2\right)^{\frac{1}{2}}$ where $x = (x_1, \ldots, x_n)$ is an element of $\mathbb{R}^N$.

- Ω : open bounded set of $\mathbb{R}^N$.
- $\partial\Omega$: boundary of Ω.

- $U \subset\subset \Omega$: means that the closure of U is compact and $\bar{U} \subset \Omega$.

- $|E|$ or meas (E) : Lebesgue measure of the subset E.

- a.e. : abbreviation for almost everywhere (with respect to the Lebesgue measure).

- s.t. : abbreviation for such that.

- V' : the dual space of V (i.e., space of linear and continuous functionals on V) where V is a Banach space.

- $\langle \cdot, \cdot \rangle$: the duality pairing between V and V'.

- $\nabla u = (D_1 u, \ldots, D_N u)\,:$ the gradient of u.

- $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}\,:$ the Laplacian of u.

- $\operatorname{div} v = \sum_{i=1}^{N} D_i v_i\,:$ the divergence of the vector $v = (v_1, \ldots, v_N)$.

- $\chi_E = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{elsewhere,} \end{cases}$ the characteristic function of the set E.

- $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)\,:$ the p-Laplacian operator for $1 < p < N$.

- $\{u \geq (\leq, <, >, =)k\} = \{x \in \Omega, u(x) \geq (\leq, <, >, =)k\}$ for a given function $u : \Omega \to \mathbb{R}$.

- $\operatorname{supp}(u) = \overline{\{x \in \Omega : u(x) \neq 0\}}\,:$ the support of a function u.

- $\operatorname{esssup}(u)$: the essential supremum of a measurable function u.

- $\operatorname{sign}(t) = \frac{t}{|t|}$: the sign of $t \neq 0$.

- $C(\Omega)\,:$ the space of continuous real-valued functions on Ω.

- $C^k(\Omega), k \in \mathbb{N}\,:$ the space of k times differentiable functions on Ω with continuity.

- $C_0^k(\Omega)$: the space of k times differentiable functions on Ω with continuity, 0 on $\partial\Omega$.

- $C_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$: the space of smooth functions with compact support in Ω.

- $\mathcal{D}'(\Omega)$: the dual space of $\mathcal{D}(\Omega)$; space of real distributions on Ω.

- $L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \text{ mesurable, } \underset{\Omega}{\operatorname{esssup}}(u) < \infty\}$.

- $C_+^0(\bar\Omega) = \big\{h \in C(\bar\Omega) : \min_{x\in\bar\Omega} h(x) > 0\big\}$ and $C_+(\bar\Omega) = \big\{h \in C(\bar\Omega) : \min_{x\in\bar\Omega} h(x) > 1\big\}$.

- $h^+ = \max_{x\in\bar{\Omega}} h(x)$ and $h^- = \min_{x\in\bar{\Omega}} h(x)$ for $h \in C^0_+(\bar{\Omega})$. We will also use the following functions

$$V_{\delta,k}(s) = \begin{cases} 1 & \text{if } s \leq k \\ \frac{k+\delta-s}{\delta} & \text{if } k < s < k+\delta, \\ 0 & \text{if } s \geq k+\delta \end{cases} \tag{1}$$

and

$$S_{\delta,k}(s) := 1 - V_{\delta,k}(s). \tag{2}$$

- For the sake of implicity we will often use the simplified notation

$$\int_\Omega f := \int_\Omega f(x)dx,$$

when referring to integrals when no ambiguity on the variable of integration is possible. If no otherwise specified, we will denote by c serval constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data (for instance c can depend on Ω, γ, N, $k, ...$) but the will never depend on the indexes of the sequences we will often introduce.

Chapter 1

Introduction

A partial differential equation (PDEs) describes a relationship between an unknown function and its partial derivatives. PDEs frequently appear in all fields of physics and engineering. In addition, in recent years we have seen a dramatic increase in the use of PDEs in fields such as biology, chemistry, computer science (especially image processing and graphics) and economics (finance). In fact, in each field where there is an interaction between several independent variables, we try to define functions in these variables and model various processes by constructing equations for these functions. When the value of where unknown functions at a certain point depend only on what happens in the neighbourhood of that point, we will generally obtain PDEs.

There are many facets to the analysis of PDEs. The classical approach that dominated the 19th century was to develop methods for finding explicit solutions. Because of the immense importance of PDEs in the various branches of physics, any mathematical development that solved a new class of PDEs was accompanied by significant progress in physics. Thus, the method of characteristics invented by Hamilton led to major advances in optics and analytical mechanics. Fourier's method solved the problem of heat transfer and wave propagation, and Green's method was instrumental in the development of the theory of electromagnetism.

The most spectacular progress in PDEs has been made in the last 50 years with the introduction of numerical methods for using computers to solve PDEs of all types, in general geometry and under arbitrary external conditions (at least in theory; in practice there are still a large number of obstacles to overcome). Technical progress has been followed by theoretical progress in understanding the structure of the solution.

The aim is to discover certain properties of the solution before calculating it, and sometimes even without a complete solution. Theoretical analysis of PDEs is not only of academic interest but has many applications. It should be pointed out that there are very complex equations that cannot be solved even with the help of supercomputers. In these cases, all we can do is try to obtain qualitative information about the solution.

In addition, a very important question concerns the formulation of the equation and the associated (limit or boundary) conditions. In general, the equation is derived from a model of a physical or engineering problem. It is not automatically evident that the model is coherent in the sense that it leads to solvable PDEs.

Furthermore, it is desirable in most cases that the solution is unique and stable, even in the presence of small perturbations in the data. A theoretical understanding of the equation allows us to check whether these conditions are fulfilled. As we will see in the following, there are many ways of solving PDEs, each of which is applicable to a certain class of equations. Therefore, it is important to carry out a thorough analysis of the equation before (or during) the solving.

The fundamental theoretical question is whether the problem constituted by the equation and its associated conditions is well-posed. The French mathematician Jacques Hadamard (1865-1963) invented the notion of well-posedness. According to his definition, a problem is said to be well-posed if it satisfies all the following criteria

1. **Existence:** "A solution exists".
2. **Uniqueness:** "The solution is unique" and
3. **Continuous dependence:** "A small change in data imply a small change in solution".

If one or more of the above conditions are not fulfilled, we say that the problem is ill-posed. It can be said that the fundamental problems of mathematical physics are all well-posed. However, in some engineering applications, we may tackle ill-posed problems. In practice, these problems are unsolvable.

Therefore, when we are faced with an ill-posed problem, the first step should be to modify it appropriately to make it well-posed.

As for as PDEs are concerned, in Physics, there is a famous equation known as " Reaction-diffusion equations". These types of equations have played an important role in the study of many different phenomena related to applications. These applications include, among many others, population dynamics (Lotka-Volterra systems), chemical reactions, combustion, morphogenesis, nerve impulses (Fitzhugh-Nagumo system), genetics, etc. Very often positive solutions are the only physically meaningful solutions or, at least, the more interesting ones. A very simple, but already interesting model problem is the semilinear parabolic equation,

$$\frac{\partial u}{\partial t} - \Delta u = g(x,u) \text{ in } \Omega \tag{1.1}$$

$$u = 0 \ , \text{ on } \partial\Omega \tag{1.2}$$

together with an initial condition. Here Ω is a smooth bounded domain in the space $\mathbb{R}^N$, the ordinary Laplacian is used to model diffusion and the nonlinearity g represents a reaction term in each physical situation. One of the main problems which is considered is the asymptotic (i.e., when time t goes to infinity) behavior of solutions to (1.1) and (1.2). Many different (and difficult to deal with) possibilities are available as, for example, traveling waves, but here we will focus on the situation where the unique positive solution to the parabolic problem (1.1) and (1.2) tends to one of the steady-state positive solutions, i.e., to a solution to the stationary elliptic problem

$$-\Delta u = g(x,u) \text{ in } \Omega, \tag{1.3}$$

$$u = 0 \text{ on } \partial\Omega. \tag{1.4}$$

In general the nonlinear term $g(x,u)$ is smooth and frequently satisfies the condition $g(x,0) \geqslant 0$. (The so-called nonpositone problems, where $g(x,0) < 0$ have been also studied recently, but they are less attractive and applicable than the former ones). Problems with nonlinearities going to infinity when $u > 0$ tends to 0 appear in some applications (see [44, 54, 55, 58], see also [45, 64]), like non-Newtonian fluids, chemical heterogeneous catalysts and nonlinear heat equations, and have intrinsic mathematical interest. These kind of problems (1.3) have been thoroughly studied during the last decades since the pioneer works by Stuart [82] and by Crandall, Rabinowitz and Tartar [35]. In the first one, the author considered a function $g(x,s)$ which "blows-up at $s=0$ " when x goes to a point belonging to the boundary of Ω. On the other hand, in the second one, the authors considered a singular function $g(x,s)=g(s)$ independent of x and they proved the existence of a solution together with some regularity properties of it. Afterwards, in 1991, Lazer and McKenna [66] studied the existence of a classical solution for the Dirichlet problem associated to the above equation in the case

$$g(x,u)=\frac{f(x)}{u^\gamma}$$

where f is an Hölder continuous function which is strictly positive in $\bar{\Omega}$ and γ is a strictly positive parameter. In particular, they proved that "If for some $0<\alpha<1$ one has that $\partial\Omega\in C^{2,\alpha}$,
$f\in C^{0,\alpha}(\bar{\Omega}), f(x)>0$ in $\bar{\Omega}$ and $\gamma>0$, then there exists an unique solution u of the Dirichlet problem

$$\begin{cases} -\Delta u=\dfrac{f(x)}{u^\gamma} & \text{in } \Omega,\\ \quad u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

such that $u\in C^{2,\alpha}(\Omega)\cap C(\bar{\Omega})$ and $u>0$ in Ω". Observe that the prescribed boundary condition in (1.5) makes the study of these singular equations hard. Actually, the assumption " $u=0$ on $\partial\Omega$" together with the singular nonlinearity implies that, for every solution u, the term $1/u(x)^\gamma$ diverges as x goes to the boundary of Ω.
In contrast with [66], we are interested in the study of distributional solutions for the problem (1.5). As usual, this means that we look for distributional solutions u of the differential equation

$$-\Delta u=\frac{f(x)}{u^\gamma}, \text{ in } \Omega, \tag{1.6}$$

which satisfy, in some sense, " $u=0$ on $\partial\Omega$". Specifically, we search for solutions $u\in W^{1,1}_{\text{loc}}(\Omega)$ such that $u>0$ a.e. in Ω, $\frac{f(x)}{u^\gamma}\in L^1_{\text{loc}}(\Omega)$ and moreover they satisfy (1.6) in a distributional sense, i.e.,

$$\int_\Omega \nabla u\nabla\phi=\int_\Omega\frac{f(x)}{u^\gamma}\phi, \quad \forall\phi\in C^1_c(\Omega).$$

With the aim of establishing what the condition " $u=0$ on $\partial\Omega$" means, we point out the surprising result obtained by Lazer and McKenna in [66]. More precisely, the authors proved that

"The unique solution u of the Dirichlet problem (5.18) belongs to the Sobolev space $W^{1,2}_0(\Omega)$ if and only if the parameter $\gamma<3$."

As a consequence, in the distributional context, one would not expect to find solutions belonging to the Sobolev space $W^{1,2}_0(\Omega)$ for any value of $\gamma>0$. Therefore, it is necessary to introduce a new concept for the condition " $u=0$ on $\partial\Omega$".

Precisely, in 2010 , Boccardo and Orsina [11] studied the existence of one distributional solution for the problem (5.18). With respect to the boundary condition " $u=0$ on $\partial\Omega$", in contrast with [33, 61] where this condition is understood under the assumption $(u-\varepsilon)^+\in W^{1,2}_0(\Omega)$ for all $\varepsilon>0$, they followed the ideas of [6]. That is, an even stronger requirement is imposed based on the fact that some positive powers of the solution of the differential equation (1.6) belong to the Sobolev space $W^{1,2}_0(\Omega)$. In this paper, the authors needed to study the cases $\gamma<1, \gamma=1$, $\gamma>1$, separately, connecting each one with the regularity of f. In particular, they proved the following result "Assume that $f\in L^m(\Omega)$ with $m\geq 1$. The following assertions hold:

- $\gamma < 1$ and $m \geq \frac{2N}{N+2+\gamma(N-2)}$, then there exists a positive solution u of (1.5) such that $u \in W_0^{1,2}(\Omega)$.

- If $\gamma = 1$ and $m = 1$, then there exists a positive solution u of (1.5) such that $u \in W_0^{1,2}(\Omega)$.

- If $\gamma > 1$ and $m = 1$, then there exists a positive solution u of (1.5) such that $u^{\frac{\gamma+1}{2}} \in W_0^{1,2}(\Omega)$.

Afterwards, L.M. De Cave studied [41] the generalization of (1.5) to the case with an operator of Leray-Lions kind, i.e., the singular elliptic problem was simplest example is the following:

$$\begin{cases} -\Delta_p u = \frac{f(x)}{u^\gamma} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this context, this thesis contributes to the study of relevant questions in the theory of quasilinear and non linear elliptic equations. In particular, most of the results we present here are stated for problems with a singular nonlinearity. By singularity, we mean that the problems that we have considered, involve a nonlinearity in the right-hand side which blows up near the boundary. This singular pattern gives rise to a lack of regularity and compactness that prevent the straighforward applications of classical methods in nonlinear analysis used for proving existence of solutions and for establishing the properties and the asymptotic behaviour of these solutions. We have shown in the Chapters 3, 4,5, 6 and 7 how to overcome these difficulties and brought new results about existence and regularity of weak solutions.

We stress that the singular problems we have studied in this thesis arise in different contexts: kinetics models in heterogeneous chemical catalysis (seeAris [7]), NonNewtonian flows models, population dynamics models. We would like to quote two nice surveys about singular problems Hernandez-Mancebo [60] and Ghergu-Radulescu [57] where a detailed bibliography and a presentation of the different physical models are available. In the present manuscript, more precisely, in Chapter 3, we discusses the existence and regularity of solutions for the following elliptic singular problems with degenerate coercivity

$$\begin{cases} -\text{div}(a(x,u,\nabla u)) = fh(u) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

with Ω a bounded open subset of $I\!R^N$, $N \geq 2$, $N > p > 1$, f is non-negative and it belongs to $L^m(\Omega)$ for some $m \geq 1$. Finally the singular sourcing $h : [0, \infty) \longrightarrow [0, \infty]$ is continuous, bounded outside the origin with $h(0) \neq 0$ and such that the following propertie hold true

$$\exists\, C,\ \gamma > 0 \quad \text{s.t} \quad h(s) \leq \frac{C}{s^\gamma} \quad \forall s \in (0, +\infty). \tag{1.8}$$

Let us give the precise assumptions on the problems that we will study. Let $a : \Omega \times I\!R \times I\!R^N \longrightarrow I\!R^N$ be Carathéodory function (that is $a(.,t,\xi)$ is measurable on Ω for every (t,ξ) in $I\!R \times I\!R^N$ and $a(x,.,.)$ is continuous on $I\!R \times I\!R^N$ for almost every x in Ω), such that the following assumptions hold :

$$a(x,t,\xi).\xi \geq b(|t|)|\xi|^p, \tag{1.9}$$

for almost every x in Ω and for every (t,ξ) in $I\!R\times I\!R^N$, where $b:(0,\infty)\longrightarrow(0,\infty)$ is a decreasing continuous. For the sake of simplicity, we take the function

$$\mathcal{B}(t)=\int_0^t b(s)^{\frac{1}{p-1}}ds, \tag{1.10}$$

is unbounded, we take in (1.9)

$$b(t)=\frac{\alpha}{(1+t)^{\theta(p-1)}}, \tag{1.11}$$

for some real number $0\le\theta\le 1$ and some $\alpha>0$.

$$|a(x,t,\xi)|\le\beta\big[a_0(x)+|t|^{p-1}+|\xi|^{p-1}\big], \tag{1.12}$$

for almost every x in Ω, for every (t,ξ) in $I\!R\times I\!R^N$, where a_0 is non-negative function in $L^{p'}(\Omega)$, with $\frac{1}{p}+\frac{1}{p'}=1$ and $\beta\ge\alpha$.

$$\big[a(x,t,\xi)-a(x,t,\xi')\big](\xi-\xi')>0, \tag{1.13}$$

for almost every x in Ω and for every t in $I\!R$, for every ξ,ξ' in $I\!R^N$, with $\xi\neq\xi'$ we will then define, for u in $W_0^{1,p}(\Omega)$ the nonlinear elliptic operator

$$A(u)=-\text{div}(a(x,u,\nabla u)).$$

The degenerate coercivity has, in some way, bad effects on both the existence and summability of the solutions. Indeed, the phenomenon of non-existence of solutions appears for large values of θ. But the presence of some lower-order terms may change the nature of the existence results. Recently, the influence of different lower-order terms in non-coercive elliptic problems was the goal of many studies, see, among others, [10, 17, 34, 36, 37, 42, 43, 62, 69, 72, 79]. Starting from paper of Croce [36], where she considered the problem

$$\begin{cases} -\text{div}(a(x,u)\nabla u)+|u|^{s-1}u=f & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{1.14}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with $N\ge 3$ and $a:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that for a.e. $x\in\Omega$ and for every $t\in\mathbb{R}$

$$\frac{\alpha}{(1+|t|)^{\gamma}}\le a(x,t)\le\beta, \tag{1.15}$$

with $\alpha>0,\beta>0$ and $\gamma\ge 0$ are constants. The author showed that, the presence of the lower-order term $|u|^{s-1}u$ not only breaks down the lack of solvability, but also can have a regularizing effects on the solutions. In particular, she obtained the existence results for Problem (1.14) without any additional restriction on γ. The main results of [36] can be summarised as follows:
1) Let $f\in L^1(\Omega)$.

- If $s>\gamma+1$, then there exists a distributional solution u to Problem (1.14) such that $u\in W_0^{1,q}(\Omega)\cap L^s(\Omega)$ with $q<\frac{2s}{s+1+\gamma}$.

- If $0 < s \leq \gamma + 1$, then there exists an entropy solution u to Problem (1.14) such that $|u|^s \in L^1(\Omega)$ and $|\nabla u| \in M^{\frac{2s}{s+1+\gamma}}(\Omega)$.

2) Let $f \in L^m(\Omega), m > 1$.

- If $s \geq \frac{\gamma+1}{m-1}$, then there exists a distributional solution u to Problem (1.14) such that $u \in H_0^1(\Omega) \cap L^{ms}(\Omega)$.

- If $\frac{\gamma}{m-1} < s < \frac{\gamma+1}{m-1}$, then there exists a distributional solution u to Problem (1.14) such that $|u|^{ms} \in L^1(\Omega)$ and $u \in W_0^{1,\frac{2ms}{s+1+\gamma}}(\Omega)$.

- If $0 < s \leq \frac{\gamma}{m-1}$, then there exists an entropy solution u to Problem (1.14) such that $|u|^{ms} \in L^1(\Omega)$ and $|\nabla u| \in M^{\frac{2ms}{s+1+\gamma}}(\Omega)$.

Subsequently, these results were extended to p-Laplacian case in [34]. Therefore, the chapter 4 generalizes some results as described before. More precisely, it deals with the existence and regularity results for distributional and entropy solutions of nonlinear singular elliptic equations with principal part having degenerate coercivity.

$$\begin{cases} -\mathrm{div}\big(a(x,u)|\nabla u|^{p-2}\nabla u\big) + |u|^{s-1}u = h(u)f & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \delta\Omega \end{cases} \tag{1.16}$$

where $1 < p < N$, Ω is bounded set in $I\!R^N$ and $a : \Omega \times I\!R \longrightarrow I\!R$ is a carathéodory function such that for a.e. $x \in \Omega$ and for every $t \in I\!R$, we have

$$a(x,t) \geq \frac{\alpha}{(1+|t|)^\theta} \tag{1.17}$$

$$a(x,t) \leq \beta, \tag{1.18}$$

for some real positive constants α, β and $0 \leq \theta \leq 1$. Moreover, f is a non negative $L^m(\Omega)$ function, with $m \geq 1$ and h satisfied (1.8).

In chapter 3 and 4 we deal with problems as in (1.7) and (1.16) possibly in presence of both a noncoercive principal operator and a general lower order term; in particular the function h may be singular and without any monotonicity property. In this case, to the best of our knowledge, there are no results in literature about existence and regularity of solutions. Our aim is to extend and improve both the existence and regularity results listed above. The existence of a solution is obtained by the means of an approximation process to (1.7) and (1.16). As one can image, the result follows by unifying truncation techniques typical of noncoercive operators with methods employed in dealing with functions possibly blowing up at the origin.

Going ahead the study of singular elliptic problems (ie. $\theta = 0$), we turn your attention to singular elliptic problems involving a Hard potential. In the paper [2], Abdellaoui et al proved an existence and summability result on the solutions of the Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda \dfrac{u^{p-1}}{|x|^p} + \dfrac{h(x)}{u^\gamma} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.19}$$

where $1 < p < N, \Omega \subset \mathbb{R}^N$ is a bounded regular domain containing the origin and $\gamma > 0$. h is a nonnegative measurable function with suitable hypotheses. Problems of the form (1.19) in the case $\gamma = 0$ are introduced as models for several physical phenomena related to the equilibrium of anisotropic continuous media which possibly are somewhere 'perfect' insulators (see [46]). One can see the results for these problems in the papers [1, 8, 29, 32] and the references therein.

When $\lambda \equiv 0$, the equations in the form of (1.19) have been widely studied in the last few decades. We refer to the papers [41] and the references therein. For $\lambda > 0$ and $p = 2$, J. Tyagi studied in [85] the existence and regularity of solutions to the following semilinear elliptic problem with a singular nonlinearity

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) - \lambda \frac{u}{|x|^2} = \frac{f(x)}{u^\theta} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.20}$$

where $\theta > 0, 0 \le f \in L^m(\Omega), 1 < m < \frac{N}{2}, 0 < \lambda < \left(\frac{N-2}{2}\right)^2$, and M is a bounded elliptic matrix. On the other hand, the authors studied in [74] the existence and regularity of solutions to the following problem :

$$\begin{cases} -\Delta_p u + u^q = \frac{f}{u^\gamma} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N (N \ge 2)$, $q > 0, \gamma \ge 0$ and f is a nonnegative function in $L^m(\Omega)$ for some $m \ge 1$. We refer the readers to Refs.[23, 49, 52, 77, 78]. Motivated by such works on this topic, in chapter 5 we study the existence and regularity of distributional solutions for the following problem, in order to improve the results obtained in [4] and [85]

$$\begin{cases} -\operatorname{div}(M(x)|\nabla u|^{p-2}\nabla u) + b|u|^{r-2}u = a\frac{u^{p-1}}{|x|^p} + \frac{f}{u^\gamma} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.21}$$

Let Ω be a bounded, open subset of $\mathbb{R}^N, N \ge 2,\ 1 < p < N$. We assume that $M : \Omega \to \mathbb{R}$, is a Lipschitz continuous function such that for some positive constants α and β

$$M(x)\xi^{p-1}\xi \ge \alpha|\xi|^p, \quad |M(x)| \le \beta \ \text{ for all } \xi \in \mathbb{R}^N \text{ and almost every } x \text{ in } \Omega. \tag{1.22}$$

Assume that

$$r > p^* \text{ and } a > 0, b > 0, \tag{1.23}$$

where p^* is the Sobolev conjugate exponent of p, that is,

$$p^* = \frac{Np}{N-p} \quad \forall p \in (1, N).$$

and

$$0 \leq f \in L^m(\Omega), \quad 1 < m < \frac{N}{p}. \tag{1.24}$$

In a natural way, one can consider another kind of non-linear singular problems presenting new challenges such as the following model problem

$$\begin{cases} -\operatorname{div} a(x, \nabla u) + \mu |u|^{p-1} u = b(x) \frac{|\nabla u|^q}{u^\theta} + \frac{f(x)}{u^\gamma} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.25}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$, f is a nonnegative $L^m(\Omega)$ function with $m \geq 1$ and, given a real number p such that $2 \leq p < N$, we have that $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that the following holds: there exist $\alpha, \beta \in \mathbb{R}^+$ such
that

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) > 0 \quad \text{for a.e. } x \in \Omega \text{ and } \forall \xi, \eta \in \mathbb{R}^N \text{ s.t. } \xi \neq \eta \tag{1.26}$$

$$a(x, \xi) \cdot \xi \geq \alpha |\xi|^p \text{ for a.e. } x \in \Omega \text{ and } \forall \xi \in \mathbb{R}^N \tag{1.27}$$

$$|a(x, \xi)| \leqslant \beta |\xi|^{p-1} \text{ for a.e. } x \in \Omega \text{ and } \forall \xi \in \mathbb{R}^N \tag{1.28}$$

and we assume that

$$0 < \gamma \leq 1, \tag{1.29}$$

$$0 \leq b(x) \in L^\infty(\Omega), \tag{1.30}$$

$$0 < \theta \leq 1, \tag{1.31}$$

and

$$0 \leq \mu, \ \ p - 1 \leq q < p. \tag{1.32}$$

The assumptions on the function a imply that the differential operator A acting between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ and defined by

$$A(u) = -\operatorname{div}(a(x, \nabla u)),$$

is coercive, monotone, surjective and satisfies the maximum principle. The simplest case is the p-Laplacian, which corresponds to the choice $a(x, \xi) = |\xi|^{p-2}\xi$.

In the literature we find several papers about elliptic problems with lower order terms having a quadratic growth with respect to the gradient (see [17, 18, 27, 76], for example and the references therein), that is, for problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = g(x, u)|\nabla u| + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In these works it is assumed that $M : \Omega \to \mathbb{R}^{N^2}$ is a bounded elliptic Carathéodory map, so that there exists $\alpha > 0$ such that $\alpha|\xi|^2 \leq M(x)\xi \cdot \xi$ for every $\xi \in \mathbb{R}^N$. Various assumptions are made on

g with no attempt of being exhaustive, we will describe some recent results where a singular g has been considered, namely $g(x,u) = b(x)\frac{1}{|u|^\theta}$. The case where $0 < \theta \leq 1$, introduced in [37], has been studied positive source $f \in L^m(\Omega)$, if $1 < m < \frac{N}{2}$ there exists a strictly positive solution $u \in L^{m^{**}}(\Omega)$, with $m^{**} = \frac{Nm}{N-2m}$, if $m > \frac{N}{2}$ then the solution u belongs to $L^\infty(\Omega)$. Furthermore, if $0 < \theta < \frac{1}{2}$, and $r = \frac{Nm}{N(1-\theta)-m(1-2\theta)}$, then

$$\frac{|\nabla u|}{u^\theta} \quad \text{belongs to} \quad \begin{cases} L^r(\Omega) & \text{if } 1 < m < \frac{2N(1-\theta)}{N+2-4\theta} \\ L^2(\Omega) & \text{if } m \geq \frac{2N(1-\theta)}{N+2-4\theta}. \end{cases}$$

Later, in [71] it is proved the existence result of solutions for the nonlinear Dirichlet problem of the type

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + \gamma u^p = B\frac{|\nabla u|^q}{u^\theta} + f & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N, N > 2\,, M(x)$ is a uniformly elliptic and bounded matrix, $\gamma > 0, B > 0, 1 \leq q < 2, 0 < \theta \leq 1$ and the source f is a nonnegative (not identicaly zero) function belonging to $L^1(\Omega)$.

Observe that, in this context. Olivia [73] studied the existence and uniqueness of nonnegative solutions to a problem which is modeled by

$$\begin{cases} -\Delta_p u = u^{-\theta}|\nabla u|^p + f u^{-\gamma} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N (N \geq 2), \Delta_p$ is the p -Laplacian operator $(1 < p < N)$,
$f \in L^1(\Omega)$ is nonnegative and $\theta, \gamma \geq 0$.

The main novelty in the chapter 6 is to show that the quadratic lower order term and the singular term has a " regularizing effect " in the sense that the problem (1.25) has a distributional solution for all $f \in L^m$ with $m \geq 1$.

Summarizing, we have shown in the previous chapters how the approximation tools combined with a priori estimates are very useful in the study of singular elliptic problems. Of course, these tools are also useful to address problems of a different nature.

The last chapter is devoted to the study the existence and regularity on the following class of singular elliptic systems

$$\begin{cases} -\operatorname{div}(a(x)\nabla u) + \psi u^{r-1} = \frac{f(x)}{u^\theta} & \text{in } \Omega, \\ -\operatorname{div}(M(x)\psi) = u^r & \text{in } \Omega, \\ u, \psi > 0 & \text{in } \Omega, \\ u = \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.33}$$

we will suppose that Ω is a bounded open set of $\mathbb{R}^N, N > 2$, that $r > 1$ and that f nonnegative (not identically zero) function belongs to $L^m(\Omega)$, for some $m > 1$, $0 < \theta < 1$. Furthermore, the function $a : \Omega \to \mathbb{R}$ will be a measurable function, such that there exist $0 < \alpha \leq \beta$ such that:

$$0 < \alpha \leq a(x) \leq \beta \quad \text{almost everywhere in } \Omega, \tag{1.34}$$

while $M : \Omega \to \mathbb{R}^{N^2}$ will be a measurable matrix, such that:

$$M(x)\xi \cdot \xi \geq \alpha|\xi|^2, \quad |M(x)| \leq \beta, \tag{1.35}$$

for almost every x in Ω, and for every ξ in $\mathbb{R}^N$.

We have been motivated by the work of Benci and Fortunato [13]. In that work the authors, investigating the eigenvalue problem for the Schrödinger operator coupled with the electromagnetic field, studied the existence for the following system of Schrödinger-Maxwell equations in $\mathbb{R}^3$

$$\begin{cases} -\frac{1}{2}\Delta u + \varphi u = \omega u \\ -\Delta\varphi = 4\pi u^2. \end{cases} \tag{1.36}$$

The existence of a solution of (1.36) is proved by using a variational approach: the equations of the system are the Euler-Lagrange equations of a suitable functional that is neither bounded from below nor from above but has a critical point of saddle type.

More recently, taking inspiration from the structure of (1.36), a series of papers (see for instance [14, 15, 47]) studied the existence and the regularizing effect of the problem

$$\begin{cases} -\operatorname{div}(a(x)u) + Bv|u|^{r-2}u = f(x) & \text{in } \Omega \\ -\operatorname{div}(M(x)v) = |u|^r & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.37}$$

where $B > 0, f \in L^m(\Omega)$ with $m > 1$, and $a(x), M(x)$ satisfies (7.2) and (7.3). One of the main feature of (1.37) is that the interplay of the two equation enhances the regularizing effect of the system with respect of the one of the single equation. The main techniques used in [14, 15, 47] are approximation scheme, a priori estimates through a test function based approach and fixed point theorems. These tools can be used for more general system that do not necessarily have a variational structure. On the other hand, the authors proved in [47] the existence of solutions for the following nonlinear elliptic system that generalizes (1.37)

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + A\varphi|u|^{r-2}u = f, & u \in W_0^{1,p}(\Omega) \\ -\operatorname{div}\left(|\nabla\varphi|^{p-2}\nabla\varphi\right) = |u|^r, & \varphi \in W_0^{1,p}(\Omega) \end{cases} \tag{1.38}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 2), 1 < p < N, A > 0, r > 1$.
Inspired by the above articles, the main novelty in the last chapter is to show that the term $\frac{1}{u^\theta}$ has a "regularizing effect" in the sense that the problem (1.33) has a distributional solution for all $f \in L^m(\Omega)$ with $m > 1$. This term provokes some mathematical difficulties, which make the study of system (1.33) particularly interesting. To our knowledge, the Schrödinger–Maxwell system with singular term has not been studied.

Chapter 2

Mathematical preliminaries

This chapter is meant to provide an overview of the real and functional analysis results that will be used afterwards. Moreover, we present some basic facts concerning the necessary function spaces.

Unless otherwise required, in this chapter, $\Omega \subset \mathbb{R}^N$ is a bounded open set equipped with N-dimensional Lebesgue measure. Note that the results in this chapter are not given in full generality, these will be presented as needed in our study.

1 Standard Lebesgue , Sobolev spaces and Marcinkiewicz spaces

In this section we recall some basic facts on classical Lebesgue and Sobolev spaces with constant exponent that we will use in the remainder of this thesis. For further details on this topic, we refer to [20, 30, 39, 53, 67].

1.1 Lebesgue spaces

We say that a measurable function $\phi : \Omega \to \mathbb{R}$ belongs to the Lebesgue space $L^p(\Omega)$, $p \in [1, \infty]$, if the quantity $||\phi||_{L^p(\Omega)}$ is finite.

$$||\phi||_{L^p(\Omega)} = \begin{cases} \inf\{C \in (0, \infty) : \quad ||\phi|| \leq C \text{ a.e. on } \Omega\} & \text{if } p = \infty \\ \left(\int_\Omega |\phi|^p\right)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases}$$

Endowed with the norm $\|\cdot\|_{L^p(\Omega)}$, $L^p(\Omega)$ is a Banach space which turns out to be separable if $p \in [1, \infty)$ and reflexive if $p \in (1, \infty)$.

For an exhaustive treatment on Lebesgue spaces we refer to [3] and [30]. We only recall the following fundamental facts.

- **Hölder's inequality**: if $p \in [1, \infty]$ and p' is the Hölder conjugate exponent of p, that is

$$p' = \begin{cases} 1 & \text{if } p = \infty \\ \frac{p}{p-1} & \text{if } p \in (1, \infty) \\ \infty & \text{if } p = 1, \end{cases}$$

then
$$\left|\int_\Omega \phi\psi\right| \le \|\phi\|_{L^p(\Omega)}\|\psi\|_{L^{p'}(\Omega)} \quad \forall \phi \in L^p(\Omega), \forall \psi \in L^{p'}(\Omega).$$

- **Young inequality**: For all non-negative real numbers a, b and every $1 < p < \infty$, the Young inequality holds
$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'},\ p' = \frac{p}{p-1},$$
which will be used in the following form: for every $\varepsilon > 0, 1 < p < \infty$ and real nonnegative numbers a, b
$$ab \le \varepsilon a^p + C_\varepsilon b^{p'} \text{ with } C_\varepsilon = \varepsilon^{\frac{-1}{p-1}}.$$

1.2 Sobolev spaces

We say that a measurable function $\phi : \Omega \to \mathbb{R}$ belongs to the local Lebesgue space $L^p_{\text{loc}}(\Omega), p \in [1, \infty]$, if $\phi \in L^p(U)$ for every open subset $U \subset\subset \Omega$.

