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LA PROPRIÉTÉ DE SCHREIER ET FACTORISATION DANS LES ANNEAUX DE SEMI-GROUPES

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THE SCHREIER PROPERTY AND FACTORIZATION IN SEMIGROUP RINGS

Résumé

Soit D un anneau commutatif intègre et Γ un monoide commutatif simplifiable et sans torsion. On désigne par $D[\Gamma]$ l'anneau de semi-groupe de Γ sur D. Au cours des dernières années, plusieurs auteurs se sont intéressés par les propriétés de factorisation dans $D[\Gamma]$. Ce qui a conduit à la construction de nouvelles classes d'exemples originaux en algèbre commutative et en théorie de factorisation.
L'objectif principal de cette thèse est l'étude de certaines propriétés de factorisation dans les anneaux de semi-groupes. Ainsi, les chapitres 2 et 3 sont consacrés à l'étude de la propriété de Schreier. Dans le chapitre 2 nous étudions la primalité et la propriété de Schreier dans le contexte plus général des anneaux gradués. Ensuite, nous appliquons cela aux anneaux de semi-groupes. Le chapitre 3 est dévoué aux anneaux de type $A + B[\Gamma^*]$, où $A \subseteq B$ est une extension d'anneaux intègres et Γ est un monoide commutatif simplifiable et sans torsion, avec $\Gamma \cap -\Gamma = \{0\}$. Nous donnons des conditions nécessaires et suffisantes pour que $A + B[\Gamma^*]$ soit de (pré–)Schreier. Dans le cas où B est un anneau de fractions de A ou Γ contient un élément irréductible, notre caractérisation généralise celle connue dans le cas des anneaux de polynômes ($\Gamma = \mathbb{N}$). D'autre part, si B n'est pas un anneau de fractions de A, nous montrons que si l'anneau $A + B[\Gamma^*]$ est de Schreier, alors Γ est un monoide sans atomes.
Dans le chapitre 4, nous essayons d'étendre, aux anneaux $A + B[\Gamma^*]$, une variété de propriétés de factorisation (atomique, ACCP, BFD) plus faibles que celles des anneaux factoriels.

Mots clés : Monoide, anneau de semi-groupe, primal, complètement primal, anneau pré–Schreier, anneau de Schreier, factorisation, atomique, UFD, ACCP, BFD.

Abstract

Let D be an integral domain and Γ a cancellative torsion–free commutative monoid. The semigroup ring of Γ over D is denoted by $D[\Gamma]$. During last few years, several authors have been interested in the factorization properties in $D[\Gamma]$. Then, with the help of these domains, they constructed crucial examples in commutative algebra and factorization theory.

The main focus of this thesis is to study certain factorization properties in semigroup rings. Chapters 2 and 3 are devoted to the study of the Schreier property. In Chapter 2, we study the concepts of primality and Schreier property in the more general context of graded domains, then we specialize to semigroup rings. This leads us to shed more light on the Schreier property in semigroup rings.

In Chapter 3, letting $A \subseteq B$ be an extension of integral domains and Γ a monoid with $\Gamma \cap -\Gamma = \{0\}$, our main result deals with the characterization of when the construction $A + B[\Gamma^*]$ is (pre–)Schreier. In the case where B is a quotient ring of A or Γ is not antimatter, our characterization recovers the case of polynomial rings ($\Gamma = \mathbb{Z}_+$). If B is not a quotient ring of A, we show that $A+B[\Gamma^*]$ is Schreier implies that the monoid Γ must be antimatter. In Chapter 4, we attempt to extend various factorization properties (Atomic, ACCP, BFD), weaker than unique factorization, to $A + B[\Gamma^*]$ domains.

Keywords : Monoid, monoid domain, primal, completely primal, pre–Schreier domain, Schreier domain, factorization, atomic, UFD, ACCP, BFD.

Author's papers involved in this thesis:

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Introduction (French)

Dans un anneau commutatif, la factorisation est une opération qui consiste à décomposer un élément en un produit d'éléments. En théorie des nombres, la factorisation a été traitée par plusieurs mathématiciens tels que Euler, Fermat, Dedekind, Gauss et d'autres.
Les grands progrès connus en cryptographie ont été source de motivation et de developpement de la recherche sur la factorisation en théorie des nombres. En effet, cela est dû au fait que la sécurité de la cryptographie à clé publique dépend de la difficulté à factoriser les nombres entiers. De plus, la capacité à factoriser des grands nombres fournirait un mécanisme pour casser les cryptages à clé publique.
D'après le théorème fondamental de l'arithmétique, tout nombre entier $n > 1$ possède une factorisation unique en produit de nombres premiers (UFD). Cependant, il existe des anneaux qui ne possèdent pas cette propriété de factorisation unique par exemple, l'anneau $\mathbb{Z}[\sqrt{-5}]$ n'est pas factoriel. Pour comprendre ce défaut de la non unicité de factorisation dans certains anneaux, les chercheurs étendirent la notion de factorisation aux polynômes et aux idéaux. En fait, la factorisation des polynômes remonte au XVII^{e} siècle avec le procédé de Newton et Leibniz qui avaient pris comme facteurs les polynômes quadratiques et linéaires. Ensuite, Bernouilli introduisit la notion de polynômes irréductibles. Vers 1876, Dedekind proposa la possibilté de factoriser un idéal en un produit d'idéaux premiers. C'est dans ce cadre que l'étude de la factorisation dans les anneaux commutatifs, trouve ses origines.

Tous les anneaux considérés dans cette thèse sont supposés commutatifs, unitaires et intègres.

Soit D un anneau intègre. Un élément $a \in D \setminus \{0\}$ est dit primal si, pour tout $x_1, x_2 \in D \setminus \{0\}$ tels que $a \mid x_1x_2$, alors il existe $a_1, a_2 \in D$, vérifiant $a = a_1a_2$, avec $a_i \mid x_i$ pour $i = 1, 2$. Cette notion remonte à 1967 avec les travaux de Cohn [14] qui a ensuite introduit la notion d'anneau de Schreier pour citer tout anneau intégralement clos dont tout élément non nul est primal. Par exemple, tout anneau à PGCD est de Schreier [14, Theorem 2.4], mais la réciproque est, en général, fausse [8, Example 2.10]. Cohn a également étudié le transfert de la propriété de Schreier aux anneaux de fractions [14, Theorem 2.6]. Ceci lui a permi d'établir la propeiété de Schreier pour les anneaux de polynômes [14, Theorem 2.7].

Vers 1987, Zafrullah dans son article [42] a défini le concept d'anneau pré–Schreier pour désigner un anneau dont tout élément non nul est primal. Donc, un anneau de Schreier est un anneau pré–Schreier qui est intégralement clos. L'étude de ces propriétés a été developpée par plusieurs auteurs [7, 8, 37].

Une partie importante de cette thèse est consacrée à l'étude de la propriété de Schreier dans les anneaux de semi-groupes. Le livre de Gilmer [25] sera notre référence principale pour la théorie des anneaux de semi-groupes.

En 1988, Matsuda a étudié la propriété de Schreier pour un anneau de semi-groupe $D[S]$, où S est un monoide commutatif simplifiable et sans torsion. Il a établit que l'anneau de groupe $D[G]$, où G est un groupe abélien, est un anneau de Schreier si, et seulement si, D est un anneau de Schreier [35, Proposition 4.5]. Ensuite, il a donné des conditions nécessaires et suffisantes pour que l'anneau de semi-groupe $D[S]$ soit de Schreier [35, Proposition 4.6].

Soit $R = \oplus_{\alpha \in \mathit{\Gamma}} R_\alpha$ un anneau $\mathit{\Gamma}$–gradué, où $\mathit{\Gamma}$ est un monoide commutatif simplifiable et sans torsion et H l'ensemble des éléments homogènes non nuls de R. Dans leur article [13] publié en 2005, les auteurs ont étudié la propriété de Schreier dans les anneaux gradués et ils ont obtenu une caractérisation en termes d'éléments homogènes [13, Theorems 2.1 et 2.2]. Dans ce contexte, ils ont introduit la notion de gr–pré–Schreier pour les anneaux gradués comme suit: l'anneau gradué R est dit gr–pre–Schreier si, tout élément $x \in H$ est gr–primal. C'est-à-dire, pour tout $y_1, y_2 \in H$ tels que $x \mid y_1 y_2$, alors il existe $x_1, x_2 \in H$, vérifiant $x = x_1 x_2$ avec $x_i \mid y_i$ pour $i = 1, 2$. Entre autre, leurs résultats recouvrent les travaux de Matsuda sur les anneaux de semi-groupes [35, 36].

Soient $A \subseteq B$ une extension d'anneaux intègres et $\mathit{\Gamma}$ un monoide commutatif (additif), simplifiable et sans torsion tel que $\mathit{\Gamma} \cap -\mathit{\Gamma} = \{0\}$. Alors $R = A + B[\mathit{\Gamma}^*]$ est un sous–anneau de l'anneau de semi-groupe $B[\Gamma]$. Notons que R peut s'obtenir comme un produit fibré avec $B[\Gamma^*]$ un ideal commun de R et $B[\mathit{\Gamma}]$. Si $\mathit{\Gamma} \cap -\mathit{\Gamma} \neq \{0\}$ ou si $A = B$, l'anneau R coincide avec $B[\mathit{\Gamma}]$. Si $\mathit{\Gamma} = \mathbb{N}$, alors $R = A + XB[X]$,

et si $\Gamma = \mathbb{N}^n$, alors $R = A + (X_1, \dots, X_n)B[X_1, \dots, X_n]$. D'après [25, Corollary 3.4], le monoide Γ admet un ordre total $\prec$ compatible avec l'addition. Puisque $\Gamma \cap -\Gamma = \{0\}$, nous pouvons supposer que $\alpha \succcurlyeq 0$ pour tout $\alpha \in \Gamma$. Donc tout élément $f \in R$ s'écrit d'une manière unique sous la forme $f = a + b_1 X^{\alpha_1} + \cdots + b_n X^{\alpha_n}$, où $a \in A$, $b_i \in B$ et $\alpha_i \in \Gamma^*$, avec $\alpha_1 \prec \cdots \prec \alpha_n$. Si $b_n \neq 0$, il est appelé le coefficient dominant de f et α_n son degré. La construction $A + B[\Gamma^*]$ a été intensivement étudiée par plusieurs auteurs et s'est avérée utile pour construire des exemples et des contre-exemples en théorie des anneaux commutatifs [17, 21, 31, 32].
Motivés par le travail fait dans [13, 20], nous considérons la question suivante; déjà posée par Zafrullah dans [43]:

> **($\mathbf{Q}_1$)** Quand est-ce que l'anneau $A + B[\Gamma^*]$ est de (pré–)Schreier?

D'autre part, à la base de toute propriété de factorisation nous trouvons la notion d'atome: on dit qu'un élément non nul et non inversible $x \in D$ est irréductible (atome) si, à chaque fois que $x = ab$ dans D, alors a ou b est inversible. L'anneau D est dit atomique si, tout élément non nul et non inversible de D se factorise en produit fini d'atomes [14]. La classe des anneaux factoriels et celle des anneaux noethériens sont des exemples importants d'anneaux atomiques.
La factorialité des anneaux de semi-groupes a été étudiée par Gilmer et Parker dans [26, Theorem 7.17]. Pour un anneau de semi-groupe $D[S]$, ils ont établi que:

> *$D[S]$ est factoriel si et seulement si D et factoriel, S est un monoide factoriel et tout élément du groupe maximal H de S est de type* $(0, 0, 0, ...)$.

Pour les anneaux atomiques, Gilmer a posé le problème suivant [25]:

> **($\mathbf{Q}_2$)** Si D et S sont atomiques, l'anneau de semi-groupe $D[S]$ est-il atomique?

La réponse négative à cette question [16, 40] a donné lieu à de nouveaux problèmes de recherche dans la théorie de la factorisation dans les anneaux de polynômes et les anneaux de semi-groupes.

Notons que dans le cadre de factorisation non unique de nouvelles propriétés ont été introduites. Parmi ces propriétés que nous allons étudier, citons la condition de chaine ascendante sur les ideaux principaux (ACCP) et la notion d'anneau à factorisation bornée (BFD): les longueurs des factorisations d'un élément donné en produits d'atomes sont bornées, voir [3]. Nous avons les implications suivantes:

$$UFD \Longrightarrow BFD \Longrightarrow ACCP \Longrightarrow \textit{Atomique}$$

Pour les anneaux de semi-groupes, ces différentes propriétés ont été étudiées dans [30, 34]. Ceci nous amène à poser la question suivante:

> **(Q$_3$)** Sous quelles conditions l'anneau $A + B[\Gamma^*]$ est atomique, satisfait la condition ACCP ou possède la propriété BFD?

Pour les anneaux $A + XB[X]$ ($\Gamma = \mathbb{N}$), cette question a été étudiée par plusieurs auteurs [6, 12, 20].

Le but de cette thèse est d'étudier et étendre certains résultats bien connus sur différents concepts de factorisation cités ci-dessus pour les anneaux de polynômes, aux anneaux de semi–groupes. Entre autre, nous allons répondre aux questions **(Q$_1$)** et **(Q$_3$)**.
La thèse est divisée en quatre chapitres recouvrant le contenu de trois articles.

Le chapitre 1 fournit les résultats de base et établit les notations nécessaires à notre étude. On y rappelle des définitions et des résultats sur la propriété de Schreier ainsi que quelques propriétés de factorisation dans les anneaux intègres.

Dans le chapitre 2, nous étudions la notion de primalité et la propriété de (pré–)Schreier dans les anneaux gradués, puis nous appliquons ces résultats aux anneaux de semi-groupes. Ainsi, dans la section 2.1, nous caractérisons les éléments primals (resp., la propriété d'être (pre–)Schreier) pour un monoide multiplicatif. Ensuite, nous étendons le théorème (Nagata type theorem) dû à Cohn [14, Theorem 2.6] au cas des monoides. La section 2.2 est réservée à l'étude de la primalité dans les domaines gradués. En fait, nous montrons quand un élément homogène non nul est primal ou complètement primal. Comme application, dans la dernière section, nous caractérisons les éléments primals dans les anneaux de semi-groupes. Les résultats de ce chapitre sont publiés dans [9].

Le chapitre 3 a pour but de caractériser les constructions de type $A + B[\Gamma^*]$ qui sont (pré–)Schreier. Ces résultats sont publiés dans [10]. Nous commençons la section 3.1 par une caractérisation de la propriété de (pré–)Schreier en termes de la propriété gr–pré–Schreier pour $A + B[\Gamma^*]$. Dans la section 3.2, nous étudions la primalité des éléments homogènes non nuls dans l'anneau $A + A_S[\Gamma^*]$, où S est une partie multiplicative de A. Ensuite, nous donnons des conditions nécessaires et suffisantes pour que cette construction soit (pré–)Schreier (Theorem 3.2.7 et Corollary 3.2.8). La section 3.3 est consacrée à l'étude de la propriété d'être (pré–)Schreier dans les anneaux de type $A+B[\Gamma^*]$, où B n'est pas nécessairement un anneau de fractions de A (Theorem 3.3.16 et Corollary 3.3.17). Enfin, pour illustrer nos résultats, nous donnons des exemples originaux d'anneaux de Schreier et pré–Schreier.

Le chapitre 4 est dédié à la généralisation de certaines propriétés de factorisation non unique aux anneaux de type $A+B[\Gamma^*]$ [11]. Ainsi, la section 4.1 est dévouée à la propriété ACCP et dans la section 4.2, sous certains conditions, nous caractérisons les anneaux $A+B[\Gamma^*]$ qui sont atomiques (Theorem 4.2.1). La section 4.3 est consacrée au concept de factorisation bornée dans les anneaux $A+B[\Gamma^*]$. Nos résultats généralisent le cas bien connu des constructions de type $A+XB[X]$.

Introduction

In a commutative ring, a factorization is an operation that consists of decomposing an element into a product. In number theory, factorization has been dealt with by several mathematicians such as Euler, Fermat, Dedekind, Gauss and others.
With the advent of public key cryptography, research in factoring integers was invigorated. The security of public key cryptography depends on the difficulty of factoring integers, and the ability to factor large integers would provide a mechanism for breaking public key ciphers.
By the fundamental theorem of arithmetic, every positive integer $n > 1$ has a unique factorization into prime elements (UFD). However, there are some integral domains that fails to be UFD such as $\mathbb{Z}[\sqrt{-5}]$. In order to understand a non-unique factorization, the mathematicians studied the factorization in the case of polynomials and ideals.

The factorization of the polynomials with integer coefficients, back to the $XVII$th century, starting by the Newton and Leibniz processes which had taken as factors the quadratic and linear polynomials, then Bernoulli introduced irreducible elements in factorization. In 1876, Dedekind proposed the possibility to factorize an ideal to product of ideals. It was in this setting that the study of factorization in integral domains arose.

All rings considered in this thesis are integral domains, that is unitary commutative rings without zero divisors.

Let D be an integral domain. In [14], Cohn called an element $a \in D\backslash\{0\}$ primal if for all $x_1, x_2 \in D\backslash\{0\}$, $a \mid x_1x_2$ implies that $a = a_1a_2$ for some $a_i \in D$ with $a_i \mid x_i$, $i = 1, 2$. Then he called D a Schreier domain, if D is integrally closed and every nonzero element of D is primal. For instance any GCD domain is Schreier [14, Theorem 2.4], but the converse is false in general, as asserted in [8, Example 2.10]. Cohn studied also the behaviour of the Schreier property under localization

[14, Theorem 2.6]. This allowed him to extend the Schreier property to polynomial rings [14, Theorem 2.7].

In his 1987 paper [42], Zafrullah called D pre–Schreier if every nonzero element of D is primal. So a Schreier domain is an integrally closed pre–Schreier domain. The study of these two concepts was continued in [7] and [37].
The purpose of this thesis is to explore some topics in factorization theory in semigroup rings. The book of Gilmer [25] will be our main reference for the theory of semigroup rings.

In 1988, Matsuda studied the Schreier property for a semigroup ring $D[S]$, where S is a cancellative torsion–free commutative monoid. He showed that $D[G]$, where G is an abelian group, is Schreier if and only if D is Schreier [35, Proposition 4.5]. Then, he established necessary and sufficient conditions for $D[S]$ to be Schreier [35, Proposition 4.6].

Let $R = \oplus_{\alpha \in \mathit{\Gamma}} R_\alpha$ be a $\mathit{\Gamma}$–graded domain, where $\mathit{\Gamma}$ is a cancellative torsion-free commutative monoid and H the set of nonzero homogeneous elements of R. In [13], the authors investigated the (pre–)Schreier property in a graded domain. They obtained a nice characterization of this property in terms of homogeneous elements [13, Theorems 2.1 and 2.2]. In this setting, they called a domain R gr–pre–Schreier if every element of H is gr–primal, that is: for each $x \in H$, if whenever $x \mid y_1 y_2$, with $y_1, y_2 \in H$, then $x = x_1 x_2$, $x_1, x_2 \in H$, where $x_i \mid y_i$, $i = 1, 2$. In the case of a semigroup rings $R = D[S]$, their results recover the work of Matsuda [35, 36].

