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The analysis of the structural properties and the spanning trees entropy of complex networks

JURY

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I would like to dedicate my thesis
to my beloved parents,
my sisters and my brothers.

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Abstract

Many real-world networks are modelled as complex networks due to their large structure and their dynamical behaviour. Graph theory provides efficient tools to understand and to analyze their mechanism. In fact, many structural properties are used such as the average path length, the diameter, the clustering coefficient, the degree distribution, the average degree, etc. Based on these features, three categories of complex networks are defined, namely scale-free networks, small-world networks and random networks. One of the important invariants to characterize their structures is the number of spanning trees of a network, which is defined as the total number of connected and acyclic subgraphs of a network having all its vertices and some or all its edges. In this work, the main objective is the calculation of the number of spanning trees of a network also known as the complexity of a network, which provides the prediction of its reliability and its robustness. However, the enumeration of spanning trees remains a challenge, particularly for complex networks. Recently, there has been much interest in finding efficient methods to obtain exact expressions of the number of spanning trees for complex networks. The primary interest of this study is to create new models for each category of complex networks based on real-networks that grow by the gradual addition of vertices and edges. Then, find their relevant structural properties to understand their mechanism. Furthermore, evaluate their complexity using combinatorial and geometric approaches. In the end, as an application, we calculate their entropy of spanning trees to quantify their robustness and compare them with other networks having the same average degree.

Keywords: Complex Networks, Graph theory, Average Path Length, Diameter, Clustering Coefficient, Degree Distribution, Average Degree, Scale-Free Network, Small-World Network, Random Network, Spanning Tree, Complexity, Reliability, Robustness, Entropy.

Résumé

L'analyse des réseaux complexes a été largement stimulée par les ressources de données massives et leur étude a été initiée pour une volonté de comprendre le comportement de divers systèmes réels. Ces réseaux complexes sont le résultat de toutes les interactions entre les composants physiques et logiques des réseaux. Pour comprendre leur mécanisme et leur comportement, de nombreuses propriétés structurelles sont utilisées, telles que la distance moyenne, le diamètre, le coefficient de clustering, la distribution des degrés et le degré moyen, etc. Ces caractéristiques définissent trois modèles de réseaux complexes notamment les réseaux sans échelle, les réseaux petit monde et les réseaux aléatoires. Ces trois modèles affichent un comportement riche, observé dans une grande variété de systèmes réels, y compris Internet, le World Wide Web, les réseaux électriques, les réseaux de neurones cérébraux et les réseaux sociaux. Pour caractériser et analyser leur structure, la théorie des graphes dispose d'un outil puissant, nommé le nombre d'arbres couvrants d'un réseau, également appelé la complexité d'un réseau. Il est défini comme le nombre total des sous-graphes connexes et sans cycles contenant tous les sommets du réseau avec le plus petit nombre possible d'arêtes. L'objectif principal de ce travail est le calcul du nombre d'arbres couvrants d'un réseau, qui permet de prédire sa fiabilité et sa robustesse. En effet, le calcul de ce nombre reste un défi, en particulier pour les réseaux complexes. Nous nous sommes intéressés à la recherche des méthodes efficaces pour obtenir la formule exacte du nombre d'arbres couvrants pour les réseaux complexes. Le premier but de cette étude est de créer de nouveaux modèles évoluant de manière dynamique dans le temps pour chaque catégorie de réseaux complexes. Ensuite, calculer leurs propriétés structurelles pertinentes pour comprendre leurs mécanismes et leurs comportements. De plus, évaluer leur complexité en utilisant et améliorant des méthodes combinatoires et géométriques. Finalement, comme application, nous calculons leur entropie afin de quantifier leur robustesse et la comparer avec d'autres réseaux ayant le même degré moyen.

Mots clés: Réseaux Complexes, Théorie des Graphes, Distance Moyenne, Diamètre, Coefficient de Clustering, Distribution des Degrés, Degré Moyen, Réseau Sans échelle, Réseau Petit Monde, Réseau Aléatoire, Arbre Couvrant, Complexité, Fiabilité, Robustesse, Entropie.

L'analyse des réseaux complexes a été largement stimulée par les ressources de données massives et leur étude a été initiée pour une volonté de comprendre le comportement de divers systèmes réels. Pour comprendre leur mécanisme, de nombreuses propriétés structurelles sont utilisées, telles que la distance moyenne, le coefficient de clustering,

la distribution des degrés, etc. Ces caractéristiques définissent trois modèles de réseaux complexes notamment les réseaux sans échelle, les réseaux petit monde et les réseaux aléatoires. Ces trois modèles affichent un comportement riche, observé dans une grande variété de systèmes réels, y compris Internet, WWW et les réseaux sociaux. Pour caractériser et analyser leur structure, la théorie des graphes dispose d'un outil puissant, nommé le nombre d'arbres couvrants d'un réseau, également appelé la complexité d'un réseau. Il est défini comme le nombre total des arbres contenant tous les sommets du réseau avec le plus petit nombre possible d'arêtes. L'objectif principal de ce travail est le calcul du nombre d'arbres couvrants d'un réseau, qui permet de prédire sa fiabilité et sa robustesse. En effet, le calcul de ce nombre reste un défi, en particulier pour les réseaux complexes. Nous nous sommes intéressés à la recherche des méthodes efficaces pour obtenir la formule exacte du nombre d'arbres couvrants pour les réseaux complexes. Le premier but de cette étude est de créer de nouveaux modèles évoluant de manière dynamique dans le temps pour chaque catégorie de réseaux complexes. Ensuite, calculer leurs propriétés structurelles pertinentes pour comprendre leurs mécanismes et leurs comportements. De plus, évaluer leur complexité en utilisant et améliorant des méthodes combinatoires et géométriques. Finalement, comme application, nous calculons leur entropie afin de quantifier leur robustesse et la comparer avec d'autres réseaux ayant le même degré moyen.

Résumé Détaillé

Contexte Général

Une grande variété de systèmes du monde réel peut être décrits et caractérisés par des réseaux complexes. Ils constituent un outil fondamental pour la modélisation de systèmes complexes dans divers domaines. Parmi les exemples de réseaux complexes fréquemment cités incluent le Web, Internet, les réseaux sociaux, les réseaux de transport, les réseaux de communication, les réseaux génétiques, les réseaux de neurones, etc. Ces réseaux complexes sont tous des systèmes d'interactions entre ses composants. La théorie de graphes fournit des outils mathématiques permettant de modéliser et analyser la structure de ces réseaux, en combinaison avec d'autres concepts et méthodes informatique et physique. L'origine de l'étude des réseaux complexes remonte à la célèbre solution adoptée par Euler en 1735 pour résoudre le problème du pont de Königsburg (Euler, 1741), qui a été cité comme la première découverte de la théorie des réseaux. A la fin des années 1950, deux mathématiciens, Erdős et Rényi, ont introduit l'un des modèles de réseaux aléatoires (Erdős et al., 1959), où chaque paire de noeuds est reliée avec la probabilité p et ils ont montré que les propriétés de tels réseaux peuvent être calculées de manière analytique. Ces dernières années, les recherches ont vécu la naissance d'un nouveau mouvement d'intérêt dans l'étude de réseaux complexes, qui ont une structure irrégulière, large et évoluant de manière dynamique dans le temps avec l'ajout des milliers ou des millions de noeuds et d'arêtes. En 1998, Watts et Strogatz ont observé qu'un petit diamètre ou une courte distance moyenne et un clustering élevé conduisent à un modèle de réseau petit monde (Watts and Strogatz, 1998), ce qui est commun dans une variété de réseaux réels. Un an plus tard, Barabási et Albert ont proposé un modèle de réseau sans échelle, avec une distribution de degrés de la loi de puissance (Barabási and Albert, 1999). Après cela, plusieurs études ont incité les chercheurs à construire des modèles de réseaux pour reproduire ou expliquer les caractéristiques communes des systèmes de la vie réelle. Parmi les objectifs de cette thèse, nous proposons de nouveaux modèles en montrant la présence de certaines propriétés génériques de divers réseaux complexes telles que le coefficient de clustering, la distribution de degrés, le diamètre, etc.

En outre, en tant qu'invariant crucial de la structure d'un réseau, le nombre d'arbres couvrants d'un réseau connexe, également considéré comme la complexité d'un réseau, est défini comme le nombre total d'arbres contenant tous les sommets du réseau et non

nécessairement toutes les arêtes (Knuth, 1997; Wu and Chao, 2004). Cela nous donne une idée sur le nombre de topologies possibles connexes et sans boucles qu'un réseau peut avoir. Ce nombre est lié aux plusieurs problèmes intéressants de réseaux, telles que sa fiabilité (Bistouni and Jahanshahi, 2017), sa robustesse (Burton and Pemantle, 1993; Lyons, 2005), l'étude des marches aléatoires (Aldous, 1990), la synchronisation (Nishikawa and Motter, 2006), etc. Vu que ses applications sont diverses dans plusieurs domaines, l'énumération des arbres couvrants a suscité une attention considérable de la part de la communauté scientifique. Elle est considérée comme un problème d'intérêt fondamental en mathématiques (Ozeki and Yamashita, 2011), en physique (Lin et al., 2011) et en informatique (Nikolopoulos et al., 2014). Pour la première fois, Kirchhoff l'a examiné dans son analyse des circuits électriques (Kirchhoff, 1847). Son théorème appelé "Matrix Tree Theorem", fournit un algorithme permettant de déterminer le nombre d'arbres couvrants de n'importe quel réseau connexe, en calculant le déterminant d'une spécifique matrice issue de la matrice laplacienne. Cependant, ce théorème n'est pas efficace pour les réseaux réels ayant un grand nombre de noeuds et d'arêtes vu qu'on sera amené à calculer le determinant d'une matrice large, ce qui est coûteux en terme de calcul. Par conséquent, la plupart des travaux récents ont tenté d'étudier ce problème, de trouver des méthodes alternatives afin d'éviter les calculs difficiles du déterminant de la matrice de Kirchhoff et de déterminer la formule exacte du nombre d'arbres couvrants pour les réseaux complexes, telles que les méthodes de suppression et de contraction Feussner (1902) et la dualité (Lang, 2002), etc.

Objectifs

Les objectifs de cette thèse sont d'abord l'étude des réseaux complexes en proposant de nouveaux modèles évoluant de manière dynamique dans le temps, basés sur des réseaux réels pour chaque catégorie de réseaux petit monde, sans échelle et aléatoires. Ensuite, l'analyse de ces modèles en déterminant leurs propriétés structurelles pertinentes pour comprendre leur mécanisme et leur comportement. Puis, le développement et l'amélioration des méthodes combinatoires et des approches géométriques facilitant le calcul du nombre d'arbres couvrants pour les réseaux proposés, prouvant que nos méthodes sont efficaces par rapport aux méthodes classiques. Enfin, proposer une application du nombre d'arbres couvrants pour quantifier la robustesse de nos réseaux étudiés et caractériser leurs structures en utilisant la mesure de la complexité asymptotique ou l'entropie d'arbres couvrants.

Contributions et Plan

La nouveauté de notre thèse est d'étudier et d'analyser de nouveaux modèles de réseaux complexes, calculer leurs propriétés structurelles et d'évaluer leur complexité en utilisant des méthodes efficaces afin de quantifier leur robustesse. Cette thèse comprend cinq chapitres: Le premier chapitre présente l'état de l'art de notre sujet. Le chapitre 2 présente les différentes méthodes de calcul du nombre d'arbres couvrants d'un réseau, y compris nos méthodes proposées et quelques exemples de leurs applications. Alors que

les chapitres 3, 4 et 5 présentent nos principales contributions pour modéliser et analyser chaque catégorie de réseaux complexes et d'évaluer leur complexité. Le plan de cette thèse est organisé comme suit:

- ***Chapitre 1: Analyse structurelle des réseaux complexes***. Dans ce chapitre, nous présentons l'état de l'art des réseaux complexes et les concepts de base de la théorie des graphes, qui sont utilisés dans la suite de ce document. Ensuite, nous discutons certaines propriétés structurelles importantes pour analyser ces réseaux complexes. Au final, nous citons les différents modèles proposés dans la littérature.

- **Chapitre 2:** ***Les méthodes de calcul du nombre d'arbres couvrants d'un réseau***. Dans ce chapitre, nous citons les différentes méthodes et techniques, connues dans la littérature, pour calculer le nombre d'arbres couvrants d'un réseau, classées comme méthodes algébriques, combinatoires et géométriques, telles que la matrice de Kirchhoff, les méthodes de suppression et de contraction, la dualité, etc. Nous examinons également leurs principaux avantages et inconvénients. Nous proposons quelques exemples d'applications, telles que l'évaluation de la complexité des réseaux Book. La limitation de ces méthodes nous pousse à utiliser et à développer des approches alternatives pour faciliter l'énumération des arbres couvrants pour les réseaux larges et complexes, tels que la méthode de contraction pour une chaîne fermée de réseaux planaires, la méthode électrique, etc. Nous traitons aussi deux applications réelles, à savoir la robustesse et la fiabilité d'un réseau. Certains résultats de ce chapitre ont été publiés dans une revue internationale (Mokhlissi et al., 2015a) et deux conférences internationales (Mokhlissi et al., 2015b, 2018a).

- **Chapitre 3:** ***L'analyse des modèles des réseaux petit monde***. Dans ce chapitre, nous proposons trois modèles de réseaux petit monde: le réseau Small-World Exponential, le réseau Koch et le réseau Farey. Nous étudions leur cas général. Nous analysons leur construction itérative et leurs propriétés structurelles telles que la distribution des degrés, le coefficient de clustering, le diamètre, la distance moyenne, etc, en montrant comment la généralisation de ces modèles affecte les propriétés du petit monde. Ensuite, nous calculons le nombre d'arbres couvrants en utilisant la méthode de décomposition généralisée suivant un noeud d'articulation pour le réseau Small-World Exponential et le réseau Koch et les transformations de la méthode électrique pour le réseau Farey. En tant qu'application, nous évaluons l'entropie des arbres couvrants pour quantifier leur robustesse et la comparons avec d'autres réseaux ayant le même degré moyen afin d'estimer le modèle le plus robuste. Le résultat principal de ce chapitre est que les généralisations des deux premiers modèles proposés ont la même robustesse malgré que leurs structures, propriétés et complexités soient différentes. Les résultats de ce chapitre ont été publiés dans une revue internationale (Mokhlissi et al., 2018b), une conférence internationale (Mokhlissi et al., 2016a) et soumis à une revue internationale (Mokhlissi et al., 2019a).

- **Chapitre 4:** ***L'analyse des modèles des réseaux sans échelle***. Dans ce chapitre, nous examinons trois modèles de réseaux sans échelle, à savoir le réseau

Flower, le réseau Mosaic et le réseau Fractal Scale-Free Lattice. Pour chaque modèle, nous proposons sa généralisation en modifiant certaines dimensions. Nous analysons sa construction et ses propriétés structurelles, en prouvant que les généralisations proposées n'affectent pas la propriété sans échelle. De plus, nous calculons le nombre d'arbres couvrants en appliquant deux approches géométriques: les approches de réduction et de bipartition pour le réseau Flower et le réseau Mosaic et nous utilisons les transformations de la méthode électrique pour le réseau Fractal Scale-Free Lattice. Enfin, afin d'évaluer la robustesse des modèles proposés, nous calculons et comparons leur entropie avec d'autres réseaux ayant le même degré moyen. Le résultat principal de ce chapitre est que les combinaisons des approches de réduction et de bipartition conduisent à la même complexité avec deux entropies différentes malgré la différence de structure et de propriétés des deux premiers modèles étudiés. Nous avons publié les résultats de ce chapitre dans une revue internationale (Mokhlissi et al., 2019b) et trois conférences internationales (Mokhlissi et al., 2016b, 2017b,a).

- **Chapitre 5:** ***L'analyse des modèles des réseaux réels et aléatoires***. Dans ce chapitre, nous étudions analytiquement certains réseaux du monde réel et tous les modèles proposés dans cette thèse et les comparons à des réseaux aléatoires ayant le même nombre de noeuds et d'arêtes. Nous discutons leurs propriétés structurelles telles que la distribution des degrés, la distance moyenne, le diamétre et le coefficient de clustering. Ensuite, nous calculons leur nombre d'arbres couvrants en utilisant les transformations de la méthode électrique, prouvant que cette technique est efficace et plus générale par rapport aux techniques classiques. En outre, nous évaluons et comparons leur entropie d'arbres couvrants afin de prédire quel réseau est le plus robuste. Le résultat principal de ce chapitre est que le modèle de réseau aléatoire peut être le mauvais modèle pour la plupart des réseaux réels, mais il est considéré comme le modèle le plus robuste à cause de son entropie élevée et de son grand nombre d'arbres couvrants. Les résultats de ce chapitre ont été soumis à une conférence internationale (Mokhlissi et al., 2020).

Pour conclure, cette thèse se termine par une conclusion générale qui résume nos contributions et décrit certaines perspectives et pistes de recherche.

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- **International journals**

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General Introduction

General context

Complex systems are ubiquitous in nature and society and their study has recently become a major topic of the multidisciplinary field of research that is spreading to many disciplines such as physics, mathematics, computer science, sociology, biology and economics. Many real systems in these different scientific fields can be modelled as complex networks, where their elements are nodes and interactions between elements are edges. The structure of these networks is irregular, complex and dynamically evolving in time with thousands or millions of nodes and edges. Examples of complex networks frequently cited include the World Wide Web, the Internet, social networks, transportation systems, communication networks, genetic networks, neural networks, etc. Regardless of their nature, complex networks share some common structural properties that define three categories of complex networks namely scale-free networks, small-world networks and random networks. Graph theory provides mathematical tools for modelling and analyzing them more adequately, in combination with other computer and physical concepts. The origin of complex network study goes back to the famous solution adopted by Euler in 1735 to solve the problem of the Königsburg bridge (Euler, 1741), which is often cited as the first true proof in the theory of networks. In the late 1950s, two mathematicians, Erdős and Rényi introduced one of the models of random networks (ER network) (Erdős et al., 1959), where each node pair is connected with the probability p and they showed that many of the properties of such networks can be calculated analytically. In recent years, researches have witnessed the birth of a new movement of interest in the study of complex networks. In 1998, Watts and Strogatz observed that a small diameter or a short average path length and a

high clustering lead to a small-world network model (Watts and Strogatz, 1998), which is common in a variety of real networks. One year later, Barabási and Albert proposed a scale-free network model, which has a degree distribution of power-law form (Barabási and Albert, 1999). After that, several studies have inspired researchers to construct network models to reproduce or explain the striking common features of real-life systems.

Besides, as a crucial invariant of the structure of a network, the number of spanning trees of a connected network, also considered as the complexity of a network, is defined as the total number of trees that connecting all the vertices of the main network and not necessary all its edges (Knuth, 1997; Wu and Chao, 2004). This gives us an idea about how many possible connected topologies that a network can have without loops. This number of spanning trees is related to several interesting network issues, such as its reliability (Bistouni and Jahanshahi, 2017), its robustness (Burton and Pemantle, 1993; Lyons, 2005), the study of random walks (Aldous, 1990), the synchronization (Nishikawa and Motter, 2006) and so on. Because of its wide range of applications in various fields, the enumeration of spanning trees has received considerable attention from the scientific community. It is a problem of fundamental interest in mathematics (Ozeki and Yamashita, 2011), physics (Lin et al., 2011) and computer science (Nikolopoulos et al., 2014). It was first considered by Kirchhoff in his analysis of electric circuits (Kirchhoff, 1847). His theorem called the "Matrix Tree Theorem", which provides a general algorithm for determining the number of spanning trees of any connected network, in terms of a determinant. However, this theorem is not efficient for real-world networks having a large value of nodes and links since we will have to calculate the determinant of a large matrix, which is costly in terms of calculation. Therefore, most of the recent works have tried to study this NP-hard problem and to find some alternative methods in order to avoid the arduous calculations of the largest determinant as needed by the algebraic method of Kirchhoff and determine the exact formula of the number of spanning trees for complex networks, such as the deletion, contraction methods Feussner (1902) and the duality (Lang, 2002), etc.

Main Objectives

The ultimate goals of this thesis are to investigate complex networks by proposing new models dynamically evolving in time based on real-networks for each category of small-world, scale-free and random networks. Then, analyze them by determining their relevant structural properties to understand their mechanism and behaviour. After that, suggest and develop some combinatorial and geometric approaches facilitating the calculation of the number of spanning trees for our proposed networks, proving that our methods are efficient compared to the classical ones. Finally, propose an application of the number of spanning trees which quantifies the robustness of our studied networks and characterizes their structures using the measure of the entropy.

Thesis Contributions and Outline

The novelty of our thesis is to study and analyze new models of complex networks and evaluate their complexity using efficient methods in order to quantify their robustness and estimate which model is more effective. This thesis consists of five chapters: The first chapter introduces the state of art of our subject. While Chapter 2 gives different methods to calculate the number of spanning trees of a network including our proposed methods and some examples of their applications and Chapter 3, 4 and 5 present our main contributions to model and analyze each category of complex networks and evaluate their complexity. The outline of this thesis is organized as follows:

- **Chapter 1:** ***The Structural analysis of complex networks***. In this chapter, we present the state of art of complex networks and the necessary background knowledge on graph theory, which is needed for the rest of this document. Then, we discuss some important structural properties to analyze these complex networks. In the end, we review the different models proposed by scientists.

- **Chapter 2:** ***The methods of calculating the number of spanning trees of a network***. In this chapter, we quote different methods and techniques, known in the literature, to calculate the number of spanning trees of a network, classed

as algebraic, combinatorial and geometric methods, such as the Kirchhoff matrix, the deletion and the contraction methods, the duality and so on. We propose also some examples of their applications such as the evaluation of the complexity of Book networks. We examine also their main advantages and issues. The restriction of these methods pushes us to use and develop alternative approaches to facilitate the enumeration of spanning trees for large and complex networks, such as the contraction method for a closed chain of planar networks, the electrically equivalent technique, etc. As applications of the number of spanning trees, we treat two real propositions, which are the robustness and the reliability of a network. Some results of this chapter were published in an international journal (Mokhlissi et al., 2015a) and two international conferences (Mokhlissi et al., 2015b, 2018a).

- **Chapter 3:** ***The analysis of models of Small-World Networks***. In this chapter, we propose three models of small-world networks: Small-World Exponential network, Koch Network and Farey network. We investigate the general case of each model. We analyze their iterative construction and their structural properties such as the degree distribution, the clustering coefficient, the diameter, the average path length, etc, showing how the generalization of these models affects the small-world properties. Then, we calculate the number of spanning trees using the generalized decomposition method following one node for the Small-World Exponential network and the Koch Network and the electrically equivalent transformations for the Farey network. As an application, we evaluate their entropy of spanning trees to quantify their robustness and compare them with other networks having the same average degree to estimate the robust model. The main result of this chapter is that the generalizations of the two first proposed models have the same robustness although their structures, properties and complexities are different. The results of this chapter were published in an international journal (Mokhlissi et al., 2018b), an international conference (Mokhlissi et al., 2016a) and submitted to an international journal (Mokhlissi et al., 2019a).

- **Chapter 4:** ***The analysis of models of Scale-Free networks***. In this chapter, we examine three models of scale-free networks, namely Flower network, Mosaic network and Fractal Scale-Free Lattice. For each model, we put forward its generalization by changing some dimensions. We analyze their construction and their structural properties, proving that the proposed generalizations do not affect the scale-free property. Furthermore, we calculate the number of spanning trees by applying two geometric approaches: The reduction and the bipartition approaches for the Flower network and the Mosaic network. While we use the electrically equivalent transformations for the Fractal Scale-Free Lattice. Finally, to evaluate the robustness of the proposed models, we compute and compare their entropy with other networks having the same average degree. The main result of this chapter is that the combinations of the reduction and the bipartition approaches lead to the same complexity with two different entropies in spite of the difference in the structure and the properties of the two first studied models. We have published the results of this chapter in an international journal (Mokhlissi et al., 2019b) and three international conferences (Mokhlissi et al., 2016b, 2017b,a).

- **Chapter 5:** ***The analysis of Real and Random networks models***. In this chapter, we study analytically some real-world networks and all proposed models in this thesis and compare them with random networks having the same number of nodes and links. We discuss their structural properties such as the degree distribution, the small-world property and the clustering coefficient. Then, we calculate their number of spanning trees by using the electrically equivalent transformations, proving that this technique is efficient and more general compared to the classical ones. Besides, we evaluate and compare their entropy of spanning trees to predict which network is more robust. The main result of this chapter is that the random network model may be the wrong model for most real networks, but it is considered as the most robust model due to its high entropy and its large number of spanning trees. The results of this chapter were submitted to an international conference (Mokhlissi et al., 2020).

To conclude, this dissertation is ended with a general conclusion that summarizes our contributions and describes some perspectives. The outline of the thesis is schematically depicted in Figure 1.

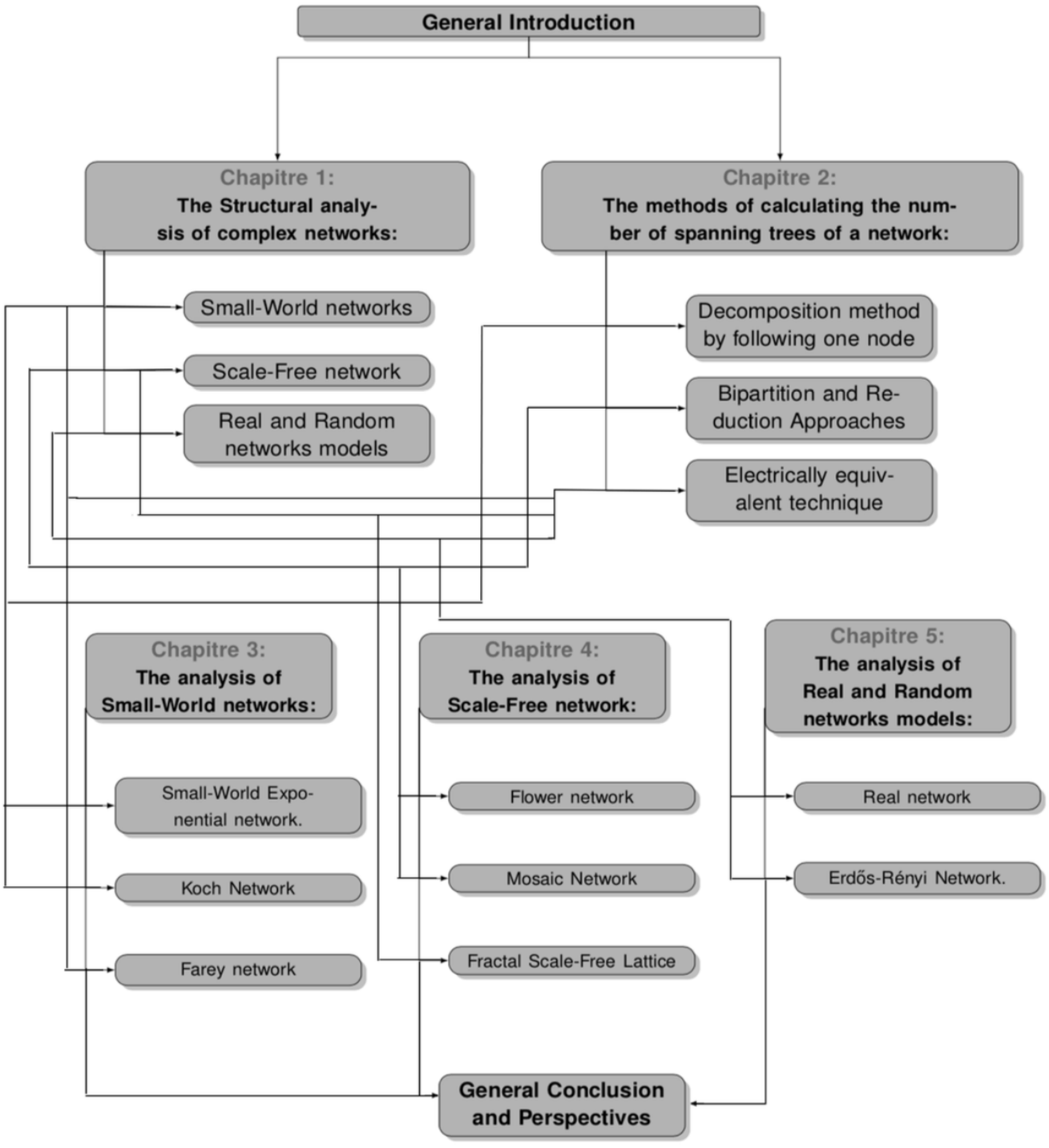


Figure 1: Schematic of the outline of this thesis.

Chapter

1

Structural analysis of complex networks

Contents

Complex networks appear in a wide range of disciplines in social and ecological to biological and technological systems. The research on these networks began with the effort of defining new features to characterize the topology and the structure of real-world networks. Mathematical graph theory is a large field that helps to understand and predict the behaviour of these complex networks. In this chapter, we present the state of the art of complex networks analysis. We establish the standard graph theory notation and complex networks terminology. We study various structural properties that are common to many networks. We discuss the main models and analytical tools, including random networks, small-world and scale-free networks and we review the important characteristics of each model. To read more about complex networks, you can usefully refer to the books (Newman, 2003, 2010).

1.1 Introduction

Networks are all around us. They are defined as a set of items, called nodes (vertices), with connections between them, called links (edges). They are used to mimic many systems in nature. Recently, real-world systems have attracted increasing attention because of their growth with a million or a billion nodes and edges from which comes the utility of their study. Many of the questions that might previously have been asked in studies of these systems as: How can we model them? why are they complex? what are their properties?... To answer to all these questions, researchers have discovered that real-world systems have common special features and there is a strong need to develop mathematical foundations, models and measures to understand and predict their behaviour and how they differ from one domain to another. Undoubtedly, they can be described by models of **complex networks**, which their structure is irregular, complex and dynamically evolving in time, where common examples include: An organization is a network of people, the Internet is a network of routers or domains, the World Wide Web (WWW) is a network of websites, the brain is a network of neurons, the global economy is a network of national economies, which are themselves networks of markets; and markets are themselves networks of interacting producers and consumers, etc (Dehmer (2010); Estrada (2015)). In this context, our work aims to analyze and model these complex networks.

1.1.1 Complex Networks in the Real World

Complex networks have been acknowledged as an invaluable tool for describing real-world systems in nature and society, much-quoted examples including the cell which is a network of chemicals linked by chemical reactions, or the Internet which is a network of routers and computers connected by physical links, etc. These networks are present in many fields as biology, sociology, psychology, computer science... They can be grouped into four categories: social networks, information networks, technological networks and biological networks. We list the most important examples in each category (See Figure 1.1) and some examples of nodes and links in particular networks (See Table 1.1) (Strogatz (2001); Albert and Barabási (2002); Newman (2003)).

- **Social networks:** A social network is, usually, a network of people, although it may sometimes be a network of groups of people, such as companies. The people or groups form the vertices of the network and the edges represent some form of social interaction between them, such as friendship between individuals, business relationships between companies or intermarriages between families... We refer to the words "social network" as online social networking services such as Facebook and MySpace and we refer to vertices as actors and the edges as ties. We will sometimes use these words when discussing social networks. Recently, studies are just beginning to arise in the structure and properties of these larger networks.

- **Information networks:** Also sometimes called "knowledge networks". They are networks that comprise people, computers, or methods organized to collect, process, transmit, and propagate data. The classic example of an information network is the network of citations between academic papers. These citations form a network in which the vertices are articles and a directed edge from article A to article B indicates that A cites B. The structure of the citation network reflects the structure of the information stored at its vertices. Another very important example of an information network is the World Wide Web, which is a network of Web pages containing information, linked together by hyperlinks from one page to another. The Web should not be confused with the Internet, which is a physical network of computers linked together by optical fiber and other data connections.

- **Technological networks:** A technological network is the physical infrastructure networks that have grown up over the last century and form the backbone of modern technological societies. Perhaps the most known such network is the Internet, the global network of data connections, electrical, optical and wireless that connect computers and other information systems together. Other important examples of technological networks, including the electric power grid which is a network of high-voltage three-phase transmission lines that spans a country or a portion of a country, transportation networks such as airline routes, roads and rail networks and a telephone network which is a network of landlines and wireless links that transmits

telephone calls.

- **Biological networks:** They are networks that occur in a number of situations in biology as a convenient representation of patterns of interactions between appropriate biological elements. Some are concrete physical networks like neural networks which are networks of connections between neurons in the brain. Another class of biological networks is that of biochemical networks, i.e., networks that represent the molecular level patterns of interaction and mechanisms of control in the biological cell such as metabolic networks, protein-protein interaction networks, and genetic regulatory networks. The food web networks are another class of biological networks of considerable importance, in which the vertices represent species in an ecosystem and a directed edge from species A to species B indicates that A feeds on B.

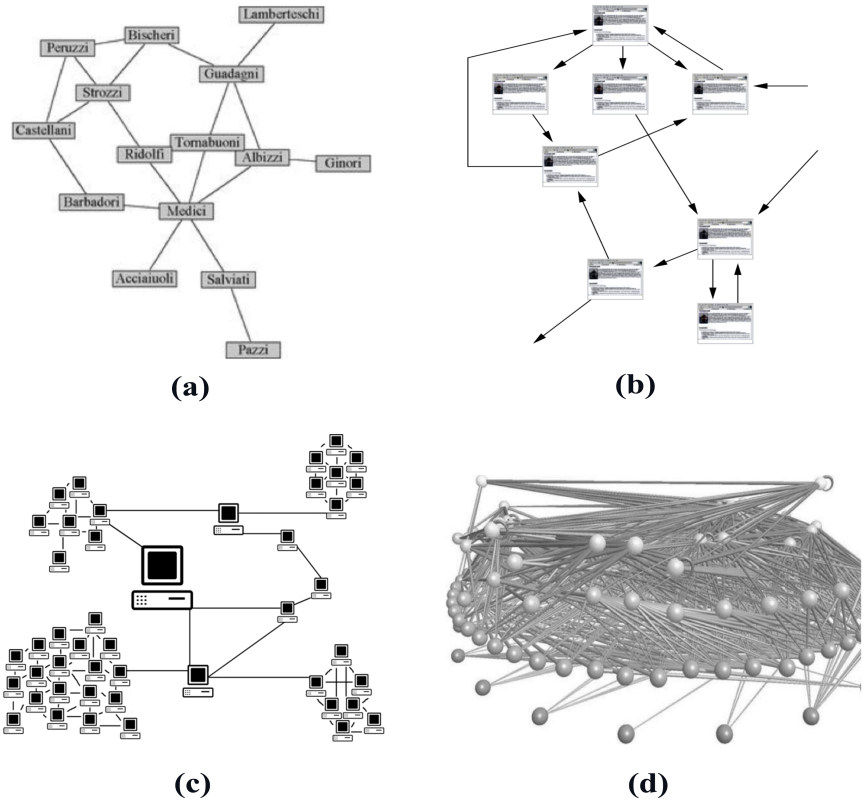


Figure 1.1: An example of a social network of a community website (a). The network structure of the WWW (b), its nodes are webpages connected by directed hyperlinks. The network structure of the Internet (c). Food web (d).

Network	Node (Vertex)	Link (Edge)
Citation network	Article	Citation
World Wide Web	Web page	Hyperlink
Internet	Computer or router	Cable or wireless data connection
Friendship network	Individual	Friendship
Food web	Species	Predation
Power grid	Station	Transmission line
Neural network	Neuron	Synapse
Metabolic network	Metabolite	Metabolic reaction

Table 1.1: Some examples of nodes and links in particular networks.

1.1.2 Modeling Complex Networks

The study of most complex networks has been initiated to understand various real systems. It has begun with the effort of defining new concepts and measures to describe the topology of real networks. In fact, these networks present specific topological features which characterize their structure and behaviour (Boccaletti et al. (2006)). The major problem is to model and analyze such networks. Their study by using Graph Theory is oftentimes a difficult procedure. Thus, there is a strong need to combine graph-theoretic methods with mathematical techniques from other scientific disciplines, such as Computer Science, Statistics and information theory, for analyzing complex networks more adequately. The primary interest of this study is to create new models based on real-networks that grow by the gradual addition of vertices and edges. Then, find and highlight their structural properties to predict their mechanisms and their behaviours and to classify them according to existing models in the literature.

1.2 Basic Concepts and Definitions on Graph Theory

The area of complex networks is a relatively recent field of study, but much of the associated terminology comes from Graph theory, which has a much longer history. Graph theory is considered as an adequate tool to represent a network as a graph, which can be used to model many types of relations and processes in physical, biological, social and information systems... In computer science, graphs are used to represent networks of communication, data organization, computational devices, etc. In this section, basic

definitions and concepts of graph theory are briefly presented. The reader may usefully refer to these two books (Wilson (1985); West (2001)).

1.2.1 Graphs

1.2.1.1 Historical Problem of Graph Theory

Historically, Graph Theory began with the Swiss mathematician Leonhard Euler (1707-1783) in his study of the Bridges of Königsburg problem (Euler, 1741). The city of Königsburg is built on a river and consists of four islands, which can be reached by means of seven bridges (See Figure 1.2.(a)). The question Euler was interested in answering is: Is it possible to go from island to island traversing each bridge only once? Euler analyzed the problem by simplifying the representation to a graph. Assume that we treat each island as a vertex and each bridge as a line edge (See Figure 1.2.(b)). The problem reduces to finding a "closed walk" in the graph which traverses each edge exactly once, this is called an Eulerian circuit. This method is considered by many to be the birth of graph theory, which has become a fundamental pillar of applied mathematics. Generally speaking, Graph Theory is a branch of combinatorics but it is closely connected to Applied Mathematics, Optimization Theory, Network Science and Computer Science.

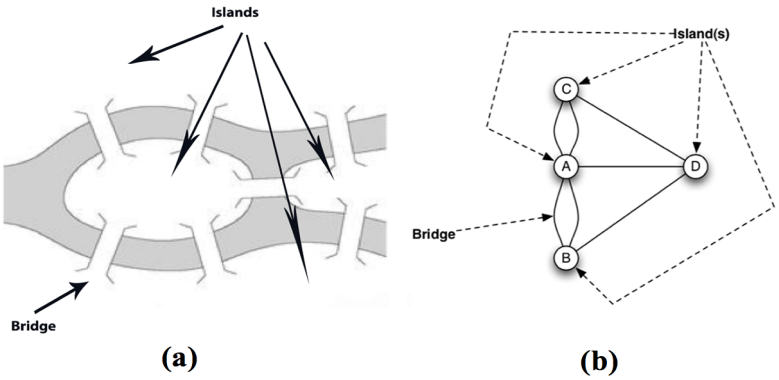


Figure 1.2: Koïigsberg Bridge Problem (a). The resulting graph (b)

1.2.1.2 Fundamental Definitions

Definition 1.2.1. ***(Graph)***

A graph is a tuple $G = (V_G, E_G)$ where V_G is a finite set of vertices and E_G is a finite collection of edges. We only consider ***undirected graphs****, meaning that there is no distinction how an edge may be directed from one vertex to another.* ***A weighted graph*** *is defined by replacing the set of edges E_G by a set of edge weights W_G. Two or more edges joining the same pair of vertices (u, v) are known as* ***multiple edges****, we call the graph having multiple edges by* ***multigraph****. An edge $e = \{v\}$ joining a vertex to itself is called a* ***loop****. A graph that is in one piece is a* ***connected graph****. A graph with no loops and no multiple edges is called a* ***simple graph*** *(Diestel (1997); Kaveh (2013); Griffin (2017)).*

Remark 1.2.1. *The number of vertices in G is called the order of G and the number of edges in G is called the size of G. The order and size of G are denoted by $|V_G|$ and $|E_G|$ respectively.*

Example 1.2.1. *Consider two graphs (a) and (b) in Figure 1.3. The first graph (a) is a simple, undirected and connected graph. The second graph (b) is a multigraph, disconnected and undirected graph with a loop e and multiple edges between (u, v).*

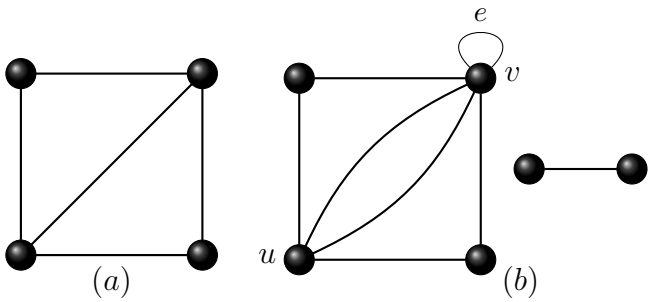


Figure 1.3: A simple graph (a) and a multigraph (b).

Definition 1.2.2. ***(Vertex degree, Average degree, Regular graph)***

The degree *of a vertex i of a graph G, denoted by k_i, is the number of edges attached to it. For a loop, we count its degree twice because it has two ends joined to that vertex.* ***The average degree*** *for the whole graph G, denoted by $< z_G >$, is the average, which is*

defined as $< z_G >= \frac{2|E_G|}{|V_G|}$ *where* $0 \leq < z_G > \leq |V_G| - 1$ *and it is not necessary an integer.* ***A*** $k-$***regular graph*** *is a graph where all its vertices have the same degree* k *(Diestel (1997)).*

Example 1.2.2. *We consider the same example as Figure 1.3 (b). The degree of* v *is* $k_v = 6$*, the degree of* u *is* $k_u = 4$ *and the average degree of (b) is* $< z_{(b)} >= \frac{2\times 8}{6} = 2,66$*.*

Definition 1.2.3. ***(Path, Distance, Cycle, Complete graphs)***

A path graph *is a simple graph whose vertices can be ordered, denoted by* $P_n = v_0, v_1, v_2, ...v_n$*. The degree of its vertices is* $k_{v_i} = 2$ *for* $i = 1, 2, ..., n-1$*.* ***The distance*** *between two vertices* (u, v) *of a network* G *is the number of edges in the shortest path connecting them, denoted by* $d(u, v)$*, called also as the geodesic distance.* ***A cycle graph****, denoted as* C_n *is a closed path* $(v_0, v_1, v_2, ..., v_n, v_0)$ *in which the degree of all its vertices is* $k_{v_i} = 2$ *for* $i = 0, 1, 2, ..., n$*.* ***A complete graph*** *with* n *vertices, denoted as* K_n*, is a simple graph that contains every possible edge between all the vertices. It has* $\frac{n(n-1)}{2}$ *edges and is a regular graph of the degree* $n-1$ *(Diestel (1997)).*

Example 1.2.3. *Consider three graphs* (a)*,* (b) *and* (c)*in Figure 1.4. The first graph* (a) *is a path graph* P_5*. The second graph* (b) *is a cycle graph* C_5 *and the distance between* (u, v) *is* 2*. The third graph* (c) *is a complete graph* K_5*.*

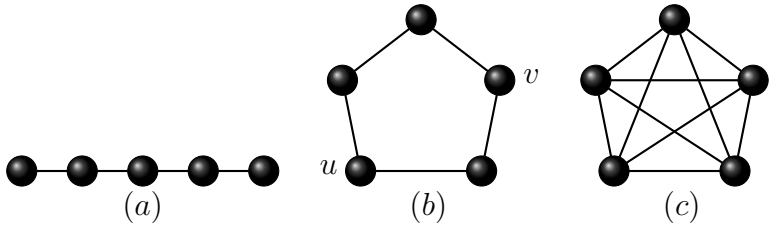


Figure 1.4: A path graph (a), a cycle graph (b) and a complete graph (c).

Definition 1.2.4. ***(isomorphism)***

An isomorphism from a simple graph G *to a simple graph* H *is a bijection between the node sets of* G *and* H*. This bijection preserves structures* $f : V(G) \to V(H)$ *such that* $(u, v) \in E(G)$ *if and only if* $f(u)f(v) \in E(H)$*. We say "*G *is isomorphic to* H*", denoted* $G \cong H$*, if there is an isomorphism from* G *to* H *(Griffin (2017)).*

Example 1.2.4. *For example, in Figure 1.5, two graphs G and H are not the same, but they are isomorphic under the correspondence $f(a) = 1$, $f(b) = 6$, $f(c) = 8$, $f(d) = 3$, $f(g) = 5$, $f(h) = 2$, $f(i) = 4$, $f(j) = 7$.*

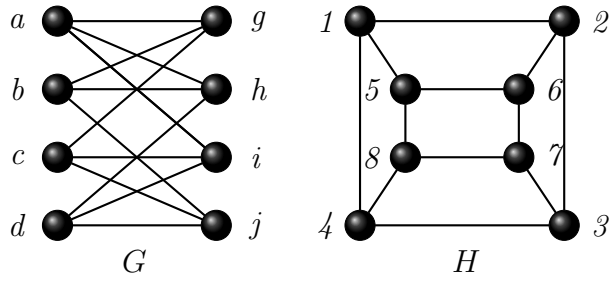


Figure 1.5: Two isomorphic graphs G and H.

Definition 1.2.5. ***(Planar graph)***

*A graph G is called **planar** if it can be drawn in the plane in such a way that no two edges cross each other (Nishizeki and Rahman (2004)).*

Example 1.2.5. *For example, in Figure 1.6, G is planar graph but the complete graph K_5 is not planar graph. It has crossing edges.*

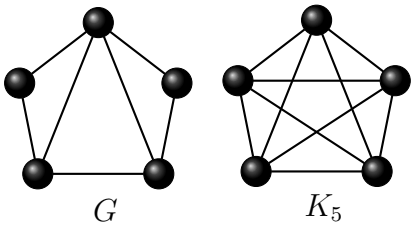


Figure 1.6: A planar graph G and a non planar graph K_5.

Euler's formula: The Euler's formula is a topological invariant, that characterized the topological properties related to the number of vertices, edges and faces (Euler (1741); Berge (1962); Moskowitz (2002)).

Corollary 1.2.1. *Let G be a connected planar graph with $|V_G|$ vertices, $|E_G|$ edges and $|F_G|$ faces. These numbers are connected by the well known Euler's relation, then:*

$$|V_G| - |E_G| + |F_G| = 2 \tag{1.1}$$

Example 1.2.6. *We consider the same example as Figure 1.6, a graph G has $|V_G| = 5$, $|E_G| = 7$ and $|F_G| = 4$. According to the Euler's formula, we get:* $5 - 7 + 4 = 2$.

1.2.1.3 Matrices Associated to a Graph

Although it is convenient to represent a graph by a diagram of vertices joined by edges, such a representation may be unsuitable if we wish to store a large graph in a computer for an automatic and algorithmic processing. There are several types of representations of graphs that will be studied including the adjacency matrix, the degree matrix and the Laplacian matrix (Clark and Holton (1991)).

Adjacency Matrix: A common representation of the topology of a graph G is through the adjacency matrix. It is a square matrix whose size is its number of vertices.

Definition 1.2.6. ***(Adjacency Matrix)*** *Let $G = (V, E)$ be a graph and assume that $V = \{v_1, v_2, ..., v_n\}$. The adjacency matrix of G is defined as $(A_{ij})_{n\times n}$:*

$$(A_{ij}) = \begin{cases} 1 & if\ \{v_i, v_j\} \in E \\ 0 & else \end{cases} \tag{1.2}$$

Remark 1.2.2. *An undirected graph corresponds to a symmetric matrix and the absence of loops corresponds to a zero diagonal. For a multiple graph, A_{ij} is defined by the number of edges between v_i and v_j.*

Degree Matrix: A second important matrix representation of a graph is the matrix of degrees that contains information about the degree of each vertex of a graph.