If $\phi \in L^1_{\text{loc}}(\Omega)$, the distributional partial derivative ϕ_{x_i} of (the Schwartzian distribution on Ω induced by) ϕ in the direction x_i is the Schwartzian distribution on Ω defined by
$$\phi_{x_i}(\zeta) = -\int_\Omega \phi\zeta_{x_i} \quad \forall \zeta \in C_c^\infty(\Omega).$$
The distributional gradient of ϕ is the vector field $\nabla\phi = (\phi_{x_1}, \ldots, \phi_{x_N})$. We recall that if $\phi \in C^1(\Omega)$, the distributional partial derivatives of ϕ coincide with the usual ones, hence the notation is consistent. We say that a measurable function $\phi : \Omega \to \mathbb{R}$ belongs to the Sobolev space $W^{1,p}(\Omega), p \in [1, \infty]$, if $\phi \in L^p(\Omega)$ and $\phi_{x_i} \in L^p(\Omega)$ for every $i \in \{1, \ldots, N\}$. Endowed with the norm
$$\|\phi\|_{W^{1,p}(\Omega)} = \|\phi\|_{L^p(\Omega)} + \||\nabla\phi|\|_{L^p(\Omega)},$$
$W^{1,p}(\Omega)$ is a Banach space which turns out to be separable if $p \in [1, \infty)$ and reflexive if $p \in (1, \infty)$. For $p \in [1, \infty)$, the closure in $W^{1,p}(\Omega)$ of the subspace $C_c^\infty(\Omega)$ will be denoted by $W_0^{1,p}(\Omega)$ and its dual space by $W^{-1,p'}(\Omega)$. Hence, $W_0^{1,p}(\Omega)$ is a separable Banach space with the same norm of $W^{1,p}(\Omega)$ and it is reflexive if $p \in (1, \infty)$. The local Sobolev space $W^{1,p}_{\text{loc}}(\Omega), p \in [1, \infty]$, consists of functions belonging to $W^{1,p}(U)$ for every open subset $U \subset\subset \Omega$. We set $H^1(\Omega) = W^{1,2}(\Omega), H_0^1(\Omega) = W_0^{1,2}(\Omega)$, $H^{-1}(\Omega) = W^{-1,2}(\Omega)$ and $H^1_{\text{loc}}(\Omega) = W^{1,2}_{\text{loc}}(\Omega)$.

For an exhaustive treatment on Sobolev spaces we refer to [3] and [30]. We only recall the following fundamental facts.

- **Sobolev's inequality**: there exists a positive constant $\mathcal{S}_0$ which depends only on N and p, such that
$$\begin{cases} \|\phi\|_{L^\infty} \le \mathcal{S}_0|\Omega|^{\frac{1}{N}-\frac{1}{p}}\||\nabla\phi|\|_{L^p(\Omega)} & \text{if } p \in (N, \infty), \\ \|\phi\|_{L^{p^*}(\Omega)} \le \mathcal{S}_0\||\nabla\phi|\|_{L^p(\Omega)} & \text{if } p \in (1, N), \end{cases} \quad \forall \phi \in W_0^{1,p}(\Omega),$$
where p^* is the Sobolev conjugate exponent of p, that is,
$$p^* = \frac{Np}{N-p} \quad \forall p \in [1, N).$$

In general, $W_0^{1,p}(\Omega)$ cannot be replaced by $W^{1,p}(\Omega)$ in the previous embedding result. However, this replacement can be made for a large class of open sets Ω, which includes for example open sets with Lipschitz boundary. More generally, if Ω satisfies a uniform interior cone condition (that is, there exists a fixed cone U_Ω of height h and solid angle ω such that each $x \in \Omega$ is the vertex of a cone $U_\Omega(x) \subset \bar{\Omega}$ and congruent to U_Ω), then there exists a positive constant $\mathcal{S}$ which depends only on N and p, such that

$$\begin{cases} \|\phi\|_{L^\infty} \leq \frac{\mathcal{S}}{\omega h^{\frac{N}{p}}} \left(||\phi||_{L^p(\Omega)} + \||\nabla\phi|\|_{L^p(\Omega)}\right) & \text{if } p \in (N,\infty), \\ \|\phi\|_{L^{p^*}(\Omega)} \leq \frac{\mathcal{S}}{\omega} \left(\frac{1}{h}||\phi||_{L^p(\Omega)} + \||\nabla\phi|\|_{L^p(\Omega)}\right) & \text{if } p \in (1,N), \end{cases} \quad \forall\phi \in W_0^{1,p}(\Omega).$$

- **Rellich-Kondrachov's Theorem**: the embedding

$$W_0^{1,p}(\Omega) \subset \begin{cases} L^\infty(\Omega) & \text{if } p \in (N,\infty), \\ L^q(\Omega) \quad \forall q \in [1,p^*) & \text{if } p \in [1,N). \end{cases}$$

is compact. Moreover, if Ω satisfies a uniform interior cone condition, then also the embedding

$$W^{1,p}(\Omega) \subset \begin{cases} L^\infty(\Omega) & \text{if } p \in (N,\infty), \\ L^q(\Omega) \quad \forall q \in [1,p^*) & \text{if } p \in (1,N). \end{cases}$$

is compact.

- **Poincaré's inequality**: there exists a positive constant $\mathcal{P}$ which depends only on N, p and Ω, such that

$$\|\phi\|_{L^p(\Omega)} \leq \mathcal{P}\||\nabla\phi|\|_{L^p(\Omega)} \quad \forall\phi \in W_0^{1,p}(\Omega).$$

Accordingly, the quantity $\||\nabla \cdot |\|_{L^p(\Omega)}$ defines a norm on $W_0^{1,p}(\Omega)$ which is equivalent to $\|\cdot\|_{W^{1,p}(\Omega)}$.

- **Stampacchia's Theorem** (see [80]): if $\Phi \in W^{1,\infty}(\mathbb{R})$ is such that $\Phi(0) = 0$, then, for every $\phi \in W_0^{1,p}(\Omega)$, the composition $\Phi(\phi)$ belongs to $W_0^{1,p}(\Omega)$ and

$$\nabla\Phi(\phi) = \Phi'(\phi)\nabla\phi \quad \text{a.e. on } \Omega.$$

Moreover, one has that

$$\nabla\phi = 0 \quad \text{a.e. on } \{\phi = \sigma\} \quad \forall\phi \in W_0^{1,p}(\Omega), \forall\sigma \in \mathbb{R}$$

Accordingly, we are able to consider compositions of functions in $W_0^{1,p}(\Omega)$ with some useful auxiliary functions, such as, for any positive σ, the truncation function at level σ, that is,

$$T_\sigma(s) = \begin{cases} s & \text{if } |s| \leq \sigma \\ \text{sign}(s)\sigma & \text{if } |s| > \sigma \end{cases}$$

and

$$G_\sigma(s) = s - T_\sigma(s) = (|s| - \sigma)^+ \,\text{sign}(s) \quad \forall s \in \mathbb{R}.$$

In particular, for every $\phi \in W_0^{1,p}(\Omega)$ and $\sigma \in (0,\infty), T_\sigma(\phi), G_\sigma(\phi)$ belong to $W_0^{1,p}(\Omega)$ and satisfy

$$\nabla T_\sigma(\phi) = \nabla\phi\chi_{\{|\phi|<\sigma\}}, \quad \nabla G_\sigma(\phi) = \nabla\phi\chi_{\{|\phi|>\sigma\}} \quad \text{a.e. on } \Omega.$$

1.3 Convergence Theorems

Throughout this subsection we provide some definitions and results on the convergence of sequences of measurable functions, which can be found, for example, in [31, 38, 39, 59, 63].

Definition 2.1. Let (u_n) be a sequence of measurable functions on Ω and u a measurable function on Ω.

1. The sequence (u_n) is said to converge almost everywhere on Ω to u if and only if meas $\{x \in \Omega : u_n(x)$ does not converge to $u(x)\} = 0$, and we write $u_n \to u$ a.e. in Ω.
2. The sequence (u_n) is said to converge in measure on Ω to u if for every $\eta > 0$
$$\lim_{n\to+\infty} \text{meas } \{x \in \Omega : |u_n(x) - u(x)| > \eta\} = 0.$$
3. The sequence (u_n) is said to be Cauchy if for every $\varepsilon > 0$ and every $\eta > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, then
$$\text{meas}\,\{x \in \Omega : |u_n(x) - u_m(x)| > \eta\} < \varepsilon.$$

The following proposition shows that if a sequence is Cauchy in measure then it must converge in measure.

Proposition 2.2. *([63]) Let (u_n) be a sequence of measurable functions on Ω, then the following statements are equivalent.*

1. *(u_n) is Cauchy in measure.*
2. *There exists a measurable function u (uniquely determined almost everywhere) such that (u_n) converges to u in measure.*

The next proposition describes the relation between different modes of convergence.

Proposition 2.3. *([39]) Let (u_n) be a sequence of measurable functions on Ω .*

1. *If $u_n \to u$ a.e. in Ω then $u_n \to u$ in measure (here Ω is bounded).*
2. *If $u_n \to u$ in measure, then there exists a subsequence (u_{n_k}) such that $u_{n_k} \to u$ a.e. in Ω as $k \to \infty$.*

We now give the definition of a Carathéodory function.

Definition 2.4. Let $m \geq 1$. A function $a = a(x, \xi) : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function if for all $\xi \in \mathbb{R}^m$ the function
$$f(\cdot, \xi) : \Omega \to \mathbb{R},$$
is measurable and for almost every $x \in \Omega$ the function
$$f(x, \cdot) : \mathbb{R}^m \to \mathbb{R},$$
is continuous.

Proposition 2.5. *([20]) Let $a = a(x, \xi) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Let u_n be a sequence of functions and u be a measurable function such that $u_n \to u$ in measure. Then $a(x, u_n) \to a(x, u)$ in measure.*

We frequently use the following convergence results.

Theorem 2.6. ***(Monotone convergence theorem** [67]) Let (u_n) be an increasing sequence of non-negative measurable functions on Ω, which converges pointwise to u. Then*

$$\int_\Omega u_n dx \longrightarrow \int_\Omega u dx \text{ when } n \to \infty.$$

Theorem 2.7. ***(Fatou's Lemma** [67]) Let (u_n) be a sequence of non-negative measurable functions on Ω. Then*

$$\int_\Omega \left(\liminf_{n\to\infty} u_n \right) dx \leq \liminf_{n\to\infty} \int_\Omega u_n dx.$$

The next result is the analog of Fatou's Lemma.

Proposition 2.8. *([38]) Let $1 \leq p < \infty$. Suppose the sequence $(u_n) \subset L^p(\Omega)$ is such that $u_n \to u$ a.e. in Ω. If*

$$\liminf_{n\to\infty} \|u_n\|_{L^p(\Omega)} < \infty,$$

then $u \in L^p(\Omega)$ and

$$\|u\|_{L^p(\Omega)} \leq \liminf_{n\to\infty} \|u_n\|_{L^p(\Omega)},$$

Theorem 2.9. ***(Lebesgue's dominated convergence Theorem** [67]) Let the sequence (u_n) of $L^p(\Omega)$ with $1 \leq p < \infty$, converge a.e. to u, and be dominated by $v \in L^p(\Omega)$, in the sense that $|u_n(x)| \leq v(x)$ a.e. in Ω. Then $u_n \to u$ (strongly) in $L^p(\Omega)$, that is, $u \in L^p(\Omega)$ and*

$$\|u_n - u\|_{L^p(\Omega)} \to 0 \text{ as } n \to \infty.$$

The partial converse of the dominated convergence theorem is stated in the following lemma.

Proposition 2.10. *([67]) If $u_n \longrightarrow u$ in $L^p(\Omega)$ with $1 \leq p < \infty$. Then we can extract a subsequence (u_{n_k}) such that $u_{n_k} \to u$ a.e. in Ω as $k \to \infty$.*

Theorem 2.11. ***(Vitali's convergence Theorem** [30]) Let (u_n) be a sequence of functions in $L^p(\Omega)$ with $1 \leq p < \infty$ such that*

- *$u_n \to u$ a.e. on Ω.*
- *(u_n) is equi-integrable, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\int_E |u_n(x)|^p \, dx \leq \varepsilon,$$

for all n and for every measurable set $E \subset \Omega$ with meas $(E) \leq \delta$.

Then $u_n \to u$ in $L^p(\Omega)$.

Proposition 2.12. *(**Weak compactness** [53]) Assume $1 < p < \infty$ and the sequence (u_n) is bounded in $L^p(\Omega)$. Then there exists a subsequence (u_{n_k}) and a function $u \in L^p(\Omega)$ such that $u_{n_k} \to u$ weakly in $L^p(\Omega)$, i.e.,*

$$\int_\Omega u_{n_k} v dx \longrightarrow \int_\Omega uv dx, \text{ as } k \to \infty, \text{ for all } v \in L^{p'}(\Omega).$$

Similarly, if (u_n) is bounded in $L^\infty(\Omega)$. Then there exists a subsequence (u_{n_k}) and a function $u \in L^\infty(\Omega)$ such that $u_{n_k} \to u$ weakly-$*$ in $L^\infty(\Omega)$, i.e.,

$$\int_\Omega u_{n_k} v dx \longrightarrow \int_\Omega uv dx, \text{ as } k \to \infty, \text{ for all } v \in L^1(\Omega).$$

We remark that when Ω is bounded, the weak-convergence of (u_n) in $L^\infty(\Omega)$ to some $u \in L^\infty(\Omega)$ implies weak convergence of (u_n) to u in any $L^p(\Omega), 1 \leq p < \infty$. It is important to note that the above theorem is false when $p = 1$, since a bounded sequence in $L^1(\Omega)$ has in general no weak convergence property. The following lemma shows the boundedness of weakly convergent sequences.

Proposition 2.13. *([53]) Let $1 \leq p \leq \infty$. Assume $u_n \to u$ weakly in $L^p(\Omega)$ (weakly * if $p = \infty$). Then*

i) (u_n) is bounded in $L^p(\Omega)$.

ii) $\|u\|_{L^p(\Omega)} \leq \liminf_{n\to\infty} \|u_n\|_{L^p(\Omega)}$.

In view of (i) it result that $u_n \to u$ weakly in $L^p(\Omega)$ (weakly- * if $p = \infty$) and $v_n \to v$ in $L^{p'}(\Omega)$ then

$$\int_\Omega u_n v_n dx \longrightarrow \int_\Omega uv dx, \text{ as } n \to \infty.$$

Proposition 2.14. *([20]) Let (u_n) be a sequence of functions in $L^p(\Omega)$ with $1 < p < \infty$. Assume that*

- *(u_n) is bounded in $L^p(\Omega)$;*
- *$u_n \to u$ a.e. on Ω.*
  Then $u_n \to u$ in $L^q(\Omega)$, for every $1 \leq q < p$ and weakly in $L^p(\Omega)$.

We have the following characterization of weak convergence in $W^{1,p}(\Omega)$.

Proposition 2.15. *([20]) A sequence (u_n) weakly converges to u in $W^{1,p}(\Omega)$, if and only if there exist $v_i \in L^p(\Omega)$ such that $u_n \to u$ weakly in $L^p(\Omega)$ and $D_i u_n \to v_i$ weakly in $L^p(\Omega), i = 1, \ldots, N$. In this case, $v_i = D_i u$.*

Proposition 2.16. *([67]) If $u \in L^p(\Omega)$, then $T_k(u) \to u$ in $L^p(\Omega)$ strong when $k \to +\infty$. If $u \in W^{1,p}(\Omega)$, then $T_k(u) \to u$ in $W^{1,p}(\Omega)$ strong.*

The following results concerns the superposition operators.

Proposition 2.17. *([67]) Let T be a globally Lipschitz continuous function from $\mathbb{R}$ to $\mathbb{R}$ piecewise C^1 and with only a finite number of points of non differentiability $c_1, c_2, \ldots, c_k$. Assume that $T(0) = 0$. Then*
1) for all $u \in W_0^{1,p}(\Omega)$:

(i) $T(u) \in W_0^{1,p}(\Omega)$.

(ii) $\nabla(T(u)) = T'(u)\nabla u$ *on* $\Omega \setminus \cup_{i=1}^k E_{c_i}$ *and* $\nabla(T(u)) = 0$ *a.e. on* $\cup_{i=1}^k E_{c_i}$, *where* $E_{c_i} = u^{-1}(c_i)$, $i = 1, \ldots, k$.

2) the mapping $u \mapsto T(u)$ *is*

(i) continuous from $W^{1,p}(\Omega)$ *strong to* $W^{1,p}(\Omega)$ *strong for all* $p < \infty$.

(ii) sequentially continuous from $W^{1,p}(\Omega)$ *weak to* $W^{1,p}(\Omega)$ *weak (weak-* for* $p = \infty$*)*.

1.4 Marcinkiewicz spaces

We say that a measurable function $\phi : \Omega \to \mathbb{R}$ belongs to the Marcinkiewicz space $M^p(\Omega), p \in (0, \infty)$, if there exists a positive constant C such that

$$|\{|\phi| > \sigma\}| \leq \frac{C}{\sigma^p} \quad \forall \sigma \in (0, \infty).$$

Endowed with the quasinorm

$$\|\phi\|_{M^p(\Omega)} = \sup_{\sigma \in (0,\infty)} \left\{ |\{|\phi| > \sigma\}|^{\frac{1}{p}} \sigma \right\},$$

$M^p(\Omega)$ is a quasi-Banach space. We recall that the Marcinkievicz spaces are intermediate spaces between Lebesgue spaces, in the sense that the following continuous embeddings hold:

$$L^p(\Omega) \subset M^p(\Omega) \subset L^{p-\epsilon}(\Omega) \quad \forall p \in (1, \infty), \epsilon \in (0, p-1].$$

Moreover, if $p \in (1, \infty)$, for every $\phi \in M^p(\Omega)$ there exists a positive constant C which depends only on p and $\|\phi\|_{M^p(\Omega)}$ such that

$$\int_U |\phi| \leq C|U|^{\frac{1}{p'}} \quad \forall \text{ measurable subset } U \subset \Omega.$$

2 Rearrangements and related properties

In this section we recall a few notions about rearrangements. Let Ω be an open bounded set of $I\!R^N$. If u is a measurable function in Ω, we define the distribution function μ of u as follows

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}|, \;\; t \geq 0.$$

Where $|E|$ denotes the Lebesgue measure of a measurable subset E of $I\!R^N$. The function μ is decreasing and right-continuous. The decreasing rearrangement of u is defined by

$$u^*(s) = \inf\{t \geq 0 : \mu_u(t) \leq s\} \qquad \text{for } s \in [0, |\Omega|].$$

Recall that the following inequality

$$u^*(\mu_u(t)) \leq t, \tag{2.1}$$

holds for every $t > 0$ (see [23],[83]). We also have (see [[84],page 3])

$$u^*(0) = \text{ess sup}\,|u|. \tag{2.2}$$

If f is any continuous increasing map from $[0, \infty]$ into $[0, \infty]$ such that $f(0) = 0$, then [84]

$$\int_\Omega f(|u(x)|)dx = \int_0^\infty f(u^*(t))dt.$$

Chapter 3

Degenerate elliptic problem with a singular nonlinearity

In this chapter we prove some existence and regularity results for a nonlinear elliptic problem with degenerate coercivity via Schauder's fixed point theorem. The results of this chapter, which have been published in [77].

1 Introduction

In this chapter we are concerned with the existence of a distributional solution for a singular degenerate elliptic problem modelled by

$$\begin{cases} -\text{div}(a(x,u,\nabla u)) = fh(u) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with Ω a bounded open subset of $I\!R^N$, $N \geq 2$, $N > p > 1$, f is non negative and it belongs to $L^m(\Omega)$ for some $m \geq 1$. Finally the singular sourcing $h : [0,\infty) \longrightarrow [0,\infty]$ is continuous, bounded outside the origin with $h(0) \neq 0$ and such that the following propertie hold true

$$\exists C,\ \gamma > 0 \quad \text{s.t} \quad h(s) \leq \frac{C}{s^\gamma} \quad \forall s \in (0,+\infty). \tag{3.2}$$

Let us give the precise assumptions on the problems that we will study. Let Ω be a bounded open subset of $I\!R^N$,$N \geq 2$, let $N > p > 1$ and let $a : \Omega \times I\!R \times I\!R^N \longrightarrow I\!R^N$ be Carathéodory function (that is $a(.,t,\xi)$ is measurable on Ω for every (t,ξ) in $I\!R \times I\!R^N$ and $a(x,.,.)$ is continuous on $I\!R \times I\!R^N$ for almost every x in Ω), such that the following assumptions hold :

$$a(x,t,\xi).\xi \geq b(|t|)|\xi|^p, \tag{3.3}$$

for almost every x in Ω and for every (t,ξ) in $I\!R \times I\!R^N$, where $b : (0,\infty) \longrightarrow (0,\infty)$ is a decreasing continuous. For the sake of simplicity, we take the function

$$\mathcal{B}(t) = \int_0^t b(s)^{\frac{1}{p-1}} ds, \tag{3.4}$$

is unbounded, for the sake of simplicity, we take in (3.3)

$$b(t) = \frac{\alpha}{(1+t)^{\theta(p-1)}}, \tag{3.5}$$

for some real number $0 \leq \theta \leq 1$ and some $\alpha > 0$.

$$|a(x,t,\xi)| \leq \beta\big[a_0(x) + |t|^{p-1} + |\xi|^{p-1}\big], \tag{3.6}$$

for almost every x in Ω, for every (t,ξ) in $I\!R \times I\!R^N$, where a_0 is non-negative function in $L^{p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\beta \geq \alpha$,

$$\big[a(x,t,\xi) - a(x,t,\xi')\big](\xi - \xi') > 0, \tag{3.7}$$

for almost every x in Ω and for every t in $I\!R$, for every ξ,ξ' in $I\!R^N$, with $\xi \neq \xi'$ we will then define, for u in $W_0^{1,p}(\Omega)$ the nonlinear elliptic operator

$$A(u) = -\mathrm{div}(a(x,u,\nabla u)).$$

In the study of problem (3.1), there are two difficulties, the first one is the fact that, due to hypothesis (3.3), the differential operator $A(u)$ though well defined between $W_0^{1,p}(\Omega)$ and its dual, but it fails to be coercive on $W_0^{1,p}(\Omega)$ when u is unbounded. Due to the lack of coercivity, the classical theory for elliptic operators acting between spaces in duality (see [70]) can not be applied even if the data f are sufficiently regular(see [76]). The second difficulty comes from the right-hand side is singular in the variable u. We overcome these difficulties by replacing operator A by another one defined by means of truncations, and approximating the singular term by non singular one.
Now, we give our definitions of solution for problem (3.1).

Definition 3.1. Let f be in $L^m(\Omega)$,$m \geq 1$. A measurable function u is a weak solution of (3.1) if $a(x,u,\nabla u) \in (L^1(\Omega))^N$, $fh(u) \in L^1(\Omega)$ and if

$$\int_\Omega a(x,u,\nabla u)\nabla\varphi dx = \int_\Omega fh(u)\varphi\, dx, \text{ for every } \varphi \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega). \tag{3.8}$$

Our first result is the following:

Theorem 3.2. *Let $f \in L^m(\Omega)$ with $m > N/p$, assume that* (3.3), (3.5), (3.6) *and* (3.7) *hold true then, there exists a function $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ solution of* (3.1).

Theorem 3.3. *Assume that* (3.3), (3.5),(3.6),(3.7) *and $0 < \gamma < 1$ hold true. Let $f \in L^m(\Omega)$ with*

$$m_1 = \left(\frac{p^*}{\theta(p-1)+1-\gamma}\right)' = \frac{Np}{Np-(N-p)[\theta(p-1)+1-\gamma]} \leq m < N/p, \tag{3.9}$$

then, there exists at least one solution u in $W_0^{1,p}(\Omega) \cap L^r(\Omega)$ of (3.1)

$$r = \frac{Nm[(p-1)(1-\theta)+\gamma]}{N-pm}. \tag{3.10}$$

Remark 3.4. If $0 < \gamma < 1$, we explicitly note that $m = m_1 \iff r = p^*$, and If $\theta = 1, \gamma \to 0$, then $m_1 \to N/p$, in this case. Observe that, for every $0 \leq \theta \leq 1$, we have $m_1 \geq (p^*)' \Rightarrow f \in W^{-1,p'}(\Omega)$, it is classical to expect a $W_0^{1,p}(\Omega)$ solution.

Theorem 3.5. *Assume that* (3.3), (3.5),(3.6),(3.7) *and* $0 < \gamma < 1$ *hold true. Let* $f \in L^m(\Omega)$ *with*

$$1 < m < \frac{Np}{Np - (N-p)[\theta(p-1) + 1 - \gamma]}, \tag{3.11}$$

then, there exists at least one solution u *in* $W_0^{1,\sigma}(\Omega)$, *that is*

$$\sigma = \frac{Nm[(p-1)(1-\theta) + \gamma]}{N - m((p-1)\theta + 1 - \gamma)}. \tag{3.12}$$

Remark 3.6. If $\gamma \to 0^+$, the result of Theorem 3.3, Theorem 3.5 coincides with regularity results for elliptic equation with coercivity (see([5],Theorem 1.3 and Theorem 1.7)).

Remark 3.7. If $0 < \gamma < 1$, under some condition on f, the summability of the solution to (3.1) is better than or equal to that of solution in ([5], Theorem 1.7 and Theorem 1.9) (see [5]).

The last result deals with the case where the source f belongs to $L^1(\Omega)$ and $0 < \gamma < 1$.

Theorem 3.8. *Let us consider* $0 < \gamma < 1$ *and* $f \in L^1(\Omega)$ *then the problem* (3.1) *admits a solution* u *belonging to* $W_0^{1,q}(\Omega)$, *with*

$$q = \frac{N[(p-1)(1-\theta) + \gamma]}{N - ((p-1)\theta + 1 - \gamma)}. \tag{3.13}$$

The chapter is organized as follows: in the next section we will give a priori estimates for solutions of approximate equation, while the third section will be devoted to the proof of the results.

2 A priori estimates

Here we provide our a priori estimates for the approximate solutions to problem (3.1).
Approximating problems. For $n \in I\!N$, let $f_n = T_n(f)$ and we consider the following problem:

$$\begin{cases} -\text{div}\,(a(x, T_n(u_n), \nabla u_n)) = f_n h_n(u_n) & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

Moreover, we set

$$h_n(s) = \begin{cases} T_n(h(s)) \text{ for } & s > 0, \\ \min(n, h(0)) & \text{otherwise,} \end{cases} \tag{3.15}$$

where $T_n(h(u_n)) \leq \frac{C}{(|u_n| + \frac{1}{n})^\gamma}$. The right hande side of (3.14) is nonnegative, that u_n is nonnegative. Observe that we have "truncated" the degenerate coercivity of the operator term and the singularity of the right hand side. The weak formulation of (3.14) is

$$\int_\Omega a(x, T_n(u_n), \nabla u_n).\nabla\varphi dx = \int_\Omega f_n h_n(u_n)\varphi dx \quad \forall\varphi \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega). \tag{3.16}$$

Proposition 3.9. *For each* $n \in \mathbb{N}$ *there exists* $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *weak solution of problem* (3.14).

Proof. The proof is based on standard Schauder's fixed point argument. Let $n \in I\!N$ be fixed and $v \in L^p(\Omega)$ be fixed. we know that the following nonsingular problem

$$\begin{cases} -\text{div}\,(a(x, T_n(w), \nabla w)) = f_n h_n(v) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ follows from the classical results (see [68] and [5]). In particular, it is well defined a map

$$G : L^p(\Omega) \to L^p(\Omega),$$

where $G(v) = w$. Again, thanks to regularity of the datum $h_n(v)f_n$, we can take w as test function and obtain

$$\int_\Omega a(x, T_n(w), \nabla w)\nabla w = \int_\Omega f_n h_n(v)w, \tag{3.17}$$

then, it follows from (3.3)

$$\alpha \int_\Omega \frac{|\nabla w|^p}{(1+n)^{\theta(p-1)}} dx \le \int_\Omega \frac{|\nabla w|^p}{(1+|T_n(w)|)^{\theta(p-1)}} dx \le n^2 \int_\Omega |w| dx$$

using the Poincaré inequality we have

$$\int_\Omega \frac{|\nabla w|^p}{(1+n)^{\theta(p-1)}} dx \le c_1 n^2 \int_\Omega |\nabla w| dx,$$

by Hölder's inequality on the right hand side, we obtain

$$\int_\Omega |\nabla w|^p dx \le c_1(1+n)^{\theta(p-1)} n^2 \int_\Omega |\nabla w| dx \le c(n)|\Omega|^{\frac{1}{p'}} \left(\int_\Omega |\nabla w|^p dx\right)^{\frac{1}{p}}$$

we deduce

$$\int_\Omega |\nabla w|^p dx \le c(n)^{p'}|\Omega|,$$

Using the Poincaré inequality on the left hand side

$$||w||_{L^p(\Omega)} \le c(n, |\Omega|)(= c^{\frac{p'}{p}}(n)|\Omega|^{\frac{1}{p}}),$$

where $c(n, |\Omega|)$ is a positive constant independent form v and w, thus, we have that the ball B of $L^p(\Omega)$ of radius $c(n, |\Omega|)$ is invariant for the map G.

Now we prove that the map G is continuous in B. Let us choose a sequence v_k that converges strongly to v in $L^p(\Omega)$, the by dominated convergence theorem

$$f_n h_n(v_k) \to f_n h_n(v) \text{ in } L^p(\Omega),$$

then we need to prove that $G(v_k)$ converge to $G(v)$ in $L^p(\Omega)$. By compactness we already know that the sequence $w_k = G(v_k)$ converge to some function w in $L^p(\Omega)$. We only need to prove that $w = G(v)$. Firstly, we have the datum $f_n h_n(v_k)$ are bounded, we have that $w_k \in L^\infty(\Omega)$ and there exists a positive constant d, independent of v_k and w_k (but possibly depending on n), such that $||w_k||_{L^\infty(\Omega)} \le d$. We know the sequence w_k is bounded in $W_0^{1,p}(\Omega)$. Hence, by uniqueness, one deduces that $G(v_k)$ converge to $G(v)$ in $L^p(\Omega)$. Lastly we need to check that the set $G(B)$ is relatively compact. Let v_k be a bounded sequence in B and let $w_k = G(v_k)$. we proved before that

$$\int_\Omega |\nabla w|^p dx = \int_\Omega |\nabla G(v)|^p dx \leq c(n, |\Omega|),$$

for any $v \in L^p(\Omega)$, then for $v = v_k$ we obtain

$$\int_\Omega |\nabla w_k|^p dx = \int_\Omega |\nabla G(v_k)|^p dx \leq c(n, |\Omega|),$$

so that $G(v)$ is relatively compact in $L^p(\Omega)$ by Rellich-kondrachov Theorem. We can then apply Schauder fixed point theorem there exist a fixed point of the map G, say u_n will exist in B such that $G(u_n) = u_n$ and we will have that $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is solution of problem (3.14). Moreover, taking $\varphi = -u_n^-$ in (3.16) and recalling that $h_n(u_n) f_n$ is nonnegative, we obtain

$$\begin{aligned}
&\frac{\alpha}{(1+n)^{\theta(p-1)}} \left\|u_n^-\right\|^p_{W_0^{1,p}(\Omega)} \leq \alpha \int_\Omega \frac{|\nabla u_n^-|^p}{(1+T_n(u_n^-))^{\theta(p-1)}} \\
&\overset{(3.5),(3.3)}{\leq} \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla\left(-u_n^-\right) = -\int_\Omega h_n(u_n) f_n u_n^- \leq 0.
\end{aligned}$$

It follows that $\|u_n^-\|_{W_0^{1,p}(\Omega)} = 0$ which means that u_n is nonnegative. □

Theorem 3.10. *Let f be in $L^m(\Omega)$ with $m > N/p$, $0 \leq \theta \leq 1$ and let u_n be solution of* (3.14)*. Then the norm of u_n in $L^\infty(\Omega)$. Indeed, we have*

$$\|u_n\|_{L^\infty(\Omega)} < \mathcal{B}^{-1}\left[\frac{C^{\frac{1}{p-1}}|\Omega|^{\frac{p'}{N}-\frac{p'}{pm}}}{(NC_N^{\frac{1}{N}})^{p'}} \frac{Nm(p-1)}{pm-N} \|f\|^{\frac{p'}{p}}_{L^m(\Omega)}\right], \tag{3.18}$$

where $\mathcal{B}^{-1}$ denotes the inverse function of $\mathcal{B}$. Furthermore, if $0 < \gamma < 1$, the norm of u_n in $W_0^{1,p}(\Omega)$ is bounded by a constant continuously depending on the norm of f in $(L^m(\Omega))^N$.