Let $A \subseteq B$ be an extension of integral domains and $\mathit{\Gamma}$ a cancellative torsion–free commutative monoid such that $\mathit{\Gamma} \cap -\mathit{\Gamma} = \{0\}$. Then $R = A + B[\Gamma^*]$ is a subring of the semigroup ring $B[\Gamma]$. Note that R can be obtained as a pullback and $B[\Gamma^*]$ is a common ideal of R and $B[\Gamma]$. If $\Gamma \cap -\Gamma \neq \{0\}$ or $A = B$, the ring R coincides with $B[\Gamma]$. If $\Gamma = \mathbb{Z}_+$, then $R = A + XB[X]$, and if $\Gamma = \mathbb{Z}_+^n$, then $R = A + (X_1, \ldots, X_n)B[X_1, \ldots, X_n]$. The monoid Γ admits a total order $\prec$ compatible with its semigroup operation [25, Corollary 3.4], and since $\Gamma \cap -\Gamma = \{0\}$, we may assume that $\alpha \succcurlyeq 0$ for all $\alpha \in \Gamma$. Hence each $f \in R$ is uniquely expressible in the form $f = a + b_1 X^{\alpha_1} + \cdots + b_n X^{\alpha_n}$, where $a \in A$, $b_i \in B$ and $\alpha_i \in \Gamma^*$, with $\alpha_1 \prec \cdots \prec \alpha_n$. If $b_n \neq 0$, it is called the leading coefficient of f and α_n the degree. The construction $A + B[\Gamma^*]$ has been studied by many authors and has proven to be useful in constructing examples and counterexamples in many areas of commutative

ring theory [17, 21, 31, 32].
Motivated by the work in [13, 20], we consider the following question:

> **(Q$_1$)** When is $A + B[\Gamma^*]$ a (pre–)Schreier domain?

Notice that this question was asked by Zafrullah in [43].

Another important factorization property that we deal with in semigroup rings is the property of being atomic. By an irreducible element or atom we mean a nonzero nonunit $x \in D$ such that $x = ab$ in D, implies that a or b is a unit. A domain D is said to be atomic if each nonzero nonunit of D is a product of a finite number of atoms of D [14]. Noetherian domains and UFDs are two important well known examples of atomic domains. The UFD property for monoid domains was studied by Gilmer and Parker in [26, Theorem 7.17]. For a semigroup ring $D[S]$, they provide the following characterization:

> *$D[S]$ is a UFD if and only if D is a UFD, the monoid S is a UFM and each element of the maximal subgroup H of S is of type (0,0,0,...).*

For the atomic property, Gilmer asked the following question [25]:

> **(Q$_2$)** Is the semigroup ring $D[S]$ atomic provided that both D and S are atomic?.

The negative answer to this question [16, 40] gave rise to new research problems in the theory of factorization in polynomial rings and semigroup rings. Various properties related to atomic domains have been studied extensively. In this work, we will mainly focus on the atomic property, the ascending chain condition on principal ideals (ACCP), and bounded factorization property (BFD), that is an atomic domain in which each nonzero nonunit element has a bound on the length of factorizations into products of atoms see [3]. We have the following implications:

$$UFD \Longrightarrow BFD \Longrightarrow ACCP \Longrightarrow \textit{Atomic}$$

For semigroup rings, these properties were studied in [30, 34]. Also, for the construction $A + XB[X]$, these questions were studied by several authors [6, 12, 20]. This leads us to ask the following question:

> **(Q$_3$)** Under which conditions is $A + B[\Gamma^*]$ an atomic domain, satisfies ACCP or has the BFD property?

The purpose of this thesis is to investigate and extend some well known results on different concepts of factorization in the case of polynomial rings, to semigroup rings. Among other things, this leads to an answer of the questions $(\mathbf{Q}_1)$ and $(\mathbf{Q}_3)$. The thesis is divided into four chapters and it recovers the results of three papers, as follows:

Chapter 1 provides the background information and sets up the notations needed in our study. We recall some definitions and results on the Schreier property and some factorization properties in an integral domain.

In Chapter 2 we investigate the primality and the (pre–)Schreier property in graded domains and then specialize to semigroup rings. Thus, in Section 2.1, we characterize the primal elements (resp., the (pre–)Schreier property) in a multiplicative monoid. Then we extend the well known Nagata type theorem for Schreier domains, due to Cohn [14, Theorem 2.6], to monoids. In Section 2.2, we study the primality in graded domains. So we determine when a nonzero homogeneous element is primal or completely primal. Then, as an application, in the last section of Chapter 2 we characterize the primal elements in semigroup rings. The results of this chapter are published in [9].

Chapter 3 has the goal of characterizing when a construction of type $A + B[\Gamma^*]$ is (pre–)Schreier and these results are published in [11]. We begin Section 3.1 by a characterization of the pre–Schreier property in terms of the gr–pre–Schreier property for $A + B[\Gamma^*]$. In Section 3.2, we investigate the primality of the nonzero homogeneous elements in $A + A_S[\Gamma^*]$ domains, where S is a multiplicative subset of A. Then, necessary and sufficient conditions for this construction to be (pre–)Schreier are given (Theorem 3.2.7 and Corollary 3.2.8). Section 3.3 is devoted to the study of the (pre–)Schreier property in $A + B[\Gamma^*]$, where B is not necessarily a quotient ring of A (Theorem 3.3.16 and Corollary 3.3.17). Lastly, we give original and new examples of (pre–)Schreier domains.

Chapter 4 focuses on generalizing some factorization properties, weaker than unique factorization, to constructions of the form $A + B[\Gamma^*]$ [10]. Thus, Section 4.1 is devoted to the ACCP property. In Section 4.2, with the assumption that Γ satisfies ACCP and of *rank*≥ 2, we characterize domains $A + B[\Gamma^*]$ that are atomic (Theorem 4.2.1). In Section 4.3, we investigate the BFD property. The results obtained generalize the case of polynomial rings $A + XB[X]$.

Chapter 1

Preliminaries

In this chapter, we recall some basic concepts and present some known results, in the literature, that we will need in this work. For more details, the references are systematically given.

1.1 Commutative monoids

A semigroup is a nonempty set closed under an associative binary operation. If $(S, \star)$ is a semigroup, then S is commutative (or abelian) if it is commutative under the operation $\star$, and S has an identity element if there exists an identity element with respect to $\star$. A semigroup with identity is called a monoid. In this work all semigroups are commutative monoids.

Let S be an additive monoid. We let S^* denote the set of all nonzero elements of S while we let $U(S)$ denote the set of invertible elements of S.

Definition 1.1.1. 1. If $U(S) = \{0\}$, we say that S is reduced.

2. S is cancellative if for $x, y, s \in S$; $x + s = y + s$ implies that $x = y$.

3. S is torsion–free if for all $x, y \in S$ and $n \in \mathbb{N}^*$; $nx = ny$ implies that $x = y$.

An ideal of S is a nonempty subset I of S such that $I \supseteq s + I = \{s + i \mid i \in I\}$ for each $s \in S$. An ideal I of S is proper if $I \neq S$.

Proposition 1.1.2 ([25, Corollary 1.5])**.** *A monoid S admits a total order compatible with its semigroup operation if and only if S is cancellative and torsion–free.*

For $M \subseteq S$, we let $\langle M \rangle$ denote the smallest submonoid of S containing M, and we call it the submonoid of S generated by M.
Let $T \subseteq S$ be an additive closed subset of S, we have the quotient monoid $S_T = \{s - t, s \in S, t \in T\}$. If $T = S$, we get $G = \langle S \rangle$ the quotient group of S.

Definition 1.1.3. We say that S is integrally closed if, for each $n \in \mathbb{N}^*$ and $x \in G$, $nx \in S$ implies that $x \in S$.

For a monoid S, the rank of S, denoted by $rank(S)$, is defined to be the rank of the quotient group $\langle S \rangle$.
Several results in multiplicative ideal theory for rings can be translated into the language of monoids. For more details, see [25].

1.2 Semigroup rings

Let S be an additive monoid and R be an integral domain.
In 1951, N. Jacobson defined in [29] the semigroup ring of S over R to be the set of functions f from S to R that are finitely nonzero with addition and multiplication defined as follows:

$$(f+g)(s) = f(s) + g(s)$$

$$(fg)(s) = \sum_{t+u=s} f(t)g(u)$$

where the symbol $\sum_{t+u=s}$ is taken over all pair (t, u) of elements of S such that $t + u = s$.
In this thesis, we adopt the notation of Northcott [39] and write $R[S]$ for the semigroup ring of S over R. If S is cancellative torsion–free monoid ordered by $\prec$, then an element f of $R[S]$ is uniquely expressible as $f = r_1X^{s_1} + r_2X^{s_2} + \cdots + r_nX^{s_n}$, where each $r_i \in R$ and $s_i \in S$ with $s_1 \prec \cdots \prec s_n$. If $r_n \neq 0$ it is called the leading coefficient of f and denoted by $lc(f)$ and s_n is called the degree.

A polynomial ring over R is a semigroup ring over R with S coincide with the set of nonnegative integers $\mathbb{Z}_+$. For more details on semigroup rings see [25, 26].

We recall the following basic properties of a semigroup ring (cf. [25]).

Proposition 1.2.1. *Let R be an integral domain and S an additive monoid.*

1. *$R[S]$ is an integral domain if and only if S is a cancellative torsion-free commutative monoid.*

2. *$R[S]$ is integrally closed if and only if both R and S are integrally closed.*

3. *The units of $R[S]$ are the units of R and the monomials aX^s where a and s are respectively units of R and S.*

1.3 Graded domains

By a graded domain $R = \oplus_{\alpha \in \Gamma} R_\alpha$, we mean an integral domain R graded by an arbitrary cancellative torsion–free monoid Γ. We denote by H the multipilcative monoid of nonzero homogeneous elements of R.
The semigroup rings $A[\Gamma]$, where A is an integral domain, constitute perhaps the most important class of Γ–graded domains with $deg(X^\alpha) = \alpha$.

If $S \subseteq H$, is a multiplicative set of R, that is, a submonoid of H. The ring of fractions R_S is graded by some quotient monoid of Γ with the nonzero homogeneous elements are of the form $\frac{h}{s}$, where $h \in H$ and $s \in S$. In particular, the quotient ring $\mathcal{H}(R) := R_H$ is a $\langle \Gamma \rangle$–graded domain, called the homogeneous quotient field of R. We have:

Proposition 1.3.1 ([2, Proposition 2.1])**.** *$\mathcal{H}(R)$ is a completely integrally closed GCD–domain.*

Let $x \in R = \oplus_{\alpha \in \Gamma} R_\alpha$, and $x = x_1 + \cdots + x_n$ be the unique representation of x as a sum of homogeneous elements. We define the content of x, denoted by $C(x)$, to be $C(x) = (x_1, ..., x_n)$. Thus $C(x)$ is a finitely generated homogeneous ideal of R.

A fractional ideal I of R is homogeneous if $uI \subseteq R$ is a homogeneous ideal of R for some $u \in H$. Clearly, a homogeneous fractional ideal of R is a submodule of $\mathcal{H}(R)$. The content of an element can be extended as follows. Let $x \in \mathcal{H}(R)$, $x = x_1 + \cdots + x_n$ with $deg(x_i) \prec deg(x_j)$ for $i < j$. The content of x is the R–submodule of $\mathcal{H}(R)$, $C(x) = (x_1, ..., x_n)$. Note that a fractional ideal $I \subseteq \mathcal{H}(R)$ of R is homogeneous if and only if $C(x) \subseteq I$ for every $x \in I$. The content satisfies the Dedekind–Mertens lemma for graded domains [38].

Proposition 1.3.2 ([1, Theorem 2.1])**.** *Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a Γ–graded integral domain. For every $x, y \in \mathcal{H}(R)$, there is a positive integer n such that $C(x)^n C(xy) = C(x)^{n+1} C(y)$.*

Following [1], we say that the graded domain R is almost normal if for each homogeneous element $x \in R_H$ of nonzero degree which is integral over R is actually in R. Notice that almost normality is weaker than the integrally closed property. We have the following proposition:

Proposition 1.3.3. *Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then R is integrally closed if and only if R is almost normal and R_0 is integrally closed in $(R_H)_0$.*

Notice that if we put all together [1, Section 1, Theorems 3.2 and 3.5] and Proposition 1.3.3, we characterize when the graded domain R is almost normal and integrally closed.
Recall that an extension of domains $A \subseteq B$ is inert if whenever $bb' \in A$ for some $b, b' \in B$, then $b = au$ and $b' = a'u^{-1}$ for some $a, a' \in A$ and u a unit of B.

Proposition 1.3.4. *Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. We consider the following statements:*

(i) *R is integrally closed;*

(ii) *For $h \in H$ and $x \in R$, $(h) : (x)$ is homogeneous;*

(iii) *R is almost normal.*

Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii). If moreover, R contains a (homogeneous) unit of nonzero degree, the three conditions are equivalent, and if $R_0 \subseteq R$ is inert, then (ii)$\Leftrightarrow$(iii).

An important case of when the extension $R_0 \subseteq R$ is inert is when the monoid Γ satisifies the condition $\Gamma \cap -\Gamma = \{0\}$ [1].

Let $A \subseteq B$ be an extension of integral domains and Γ a cancellative torsion–free commutative monoid. Let $\Gamma^* = \Gamma \backslash \{0\}$. Assume that $\Gamma \cap -\Gamma = \{0\}$. Then $R = A + B[\Gamma^*]$ is a subring of $B[\Gamma]$. Note that R can be obtained as pullback with $B[\Gamma^*]$ is a common ideal of R and $B[\Gamma]$. If $\Gamma = \mathbb{Z}_+$, then $R = A + XB[X]$, and if $\Gamma = \mathbb{Z}_+^n$, then $R = A + (X_1, ..., X_n)B[X_1, ..., X_n]$. By Proposition 1.1.2, Γ admits a total order $\preccurlyeq$, so we may assume that $\alpha \succcurlyeq 0$ for all $\alpha \in \Gamma$. Hence, each $f \in R$ is uniquely expressible in the form $f = a + b_1X^{\alpha_1} + \cdots + b_nX^{\alpha_n}$, where $a \in A$, $b_i \in B$ and $\alpha_i \in \Gamma^*$, with $\alpha_1 \prec \cdots \prec \alpha_n$. If $b_n \neq 0$, it is called the leading coefficient of f and α_n the degree. These constructions were studied by several authors [21, 31, 32].

The construction $R = A + B[\Gamma^*]$ is an interesting example of graded domains, with the monoid of nonzero homogeneous elements is $H = A^* \cup \{bX^\alpha \mid b \in B^*, \ \alpha \in \Gamma^*\}$.

1.4 Schreier domains

Let R be an integral domain. Following Cohn [14], an element $x \in R$ is called:

- Primal, if whenever $x \mid a_1a_2$ with $a_1, a_2 \in R$, x can be written in R as $x = x_1x_2$ such that $x_i \mid a_i$, $i = 1, 2$.
- Completely primal, if every factor of x is primal.

In 1987, Zafrullah [42] introduced a pre–Schreier domain as a domain in which every element is (completely) primal. Thus an integrally closed pre–Schreier domain is called a Schreier domain [14]. The Schreier property generalizes the GCD property.

The behaviour of Schreier rings under localization was described by the following:

Proposition 1.4.1. ([14, Theorem 2.6]) Let R be an integrally closed integral domain and S a multiplicative subset of R.

1. If R is a Schreier domain, so is R_S.
2. If R_S is a Schreier domain and S is generated by the completely primal elements of R, then R is a Schreier domain (Nagata type Theorem).

As an application of the Proposition 1.4.1, Cohn proved the following result:

Proposition 1.4.2. ([14, Theorem 2.7]) Let R be a Schreier domain and X an indeterminate, then $R[X]$ is a Schreier domain.

1.5 Factorization properties in an integral domain

The aim of this section is to recall some definitions about some factorization properties that we will deal with in this thesis.

Let R be an integral domain. We denote by $U(R)$ the multiplicative group of units of R.

A nonzero nonunit $a \in R$ is said to be irreducible or atom if $a = bc$ implies $b \in U(R)$ or $c \in U(R)$. Two elements $a, b \in R$ are said to be associated, denoted $a \sim b$, if $a \mid b$ and $b \mid a$. Note that $a \sim b$ if and only if $b = ua$ for some $u \in U(R)$.
A nonzero element $p \in R$ is called prime if $p \mid ab$ implies $p \mid a$ or $p \mid b$, for $a, b \in R$.

Every prime element is irreducible but an irreducible element need not be prime. A domain in which the notions of irreducible and prime coincide is called AP-domain [18]. Examples of such domains are GCDs and pre–Schreier domains.

Recall that R is said to be a unique factorization domain (UFD) if every nonzero, nonunit has a unique factorization into a finite product of irrecucibles, up to order of the factors and associates.

Definition 1.5.1.

1. We say that R is atomic if every nonzero nonunit element of R has a factorization into a finite number of irreducibles (atoms).
2. We say that R satisfies the ascending chain condition on principal ideals (ACCP) if there does not exist an infinite strictly ascending chain of principal ideals of R.
3. Wa say that R is a bounded factorization domain (BFD) if it is atomic and for each nonzero element there is a bound on the lengths of factorizations into products of atoms.

In general, we have the following implications [3]:

$$UFD \Longrightarrow BFD \Longrightarrow ACCP \Longrightarrow Atomic$$

The converse of these implications do not hold in general [3, 4, 27, 41].

Important examples of atomic domains are UFDs and Noetherian domains.

Notice that an atomic Schreier domain is a UFD [14, Theorem 2.3].

Finally, a domain with no atoms will be called an antimatter domain. These domains were studied in [15].

Chapter 2

The Schreier property in semigroup rings

2.0 Introduction

Let A be an integral domain. Following P.M. Cohn [14], an element $x \in A$ is primal if whenever $x \mid a_1a_2$ with $a_1, a_2 \in A$, x can be written as $x = x_1x_2$ such that $x_i \mid a_i$, $i = 1, 2$, and x is completely primal if every factor of x is primal. A ring in which every element is (completely) primal is called a pre-Schreier domain [42] and an integrally closed pre-Schreier domain is called a Schreier domain [14]. The Schreier property generalizes the GCD property.

The primality of an element in a domain depends only on the multiplicative semigroup of nonzero elements of that domain. This led several authors to study the primality in the more general context of semigroups. Let S be a commutative multiplicative cancellative monoid. For $s, t \in S$, $s \mid t$ if $t = sr$ for some $r \in S$. An element $s \in S$ is primal if for $t_1, t_2 \in S$, $s \mid t_1t_2$ implies $s = s_1s_2$ where $s_1, s_2 \in S$ and $s_i \mid t_i$ for $i = 1, 2$. Completely primal elements and the (pre–)Schreier property for monoids are defined similarly.

In a polynomial ring, in the indeterminate X, over a ring A, the powers X^n, $n \in \mathbb{N}$, are primal, i.e., $X^n \mid fg$ for some $f, g \in A[X]$, then $f = X^r f_1$ and $g = X^{n-r} g_1$ for some $r \in \mathbb{N}$, $f_1, g_1 \in A[X]$. This fact is crucial when working with polynomials. This raises the question of whether this result can be extended to powers X^α, $\alpha \in S$, S a semigroup. Note that in this case X^α is not necessarily a power of a prime element like in the polynomial rings. On the other hand, an intersting work on the

Schreier property for semigroup rings was made by Matsuda [36], and Brookfield and Rush [13] . In [13], the authors showed that a semigroup ring is pre-Schreier if and only if it is Schreier.
The aim of this chapter is to deepen and shed new light on primality in semigroup rings. In Section 2.1, we write some well known results on primal elements and Schreier property, in domains and ordered groups, in the language of monoids. In Section 2.2, we study primality in the more general context of graded domains. In [13] it was shown that in graded domains the investigation of the Schreier property can be reduced to the study of the primality of the homogeneous elements. In this section we characterize the (completely) primality of an homogeneous element in terms of its (completely) primality in the multiplicative semigroup of nonzero homogeneous elements. In the integrally closed case, we get an equivalence between these two primalities. As an application, in Section 2.3, we characterize primal elements in semigroup rings. In particular, we investigate the primality of the powers X^α in a semigroup ring and recover the case of polynomial rings.