Definition 1.2.7. ***(Degree Matrix)*** *Let $G = (V, E)$ be a graph and assume that $V = \{v_1, v_2, ..., v_n\}$ and k_{v_i} is the degree of the vertex v_i. The degree matrix of G is a diagonal matrix, which is defined as $(D_{ij})_{n\times n}$:*

$$(D_{ij}) := \begin{cases} k(v_i) & if\ i = j \\ 0 & else \end{cases} \tag{1.3}$$

Laplacian Matrix: (Kenyon (2012)) Another important matrix representation of a graph is through its Laplacian matrix, which is the difference of the degree matrix and the adjacency matrix of a graph. The Laplacian matrix can be used to find many other properties of a graph, see e.g., spectral graph theory (Chaiken and Kleitman (1978)), Matrix-Tree Theorem (Kirchhoff, 1847).

Definition 1.2.8. ***(Laplacian matrix)*** *Let $G = (V, E)$ be a graph, assume that $V = \{v_1, v_2, ..., v_n\}$ and k_{v_i} is the degree of the vertex v_i. The Laplacian matrix of G is defined as $L_{ij} = D_{ij} - A_{ij}$ with D_{ij} is the degree matrix of G and A_{ij} is its adjacency matrix:*

$$(L_{ij}) = \begin{cases} k(v_i) & if\ i = j \\ -1 & if\ i \neq j, \{v_i, v_j\} \in E \\ 0 & else \end{cases} \tag{1.4}$$

More formally, an undirected graph $G = (V, E)$ with $V = \{v_1, v_2, ..., v_n\}$, weighted by the weight function at any edge (v_i, v_j) associated weight $w(v_i, v_j)$. The Laplacian matrix of G verifies:

$$(L_{ij}) = \begin{cases} w(v_i) & if\ i = j \\ -w(v_i, v_j) & if\ i \neq j, \{v_i, v_j\} \in E_G \\ 0 & else \end{cases} \tag{1.5}$$

The weight $w(v_i) = \sum_{j \sim i} w(v_i, v_j)$ is the diagonal elements of matrix 1.5, which is the sum of the weights of edges incident on vertex i and the off-diagonal elements are $-w(v_i, v_j)$, if an edge connects the two vertices, and 0 otherwise.

Example 1.2.7. *Consider the simple and undirected graph G in Figure 1.7, which has the adjacency matrix A, the degree matrix D and the Laplacian matrix L as follows:*

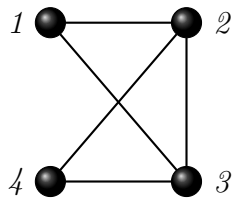


Figure 1.7: A simple and undirected graph G.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \mathbf{L} = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

1.2.2 Trees

1.2.2.1 Trees

The concept of a tree is one of the most important and commonly used ideas in graph theory, especially in the applications of computer science as data storage, searching and communication. Although trees are relatively simple structures, they form the basis of many of practical techniques used to model and to design large and complex networks.

Definition 1.2.9. ***(Tree)***
A tree is a connected graph without cycles, denoted as $T = (V, E)$. In other words, it is a graph in which any two vertices are connected by exactly one path (Diestel (1997); Kaveh (2013)). It has $|V| - 1$ edges. There are two types of vertices in a tree:

- *Leaves with the degree 1.*
- *Internal nodes whose degree is greater than 1.*

Example 1.2.8. *In Figure 1.8. The following 4 graphs are trees.*

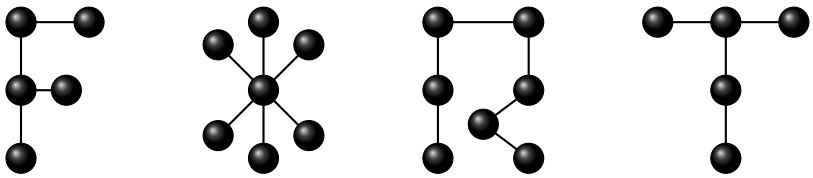


Figure 1.8: Simple trees

1.2.2.2 Spanning Trees

In graph theory, one of the major uses of trees is to find spanning trees of a graph (Knuth, 1997; Wu and Chao, 2004). This type of trees has always been of great interest in various

areas of computer science. A spanning tree is an important concept that we need later. In Chapter 2, we explain in details how to calculate the number of spanning trees of a graph.

Definition 1.2.10. ***(Spanning Trees)***

Spanning Tree is defined as an acyclic and a connected subgraph which has all the vertices of a graph covered with minimum possible number of edges.

Example 1.2.9. *For example, Figure 1.9 shows a graph G and some of its spanning trees.*

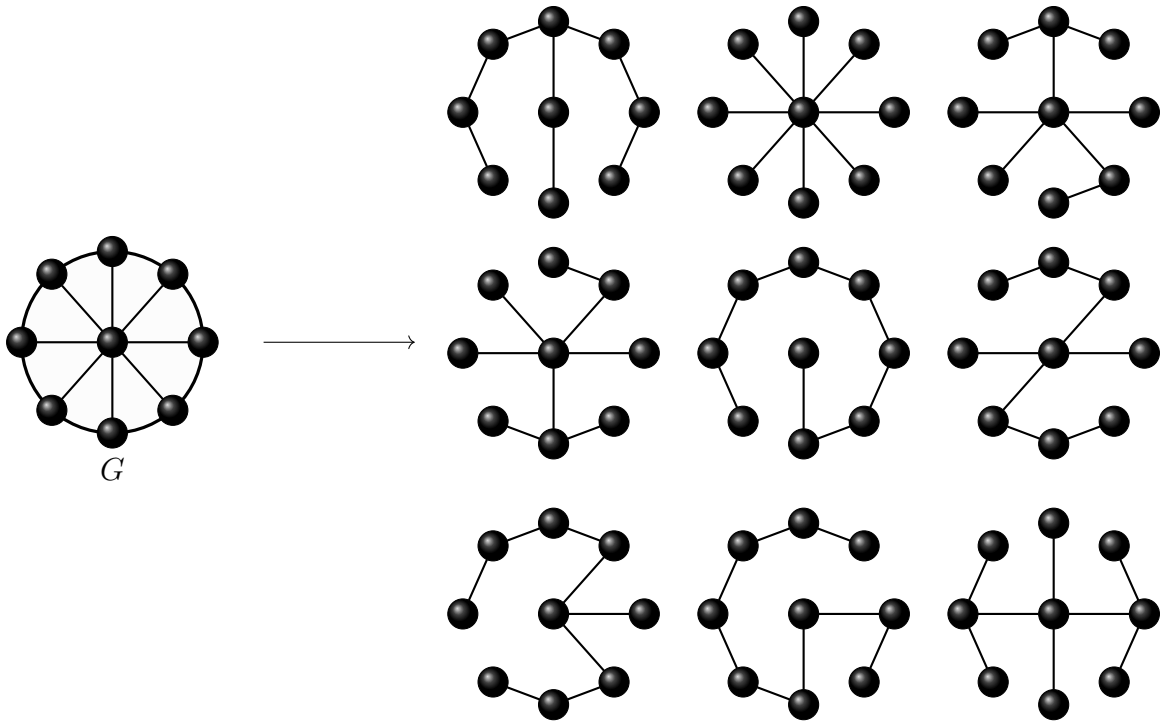


Figure 1.9: A graph G with some of its spanning trees.

1.3 Structural Properties of Complex Networks

The analysis of relevant structural properties is one of the major objectives that guide the research on complex networks. A structural property is any inherently graph-theoretical property that preserves under all possible topological changes of a network. This means that networks with the same structural property naturally define a certain family or type of networks (See Section 1.4). However, the structural properties are defined and studied

to understand the mechanism and the behaviour of real-world networks. Among these properties, three concepts have attracted lots of attention in the research of complex networks. These concepts are the clustering coefficient, the degree distribution and the small-world effect. They play a crucial role to classify complex networks. In this section, we discuss some important properties that appear to be common to many networks.

1.3.1 Clustering Coefficient or Transitivity

The clustering coefficient C, introduced by Watts and Strogatz (1998), is a measure of grouping nodes in a network. More precisely, this coefficient measures how close the neighbourhood of a vertex is. In many networks, it is found that if vertex A is connected to vertex B and vertex B to vertex C, then there is a heightened probability that vertex A will also be connected to vertex C. It can be defined in two different versions: the global and the local. The global clustering coefficient measures the density of triangles in a network, often called transitivity (Newman et al. (2002); Newman (2003)). In social networks, it measures the fraction of the total pairs of friends of an individual that are each other friends (Wasserman and Faust (1994)). That means also the probability that the friend of your friend is also your friend. The global clustering coefficient is defined as:

$$C = \frac{3 \times Number\ of\ triangles}{Number\ of\ all\ triplets} \tag{1.6}$$

where a triplet means three nodes that are connected by either two or three undirected links i.e., subgraphs isomorphic to a path on 3 vertices and the factor of "3" in the numerator accounts for the fact that each triangle contributes to three triples. Whereas the local clustering coefficient of a node i is the ratio between the number of links e_i connecting the nearest neighbours of i and the total number of maximum possible links between these neighbours, which defined as:

$$C_i = \frac{2e_i}{k_i(k_i - 1)} \tag{1.7}$$

where "k_i" is the degree of a node i. As an alternative to the local clustering coefficient, the overall level of clustering in a network G is measured as the average of the local

clustering coefficients of all the nodes, which has been extensively used in the analysis of complex networks, defined as:

$$C_G = \frac{1}{|V_G|} \sum_{i=1}^{|V_G|} C_i \tag{1.8}$$

The value of the clustering coefficient of a network is between $0 \leq C \leq 1$. For the star network or the tree, their clustering coefficient is $C = 0$ and for a complete network, its clustering coefficient is $C = 1$.

Example 1.3.1. *From Figure 1.10, we illustre the definition of the clustering coefficient C. According to the global definition in Equation 1.6, this network has one triangle and* 8 *connected triples. Hence, it has a clustering coefficient of* $3 \times \frac{1}{8} = \frac{3}{8} = 0.375$. *According to the local clustering coefficient in Equation 1.7, we get for each vertex:* $1, 1, \frac{1}{6}, 0$ *and* 0. *For the clustering coefficient of the whole network according to Equation 1.8, we get:* $C = \frac{13}{30} = 0.433$.

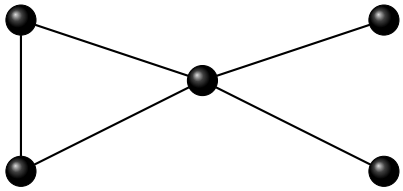


Figure 1.10: Illustration of the definition of the clustering coefficient C.

1.3.2 Degree Distribution

In network theory, an important property for the study of a complex network is its degree distribution, denoted by $P(k)$, which is the probability that a randomly selected node has exactly a degree k (Dorogovtsev and Mendes (2004)). It is defined to be the fraction of nodes in the network with a degree k, where $|V_k|$ is the number of nodes having the degree k. We have

$$P(k) = \frac{|V_k|}{|V_G|}. \tag{1.9}$$

Example 1.3.2. *We consider the same example as Figure 1.10. This graph has* $|V_G| = 5$ *vertices, of which 2 have degree 1, 2 have degree 2 and 1 have degree 4. Thus the values of* $P(k)$ *for* $k = 1, 2, 4$ *are:* $P(1) = \frac{2}{5}, P(2) = \frac{2}{5}, P(4) = \frac{1}{5}$.

It is often illuminating to make a plot of the degree distribution $P(k)$ of a large network by a histogram of the degrees of nodes. Another way to present the degree data is to plot a graph of the cumulative distribution function (Price (1976)). It has the advantage that all the original data are represented. It is defined as the probability that the degree of nodes is greater than or equal to k:

$$P_{cum}(k) = \sum_{k'=k}^{\infty} P(k') \tag{1.10}$$

In a random network of the type studied by Erdős and Rényi (1961) (See Section 1.4.3), each edge is present or absent with the equal probability p and its degree distribution follows the binomial distribution or Poisson in the limit of large network size (See Figure 1.11 (a)).

$$P(k) = \binom{|V_G| - 1}{k} p^k (1-p)^{|V_G|-1-k} \sim \frac{< z >^k \, \mathrm{e}^{-<z>}}{k!} \tag{1.11}$$

Where $< z >$ is the average degree of a network. Generally, it was believed that the degree distribution in real-world networks follows a Poisson distribution but in reality, the degree distribution of real-world networks is unlike those of the random networks. These real networks have a highly skewed degree distribution following power-laws (Simon (1955)) (See Figure 1.11 (b)). They are named as scale-free networks (See Section 1.4.2).

$$P(k) \sim k^{-\gamma} \tag{1.12}$$

Where γ is a constant which belongs to the interval $[2, 3]$. Note that such power-law distributions show up as power laws in the cumulative distributions also (Newman (2003)), but with exponent $\gamma - 1$ rather than γ:

$$P_{cum}(k) = \sum_{k'=k}^{\infty} k'^{-\gamma} \sim k^{-(\gamma-1)} \tag{1.13}$$

Recent interest in networks with power-law degree distribution started with Barabási and Albert (1999), who proposed a mechanism to explain the appearance of the power-law distribution, which they called preferential attachment (See Section 1.4.2). This

mechanism is based on some nodes, which they called hubs, had many more connections than others and that the network as a whole had a power-law distribution of the number of links connecting to a node. This concept is used to refer to the principle of "the rich gets richer" (Jackson (1935); Merton (1968)). In the language of social networks, it refers to the more friends you have, the easier it is to make new ones. In terms of network theory, all these concepts refer to the idea that if a node has a high degree, it has a higher probability to attract more connections and thus its connectivity grows at a faster rate than other nodes with low connectivity. There are other real-networks that their degree distribution follows an exponential distribution (Newman (2003)):

$$P(k) \sim \mathrm{e}^{-\alpha k} \tag{1.14}$$

These also give exponentials in the cumulative distribution with the same exponent:

$$P_{cum}(k) = \sum_{k'=k}^{\infty} \mathrm{e}^{-\alpha k'} \sim \mathrm{e}^{-\alpha k} \tag{1.15}$$

In Figure 1.11, we show the representation of the degree distribution of two networks. Binomial distribution for random networks (a). Power-law degree distribution for scale-free networks (b).

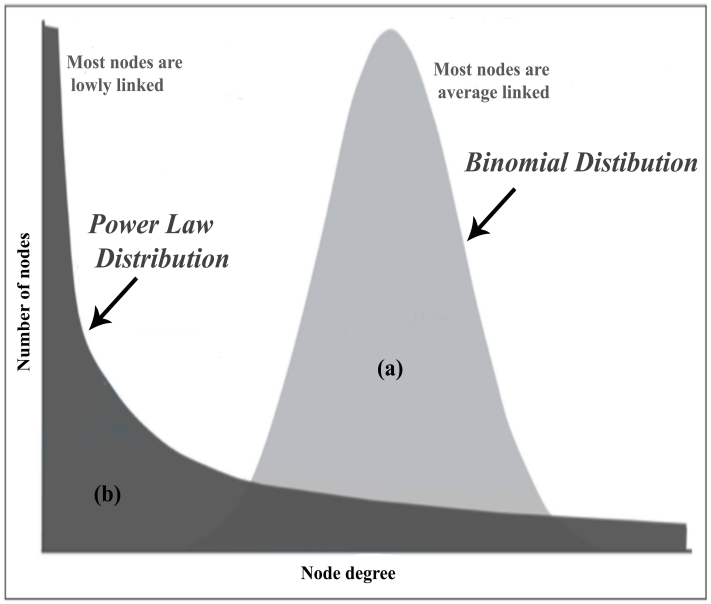


Figure 1.11: The degree distribution of a random network (a) and a scale-free network (b).

1.3.3 Small-World Effect

In general, a network that has the small-world effect in which all nodes can be reached to the other nodes through a small number of hops. This property was found in many real-world networks. For example, in social networks, the average distance from one individual to another is small compared to the size of a network. Indeed, In the 1960s, social psychologist Stanley Milgram conducted some experiments which are referred to as, the small-world experiment (Milgram (1967)). He gave 300 letters to participants living in the cities of United States, Boston and Omaha, along with instructions to deliver them to one particular target person by mailing the letter to an acquaintance, they considered to be closer to the target. That person then got the same set of instructions, which, therefore, set up a chain. Milgram found that the average path length that separates any two people in the world was about six hops, which is called also the small-world phenomenon. He suggested that human society is a small-world type network characterized by short path lengths (See Section 1.4.1) and the experiments are often associated with the phrase "six degrees of separation" (Watts (2004)). In literature, this concept is often referred to as the average path length of a network.

1.3.3.1 Average path length

The average path length (APL) characterizes the small-world effect. It is a measure of the efficiency of information or mass transport on a network. It gives an idea of, on average, how far apart any two nodes lie in a network. It is defined as the average of the shortest distance between all nodes of a network G, denoted by l_G (Newman (2003)). For an undirected network, the average path length is calculated as:

$$l_G = \frac{1}{|V_G|(|V_G|-1)} \sum_{u \neq v} d(u,v) \tag{1.16}$$

Where $d(u,v)$ is the geodesic distance from vertex u to vertex v. For small-world networks, the distance between any two nodes scales as the logarithm of the number of nodes, suggesting that the APL between any two nodes in the network is quite low. For random networks, they have also the small-world effect, their average distance also scales

as the logarithm of the number of nodes (Albert and Barabási (2002)). The small-world effect can be quite useful in different networks. For example, if one considers the spread of information across a network, the small-world effect implies that the spread will be fast on most real-world networks. If it takes only six steps for a rumour to spread from any person to any other, for instance, then the rumour will spread much faster. To more technical applications such as estimating the number of hops required for an information packet to get from one computer to another on the Internet (Zhang et al. (2004)). Another example to study how to control and take precautions against an epidemic spread in social networks (Moore and Newman (2000)), etc. The small-world effect also underlies some well-known parlor games, particularly the calculation of Erdòs numbers (De Castro and Grossman (1999)) and Bacon numbers. In some cases, it is difficult to obtain the analytic solution of APL, especially for large networks. Therefore, researchers adopt "diameter" as an alternative parameter to demonstrate the short distance between any two nodes of a network.

1.3.3.2 Diameter

By definition, the diameter of the network G, denoted as D_G, is the longest geodesic, i.e., the maximum of the shortest distance between any two nodes (u, v) of a network G:

$$D_G = max_{u,v} d(u, v) \tag{1.17}$$

The diameter characterizes the maximal communication delay in a network. If a network is with a small diameter, it is undoubtedly with a short APL, i.e., if the diameter of a network G scales as $D_G = \mathcal{O}(\log V_G)$ and we have always the APL is smaller than the diameter $l_G \leq D_G$, then it is undoubtedly with a short APL: $l_G = \mathcal{O}(\log V_G)$. So the APL should increase more slowly (Gu et al. (2013); Takes and Kosters (2011)). These two structural properties (The APL and the diameter) define the small-world effect, because one of them is related to the other indirectly.

Example 1.3.3. *We consider the same example as Figure 1.10. This graph has* $|V_G| = 5$ *vertices. Then, its APL is:* $l = \frac{1+2+2+2+1+2+2+2+1+1+1+1+1+2+2+1+1+2+2}{5\times 4} = \frac{3}{2} = 1.5$ *and its diameter is:* $D = 2$.

1.3.4 Fractals Dimension

Another fundamental measurement characterizing a specific type of complex networks, it measures the dimension of the fractality, named as Fractals Dimension, which is the scaling rule from knowing how something scales. It is an index for characterizing fractal patterns. So, what is a fractal? A fractal is an irregular geometric object with an infinite nesting of the structure at all scales, it is realized by repeated construction of an elementary shape on progressively smaller length scales. They display self-similarity in all scales. The concept of fractals first introduced by Mandelbrot (1982), characterizes many real-life systems in nature and society and has received tremendous interest from the scientific community (Mandelbrot (1982); Aguirre et al. (2009)). Fractals have been observed in complex networks including the World Wide Web (WWW), protein interaction networks and metabolism, etc. A network is said to be fractal if it has a finite fractal dimension. There are many types of fractal dimension (Costa et al. (2007)). Generally, in order to measure the fractal dimension of complex networks, **a box-counting method** has been proposed by Song (2005). In the former, the network is covered with N_B boxes, where all vertices in each of them are connected by a minimum distance smaller than l_B. Then, the fractal box dimension of the network, denoted by d_B ($0 < d_B < \infty$), is defined by

$$d_B \approx \frac{\ln N_B}{\ln l_B} \tag{1.18}$$

A self-similar network is exactly or approximately similar to a part of itself (Song et al. (2005)). Note that fractality and self-similarity do not always imply each other. A fractal network is always self-similar, but a self-similar network may be not fractal. Another scaling relation is found with a renormalization procedure based on the box counting method (Song (2005)). A renormalized network is created with each box of the original network transformed into a vertex and two new vertices are connected if at least one edge exists between vertices of the corresponding boxes in the original network (Song et al. (2006)). By considering the degree k' of each vertex of the renormalized network versus the maximum degree k in each box of the original network, we have that:

$$k' \approx l_B^{-d_k} k \tag{1.19}$$

Where d_k is the degree exponent of boxes. The exponents d_B, d_k and γ (of the power law of the degree distribution (See Section 1.4.2)) are related by

$$\gamma = 1 + \frac{d_B}{d_k} \tag{1.20}$$

Thus, scale-free networks, characterized by the exponent, can also be described by the two length invariant exponents d_B and d_k.

1.4 Models of Complex Networks

In order to mimic the real-world systems, a wide variety of models have been proposed, including the small-world networks, scale-free networks and random networks. A small-world network model is defined by Watts and Strogatz, named WS model (Watts and Strogatz, 1998). It started an avalanche of research on small-world networks. A scale-free network model is defined by Barabási and Albert, named BA model (Barabási and Albert, 1999). It has attracted an exceptional amount of attention within the physics community. While a random network model is introduced by Erdős and Reńyi, named ER model (Erdős and Rényi, 1960, 1961). It is considered the most basic model to analyse real-world networks. Modeling complex networks with small-world, scale-free and random networks properties is still an important issue to explain how networks with these properties appear in the real world. This section introduces those three commonly cited models, which will be used throughout this thesis as a tool to classify other new models of networks.

1.4.1 Small-World Networks

The small-world model describes the fact that despite the large graph size in most real-world networks, there is a relatively short path between any two nodes and large clustering coefficients. There are different realizations of the small-world model but the original model as proposed by Watts and Strogatz (1998) is by far the most widely studied. It starts with a ring of n vertices in which each vertex is connected to its k nearest neighbours, for a given k. This forms a regular graph as shown in Figure 1.12(a). Then, each edge

is rewired with a given probability p by choosing randomly a new vertex to connect, except that no double links or loops are allowed. In a regular graph, since neighbours are connected to each other, the overall clustering coefficient C is very high. On the other hand, the average path length l is very high as vertices are only connected to their neighbours. The random rewiring a few nodes introduces edges that connect nodes lying at long distances, which in turn, reduces the overall average path length. Since many vertices are connected to their neighbours, the overall clustering coefficient C remains high whereas the average path length l is reduced, giving us the properties of a small world network (See Figure 1.12(b)). Thus l scales logarithmically with the network order V and the clustering coefficient decreases with V. The clustering coefficient for this small-world model can be calculated relatively easily as:

$$C = \frac{3(k-1)}{2(2k-1)}(1-p)^3 \tag{1.21}$$

And the average path length l does not begin to decrease until $p \geq 2/Vk$. This implies that there exists a p-dependent crossover length V^* such that if $V < V^*, l \sim V$, but if $V > V^*, l \sim \ln(V)$. The concept of the crossover length was introduced by Barthélémy and Amaral (1999), who conjectured that the characteristic path length scales as

$$l(V,p) \sim V^* F\left(\frac{V}{V^*}\right), \tag{1.22}$$

Where

$$F(u) = \begin{cases} u, & \text{if } u \ll 1 \\ \ln(u), & \text{if } u \gg 1 \end{cases} \tag{1.23}$$

If the process of random rewiring continues, we eventually end up rewiring every node which results in a random graph as vertices no longer share common neighbours (See Figure 1.12(c)). It is important to note that networks produced using this model do not have scale-free degree distribution, it is similar to that of random networks. Since every vertex in the network initially has a fix k degree, random rewiring of only a few vertices does not affect the overall behaviour of the degree distribution. Thus, the topology

of the network is relatively homogeneous and all nodes having approximately the same number of edges. This original model has been studied at some length in the mathematical and physical literature (Newman, 2000). Other models have been proposed to produce networks with small-world properties without using this basic model such as (Newman and Watts, 1999; Mathias and Gopal, 2001; Guillaume and Latapy, 2006).

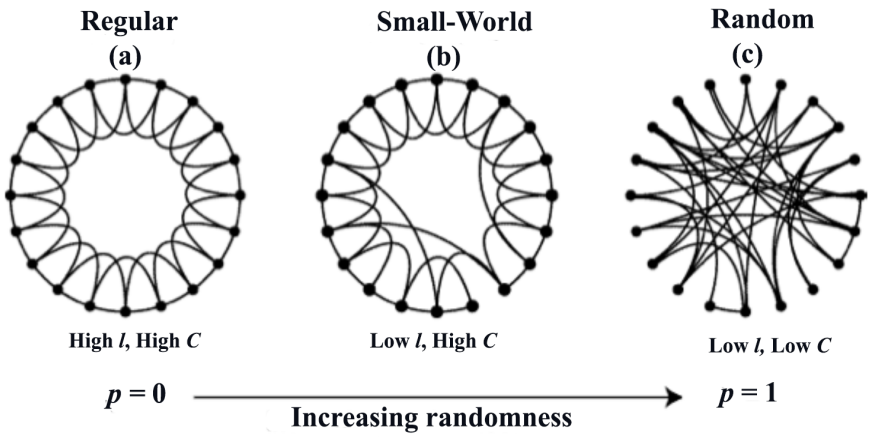


Figure 1.12: Schematic representation of the evolution of the rewiring process in the Watts-Strogatz model: From a regular network (a) with $p = 0$ to a random Network (c) with $p = 1$, where random rewiring of few edges in a regular network produces a small-world network (b) with $0 < p < 1$, high clustering coefficient and low average path length. The figure is taken from Watts and Strogatz (1998).

1.4.2 Scale-Free Networks

After Watts and Strogatz's model, Barabási and Albert (1999) showed that many real systems are characterized by an uneven distribution. Some vertices are highly connected, called *hubs* while others have few connections. Specifically, the degree distribution follows a power law for a large k (See Equation 1.12). These networks are called **scale-free networks**. They are characterized by the non-homogeneity of their structure, or what is known by the heterogeneity .i.e., very few nodes have many edges but most nodes have

very few edges. Indeed, we can define scale-free networks as dynamic systems that evolve through the subsequent addition and deletion of nodes and links. Barabási and Albert (1999) proposed a model, called as Barabási-Albert (BA) network model to reproduce the important characteristic of scale-free networks. This model of network growth is inspired by the formation of the World Wide Web and is based on two basic ingredients: growth and preferential attachment. The growth model controls how a network grows over time and using the preferential attachment as a guide to growing the network, where new nodes prefer to attach to well-connected nodes. The BA model is generated starting with a set of m_0 vertices. Then, at each step of the construction, the network grows with the addition of new vertices. For each new vertex, $m \leq m_0$ new edges are inserted between the new vertex and some previous vertices. The vertices which receive the new edges are chosen following a linear preferential attachment rule, i.e. the probability of the new vertex i to connect with an existing vertex j is proportional to the degree of j:

$$p(i \to j) = \frac{k_j}{\sum_u k_u} \tag{1.24}$$

Thus, the most connected vertices have a greater probability of receiving new vertices. This is known as "the rich get richer" paradigm. After t timesteps, this procedure results in a network with $V = t + m_0$ nodes and $E = mt$ edges, corresponding to an average degree $< z >= 2m$ for large times. This network evolves into a scale-invariant state with the probability that a node has k edges following a power-law with an exponent $\gamma = 3$. The scaling exponent is independent of m. The clustering coefficient of the scale-free network decreases with the network size following approximately a power-law

$$C \sim V^{-0.75} \tag{1.25}$$

And its average path length increases approximately logarithmically with V, the best fit following a generalized logarithmic form

$$l \sim \frac{\log V}{\log(\log V)} \tag{1.26}$$

Mathematical results for scale-free networks have been studied by several researchers such as (Bollobás and Riordan, 2003; Bollt and ben Avraham, 2005). Other alternative models for obtaining power-law degree distributions with different exponents γ can be found in the literature (Aiello et al., 2001; Caldarelli et al., 2002; Dorogovtsev and Mendes, 2013). Figure 1.13 shows an example of a Barabási-Albert network.

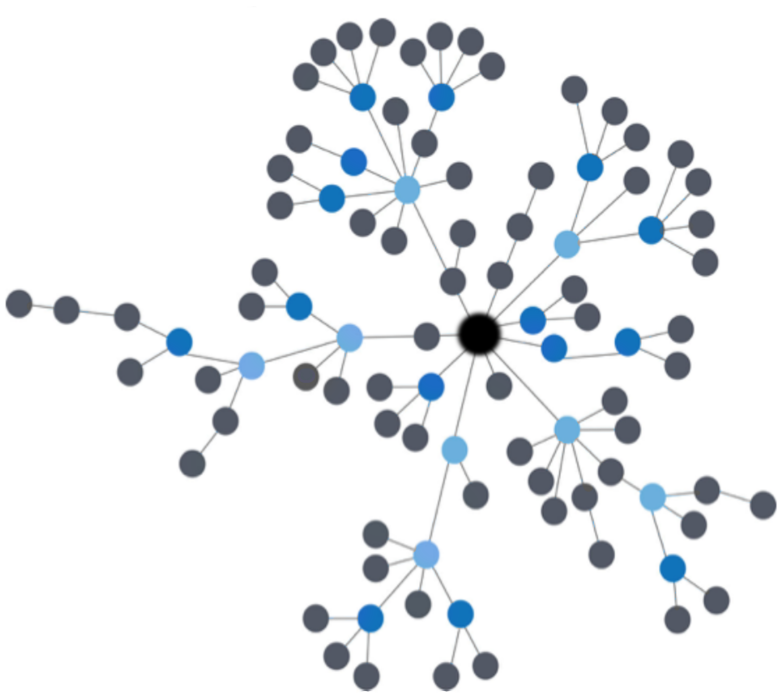


Figure 1.13: An example of the scale-free network of Barabási and Albert.

1.4.3 Random Networks

The systematic study of random graphs was initiated by two Hungarian mathematicians Paul Erdős and Alfred Rényi in 1959. They published a celebrated series of papers about the random graph model in the late 1950s and early 1960s (Erdős et al., 1959; Erdős and Rényi, 1960, 1961). They proposed a model $G(V, E)$ to generate random graphs consisting of V vertices and E edges. Starting with V disconnected vertices, the network is constructed by the addition of E edges at random, avoiding multiple and self connections. Another similar model $G(V, p)$ defines V vertices which are connected with the probability $0 \leq p \leq 1$. The latter model is widely known as Erdős-Rényi network

(ER). It is also sometimes called the "Poisson random graph" or the "Bernoulli random graph", names that refer to the distributions of degrees and edges in the model. There are many random graph models, but $G(V,p)$ is the most fundamental and widely studied of them. Here, we will discuss $G(V,p)$. In Figure 1.14, we illustrate some examples of Erdős-Rényi random networks with the same number of nodes (30 nodes) and different probability values. In fact, technically, $G(V,p)$ is the ensemble of all such graphs in which a graph having $E = p\frac{V(V-1)}{2}$ edges. It exhibits some important properties (Solomonoff and Rapoport, 1951; Bollobás, 1998b; Newman et al., 2001; Newman, 2003). For example, we can get the connected random network if the probability $p \geq \frac{\ln V}{V}$. Its average degree is $< z >= p(V-1) \simeq pV$. However, in almost all other respects, the properties of the random graph do not match those of networks in the real world. It has a low clustering coefficient: the probability of connection of two vertices is p regardless of whether they have a common neighbour, and hence $C = p << 1$, which tends to zero as n^{-1} in the limit of large system size. The ER network is characterized by a small-world effect: The diameter varies in a small range of values around $D = \frac{\ln V}{\ln(pV)} = \frac{\ln V}{\ln<z>}$. The average shortest path length l has the same behaviour as a function of V as the diameter, $l \sim \frac{\ln V}{\ln<z>}$. Its degree distribution follows the binomial distribution (See Equation 1.11), because the maximal degree of a node i is $V-1$, the probability that the vertex has k links is $p^k(1-p)^{V-1-k}$ and there are $\binom{V-1}{k}$ possibilities to choose k links from $V-1$ nodes. In the limit $V \to \infty$ becomes $P(k) = \frac{<z>^k e^{-<z>}}{k!}$. This means that the degree distribution of a node in a random network can be approximated by the Poisson distribution for large V. For this reason, random networks are also called Poisson random networks.

The expected structure of the random graph varies with the value of p. The edges join vertices together to form components, i.e., subsets of vertices that are connected by paths through the network. The ER model possesses a phase transition: from a low p value for which there are few links and many small components to a high p value for which an extensive fraction of all nodes are joined together in a single giant component (See Figure 1.14).

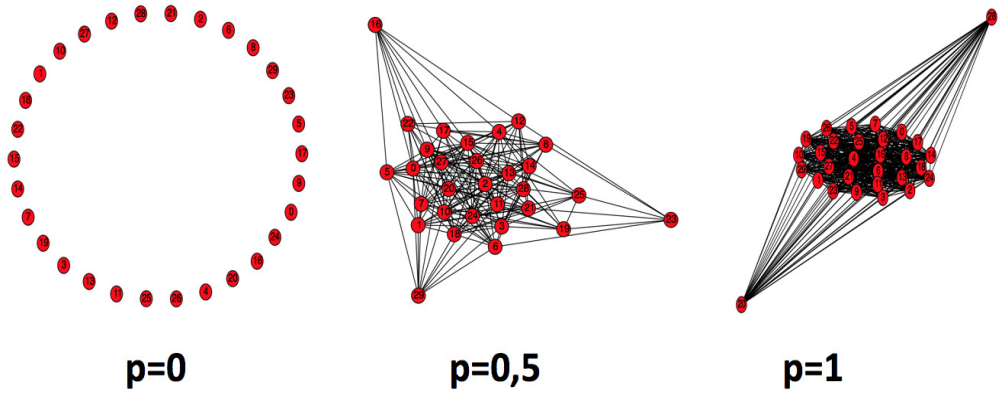


Figure 1.14: Erdős-Rényi network with 30 nodes and the probability values 0, 0.5 and 1

1.5 Summary

This chapter is an elementary introduction to the field of complex networks. The important networks models and properties are reviewed. The focus of this chapter is to analyze the relevance of topological properties that are the basics of classifying complex networks. Besides, the basic concepts of graph theory are also discussed. Indeed, graphs are used in several disciplines. For example, many problems in computer science have been solved by using graph theory due to the importance of its algorithmic aspect. A very important tool of this theory is a spanning tree of a network, which has been the remedy of several problems including the switching loops in computer networks (Spanning Tree Protocol: STP), the prediction of the reliability and the robustness of networks, etc. All these different applications have led us to study different spanning trees derived from a network and find several methods to enumerate them. In the next chapter, we will discuss the notion of the number of spanning trees of a network and we will treat the different techniques and approaches to calculate it.

Chapter

2 Methods of calculating the number of spanning trees of a network

Contents

Spanning trees have been widely studied in many aspects of mathematics and theoretical computer science. An interesting issue is the calculation of the number of spanning trees, which has a lot of connections with networks (Wu, 1982; Kim et al., 2007). In view of its wide range of applications, the enumeration of spanning trees has received considerable attention from the scientific community. In this chapter, first, we investigate two important applications of enumerating the spanning trees such as the reliability and the robustness of a network. Then, we present the different methods to obtain the exact formula of the number of spanning trees for a network, classed as algebraic, combinatorial and geometric methods, especially the Kirchhoff theorem, Fussner's formula, the decomposition method, the suppression and contraction method and the duality, etc. Finally, we propose some applications of these methods such as the evaluation of the complexity of Book networks. We note that the results of this chapter were published in an international journal (Mokhlissi et al., 2015a) and two international conferences (Mokhlissi et al., 2015b, 2018a).

2.1 Introduction

A spanning tree is relevant to various aspects of networks. It is a powerful tool to model and analyze the structure of a connected network. According to Definition 1.2.10, a spanning tree is a tree containing all the vertices and some or all edges of the main network. The most classical theory of interest concerning spanning trees is the number of spanning trees of a graph G, also known as the complexity of G, denoted as $\tau(G)$. Generally, the number of different spanning trees of a graph G turns out to be a very useful number leading to interesting results and applications. The research of this number has a long history. Firstly, it was initiated by the physicist Kirchhoff (1847) who proposed a theorem called the "Matrix-Tree Theorem". Then, many researches are devoted to finding the exact formula of the number of spanning trees of a given graph.

2.1.1 Problematic

For a connected graph G, it is easy to find a tree that contains all nodes and some or all edges of G, which is named a spanning tree of G. We can also find two or three others. In general, a graph can have several spanning trees, but the connectivity remains an essential condition for spanning trees. Now, the difficult question is how many spanning trees can contain this graph? Let's imagine that we have a network with millions or billions of nodes, can we calculate its number of spanning trees easily? It is difficult even impossible to obtain it by hand. In the literature, several researchers have found an answer to this question by proposing several methods and techniques. The main objective of this chapter is to quote almost all the known methods in the literature in order to calculate the number of spanning trees of a graph. It will be divided into three types of methods: Algebraic, combinatorial and geometric.

2.1.2 Applications of the number of spanning trees of a network

The number of spanning trees or what is called the complexity of a network is a very important invariant in graph theory, which has a lot of applications. In particular, in communication networks, the number of spanning trees is used to generate spanning trees

in order to build a loop-free logical topology for Ethernet networks and prevent bridge loops and the broadcast radiation that results from them. This algorithm is the spanning tree protocol (STP) defined in the IEEE 802.1D standard (Perlman, 1985; Group et al., 2004). Other uses for the number of spanning trees including the synchronization (Nishikawa and Motter, 2006), the percolation (Dhar, 2006), the analysis of social networks (Smith and Christakis, 2008), the study of random walks (Marchal et al., 2000), etc. Recently, two important applications of the complexity of a network are mostly treated, which are the robustness and the reliability of a network.

2.1.2.1 Robustness of a network

As an application of the number of spanning trees of a network G_n, we use the entropy of spanning trees or what is called the asymptotic complexity, denoted as ρ_{G_n}, to characterize the structure of a network and to evaluate its robustness. This latter is related to the capacity of a network to withstand random changes, failures and perturbations in its structure over evolutionary time. The best known mathematical model of the network robustness is offered by the percolation theory (Stauffer et al., 1993). In this work, we propose a new measure of the entropy to quantify the robustness of a network. This entropy of spanning trees is considered as a quantitative measure of the graph itself. If the number of spanning trees $\tau(G_n)$ grows exponentially with the network order V_{G_n}, then there exists a constant ρ_{G_n}, describing this exponential growth (Burton and Pemantle, 1993; Lyons, 2005):

$$\rho_{G_n} = \lim_{V_{G_n}\to\infty} \frac{\ln|\tau(G_n)|}{|V_{G_n}|} = \lim_{n\to\infty} \frac{\ln|\tau(G_n)|}{|V_{G_n}|} \tag{2.1}$$

where G_n is an increasing sequence of graphs approaching an infinite graph and depending on the number of iterations n, called an iterative graph. By calculating this entropy, we can estimate how the network will evolve to infinity. The most robust network is the network that has the highest entropy of spanning trees. According to Equation 2.1, when we have a high entropy value, this means that we have a large number of spanning trees, because the increase of the number of spanning trees provides more possibilities of connecting two nodes related by defective links, that ensures a good robustness and

availability of communication networks. So, if the number of spanning trees of a network increases, its robustness can improve. This entropy of spanning trees was calculated for many networks such as the Small-World Exponential Network (Mokhlissi et al., 2016a), the Koch network (Mokhlissi et al., 2018b), the prism and anti-prism (Sun et al., 2016), the contact graphs of disk packings (Qin et al., 2015), the pseudofractal scale-free web (Zhang et al., 2010b), the Sierpinski gasket (Chang et al., 2007), the lattice (Wu, 1977) and the Farey graph (Zhang et al., 2012), etc. In the next chapters, we calculate the entropy for many complex networks and we compare the results with other networks having the same average degree.

2.1.2.2 Reliability of a network

Reliability is defined as the probability of a successful connection of all network nodes to each other. In other words, an arbitrary node of an active network should have access to all other nodes. A spanning tree of a network can provide this concept. A network can have multiple spanning trees. However, the network will be successful if at least one of the spanning trees is available. To analyze the reliability of a network, a novel approach named the *Spanning Tree Set Method*, based on the number of spanning trees, is treated to determine the exact value of network reliability for predicting the weakness of a system and giving suitable emergency management operations under faulty situations. This method is proposed by Bistouni and Jahanshahi (2017). Suppose a network G with V nodes and E edges. Also, suppose that the set of spanning trees in the network is $S = \{S_1, S_2, ..., S_{\tau(G)}\}$, where $\tau(G)$ is the number of spanning trees of G. We assume that each link in the network may fail and the failures are assumed to be exponentially distributed. Let λ be as the failure rate of a link, then the link reliability is given by $R_E(t) = e^{-\lambda t}$. Operating time t should be considered. The reliability of the entire network is the probability of the union of its spanning trees:

$$\begin{aligned} R_G &= P(S_1 \cup S_2 \cup ... \cup S_{\tau(G)}) \\ &= \sum_{i=1}^{\tau(G)} P(S_i) - \sum_{i<j} P(S_i \cap S_j) + \sum_{i<j<k} P(S_i \cap S_j \cap S_k) - ... + (-1)^{\tau(G)-1} P\left(\bigcap_{i=1}^{\tau(G)} S_i\right). \end{aligned} \tag{2.2}$$

The main advantage of this approach is that it divides the main problem into solvable subproblems and it is applicable to any type of network with any amount of topological complexity. The important issue of this method is about the computational complexity. In the calculation of the reliability, it should be noted that the main function in this method is related to counting the number of spanning trees of the network. Thanks to a lot of research works done in this field, especially in the domain of complex networks, the spanning tree enumeration can be done in a linear complexity of $O(n)$ where n is the number of nodes in the network. In this chapter, we will investigate almost all the methods of the calculation of the number of spanning trees of a network with different complexities.

Example 2.1.1. *We consider a network G in Figure 2.1 (a), which is a network with a relatively simple structure. This network has three spanning trees that have been shown in Figure 2.1 (b): S_1, S_2, S_3. This example is taken from (Bistouni and Jahanshahi, 2017).*

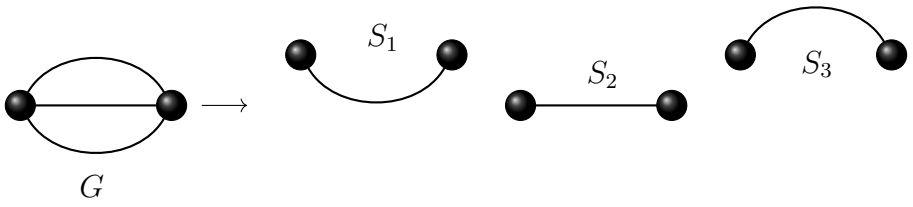


Figure 2.1: (a) A network G with three links and (b) The three spanning trees of G.

Therefore, considering 2.2, and assuming that $R_E(t) = e^{-\lambda t}$, we have:

$$\begin{aligned} R_G &= P(S_1 \cup S_2 \cup S_3) \\ &= P(S_1) + P(S_2) + P(S_3) - P(S_1 \cap S_2) - P(S_1 \cap S_3) - P(S_2 \cap S_3) + P(S_1 \cap S_2 \cap S_3) \\ &= 3e^{-\lambda t} - 3e^{-2\lambda t} + e^{-3\lambda t}. \end{aligned} \tag{2.3}$$

let's assume that the link failure rate (λ) is equal to $0,000001$ per hour. Then, the link reliability is $R_E(t) = 0.9999990000005$. Therefore, according to 2.3, the exact value of the reliability of G is: $R_G = 1$.

2.2 Algebraic methods for enumerating spanning trees of a network

Algebraic graph theory is a branch of mathematics that studies graphs by using algebraic properties of associated matrices. There are some known algebraic methods for calculating the number of spanning trees of a network.

2.2.1 Matrix-Tree Theorem

A general method to calculate the number of spanning trees of a network G is using linear algebra. The first study of the number of spanning trees was initiated by the physicist Kirchhoff (1847) who proposed a theorem called the "Matrix-Tree Theorem" based on the determinant of a submatrix of the Laplacian matrix corresponding to the network (Chaiken and Kleitman, 1978; Merris, 1994) defined by $L_G = D_G - A_G$ with A_G is the adjacency matrix of G and D_G its degree matrix (See Section 1.2.1.3).

Theorem 2.2.1. *(Kirchhoff, 1847) The Kirchhoff Matrix-Tree Theorem computes the number of spanning trees of a graph G by:*

$$\tau(G) = (-1)^{i+j} \det L^*(G). \tag{2.4}$$

Where $L^*(G)$ is a matrix obtained by deleting row i and column j of the Laplacian matrix $L(G)$.

➥ **Advantages and constraints of Kirchhoff Matrix-Tree Theorem:**

Matrix-Tree Theorem is the first and general method to calculate the number of spanning trees of a network, it provides the exact value of this number and gives useful results for networks containing a small number of vertices, but it is practically not efficient for real networks having a large number of nodes and edges, because the calculation of the determinant of their Laplacian matrix is very difficult and even impossible. Recently, there has been much interest in studying this problem and finding alternative methods to avoid the calculations of the largest determinant of the Kirchhoff matrix. Some of these methods have been proposed in this chapter: Combinatorial methods and geometric approaches.

2.2.2 Eigenvalues method

Another interpretation of Kirchhoff's Theorem is given by the set of non-zero eigenvalues of the Laplacian matrix of the graph G. It is simple and easy to calculate the number of spanning trees of graphs using the eigenvalues method just for simple graphs with a small number of vertices. However, for a large and complex network, this method becomes expensive or sometimes impossible.

Theorem 2.2.2. *(Kelmans and Chelnokov, 1974) For a given undirected connected graph G with n vertices, let $\lambda_1, ..., \lambda_{n-1}$ be the non-zero eigenvalues of $L(G)$. Then, the number of spanning trees $\tau(G)$ is equal to:*

$$\tau(G) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i \tag{2.5}$$

Remark 2.2.1. *There are some faster methods to count the number of spanning trees in some special graphs:*

- *If G is a graph with loops, then to count $\tau(G)$, we first neglect all the loops in G, because they have no contribution to the construction of a spanning tree.*
- *If G is a tree, then $\tau(G) = 1$.*
- *if G is a disconnected graph, then $\tau(G) = 0$.*
- *If G is a cyclic graph with n vertices, then $\tau(G) = n$.*
- *If G is a complete graph with n vertices, Cayley's formula can be used: $\tau(K_n) = n^{n-2}$ (Cayley, 1889). As an example, the number of spanning trees of the complete graph K_4 is $\tau(k_4) = 4^{4-2} = 16$.*

2.3 Combinatorial methods for enumerating spanning trees of a network

In this section, we investigate some combinatorial methods that facilitate the computation of the complexity of a network containing a large number of vertices and edges without using the determinant of the Laplacian matrix.

2.3.1 Deletion and Contraction methods

The deletion and the contraction methods derive recursive functions that enumerate spanning trees of a planar graph containing a simple edge e. In this section, we will present the results, the demonstrations of this method and its generalization.