Proof. For $\varepsilon > 0$ and $t > 1$, we use in the formulation (3.16). Let the test function $v = T_\varepsilon(G_t(u_n))$ where $\{t < |u_n| < t+\varepsilon\}$ denotes the test set $\{x \in \Omega : t < |u_n(x)| < t+\varepsilon\}$. Assumption (3.3) yields

$$\begin{aligned}
&\alpha \int_{\{t<|u_n|<t+\varepsilon\}} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}} dx \leq \varepsilon \int_{\{t<|u_n(x)|\}} f_n h_n(u_n) dx \\
&\leq \varepsilon \sup_{u_n \in [t,+\infty]} (h_n(u_n)) \int_{t<|u_n(x)|} f_n dx \\
&\leq \varepsilon \sup_{u_n \in [t,+\infty]} \left(\frac{C}{(|u_n(x)| + \frac{1}{n})^\gamma}\right) \int_{\{t<|u_n(x)|\}} f dx,
\end{aligned}$$

in the set $\{t < |u_n(x)|\}$, we have that $|u_n(x)| + \frac{1}{n} > t > 1$ and dividing both sides by ε we get

$$\frac{\alpha}{\varepsilon} \int_{\{t<|u_n(x)|<t+\varepsilon\}} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}} dx \leq C \int_{\{t<|u_n(x)|\}} f dx.$$

The above inequality and Hölder's inequality

$$\left(\frac{\alpha}{\varepsilon}\int_{\{t<|u_n(x)|<t+\varepsilon\}}\frac{|\nabla u_n|}{(1+|u_n|)^{\theta(p-1)}}dx\right)^p$$
$$\leq C\left(\frac{\alpha}{\varepsilon}\int_{\{t<|u_n(x)|<t+\varepsilon\}}\frac{1}{(1+|u_n|)^{\theta(p-1)}}dx\right)^{p-1}\int_{\{t<|u_n(x)|\}}fdx. \tag{3.19}$$

We can pass to the limit as ε goes to 0^+ in (3.19) to get, after simplification

$$\frac{\alpha}{(1+t)^{\theta(p-1)}}\left(\frac{d}{dt}\int_{\{|u_n|\leq t\}}|\nabla u_n|dx\right)^p \leq C(-\mu'_{u_n(t)})^{p-1}\left(\int_0^{\mu_{u_n}(t)}|f_n^*(\tau)|d\tau\right). \tag{3.20}$$

On the other hand, from Fleming-Rishel Coera Formula and isoperimetric inequality (see [83] and [84]) we have for almost every $t>0$

$$NC_N^{\frac{1}{N}}(\mu_{u_n}(t))^{\frac{N-1}{N}} \leq \frac{d}{dt}\int_{\{|u_n|\leq t\}}|\nabla u_n|dx, \tag{3.21}$$

where C_N is the measure of the unit ball in $I\!R^N$. Then (3.20) and (3.21) give

$$\frac{\alpha^{\frac{1}{p-1}}}{(1+t)^{\theta}} \leq \frac{C^{\frac{1}{p-1}}}{\left(NC_N^{\frac{1}{N}}\right)^{p'}}\left(\int_0^{\mu_{u_n}(t)}f^*(\tau)d\tau\right)^{\frac{p'}{p}}\frac{-\mu'_{u_n}(t)}{(\mu_{u_n}(t))^{(1-\frac{1}{N})p'}}. \tag{3.22}$$

Integrating both sides of (3.22) between 0 and σ, we obtain

$$\mathcal{B}(\sigma) \leq \frac{C^{\frac{1}{p-1}}}{\left(NC_N^{\frac{1}{N}}\right)^{p'}}\int_0^{\sigma}\left(\int_0^{\mu_{u_n}(t)}f^*(\tau)d\tau\right)^{\frac{p'}{p}}\frac{-\mu'_{u_n}(t)}{(\mu_{u_n}(t))^{(1-\frac{1}{N})p'}}dt. \tag{3.23}$$

Making a change of variables in the last integral, we get

$$\mathcal{B}(\sigma) \leq \frac{C^{\frac{1}{p-1}}}{\left(NC_N^{\frac{1}{N}}\right)^{p'}}\int_{u_n(\sigma)}^{|\Omega|}\left(\int_0^{\rho}f^*(\tau)d\tau\right)^{\frac{p'}{p}}\frac{d\rho}{\rho^{(1-\frac{1}{N})p'}}d\rho. \tag{3.24}$$

Let us denote by u_n^* the decreasing rearrangement of u_n. By (2.1), one has

$$\mathcal{B}(u_n^*(v)) \leq \frac{C^{\frac{1}{p-1}}}{\left(NC_N^{\frac{1}{N}}\right)^{p'}}\int_v^{|\Omega|}\left(\int_0^{\rho}f^*(\tau)d\tau\right)^{\frac{p'}{p}}\frac{d\rho}{\rho^{p'(1-\frac{1}{N})}}.$$

Taking into account (2.2), by evaluating $\mathcal{B}(u_n^*(0))$ we get(3.18). Let us denote in what follows by c_∞ the constant on the right in (3.18), that is

$$||u_n||_\infty \leq c_\infty, \tag{3.25}$$

it is easy to get an estimation in $W_0^{1,p}(\Omega)$. Taking u_n as test function in formulation (3.8) then using (3.3),(3.25) and Hölder inequality, we get

$$b^p(c_\infty)\int_\Omega |\nabla u_n|^p dx \le \int_\Omega f u_n^{1-\gamma} dx \le ||u_n^{1-\gamma}||_{L^\infty(\Omega)} \int_\Omega f dx \le c_\infty \int_\Omega f dx$$
$$\le c_\infty |\Omega|^{1-\frac{1}{m}} ||f||_{L^m(\Omega)},$$

then

$$\int_\Omega |\nabla u_n|^p dx \le \frac{c_\infty |\Omega|^{1-\frac{1}{m}}}{b^p(c_\infty)} ||f||_{L^m(\Omega)}. \tag{3.26}$$

□

Theorem 3.11. *On suppose that* $0 < \gamma < 1$ *and the datum* $f \in L^m(\Omega)$, *with*

$$\frac{Np}{Np-(N-p)[\theta(p-1)+1-\gamma]} \le m < N/p,$$

let

$$r = \frac{Nm[(p-1)(1-\theta)+\gamma]}{N-pm}.$$

Then, the solution u_n *to* (3.16) *are uniformly bounded in* $L^r(\Omega)\cap W_0^{1,p}(\Omega)$.

Proof. Let us choose $(1+u_n)^\nu - 1$ as a test function by the hypotheses on a, one has

$$\nu\left(\frac{p}{(p-1)(1-\theta)+\nu}\right)^p \int_\Omega \left|\nabla[(1+u_n)^{\frac{-\theta(p-1)+\nu+p-1}{p}}]\right|^p dx$$
$$= \nu\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{\theta(p-1)-\nu+1}} dx \le \nu \int_\Omega \frac{|\nabla u_n|^p}{(1+T_n(u_n))^{\theta(p-1)}}(1+u_n)^{\nu-1} dx$$
$$\le C\int_\Omega \frac{T_n(f)}{(u_n+\frac{1}{n})^\gamma}\big((u_n+1)^\nu - 1\big)\,dx \le C + C\int_\Omega \frac{|f|}{(u_n+1)^{-\nu+\gamma}} dx. \tag{3.27}$$

By Sobolev's inequality on the left hand side and Hölder's inequality on the right one we have

$$\left(\int_\Omega ((1+u_n)^{\frac{-\theta(p-1)+\nu+p-1}{p}})^{p^*} dx\right)^{\frac{p}{p^*}}$$
$$\le C||f||_{L^m(\Omega)}\left(\int_\Omega (u_n+1)^{m'(\nu-\gamma)} dx\right)^{\frac{1}{m'}}. \tag{3.28}$$

Let $\nu > \gamma$ be such that

$$\frac{-\theta(p-1)+\nu+p-1}{N-p}N = (\frac{\nu-\gamma}{m-1})m$$

and $\frac{p}{p^*} > \frac{1}{m'}$, that is

$$\nu = \frac{N(m-1)(1-\theta)(p-1)+\gamma m(N-p)}{N-pm}$$

and $m < \frac{N}{p}$, we observe that

$$\frac{p^*}{p}(-\theta(p-1)+\nu+p-1) = \frac{mN}{N-pm}[(p-1)(1-\theta)+\gamma] = r > 1$$

This implies that u_n is bounded in $L^r(\Omega)$.
By (3.27), (3.28) and $\nu \geq 1+\theta(p-1)$ $\left(\Leftrightarrow \frac{Np}{Np-(N-p)[\theta(p-1)+1-\gamma]} \leq m\right)$, we get

$$\int_\Omega |\nabla u_n|^p dx \leq \int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{\theta(p-1)-\nu+1}} dx \leq C||f||_{L^m(\Omega)} \int_\Omega |u_n|^r dx \leq Cst.$$

□

Theorem 3.12. *On suppose that* $0 < \gamma < 1$ *and* (3.11) *holds true. Let* σ *be as in* (3.12) *then the solution* u_n *to* (3.16) *are uniformly bounded in* $W_0^{1,\sigma}(\Omega)$.

Proof. Let us choose $(1+u_n)^\lambda - 1$ with $\lambda = \frac{N(m-1)(1-\theta)(p-1)+\gamma m(N-p)}{N-pm}$ as a test function in (3.16) with the summer arguments as before we have

$$\left(\int_\Omega [(1+u_n)^{\frac{-\theta(p-1)+\lambda+p-1}{p}}]^{p^*} dx\right)^{\frac{p}{p^*}} \leq C \int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{\theta(p-1)-\lambda+1}} dx$$
$$\leq C||f||_{L^m(\Omega)} \left(\int_\Omega (1+u_n)^{m'(\lambda-\gamma)} dx\right)^{\frac{1}{m'}}.$$

As above, we infer that u_n is bounded in $L^{\frac{N((p-1)(1-\theta)+\lambda)}{N-p}}(\Omega)$. This choice of λ gives that $\lambda > \gamma$ thanks to the fact that $\theta(p-1)-\lambda+1 > 0$ and $0 < \sigma = \frac{Nm[(p-1)(1-\theta)+\gamma]}{N-m(\theta(p-1)+1-\gamma)}$, by the assumptions on m, writing

$$\int_\Omega |\nabla u_n|^\sigma dx = \int_\Omega \frac{|\nabla u_n|^\sigma}{(1+u_n)^{\frac{\theta(p-1)-\lambda+1}{p}}} (1+u_n)^{\frac{\theta(p-1)-\lambda+1}{p}} dx.$$

Now let $1 < \sigma < p$ and using Hölder's inequality with exponent $\frac{p}{\sigma}$, we obtain

$$\int_\Omega |\nabla u_n|^\sigma dx \leq \left[\int_\Omega \frac{|\nabla u_n|^\sigma}{(1+u_n)^{\theta(p-1)-\lambda+1}} dx\right]^{\frac{\sigma}{p}} \left[\int_\Omega (1+u_n)^{\sigma\frac{\theta(p-1)-\lambda+1}{p-\sigma}} dx\right]^{\frac{p-\sigma}{p}}.$$

The above estimates imply that the sequences u_n is bounded in $W_0^{1,\sigma}(\Omega)$ if

$$\sigma\frac{\theta(p-1)-\lambda+1}{p-1} = \frac{N[(p-1)(1-\theta)+\lambda]}{N-p},$$

that is

$$\sigma = \frac{Nm[(p-1)(1-\theta)+\gamma]}{N-m[\theta(p-1)+1-\gamma]}.$$

By virtue of $\lambda < 1+\theta(p-1)$ or $\sigma < p$, therefore we have $m < Np/[Np-(N-p)(\theta(p-1)+1-\gamma)]$. □

Theorem 3.13. *Assume that* $0 < \gamma < 1$ *and* $f \in L^1(\Omega)$, *let* q *be as in* (3.13), *then the solution* u_n *to* (3.14) *are uniformly bounded in* $W_0^{1,q}(\Omega)$.

Proof. Choosing $\varphi = (1+u_n)^\gamma - 1$ in (3.16). Using assumption (3.3), we can write

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta(p-1)-\gamma}} dx \leq c \int_\Omega |f| dx. \tag{3.29}$$

Let $q < p$, writing

$$\begin{aligned} \int_\Omega |\nabla u_n|^q dx &= \int_\Omega \frac{|\nabla u_n|^q}{(1+u_n)^{\frac{q(1+\theta(p-1)-\gamma)}{p}}} (1+u_n)^{\frac{q(1+\theta(p-1)-\gamma)}{p}} dx. \\ &\leq \left[\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{(1+\theta(p-1)-\gamma)}} dx \right]^{\frac{q}{p}} \left[\int_\Omega (1+u_n)^{\frac{r(1+\theta(p-1)-\gamma)}{p-q}} dx \right]^{1-\frac{q}{p}} \\ &\leq c \left[\int_\Omega (1+u_n)^{\frac{q(1+\theta(p-1)-\gamma)}{p-q}} dx \right]^{1-\frac{q}{p}}. \end{aligned} \tag{3.30}$$

By Sobolev inequality, we obtain

$$\left(\int_\Omega u_n^{q^*} dx \right)^{\frac{q}{q^*}} \leq \int_\Omega |\nabla u_n|^q dx \leq c \left[\int_\Omega (1+u_n)^{\frac{q(1+\theta(p-1)-\gamma)}{p-q}} dx \right]^{1-\frac{q}{p}}. \tag{3.31}$$

Now choose q in order to have

$$q^* = \frac{q(1+\theta(p-1)-\gamma)}{p-q}, \tag{3.32}$$

we point out that $\frac{q}{q^*} > \frac{p-q}{p}$, thus from (3.31), (3.30) and (3.32), we deduce that the sequence u_n is bounded in $W_0^{1,q}(\Omega)$. □

3 Proof of the results

In this section we are going to use the results of section 2 in order to prove Theorem 3.2, Theorem 3.3 and Theorem 3.5.

Proof of Theorem 3.2.
Step 1: We prove that

$$\lim_{n\to+\infty} \int_\Omega h_n(u_n) f_n \varphi dx = \int_\Omega h(u) f \varphi dx, \tag{3.33}$$

for all non negative $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. First we observe that from the Young inequality and the hypotheses in (3.6), one gets

$$\int_\Omega h_n(u_n)f_n\varphi = \int_\Omega a(x,T_n(u_n),\nabla u_n)\nabla\varphi dx \leq \int_\Omega a_0(x)\nabla\varphi dx$$
$$+\int_\Omega |u_n|^{p-1}\nabla\varphi dx + \int_\Omega |\nabla u_n|^{p-1}\nabla\varphi dx \leq \int_\Omega a_0(x)\nabla\varphi dx + \frac{p-1}{p}\int_\Omega |u_n|^p$$
$$+\frac{1}{p}\int_\Omega |\nabla\varphi|^p dx + \frac{p-1}{p}\int_\Omega |\nabla u_n|^p dx + \frac{1}{p}\int_\Omega |\nabla\varphi|^p dx \leq \frac{1}{p'}\int_\Omega a_0(x)^{p'}dx +$$
$$\frac{1}{p}\int_\Omega |\nabla\varphi|^p dx + \frac{p-1}{p}\int_\Omega |u_n|^p + \frac{1}{p}\int_\Omega |\nabla\varphi|^p dx + \frac{p-1}{p}\int_\Omega |\nabla u_n|^p dx$$
$$+\frac{1}{p}\int_\Omega |\nabla\varphi|^p dx \leq c + c\left[\int_\Omega |\nabla\varphi|^p dx + \int_\Omega |u_n|^p dx + \int_\Omega |\nabla u_n|^p dx\right],$$

then

$$\int_\Omega h_n(u_n)f_n\varphi \leq c + c[||\varphi||_{W_0^{1,p}(\Omega)} + ||u_n||_{W_0^{1,p}(\Omega)}]. \tag{3.34}$$

From now we consider a non negative $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. An application of the Fatou Lemma in (3.34) with respect to n gives

$$\int_\Omega h(u)f\varphi \leq c, \tag{3.35}$$

where c does not depend on n. Hence $fh(u)\varphi \in L^1(\Omega)$ for any non negative $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. As a consequence, if $h(s)$ is unbounded as s tends to 0, we deduce that

$$\{u=0\}\subset\{f=0\}, \tag{3.36}$$

up to a set of zero Lebesgue measure.
From now on, we assume that $h(s)$ is unbounded as s tends to 0. Let φ be a non negative function in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, choosing it as test function in the weak formulation of (3.14), we have

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\nabla\varphi dx = \int_\Omega f_n h_n(u_n)\varphi dx. \tag{3.37}$$

We want to pass to the limit in the right hand side of (3.37) as n tends to infinity. we fix $\delta>0$, and we decompose the right hand side in the following way

$$\int_\Omega h_n(u_n)f_n\varphi dx = \int_{\{u_n\leq\delta\}} h_n(u_n)f_n\varphi dx + \int_{\{u_n>\delta\}} h_n(u_n)f_n\varphi dx. \tag{3.38}$$

Therefore we have, thanks to Lemma 1.1 contained in [71], that $V_\delta(u_n)$ belongs to $W_0^{1,p}(\Omega)$, where V_δ is defined by

$$V_\delta(s) = \begin{cases} 1 & \text{if } s\leq\delta \\ \frac{2\delta-s}{\delta} & \text{if } \delta<s<2\delta, \\ 0 & \text{if } s\geq 2\delta. \end{cases} \tag{3.39}$$

So we take it is test function in the weak formulation of (3.14), using (3.39), (3.3) and (3.6) we obtain

$$\int_{\{u_n \le \delta\}} h_n(u_n) f_n \varphi dx \le \int_\Omega h_n(u_n) f_n V_\delta(u_n) \varphi dx$$
$$= \int_\Omega a(x, T_n(u_n), \nabla u_n) \nabla\varphi V_\delta(u_n) dx - \frac{1}{\delta} \int_{\{\delta < u_n < 2\delta\}} a(x, T_n(u_n), \nabla u_n) \varphi \nabla u_n dx,$$

by using (3.3) and (3.6), we have

$$\int_{\{u_n \le \delta\}} h_n(u_n) f_n \varphi dx \le \beta \int_\Omega \left[a_0(x) + |u_n|^{p-1} + |\nabla u_n|^{p-1}\right] \nabla\varphi V_\delta(u_n) dx$$
$$- \frac{1}{\delta (n+1)^{\theta(p-1)}} \int_{\{\delta < u_n < 2\delta\}} |\nabla u_n|^p \varphi dx$$
$$\le \beta \int_\Omega \left[a_0(x) + |u_n|^{p-1} + |\nabla u_n|^{p-1}\right] \nabla\varphi V_\delta(u_n) dx.$$

Using that V_δ is bounded we deduce that $|\nabla u_n|^{p-1} \nabla\varphi V_\delta(u_n)$ converges to $|\nabla u|^{p-1} \nabla\varphi V_\delta(u)$ weakly in $L^{p'}(\Omega)^N$ as n tends to infinity. This implies that

$$\lim_{n \to +\infty} \int_{\{u_n \le \delta\}} h_n(u_n) f_n \varphi dx \le \beta \int_\Omega \left[a_0(x) + |u|^{p-1} + |\nabla u|^{p-1}\right] \nabla\varphi V_\delta(u) dx. \tag{3.40}$$

Since $V_\delta(u)$ converges to $\chi_{\{u=0\}}$ a.e in Ω as δ tends to 0 and since $u \in W_0^{1,p}(\Omega)$, then $\left[a_0(x) + |u|^{p-1} + |\nabla u|^{p-1}\right] \nabla\varphi V_\delta(u)$ converges to 0 a.e. in Ω as δ tends to 0. Applying the Lebesgue Theorem on the right hand side of (3.40) we obtain that

$$\lim_{\delta \to 0^+} \lim_{n \to +\infty} \int_{\{u_n \le \delta\}} h_n(u_n) f_n \varphi dx = 0. \tag{3.41}$$

As regards the second term in the right hand side of (3.38) we have

$$0 \le h_n(u_n) f_n \varphi \chi_{\{u_n > \delta\}} \le \sup_{s \in]\delta, \infty)} [h(s)] f \varphi \in L^1(\Omega), \tag{3.42}$$

we remark that we need to choose $\delta \neq \{\eta ; |u = \eta| > 0\}$, which is at most a countable set. As a consequence $\chi_{\{u_n > \delta\}}$ converges to $\chi_{\{u > \delta\}}$ a.e in Ω, we deduce first that $h_n(u_n) f_n \chi_{\{u_n > \delta\}} \varphi$ converges to $h(u) f \chi_{\{u > \delta\}} \varphi$ strongly in $L^1(\Omega)$ as n tends to infinity, then, since $h(u) f \chi_{\{u > \delta\}} \varphi$ belongs to $L^1(\Omega)$, that $f h(u) \chi_{\{u > \delta\}} \varphi$ converges to $f h(u) \chi_{\{u > 0\}} \varphi$ strongly in $L^1(\Omega)$ as δ tend to 0.
and then, once again by the Lebesgue Theorem, one gets

$$\lim_{\delta \to 0^+} \lim_{n \to +\infty} \int_{\{u_n > \delta\}} h_n(u_n) f_n \varphi dx = \int_{\{u > 0\}} h(u) f \varphi dx. \tag{3.43}$$

By (3.43) and (3.41), we deduce that

$$\lim_{n \to +\infty} \int_\Omega h_n(u_n) f_n \varphi dx = \int_\Omega h(u) f \varphi dx \quad \forall\ 0 \le \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{3.44}$$

Moreover, decomposing any $\varphi = \varphi^+ - \varphi^-$, with $\varphi^+ = \max(\varphi, 0)$, $\varphi^- = -\min(\varphi, 0)$
the positive and the negative part of a function φ, and using that (3.44) is linear in φ, we deduce that (3.44) holds for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We treated $h(s)$ unbounded as s tends to 0, as regards bounded function h the proof is easier and only difference deals with the passage to the limit in the left hand side of (3.44). We can avoid introducing δ and we can substitute (3.42) with

$$0 \leq f_n h_n(u_n)\varphi \leq f||h||_{L^\infty(\Omega)}\varphi.$$

Using the same argument above we have that $f_n h_n(u_n)\varphi$ converges to $fh(u)\varphi$ strongly in $L^1(\Omega)$ as n tends to infinity. This concludes (6.4).

Step 2: Thanks to (3.26), the sequence u_n is bounded in $W_0^{1,p}(\Omega)$. Therefore, there exist a subsequence of u_n still denoted by u_n, and a measurable function u such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega) \text{ and a.e in } \quad \Omega. \tag{3.45}$$

We shall prove that, up to a subsequence

$$u_n \longrightarrow u \quad \text{strongly in} \quad W_0^{1,p}(\Omega). \tag{3.46}$$

We take $u_n - u$ test function in the weak formulation of (3.16), we obtain for $n > c_\infty$

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla(u_n - u)dx = \int_\Omega f_n h_n(u_n)(u_n - u)dx, \tag{3.47}$$

the right hand side tends to zero when n tends to infinity. On the other hand we write

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)\nabla(u_n - u) \\ &= \int_\Omega a(x, u_n, \nabla u_n)\nabla(u_n - u)dx - \int_\Omega a(x, u_n, \nabla u)\nabla(u_n - u), \end{aligned} \tag{3.48}$$

by (3.45) one has

$$\lim_{n\to+\infty} \int_\Omega a(x, u_n, \nabla u_n)\nabla(u_n - u)dx = 0,$$

As regards the second term on the right in (3.48) and see step 1 in the proof of Theorem 3.2, using (3.6) and Vitali's Theorem we obtain that

$$a(x, u_n, \nabla u) \longrightarrow a(x, u, \nabla u) \ \text{ strongly in } \ (L^{p'}(\Omega))^N.$$

Therefore, we obtain

$$\lim_{n\to+\infty} \int_\Omega (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u))\nabla(u_n - u)dx = 0, \tag{3.49}$$

thanks to (3.7), the integrand function in the left hand side in (3.49) is non negative, therefore

$$(a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u))\nabla(u_n - u) \longrightarrow 0 \ \text{ strongly in } L^1(\Omega).$$

Thus, up a subsequence still indexed by u_n, one has

$$a((x,u_n,\nabla u_n) - a(x,u_n,\nabla u))\nabla(u_n - u) \longrightarrow 0,$$

for almost every x in Ω, there exists a subset Z of Ω zero measure, such that for all x in $\Omega\backslash Z$ we have

$$D_n(x) = (a(x,u_n(x),\nabla u_n(x)) - a(x,u_n(x),\nabla u(x)\nabla(u_n - u)(x))) \longrightarrow 0 \tag{3.50}$$

$|u(x)| < \infty$,$|\nabla u(x)| < \infty$,$|a_0(x)| < \infty$ and $u_n(x) \longrightarrow u(x)$, then by the growth condition (3.6),(3.3) and $||u_n||_\infty \leq c$

$$D_n(x) \geq \frac{1}{(1+c)^{\theta(p-1)}}|\nabla u_n(x)|^{p-1} - c(x)\big(1 + |\nabla u_n(x)| + |\nabla u_n(x)|^{p-1}\big),$$

where $c(x)$ is a constant depends on x but does not depend on n, which schows thanks to (3.50), that the sequence $|\nabla u_n(x)|$ is unformly bounded in $\mathbb{R}^N$, with respect to n, we argue simililary as in Lemma 5 in [21], to obtain (3.46).

We can now pass to the limit going back to the equation (3.16), to do this, let $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. For every $n > c_\infty$ one has

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\varphi dx = \int_\Omega h_n(u_n)f_n\varphi dx, \tag{3.51}$$

by (3.46), we have $\nabla u_n \longrightarrow \nabla u$ strongly in $(L^p(\Omega))^N$ and a.e in Ω, so that Vitali's Theorem implies that

$$a(x,u_n,\nabla u_n) \longrightarrow a(x,u,\nabla u) \text{ strongly in } (L^{p'}(\Omega))^N.$$

Then, passing to the limit in (3.51) and using the result in the Step 1, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla\varphi dx = \int_\Omega fh(u)\varphi dx,$$

for all φ in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, moreover, from (3.25), we have

$$u \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).$$

□

Proof of Theorems 3.3 and 3.5. Because the proofs of Theorems 3.5 are similar to that of Theorem 3.3 , we restrict to the proof of Theorem 3.3 □

Proof of Theorems 3.3. As consequence of Theorem 3.11 there exist a subsequence, still indexed by n, and a measurable function u in $W_0^{1,p}(\Omega)\cap L^r(\Omega)$ such that u_n converges weakly to u. Moreover, by Rellich Theorem we have

$$u_n \longrightarrow u \quad \text{a.e in } \Omega. \tag{3.52}$$

Fix $k > 0$, we will prove that

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega). \tag{3.53}$$

By Theorem 3.11, the sequence $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$. Therefore, by (3.52) we get

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega). \tag{3.54}$$

Using $T_k(u_n) - T_k(u)$, which belongs to $W_0^{1,p}(\Omega)$, as test function in formulation (3.51), we get

$$\int_\Omega a(x, T_n(u_n), \nabla u_n)\nabla(T_k(u_n) - T_k(u))dx = \int_\Omega h_n(u_n)f_n(T_k(u_n) - T_k(u))dx.$$

Thanks to (3.54) and (3.44), we have

$$\lim_{n\to+\infty} \int_\Omega a(x, T_n(u_n), \nabla u_n)\nabla(T_k(u_n) - T_k(u))dx = 0. \tag{3.55}$$

By the growth condition (3.6) and Theorem 3.11, the sequence $a(x, T_n(u_n), \nabla u_n)$ is bounded in $(L^{p'}(\Omega))^N$. Then, it converges weakly to some l in $(L^{p'}(\Omega))^N$ and we obtain

$$\lim_{n\to+\infty} \int_{|u_n|\geq k} a(x, T_n(u_n), \nabla u_n)\nabla T_k(u)dx = \int_{|u|\geq k} l\nabla T_k(u)dx = 0. \tag{3.56}$$

The continuity of the function a, (3.52) and Vitali's theorem allow us to have

$$a(x, T_n(u_n), \nabla T_k(u)) \longrightarrow a(x, u, \nabla T_k(u)) \quad \text{strongly in} \quad (L^{p'}(\Omega))^N.$$

Therefore, by Theorem 3.11 and (3.54) we get

$$\lim_{n\to+\infty} \int_\Omega a(x, T_n(u_n), \nabla T_k(u_n))\nabla(T_k(u_n) - T_k(u))dx = 0. \tag{3.57}$$

On the other hand, we write for $n > k$

$$\begin{aligned}
&\int_\Omega (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)))\, \nabla(T_k(u_n) - T_k(u))dx \\
&= \int_\Omega (a(x, T_k(u_n), \nabla T_k(u_n))\nabla(T_k(u_n) - T_k(u))dx \\
&- \int_\Omega (a(x, T_k(u_n), \nabla T_k(u))\nabla(T_k(u_n) - T_k(u))dx \\
&= \int_{|u_n|<k} (a(x, T_n(u_n), \nabla u_n)\nabla(T_k(u_n) - T_k(u))dx \\
&- \int_\Omega (a(x, T_k(u_n), \nabla T_k(u))\nabla(T_k(u_n) - T_k(u))dx \\
&= \int_\Omega (a(x, T_n(u_n), \nabla u_n)\nabla(T_k(u_n) - T_k(u))dx \\
&- \int_{|u_n|\geq k} (a(x, T_n(u_n), \nabla u_n)\nabla(T_k(u_n) - T_k(u))dx \\
&- \int_\Omega (a(x, T_k(u_n), \nabla T_k(u))\nabla(T_k(u_n) - T_k(u))dx.
\end{aligned}$$

Observing that $\nabla T_k(u_n) = 0$ on the set $|u_n| \geq k$, we get

$$\begin{aligned}
&\int_\Omega (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)))\nabla(T_k(u_n) - T_k(u))dx \\
&= \int_\Omega (a(x, T_n(u_n), \nabla u_n)\nabla(T_k(u_n) - T_k(u))dx \\
&+ \int_{|u_n|\geq k} (a(x, T_n(u_n), \nabla u_n)\nabla(T_k(u))dx \\
&- \int_\Omega (a(x, T_k(u_n), \nabla T_k(u))\nabla(T_k(u_n) - T_k(u))dx.
\end{aligned}$$

Thus,it follows from (3.55),(3.56) and (3.57) that

$$\int_\Omega (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)))\nabla(T_k(u_n) - T_k(u))dx \to 0.$$

when n tends to $+\infty$. By Lemma 5 of [21], we obtain (3.53). The strong convergence (3.53) implies, for some subsequence still indexed by n, that

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e in} \quad \Omega,$$

which yields , since $(a(x, T_n(u_n), \nabla u_n)$ is bounded in $(L^{p'}(\Omega))^N$, that

$$a(x, T_n(u_n), \nabla u_n) \rightharpoonup a(x, u, \nabla u) \quad \text{weakly in} \quad (L^{p'}(\Omega))^N.$$

Therefore ,passing to the limit in (3.51) we obtain (3.8). □

Proof of Theorem 3.8. By Theorem 3.13 the sequence u_n is uniformly bounded in $W_0^{1,q}(\Omega)$. Therefore we can obtain a solution passing to the limit, namely arguing exactly as in Theorem 3.3. □

Chapter 4

Regularizing effect of absorption terms in singular and degenerate elliptic problems

In this chapter, we study a nonlinear singular elliptic equation with degenerate coercivity, lower order term and non-regular data. We discuss the existence and regularity of solutions in Sobolev spaces. The results of this chapter generalize the corresponding ones in the coercive case, given in [34].

1 Introduction

Let us consider the following problem

$$\begin{cases} -\mathrm{div}\big(a(x,u)|\nabla u|^{p-2}\nabla u\big) + |u|^{s-1}u = h(u)f & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $1 < p < N$, Ω is bounded open subset in $\mathbb{R}$ and $a : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$, we have

$$a(x,s) \geq \frac{\alpha}{(1+|s|)^{\theta}} \tag{4.2}$$

$$a(x,s) \leq \beta, \tag{4.3}$$

for some real positive constants α, β and $0 \leq \theta \leq 1$. Moreover, f is a non negative $L^m(\Omega)$ function, with $m \geq 1$ and the term $h : [0,\infty) \longrightarrow [0,\infty)$ is continuous, bounded outside the origin with $h(0) \neq 0$ and such that the following propertie hold true

$$\exists c,\, \gamma > 0 \text{ s.t } h(s) \leq \frac{c}{s^{\gamma}} \quad \forall s \in (0,+\infty), \tag{4.4}$$

for some real number γ such that $0 < \gamma \leq 1$.

Concerning the non-singular elliptic problems with lower order terms we have introduced in the introduction this type of problems, now we turn our attention to recalling some results when the authors had added the singular sourcing term. Problems of the p-Laplacian type (i.e $\theta = 0$), have been well studied in both the existence and regularity aspects with f having different summability (see [41]). This framework has been extended to the problems with a lower order, considering

$$\begin{cases} -\Delta u + u^s = \frac{f}{u^\gamma} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

with $f \in L^m(\Omega)$, $m \geq 1$, $0 < \gamma \leq 1$. Existence and regularity were established in [40]. Recently Oliva [74] have proved the existence and regularity of the solution to the problem

$$\begin{cases} -\Delta_p u + g(u) = h(u)f & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

f is a nonnegative and it belongs to $f \in L^m(\Omega)$, $m \geq 1$, for some $0 \leq \gamma < 1$. While $g(s)$ is continuous, $g(0) = 0$ and, as $s \to \infty$, could act as s^q with $q \geq -1$, the p-Laplacian operator is $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ and h is continuous, it possibly blows up at the origin and it is bounded at infinity. In chpiter 3, we studied the degenerate elliptic problem with a singular nonlinearity. Following this way in this work, we are interested again in the regularity results. By adding the singular term to the right of (3.1), we investigate the regularity of solutions of problems of kind (4.1) in light of the influence of some lower order terms.

We will prove in section(3) that these problems admit a bounded $W_0^{1,p}(\Omega)$ solution u_n, $n \in \mathbb{N}$ by using Schauder's fixed point theorem. In section 4 we will get some a priori estimates and convergence results on the sequence of approximating solutions. In the end, we pass to the limit in the approximate problems.