2.1 Primal elements in Monoids

Throughout this section a monoid means a multiplicative cancellative unitary commutative semigroup. Let S be a monoid. If $T \subseteq S$ is a multiplicative subset of S, then we get the fraction monoid $S_T := \{s/t \mid s \in S, t \in T\}$. If $T = S$, we have the quotient group of S, $G = \langle S \rangle$.

The aim of this section is to translate and adapt the proofs of some well known results on primality and the Schreier property in domains and partially ordered groups, by using the language of monoids. These results will be needed next in the case of graded domains and semigroup rings.

Let $s, t \in S$. We say that s divides t, denoted $s \mid t$ if $t = sr$ for some $r \in S$. We make use of the preorder on S: $s \leq t$ if $s \mid t$. An element $s \in S$ is called primal if for $t_1, t_2 \in S$, $s \leq t_1t_2$ implies $s = s_1s_2$, where $s_1, s_2 \in S$ and $s_i \leq t_i$, $i = 1, 2$, and s is completely primal if every factor of s is primal. As for domains, a monoid in which every element is (completely) primal is called a pre–Schreier monoid and an integrally closed pre–Schreier monoid is called a Schreier monoid. Note that in the case of a domain A, the monoid in question is the multiplicative monoid A^*, and in the case of an ordered group G, it is the positive cone G^+.

In [7], the authors believe that completely primal elements are the building blocks of the Schreier property. In what follows we give some characterizations of completely primal elements in monoids. For $x_1, ..., x_n \in S$, let $U(x_1, ..., x_n) = \{g \in S \mid g \geq x_1, ..., x_n\}$. A nonempty subset $U \subseteq S$ is lower directed if for $s_1, s_2 \in U$, there exists $s \in U$ with $s \leq s_1, s_2$. The following lemma is well known in ordered groups [7, Theorem 2.1].

Lemma 2.1.1. *Let S be a monoid. An element s of S is completely primal if and only if for each $x \in S$, the set $U(s, x)$ is lower directed. Moreover, if $\{s_1, ..., s_n\}$ is a set of completely primal elements of S, then $U(s_1, ..., s_n)$ is lower directed.*

Proof. The proof of the first part is similar to that of [7, Theorem 2.1 (1)⇔(2)]. For the second part, note that the case $n = 1$ is clear and $n = 2$ follows from the first part. Suppose that $U(s_1, ..., s_{n-1})$ is lower directed. Let $r_1, r_2 \in U(s_1, ..., s_n)$. Then $r_1, r_2 \in U(s_1, ..., s_{n-1})$ and by induction there exists $t \in U(s_1, ..., s_{n-1})$ such that $t \leq r_1, r_2$. But then $r_1, r_2 \in U(t, s_n)$ and since s_n is completely primal there is $s \in U(t, s_n)$ such that $s \leq r_1, r_2$. Hence $s \in U(t, s_n) \subseteq U(s_1, ..., s_n)$. □

The following key characterization of completely primal elements in monoids was proved in [8, Lemma 4.6] for domains. Here we give a short proof in the case of monoids.

Proposition 2.1.2. *Let S be a monoid. An element s of S is completely primal if and only if $s \leqslant r_i t_j$, $r_i, t_j \in S$, for $i = 1, ..., m$ and $j = 1, ..., n$ implies that $s = s_1 s_2$ where $s_1 \leqslant r_i$ for each i and $s_2 \leqslant t_j$ for each j.*

Proof. Let s be a completely primal element and $s \leqslant r_i t_j$ for $i = 1, ..., m$ and $j = 1, ..., n$. Then, for each $i = 1, ..., m$, $s = r_{ij} t_{ji}$, where $r_{ij} \leqslant r_i$ and $t_{ji} \leqslant t_j$ for $j = 1, ..., n$. Since for each i, r_{ij} is completely primal (a factor of s), and $s, r_i \in U(r_{i1}, ..., r_{in})$, there exists $d_i \in S$ such that $r_{ij} \leqslant d_i \leqslant r_i, s$ for every $j = 1, ..., n$. Now, $s, t_1, ..., t_n \in U(s/d_1, ..., s/d_n)$, then there exists $t \in S$ such that $s/d_1, ..., s/d_n \leqslant t \leqslant s, t_1, ..., t_n$. Let $r \in S$ such that $s = rt$. One can easily check that $r \leqslant r_i$ for $i = 1, ..., m$.

For the converse, let $s \in S$ satisfying the condition as in the proposition, and let $x \in S$. We show that $U(s, x)$ is lower directed. Let $r_1, r_2 \in U(s, x)$. For $i = 1, 2$, write $r_i = x t_i$, so $s \leq x t_i$. By our hypothesis $s = s_1 s_2$ such that $s_1 \leq x$ and $s_2 \leq t_i$

for $i = 1, 2$. But $d = xs_2 \in U(s, x)$ and $d \leq r_1, r_2$. Thus $U(s, x)$ is lower directed and by the previous lemma s is completely primal. $\square$

To sum up, we get the following characterization of pre-Schreier monoids, see [42, Theorem 1.1].

Corollary 2.1.3. *Let S be a monoid. The following are equivalent:*

(i) *S is a pre-Schreier monoid;*

(ii) *For all $s, t, x, y \in S$ with $s, t \mid x, y$ there exists $r \in S$ such that $s, t \mid r \mid x, y$;*

(iii) *For all $s_1, ..., s_m \in S$ and $t_1, ..., t_n \in S$ such that $s_i \mid t_j$, for $i = 1, ..., m$ and $j = 1, ..., n$, then there exists $r \in S$ such that $s_i \mid r \mid t_j$ for each i, j;*

(iv) *For all $s, r_1, ..., r_m \in S$ and $t_1, ..., t_n \in S$ such that $s \mid r_i t_j$ for each $i = 1, ..., m$ and $j = 1, ..., n$, then $s = s_1 s_2$ for some $s_1, s_2 \in S$ such that $s_1 \mid r_i$ and $s_2 \mid t_j$, for each i, j.*

We end this section by translating to monoids the well known Nagata type theorem for Schreier domains due to Cohn [14, Theorem 2.6]. Our proof is slightly different from that in [14], for we use the characterization of completely primal elements in Proposition 2.1.2.

Let S be a monoid and T a multiplicative subset of S. The set T is called divisor-closed if T is saturated.

Proposition 2.1.4. *Let S be a monoid and T a multiplicative subset of S.*

(1) *If S is pre-Schreier, then S_T is pre-Schreier.*

(2) *Assume that T is a divisor-closed subset of S such that every element of T is primal in S. If the monoid S_T is pre-Schreier, then S is pre-Schreier.*

Proof. (1) Similar to the case of domains [42, Corollary 1.3].

(2) Assume that S_T is pre-Schreier and let $s, x_1, x_2 \in S$ such that $s \leqslant x_1 x_2$ in S. So $s \leqslant x_1 x_2$ in S_T. Since S_T is pre-Schreier, s is completely primal in S_T. Then $s = (s_1 t_1^{-1})(s_2 t_2^{-1})$ for some $s_1, s_2 \in S$ and $t_1, t_2 \in T$ such that $s_1 t_1^{-1} \leqslant x_1$ and $s_2 t_2^{-1} \leqslant x_2$ in S_T. So $x_1 = (s_1 t_1^{-1})(s_1' r_1^{-1})$ and $x_2 = (s_2 t_2^{-1})(s_2' r_2^{-1})$ for some $s_1', s_2' \in S$ and $r_1, r_2 \in T$. We put $r = t_1 r_1 t_2 r_2$, then r is an element of T which satisfies :

$$rs = (s_1 r_2)(s_2 r_1)$$

$$rx_1 = (s_1r_2)(s'_1t_2)$$

$$rx_2 = (s_2r_1)(s'_2t_1)$$

$$r((x_1x_2)/s) = (s'_1t_2)(s'_2t_1)$$

So $r \leqslant$ to the elements in the set product $\{s_1r_2, s'_2t_1\}\{s_2r_1, s'_1t_2\}$. As r is completely primal in S and by Proposition 2.1.2, there exist $u, v \in S$ such that $r = uv$ with $u \leqslant s_1r_2, s'_2t_1$ and $v \leqslant s_2r_1, s'_1t_2$. Then $s = (s_1r_2u^{-1})(s_2r_1v^{-1})$ with $s_1r_2u^{-1} \leqslant x_1$ and $s_2r_1v^{-1} \leqslant x_2$, hence s is primal in S. Consequently, S is pre-Schreier.

□

2.2 Primal elements in a graded domain

Throughout, a monoid means a torsionless grading monoid, that is, a (additive) cancellative torsion–free commutative semigroup.

In this section, we study primality in a graded integral domain $R = \oplus_{\alpha\in\Gamma} R_\alpha$, graded by a torsionless grading monoid Γ. We denote by H the multiplicative set (monoid) of nonzero homogeneous elements of R.

Recall that an element $x \in H$ is called:

gr-primal [13] if whenever $x \mid y_1y_2$ with $y_1, y_2 \in H$, then $x = x_1x_2$, where $x_1, x_2 \in H$ and $x_i \mid y_i$, $i = 1, 2$, and x is completely gr-primal if every homogeneous factor of x is gr-primal.

These two definitions are equivalent, respectively, to x primal and completely primal in the multiplicative monoid H.
The graded domain R is called gr-pre-Schreier if every element of H is (completely) gr-primal.

In [13], Brookfield and Rush introduced gr-pre-Schreier domains and characterized pre-Schreier graded domains in terms of the gr-pre-Schreier property. In the integrally closed case, they showed that the Schreier property is equivalent to the gr-pre-Schreier property.
For an integral domain A with quotient field K and fractional ideals I, J, define $[I : J] = \{x \in K, xJ \subseteq I\}$, $I^{-1} = [A : I]$ and $I : J = [I : J] \cap A$. A homogeneous

(fractional) ideal I of the graded domain R is called H-locally cyclic if every finite subset of homogeneous elements of I is contained in a (homogeneous) principal sub-ideal of I. We start this section with some characterization of gr-pre-Schreier domains.

Proposition 2.2.1. *Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a graded domain. The following statements are equivalent:*

(i) *R is a gr-pre-Schreier domain;*

(ii) *H is a pre-Schreier monoid;*

(iii) *For every nonzero homogeneous element $u \in \mathcal{H}(R)$, $(1, u)^{-1}$ is H–locally cyclic;*

(iv) *For every nonzero $x \in \mathcal{H}(R)$, $C(x)^{-1}$ is H–locally cyclic.*

Proof. (i)⇔(ii) is obvious. For (ii)⇔(iii), note that for $a, b \in H$, we have $(a, b)^{-1} = (ab)^{-1}(aR \cap bR)$, and for a homogeneous element $u \in \mathcal{H}(R)$, $u = a/b$ for some $a, b \in H$. Then apply Corollary 2.1.3 (ii) in H. For (iii)⇔(iv), note that $C(x)^{-1}$ is a finite intersection of homogeneous principal fractional ideals. □

Also, we get the following Nagata type theorem for gr-pre-Schreier domains analogue to that of Schreier property due to P. M. Cohn [14, Theorem 2.6].

Proposition 2.2.2. *Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a graded domain and $S \subseteq H$ a multiplicative subset of R. Then:*

(1) *If R is a gr-pre-Schreier domain, then R_S is a gr-pre-Schreier domain.*

(2) *Assume that S is saturated, generated by completely gr-primal elements, and R_S is a gr-pre-Schreier domain, then R is a gr-pre-Schreier domain.*

Proof. Apply Proposition 2.1.4 to the quotient monoid H_S. □

Example 2.2.3. (1) Let $R = A[X]$ be the polynomial ring over a ring A. One can easily see that $A[X]$ is gr-pre-Schreier if and only if A is pre-Schreier. By [13, Theorem 3.2], $A[X]$ is pre-Schreier if and only if it is Schreier, if and only if A is Schreier.

(2) Let $A \subseteq B$ be an extension of integral domains and set $R = A + XB[X]$. Primality and the Schreier property for $A + XB[X]$ domains were studied in [19, 20].

We claim that $R = A + XB[X]$ is gr-pre-Schreier if and only if A is pre-Schreier and $B = A_S$, where $S = U(B) \cap A$, $U(B)$ denotes the set of invertible elements of B. Suppose that R is gr-pre-Schreier. Clearly A is pre-Schreier. On the other hand, by using the primality of X and the fact that $X \mid (bX)^2$, $b \in B$, it was shown in [19, Remark 1.1] that $B = A_S$, where $S = U(B) \cap A$.
Conversely, we use the Proposition 2.2.2. The quotient ring $R_S = A_S[X]$ is gr-pre-Schreier since A, and hence A_S, is pre-Schreier. The elements of S are gr-primal in $R = A + XA_S[X]$. Indeed, let $a \in S$ and $h_1, h_2 \in H$ such that $a \mid h_1h_2$. Since A is pre-Schreier, the case where $h_1, h_2 \in A$ is clear. Assume that $h_2 = bX^n$ for some $b \in A_S$ and $n \neq 0$. Then $a \mid h_2$ in R, and write $a = 1 \times a$.

By [20, Theorem 2.7 and Corollary 2.9], R is a pre-Schreier (resp., Schreier) domain if and only if A is a pre-Schreier (resp., Schreier) domain, $B = A_S$, where $S = U(B) \cap A$, and A_S is a Schreier domain.

Inspired by the work in [13], in the following we study (completely) primal elements in a graded domain in terms of (completely) gr-primality.

Let $h \in H$; we say that h is degree gr-primal if $h \mid x_iy_j$, $x_i, y_j \in H$, for $i = 1, ..., m$ and $j = 1, ..., n$, with $deg(x_k) < deg(x_l)$ and $deg(y_k) < deg(y_l)$ for all $k < l$, then $h = h_1h_2$ such that $h_1 \mid x_i$ for each i and $h_2 \mid y_j$ for each j. The degree gr-primality is a weak form of the completely gr-primality in H.

Theorem 2.2.4. *Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a graded domain, and $h \in H$. Then:*

(1) *h is primal in R if and only if h is degree gr-primal and $(h) : (x)$ is homogeneous for each $x \in R$.*

(2) *h is completely primal in R if and only if h is completely gr-primal and $(h) : (x)$ is homogeneous for each $x \in R$.*

Proof. (1) For the "only if" condition, assume that $h \mid x_iy_j$, $x_i, y_j \in H$, for $i = 1, ..., m$ and $j = 1, ..., n$, with $deg(x_k) < deg(x_l)$ and $deg(y_k) < deg(y_l)$ for all $k < l$. Then $h \mid xy$ in R, where $x = x_1 + \cdots + x_m$ and $y = y_1 + \cdots + y_n$. By the primality $h = h_1h_2$, $h_1, h_2 \in H$, with $h_1 \mid x$ and $h_2 \mid y$. Clearly, $h_1 \mid x_i$ and $h_2 \mid y_j$ for each i, j. To see that $(h) : (x)$ is homogeneous, let $y \in (h) : (x)$. Then $h \mid xy$. Now, $h = h_1h_2$, $h_1, h_2 \in H$, with $h_1 \mid x$ and $h_2 \mid y$. It follows that $C(y) \subseteq (h) : (x)$.

For the "if" condition, let $x = x_1 + \cdots + x_m$ and $y = y_1 + \cdots + y_n$ be two nonzero elements of R, with $deg(x_k) < deg(x_l)$ and $deg(y_k) < deg(y_l)$ for all

$k < l$, such that $h \mid xy$. Now, $y \in (h) : (x)$, a homogeneous ideal, then $h \mid x_i y_j$ for $i = 1, ..., m$ and $j = 1, ..., n$. On the other hand, h is degree gr-primal implies that $h = h_1 h_2$, where $h_1 \mid x_i$ and $h_2 \mid y_j$ for each i, j. Then $h = h_1 h_2$ with $h_1 \mid x$ and $h_2 \mid y$, so h is primal in R.

(2) For the "only if"condition, clearly, if h is completely primal in R it is completely gr-primal. The remainder is similar to (1). For the "if"condition, by the same argument as in the proof of (1), h is primal in R. To prove that h is completely primal in R, let k be a factor of h. Necessarily, $k \in H$. Then k is completely gr-primal and $h = kk'$ for some $k' \in H$. Let $x \in R$ and $y \in (k) : (x)$, with $y = y_1 + \cdots + y_n$ and $deg(y_i) < deg(y_j)$ for all $i < j$. Then $k'y \in (kk') : (x) = (h) : (x)$. Since $(h) : (x)$ is homogeneous, then, for each i, $k'y_i \in (h) : (x)$, so $y_i \in (k) : (x)$. Thus $(k) : (x)$ is homogeneous. Hence, like h, k is primal in R. Therefore, h is completely primal in R.

□

Example 2.2.5. We give an example of a degree gr-primal element which is not completely gr-primal.
Let $R = \mathbb{Z} + X\mathbb{R}[X]$. By [19, Example 1.7(ii)], X^2 is primal in R, but X is not primal in R, so X^2 not completely primal. By Theorem 2.2.4, X^2 is degree gr-primal but not completely gr-primal.

Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a graded domain, $h \in H$, and let R_h be the quotient ring of R with respect to the multiplicative set generated by h. Note that R_h is a graded subring of $\mathcal{H}(R)$.

Definition 2.2.6. Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a graded domain, and $h \in H$.

(1) We say that R is R_h–almost normal if every homogeneous element $x \in R_h$ of nonzero degree which is integral over R is actually in R.

(2) We say that R is R_h–integrally closed if R is integrally closed in R_h.

Note that R is R_h–integrally closed if and only if R is integrally closed in R_h with respect to the homogeneous elements of R_h. Thus, R is R_h–integrally closed if and only if R is R_h–almost normal and R_0 is integrally closed in $(R_h)_0$.

Note that almost normality defined in [1], is a globalization of R_h-almost normality, $h \in H$. Thus, R is almost normal if and only if R is R_h-almost normal for every $h \in H$. A similar statement is true for the integrally closed case.

Recall that an extension of domains $A \subseteq B$ is inert if whenever $bb' \in A$ for some $b, b' \in B$, then $b = au$ and $b' = a'u^{-1}$ for some $a, a' \in A$ and u a unit of B.

Proposition 2.2.7. *Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a graded domain, and $h \in H$. Consider the following statements:*

(i) *R is R_h–integrally closed.*

(ii) *$(h) : (x)$ is homogeneous for each $x \in R_h$.*

(iii) *R is R_h–almost normal.*

Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii). *If moreover, R contains a (homogeneous) unit of nonzero degree the three conditions are equivalent, and if $R_0 \subseteq R$ is inert, then* (ii)$\Leftrightarrow$(iii)*.*

Proof. The proof is inspired from [1].

(i)$\Rightarrow$(ii). Let $x \in R_h$ and $y \in R$ such that $C(xy) \subseteq (h)$. Then $C(x)^n C(xy) \subseteq hC(x)^n$ implies $C(x)^{n+1}C(y) \subseteq hC(x)^n$, for some integer n in the Dedekind-Mertens lemma. Thus $\frac{1}{h}C(x)C(y) \subseteq [C(x)^n : C(x)^n] \cap R_h = R$, since R is R_h-integrally closed. Hence $C(x)C(y) \subseteq (h)$. Therefore, $(h) : (x)$ is homogeneous.