Definition 2.3.1. *Let G be a planar graph and $e = v_1v_2 \in E(G)$ an edge of G:*

- ***Deletion method:*** *The graph $G - e$ is the result of the deletion of the edge e of G.*
- ***Contraction method:*** *The graph $G.e$ is obtained after the contraction of the edge e of G (See Figure 2.2).*

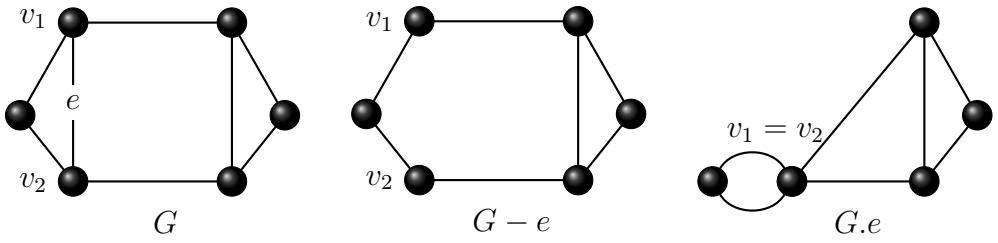


Figure 2.2: Graphs G, $G - e$ and $G.e$.

2.3.1.1 Fussner's formula

The deletion and the contraction methods are important results in the enumerative combinatorics, that are due to the origin Feussner (1902), which allows the enumeration of spanning trees of a planar graph.

Theorem 2.3.1. ***(Fussner's formula)*** *Feussner (1902) Let G be a planar graph. If $e = v_1v_2 \in E_G$ (See Figure 2.2). The number of spanning trees of G is given by the following formula:*

$$\tau(G) = \begin{cases} 1 & \text{If } G \text{ does not contain any edges,} \\ \tau(G - e) & \text{If } e \text{ is a loop,} \\ \tau(G.e) & \text{If } e \text{ is a deletion edge,} \\ \tau(G - e) + \tau(G.e) & \text{If } e \text{ is a simple edge.} \end{cases} \tag{2.6}$$

Example 2.3.1. *Consider the graph G shown in Figure 2.3. To apply Fussner's formula, we recursively remove and contract edges until we get tree or single vertex (loops are not considered). We follow the process of removing and contracting edges as shown in Figure 2.3. The total number of these trees and single vertices is equal to the total number of spanning trees of the graph G. According to this example, the graph G has 5 spanning trees.*

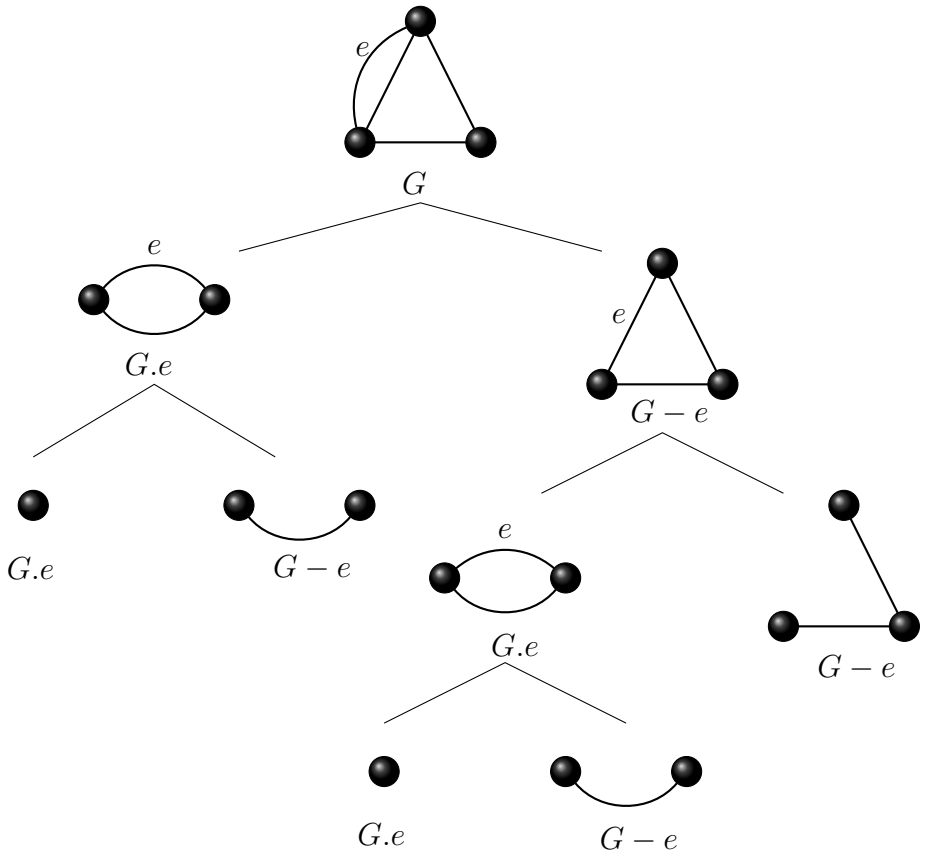


Figure 2.3: The process of removing and contracting edges of the graph G.

➥ Advantages and constraints of deletion and contraction methods

The deletion and the contraction methods are used to calculate the number of spanning trees of several planar graphs such as Fan graph, Wheel graph, etc. This method has several advantages such that it is easy to use because it relies on simple operations applied to the graph, the deletion or contraction edge can be any edge of the graph and the computational complexity of this method is linear $O(n)$. However, the only disadvantage is that the use of this method concerns only graphs containing a simple edge. Therefore, we investigate its generalization.

2.3.1.2 Generalization of Fussner's formula

In the paper of Modabish and El Marraki (2011), they gave a generalization of the deletion and contraction methods by considering a planar graph containing a simple path, which all its vertices have strictly a degree equal two, except the two vertices of ends (See Figure 2.4).

Definition 2.3.2. *Let G be a planar graph and $P = \{v_0, v_1, ..., v_k\}$ be a simple path of G with length k:*

- ***Deletion method:*** *The graph $G - P$ is the result of the deletion of the path P.*
- ***Contraction method:*** *The graph $G.P$ is obtained after contracting the path P (the contraction of adjacent vertices v_0 and v_k) (See Figure 2.4).*

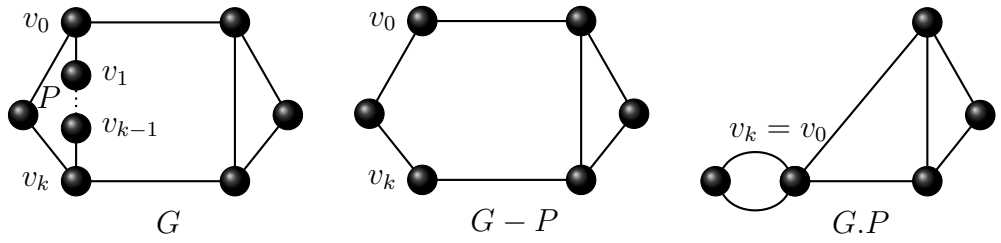


Figure 2.4: Graphs G, $G - P$ and $G.P$.

Theorem 2.3.2. ***Generalization of Fussner's formula*** *(Modabish and El Marraki, 2011) Let G be a planar graph. If P is a path with vertices $(v_0, v_1, ..., v_k)$, (See Figure 2.4). Then, the number of spanning trees of G is given by the following formula:*

$$\tau(G) = k \times \tau(G - P) + \tau(G.P) \tag{2.7}$$

2.3.2 Decomposition methods for enumerating spanning trees of a network

In this section, we rely on the principle of a process of "Divide and Conquer", which divides a problem recursively in sub-problems, solves each of this sub-problems and then merges the partial results for a general solution. An example of this technique is the decomposition method: to calculate the number of spanning trees of a planar network, we follow this algorithm:

1. First, we decompose the original graph into different subgraphs that are connected by one node, two nodes, an edge or a path.

2. Then, we calculate the number of spanning trees for each subgraph.

3. Finally, we collect the results to obtain the complexity of the original graph.

The use of this technique is due to its ease to discover the spanning trees of a complex network. We must search the best way to decompose the graph. We have several possibilities to do it, as we can decompose the graph into different subgraphs according to certain constraints: by following one node, two nodes, an edge, a path, etc. In this section, we will study each case.

2.3.2.1 Number of spanning trees of a planar graph of type $G = C_1 \bullet C_2$

First, we begin by studying the case where the subgraphs C_1 and C_2 are connected by a single vertex.

Definition 2.3.3. *Let $G = C_1 \bullet C_2$ be a planar graph obtained by connecting C_1 and C_2 with one vertex v_1. i.e., C_1 and C_2 are connected subgraphs which intersect exactly in one vertex v_1. We say that v_1 is an articulation node if its deletion disconnects the subgraphs C_1 and C_2 (See Figure 2.5).*

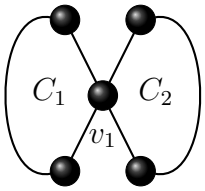


Figure 2.5: A graph $G = C_1 \bullet C_2$ with an articulation node v_1

Property 2.3.1. *Let G be a planar graph of type $G = C_1 \bullet C_2$:*

- *C_1 and C_2 have a common vertex v_1 and a common face (the external face).*
- *$V_G = V_{C_1} + V_{C_2} - 1$, $E_G = E_{C_1} + E_{C_2}$ and $F_G = F_{C_1} + F_{C_2} - 1$.*
- *If we remove the vertex v_1 of the graph G, the resulting graph is not connected.*
- *A path from a vertex of C_1 to a vertex of C_2 must pass through v_1.*

Theorem 2.3.3. *(Modabish and El Marraki, 2011) Let $G = C_1 \bullet C_2$ be a planar graph. C_1 and C_2 two subgraphs are connected with one vertex v_1 (See Figure 2.5). Then, the number of spanning trees of G is given by:*

$$\tau(G) = \tau(C_1 \bullet C_2) = \tau(C_1) \times \tau(C_2). \tag{2.8}$$

2.3.2.2 Number of spanning trees of a planar graph of type $G = C_1 \bullet C_2 \bullet ... \bullet C_n$

Now, we are interested in a planar graph G composed in many subgraphs C_i, which are connected by one vertex. This method is considered as the generalized decomposition method following one articulation node.

Theorem 2.3.4. ***(Generalization of Theorem 2.3.3)*** *(Modabish et al., 2011) Let G be a chain of planar graphs defined by $G = C_1 \bullet C_2 \bullet ... \bullet C_n$ (See Figure 2.6). The number of spanning trees of G is given by the following formula:*

$$\tau(G) = \prod_{i=1}^{n} \tau(C_i). \tag{2.9}$$

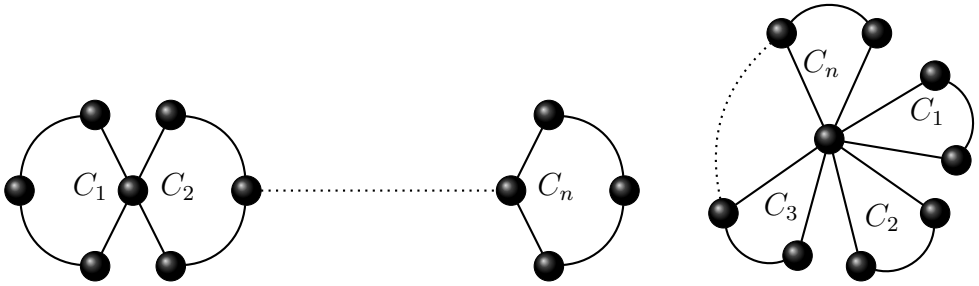


Figure 2.6: Chain network and Star network

2.3.2.3 Number of spanning trees of a planar graph of type $G = C_1 : C_2$

In this section, we are interested in a planar graph G composed in two subgraphs having two common vertices. Let G be a planar graph as $G = C_1{:}C_2$ with C_1 and C_2 two subgraphs which are connected by two vertices v_1, v_2, such that if a path connects a vertex of C_1 and a vertex of C_2, it must pass through one of the two common vertices v_1 and v_2 (See Figure 2.7).

Property 2.3.2. *Let G be a planar graph of type $G = C_1{:}C_2$:*

- *C_1 and C_2 have two common vertices v_1, v_2 and a common face (the external face).*
- *$V_G = V_{C_1} + V_{C_2} - 2$, $E_G = E_{C_1} + E_{C_2}$ and $F_G = F_{C_1} + F_{C_2}$.*
- *If we remove the vertex v_1 and v_2 of the graph G, the resulting graph is not connected (Nishizeki and Rahman, 2004).*
- *A path that connects a vertex of C_1 and a vertex of C_2 must pass through v_1 or v_2.*

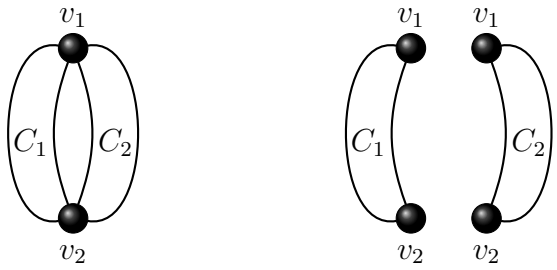


Figure 2.7: A planar graph $G = C_1{:}C_2$

Theorem 2.3.5. *(Modabish and El Marraki, 2011) Let $G = C_1{:}C_2$ be a planar graph where ":" represents a common pair (v_1, v_2) between two subgraphs C_1 and C_2 . The number of spanning trees of G is given by the following formula:*

$$\tau(G) = \tau(C_1) \times \tau(C_2.v_1v_2) + \tau(C_1.v_1v_2) \times \tau(C_2) \tag{2.10}$$

2.3.2.4 Number of spanning trees of a planar graph of type $G = C_1|C_2$

Now, we are interested in the graph of type $G = C_1|C_2$, such that v_1 and v_2 two vertices of G connected by an edge e (See Figure 2.8).

Property 2.3.3. *Let G be a planar graph of type $G = C_1|C_2$:*

- *C_1 and C_2 have two common vertices v_1, v_2 a common edge e and a common face (the external face).*
- *$V_G = V_{C_1} + V_{C_2} - 2$, $E_G = E_{C_1} + E_{C_2} - 1$ and $F_G = F_{C_1} + F_{C_2} - 1$.*

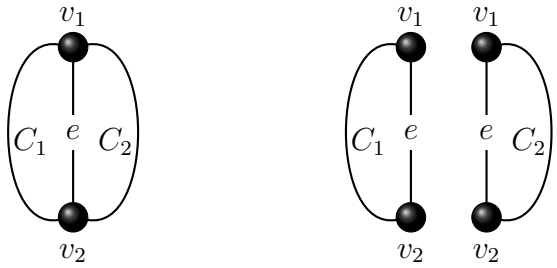


Figure 2.8: A planar graph $G = C_1|C_2$

Theorem 2.3.6. *(Modabish and El Marraki, 2011) Let G be a planar graph of type $G = C_1|C_2$ and v_1 and v_2 two vertices of C_1 and C_2 connected by an edge e (See Figure 2.8). Then, the number of spanning trees of G is given by the following formula:*

$$\tau(G) = \tau(C_1) \times \tau(C_2) - \tau(C_1 - e) \times \tau(C_2 - e). \tag{2.11}$$

2.3.2.5 Number of spanning trees of a planar graph of type $G = C_1 \ddagger C_2$

Now, we are interested in the planar graph G, which contains a simple path P with length k in common between C_1 and C_2, we denote $G = C_1 \ddagger C_2$ (See Figure 2.9).

Property 2.3.4. *Let G be a planar graph of type $G = C_1 \ddagger C_2$ where $\ddagger$ a simple path that contains $k+1$ vertices and k edges, then:*

- *C_1 and C_2 have $k+1$ common vertices $v_1, v_2, ..., v_k, v_{k+1}$, k common edges (simple path P) and a common face (the external face).*
- *$V_G = V_{C_1} + V_{C_2} - (k+1)$, $E_G = E_{C_1} + E_{C_2} - k$ and $F_G = F_{C_1} + F_{C_2} - 1$.*

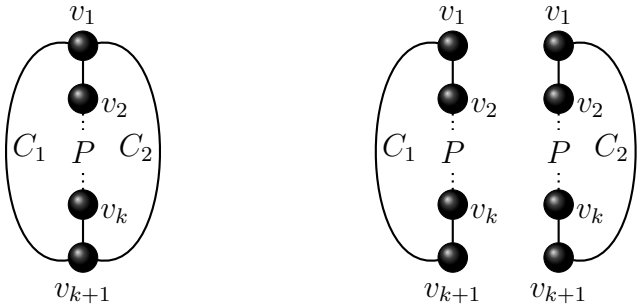


Figure 2.9: A planar graph $G = C_1 \ddagger C_2$

Theorem 2.3.7. ***(Generalization of Theorem 2.3.6)*** *(Lotfi et al., 2012) Let G be a planar graph of type $G = C_1 \ddagger C_2$. v_1 and v_{k+1} two vertices of G connected by a simple path $P = \{v_1, v_2, ..., v_k, v_{k+1}\}$ that contains k edges (See Figure 2.9). Then, the number of spanning trees of G is given by the following formula:*

$$\tau(G) = \tau(C_1) \times \tau(C_2) - k^2 \tau(C_1 - P) \times \tau(C_2 - P). \tag{2.12}$$

➥ **Advantages and constraints of decomposition methods**

The decomposition method is a method of "divide and conquer" based on simple operations. It allows the derivation of recursive functions that evaluate the complexity of a large planar graph and the only disadvantage is that this method is limited for planar graphs containing one node, two nodes, an edge or a path in common between subgraphs.

2.3.2.6 Application: A Book network with a common edge e

In general, the Book network is defined as the graph cartesian product $S_{n+1} \times P_2$ where S_{n+1} is a star network with n vertices of degree 1, one vertex of degree n and P_2 is the path graph of 2 vertices. There are two types of a Book network: a Book network with a common edge and a Book network with a common path. In this section, as an application of the decomposition methods, we focus on a Book network with a common edge, which is composed of a number of subnets having arbitrary two nodes in common and one edge e, denoted as $B_{m,p}$ where m number of p-cycles ($m \geq 1$, $p \geq 3$) with a common edge e (See Figure 2.10).

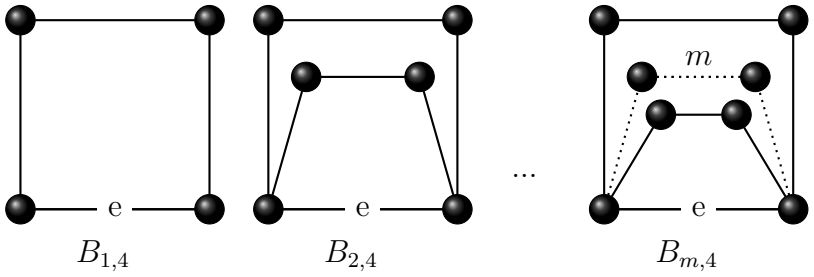


Figure 2.10: The representation of Book networks with a common edge e.

We calculate its number of spanning trees based on some combinatorial methods: deletion and contraction methods by deleting an edge and contracting two vertices and

the decomposition method by following two nodes. In the end, we enumerate the spanning trees of a chain of Book network $C_{m,p,n}$ using the generalized decomposition method by following one node (See Figure 2.11). The results of this application were presented in (Mokhlissi et al., 2015a)

Theorem 2.3.8. *Let $B_{m,p}$ be a Book network where m number of p-cycles ($m \geq 1$, $p \geq 3$). The number of spanning trees of the Book network $B_{m,p}$ is given by:*

$$\tau(B_{m,p}) = \frac{-m(p-2)(p-1) + (p-1)^{(m+1)} - (p-1)}{(p-2)^2} + (p-1)^{(m-1)} + m(p-1) \quad (2.13)$$

Proof: We use Equation 2.3.1, we obtain $\tau(B_{m,p}) = \tau(B_{m,p} - e) + \tau(B_{m,p}.e)$, then:

$$\tau(B_{m,p}) = \tau(B_{m,p} - e) + m \times (p-1) \quad (2.14)$$

We calculate the complexity of $B_{m,p} - e$ by using Theorem 2.3.5, we obtain $\tau(B_{m,p} - e) = \tau(B_{1,p} - e) \times \tau(B_{m-1,p}.e) + \tau(B_{m-1,p} - e) \times \tau(B_{1,p}.e)$, then:

$$\begin{aligned}
\tau((B_{m,p} - e) &= (m-1)(p-1) + (p-1) \times \tau(B_{m-1,p} - e) \\
\tau(B_{m-1,p} - e) &= (m-2)(p-1) + (p-1) \times \tau(B_{m-2,p} - e) \\
&\vdots \\
\tau(B_{2,p} - e) &= (p-1) + (p-1) \times \tau(B_{1,p} - e) \; with \; \tau(B_{1,p} - e) = 1
\end{aligned}$$

We multiply the equation of $\tau(B_{m-1,p} - e)$ by $(p-1)$, the equation of $\tau(B_{m-2,p} - e)$ by $(p-1)^2$ and so on until the last equation $\tau(B_{2,p} - e)$, which will be multiplied by $(p-1)^{(m-2)}$. Summing all the obtained equations, we can find this formula.

$$\begin{aligned}
\tau(B_{m,p} - e) &= (m-1)(p-1) + (m-2)(p-1)^2 + (m-3)(p-1)^3 + ... + \\
&\quad 2(p-1)^{(m-2)} + (p-1)^{(m-1)} + (p-1)^{(m-1)}. \\
\tau(B_{m,p} - e) &= (p-1)^{(m-1)} \left[\frac{1}{(p-1)^0} + \frac{2}{(p-1)^1} + \frac{3}{(p-1)^2} + ... + \right. \\
&\quad \left. \frac{(m-3)}{(p-1)^{(m-4)}} + \frac{(m-2)}{(p-1)^{(m-3)}} + \frac{(m-1)}{(p-1)^{(m-2)}} \right] + (p-1)^{(m-1)}.
\end{aligned}$$

We have $g(x) = 1 + x + x^2 + x^3 + ... + x^{n-1} = \frac{1-x^n}{1-x}$. Then, $g'(\frac{1}{x}) = 1 + \frac{2}{x} + \frac{3}{x^2} + ... + \frac{n-1}{x^{n-2}}$

We can see that $\tau(B_{m,p}-e)=(p-1)^{(m-1)}\times g'(\frac{1}{p-1})+(p-1)^{(m-1)}$, with $g'(x)=\frac{-nx^{n-1}(1-x)+1-x^n}{(1-x)^2}$. First, we calculate $g'(\frac{1}{p-1})$, and we replace it in $\tau(B_{m,p}-e)$:

$$\tau(B_{m,p}-e)=\frac{-m(p-2)(p-1)+(p-1)^{(m+1)}-(p-1)}{(p-2)^2}+(p-1)^{(m-1)}, m\geq 1, p\geq 3 \tag{2.15}$$

We replace the formula 2.15 of $\tau(B_{m,p}-e)$ in $\tau(B_{m,p})$ of Equation 2.14, then the result.

Let $C_{m,p,n}$ be a chain of Book networks, it is composed of n Book networks connected by an articulation point where m number of p-cycles ($m\geq 1$, $p\geq 3$, $n\geq 2$) (See Figure 2.11).

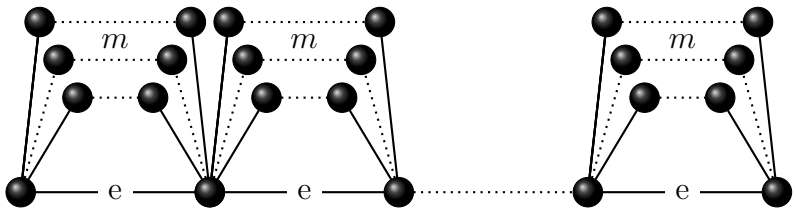


Figure 2.11: A chain of Book networks $C_{m,p,n}$

Theorem 2.3.9. *Let $C_{m,p,n}$ be a chain, composed of n Book networks with m number of p-cycles. The number of spanning trees of $C_{m,p,n}$ is given by the following formula, for $m\geq 1$, $p\geq 3$, $n\geq 2$:*

$$\tau(C_{m,p,n})=\left[\frac{-m(p-2)(p-1)+(p-1)^{(m+1)}-(p-1)}{(p-2)^2}+(p-1)^{(m-1)}+m(p-1)\right]^n \tag{2.16}$$

Proof: We use Equation 2.3.4. We obtain $\tau(C_{m,p,n})=(\tau(B_{m,p}))^n$. We replace in $\tau(C_{m,p,n})$ the formula of $\tau(B_{m,p})$ of Theorem 2.3.8, then the result.

2.3.3 Contraction methods for enumerating spanning trees of a network

The contraction method is defined by removing an edge and connecting two vertices. It is used to derive recursive functions to facilitate the enumeration of spanning trees of several classes of planar graphs. In this section, we propose a technique based on the contraction of vertices to calculate the number of spanning trees for a closed chain of planar networks.

We study two families of a closed chain: a closed chain of the same planar networks L_n (See Figure 2.12) and a closed chain of different types of planar networks G_n (See Figure 2.14). We consider this latter as the generalization of the first closed chain. The results of this section were presented in (Mokhlissi et al., 2015b).

2.3.3.1 Contraction method for a closed chain of the same planar networks

We consider a closed chain L_n of the same planar networks C (See Figure 2.12). It is defined by n planar networks C connected by articulation points. This type of representation has been used for many types of networks, such as Star Flower networks (Modabish and El Marraki, 2012), maximal planar networks (Lotfi et al., 2014), Pseudofractal Scale-Free networks (Lotfi et al., 2014), Fan networks (Mokhlissi and El Marraki, 2014) and Wheel networks (Mokhlissi and El Marraki, 2014). We propose a recursive approach based on the contraction of nodes. As an application, we obtain the explicit expression for the number of spanning trees in the closed chain of the same cyclic network (See Figure 2.13).

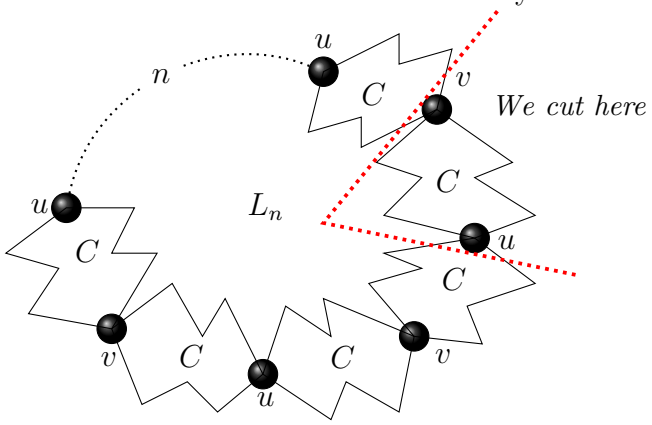


Figure 2.12: A closed chain L_n of the same planar networks C.

Theorem 2.3.10. *(Lotfi et al., 2014) Let L_n be a closed chain with the same planar networks. It's formed by a set of n planar networks C connected by articulation points with u and v two cut vertices of C (See Figure 2.12). The number of spanning trees of L_n is given by the following formula:*

$$\tau(L_n) = n \times \tau(C)^{n-1} \times \tau(C.uv) \tag{2.17}$$

2.3.3.1.1 Application: A closed chain of the same cyclic networks

As an application of the contraction method for a closed chain of the same planar networks, we provide a theoretical study of the number of spanning trees for a closed chain of the same cyclic networks $L_{n,m}$, which is a set of cyclic networks C connected by cut vertices (See Figure 2.13).

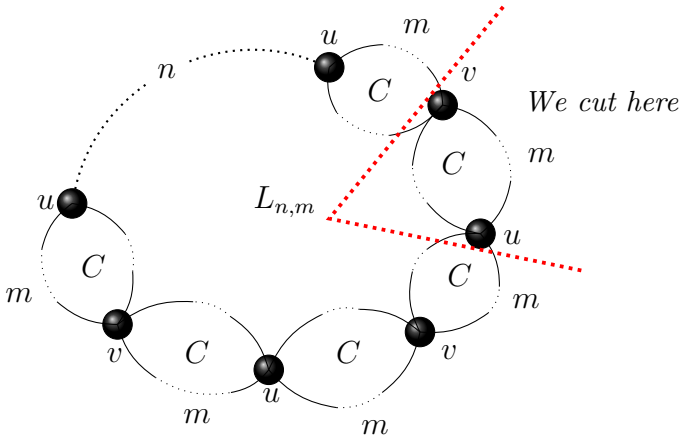


Figure 2.13: A closed chain $L_{n,m}$ of the same cyclic networks

Theorem 2.3.11. *(Mokhlissi et al., 2015b) Let $L_{n,m}$ be a closed chain composed by n of $m-$cyclic networks C connected by articulation points and u and v two cut vertices of C. The number of spanning trees of $L_{n,m}$ is given by the following formula:*

$$\tau(L_{n,m}) = n \times m^{n-1} \times (m-1). \tag{2.18}$$

Proof: Let $L_{n,m}$ be a closed chain formed by n of $m-$cyclic networks C with u and v two cut vertices of $L_{n,m}$. Using Theorem 2.3.10, we get: $\tau(L_{n,m}) = n \times \tau(C)^{n-1} \times \tau(C.uv)$. C is a $m-$cyclic network, then its number of spanning trees is equal to its length: $\tau(C) = m$, then the result.

2.3.3.2 Contraction method for a closed chain of different types of planar networks

In this section, we study another family of a closed chain G_n with n different types of planar networks C_i connected by articulation points (See Figure 2.14). This type of network presents a generalization of the representation of the precedent closed chain (See Figure 2.12). Then, we propose our theoretical analysis based on the contraction method

to enumerate spanning trees for this family of a closed chain. As an application, we obtain the exact formula of the number of spanning trees for a closed chain with different multigraphs (See Figure 2.15).

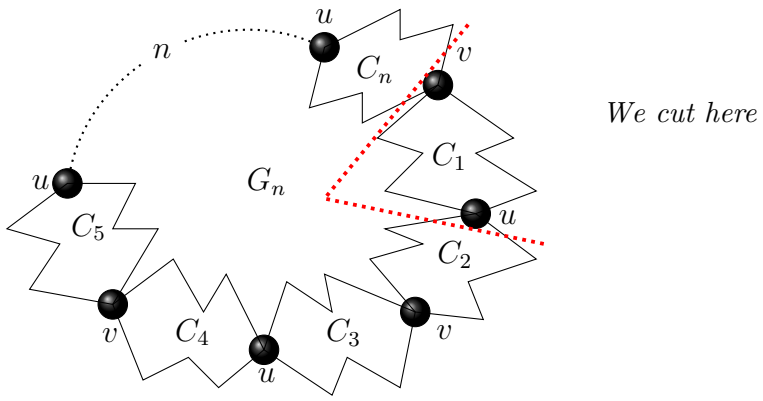


Figure 2.14: A closed chain G_n of different types of planar networks C_i

Theorem 2.3.12. *(Mokhlissi et al., 2015b) Let G_n be a closed chain with n different types of planar networks $C_1, C_2, C_3, ..., C_n$ having two vertices u and v in common (See Figure 2.14). The number of spanning trees of G_n is given by the following formula:*

$$\tau(G_n) = \tau(C_1.uv) \times \left[\sum_{k=0}^{n-1} \prod_{i=2}^{n-k} \tau(C_i) \times \tau(C_1)^k\right] \tag{2.19}$$

Proof: Let G_n be a closed chain composed of n different types of planar networks $C_1, C_2, C_3..., C_n$. We use Theorems 2.3.5 and 2.3.4 by cutting C_1 of G_n and applying the contraction method (See Figure 2.14). We get

$$\begin{aligned}
\tau(G_n) &= \tau(C_1) \times \tau(G_{n-1}) + \tau(C_1.uv) \times \prod_{i=2}^{n} \tau(C_i) \\
\tau(G_{n-1}) &= \tau(C_1) \times \tau(G_{n-2}) + \tau(C_1.uv) \times \prod_{i=2}^{n-1} \tau(C_i) \\
&\vdots \\
\tau(G_2) &= \tau(C_1) \times \tau(G_1) + \tau(C_1.uv) \times \tau(C_2) \\
\tau(G_1) &= \tau(C_1.uv).
\end{aligned}$$

We multiply the equation of $\tau(G_{n-1})$ by $\tau(C_1)$, the equation of $\tau(G_{n-2})$ by $\tau(C_1)^2$ and so on until the last equation $\tau(G_3)$ which will be multiplied by $\tau(C_1)^{n-3}$ and the equation

of $\tau(G_2)$ which will be multiplied by $\tau(C_1)^{n-2}$. Summing all the obtained equations, we can find this formula.

$$\begin{aligned}\tau(G_n) =&\tau(C_1)^{n-1}\tau(C_1.uv) + \tau(C_1.uv)\Big[\prod_{i=2}^{n}\tau(C_i)\tau(C_1)^0 + \prod_{i=2}^{n-1}\tau(C_i)\tau(C_1)^1 + ...+ \\ &\tau(C_2)\tau(C_1)^{n-2}\Big] \\ =&\tau(C_1.uv)\left[\prod_{i=2}^{n}\tau(C_i)\tau(C_1)^0 + \prod_{i=2}^{n-1}\tau(C_i)\tau(C_1)^1 + ... + \tau(C_2)\tau(C_1)^{n-2} + \tau(C_1)^{n-1}\right]\end{aligned}$$

Then, the result.

2.3.3.2.1 Application: A closed chain with different multigraphs

As an application of the contraction method for a closed chain of different types of planar networks, we examine the number of spanning trees for a closed chain K_n, which is composed of n multigraphs $C_1, C_2, C_3, ..., C_n$ having only two vertices u and v in common. Each multigraph is characterized by a number of distinct edges joining the same pair of common nodes. The transition from one to the next multigraph is done by adding an edge connecting u and v (See Figure 2.15).

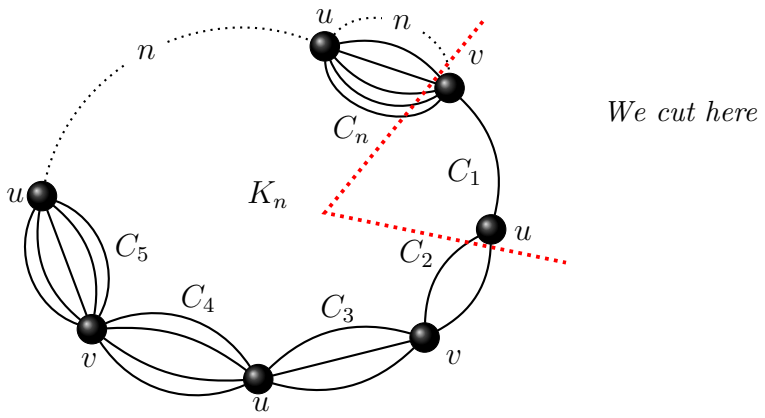


Figure 2.15: A closed chain K_n with different multigraphs

Theorem 2.3.13. *(Mokhlissi et al., 2015b) Let K_n be a closed chain, composed of n multigraphs $C_1, C_2, C_3, ..., C_n$ having only two vertices u and v in common (See Figure 2.15). The formula to calculate the number of spanning trees of K_n is as follow:*

$$\tau(K_n) = \sum_{i=1}^{n} i! \tag{2.20}$$

Proof: Let K_n be a closed chain, composed of n multigraphs $C_1, C_2, C_3, ..., C_n$ with u and v two cut vertices of C_i. Using Theorem 2.3.12, we get $\tau(K_n) = n! + (n-1)! + (n-2)! + (n-3)! + \cdots + 3! + 2! + 1!$, with $\tau(C_1) = 1$. Then the result.

➥ **Advantages and constraints of the contraction method**

The contraction method allows deriving recursive functions calculating the number of spanning trees of a large family of planar graphs, but it requires the presence of a pair of separation. Therefore, we search more genericity by applying geometric transformations.

2.4 Geometric approaches for enumerating spanning trees of a network

In this section, we propose four approaches that are based on geometrical transformations of the original network to facilitate the calculation of the number of spanning trees for complex and large networks, namely duality, bipartition, reduction approaches and the electrically equivalent technique. They change the geometric nature of a network and its complexity is calculated according to that of the network obtained after some geometric transformations. We begin by enumerating the spanning trees of the dual of a planar network. Then the bipartite, the k-partite network, the reduced and the k-reduced network according to that of the initial network. Finally, we show the relevance of these approaches by applying them to a planar network such as the Book network with common path l. In addition, we investigate the technique of electrically equivalent transformations, we give its definition, we illustrate it by an example and we explain its methodology by providing an algorithm for each transformation. The aim of these approaches is the evaluation of the number of spanning trees of large and complex networks which cannot be found by using the existing methods such as the contraction, the deletion and the decomposition, etc. Some results of this section were presented in (Mokhlissi et al., 2018a).

2.4.1 Duality

Duality is a very important mathematical concept that has several uses including linear algebra, functional analysis, group theory and graph theory (Lang, 2002; Nishizeki and Rahman, 2004; Caspard et al., 2007). It is a geometric transformation of a graph into a

dual graph. It concerns the family of planar graphs and facilitates the calculation of their number of spanning trees because there is an equality relation between the complexity of a planar graph and that of its dual. We begin by defining this approach and determining its structural properties. Then, we show the relationship between the number of spanning trees of a graph and that of its dual. Finally, we illustrate this concept by an example.

Definition 2.4.1. *Let G be a planar graph, we can construct the dual of G, denoted as G^*, as follows:*

- *To construct the set of vertices V^* of G^*, a vertex v_i^* will be placed in each middle of faces F_i of the graph G.*
- *For each edge e of G, we draw an edge e^* of G^*, which intersects the edge e (but not another edge of G) and connects the vertices v_i^*, which are in the faces F_i delimited by e. The edges e^* of G^* are called the dual edges of e of G.*

Property 2.4.1. *Let $G^*(V_G^*, E_G^*, F_G^*)$ be the dual graph of a planar graph $G(V_G, E_G, F_G)$.*

- *For each vertex v of G, there is a corresponding face in G^*. Then, $|V_G| = |F_G^*|$.*
- *For each edge e of G, there is a corresponding edge in G^*. Then, $|E_G| = |E_G^*|$.*
- *For each face f of G, there exists a corresponding vertex in G^*. Then, $|F_G| = |V_G^*|$.*
- *The dual graph of a planar graph is also planar.*
- *A dual graph is always connected.*
- *A dual graph may have loops and multiple edges even if the initial graph is simple.*
- *A planar graph may not have a unique dual if it has several planar representations.*

Theorem 2.4.1. *(Lotfi et al., 2015) Let $G^*(V_G^*, E_G^*, F_G^*)$ be the dual graph of $G(V_G, E_G, F_G)$. The number of spanning trees of G is equal to the number of spanning trees of G^*:*

$$\tau(G) = \tau(G^*) \tag{2.21}$$

Example 2.4.1. *Let $G(V_G, E_G, F_G)$ be a planar graph. The dual graph $G^*(V_G^*, E_G^*, F_G^*)$ is represented as follows: the edges are illustrated by the red dashes and the vertices are drawn by red circles (See Figure 2.16)*

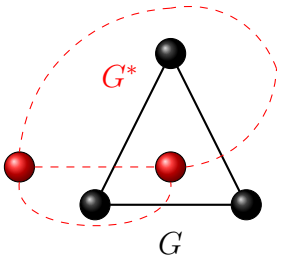


Figure 2.16: A graph G and its dual G^*.

2.4.2 Bipartition Approach

The bipartite graphs belong to one of the simplest families in graph theory. They are used in various areas such as Semantic Web, Data Mining and segmentation of images (Zha et al., 2001; Rege et al., 2006; Li et al., 2012). The bipartition approach is characterized by the presence of nodes. Its main objective is to reduce the number of nodes to simplify the enumeration of spanning trees for planar networks having a large number of nodes. We start by defining this approach and determining its properties. Then, we propose a theorem to calculate the number of spanning trees of the bipartite network according to that of the origin network and its number of faces. Finally, we generalize these results for the case of the k-partite network.

Definition 2.4.2. *A graph becomes bipartite when we add a new vertex between two directly connected vertices, denoted by $B_2(G)$ (See Figure 2.17).*

Property 2.4.2. *Let G be a planar graph and $B_2(G)$ its bipartite graph.*

- *The number of vertices in $B_2(G)$ is given by $|V_{B_2(G)}| = |V_G| + |E_G|$,*
- *The number of edges is given by $|E_{B_2(G)}| = 2|E_G|$,*
- *The number of faces is $|F_{B_2(G)}| = |F_G|$,*
- *The average degree is $< z >_{B_2(G)} = \frac{2|E_{B_2(G)}|}{|V_{B_2(G)}|} = \frac{4|E_G|}{|V_G|+|E_G|}$.*

Theorem 2.4.2. *(Lotfi et al., 2015) Let $B_2(G)$ be a bipartite graph of a planar graph G. The number of spanning trees in $B_2(G)$ is given by:*

$$\tau(B_2(G)) = 2^{|F_G|-1}\tau(G) \tag{2.22}$$

Definition 2.4.3. ***(Generalization of the bipartition approach)*** *A k-partite graph of a planar graph G is defined by adding $k-1$ new vertices in each edge to get k new edges, denoted by $B_k(G)$ (See Figure 2.17).*

Property 2.4.3. *Let G be a planar graph and $B_k(G)$ its k-partite graph.*

- *The number of vertices in $B_k(G)$ is given by $|V_{B_k(G)}| = |V_G| + (k-1)|E_G|$,*
- *The number of edges is given by $|E_{B_k(G)}| = k|E_G|$,*
- *The number of faces is $|F_{B_k(G)}| = |F_G|$,*
- *The average degree is $< z >_{B_k(G)} = \frac{2|E_{B_k(G)}|}{|V_{B_k(G)}|} = \frac{2k|E_G|}{|V_G|+(k-1)|E_G|}$.*

Theorem 2.4.3. *(Lotfi et al., 2015) Let $B_k(G)$ be a k-partite graph of a planar graph G. The number of spanning trees of $B_k(G)$ is given by:*

$$\tau(B_k(G)) = k^{|F_G|-1}\tau(G) \tag{2.23}$$

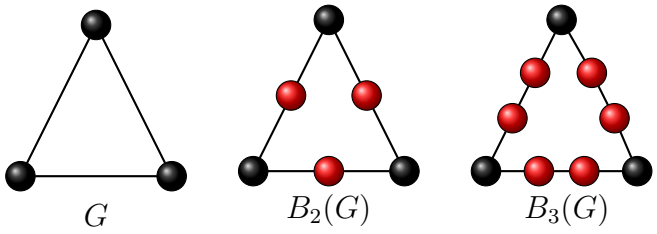


Figure 2.17: A graph G, its bipartite graph and its 3-partite graph

2.4.3 Reduction Approach

The reduction approach is another concept that is characterized by the presence of multiple edges. Its main purpose is to reduce the number of edges to facilitate the calculation of the number of spanning trees for planar networks having a large number of edges. We

begin by a definition of the reduction approach and we provide its properties. Then, we present the formula of the number of spanning trees of the reduced network according to that of the origin network and its number of nodes. Finally, we generalize these results for the case of the k-reduced network.

Definition 2.4.4. *A graph becomes reduced when we add a new edge connecting two existing vertices of a planar graph G. It is denoted by $R_2(G)$ (See Figure 2.18).*

Property 2.4.4. *Let G be a planar graph and $R_2(G)$ its reduced graph.*

- *The number of vertices in $R_2(G)$ is given by $|V_{R_2(G)}| = |V_G|$,*
- *The number of edges is given by $|E_{R_2(G)}| = 2|E_G|$,*
- *The number of faces is $|F_{R_2(G)}| = |F_G| + |E_G|$,*
- *The average degree is $< z >_{R_2(G)} = \frac{2|E_{R_2(G)}|}{|V_{R_2(G)}|} = \frac{4|E_G|}{|V_G|}$.*

Theorem 2.4.4. *(Lotfi et al., 2015) Let $R_2(G)$ be a reduced graph of a planar graph G. The number of spanning trees of $R_2(G)$ is given by:*

$$\tau(R_2(G)) = 2^{|V_G|-1}\tau(G) \tag{2.24}$$

Definition 2.4.5. ***(Generalization of the reduction approach)*** *Let G be a planar graph. The k-reduced graph of G, denoted $R_k(G)$, is obtained when for each pair of vertices of G, we have k multiple edges connecting them (See Figure 2.18).*

Property 2.4.5. *Let G be a planar graph and $R_k(G)$ its k-reduced graph.*

- *The number of vertices in $R_k(G)$ is given by $|V_{R_k(G)}| = |V_G|$,*
- *The number of edges is given by $|E_{R_k(G)}| = k|E_G|$,*
- *The number of faces is $|F_{R_k(G)}| = |F_G| + (k-1)|E_G|$,*
- *The average degree is $< z >_{R_k(G)} = \frac{2|E_{R_k(G)}|}{|V_{R_k(G)}|} = \frac{2k|E_G|}{|V_G|}$.*

Theorem 2.4.5. *(Lotfi et al., 2015) Let $R_k(G)$ be a k-reduced graph of a planar graph G. The number of spanning trees of $R_k(G)$ is given by:*

$$\tau(R_k(G)) = k^{|V_G|-1}\tau(G) \tag{2.25}$$

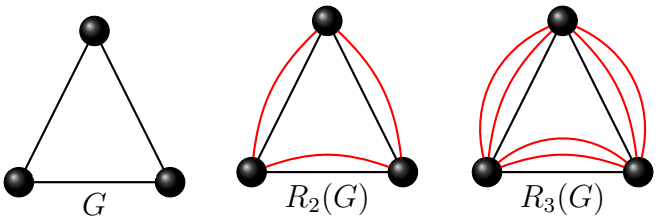


Figure 2.18: A graph G, its reduced graph and its 3-reduced graph

➥ **Advantages and constraints of the duality, reduction and bipartition approaches**

These geometric approaches facilitate the calculation of the number of spanning trees of a planar network by changing its topological nature and decreasing its number of vertices and edges, but their application concerns just specific types of planar networks. Therefore, we propose a general technique based on some geometric transformations to calculate the number of spanning trees of any network whatever its size and its structure.

2.4.3.1 Application: A Book network with a common path l_k

In this section, we concentrate on a Book network with a common path l_k, denoted as $P_{m,k}$ where m is the number of paths between two common vertices u and v and k is the size of the path l_k ($m \geq 2$, $k \geq 2$). It contains the same number of vertices in each cycle (See Figure 2.19). Then, we calculate its number of spanning trees based on the combination of two geometric approaches: the bipartition and the reduction approaches. Finally, we enumerate the spanning trees of a chain of Book network $C_{m,k,n}$ using the same combination (See Figure 2.21). The results of this application were presented in (Mokhlissi et al., 2018a).

Theorem 2.4.6. *(Mokhlissi et al., 2018a) Let $P_{m,k}$ be a Book network with a common path l_k, where m is the number of paths between two common vertices u and v and k is*

the size of the path l_k ($m \geq 2$, $k \geq 2$). The number of spanning trees of $P_{m,k}$ is given by:

$$\tau(P_{m,k}) = m \times k^{m-1} \tag{2.26}$$

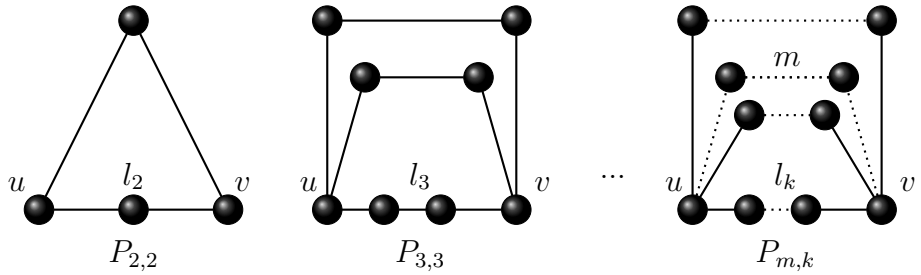


Figure 2.19: The representation of Book networks with a common path l_k .