2 Statement of definitions and the main results

2.1 Statement of definitions

In this context we deal with some class of solutions

Definition 4.1. A nonnegative measurable function u is a weak solution to problem (4.1) if $u \in W_0^{1,1}(\Omega)$ and if

$$a(x,u)|\nabla u|^{p-2}\nabla u \in (L^1(\Omega))^N, \;\; h(u)f \in L^1(\Omega), \;\; |u|^{s-1}u \in L^1(\Omega),$$

$$\int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u \nabla\varphi dx + \int_\Omega |u|^{s-1}u\varphi = \int_\Omega fh(u)\varphi \;\; \forall\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{4.7}$$

Definition 4.2. A nonnegative measurable function u is an entropy solution to problem (4.1) if $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$ and

$$a(x, T_k(u))|\nabla T_k(u)|^{p-2}\nabla T_k(u) \in (L^1(\Omega))^N, |u|^s \in L^1(\Omega), \; h(u)f \in L^1(\Omega),$$

and if

$$\int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u \nabla T_k(u-\varphi)dx + \int_\Omega |u|^{s-1}uT_k(u-\varphi)dx \tag{4.8}$$
$$\leq \int_\Omega fh(u)T_k(u-\varphi)dx,$$

for every $k > 0$ and for any $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Let

$$p_0 := 1 + \frac{(1+\theta-\gamma)(N-1)}{N(1-\gamma)+\gamma}. \tag{4.9}$$

2.2 Statement of the main results

The main results of this paper are stated as follows:

Theorem 4.3. *Let a satisfy* (4.2) *and* (4.3)*. Let h satisfy* (4.4) *with* $0 < \gamma \leq 1$ *and let f be a positive function in* $L^m(\Omega)$*,* $m > 1$*,* $1 < p < N$*.*

i) If $s \geq \frac{1+\theta-\gamma}{m-1}$*, then there exists a weak solution u to problem* (4.1) *such that*

$$u \in W_0^{1,p}(\Omega) \cap L^{ms+\gamma}(\Omega).$$

ii) If $\frac{1+\theta-\gamma}{pm-1} < s < \frac{1+\theta-\gamma}{m-1}$*, then there exists a weak solution u to problem* (4.1) *such that*

$$u^{ms+\gamma} \in L^1(\Omega) \quad \textit{and} \quad u \in W_0^{1,\sigma}(\Omega) \quad, 1 < \sigma = \frac{pms}{1+\theta+s-\gamma}.$$

iii) If $0 < s \leq \frac{1+\theta-\gamma}{pm-1}$*, then there exists an entropy solution u to problem* (4.1) *such that*

$$u^{ms+\gamma} \in L^1(\Omega) \quad \textit{and} \quad |\nabla u| \in M^{\frac{pms}{1+\theta+s-\gamma}}(\Omega).$$

Remark 4.4. If $p = 2$ and $\gamma = 0$; the result of Theorem 4.3 coincides with regularity results in the case of an elliptic operator with degenerate coercivity (see [21], Theorem 1.5).

Theorem 4.5. *Under the assumptions* (4.2)-(4.3) *and h satisfy* (4.4)*, with* $0 < \gamma \leq 1$ *and let* $f \in L^m(\Omega)$ *be non negative function, with,* $m > 1$*,* $p_0 < p < N$*.*

i) If $0 < s \leq \frac{N(1-\gamma)+\gamma}{m(N-1)}$*, then there exists a weak solution u to problem* (4.1) *such that* $u^{ms+\gamma} \in L^1(\Omega)$ *and* $u \in W_0^{1,\sigma}(\Omega)$*, where* $1 < \sigma = \frac{N[p+s(m-1)-1-\theta+\gamma]}{N+s(m-1)-1-\theta+\gamma}$.

ii) If $s \geq \frac{N(1-\gamma)+\gamma}{m(N-1)}$*, then item(ii) of Theorem 4.3 holds.*

Remark 4.6. If $\gamma = 0$; the result of Theorem 4.3 coincides with regularity results in the case of an elliptic operator with degenerate coercivity (see [34], Theorem 3) and Theorem 4.5 coincides with ([34], Theorem 4).

Theorem 4.7. *Let a satisfy* (4.2) *and* (4.3)*. Let h satisfy* (4.4) *with* $0 < \gamma \leq 1$ *and let f be a positive function in* $L^1(\Omega)$*,* $1 < p < N$*.*

a) If $s > \frac{1+\theta-\gamma}{p-1}$*, then there exists a weak solution u to problem* (4.1) *such that* $u^{s+\gamma} \in L^1(\Omega)$ *and* $u \in W_0^{1,r}(\Omega)$*, where* $1 < r < \frac{ps}{s+1+\theta-\gamma}$.

b) If $0 < s \leq \frac{1+\theta-\gamma}{p-1}$, *then there exists an entropy solution* u *to problem* (4.1) *such that*

$$u^{s+\gamma} \in L^1(\Omega) \quad and \quad |\nabla u| \in M^{\frac{ps}{s+1+\theta-\gamma}}(\Omega).$$

Remark 4.8. If $p = 2$ and $\gamma = 0$; the result of Theorem 4.7 coincides with regularity results in the case of an elliptic operator with degenerate coercivity (see [36], Theorem 1.4).

Theorem 4.9. *Assume that* (4.2) *and* (4.3) *hold true. Let* h *satisfy* (4.4) *with* $0 < \gamma \leq 1$. *Let* $p_0 = 1 + \frac{(1+\theta-\gamma)(N-1)}{N(1-\gamma)+\gamma} < p < N$ *and let* f *be positive function in* $L^1(\Omega)$. *Then there exists a weak solution* u *to problem* (4.1) *such that if* $0 < s \leq \frac{N(1-\gamma)+\gamma}{N-1}$

$$\textit{then } u \textit{ belong to } W_0^{1,r}(\Omega), \textit{ with } 1 < r < \frac{N[p-\theta-1+\gamma]}{N-\theta+\gamma-1}.$$

Remark 4.10. If $\gamma = 0$; the result of Theorem 4.7 coincides with regularity results in the case of an elliptic operator with degenerate coercivity (see [34], Theorem 1) and Theorem 4.9 coincides with ([34], Theorem 2).

3 A priori estimates and Preliminary facts

Let us introduce the following scheme of approximation

$$\begin{cases} -\operatorname{div}\big(a(x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n\big) + |u_n|^{s-1}u_n = h_n(u_n)f_n, & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.10}$$

where $f_n = T_n(f)$. Moreover, we set

$$h_n(s) = \begin{cases} T_n(h(s)) & \text{for} \quad s > 0, \\ \min(n, h(0)) & \text{otherwise.} \end{cases} \tag{4.11}$$

The right hand side of (4.10) is non negative, that u_n is non negative. The existence of weak solution $u_n \in W_0^{1,p}(\Omega)$ is guaranteed by the following lemma.

Lemma 4.11. *Problem* (4.10) *has a non negative solution* u_n *in* $W_0^{1,p}(\Omega)$, *such that*

$$\int_\Omega |u_n|^{ms+\gamma}dx \leq c\int_\Omega |f|^m dx \tag{4.12}$$

and the solution u_n *satisfies*

$$\int_\Omega a(x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx + \int_\Omega |u_n|^{s-1}u_n\varphi = \int_\Omega f_n h_n(u_n)\varphi, \tag{4.13}$$

where $0 < \gamma \leq 1$ *and* φ *in* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Proof. This proofs derived from Schauder's fixed point argument in [75]. For fixed $n \in I\!N$ let us define a map

$$G : L^p(\Omega) \to L^p(\Omega),$$

such that, for any v be a function in $L^p(\Omega)$ gives the weak solution w to the following problem

$$-\,div(a(x, T_n(w))|\nabla w|^{p-2}\nabla w) + |w|^{s-1}w = f_n h_n(v). \tag{4.14}$$

The existence of a unique $w \in W_0^{1,p}(\Omega)$ corresponding to a $v \in L^p(\Omega)$ follows from the classical result of [[5], [68]]. Moreover, since the datum $f_n h_n(v)$ bounded, we have that $w \in L^\infty(\Omega)$ and there exists a positive constant d_1, independents of v and w (but possibly depending in n), such that
$||w||_{L^\infty(\Omega)} \le d_1$. Again, thanks to the regularity of the datum $f_n h_n(v)$, we have can choose w as test function in the weak formulation (4.13), we have

$$\int_\Omega a(x, T_n(w))|\nabla w|^{p-2}\nabla w \nabla w + \int_\Omega |w|^{s-1}w.w = \int_\Omega f_n h_n(v)w, \tag{4.15}$$

then, it follows from (4.2)

$$\alpha \int_\Omega \frac{|\nabla w|^p}{(1+n)^\theta} dx \le n^2 \int_\Omega |w| dx,$$

using the Poincaré inequality we have

$$\int_\Omega \frac{|\nabla w|^p}{(1+n)^\theta} dx \le \frac{c_1}{\alpha} n^2 \int_\Omega |\nabla w| dx,$$

then

$$\int_\Omega |\nabla w|^p dx \le \frac{c_1}{\alpha}(1+n)^{\theta+2} \int_\Omega |\nabla w| dx \le c(n,\alpha)|\Omega|^{\frac{1}{p'}} \left(\int_\Omega |\nabla w|^p dx \right)^{\frac{1}{p}}, \tag{4.16}$$

we obtain

$$\int_\Omega |\nabla w|^p dx \le c^{p'}(n,\alpha)|\Omega|,$$

using the Poincaré inequality on the left hand side

$$||w||_{L^p(\Omega)} \le c^{\frac{p'}{p}} |\Omega|^{\frac{1}{p}} = c(n,\alpha,|\Omega|), \tag{4.17}$$

where $c(n,\alpha,|\Omega|)$ is a positive constant independent form v, thus, we have that the ball S of radius $c(n,\alpha,|\Omega|)$ is invariant for G.
Now, we are going to prove that the map G is continuous in S. Consider a sequence (v_k) that converges to v in $L^p(\Omega)$. We recall that $w_k = f_n h_n(v_k)$ are bounded, we have that $w_k \in L^\infty(\Omega)$ and there exists a positive constant d, independent of v_k and w_k, such that $||w_k||_{L^\infty(\Omega)} \le d$. Then by dominated convergence theorem

$$||f_n h_n(v_k) - f_n h_n(v)||_{L^p(\Omega)} \longrightarrow 0.$$

Hence, by the uniqueness of the weak solution, we can say that $w_k = G(v_k)$ converges to $w = G(v)$ in $L^p(\Omega)$. Thus G is continuous over $L^p(\Omega)$.

What finally needs to be checked is that $G(S)$ is relatively compact in $L^p(\Omega)$. Let v_k be a bounded sequence, and let $w_k = G(v_k)$. Reasoning as to obtain (4.17), we have

$$\int_\Omega |\nabla w_k|^p dx = \int_\Omega |\nabla G(v_k)|^p dx \leq c(n, \alpha, \gamma),$$

where c is clearly independent from v_k , so that, $G(L^p(\Omega))$ is relatively compact in $L^p(\Omega)$. Now, applying the Schauder's fixed point theorem that G has a fixed point $u_n \in S$ that is solution to (4.10) in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.
To show (4.12), we will consider the cases $m > 1$ and $m = 1$.
Case $m > 1$, choosing $\varphi = |u_n|^{s(m-1)+\gamma}$ in (4.13), we have

$$\int_\Omega |u_n|^{sm+\gamma} dx \leq \int_\Omega |f||u_n|^{s(m-1)} dx,$$

therefore

$$\int_\Omega |u_n|^{sm+\gamma} \leq c \left(\int_\Omega |f|^m\right)^{\frac{1}{m}} \left(\int_\Omega |u_n|^{sm+\gamma}\right)^{1-\frac{1}{m}},$$

wish implies (4.12).

Case $m = 1$. Choosing $\varphi = u_n^\gamma$, then

$$\int_\Omega |u_n|^{s-1} u_n u_n^\gamma dx \leq \int_\Omega \frac{f}{u_n^\gamma} u_n^\gamma dx \leq f dx,$$

which the estimate (4.12), as desired.

Lemma 4.12. *[76] Let u be a measurable function in $M^r(\Omega)$, $r > 0$, and suppose that there exists a positive constant $\rho > 0$ such that*

$$\int_\Omega |\nabla T_k(u)|^p dx \leq C k^\rho \quad \forall k > 0.$$

Then $|\nabla u| \in M^{\frac{pr}{\rho+r}}(\Omega)$.

Proof. Let λ be fixed positive real number. For every $k > 0$, we have

$$\begin{aligned} meas\{|\nabla u| > \lambda\} &= meas\{|\nabla u| > \lambda, |u| \leq k\} + meas\{|\nabla u| > \lambda, |u| > k\} \\ &\leq meas\{|\nabla u| > \lambda, |u| \leq k\} + meas\{|u| > k\} \end{aligned}$$

and

$$meas\{|\nabla u| > \lambda, |u| \leq k\} \leq \frac{1}{\lambda^p} \int_\Omega |\nabla T_k(u)|^p dx \leq C \frac{k^\rho}{\lambda^p}.$$

Since $u \in M^r(\Omega)$, it follows that

$$meas\{|\nabla u| > \lambda\} = C \frac{k^\rho}{\lambda^p} + \frac{C}{k^r},$$

and this latter inequality holds for every $k > 0$. Minimizing with respect to k, we easily obtain

$$meas\{|\nabla u| > \lambda\} = \frac{C}{\lambda^{\frac{pr}{\rho+r}}}.$$

Thus, $|\nabla u| \in M^{\frac{pr}{\rho+r}}(\Omega)$.

Lemma 4.13. *Let u_n be a sequence of measurable functions such that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ for every $k > 0$. Then there exists a measurable function u such that $T_k(u) \in W_0^{1,p}(\Omega)$ and, moreover,*

$$T_k(u_n) \longrightarrow T_k(u) \;\; \textit{weakly in } W_0^{1,p}(\Omega) \;\; \textit{and} \;\; u_n \longrightarrow u \;\; \textit{a.e.in } \Omega.$$

Proof. Let us prove that $u_n \longrightarrow u$ locally in measure. To begin with, we observe that, for $t, \varepsilon > 0$, we have

$$\{|u_n - u_m| > t\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > t\}.$$

Therefore,

$$meas\{|u_n - u_m| > t\} \leq meas\{|u_n| > k\} + meas\{|u_m| > k\}$$
$$+meas\{|T_k(u_n) - T_k(u_m)| > t\}.$$

Choosing k large enough, we obtain

$$meas\{|u_n| > k\} < \varepsilon \;\text{ and }\; meas\{|u_m| > k\} < \varepsilon.$$

We can assume that $\{T_k(u_n)\}$ is a Cauchy sequence in $L^q(\Omega)$ for every $q < p^* = \frac{Np}{N-p}$. Then $\forall n, m \geq n_0(k,t)$:

$$meas\{|T_k(u_n) - T_k(u_m)| > t\} \leq t^{-q} \int_\Omega |T_k(u_n) - T_k(u_m)|^q dx \leq \varepsilon$$

This proves that $\{u_n\}$ is a Cauchy sequence in measure in Ω. Therefore, there exists a measurable function u such that $u_n \longrightarrow u$ in measure. Hence that $u_n \longrightarrow u$ a.e.in Ω, and so

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{weakly in} \quad W_0^{1,p}(\Omega).$$

4 Proof of the results

This section is devoted to proving theorems cited above. We start with

Proof of Theorem 4.3. We separate our proof into three parts, according to the values of s

Part I. Let $s \geq \frac{1+\theta-\gamma}{m-1}$ and we take $\varphi = (1+u_n)^{1+\theta} - 1$ as a test function in (4.13), using (4.2), we obtain

$$\alpha \int_\Omega |\nabla u_n|^p dx + \int_\Omega u_n^s[(1+u_n)^{1+\theta} - 1] \leq c \int_\Omega f u_n^{1+\theta-\gamma} dx.$$

In the following, we dropping the second positive term, and we get

$$\alpha \int_\Omega |\nabla u_n|^p dx \leq c \int_\Omega f u_n^{1+\theta-\gamma} dx.$$

Applying Hölder's inequality in the right-hand side of above estimate, we obtain

$$\int_\Omega f u_n^{1+\theta-\gamma} dx \leq c \left[\int_\Omega u_n^{\frac{m(1+\theta-\gamma)}{m-1}} dx\right]^{1-\frac{1}{m}}.$$

Then, we get

$$\int_\Omega |\nabla u_n|^p dx \leq c \left[\int_\Omega u_n^{\frac{m(1+\theta-\gamma)}{m-1}} dx \right]^{1-\frac{1}{m}}. \tag{4.18}$$

The condition $\frac{m(1+\theta-\gamma)}{m-1} \leq ms$, ensure that $s \geq \frac{1+\theta-\gamma}{m-1}$. Then by (4.12) the right-hand side of (4.18) is uniformly bounded, so we can get

$$\int_\Omega |\nabla u_n|^p dx \leq c. \tag{4.19}$$

In order to prove that the limit function u is a solution of (4.1) in the sense of Definition 4.1, we need to show that we can pass to the limit in the weak formulation of the approximating problems (4.10).

Now we focus on the left-hand side of (4.13), by (4.19) we conclude that there exists a subsequence, still indexed by n, and a measurable function u in $W_0^{1,p}(\Omega)$, such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $u_n \longrightarrow u$ a.e in Ω. Fatou's lemma implies $u \in L^{ms+\gamma}(\Omega)$. We see that (see [[21], Lemma 5] and see also [48, 77])

$$\nabla u_n \longrightarrow \nabla u \text{ a.e in } \Omega. \tag{4.20}$$

Next, we pass to the limit in (4.13). By (4.20), we can easily obtain

$$|\nabla u_n|^{p-2}.|\nabla u_n| \rightharpoonup |\nabla u|^{p-2}.|\nabla u| \text{ weakly in } L^{p'}(\Omega).$$

Moreover,

$$a(x, T_n(u_n))\nabla\varphi \longrightarrow a(x,u)\nabla\varphi \text{ in } L^p(\Omega).$$

Consequently, we have

$$\int_\Omega a(x, T_n(u_n))|\nabla u_n|^{p-2}.\nabla u_n \nabla\varphi dx \longrightarrow \int_\Omega a(x,u)|\nabla u|^{p-2}.\nabla u\nabla\varphi dx.$$

Therefore, we can pass to the limit in the first term of the left-hand side of (4.13). We will show that

$$|u_n|^{s-1}u_n \to |u|^{s-1}u \text{ in } L^1(\Omega). \tag{4.21}$$

We take $S_{\eta,k}(u_n)$ as a test function in the weak formulation (4.10), we deduce

$$\begin{aligned} &\int_\Omega a(x, T_n(u_n))|\nabla u_n|^p S'_{\eta,k}(u_n)dx + \int_\Omega |u_n|^{s-1}u_n S_{\eta,k}(u_n)dx \\ &\leq \sup_{s\in[k,\infty)}[h(s)] \int_\Omega f_n S_{\eta,k}(u_n), \end{aligned}$$

which, observing that the first term on the left hand side is non negative and taking the limit with respect to $\eta \to 0$, implies

$$\int_{\{u_n\geq k\}} |u_n|^{s-1}u_n dx \leq \sup_{s\in[k,\infty)}[h(s)] \int_{\{u_n\geq k\}} f_n dx,$$

which, since f_n converges to f in $L^m(\Omega)$, easily implies that $|u_n|^{s-1}u_n$ is equi-integrable and so it converges to $|u|^{s-1}u$ in $L^1(\Omega)$, this concludes (4.21).

The next step we want to pass to the limit in the right hand side of (4.13).
Let us take $0 \leq \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function in the weak formulation of (4.10), by using the young inequality and the hypotheses in (4.2) and (4.3), we have

$$\begin{aligned}
&\int_\Omega h_n(u_n) f_n \varphi = \int_\Omega a(x, T_n(u_n))|\nabla u_n|^{p-2}.\nabla u_n \nabla\varphi dx + \int_\Omega u_n^{s-1} u_n \varphi dx \\
&\leq C||\varphi||_{L^\infty(\Omega)} + \beta \int_\Omega |\nabla u_n|^{p-1} \nabla\varphi dx + \frac{1}{s}\int_\Omega u_n^s dx + \frac{1}{s'}\int_\Omega \varphi^{s'} dx \\
&\leq C||\varphi||_{L^\infty(\Omega)} + \beta\frac{p-1}{p}\int_\Omega |\nabla u_n|^p dx + \beta\frac{1}{p}\int_\Omega |\nabla\varphi|^p dx + \frac{1}{s}\int_\Omega u_n^s dx + \frac{1}{s'}\int_\Omega \varphi^{s'} dx \\
&\leq C||\varphi||_{L^\infty(\Omega)} + C\left[\int_\Omega |\nabla\varphi|^p dx + \int_\Omega |\nabla u_n|^p dx\right] + \frac{1}{s}\|u_n\|_{L^s(\Omega)} + \frac{1}{s'}\|\varphi\|_{L^{s'}(\Omega)},
\end{aligned}$$

then

$$\int_\Omega h_n(u_n) f_n \varphi \leq C||\varphi||_{L^\infty(\Omega)} + C[||\varphi||_{W_0^{1,p}(\Omega)} + ||u_n||_{W_0^{1,p}(\Omega)}] + \frac{1}{s}\|u_n\|_{L^s(\Omega)} + \frac{1}{s'}\|\varphi\|_{L^{s'}(\Omega)}. \tag{4.22}$$

From now on, we assume that $h(s)$ is unbounded as s tends to 0. An application of the Fatou Lemma in (4.22) with respect to n gives

$$\int_\Omega h(u) f \varphi \leq c, \tag{4.23}$$

where c does not depend on n.
Hence $fh(u)\varphi \in L^1(\Omega)$ for any nonnegative $\varphi \in W_0^{1,p}(\Omega)$. As a consequence, if $h(s)$ is unbounded as s tends to 0, we deduce that

$$\{u = 0\} \subset \{f = 0\} \tag{4.24}$$

up to a set of zero Lebesgue measure.
Now, for $\delta > 0$, we split the right hand side of (4.13) as

$$\int_\Omega h_n(u_n) f_n \varphi dx = \int_{\{u_n \leq \delta\}} h_n(u_n) f_n \varphi dx + \int_{\{u_n > \delta\}} h_n(u_n) f_n \varphi dx, \tag{4.25}$$

and we pass to limit as $n \to +\infty$ and then $\delta \to 0$, we remark that we need to choose $\delta \neq \{\eta; |u = \eta| > 0\}$, which is at most a countable set, for the second term (4.25) we have

$$0 \leq h_n(u_n) f_n \varphi \chi_{\{u_n > \delta\}} \leq \sup_{s \in]\delta, \infty)} [h(s)] f\varphi \in L^1(\Omega), \tag{4.26}$$

which precis to apply the Lebesgue Theorem with respect n. Hence on has

$$\lim_{n\to+\infty} \int_{\{u_n > \delta\}} h_n(u_n) f_n \varphi dx = \int_{\{u > \delta\}} h(u) f \varphi dx.$$

Moreover it follows by (5.21)that

$$\lim_{\delta\to 0^+} \lim_{n\to+\infty} \int_{\{u_n > \delta\}} h_n(u_n) f_n \varphi dx = \int_{\{u > 0\}} h(u) f \varphi dx. \tag{4.27}$$

Now in order to get rid of the first term of the right hand side of (4.25), we take $V_\delta(u_n)\varphi$ is a test function in the weak formulation of (4.10),where
$V_\delta(u_n) := V_{\delta,\delta}(u_n)$ is defined in (3.39) and by Lemma 1.1 contained in [34], we have $V_\delta(u_n)$ belongs to $W_0^{1,p}(\Omega)$, then (recall $V_\delta'(u_n) \leq 0 \ for \ s \geq 0$)

$$\begin{aligned}&\int_{\{u_n\leq\delta\}} h_n(u_n)f_n\varphi dx \leq \int_\Omega h_n(u_n)f_nV_\delta(u_n)\varphi dx\\&= \int_\Omega a(x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi V_\delta(u_n)dx\\&-\frac{1}{\delta}\int_{\{\delta<u_n<2\delta\}} a(x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n\varphi\nabla u_n dx + \int_\Omega |u_n|^{s-1}u_nV_\delta(u_n)\varphi dx,\end{aligned}$$

by using (4.2) and (4.3), we have

$$\begin{aligned}\int_{\{u_n\leq\delta\}} h_n(u_n)f_n\varphi dx &\leq \beta\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi V_\delta(u_n)dx\\&+\int_\Omega |u_n|^{s-1}u_nV_\delta(u_n)\varphi dx,\end{aligned}$$

using that V_δ is bounded we deduce that $|\nabla u_n|^{p-2}\nabla u_n V_\delta(u_n)$ converges to $|\nabla u|^{p-2}\nabla u V_\delta(u)$ weakly in $L^{p'}(\Omega)^N$ as n tends to infinity. This implies that

$$\begin{aligned}\lim_{n\to+\infty}\int_{\{u_n\leq\delta\}} h_n(u_n)f_n\varphi dx &\leq \beta\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi V_\delta(u)dx\\&+\int_\Omega |u|^{s-1}uV_\delta(u)\varphi dx.\end{aligned}\tag{4.28}$$

Since $V_\delta(u)$ converges to $\chi_{\{u=0\}}$ a.e in Ω as δ tends to 0 and since $u \in W_0^{1,p}(\Omega)$, then $|\nabla u|^{p-2}\nabla u\nabla\varphi V_\delta(u)$ converges to 0 a.e. in Ω as δ tends to 0. Applying the Lebesgue Theorem on the right hand side of (4.28) we obtain that

$$\begin{aligned}&\lim_{\delta\to0^+}\lim_{n\to+\infty}\int_{\{u_n\leq\delta\}} h_n(u_n)f_n\varphi dx\\&\leq \beta\int_{\{u=0\}} |\nabla u|^{p-2}\nabla u\nabla\varphi dx + \int_{\{u=0\}} |u|^{s-1}u\varphi dx = 0,\end{aligned}\tag{4.29}$$

by (4.27) and (4.29), we deduce that

$$\lim_{n\to+\infty}\int_\Omega h_n(u_n)f_n\varphi dx = \int_\Omega h(u)f\varphi dx \quad \forall 0\leq\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).\tag{4.30}$$

Moreover, decomposing any $\varphi = \varphi^+ - \varphi^-$, and using that (4.30) is linear in φ, we deduce that (4.30) holds for every $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

We treated $h(s)$ unbounded as s tends to 0, as regards bounded function h the proof is easier and only difference deals with the passage to the limit in the left hand side of (4.30). We can avoid introducing δ and we can substitute (4.26) with

$$0 \leq f_n h_n(u_n)\varphi \leq f||h||_{L^\infty(\Omega)}\varphi.$$

Using the same argument above we have that

$$\lim_{n\to+\infty}\int_\Omega f_n h_n(u_n)\varphi dx = \int_\Omega f h(u)\varphi dx, \tag{4.31}$$

whence one deduces (4.7). This concludes the proof of part I.

Part II. Let

$$\frac{1+\theta-\gamma}{pm-1} < s < \frac{1+\theta-\gamma}{m-1}.$$

Taking $\varphi = (1+u_n)^{s(m-1)+\gamma} - 1$ as a test function in (4.13). Using assumption (4.2) and dropping the nonnegative term, we get

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-s(m-1)-\gamma}}dx \leq c\int_\Omega |f||u_n|^{s(m-1)}dx. \tag{4.32}$$

Applying Hölder's inequality in the right-hand side of the estimate (4.32), we get

$$\int_\Omega |f||u_n|^{s(m-1)}dx \leq c\left[\int_\Omega u_n^{ms+\gamma}dx\right]^{1-\frac{1}{m}} \leq c.$$

Then, we obtain

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-s(m-1)-\gamma}}dx \leq c. \tag{4.33}$$

Let $1 \leq \sigma < p$. Writing

$$\int_\Omega |\nabla u_n|^\sigma dx = \int_\Omega \frac{|\nabla u_n|^\sigma (1+u_n)^{\frac{\sigma}{p}(1+\theta-s(m-1)-\gamma)}}{(1+u_n)^{\frac{\sigma}{p}(1+\theta-s(m-1)-\gamma)}}dx$$

and using Hölder's inequality, we get

$$\int_\Omega \frac{|\nabla u_n|^\sigma (1+u_n)^{\frac{\sigma}{p}(1+\theta-s(m-1)-\gamma)}}{(1+u_n)^{\frac{\sigma}{p}(1+\theta-s(m-1)-\gamma)}}dx \leq c\left[\int_\Omega (1+u_n)^{\frac{\sigma}{p-\sigma}[1+\theta-s(m-1)-\gamma]}dx\right]^{1-\frac{\sigma}{p}}.$$

Then by (4.33), we arrive at

$$\int_\Omega |\nabla u_n|^\sigma dx \leq c\left[\int_\Omega (1+u_n)^{\frac{\sigma}{p-\sigma}[1+\theta-s(m-1)-\gamma]}dx\right]^{1-\frac{\sigma}{p}}. \tag{4.34}$$

We now choose σ in order to have

$$\frac{\sigma}{p-\sigma}[1+\theta-s(m-1)-\gamma] \leq ms.$$

The last inequality is equivalent to

$$\sigma \leq \frac{pms}{s+1+\theta-\gamma}.$$

Thanks to Lemma 7.6, if implies that

$$\frac{1+\theta-\gamma}{pm-1} < s \text{ implies } \frac{pms}{s+1+\theta-\gamma} > 1.$$

In that case, the right- hand side of (4.34) is uniformly bounded and so we have

$$\int_\Omega |\nabla u_n|^\sigma dx \leq c,\ \sigma = \frac{pms}{1+\theta+s-\gamma}.$$

Up to a subsequence, there exists a function $u \in W_0^{1,\sigma}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ weakly in}\, W_0^{1,\sigma}(\Omega) \text{ and } u_n \longrightarrow u \text{ a.e } in\ \ \Omega.$$

By Lemma 5 (see[21]), we have $\nabla u_n \longrightarrow \nabla u\ \ a.e\ \ in\ \Omega$. Fatou's Lemma implies $u^{ms+\gamma} \in L^1(\Omega)$ we will now pass to the limit in (4.13). We can easily obtain

$$|\nabla u_n|^{p-2}\nabla u_n \longrightarrow |\nabla u|^{p-2}\nabla u \text{ weakly in } L^{\frac{\sigma}{p-1}}(\Omega),$$

and

$$a(x, T_n(u_n))\nabla\varphi \longrightarrow a(x,u)\nabla\varphi, \text{ in } L^{(\frac{\sigma}{p-1})'}(\Omega).$$

Therefore,we have

$$\int_\Omega a(x, T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx \longrightarrow \int_\Omega a(x,u)|\nabla u|^{p-2}\nabla u\nabla\varphi dx.$$

The remaining two parts in (4.13) are the same as part I.
PART III. Suppose that $0 < s \leq \frac{1+\theta-\gamma}{pm-1}$. We show that there exists an entropy solution to problem (4.10). Estimate (4.33) implies that

$$\int_{\Omega\cap\{|u_n|<k\}} \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-s(m-1)-\gamma}} dx \leq c,$$

and consequently

$$\int_\Omega |\nabla T_k(u_n)|^p dx = \int_{\Omega\cap\{|u_n|<k\}} |\nabla T_k(u_n)|^p dx \leq c(1+k)^{1+\theta-s(m-1)-\gamma}. \tag{4.35}$$

Thanks to Lemma 4.13 there exists a function u such that $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, besides, passing if necessary to subsequence, we obtain

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega) \text{ and } a.e.\text{in } \Omega.$$

Then, we can pass to the limit in (4.35), to get

$$\int_\Omega |\nabla T_k(u)|^p dx \leq c(1+k)^{1+\theta-s(m-1)-\gamma}.$$

Lemma 4.12 gives us, if

$$s \leq \frac{1+\theta-\gamma}{pm-1} \leq \frac{1+\theta-\gamma}{m-1},$$

then, we obtain

$$|\nabla u| \in M^{\frac{pms}{1+\theta+s-\gamma}}(\Omega).$$

Since $|u_n|^{ms+\gamma}$ is uniformly bounded in $L^1(\Omega)$, by applying Fatou Lemma implies that $|u|^{ms+\gamma} \in L^1(\Omega)$. We will show that u is an entropy solution of (4.1). Indeed, let us choose

$$T_k(u_n-\varphi), \quad \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

as a test function in (4.13), then we have

$$\int_\Omega a(x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n \nabla T_k(u_n-\varphi)dx + \int_\Omega |u_n|^{s-1}u_n T_k(u_n-\varphi)$$
$$= \int_\Omega f_n h_n(u_n) T_k(u_n-\varphi). \tag{4.36}$$

Let us pass to the limit in (4.36). For the second term on the left-hand side and for the right-hand side, we can use (4.31)to obtain the limit. For the first term on the left-hand side, we will first show that $\nabla T_k(u_n) \to \nabla T_k(u)$ a.e. in Ω. Let $\varphi = T_k(u_n) - T_k(u)$ in (4.13), then we obtain

$$\int_\Omega a(x,T_n(T_k(u_n)))|\nabla u_n|^{p-2}\nabla u_n[\nabla T_k(u_n) - \nabla T_k(u)]dx$$
$$+\int_\Omega |u_n|^{s-1}u_n[T_k(u_n)-T_k(u)] = \int_\Omega f_n h_n(u_n)[T_k(u_n)-T_k(u)].$$

As a consequence, we have

$$\int_\Omega a(x,T_n(T_k(u_n)))[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)]$$
$$\times[\nabla T_k(u_n) - \nabla T_k(u)]dx$$
$$= \int_\Omega f_n h_n(u_n)[T_k(u_n)-T_k(u)]dx - \int_\Omega |u_n|^{s-1}u_n[T_k(u_n)-T_k(u)]dx$$
$$- \int_\Omega a(x,T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2}\nabla T_k(u)][\nabla T_k(u_n) - \nabla T_k(u)]dx. \tag{4.37}$$

We are going to show that the three terms of the right-hand side in (4.37) all converge to zero. For the first term, we can use the (4.31) to take the limit. As the result of the proof in part one, we obtain

$$u_n^s \to u^s \text{ in } L^1(\Omega). \tag{4.38}$$

Again, by (4.38), we deduce

$$\int_\Omega u_n^s[T_k(u_n)-T_k(u)]dx \to 0 \text{ as } n \to \infty.$$

We can easily know the fact that $a(x, T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2}\nabla T_k(u) \in L^{p'}(\Omega)$. Thus, for every measurable set $E \subset \Omega$, we can write

$$\int_E |a(x, T_n(T_k(u_n)))|\nabla T_k(u)|^{p-1}|^{p'} dx \to 0 \,\text{as}\, meas E \to 0.$$

Because

$$a(x, T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2}\nabla T_k(u) \to a(x, T_k(u))|\nabla T_k(u)|^{p-2}\nabla T_k(u) \,\text{a.e.in}\, \Omega,$$

using Vitali's Theorem, we obtain

$$a(x, T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2}\nabla T_k(u) \to a(x, T_k(u))|\nabla T_k(u)|^{p-2}\nabla T_k(u) \text{ in } L^{p'}(\Omega).$$

Hence one can apply By Lemma 4.13, obtaining that

$$\nabla T_k(u_n) - \nabla T_k(u) \rightharpoonup 0 \;\text{ weakly in }\; L^p(\Omega).$$

Therefore,

$$\int_\Omega a(x, T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2}\nabla T_k(u)][\nabla T_k(u_n) - \nabla T_k(u)]dx \to 0 \;\text{ as }\; n \to \infty.$$

Therefore from the previous we deduce

$$\int_\Omega a(x, T_n(T_k(u_n)))|\nabla T_k(u)|^{p-2}\nabla T_k(u)][\nabla T_k(u_n) - \nabla T_k(u)]dx \to 0.$$

So we can apply Lemma 5 in [21] (see also [77, 48]) obtaining $\nabla T_k(u_n) \to \nabla T_k(u)$ in $L^p(\Omega)$. Therefore,

$$\nabla T_k(u_n) \to \nabla T_k(u) \;\text{ a.e. in }\; \Omega.$$

Let $m = k + |\varphi|$. The first term on the left-hand side in (4.36) can be rewritten as

$$\int_\Omega a(x, T_n(u_n))|\nabla T_m(u)|^{p-2}\nabla T_m(u)\nabla T_k(u_n - \varphi)dx.$$

Since $\nabla T_m(u_n) \to \nabla T_m(u)$ a.e.in Ω, as a result of the Fatou's Lemma, we have

$$\begin{aligned}
\liminf_{n\to\infty} &\int_\Omega a(x, T_n(u_n))|\nabla T_m(u)|^{p-2}\nabla T_m(u)\nabla T_k(u_n - \varphi)dx \\
&\geq \int_\Omega a(x, u)|\nabla T_m(u)|^{p-2}\nabla T_m(u)\nabla T_k(u_n - \varphi)dx \\
&= \int_\Omega a(x, u)|\nabla u|^{p-2}\nabla u\nabla T_k(u_n - \varphi)dx.
\end{aligned}$$

So we see that u is an entropy solution of (4.1).