(ii)$\Rightarrow$(iii). Let $x = a/h^k \in R_h$, $a \in H$, a homogeneous element of nonzero degree which is integral over R. Let $f(Y) = Y^n + a_{n-1}Y^{n-1} + \cdots + a_0$ with coefficients in R such that $f(x) = 0$. Since x is homogeneous, we may assume that we have an equation of the form $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ with the a_i's homogeneous and $deg(a_i) = (n-i)deg(x)$. Then $f(Y) = (Y - x)g(Y)$ with $g(Y) = Y^{n-1} + b_{n-2}Y^{n-2} + \cdots + b_0$. We may assume that the elements $b_i \in R_h$ are homogeneous of distinct nonzero degree. From $f(1) = (1 - x)g(1)$, it follows that $(1 - x)g(1) \in R$. Now, $(h^k - a)(g(1)/h^{k-1}) \in hR$ implies $h^k - a \in (h) : (g(1)/h^{k-1})$, which is homogeneous. Since $1/h^{k-1} \in C(g(1)/h^{k-1})$, it follows that $(1/h^{k-1})(h^k - a) \in hR$. So $1 - x \in R$. Hence $x \in R$.

For the moreover statements, assume that R contains a (homogeneous) unit u of nonzero degree. If $x \in R_h$ is a homogeneous element of zero degree which is integral over R, then $ux \in R_h$ is a homogeneous element of nonzero degree which is integral over R. If R is R_h-almost normal, then $ux \in R$. Hence $x \in R$. This proves that (iii)$\Rightarrow$(i). For the last statement, we proceed as in [1, Theorem 3.7(2)]. $\square$

Corollary 2.2.8. *Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a graded domain. Assume that R is integrally closed or $R_0 \subseteq R$ is inert and R is almost normal. Then:*

(1) *A homogeneous element is primal in R if and only if it is degree gr–primal.*

(2) *A homogeneous element is completely primal in R if and only if it is completely gr-primal.*

Proof. This follows from Theorem 2.2.4 and Proposition 2.2.7. □

Remark 2.2.9. (1) In [33, Section 3], the author gave an example which show that R may be an almost normal graded domain, that is, R is R_h–almost normal for every $h \in H$, but there exist $h \in H$ and $x \in R$ such that $(h) : (x)$ is not homogeneous.

(2) Let $h \in H$. In Theorem 2.2.4, we can check that h is primal (resp., completely primal) if and only if h is degree (resp., completely) gr-primal and $(h) : (x)$ is homogeneous for every $x \in R_h$.

Example 2.2.10. (1) Let A be an integral domain with quotient field K. Let $R = A[X]$, a polynomial ring. Note that the extension $A \subseteq A[X]$ is inert. If every element of A is primal in $A[X]$, then, by Cohn's Nagata type theorem for Schreier domains [14, Theorem 2.6], $A[X]$ is Schreier since $K[X] = A[X]_S$, where $S = A \setminus \{0\}$, is Schreier (UFD). The above results shed more light on the primality of elements of A in $A[X]$. Let $0 \neq a \in A$. Clearly, a is degree gr-primal if and only if a is completely gr-primal, if and only if a is completely primal in A. Thus a is (completely) primal in $A[X]$ if and only if a is completely primal in A and A is integrally closed in A_a. For more details, see the next section.

(2) For an extension of integral domains $A \subseteq B$, consider the pullback $R = A + XB[X]$. Since the extension $A \subseteq R$ is inert, then, by Theorem 2.2.4 and Proposition 2.2.7, $h = aX^n$ is primal (resp., completely primal) in R if and only if h is degree (resp., completely) gr–primal and B is integrally closed in B_a (Here $R_h = B_a[X, X^{-1}]$ if $n \geq 1$, and $R_h = A_a + XB_a[X]$ if $n = 0$).

As a corollary of Theorem 2.2.4, Proposition 2.2.7, and Cohn's Nagata type theorem for Schreier domains, we reobtain a characterization of the Schreier property in graded domains.

Corollary 2.2.11 ([13, Theorem 2.2]). *Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a graded domain. Then the following statements are equivalent:*

(i) *R is Schreier;*

(ii) *R is pre-Schreier and R_0 is integrally closed in $(R_H)_0$;*

(iii) *R is gr–pre–Schreier and integrally closed.*

2.3 Primal elements in semigroup rings

As an application of the previous sections, we study the primality in semigroup rings. Throughout this section, Γ denotes a nonzero cancellative torsion–free commutative monoid (written additively) with quotient group G, and A is an integral domain with quotient field K. Let $A[\Gamma]$ be the semigroup ring of Γ over A. Then $A[\Gamma]$ is a Γ–graded integral domain and each nonzero element $f \in A[\Gamma]$ can be written uniquely as $f = a_1 X^{s_1} + \cdots + a_n X^{s_n}$ where $0 \neq a_i \in A$ and $s_i \in \Gamma$ with $s_1 \prec \cdots \prec s_n$. Note that here, $H = \{aX^\alpha \mid 0 \neq a \in A, \alpha \in \Gamma\}$ and $A[\Gamma]_H = K[G]$. For more on semigroup rings, see [25].

Proposition 2.3.1. *Let $A[\Gamma]$ be the semigroup ring of Γ over A, and consider the element of the form aX^α where $0 \neq a \in A$ and $\alpha \in \Gamma$. The following statements are equivalent:*

(i) *aX^α si primal in $A[\Gamma]$;*

(ii) *a and X^α are both primal in $A[\Gamma]$.*

Proof. (i)⇒(ii). Suppose that aX^α is primal in $A[\Gamma]$. Let $f, g \in A[\Gamma]$ such that $a \mid fg$, then $aX^\alpha \mid f(gX^\alpha)$. Since aX^α is primal $aX^\alpha = a_1 X^{\alpha_1} a_2 X^{\alpha_2}$ where $a_1 X^{\alpha_1} \mid f$ and $a_2 X^{\alpha_2} \mid gX^\alpha$, so $a_1 \mid f$ and $a_2 \mid g$. Thus $a = a_1 a_2$ such that $a_1 \mid f$ and $a_2 \mid g$, so a is primal in $A[\Gamma]$.
To prove that X^α is primal in $A[\Gamma]$, let $f, g \in A[\Gamma]$ such that $X^\alpha \mid fg$, then $aX^\alpha \mid (af)g$. Thus $aX^\alpha = a_1 X^{\alpha_1} a_2 X^{\alpha_2}$, where $a_1 X^{\alpha_1} \mid af$ and $a_2 X^{\alpha_2} \mid g$. Hence $X^\alpha = X^{\alpha_1} X^{\alpha_2}$ with $X^{\alpha_1} \mid f$ and $X^{\alpha_2} \mid g$.

(ii)⇒(i). Assume that a and X^α are both primal in $A[\Gamma]$ and let $f, g \in A[\Gamma]$ such that $aX^\alpha \mid fg$. Then $a \mid fg$ and $X^\alpha \mid fg$. Since a and X^α are primal in $A[\Gamma]$, we have $a = a_1 a_2$ such that $a_1 \mid f$ and $a_2 \mid g$ for some $a_1, a_2 \in A$; and $X^\alpha = X^{\alpha_1} X^{\alpha_2}$

such that $X^{\alpha_1} \mid f$ and $X^{\alpha_2} \mid g$ for some $\alpha_1, \alpha_2 \in \Gamma$. Hence $aX^\alpha = a_1X^{\alpha_1}a_2X^{\alpha_2}$, where $a_1X^{\alpha_1} \mid f$ and $a_2X^{\alpha_2} \mid g$, so aX^α is primal in $A[\Gamma]$. $\square$

For a semigroup ring $A[\mathit{\Gamma}]$, let $h = aX^\alpha \in H$. Then $A[\mathit{\Gamma}]_h = A_a[\mathit{\Gamma}_\alpha]$, where $\mathit{\Gamma}_\alpha$ is the quotient monoid with respect to the additive set generated by α. Note that $A[\mathit{\Gamma}]$ is integrally closed in $A_a[\mathit{\Gamma}_\alpha]$ if and only if A is integrally closed in A_a and $\mathit{\Gamma}$ is integrally closed in $\mathit{\Gamma}_\alpha$.

Proposition 2.3.2. *Let $A[\mathit{\Gamma}]$ be the semigroup ring of $\mathit{\Gamma}$ over A and $h = aX^\alpha \in H$. The following statements are equivalent:*

(i) *$A[\mathit{\Gamma}]$ is $A_a[\mathit{\Gamma}_\alpha]$–integrally closed;*

(ii) *$(h) : (f)$ is homogeneous for each $f \in A_a[\mathit{\Gamma}_\alpha]$;*

(iii) *$A[\mathit{\Gamma}]$ is $A_a[\mathit{\Gamma}_\alpha]$–almost normal.*

Proof. By Proposition 2.2.7, it remains to show that (iii)⇒(i). Let $\lambda \in A_a$ be integral over $A[\mathit{\Gamma}]$. Take $0 \neq \gamma \in \mathit{\Gamma}$. Then $\lambda X^\gamma \in A_a[\mathit{\Gamma}_\alpha]$ is a homogeneous element of nonzero degree which is integral over $A[\mathit{\Gamma}]$. So $\lambda X^\gamma \in A[\mathit{\Gamma}]$, hence $\lambda \in A$. Now, by the $A_a[\mathit{\Gamma}_\alpha]$–almost normality, $A[\mathit{\Gamma}]$ is $A_a[\mathit{\Gamma}_\alpha]$–integrally closed. $\square$

The following lemmas characterize degree (resp., completely) gr-primality in semigroup rings.

Lemma 2.3.3. *Let $A[\mathit{\Gamma}]$ be the semigroup ring of Γ over A and $0 \neq a \in A$. The following statements are equivalent:*

(i) *a is completely gr–primal;*

(ii) *a is degree gr–primal;*

(iii) *a is completely primal in A.*

Proof. (i)⇒(ii). This is clear.

(ii)⇒(iii). Suppose that $a \mid b_ic_j$ in A for $i = 1, ..., m$ and $j = 1, ..., n$. Let $0 \neq \alpha \in \mathit{\Gamma}$; set $\beta_i = i\alpha$ and $\gamma_j = j\alpha$ for $i = 1, ..., m$ and $j = 1, ..., n$. Then $a \mid (b_iX^{\beta_i})(c_jX^{\gamma_j})$, in $A[\Gamma]$, for $i = 1, ..., m$ and $j = 1, ..., n$. By (ii), there exist $a_1, a_2 \in A$ such that $a = a_1a_2$ where $a_1 \mid b_i$ for each i and $a_2 \mid c_j$ for each j. Hence a is completely primal in A (cf. Proposition 2.1.2).

(iii)$\Rightarrow$(i). Assume that $a \mid b_iX^{\beta_i}c_jX^{\gamma_j}$ in $A[\Gamma]$ for $i = 1, ..., m$ and $j = 1, ..., n$. Then $a \mid b_ic_j$ in A for each i, j. So $a = a_1a_2$, where $a_1 \mid b_i$ for each i and $a_2 \mid c_j$ for each j. Thus $a = a_1a_2$ such that $a_1 \mid b_iX^{\beta_i}$ for each i and $a_2 \mid c_jX^{\gamma_j}$ for each j. This proves that a is completely gr-primal. $\square$

Lemma 2.3.4. *Let $A[\Gamma]$ be the semigroup ring of Γ over A and $\alpha \in \Gamma$. The following statements are equivalent:*

(i) *X^α is completely gr–primal;*

(ii) *X^α is degree gr–primal;*

(iii) *α is completely primal in Γ.*

Proof. (i)$\Rightarrow$(ii). This is clear.

(ii)$\Rightarrow$(iii). Suppose that $\alpha \mid \beta_i + \gamma_j$ for $i = 1, ..., m$ and $j = 1, ..., n$. We may assume that $\beta_1 \prec \cdots \prec \beta_m$ and $\gamma_1 \prec \cdots \prec \gamma_n$. Then $X^\alpha \mid X^{\beta_i}X^{\gamma_j}$ for $i = 1, ..., m$ and $j = 1, ..., n$. By (ii), there exist $\alpha_1, \alpha_2 \in \Gamma$ such that $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \mid \beta_i$ for each i and $\alpha_2 \mid \gamma_j$ for each j. Hence α is completely primal in Γ.

(iii)$\Rightarrow$(i). Assume that $X^\alpha \mid b_iX^{\beta_i}c_jX^{\gamma_j}$ in $A[\Gamma]$ for $i = 1, ..., m$ and $j = 1, ..., n$. Then $\alpha \mid \beta_i + \gamma_j$ in Γ for each i, j. So $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \mid \beta_i$ for each i and $\alpha_2 \mid \gamma_j$ for each j. Thus $X^\alpha = X^{\alpha_1}X^{\alpha_2}$ such that $X^{\alpha_1} \mid b_iX^{\beta_i}$ for each i and $X^{\alpha_2} \mid c_jX^{\gamma_j}$ for each j. This proves (i). $\square$

From Proposition 2.3.1 and Lemmas 2.3.3 and 2.3.4 we get:

Proposition 2.3.5 ([13, Lemma 3.1])**.** *Let $A[\Gamma]$ be the semigroup ring of Γ over A. The following statements are equivalent:*

(i) *$A[\Gamma]$ is gr–pre–Schreier;*

(ii) *A and Γ are pre–Schreier.*

Next, we state our main result of this section.

Theorem 2.3.6. *Let $A[\Gamma]$ be the semigroup ring of Γ over A, and let $0 \neq a \in A$ and $\alpha \in \Gamma$. Then:*

(1) *a is (completely) primal in $A[\Gamma]$ if and only if a is completely primal in A and A is integrally closed in A_a.*

(2) *X^α is (completely) primal in $A[\Gamma]$ if and only if α is completely primal in Γ and Γ is integrally closed in Γ_α.*

Proof. This follows from Theorem 2.2.4, Remark 2.2.9 (2), Proposition 2.3.2, and Lemmas 2.3.3 and 2.3.4. □

From Theorem 2.3.6 and Corollary 2.2.8, we get:

Corollary 2.3.7. *Let $A[\Gamma]$ be the semigroup ring of Γ over A, and let $0 \neq a \in A$ and $\alpha \in \Gamma$. Then:*

(1) *Assume that A is integrally closed. Then a is (completely) primal in $A[\Gamma]$ if and only if a is completely primal in A.*

(2) *Assume that Γ is integrally closed. Then X^α is (completely) primal in $A[\Gamma]$ if and only if α is completely primal in Γ.*

Corollary 2.3.8 ([13, Theorem 3.2])**.** *Let $A[\Gamma]$ be the semigroup ring of Γ over A. The following statements are equivalent:*

(i) *$A[\Gamma]$ is pre-Schreier;*

(ii) *$A[\Gamma]$ is Schreier;*

(iii) *A and Γ are Schreier.*

Proof. For (i)⇒(ii)⇒(iii) use Proposition 2.3.1 and Theorem 2.3.6, and remark that A (resp., Γ) is integrally closed if and only if A (resp., Γ) is integrally closed in A_a (resp., Γ_α) for each $0 \neq a \in A$ (resp., $0 \neq \alpha \in \Gamma$). For (iii)⇒(i), we need the Cohn's Nagata type theorem for Schreier domains. □

In the case of polynomial rings, we recover some results established in [5, Proposition 6] and [8, Lemma 4.7]. Note that in a polynomial ring the powers of X are primary, so they are primal. Thus by Proposition 2.3.1, a nonzero homogeneous element of the form aX^n, $a \in A$, is primal in $A[X]$ if and only if a is primal in $A[X]$.

Corollary 2.3.9. *Let A be an integral domain and X an indeterminate. Then:*

(1) *a is (completely) primal in $A[X]$ if and only if a is completely primal in A and A is integrally closed in A_a.*

(2) *$A[X]$ is Schreier if and only if $A[X]$ is pre–Schreier, if and only if A is Schreier.*

Chapter 3

The Schreier property and the composite semigroup ring $A + B[\Gamma^*]$

3.0 Introduction

All rings considered in this chapter are commutative. Let D be an integral domain and $0 \neq x \in D$. Following P. M. Cohn [14] we say that x is primal (resp., completely primal) if for each $a_1, a_2 \in D$; $x \mid a_1 a_2$ in D implies that x can be factorized, in D, as $x = x_1 x_2$, where $x_i \mid a_i$, $i = 1, 2$ (resp., every factor of x in D is primal). The domain D is said to be pre-Schreier if every nonzero element of D is (completely) primal [42], and an integrally closed pre-Schreier domain is called Schreier [14]. The Schreier property generalizes the GCD property.

Let $A \subseteq B$ be an extension of integral domains and Γ a commutative, additive, cancellative torsion-free monoid. Let $B[\Gamma]$ be the semigroup ring of Γ over B and set $\Gamma^* = \Gamma \setminus \{0\}$. Suppose that $\Gamma \cap -\Gamma = \{0\}$. Then $R = A + B[\Gamma^*]$ is a subring of $B[\Gamma]$. Note that R can be obtained as a pullback with $B[\Gamma^*]$ a common ideal of R and $B[\Gamma]$. If $\Gamma \cap -\Gamma \neq \{0\}$ or $A = B$, the ring R coincides with $B[\Gamma]$. If $\Gamma = \mathbb{Z}_+$, then $R = A + XB[X]$, and if $\Gamma = \mathbb{Z}_+^n$, then $R = A + (X_1, \ldots, X_n)B[X_1, \ldots, X_n]$. The monoid Γ admits a total order $\preceq$ compatible with its semigroup operation [25, Corollary 3.4], and since $\Gamma \cap -\Gamma = \{0\}$, we may assume that $\alpha \succeq 0$ for all $\alpha \in \Gamma$. Hence, each $f \in R$ is uniquely expressible in the form $f = a + b_1 X^{\alpha_1} + \cdots + b_n X^{\alpha_n}$, where $a \in A$, $b_i \in B$ and $\alpha_i \in \Gamma^*$, with $\alpha_1 \prec \cdots \prec \alpha_n$. If $b_n \neq 0$, it is called the leading coefficient of f and α_n the degree. The construction $A + B[\Gamma^*]$ has been studied by many authors and has proven to be useful in constructing examples and

counterexamples in many areas of commutative ring theory [17, 21, 31, 32].
In [13], the authors characterize the Schreier property in semigroup rings. Their results extend the case of polynomial rings. The Schreier property in $A + XB[X]$ domains has been studied in [19, 20]. Also, in [31], the author investigates the GCD property in $A + B[\Gamma^*]$. The extension of these works to the Schreier property in $A + B[\Gamma^*]$ domain was left open, in fact, in [43] it was asked:

When is $A + B[\Gamma^*]$ a pre–Schreier domain?

Our purpose in this chapter is to investigate this question. In Section 3.1, we present some preliminary results about the Schreier property in monoids and graded domains. Then apply this to the $A + B[\Gamma^*]$ domains naturally graded by Γ. In Section 3.2, we study primal homogeneous elements and, as an application, we characterize the Schreier property in $A + A_S[\Gamma^*]$, where S is a multiplicative subset of the domain A. As we show in Section 3.3, this particular case of our construction is crucial; it includes the pre-Schreier domains of the general form $A+B[\Gamma^*]$ with Γ not antimatter. In contrast, the Schreier property in the general case when B is not a quotient ring of A forces Γ to be an antimatter monoid. Also in this latter context, a characterization of the Schreier property is provided. The results obtained extend those of $A + XB[X]$ domains and lead to new examples of (pre–)Schreier domains.

Notation and terminology used in this chapter are standard as in [24, 25].

3.1 Preliminary results

Throughout this chapter a monoid means a unitary commutative cancellative torsion-free semigroup. Let S be a multiplicative monoid. If $T \subseteq S$ is a multiplicatively closed subset of S, then we get the quotient monoid $S_T := \{s/t, s \in S, t \in T\}$. If $T = S$, we have the quotient group of S, $G := \langle S \rangle$. The monoid S is called integrally closed if, $x^n \in S$ for $n \in \mathbb{Z}_+$ and $x \in G$ implies $x \in S$.