Proof: Let e be an edge connecting two vertices u and v, $R_m(e)$ is the m-reduced graph of e and $B_k(R_m(e))$ is the k-partite graph of the m-reduced graph of the edge e. This geometrical transformation of the edge e is used to form the Book network $P_{m,k}$ with a common path l_k. Then $\tau(P_{m,k}) = \tau(B_k(R_m(e)))$. We apply Theorem 2.4.5 to the edge e, we obtain $\tau(R_m(e)) = m$, since $V_e = 2$ and we apply Theorem 2.4.3 to $R_m(e)$, we obtain $\tau(B_k(R_m(e))) = k^{|F_{R_m(e)}|-1} \times \tau(R_m(e))$ and $|F_{R_m(e)}| = m$, hence $\tau(P_{m,k}) = m \times k^{m-1}$. We illustrate this demonstration in Figure 2.20 for $P_{3,3}$.

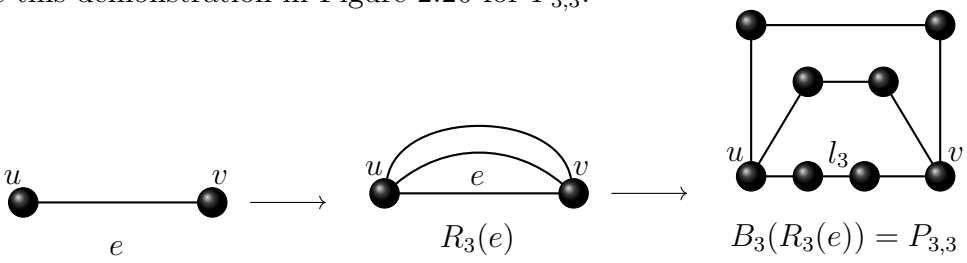


Figure 2.20: The demonstration of $P_{3,3}$.

Let $C_{m,k,n}$ be a chain of Book networks $P_{m,k}$ with a common path l_k. It is composed of n Book networks connected by an articulation point where m is the number of paths between two common vertices and k is the size of the path l_k ($m \geq 2$, $k \geq 2$, $n \geq 2$). We use the same combination of two geometric approaches: the bipartition and the reduction approaches to calculate its number of spanning trees (See Figure 2.21).

Theorem 2.4.7. *(Mokhlissi et al., 2018a) Let $C_{m,k,n}$ be a chain of Book networks $P_{m,k}$ with a common path l_k where n is the number of Book networks, m is the number of paths*

between two common vertices and k is the size of the path l_k ($m \geq 2$, $k \geq 2$, $n \geq 2$). Then, the number of spanning trees of $C_{m,k,n}$ is given by:

$$\tau(C_{m,k,n}) = m^n \times k^{n(m-1)} \tag{2.27}$$

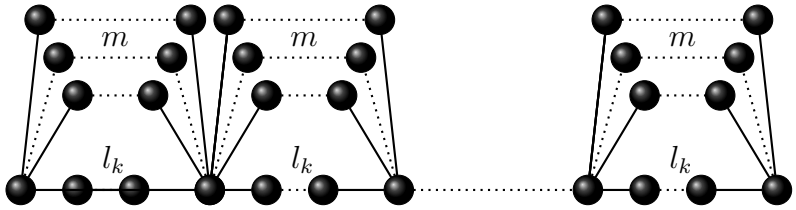


Figure 2.21: A chain of Book networks $C_{m,k,n}$

Proof: Let t be a path, $R_m(t)$ is the m-reduced graph of t and $B_k(R_m(t))$ is the k-partite graph of the m-reduced graph of t. This geometrical transformation of t constructs the chain $C_{m,k,n}$ of n Book networks $P_{m,k}$. Then $\tau(C_{m,k,n}) = \tau(B_k(R_m(t)))$. We apply Theorem 2.4.5 to the path t, we obtain $\tau(R_m(t)) = m^n$ and we apply Theorem 2.4.3 to $R_m(t)$, we obtain $\tau(B_k(R_m(t))) = k^{|F_{R_m(t)}|-1} \times \tau(R_m(t))$ and $|F_{R_m(t)}| - 1 = (m-1)n$, hence $\tau(C_{m,k,n}) = m^n \times k^{n(m-1)}$.

2.4.4 Electrically equivalent technique for enumerating spanning trees of a network

In this section, we use some notions of the electrical networks (Teufl and Wagner, 2010b,a), we study the technique of the electrically equivalent transformations and we analyze the characteristics of each one. Finally, we propose an algorithm for each transformation and we provide the main algorithm which defines the number of spanning trees of a given network.

2.4.4.1 Definition

The electrically equivalent technique is a transformation of the network structure (Teufl and Wagner, 2010b,a). This technique simplifies the network structure by reducing the number of edges and vertices and changes the weight of the edges. An edge-weighted network is an electrical network such that the weights represent the conductances of the corresponding edges. Furthermore, the number of spanning trees changes under the five

transformations of this technique. It is proved that if a subgraph of a graph G is replaced by an electrically equivalent graph, the number of spanning trees only changes by a factor that does not depend on G, which is very important to determine the number of spanning trees of a network.

Theorem 2.4.8. *(Teufl and Wagner, 2010a) Suppose that a graph G can be partitioned into two "edge-disjoint" subgraphs S_1 and S_2 and the vertex set of G satisfies $V_{S_1} \cup V_{S_2} = V_G$ and $V_{S_1} \cap V_{S_2} = S^*$. Suppose that S_2' is a graph with $E_{S_1} \cap E_{S_2'} = \varnothing$ and $V_{S_1} \cap V_{S_2'} = S^*$. Suppose that S_2 and S_2' are electrically equivalent with respect to S^*. Finally, let $G' = S_1 \cup S_2'$. If $\tau(G) \neq 0$ and $\tau(S_2) \neq 0$. Then, the following formula holds:*

$$\frac{\tau(G')}{\tau(G)} = \frac{\tau(S_2')}{\tau(S_2)} \tag{2.28}$$

As a consequence of this theorem, we can use simplification techniques for networks in order to calculate their number of spanning trees. The electrically equivalent technique has been applied to many networks as a prism and an antiprism (Sun et al., 2016), the contact graphs of disk packings (Qin et al., 2015), Apollonian networks (Sun et al., 2016) and the generalized pseudofractal networks (Xiao et al., 2015), etc. In Table 2.1, we consider the effect of five simple transformations of the electrically equivalent technique on the weight of each edge and on the number of spanning trees. Let G be a weighted graph and $\tau(G)$ be its weighted number of spanning trees. Let G' be the corresponding electrically equivalent graph and $\tau(G')$ be its weighted number of spanning trees. These five transformations are as follows: Parallel edges, Serial edges, Wye−Delta transform, Delta−Wye transform and Star−Mesh transform. This last transformation is a generalization of Serial edges and Wye−Delta transform. As a matter of fact, if we have a star S_n with n vertices and the weights $a_1, a_2, ..., a_n$, it will be transformed to a complete graph K_n with a new weight $x_{i,j} = \frac{a_i a_j}{\sum_{k=1}^{n} a_k}$ for each edge $v_i v_j$ $(i \neq j)$. Its number of spanning trees is given by $\tau(K_n) = \frac{1}{\sum_{k=1}^{n} a_k} \tau(S_n)$. As an example, we consider the fifth illustration in Table 2.1, which presents the transformation of the star S_4 to the complete graph K_4.

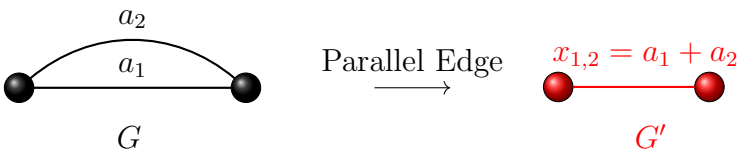
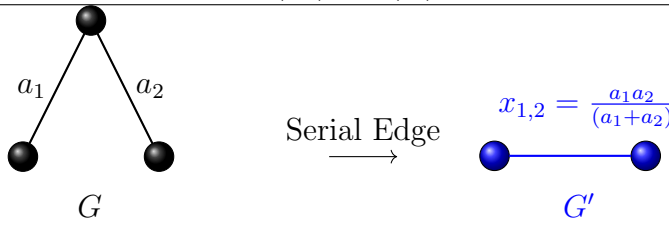
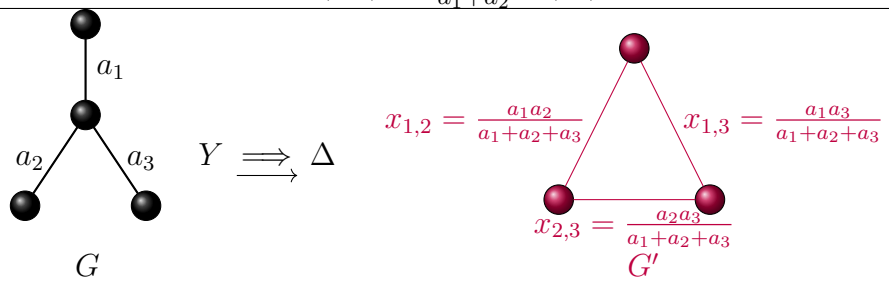
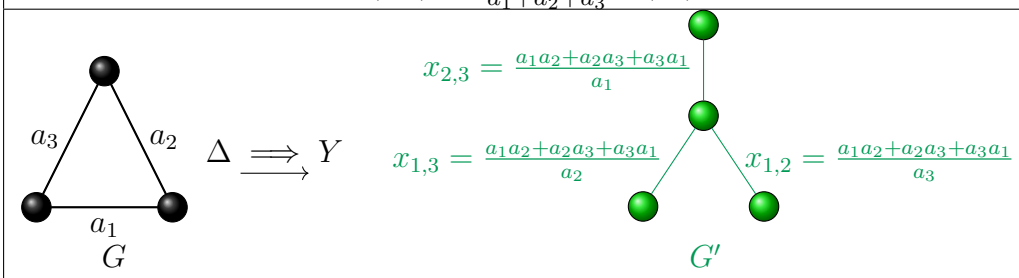
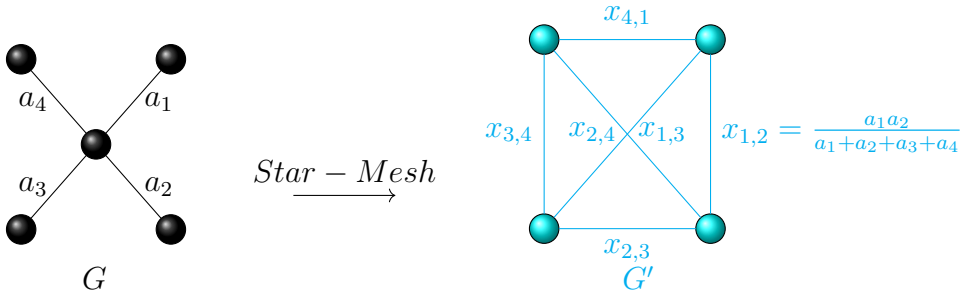
Graph G, its electrically equivalent graph G' and its complexity
 $\tau(G') = \tau(G)$
 $\tau(G') = \frac{1}{a_1+a_2}.\tau(G)$
 $\tau(G') = \frac{1}{a_1+a_2+a_3}.\tau(G)$
 $\tau(G') = \frac{(a_1 a_2 + a_2 a_3 + a_3 a_1)^2}{a_1 a_2 a_3}.\tau(G)$
 $\tau(G') = \frac{1}{a_1+a_2+a_3+a_4}\tau(G)$

Table 2.1: The five electrically equivalent transformations.

2.4.4.2 Example

Figure 2.22 shows an example of the application of the electrically equivalent transformations. We calculate the number of spanning trees of the network G by applying four electrically equivalent transformations. The corresponding calculations of the conductances are as follows:

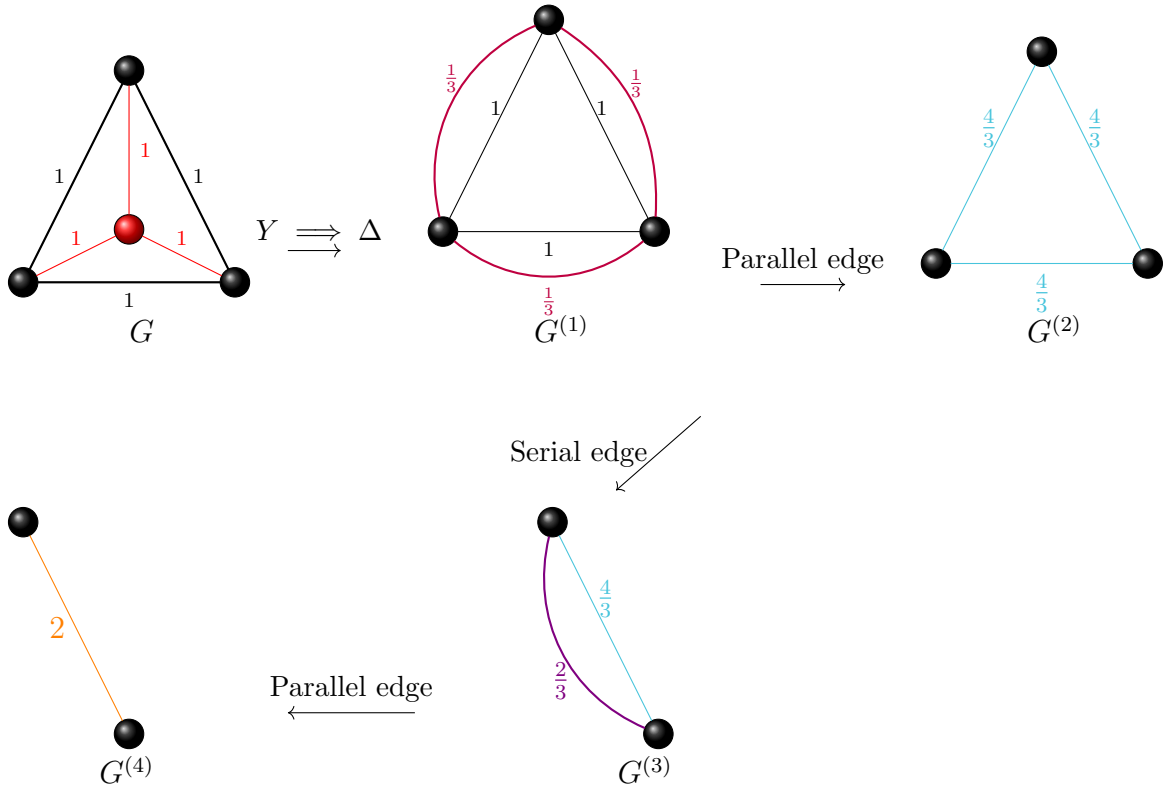


Figure 2.22: An example of electrically equivalent transformations.

1. Note the conductance of each edge of the original network is 1, then the corresponding conductance of the resulting network for Wye−Delta transformation is $\frac{1.1}{1+1+1} = \frac{1}{3}$.

2. For two parallel edges with conductances 1 and $\frac{1}{3}$, the conductance of the new edge is the sum of two original conductances, i.e. $1 + \frac{1}{3} = \frac{4}{3}$.

3. When two serial edges with conductances $\frac{4}{3}$ and $\frac{4}{3}$ are merged into a new edge, its conductance is $\frac{\frac{4}{3}\times\frac{4}{3}}{\frac{4}{3}+\frac{4}{3}} = \frac{2}{3}$.

4. For two parallel edges, the conductance of the new edge is $\frac{2}{3} + \frac{4}{3} = 2$.

   The weighted number of spanning trees in the network G is calculated as:

(a) $\tau(G) = \tau(G^{(0)})$.

(b) $\tau(G) = 3 \times \tau(G^{(1)}) \rightarrow Y \implies \Delta$.

(c) $\tau(G) = 3 \times \tau(G^{(2)}) \rightarrow$ *Parallel edge.*

(d) $\tau(G) = 3 \times \frac{8}{3} \times \tau(G^{(3)}) \rightarrow$ *Serial edge.*

(e) $\tau(G) = 8 \times \tau(G^{(4)}) \rightarrow$ *Parallel edge.*

(f) $\tau(G) = 8 \times 2 = 16$.

So, the original network G has 16 spanning trees according to the factors of these transformations.

2.4.4.3 Electrically equivalent transformations algorithms.

In general, a network can be simplified by identifying edges in series, parallel, Wye-Delta, Delta-Wye and Star-Mesh. The main task is which transformation must be applied first. Two properties can be used to provide an order for these transformations: The degree of each node and the weights of the equivalent edges of each transformation. In literature, the resistances in series and parallel are the most used transformations (Teufl and Wagner, 2010a)).

1. **Parallel Edge:** We start with the parallel edge transformation to reduce the number of edges in a network. We calculate the weight for the new edge. This transformation does not change the complexity of the obtained graph. The algorithm 1 presents the transformation of parallel edges.
2. **Serial Edge:** The second transformation is the serial edge. We look for a node with a degree '2'. This transformation allows to reduce the number of vertices and edges of a network. The algorithm 2 presents the transformation of serial edges.
   Some networks cannot be simplified either by the serial edge or by the parallel edge transformations. For this reason, three new transformations have been proposed namely Wye-Delta, Delta-Wye or Star-Mesh.
3. **Wye-Delta transformation:** We call the transformations of "Wye" and "Delta" according to the form of schemas, which look like letters (Y, Δ). The Wye-Delta transformation has priority because it reduces the number of nodes and simplifies the structure of large networks. For every four nodes of a Wye, we will have only

three nodes of a Delta and the number of edges remains unchangeable. We look for a node with a degree '3' in a network. The algorithm 3 presents the transformation of Wye-Delta.

4. **Delta-Wye transformation:** This transformation is the inverse of the Wye−Delta transformation. Its application generates a node of degree '3' and the degrees of Delta nodes will be decreased by 1. The algorithm 4 presents the transformation of Delta-Wye.

5. **Star-Mesh transformation:** For a network that has no parallel edges and all its nodes have a degree greater than '3', the transformations mentioned above are not appropriate, so we will use the Star-Mesh transformation. We look for a node with a degree $n > 3$. This transformation reduces the number of vertices by deleting the vertices having the degree n. The algorithm 5 presents the transformation of Star-Mesh.

Algorithm 1 The algorithm of Parallel edge transformation

function $ParallelEdge(G)$
  $L1$: List of pairs of nodes that are connected by multiple edges;
  $n1$: The length of $L1$;
  for $i = 1 \rightarrow n1$ **do**
    $x_{1,2}$: The sum of the weights of all multiple edges between the pair of nodes of $L1[i]$;
    Remove all multiple edges between the pair of nodes of $L1[i]$;
    Add a new edge between the pair of nodes of $L1[i]$;
    Assign the weight $x_{1,2}$ to the new edge in $L1[i]$;
    $\tau(G) \leftarrow 1 * \tau(G)$;
  end for
  Empty $L1$;
end function

Algorithm 2 The algorithm of Serial edge transformation

function $SerialEdge(G)$
  $L2$: List of nodes having degree 2 and their two neighbors;
  $n2$: The length of $L2$;
  for $i = 1 \rightarrow n2$ **do**
    a_1, a_2 : Weights of two edges between nodes having the degree 2 and their neighbors in $L2[i]$;
    $x_{1,2} \leftarrow (a_1 * a_2)/(a_1 + a_2)$;
    Remove the node having the degree 2 from $L2[i]$;
    Add a new edge between two neighbors to $L2[i]$;
    Assign the weight $x_{1,2}$ to the new edge created between two neighbors in $L2[i]$;
    $\tau(G) \leftarrow (a_1 + a_2) * \tau(G)$;
  end for
  Empty $L2$;
end function

Algorithm 3 The algorithm of Wye-Delta transformation

function $Wye - Delta(G)$
  $L3$: List of nodes having degree 3 and their three neighbors;
  $n3$: The length of $L3$;
  for $i = 1 \rightarrow n3$ **do**
    a_1, a_2, a_3: Weights of three edges between nodes having the degree 3 and their neighbors in $L3[i]$;
    $x_{1,2} \leftarrow (a_1 * a_2)/(a_1 + a_2 + a_3)$;
    $x_{1,3} \leftarrow (a_1 * a_3)/(a_1 + a_2 + a_3)$;
    $x_{2,3} \leftarrow (a_2 * a_3)/(a_1 + a_2 + a_3)$;
    Remove the node having the degree 3 from $L3[i]$;
    Add three new edges between the three neighbors to $L3[i]$;
    Assign the weights $x_{1,2}$, $x_{1,3}$ and $x_{2,3}$ to the new three edges created between the edges having the weights a_1 and a_2, the weights a_1 and a_3 and the weights a_2 and a_3, respectively;
    $\tau(G) \leftarrow (a_1 + a_2 + a_3) * \tau(G)$;
  end for
  Empty $L3$;
end function

Algorithm 4 The algorithm of Delta-Wye transformation

function $Delta - Wye(G)$
  $L4$: list of triangles in G;
  $n4$: The length of $L4$;
  for $i = 1 \rightarrow n4$ **do**
    a_1, a_2, a_3: Weights of the three edges of a triangle of $L4[i]$;
    $x_{2,3} \leftarrow (a_1 * a_2 + a_2 * a_3 + a_1 * a_3)/a_1$;
    $x_{1,3} \leftarrow (a_1 * a_2 + a_2 * a_3 + a_1 * a_3)/a_2$;
    $x_{1,2} \leftarrow (a_1 * a_2 + a_2 * a_3 + a_1 * a_3)/a_3$;
    Remove all the three edges of a triangle of $L4[i]$;
    Add a new node in G;
    Add three new edges between the new node and the three nodes of a triangle of $L4[i]$;
    Assign the weights $x_{2,3}$, $x_{1,3}$ and $x_{1,2}$ to the new three edges created between the new node and the node between two edges having the weights a_2 and a_3, a_1 and a_3, a_1 and a_2, respectively;
    $\tau(G) \leftarrow (a_1 * a_2 * a_3)/(a_1 * a_2 + a_2 * a_3 + a_1 * a_3)\hat{}2 * \tau(G)$;
  end for
  Empty $L4$;
end function

Algorithm 5 The algorithm of Star-Mesh transformation

```
1: function Star − Mesh(G)
2:     L5: Sorted list of nodes having the degree > 3 with their neighbors;
3:     n5: The length of L5;
4:     for i = 1 → n5 do
5:         t: The size of L5[i];
6:         s ← 0;
7:         for j = 2 → t do
8:             a(j) : Weight of the edge between nodes having the degree greater than 3 and their
9:                 neighbor j;
10:            s ← s + a(j)                          ▷ Summing all the weights a(j)
11:        end for
12:        for j = 1 → t − 1 do
13:            for k = j + 1 → t do
14:                x(j)(k) ← a(j) ∗ a(k)/s ;
15:                Add new edge between two edges that have the weight a(j) and a(k);
16:                Assign the weight x(j)(k) to the new edge;
17:            end for
18:        end for
19:        Remove the node having the degree greater than 3 in L5[i];
20:        τ(G) ← s ∗ τ(G) ;
21:    end for
22:    Empty L5;
23: end function
```

2.4.4.4 Methodology

To calculate the number of spanning trees of a network using the electrically equivalent transformations, we follow the steps below:

1. First, if we have an unweighted network G, we add a weight (conductance) for each edge. We often assign a weight "1" to each edge of a network.

2. We apply the electrically equivalent transformations on the weighted network G using the cited algorithms and Algorithm 6 to obtain the conductances of all the edges of the transformed network G'. We verify the isomorphism between two successive iterations of a network to deduce the relationship between their number of spanning trees and the exact value of the number of spanning trees in each iteration.

3. Finally, we deduce the exact formula of the number of spanning trees of the original network.

The proposed algorithm 6 presents the order of electrically equivalent transformations with the original network G_n and the transformed network G_{n-1}.

Algorithm 6 The algorithm of the electrically equivalent transformations

Input: G_n: A connected network with n iterations, G_{n-1}: its transformed network;
N: The number of nodes of G_n, E: The number of edges of G_n;
Output: $\tau(G_n)$: The number of spanning trees of G_n;
eq: The equation between $\tau(G_n)$ and $\tau(G_{n-1})$;

$\tau(G_n) \leftarrow 1$;
while not $N = 2$ and $E = 1$ **do** ▷ The last transformed network
  if G_n is isomorphic to G_{n-1} **then**
    $eq = \tau(G_n)$;
  end if
  if Existence of multiple edges between two nodes **then**
    $ParallelEdge(G_n)$ function; ▷ Algorithm 1
  else if Existence of nodes having the degree 2 **then**
    $SerialEdge(G_n)$ function; ▷ Algorithm 2
  else if Existence of nodes having the degree 3 **then**
    $Wye - Delta(G_n)$ function; ▷ Algorithm 3
  else if Existence of triangles in (G_n) **then**
    $Delta - Wye(G_n)$ function; ▷ Algorithm 4
  else if Existence of nodes having the degree higher than 3 **then**
    $Star - Mesh(G_n)$ function; ▷ Algorithm 5
  else
    break;
  end if
end while
$\tau(G_n) = \tau(G_n) *$ Weight of the last transformed network ($N = 2$ and $E = 1$).

➥ Advantages and constraints of the electrically equivalent technique

The electrically equivalent technique to enumerate spanning trees is general and can be easily extended to any complex networks, because if any subgraph of the main graph is replaced by an electrically equivalent graph, the number of spanning trees only changes by a factor that is independent of the rest of the graph. Based on this technique, we can determine the relationship between the number of spanning trees of the original and the transformed networks. It provides the calculation of the edges' weights of the transformed network and the exact value of the number of spanning trees of the original network.

2.5 Summary

In this chapter, we have studied the different methods and techniques of calculating the number of spanning trees of a network, notably algebraic methods such as Matrix-Tree Theorem, combinatorial methods such as the deletion and the contraction methods, the decomposition methods and geometric approaches such as the electrically equivalent technique, etc. All these methods are efficient to calculate the number of spanning trees. However, some of them have restrictions. As the first method proposed by Kirchhoff, which calculates the number of spanning trees for all graphs, but if we consider a large and complex network which has a very large number of vertices, it will not be easy to calculate the determinant of a very large matrix. As a remedy to this problem, researchers have developed techniques and approaches without going through the determinant. In this chapter, we have treated almost all solutions to obtain the exact formula of the number of spanning trees. We have proposed many generalizations of some methods known in the literature as the contraction method and we have shown that the electrically equivalent technique is considered as the general method and can be easily applied to any complex network whatever its size. Furthermore, we have explored two important real applications of enumerating spanning trees such as the reliability and the robustness of a network. In the next chapters, we study the three types of complex networks: small-world, scale-free and random networks, we propose new models for each type, we analyze their structure and their topological properties, we calculate their complexity using some methods mentioned in this chapter and we evaluate their entropy to estimate the most robust network. We start in Chapter 3 by investigating three models of the small-world network: The Koch Network, the Small-World Exponential network, the Farey network and we perform a comparative study between them.

Chapter

3

Analysis of models of Small-World Networks

Contents

Small-world networks are a class of networks that are highly clustered, yet have the small-world effect. They display rich behaviour as observed in a large variety of real systems. Social networks are intuitive examples of this type of networks. In this chapter, we discuss three categories of small-world networks, namely Small-World Exponential network, Koch Network and Farey network. For each network, we propose its generalization. We investigate their iterative construction. We focus on the analysis of their structural properties such as the degree distribution, the clustering coefficient, the diameter, etc, and deduce the complicated analytical results from the growth mechanism used in these models. Then, we calculate their number of spanning trees using some methods proposed in Chapter 2. Finally, as an application, we evaluate their entropy of spanning trees to quantify their robustness and compare them with other networks having the same average degree to estimate the robust one. The results of this chapter were published in an international journal (Mokhlissi et al., 2018b), an international conference (Mokhlissi et al., 2016a) and submitted to an international journal (Mokhlissi et al., 2019a).

3.1 Why be interested in small-world networks?

Many scientists care about small-world networks because they appear to be everywhere. For example, they can be found in social networks (Scott, 2000), economic networks (Kirman, 1997), transportation systems (Guimera et al., 2005), epidemic spreading (Pastor-Satorras and Vespignani, 2001), metabolic networks (Jeong et al., 2000), food web (Montoya and Solé, 2002) and so on. Small-world networks have two structural properties, the small-world effect and high clustering. The small-world effect is the concept where any two nodes in a network are connected to each other through a small path. This property minimizes the number of connections (links) in a network. The second property of small-world networks is the high clustering coefficient where two nodes having a common neighbour have a high tendency to be connected to each other. This feature maximizes the connectivity in a network. In order to mimic real-world networks, many models have been proposed to generate networks with small-world properties to understand, improve and manipulate them for useful applications (Zaidi, 2013).

3.2 Analysis of Small-World Exponential network

In this section, we introduce a well known family of small-world network: *The Small-World Exponential network* (See Figure 3.1) (Mokhlissi et al., 2016a; Liu et al., 2015; Barriere et al., 2009). Its construction is based on triangles and it has an exponential form of degree distribution. The Small-World Exponential network has been observed from some real-life systems as tensor networks (Marti et al., 2010), social networks (Bonneau et al., 2009), quantum walks (Hillery et al., 2010), etc. We present its construction which is built in an iterative way, determine their structural properties and calculate its number of spanning trees using the generalized decomposition method following one articulation node. Then, we propose a new family of *generalized Small-World Exponential networks* (Mokhlissi et al., 2018b)(See Figure 3.2), where the difference relies on the size and the dimension of the added cyclic subgraph. We also investigate its construction and their structural properties, indicating that the generalization of this small-world network affects the properties of small-world networks and its behaviour follows the characteristics

of scale-free networks. We calculate its number of spanning trees using the same method. In the end, we calculate its entropy of spanning trees to evaluate its robustness.

3.2.1 Small-World Exponential network $G_{1,3,n}$

3.2.1.1 Construction of the Small-World Exponential network $G_{1,3,n}$

The Small-World Exponential network is denoted by $G_{1,3,n}$ with n is the current generation, '1' refers to the number of the added triangles for each node of each triangle added to the previous iteration and '3' refers to the size of triangles. $G_{1,3,n}$ is constructed by following this algorithm: At $n = 0$, we have a simple node. At first generation, $G_{1,3,1}$ is a simple triangle. For $n > 1$, each node of each triangle added to the previous iteration is replaced by a new triangle. Thus, each of the newly appeared triangles contains exactly one node of the network of the previous iteration. The degree of the articulation nodes of the first iteration (The red nodes in Figure 3.1) is $2n$. The growth process to the next generations continues in a similar way. In Figure 3.1, the first three iterations of the Small-World Exponential network $G_{1,3,n}$ are illustrated.

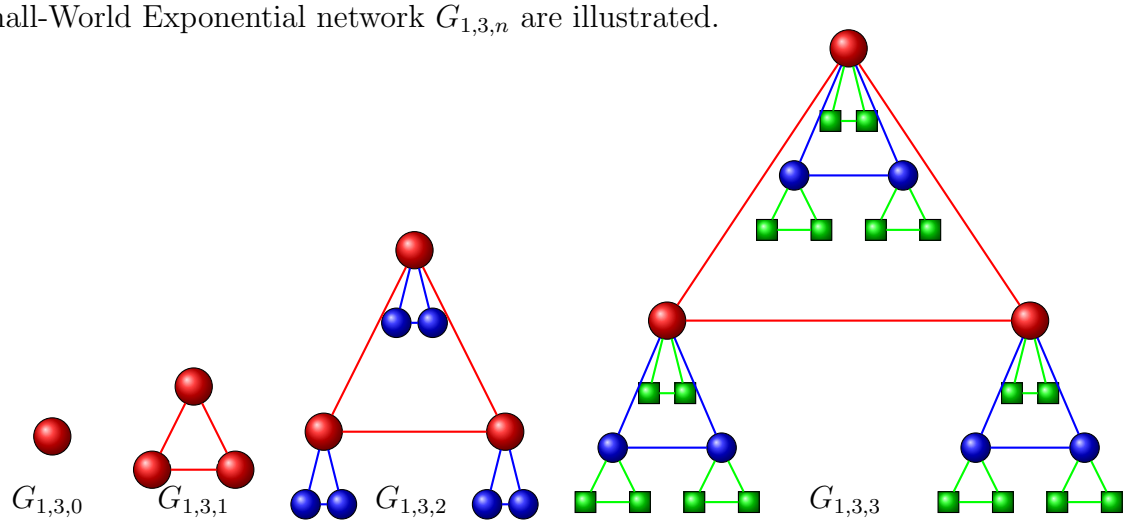


Figure 3.1: The first three generations of the Small-World Exponential network $G_{1,3,n}$

Next, we compute the numbers of nodes, edges, faces and the average degree of the Small-World Exponential network. According to the construction of $G_{1,3,n}$, we get:

- Let $V_{G_{1,3,n}}$ be the number of nodes created at step n. For $n \geq 0$, we notice $V_{G_{1,3,n}} = 3V_{G_{1,3,n-1}} = 3^2V_{G_{1,3,n-2}} = 3^3V_{G_{1,3,n-3}} = ... = 3^{n-1}V_{G_{1,3,1}} = 3^nV_{G_{1,3,0}}$. Thus,

the number of nodes of $G_{1,3,n}$ is:

$$V_{G_{1,3,n}} = 3^n, \quad n \geq 0. \tag{3.1}$$

- Similarly, we find **the number of edges of $G_{1,3,n}$:**

$$E_{G_{1,3,n}} = 3 \times \frac{3^n - 1}{2}, \quad n \geq 0. \tag{3.2}$$

- **The number of faces of $G_{1,3,n}$ is:**

$$F_{G_{1,3,n}} = \frac{3^n + 1}{2}, \quad n \geq 0. \tag{3.3}$$

- Let $< z >_{G_{1,3,n}}$ be the average degree of $G_{1,3,n}$. It is calculated as follows: $< z >_{G_{1,3,n}} = \frac{2E_{G_{1,3,n}}}{V_{G_{1,3,n}}}$. Thus **the average degree of $G_{1,3,n}$** is (which is approximately 3 for large n):

$$< z >_{G_{1,3,n}} = \frac{3^n - 1}{3^{(n-1)}}, \quad n \geq 0 \tag{3.4}$$

3.2.1.2 *Structural properties of the Small-World Exponential network $G_{1,3,n}$*

- **The degree distribution**

We denote $k_u(n)$ as the degree of a node u at the step n. When a node u is added to the network at step $n_u \geq 0$, it has a degree $k_u(n_u) = 2$. To determine $k_u(n)$, we first determine the number of triangles involving the node u at step n that is represented by $M_{\Delta_{(u,n)}}$. These triangles will create new nodes connected to the node u at step $n+1$. Then, at step n_u, $M_{\Delta_{(u,n_u)}} = 1$. By construction, $M_{\Delta_{(u,n)}} = M_{\Delta_{(u,n-1)}} + 1$. We can derive $M_{\Delta_{(u,n)}} = n - n_u + 1$. Note that the relation between $k_u(n)$ and $M_{\Delta_{(u,n)}}$ satisfies

$$k_u(n) = 2M_{\Delta_{(u,n)}} = 2(n - n_u + 1). \tag{3.5}$$

In this way, at step n the degree of node u has been computed explicitly. From Equation 3.5, we can see that at each step the degree of a node is:

$$k_u(n) = k_u(n-1) + 2. \tag{3.6}$$

Then, the cumulative degree distribution is given by

$$P_{cum}(k) = \frac{1}{V_{G_{1,3,n}}} \sum_{i \leq n_u} M_v(i) = \frac{3^{n_u}}{3^n}. \tag{3.7}$$

with $M_v(n)$ is the number of new nodes at step n and $V_{G_{1,3,n}} = \sum_{i=0}^{n} M_v(i)$. From Equation 3.5, we obtain $n_u = n + 1 - \frac{k}{2}$. Then, the cumulative degree distribution of $G_{1,3,n}$ will be:

$$P_{cum}(k) = 3^{-\frac{k}{2}+1} \tag{3.8}$$

For large n, we can obtain:

$$P_{cum}(k) \sim 3^{-\frac{k}{2}}. \tag{3.9}$$

So the degree distribution of the Small-World Exponential network $G_{1,3,n}$ follows an exponential distribution.

- **Clustering coefficient**

Now, we calculate the clustering coefficient of the Small-World Exponential network $G_{1,3,n}$ for any node u, which is given by $C_u = 2e_u/[k_u(k_u-1)]$, where e_u is the number of existing links between all the k_u neighbors of node u. From Equation 3.5, we have $k_u = 2(n - n_u + 1)$. Among the $2(n - n_u + 1)$ neighbors, 2 nodes that belong to the same triangle are connected to each other, leading to the total number of links $e_u = n - n_u + 1$. Thus, the C_u is given by:

$$C_u = \frac{1}{2(n - n_u + 1) - 1} \tag{3.10}$$

Based on Equation 3.10, we can list the correspondence between each kind of clustering coefficient and the corresponding amount of nodes:

$$C_u = \begin{cases} 1, & \text{for } 2 \times 3^{n-1} \text{ nodes,} \\ \frac{1}{3}, & \text{for } 2 \times 3^{n-2} \text{ nodes,} \\ \vdots & \vdots \\ \frac{1}{2n-1}, & \text{for } 2 \times 3^{0} + 1 \text{ nodes,} \end{cases} \tag{3.11}$$

Where the last case represents the center of the whole network. Then, we can obtain the average clustering coefficient of all the nodes,

$$C_{G_{1,3,n}} = \frac{1}{3^n}\left[\frac{3}{2n-1} + \sum_{i=1}^{n} 2 \times 3^{n-i} \times \frac{1}{2i-1}\right] \tag{3.12}$$

For large n, the clustering coefficient of the Small-World Exponential network $G_{1,3,n}$ converges to a nonzero value $C_{G_{1,3,n}} \approx 0.76$. Therefore, the clustering coefficient of the Small-World Exponential network $G_{1,3,n}$ is high.

- **Diameter**

  Most real networks are small-world and their average path length grows logarithmically with the network order. For a general network, it is not easy to derive a closed formula for its average path length. However, the small diameter is also consistent with the concept of small-world. Let $D_{G_{1,3,n}}$ be the diameter of $G_{1,3,n}$. From Figure 3.1, it has been noticed that at iteration $n = 1$, the diameter $D_{G_{1,3,1}} = 1$. For $n > 1$, the diameter of $G_{1,3,n}$ increases by 2 at most. So for i from 2 to n, we have: $D_{G_{1,3,i}} = D_{G_{1,3,i-1}} + 2$. Summing all the obtained equations, we find: $D_{G_{1,3,n}} = D_{G_{1,3,1}} + 2(n-1)$. So, **the diameter of $G_{1,3,n}$ is:**

$$D_{G_{1,3,n}} = 2n - 1, n \geq 1 \tag{3.13}$$

  We can present this diameter by another formula $D_{G_{1,3,n}} = 2log_{(3)}(V_{G_{1,3,n}}) - 1$, which grows logarithmically with the network order, indicating that $G_{1,3,n}$ is a small-world network.

☞ **Discussion**

According to the above results, the Small-World Exponential network $G_{1,3,n}$ has specific properties, its degree distribution follows an exponential distribution, a high clustering coefficient and a small diameter, which indicate that $G_{1,3,n}$ is a small-world network.

3.2.1.3 Number of spanning trees of the Small-World Exponential network $G_{1,3,n}$

The number of spanning trees of complex networks is difficult to evaluate using classical approaches such as calculating the determinant of its Laplacian matrix. To remedy this problem, we use the generalized decomposition method following one articulation node mentioned in 2.3.2.2, which facilitates the computation of the number of spanning trees of the Small-World Exponential network by obtaining recursive expressions.

Theorem 3.2.1. *(Mokhlissi et al., 2016a) Let $G_{1,3,n}$ be the Small-World Exponential network. The number of spanning trees of $G_{1,3,n}$ is given by the following formula:*

$$\tau(G_{1,3,n}) = 3^{\frac{3^n-1}{2}}, n \geq 1. \tag{3.14}$$

Proof: The Small-World Exponential network $G_{1,3,n}$ can be decomposed into subgraphs according to the articulation nodes. These subgraphs are triangles $\Delta_{(n)}$. Using Equation 2.9, we obtain: $\tau(G_{1,3,n}) = \prod^{M_{\Delta_{(n)}}} \tau(\Delta_{(n)}) = \tau(\Delta_{(n)})^{M_{\Delta_{(n)}}}$ with $M_{\Delta_{(n)}}$ is the number of triangles in $G_{1,3,n}$. In order to calculate the number of spanning trees of $G_{1,3,n}$, we need to find the number of triangles in $G_{1,3,n}$. From our network, for i from 1 to n, we see: $M_{\Delta_{(i)}} = 3 \times M_{\Delta_{(i-1)}} + 1$. Then, we multiply the equation of $M_{\Delta_{(n-1)}}$ by 3 , the equation of $M_{\Delta_{(n-2)}}$ by 3^2 and so on until the last equation $M_{\Delta_{(1)}}$ which will be multiplied by 3^{n-1}. Summing all the obtained equations: $\sum_{i=0}^{n-1} 3^i M_{\Delta_{(n-i)}} = \sum_{i=0}^{n-1} 3^{i+1} M_{\Delta_{(n-i-1)}} + \sum_{i=0}^{n-1} 3^i$. We find: $M_{\Delta_{(n)}} = \sum_{i=0}^{n-1} 3^i$. So the number of triangles in $G_{1,3,n}$ is: $M_{\Delta_{(n)}} = \frac{3^n-1}{2}$. We replace it in the equation of $\tau(G_{1,3,n})$, hence we obtain: $\tau(G_{1,3,n}) = 3^{\frac{3^n-1}{2}}$.

3.2.2 Generalized Small-World Exponential network $G_{m,l,n}$

3.2.2.1 Construction of the generalized Small-World Exponential network $G_{m,l,n}$

The generalized Small-World Exponential network is denoted by $G_{m,l,n}$ with two controllable parameters: $l > 3$ is the size of the cyclic subgraph and $m > 1$ is the dimension of the cyclic subgraph i.e the number of the added cyclic subgraphs for each node of each cyclic subgraph added to the previous iteration. The construction of $G_{m,l,n}$ follows this algorithm: At $n = 0$, we have a simple node. At first generation, $G_{m,l,1}$ is a cyclic graph with the size l. For $n > 1$, each node of each cyclic subgraph added to the previous iteration is replaced by m new cyclic subgraphs having the size l. Thus, each of the newly appeared cyclic subgraph contains exactly one node of the network of the previous iteration. The degree of the articulation nodes of the first iteration (The red nodes in Figure 3.2) is $\frac{2(m^n-1)}{m-1}$. The same process is used for the other iterations. In Figure 3.2, the first three iterations of the generalized Small-World Exponential network $G_{m,l,n}$ are illustrated, with $m = 2$ and $l = 4$. Let us compute the order, the size, the number of faces and the average degree of the generalized Small-World Exponential network.

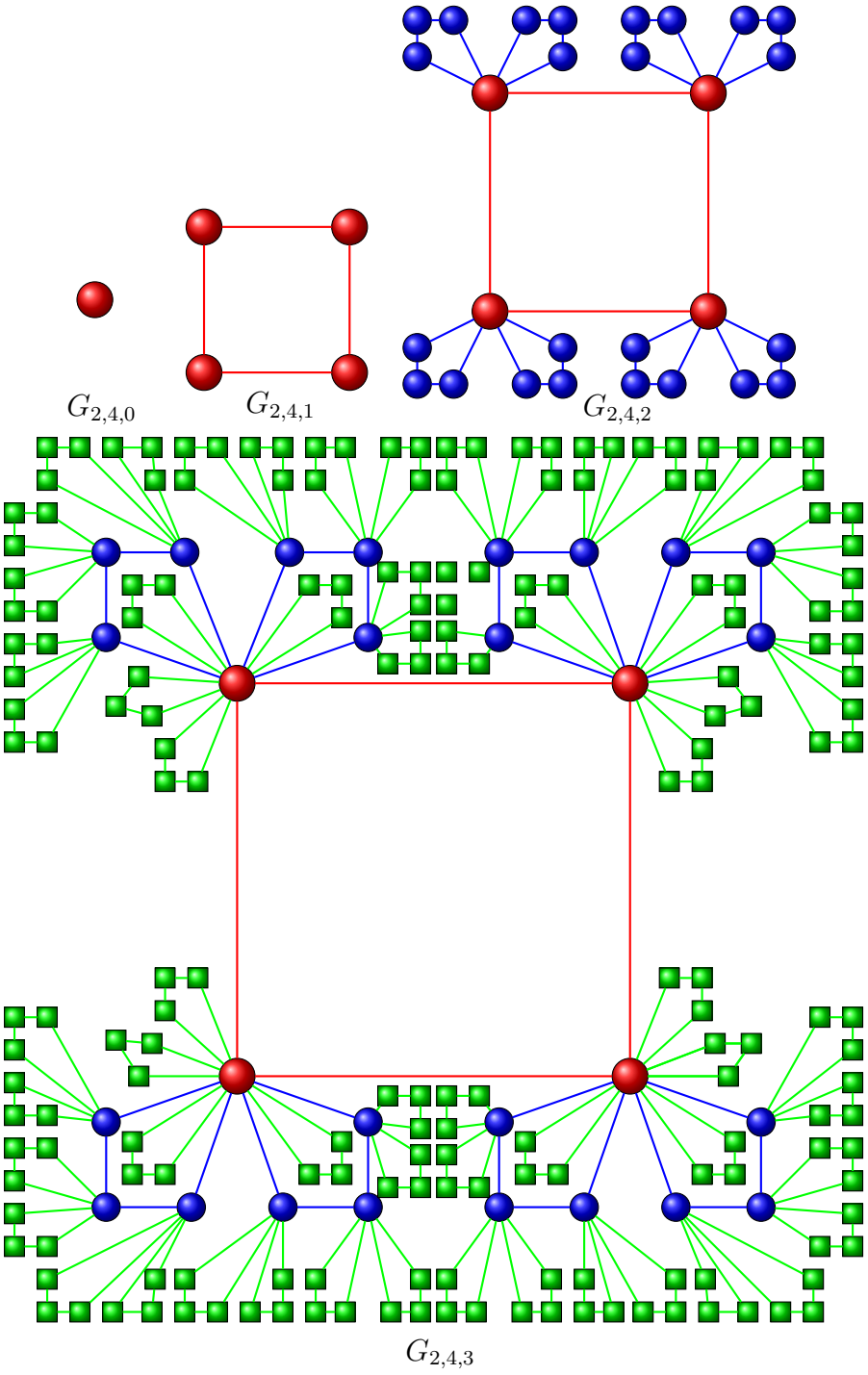


Figure 3.2: The first three generations of the generalized Small-World Exponential network $G_{2,4,n}$

According to the construction of $G_{m,l,n}$, we get:

- Let $V_{G_{m,l,n}}$ be the number of nodes created at n. For i from 1 to n, we notice $V_{G_{m,l,i}} = lm \times V_{G_{m,l,i-1}} - (m-1)l$. Then, we multiply the equation of $V_{G_{m,l,n-1}}$ by (lm), the equation of $V_{G_{m,l,n-2}}$ by $(lm)^2$ and so on until the last equation $V_{G_{m,l,1}}$

which will be multiplied by $(lm)^{(n-1)}$. Summing all the obtained equations: $\sum_{i=0}^{n-1}(lm)^i V_{G_{m,l,n-i}} = \sum_{i=0}^{n-1}(lm)^{i+1}V_{G_{m,l,n-i-1}} - (m-1)l\sum_{i=0}^{n-1}(lm)^i$. We find the following results: $V_{G_{m,l,n}} = (lm)^n V_{G_{m,l,0}} - (m-1)l\sum_{i=0}^{n-1}(lm)^i$ with $V_{G_{m,l,0}} = 1$. Thus, **the number of nodes of $G_{m,l,n}$ is:**

$$V_{G_{m,l,n}} = \frac{(lm)^n(l-1)+(m-1)l}{lm-1}, n \geq 0. \tag{3.15}$$

- Similarly, we find **the number of edges of $G_{m,l,n}$:**

$$E_{G_{m,l,n}} = l \times \frac{(lm)^n - 1}{(lm)-1}, n \geq 0. \tag{3.16}$$

- **The number of faces of $G_{m,l,n}$ is:**

$$F_{G_{m,l,n}} = \frac{(lm)^n + (lm-2)}{lm-1}, n \geq 0. \tag{3.17}$$

- **The average degree of** $G_{m,l,n}$ is (which is approximately 3 for large n):

$$< z >_{G_{m,l,n}} = \frac{2E_{G_{m,l,n}}}{V_{G_{m,l,n}}} = \frac{2l \times ((lm)^n - 1)}{(lm)^n(l-1)+(m-1)l}, n \geq 0. \tag{3.18}$$

3.2.2.2 Structural properties of the generalized Small-World Exponential network $G_{m,l,n}$

- **The degree distribution**

To calculate the degree distribution of the generalized Small-World Exponential network $G_{m,l,n}$, we use the same process applied on the degree distribution of the small-world exponential network $G_{1,3,n}$. Then, the cumulative degree distribution of $G_{m,l,n}$ is given by:

$$P_{cum}(k) = \frac{(lm)^n \times (lm) \times (\frac{k}{2}(m-1)+1)^{-\frac{\ln(lm)}{\ln(m)}} \times (l-1)+(m-1)l}{(lm)^n(l-1)+(m-1)l}. \tag{3.19}$$

For large n, we can obtain:

$$P_{cum}(k) \sim \frac{(m-1)^{-\frac{\ln(lm)}{\ln(m)}}}{2} \times k^{-\frac{\ln(lm)}{\ln(m)}} \tag{3.20}$$

So the degree distribution of the generalized Small-World Exponential network

$G_{m,l,n}$ follows a power-law form with the exponent $\gamma = 1 + \frac{\ln(lm)}{\ln(m)}$. For $m \geq l$, the exponent γ belongs to the interval $[2, 3]$, which means that $G_{m,l,n}$ is a scale-free network.