Proof of Theorem 4.5. We separate our proof in two parts, according to the values of s

Part I. Suppose $0 < s \leq \frac{N(1-\gamma)+\gamma}{m(N-1)}$. It is obvious that $s \leq \frac{N(1-\gamma)+\gamma}{m(N-1)}$, which implies $ms + \gamma \leq \sigma^*$ for $\sigma \geq 1$. Thus, from (4.34) and applying Sobolev's embedding Theorem, we obtain

$$\int_\Omega |u_n|^{\sigma^*} dx \leq c\left[\int_\Omega (1 + u_n)^{\frac{\sigma}{p-\sigma}[1+\theta-s(m-1)-\gamma]}dx\right]^{\frac{(p-\sigma)\sigma^*}{p\sigma}}.$$

On other hand, if $\frac{\sigma}{p-\sigma}[1+\theta-s(m-1)-\gamma] \leq \sigma^*$ implies that

$$\sigma \leq \frac{N[p+s(m-1)-1-\theta+\gamma]}{N+s(m-1)-1-\theta+\gamma}.$$

Therefore, since $m>1$ and $p>p_0>1+\frac{(N-1)[1+\theta-s(m-1)-\gamma]}{N}$, we have

$$\frac{N[p+s(m-1)-1-\theta+\gamma]}{N+s(m-1)-1-\theta+\gamma} > 1.$$

$$\int_\Omega |u_n|^{\sigma^*} dx \leq c + c\left(\int_\Omega |u_n|^{\sigma^*} dx\right)^{\frac{(p-\sigma)\sigma^*}{p\sigma}}. \tag{4.39}$$

In other hand by Young's inequality and from (4.39), can get

$$\int_\Omega |u_n|^{\sigma^*} dx \leq c.$$

We now observe that, by (4.34) and since $\frac{\sigma}{p-\sigma}[1+\theta-s(m-1)-\gamma] \leq \sigma^*$, we have

$$\int_\Omega |\nabla u_n|^\sigma dx \leq c, \quad \sigma \leq \frac{N[p+s(m-1)-1-\theta+\gamma]}{N+s(m-1)-1-\theta+\gamma}.$$

The remaining proof of this part is the same as part II in Theorem 4.3, we have can show that u is a distributional solution to problem (4.1).

Part II. Let $s \geq \frac{N(1-\gamma)+\gamma}{m(N-1)}$. Since $p > p_0$, it follows that

$$\frac{N(1-\gamma)+\gamma}{m(N-1)} > \frac{1+\theta-\gamma}{pm-1},$$

thus, we can show that u is a distributional solution to the problem (4.1) by the same method as in Part II of Theorem 4.3.

Proof of Theorem 4.7. We separate our proof in two parts, according to the values of s

Part a. Let $s > \frac{1+\theta-\gamma}{p-1}$. If we choose $\varphi = (1+u_n)^\gamma - 1$ as a test function in (4.13). Using assumption (4.2) and dropping the non negative term, we can write

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-\gamma}} dx \leq c + c\int_\Omega |f| dx \leq c. \tag{4.40}$$

From the other hand, let $r < p$, writing

$$\int_\Omega |\nabla u_n|^r dx = \int_\Omega \frac{|\nabla u_n|^r}{(1+u_n)^{\frac{r(1+\theta-\gamma)}{p}}} (1+u_n)^{\frac{r(1+\theta-\gamma)}{p}} dx.$$

$$\leq \left(\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{(1+\theta-\gamma)}} dx\right)^{\frac{r}{p}} \left(\int_\Omega (1+u_n)^{\frac{r(1+\theta-\gamma)}{p-r}} dx\right)^{1-\frac{r}{p}}$$

$$\leq c\left(\int_\Omega (1+u_n)^{\frac{r(1+\theta-\gamma)}{p-r}}dx\right)^{1-\frac{r}{p}}.$$

Thanks to Lemma 7.6, if

$$\frac{r}{p-r}(1+\theta-\gamma)\leq s\ ,ie\ \ r<\frac{ps}{1+\theta+s-\gamma}.$$

Then

$$s>\frac{1+\theta-\gamma}{p-1}\quad \text{implies}\ ,\ \frac{ps}{1+\theta+s-\gamma}>1.$$

In that case, the right-hand sides is uniformly bounded and so we get

$$\int_\Omega |\nabla u_n|^r dx \leq c\quad ,r<\frac{ps}{1+s+\theta-\gamma}.$$

As a consequence, there exists a function $u \in W_0^{1,r}(\Omega)$ such that

$$u_n \rightharpoonup u\ \text{ weakly in } W_0^{1,r}(\Omega) \text{ and } u_n \longrightarrow u\ \text{ a.e in } \Omega.$$

Let

$$g_n = f_n h_n(u_n) - T_n(|u_n|^{s-1}u_n).$$

Because g_n is bounded in $L^1(\Omega)$, and u_n is a solution of

$$\begin{cases} -\text{div}\big(a(x,T_n(u_n))|\nabla u_n|^{p-2}\nabla u_n\big) = g_n, \\ u_n \in W_0^{1,p}(\Omega), \end{cases}$$

then use the argument of Lemma 1 (see[22]), one may get

$$\nabla u_n \longrightarrow \nabla u\ \ a.e\ \ in\ \Omega. \tag{4.41}$$

We are going to show that u is a distributional solution to problem (4.1) by passing to the limit in (4.13). We suppose that $\varphi \in C_0^\infty(\Omega)$. Since
$|\nabla u_n|^{p-2}\nabla u_n \in L^{\frac{r}{p-1}}(\Omega)$ and (4.41) hold, we have

$$|\nabla u_n|^{p-2}\nabla u_n \longrightarrow |\nabla u|^{p-2}\nabla u\ \text{ weakly in }\ L^{\frac{r}{p-1}}(\Omega).$$

Using Vitali's Theorem, we obtain

$$a(x,T_n(u_n))\nabla\varphi \longrightarrow a(x,u)\nabla\varphi\quad \text{in } L^{(\frac{r}{p-1})'}(\Omega),$$

where $(\frac{r}{p-1})' = \frac{p-1-r}{p-1}$. Therefore, we can pass to the limit in the first term on the left-hand side of (4.13). For the second term on the left hand-side and the first term on the right-hand side in (4.13) we can namely arguing exactly as part I in Theorem 4.3. Therefore, we conclude that u is a distributional solution to problem (4.1).

Part b. Let $0 < s \leq \frac{1+\theta-\gamma}{p-1}$. Let us choose $T_k(u_n)$ as a test function in (4.13), using assumption (4.2) and removing the second term non negative, we get

$$\int_\Omega |\nabla T_k(u_n)|^p dx \leq ck^{1-\gamma}(1+k)^\theta \leq c(1+k)^{1-\gamma+\theta}. \tag{4.42}$$

Now by Lemma 4.13, there exists a function u such that $T_k(u) \in W_0^{1,p}(\Omega)$. Moreover,

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega)\ \forall k > 0 \quad \text{and } u_n \longrightarrow u \text{ a.e in } \Omega.$$

Fatou's Lemma implies that $|u|^{s+\gamma} \in L^1(\Omega)$. We can pass to the limit in (4.42),to obtain

$$\int_\Omega |\nabla T_k(u)|^p dx \leq c(1+k)^{1+\theta-\gamma}.$$

As a result of the Lemma 4.12, we obtain $|\nabla u| \in M^{\frac{ps}{1+\theta+s-\gamma}}(\Omega)$.

By the same method as in part II of Theorem 4.3, we can show that u is an entropy solution of (4.1).

Proof of Theorem 4.9. Let $0 < s \leq \frac{N(1-\gamma)+\gamma}{N-1}$. Then $s \leq \frac{N(1-\gamma)+\gamma}{N-1}$ implies $s+\gamma \leq r^*$. Using (4.40), we get

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-\gamma}} dx \leq c.$$

Let $1 \leq r < p$, let us write

$$\int_\Omega |\nabla u_n|^r dx = \int_\Omega \frac{|\nabla u_n|^r}{(1+u_n)^{\frac{r(1+\theta-\gamma)}{p}}} (1+u_n)^{\frac{r(1+\theta-\gamma)}{p}} dx.$$

Then it follows from Hölder's inequality that

$$\int_\Omega |\nabla u_n|^r dx \leq c \left(\int_\Omega (1+u_n)^{\frac{r(1+\theta-\gamma)}{p-r}} dx \right)^{1-\frac{r}{p}}. \tag{4.43}$$

On other hand by Sobolev embedding Theorem and from (4.43), we can get

$$\left(\int_\Omega |u_n|^{r^*} dx \right)^{\frac{1}{r^*}} \leq c \left(\int_\Omega |\nabla u_n|^r dx \right)^{\frac{1}{r}}, r^* = \frac{Nr}{N-r}.$$

Which implies,

$$\left(\int_\Omega |u_n|^{r^*} dx \right)^{\frac{1}{r^*}} \leq c \left(\int_\Omega (1+u_n)^{\frac{r(1+\theta-\gamma)}{p-r}} dx \right)^{\frac{(p-r)r^*}{pr}}.$$

being $\frac{r(1+\theta-\gamma)}{p-r} \leq r^*$, so that

$$r \leq \frac{N[p-1-\theta+\gamma]}{N-1-\theta+\gamma}.$$

Then, by $m=1$ we have $p_0 = 1 + \frac{(1+\theta-\gamma)(N-1)}{N(1-\gamma)+\gamma}$. By virtue of $p > p_0 > 1 + \frac{(N-1)[1+\theta-\gamma]}{N}$, ensures that

$$\frac{N[p-1-\theta+\gamma]}{N-1-\theta+\gamma} > 1.$$

Then, we can obtain

$$\int_\Omega u_n^{r^*} dx \leq c \left(\int_\Omega (1+u_n)^{r^*} dx \right)^{\frac{(p-r)r^*}{p-r}} \leq c + c \left(\int_\Omega u_n^{r^*} dx \right)^{\frac{(p-r)r^*}{pr}}.$$

Using Young inequality in the above estimate gives

$$\int_\Omega |u_n|^{r^*} dx \leq c.$$

Which together with (4.43) and $\frac{r}{p-r}(\theta+1-\gamma) \leq r^*$ implies

$$\int_\Omega |\nabla u_n|^r dx \leq c \quad , r < \frac{N[p-1-\theta+\gamma]}{N-1-\theta+\gamma}.$$

Just as in the proof of part I in the Theorem 4.7, we can conclude that u is a distributional solution of (4.1).

Chapter 5

Singular elliptic problem involving a Hardy potential and lower order term

1 Introduction and main results

In the present chapter we will analyze the interaction between $b|u|^{r-2}u$ as an absorption term in the equation and the term $a\,(u^{p-1}/|x|^p)$ involving the Hardy potential in order to prove the existence and regularity of solution to problem (5.1), for every $a > 0$ (and not only for a smaller than the Hardy constant).

Let us consider the following singular elliptic problem

$$\begin{cases} -\operatorname{div}(M(x)|\nabla u|^{p-2}\nabla u) + b|u|^{r-2}u = a\frac{u^{p-1}}{|x|^p} + \frac{f}{u^\gamma} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $1 < p < N$; $\Omega \subset \mathbb{R}^N$ is a bounded regular domain containing the origin and $0 < \gamma \leq 1$.

We assume that $M : \Omega \to \mathbb{R}$, is a Lipschitz continuous function such that for some positive constants α and β

$$M(x)\xi^{p-1}\xi \geq \alpha|\xi|^p, \quad |M(x)| \leq \beta \text{ for all } \xi \in \mathbb{R}^N \text{ and almost every } x \text{ in } \Omega. \tag{5.2}$$

Assume that

$$r > p^* \text{ and } a > 0, b > 0 \tag{5.3}$$

$$0 \leq f \in L^m(\Omega), \quad 1 < m < \frac{N}{p}, \tag{5.4}$$

Now, we give our definition of solution for problem (5.1)

Definition 5.1. We say that $u \in W_0^{1,1}(\Omega)$ is a distributional solution to problem (5.1) if

$$|\nabla u|^{p-1} \in L^1_{loc}(\Omega), a\frac{u^{p-1}}{|x|^p} + \frac{f}{u^\gamma} \in L^1_{loc}(\Omega), b|u|^{r-2}u \in L^1_{loc}(\Omega) \tag{5.5}$$

and for all $\varphi \in \mathcal{C}_c^1(\Omega)$, we have

$$\int_\Omega M(x)|\nabla u|^{p-2}\nabla u \nabla \varphi dx + b\int_\Omega |u|^{r-2}u\varphi = \int_\Omega \left(a\frac{u^{p-1}}{|x|^p} + \frac{f}{u^\gamma}\right)\varphi\, dx. \tag{5.6}$$

The main result of this paper is the following theorems:

Theorem 5.2. *Assume that* (5.2), (5.3) *and* $0 < \gamma \leq 1$ *holde true. Let* f *be nonnegative function in* $L^m(\Omega)$, *with*

$$\frac{r}{r-1+\gamma} \leq m < \frac{N}{p}\frac{r-p}{r-1+\gamma}.$$

Then there exists a distributional solution u *of* (5.1), *which belongs to* $W_0^{1,p}(\Omega) \cap L^{m(r-1+\gamma)}(\Omega)$.

Remark 5.3. Observe that the interval $\left[\frac{r}{r-1+\gamma}, \frac{N}{p}\frac{r-p}{r-1+\gamma}\right)$ is not empty if $r > p^*$.

Remark 5.4. Thanks to the presence of the lower order term $b|u|^{r-2}u$, the result of Theorem 2.1 improves that of [85] (where $b = 0$ and $p = 2$) in several directions. First of all, if

$$\frac{r}{r-1+\gamma} \leq m < \frac{2N}{N+2+\gamma(N-2)},$$

we have finite energy solutions (instead of infinite energy ones). Furthermore, we have that solutions exist for every $a > 0$ (and not only for a smaller than the Hardy constant). Finally, the summability in Lebesgue spaces, $m(r-1+\gamma)$, is better than the summability $(1+\gamma)m^{**}$ obtained in [85].

Theorem 5.5. *Assume that* (5.2), (5.3) *and* $0 < \gamma \leq 1$ *hold true. Let* f *be nonnegative function in* $L^m(\Omega)$, *with*

$$1 < m < \frac{r}{r-1+\gamma}.$$

Then there exists a distributional solution u *of* (5.1), *which belongs to* $W_0^{1,q}(\Omega) \cap L^{m(r-1+\gamma)}(\Omega)$, *where* q *is given by*

$$q = pm\frac{r-1+\gamma}{r}.$$

Remark 5.6. In the case $p = 2$ we can observe also that the result of Theorem 5.5 improves the result of [85]. Once again we have solutions for every $a > 0$, and the summability of the gradients in Lebesgue spaces, i.e., $\frac{2m(r-1+\gamma)}{r}$, is better than the summability $q = \frac{Nm(\theta+1)}{m(1+\theta)+(N-2m)}$ obtained in [85], since $m < \frac{N}{2}\frac{r-2}{r-1+\gamma}$. Note that $q < 2$ because $m < \frac{r}{r-1+\gamma}$.

Remark 5.7. If $p = 2$ and $\gamma \to 0^+$, the result of Theorem 5.2, Theorem 5.5 coincides with regularity results for elliptic equation problems involving Hardy potential (see([4],Theorem 2.1 and Theorem 3.1)).

We organize the work as follows. In Section 2, we introduce an approximation of problem (5.1), we prove the uniform positivity of the approximating solutions. In Section 3, we give the a priori estimates valid for the case of finite and infinite energy solutions. Finally, in Section 4, we prove Theorems 5.2 and 5.5.

2 The approximation scheme

To prove our existence results, we work with an approximation of (5.1). Let $n \in \mathbb{N}$, $f_n(x) := T_n(f)$. Let us consider the approximate problem:

$$\begin{cases} -\operatorname{div}\left(M(x)|\nabla u_n|^{p-2}\nabla u_n\right) + b|u_n|^{r-2}u_n = a\frac{(T_n(|u_n|))^{p-1}}{(|x|^p+\frac{1}{n})} + \frac{f_n(x)}{\left(|u_n|+\frac{1}{n}\right)^{\gamma}} & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.7}$$

Lemma 5.8. *For each integer $n \in \mathbb{N}$, the problem* (5.7) *admits a non-negative solution $W_0^{1,p}(\Omega)$ for all $1 < p < N$.*

Proof. Let $n \in \mathrm{N}$ be fixed and let $v \in L^p(\Omega)$. We define the map

$$\begin{aligned} S : L^p(\Omega) &\to L^p(\Omega) \\ v \quad &\mapsto S(v), \end{aligned}$$

where $w = S(v)$ is the weak solution to the following problem

$$\begin{cases} -\operatorname{div}\left(M(x)|\nabla w|^{p-2}\nabla w\right) + b|w|^{r-2}w = a\frac{(T_n(|w|))^{p-1}}{(|x|^p+\frac{1}{n})} + \frac{f_n(x)}{\left(|v|+\frac{1}{n}\right)^{\gamma}} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

The existence of a solution $w \in W_0^{1,p}(\Omega)$ follows from the classical results of [68]. Let us take w as a test function and by (5.2), we get

$$\int_\Omega M(x)|\nabla w|^{p-2}\cdot\nabla w dx + b\int_\Omega |w|^{r-2}w = \int_\Omega a\frac{(T_n(|w|))^p}{(|x|^p+\frac{1}{n})} + \int_\Omega \frac{f_n(x)\cdot w}{\left(|v|+\frac{1}{n}\right)^{\gamma}}dx,$$

dropping the second positive order term, we obtain

$$\alpha\int_\Omega |\nabla w|^p dx \le \int_\Omega a\frac{(T_n(|w|))^{p-1}}{(|x|^p+\frac{1}{n})} + \int_\Omega \frac{f_n(x)\cdot w}{\left(|v|+\frac{1}{n}\right)^{\gamma}}dx.$$

Therefore, using the Sobolev inequality on the left hand side and the Hölder inequality on the right hand side one has

$$\left[\int_\Omega |w|^{p^*}\right]^{p/p^*} \le C(a,n,\alpha,\gamma)\left(\int_\Omega |w|^{p^*}\right)^{\frac{1}{p^*}},$$

for some constant C independent on v. This implies

$$\|w\|_{L^{p^*}(\Omega)} \le C(a,n,\alpha,\gamma) = R,$$

so that the ball of $L^{p^*}(\Omega)$ of radius R is invariant for S. It is easy to prove, using the Sobolev embedding, that S is both continuous and compact on $L^{p^*}(\Omega)$, so that by Schauder's fixed point theorem there there exists $u_n \in W_0^{1,p}(\Omega)$ such that $u_n = T(u_n)$, i.e., u_n solves

$$\begin{cases} -\operatorname{div}\left(M(x)|\nabla u_n|^{p-2}\nabla u_n\right) + b|u_n|^{r-2}u_n = a\frac{(T_n(|u_n|))^{p-1}}{(|x|^p+\frac{1}{n})} + \frac{f_n(x)}{\left(|u_n|+\frac{1}{n}\right)^{\gamma}} & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, since $\frac{f_n}{\left(|u_n|+\frac{1}{n}\right)^\gamma} \geq 0$, taking $u_n^- = \min(u_n, 0)$ test function in (5.7) and using (5.2), then we get

$$\alpha \int_\Omega |\nabla u_n^-|^p + \int_\Omega |u_n|^{r-2} (u_n^-)^2 = a \int_\Omega \frac{(T_n(|u_n|))^{p-1}}{(|x|^p + \frac{1}{n})} u_n^- + \int_\Omega \frac{f_n}{\left(|u_n| + \frac{1}{n}\right)^\gamma} u_n^- \leq 0,$$

we obtain

$$\alpha \int_\Omega |\nabla u_n^-|^p \leq 0,$$

so that $u_n \geq 0$ almost everywhere in Ω.
The next step consists in the proof that u_n is uniformly bounded from below on the compact subsets of Ω. □

Lemma 5.9. *Let u_n be a solution of* (5.7)*. Then for every subset $\omega \subset\subset \Omega$ there exists a positive constant c_ω, independent on n, such that*

$$u_n(x) \geq c_\omega > 0, \quad \textit{for every } x \in \omega \textit{ and for every } n \in \mathrm{N}.$$

Proof. Since u_n solution of (5.7), then

$$-\operatorname{div}\left(M(x)|\nabla u_n|^{p-2}\nabla u_n\right) + bu_n^{r-1} = a\frac{(T_n(u_n))^{p-1}}{|x|^p + \frac{1}{n}} + \frac{f_n}{\left(u_n + \frac{1}{n}\right)^\gamma},$$

as $a > 0$, then we obtain

$$-\operatorname{div}\left(M(x)|\nabla u_n|^{p-2}\nabla u_n\right) + bu_n^{r-1} \geq \frac{f_n}{\left(u_n + \frac{1}{n}\right)^\gamma},$$

this implies that the sequence u_n is a supersolution to problem

$$\begin{cases} -\operatorname{div}\left(M(x)\,|\nabla v|^{p-2}\,\nabla v\right) + b\,|v|^{r-2}\,v = \frac{f_n}{\left(|v|+\frac{1}{n}\right)^\gamma} & \text{in } \Omega \\ v > 0 & \text{in } \partial\Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Thanks to Lemma 2.2 in [23], $\exists c_\omega > 0$ (independent of n) such that $v \geq c_w$ in $\omega, \forall n \in \mathbb{N}, \forall \omega \subset\subset \Omega$, since $u_n \geq v$, so

$$u_n \geq c_\omega \text{ in } \omega, \forall n \in \mathbb{N}, \forall \omega \subset\subset \Omega.$$

□

3 A priori estimates

In order to prove the existence of solutions for problem (5.1), we first need some a priori estimates on u_n. We start by proving the following lemma

Lemma 5.10. *Let u_n be the solution of problem* (5.7)*, with $0 < \gamma \leq 1$. Assume that f be a nonnegative function in $L^m(\Omega)$ with $\frac{r}{r-1+\gamma} \leq m < \frac{N}{p}\frac{r-p}{r-1+\gamma}$ and suppose that* (5.2) *and* (5.3) *hold. Then, u_n is bounded in $W_0^{1,p}(\Omega) \cap L^{m(r-1+\gamma)}(\Omega)$.*

Proof. Let u_n be a solution of (5.7). We use $u_n^{\lambda+1}$ with $\lambda = (m-1)(r-1)-1+\gamma m$ ($\lambda \geq 0$, since $m \geq \frac{r}{r-1+\gamma}$) as test function in (5.7) and using (5.2), we get

$$\alpha(\lambda+1)\int_\Omega |\nabla u_n|^p\, u_n^\lambda + b\int_\Omega u_n^{\lambda+r} \leq a\int_\Omega \frac{u_n^{\lambda+p}}{|x|^p+\frac{1}{n}} + \int_\Omega f u_n^{\lambda+1-\gamma}, \tag{5.8}$$

dropping the positive first term, then the last inequality becomes

$$b\int_\Omega u_n^{\lambda+r} \leq a\int_\Omega \frac{u_n^{\lambda+p}}{|x|^p+\frac{1}{n}} + \int_\Omega f u_n^{\lambda+1-\gamma}. \tag{5.9}$$

In addition, using the Hölder inequality with exponent m and taking into account that $(\lambda+1-\gamma)m' = \lambda+r$, we arrive at

$$\int_\Omega f u_n^{\lambda+1-\gamma} \leq \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{(\lambda+1-\gamma)m'}\right)^{\frac{1}{m'}} = \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{\lambda+r}\right)^{\frac{1}{m'}}.$$

Under condition, $r > p^*$ we have $\lambda+p < \lambda+r$. Thus, since

$$\left(\frac{\lambda+r}{\lambda+p}\right)' = \frac{\lambda+r}{r-p},$$

we can write

$$\int_\Omega \frac{u_n^{\lambda+p}}{|x|^p+\frac{1}{n}} \leq \left(\int_\Omega u_n^{\lambda+r}\right)^{\frac{\lambda+p}{\lambda+r}}\left(\int_\Omega \frac{1}{|x|^{p\frac{\lambda+r}{r-p}}}\right)^{\frac{r-p}{\lambda+r}}.$$

Since $p\frac{\lambda+r}{r-p} < N$, gives $m < \frac{N}{p}\frac{r-p}{r-1+\gamma}$. Therefore, we deduce that

$$\int_\Omega \frac{u_n^{\lambda+p}}{|x|^p+\frac{1}{n}} \leq C\left(\int_\Omega u_n^{\lambda+r}\right)^{\frac{\lambda+p}{\lambda+r}}.$$

By the fact that $\lambda+r = m(r-1+\gamma)$, removing the positive first term of (5.8) and using the fact that $\lambda+r = m(r-1+\gamma)$, we find that

$$b\int_\Omega u_n^{m(r-1+\gamma)} \leq aC\left(\int_\Omega u_n^{m(r-1+\gamma)}\right)^{\frac{\lambda+p}{\lambda+r}} + \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{m(r-1+\gamma)}\right)^{\frac{1}{m'}},$$

the above estimate implies

$$\int_\Omega u_n^{m(r-1+\gamma)} \leq C. \tag{5.10}$$

By (5.10) and going to back to (5.8) we conclude that

$$\int_\Omega |\nabla u_n|^p\, u_n^\lambda \leq C.$$

Since $\lambda \geq 0$, we obtain

$$\int_{\{u_n\geq 1\}} |\nabla u_n|^p \leq C. \tag{5.11}$$

Observe now that, since $\{u_n\}$ is bounded in $L^{m(r-1+\gamma)}(\Omega)$, and since $\frac{1}{|x|^p}$ belongs to $L^\rho(\Omega)$ for every $\rho < \frac{N}{p}$, the sequence $\left\{(T_n(u_n))^{p-1}/\left(|x|^p + \frac{1}{n}\right)\right\}$ is bounded in $L^s(\Omega)$ for every s such that

$$\frac{1}{s} > \frac{p}{N} + \frac{p-1}{m(r-1+\gamma)},$$

that, implies

$$s < \frac{Nm(r-1+\gamma)}{N(p-1)+pm(r-1+\gamma)}.$$

Taking into consideration that

$$\frac{Nm(r-1+\gamma)}{N(p-1)+pm(r-1+\gamma)} > 1,$$

is equivalent to

$$m > \frac{N}{N-p}\frac{p-1}{r-1+\gamma},$$

which is true, since $\frac{N}{N-p}\frac{p-1}{r-1+\gamma} < 1$, by the assumption $r > p^*$. Therefore,

$$\text{the sequence } \left\{(T_n(u_n))^{p-1}/\left(|x|^p + \frac{1}{n}\right)\right\} \text{ is bounded in } L^1(\Omega). \tag{5.12}$$

On other choosing $T_k(u_n)$ as test function in (5.7), we obtain, dropping a positive term,

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p \leq C, \tag{5.13}$$

which, together with (5.11), yields that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ for every $a > 0$.

We now deal with the case of f belonging to $L^m(\Omega), 1 < m < \frac{r}{r-1+\gamma}$. In this case, one cannot expect to have solutions in $W_0^{1,p}(\Omega)$, but in a larger space □

Lemma 5.11. *Let u_n be the solution of problem (5.7), with $0 < \gamma \leq 1$. Assume that f be a nonnegative function in $L^m(\Omega)$ with $1 < m < \frac{r}{r-1+\gamma}$ and suppose that (5.2) and (5.3) hold. Then, u_n is bounded in $W_0^{1,q}(\Omega) \cap L^{m(r-1+\gamma)}(\Omega)$, where*

$$q = pm\frac{r-1+\gamma}{r}.$$

Proof. As in the proof of Lemma 6.10, we consider the approximate problems (5.7). Let $\varepsilon > 0$, since $1 < m < \frac{r}{r-1+\gamma}$, we have $0 < \mu := 1 - \gamma m - (m-1)(r-1) < 1$ and define

$$v = \frac{u_n}{(u_n+\varepsilon)^\mu}, \tag{5.14}$$

then, we can write

$$\nabla v = \frac{\nabla u_n}{(u_n+\varepsilon)^\mu} - \mu\frac{\nabla u_n u_n}{(u_n+\varepsilon)^{\mu+1}} = \frac{(1-\mu)u_n+\varepsilon}{(u_n+\varepsilon)^{\mu+1}}\nabla u_n,$$

by using (5.2), we deduce that

$$M(x)|\nabla u_n|^{p-2}\nabla u_n \nabla v \geq \alpha\frac{(1-\mu)u_n+\varepsilon}{(u_n+\varepsilon)^{\mu+1}}|\nabla u_n|^p \geq \alpha(1-\mu)\frac{|\nabla u_n|^p}{(u_n+\varepsilon)^\mu}. \tag{5.15}$$

Now testing (5.7) by(5.14) and observe that (5.15), we obtain

$$\alpha(1-\mu)\int_{\Omega}\frac{|\nabla u_n|^p}{(u_n+\varepsilon)^{\mu}}+b\int_{\Omega}\frac{u_n^r}{(u_n+\varepsilon)^{\mu}}\leq a\int_{\Omega}\frac{u_n^{p-\mu}}{|x|^p+\frac{1}{n}}+\int_{\Omega}fu_n^{1-\mu-\gamma}. \tag{5.16}$$

Dropping the positive first term, and then letting ε tend to zero, we thus have

$$b\int_{\Omega}u_n^{r-\mu}\leq a\int_{\Omega}\frac{u_n^{p-\mu}}{|x|^p+\frac{1}{n}}+\int_{\Omega}fu_n^{1-\mu-\gamma},$$

which is nothing but (5.9), since $\mu=-\lambda$. Starting from this inequality, and working as in the proof of Lemma 6.10, we prove the boundedness of $\{u_n\}$ in $L^{m(r-1+\gamma)}(\Omega)$. Using this fact and (5.16), we obtain

$$\int_{\Omega}\frac{|\nabla u_n|^p}{(u_n+\varepsilon)^{\mu}}\leq C. \tag{5.17}$$

Let $q<p$, applying Hölder inequality and by (5.17), we find

$$\int_{\Omega}|\nabla u_n|^q=\int_{\Omega}\frac{|\nabla u_n|^q}{(u_n+\varepsilon)^{\frac{\mu q}{p}}}(u_n+\varepsilon)^{\frac{\mu q}{p}}\leq C\left(\int_{\Omega}(u_n+\varepsilon)^{\frac{\mu q}{p-q}}\right)^{\frac{p-q}{p}}.$$

Finally we choose q such that $\frac{\mu q}{p-q}=m(r-1+\gamma)$, it is easy to verify that $q=pm\frac{r-1+\gamma}{r}$. Therefore, $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$, with $q=pm\frac{r-1+\gamma}{r}$. □

4 Proofs of Theorems 5.2 and 5.5

We are ready to prove the existence of at least a solution of (5.1) in the sense of Definition 7.1
Proof of Theorem 5.2. Thanks to Lemma 5.10, the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Therefore, there exists a function $u\in W_0^{1,p}(\Omega)$ such that (up to a subsequence)

$$\begin{cases} u_n\rightharpoonup u & \text{in } W_0^{1,p}(\Omega)\\ u_n\to u & \text{a.e. in } \Omega.\end{cases} \tag{5.18}$$

We use the fact that, thanks to (5.12), (5.10)) and Lemma 5.9, we have that the right-hand side

$$a\frac{(T_n(|u_n|))^{p-1}}{(|x|^p+\frac{1}{n})}+\frac{f_n(x)}{(|u_n|+\frac{1}{n})^{\gamma}}-b|u_n|^{r-2}u_n \text{ is bounded in } L^1_{\text{loc}}(\Omega).$$

Therefore, thanks to Remark 2.2 after Theorem 2.1 of [26], we have that

$$\nabla u_n\to\nabla u \text{ a.e. in } \Omega. \tag{5.19}$$

For the first term, by (5.19) we have that

$$M(x)\,|\nabla u_n|^{p-2}\,\nabla u_n\to M(x)|\nabla u|^{p-2}\nabla u \quad \text{a.e. in }\Omega,$$

furthermore $M(x)\left|\nabla u_n\right|^{p-2}\nabla u_n$ is majorette by $\beta\left|\nabla u_n\right|^{p-1}$ and by Vitali's Theorem, we have

$$\lim_{n\to\infty}\int_\Omega M(x)\left|\nabla u_n\right|^{p-2}\nabla u_n\cdot\nabla\varphi=\int_\Omega M(x)|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi.$$

On other hand, by (5.18) we can see also that, the sequence $\{u_n\}$ converges to u strongly in $L^p(\Omega)$ and almost everywhere in Ω. As for the sequence $\left\{T_n(u_n)^{p-1}/\left(|x|^p+\frac{1}{n}\right)\right\}$, since it is bounded in $L^s(\Omega)$ for some $s>1$, it strongly converges to $\frac{u^{p-1}}{|x|^p}$ in $L^1(\Omega)$. Since $\{u_n\}$ is bounded in $L^{m(r-1+\gamma)}(\Omega)$ and $m>1$, we also have that

$$\left|u_n\right|^{r-2}u_n \text{ strongly converges to } |u|^{r-2}u \text{ in } L^1(\Omega). \tag{5.20}$$

Next, let $\omega=\{\varphi\neq 0\}$ the by Lemma 5.9, one has, for every $\varphi\in C_c^1(\Omega)$

$$\left|\frac{f_n\varphi}{\left(u_n+\frac{1}{n}\right)^\gamma}\right|\leqslant\frac{\|\varphi\|_{L^\infty(\Omega)}}{c_\omega^\gamma}f,$$

then from the later estimate, (5.18) and applying Lebesgue Theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega\frac{f_n\varphi}{\left(u_n+\frac{1}{n}\right)^\gamma}=\int_\Omega\frac{f\varphi}{u^\gamma}. \tag{5.21}$$

Therefore, if φ belongs to $C_c^1(\Omega)$, we can pass to the limit in the identities

$$\int_\Omega M(x)|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi+b\int_\Omega\left|u_n\right|^{r-2}u_n\varphi$$

$$=a\int_\Omega\frac{(T_n(u_n))^{p-1}}{|x|^p+\frac{1}{n}}\varphi+\int_\Omega\frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi.$$

Hence, we conclude that the solution u satisfes the conditions (5.5) and (5.6) of Definition 7.1, so that the proof of Theorem 5.2 is now completed.