The Schreier property can be defined in the more general context of monoids. Let S be a multiplicative monoid. For $s, t \in S$, we say that s divides t, denoted by $s \leq t$, if $t = sr$ for some $r \in S$. An element $s \in S$ is primal (resp., completely primal) if whenever $s \leq t_1 t_2$, $t_1, t_2 \in S$, s can be written as $s = s_1 s_2$ such that $s_i \leq t_i$, $i = 1, 2$ (resp., every factor of s in S is primal). A monoid S is called pre-Schreier if every element of S is primal, and Schreier when it is pre-Schreier and integrally closed. Note that a domain is pre-Schreier if the multiplicative monoid of its nonzero elements is pre-Schreier. Thus, results about pre-Schreier domains can be written in the language of monoids. For instance, the Nagata type theorem for Schreier domains due to Cohn [14, Theorem 2.6] can be stated for monoids.

Proposition 3.1.1. *Let S be a multiplicative monoid and T a multiplicative subset of S.*

(1) *If S is pre-Schreier, then S_T is pre-Schreier.*

(2) *Assume that T is a divisor-closed subset of S such that every element of T is primal in S. If the monoid S_T is pre-Schreier, then S is pre-Schreier.*

Remark 3.1.2. GCD monoids are Schreier. As a generalization of lattice ordered groups, in [22] the author studied the Schreier property for ordered groups, such groups were called Riesz groups. A directed ordered group G is a lattice ordered group (resp., a Riesz group) if and only if its monoid of positive elements G^+ is a GCD (resp., Schreier) monoid [22, 23]. Examples of Schreier monoids that are not GCD monoids can be easily constructed, see [22, Section 3]. For an elementary example, consider the monoid $S = \{(s,t) \in \mathbb{Q}^2, s,t > 0\} \cup \{(0,0)\}$ with the relation $(s_1,t_1) < (s_2,t_2)$ if and only if $s_1 < s_2$ and $t_1 < t_2$.

We next recall some results about the Schreier property for graded domains. Let $R = \oplus_{\alpha\in\Gamma} R_\alpha$ be a graded domain, graded by an additive monoid Γ. We denote by H the multiplicative monoid of nonzero homogeneous elements of R. If $T \subseteq H$ is a multiplicative set of R, that is, a submonoid of H, the ring of quotient R_T is graded by some quotient monoid of Γ whose set of nonzero homogeneous elements is H_T. In particular, $\mathcal{H}(R) := R_H$ is a $\langle\Gamma\rangle$-graded domain, called the homogeneous quotient field of R. Note that $\mathcal{H}(R)$ is a completely integrally closed GCD domain (cf. [2]).

We say that the graded domain R is almost normal if every homogeneous element $x \in \mathcal{H}(R)$ of nonzero degree which is integral over R is actually in R. Note that R is integrally closed if and only if R is almost normal and R_0 is integrally closed in $\mathcal{H}(R)_0$ [1].

An homogeneous element $x \in H$ is called gr-primal in the graded domain R [13], if whenever $x \mid y_1y_2$ with $y_1, y_2 \in H$, then $x = x_1x_2$, $x_1, x_2 \in H$, where $x_i \mid y_i$, $i = 1,2$, and x is completely gr-primal if every (homogeneous) factor of x in R is gr-primal. These two definitions are equivalent, respectively, to x primal and completely primal in the multiplicative monoid H. The graded domain R is called gr-pre-Schreier if every element of the monoid H is (completely) gr-primal, that is H is a pre-Schreier monoid. In [13], the authors studied (pre-)Schreier graded domains and characterized them in terms of the gr-pre-Schreier property. In the integrally closed case, they proved that, in graded domains, the Schreier property is equivalent to the gr-pre-Schreier property. For a monoid domain $D[\Gamma]$ over a domain D, naturally Γ-graded, they showed that $D[\Gamma]$ is pre-Schreier if and only if it is Schreier, if and only if the domain D and the monoid Γ are Schreier.

The following proposition is the Nagata type theorem for gr-pre-Schreier domains.

Proposition 3.1.3. *Let $R = \oplus_{\alpha \in \Gamma} R_\alpha$ be a graded domain and $T \subseteq H$ a multiplicative set of R. Then:*

(i) *If R is a gr-pre-Schreier domain, then R_T is a gr-pre-Schreier domain.*

(ii) *If R_T is a gr-pre-Schreier domain and T is saturated consisting of gr-primal elements of R, then R is a gr-pre-Schreier domain.*

Proof. Apply Proposition 3.1.1 to the quotient monoid H_T. □

Let $A \subseteq B$ be an extension of integral domains and Γ an additive monoid such that $\Gamma \cap -\Gamma = \{0\}$. Then $R = A + B[\Gamma^*]$ is naturally graded by Γ with the set of nonzero homogeneous elements $H = A^* \cup \{bX^\alpha, b \in B^*, \alpha \in \Gamma^*\}$. The following proposition characterizes the pre-Schreier property in terms of the gr-pre-Schreier property for $A + B[\Gamma^*]$ domains.

Proposition 3.1.4. *Let $A \subseteq B$ be an extension of integral domains and $R = A + B[\Gamma^*]$. The following statements are equivalent.*

(i) *R is a pre–Schreier domain.*

(ii) *R is an almost normal gr-pre-Schreier domain.*

(iii) *R is a gr-pre-Schreier domain and both Γ and B are integrally closed.*

Proof. (i)⇔(ii) The "only if" condition follows from [13, Theorem 2.1] and [1, Theorems 3.2 and 3.5]. The converse follows from [13, Theorem 2.1] and [1, Section 2, Theorems 3.2 and 3.7(2)], since $\Gamma \cap -\Gamma = \{0\}$.

(ii)$\Leftrightarrow$(iii) We show that R is almost normal if and only if Γ and B are integrally closed. Assume that R is almost normal. To prove that Γ is integrally closed, let g be a nonzero element of $G = \langle \Gamma \rangle$ such that $ng \in \Gamma$ for some integer $n \geq 1$. Then $X^{ng} \in R$. Now, let $f = Y^n - X^{ng} \in R[Y]$, then $f(X^g) = 0$. That is X^g is integral over R. Since R is almost normal, then $X^g \in R$. It follows that $g \in \Gamma$, and hence Γ is integrally closed. We next show that B is integrally closed. For, we consider an element $c \in K = qf(B)$ integral over B. Then there exists an integer $n \geq 1$ and $b_0, \ldots, b_{n-1} \in B$ such that $c^n + b_{n-1}c^{n-1} + \cdots + b_1 c + b_0 = 0$. Multiplying by $X^{n\gamma}$, $\gamma \in \Gamma^*$, we get $(cX^\gamma)^n + b_{n-1}X^\gamma(cX^\gamma)^{n-1} + \cdots + b_1 X^{(n-1)\gamma}(cX^\gamma) + b_0 X^{n\gamma} = 0$, so cX^γ is integral over R. Hence $cX^\gamma \in R$, because R is almost normal. Thus $c \in B$. Hence B is integrally closed.
Conversely, assume that Γ and B are integrally closed. Note that $\mathcal{H}(R) = K[G]$, where $K = qf(B)$ and $G = \langle \Gamma \rangle$. Let $0 \neq b \in K$ and $0 \neq g \in G$ such that bX^g is integral over R. Then bX^g is also integral over $B[\Gamma]$ since $R \subseteq B[\Gamma]$. So $bX^g \in B[\Gamma]$ as $B[\Gamma]$ is integrally closed. Hence $bX^g \in B[\Gamma^*] \subseteq R$, and we are done. $\square$

Note that by the proof of Proposition 3.1.4, the construction $R = A + B[\Gamma^*]$ is integrally closed if and only if both Γ and B are integrally closed and A is integrally closed in B.

Corollary 3.1.5. *Let $A \subseteq B$ be an extension of integral domains and $R = A + B[\Gamma^*]$. The following statements are equivalent.*

(i) *R is a Schreier domain;*

(ii) *R is an integrally closed gr-pre-Schreier domain.*

3.2 The Schreier property in $A + A_S[\Gamma^*]$

Throughout this section, A is an integral domain, S is a multiplicatively closed subset of A, and Γ is a commutative, additive, cancellative torsion–free monoid such that $\Gamma \cap -\Gamma = \{0\}$.

Let $\Gamma^* = \Gamma \backslash \{0\}$, in this section, we study the Schreier property in the domains of type $A + A_S[\Gamma^*]$. We first characterize (completely) gr-primal elements in the ring $A + A_S[\Gamma^*]$ naturally graded by Γ, with the set of nonzero homogeneous elements $H = A^* \cup \{cX^\alpha, c \in A_S^*, \alpha \in \Gamma^*\}$.

Proposition 3.2.1. *Let A be an integral domain, S a multiplicative subset of A, and $0 \neq a \in A$. Then a is gr–primal in $A + A_S[\Gamma^*]$ if and only if a is primal in A.*

Proof. Clearly, if a is gr-primal in $A + A_S[\Gamma^*]$, then a is primal in A. For the converse, assume that a is primal in A. Let $b, c \in A_S$ be nonzero elements and $\alpha, \beta \in \Gamma$ such that $a \mid (bX^\alpha)(cX^\beta)$ in $A + A_S[\Gamma^*]$.
Case 1: $\alpha = 0$ or $\beta = 0$
If $\alpha = \beta = 0$, this is clear. Assume that $\alpha = 0$ and $\beta \neq 0$. Then $a \mid b(cX^\beta)$. So $a \mid bc$ in A_S implies $bc = a\frac{a'}{s}$ for some $a' \in A$ and $s \in S$. Let $c = \frac{c'}{t}$ for some $c' \in A$ and $t \in S$. Then $bsc' = aa't$ which implies that $a \mid bsc'$ in A. Hence $a = a_1a_2$ for some $a_1, a_2 \in A$, $a_1 \mid b$ and $a_2 \mid sc'$ in A. Now, $cX^\beta = a_2(\frac{a_2'}{st}X^\beta)$, where $a_2' \in A$ such that $sc' = a_2a_2'$. So $a = a_1a_2$ with $a_1 \mid b$ and $a_2 \mid cX^\beta$.
The case $\alpha \neq 0$ and $\beta = 0$ is similar.
Case 2: $\alpha \neq 0$ and $\beta \neq 0$
Then, $a \mid (bX^\alpha)(cX^\beta)$ implies that $a \mid bc$ in A_S, so $bc = a\frac{a'}{s}$ for some $a' \in A$ and $s \in S$. Since $b, c \in A_S$, there exist $b', c' \in A$ and $t_1, t_2 \in S$ such that $b = \frac{b'}{t_1}$ and $c = \frac{c'}{t_2}$. So $b'sc' = at_1t_2a'$ implies $a \mid b'sc'$ in A. Since a is primal in A, then $a = a_1a_2$ for some $a_1, a_2 \in A$, $a_1 \mid b's$ and $a_2 \mid c'$ in A. Hence $bX^\alpha = a_1(\frac{a_1'}{st_1}X^\alpha)$ and $cX^\beta = a_2(\frac{a_2'}{t_2}X^\beta)$, where $b's = a_1a_1'$ and $c' = a_2a_2'$. Therefore, $a = a_1a_2$ such that $a_1 \mid bX^\alpha$ and $a_2 \mid cX^\beta$, in $A + A_S[\Gamma^*]$. □

Corollary 3.2.2. *Let A be an integral domain, S a multiplicative subset of A, and $0 \neq a \in A$. Then a is completely gr–primal in $A + A_S[\Gamma^*]$ if and only if a is completely primal in A.*

Next, we explore when X^α, for some $\alpha \in \Gamma^*$, is (completely) gr–primal in $A + A_S[\Gamma^*]$.

Definition 3.2.3. Let A be an integral domain, S a saturated multiplicative subset of A. S is called good if for each $s \in S$, $a, b \in A$ and $s \mid ab$, there exists $t \in S$ such that $t \mid a$ and $s \mid tb$.

By [19, Remark 1.3(b)], if S is consisting of (completely) primal elements in A, then S is good.

Proposition 3.2.4. *Let A be an integral domain, S a saturated multiplicative subset of A, and $0 \neq \alpha \in \Gamma$. Let $R = A + A_S[\Gamma^*]$, then the following statements are equivalent:*

(i) X^α *is gr–primal in R;*

(ii) α *is primal in* Γ *and* S *is good.*

Proof. (i)$\Rightarrow$ (ii) Assume that X^α is gr-primal in R. Let $\beta_1, \beta_2 \in \Gamma$ such that $\alpha \leq \beta_1+\beta_2$, then $X^\alpha \mid X^{\beta_1}X^{\beta_2}$ in R. Since X^α is gr-primal, then $X^\alpha = a_1X^{\alpha_1}a_2X^{\alpha_2}$, $a_1X^{\alpha_1} \mid X^{\beta_1}$ and $a_2X^{\alpha_2} \mid X^{\beta_2}$ for some $a_1, a_2 \in A_S$ and $\alpha_1, \alpha_2 \in \Gamma$. So $\alpha = \alpha_1 + \alpha_2$, $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$, hence α is primal in Γ.
For the second part of (ii), let $s \in S$ and $a, b \in A$ such that $s \mid ab$ in A. Then $X^\alpha \mid a(\frac{bX^\alpha}{s})$ in R. Since X^α is gr-primal, $X^\alpha = a_1(b_1X^\alpha)$ for some $a_1 \in A$ and $b_1 \in A_S$ such that $a_1 \mid a$ and $b_1X^\alpha \mid \frac{bX^\alpha}{s}$. So $a_1b_1 = 1$. Let $b_1 = \frac{a_1'}{t}$ for some $a_1' \in A$ and $t \in S$. Then $a_1a_1' = t \in S$ implies $a_1 \in S$, since S is saturated. Set $s' = a_1$, so $b_1 = \frac{1}{s'}$. Then $s' \mid a$ and $s \mid s'b$. Hence S is good.

(ii)$\Rightarrow$ (i) Suppose that α is primal in Γ and S is good. To prove (i), let $b_1, b_2 \in A_S$ be nonzero elements and $\beta_1, \beta_2 \in \Gamma$ such that $X^\alpha \mid b_1X^{\beta_1}b_2X^{\beta_2}$. Then $\alpha \leq \beta_1 + \beta_2$, so $\alpha = \alpha_1 + \alpha_2$ for some $\alpha_1, \alpha_2 \in \Gamma$ such that $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$. We consider the following cases:

Case 1: $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$
Then $X^\alpha = X^{\alpha_1}X^{\alpha_2}$, $X^{\alpha_1} \mid b_1X^{\beta_1}$ and $X^{\alpha_2} \mid b_2X^{\beta_2}$ in R.

Case 2: $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$
Note that since $\alpha \neq 0$, then $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. Assume that $\alpha_1 \neq 0$. Let $s \in S$ such that $sb_1 \in A$, then $X^\alpha = (\frac{X^{\alpha_1}}{s})(sX^{\alpha_2})$ with $\frac{X^{\alpha_1}}{s} \mid b_1X^{\beta_1}$ and $sX^{\alpha_2} \mid b_2X^{\beta_2}$. If $\alpha_1 = \beta_1 = 0$, we consider the trivial factorization $X^\alpha = 1.X^{\alpha_2}$. Note that $X^{\alpha_2} \mid b_2X^{\beta_2}$ since $\alpha_2 < \beta_2$.

Case 3: $\alpha_1 = \beta_1 \neq 0$ and $\alpha_2 = \beta_2 \neq 0$
$X^\alpha \mid b_1X^{\beta_1}b_2X^{\beta_2}$ implies $b_1b_2 \in A$. Take $b_1 = \frac{a_1}{s_1}$ and $b_2 = \frac{a_2}{s_2}$ for some $a_1, a_2 \in A$ and $s_1, s_2 \in S$. Let $s = s_1s_2$, then $\frac{a_1a_2}{s} \in A$ implies $s \mid a_1a_2$ in A, so there exists $t \in S$ such that $t \mid a_1$ and $s \mid ta_2$ in A. Hence $X^\alpha = (\frac{t}{s_1}X^{\beta_1})(\frac{s_1}{t}X^{\beta_2})$, $\frac{t}{s_1}X^{\beta_1} \mid b_1X^{\beta_1}$ and $\frac{s_1}{t}X^{\beta_2} \mid b_2X^{\beta_2}$.

Case 4: $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2$
Then, $X^\alpha = X^{\beta_2} \mid b_1(b_2X^{\beta_2})$ with $b_1 \in A$, so $b_1b_2 \in A$. Since $b_2 \in A_S$, there exist $a_2 \in A$ and $s \in S$ such that $b_2 = \frac{a_2}{s}$. Then $b_1b_2 = \frac{b_1a_2}{s} \in A$, so $s \mid b_1a_2$. Since S is good, there exists $t \in S$ such that $t \mid b_1$ and $s \mid ta_2$ in A. Therefore, $X^\alpha = X^{\beta_2} = t(\frac{X^{\beta_2}}{t})$, $t \mid b_1$, and $\frac{X^{\beta_2}}{t} \mid b_2X^{\beta_2}$ since $b_2X^{\beta_2} = \frac{(ta_2)}{s}(\frac{X^{\beta_2}}{t})$.

By commutativity, the remaining cases also hold. $\square$

Let A be an integral domain and S a multiplicative subset of A. For $\alpha \in \Gamma$, note that the divisors of X^α in $R = A + A_S[\Gamma^*]$ are the elements of S and the elements uX^β, where u is a unit in A_S and $\beta \leq \alpha$, $\beta \in \Gamma$. On the other hand, as we shall see in a general case (Lemma 3.3.12), one can check that if X^α is gr-primal in R, then uX^α is gr-primal in R for every unit u in A_S such that $uX^\alpha \in R$. Thus by Propositions 3.2.1 and 3.2.4 and Lemma 3.3.12, we get:

Corollary 3.2.5. *Let A be an integral domain, S a multiplicative subset of A, and α a nonzero element of Γ. Let $R = A + A_S[\Gamma^*]$, then the following statements are equivalent:*

(i) *X^α is completely gr–primal in R;*

(ii) *α is completely primal in Γ and S consists of primal elements of A.*

As a consequence of the above results, we characterize the gr-pre-Schreier property in $A + A_S[\Gamma^*]$ domains graded naturally by Γ.

Theorem 3.2.6. *Let A be an integral domain and S a multiplicative subset of A. The following statements are equivalent:*

(i) *$R = A + A_S[\Gamma^*]$ is gr–pre–Schreier;*

(ii) *A and Γ are pre–Schreier.*

Proof. Let $\overline{S}$ be the saturation of S in A. Then $A_{\overline{S}} = A_S$, so we may assume that S is saturated.
(i)⇒(ii). This follows from Propositions 3.2.1 and 3.2.4.
(ii)⇒(i). We use Proposition 3.1.3. We have $R_S = A_S[\Gamma]$ is gr–pre–Schreier (cf. Proposition 2.3.5). On the other hand, by Proposition 3.2.1, the elements of S are gr–primal in R. Hence R is gr–pre–Schreier. □

By Proposition 3.1.4 and Theorem 3.2.6 , we get our main results of this section. This recovers the case of $A + XA_S[X]$ domains [19, 20].