- **Clustering coefficient**

  Another property of interest is the clustering coefficient, which is a measure of the likelihood for neighbors of a node to be neighbors of one another (See Section 1.3.1). The generalized Small-World Exponential network $G_{m,l,n}$ with $m > 1, l > 3, n \geq 0$ have zero clustering because the neighbors of any node are never neighbors of one other. The main cause is the absence of triangles in $G_{m,l,n}$ because the size l of the cyclic subgraphs is greater than 3. Therefore, one of the properties of the small-world network is not verified.

- **Diameter:**

  Let $D_{G_{m,l,n}}$ be the diameter of $G_{m,l,n}$ created at generation n. This diameter can be calculated in two cases:

  – If the size of cyclic subgraphs l is even, we can calculate the diameter as follows: At iteration $n = 1$, the diameter $D_{G_{m,l,1}} = \frac{l}{2}$. For $n > 1$, the diameter of $G_{m,l,n}$ increases by l at most.

  – If the size of cyclic subgraphs l is odd, we can calculate the diameter as follows: At iteration $n = 1$, the diameter $D_{G_{m,l,1}} = \left\lfloor \frac{l}{2} \right\rfloor$. For $n > 1$, the diameter of $G_{m,l,n}$ increases by $(l - 1)$ at most.

  So **the diameter of $G_{m,l,n}$ is:**

$$D_{G_{m,l,n}} = \frac{l - \epsilon}{2} + (l - \epsilon)(n - 1) with \begin{cases} \epsilon = 0, & \text{if } l \text{ is even,} \\ \epsilon = 1, & \text{if } l \text{ is odd} \end{cases} \tag{3.21}$$

  For large n, the number of nodes will be $V_{G_{m,l,n}} = (lm)^n$. Then, the diameter can be presented by another formula which grows logarithmically with the network order indicating that $G_{m,l,n}$ is a small-world network.

$$D_{G_{m,l,n}} = \frac{l - \epsilon}{2} + (l - \epsilon)\left[log_{(lm)}(V_{G_{m,l,n}}) - 1\right] with \begin{cases} \epsilon = 0, & \text{if } l \text{ is even,} \\ \epsilon = 1, & \text{if } l \text{ is odd} \end{cases} \tag{3.22}$$

☞ **Discussion**

According to the above analysis, the generalized Small-World Exponential network $G_{m,l,n}$ is a scale-free network, because its degree distribution follows a power-law form, its clustering coefficient is missing and a short diameter, which satisfy the properties for scale-free networks. We deduce that the model of the generalization of Small-World Exponential network affects the properties of small-world networks and it is merged by the scale-free characteristics.

3.2.2.3 Number of spanning trees of the generalized Small-World Exponential network $G_{m,l,n}$

The enumeration of spanning trees is a fundamental issue in many problems encountered in the network analysis. However, explicitly determining this interesting quantity in networks is a theoretical challenge, especially for complex networks. Fortunately, the construction of the generalized Small-World Exponential network $G_{m,l,n}$ makes it possible to derive the exact formula of this number using the generalized decomposition method following one articulation node.

Theorem 3.2.2. *(Mokhlissi et al., 2018b) Let $G_{m,l,n}$ be the generalized Small-World Exponential networks. The complexity of $G_{m,l,n}$ is given by the following formula:*

$$\tau(G_{m,l,n}) = l^{\frac{(lm)^n-1}{lm-1}}, \quad n \geq 1. \tag{3.23}$$

Proof: The generalized Small-World Exponential network $G_{m,l,n}$ can be decomposed into cyclic subgraphs $\Theta_{(n)}$ according to the articulation nodes. Using Equation 2.9, we obtain: $\tau(G_{m,l,n}) = \prod^{M_{\Theta_{(n)}}} \tau(\Theta_{(n)}) = \tau(\Theta_{(n)})^{M_{\Theta_{(n)}}}$ with $M_{\Theta_{(n)}}$ is the number of cyclic subgraphs in $G_{m,l,n}$. In order to calculate the number of spanning trees of $G_{m,l,n}$, we need to find firstly the number of cyclic subgraphs in $G_{m,l,n}$. From our network, for i from 1 to n, we see: $M_{\Theta_{(i)}} = lm \times M_{\Theta_{(i-1)}} + 1$. Then, we multiply the equation of $M_{\Theta_{(n-1)}}$ by (lm), the equation of $M_{\Theta_{(n-2)}}$ by $(lm)^2$ and so on until the last equation $M_{\Theta_{(1)}}$ which will be multiplied by $(lm)^{n-1}$. Summing all the obtained equations: $\sum_{i=0}^{n-1}(lm)^i M_{\Theta_{(n-i)}} = \sum_{i=0}^{n-1}(lm)^{i+1} M_{\Theta_{(n-i-1)}} + \sum_{i=0}^{n-1}(lm)^i$. We find the number of cyclic subgraphs in $G_{m,l,n}$: $M_{\Theta_{(n)}} = \frac{(lm)^n-1}{(lm)-1}$. We replace it in the equation of $\tau(G_{m,l,n})$ with $\tau(\Theta_{(n)}) = l$, hence we obtain: $\tau(G_{m,l,n}) = l^{\frac{(lm)^n-1}{(lm-1)}}$.

3.2.2.4 Entropy of spanning trees of the generalized Small-World Exponential network $G_{m,l,n}$

The number of spanning trees of the generalized Small-World Exponential network grows exponentially, so we can calculate its entropy of spanning trees according to the definition of the entropy in Equation 2.1. Let $\rho_{G_{m,l,n}}$ be the entropy of spanning trees for the generalized Small-World Exponential network.

Corollary 3.2.1. *(Mokhlissi et al., 2018b)The entropy of spanning trees of the generalized Small-World Exponential network $G_{m,l,n}$ is given by:*

$$\rho_{G_{m,l,n}} = \frac{ln(l)}{(l-1)} \tag{3.24}$$

Proof: By applying Equation 2.1:

$\rho_{G_{m,l,n}} = \lim_{V_{G_{m,l,n}} \to \infty} \frac{\ln|\tau(G_{m,l,n})|}{|V_{G_{m,l,n}}|} = \lim_{n\to\infty} \frac{\ln|\tau(G_{m,l,n})|}{|V_{G_{m,l,n}}|} = \lim_{n\to\infty} \frac{\ln(l^{\frac{(lm)^n-1}{(lm)-1}})}{\frac{(lm)^n(l-1)+(m-1)l}{lm-1}} = \lim_{n\to\infty} \frac{1}{(lm-1)} \times \frac{\ln(l)}{1-\frac{(m-1)l}{(lm)-1}}$. Then, the result. The entropy $\rho_{G_{m,l,n}}$ depends just on the size of the cyclic subgraphs l and not on the dimension of the cyclic subgraphs m.

3.3 Analysis of Koch Network

In this section, another class of small-world networks called the *Koch network* is studied analytically (See Figure 3.3) (Zhang et al., 2010a, 2009). This network is derived from the class of Koch curves, which are one of the interesting families of fractals that helps to understand the geometric fractals in real systems (Von Koch, 1906). The construction of the Koch network is based on triangles, so we propose an iterative algorithm to investigate its construction. We calculate its topological properties, showing that the Koch network simultaneously exhibits scale-free behaviour and small-world properties and we enumerate the exact number of its spanning trees using the generalized decomposition method following one articulation node. Besides, we put forward a family of the *generalized koch network* (Mokhlissi et al., 2018b) (See Figure 3.4), where the difference relies on the size and the dimension of the added cyclic subgraphs. We propose analytically an iterative algorithm of its construction, we determine its properties, indicating that the generalized Koch network is characterized by scale-free properties and we calculate its number of spanning trees using the same method. In the end, we evaluate its entropy of spanning trees to quantify its robustness.

3.3.1 Koch network $C_{1,3,n}$

3.3.1.1 Construction of the Koch network $C_{1,3,n}$

According to the construction process of the famous Koch curve (Von Koch, 1906), we introduce an iterative algorithm to construct the Koch network, denoted by $C_{1,3,n}$ after n generation, '1' refers to the number of the added triangles for each node of every existing triangle of the previous iteration and '3' refers to the size of triangles. The algorithm is presented as follows: Initially ($n = 0$), $C_{1,3,0}$ is a simple triangle. For $n \geq 1$, $C_{1,3,n}$ is obtained from $C_{1,3,n-1}$ by adding one triangle for each of the three nodes of every existing triangles in $C_{1,3,n-1}$. The degree of the articulation nodes of the first iteration (The green nodes in Figure 3.3) is 2^{n+1}. The growth process to the next iteration keeps on in a similar way. Figure 3.3 shows the growth process for two generations of the Koch network $C_{1,3,n}$.

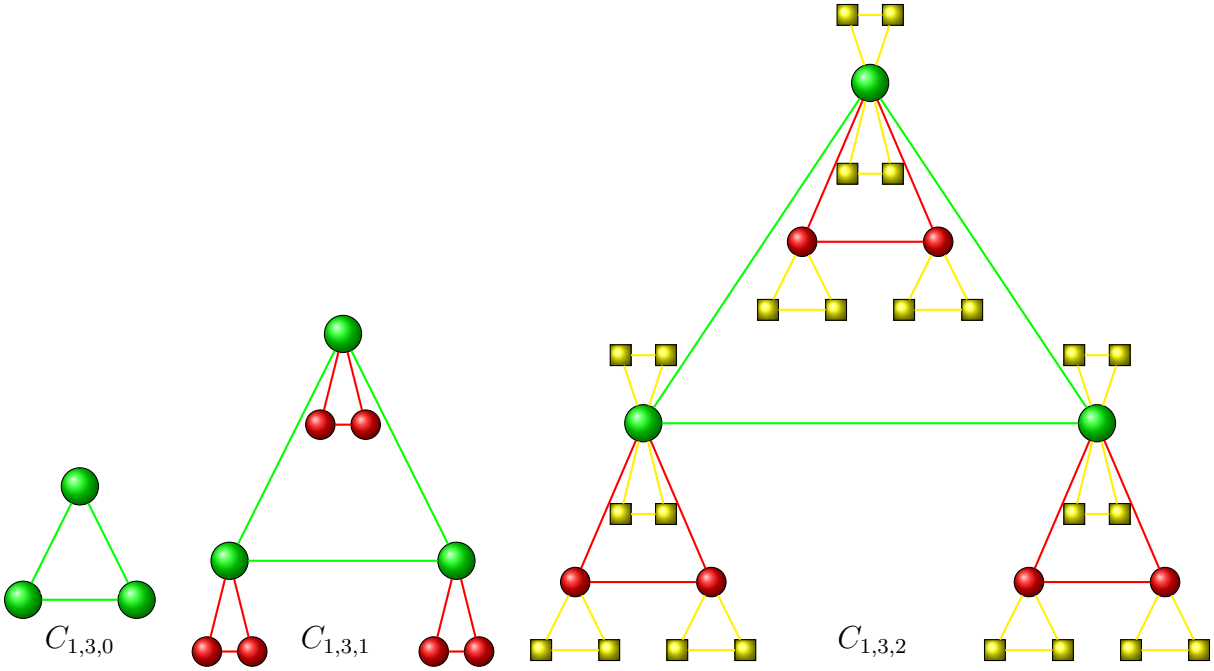


Figure 3.3: The first two generations of the Koch network $C_{1,3,n}$

Next, we determine the number of nodes, edges, faces and the average degree of the Koch network. According to the construction of $C_{1,3,n}$, we get:

- Let $V_{C_{1,3,n}}$ be the number of nodes of $C_{1,3,n}$. For i from 1 to n, we notice $V_{C_{1,3,i}} = 4V_{C_{1,3,i-1}} - 3$. Then, the equation of $V_{C_{1,3,n-1}}$ is multiplied by 4, the equation of $V_{C_{1,3,n-2}}$ by 4^2 and so on until the last equation $V_{C_{1,3,1}}$ which is multiplied by

4^{n-1}. Summing all the obtained equations: $\sum_{i=0}^{n-1} 4^i V_{C_{1,3,n-i}} = \sum_{i=0}^{n-1} 4^{i+1} V_{C_{1,3,n-i-1}} - 3\sum_{i=0}^{n-1} 4^i$. We find: $V_{C_{1,3,n}} = 4^n V_{C_{1,3,0}} - 3\sum_{i=0}^{n-1} 4^i$ with $V_{C_{1,3,0}} = 3$. Thus, **the number of nodes of $C_{1,3,n}$ is:**

$$V_{C_{1,3,n}} = 2 \times 4^n + 1, n \geq 0. \tag{3.25}$$

- Similarly, we find **the number of edges of $C_{1,3,n}$:**

$$E_{C_{1,3,n}} = 3 \times 4^n, n \geq 0. \tag{3.26}$$

- **The number of faces of $C_{1,3,n}$ is:**

$$F_{C_{1,3,n}} = 4^n + 1, n \geq 0. \tag{3.27}$$

We can obtain the number of faces of $C_{1,3,n}$ also by using Corollary 1.2.1.

- **The average degree of** $C_{1,3,n}$ is (which is approximately 3 for large n):

$$< z >_{C_{1,3,n}} = \frac{2E_{C_{1,3,n}}}{V_{C_{1,3,n}}} = \frac{6 \times 4^n}{2 \times 4^n + 1}, n \geq 0 \tag{3.28}$$

3.3.1.2 Structural properties of the Koch network $C_{1,3,n}$

- **The degree distribution**

We denote $k_u(n)$ as the degree of a node u at the step n. When a node u is added to the network at step $n_u \geq 0$, it has a degree $k_u(n_u) = 2$. To determine $k_u(n)$, we first determine the number of triangles involving the node u at step n that is represented by $M_{\Delta_{(u,n)}}$. These triangles will create new nodes connected to the node u at step $n+1$. Then, at step $n_u, M_{\Delta_{(u,n_u)}} = 1$. By construction, $M_{\Delta_{(u,n)}} = 2M_{\Delta_{(u,n-1)}}$. We can derive $M_{\Delta_{(u,n)}} = 2^{n-n_u}$. Note that the relation between $k_u(n)$ and $M_{\Delta_{(u,n)}}$ satisfies

$$k_u(n) = 2M_{\Delta_{(u,n)}} = 2^{n-n_u+1}. \tag{3.29}$$

In this way, at step n the degree of node u has been computed explicitly. From Equation 3.29, we can see that at each step the degree of a node doubles:

$$k_u(n) = 2k_u(n-1). \tag{3.30}$$

Then, the cumulative degree distribution is given by

$$P_{cum}(k) = \frac{1}{V_{C_{1,3,n}}} \sum_{i \leq n_u} M_v(i) = \frac{2 \times 4^{n_u} + 1}{2 \times 4^n + 1}. \tag{3.31}$$

with $M_v(n)$ is the number of new nodes at step n. We can calculate it as follows: by construction, we notice that for $n \geq 1$, each triangle in the network will lead to an addition of six new nodes and nine new edges at step n, then we can easily obtain the following relation: $M_v(n) = 6M_{\Delta_{(n-1)}}$ with $M_{\Delta_{(n-1)}}$ is the total number of triangles existing at step $n-1$. So, $M_{\Delta_{(n)}} = 4M_{\Delta_{(n-1)}}$. Considering the initial condition, $M_{\Delta_{(0)}} = 1$, it follows that $M_{\Delta_{(n)}} = 4^n$. Then, $M_v(n) = 6 \times 4^{n-1}$ and $V_{C_{1,3,n}} = \sum_{i=0}^{n} M_v(i)$. From Equation 3.29, we obtain $n_u = n + 1 - \frac{\ln k}{\ln 2}$. Then, the cumulative degree distribution of $C_{1,3,n}$ will be:

$$P_{cum}(k) = \frac{2 \times 4^n \times 4 \times k^{-\frac{\ln 4}{\ln 2}} + 1}{2 \times 4^n + 1} \tag{3.32}$$

For large n, we can obtain:

$$P_{cum}(k) \sim 4 \times k^{-2}. \tag{3.33}$$

Consequently, for the cumulative degree distribution $P_{cum}(k) \sim k^{-\gamma+1}$ means that the degree distribution follows a power-law form $P(k) \sim k^{-\gamma}$. So, the degree distribution of the Koch network $C_{1,3,n}$ follows a power-law form with $\gamma = 3$.

- **Clustering coefficient**

The clustering coefficient of the Koch network $C_{1,3,n}$ is calculated analytically as follows: By construction, for any given node u having a degree k_u, we have obtained $k_u = 2e_u$ for all nodes at all steps, with e_u is the number of existing triangles attached to node u. So, there is a one-to-one correspondence between the clustering coefficient of a node and its degree. According to Equation 1.7 of the clustering coefficient, for a node u of the degree k_u, we have

$$C_u = \frac{1}{k_u - 1} \tag{3.34}$$

Which is inversely proportional to k_u in the limit of large k_u. The scaling of $C_u \sim k^{-1}$ has been observed in many real-world scale-free networks. After n iteration evolutions, the clustering coefficient $C_{C_{1,3,n}}$ of the whole network is given by

$$C_{C_{1,3,n}} = \frac{1}{V_{C_{1,3,n}}} \sum_{i=0}^{n} \frac{1}{k_u(i) - 1} M_v(i) \tag{3.35}$$

Where the sum runs over all the nodes and $k_u(i)$ is the degree of those nodes created

at step i, which is given by Equation 3.29. For large n, the clustering coefficient of the Koch network $C_{1,3,n}$ converges to a nonzero value $C_{C_{1,3,n}} = 0.82008$. Therefore, the Koch network $C_{1,3,n}$ has a high clustering coefficient.

- **Diameter:**

  The diameter of $C_{1,3,n}$ is calculated as follows: From Figure 3.3, it is clear that $D_{C_{1,3,0}} = 1$. At each iteration $n \geq 1$, the diameter of $C_{1,3,n}$ increases by 2 at most. So for i from 1 to n, we have: $D_{C_{1,3,i}} = D_{C_{1,3,i-1}} + 2$ Summing all the obtained equations, we find: $D_{C_{1,3,n}} = D_{C_{1,3,0}} + 2n$. Thus, **the diameter of $C_{1,3,n}$ is:**

$$D_{C_{1,3,n}} = 2n + 1. \tag{3.36}$$

  For large n, the number of nodes will be $V_{C_{1,3,n}} = 2.4^n$. Then, the diameter can be presented by another formula $D_{C_{1,3,n}} = 2log_{(4)}(\frac{V_{C_{1,3,n}}}{2}) + 1$, which grows logarithmically with the network order indicating that $C_{1,3,n}$ is a small-world network.

☞ **Discussion**

According to the above results, the Koch network $C_{1,3,n}$ has specific properties characterizing a majority of real-life network systems: its degree distribution follows a power-law form, a high clustering coefficient and a small diameter, which indicate that $C_{1,3,n}$ is simultaneously a small-world network and scale-free network.

3.3.1.3 Number of spanning trees of the Koch network $C_{1,3,n}$

Using the same method applied on the first small-world network (the generalized decomposition method following one articulation node), the exact number of spanning trees of the Koch network $C_{1,3,n}$ is obtained.

Theorem 3.3.1. *Let $C_{1,3,n}$ be the Koch network. The number of spanning trees of $C_{1,3,n}$ is given by the following formula:*

$$\tau(C_{1,3,n}) = 3^{4^n}, \quad n \geq 0. \tag{3.37}$$

Proof: From Figure 3.3, we see that $C_{1,3,n}$ contains several triangles $\Delta_{(n)}$. Using Equation 2.9, $\tau(C_{1,3,n}) = \prod^{M_{\Delta_{(n)}}} \tau(\Delta_{(n)}) = \tau(\Delta_{(n)})^{M_{\Delta_{(n)}}}$ with $M_{\Delta_{(n)}} = 4^n$ is the number of triangles in $C_{1,3,n}$. Replacing this result in the equation of $\tau(C_{1,3,n})$ with $\tau(\Delta_{(n)}) = 3$, hence we obtain: $\tau(C_{1,3,n}) = 3^{4^n}, n \geq 0$.

3.3.2 Generalized Koch Network $C_{m,l,n}$

3.3.2.1 Construction of the generalized Koch network $C_{m,l,n}$

Inspired by the algorithm of the Koch network, we propose a family of the generalized Koch network, denoted as $C_{m,l,n}$ with two integer parameters: $l > 3$ is the size of the cyclic subgraph and $m > 1$ is the dimension of the cyclic subgraph i.e the number of the added cyclic subgraphs for each node of every existing cyclic subgraph of the previous iteration. The iterative algorithm of its construction is as follows: Initially ($n = 0$), $C_{m,l,0}$ is a cyclic graph with the size l. For $n \geq 1$, $C_{m,l,n}$ is obtained from $C_{m,l,n-1}$ by adding m new cyclic subgraphs having the size l for each node of every existing cyclic subgraphs in $C_{m,l,n-1}$. The growth process of the generalized Koch network to the next generation keeps on in a similar way. The degree of the articulation nodes of the first iteration (The green nodes in Figure 3.4) is $2(m+1)^n$. Figure 3.4 illustrates the growing process of the generalized Koch network for the first two generations with $m = 2$ and $l = 4$. Then, the explicit results for the number of nodes, number of edges, number of faces and average degree of $C_{m,l,n}$ are stated. According to its construction, we get:

- The number of nodes of $C_{m,l,n}$ is calculated as follows: For i from 1 to n, we notice: $V_{C_{m,l,i}} = (lm+1) \times V_{C_{m,l,i-1}} - lm$. Then, we multiply the equation of $V_{C_{m,l,n-1}}$ by $(lm+1)$, the equation of $V_{C_{m,l,n-2}}$ by $(lm+1)^2$ and so on until the last equation $V_{C_{m,l,1}}$, which will be multiplied by $(lm+1)^{(n-1)}$. Summing all the obtained equations: $\sum_{i=0}^{n-1}(lm+1)^i V_{C_{m,l,n-i}} = \sum_{i=0}^{n-1}(lm+1)^{i+1}V_{C_{m,l,n-i-1}} - lm\sum_{i=0}^{n-1}(lm+1)^i$. We find: $V_{C_{m,l,n}} = (lm+1)^n V_{C_{m,l,0}} - lm\sum_{i=0}^{n-1}(lm+1)^i$ with $V_{C_{m,l,0}} = l$. Thus, **the number of nodes of $C_{m,l,n}$ is:**

$$V_{C_{m,l,n}} = (l-1)(lm+1)^n + 1, n \geq 0. \tag{3.38}$$

- Similarly, we find **the number of edges of $C_{m,l,n}$:**

$$E_{C_{m,l,n}} = l(lm+1)^n, n \geq 0. \tag{3.39}$$

- **The number of faces of $C_{m,l,n}$ is:**

$$F_{C_{m,l,n}} = (lm+1)^n + 1, n \geq 0. \tag{3.40}$$

We can obtain the number of faces of $C_{m,l,n}$ also by using Corollary 1.2.1.

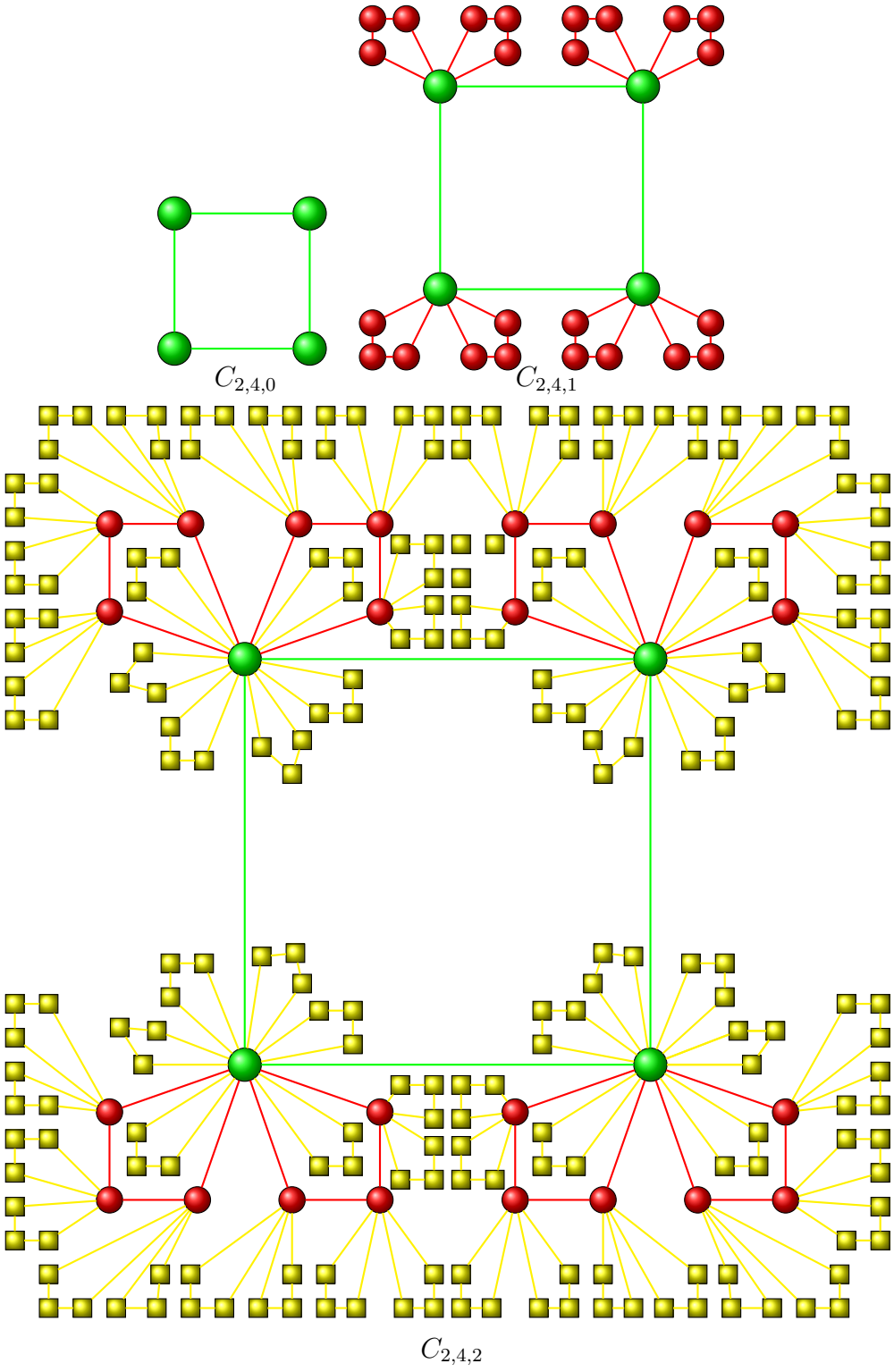


Figure 3.4: The first two generations of the generalized Koch Network $C_{2,4,n}$

- **The average degree of** $C_{m,l,n}$ is (which is approximately 3 for large n):

$$< z >_{C_{m,l,n}} = \frac{2E_{C_{m,l,n}}}{V_{C_{m,l,n}}} = \frac{2l(lm+1)^n}{(l-1)(lm+1)^n+1}, n \geq 0 \tag{3.41}$$

3.3.2.2 Structural properties of the generalized Koch network $C_{m,l,n}$

- **The degree distribution**

  To calculate the degree distribution of the generalized Koch network $C_{m,l,n}$, we use the same process applied on the degree distribution of the Koch network $C_{1,3,n}$. Then, the cumulative degree distribution of $C_{m,l,n}$ is given by:

$$P_{cum}(k) = \frac{(l-1)(lm+1)^n \times (\frac{k}{2})^{-\frac{\ln(lm+1)}{\ln(m+1)}} + 1}{(l-1)(lm+1)^n + 1}. \tag{3.42}$$

  For large n, we can obtain:

$$P_{cum}(k) \sim 2^{\frac{\ln(lm+1)}{\ln(m+1)}} \times k^{-\frac{\ln(lm+1)}{\ln(m+1)}}. \tag{3.43}$$

  So the degree distribution of the generalized Koch network $C_{m,l,n}$ follows a power-law form with the exponent $\gamma = 1 + \frac{\ln(lm+1)}{\ln(m+1)}$. For $m \geq l$, the exponent γ belongs to the interval $[2, 3]$, which means that $C_{m,l,n}$ is a scale-free network.

- **Clustering coefficient**

  From the construction of the generalized Koch network $C_{m,l,n}$ with $m > 1, l > 3$ and $n \geq 0$, it is obvious that $C_{m,l,n}$ have zero clustering because the neighbors of any node are never neighbors of one other. The main reason is the absence of triangles in $C_{m,l,n}$ because the size l of the cyclic subgraphs is greater than 3. Therefore, one of the properties of the small-world network is missing.

- **Diameter:**

  Let $D_{C_{m,l,n}}$ be the diameter of $C_{m,l,n}$ created at generation n. **This diameter** can be presented by the following formula for $n \geq 0$:

$$D_{C_{m,l,n}} = \frac{l-\epsilon}{2} + n(l-\epsilon) with \begin{cases} \epsilon = 0, & \text{if } l \text{ is even,} \\ \epsilon = 1, & \text{if } l \text{ is odd} \end{cases} \tag{3.44}$$

For large n, the number of nodes will be $V_{C_{m,l,n}} = (l-1)(lm+1)^n$. Then, we can present the diameter by another formula which grows logarithmically with the network order indicating that $C_{m,l,n}$ is a small-world network.

$$D_{C_{m,l,n}} = \frac{l-\epsilon}{2} + (l-\epsilon)\left[log_{(lm+1)}\left(\frac{V_{C_{m,l,n}}}{l-1}\right)\right] with \begin{cases} \epsilon = 0, & \text{if } l \text{ is even,} \\ \epsilon = 1, & \text{if } l \text{ is odd} \end{cases}$$

☞ **Discussion**

According to the above discussion, the generalized Koch network $C_{m,l,n}$ incorporates some specific properties: its degree distribution follows a power-law form, a zero clustering coefficient and short diameter, indicating that the generalized Koch network $C_{m,l,n}$ is a scale-free network. We deduce that the model of the generalization of Koch network is limited just to scale-free property and not small-world properties.

3.3.2.3 Number of spanning trees of the generalized Koch Network $C_{m,l,n}$

In order to calculate the number of spanning trees of the generalized Koch Network $C_{m,l,n}$, we use the same method as the last network studied before: The generalized decomposition method following one articulation node.

Theorem 3.3.2. *(Mokhlissi et al., 2018b) Let $C_{m,l,n}$ be the generalized Koch network. The number of spanning trees of $C_{m,l,n}$ is given by the following formula:*

$$\tau(C_{m,l,n}) = l^{(lm+1)^n}, \quad n \geq 0 \tag{3.45}$$

Proof: From Figure 3.4, we see that $C_{m,l,n}$ contains several cyclic subgraphs $\Theta_{(n)}$. Using Equation 2.9, $\tau(C_{m,l,n}) = \prod^{M_{\Theta_{(n)}}} \tau(\Theta_{(n)}) = \tau(\Theta_{(n)})^{M_{\Theta_{(n)}}}$ with $M_{\Theta_{(n)}}$ is the number of cyclic subgraphs in $C_{m,l,n}$. For i from 1 to n, we notice: $M_{\Theta_{(i)}} = (lm+1)M_{\Theta_{(i-1)}}$. So, the number of cyclic subgraphs in $C_{m,l,n}$ is: $M_{\Theta_{(n)}} = (lm+1)^n$. Replacing this result in the equation of $\tau(C_{m,l,n})$ with $\tau(\Theta_{(n)}) = l$, hence we obtain: $\tau(C_{m,l,n}) = l^{(lm+1)^n}, n \geq 0$.

3.3.2.4 Entropy of spanning trees of the generalized Koch network $C_{m,l,n}$

The number of spanning trees of the generalized Koch network grows exponentially, so we can calculate its entropy of spanning trees according to the definition of the entropy in Equation 2.1. Let $\rho_{C_{m,l,n}}$ be the entropy of spanning trees for the generalized Koch network.

Corollary 3.3.1. *(Mokhlissi et al., 2018b) The entropy of spanning trees of the generalized Koch network $C_{m,l,n}$ is:*

$$\rho_{C_{m,l,n}} = \frac{ln(l)}{(l-1)} \tag{3.46}$$

Proof: By applying Equation 2.1:

$$\rho_{C_{m,l,n}} = \lim_{V_{C_{m,l,n}} \to \infty} \frac{\ln|\tau(C_{m,l,n})|}{|V_{C_{m,l,n}}|} = \lim_{n\to\infty} \frac{\ln|\tau(C_{m,l,n})|}{|V_{C_{m,l,n}}|} = \lim_{n\to\infty} \frac{\ln(l^{(lm+1)^n})}{(l-1)(lm+1)^n+1} = \lim_{n\to\infty} \frac{\ln(l)}{(l-1)+\frac{1}{(lm+1)^n}}.$$

Then, the result. The entropy $\rho_{C_{m,l,n}}$ depends just on the size of the cyclic subgraphs l and not on the dimension of the cyclic subgraphs m.

☞ **Comparison between the entropy of spanning trees of the Small-World Exponential network and the Koch network.**

From the results of Corollary 3.2.1 and 3.3.1, we find that the generalized Small-World Exponential network and the generalized Koch network have the same entropy $\rho_{G_{m,l,n}} = \rho_{C_{m,l,n}} = \frac{ln(l)}{(l-1)}$. It means that the generalized Small-World Exponential network and the generalized Koch network have the same robustness despite the fact that their structures, properties and complexities are different. From these results, we can deduce that entropy of spanning trees is one of the parameters that characterize the performance of a topology of a network.

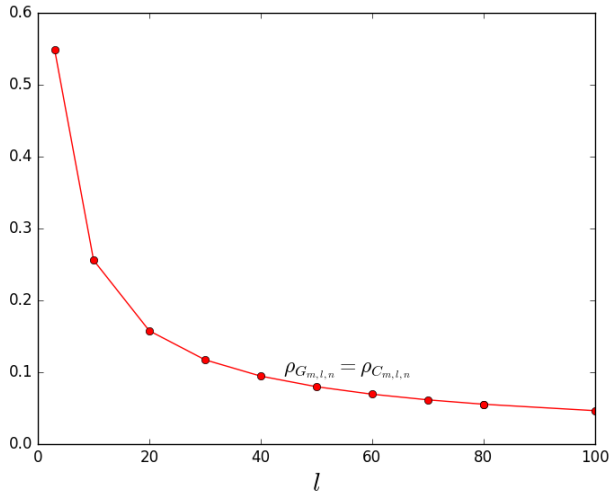


Figure 3.5: The entropy of spanning trees of the generalized Small-World Exponential network $G_{m,l,n}$ and the generalized Koch network $C_{m,l,n}$

Figure 3.5 shows the increase of the size of the cyclic subgraphs l leads to the decreasing of the entropy of spanning trees of $G_{m,l,n}$ and $C_{m,l,n}$. This result proves that these networks having a low value of l are more robust than those having a high value of l.

From Table 3.1, we compare the entropy of spanning trees of the Small-World Exponential network $G_{1,3,n}$ and the Koch network $C_{1,3,n}$ (0.549) with those of other networks having the same average degree 3. We notice that the value of their entropy of spanning trees is the smallest known for networks with the average degree 3. This reflects the fact that the Koch network and the Small-World Exponential network are less robust and their structural topology is more vulnerable to a random breakdown of links than other networks having the same average degree.

Type of network	$< z >$	ρ
Koch network (Mokhlissi et al., 2018b)	**3**	**0.549**
Small-World Exponential network (Mokhlissi et al., 2016a, 2018b)	**3**	**0.549**
The Hanoi network (Zhang et al., 2016)	3	0.677
The 2-Flower network(Mokhlissi et al., 2017a)	3	0.6931
The 3-2-12 lattices (Wu, 1977)	3	0.721
The 4-8-8 bathroom tile (Wu, 1977)	3	0.787
Honeycomb lattice (Shrock and Wu, 2000)	3	0.807

Table 3.1: The entropy of spanning trees of several networks having the same average degree 3.

3.4 Analysis of Farey network

In this section, we focus on the *Farey network* (See Figure 3.6) (Zhang and Comellas, 2011; Xiao and Zhao, 2013; Jiang et al., 2018). It was first introduced by Matula and Kornerup in 1979 (Matula and Kornerup, 1979), also studied by Colbourn in 1982 (Colbourn, 1982) and further was used as a model of the small-world network by Zhang and Comellas (Zhang and Comellas, 2011) due to its remarkable properties of real networks. The Farey network is derived from the famous Farey sequence (Hardy et al., 1979). This network shows the small-world effect: a small diameter or a short average distance and a large clustering coefficient. We propose a generalization of this model by adding q nodes to every iterative edge in each step (See Figure 3.9) (Mokhlissi et al., 2019a). We analyze the structural properties of the Farey network and its generalization such as the size, the order, the number of faces, the average degree, the diameter, the degree distribution and the clustering coefficient, proving that the *generalized Farey network* affects the properties of small-world networks, it combines the small-world and the scale-free properties. Then, we derive the exact expression of their number of spanning trees using the electrically equivalent transformations. Finally, we further calculate the entropy of spanning trees

and we compare it with those for other networks having the same average degree.

3.4.1 Farey network $F_{1,n}$

3.4.1.1 Construction of the Farey network $F_{1,n}$

The construction of the Farey network can mimic processes that drive the real networks evolution through time. For example, in social, collaborative and some technological and biological networks, it is very likely that a new node will join to nodes that are already adjacent. The Farey network is denoted by $F_{1,n}$ with $n \geq 0$ iterations and '1' refers to the added node for every iterative edge. Its construction follows this algorithm: At $n = 0$, $F_{1,0}$ has two nodes related by an edge. For $n \geq 1$, $F_{1,n}$ is obtained from $F_{1,n-1}$ by adding to every iterative edge introduced at the previous iteration a new node adjacent to the two end nodes of this edge. The growth process of the Farey network to the next iterations continues in a similar way. Figure 3.6 illustrates the first three iterative steps of the Farey network $F_{1,n}$.

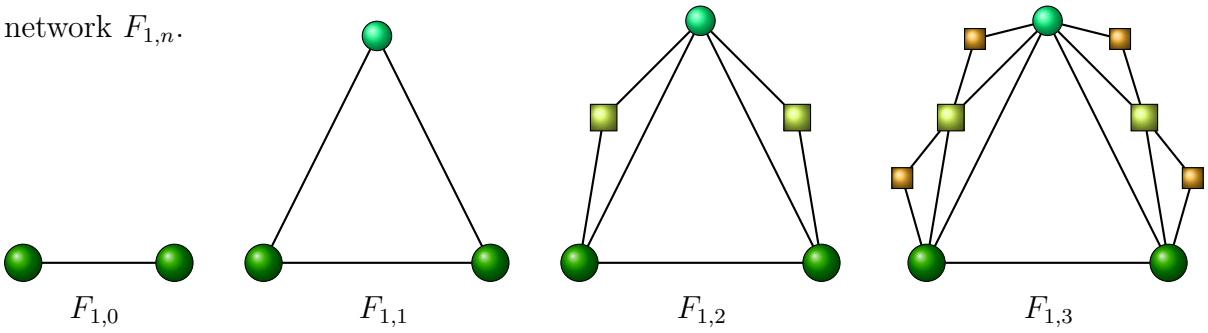


Figure 3.6: The first three generations of the Farey network $F_{1,n}$

Next, we calculate the number of nodes, edges, faces and the average degree of the Farey network. According to the construction of $F_{1,n}$, we get:

- Let $V_{F_{1,n}}$ be the number of vertices of $F_{1,n}$. For i from 1 to n, we have $V_{F_{1,i}} = 2V_{F_{1,i-1}} - 1$. Then, we multiply the equation of $V_{F_{1,n-1}}$ by 2, the equation of $V_{F_{1,n-2}}$ by 2^2 and so on until the last equation $V_{F_{1,1}}$ which will be multiplied by 2^{n-1}. Summing all the obtained equations, we get $\sum_{i=0}^{n-1} 2^i V_{F_{1,n-i}} = \sum_{i=0}^{n-1} 2^{i+1} V_{F_{1,n-i-1}} - \sum_{i=0}^{n-1} 2^i$. We find the following results: $V_{F_{1,n}} = 2^n V_{F_{1,0}} - \sum_{i=0}^{n-1} 2^i$ with $V_{F_{1,0}} = 2$. Thus, **the number of nodes of $F_{1,n}$ is:**

$$V_{F_{1,n}} = 2^n + 1, \quad n \geq 0. \tag{3.47}$$

- Similarly, we find **the number of edges of $F_{1,n}$:**

$$E_{F_{1,n}} = 2^{n+1} - 1, \quad n \geq 0. \tag{3.48}$$

- **The number of faces of $F_{1,n}$ is:**

$$F_{F_{1,n}} = 2^n, \quad n \geq 0. \tag{3.49}$$

- **The average degree of** $F_{1,n}$ is (which tends to 4 when the network size is large enough)

$$< z >_{F_{1,n}} = \frac{2E_{F_{1,n}}}{V_{F_{1,n}}} = \frac{2(2^{n+1}-1)}{2^n+1}, \quad n \geq 0. \tag{3.50}$$

3.4.1.2 Structural properties of the Farey network $F_{1,n}$

- **The degree distribution**

We calculate the degree distribution of the Farey network $F_{1,n}$ as follows: at $n = 0$, the network has two nodes of degree 1. At step n_u, a new node u is added to the network. this node has degree $k_u(n_u) = 2$ and it is related to two generating edges. From the construction of the Farey network, all its nodes, except the initial two nodes, are always related to 2 generated edges and will increase their degrees by 2 at the next iteration. We denote the degree of the node u at step n by $k_u(n)$. By construction, we have:

$$\begin{cases} k_u(n) = k_u(n-1) + 2, & u \in V_{F_{1,n}}, u \neq \frac{0}{1}, \frac{1}{1} \\ k_{\frac{0}{1}}(n) = k_{\frac{1}{1}}(n) = n + 1. \end{cases} \tag{3.51}$$

Which leads to:

$$k_u(n) = 2(n - n_u + 1) \tag{3.52}$$

Which is linear. Therefore, the degree distribution of nodes of the Farey network $F_{1,n}$ is as follows: the number of nodes of degree $2 \times 1, 2 \times 2, 2 \times 3, ..., 2 \times n$, equals, respectively $2^{n-1}, 2^{n-2}, ..., 2, 1$ and the two initial nodes have degree $n + 1$. Then, the cumulative degree distribution of $F_{1,n}$ is given by:

$$P_{cum}(k) = \sum_{i=0 \leq n_u} \frac{M_v(i)}{V_{F_{1,n}}} = \frac{2}{2^n+1} + \sum_{i=1 \leq n_u} \frac{2^{i-1}}{2^i+1} \tag{3.53}$$

with $M_v(n)$ the number of new nodes at step n. We can calculate it as follows: by

construction, we notice that for $n \geq 1$, $M_v(n) = M_e(n-1)$ with $M_e(n-1)$ is the number of new links at step $n-1$. Considering the initial condition $M_v(0) = 2$ and $M_e(0) = 1$, it follows that $M_v(n) = 2^{n-1}$ and $V_{F_{1,n}} = \sum_{i=0}^{n} M_v(i)$. From Equation 3.52, we obtain $n_u = n - \frac{k-2}{2}$. Then, **the cumulative degree distribution of** $F_{1,n}$ will be for large n:

$$P_{cum}(k) = \frac{1 + 2^{n-\frac{k-2}{2}}}{1 + 2^n} \sim (2)^{-\frac{k}{2}} \tag{3.54}$$

So the degree distribution of $F_{1,n}$ follows an exponential distribution.

- **Clustering coefficient**

We can calculate the clustering coefficient for the Farey network $F_{1,n}$ according to Equation 1.7 as follows: When a new node u is added to the network at step n_u, its degree is $k_u = 2$ and $e_u = 1$. By construction, for all nodes at all steps, we have $e_u = k_u - 1$. So, the expression for the clustering coefficient is $C_u = \frac{2}{k_u}$. Then, the clustering coefficient of the Farey network $F_{1,n}$ is easily obtained for any arbitrary step n:

$$\begin{aligned} C_{F_{1,n}} =& \frac{1}{V_{F_{1,n}}} \left[\sum_{i=1}^{n} \frac{1}{i} .2^{i-1} + \frac{2}{n+1} .2 \right] \\ =& \frac{1}{2^n + 1} \left[2^n \ln 2 - \frac{1}{2} \Phi\left(\frac{1}{2}, 1, 1+n\right) + \frac{4}{n+1} \right] \end{aligned} \tag{3.55}$$

where $\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}$ denotes the Lerch transcendent function. The clustering coefficient of the Farey network $F_{1,n}$ tends to $\ln 2$ for large n. Thus, the clustering coefficient of $F_{1,n}$ is high.

- **Diameter:**

Let $D_{F_{1,n}}$ be the diameter of $F_{1,n}$. It can be calculated as follows: Clearly, at steps $n = 0$, the diameter is 1. For $n \geq 1$, the distance between two nodes is less than or equal to n to get from one node to the other in each step. Thus, the diameter of $F_{1,n}$ is:

$$D_{F_{1,n}} = n, \quad n \geq 1. \tag{3.56}$$

As $\ln(V_{F_{1,n}}) \sim n \times \ln 2$, for large n, we have diameter $D_{F_{1,n}} \sim \ln |V_{F_{1,n}}|$. It grows logarithmically with the number of nodes of the network. Since this network has a

small diameter, it is undoubtedly with a short average path length.