Proof of Theorem 5.5. In virtue of the Lemma 5.11, the sequence of approximated solutions u_n is bounded in $W_0^{1,q}(\Omega)$, with $q=pm\frac{r-1+\gamma}{r}$, so that it weakly converges (up to sub-sequences) to a function u in the same space. Observing that $p-1<q$. Therefore, we can obtain a solution passing to the limit, namely arguing exactly as in Theorem 5.2.

Chapter 6

Existence and Regularity of solutions to a singular elliptic equation with natural growth in the gradient

1 Introduction

In this chapter we investigate the interaction between two regularizing terms in the following nonlinear elliptic equation

$$\begin{cases} -\operatorname{div} a(x,\nabla u) + \mu|u|^{p-1}u = b(x)\frac{|\nabla u|^q}{u^\theta} + \frac{f(x)}{u^\gamma} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$, f is a nonnegative $L^m(\Omega)$ function with $m \geq 1$ and, given a real number p such that $2 \leq p < N$, we have that $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that the following holds: there exist $\alpha, \beta \in \mathbb{R}^+$ such that

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) > 0 \quad for a.e.\ x \in \Omega\ and\ \forall \xi, \eta \in \mathbb{R}^N\ s.t.\ \xi \neq \eta \tag{6.2}$$

$$a(x,\xi) \cdot \xi \geqslant \alpha|\xi|^p \tag{6.3}$$

for a.e. $x \in \Omega$ and $\forall \xi \in \mathbb{R}^N$

$$|a(x,\xi)| \leqslant \beta|\xi|^{p-1} \tag{6.4}$$

for a.e. $x \in \Omega$ and $\forall \xi \in \mathbb{R}^N$ and we assume that

$$0 < \gamma \leq 1, \tag{6.5}$$

$$0 \leq b(x) \in L^\infty(\Omega), \tag{6.6}$$

$$0 < \theta \leq 1, \tag{6.7}$$

and

$$0 \leq \mu, \ \ p - 1 \leq q < \frac{p(p+\beta)}{p+1}. \tag{6.8}$$

The assumptions on the function a imply that the differential operator A acting between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ and defined by

$$A(u) = -\operatorname{div}(a(x, \nabla u))$$

is coercive, monotone, surjective and satisfies the maximum principle. The simplest case is the p-Laplacian, which corresponds to the choice $a(x,\xi) = |\xi|^{p-2}\xi$.

In section 2 we construct an approximate problem of (1), the existence of weak solution of the last one is proved by Schauder's fixed point Theorem. In section 3.1 is devoted to prove to the existence and regularity results both in case $q = p-1$,$\mu = 0$ and $f \in L^m(\Omega)$with $m > 1$. In the last section we deal with the case $p-1 < q < p$, $\mu > 0$ and $f \in L^1(\Omega)$, we prove the existence of solution of problem (6.1). Note that the presence of the lower order term $\mu|u|^{p-1}u$ is crucial in the sense that it guarantees the existence of solution when the data f belongs only in $L^1(\Omega)$.

2 A priori estimates

We will prove the existence of solutions of problem (6.1) by a standard approximation procedure which avoids singularities. To this end, we consider for $n \in \mathbb{N}$ the following approximate problem

$$\begin{cases} -\operatorname{div}\left(a\left(x, \nabla u_n\right)\right) + \mu|u_n|^{p-1}u_n & \\ = b(x)\frac{|\nabla u_n|^q}{\left(1+\frac{1}{n}|\nabla u_n|^q\right)\left(\frac{1}{n}+u_n\right)^\theta} + \frac{f_n}{(\frac{1}{n}+u_n)^\gamma} & \text{in}\Omega \\ u_n \geq 0 & \text{in}\Omega \\ u_n = 0 & \text{on}\partial\Omega \end{cases} \tag{6.9}$$

where $f_n = T_n(f)$. The weak formulation of (6.9) is

$$\int_\Omega a(x, \nabla u_n)\nabla\varphi + \int_\Omega \mu|u_n|^{p-1}u_n\varphi = \int_\Omega b(x)\frac{|\nabla u_n|^q}{\left(1+\frac{1}{n}|\nabla u_n|^q\right)\left(\frac{1}{n}+u_n\right)^\theta}\varphi \tag{6.10}$$

$$+ \int_\Omega \frac{f_n}{(\frac{1}{n}+u_n)^\gamma}\varphi, \quad \forall\varphi \in C_c^1(\Omega).$$

Now, we briefly sketch how to deduce the existence of a nonnegative solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of problem (6.9) . Firstly, let us observe that it follows from [22] that there exists a nonnegative solution to

$$\begin{cases} -\operatorname{div}\left(a\left(x, \nabla w\right)\right) + \mu|w|^{p-1}w & \\ = b(x)\frac{|\nabla w|^q}{\left(1+\frac{1}{n}|\nabla w|^q\right)\left(\frac{1}{n}+w\right)^\theta} + \frac{f_n}{(\frac{1}{n}+v)^\gamma} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{6.11}$$

for any nonnegative v belonging to $L^p(\Omega)$ and such that $\|w\|_{L^\infty(\Omega)} \leq c_n$ for some positive constant c_n which does not depend on v. Now, through an application of the Schauder theorem, one can show that the application $T : L^p(\Omega) \mapsto L^p(\Omega)$ such that $T(v) = w$ admits a fixed point. Hence, let us show an invariant ball for T on which the application is both continuous and compact. Indeed, taking w as a test function in (6.11), one has

$$\alpha\int_\Omega |\nabla w|^p \leq c_p||b||_{L^\infty(\Omega)}c(n,\theta,\gamma)|\Omega|^{p'}\big(\int_\Omega |\nabla w|^p dx\big)^{\frac{1}{p}} \tag{6.12}$$

Then, an application of the Poincaré inequality gives that

$$\|w\|_{L^p(\Omega)} \leq \left(\frac{c_p ||b||_{L^\infty(\Omega)} c(n,\theta,\gamma) |\Omega|^{p'}}{\alpha} \right)^{p'} = r$$

where c_p is the Poincaré constant. Therefore the ball of the radius r is invariant for T. Now, let v_k a sequence in the ball of radius r which converges to v in $L^p(\Omega)$ as $k \to \infty$ and let $w_k = T(v_k)$. Then, in order to show the continuity of T, one needs to prove that w_k converges to $w = T(v)$ in $L^p(\Omega)$ as $k \to \infty$. To this aim, let us observe that an application of (6.12) gives that w_k is bounded in $W_0^{1,p}(\Omega)$ with respect to k; moreover, it follows from Lemma 2 of [21] that w_k is also bounded in $L^\infty(\Omega)$ with respect to k. Now, under the above assumptions, Lemma 4 of [21] gives that, up to subsequences, w_k converges to a function w in $W_0^{1,p}(\Omega)$. This is sufficient to pass to the limit as $k \to \infty$ the weak formulation of the equation solved by w_k in order to deduce that $w = T(v)$. For the compactness, it is sufficient to underline that if v_k is bounded in $L^p(\Omega)$ then one can recover that w_k is bounded in $W_0^{1,p}(\Omega)$ with respect to k thanks to (6.12); this implies that, up to subsequences, it converges to a function in $L^p(\Omega)$. Then, we are in position to apply the Schauder theorem in order to deduce the existence of u_n. Moreover, due to the fact that the right-hand side is positive and that the operator A satisfies the maximum principle, we can conclude that $u_n \geq 0$.

Lemma 6.1. *let $u_n$be a solution to* (6.9) *then for every $\omega \subset\subset \Omega$ there exists a constant $c_\omega > 0$ which does not depend on n and such that*

$$u_n \geq c_\omega \ \text{a.e. in } \omega \tag{6.13}$$

Proof. Let $\mu \geq 0$, $0 \leq f_n$ and let us now consider $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ solution to

$$\begin{cases} -\operatorname{div}(a(x,\nabla v)) + \mu |v|^{p-1} v = \frac{f}{v^\gamma} & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

and we observe that by Lemma 2.2 of [23] one has that for any $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ such that

$$v \geq c_\omega \text{ a.e. in } \omega \tag{6.14}$$

Now, let us take $(v - u_n)^+$ as a test function in the difference of weak formulations solved by v and u_n; it yields

$$\begin{aligned} \textstyle\int_\Omega (a(x,\nabla v) - a(x,\nabla u_n)) \cdot \nabla (v-u_n)^+ + \int_\Omega (|v|^{p-1}v - |u_n|^{p-1}u_n)(v-u_n)^+ \\ \leq \textstyle\int_\Omega f_n \frac{\left(u_n+\frac{1}{n}\right)^\gamma - \left(v+\frac{1}{n}\right)^\gamma}{\left(u_n+\frac{1}{n}\right)^\gamma \left(v+\frac{1}{n}\right)^\gamma} (v-u_n)^+ \\ - \textstyle\int_\Omega \frac{|\nabla u_n|^q}{(\frac{1}{n}+u_n)^\gamma (1+\frac{1}{n}|\nabla u_n|^q)} (v-u_n)^+ \leq 0 \end{aligned}$$

which, since the second term on the left hand side is nonnegative, implies

$$\int_\Omega (a(x,\nabla v) - a(x,\nabla u_n)) \cdot \nabla (v-u_n)^+ \leq 0$$

and this gives that $u_n \geq v$ almost everywhere in Ω. Consequently, the desired conclusion is a direct consequence of (6.14). □

3 The main results and their proof

3.1 The case $q = p-1$, $\mu = 0$ and $f \in L^m(\Omega)$ with $m > 1$

In this subsection, we want to analyse the case $0 < \gamma \leq 1$, $\mu = 0$, $0 \leq f \in L^m(\Omega)(m > 1)$. We first give the definition of a distributional solution to problem (6.1)

Definition 6.2. Let f be a nonnegative (not identically zero) function in $L^m(\Omega)$ function, with $m > 1$. A positive and measurable function u is a distributional solution to problem (6.1) if $u \in W_0^{1,1}(\Omega)$, if $|a(x,\nabla u)|$, $\frac{|\nabla u|^{p-1}}{u^\theta} \in L^1_{\text{loc}}(\Omega)$,

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0 : u \geq c_\omega \text{ in } \omega \tag{6.15}$$

and if

$$\int_\Omega a(x,\nabla u)\nabla\varphi = \int_\Omega b(x)\frac{|\nabla u|^{p-1}}{u^\theta}\varphi + \int_\Omega \frac{f(x)}{u^\gamma}\varphi, \quad \forall \varphi \in C^1_c(\Omega). \tag{6.16}$$

The main results of this subsection are as follows:

Theorem 6.3. *Assume* (6.3),(6.4) *and* (6.5). *Then, if* $m_1 = \frac{mN(p-1+\gamma)}{N-pm}$ *and* $\tilde{m} = \frac{Nm(p-1+\gamma)}{N+m(1-\gamma)}$ *there exists a distributional solution* u *of* (7.4)

$$u \in \begin{cases} L^{m_1}(\Omega) & \text{if } 1 < m < N/p, \\ L^\infty(\Omega) & \text{if } m > N/p, \end{cases}$$

$$|\nabla u| \in \begin{cases} L^{\tilde{m}}(\Omega) & \text{if } 1 < m < pN/[N(p-1+\gamma)+p(1-\gamma)] \\ L^p(\Omega) & \text{if } m \geq pN/[N(p-1+\gamma)+p(1-\gamma)] \end{cases}$$

and if $r = \frac{\tilde{m}}{p-1}$, *we have*

$$\frac{|\nabla u|^{p-1}}{u^\theta} \in \begin{cases} L^r_{\text{loc}}(\Omega) & \text{if } 1 < m < pN/[N(p-1+\gamma)+p(1-\gamma)] \\ L^{p'}_{\text{loc}}(\Omega) & \text{if } m \geq pN/[N(p-1+\gamma)+p(1-\gamma)]. \end{cases}$$

Furthermore, if $0 < \theta < (p-1)(1-\gamma)/p$ *and* $r = Nm(p-1+\gamma)/[N(p-1-\theta) - m[(p-1)(1-\gamma)-p\theta]$, *then*

$$\frac{|\nabla u|^{p-1}}{u^\theta} \in \begin{cases} L^r(\Omega) \ \text{if } 1 < m < \frac{p(p-1)N(1-\theta)}{N(p-1)(p-1+\gamma)+p(p-1)(1-\gamma)-p^2\theta} \\ L^{p'}(\Omega) \ \text{if } m \geq \frac{p(p-1)N(1-\theta)}{N(p-1)(p-1+\gamma)+p(p-1)(1-\gamma)-p^2\theta}. \end{cases}$$

Remark 3.1. In the case where the lower order term does not exist (i.e., $b(x) = 0$), the results of previous theorem coincide with regularity results obtained in ([41, Theorem 4.4]).

Remark 3.2. If $p = 2$ and $\gamma = 0$; the result of Theorem 6.3 coincides with regularity results of [24].

Now, we can prove the following existence and regularity result

Lemma 6.4. *Let* u_n *be a solution of problem* (6.9) *and suppose that* (6.3)–(6.7) *hold true, let* f *be a nonnegative function in* $L^m(\Omega)$, *with* $1 < m < N/p$, $\sigma = \min(\tilde{m}, p)$, $r = \frac{\tilde{m}}{p-1}$. *Then we have*

- *The sequence* $\{u_n\}$ *is bounded in* $L^{m_1}(\Omega) \cap W_0^{1,\sigma}(\Omega)$. (6.17)
- *The sequence* $\left\{\frac{|\nabla u_n|^{p-1}}{u_n^\theta}\right\}$ *is bounded in* $L^r_{\text{loc}}(\Omega) \cap L^{p'}_{\text{loc}}(\Omega)$, (6.18)

with $\tilde{m}$ *and* m_1 *are defined in the Theorem 6.3.*

Proof. Here, and in the following, we will denote by C the generic constant which is independent of $n \in \mathbb{N}$. Define, for $k > 0$ and $s > 0$

$$\eta_k(s) = \frac{1}{k} T_k\left(G_1(s)\right) = \begin{cases} 0, & \text{if } 0 \leq s < 1 \\ \frac{s-1}{k}, & \text{if } 1 \leq s < 1+k \\ 1, & \text{if } s \geq 1+k. \end{cases}$$

We choose $v_n = u_n^{p\lambda-(p-1)} \eta_k(u_n)$ as test function in the weak formulation of (6.10) (this choice is possible since every u_n belong to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$). Noting that since $f_n \leq f$ and let $\lambda > 1/p'$, dropping a first non negative term, we obtain

$$\begin{aligned} &\alpha(p\lambda-(p-1)) \int_\Omega |\nabla u_n|^p u_n^{p\lambda-p} \eta_k(u_n) \\ &\leq \int_\Omega b(x) \frac{|\nabla u_n|^{p-1} u_n^{p\lambda-(p-1)}}{\left(1+\frac{1}{n}|\nabla u_n|^{p-1}\right)\left(\frac{1}{n}+u_n\right)^\theta} \eta_k(u_n) + \int_\Omega f_n u_n^{p\lambda-(p-1)-\gamma} \eta_k(u_n) \\ &\leq \|b\|_{L^\infty(\Omega)} \int_\Omega |\nabla u_n|^{p-1} u_n^{(p-1)(\lambda-1)} \eta_k(u_n) u_n^{\lambda-\theta} + \int_\Omega f u_n^{p\lambda-(p-1)-\gamma} \eta_k(u_n). \end{aligned}$$

Let $\varepsilon > 0$ be such that $0 < \varepsilon\|b\|_{L^\infty(\Omega)} < \alpha(p\lambda-(p-1))$. By Young inequality with ε, we deduce that

$$\begin{aligned} &\left[\alpha(p\lambda-(p-1)) - \|b\|_{L^\infty(\Omega)}\right] \int_\Omega |\nabla u_n|^p u_n^{p\lambda-p} \eta_k(u_n) \\ &\leq C\|b\|_{L^\infty(\Omega)}^p \int_\Omega \eta_k(u_n) u_n^{p(\lambda-\theta)} + \int_\Omega f u_n^{p\lambda-(p-1)-\gamma} \eta_k(u_n). \end{aligned}$$

Letting k tend to zero, and Lebesgue Theorem in the right-hand side using and Fatou Lemma in the left-hand side, we get

$$C \int_{\{u_n \geq 1\}} |\nabla u_n|^p u_n^{p\lambda-p} \leq \int_{\{u_n \geq 1\}} u_n^{p(\lambda-\theta)} + \int_{\{u_n \geq 1\}} f u_n^{p\lambda-(p-1)-\gamma}. \tag{6.19}$$

We now remark that for every $t \geq 1$ and $\delta > 0$, there exists $C_\delta > 0$ such that

$$t^{p(\lambda-\theta)} \leq \delta t^{p\lambda} + C_\delta. \tag{6.20}$$

The inequality is trivially true if $\theta \geq \lambda$, while is a consequence of Young inequality if $\lambda > \theta$. Recall that the estimate (6.19), we have

$$\int_{\{u_n \geq 1\}} |\nabla u_n|^p u_n^{p\lambda-p} \leq \delta \int_{\{u_n \geq 1\}} u_n^{p\lambda} + |\Omega| C_\delta + \int_{\{u_n \geq 1\}} f u_n^{p\lambda-(p-1)-\gamma}. \tag{6.21}$$

Taking into account that $0 \leq u_n = T_1(u_n) + G_1(u_n) \leq 1 + G_1(u_n)$, and using Poincaré inequality, we conclude that

$$\begin{aligned} C \int_\Omega \left|\nabla G_1(u_n)^\lambda\right|^p &\leq \delta \int_\Omega G_1(u_n)^{p\lambda} + C + \int_\Omega f G_1(u_n)^{p\lambda-(p-1)-\gamma} \\ &\leq \frac{\delta}{\lambda_1} \int_\Omega \left|\nabla G_1(u_n)^\lambda\right|^p + C + \int_\Omega f G_1(u_n)^{p\lambda-(p-1)-\gamma}, \end{aligned}$$

where λ_1 is the Poincaré constant for Ω (i.e. the first eigenvalue of the Laplacian with homogeneous Dirichlet boundary conditions). Choosing δ small enough, we thus have

$$\int_\Omega |\nabla G_1(u_n)^\lambda|^p \le C + C\int_\Omega f G_1(u_n)^{p\lambda-(p-1)-\gamma}.$$

Following the same technique as in [22], choosing $\lambda = \frac{m_1}{p^*}$, it is easy to see that if $\lambda = \frac{m(N-p)[p-1+\gamma]}{p(N-pm)} > \frac{(N-p)[p-1+\gamma]}{p(N-p)} = \frac{p-1+\gamma}{p}$ if only if $m > 1$. Note that with such a choice, we have that $\lambda p^* = m_1$, and $(p\lambda-(p-1)-\gamma)m' = \lambda p^* = m_1 = \frac{Nm[p-1+\gamma]}{N-pm}$. Therefore, using Sobolev and Hölder inequalities, we get

$$\begin{aligned}\mathcal{S}\left(\int_\Omega G_1(u_n)^{m_1}\right)^{\frac{p}{p^*}} &\le \int_\Omega \left|\nabla G_1(u_n)^\lambda\right|^p \le C + C\int_\Omega f G_1(u_n)^{p\lambda-(p-1)-\gamma}\\ &\le C + C\|f\|_{L^m(\Omega)}\left(\int_\Omega G_1(u_n)^{m_1}\right)^{\frac{1}{m'}},\end{aligned}$$

where $\mathcal{S}$ is the Sobolev constant, thanks to the assumption $m < N/p$, we have $p/p^* > 1/m'$, putting to gather all the previous estimates we conclude that

$$\|G_1(u_n)\|_{L^{m_1}(\Omega)} \le C\|f\|_{L^m(\Omega)}. \tag{6.22}$$

Note that from the boundedness of $\{G_1(u_n)\}$ in $L^{m_1}(\Omega)$ it trivially follows the boundedness of $\{u_n\}$ in $L^{m_1}(\Omega)$ since, as before, $0 \le u_n \le 1 + G_1(u_n)$.
Now we point out that $m \ge \frac{pN}{N(p-1)+p(1-\gamma)+\gamma N}$, since $\lambda \ge 1$. Therefore from (6.21) and (6.22) (note that the right-hand side is bounded), we have that

$$\int_\Omega |\nabla G_1(u_n)|^p \le \int_{\{u_n\ge 1\}} |\nabla u_n|^p u_n^{p\lambda-p} \le C,$$

we deduce that the sequence $\{G_1(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$. If on the other hand $1 < m < \frac{pN}{N(p-1)+p(1-\gamma)+\gamma N}$, then $\lambda < 1$ and we have to proceed differently. Let now σ be such that the use of by Hölder inequality, $\sigma < p$ we obtain

$$\begin{aligned}\int_\Omega |\nabla G_1(u_n)|^\sigma &= \int_\Omega \frac{|\nabla G_1(u_n)|^\sigma}{u_n^{\sigma(1-\lambda)}} u_n^{\sigma(1-\lambda)}\\ &\le \left(\int_{\{u_n\ge 1\}} |\nabla u_n|^p u_n^{p\lambda-p}\right)^{\frac{\sigma}{p}} \left(\int_{\{u_n\ge 1\}} u_n^{\frac{p\sigma(1-\lambda)}{p-\sigma}}\right)^{\frac{p-\sigma}{p}}.\end{aligned}$$

Imposing $\sigma = \frac{Nm(p+\gamma-1)}{N-m(1-\gamma)} (= \tilde{m})$, we obtain $\frac{p\sigma(1-\lambda)}{p-\sigma} = m_1$, so that the above inequality becomes, thanks to (6.21) and (6.22)

$$\int_\Omega |\nabla G_1(u_n)|^{\tilde{m}} \le C.$$

Summing up, we have therefore proved that the sequence:

$$\{G_1(u_n)\} \text{ is bounded in } L^{m_1}(\Omega)\cap W_0^{1,\sigma}(\Omega), \sigma = \min(\tilde{m}, p). \tag{6.23}$$

On the other hand, taking $T_1(u_n)$ as test function in (6.9) , we have

$$\begin{aligned}\alpha \int_\Omega |\nabla T_1(u_n)|^p \leq& \|b\|_{L^\infty(\Omega)} \int_\Omega \frac{|\nabla T_1(u_n)|^{p-1}}{\left(\frac{1}{n}+u_n\right)^\theta} T_1(u_n) \\ &+ \|b\|_{L^\infty(\Omega)} \int_\Omega |\nabla G_1(u_n)|^{p-1} + \int_\Omega f \\ &\leq \|b\|_{L^\infty(\Omega)} \int_\Omega |\nabla T_1(u_n)|^{p-1} \\ &+ \|b\|_{L^\infty(\Omega)} \int_\Omega |\nabla G_1(u_n)|^{p-1} + \int_\Omega f,\end{aligned}$$

which implies (thanks to (6.23)) that the sequence $\{T_1(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$. This estimate and the estimate (6.23) give (6.17). First case: The proof of (6.18) is then a simple consequence of (6.13) and (6.17), if $w \subset\subset \Omega$, then

$$\int_w \left(\frac{|\nabla u_n|^{p-1}}{u_n^\theta}\right)^{p'} \leq \frac{1}{c_w^{p'\theta}} \int_\Omega |\nabla u_n|^p \leq C \tag{6.24}$$

In the second case, we take $r = \frac{\tilde{m}}{p-1}$, then by (6.13) and (6.17), we have

$$\int_w \left(\frac{|\nabla u_n|^{p-1}}{u_n^\theta}\right)^{r} \leq \frac{1}{c_w^{r\theta}} \int_\Omega |\nabla u_n|^{\tilde{m}} \leq C \tag{6.25}$$

Using (6.24) and (6.25), we deduce that (6.18) holds true. □

Lemma 6.5. *Let u_n be a solution of* (6.9) *under assumptions*(6.3)–(6.7) *and let f be a nonnegative function in $L^m(\Omega)$. Then, if $m > N/p$*

$$\text{The sequence}\, \{u_n\} \text{ is bounded in}\, L^\infty(\Omega) \cap W_0^{1,p}(\Omega) \tag{6.26}$$

$$\text{The sequence} \left\{\frac{|\nabla u_n|^{p-1}}{u_n^\theta}\right\} \text{ is bounded in}\, L^{p'}_{\text{loc}}(\Omega). \tag{6.27}$$

Proof. We take $v_n = G_k(u_n)$ as test function in (6.9). We obtain, using (6.3),(6.4) and (6.5)

$$\begin{aligned}\alpha \int_{\{u_n \geq k\}} |\nabla u_n|^p &\leq \|b\|_{L^\infty(\Omega)} \int_{\{u_n \geq k\}} |\nabla u_n|^{p-1} \frac{G_k(u_n)}{u_n^\theta} + \int_{\{u_n \geq k\}} \frac{fG_k(u_n)}{(u_n+\frac{1}{n})^\gamma} \\ &\leq \tfrac{1}{k^\theta}\|b\|_{L^\infty(\Omega)} \int_{\{u_n \geq k\}} |\nabla u_n|^{p-1} G_k(u_n) + \int_{\{u_n \geq k\}} \frac{f(x)G_k(u_n)}{(u_n+\frac{1}{n})^\gamma}.\end{aligned}$$

Noting that $u_n + \frac{1}{n} \geq k \geq 1$ on the set $A_{n,k}$, where $G_k(u_n)$, we have

$$\alpha \int_{\{u_n \geq k\}} |\nabla u_n|^p \leq \frac{1}{k^\theta}\|b\|_{L^\infty(\Omega)} \int_{\{u_n \geq k\}} |\nabla u_n|^{p-1} G_k(u_n) + \int_{\{u_n \geq k\}} fG_k(u_n).$$

and by Young and Poincaré inequalities, we have that

$$\begin{aligned}\int_{\{u_n\geq k\}} |\nabla u_n|^{p-1} G_k(u_n) &\leq \frac{1}{p'}\int_{\{u_n\geq k\}} |\nabla u_n|^p + \frac{1}{p}\int_{\{u_n\geq k\}} G_k(u_n)^p \\ &\leq \frac{1+\lambda_1(p-1)}{p\lambda_1}\int_{\{u_n\geq k\}} |\nabla u_n|^p .\end{aligned}$$

Therefore,

$$\left(\alpha - \frac{1}{k^\theta}\frac{\|b\|_{L^\infty(\Omega)}^{(1+\lambda_1(p-1))}}{p\lambda_1}\right)\int_{\{u_n\geq k\}} |\nabla u_n|^p \leq \int_{\{u_n\geq k\}} fG_k(u_n) .$$

Next, we can take $k > k_0$, with

$$k_0 = \left(\frac{\|b\|_{L^\infty(\Omega)}(1+\lambda_1(p-1))}{\alpha\lambda_1}\right)^{\frac{1}{\theta}}, \tag{6.28}$$

we have

$$\frac{\alpha}{p'}\int_{\{u_n\geq k\}} |\nabla u_n|^p \leq \int_{\{u_n\geq k\}} fG_k(u_n) .$$

From this point outwards we can proceed as in the proof of [25, Theorem 1.1], to prove that the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$, as desired and the proof of (6.27) is essentially the same technique used in (6.24). □

If $0 < \theta < (1-\gamma)/p'$, the estimates on the right-hand side $\frac{|\nabla u_n|^{p-1}}{u_n^\theta}$ are not only local but also global.

Lemma 6.6. *Let u_n be a solution of* (6.9), *let us assume that* (6.3)–(6.6) *and* $0 < \theta < (1-\gamma)/p'$, *hold true and that f be a nonnegative function in $L^m(\Omega)$, with*

$$m \geq \frac{p(p-1)N(1-\theta)}{N(p-1)(p-1+\gamma) + p(p-1)(1-\gamma) - p^2\theta} \tag{6.29}$$

then,

$$\textit{The sequence} \left\{\frac{|\nabla u_n|^{p-1}}{u_n^\theta}\right\} \textit{ is bounded in } L^{p'}(\Omega). \tag{6.30}$$

Proof. We fix $\lambda > (p-1+\gamma)/p$, let $0 < \varepsilon < 1/n$, and choose $v_n = (u_n+\varepsilon)^{p\lambda-(p-1)} - \varepsilon^{p\lambda-(p-1)}$ as test function in (6.9) this choice is possible since every u_n belong to $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. We obtain, dropping some negative terms

$$\begin{aligned}&\alpha(p\lambda-(p-1))\int_\Omega |\nabla u_n|^p (u_n+\varepsilon)^{p\lambda-p} \\ &\leq \int_\Omega b(x)\frac{|\nabla u_n|^{p-1}(u_n+\varepsilon)^{p\lambda-(p-1)}}{\left(1+\frac{1}{n}|\nabla u_n|^{p-1}\right)\left(\frac{1}{n}+u_n\right)^\theta} + \int_\Omega f_n (u_n+\varepsilon)^{p\lambda-(p-1)-\gamma} \\ &\leq \|b\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_n|^{p-1}(u_n+\varepsilon)^{(p-1)(\lambda-1)+(\lambda-\theta)} + \int_\Omega f(u_n+\varepsilon)^{p\lambda-(p-1)-\gamma} .\end{aligned}$$

In view of the latter estimate we have used that $0 \leq f_n \leq f$. We can apply Young inequality, we thus obtain

$$\begin{aligned} & c\alpha(p\lambda-(p-1))/p \int_\Omega |\nabla u_n|^p (u_n+\varepsilon)^{p\lambda-p} \\ & \leq C\int_\Omega (u_n+\varepsilon)^{p(\lambda-\theta)} + C\int_\Omega f\,(u_n+\varepsilon)^{p\lambda-(p-1)-\gamma}. \end{aligned}$$

Letting ε tend to zero, and using Lebesgue theorem (in the right one, recall that u_n is in $L^\infty(\Omega)$) and Fatou Lemma (in the left-hand side), we arrive at

$$\int_\Omega |\nabla u_n|^p u_n^{p\lambda-p} \leq C\int_\Omega u_n^{p(\lambda-\theta)} + C\int_\Omega f u_n^{p\lambda-(p-1)-\gamma},$$

since now our assumption is $0<\theta<(p-1+\gamma)/p$ and $\lambda>(p-1+\gamma)/p$, we have that $\lambda>\theta$; thus, using Young inequality we have that, for $\delta>0$

$$\begin{aligned} \int_\Omega |\nabla u_n|^p u_n^{p\lambda-p} & \leq \delta\int_\Omega u_n^{p\lambda} + |\Omega| C_\delta + C\int_\Omega f u_n^{p\lambda-(p-1)-\gamma} \\ & \leq \frac{\delta}{\lambda_1}\int_\Omega |\nabla u_n|^p u_n^{p\lambda-p} + C + C\int_\Omega f u_n^{p\lambda-(p-1)-\gamma}, \end{aligned}$$

where in the last inequality we have used Poincaré inequality. Thus if δ is small enough, we have

$$\int_\Omega |\nabla u_n|^p u_n^{p\lambda-p} \leq C + C\int_\Omega f u_n^{p\lambda-(p-1)-\gamma}.$$

If $1<m<\frac{pN}{N(p-1)+p(1-\gamma)+\gamma N}$, the choice $\lambda(m)=\frac{m(N-p)(p-1+\gamma)}{p(N-pm)}$ implies $\frac{p-1+\gamma}{p}<\lambda(m)<1$ and (reasoning as in the proof of Lemma 6.4)

$$\int_\Omega \frac{|\nabla u_n|^p}{u_n^{p(1-\lambda(s))}} \leq C\left(\|f\|_{L^m(\Omega)}\right). \tag{6.31}$$

Let $\bar{m}$ be a real number, such that

$$\bar{m} = \frac{pN(1-\theta)}{N(p-1)(p-1+\gamma)+p(p-1)(1-\gamma)-p^2\theta},$$

we have that $\lambda(m)=1-\frac{\theta}{p-1}$, and so (6.31) becomes

$$\int_\Omega \left(\frac{|\nabla u_n|^{p-1}}{u_n^\theta}\right)^{p'} \leq C\|f\|_{L^{\bar{m}}(\Omega)}, \tag{6.32}$$

which is (6.30) if $m=\bar{m}$. Since Ω has finite measure, if $m>\bar{m}$ and if f belong to $L^m(\Omega)$, then it is also in $L^{\bar{m}}(\Omega)$, so that (6.32) still holds for these values of m. □

Lemma 6.7. *Let u_n be a solution of* (6.9)*. Suppose that* (6.3)*–*(6.6) *and* $0 < \theta < (1-\gamma)/p'$ *hold true. Then if* $r = \frac{Nm(p-1+\gamma)}{N(p-1-\theta)-m[(p-1)(1-\gamma)-p\theta]}$ *and that* $0 \le f \in L^m(\Omega)$*, with*

$$1 < m < \frac{pN(p-1-\theta)}{N(p-1)(p-1+\gamma)+p(p-1)(1-\gamma)-p^2\theta}, \tag{6.33}$$

then,

$$\textit{The sequence} \left\{ \frac{|\nabla u_n|^{p-1}}{u_n^\theta} \right\} \textit{ is bounded in } L^r(\Omega). \tag{6.34}$$

Proof. Let $\theta > 0$ and $N > p$, we have $m < \frac{pN}{N(p-1)+p(1-\gamma)+\gamma N}$.
Let $1 < r < p'$; then, we used Hölder inequality with exponents $\frac{p'}{r}$ and $\frac{p'}{p'-r}$, we obtain

$$\int_\Omega \left(\frac{|\nabla u_n|^{p-1}}{u_n^\theta} \right)^r = \int_\Omega \frac{|\nabla u_n|^{r(p-1)}}{u_n^{r(p-1)(1-\lambda(m))}} u_n^{r(p-1)(1-\lambda(m)-\frac{\theta}{p-1})}$$

$$\le \left(\int_\Omega \frac{|\nabla u_n|^p}{u_n^{p(1-\lambda(m))}} \right)^{\frac{r}{p'}} \left(\int_\Omega u_n^{\frac{pr(1-\lambda(m)-\frac{\theta}{p-1})}{p'-r}} \right)^{\frac{p'-r}{p'}}.$$

Moreover, using (6.31) which is admissible since $m < \frac{pN}{N(p-1)+p(1-\gamma)+\gamma N}$, we thus obtain

$$\int_\Omega \left(\frac{|\nabla u_n|^{p-1}}{u_n^\theta} \right)^r \le C\|f\|_{L^m(\Omega)} \left(\int_\Omega u_n^{\frac{pr(1-\lambda(m)-\frac{\theta}{p-1})}{p'-r}} \right)^{\frac{p'-r}{p'}}. \tag{6.35}$$

Taking $r = r(m)$ such that $\frac{pr(m)(1-\lambda(m)-\frac{\theta}{p-1})}{p'-r(m)} = \frac{Nm(p-1+\gamma)}{N-pm}$, that is $r(m) = \frac{Nm(p-1+\gamma)}{N(p-1-\theta)-m[(p-1)(1-\gamma)-p\theta]}$; the assumptions on m, and the fact that $r(m)$ is increasing, imply that
$1 < \frac{N(p-1+\gamma)}{N(1-\theta)-(1-\gamma-p\theta)} < r(m) < r\left(\frac{pN(p-1-\theta)}{N(p-1)(p-1+\gamma)+p(p-1)(1-\gamma)-p^2\theta} \right) = p'$, hence by (6.35) we derive that

$$\int_\Omega \left(\frac{|\nabla u_n|^{p-1}}{u_n^\theta} \right)^r \le C\|f\|_{L^m(\Omega)},$$

as desired. □

Now, we are going to prove Theorem 6.3.