Theorem 3.2.7. *Let A be an integral domain and S a multiplicative subset of A. The following statements are equivalent:*

(i) *$R = A + A_S[\Gamma^*]$ is pre–Schreier;*

(ii) *A is pre–Schreier and both A_S and Γ are Schreier.*

Corollary 3.2.8. *Let A be an integral domain and S a multiplicative subset of A. The following statements are equivalent:*

(i) *$R = A + A_S[\Gamma^*]$ is Schreier;*

(ii) *A and Γ are Schreier.*

Example 3.2.9. Let A be a pre–Schreier domain which is not Schreier, see for instance, [37]. Let Γ be a Schreier monoid (e.g., $\Gamma = \mathbb{Z}_+^n$ a GCD monoid.) Then by the above results, $A + K[\Gamma^*]$, K the quotient field of A, is an almost normal pre–Schreier domain which is not Schreier.

3.3 The general case

In this section, $A \subseteq B$ is an extension of integral domains, $S = U(B) \cap A$, and Γ is a commutative, additive, cancellative torsion–free monoid such that $\Gamma \cap -\Gamma = \{0\}$. We next investigate the Schreier property in the general case of $A+B[\Gamma^*]$ domains. For, we study the (completely) gr–primal elements in $R = A+B[\Gamma^*]$ graded naturally by Γ.

The following proposition characterizes gr-primal constant elements (elements of A) in $R = A + B[\Gamma^*]$.

Proposition 3.3.1. *Let $A \subseteq B$ be an extension of integral domains. Let $R = A + B[\Gamma^*]$ and $0 \neq a \in A$. The following statements are equivalent:*

(1) *a is gr–primal in R;*

(2) (i) *a is primal in A and in B;*

   (ii) *For every $d \in A$ such that $d \mid a$, say $a = dd'$ for some $d' \in A$, whenever $db \in A$ for some $b \in B$, there exists $0 \neq c \in B$ such that $cd, c^{-1}d', c^{-1}b \in A$;*

(iii) *Whenever* $a = b_1b_2$ *for some* $b_1, b_2 \in B$, *there exists* $u \in U(B)$ *such that* $b_1u, b_2u^{-1} \in A$.

Proof. (1)$\Rightarrow$(2). Assume that a is gr–primal in R. (i) is clear. To prove (ii), let $d, d' \in A$ such that $a = dd'$ and $db \in A$, $b \in B$. So $a \mid (db)(d'X^{\alpha})$, $\alpha \in \Gamma^*$. Then, there exist $a_1, a_2 \in A$ such that $a = a_1a_2$ with $a_1 \mid db$ and $a_2 \mid d'X^{\alpha}$ in R. Let $a'_1 \in A$ and $a'_2 \in B$ such that $db = a_1a'_1$ and $d' = a_2a'_2$. Then $b = a'_1a'_2$. One can easily check that $c = a'_2$ satisfies the desired conditions.
For (iii), let $b_1, b_2 \in B$ such that $a = b_1b_2$. In R, we have $a \mid (b_1X^{\alpha})(b_2X^{\alpha})$, $\alpha \in \Gamma^*$. Since a is gr–primal in R, there exist $a_1, a_2 \in A$ such that $a = a_1a_2$; $a_1 \mid b_1X^{\alpha}$ and $a_2 \mid b_2X^{\alpha}$ in R. So $b_1 = a_1b'_1$ and $b_2 = a_2b'_2$ for some $b'_1, b'_2 \in B$. Then $b_1b_2 = a_1a_2b'_1b'_2$ implies $a = ab'_1b'_2$, so $b'_1b'_2 = 1$. Hence $b'_1, b'_2 \in U(B)$ with $b_1b'_2 = a_1 \in A$ and $b_2b'_1 = a_2 \in A$.

(2)$\Rightarrow$(1). Assume that the conditions in (2) hold. Let $b_1, b_2 \in B\backslash\{0\}$ and $\beta_1, \beta_2 \in \Gamma$ such that $a \mid (b_1X^{\beta_1})(b_2X^{\beta_2})$. To prove the gr–primality of a, we consider the following cases.

Case 1: $\beta_1 \neq 0$ and $\beta_2 \neq 0$.

We have $a \mid (b_1X^{\beta_1})(b_2X^{\beta_2})$, then $a \mid b_1b_2$ in B. As a is primal in B, there exist $c_1, c_2 \in B$ such that $a = c_1c_2$, with $c_1 \mid b_1$ and $c_2 \mid b_2$ in B. Since (iii) holds and $a = c_1c_2 \in A$, we can choose c_1 and c_2 in A with $c_1 \mid b_1$ and $c_2 \mid b_2$ in B. Hence $c_1 \mid b_1X^{\beta_1}$ and $c_2 \mid b_2X^{\beta_2}$ in R.

Case 2: $\beta_1 = 0$ or $\beta_2 = 0$

If $\beta_1 = \beta_2 = 0$, then $b_1, b_2 \in A$ and $a \mid b_1b_2$ in A. Since a is primal in A, we are done.
Suppose that $\beta_1 \neq 0$ and $\beta_2 = 0$. In this case, $a \mid (b_1X^{\beta_1})b_2$ with $b_2 \in A$. Then $a \mid b_1b_2$ in B. Since a is primal in B, there exist $a_1, a_2 \in B$ such that $a = a_1a_2$, $a_1 \mid b_1$ and $a_2 \mid b_2$ in B. Again by (iii), a_1 and a_2 can be chosen in A. Let $b_1 = a_1a'_1$ and $b_2 = a_2a'_2$ for some $a'_1, a'_2 \in B$. Now, apply (ii) for $d = a_2$, $d' = a_1$ and $b = a'_2$, we get $c \in B$ such that $ca_2, c^{-1}a_1, c^{-1}a'_2 \in A$. Write $a = (c^{-1}a_1)(ca_2)$, also $b_1 = (c^{-1}a_1)(ca'_1)$ and $b_2 = (ca_2)(c^{-1}a'_2)$. Thus $a = c_1c_2$, $c_1 = c^{-1}a_1$, $c_2 = ca_2 \in A$, with $c_1 \mid b_1X^{\beta_1}$ and $c_2 \mid b_2$ in R.
By commutativity, the case $\beta_1 = 0$ and $\beta_2 \neq 0$ is similar. $\square$

As a consequence of the previous proposition, we next characterize completely gr-primal constant element in $R = A + B[\Gamma^*]$.

Definition 3.3.2. Let $A \subseteq B$ be an extension of integral domains and $a \in A$. We say that a is A–completely primal in B if every $d \in A$ such that $d \mid a$ in A, d is primal in B.

Corollary 3.3.3. *Let $A \subseteq B$ be an extension of integral domains and $S = U(B) \cap A$. Let $R = A + B[\Gamma^*]$ and $0 \neq a \in A$. The following statements are equivalent:*

(1) *a is completely gr–primal in R;*

(2) (i) *a is completely primal in A and A–completely primal in B;*

(ii) *For every $a' \in A$ such that $a' \mid a$, whenever $a'b \in A$ for some $b \in B$, there exists $t \in S$ such that $t^{-1}a', tb \in A$;*

(iii) *For every $a' \in A$ such that $a' \mid a$ in A, whenever $a' = b_1 b_2$ for some $b_1, b_2 \in B$, there exists $u \in U(B)$ such that $b_1 u, b_2 u^{-1} \in A$.*

Proof. This follows from Proposition 3.3.1. For (1)⇒(2) (ii), remark that a' is gr–primal in R and apply Proposition 3.3.1 (2) (ii) for a' with $d = a'$ and $d' = 1$. Then we get $0 \neq c \in B$ such that $cd, c^{-1}d', c^{-1}b \in A$. So $c^{-1} \in A$. Hence $t = c^{-1} \in S$ and $t^{-1}a', tb \in A$. □

The next result gives necessary and sufficient conditions for X^α, $\alpha \in \Gamma^*$, to be gr–primal in $A + B[\Gamma^*]$. We start with the following lemma which recovers the case of $A + XB[X]$ domains [19].

Recall that a monoid is called antimatter if it does not contain any atoms. The additive monoid of nonnegative rationals is an antimatter monoid. For examples of non antimatter monoids, consider the monoids $\mathbb{Z}_+^n$, $n \geq 1$ an integer.

Lemma 3.3.4. *Let $A \subseteq B$ be an extension of integral domains and $R = A + B[\Gamma^*]$. Suppose that Γ is not an antimatter monoid and let α be an atom of Γ. If X^α is gr-primal in R, then $B = A_S$ where $S = U(B) \cap A$.*

Proof. Assume that X^α is gr-primal in R and let $b \in B$. Then $X^\alpha \mid (bX^\alpha)(bX^\alpha)$. Hence there exist $g, h \in R$ such that $X^\alpha = gh$, $g \mid bX^\alpha$ and $h \mid bX^\alpha$. Since α is an atom, we may assume that $g = a \in A$ and $h = b'X^\alpha$ for some $b' \in B$. Then $X^\alpha = ab'X^\alpha$, with $b'X^\alpha \mid bX^\alpha$. Thus $1 = ab'$, so $a \in S$. Let $a' \in A$ such that $bX^\alpha = a'(b'X^\alpha)$. Then, $b = a'b' = a'a^{-1} \in A_S$, so $B \subseteq A_S$. Hence $B = A_S$. □

Corollary 3.3.5 (cf. [20, Theorem 2.7 and Corollary 2.9]). *Let $A \subseteq B$ be an extension of integral domains, $S = U(B) \cap A$, and $R = A + B[\Gamma^*]$. Suppose that Γ is not an antimatter monoid. Then R is a pre-Schreier domain (resp., Schreier domain) if and only if A is pre-Schreier, $B = A_S$, and both A_S and Γ are Schreier (resp., $B = A_S$ and both A and Γ are Schreier).*

Proof. This follows from Lemma 3.3.4, Theorem 3.2.7 and Corollary 3.2.8. □

The characterization of the gr-primality of X^α, $\alpha \in \Gamma^*$, in $A + B[\Gamma^*]$ domains depends on whether B is a quotient ring of A or not. Note that B is a quotient ring of A if and only if $B = A_S$, where $S = U(B) \cap A$. The case where B is a quotient ring of A was studied in Section 3.2, we next focus on the case $B \neq A_S$, $S = U(B) \cap A$.

To state our next results we need some definitions and notations.

Definition 3.3.6. Let $A \subseteq B$ be an extension of integral domains and $S = U(B) \cap A$. We say that the extension $A \subseteq B$ is A–inert if whenever $ab \in A$ for some nonzero elements $a \in A$, $b \in B$, there exists $t \in S$ such that $at^{-1}, bt \in A$.

Definition 3.3.7. Let $A \subseteq B$ be an extension of integral domains. We say that B is associate to A if every element of B is associate to an element of A. That is, for every $b \in B$, there exists $u \in U(B)$ and an element $a \in A$ such that $b = ua$.

Remark 3.3.8. (1) Let $A \subseteq B$ be an extension of rings with $B = A_S$, where $S = U(B) \cap A$. Then, one can check that, the extension $A \subseteq A_S$ is A-inert if and only if S is good, and in this case $A \subseteq A_S$ is also inert.

(2) For a domain A every quotient ring of A is associate to A. For an example of an associate extension which is not a quotient ring, let A be an integral domain and K a field strictly containing the quotient field of A. Let T be a domain of the form $T = K + I$, where I is an ideal of T such that $I \cap A = \{0\}$. Then the classical pullback $R = A + I$ is a subring of T with T associate to R.

Let $\alpha \in \Gamma^*$. Recall that α is called reducible in Γ if $\alpha = \alpha_1 + \alpha_2$ for some $\alpha_1, \alpha_2 \in \Gamma^*$. The element α will be called completely reducible if every divisor of α is reducible. We say that α is Γ-lower directed (resp., completely Γ-lower directed) if for each $\beta \in \Gamma^*$, α (resp., every nonzero divisor of α in Γ) and β have a common divisor in Γ^*. Finally, the monoid Γ will be called lower directed if every two elements of Γ^* have a common divisor in Γ^*.

Lemma 3.3.9. *Let $A \subseteq B$ be an extension of integral domains and let $\alpha \in \Gamma^*$ such that X^α is gr–primal in $A + B[\Gamma^*]$. Then:*

(1) *Either α is completely reducible or B is associate to A.*

(2) *Either α is completely Γ–lower directed or B is associate to A.*

Proof. (1) Assume that there exits an atom $\gamma \leq \alpha$. Then $\alpha = \gamma + \gamma'$ for some $\gamma' \in \Gamma$. Let $0 \neq b \in B$. Then $X^\alpha \mid (bX^\gamma)(bX^\alpha)$. Since X^α is gr-primal, $X^\alpha = (b_1 X^{\alpha_1})(b_2 X^{\alpha_2})$ for some $b_1, b_2 \in B$ and $\alpha_1, \alpha_2 \in \Gamma$, with $b_1 X^{\alpha_1} \mid bX^\gamma$ and $b_2 X^{\alpha_2} \mid bX^\alpha$. Then $\alpha_1 \leq \gamma$. So $\alpha_1 = 0$ or $\alpha_1 = \gamma$.
If $\alpha_1 = 0$, then $b_1 \in A$ and $\alpha_2 = \alpha$. Hence $b = ab_2$ for some $a \in A$, with $b_1 b_2 = 1$, and we are done in this case.
If $\alpha_1 = \gamma$, clearly $b = a' b_1$ for some $a' \in A$. Hence, also in this case b is associate to an element of A.

(2) Assume that α is not completely Γ-lower directed. Let $\gamma \leq \alpha$ such that γ is prime to some $\delta \in \Gamma^*$. Let $0 \neq b \in B$ and $\gamma' \in \Gamma$ such that $\alpha = \gamma + \gamma'$. Then $X^\gamma \mid (bX^\gamma)X^\delta$ implies $X^\alpha \mid (bX^\gamma)X^{\delta+\gamma'}$. Since X^α is gr-primal, $X^\alpha = (b_1 X^{\alpha_1})(b_2 X^{\alpha_2})$ for some $b_1, b_2 \in B$ and $\alpha_1, \alpha_2 \in \Gamma$, with $b_1 X^{\alpha_1} \mid bX^\gamma$ and $b_2 X^{\alpha_2} \mid X^{\delta+\gamma'}$. We have $\alpha = \alpha_1 + \alpha_2 = \gamma + \gamma'$, so $\alpha_2 - \gamma' = \gamma - \alpha_1 \in \Gamma$, since $b_1 X^{\alpha_1} \mid bX^\gamma$. Thus $\alpha_2 - \gamma'$ is a common divisor of γ and δ. So $\alpha_2 = \gamma'$ and $\alpha_1 = \gamma$. Now, $b_1 X^\gamma \mid bX^\gamma$ implies $b = b_1 a$ for some $a \in A$, with $b_1 b_2 = 1$. Hence B is associate to A. □

Proposition 3.3.10. *Let $A \subseteq B$ be an extension of integral domains such that B is not a quotient ring of A. Let $S = U(B) \cap A$ and $\alpha \in \Gamma^*$. Then X^α is gr–primal in $R = A + B[\Gamma^*]$ if and only if the following conditions hold:*

(a) *α is a reducible primal element of Γ;*

(b) *α is Γ–lower directed;*

(c) *Either α is completely reducible or B is associate to A;*

(d) *Either α is completely Γ–lower directed or B is associate to A;*

(e) *The extension $A \subseteq B$ is inert and A–inert.*

Proof. ($\Rightarrow$): Assume that X^α is gr–primal in R.

(a) Clearly, α is primal in Γ, and by Lemma 3.3.4, α is reducible.

(b) Suppose that α is not Γ–lower directed. Then there exists $\beta \in \Gamma^*$ which is prime to α in Γ. Let $0 \neq b \in B$. Then $X^\alpha \mid (bX^\alpha)X^\beta$. Since X^α is gr–primal, $X^\alpha = (b_1X^{\alpha_1})(b_2X^{\alpha_2})$ for some $b_1, b_2 \in B$ and $\alpha_1, \alpha_2 \in \Gamma$, $b_1X^{\alpha_1} \mid bX^\alpha$ and $b_2X^{\alpha_2} \mid X^\beta$. Thus α_2 is a common divisor of α and β, so $\alpha_2 = 0$ and $\alpha = \alpha_1$. Hence $X^\alpha = (b_1X^\alpha)b_2$, with $b_2 \in A$. Note that $b_2 \in S$ since $b_1b_2 = 1$. Now, $b = b_1b_1'$ for some $b_1' \in A$, so $b = b_2^{-1}b_1' \in A_S$. Hence $B = A_S$, a contradiction.

(c)–(d) This is Lemma 3.3.9.

(e) To prove that $A \subseteq B$ is an inert extension, let b_1, b_2 be nonzero elements of B such that $b_1b_2 \in A$. Since α is reducible, then $\alpha = \beta_1 + \beta_2$ with $\beta_i \in \Gamma^*$ for $i = 1, 2$. So $X^\alpha \mid (b_1X^{\beta_1})(b_2X^{\beta_2})$. As X^α is gr–primal, $X^\alpha = (c_1X^{\alpha_1})(c_2X^{\alpha_2})$ for some $c_1, c_2 \in B$ and $\alpha_1, \alpha_2 \in \Gamma$, $c_1X^{\alpha_1} \mid b_1X^{\beta_1}$ and $c_2X^{\alpha_2} \mid b_2X^{\beta_2}$.
Now $\alpha = \alpha_1 + \alpha_2 = \beta_1 + \beta_2$ with $\alpha_i \leq \beta_i$, for $i = 1, 2$, hence $\alpha_i = \beta_i$, $i = 1, 2$. This implies $b_1X^{\beta_1} = a_1(c_1X^{\beta_1})$ and $b_2X^{\beta_2} = a_2(c_2X^{\beta_2})$ for some $a_1, a_2 \in A$. Thus $b_1 = a_1c_1$ and $b_2 = a_2c_2$ with $c_1c_2 = 1$. Hence $A \subseteq B$ is an inert extension.
It remains to show that the extension $A \subseteq B$ is A–inert. For, let $a \in A$ and $b \in B$ be nonzero elements such that $ab \in A$. Then $X^\alpha \mid a(bX^\alpha)$ in R. So $X^\alpha = t(b'X^\alpha)$ for some $t \in A$ and $b' \in B$, $t \mid a$ and $b'X^\alpha \mid bX^\alpha$ in R. We have $tb' = 1$, so $t \in S$. Clearly, $at^{-1} \in A$. Moreover, $t^{-1}X^\alpha \mid bX^\alpha$ in R implies that $bt \in A$. Hence, the extension $A \subseteq B$ is A–inert.

($\Leftarrow$): Suppose that $X^\alpha \mid (b_1X^{\beta_1})(b_2X^{\beta_2})$ for some $b_1, b_2 \in B^*$ and $\beta_1, \beta_2 \in \Gamma$. Then $\alpha \leq \beta_1 + \beta_2$. Thus $\alpha = \alpha_1 + \alpha_2$ for some $\alpha_1, \alpha_2 \in \Gamma$, $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$. We have many cases.

Case 1: $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$

Then $X^\alpha = X^{\alpha_1}X^{\alpha_2}$, $X^{\alpha_1} \mid b_1X^{\beta_1}$ and $X^{\alpha_2} \mid b_2X^{\beta_2}$.

Case 2: $\alpha_1 = \beta_1 \neq 0$ and $\alpha_2 < \beta_2$

2.1 Suppose that B is associate to A.

Assume that $\alpha_2 \neq 0$, and let $a_1 \in A$ such that $b_1 = u_1a_1$ for some $u_1 \in U(B)$. Then $X^\alpha = (u_1X^{\alpha_1})(u_1^{-1}X^{\alpha_2})$ with $u_1X^{\alpha_1} \mid b_1X^{\beta_1}$ and $u_1^{-1}X^{\alpha_2} \mid b_2X^{\beta_2}$.