- **Average path length**

We calculate the average path length $l_{F_{1,n}}$ of the Farey network $F_{1,n}$ as follows: As shown in Figure 3.7, the Farey network $F_{1,n}$ may be obtained by joining at three boundary vertices (X, Y, and Z) two copies of $F_{1,n}$ that we will label as $F_{1,n}^{(\eta)}$ with $\eta = 1, 2$.

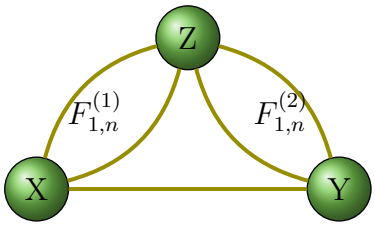


Figure 3.7: Schematic illustration of the recursive construction of the Farey network $F_{1,n}$

According to the recursive construction, the sum of distances $\sigma(F_{1,n})$ satisfies the recursive relation:

$$\sigma(F_{1,n+1}) = 2\sigma(F_{1,n}) + S_n \tag{3.57}$$

Where S_n denotes the sum of distances of pairs of nodes which are not both in the same $F_{1,n}^{(\eta)}$ subgraph. its final expression is:

$$S_n = \frac{1}{18}\Big[-5 - 3(-1)^n + 12 \times 2^n + 14 \times 2^{2n} + 12n \times 4^n\Big]. \tag{3.58}$$

Inserting Equation 3.58 into Equation 3.57 and using the initial condition $\sigma(F_{1,0}) = 1$. Then, Equation 3.57 can be solved inductively:

$$\sigma(F_{1,n}) = \frac{1}{18}\Big[5 + (-1)^n + (6n+17)2^n + (6n-5)4^n\Big]. \tag{3.59}$$

which together with the number of nodes leads to the average path length of $F_{1,n}$:

$$l_{F_{1,n}} = \frac{2^{2n}(6n-5) + 2^n(6n+17) + 5 + (-1)^n}{9 \times 2^{2n} + 9 \times 2^n} \tag{3.60}$$

For a large step n, $l_{F_{1,n}} \sim n \sim \ln|V_{F_{1,n}}|$, which shows a logarithmic scaling of the average path length with the order of the Farey network $F_{1,n}$.

☞ **Discussion**

According to the above results, the Farey network has specific properties, its degree distribution follows an exponential distribution, a high clustering coefficient, a small

diameter and a small average path length, which indicate that $F_{1,n}$ is a small-world network.

3.4.1.3 Number of spanning trees of the Farey network $F_{1,n}$

We employ the electrically equivalent technique to derive the exact expression of the number of spanning trees of Farey network $F_{1,n}$ by following the steps of Section 2.4.4.4:

1. Firstly, we start by constructing the Farey network $F_{1,n}$ and adding the weight a_0 on each of its edge.

2. Then, we apply the algorithms of the electrically equivalent transformations to obtain the change of weights of all edges of $F_{1,n-1}$ from $F_{1,n}$, the relationship between $\tau(F_{1,n})$ and $\tau(F_{1,n-1})$ and the exact value of the number of spanning trees of $F_{1,n}$ in each step. We deduce $\tau(F_{1,n}) = (2a_0)^{2^{n-1}}.\tau(F_{1,n-1})$. To prove mathematically this formula, we use the method of the mathematical induction:

   Basic step: We show that $\tau(F_{1,1}) = (2a_0)\tau(F_{1,0})$. We denote $F_{1,1}^{(0)}$ as the Farey network at iteration 1 in the transformation 0 with all its edges having the weight a_0. We show the electrically equivalent evolving process from $F_{1,1}^{(0)}$ to $F_{1,0}^{(0)}$ in Figure 3.8.

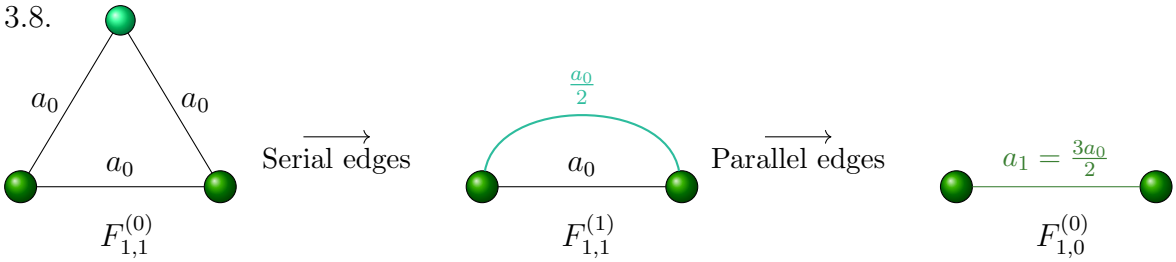


Figure 3.8: The transformations from $F_{1,1}$ to $F_{1,0}$.

Two transformations are used in sequence and the corresponding weights of the resulting edges are calculated as follows:

(a) *Serial edge:* We replace the serial edges with conductances a_0 by a new edge with the conductance $\frac{a_0.a_0}{a_0+a_0} = \frac{a_0}{2}$ to obtain a new subgraph $F_{1,1}^{(1)}$.

(b) *Parallel edge:* We merge the parallel edges having the conductance $\frac{a_0}{2}$ and a_0 to obtain a new subgraph $F_{1,0}^{(0)}$. The corresponding conductance of a new edge is $\frac{a_0}{2} + a_0 = a_1$.

The application of these two electrically equivalent transformations are used to find the formula of the number of spanning trees between $F_{1,1}^{(0)}$ and $F_{1,0}^{(0)}$. Then
$\tau(F_{1,1}) = \tau(F_{1,1}^{(0)})$
$\tau(F_{1,1}) = (2a_0)\tau(F_{1,1}^{(1)}) \rightarrow$ $(Serial\ edges)$.
$\tau(F_{1,1}) = (2a_0) \times 1 \times \tau(F_{1,1}^{(2)}) \rightarrow$ $(Parallel\ edges)$.
$\tau(F_{1,1}) = (2a_0)\tau(F_{1,0}^{(0)})$.

Induction step: We assume that $\tau(F_{1,n}) = (2a_0)^{2^{n-1}}\tau(F_{1,n-1})$. We show that $\tau(F_{1,n+1}) = (2a_0)^{2^n}\tau(F_{1,n})$.

The number of triangles added in each iteration in $F_{1,n}$ is 2^{n-1}. Note that the number of spanning trees depends on those added triangles. So, for $F_{1,n+1}$, the number of added triangles in each step is 2^n. Then, the number of spanning trees of $F_{1,n+1}$ is $\tau(F_{1,n+1}) = (2a_0)^{2^n}.\tau(F_{1,n})$.

3. Finally, we deduce the exact formula of the number of spanning trees of $\tau(F_{1,n})$.

$$\begin{aligned} \tau(F_{1,n}) &= (2a_0)^{2^{n-1}}.\tau(F_{1,n-1}) \\ &= (2a_0)^{2^{n-1}}.(2a_1)^{2^{n-2}}...(2a_{n-1})^{2^0}.\tau(F_{1,0})\ with\ \tau(F_{1,0}) = a_n \\ &= a_n.\left[(2a_0)^{2^{n-1}}.(2a_1)^{2^{n-2}}...(2a_{n-1})^{2^0}\right] \end{aligned} \tag{3.61}$$

In order to calculate this formula, first we should find the equation of a_n according to a_0. By construction, we start by the weight a_0 at step n. After applying the electrically equivalent transformations, we get the weight a_n at step 0 (As Figure 3.8, we start by the weight a_0 at step 1 and we finish by the weight a_1 at step 0). From this result, we deduce that $a_n = a_0 + \frac{a_{n-1}}{2}$. Then, $a_n = a_0[1 + \frac{1}{2} + (\frac{1}{2})^2 + ... + (\frac{1}{2})^n]$. Thus, the equation will be:

$$a_n = 2a_0.\left[1 - \left(\frac{1}{2}\right)^{n+1}\right] \tag{3.62}$$

We replace Equation 3.62 in Equation 3.61 to get:
$\tau(F_{1,n}) = 2a_0.\left[1-\left(\frac{1}{2}\right)^{n+1}\right] \times \left[2a_0\right]^{2^{n-1}} \times \left[2.2a_0.(1-(\frac{1}{2})^2)\right]^{2^{n-2}} \times ... \times \left[2.2a_0.(1-(\frac{1}{2})^n)\right]^{2^0}$
We simplify this equation to obtain the exact formula of the number of spanning trees of the Farey network:

$$\tau(F_{1,n}) = \left(2a_0\right)^{2^n} \times \left[1 - \left(\frac{1}{2}\right)^{n+1}\right] \times \prod_{i=2}^{n} \left[2.\left(1 - \left(\frac{1}{2}\right)^{i}\right)\right]^{2^{(n-i)}} \tag{3.63}$$

3.4.2 Generalized Farey network $F_{q,n}$

3.4.2.1 Construction of the generalized Farey network $F_{q,n}$

The general case of the Farey network is denoted by $F_{q,n}$ with $q > 1$ the number of added nodes to every iterative edge in each step and $n \geq 0$ generations. The construction of the generalized Farey network follows this algorithm: At $n = 0$, $F_{q,0}$ has two nodes related by an edge. For $n \geq 1$, $F_{q,n}$ is obtained from $F_{q,n-1}$ by adding to every iterative edge introduced at the previous iteration q new nodes adjacent to the two end nodes of this edge. The growth process of the generalized Farey network to the next iterations continues in a similar way. Figure 3.9 illustrates the first two iterative steps of the generalized Farey network with $q = 2$.

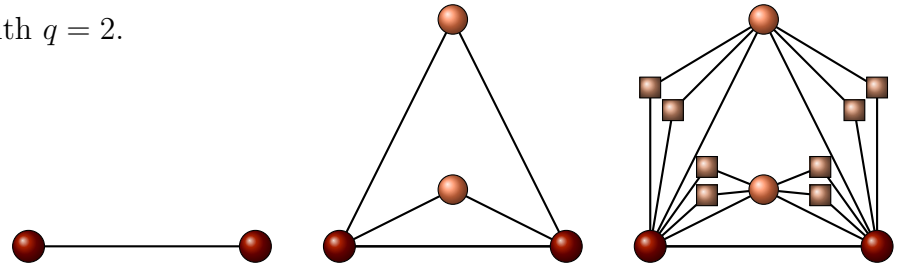


Figure 3.9: The first two iterations of the generalized Farey network with $q = 2$.

Now we compute the number of nodes, edges, faces and the average degree of the generalized Farey network $F_{q,n}$. According to the construction of $F_{q,n}$, we get:

- We denote the number of nodes by $V_{F_{q,n}}$. For i from 1 to n, we have $V_{F_{q,i}} = 2qV_{F_{q,i-1}} - (3q-2)$. Then, we multiply the equation of $V_{F_{q,n-1}}$ by $(2q)$, the equation of $V_{F_{q,n-2}}$ by $(2q)^2$ and so on until the last equation $V_{F_{q,1}}$ which will be multiplied by $(2q)^{n-1}$. Summing all the obtained equations, we get $\sum_{i=0}^{n-1}(2q)^i V_{F_{q,n-i}} = \sum_{i=0}^{n-1}(2q)^{i+1}V_{F_{q,n-i-1}} - (3q-2)\sum_{i=0}^{n-1}(2q)^i$. We find the following results: $V_{F_{q,n}} = (2q)^n V_{F_{q,0}} - (3q-2)\sum_{i=0}^{n-1}(2q)^i$ with $V_{F_{q,0}} = 2$. Thus, **the number of nodes of $F_{q,n}$ is:**
$$V_{F_{q,n}} = \frac{q(2q)^n + 3q - 2}{2q-1}, \quad n \geq 0. \tag{3.64}$$
- Similarly, we find **the number of edges of $F_{q,n}$:**
$$E_{F_{q,n}} = \frac{(2q)^{n+1} - 1}{2q-1}, \quad n \geq 0. \tag{3.65}$$

- **The number of faces of $F_{q,n}$ is:**

$$F_{F_{q,n}} = \frac{(2q)^{n+1} - q(2q)^n + (q-1)}{2q-1}, \quad n \geq 0. \tag{3.66}$$

- **The average degree of** $F_{q,n}$ is (which tends to 4 when the network size is large enough)

$$< z >_{F_{q,n}} = \frac{2E_{F_{q,n}}}{V_{F_{q,n}}} = \frac{2(2q)^{n+1} - 2}{q(2q)^n + 3q - 2}, \quad n \geq 0. \tag{3.67}$$

3.4.2.2 Structural properties of the generalized Farey network $F_{q,n}$

- **The degree distribution**

We calculate the degree distribution of the generalized Farey network $F_{q,n}$ using the same process as the degree distribution of the Farey network $F_{1,n}$. Then, the cumulative degree distribution of $F_{q,n}$ is given by:

$$P_{cum}(k) = \frac{q(2q)^n.(2q).(1 + k\frac{(q-1)}{2})^{-\frac{\ln 2q}{\ln 2}} + 3q - 2}{q(2q)^n + 3q - 2} \tag{3.68}$$

When n is large enough, **the cumulative degree distribution of** $F_{q,n}$ will be:

$$P_{cum}(k) \sim (\frac{q-1}{2})^{-\frac{\ln 2q}{\ln q}}.k^{-\frac{\ln 2q}{\ln q}} \tag{3.69}$$

We get that the degree distribution of the generalized Farey network $F_{q,n}$ follows a power-law form with the exponent $\gamma \sim 1 + \frac{\ln 2q}{\ln q}$, lies between 2 and 3. For the scale-free networks, their exponent of degree distribution lies in the same interval between 2 and 3. This result proves that the scale-free characteristic emerges from the generalized Farey network for larger q. However, this property does not exist in the particular case of Farey network when $q = 1$. In Figure 3.10, we report the simulation results of the degree distribution of the generalized Farey network for different values of q. We can see that this degree distribution follows a power law.

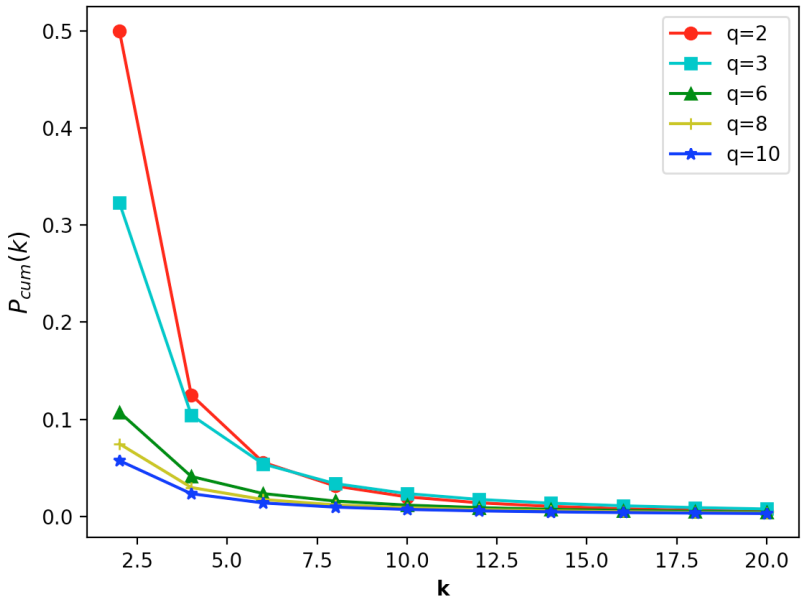


Figure 3.10: The cumulative degree distribution of $F_{q,n}$ for different values of q

- **The clustering coefficient**

  We calculate the clustering coefficient for the generalized Farey network $F_{q,n}$ using the same process as the clustering coefficient of the Farey network $F_{1,n}$: When a new node u is added to the network at step n_u, its clustering coefficient is $C_u = \frac{2}{k_u}$. Then, the clustering coefficient of the whole network $C_{F_{q,n}}$ is given by:

$$C_{F_{q,n}} = \frac{2q-1}{q(2q)^n + 3q - 2} \times \left[2.\frac{2(1-q)}{1-q^{n+1}} + \sum_{i=1}^{n} \frac{1-q}{1-q^{(n-i+1)}} .2^{i-1}.q^i \right] \tag{3.70}$$

  For large value of n, the clustering coefficient $C_{F_{q,n}}$ tends to $\frac{2q(q-1)}{2q^2-1}$. Therefore, for large value of q and n, the clustering coefficient of the generalized Farey network approaches a constant value 1. However, the value of the clustering coefficient for the particular case of Farey network ($q = 1$) is $\ln 2$. As a consequence, the generalized Farey network $F_{q,n}$ has higher clustering coefficient than the Farey network $F_{1,n}$. This result gets back at the creation of new closed triplets and triangles of nodes at each step, that causes a high clustering coefficient. Figure 3.11 shows the clustering coefficient of $F_{q,n}$ with different values of q. We can see in this figure that the clustering coefficient approaches a value 1 for large values of q.

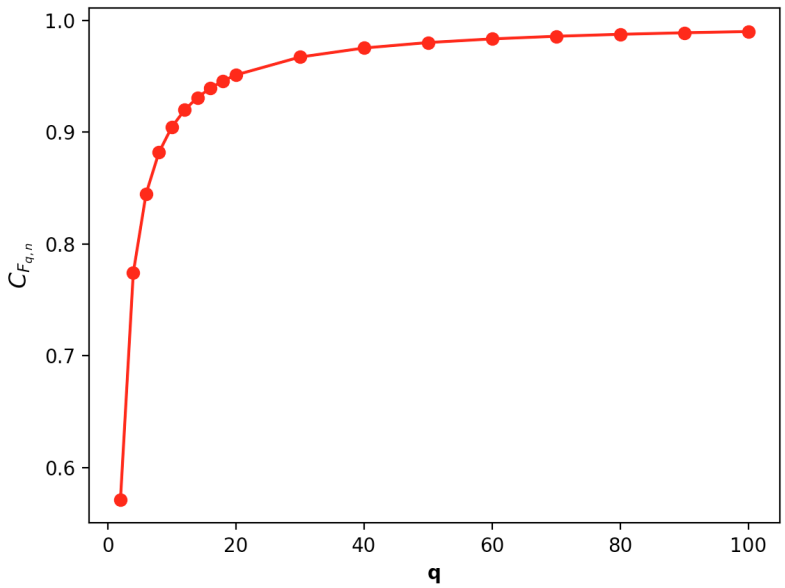


Figure 3.11: The clustering coefficient of the generalized Farey network $F_{q,n}$

- **The diameter**

  Let $D_{F_{q,n}}$ be the diameter of $F_{q,n}$. This diameter can be calculated as follows: Clearly, at steps $n = 0$, the diameter is 1. For $q > 1$ and $n \geq 1$, the distance between two nodes is less than or equal to $n + 1$ to get from one node to the other in each step. Thus, **the diameter of** $F_{q,n}$ is:

$$D_{F_{q,n}} = n + 1, \quad n \geq 1, q > 1. \tag{3.71}$$

  For large n, the number of nodes will be $q(2q)^n$. Thus, the diameter of $F_{q,n}$ scales as $D_{F_{q,n}} \sim \log_{(2q)} (\frac{V_{F_{q,n}}}{q})$ for $q \geq 1$. It grows logarithmically with the number of nodes of the network. Since this network has a small diameter, it is undoubtedly with a short average path length.

☞ **Discussion**

According to the above results, the generalized Farey network has specific properties: its degree distribution follows a power-law form, while the degree distribution is exponential in the Farey network. The difference of topological properties among them are rooted in their different growth mechanisms. In other words, scale-free originates from the exponential increase mechanism of nodes' degrees in generalized Farey networks, while exponential degree distribution is caused by the linear incre-

mental in Farey network. The clustering coefficient of the generalized Farey network approaches a constant value 1, but for the Farey network is $\ln 2$. Nodes' degrees increase exponentially in the generalized Farey network, while that growth in the Farey network is linear, which leads to the generalized Farey network having higher clustering coefficient than Farey network. The basic motif in each growth step is a triangle, which causes high clustering coefficients both in the generalized Farey network and the Farey network. The diameter and the average path length are all small and grow logarithmically with the number of nodes for the generalized Farey network and the Farey network. From this analysis, the generalized Farey network $F_{q,n}$ is a scale-free and a small-world network because it combines the three characteristics: a small diameter, a high clustering coefficient and its degree distribution follows a power law form. However, the Farey network $F_{1,n}$ is just a small-world network. We see this type of networks in many real-life networks.

3.4.2.3 Number of spanning trees of the generalized Farey network $F_{q,n}$

To calculate the number of spanning trees of the generalized Farey network $F_{q,n}$ using the electrically equivalent transformations, we follow the steps of Section 2.4.4.4:

1. Firstly, we start by constructing the generalized Farey network $F_{q,n}$ and adding the weight a_0 on each of its edge.

2. Then, we apply the algorithms of the electrically equivalent transformations to obtain the new weights of all edges of $F_{q,n-1}$ from $F_{q,n}$, the relationship between $\tau(F_{q,n})$ and $\tau(F_{q,n-1})$ and the exact value of the number of spanning trees of $F_{q,n}$ in each step. We deduce $\tau(F_{q,n}) = (2a_0)^{q^n \times 2^{n-1}}.\tau(F_{q,n-1})$. To prove mathematically this formula, we use the method of the mathematical induction:

   Basic step: We show that $\tau(F_{q,1}) = (2a_0)^q \tau(F_{q,0})$. We denote $F_{q,1}^{(0)}$ as the generalized Farey network at iteration 1 in the transformation 0 with all its edges having the weight a_0. We take as an example of the generalized Farey network with $q = 2$ to show the electrically equivalent evolving process from $F_{2,1}^{(0)}$ to $F_{2,0}^{(0)}$ (See Figure 3.12).

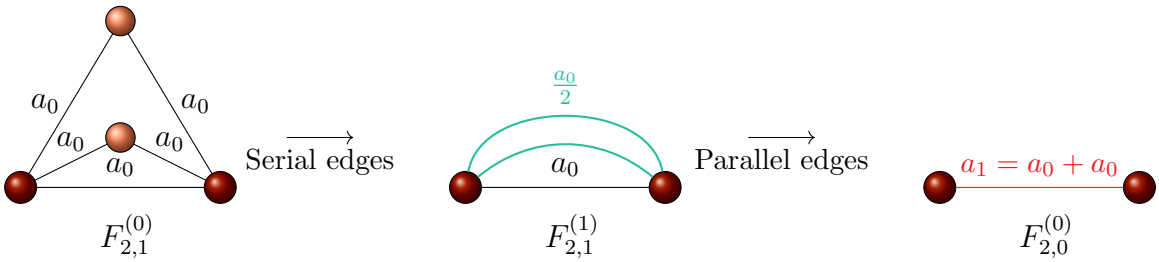


Figure 3.12: The transformations from $F_{2,1}$ to $F_{2,0}$.

Two transformations are used in sequence and the corresponding weights of the resulting edges are calculated as follows:

(a) *Serial edge:* We replace all the serial edges with conductances a_0 by a new edge with the conductance $\frac{a_0.a_0}{a_0+a_0} = \frac{a_0}{2}$ to obtain a new subgraph $F_{q,1}^{(1)}$.

(b) *Parallel edge:* We merge q parallel edges having the same conductance $\frac{a_0}{2}$ with the edge that has the weight a_0 to obtain a new subgraph $F_{q,0}^{(0)}$. The corresponding conductance of a new edge is $q.\frac{a_0}{2} + a_0 = a_1$.

The application of these two electrically equivalent transformations are used to find the formula of the number of spanning trees between $F_{q,1}^{(0)}$ and $F_{q,0}^{(0)}$. Then

$\tau(F_{q,1}) = \tau(F_{q,1}^{(0)})$

$\tau(F_{q,1}) = (2a_0)^q \tau(F_{q,1}^{(1)}) \rightarrow$ $(Serial\ edges)$.

$\tau(F_{q,1}) = (2a_0)^q \times 1^q \times \tau(F_{q,1}^{(2)}) \rightarrow$ $(Parallel\ edges)$.

$\tau(F_{q,1}) = (2a_0)^q \tau(F_{q,0}^{(0)})$.

Induction step: We assume that $\tau(F_{q,n}) = (2a_0)^{q^n \times 2^{n-1}} \tau(F_{q,n-1})$. We show that $\tau(F_{q,n+1}) = (2a_0)^{q^{(n+1)} \times 2^n} \tau(F_{q,n})$.

The number of triangles added in each iteration in $F_{q,n}$ is $q^n 2^{n-1}$. Note that the number of spanning trees depends on those added triangles. So, for $F_{q,n+1}$, the number of added triangles in each step is $q^{n+1}2^n$. Then, the number of spanning trees of $F_{q,n+1}$ is $\tau(F_{q,n+1}) = (2a_0)^{q^{(n+1)} \times 2^n}.\tau(F_{q,n})$.

3. Finally, we deduce the exact formula of the number of spanning trees of $\tau(F_{q,n})$.

$$\begin{aligned} \tau(F_{q,n}) &= (2a_0)^{q^n \times 2^{n-1}}.\tau(F_{q,n-1}) \\ &= (2a_0)^{q^n \times 2^{n-1}}.(2a_1)^{q^{(n-1)} \times 2^{n-2}} ... (2a_{n-1})^{q^1 \times 2^0}.\tau(F_{q,0}) \ with\ \tau(F_{q,0}) = a_n \\ &= a_n.\left[(2a_0)^{q^n \times 2^{n-1}}.(2a_1)^{q^{(n-1)} \times 2^{n-2}} ... (2a_{n-1})^{q^1 \times 2^0}\right] \end{aligned} \tag{3.72}$$

In order to calculate this formula, first we should find the equation of a_n according to a_0. By construction, we start by the weight a_0 at step n. After applying the electrically equivalent transformations, we get the weight a_n at step 0 (As Figure 3.12, we start by the weight a_0 at step 1 and we finish by the weight a_1 at step 0). From this result, we deduce that $a_n = a_0 + q.\frac{a_{n-1}}{2}$. Then, $a_n = a_0[1 + \frac{q}{2} + (\frac{q}{2})^2 + ... + (\frac{q}{2})^n]$. Thus, the equation is:

$$a_n = 2a_0.\left[\frac{1-(\frac{q}{2})^{n+1}}{2-q}\right], q \neq 2. \tag{3.73}$$

We replace Equation (3.73) in Equation (3.72) to get:

$$\tau(F_{q,n}) = 2a_0.(\tfrac{1-(\frac{q}{2})^{n+1}}{2-q})\times\left[(2a_0)^{q^n\times 2^{n-1}}.(2.2a_0.(\tfrac{1-(\frac{q}{2})^2}{2-q}))^{q^{(n-1)}\times 2^{(n-2)}}...(2.2a_0.(\tfrac{1-(\frac{q}{2})^n}{2-q}))^{q^1\times 2^0}\right].$$

We simplify this equation to obtain the exact formula of the number of spanning trees of the generalized Farey network:

$$\tau(F_{q,n}) = \left[\frac{1-(\frac{q}{2})^{n+1}}{2-q}\right]\times\left(2a_0\right)^{1+\sum_{i=1}^{n} q^i.2^{(i-1)}}\times\prod_{i=2}^{n}\left[2.(\frac{1-(\frac{q}{2})^i}{2-q})\right]^{q^{(n-i+1)}.2^{(n-i)}} \tag{3.74}$$

3.4.2.4 Entropy of spanning trees of the generalized Farey network $F_{q,n}$

The number of spanning tree of the generalized Farey network grows exponentially, so we can calculate its entropy of spanning trees. Let $\rho_{F_{q,n}}$ be the entropy of spanning trees for the generalized Farey network.

Corollary 3.4.1. *(Mokhlissi et al., 2019a) The entropy of spanning trees of the generalized Farey network $F_{q,n}$ is:*

$$\rho_{F_{q,n}} = \frac{1}{2q}.\left(2q.\ln 2 + \ln(\frac{2}{2-q})\right), \quad q < 2. \tag{3.75}$$

Proof: Computing the entropy of spanning trees of the generalized Farey network can be done analytically by using Equations 3.74, 2.1 and 3.64. From Equation 3.74, we put $a_0 = 1$. We follow the demonstration below:

$$\begin{aligned}\rho_{F_{q,n}} &= \lim_{V_{F_{q,n}}\to\infty}\frac{\ln|\tau(F_{q,n})|}{|V_{F_{q,n}}|} = \lim_{n\to\infty}\frac{\ln|\tau(F_{q,n})|}{|V_{F_{q,n}}|}\\ &= \lim_{n\to\infty}\frac{\ln\left[[\frac{1-(\frac{q}{2})^{n+1}}{2-q}]\times(2)^{1+\sum_{i=1}^{n} q^i.2^{(i-1)}}\times\prod_{i=2}^{n}[2.(\frac{1-(\frac{q}{2})^i}{2-q})]^{q^{(n-i+1)}.2^{(n-i)}}\right]}{\frac{q(2q)^n+3q-2}{2q-1}}\end{aligned}$$

$$= \lim_{n\to\infty} \frac{(2q-1).\ln\left(\frac{1-(\frac{q}{2})^{n+1}}{2-q}\right)}{q(2q)^n+3q-2} + \frac{(2q-1).(1+\sum_{i=1}^{n} q^i.2^{(i-1)})\ln 2}{q(2q)^n+3q-2} + \frac{(2q-1).\sum_{i=2}^{n}\ln\left(2.(\frac{1-(\frac{q}{2})^i}{2-q})\right)^{q^{(n-i+1)}.2^{(n-i)}}}{q(2q)^n+3q-2}. \quad (3.76)$$

- We calculate each part separately, the first part tends to 0:
  $\lim_{n\to\infty} \frac{(2q-1).\ln\left(\frac{1-(\frac{q}{2})^{n+1}}{2-q}\right)}{q(2q)^n+3q-2} = \lim_{n\to\infty} \frac{(2q-1)\ln(\frac{q}{2})}{q\ln(2q)} \times \frac{q}{(2^{n+1}-q^{n+1}).2^n} = 0.$

- The second part tends to $\ln 2$:
  $\lim_{n\to\infty} \frac{(2q-1).(1+\sum_{i=1}^{n} q^i.2^{(i-1)})\ln 2}{q(2q)^n+3q-2} = \lim_{n\to\infty} \frac{2^n.q^{n+1}.\ln 2-q\ln 2}{2^n.q^{n+1}+3q-2} = \ln 2.$

- The last part tends to $\frac{\ln(\frac{2}{2-q})}{2q}$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{(2q-1).\sum_{i=2}^{n}\ln\left(2.(\frac{1-(\frac{q}{2})^i}{2-q})\right)^{q^{(n-i+1)}.2^{(n-i)}}}{q(2q)^n+3q-2} &= \frac{(2q-1)\ln 2}{q(2q)^n+3q-2}.\sum_{i=2}^{n}\frac{q(2q)^n}{(2q)^i} + \\
\frac{(2q-1)}{q(2q)^n+3q-2}.\sum_{i=2}^{n}\frac{q(2q)^n}{(2q)^i}\ln(\frac{1-(\frac{q}{2})^i}{2-q}) &= (2q-1)\ln 2.\sum_{i=2}^{n}\frac{1}{(2q)^i} + \frac{q(2q)^n(2q-1)}{q(2q)^n+3q-2}. \\
\left[\sum_{i=2}^{n}\frac{\ln(\frac{2^i-q^i}{2^i})}{(2q)^i} - \ln(2-q).\sum_{i=2}^{n}\frac{1}{(2q)^i}\right] &= \frac{\ln 2}{2q} + \left[\sum_{i=2}^{n}\left(\frac{\ln(2^i-q^i)}{(2q)^i} - \frac{\ln(2^i)}{(2q)^i}\right) - \right. \\
\left.\ln(2-q).\frac{1}{(2q)^2}.\frac{1-(\frac{1}{2q})^{n-1}}{1-\frac{1}{2q}}\right].(2q-1) &= \frac{\ln 2}{2q} - \frac{\ln(2-q)}{2q} \\
&= \frac{\ln(\frac{2}{2-q})}{2q}, q<2.
\end{aligned}$$

We replace all these results in Equation (3.76). Then, the results. The obtained entropy of spanning trees of $F_{q,n}$ depends on q, but we can use the entropy of $\rho_{F_{q,n}}$ with $q < 2$. For $q = 1$, $\rho_{F_{1,n}} = 0.9457$. In Table 3.2, we compare the entropy of spanning trees of the Farey network $F_{1,n}$ (0.9457) with those of other networks having the same average degree 4. We notice that the value of this entropy is bigger than those of the Pseudo-fractal web and smaller than those of the Fractal Scale-Free Lattice, the 2-dimensional Sierpinski gasket, the square lattice and the 2-Mosaic networks. This result proves that the structural topology of the Farey network is more robust than the Pseudo-fractal web

and less robust than the other networks.

Type of network	$< z >$	ρ
Pseudo-fractal web (Zhang et al., 2010b)	4	0.8959
Farey network (Mokhlissi et al., 2019a)	**4**	**0.9457**
Fractal Scale-Free Lattice (Mokhlissi et al., 2019b)	4	1.0397
The 2-dimensional Sierpinski gasket (Chang et al., 2007)	4	1.0486
Square lattice (Wu, 1977)	4	1.1662
The 2-Mosaic networks (Mokhlissi et al., 2017b)	4	1.3862

Table 3.2: The entropy of spanning trees of several networks having the same average degree.

3.5 Summary

To conclude, we have investigated three important models of small-world networks, which display rich structural behaviours in real-world networks:

- Small-World Exponential network and Koch Network: They are based on triangles. For the first network, its degree distribution follows an exponential distribution. For the second network, its degree distribution follows a power law-form and both they have a high clustering coefficient and small diameter. We have proposed two new models based on the generalization of the Small-World Exponential network and the generalization of the Koch network. These models rely on the size and dimension of the added cyclic subgraph and they are characterized by the scale-free characteristics. Then, we have calculated their number of spanning trees using the decomposition method following one articulation node. In the end, we have calculated and compared their entropy of spanning trees with other networks having the same average degree, indicating that the generalizations of these networks have the same robustness although their structures, properties and complexities are different.

- Farey network: It shows the small-world effect: a small diameter or a short average distance and a large clustering coefficient. We have suggested a generalization of this model by adding nodes to every iterative edge in each step. We have analyzed its structural properties, showing that the generalized Farey network combines the small-world and the scale-free properties. Then, we have evaluated its complexity using the electrically equivalent transformations. Finally, we have calculated its

asymptotic complexity and compared it with those for other networks having the same average degree.

In the next chapter, we examine three models of scale-free networks: Flower network, Mosaic network and Fractal Scale-Free Lattice. We propose their generalizations, we analyze their structure and their topological properties, we calculate their complexity using some geometric methods mentioned in Chapter 2 and we evaluate their entropy to estimate the most robust network.

Chapter

4

Analysis of models of Scale-Free networks

Contents

Scale-free networks are ubiquitous in science and in everyday life. They have been the focus of intense interest, because many large complex networks, such as the Internet and the World Wide Web and networks of interactions between proteins are scale-free. Their degree distribution follows a power-law form. In this chapter, we investigate three categories of scale-free networks, namely Flower network, Mosaic network and Fractal Scale-Free Lattice. We propose three new models based on the generalization of these mentioned networks. We study their structural properties, proving that the proposed generalizations do not affect the scale-free property. In addition, we use three methods proposed in Chapter 2, based on transformations by changing the geometric nature of the models in order to count their number of spanning trees. Finally, in order to evaluate the robustness of these generalized networks, we compute and compare their entropy with other networks having the same average degree. The results of this chapter were published in an international journal (Mokhlissi et al., 2019b) and three international conferences (Mokhlissi et al., 2016b, 2017b,a).

4.1 Why scale-free networks are very important to study?

Researchers have observed that a lot of real-world networks like the Internet or social networks have a topology that comes close to the scale-free networks and they have also demonstrated that several naturally occurring scale-free networks exhibit fractal scaling. This type of networks is known by its degree distribution which decays like a power law (Barabási and Bonabeau, 2003; Lenaerts, 2011). This means that the vast majority of nodes have very few connections, while a few important nodes, called by hubs, have a huge number of connections. The nodes with a large degree have a high probability of attracting more links and their connectivity increases rapidly compared to other nodes, which explains the principle of the "Rich get richer dynamics". We can see the property of the scale-free networks on the Web where the major websites like Google or Facebook are hubs, they dominate the network, while there are millions of smaller websites with very few connections. In order to understand the dynamics of growing real-world networks, many models have been proposed to generate networks with the scale-free property (Barabási and Albert, 1999; Aiello et al., 2001).

4.2 Analysis of Flower network

In this section, we present a well-known family of scale-free networks, called the *Flower network*, which displays some remarkable properties observed in real-life networks (Rozenfeld et al., 2007; Rozenfeld and Ben-Avraham, 2007; Lin et al., 2011). This network exhibits the self-similarity and the fractality. We study two types of Flower networks: The particular case: *The* 2*-Flower network*, which is based on the combination of the reduction and the bipartition approaches (See Figure 4.1) and the general case: *The* m*-Flower network*, which is based on the combination of the m-reduction and the m-partition approaches (See Figure 4.2) (Mokhlissi et al., 2016b, 2017a). We introduce their construction by applying firstly the reduced approach then, the bipartite approach for *the* 2*-Flower network* and m-reduced approach then, the m-partite approach for *the* m*-Flower network*. Then, we discuss their topological properties indicating that the generalization of the Flower network is also a scale-free network. After that, we calculate their number of spanning trees using the combination of the geometric approaches: The reduction and

the bipartition approaches, the m-reduction and the m-partition approaches. Finally, we evaluate the entropy of spanning trees and compare it with those for other studied networks with the same average degree in order to estimate the most robust network between them.

4.2.1 2-Flower network

4.2.1.1 Construction of the 2-Flower network

The 2-Flower network is denoted by $F_{2,n}$ after n generations of evolution and '2' refers to the dimension of the Flower network. It can be created using the following iterative way: For the iteration $n = 0$, we have a simple edge that connects two vertices. For $n \geq 1$, first, we apply the reduced approach by replacing each exiting edge in $F_{2,n-1}$ by two links.Then, we apply the bipartite approach to this last obtained graph by adding a new vertex in the middle of each of its edges in order to get two links for each existing edge of the reduced graph. We can denote this process by $F_{2,n} = B_2(R_2(F_{2,n-1}))$. For illustration, in Figure 4.1, we present the first two iterations of the 2-Flower network.

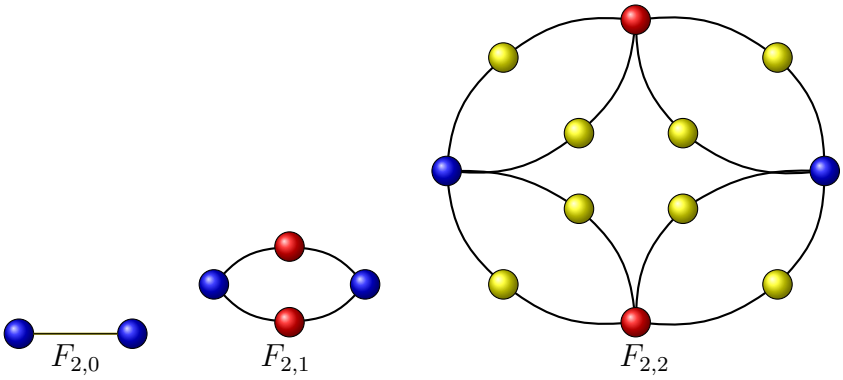


Figure 4.1: The first two iterations of the 2-Flower network $F_{2,n}$.

Next, we compute the numbers of nodes, edges, faces and the average degree of the 2-Flower network. According to the construction of $F_{2,n}$, we get:

- Let $E_{F_{2,n}}$ be the number of edges created at iteration n. We have $E_{F_{2,n}} = E_{B_2(R_2(F_{2,n-1}))}$. Using Property 2.4.2, we obtain: $E_{F_{2,n}} = 2E_{R_2(F_{2,n-1})}$ and using Property 2.4.4, we obtain: $E_{F_{2,n}} = 4E_{F_{2,n-1}} = 4^2E_{F_{2,n-2}} = 4^3E_{F_{2,n-3}} = ... = 4^nE_{F_{2,0}}$. So, **the number of edges of** $F_{2,n}$ is:

$$E_{F_{2,n}} = 2^{2n}, \quad n \geq 0. \tag{4.1}$$

- Similarly, we find **the number of vertices of** $F_{2,n}$:

$$V_{F_{2,n}} = 2 + \frac{2}{3}(2^{2n} - 1), \quad n \geq 0. \tag{4.2}$$

- **The number of faces of** $F_{2,n}$ is:

$$F_{F_{2,n}} = 1 + \frac{2^{2n} - 1}{3}, \quad n \geq 0. \tag{4.3}$$

- Let $< z >_{F_{2,n}}$ be the average degree of $F_{2,n}$. It is calculated as follows: $< z >_{F_{2,n}} = \frac{2E_{F_{2,n}}}{V_{F_{2,n}}}$. Thus, **the average degree of** $F_{2,n}$ is (which is approximately 3 for large n):

$$< z >_{F_{2,n}} = \frac{2^{2n+1}}{\frac{4}{3} + \frac{2^{2n+1}}{3}}, \quad n \geq 0. \tag{4.4}$$

4.2.1.2 *Topological properties of the* 2*-Flower network*

- **The degree distribution**

We calculate the degree distribution of the 2-Flower network $F_{2,n}$ as follows: At $n = 0$, the network has two nodes of degree 1. At the step n_u, $k_u(n_u) = 2$ is the degree of a new added node u. We denote the degree of the node u at step n by $k_u(n)$. From the construction of $F_{2,n}$, the degree of its nodes is:

$$k_u(n) = 2k_u(n-1), \quad n \geq 0. \tag{4.5}$$

Which leads to:

$$\begin{cases} k_u(n) = 2^{n-n_u+1}, & u \in V_{F_{2,n}}, u \neq \frac{0}{1}, \frac{1}{1} \\ k_{\frac{0}{1}}(n) = k_{\frac{1}{1}}(n) = 2^n. \end{cases} \tag{4.6}$$

Then, the cumulative degree distribution is given by:

$$P_{cum}(k) = \frac{1}{V_{F_{2,n}}} \sum_{i=0}^{n_u} M_v(i) = \frac{4^{n_u} + 2}{4^n + 2} \tag{4.7}$$

with $M_v(n)$ is the number of new nodes at step n and $V_{F_{2,n}} = \sum_{i=0}^{n} M_v(i)$. From Equation 4.6, we obtain $n_u = n + 1 - \frac{\ln k}{\ln 2}$. Then, the cumulative degree distribution of $F_{2,n}$ will be:

$$P_{cum}(k) = \frac{4^{n+1} \times k^{-\frac{\ln 4}{\ln 2}} + 2}{4^n + 2} \tag{4.8}$$

For large n, we can obtain:

$$P_{cum}(k) \sim k^{-2} \tag{4.9}$$

So the degree distribution of the 2-Flower network $F_{2,n}$ follows a power-law form with the exponent $\gamma = 3$.

- **Clustering coefficient**

  The 2-Flower network $F_{2,n}$ have zero clusterings because the neighbors of any node are never neighbors of one other. The main cause is the absence of triangles in $F_{2,n}$. Therefore, one of the properties of the small-world network is missing.

- **Diameter**

  Let $D_{F_{2,n}}$ be the diameter of $F_{2,n}$ created at generation n. This diameter can be calculated as follows: According to the construction of $F_{2,n}$, we get: $D_{F_{2,n}} = 2D_{F_{2,n-1}} = 4D_{F_{2,n-2}} = ... = 2^n D_{F_{2,0}}$, with $D_{F_{2,0}} = 1$. Thus, **the diameter of** $F_{2,n}$ is:

$$D_{F_{2,n}} = 2^n, \quad n \geq 0. \tag{4.10}$$

  for large n, the number of vertices will be $V_{F_{2,n}} \sim 2^{2n}$. Then, $D_{F_{2,n}} = V_{F_{2,n}}^{(\frac{1}{2})} = \sqrt{V_{F_{2,n}}}$. It grows as a square power of the number of nodes in the network, which indicates that the 2-Flower network $F_{2,n}$ is not a small world.

- **Fractal dimension**

  Fractals are an irregular geometric object with an infinite nesting of the structure at all scales. This type of network is characterized by the fractal dimension, which is the scaling rule d_f, from knowing how something scales. For $F_{2,n}$, we follow the mathematical framework presented in (Song et al., 2006). By construction, in the infinite n limit, the different quantities of the network grow as:

$$\begin{cases} V_{F_{2,n}} \simeq 4.V_{F_{2,n-1}}, \\ k_u(n) = 2.k_u(n-1), \\ D_{F_{2,n}} = 2.D_{F_{2,n-1}}. \end{cases} \tag{4.11}$$

  where $V_{F_{2,n}}$, $k_u(n)$ and $D_{F_{2,n}}$ are the number of vertices, the degree of a node u and the diameter of $F_{2,n}$ respectively. From Equation (4.11), it is obvious that $V_{F_{2,n}}$,

$k_u(n)$ and $D_{F_{2,n}}$ increase by a factor of $f_V = 4$, $f_k = 2$, and $f_D = 2$, respectively. So, we can derive the scaling exponents in terms of the microscopic parameters (Song et al., 2006): **The fractal dimension** is $d_f = \frac{\ln f_V}{\ln f_D} = 2$, and the degree exponent of boxes is $d_k = \frac{\ln f_k}{\ln f_D} = 1$. The exponent of the degree distribution satisfies $\gamma = 1 + \frac{d_f}{d_k} = 1 + \frac{\ln f_V}{\ln f_k} = 3$, giving the same γ as that obtained in the direct calculation of the degree distribution. So, the 2-Flower network $F_{2,n}$ is a fractal.

☞ **Discussion**

According to the above results, the 2-Flower network $F_{2,n}$ has specific properties: its degree distribution follows a power law form, a zero clustering coefficient, its diameter does not scale logarithmically with the number of nodes of the network and it is characterized by a finite fractal dimension, which indicate that $F_{2,n}$ is a scale-free network and a fractal.

4.2.1.3 Number of spanning trees of the 2-Flower network

Due to the large size of complex networks, their complexity is very difficult to compute, even if we use the theorem of Kirchhoff. For this reason, we use two geometric approaches: The reduction and the bipartition approaches (See Section 2.4.3 and Section 2.4.2) that facilitate the computation. We combine Our approaches: First, we apply the reduction approach, then the bipartite approach to find the exact number of spanning trees for the 2-Flower network.

Theorem 4.2.1. *(Mokhlissi et al., 2016b, 2017a) Let $F_{2,n}$ denote a 2-Flower network where n is the number of iterations. The number of spanning trees of $F_{2,n}$ is given by the following formula:*

$$\tau(F_{2,n}) = 2^{2[\frac{2^{2n}-1}{3}]} \tag{4.12}$$

Proof: This process can be presented as $\tau(F_{2,n}) = \tau(B_2(R_2(F_{2,n-1})))$. Using Theorem 2.4.2 of the bipartition approach, we obtain: $\tau(F_{2,n}) = 2^{F_{R_2(F_{2,n-1})}-1} \times \tau(R_2(F_{2,n-1}))$ with $F_{R_2(F_{2,n-1})} = F_{F_{2,n}}$. Using Theorem 2.4.4 of the reduction approach, we obtain: $\tau(F_{2,n}) = 2^{F_{F_{2,n}}-1} \times 2^{V_{F_{2,n-1}}-1} \times \tau(F_{2,n-1})$. Using Equation 4.2 and 4.3 , we obtain:

$$\tau(F_{2,n}) \;=\; 2^{\frac{2^{2n}-1}{3}} \times 2^{1+\frac{2}{3}(2^{2(n-1)}-1)} \times \tau(F_{2,n-1})$$

$$\begin{aligned}\tau(F_{2,n}) &= 2^{2^{2n-1}} \times \tau(F_{2,n-1}) \\ \tau(F_{2,n}) &= 2^{2^{2n-1}+2^{2n-3}+2^{2n-5}+\ldots+2^1} \times \tau(F_{2,0}) \; with \; \tau(F_{2,0}) = 1 \\ \tau(F_{2,n}) &= 2^{2[\frac{2^{2n}-1}{3}]}.\end{aligned}$$

Hence, the result.