Proof of Theorem 6.3. Thanks to (6.17) (or (6.26)), the sequence $\{u_n\}$ of solutions of (6.9) is bounded in $W_0^{1,\sigma}(\Omega)$, with $\sigma = \min(\tilde{m}, p)$. Thus, up to subsequences, u_n weakly converges to some function u in $W_0^{1,\sigma}(\Omega)$, with σ as above and therefore u satisfies the boundary condition. However, due to the nonlinear nature of the lower order term, the weak convergence of u_n is not enough to pass to the limit in the distributional formulation of (6.9). In order to proceed, we use the fact that, thanks to (6.18) (or (6.27)), we have that the right-hand side

$$b(x) \frac{|\nabla u_n|^{p-1}}{\left(1 + \frac{1}{n}|\nabla u_n|^{p-1}\right)\left(\frac{1}{n} + u_n\right)^\theta} \quad \text{is bounded in (at least) } L^1_{\text{loc}}(\Omega).$$

Therefore, thanks to Remark 2.2 after Theorem 2.1 of [26] (see also [23] and [68]), we have that $\nabla u_n(x)$ almost everywhere converges to $\nabla u(x)$ in Ω; this implies that

$$\lim_{n\to+\infty}\frac{|\nabla u_n|^{p-1}}{\left(1+\frac{1}{n}|\nabla u_n|^{p-1}\right)\left(\frac{1}{n}+u_n\right)^{\theta}}=\frac{|\nabla u|^{p-1}}{u^{\theta}}\quad\text{almost everywhere in }\Omega.$$

This almost everywhere convergence, and the local boundedness of the sequence in $L^r(\Omega)$, with

$$r=\frac{\tilde{m}}{p-1}\ \text{or}\ r=p',\ \text{yield that}$$

$$\lim_{n\to+\infty}\frac{|\nabla u_n|^{p-1}}{\left(1+\frac{1}{n}|\nabla u_n|^{p-1}\right)\left(\frac{1}{n}+u_n\right)^{\theta}}=\frac{|\nabla u|^{p-1}}{u^{\theta}}\quad\text{locally weakly in}\quad L^r(\Omega).$$

Next we note that, for all $0<\gamma\le 1$ and $\varphi\in C_0^1(\Omega)$, if $\omega=\{x\in\Omega:|\varphi|>0\}$, we have

$$\left|\frac{f_n\varphi}{(u_n+1/n)^{\gamma}}\right|\leqslant\frac{\|\varphi\|_\infty f}{c_\omega^{\gamma}}\in L^1(\Omega)$$

and that, for $n\to\infty$

$$\frac{f_n\varphi}{(u_n+1/n)^{\gamma}}\longrightarrow\frac{f\varphi}{u^{\gamma}}\ \text{a.e in }\Omega.$$

Here we use the convention that if $u=+\infty$, then $\frac{f\varphi}{u^{\gamma}}=0$. Therefore, by Lebesgue Theorem, it follows that

$$\lim_{n\to\infty}\int_\Omega\frac{f_n\varphi}{(u_n+1/n)^{\gamma}}=\int_\Omega\frac{f\varphi}{u^{\gamma}}.\tag{6.36}$$

Concerning the left hand side of (6.10), we can use the assumption (6.4) on a and the generalized Lebesgue Theorem, we can pass to the limit for $n\longrightarrow\infty$ obtaining

$$\lim_{n\to\infty}\int_\Omega a(x,\nabla u_n)\nabla\varphi=\int_\Omega a(x,\nabla u)\nabla\varphi.$$

We now take φ in $C_c^1(\Omega)$ as test function in (6.9), to have that

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla\varphi=\int_\Omega b(x)\frac{|\nabla u_n|^{p-1}}{\left(1+\frac{1}{n}|\nabla u_n|^{p-1}\right)\left(\frac{1}{n}+u_n\right)^{\theta}}\varphi+\int_\Omega\frac{f_n}{(\frac{1}{n}+u_n)^{\gamma}}\varphi.$$

Passing to the limit in n we obtain

$$\int_\Omega a(x,\nabla u)\cdot\nabla\varphi=\int_\Omega b(x)\frac{|\nabla u|^{p-1}}{u^{\theta}}\varphi+\int_\Omega\frac{f}{u^{\gamma}}\varphi,$$

for every φ in $C_c^1(\Omega)$, so that u is a solution in the sense of distributions. □

3.2 The case $p-1 \le q < \frac{p(p+\beta)}{p+1}, \mu > 0$ and $0 \le f \in L^1(\Omega)$.

In this subsection, we treat the case where $0 \le f \in L^1(\Omega)$, $\mu > 0$, $\beta = \min(\theta,\gamma)$ and $p-1 \le q < \frac{p(p+\beta)}{p+1}$. Here, we give our main existence result for this subsection

Theorem 6.8. *Assume that* (6.3)–(6.7) *hold true and let f be a nonnegative function in $L^1(\Omega)$. Then there exists a solution u for* (6.1)*, in the sense that: $u \in W_0^{1,r}(\Omega) \cap L^{p+\beta}(\Omega)$, with $\beta = \min(\theta,\gamma)$, $1 \le r < \frac{p(p+\beta)}{p+1}$, $\frac{|\nabla u|^q}{u^\theta} \in L^1_{loc}(\Omega)$*

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0 : u \ge c_\omega \ in\ \omega \tag{6.37}$$

and that

$$\int_\Omega a(x,\nabla u)\nabla\varphi + \mu\int_\Omega u^p\varphi = \int_\Omega b(x)\frac{|\nabla u|^q}{u^\theta}\varphi + \int_\Omega \frac{f}{u^\gamma}\varphi, \quad \forall \varphi \in C_c^1(\Omega).$$

In the next Lemma well be used in the proof of Theorem 6.8, we state some a priori estimates on the solution u_n and on the lower order term of the approximate problem (6.9)

Lemma 6.9. *Let u_n be a solution of* (6.9)*. Suppose that f be a nonnegative function in $L^1(\Omega)$ and* (6.3)–(6.7) *hold true. Then the sequence u_n is bounded in $W_0^{1,r}(\Omega) \cap L^{p+\beta}(\Omega)$, with $\beta = \min(\theta,\gamma)$, $1 \le r < \frac{p(p+\beta)}{p+1}$ and $\frac{|\nabla u_n|^q}{u_n^\theta}$ is bounded in $L^1_{loc}(\Omega)$.*

Proof. In the case $\theta \ge \gamma$, let $\big(G_1(u_n)\big)^\gamma$ as test function in (6.9), using (6.3),(6.4) and the fact that $0 \le f_n \le f$, we thus have

$$\gamma\alpha\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^p}{u_n^{1-\gamma}} + \int_{\{u_n\ge 1\}}u_n^{p+\gamma} \le ||b||_{L^\infty(\Omega)}\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^q}{u_n^{\theta-\gamma}} + \int_\Omega f \tag{6.38}$$

and then, by Young inequality, we deduce that

$$\begin{aligned} ||b||_{L^\infty(\Omega)}\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^q}{u_n^{\theta-\gamma}} &\le ||b||_{L^\infty(\Omega)}\int_{\{u_n\ge 1\}}|\nabla u_n|^q \\ = ||b||_{L^\infty(\Omega)}\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^q}{u_n^{\frac{q(1-\gamma)}{p}}}u_n^{\frac{q(1-\gamma)}{p}} &\le \frac{\gamma\alpha}{p}\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^p}{u_n^{1-\gamma}} + C\int_{\{u_n\ge 1\}}u_n^{\frac{q(1-\gamma)}{p-q}}, \end{aligned}$$

which implies from (6.38) that

$$\frac{\gamma\alpha}{p'}\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^p}{u_n^{1-\gamma}} + \int_{\{u_n\ge 1\}}u_n^{p+\gamma} \le C\int_{\{u_n\ge 1\}}u_n^{\frac{q(1-\gamma)}{p-q}} + \int_{\{u_n\ge 1\}}f, \tag{6.39}$$

thanks to (6.39) we have

$$\frac{\gamma\alpha}{p'}\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^p}{u_n^{1-\gamma}} + \frac{1}{p}\int_{\{u_n\ge 1\}}u_n^{p+\gamma} \le C\int_{\{u_n\ge 1\}}u_n^{\frac{q(1-\gamma)}{p-q}} + C.$$

Since, $\frac{q(1-\gamma)}{p-q} < p+\gamma$ the above estimate implies that

$$\int_{\{u_n\ge 1\}}\frac{|\nabla u_n|^p}{u_n^{1-\gamma}} \le C \tag{6.40}$$

and

$$\int_{\{u_n \geq 1\}} u_n^{p+\gamma} \leq C. \tag{6.41}$$

Now we choose $\varepsilon < 1/n$ and use $(T_1(u_n)+\varepsilon)^\theta - \varepsilon^\theta$ as test function, dropping the positive term and using (6.3), (6.4) we obtain

$$\begin{aligned}\alpha\theta \int_\Omega \frac{|\nabla T_1(u_n)|^p}{(T_1(u_n)+\varepsilon)^{1-\theta}} &\leq ||b||_{L^\infty(\Omega)} \int_\Omega \frac{|\nabla u_n|^q}{\left(u_n+\frac{1}{n}\right)^\theta} (T_1(u_n)+\varepsilon)^\theta \\ + \int_\Omega f_n (T_1(u_n)+\varepsilon)^{\theta-\gamma} &\leq ||b||_{L^\infty(\Omega)} \int_{\{u_n\geq 1\}} |\nabla u_n|^q \\ + ||b||_{L^\infty(\Omega)} \int_{\{u_n<1\}} |\nabla u_n|^q &+ (1+\varepsilon)^{\theta-\gamma} \int_\Omega f.\end{aligned} \tag{6.42}$$

Using Young inequality together with (6.40) and (6.41) and the fact that $\frac{q(1-\gamma)}{p-q} < p+\gamma$, yield that

$$\int_{\{u_n\geq 1\}} |\nabla u_n|^q = \int_{\{u_n\geq 1\}} \frac{|\nabla u_n|^q}{u_n^{\frac{(1-\gamma)q}{p}}} u_n^{\frac{(1-\gamma)q}{p}} \leq C \int_{\{u_n\geq 1\}} \frac{|\nabla u_n|^p}{u_n^{(1-\gamma)}} + C\int_\Omega u_n^{p+\gamma} \leq C.$$

Then we deduce from (6.42) and the above estimate, using again young inequality, we obtain

$$\begin{aligned}&\alpha\theta \int_\Omega \frac{|\nabla T_1(u_n)|^p}{(T_1(u_n)+\varepsilon)^{1-\theta}} \\ &\leq ||b||_{L^\infty(\Omega)} \int_\Omega \frac{|\nabla T_1(u_n)|^q}{(T_1(u_n)+\varepsilon)^{\frac{q}{p}(1-\theta)}} (T_1(u_n)+\varepsilon)^{\frac{q}{p}(1-\theta)} + (1+\varepsilon)^{\theta-\gamma}\int_\Omega f + C \\ &\leq \frac{\alpha\theta}{p} \int_\Omega \frac{|\nabla T_1(u_n)|^p}{(T_1(u_n)+\varepsilon)^{1-\theta}} + C(1+\varepsilon)^{\frac{q}{p-q}(1-\theta)} + (1+\varepsilon)^{\theta-\gamma}\int_\Omega f + C,\end{aligned} \tag{6.43}$$

it follows that

$$\int_\Omega \frac{|\nabla T_1(u_n)|^p}{(T_1(u_n)+\varepsilon)^{1-\theta}} \leq C\left((1+\varepsilon)^{\frac{q}{p-q}(1-\theta)} + (1+\varepsilon)^{\theta-\gamma}\right).$$

Thus, we obtain

$$\begin{aligned}\int_\Omega |\nabla T_1(u_n)|^p &= \int_\Omega \frac{|\nabla T_1(u_n)|^p}{(T_1(u_n)+\varepsilon)^{1-\theta}} (T_1(u_n)+\varepsilon)^{1-\theta} \\ &\leq C(1+\varepsilon)^{1-\theta}\left((1+\varepsilon)^{\frac{q}{p-q}(1-\theta)} + (1+\varepsilon)^{\theta-\gamma}\right).\end{aligned}$$

Hence, taking ε tends to 0, we deduce that

$$\int_\Omega |\nabla T_1(u_n)|^p \leq C, \tag{6.44}$$

from (6.40) and (6.44) we conclude that

$$\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1-\gamma}} \leq \int_{\{u_n\geq 1\}} \frac{|\nabla u_n|^p}{u_n^{1-\gamma}} + \int_\Omega |\nabla T_1(u_n)|^p \leq C. \tag{6.45}$$

Let $1 \leq r < p$, using the estimate (6.45) together with Hölder inequality we arrive at

$$\int_\Omega |\nabla u_n|^r \leq \int_\Omega \frac{|\nabla u_n|^r}{(1+u_n)^{\frac{r(1-\gamma)}{p}}} (1+u_n)^{\frac{r(1-\gamma)}{p}} \leq C \left(\int_\Omega (1+u_n)^{\frac{r(1-\gamma)}{p-r}} \right)^{1-\frac{r}{p}}, \tag{6.46}$$

starting from (6.46) and thanks to (6.41) noticing that $\frac{r(1-\gamma)}{p-r} \leq p+\gamma$ is equivalent to $r \leq \frac{p(p+\gamma)}{p+1}$, we Thus obtain

$$\int_\Omega |\nabla u_n|^r \leq C, \quad \forall 1 \leq r \leq \frac{p(p+\gamma)}{p+1} < p \tag{6.47}$$

Thus, recalling (6.13),(6.5), estimate (6.47) and by means of Hölder inequality, it follows for every $\omega \subset\subset \Omega$ that

$$\int_\omega \frac{|\nabla u_n|^q}{u_n^\theta} \leq \frac{|\Omega|^{\frac{r-q}{r}}}{c_\omega^\theta} \|u_n\|^q_{W_0^{1,r}(\Omega)} \leq C. \tag{6.48}$$

In the case $\gamma \geq \theta$, we can obtaining the results, changing γ by θ in the exponents of the test functions and namely arguing exactly as above. Then Lemma 6.9 is completely proved. □

We prove now the following convergence result.

Proposition 6.10. *Under assumption* (6.3)*, we have*

$$u_n^p \to u^p \text{ strongly in } L^1(\Omega).$$

Proof. We take $T_1(u_n - T_h(u_n))$ as test function in (6.9) dropping the positive term, using (6.3), (6.4) and we then have

$$\alpha \int_{\{h \leq u_n \leq h+1\}} |\nabla u_n|^p + \mu \int_{\{u_n \geq h+1\}} u_n^p \leq ||b||_{L^\infty(\Omega)} \int_{\{h \leq u_n \leq h+1\}} |\nabla u_n|^q$$
$$+ ||b||_{L^\infty(\Omega)} \int_{\{u_n > h+1\}} |\nabla u_n|^q + \frac{1}{h^\gamma} \int_{\{u_n \geq h\}} f,$$

which implies using (6.47), Young together with Hölder inequalities that

$$\begin{aligned} &\frac{\alpha}{p} \int_{\{h \leq u_n \leq h+1\}} |\nabla u_n|^p + \mu \int_{\{u_n \geq h+1\}} u_n^p \\ &\leq C|u_n > h|^{1-\frac{q}{p}} + ||b||_{L^\infty(\Omega)} \|u_n\|^q_{W_0^{1,r}(\Omega)} |u_n > h|^{\frac{r-q}{r}} + \frac{1}{h^\gamma} \int_{\{u_n \geq h\}} f \\ &\leq C|u_n > h|^{1-\frac{q}{p}} + C|u_n > h|^{\frac{r-q}{r}} + \frac{1}{h^\gamma} \int_{\{u_n \geq h\}} f. \end{aligned}$$

Letting $n \to +\infty$ and then $h \to +\infty$, we obtain

$$\int_{\{u_n \geq h+1\}} u_n^p \leq w(n,h), \tag{6.49}$$

where $w(n,h)$ tends to zero when $n \to +\infty$ and $h \to +\infty$. Let E be a measurable subset of Ω, we have

$$\int_E u_n^p \leq \int_{\{u_n > h\}} u_n^p + h^p |E|.$$

Then, thanks to (6.49), we take the limit as $|E|$ tends to zero, h tends to infinity and since u_n^p converges to u^p almost everywhere, we easily conclude by Vitali's Theorem the proof of Proposition 6.10. □

Proof of Theorem 6.8. Using Proposition 6.10 and Lemma 6.9, we can obtain a solution passing to the limit, namely arguing exactly as in Theorem 6.3. □

Chapter 7

Existence and regularity of positive solutions for Schrodinger-Maxwell system with singularity

1 Introduction and main results

In this chapter, we consider the following Schrödinger-Maxwell system with singular term

$$\begin{cases} -\operatorname{div}(a(x)\nabla u) + \psi u^{r-1} = \frac{f(x)}{u^\theta} & \text{in } \Omega \\ -\operatorname{div}(M(x)\nabla \psi) = u^r & \text{in } \Omega \\ u, \psi > 0 & \text{in } \Omega \\ u = \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

we will suppose that Ω is a bounded open set of $\mathbb{R}^N, N > 2$, that $r > 1$ and that f nonnegative (not identically zero) function belongs to $L^m(\Omega)$, for some $m > 1$, $0 < \theta < 1$. Furthermore, the function $a : \Omega \to \mathbb{R}$ will be a measurable function, such that there exist $0 < \alpha \leq \beta$ such that:

$$0 < \alpha \leq a(x) \leq \beta \ \text{ almost everywhere in } \ \Omega, \tag{7.2}$$

while $M : \Omega \to \mathbb{R}^{N^2}$ will be a measurable matrix, such that:

$$M(x)\xi \cdot \xi \geq \alpha|\xi|^2, \quad |M(x)| \leq \beta, \tag{7.3}$$

for almost every x in Ω, and for every ξ in $\mathbb{R}^N$.

In this work we want to prove existence and regularity results for problem (7.1) in case Ω is an open bounded subset of $\mathbb{R}^N$ $(N > 2)$, $0 < \theta < 1$, $r > 1$ and $0 \leq f \in L^m(\Omega)$, with $m > 1$. In particular, we show how the coupling between the equations in the system gives rise to a regularizing effect producing the existence of finite energy solutions.

In the last part of this Section, we prove the existence of a saddle point (u, φ) of the following functional

$$J(u,\varphi) = \begin{cases} \frac{1}{2}\int_\Omega a(x)|\nabla u|^2 - \frac{1}{2r}\int_\Omega M(x)\nabla\varphi\nabla\varphi \\ +\frac{1}{r}\int_\Omega \varphi^+|u|^r - \frac{1}{1-\theta}\int_\Omega f(u^+)^{1-\theta} & \text{if } \varphi^+|u|^r \in L^1(\Omega), \\ +\infty, & \text{otherwise,} \end{cases} \tag{7.4}$$

defined on $W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$. Finally, in the Appendix, we give the proof of an existence result for the first equation of approximating system and some results allowing to prove the existence of system (7.1).

Now, we give our definition of solution for problem (7.1).

Definition 7.1. A couple of functions $(u,\psi)\in W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$ is a energy solution to system (7.1) if

$$u, \psi > 0 \quad \text{a.e in } \Omega, \tag{7.5}$$

$$\frac{f\phi}{u^\theta} \in L^1_{loc}(\Omega) \quad \forall \phi \in C^1_c(\Omega), \tag{7.6}$$

and hold

$$\begin{cases} \int_\Omega a(x)\nabla u\nabla\phi + \int_\Omega \psi u^{r-1}\phi = \int_\Omega \frac{f\phi}{u^\theta} & \forall\phi\in C_c^1(\Omega) \\ \int_\Omega M(x)\nabla\psi\nabla v = \int_\Omega u^r v \quad \forall v\in W_0^{1,2}(\Omega). \end{cases} \tag{7.7}$$

Our main result is the following.

Theorem 7.2. *Let us assume* (7.2) *and* (7.3). *Given* $0<\theta<1$, $r>1, m^{**}=\frac{Nm}{N-2m}$ *and let* f *be a nonnegative (not identically zero) function in* $L^m(\Omega)$. *We have the following*

(i) *if* $r\geq\frac{2N}{\theta(N-2)+N+2}$, *and if* $m\geq\left(\frac{r+1}{1-\theta}\right)'$, *there exist* u *and* ψ *in* $W_0^{1,2}(\Omega)$, *solutions of* (7.1) *in the sense of Definition 7.1, furthermore*

- **(a)** *if* $m>\frac{N}{2}$, *then* u *belongs to* $L^\infty(\Omega)$,
- **(b)** *if* $\left(\frac{r+1}{1-\theta}\right)'\leq m<\frac{N}{2}$, *then* u *belongs to* $L^\sigma(\Omega)$, *with*

$$\sigma = \max\left((1+\theta)m^{**}, \frac{m(2r+1+\theta)}{m+1}\right).$$

(ii) *if* $1<r<\frac{2N}{\theta(N-2)+N+2}$, *and if* $m\geq\frac{2N}{\theta(N-2)+N+2}$, *there exist* u *and* ψ *in* $W_0^{1,2}(\Omega)$, *solutions of* (7.1) *in the sense of Definition 7.1, furthermore*

- **(a)** *if* $m>\frac{N}{2}$, *then* u *belongs to* $L^\infty(\Omega)$,
- **(b)** *if* $\frac{2N}{\theta(N-2)+N+2}\leq m<\frac{N}{2}$, *then* u *belongs to* $L^{(1+\theta)m^{**}}(\Omega)$.

2 The Approximated Problem and a priori estimates

2.1 The Approximated Problem

Let n in $\mathbb{N}$, and let $f_n = T_n(f)$, so that $\{f_n\}$ is a sequence of $L^\infty(\Omega)$ functions, which strongly converges to f in $L^m(\Omega)$, and satisfies the inequality $0 \leq f_n \leq f$. Thanks to Theorem 7.9 (see the Appendix), for every n in $\mathbb{N}$, there exist weak solutions u_n and ψ_n in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$(with $u_n \geq 0$ and $\psi_n \geq 0$) of the approximate system:

$$\begin{cases} -\operatorname{div}(a(x)\nabla u_n) + \psi_n u_n^{r-1} = \frac{f_n}{(\frac{1}{n}+u_n)^\theta} & \text{in } \Omega \quad (I) \\ -\operatorname{div}(M(x)\nabla \psi_n) = u_n^r & \text{in } \Omega \quad (II) \\ u_n, \psi_n \geq 0 & \text{in } \Omega \\ u_n = \psi_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.8}$$

2.2 A Priori Estimates

We are now going to prove some a priori estimates on the sequence of approximated solutions u_n.

Lemma 7.3. *Let $k > 0$ be fixed. The sequence $\{T_k(u_n)\}$, where u_n is a solution to (I) of* (7.8), *is bounded in $W_0^{1,2}(\Omega)$.*

Proof. Taking $T_k(u_n)$ as a test function in (I) of problems (7.8) and using the assumption (7.2) and $f_n \leq f$, we obtain

$$\alpha \int_\Omega |\nabla T_k(u_n)|^2 + \int_\Omega \psi_n u_n^{r-1} T_k(u_n) \leqslant \int_\Omega f\,(T_k(u_n))^{1-\theta}. \tag{7.9}$$

Dropping the second nonnegative terms in the left hand side of (7.9), we have

$$\alpha \int_\Omega |\nabla T_k(u_n)|^2 \leqslant \int_\Omega f\,(T_k(u_n))^{1-\theta}.$$

Since $\theta < 1$, we have

$$\alpha \int_\Omega |\nabla T_k(u_n)|^2 \leqslant k^{1-\theta} \int_\Omega f,$$

so that $T_k(u_n)$ is bounded in $W_0^{1,2}(\Omega)$ with respect to n. □

Lemma 7.4. *Assume that $0 < \theta < 1$, $r > 1$ and let f be a nonnegative function in $L^m(\Omega)$ with $m \geq \max\left(\left(\frac{r+1}{1-\theta}\right)', \frac{2N}{\theta(N-2)+N+2}\right)$. Let (u_n, ψ_n) be a couple solutions of* (7.8), *then :*

- *The sequences $\{u_n\}$ and $\{\psi_n\}$ are bounded in $W_0^{1,2}(\Omega)$.*
- *The sequence $\{u_n\}$ is bounded in $L^\sigma(\Omega)$, where σ is defined in Theorem 7.2 if $m < \frac{N}{2}$, and $\sigma = +\infty$ if $m > \frac{N}{2}$.*

Proof. **$L^\infty(\Omega)$ estimate.** Suppose that $m > \frac{N}{2}$, let $k > 1$. Choosing $G_k(u_n)$ as test function in (7.8), we obtain, recalling (7.2),

$$\alpha \int_\Omega |\nabla G_k(u_n)|^2 \leq \int_\Omega M(x)\nabla G_k(u_n) \times \nabla G_k(u_n)$$

$$=\int_{\Omega}\frac{f_n G_k(u_n)}{(u_n+\frac{1}{n})^{\theta}}\leq\int_{\Omega} f G_k(u_n), \tag{7.10}$$

where in the last passage we have used that $u_n+\frac{1}{n}\geq k\geq 1$, on the set $\{u_n\geq k\}$ where $G_k(u_n)\neq 0$. Starting from inequality (7.10) and arguing as in [80], Theorem 4.2, we have that there exists a constant C (independent on n), such that

$$\|u_n\|_{L^\infty(\Omega)}\leq C\|f\|_{L^m(\Omega)}.$$

Since u_n is bounded in $L^\infty(\Omega)$, as well.□

Estimates using the lower order term.

In this step, we will suppose that $m\geq\left(\frac{r+1}{1-\theta}\right)'$. Taking u_n as test function in the first equation of (7.8), using (7.2) and dropping a positive term, we obtain

$$\alpha\int_{\Omega}|\nabla u_n|^2+\int_{\Omega}\psi_n u_n^r\leq\int_{\Omega} f_n u_n^{1-\theta},$$

while using ψ_n as test function in (II) and (7.3), we can see that

$$\alpha\int_{\Omega}|\nabla\psi_n|^2\leq\int_{\Omega}\psi_n u_n^r.$$

Thus we have, once again, that

$$\alpha\int_{\Omega}|\nabla u_n|^2+\alpha\int_{\Omega}|\nabla\psi_n|^2\leq\int_{\Omega} f_n u_n^{1-\theta}. \tag{7.11}$$

We now follow [76], let $\gamma\geq 1$ to be determined later, and choose $u_n^{2\gamma-1}$ as test function in the first equation of (7.8); using (7.2), and dropping two positive terms, we obtain, since $f_n\leq f$

$$\alpha(2\gamma-1)\int_{\Omega}|\nabla u_n|^2 u_n^{2\gamma-2}\leq\int_{\Omega} f_n u_n^{2\gamma-1-\theta}\leq\int_{\Omega} f u_n^{2\gamma-1-\theta}. \tag{7.12}$$

On the other hand, taking u_n^γ as a test function in (II), by estimate (7.3) and using Young inequality, we can see that

$$\begin{aligned}\int_{\Omega}u_n^{r+\gamma}&=\gamma\int_{\Omega}M(x)\nabla\psi_n\nabla u_n u_n^{\gamma-1}\\&\leq\beta\gamma\int_{\Omega}|\nabla\psi_n\|\nabla u_n\|\,u_n|^{\gamma-1}\\&\leq C\int_{\Omega}|\nabla\psi_n|^2+C\int_{\Omega}|\nabla u_n|^2 u_n^{2\gamma-2}.\end{aligned}$$

Using (7.11) and (7.12) with this inequality, we deduce that

$$\int_{\Omega}u_n^{r+\gamma}\leq C\int_{\Omega} f u_n^{1-\theta}+C\int_{\Omega} f u_n^{2\gamma-1-\theta},$$

so that we have

$$\int_{\Omega}u_n^{r+\gamma}\leq C\int_{\Omega} f u_n^{1-\theta}+C\int_{\Omega} f u_n^{2\gamma-1-\theta}, \tag{7.13}$$

where in the last passage, we have used that $2\gamma-1-\theta \geq 1-\theta$, since $\gamma \geq 1$. We now choose $\gamma = \frac{r(m-1)+m(\theta+1)}{m+1}$, so that $\gamma \geq 1$ since $m \geq \frac{r+1}{r+\theta} = (\frac{r+1}{1-\theta})'$. With this choice of γ, we obtain $r+\gamma = \frac{m(2r+1+\theta)}{m+1} = (2\gamma-1-\theta)m'$, so by Hölder inequality, we deduce from (7.13) that

$$\int_\Omega u_n^{\frac{m(2r+1+\theta)}{m+1}} \leq C\|f\|_{L^m(\Omega)} \left(\left[\int_\Omega u_n^{\frac{m(1-\theta)}{m-1}}\right]^{\frac{1}{m'}} + \left[\int_\Omega u_n^{\frac{m(2r+1+\theta)}{m+1}}\right]^{\frac{1}{m'}} \right).$$

Thanks to the fact that $m > 1$, we therefore obtain (after simplifying equal terms) that:

$$\left[\int_\Omega u_n^{\frac{m(2r+1+\theta)}{m+1}}\right]^{\frac{1}{m}} \leq C\|f\|_{L^m(\Omega)},$$

that is, the sequence $\{u_n\}$ is bounded in $L^s(\Omega)$, with $s = \frac{m(2r+1+\theta)}{m+1}$. As a consequence of this estimate, and of the fact that $s \geq m'$, we have that

$$\int_\Omega f u_n^{1-\theta} \leq C,$$

from the last inequality and going back to (7.11), we obtain that the sequences $\{u_n\}$ and $\{\psi_n\}$ are bounded in $W_0^{1,2}(\Omega)$. □

Estimates not using the lower order term.