We next assume that $\alpha_2 = 0$. Since α is Γ–lower directed, there exists $\gamma \in \Gamma^*$ such that $\gamma \leq \alpha$ and $\gamma \leq \beta_2$.

(i) If $\gamma \neq \alpha$, let $a_2 \in A$ such that $b_2 = u_2a_2$, for some $u_2 \in U(B)$. Then $X^\alpha = (u_2^{-1}X^{\alpha-\gamma})(u_2X^\gamma)$, with $u_2^{-1}X^{\alpha-\gamma} \mid b_1X^{\beta_1}$ and $u_2X^\gamma \mid b_2X^{\beta_2}$.

(ii) Suppose that $\gamma = \alpha$. Since α is reducible, then $\alpha = \gamma_1 + \gamma_2$ for some $0 < \gamma_1 < \alpha$ and $0 < \gamma_2 < \alpha$. Thus $X^\alpha = X^{\gamma_1} X^{\gamma_2}$ with $X^{\gamma_1} \mid b_1 X^{\beta_1}$ and $X^{\gamma_2} \mid b_2 X^{\beta_2}$.

2.2 Assume that B is not associate to A.

Since $\alpha_2 < \beta_2$, there exists $\alpha_2' \in \Gamma^*$ such that $\beta_2 = \alpha_2 + \alpha_2'$. By (d), there exists $\gamma \in \Gamma^*$ such that $\gamma \leq \beta_1$ and $\gamma \leq \alpha_2'$. So $\beta_1 = \gamma + \gamma'$ for some $\gamma' \in \Gamma$. Moreover, by (c) γ is reducible since $\gamma \leq \alpha$. Then $\gamma = \gamma_1 + \gamma_2$, for some $\gamma_1, \gamma_2 \in \Gamma^*$. Set $\delta_1 = \gamma_1 + \gamma'$ and $\delta_2 = \gamma_2 + \alpha_2$. One can easily check that $\delta_1 < \beta_1$ and $\delta_2 < \beta_2$. Hence $X^\alpha = X^{\delta_1} X^{\delta_2}$, $X^{\delta_1} \mid b_1 X^{\beta_1}$ and $X^{\delta_2} \mid b_2 X^{\beta_2}$.

Case 3: $\alpha_1 = \beta_1 = 0$ and $\alpha_2 < \beta_2$

Consider the trivial factorization $X^\alpha = 1.X^\alpha = 1.X^{\alpha_2}$.

Case 4: $\alpha_1 = \beta_1 \neq 0$ and $\alpha_2 = \beta_2 \neq 0$

Note that $X^\alpha \mid (b_1 X^{\beta_1})(b_2 X^{\beta_2})$ implies $b_1 b_2 \in A$. Since $A \subseteq B$ is inert, there exists $u \in U(B)$ such that $b_1 u, b_2 u^{-1} \in A$. So $X^\alpha = (u^{-1} X^{\beta_1})(u X^{\beta_2})$, $u^{-1} X^{\beta_1} \mid b_1 X^{\beta_1}$ and $u X^{\beta_2} \mid b_2 X^{\beta_2}$.

Case 5: $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2$

We have $X^\alpha \mid b_1(b_2 X^{\beta_2})$, so $b_1 b_2 \in A$ with $b_1 \in A$. Since the extension $A \subseteq B$ is A–inert, there exists $t \in S$ such that $b_1 t^{-1}, b_2 t \in A$. Hence $X^\alpha = t(t^{-1} X^{\beta_2})$ such that $t \mid b_1$ and $t^{-1} X^{\beta_2} \mid b_2 X^{\beta_2}$ in R.

Since our rings are commutative, the other cases hold. Finally, X^α is gr–primal. $\square$

As a corollary of Proposition 3.3.10, we next characterize when X^α, $\alpha \in \Gamma$, is completely gr-primal in $A + B[\Gamma^*]$. For this, we need the following lemmas.

Lemma 3.3.11. *Let $A \subseteq B$ be an extension of domains and $S = U(B) \cap A$. Then $a \in S$ is gr-primal in $A + B[\Gamma^*]$ if and only if a is primal in A.*

Proof. Clearly, if a is gr-primal in $A + B[\Gamma^*]$, then a is primal in A. Conversely, assume that a is primal in A. Let $b_1, b_2 \in B^*$ and $\beta_1, \beta_2 \in \Gamma$ such that $a \mid (b_1 X^{\beta_1})(b_2 X^{\beta_2})$. If $\beta_1 \neq 0$, for the gr-primality of a, we consider the trivial factorization $a = a.1$. The case $\beta_2 \neq 0$ is similar. Suppose that $\beta_1 = \beta_2 = 0$. Then $b_1, b_2 \in A^*$ and $a \mid b_1 b_2$. Thus, also in this case, we are done by using the primality of a in A. $\square$

Lemma 3.3.12. *Let $A \subseteq B$ be an extension of domains and $S = U(B) \cap A$. Let $\alpha \in \Gamma^*$ such that X^α is gr-primal in $A+B[\Gamma^*]$. Then uX^α is gr-primal in $A+B[\Gamma^*]$ for every $u \in U(B)$.*

Proof. Let $u \in U(B)$ such that $uX^\alpha \mid (b_1X^{\beta_1})(b_2X^{\beta_2})$, where $b_1, b_2 \in B^*$ and $\beta_1, \beta_2 \in \Gamma$. We consider the following cases:

Case 1: $\beta_1 \neq 0$ and $\beta_2 \neq 0$

Then $X^\alpha \mid (u^{-1}b_1X^{\beta_1})(b_2X^{\beta_2})$ and $X^\alpha \mid (b_1X^{\beta_1})(u^{-1}b_2X^{\beta_2})$. Since X^α is gr-primal, then $X^\alpha = (c_{11}X^{\alpha_{11}}).(c_{12}X^{\alpha_{12}}) = (c_{21}X^{\alpha_{21}}).(c_{22}X^{\alpha_{22}})$, where $c_{ij} \in B$, $\alpha_{ij} \in \Gamma$, $i, j \in \{1, 2\}$, with $c_{ij}X^{\alpha_{ij}} \mid (u^{-1}b_j)X^{\beta_j}$ if $i = j$, and $c_{ij}X^{\alpha_{ij}} \mid b_jX^{\beta_j}$ if $i \neq j$. If $\alpha_{11} \neq 0$, we get the factorization $uX^\alpha = (uc_{11}X^{\alpha_{11}}).(c_{12}X^{\alpha_{12}})$. Similarly, if $\alpha_{22} \neq 0$, we consider the factorization $uX^\alpha = (c_{21}X^{\alpha_{21}}).(uc_{22}X^{\alpha_{22}})$. We next suppose that $\alpha_{11} = \alpha_{22} = 0$. Then $\alpha = \alpha_{12} = \alpha_{21}$. We consider two cases. (i) Assume that $B = A_S$. Set $u = \frac{a}{s}$ for some $a \in A$ and $s \in S$. Thus $uX^\alpha = (ac_{11}).(\frac{c_{12}}{s}X^\alpha)$. One can easily check that $ac_{11} \mid b_1X^{\beta_1}$ and $\frac{c_{12}}{s}X^\alpha \mid b_2X^{\beta_2}$. (ii) If $B \neq A_S$, by Proposition 3.3.10, α is reducible. Then there exist $\alpha_1, \alpha_2 \in \Gamma^*$ such that $\alpha = \alpha_1 + \alpha_2$. Note that, for $i = 1, 2$, $\alpha_i < \alpha \leq \beta_i$. Thus $uX^\alpha = (uX^{\alpha_1}).X^{\alpha_2}$, $uX^{\alpha_1} \mid b_1X^{\beta_1}$ and $X^{\alpha_2} \mid b_2X^{\beta_2}$.

Case 2: $\beta_1 = 0$

Then $\beta_2 \neq 0$ and $X^\alpha \mid b_1(u^{-1}b_2X^{\beta_2})$. Since X^α is gr-primal, then $X^\alpha = c_1.(c_2X^\alpha)$, where $c_1 \in A$ and $c_2 \in B$, with $c_1 \mid b_1$ and $c_2X^\alpha \mid u^{-1}b_2X^{\beta_2}$. Thus $uX^\alpha = c_1.(uc_2X^\alpha)$, $c_1 \mid b_1$ and $uc_2X^\alpha \mid b_2X^{\beta_2}$.

Case 3 : $\beta_2 = 0$

This case is similar to the Case 2.

□

Let $A \subseteq B$ be an extension of integral domains and $S = U(B) \cap A$. The divisors of X^α, $\alpha \in \Gamma^*$, in $R = A + B[\Gamma^*]$ are the elements of S and terms of the form uX^β, where u is a unit in B and $\beta \in \Gamma$, $\beta \leq \alpha$. Thus, from Proposition 3.3.10 and Lemmas 3.3.11 and 3.3.12, we get the following proposition:

Proposition 3.3.13. *Let $A \subseteq B$ be an extension of integral domains and $S = U(B) \cap A$ such that $B \neq A_S$. Let $\alpha \in \Gamma^*$, the following statements are equivalent:*

(i) X^α *is completely gr-primal in* $A + B[\Gamma^*]$*;*

(ii) α *is a completely, reducible,* Γ*–lower directed, primal element of* Γ*,* S *consists of primal elements of* A*, and the extension* $A \subseteq B$ *is inert and* A*–inert.*

Remark 3.3.14. 1) Let $A \subseteq B$ be an extension of domains such that B is not a quotient ring of A. As an application of the above results, in $A + XB[X]$ domains, X^n, $n \geq 2$ an integer, is gr-primal if and only if B is associate to A and the extension $A \subseteq B$ is inert and A-inert, see [19, Theorem 1.5].

2) Let $A \subseteq B$ be an extension of domains such that B is not associate to A. If in the monoid Γ, there exists an α which has an atom as a divisor, then X^α is never (gr-) primal in $A + B[\Gamma^*]$.

3) Consider the construction $R = \mathbb{Z} + X\mathbb{R}[X]$. Since the integer 1 is an atom in the additive monoid $\mathbb{Z}_+$, X is not gr-primal. However, X^2 is gr-primal, but not completely gr-primal, see [19, Examples 1.7(ii)].

Theorem 3.3.15. *Let* $A \subseteq B$ *be an extension of integral domains such that* B *is not a quotient ring of* A*. The following statements are equivalent:*

(i) $R = A + B[\Gamma^*]$ *is gr–pre–Schreier;*

(ii) A *is pre–Schreier,* Γ *is an antimatter lower directed pre–Schreier monoid,* B *is pre–Schreier and the extension* $A \subseteq B$ *is inert and* A*–inert.*

Proof. (i)⇒(ii). Almost all of the results in (ii) follow from Propositions 3.3.1 and 3.3.10. We need only to show that B is pre–Schreier. Consider the multiplicative set of R, $T = \{X^\alpha, \alpha \in \Gamma\}$. Then $R_T = B[\langle\Gamma\rangle]$ is gr–pre–Schreier. Hence B is pre–Schreier (cf. Proposition 2.3.5).
(ii)⇒(i). We use the Nagata type theorem for gr-pre-Schreier domains (cf. Proposition 3.1.3). Consider the saturated multiplicative set of R, $T = S \cup \{uX^\alpha, u \in U(B), \alpha \in \Gamma^*\}$, where $S = U(B) \cap A$. By Propositions 3.3.1 and 3.3.10 and Lemma 3.3.12, T consists of gr–primal elements of R. The quotient ring $R_T = B[\langle\Gamma\rangle]$ is gr–pre–Schreier. Hence by Proposition 3.1.3, R is gr–pre–Schreier. □

Theorem 3.3.16. *Let* $A \subseteq B$ *be an extension of integral domains such that* B *is not a quotient ring of* A*. The following statements are equivalent:*

(i) $R = A + B[\Gamma^*]$ *is pre–Schreier;*

(ii) *A is pre–Schreier, Γ is an antimatter lower directed Schreier monoid, B is Schreier and the extension $A \subseteq B$ is inert and A–inert.*

Proof. This follows from Proposition 3.1.4 and Theorem 3.3.15. □

Corollary 3.3.17. *Let $A \subseteq B$ be an extension of integral domains such that B is not a quotient ring of A. The following statements are equivalent:*

(i) *$R = A + B[\Gamma^*]$ is Schreier;*

(ii) *A and B are Schreier with A integrally closed in B, Γ is an antimatter lower directed Schreier monoid and the extension $A \subseteq B$ is inert and A–inert.*

Corollary 3.3.18. *Let $A \subseteq B$ be an extension of integral domains such that B is an overring of A which is not a quotient ring of A. The following statements are equivalent:*

(i) *$R = A + B[\Gamma^*]$ is Schreier;*

(ii) *A and B are Schreier, Γ is an antimatter lower directed Schreier monoid and the extension $A \subseteq B$ is inert and A–inert.*

Example 3.3.19. 1) Let $K \subset L$ be an extension of fields. Then $R = K + L[\Gamma^*]$ is pre–Schreier if, and only if, Γ is an antimatter lower directed Schreier monoid. For an illustration, consider the additive monoid $\Gamma = \mathbb{Q}_+$.

2) Let $\Gamma = \{(s,t) \in \mathbb{Q}^2, s,t > 0\} \cup \{(0,0)\}$ with the relation as in Remark 3.1.2. Then Γ is an antimatter lower directed Schreier monoid. Consider the domain $R = \mathbb{Z} + \mathbb{Z}[Y][\Gamma^*]$, Y an indeterminate. By the above results, R is a Schreier domain. However, R is not a GCD domain [31].

Chapter 4

Some factorization properties in $A + B[\Gamma^*]$ domains

4.0 Introduction

We adopt the following definitions and notation. A monoid means a commutative cancellative torsion-free monoid. Let $A \subseteq B$ be an extension of integral domains and Γ a commutative, additive, cancellative torsion-free monoid. Let $B[\Gamma]$ be the semigroup ring of Γ over B and set $\Gamma^* = \Gamma \setminus \{0\}$. Suppose that $\Gamma \cap -\Gamma = \{0\}$. Then $R = A + B[\Gamma^*]$ is a subring of $B[\Gamma]$. Note that R can be obtained as a pullback and $B[\Gamma^*]$ is a common ideal of R and $B[\Gamma]$. If $\Gamma \cap -\Gamma \neq \{0\}$ or $A = B$, the ring R coincides with $B[\Gamma]$. If $\Gamma = \mathbb{Z}_+$, then $R = A + XB[X]$, and if $\Gamma = \mathbb{Z}_+^n$, then $R = A + (X_1, \ldots, X_n)B[X_1, \ldots, X_n]$. The monoid Γ admits a total order $\prec$ compatible with its semigroup operation [25, Corollary 3.4], and since $\Gamma \cap -\Gamma = \{0\}$, we may assume that $\alpha \succcurlyeq 0$ for all $\alpha \in \Gamma$. Hence each $f \in R$ is uniquely expressible in the form $f = a + b_1 X^{\alpha_1} + \cdots + b_n X^{\alpha_n}$, where $a \in A$, $b_i \in B$ and $\alpha_i \in \Gamma^*$, with $\alpha_1 \prec \cdots \prec \alpha_n$. If $b_n \neq 0$, it is called the leading coefficient of f and α_n the degree. The construction $A + B[\Gamma^*]$ has been studied by many authors and has proven to be useful in constructing examples and counterexamples in many areas of commutative ring theory [17, 21, 31, 32].

Let S denote a multiplicative monoid and $U(S)$ the set of units of S. We will freely use results about multiplicative ideal theory in semigroups analogues of those of commutative rings. For more details, the reader can refer to [25]. For $s, t \in S$, $s \mid t$ if $t = sr$ for some $r \in S$. If $s_1, s_2 \in S$ with $s_1 \mid s_2$ and $s_2 \mid s_1$, then $s_2 = us_1$ for $u \in U(S)$, in which case we say that s_1 and s_2 are associate. An irreducible element (or atom) of S is an element $s \in S$ such that if $s = tr$ in S, then either t or r is a unit of S. The monoid S is called atomic if each nonunit of S is a product of a finite number of irreducible elements of S. We say that S satisfies the ascending chain condition on principal ideals (ACCP) if there does not exist an infinite strictly ascending chain of principal ideals of S. The monoid S is a bounded factorization monoid (BFM) if S is atomic and for each element there is a bound on the length of factorizations into products of atoms.

For a domain D, the set of its nonzero elements, units and nonzero nonunits are denoted by D^*, $U(D)$ and $D^\#$, respectively. If the multiplicative monoid D^* is atomic (resp., satisfies the ACCP, BFM), the domain D is called an atomic domain (resp., satisfies the ACCP, BFD). Atomic domains where defined in [14] and BFDs were introduced in [3]. For a monoid S, unique factorization property (UF) and the properties BF, ACCP, and atomic satisfy the implications:

$$UF \Rightarrow BF \Rightarrow ACCP \Rightarrow Atomic.$$

Examples given in [3] for domains show that the reverse of these implications is not possible. For more details on these factorization properties, we refer the reader to [3] for domains and to [28] for monoids.

The above factorization properties for $\Gamma = \mathbb{Z}_+$, i.e., $R = A + XB[X]$, were intensively studied by several authors, see for instance, [6, 12]. In [30], Kim investigated these factorization concepts in the monoid domains. The purpose of this chapter is to determine necessary and sufficient conditions for the domain $R = A + B[\Gamma^*]$, where Γ is an additive monoid such that $\Gamma \cap -\Gamma = \{0\}$, to be atomic, satisfy the ACCP- or BF-property. The results obtained extend and recover the case of $A + XB[X]$ domains.

General references for any undefined terminology or notation are [24, 25].

4.1 ACCP condition

The ACCP property for $A + XB[X]$ domains was studied in [6, 12, 20]. In [20, Proposition 1.2], the authors showed that $A + XB[X]$ satisfies the ACCP if and only if $\bigcap_{n\geq 1} a_1 \cdots a_n B = (0)$ for each infinite sequence $(a_n)_{n\geq 1}$ of nonunits of A. This result was extended to $A + B[\Gamma^*]$ domains in [32, Theorem 3.4], where it was showed that $R = A + B[\Gamma^*]$ satisfies the ACCP if and only if Γ satisfies the ACCP and for each infinite sequence $(a_n)_{n\geq 1}$ of nonunits of A

$$\bigcap_{n\geq 1} a_1 \cdots a_n B = (0).$$

In the following, we give a characterization of the ACCP property for $A + B[\Gamma^*]$ in the spirit of [6, Proposition 1.1].

Let $A \subseteq B$ be an extension of integral domains. We say that B satisfies the A-ACCP property if every chain of cyclic A-submodules of B terminates. Note that the ACCP for B coincides with the B-ACCP. We have the following lemma whose proof is similar to [20, Remark 1.1].

Lemma 4.1.1. *Let $A \subseteq B$ be an extension of integral domains. Then the following statements are equivalent:*

(i) *B satisfies the A-ACCP condition.*

(ii) *For every infinite sequence $(a_n)_{n\geq 1}$ of nonunits of A,*

$$\bigcap_{n\geq 1} a_1 \cdots a_n B = (0).$$

In the following proposition, the equivalence (i)⇔(ii) extends [6, Proposition 1.1] and the equivalence (i)⇔(iii) is [32, Theorem 3.4].