4.2.2 m-Flower network

4.2.2.1 Construction of the m-Flower network

The m-Flower network is denoted by $F_{m,n}$ with n generations and 'm' refers to the dimension of the Flower network. It can be created using the following iterative way: For $n = 0$, we have a simple edge that connects two vertices. For $n \geq 1$, first, we apply the m-reduced approach to obtain m multiple edges connecting each pair of vertices of $F_{m,n-1}$. Then, we apply the m-partite approach to this last obtained graph by adding $m-1$ vertices in each edge to get m links for each existing edge of the reduced graph.This process can be denoted by $F_{m,n} = B_m(R_m(F_{m,n-1}))$. In Figure 4.2, we illustrate the first two iterations of the m-Flower network with $m = 3$.

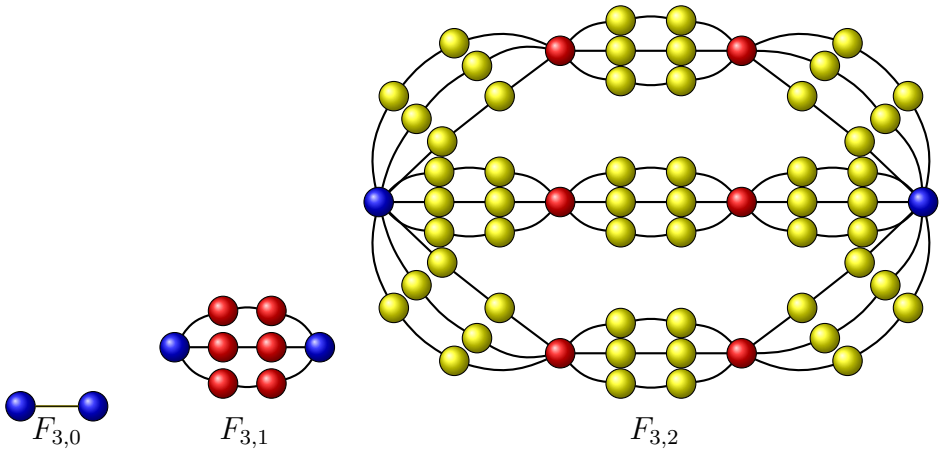


Figure 4.2: The first two iterations of the 3-Flower network.

Then, we calculate the numbers of nodes, edges, faces and the average degree of the m-Flower network. According to the construction of $F_{m,n}$, we get:

- Let $E_{F_{m,n}}$ be the number of edges created at iteration n. We have $E_{F_{m,n}} = E_{B_m(R_m(F_{m,n-1}))}$. Using Property 2.4.3, we obtain: $E_{F_{m,n}} = m \times E_{R_m(F_{m,n-1})}$ and using Property 2.4.5,

we obtain: $E_{F_{m,n}} = m^2 \times E_{F_{m,n-1}} = m^4 \times E_{F_{m,n-2}} = m^6 \times E_{F_{m,n-3}} = ... = m^{2n} \times E_{F_{m,0}}$. So, **the number of edges of** $F_{m,n}$ is:

$$E_{F_{m,n}} = m^{2n}, \quad n \geq 0. \tag{4.13}$$

- Similarly, we find **the number of vertices of** $F_{m,n}$:

$$V_{F_{m,n}} = 2 + \frac{m(m^{2n}-1)}{m+1}, \quad n \geq 0. \tag{4.14}$$

- **The number of faces of** $F_{m,n}$ is:

$$F_{F_{m,n}} = 1 + \frac{m^{2n}-1}{m+1}, \quad n \geq 0. \tag{4.15}$$

- Let $< z >_{F_{m,n}}$ be the average degree of $F_{m,n}$. It is calculated as follows: $< z >_{F_{m,n}} = \frac{2E_{F_{m,n}}}{V_{F_{m,n}}}$. Thus, **the average degree of** $F_{m,n}$ is (which is approximately 3 for large n):

$$< z >_{F_{m,n}} = \frac{2m^{2n}}{2 + \frac{m(m^{2n}-1)}{m+1}}, \quad n \geq 0. \tag{4.16}$$

4.2.2.2 *Topological properties of the m-Flower network*

- **The degree distribution**

To calculate the degree distribution of the m-Flower network $F_{m,n}$, we use the same process applied on the degree distribution of the 2-Flower network $F_{2,n}$. Then, the cumulative degree distribution of $F_{m,n}$ is given by:

$$P_{cum}(k) = \frac{m^{2n+1} \times (\frac{k}{2})^{-2} + m + 2}{m^{2n+1} + m + 2} \tag{4.17}$$

For large n, we can obtain:

$$P_{cum}(k) \sim 4 \times k^{-2} \tag{4.18}$$

So the degree distribution of the m-Flower network $F_{m,n}$ follows a power-law form with the exponent $\gamma = 3$.

- **Clustering coefficient**

The m-Flower network $F_{m,n}$ have also zero clusterings because the neighbors of any node are never neighbors of one other. Therefore, one of the properties of the small-world network is missing.

- **Diameter**

  Let $D_{F_{m,n}}$ be the diameter of $F_{m,n}$ created at generation n. This diameter can be calculated as follows: According to the construction of $F_{m,n}$, we get: $D_{F_{m,n}} = mD_{F_{m,n-1}} = m^2 D_{F_{m,n-2}} = ... = m^n D_{F_{m,0}}$, with $D_{F_{m,0}} = 1$. Thus, **the diameter of** $F_{m,n}$ is:

  $$D_{F_{m,n}} = m^n, \quad n \geq 0. \tag{4.19}$$

  for large n, the number of nodes will be $V_{F_{m,n}} \sim m^{2n}$. Then, $D_{F_{m,n}} = V_{F_{m,n}}^{(\frac{1}{2})} = \sqrt{V_{F_{m,n}}}$. It grows as a square power of the number of nodes in the network, which indicates that the m-Flower network $F_{m,n}$ is not a small world.

- **Fractal dimension**

  We calculate the fractal dimension of the m-Flower network $F_{m,n}$ as follows: By construction, in the infinite n limit, the different quantities of the network grow as:

  $$\begin{cases} V_{F_{m,n}} \simeq m^2.V_{F_{m,n-1}}, \\ k_u(n) = m.k_u(n-1), \\ D_{F_{m,n}} = m.D_{F_{m,n-1}}. \end{cases} \tag{4.20}$$

  where $V_{F_{m,n}}$, $k_u(n)$ and $D_{F_{m,n}}$ are the number of nodes, the degree of a node u and the diameter of $F_{m,n}$ respectively. From Equation (4.20), it is obvious that $V_{F_{m,n}}$, $k_u(n)$ and $D_{F_{m,n}}$ increase by a factor of $f_V = m^2$, $f_k = m$, and $f_D = m$, respectively. So, **the fractal dimension** is $d_f = \frac{\ln f_V}{\ln f_D} = 2$, and the degree exponent of boxes is $d_k = \frac{\ln f_k}{\ln f_D} = 1$. The exponent of the degree distribution satisfies $1 + \frac{d_f}{d_k} = 1 + \frac{\ln f_V}{\ln f_k} = 3$, giving the same γ as that obtained in the direct calculation of the degree distribution. So, the m-Flower network $F_{m,n}$ is a fractal.

☞ **Discussion**

According to the above results, the m-Flower network $F_{m,n}$ incorporates some specific properties: its degree distribution follows a power law form, a zero clustering coefficient, its diameter does not scale logarithmically with the number of nodes of the network and it has a finite fractal dimension, which indicate that $F_{m,n}$ is a scale-free network and a fractal.

4.2.2.3 Number of spanning trees of the m-Flower network

In order to calculate the number of spanning trees of the m-Flower network, two approaches are combined: First, the reduction approach, then the bipartite approach are applied.

Theorem 4.2.2. *(Mokhlissi et al., 2016b, 2017a) The number of spanning trees of the m-Flower network is given by:*

$$\tau(F_{m,n}) = m^{m[\frac{m^{2n}-1}{m^2-1}]} \tag{4.21}$$

Proof: This process can be presented as $\tau(F_{m,n}) = \tau(B_m(R_m(F_{m,n-1})))$. Using Theorem 2.4.3 of the m-partition approach, we obtain: $\tau(F_{m,n}) = m^{F_{R_m(F_{m,n-1})}-1} \times \tau(R_m(F_{m,n-1}))$ with $F_{R_m(F_{m,n-1})} = F_{F_{m,n}}$. Using Theorem 2.4.5 of the m-reduction approach, we obtain: $\tau(F_{m,n}) = m^{F_{F_{m,n}}-1} \times m^{V_{F_{m,n-1}}-1} \times \tau(F_{m,n-1})$. Using Equation 4.14 and 4.15, we obtain:

$$\begin{aligned}
\tau(F_{m,n}) &= m^{\frac{m^{2n}-1}{m+1}} \times m^{1+\frac{m(m^{2(n-1)}-1)}{m+1}} \times \tau(F_{m,n-1}) \\
\tau(F_{m,n}) &= m^{m^{2n-1}} \times \tau(F_{m,n-1}) \\
\tau(F_{m,n}) &= m^{m^{2n-1}+m^{2n-3}+m^{2n-5}+...+m^1} \times \tau(F_{m,0}) \; with \; \tau(F_{m,0}) = 1 \\
\tau(F_{m,n}) &= m^{m[\frac{m^{2n}-1}{m^2-1}]}.
\end{aligned}$$

4.2.2.4 Entropy of spanning trees of the m-Flower network

Since the number of spanning trees for the m-Flower network $\tau(F_{m,n})$ grows exponentially with the network order $V_{F_{m,n}}$, so we can calculate its entropy of spanning trees according to the definition of the entropy of Equation 2.1 in Section 2.1.2.1:

Corollary 4.2.1. *(Mokhlissi et al., 2017a) The entropy of spanning trees of the m-Flower network is given by:*

$$\rho_{F_{m,n}} = \frac{ln(m)}{m-1} \tag{4.22}$$

Proof: We calculate the entropy of spanning trees of a m-Flower network as follows:

$$\rho_{F_{m,n}} = \lim_{V_{F_{m,n}} \to \infty} \frac{\ln|\tau(F_{m,n})|}{|V_{F_{m,n}}|} = \lim_{n\to\infty} \frac{\ln|\tau(F_{m,n})|}{|V_{F_{m,n}}|} = \lim_{n\to\infty} \frac{\ln(m^{m[\frac{m^{2n}-1}{m^2-1}]})}{2+\frac{m(m^{2n}-1)}{m+1}} = \lim_{n\to\infty} \frac{\ln(m)\times(m+1)}{m^2-1}.$$

Then, the result. Similarly, we can find the entropy of the 2-Flower network: $\rho_{F_{2,n}} = \ln 2 = 0.6931$. In Figure 4.3, we show that the entropy of spanning trees of the m-Flower

network varies with the dimension m and the increasing of this value leads to decrement the entropy of spanning trees. From this result, we deduce that the m-Flower networks with a larger dimension are less robust than those with a smaller dimension.

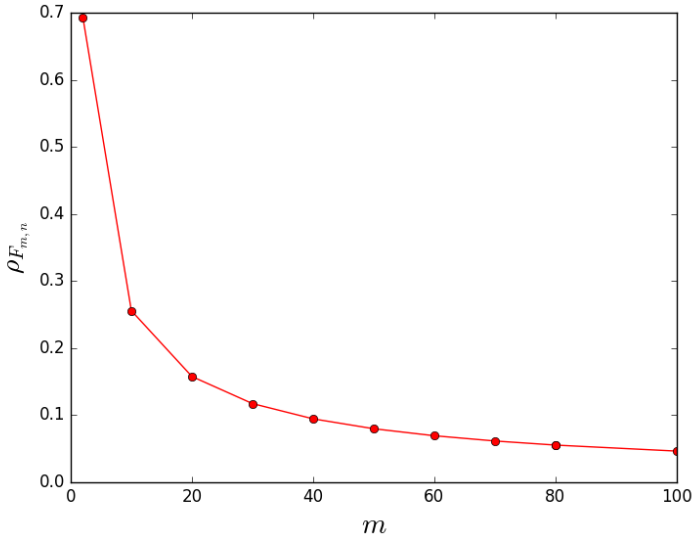


Figure 4.3: The entropy of spanning trees of the m-Flower network

Type of network	$< z >$	ρ
Koch network (Mokhlissi et al., 2018b)	3	0.549
Small-World Exponential network (Mokhlissi et al., 2018b, 2016a)	3	0.549
Hanoi network (Zhang et al., 2016)	3	0.677
2-Flower network(Mokhlissi et al., 2017a)	**3**	**0.6931**
The 3-2-12 lattice (Wu, 1977)	3	0.721
The 4-8-8 bathroom tile (Wu, 1977)	3	0.787
Honeycomb lattice (Shrock and Wu, 2000)	3	0.807

Table 4.1: The entropy of spanning trees of several networks having the same average degree.

From Table 4.1, we compare the entropy of spanning trees of the 2-Flower network $\rho_{F_{2,n}}$ with those of other networks with the same average degree. We notice that the value of this entropy is the highest reported for the Koch network, the Small-World Exponential network and the Hanoi network and it is the lowest reported for the 3-2-12 lattice, the 4-8-8 bathroom tile and Honeycomb lattice. This reflects the fact that the 2-Flower network has an average spanning tree rate compared to other networks with the same average degree. This result proves that the 2-Flower network is more robust than the Koch network, the Small-World Exponential network and the Hanoi network. On the other hand, the 2-Flower network is less robust than the 3-2-12 lattices, the 4-8-8 bathroom tile and Honeycomb lattice.

4.3 Analysis of Mosaic network

In this section, we introduce another family of scale-free networks, named the *Mosaic network*, which incorporates some key properties characterizing a majority of real-life networked systems. It consists of self-repeating patterns on all length scales. This type of networks highlights the concepts of self-similarity and fractality. Two types of the Mosaic network are examined: The particular case: *The* 2*-Mosaic network*, based on the bipartite and the reduced approaches (See Figure 4.4) and the general case: *The* m*-Mosaic network*, based on the m-partition and the m-reduction approaches (See Figure 4.5) (Mokhlissi et al., 2016b, 2017b). We investigate their construction. For *the* 2*-Mosaic network*, we apply firstly the bipartite approach then, the reduced approach and for *the* m*-Mosaic network*, we apply firstly the m-partite approach then, the m-reduced approach. Then, we analyze their topological properties, showing that the generalization of the Mosaic network is also a scale-free network. After that, we evaluate their complexity by combining our geometric approaches: The bipartition and the reduction approaches, the m-partition and the m-reduction approaches. Finally, we calculate the entropy of spanning trees and compare it with those for other networks having the same average degree of nodes in order to determine the most robust network between them.

4.3.1 2-Mosaic network

4.3.1.1 Construction of the 2*-Mosaic network*

The 2-Mosaic network is denoted by $M_{2,n}$ with n iteration and '2' refers to the dimension of the Mosaic network. It is constructed as follows: At first, $M_{2,0}$ is a simple edge that connects two nodes. At the next generation, the biparite approach is applied by adding a new node in the simple edge of $M_{2,0}$ to get two edges. Then the reduced approach is applied by adding a new multiple edge for each existing edge of the bipartite network. The growth process to the next iterations continues in a similar way. This process is denoted by $M_{2,n} = R_2(B_2(M_{2,n-1}))$. This process is the opposite of that of the 2-Flower network. For illustration, in Figure 4.4, the first three generations of the 2-Mosaic network are presented. Next, we calculate the numbers of nodes, edges, faces and the average degree of the 2-Mosaic network. According to the construction of $M_{2,n}$, we get:

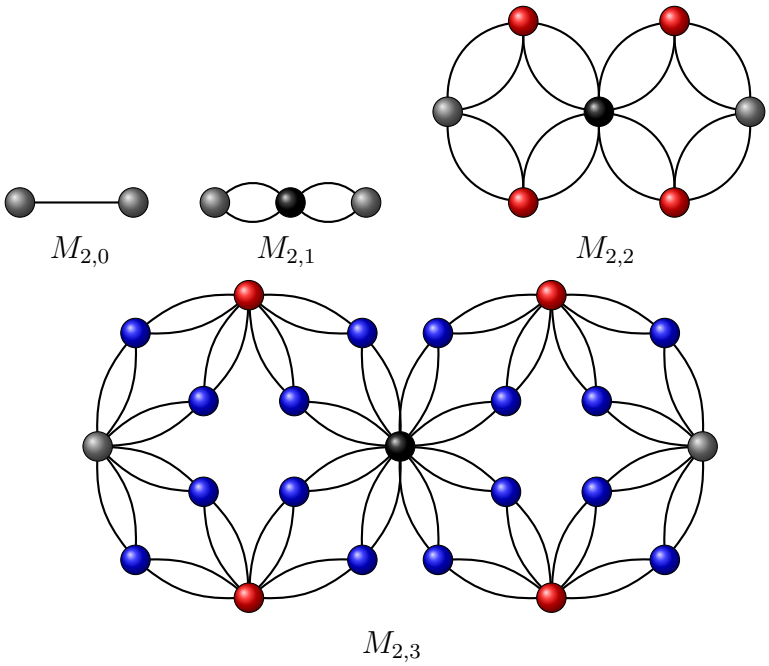


Figure 4.4: The first three generations of the 2-Mosaic network.

- Let $E_{M_{2,n}}$ be the number of edges created at iteration n. We have $E_{M_{2,n}} = E_{R_2(B_2(M_{2,n-1}))}$. Using Property 2.4.4, we obtain: $E_{M_{2,n}} = 2E_{B_2(M_{2,n-1})}$ and using Property 2.4.2, we obtain: $E_{M_{2,n}} = 4E_{M_{2,n-1}} = 4^2E_{M_{2,n-2}} = 4^3E_{M_{2,n-3}} = ... = 4^nE_{M_{2,0}}$. Thus, **the number of edges of** $M_{2,n}$ is:

$$E_{M_{2,n}} = 2^{2n}, \quad n \geq 0. \tag{4.23}$$

- Similarly, we find **the number of vertices of** $M_{2,n}$:

$$V_{M_{2,n}} = 2 + \frac{2^{2n}-1}{3}, \quad n \geq 0. \tag{4.24}$$

- **The number of faces of** $M_{2,n}$ is:

$$F_{M_{2,n}} = 1 + \frac{2}{3}(2^{2n}-1), \quad n \geq 0. \tag{4.25}$$

- Let $< z >_{M_{2,n}}$ be the average degree of $M_{2,n}$. It is calculated as follows: $< z >_{M_{2,n}} = \frac{2E_{M_{2,n}}}{V_{M_{2,n}}}$. Thus, **the average degree of** $M_{2,n}$ is (which is approximately 4 for large n):

$$< z >_{M_{2,n}} = \frac{2^{2n+1}}{2 + \frac{2^{2n}-1}{3}}, \quad n \geq 0. \tag{4.26}$$

4.3.1.2 Structural properties of the 2-Mosaic network

- **The degree distribution**

The degree distribution of the 2-Mosaic network $M_{2,n}$ is calculated as follows: At $n = 0$, the network has two nodes of degree 1. At the step n_u, a new added node u is created and it has the degree $k_u(n_u) = 4$. Let $k_u(n)$ be the degree of the node u at step n. From the construction of $M_{2,n}$, the degree of its nodes is:

$$k_u(n) = 2k_u(n-1), \quad n \geq 0. \tag{4.27}$$

Which leads to:

$$\begin{cases} k_u(n) = 2^{n-n_u+2}, & u \in V_{M_{2,n}}, u \neq \frac{0}{1}, \frac{1}{1} \\ k_{\frac{0}{1}}(n) = k_{\frac{1}{1}}(n) = 2^n. \end{cases} \tag{4.28}$$

Then, the cumulative degree distribution is given by:

$$P_{cum}(k) = \frac{1}{V_{M_{2,n}}} \sum_{i=0}^{n_u} M_v(i) = \frac{4^{n_u}+5}{4^n+5} \tag{4.29}$$

with $M_v(n)$ is the number of new nodes at step n and $V_{M_{2,n}} = \sum_{i=0}^{n} M_v(i)$. From Equation 4.28, we obtain $n_u = n+2-\frac{\ln k}{\ln 2}$. Then, the cumulative degree distribution of $M_{2,n}$ will be:

$$P_{cum}(k) = \frac{4^{n+2} \times k^{-\frac{\ln 4}{\ln 2}} + 5}{4^n+5} \tag{4.30}$$

For large n, we can obtain:

$$P_{cum}(k) \sim k^{-2} \tag{4.31}$$

So the degree distribution of the 2-Mosaic network $M_{2,n}$ follows a power-law form with the exponent $\gamma = 3$.

- **Clustering coefficient**

The 2-Mosaic network $M_{2,n}$ have zero clusterings because there is no link between neighbors of any node. Therefore, one of the properties of the small-world network is not verified.

- **Diameter**

Let $D_{M_{2,n}}$ be the diameter of $M_{2,n}$ created at generation n. We can calculate this diameter as follows: According to the construction of $M_{2,n}$, we get: $D_{M_{2,n}} =$

$2D_{M_{2,n-1}} = 4D_{M_{2,n-2}} = ... = 2^n D_{M_{2,0}}$, with $D_{M_{2,0}} = 1$. Thus, **the diameter of** $M_{2,n}$ is:

$$D_{M_{2,n}} = 2^n, \quad n \geq 0. \tag{4.32}$$

for large n, the number of vertices will be $V_{M_{2,n}} \sim 2^{2n}$. Then, $D_{M_{2,n}} = V_{M_{2,n}}^{(\frac{1}{2})} = \sqrt{V_{M_{2,n}}}$. It grows as a square power of the number of nodes in the network, which indicates that the 2-Mosaic network $M_{2,n}$ is not a small world.

- **Fractal dimension**

For 2-Mosaic network $M_{2,n}$, we follow the same process as the Flower network to calculate the fractal dimension. By construction, in the infinite n limit, the different quantities of the network grow as:

$$\begin{cases} V_{M_{2,n}} \simeq 4.V_{M_{2,n-1}}, \\ k_u(n) = 2.k_u(n-1), \\ D_{M_{2,n}} = 2.D_{M_{2,n-1}}. \end{cases} \tag{4.33}$$

where $V_{M_{2,n}}$, $k_u(n)$ and $D_{M_{2,n}}$ are the number of nodes, the degree of a node u and the diameter of $M_{2,n}$ respectively. From Equation (4.33), it is obvious that $V_{M_{2,n}}$, $k_u(n)$ and $D_{M_{2,n}}$ increase by a factor of $f_V = 4$, $f_k = 2$, and $f_D = 2$, respectively. So, **The fractal dimension** is $d_f = \frac{\ln f_V}{\ln f_D} = 2$ and the degree exponent of boxes is $d_k = \frac{\ln f_k}{\ln f_D} = 1$. The exponent of the degree distribution satisfies $\gamma = 1 + \frac{d_f}{d_k} = 1 + \frac{\ln f_V}{\ln f_k} = 3$, giving the same γ as that obtained in the direct calculation of the degree distribution. So, the 2-Mosaic network $M_{2,n}$ is a fractal.

☞ **Discussion**

According to the found results, the 2-Mosaic network $M_{2,n}$ exhibits many interesting structural features: its degree distribution follows a power law form, its clustering coefficient is missing, its diameter does not scale logarithmically with the number of nodes of the network and it has a finite fractal dimension, which indicate that $M_{2,n}$ is a scale-free network and a fractal.

4.3.1.3 Number of spanning trees of the 2-Mosaic network

The 2-Mosaic network is characterized by the self-similarity and a large number of nodes and edges. For this reason, we can not calculate its complexity using the Kirchhoff Theorem because we need to calculate the determinant of a very large matrix. Therefore, we combine two geometric approaches by applying first the bipartite approach, then the reduction approach to find the exact formula of the number of spanning trees of $M_{2,n}$.

Theorem 4.3.1. *(Mokhlissi et al., 2016b, 2017b) The number of spanning trees of 2-Mosaic network $M_{2,n}$ is given by the following formula:*

$$\tau(M_{2,n}) = 2^{2[\frac{2^{2n}-1}{3}]} \tag{4.34}$$

Proof: The result is demonstrated as follows: $\tau(M_{2,n}) = \tau(R_2(B_2(M_{2,n-1})))$. Using Theorem 2.4.4, we obtain: $\tau(M_{2,n}) = 2^{V_{B_2(M_{2,n-1})}-1} \times \tau(B_2(M_{2,n-1}))$ given that $V_{B_2(M_{2,n-1})} = V_{M_{2,n}}$. Using Theorem 2.4.2, we obtain: $\tau(M_{2,n}) = 2^{V_{M_{2,n}}-1} \times 2^{F_{M_{2,n-1}}-1} \times \tau(M_{2,n-1})$. Using Equation 4.24 and 4.25, we obtain:

$$\begin{aligned}
\tau(M_{2,n}) &= 2^{1+\frac{2^{2n}-1}{3}} \times 2^{\frac{2}{3}(2^{2(n-1)}-1)} \times \tau(M_{2,n-1}) \\
\tau(M_{2,n}) &= 2^{2^{2n-1}} \times \tau(M_{2,n-1}) \\
\tau(M_{2,n}) &= 2^{2^{2n-1}+2^{2n-3}+2^{2n-5}+\ldots+2^1} \times \tau(M_{2,0}) \\
\tau(M_{2,n}) &= 2^{2[\frac{2^{2n}-1}{3}]}.
\end{aligned}$$

4.3.2 m-Mosaic Network

4.3.2.1 Construction of the m-Mosaic network

The m-Mosaic network, denoted by $M_{m,n}$, is characterized by two parameters m and n where m stands for the dimension of the network and n for the current iteration. It is created in the following iterative way: For the first generation, a simple link connects two nodes. For the next generations, the m-partite approach is applied to the network in the previous iteration by adding $m-1$ nodes in each link. Then, the m-reduced approach is applied to the last obtained network to get m multiple links connecting each pair of nodes. This process is denoted by $M_{m,n} = R_m(B_m(M_{n-1}))$. For illustration, in Figure 4.5, the first two iterations of the m-Mosaic network with $m = 3$ are shown.

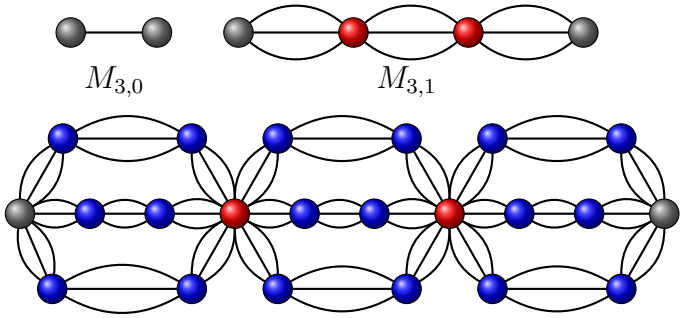


Figure 4.5: The first two generations of the 3-Mosaic network.

The exact values for the number of nodes, edges, faces and the average degree of the m-Mosaic network are presented as follows. According to its construction, we get:

- Let $E_{M_{m,n}}$ be the number of edges created at iteration n. We have $E_{M_{m,n}} = E_{R_m(B_m(M_{m,n-1}))}$. Using Property 2.4.5, we obtain $E_{M_{m,n}} = mE_{B_m(M_{m,n-1})}$. Using Property 2.4.3, we obtain $E_{M_{m,n}} = m^2E_{M_{m,n-1}} = m^4E_{M_{m,n-2}} = m^6E_{M_{m,n-3}} = ... = m^{2n}E_{M_{m,0}}$. Thus, **the number of edges of** $M_{m,n}$ is:

$$E_{M_{m,n}} = m^{2n}, \quad n \geq 0. \tag{4.35}$$

- Similarly, we find **the number of vertices of** $M_{m,n}$ is:

$$V_{M_{m,n}} = 2 + \frac{m^{2n} - 1}{m + 1}, \quad n \geq 0. \tag{4.36}$$

- **The number of faces of** $M_{m,n}$ is:

$$F_{M_{m,n}} = 1 + \frac{m(m^{2n} - 1)}{m + 1}, \quad n \geq 0. \tag{4.37}$$

- Let $< z >_{M_{m,n}}$ be the average degree of $M_{m,n}$. It is calculated as follows: $< z >_{M_{m,n}} = \frac{2E_{M_{m,n}}}{V_{M_{m,n}}}$. Thus, **the average degree of** $M_{m,n}$ is (which is approximately 4 for large n):

$$< z >_{M_{m,n}} = \frac{2E_{M_n}}{V_{M_n}} = \frac{2m^{2n}}{2 + \frac{m^{2n}-1}{m+1}}, \quad n \geq 0. \tag{4.38}$$

4.3.2.2 *Structural properties of the m-Mosaic network*

- **The degree distribution**

  To calculate the degree distribution of the m-Mosaic network $M_{m,n}$, we use the same process applied on the degree distribution of the 2-Mosaic network $M_{2,n}$. Then, the

cumulative degree distribution of $M_{m,n}$ is given by:

$$P_{cum}(k) = \frac{m^{2n+2} + (\frac{k}{2})^{-2}}{m^{2n} + 2m + 1} \tag{4.39}$$

For large n, we can obtain:

$$P_{cum}(k) \sim 4 \times k^{-2} \tag{4.40}$$

So the degree distribution of the m-Mosaic network $M_{m,n}$ follows a power-law form with the exponent $\gamma = 3$.

- **Clustering coefficient**

  The m-Mosaic network $M_{m,n}$ have also zero clusterings because there is no link between neighbors of any node. Therefore, one of the properties of the small-world network is missing.

- **Diameter**

  Let $D_{M_{m,n}}$ be the diameter of $M_{m,n}$ created at generation n. We can calculate this diameter as follows: According to the construction of $M_{m,n}$, we get: $D_{M_{m,n}} = mD_{M_{m,n-1}} = m^2 D_{M_{m,n-2}} = ... = m^n D_{M_{m,0}}$, with $D_{M_{m,0}} = 1$. Thus, **the diameter of** $M_{m,n}$ is:

$$D_{M_{m,n}} = m^n, \quad n \geq 0. \tag{4.41}$$

  for large n, the number of nodes will be $V_{M_{m,n}} \sim m^{2n}$. Then, $D_{M_{m,n}} = V_{M_{m,n}}^{(\frac{1}{2})} = \sqrt{V_{M_{m,n}}}$. It grows as a square power of the number of nodes in the network, which indicates that the m-Mosaic network $M_{m,n}$ is not a small world.

- **Fractal dimension**

  We calculate the fractal dimension of the m-Mosaic network $M_{m,n}$ as follows: By construction, in the infinite n limit, the different quantities of the network grow as:

$$\begin{cases} V_{M_{m,n}} \simeq m^2.V_{M_{m,n-1}}, \\ k_u(n) = m.k_u(n-1), \\ D_{M_{m,n}} = m.D_{M_{m,n-1}}. \end{cases} \tag{4.42}$$

  where $V_{M_{m,n}}$, $k_u(n)$ and $D_{M_{m,n}}$ are the number of nodes, the degree of a node u

and the diameter of $M_{m,n}$ respectively. From Equation (4.42), it is clear that $V_{M_{m,n}}$, $k_u(n)$ and $D_{M_{m,n}}$ increase by a factor of $f_V = m^2$, $f_k = m$, and $f_D = m$, respectively. So, **the fractal dimension** is $d_f = \frac{\ln f_V}{\ln f_D} = 2$, and the degree exponent of boxes is $d_k = \frac{\ln f_k}{\ln f_D} = 1$. The exponent of the degree distribution satisfies $1+\frac{d_f}{d_k} = 1+\frac{\ln f_V}{\ln f_k} = 3$, giving the same γ as that obtained in the direct calculation of the degree distribution. So, the m-Mosaic network $M_{m,n}$ is a fractal.

☞ **Discussion**

According to the found results, the m-Mosaic network $M_{m,n}$ presents many structural characteristics: its degree distribution follows a power law form, a zero clustering coefficient, its diameter does not scale logarithmically with the number of nodes of the network and it has a finite fractal dimension, which indicate that $M_{m,n}$ is a scale-free network and a fractal.

4.3.2.3 Number of spanning trees of the m-Mosaic network

To evaluate the complexity of the m-Mosaic network, two approaches are combined. First, the m-partite approach, then the m-reduced approach are applied.

Theorem 4.3.2. *(Mokhlissi et al., 2016b, 2017b) The number of spanning trees of m-Mosaic network $M_{m,n}$ is given by the following formula:*

$$\tau(M_{m,n}) = m^{m[\frac{m^{2n}-1}{m^2-1}]} \tag{4.43}$$

Proof: Theorem is demonstrated as follows: $\tau(M_{m,n}) = \tau(R_m(B_m(M_{m,n-1})))$. Using Theorem 2.4.5, we obtain: $\tau(M_{m,n}) = m^{V_{B_m(M_{m,n-1})}-1} \times \tau(B_m(M_{m,n-1}))$ with $V_{B_m(M_{m,n-1})} = V_{M_{m,n}}$. Using Theorem 2.4.3, we obtain: $\tau(M_{m,n}) = m^{V_{M_{m,n}}-1} \times m^{F_{M_{m,n-1}}-1} \times \tau(M_{m,n-1})$. Using Equation 4.36 and 4.37, we obtain:

$$\begin{aligned}
\tau(M_{m,n}) &= m^{1+\frac{m^{2n}-1}{m+1}} \times m^{\frac{m(m^{2(n-1)}-1)}{m+1}} \times \tau(M_{m,n-1}) \\
\tau(M_{m,n}) &= m^{m^{2n-1}} \times \tau(M_{m,n-1}) \\
\tau(M_{m,n}) &= m^{m^{2n-1}+m^{2n-3}+m^{2n-5}+\ldots+m^1} \times \tau(M_{m,0}) \\
\tau(M_{m,n}) &= m^{m[\frac{m^{2n}-1}{m^2-1}]}.
\end{aligned}$$

4.3.2.4 Entropy of spanning trees of the m-Mosaic network

The number of spanning trees of the m-Mosaic network $\tau(M_{m,n})$ grows exponentially with the network order $V_{M_{m,n}}$, then we can calculate its entropy of spanning trees using Equation (2.1).

Corollary 4.3.1. *(Mokhlissi et al., 2017b) The entropy of spanning trees of the m-Mosaic network is:*

$$\rho_{M_{m,n}} = m\frac{ln(m)}{m-1} \tag{4.44}$$

Proof: The entropy of spanning trees of the m-Mosaic network is calculated as follows: $\rho_{M_{m,n}} = \lim\limits_{V_{M_{m,n}}\to\infty} \frac{\ln|\tau(M_{m,n})|}{|V_{M_{m,n}}|} = \lim\limits_{n\to\infty} \frac{\ln|\tau(M_{m,n})|}{|V_{M_{m,n}}|} = \lim\limits_{n\to\infty} \frac{\ln(m^{m[\frac{m^{2n}-1}{m^2-1}]})}{2+\frac{m^{2n}-1}{m+1}} = \lim\limits_{n\to\infty} \frac{m\times m^{2n}}{(m-1)\times(m+1)}\times \frac{\ln(m)(m+1)}{m^{2n}} = m\frac{ln(m)}{m-1}$. Then, the result.

Similarly, we can find the entropy of the 2-Mosaic network: $\rho_{M_{2,n}} = 2\times\ln 2 = 1,3862$. According to Figure 4.6, the entropy of spanning trees of the m-Mosaic network is bigger than 1 depending on the dimension m and the increasing of the value of this dimension leads to the increase of the entropy of spanning trees. From this result, we deduce that the m-Mosaic networks with a smaller dimension are less robust than those with a larger dimension.

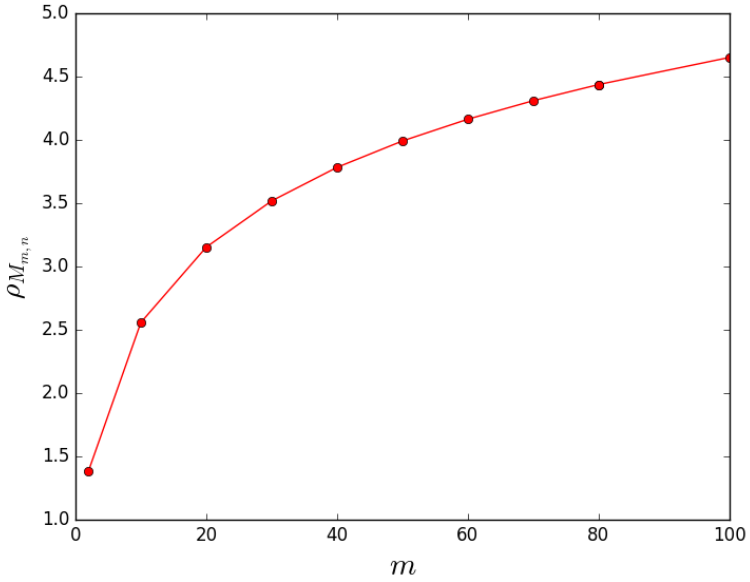


Figure 4.6: The entropy of spanning trees of the m-Mosaic network.

From Table 4.2, the entropy of the 2-Mosaic network is larger than that of the Apollonian network with the same average degree and it is smaller than that of the contact graph of disk packings, the three-dimensional Sierpinski graph and the three-hypercubic lattice and also the increasing of the value of the dimension m of m-Mosaic networks leads to the increase of their entropy of spanning trees. This proves that the value of the entropy of the spanning trees of $M_{m,n}$ with $m > 3$ is the biggest known for networks with an average degree of 6. It is demonstrated that the m-Mosaic network with $m > 3$ has much more spanning trees compared to other networks with the same average degree. The increase in the number of spanning trees overall networks with the identical average degree can lead to a more robust network. This means that the m-Mosaic network with $m > 3$ is the most robust network compared to other networks having the same average degree.

Type of network	$< z >$	ρ
Apollonian network (Zhang et al., 2014)	6	1.3540
2-Mosaic networks (Mokhlissi et al., 2017b)	**6**	**1,3862**
The contact graph of disk packings (Qin et al., 2015)	6	1.4354
Three-dimensional Sierpinski graph (Chang et al., 2007)	6	1.5694
3-Mosaic networks (Mokhlissi et al., 2017b)	**6**	**1.6479**
Three-hypercubic lattice (Felker and Lyons, 2003)	6	1.6734
4-Mosaic networks (Mokhlissi et al., 2017b)	**6**	**1.8483**

Table 4.2: The entropy of spanning trees of several networks having the same average degree.

☞ **Comparison between the entropy and the number of spanning trees of the Flower network and the Mosaic network.**

From the results above, we notice that the m-Flower network and the m-Mosaic network have the same complexity(Mokhlissi et al., 2016b):

$$\tau(F_{m,n}) = \tau(M_{m,n}) = \tau(B_m(R_m(F_{m,n-1}))) = \tau(R_m(B_m(M_{m,n-1})))$$

If we start with the application of the m-partition approach, then the m-reduction approach or the opposite process, we get the same complexity. This reflects that the combination of the m-partition and the m-reduction approaches with two different ways leads to the same complexity in spite of the difference of the structure and the properties

of the studied networks. Otherwise, they have two different entropies, although they have the same complexity. The m-Mosaic network with $m > 3$ has the largest entropy, so it is the most robust network. From these results, it is deduced that the complexity is not sufficient to characterize the performance of a topology of a network, but the entropy must be calculated to quantify its robustness and characterize its structure.

4.4 Analysis of Fractal Scale-Free Lattice

In this section, we are interested in the Fractal Scale Free Lattice (Zhang et al., 2008b, 2011; Mokhlissi et al., 2019b). It is a scale-free network with fractality that follows a power-law degree distribution. It is characterized by the self-similar property that highlights its topological structure. We investigate two cases of the Fractal Scale-Free Lattice: The particular case and the general case. The difference between these cases is the number of the connected clusters "q" that replaces each iterative edge (See Figure 4.7). For each case, we examine the construction of the network and its structural properties, proving that the generalization of the Fractal Scale-Free Lattice network does not affect the scale-free property. Then, we evaluate its number of spanning trees using the electrically equivalent transformations. In the end, we calculate its entropy of spanning trees and we compare it with other studied networks having the same average degree. The results show that this value is constant and does not depend on any parameter.

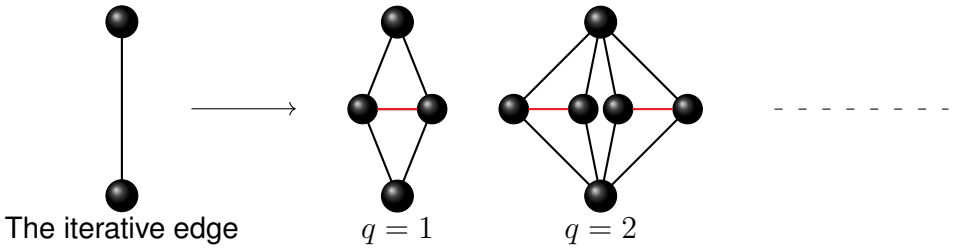


Figure 4.7: The difference between the particular case and the general case of the Fractal Scale-Free Lattice. Each iterative edge is replaced by q connected clusters. The red edge is a noniterated edge.

4.4.1 Fractal Scale-Free Lattice $L_{1,n}$

4.4.1.1 Construction of the Fractal Scale-Free Lattice $L_{1,n}$

The Fractal Scale-Free Lattice is denoted by $L_{1,n}$ with '1' refers to the number of the connected clusters and $n \geq 0$ iterations. The construction of this studied network follows this algorithm: At $n = 0$, we have an iterative edge connecting two nodes. For $n \geq 1$, we get $L_{1,n}$ from $L_{1,n-1}$. We replace each iterative edge in $L_{1,n-1}$ by one connected cluster. The growing process is repeated in a similar way n times. Figure 4.8 illustrates the first three generations of the Fractal Scale-Free Lattice $L_{1,n}$

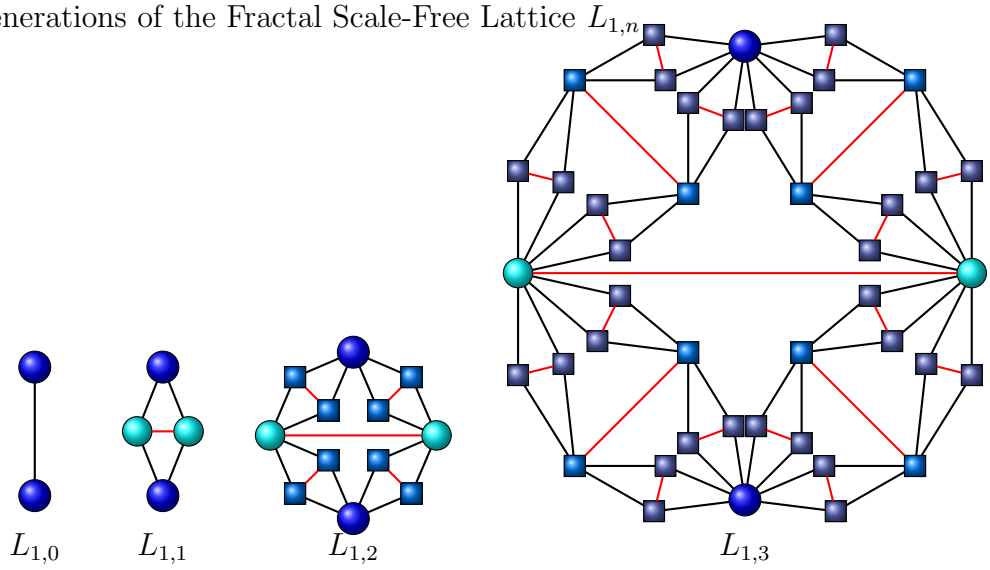


Figure 4.8: The first three generations of the Fractal Scale-Free Lattice $L_{1,n}$.

Next, we compute the total number of nodes, edges, faces and average degrees of $L_{1,n}$. According to its construction, we get:

- Let $V_{L_{1,n}}$ be the number of nodes. For i from 1 to n, we have $V_{L_{1,i}} = 4V_{L_{1,i-1}} - 4$. Then, we multiply the equation of $V_{L_{1,n-1}}$ by 4, the equation of $V_{L_{1,n-2}}$ by 4^2 and so on until the last equation $V_{L_{1,1}}$ which will be multiplied by 4^{n-1}. Summing all the obtained equations $\sum_{i=0}^{n-1} 4^i V_{L_{1,n-i}} = \sum_{i=0}^{n-1} 4^{i+1} V_{L_{1,n-i-1}} - 4\sum_{i=0}^{n-1} 4^i$. We find $V_{L_{1,n}} = 4^n V_{L_{1,0}} - 4\sum_{i=0}^{n-1} 4^i$ with $V_{L_{1,0}} = 2$. Thus **the number of nodes of $L_{1,n}$ is:**

$$V_{L_{1,n}} = \frac{2 \times 4^n + 4}{3}, \quad n \geq 0. \tag{4.45}$$

- Similarly, we find **the number of edges of $L_{1,n}$:**

$$E_{L_{1,n}} = \frac{4^{n+1} - 1}{3}, \quad n \geq 0. \tag{4.46}$$

- **The number of faces of $L_{1,n}$ is:**

$$F_{L_{1,n}} = \frac{2 \times 4^n + 1}{3}, \quad n \geq 0. \tag{4.47}$$

- Let $< z >_{L_{1,n}}$ be the average degree of $L_{1,n}$. It is calculated as follows: $< z >_{L_{1,n}} = \frac{2E_{L_{1,n}}}{V_{L_{1,n}}}$. Thus **the average degree of** $L_{1,n}$ is (which is approximately 4 for large n):

$$< z >_{L_{1,n}} = \frac{4^{n+1} - 1}{4^n + 2}, \quad n \geq 0 \tag{4.48}$$

4.4.1.2 *Structural properties of the Fractal Scale-Free Lattice* $L_{1,n}$

- **The degree distribution**

The degree distribution of the Fractal Scale-Free Lattice $L_{1,n}$ is calculated as follows (Zhang et al., 2011, 2008b,a): When a new node u is added to the network at $n_u \geq 1$, it has three links: two iterative links and one noniterated link (The red links in Figure 4.8). Let $M_e(u, n)$ be the number of iterative links at iteration n that will create new nodes connected to the node u at $n + 1$. Then, at n_u, $M_e(u, n_u) = 2$. From the construction of the network $L_{1,n}$, we can see that at any step, each iterative link of u is broken and generates two new iterative links connected to u. Let $k_u(n)$ be the degree of node u at step n. So, the relation between $k_u(n)$ and $M_e(u, n)$ satisfies:

$$k_u(n) = M_e(u, n) + 1 \tag{4.49}$$

where "1" represents the only noniterated link of node u. Now, we calculate $M_e(u, n)$. We see from the construction of $L_{1,n}$: $M_e(u, n) = 2M_e(u, n-1) = 2^2 M_e(u, n-2) = ... = 2^{n-n_u} M_e(u, n_u) = 2^{n-n_u+1}$. Then

$$k_u(n) = 2^{n-n_u+1} + 1 \tag{4.50}$$

We mention that the initial two nodes (The dark blue nodes in Figure 4.8) created at iteration 0 are different from the other nodes. Since the initial two nodes have no noniterated link, their degree equals the number of iterative links connecting to them 2^n. Then, the cumulative degree distribution of $L_{1,n}$ is given by:

$$P_{cum}(k) = \sum_{i \leq n_u} \frac{M_v(i)}{V_{L_{1,n}}} = \frac{2 \times 4^{n_u} + 4}{2 \times 4^n + 4} \tag{4.51}$$

with $M_v(n)$ is the number of new nodes at step n. By construction, we notice that for $n \geq 1$, $M_v(n) = 2M_e(n-1)$ with $M_e(n-1)$ is the number of iterative links at step $n-1$. Considering the initial condition $M_v(0) = 2$ and $M_e(0) = 1$, it follows that $M_v(n) = 2.4^{n-1}$. From the Equation (**??**), we obtain $n_u = n + 1 - \frac{\ln(k-1)}{\ln 2}$. So, Equation (4.51) will be

$$P_{cum}(k) = \frac{2 \times 4^n \times 4(k-1)^{-(\ln 4/\ln 2)} + 4}{2 \times 4^n + 4} \tag{4.52}$$

When n is large enough, **the cumulative degree distribution of** $L_{1,n}$ will be

$$P_{cum}(k) \sim 4(k-1)^{-2}. \tag{4.53}$$

From Equation (4.53), we get that the degree distribution of the Fractal Scale-Free Lattice $L_{1,n}$ follows a power-law form with the exponent $\gamma = 3$, indicating that $L_{1,n}$ is a scale free network.