In this step, we will suppose that $m \geq \frac{2N}{\theta(N-2)+N+2}$. Let u_n and ψ_n be solutions of (7.8), let $\gamma \geq 1$, and take $u_n^{2\gamma-1}$ as test function in (I) of (7.8), we have, dropping two positive terms, and using (7.2),

$$\alpha(2\gamma-1)\int_\Omega |\nabla u_n|^2 u_n^{2\gamma-2} \leq \int_\Omega f_n u_n^{2\gamma-2} u_n^{1-\theta}.$$

By exploiting Sobolev and Hölder inequalities, and since $f_n \leq f$, we deduce

$$\begin{aligned} \frac{\alpha\mathcal{S}(2\gamma-1)}{\gamma^2}\left[\int_\Omega u_n^{2^*\gamma}\right]^{\frac{2}{2^*}} &\leq \alpha(2\gamma-1)\int_\Omega |\nabla u_n|^2 u_n^{2\gamma-2} \\ &\leq \int_\Omega f_n u_n^{2\gamma-2} u_n^{1-\theta} \\ &\leq \|f\|_{L^m(\Omega)} \left[\int_\Omega u_n^{(2\gamma-1-\theta)m'}\right]^{\frac{1}{m'}}, \end{aligned}$$

where $\mathcal{S}$ is the constant of the Sobolev embedding. Imposing $2^*\gamma = (1+\theta)m^{**}$, we have $\gamma = \frac{(1+\theta)m^{**}}{2^*}$, so that $\gamma \geq 1$ (since $(1+\theta)m^{**} \geq 2^*$) and $(2\gamma-1-\theta)m' = (1+\theta)m^{**} = s$, we have

$$\left[\int_\Omega u_n^s\right]^{\frac{2}{2^*}} \leq C\|f\|_{L^m(\Omega)} \left[\int_\Omega u_n^s\right]^{\frac{1}{m'}},$$

so that

$$\left[\int_\Omega u_n^s\right]^{\frac{1}{s}} \leq C\|f\|_{L^m(\Omega)}.$$

Thus, the sequence $\{u_n\}$ is bounded in $L^s(\Omega)$, being $m \geq \frac{2N}{N+2+\theta(N-2)}$, we have that the sequence $\{f_n u_n^{1-\theta}\}$ is bounded in $L^1(\Omega)$. Taking, u_n as test function in the equation (I) of (7.8), to obtain, after using (7.2) and dropping a positive term,

$$\alpha \int_\Omega |\nabla u_n|^2 + \int_\Omega \psi_n u_n^r \leq \int_\Omega f_n u_n^{1-\theta} \leq C,$$

so that, the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, and the sequence $\{\psi_n u_n^r\}$ is bounded in $L^1(\Omega)$. Choosing ψ_n as test function in (II) of (7.8), and using (7.3), we thus have:

$$\alpha \int_\Omega |\nabla \psi_n|^2 \leq \int_\Omega \psi_n u_n^r \leq C,$$

so that also the sequence $\{\psi_n\}$ is bounded in $W_0^{1,2}(\Omega)$. □

2.3 Proof of Theorem 7.2

In virtue of the Lemma 7.4, the sequence of approximated solutions u_n is bounded in $W_0^{1,2}(\Omega) \cap L^\sigma(\Omega)$. Therefore, there exists a function u belongs to $W_0^{1,2}(\Omega) \cap L^\sigma(\Omega)$ such that, up to subsequences, u_n converges, weakly in $W_0^{1,2}(\Omega)$, weakly in $L^\sigma(\Omega)$, and almost everywhere in Ω, to some function u, while ψ_n converges, weakly in $W_0^{1,2}(\Omega)$ and almost everywhere in Ω, to some function ψ. Since the sequence $\{u_n^r\}$ is bounded in $L^\rho(\Omega)$, with $\rho = \frac{\sigma}{r} > 1$, it is weakly convergent in the same space to u^r. Therefore, one can pass to the limit in the identities

$$\int_\Omega M(x)\nabla\psi_n \nabla w = \int_\Omega u_n^r w, \quad \forall w \in W_0^{1,2}(\Omega),$$

to have that ψ and u are such that:

$$\int_\Omega M(x)\nabla\psi \nabla w = \int_\Omega u^r w, \quad \forall w \in W_0^{1,2}(\Omega).$$

Choosing $w = T_k(v)$, with $v \geq 0$ in $W_0^{1,2}(\Omega)$, we arrive at

$$\int_\Omega M(x)\nabla\psi \nabla T_k(v) = \int_\Omega u^r T_k(v), \quad \forall k > 0.$$

Letting k tend to infinity, using Lebesgue Theorem in the left-hand side (recall that ψ belongs to $W_0^{1,2}(\Omega)$,) and Beppo Levi Theorem in the right-hand side, we deduce that

$$\int_\Omega M(x)\nabla\psi \nabla v = \int_\Omega u^r v, \quad \forall v \in W_0^{1,2}(\Omega), v \geq 0.$$

If v belongs to $C_c^1(\Omega)$, writing $v = v^+ - v^-$, and subtracting the above identities written for v^+ and v^- (not that both terms are finite, because the left-hand side is finite), we have that

$$\int_\Omega M(x)\nabla\psi \nabla v = \int_\Omega u^r v, \quad \forall v \in W_0^{1,2}(\Omega),$$

that is, ψ is a weak solution of the second equation. We study now the first equation: We want to prove that $\psi_n u_n^{r-1}$ strongly converges to ψu^{r-1} in $L^1(\Omega)$. First of all, let $\varepsilon > 0, k > 0$, and choose $\frac{1}{\varepsilon} u_n^+ T_\varepsilon(G_k(u_n))$ as test function in the first equation of the system. Dropping two positive terms (those coming from the differential part of the equation), and using that $f_n \leq f$, we obtain

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\{u_n\geq k\}} \psi_n \left[u_n^+\right]^r T_\varepsilon(G_k(u_n)) &\leq \frac{1}{\varepsilon}\int_{\{u_n\geq k\}} f_n u_n^{1-\theta} T_\varepsilon(G_k(u_n)) \\ &\leq \frac{1}{\varepsilon}\int_{\{u_n\geq k\}} f u_n^{1-\theta} T_\varepsilon(G_k(u_n)).\end{aligned}$$

Letting ε tend to zero, using Fatou Lemma on the left-hand side, and Lebesgue Theorem on the right-hand one (recall that every u_n is a function in $L^\infty(\Omega)$), we have that

$$\begin{aligned}\int_{\{u_n\geq k\}} \psi_n \left[u_n^+\right]^r \leq \int_{\{u_n\geq k\}} f u_n &\leq \left[\int_{\{u_n\geq k\}} f^m\right]^{\frac{1}{m}} ||u_n^{1-\theta}||_{L^{m'}(\Omega)}. \\ &\leq C\left[\int_{\{u_n\geq k\}} f^m\right]^{\frac{1}{m}},\end{aligned}$$

since the sequence $\{u_n\}$ is bounded in $L^{m'}(\Omega)$ being $\sigma \geq (1-\theta)m'$. Then

$$\int_{\{u_n\geq k\}} \psi_n u_n^r \leq C\left[\int_{\{u_n\geq k\}} f^m\right]^{\frac{1}{m}}.$$

Let now E be a measurable subset of Ω. So that

$$\begin{aligned}\int_E \psi_n u_n^r &= \int_{E\cap\{u_n\leq k\}} \psi_n u_n^r + \int_{E\cap\{u_n\geq k\}} \psi_n u_n^r \\ &\leq k^r \int_E \psi_n + C\left[\int_{\{u_n\geq k\}} f^m\right]^{\frac{1}{m}}.\end{aligned}$$

Now we choose $\varepsilon > 0$, and let k large enough, we obtain

$$C\left[\int_{\{u_n\geq k\}} f^m\right]^{\frac{1}{m}} \leq \varepsilon, \quad \forall n \in \mathbb{N}.$$

Such a choice of k is possible, since the measure of $\{u_n \geq k\}$ tends to zero as k tends to infinity, uniformly in n, as a consequence of the boundedness of $\{u_n\}$ in (for example) $L^1(\Omega)$, and since f^m belongs to $L^1(\Omega)$. Once k has been chosen, let $\delta > 0$ be such that meas $(E) \leq \delta$ implies that:

$$k^r \int_E \psi_n \leq \varepsilon, \quad \forall n \in \mathbb{N}.$$

Such a choice of δ is possible thanks to Vitali theorem, since the sequence $\{\psi_n\}$ is strongly convergent in (at least) $L^1(\Omega)$ being bounded in $W_0^{1,2}(\Omega)$. Thus, the sequence $\{\psi_n u_n^r\}$ is uniformly equi-integrable. Since it is almost everywhere convergent, Vitali theorem implies that:

$$\psi_n u_n^r \text{ strongly converges to } \psi u^r \text{ in } L^1(\Omega).$$

With the same technique, one can prove that the sequence $\{\psi_n u_n^{r-1}\}$ is uniformly equi-integrable, so that $\psi_n u_n^{r-1}$ strongly converges to ψu^{r-1} in $L^1(\Omega)$. To conclude, the proof theorem we just need to pass to the limit in (I) of (7.8). Now, by adapting a suitable way used in [28], we can pass to the limit in the right hand side of (I) in (7.8), then let $w = \{\varphi \neq 0\}$ and by Lemma 7.7 (see the Appendix), one has, for φ in $C_c^1(\Omega)$, we have that

$$0 \leq \left| \frac{f_n \varphi}{(u_n + \frac{1}{n})^\theta} \right| \leq \frac{\|\varphi\|_{L^\infty(\Omega)}}{c_\omega^\theta} f.$$

Therefore, by Lebesgue convergence Theorem, we obtain

$$\lim_{n\to+\infty} \int_\Omega \frac{f_n \varphi}{(u_n + \frac{1}{n})^\theta} = \int_\Omega \frac{f\varphi}{u^\theta}.$$

On other hand, by Lemma 7.3, we deduce $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,2}(\Omega)$. Then by Proposition 4.1 in [23] and Theorem 2.3 in [41], we obtain ∇u_n converges to ∇u almost everywhere in Ω. Now, we can pass to the limit in the identities:

$$\int_\Omega a(x)\nabla u_n \nabla \eta + \int_\Omega \psi_n u_n^{r-1} \eta = \int_\Omega \frac{f_n}{(\frac{1}{n} + u_n)^\theta} \eta, \quad \forall \eta \in C_c^1(\Omega),$$

to have that

$$\int_\Omega a(x)\nabla u \nabla \eta + \int_\Omega \psi u^{r-1} \eta = \int_\Omega \frac{f}{u^\theta} \eta, \quad \forall \eta \in C_c^1(\Omega),$$

as desired.

3 Saddle points

In this section, we can prove that the energy solution (u, ψ) of system (7.1) given by Theorem 7.2 can be seen (under some assumptions on r and f) as a saddle point of a suitable functional.

Remark 3.1. If $1 < r \leq \frac{N+2+(N-2)\theta}{N-2}$, and f nonnegative function belongs to $L^m(\Omega)$, with $m \geq (\frac{2^*}{1-\theta})'$, then not only ψ but also u is a energy solution of the first equation of (7.8). Indeed, since both u and ψ belong to $L^{2^*}(\Omega)$ (being $W_0^{1,2}(\Omega)$ functions), we have that: $\psi u^{r-1} \in L^\rho(\Omega), \rho = \frac{2^*}{r}$. Since, by the assumptions on r : $\frac{2^*}{r} \geq \frac{2N}{N-2} \frac{N-2}{N+2+(N-2)\theta} = (\frac{2^*}{1-\theta})'$, the function ψu^{r-1} belongs to the dual of $W_0^{1,2}(\Omega)$; therefore, one has (by density of $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ in $W_0^{1,2}(\Omega)$)

$$\int_\Omega a(x)\nabla u \nabla \varphi + \int_\Omega \psi u^{r-1} \varphi = \int_\Omega \frac{f}{u^\theta} \varphi, \quad \forall \varphi \in C_c^1(\Omega),$$

as desired.

Thanks to this remark, we have the following theorem:

Theorem 7.5. *Suppose that a and M satisfy* (7.2) *and* (7.3), *and that M is symmetric. Let $1 < r \leq \frac{2N}{N+2+(N-2)\theta}$ and let f nonnegative function belongs to $L^m(\Omega)$, with $m \geq (\frac{2^*}{1-\theta})'$. Then, the energy solution (u, ψ) of system* (7.1) *given by Theorem 7.2 is a saddle point of the functional J defined in* (7.4); *that is*

$$J(u, \varphi) \leq J(u, \psi) \leq J(v, \psi), \forall v, \varphi \in W_0^{1,2}(\Omega) \text{ such that } \psi|v|^r \in L^1(\Omega). \tag{7.14}$$

Proof. We begin with the second equation of (7.1); by Theorem 7.2, ψ is a weak solution of the second equation of (7.1). Choosing $\frac{\psi-\varphi^+}{r}$, with φ in $W_0^{1,2}(\Omega)$, as test function, we get

$$\frac{1}{r}\int_\Omega M(x)\nabla\psi\nabla\left(\psi-\varphi^+\right)=\frac{1}{r}\int_\Omega |u|^r\left(\psi-\varphi^+\right).$$

Adding and subtracting the term

$$\frac{1}{2r}\int_\Omega M(x)\nabla\varphi^+\nabla\varphi^+,$$

we have, after straightforward passages

$$\frac{1}{2r}\int_\Omega M(x)\nabla\left(\psi-\varphi^+\right)\nabla\left(\psi-\varphi^+\right)+\frac{1}{2r}\int_\Omega M(x)\nabla\psi\nabla\psi-\frac{1}{r}\int_\Omega \psi|u|^r$$
$$=\frac{1}{2r}\int_\Omega M(x)\nabla\varphi^+\nabla\varphi^+-\frac{1}{r}\int_\Omega \varphi^+|u|^r$$

since the first term is positive, we, therefore, have that (recall that $\psi\geq 0$, so that $\psi=\psi^+$)

$$\frac{1}{2r}\int_\Omega M(x)\nabla\psi\nabla\psi-\frac{1}{r}\int_\Omega \psi^+|u|^r\leq\frac{1}{2r}\int_\Omega M(x)\nabla\varphi^+\nabla\varphi^+-\frac{1}{r}\int_\Omega \varphi^+|u|^r,$$

for every φ in $W_0^{1,2}(\Omega)$. Changing sign to this identity, and adding to both sides the (finite, thanks to the assumptions on f and to the fact that u belongs to $W_0^{1,2}(\Omega)$) term

$$\frac{1}{2}\int_\Omega a(x)|\nabla u|^2-\frac{1}{1-\theta}\int_\Omega f(u^+)^{1-\theta},$$

we arrive

$$J(u,\varphi)\leq J(u,\psi),\quad \forall\varphi\in W_0^{1,2}(\Omega),$$

which is the first half of (7.14). As for the second, by Remark 3.1, we obtain that u is a weak solution of the first equation of (7.1). Fix $\psi\in W_0^{1,2}(\Omega)$ and let I be the functional defined on $W_0^{1,2}(\Omega)$ as $I(v):=J(v,\psi)$. If the matrix $M(x)$ and a(x) is symmetric, and if f nonnegative function belongs to $L^m(\Omega)$, with $m>\left(\frac{2^*}{1-\theta}\right)'$ the solution of (7.1) given by Theorem 7.2 is the minimum of the functional

$$I(v)=\frac{1}{2}\int_\Omega a(x)\nabla v\times\nabla v-\frac{1}{2r}\int_\Omega M(x)\nabla\psi\nabla\psi$$
$$+\frac{1}{r}\int_\Omega \psi^+|v|^r-\frac{1}{1-\theta}\int_\Omega f(v^+)^{1-\theta},\quad v\in W_0^{1,2}(\Omega)$$

which is well defined since $\theta<1$. Indeed, if we consider the functional

$$I_n(v)=\frac{1}{2}\int_\Omega a(x)\nabla v\times\nabla v-\frac{1}{2r}\int_\Omega M(x)\nabla\psi\nabla\psi$$
$$+\frac{1}{r}\int_\Omega \psi^+|v|^r-\frac{1}{1-\theta}\int_\Omega f_n\left(v^++\frac{1}{n}\right)^{1-\theta},\quad v\in W_0^{1,2}(\Omega)$$

with $f_n = \min(f(x), n)$, then there exists a minimum u_n of I_n. From the inequality $I_n(u_n) \le I_n(u_n^+)$ one can prove that $u_n \ge 0$, so that u_n is a solution of the Euler equation for I_n, i.e., of (7.1). Therefore, by Lemma 7.7 and Remark 7.8, u_n is unique and increasing in n, satisfies (7.19) and, from the inequality $I(u_n) \le I_n(0) \le C$, it is bounded in $W_0^{1,2}(\Omega)$ (with the same proof of Lemma 7.4). If u is the limit of u_n, letting n tend to infinity in the inequalities $I_n(u_n) \le I_n(v)$, one finds that $I(u) \le I(v)$, so that u is a minimum of I, and u is a solution of (7.1) (by Theorem 7.2). Since u satisfies (7.19), Eq. (7.1) can be seen as the Euler equation for I; note that I is not differentiable on $W_0^{1,2}(\Omega)$. We obtain that:

$$J(u, \psi) \le J(v, \psi), \quad \forall v \in W_0^{1,2}(\Omega) \text{ such that } \psi|v|^r \in L^1(\Omega),$$

which is the second part of (7.14).

4 Appendix: Basic Results and Existence for Bounded Data

In this Appendix, we will prove some results concerning the first equation of system (7.1), and the whole system in the case of bounded data.
Now we prove the existence of a solution to the following approximating problem:

$$\begin{cases} -\operatorname{div}(a(x)\nabla u_n) + g(x)\left|u_n\right|^{r-2} u_n = \frac{f_n(x)}{\left(|u_n|+\frac{1}{n}\right)^\theta} & \text{in } \Omega \\ u_n \ge 0 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.15}$$

where Ω is a bounded open subset of $\mathbb{R}^N, N \ge 2, f$ is a positive (that is $f(x) \ge 0$ and not zero a.e.) function in $L^m(\Omega)$, with $m \ge 1$, $0 < \theta < 1$ and $g(x) \in L^1(\Omega)$, with

$$0 \le \lambda \le g(x). \tag{7.16}$$

Due to the nature of the approximation, the sequence u_n will be increasing with n, so that the (strict) positivity of the limit will be derived from the (strict) positivity of any of the u_n (which in turn will follow by the standard maximum principle for elliptic equations).

Lemma 7.6. *Problem* (7.15) *has a nonnegative solution u_n in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.*

In order to prove Lemma 7.6, we will work by approximation, namely by introducing the following

$$\begin{cases} -\operatorname{div}(a(x)u_{n,k}) + g(x)T_k\left(\left|u_{n,k}\right|^{r-2} u_{n,k}\right) = \frac{f_n(x)}{\left(\left|u_{n,k}\right|+\frac{1}{n}\right)^\theta} & \text{in } \Omega \\ u_{n,k} = 0 & \text{on } \Omega \end{cases} \tag{7.17}$$

where $n, k \in \mathbb{N}, 0 \le f_n(x) := T_n(f(x)) \in L^\infty(\Omega), 0 \le \theta < 1$ and $r \ge 1$. Thanks to [80, Thoerem 2], we know that there exists $u_{n,k} \in W_0^{1,2}(\Omega)$ weak solution to (7.17) for each $n, k \in \mathbb{N}$ fixed. Moreover $u_{n,k} \in L^\infty(\Omega)$ for all $n, k \in \mathbb{N}$ since, if $m \ge 1$ is fixed, taking $G_m(u_{n,k}) \in W_0^{1,2}(\Omega)$ as test function in (7.17) and using that $G_m(u_{n,k})$ and $T_k\left(\left|u_{n,k}\right|^{r-2} u_{n,k}\right)$ have the same sign of $u_{n,k}$, we immediately find that

$$\alpha \int_\Omega \left|\nabla G_m(u_{n,k})\right|^2 \le \int_\Omega f_n G_m(u_{n,k})$$

and so we can proceed as in [81] to end up with $u_{n,k} \in L^\infty(\Omega)$. Moreover the previous L^∞ estimate is independent from $k \in \mathbb{N}$. Now taking $u_{n,k}$ as a test function in the weak formulation of (7.17), we find that $u_{n,k}$ is bounded in $W_0^{1,2}(\Omega)$ with respect to k for $n \in \mathbb{N}$ fixed. Since $u_{n,k}$ is bounded in $L^\infty(\Omega)$ independently on k, for each $n \in \mathbb{N}$ fixed we choose k_n large enough to obtain the following scheme of approximation

$$\begin{cases} -\operatorname{div}(a(x)u_n) + g(x)\,|u_n|^{r-2}\,u_n = \frac{f_n(x)}{\left(|u_n|+\frac{1}{n}\right)^\theta} & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{7.18}$$

where $u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is given by u_{n,k_n}. As concerns the sign of u_n, taking $u_n^- := \min(u_n, 0) \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ as test function in (7.18), we find

$$\int_\Omega a(x)\left|\nabla u_n^-\right|^2 + \int_\Omega g(x)\,|u_n|^{r-2}\left(u_n^-\right)^2 = \int_\Omega \frac{f_n}{\left(|u_n| + \frac{1}{n}\right)^\theta} u_n^- \leq 0$$

and so that $u_n \geq 0$ almost everywhere in Ω.

Lemma 7.7. *The sequence u_n is increasing with respect to $n, u_n > 0$ in Ω, and for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$ (independent on n) such that*

$$u_n(x) \geq c_\omega > 0 \quad \text{for every } \; x \; \text{ in } \omega, \; \text{for every } n \text{ in } \mathbb{N}. \tag{7.19}$$

Moreover there exists the pointwise limit $u \geq c_\omega$ of the sequence u_n.

Proof. Since $0 \leq f_n \leq f_{n+1}$ and $\theta > 0$, one has

$$-\operatorname{div}\left(a(x)\nabla u_n\right) + g(x)\,|u_n|^{r-2}\,u_n = \frac{f_n}{\left(u_n + \frac{1}{n}\right)^\theta} \leq \frac{f_{n+1}}{\left(u_n + \frac{1}{n+1}\right)^\theta},$$

so that

$$\begin{aligned} &-\operatorname{div}\left(a(x)\left(\nabla u_n - \nabla u_{n+1}\right)\right) + g(x)\left(|u_n|^{r-2}\,u_n - |u_{n+1}|^{r-2}\,u_{n+1}\right) \\ &\leq f_{n+1} \frac{\left(u_{n+1}+\frac{1}{n+1}\right)^\theta - \left(u_n+\frac{1}{n+1}\right)^\theta}{\left(u_n+\frac{1}{n+1}\right)^\theta\left(u_{n+1}+\frac{1}{n+1}\right)^\theta}. \end{aligned}$$

We now choose $(u_n - u_{n+1})^+$ as test function and taking into account the monotonicity of the function $t \to |t|^{r-2}t$. For the right hand side we observe that

$$\left[\left(u_{n+1} + \frac{1}{n+1}\right)^\theta - \left(u_n + \frac{1}{n+1}\right)^\theta\right](u_n - u_{n+1})^+ \leq 0,$$

recalling that $f_{n+1} \geq 0$, we obtain

$$0 \leq \alpha \int_\Omega \left|\nabla\left(u_n - u_{n+1}\right)^+\right|^2 \leq 0.$$

Therefore $(u_n - u_{n+1})^+ = 0$ almost everywhere in Ω, which implies $u_n \leq u_{n+1}$. Since u_1 belongs to $L^\infty(\Omega)$, and there exists a constant (only depending on Ω and N) such that

$$\|u_1\|_{L^\infty(\Omega)} \leq C\,\|f_1\|_{L^\infty(\Omega)} \leq C,$$

one has
$$- \operatorname{div}(a(x)\nabla u_1) + g(x)\,|u_1|^{r-2}\,u_1 = \tfrac{f_1}{(u_1+1)^\theta} \geq \tfrac{f_1}{\left(\|u_1\|_{L^\infty(\Omega)}+1\right)^\theta} \geq \tfrac{f_1}{(C+1)^\theta}.$$

Since $\frac{f_1}{(C+1)^\theta}$ is not identically zero, the strong maximum principle implies that $u_1 > 0$ in Ω (see [86]; observe that u_1 is differentiable by Chapter 4 of [65] , and that (7.19) holds for u_1 (with c_ω only depending on ω, N, f_1 and θ). Since $u_n \geq u_1$ for every n in $\mathbb{N}$, (7.19) holds for u_n (with the same constant c_ω which is then independent on n).

Remark 7.8. If u_n and v_n are two solutions of (7.18), repeating the argument of the first part of the proof of Lemma 7.7 shows that $u_n \leq v_n$. By symmetry, this implies that the solution of (7.18) is unique.

Theorem 7.9. *Let $n \in \mathbb{N}$, $0 < \theta < 1$, f be a positive function in $L^\infty(\Omega)$, and let $r > 1$. Then, there exist a solutions $(u_n, \varphi_n) \in \left(W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\right)^2$ of the following system*

$$\begin{cases} -\operatorname{div}(a(x)\nabla u_n) + \varphi_n u_n^{r-1} = \frac{f}{(u_n+\frac{1}{n})^\theta} & \text{in } \Omega, \\ -\operatorname{div}(M(x)\nabla \varphi_n) = u_n^r & \text{in } \Omega \\ u_n, \varphi_n \geq 0 & \text{in } \Omega, \\ u_n = \varphi_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.20}$$

Proof. Fix $\psi_n \in W_0^{1,2}(\Omega)$, let $n \in \mathbb{N}$ and we define $S : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ as the operator such that $v_n = S(\psi_n)$. By the maximum principle, $\psi_n \geq 0$, taking account Lemma 7.6 and Remark 7.8, there, exists a unique solution v_n of:

$$- \operatorname{div}(a(x)\nabla v_n) + \psi_n |v_n|^{r-2} v_n = \frac{f}{(v_n + \frac{1}{n})^\theta}. \tag{7.21}$$

Since, by Lemma 7.6, one has

$$\|v_n\|_{W_0^{1,2}(\Omega)} \leq C_1 \|f\|_{L^\infty(\Omega)}, \quad \|v_n\|_{L^\infty(\Omega)} \leq C_1 \|f\|_{L^\infty(\Omega)}. \tag{7.22}$$

Now we define $T : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ as the operator such that $\zeta_n = T(v_n) = T(S(\psi_n))$. Thanks to the results in [34], ζ_n is the unique weak solution of the Euler-Lagrange equation

$$- \operatorname{div}(M(x)|\nabla \zeta_n|) = |v_n|^r, \quad \zeta_n \in W_0^{1,2}(\Omega). \tag{7.23}$$

Following [5], we thus have
$$\|\zeta_n\|_{W_0^{1,2}(\Omega)} + \|\zeta_n\|_{L^\infty(\Omega)} \leq C_2 \|v_n\|^r_{L^\infty(\Omega)},$$
using (7.22), we deduce that,

$$\|\zeta_n\|_{W_0^{1,2}(\Omega)} + \|\zeta_n\|_{L^\infty(\Omega)} \leq C\|f\|_{L^\infty(\Omega)} =: R, \tag{7.24}$$

where C_1 and C_2 are positive constants not depending on v_n.

We want to prove that $T \circ S$ has a fixed point by Schauder's fixed point theorem. By (7.24) we have that $\overline{B_R(0)} \subset W_0^{1,2}(\Omega)$ is invariant for $T \circ S$. Let $\psi_k =: (\psi_{n,k})_k \subset W_0^{1,2}(\Omega)$ be a sequence weakly convergent to some ψ and let $v_k =: (v_{n,k})_k = S(\psi_k)$. As a consequence of (7.22), there exists a subsequence indexed by v_k such that

$$v_k \to v \text{ weakly in } W_0^{1,2}(\Omega), \text{ and a.e. in } \Omega \tag{7.25}$$

$$v_k \to v \text{ weakly}-* \text{ in } L^\infty(\Omega).$$

Moreover, we have

$$-\operatorname{div}(a(x)\nabla v_k) = \frac{f}{(\frac{1}{n}+v_k)^\theta} - (\psi_k^+)\,|v_k|^{r-2}\, v_k =: g_k$$

and, using Hölder's inequality, the Poincaré inequality and (7.22), we obtain

$$\|g_k\|_{L^1(\Omega)} \le C\|f\|_{L^\infty(\Omega)} + \|v_k\|^{r-1}_{L^\infty(\Omega)}\,\|\psi_k\|_{L^1(\Omega)}$$

$$\le C\|f\|_{L^\infty(\Omega)} + C_1\|f\|^{r-1}_{L^\infty(\Omega)}\,\|\psi_k\|_{W_0^{1,2}(\Omega)} \le C.$$

Then, by Theorem 2.1 in [26], we obtain that ∇v_k converges to ∇v_n almost everywhere in Ω. Since

$$\|\nabla v_k\|_{(L^2(\Omega))^N} = \|v_k\|_{W_0^{1,2}(\Omega)} \le C_1\|f\|_{L^m(\Omega)},$$

thus, we conclude that

$$\nabla v_k \to \nabla v_n \text{ weakly in } \left(L^2(\Omega)\right)^N. \tag{7.26}$$

We recall that v_k satisfies

$$\int_\Omega a(x)\nabla v_k\cdot\nabla w + \int_\Omega \psi_k\,|v_k|^{r-2}\, v_k w = \int_\Omega \frac{f}{(\frac{1}{n}+v_k)^\theta} w, \quad \forall w\in C_c^1(\Omega).$$

Letting k tend to infinity, by (7.25),(7.26) and Vitali's theorem, we have that

$$\int_\Omega |\nabla v_n|\cdot\nabla w + \int_\Omega \psi_n|v_n|^{r-2}v_n w = \int_\Omega \frac{f}{(\frac{1}{n}+v_n)^\theta} w, \quad \forall w\in C_c^1(\Omega),$$

so that v is the unique weak solution of (7.21) and it does not depend on the subsequence. Hence $v_k = S(\psi_k)$ converges to $v_n = S(\psi_n)$ weakly in $W_0^{1,2}(\Omega)$ and weakly$-*$ in $L^\infty(\Omega)$. Then

$$|v_k|^r \to |v_n|^r \text{ strongly in } L^q(\Omega)\ \forall q<+\infty \text{ and } \||v_k|^r\|_{L^1(\Omega)} \le C. \tag{7.27}$$

Thanks to (7.24), (7.27) and proceeding in the same way, we get

$$\zeta_k := \zeta_{n,k} = T(v_k) \to \zeta_n = T(v_n) \text{ weakly in } W_0^{1,2}(\Omega), \text{weakly}-* \text{ in } L^\infty(\Omega), \tag{7.28}$$

$$|\nabla\zeta_k|\,\nabla\zeta_k \to |\nabla\zeta_n|\nabla\zeta_n \text{ weakly in } \left(L^2(\Omega)\right)^N,$$

and ζ is the unique weak solution of (7.23). Now we want to prove that ζ_k converges to ζ strongly in $W_0^{1,2}(\Omega)$. In order to obtain this, by Lemma 5 in [22], it is sufficient to prove the following

$$\lim_{k\to\infty}\int_\Omega |\nabla(\zeta_k-\zeta_n)|^2 = 0. \tag{7.29}$$

We have that

$$\int_\Omega (|\nabla \zeta_k| - |\nabla \zeta_n|) \cdot \nabla (\zeta_k - \zeta_n) = \int_\Omega |\nabla \zeta_k|^2 - \int_\Omega |\nabla \zeta_n| \cdot \nabla \zeta_k - \int_\Omega |\nabla \zeta_k| \cdot \nabla \zeta_n + \|\zeta_n\|^2_{W_0^{1,2}(\Omega)} \tag{7.30}$$

The second and the third term on the right hand side of (7.30) converge, by (7.28), to $\|\zeta_n\|^2_{W_0^{1,2}(\Omega)}$. Then it is sufficient to prove that

$$\lim_{k\to\infty} \|\zeta_k\|^2_{W_0^{1,2}(\Omega)} = \|\zeta_n\|^2_{W_0^{1,2}(\Omega)}. \tag{7.31}$$

Since ζ_k is equal to $T(v_k) \geq 0$, we deduce that

$$\int_\Omega |\nabla \zeta_k|^2 = \int_\Omega |v_k|^r \zeta_k.$$

Using Vitali's Theorem and (7.27), we have that

$$\lim_{k\to\infty} \int_\Omega |v_k|^r \zeta_k = \int_\Omega |v_n|^r \zeta = \|\zeta_n\|^2_{W_0^{1,2}(\Omega)},$$

so that (7.31) is true and (7.29) is proved. Hence we have proved that if ψ_k converges to ψ_n weakly in $W_0^{1,2}(\Omega)$ then $\zeta_k = T(S(\psi_k))$ converges to $\zeta_n = T(S(\psi_n))$ strongly in $W_0^{1,2}(\Omega)$. As a consequence we have that $T \circ S$ is a continuous operator and that $T\left(S\left(\overline{B_R(0)}\right)\right) \subset W_0^{1,2}(\Omega)$ is a compact subset. Then there exists, by Schauder's fixed point Theorem, a function φ_n in $W_0^{1,2}(\Omega)$ such that $\varphi_n = T(S(\varphi_n))$ and, since $T(v_n) \geq 0$ for every v_n in $W_0^{1,2}(\Omega)$, φ_n is nonnegative. Moreover let $u_n = S(\varphi_n)$, we have that (u_n, φ_n) is a solution of (7.20) .

Conclusion and Further Prospects

In this thesis, we have talked about the existence and regularity of solutions for a certain class of elliptic problems. In particular, most of the results we present here are stated for problems with a singular nonlinearity. There are several motivations for our work coming not only from problems in applied mathematics but also from physic alphenomena and applied economical models. For instance, nonlinear singular boundary values problems arise in the context of heterogeneous chemical catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids and boundary layer phenomena for viscous fluids. Moreover, nonlinear singular elliptic equations are also encountered in glacial advance, in the transport of coal slurries down conveyor belts and in several other geophysical and industrial contents.

This work raises a number of questions that researchers will need to explore in further studies, for example, these problems can be generalised to the following spaces: Sobolev spaces with variable exponent, anisotropic Sobolev spaces and fractional Sobolev spaces.

We hope this thesis will contribute to the theory of elliptic operators and will be useful for researchers who wish to work in this field.

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