Proposition 4.1.2. *Let $A \subseteq B$ be an extension of integral domains and Γ an additive monoid such that $\Gamma \cap -\Gamma = \{0\}$. The following statements are equivalent:*

(i) *$R = A + B[\Gamma^*]$ satisfies the ACCP.*

(ii) *Γ satisfies the ACCP and B satisfies the A-ACCP condition.*

(iii) *Γ satisfies the ACCP and for every infinite sequence $(a_n)_{n\geq 1}$ of nonunits of A,*

$$\bigcap_{n\geq 1} a_1 \cdots a_n B = (0).$$

Proof. (i)⇒(ii). Suppose that R satisfies the ACCP and let $b_1 A \subseteq b_2 A \subseteq \cdots$ with each $b_n \in B$. For $\alpha \in \Gamma^*$, we have $b_1 X^\alpha R \subseteq b_2 X^\alpha R \subseteq \cdots$ is a chain of principal ideal of R. Since R satisfies the ACCP, this chain terminates. Hence the chain $\{b_n A\}_{n\geq 1}$ terminates. Therefore, B satisfies the A-ACCP condition. By a similar argument, one can easily show that Γ satisfies the ACCP.

(ii)$\Rightarrow$(i). Assume that (ii) holds. Let $f_1R \subseteq f_2R \subseteq \cdots$ be an ascending chain of principal ideals of R. For each $n \geq 1$, let α_n be the degree of f_n. Then $(\alpha_1) \subseteq (\alpha_2) \subseteq \cdots$ is an ascending chain of principal ideals of Γ which satisfies the ACCP. Moreover, since $\Gamma \cap -\Gamma = \{0\}$, there exists $k \geq 1$ such that $\alpha_n = \alpha_k$ for every $n \geq k$. Then we may assume that all f_n have the same degree.
On the other hand, let b_n be the leading coefficient of f_n, so $b_1A \subseteq b_2A \subseteq \cdots$ is an ascending chain which terminates by hypothesis. Hence there exists $k \geq 1$ such that $\frac{b_n}{b_{n+1}} \in U(A)$ for all $n \geq k$. Consequently, $f_{n+1}R = f_nR$ for all $n \geq k$.

(ii)$\Leftrightarrow$(iii). This follows from Lemma 4.1.1. □

Corollary 4.1.3. *Let X be a set of indeterminates over B. Then $A + XB[X]$ satisfies the ACCP if and only if B satisfies the A-ACCP.*

Corollary 4.1.4. *Let B be an integral domain and Γ an additive monoid such that $\Gamma \cap -\Gamma = \{0\}$. Then $B[\Gamma]$ satisfies the ACCP if and only if Γ satisfies the ACCP and B satisfies the ACCP.*

Remark 4.1.5.
(i) If R satisfies the ACCP, then by Proposition 4.1.2 (ii) one can easily check that $U(B) \cap A = U(A)$.

(ii) Assume that $U(B) \cap A = U(A)$. If B satisfies the ACCP, then $A + B[\Gamma^*]$ satisfies the ACCP if and only if Γ satisfies the ACCP. But $A + B[\Gamma^*]$ may satisfy the ACCP and B not. For example, by Proposition 4.1.2, if A is a field and Γ satisfies the ACCP, then $A + B[\Gamma^*]$ satisfies the ACCP for every extension rings $A \subseteq B$. Note that in this case the condition $U(B) \cap A = U(A)$ is satisfied.

(iii) Corollary 4.1.4 is obtained in [26], where the authors left open the difficult question of when an arbitrary semigroup ring satisfies the ACCP.

4.2 Atomicity condition

In this section, we study the atomic property in $A + B[\Gamma^*]$ domains.

Let S denote a multiplicative monoid and $X \subseteq S$. Let $\mathrm{CD}_S(X) = \{s \in S, \forall x \in X, s \mid x\}$, the set of all common divisors of the elements of X. An element $c \in \mathrm{CD}_S(X)$ is called a maximal common divisor (MCD) of X in S if whenever $d \in$

$\mathrm{CD}_S(X)$ and $c \mid d$, then c and d are associate. We denote by $\mathrm{MCD}_S(X)$ the set of all MCDs of X in S (resp. the multiplicative monoid S^*, for a domain S). A monoid (resp., a domain) S is called an *MCD* monoid (resp., *MCD* domain) if $\mathrm{MCD}_S(X) \neq \emptyset$ for every finite set $X \subseteq S$ (resp., $X \subseteq S^*$). Note that the ACCP property implies the *MCD* property. *MCD* domains were studied in [40].

Let $A \subseteq B$ be an extension of integral domains and $X \subseteq B^*$. An element $a \in A$ is called an A-MCD of X if $a \in \mathrm{CD}_{B^*}(X)$ and whenever $a' \in \mathrm{CD}_{B^*}(X) \cap A$ and $a \mid a'$ in A, then a and a' are associate in A. We denote by A-MCD(X) the set of all A-MCDs of X. We say that B is an A-*MCD* domain if A-MCD$(X) \neq \emptyset$ for every finite set $X \subseteq B^*$.

For a monoid S, the rank of S, denoted by $rank(S)$, is the rank of the quotient group $\langle S \rangle$.

Theorem 4.2.1. *Let $A \subseteq B$ be an extension of integral domains and Γ an additive monoid such that $\Gamma \cap -\Gamma = \{0\}$.*
Assume that Γ satisfies the ACCP and $rank(\Gamma) \geq 2$. Then $R = A + B[\Gamma^*]$ is atomic if and only if A is atomic, $U(B) \cap A = U(A)$ and B is an A-MCD-domain.

Proof. For the "only if"condition. Clearly an element $a \in A^{\#}$ is an atom of A if and only if it is an atom of R. So A is atomic. Next, let α be an atom of Γ. Then $X^\alpha = a_1 \cdots a_r (bX^\alpha)$ with $a_1, \ldots, a_r \in A$ are atoms and bX^α is irreducible in R, since R is atomic. Now, let $a \in U(B) \cap A$, then $bX^\alpha = a((a^{-1}b)X^\alpha)$, so $a \in U(A)$. Hence $U(B) \cap A = U(A)$. We next show that B is an A-*MCD* domain. Let $b_1, \ldots, b_n \in B^*$. To show that A-MCD$(b_1, \ldots, b_n) \neq \emptyset$, we consider two cases.

Case 1: *Γ has finitely many atoms up to associates.*
Let $\gamma_1, \ldots, \gamma_k$, $k \geq 2$, be the (non-associate) atoms of Γ. Then $\Gamma = \mathbb{Z}_+\gamma_1 + \cdots + \mathbb{Z}_+\gamma_k$, a finitely generated monoid. Note that the quotient group $\langle \Gamma \rangle$ is a finitely generated free abelian group, so $\langle \Gamma \rangle = \mathbb{Z}^m$ for some integer $m \geq 2$. Since $rank(\Gamma) \geq 2$, there exist two non-associate atoms $\alpha_1, \alpha_2 \in \Gamma$ which are $\mathbb{Z}$-linearly independent. In addition, as elements of $\mathbb{Z}^m$, we may assume that the gcd of the components of α_i, $i = 1, 2$, is 1. By using the Hermite Normal form for matrices over the integers, $\{\alpha_1, \alpha_2\}$ can be extended into a $\mathbb{Z}$-basis $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ of $\langle \Gamma \rangle$. Now, let $Y_i = X^{\alpha_i}$, $i = 1, \ldots, m$, and denote by K the quotient field of B. Therefore,

$$K[\langle \Gamma \rangle] = K[\mathbb{Z}^m] = K[Y_1, \ldots, Y_m, Y_1^{-1}, \ldots, Y_m^{-1}].$$

Let $f = b_1 Y_1 + \cdots + b_{n-1} Y_1^{n-1} + b_n Y_2$. Then $f \in R$ is irreducible in $K[Y_1, \ldots, Y_n]$, and hence in $K[\langle \Gamma \rangle]$. Since R is atomic, we have

$$f = a_1 \cdots a_r (c_1 Y_1 + \cdots + c_{n-1} Y_1^{n-1} + c_n Y_2),$$

where $a_1, \ldots, a_r \in A$ are atoms in A, and $c_1, \ldots, c_n \in B$ with no nontrivial common factors belonging to A. Then $a = a_1 \cdots a_r \in A\text{-MCD}(b_1, \ldots, b_n)$.

Case 2: *Γ has infinitely many non-associate atoms.*
Let $\gamma_1, \ldots, \gamma_n$ be n non-associate atoms of Γ, and let $f = b_1 X^{\gamma_1} + \cdots + b_n X^{\gamma_n}$. Since the exponents of f are atoms, $f \in R$ has no non-constant factor in R. As above, we have

$$f = a(c_1 X^{\gamma_1} + \cdots + c_n X^{\gamma_n}),$$

with $a \in A$, and $c_1, \ldots, c_n \in B$ with no common factors in A. Thus, also A-MCD$(b_1, \ldots, b_n) \neq \emptyset$ in this case.
For the "if"condition, note that since $U(B) \cap A = U(A)$, $\alpha \in \Gamma$ is an atom in Γ if and only if X^α is an atom in R. Now, let $f = a_0 + b_1 X^{\alpha_1} + \cdots + b_n X^{\alpha_n} \in R^\#$ with $0 \prec \alpha_1 \prec \cdots \prec \alpha_n$. Let $a \in A\text{-MCD}(a_0, b_1, \ldots, b_n)$ and $\alpha \in \text{MCD}_\Gamma(\alpha_1, \ldots, \alpha_n)$. Then $f = aX^\beta f_1$, where $\beta = \alpha$ if $a_0 = 0$, and $\beta = 0$ otherwise. Note that since A and Γ are atomic, aX^β is a product of atoms in R. We next show that if $f_1 \in R^\#$, it is also a product of finitely many atoms. Note that by the MCD property, f_1 has no nontrivial factor of the form cX^γ with $c \in A^\#$ or $\gamma \in \Gamma \setminus \{0\}$. Assume that f_1 is reducible. A nontrivial factorization of f_1 gives rise to a strictly decreasing sequence $0 < \cdots \gamma_k \mid \cdots \gamma_2 \mid \gamma_1$ of divisors of $\delta = deg(f_1)$. Then

$$\gamma_1 + \Gamma \subseteq \gamma_2 + \Gamma \subseteq \cdots$$

is a chain of principal ideals in Γ that terminates. Since $\Gamma \cap -\Gamma = \{0\}$, there exists an integer k_0 such that $\gamma_k = \gamma_{k_0}$ for every $k \geq k_0$. It follows that there exists a factorization of f_1 with a maximal number of nonunit factors, hence a factorization into atoms, in R. Hence R is atomic. $\square$

Corollary 4.2.2. *Let X be a set of indeterminates over B with $\mid X \mid \geq 2$. Then $A + XB[X]$ is atomic if and only if A is atomic, $U(B) \cap A = U(A)$ and B is an A-MCD domain.*

Corollary 4.2.3. *Let B be an integral domain and Γ an additive monoid such that $\Gamma \cap -\Gamma = \{0\}$. Assume that Γ satisfies the $ACCP$ and $rank(\Gamma) \geq 2$. Then $B[\Gamma]$ is atomic if and only if B is atomic and B is an MCD-domain.*

Remark 4.2.4. (i) Note that in the proof of the "if" condition of Theorem 4.2.1,we do not need the hypothesis $rank(\Gamma) \geq 2$. For instance, if A is atomic, $U(B) \cap A = U(A)$ and B is an A-MCD domain, then $R = A + XB[X]$ is atomic (X an indeterminate and $\Gamma = \mathbb{Z}_+$). However, the converse of this last statement about $A + XB[X]$ domains is an open problem even for a polynomial ring $A[X]$ ($A = B$), see [40].

(ii) In Corollary 4.2.3, if we drop the condition $\Gamma \cap -\Gamma = \{0\}$, things get worse: Let $\Gamma = \mathbb{Q} \times \mathbb{Q}$ and K a field. Clearly, the additive group Γ satisfies trivially the ACCP and $rank(\Gamma) = 2$. But $K[\Gamma]$ is not atomic since it is a GCD domain [25, Theorem 14.2] but does not satisfy the ACCP [25, Theorem 14.17].

Example 4.2.5. For the ACCP hypothesis for Γ in Theorem 4.2.1, note that if $A + B[\Gamma^*]$ is atomic, then Γ is at least atomic. The following example shows that if Γ is an atomic monoid that does not satisfy the ACCP, Theorem 4.2.1 does not hold in general. Let p be a fixed prime number and $(p_n)_{n\geq 1}$ a strictly increasing sequence of prime numbers. Consider the Puiseux monoid, see [16, Example 4.2], $\Gamma_p = \langle \frac{1}{p^n p_n} \mid p_n \neq p \rangle$. By an elementary argument of divisibility, on can easily check that Γ_p is atomic. The monoid Γ_p does not satisfy the ACCP since the chain of principal ideals $\{\frac{1}{p^n} + \Gamma_p\}_n$ does not terminate. Atomic domains that do not satisfy the ACCP are hard to come by. The first such example is due to A. Grams [27]. Now, let $\Gamma = \Gamma_p \times \Gamma_p$ ($rank(\Gamma) = 2$) and K a field of characteristic p. Like Γ_p, Γ is an atomic monoid that does not satisfy the ACCP. By [16, Theorem 4.3], the semigroup ring $K[\Gamma]$ is not atomic.

4.3 BF property

In this section, we investigate bounded factorization domains. Recall that, a multiplicative monoid S or a domain D ($S = D^*$) has the BF-property if there is a length function $l : S \to \mathbb{Z}_+$ such that:

(i) $l(s) = 0$ if and only if $s \in U(S)$, and

(ii) $l(st) \geq l(s) + l(t)$ for all $s, t \in S$.

For an extension of integral domains $A \subseteq B$, we say that B is a bounded factorization domain with respect to A (A-BFD) if for each $b \in B^{\#}$, there is a positive integer $N_A(b)$ such that whenever $b = b_1 b_2 \cdots b_n$ with each $b_i \in B^{\#}$, then at most $N_A(b)$ of the b_i's are in A [6]. We say that $l_A : B^* \to \mathbb{Z}_+$ is an A-length function if :

(i) $l_A(b) = 0$ if and only if $b \in U(B)$ or b has no nontrivial factors in A, and

(ii) $l_A(bc) \geqslant l_A(b) + l_A(c)$ for all $b, c \in B^*$.

Note that if $A = B$, then l_A is a length function on A. Clearly, B is an A-BFD if and only if B has an A-length function. Note that for the domain B, A-BFD implies A-ACCP.

Theorem 4.3.1. *Let $A \subseteq B$ be an extension of integral domains and Γ an additive monoid such that $\Gamma \cap -\Gamma = \{0\}$. The following statements are equivalent:*

(i) *$R = A + B[\Gamma^*]$ is a BFD.*

(ii) *Γ is a BFM, $U(B) \cap A = U(A)$ and B is an A–BFD.*

Proof. (i)⇒(ii). Suppose that R is a BFD. Clearly, Γ is a BFM, and hence has at least one atom α. Also, $U(B) \cap A = U(A)$ since R satisfies the ACCP. Let $b \in B^{\#}$ with a factorization $b = a_1 a_2 \cdots a_m b_1 b_2 \cdots b_n$ in B where each $a_i \in B^{\#} \cap A$ and $b_j \in B^{\#} \setminus A$. Then, this gives rise to a factorization $bX^\alpha = (a_1 a_2 \cdots a_m)(b_1 \cdots b_n X^\alpha)$ of $m+1$ nonunit factors of R. Since R is a BFD, there exists a positive integer $N(b)$ such that $m \leq N(b)$. So B is an A-BFD.

(ii)⇒(i). Denote by l_Γ the length function of Γ and by l_A the A-length function of B. Let $f = b_0 + b_1 X^{\alpha_1} + \cdots + b_n X^{\alpha_n} \in R$ with $b_n \neq 0$ and $0 \prec \alpha_1 \prec \cdots \prec \alpha_n$. Then $l(f) = l_A(b_n) + l_\Gamma(\alpha_n)$ is a length function on R. So R is a BFD. □

Corollary 4.3.2. *Let X be a set of indeterminates over B. Then $A + XB[X]$ is a BFD if and only if $U(B) \cap A = U(A)$ and B is an A-BFD.*

Corollary 4.3.3. *Let B be an integral domain and Γ an additive monoid such that $\Gamma \cap -\Gamma = \{0\}$. Then $B[\Gamma]$ is a BFD if and only if Γ is a BFM and B is a BFD.*

Remark 4.3.4. (i) If B is a BFD, then B is an A-BFD. Thus, by Theorem 4.3.1, $R = A + B[\Gamma^*]$ is a BFD if and only if $U(B) \cap A = U(A)$ and Γ is a BFM. But if R is a BFD, B need not be a BFD. To see this, take A a field and Γ a BFM, then $R = A + B[\Gamma^*]$ is a BFD for any extension of domains $A \subseteq B$ (Theorem 4.3.1).

(ii) If the condition $\Gamma \cap -\Gamma = \{0\}$ is not satisfied, Corollary 4.3.3 does not hold. For instance, take $B = K$, a field, and $\Gamma = \mathbb{Q}$. Then B and Γ satisfy trivially the BF-property, but $K[\mathbb{Q}]$ is not even atomic [25, Theorems 14.2 and 14.17].

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ملخص الأطروحة

لتكن D حلقة تبادلية تامة و Γ وحدية تبادلية مبسطة و خالية من الالتواء. نرمز بـ $D[\Gamma]$ لحلقات الشبه زمرة من Γ على D. في السنوات الأخيرة، إهتم العديد من الباحثين بخاصيات التعميل في $D[\Gamma]$ ما ساعد على بناء فصول جديدة من الأمثلة الأصلية في الجبر التبادلي و نظرية التعميل.

الهدف الرئيسي من هذه الأطروحة هو دراسة بعض خاصيات التعميل في حلقات الشبه زمرة. و هكذا تم تخصيص الفصلين الثاني و الثالث لدراسة خاصية شريير (Schreier). ضمن الفصل الثاني سندرس مفهوم البدائية و كذلك خاصية شريير في سياق أكثر شمولية على الحلقات المتدرجة ثم نطبق ما توصلنا إليه على حلقات الشبه زمرة أما الفصل الثالث فنتناول خلاله الحلقات من صنف $A + B[\Gamma^*]$، حيث $A \subseteq B$ امتداد لحلقتين تامتين و Γ وحدية مبسطة خالية من الإلتواء مع الشرط $\Gamma \cap -\Gamma = \{0\}$. في هذا الإطار، توصلنا إلى تحديد الشروط اللازمة و الكافية لكي تستوفي الحلقة $A + B[\Gamma^*]$ خاصية شريير، كما يلي: في حالة ما إذا كانت B حلقة كسور لـ A أو أن Γ تحتوي على عنصر غير قابل للإختزال، فإن توصيفنا يعمم نظيره على حلقات الحدوديات ($\Gamma = \mathbb{N}$). من جهة أخرى، إن لم تكن B حلقة كسور للحلقة A فقد أثبتنا أن الحلقة $A + B[\Gamma^*]$ تستوفي خاصية شريير فقط إذا كانت جميع عناصر Γ قابلة للإختزال.

في الفصل الرابع سنوسع دراستنا كي تشمل باقة من خاصيات التعميل و أثرها على الحلقات من صنف $A + B[\Gamma^*]$، و سنزكي ما توصلنا إليه بمجموعة من الأمثلة الجديدة و المتنوعة في بنيتها.

لائحة بعض المصطلحات

Ring	حلقة
Integral domain	حلقة تامة
Atomic domain	حلقة ذرية
Graded domain	حلقة متدرجة
Semigroup	شبه زمرة
Monoid	وحدية
Semigroup ring	حلقة الشبه زمرة
Primal element	عنصر بدائي
Completely primal element	عنصر بدائي كليا
Integrally closed domain	حلقة مغلقة بالكامل
Quotient ring	حلقة كسور
Overring	حلقة فوقية