- **Clustering coefficient**

  By construction of $L_{1,n}$, it is easy to calculate analytically the clustering coefficient C_u for each node u with the degree k (Zhang et al., 2008a, 2011, 2008b):
  - For the two nodes created at iteration $n = 0$, their degree is $k = 2^n$, and the existing edges among these neighbors is $e = \frac{k}{2} = 2^{n-1}$, all of which are noniterated edges.
  - For those nodes born at step $r(0 < r < n)$, their degree is $k = 2^{n-r+1} + 1$ (See Equation (4.50)) and there are only $e = \frac{k-1}{2} = 2^{n-r}$ edges that actually exist among the neighbor nodes.
  - For the smallest nodes created at step n, their degree is $k = 3$ and the existing number of links between the neighbors of each node is $e = 2$.

  Thus, **the clustering coefficient** C_u of the node u and its degree k:

$$C_u = \begin{cases} 1/(k-1) & \text{for } k = 2^n, \\ 1/k & \text{for } k = 2^{n-r+1} + 1, \ (0 < r < n), \\ 2/k & \text{for } k = 2^1 + 1. \end{cases} \tag{4.54}$$

which is inversely proportional to k in the limit of large k. The local clustering scales as $C_u \sim k^{-1}$. It is interesting to notice that a similar scaling has been observed in several real-life scale free networks. Let $C_{L_{1,n}}$ be the clustering coefficient of the whole network. Using Equation (4.54), we can obtain the average clustering coefficient of the Fractal Scale-Free Lattice $L_{1,n}$:

$$C_{L_{1,n}} = \frac{1}{V_{L_{1,n}}}\left[\frac{M_v(o)}{k_o - 1} + \sum_{r=1}^{n-1}\frac{M_v(r)}{k_r} + \frac{2M_v(n)}{k_n}\right] \tag{4.55}$$

where k_r is the degree of a node at step n, which was created at step r. Then, Equation (4.55) will be

$$C_{L_{1,n}} = \frac{3}{2.4^n + 4}\left[\frac{2}{2^n - 1} + \sum_{r=1}^{n-1}\frac{2.4^{r-1}}{2^{(n-r+1)} + 1} + \frac{4^n}{3}\right] \tag{4.56}$$

When ($V_{L_{1,n}} \to \infty$), Equation (4.56) converges to a nonzero value $C_{L_{1,n}} = 0.5435$. Therefore, the average clustering coefficient of the Fractal Scale-Free Lattice $L_{1,n}$ is high.

- **Diameter**

Let $D_{L_{1,n}}$ be the diameter of the Fractal Scale-Free Lattice $L_{1,n}$. It can be calculated as follows: For i from 1 to n, we have $D_{L_{1,i}} = 2D_{L_{1,i-1}}$. Then, we multiply the equation of $D_{L_{1,n-1}}$ by 2, the equation of $D_{L_{1,n-2}}$ by 2^2 and so on until the last equation $D_{L_{1,1}}$, which will be multiplied by 2^{n-1}. Summing all the obtained equations $\sum_{i=0}^{n-1} 2^i D_{L_{1,n-i}} = \sum_{i=0}^{n-1} 2^{i+1} D_{L_{1,n-i-1}}$. We find $D_{L_{1,n}} = 2^n D_{L_{1,0}}$ with $D_{L_{1,0}} = 1$. Thus, **the diameter of $L_{1,n}$ is:**

$$D_{L_{1,n}} = 2^n, \quad n \geq 0. \tag{4.57}$$

For large n, the number of nodes will be 4^n. Thus, the diameter of $L_{1,n}$ scales as $D_{L_{1,n}} = \sqrt{V_{L_{1,n}}}$. It grows as a square power of the number of nodes in the network, which indicates that the Fractal Scale-Free Lattice $L_{1,n}$ is not a small world.

- **Fractal dimension**

We calculate the dimension of the Fractal Scale-Free Lattice $L_{1,n}$ as follows: By construction, in the infinite n limit, the different quantities of the network grow

as (Zhang et al., 2011, 2008b):

$$\begin{cases} V_{L_{1,n}} \simeq 4.V_{L_{1,n-1}}, \\ k_u(n) \simeq 2.k_u(n-1), \\ D_{L_{1,n}} = 2.D_{L_{1,n-1}}. \end{cases} \tag{4.58}$$

where $V_{L_{1,n}}$, $k_u(n)$ and $D_{L_{1,n}}$ are the number of vertices, the degree of a node u and the diameter of $L_{1,n}$ respectively. From Equation (4.58), it is obvious that $V_{L_{1,n}}$, $k_u(n)$ and $D_{L_{1,n}}$ increase by a factor of $f_V = 4$, $f_k = 2$, and $f_D = 2$, respectively. So, **the fractal dimension** is $d_f = \frac{\ln f_V}{\ln f_D} = 2$, and the degree exponent of boxes is $d_k = \frac{\ln f_k}{\ln f_D} = 1$. The exponent of the degree distribution satisfies $\gamma = 1 + \frac{d_f}{d_k} = 1 + \frac{\ln f_V}{\ln f_k} = 3$, giving the same γ as that obtained in the direct calculation of the degree distribution.

- **Average path length**

The Fractal Scale-Free Lattice $L_{1,n}$ has a self-similar structure allowing us to calculate the average path length (APL) $l_{L_{1,n}}$ analytically (Zhang et al., 2011, 2008b,a):

$$l_{L_{1,n}} = \frac{\sigma(L_{1,n})}{|V_{L_{1,n}}|(|V_{L_{1,n}}| - 1)} \tag{4.59}$$

where $\sigma(L_{1,n}) = \sum_{u \neq v} d(u, v)$. As shown in Figure 4.9, the network $L_{1,n+1}$ may be obtained by the juxtaposition of four copies of $L_{1,n}$, which are labeled as $L^{\alpha}_{1,n}$ with $\alpha = 1, 2, 3, 4$. Then:

$$\sigma(L_{1,n+1}) = 4\sigma(L_{1,n}) + \Delta_n. \tag{4.60}$$

where Δ_n is the sum over all shortest paths whose endpoints are not in the same $L_{1,n}$ branch. Its expression is found below:

$$\Delta_n = \frac{1}{189}\left[45 - 119.2^{n+1} + 15.2^{3n+2} + 7.2^{5n+6} + 63.4^{2n+1} + 21n.2^{3n+2}\right]. \tag{4.61}$$

Substituting Equation 4.61 for Δ_n into Equation 4.60 and using $\sigma(L_{1,0}) = 1$, then

$$\sigma(L_{1,n}) = \frac{1}{189}\left[2^{5n+4} + 21.2^{4n} + 21n.2^{3n} - 27.2^{3n} + 75.2^{2n} + 119.2^{n} - 15\right]. \tag{4.62}$$

Inserting Equation 4.62 into Equation 4.59, we can obtain the expression for $l_{L_{1,n}}$:

$$l_{L_{1,n}} = \frac{(16.2^n + 21)16^n + (21n - 27)8^n + 75.4^n + 119.2^n - 15}{21(2 + 5.4^n + 2.16^n)} \tag{4.63}$$

For large n, $l \sim \frac{8}{21}.2^n$. Note that in the infinite n limit, $V_{L_{1,n}} \sim 4^n$. So, the APL scales as $l \sim V_{L_{1,n}}^{1/2}$, which indicates that the Fractal Scale-Free Lattice $L_{1,n}$ is not a small-world.

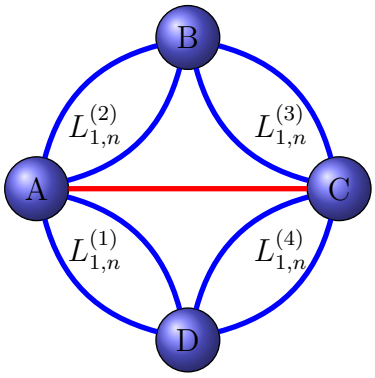


Figure 4.9: Schematic illustration of the recursive construction of the Fractal Scale-Free Lattice $L_{1,n}$. The red link is noniterated edge.

☞ Discussion

According to the above results, the Fractal Scale-Free Lattice $L_{1,n}$ has some interesting properties: its degree distribution follows a power law form, a high clustering coefficient, its diameter and its average path length do not scale logarithmically with the number of nodes of the network and it has a finite fractal dimension, which indicate that $L_{1,n}$ is a scale-free network and a fractal.

4.4.1.3 Number of spanning trees of the Fractal Scale-Free Lattice $L_{1,n}$

By applying the electrically equivalent technique, we can easily obtain the number of spanning trees of the Fractal Scale-Free Lattice $L_{1,n}$, because if any subgraph of $L_{1,n}$ is replaced by an electrically equivalent network, the number of spanning trees only changes by a factor that is independent of the rest of the network. To calculate its number of spanning trees using this technique, we follow the steps of Section 2.4.4.4:

1. First, we add the weight a_0 on each edge of $L_{1,n}$.

2. Then, we use the electrically equivalent transformations to obtain the change of edges conductances of the $n-1$ generation from $L_{1,n}$, the relationship between the number of spanning trees of $\tau(L_{1,n})$ and $\tau(L_{1,n-1})$ and the exact value of the number of spanning trees of $L_{1,n}$ in each step.

   By applying the algorithms of the electrically equivalent transformations, we deduce $\tau(L_{1,n}) = (8a_0^2)^{4^{n-1}}\tau(L_{1,n-1})$. To prove mathematically this formula, we use the mathematical induction:

   Basic step: We prove that $\tau(L_{1,1}) = 8a_0^2\tau(L_{1,0})$. We denote $L_{1,1}^{(0)}$ as the Fractal Scale-Free Lattice at iteration 1 in the transformation 0 with all its edges having the conductance a_0. Figure 4.10 shows the electrically equivalent evolving process from $L_{1,1}^{(0)}$ to $L_{1,0}^{(0)}$.

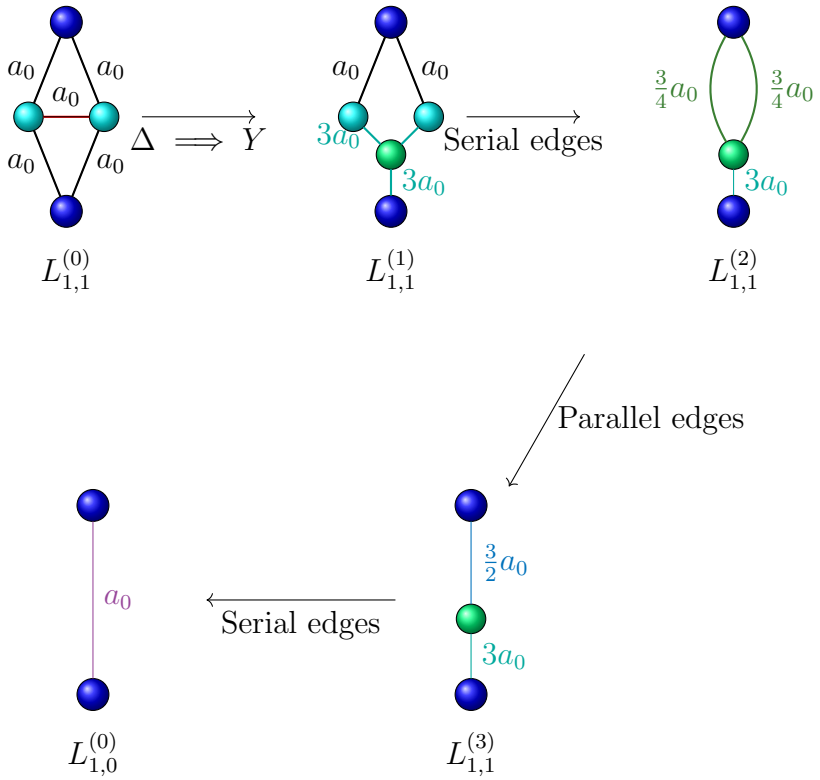


Figure 4.10: The transformations from $L_{1,1}$ to $L_{1,0}$.

Four transformations are used in sequence and the corresponding weights of the resulting edges are calculated as follows:

(a) *Delta− Wye transformation:* We replace one triangle in the connected cluster of $L_{1,1}^{(0)}$ (Δ network) with its electrically equivalent component (Y network). We get $L_{1,1}^{(1)}$ as the resulting graph with three new edges with a new conductance $\frac{a_0^2+a_0^2+a_0^2}{a_0} = 3a_0$.

(b) *Serial edge:* We replace all the serial edges with conductances a_0 and $3a_0$ by a new edge with the conductance $\frac{3a_0.a_0}{3a_0+a_0} = \frac{3}{4}a_0$ to obtain a new subgraph $L_{1,1}^{(2)}$.

(c) *Parallel edge:* We merge two parallel edges having the same conductance $\frac{3}{4}a_0$ of $L_{1,1}^{(2)}$ to obtain a new subgraph $L_{1,1}^{(3)}$. The corresponding conductance of a new edge is $\frac{3}{4}a_0 + \frac{3}{4}a_0 = \frac{3}{2}a_0$.

(d) *Serial edge:* We replace two serial edges with conductances $3a_0$ and $\frac{3}{2}a_0$ by a new edge with the conductance $\frac{3a_0.\frac{3}{2}a_0}{3a_0+\frac{3}{2}a_0} = a_0$ to get $L_{1,0}^{(0)}$.

Noting that those four applications of the electrically equivalent transformations suffice to find the conversion formula of spanning trees between $L_{1,1}^{(0)}$ and $L_{1,0}^{(0)}$. Then

$\tau(L_{1,1}) = \tau(L_{1,1}^{(0)})$.

$\tau(L_{1,1}) = \frac{a_0^3}{(3a_0^2)^2}\tau(L_{1,1}^{(1)}) = \frac{1}{9a_0}\tau(L_{1,1}^{(1)}) \rightarrow (\Delta \implies Y)$.

$\tau(L_{1,1}) = \frac{1}{9a_0}(3a_0 + a_0)^2\tau(L_{1,1}^{(2)}) = \frac{4^2a_0}{9}\tau(L_{1,1}^{(2)}) \rightarrow \ (Serial\ edge)$.

$\tau(L_{1,1}) = \frac{4^2a_0}{9} \times 1 \times \tau(L_{1,1}^{(3)}) \rightarrow \ (Parallel\ edge)$.

$\tau(L_{1,1}) = \frac{4^2a_0}{9}(\frac{3}{2}a_0 + 3a_0)\tau(L_{1,1}^{(4)}) = \frac{4^2a_0}{9}.\frac{9a_0}{2}\tau(L_{1,1}^{(4)}) \rightarrow \ (Serial\ edge)$.

$\tau(L_{1,1}) = 8a_0^2\tau(L_{1,0}^{(0)})$.

Induction step: We assume that $\tau(L_{1,n}) = (8a_0^2)^{4^{n-1}}\tau(L_{1,n-1})$. We show that $\tau(L_{1,n+1}) = (8a_0^2)^{4^n}\tau(L_{1,n})$.

The number of connected clusters in $L_{1,n}$ is 4^{n-1}. Since the number of spanning trees depends on the number of connected clusters, this means in $L_{1,n+1}$, the number of connected cluster is 4^n. Then, the number of spanning trees of $L_{1,n+1}$ is $\tau(L_{1,n+1}) = (8a_0^2)^{4^n}\tau(L_{1,n})$.

3. Finally, we deduce the exact formula of the number of spanning trees of $\tau(L_{1,n})$.

$$\begin{aligned}
\tau(L_{1,n}) &= (8a_0^2)^{4^{n-1}}\tau(L_{1,n-1}) \\
\tau(L_{1,n}) &= (8a_0^2)^{4^{n-1}} \times (8a_0^2)^{4^{n-2}} \times ... \times (8a_0^2)^{4^0}\tau(L_{1,0}) \\
\tau(L_{1,n}) &= (8a_0^2)^{4^{n-1}+4^{n-2}+...+4^0}\tau(L_{1,0}) \ with\ \tau(L_{1,0}) = a_0
\end{aligned}$$

Hence, the number of spanning trees of $L_{1,n}$ is given by this formula:

$$\tau(L_{1,n}) = (8a_0^2)^{\frac{4^n-1}{3}} a_0, \quad n \geq 0. \tag{4.64}$$

4.4.2 Generalized Fractal Scale-Free Lattice $L_{q,n}$

4.4.2.1 Construction of the Generalized Fractal Scale-Free Lattice $L_{q,n}$

The generalized Fractal Scale-Free Lattice is denoted by $L_{q,n}$ with $q \geq 1$ is the number of connected clusters and $n \geq 0$ iterations. The construction of the generalized Fractal Scale-Free Lattice network follows this algorithm: At $n = 0$, we have an iterative edge connecting two nodes. For $n \geq 1$, we get $L_{q,n}$ from $L_{q,n-1}$. We replace each iterative edge in $L_{q,n-1}$ by q connected clusters. The growth process of the generalized Fractal Scale-Free Lattice to the next generations keeps going on in a similar way. Figure 4.11 illustrates the growing process of the generalized Fractal Scale-Free Lattice $L_{2,n}$ where $q = 2$.

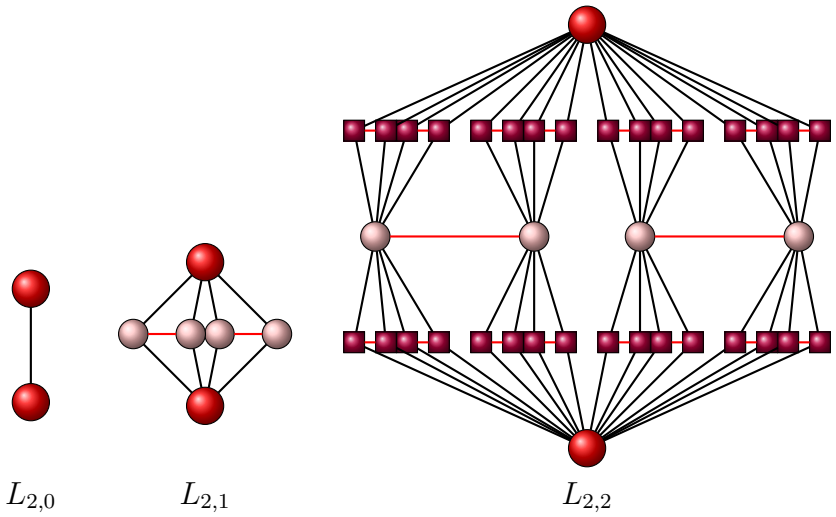


Figure 4.11: The first two generations of the generalized Fractal Scale-Free Lattice $L_{2,n}$.

Next, we compute the number of total nodes, edges, faces and the average degree of the generalized Fractal Scale-Free Lattice. According to the construction of $L_{q,n}$, we get:

- Let $V_{L_{q,n}}$ be the number of nodes created at the iteration n. For i from 1 to n, we have $V_{L_{q,i}} = 4qV_{L_{q,i-1}} - 6q + 2$. Then, we multiply the equation of $V_{L_{q,n-1}}$ by $(4q)$, the equation of $V_{L_{q,n-2}}$ by $(4q)^2$ and so on until the last equation of $V_{L_{q,1}}$ which will be multiplied by $(4q)^{n-1}$. Summing all the obtained equations $\sum_{i=0}^{n-1}(4q)^i V_{L_{q,n-i}} = \sum_{i=0}^{n-1}(4q)^{i+1}V_{L_{q,n-i-1}} + (-6q+2)\sum_{i=0}^{n-1}(4q)^i$. We find the following results $V_{L_{q,n}} =$

$(4q)^n V_{L_{q,0}} + (-6q+2)\sum_{i=0}^{n-1}(4q)^i$ with $V_{L_{q,0}} = 2$. Thus, **the number of nodes of $L_{q,n}$ is:**

$$V_{L_{q,n}} = \frac{(4q)^n 2q + 6q - 2}{4q-1}, \quad n \geq 0. \tag{4.65}$$

- Similarly, we find **the number of edges of $L_{q,n}$:**

$$E_{L_{q,n}} = \frac{(4q)^n(5q-1) - q}{4q-1}, \quad n \geq 0. \tag{4.66}$$

- **the number of faces of $L_{q,n}$ is:**

$$F_{L_{q,n}} = \frac{(4q)^n(3q-1) + q}{4q-1}, \quad n \geq 0. \tag{4.67}$$

- **The average degree of** $L_{q,n}$ is (which is approximately 4 for large n):

$$< z >_{L_{q,n}} = \frac{2E_{L_{q,n}}}{V_{L_{q,n}}} = \frac{(4q)^n(5q-1) - q}{(4q)^n q + 3q - 1}, \quad n \geq 0 \tag{4.68}$$

4.4.2.2 *Structural properties of the Generalized Fractal Scale-Free Lattice $L_{q,n}$*

- **The degree distribution**

To calculate the degree distribution of the generalized Fractal Scale-Free Lattice $L_{q,n}$, we use the same process applied on the degree distribution of the Fractal Scale-Free Lattice $L_{1,n}$ (Mokhlissi et al., 2019b). Then, the cumulative degree distribution of $L_{q,n}$ is given by:

$$P_{cum}(k) = \frac{(4q)^n \times (k-1)^{-(\frac{\ln(4q)}{\ln(2q)})} \times (4q)^{\frac{\ln 2}{\ln(2q)}} \times 2q + 6q - 2}{(4q)^n 2q + 6q - 2} \tag{4.69}$$

When n is large enough, **the cumulative degree distribution of $L_{q,n}$** will be

$$P_{cum}(k) \sim (4q)^{\frac{\ln 2}{\ln(2q)}} \times (k-1)^{-(\frac{\ln(4q)}{\ln(2q)})}. \tag{4.70}$$

From Equation 4.70, we get that the degree distribution follows a power-law form with degree exponent continuously $\gamma \sim 1 + \frac{\ln(4q)}{\ln(2q)}$ which belongs to the interval $[2, 3]$, indicating that $L_{q,n}$ is a scale-free network. Figure 4.12 shows the scaling behavior of the cumulative degree distribution $P_{cum}(k)$ of $L_{q,n}$ for different values of k and q.

- **Clustering coefficient**

We calculate the clustering coefficient of the generalized Fractal Scale-Free Lattice $L_{q,n}$ using the same process as the clustering coefficient of the Fractal Scale-Free

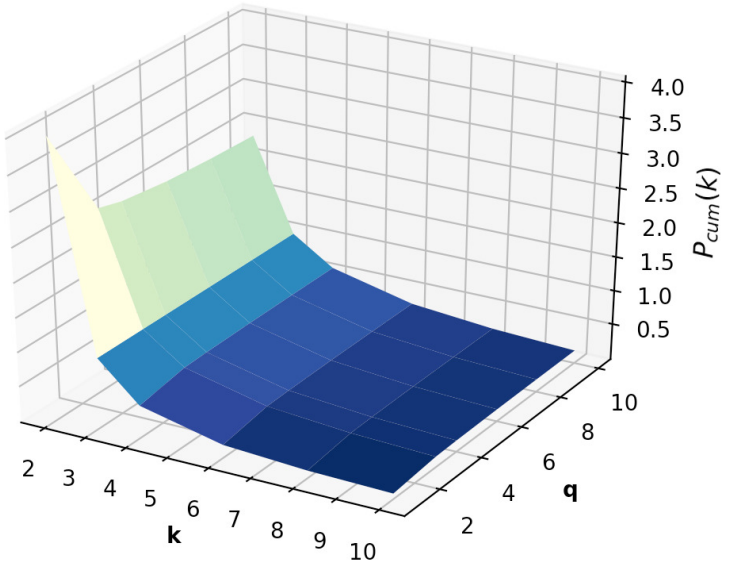


Figure 4.12: The cumulative degree distribution of $L_{q,n}$

Lattice $L_{1,n}$ (Mokhlissi et al., 2019b). Then, the clustering coefficient C_u of a node u with the degree k:

$$C_u = \begin{cases} 1/(k-1) & \text{for } k = (2q)^n, \\ 1/k & \text{for } k = 2(2q)^{n-r} + 1,\ (0 < r < n), \\ 2/k & \text{for } k = 2^1 + 1. \end{cases} \tag{4.71}$$

Thus, the average clustering coefficient $C_{L_{q,n}}$ of the whole network $L_{q,n}$ is given by:

$$C_{L_{q,n}} = \frac{1}{V_{L_{q,n}}} \left[\frac{M_v(o)}{k_o - 1} + \sum_{r=1}^{n-1} \frac{M_v(r)}{k_r} + \frac{2M_v(n)}{k_n} \right] \tag{4.72}$$

where k_r is the degree of a node at step n, which was created at step r. Then

$$C_{L_{q,n}} = \frac{4q-1}{(4q)^n 2q + 6q - 2} \left[\frac{2}{(2q)^n - 1} + \sum_{r=1}^{n-1} \frac{(2q)(4q)^{r-1}}{2(2q)^{(n-r)} + 1} + \frac{(4q)^n}{3} \right] \tag{4.73}$$

The clustering coefficient of the generalized Scale-Free Lattice $L_{q,n}$ approaches a constant value 1. As demonstrated in Figure 4.13, for large value of n, the clustering coefficient will be high with different value of q.

- **Diameter**

The diameter of the generalized Fractal Scale-Free Lattice $D_{L_{q,n}}$ is still the same as

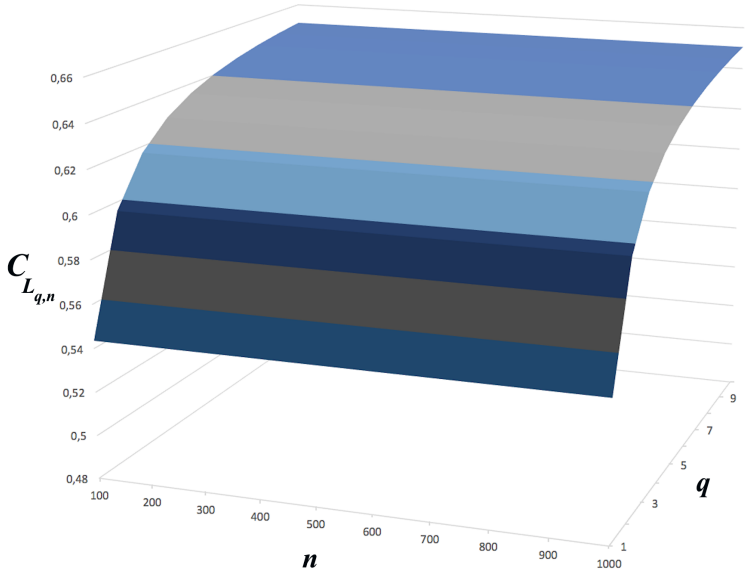


Figure 4.13: The clustering coefficient of $L_{q,n}$

the particular case of the Fractal Scale-Free Lattice for each step (Mokhlissi et al., 2019b).

$$D_{L_{q,n}} = 2^n, n \geq 0. \tag{4.74}$$

Then, we deduce that $D_{L_{q,n}}$ does not depend on the number of clusters q. For large n, the diameter of $L_{q,n}$ grows as a square power of the number of nodes in the network: $D_{L_{q,n}} = \sqrt{\frac{V_{L_{q,n}}}{q^n}}$, which indicates that the generalized Fractal Scale-Free Lattice $L_{q,n}$ is not a small world.

- **Fractal dimension**

  For the generalized Fractal Scale-Free Lattice $L_{q,n}$, we calculate its fractal dimension as follows: By construction, in the infinite n limit, the different quantities of the network grow as (Mokhlissi et al., 2019b):

$$\begin{cases} V_{L_{q,n}} \simeq 4q.V_{L_{q,n-1}}, \\ k_u(n) \simeq 2q.k_u(n-1), \\ D_{L_{q,n}} = 2.D_{L_{q,n-1}}. \end{cases} \tag{4.75}$$

  where $V_{L_{q,n}}$, $k_u(n)$ and $D_{L_{q,n}}$ are the number of vertices, the degree of a node u and the diameter of $L_{q,n}$ respectively. From Equation 4.75, it is obvious that $V_{L_{q,n}}$, $k_u(n)$ and $D_{L_{q,n}}$ increase by a factor of $f_V = 4q$, $f_k = 2q$, and $f_D = 2$, respectively. So, **the fractal dimension** is $d_f = \frac{\ln f_V}{\ln f_D} = 2 + \frac{\ln q}{\ln 2}$, and the degree exponent of

boxes is $d_k = \frac{\ln f_k}{\ln f_D} = 1 + \frac{\ln q}{\ln 2}$. The exponent of the degree distribution satisfies $\gamma = 1 + \frac{d_f}{d_k} = 1 + \frac{\ln f_V}{\ln f_k} = 1 + \frac{\ln 4q}{\ln 2q}$, giving the same γ as that obtained in the direct calculation of the degree distribution.

☞ **Discussion**

After calculating the structural properties of the generalized Fractal Scale-Free Lattice, we get that the degree distribution follows a power-law form, a high clustering coefficient, the diameter does not scale logarithmically with the number of nodes of the network and it has a finite fractal dimension. This proves that $L_{q,n}$ is a scale-free network and a fractal as the Fractal Scale-Free Lattice $L_{1,n}$. We deduce from these results that the generalization of the Fractal Scale-Free Lattice network does not affect the scale-free property and the fractality.

4.4.2.3 Number of spanning trees of the generalized Fractal Scale-Free Lattice $L_{q,n}$

In order to calculate the number of spanning trees of $L_{q,n}$ using the electrically equivalent transformations (Mokhlissi et al., 2019b), we follow the steps of Section 2.4.4.4:

1. First, we put the weight a_0 on each edge of $L_{q,n}$.
2. Then, we use the electrically equivalent technique to transform $L_{q,n}$ to $L_{q,n-1}$. We get the relationship between the number of spanning trees $\tau(L_{q,n})$ and $\tau(L_{q,n-1})$ and the exact value of the number of spanning trees of $L_{q,n}$ in each step.

   By applying the algorithms of the electrically equivalent transformations, we deduce $\tau(L_{q,n}) = (8a_0^2)^{q^n 4^{n-1}} \tau(L_{q,n-1})$. To prove mathematically this formula, we use the mathematical induction:

   Basic step: We prove that $\tau(L_{q,1}) = (8a_0^2)^q \tau(L_{q,0})$. We denote $L_{q,1}^{(0)}$ as the generalized Fractal Scale-Free Lattice at iteration 1 with q connected clusters in the transformation 0 with all its edges having the weights a_0. We take as an example the generalized Fractal Scale-Free Lattice with $q = 2$. Figure 4.14 shows the electrically equivalent evolving process from $L_{2,1}^{(0)}$ to $L_{2,0}^{(0)}$.

   Five transformations are used in sequence and the corresponding weights of the resulting edges are calculated as follows:

(a) *Delta– Wye transformation:* We replace one triangle in each connected cluster of $L_{q,1}^{(0)}$ (Δ network) with its electrically equivalent component (Y network). We get $L_{q,1}^{(1)}$ as the resulting graph with three new edges with a new conductance $\frac{a_0^2+a_0^2+a_0^2}{a_0} = 3a_0$.

(b) *Serial edge:* We replace all the serial edges with conductances a_0 and $3a_0$ by a new edge with the conductance $\frac{3a_0.a_0}{3a_0+a_0} = \frac{3}{4}a_0$ to obtain a new subgraph $L_{q,1}^{(2)}$.

(c) *Parallel edge:* We merge two parallel edges having the same conductance $\frac{3}{4}a_0$ of $L_{q,1}^{(2)}$ to obtain a new subgraph $L_{q,1}^{(3)}$. The corresponding conductance of a new edge is $\frac{3}{4}a_0 + \frac{3}{4}a_0 = \frac{3}{2}a_0$.

(d) *Serial edge:* We replace two serial edges with conductances $3a_0$ and $\frac{3}{2}a_0$ by a new edge with the conductance $\frac{3a_0.\frac{3}{2}a_0}{3a_0+\frac{3}{2}a_0} = a_0$ to get $L_{q,1}^{(4)}$.

(e) *Parallel edge:* We merge two parallel edges having the same conductance a_0 of $L_{q,1}^{(4)}$ to obtain a new subgraph $L_{q,0}^{(0)}$. The conductance of the new edge is $a_0 + a_0 = 2a_0$.

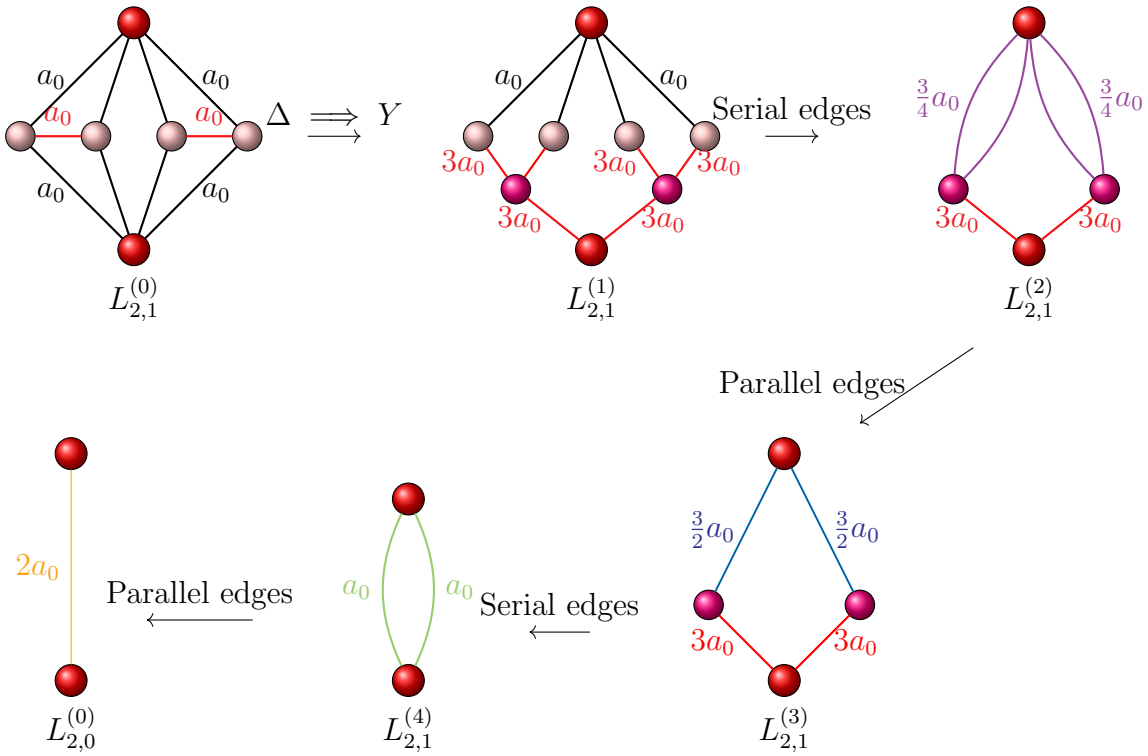


Figure 4.14: The transformations from $L_{2,1}$ to $L_{2,0}$.

According to these five applications of the electrically equivalent transformations, we determine the expression of the number of spanning trees between $L_{q,1}^{(0)}$ and $L_{q,0}^{(0)}$. Then $\tau(L_{q,1}) = \tau(L_{q,1}^{(0)})$.
$\tau(L_{q,1}) = (\frac{a_0^3}{(3a_0^2)^2})^q \tau(L_{q,1}^{(1)}) = (\frac{1}{9a_0})^q \tau(L_{q,1}^{(1)}) \rightarrow (\Delta \implies Y)$.
$\tau(L_{q,1}) = (\frac{1}{9a_0})^q (3a_0 + a_0)^{2q} \tau(L_{q,1}^{(2)}) = (\frac{4^2}{9} a_0)^q \tau(L_{q,1}^{(2)}) \rightarrow$ $(Serial\ edge)$.
$\tau(L_{q,1}) = (\frac{4^2}{9} a_0)^q \times 1^q \times \tau(L_{q,1}^{(3)}) \rightarrow$ $(Parallel\ edge)$.
$\tau(L_{q,1}) = (\frac{4^2}{9} a_0)^q (\frac{3}{2} a_0 + 3a_0)^q \tau(L_{q,1}^{(4)}) = (8a_0^2)^q \tau(L_{q,1}^{(4)}) \rightarrow$ $(Serial\ edge)$.
$\tau(L_{q,1}) = (8a_0^2)^q \times 1^q \times \tau(L_{q,1}^{(5)}) \rightarrow$ $(Parallel\ edge)$.
$\tau(L_{q,1}) = (8a_0^2)^q \tau(L_{q,0}^{(0)})$.

Induction step: We assume that $\tau(L_{q,n}) = (8a_0^2)^{q^n 4^{n-1}} \tau(L_{q,n-1})$. We show that $\tau(L_{q,n+1}) = (8a_0^2)^{q^{n+1} 4^n} \tau(L_{q,n})$.

The number of connected clusters in $L_{q,n}$ is $q^n 4^{n-1}$. Note the number of spanning trees depends on the number of connected clusters. The number of connected clusters in $L_{q,n+1}$ is $q^{n+1} 4^n$. Then, the number of spanning trees of $L_{q,n+1}$ is $\tau(L_{q,n+1}) = (8a_0^2)^{q^{n+1} 4^n} \tau(L_{q,n})$.

3. Finally, we deduce the exact formula of the number of spanning trees of $\tau(L_{q,n})$.

$$\begin{aligned}
\tau(L_{q,n}) &= (8a_0^2)^{q^n 4^{n-1}} \tau(L_{q,n-1}) \\
\tau(L_{q,n}) &= (8a_0^2)^{q^n 4^{n-1} + q^{n-1} 4^{n-2} + \ldots + q^1 4^0} \tau(L_{q,0}) \ with\ \tau(L_{q,0}) = qa_0 \\
\tau(L_{q,n}) &= qa_0 (8a_0^2)^{q \sum_{j=0}^{n-1} (4q)^j}
\end{aligned}$$

Hence, the number of spanning trees of $L_{q,n}$ is given by the following formula (Mokhlissi et al., 2019b):

$$\tau(L_{q,n}) = qa_0 (8a_0^2)^{q \frac{(4q)^n - 1}{4q-1}}, n \geq 0. \tag{4.76}$$

4.4.2.4 Entropy of spanning trees of the generalized Fractal Scale-Free Lattice $L_{q,n}$

Since the number of spanning trees of the generalized Fractal Scale-Free Lattice grows exponentially with $V_{L_{q,n}}$, we can calculate its entropy of spanning trees according to the definition of the entropy of Equation 2.1 in Section 2.1.2.1. Let $\rho_{L_{q,n}}$ be the entropy of

spanning trees for $L_{q,n}$. From Equation 4.76, we put $a_0 = 1$. We obtain:

$$\begin{aligned}\rho_{L_{q,n}} &= \lim_{n\to\infty} \frac{\ln|\tau(L_{q,n})|}{|V_{L_{q,n}}|} = \lim_{n\to\infty} \frac{\ln(q(8)^{q\frac{(4q)^n-1}{4q-1}})}{(4q)^n 2q + 6q - 2} \times (4q-1) \\ &= \lim_{n\to\infty} \frac{\ln(q) + 3q\frac{(4q)^n-1}{4q-1}.\ln(2)}{(4q)^n 2q + 6q - 2} \times (4q-1) \\ &= \frac{3\ln(2)}{2} \approx 1,0397.\end{aligned}$$

The obtained entropy of $L_{q,n}$ is a constant value that does not depend on q. So, $\rho_{L_{q,n}}$ remains unchanged by the addition of the connected clusters. It means no matter how it gets started, in the long run, the entropy of spanning trees remains the same.

Type of network	$< z >$	ρ
Pseudo-fractal web (Zhang et al., 2010b)	4	0.8959
Farey network (Mokhlissi et al., 2019a)	4	0.9457
Generalized Fractal Scale-Free Lattice (Mokhlissi et al., 2019b)	**4**	**1.0397**
The 2-dimensional Sierpinski gasket (Chang et al., 2007)	4	1.0486
Square lattice (Wu, 1977)	4	1.1662
The 2-Mosaic networks (Mokhlissi et al., 2017b)	4	1.3862

Table 4.3: The entropy of spanning trees of several networks having the same average degree.

In Table 4.3, we compare the entropy of the generalized Fractal Scale-Free Lattice $L_{q,n}$ (1.0397) with those of other networks having the same average degree 4. We notice that this value of the entropy is bigger than those of the Pseudo-fractal web and the Farey network. While for the 2-dimensional Sierpinski gasket, the square lattice and the 2-Mosaic networks, the entropies of spanning trees are greater than 1.0397. This result proves that the generalized Fractal Scale-Free Lattice is more robust than the Pseudo-fractal web and the Farey network and less robust than the 2-dimensional Sierpinski gasket, the square lattice and the 2-Mosaic networks.

4.5 Summary

In summary, diverse real-life networks have power-law degree distribution, which defines the type of scale-free networks. In this chapter, we have examined three important models of scale-free networks:

- Flower network and Mosaic network: They exhibit the concepts of self-similarity and fractality. For the construction of the Flower network, first, we have applied the reduction approach, then the bipartition approach. We have inverted this combination by applying the bipartition approach, then the reduction approach to construct the Mosaic network. We have analyzed their topological properties and calculated their number of spanning trees using the same combinations of geometric approaches. Then, we have proposed the generalization of these networks by changing their dimension. We have used the same combinations of the m-reduction and m-partition approaches to reveal the mechanism of their construction and to calculate their complexity. We have determined their topological properties, showing that the generalization of these networks does not affect the scale-free property. Finally, we have evaluated their robustness by calculating their entropy.
- Fractal Scale-Free Lattice: It is a scale-free network with fractality that follows a power-law degree distribution. We have proposed the generalization of the lattice structure by multiplying the number of the connected clusters for each iterative edge. In addition, we have determined its topological properties, proving that the generalization of the Lattice network affects neither the scale-free property nor the fractality. Using the electrically equivalent transformations, we have obtained the exact formula of its number of spanning trees. Finally, we have evaluated and compared its entropy with other networks having the same average degree. The results show that this value is constant and does not depend on any parameter.

In the last chapter, we focus on the analytical study of some real-world networks and random networks having the same number of nodes and links. We discuss their structural properties. Then, we calculate their complexity by using the electrically equivalent technique. Finally, we evaluate and compare their entropy in order to predict which network is more robust.

Chapter

5

Analysis of Real and Random networks models

Contents

Recently, the analysis of complex networks such as the Internet, World Wide Web, telecommunication networks, protein interaction networks, social networks has attracted much attention due to their quick growth. The study of these real-world networks requires the use of random models. Such models would allow researchers to better understand real-world networks and to predict their behavior. In this chapter, we propose an analytical study of some real-world networks and random networks having the same number of nodes and links. We discuss their structural properties such as the diameter, the average path length, the clustering coefficient and the degree distribution. Then, we calculate their number of spanning trees by using the electrically equivalent transformations, proving that this technique is efficient and more general compared to the classical ones. Finally, we evaluate and compare their entropy of spanning trees in order to predict which network is more robust. The results of this chapter were accepted at an international conference (Mokhlissi et al., 2020).

5.1 Why use random networks?

The random network is one of the oldest and most studied models of a network. It can not mimic real networks, but it is considered as a guide or a reference to compare with the real network's behaviour. When we explore a property of the real network, we can use a random network as a model to realize if this property carries information or if it is random. The best model of the random network is that of the ER network, presented by Erdős and Rényi (Erdős et al., 1959). They showed that many of the properties of such networks can be calculated analytically. The ER network differs from real networks in two important features: it lacks the clustering coefficient and it has an unrealistic Poissonian degree distribution. Such a model allows researchers to better understand the mechanism of real-world networks and to predict their behaviour. Therefore, the random network model may be the wrong model for most real networks, but it remains quite relevant for network science (Barabási, 2013).

5.2 Structural Properties of real and random Networks

In this section, we introduce an analytical comparison of the structural properties between real-world networks such as Zachary's karate club (Zachary, 1977), Dolphin social network (Lusseau et al., 2003), Les Miserables (Knuth, 1993), Books about US politics (Krebs, 2004), Word adjacencies (Newman, 2006) and American College football (Girvan and Newman, 2002), all other models of scale-free and small-world networks studied in the previous chapters such as the Flower network (See Section 4.2) (Mokhlissi et al., 2017a), the Mosaic network (See Section 4.3) (Mokhlissi et al., 2017b), the Koch Network (See Section 3.3) (Mokhlissi et al., 2018b), the Small-World Exponential Network (See Section 3.2) (Mokhlissi et al., 2018b, 2016a), the Fractal Scale-Free Lattice (See Section 4.4) (Mokhlissi et al., 2019b) and the Farey network (See Section 3.4) (Mokhlissi et al., 2019a) and the first model of small-world networks as the Watts-Strogatz Network (Watts and Strogatz, 1998), the first model of scale-free network as the barabasi-albert Network (Barabási and Albert, 1999)) and the first model of random networks as the Erdos-Renyi network (Erdős and Rényi, 1960, 1961) having the same number of

vertices and edges.

5.2.1 Dataset

We start by presenting several datasets used as examples for real networks:

- **Zachary's karate club**

  This dataset contains 34 members of a karate club at a US university, as described by Wayne Zachary in 1977. This network contains 78 pairwise links between members.

- **Dolphin social network**

  This dataset contains the network of frequent associations between 62 dolphins in a community living off Doubtful Sound, New Zealand. It contains 159 edges that indicate a frequent association between them.

- **Les Miserables**

  The co-occurrences of characters in Victor Hugo's novel 'Les Miserables'. It contains 77 characters that represent the nodes and the two characters appeared in the same chapter of the book represents an edge between two nodes.

- **Books about US politics**

  A set of books about US politics published around the time of the 2004 presidential election. It consists of 105 books that represent the nodes and 441 edges between books represent frequent co-purchasing of books by the customer from the same merchant.

- **Word adjacencies**

  The adjacency network of common adjectives and nouns in the novel David Copperfield by Charles Dickens. It consists of 112 nodes that represent the most commonly occurring adjectives and nouns in the book and 425 edges connect any pair of words that occur in adjacent positions in the text of the book

- **American College football**

  This dataset consists of 115 teams considered as nodes and 613 edges correspond to games played by the teams against each other during the regular season of fall 2000.

We discuss the structural properties of real networks such as the clustering coefficient, the degree distribution, the diameter, the average path length, with values expected for random networks.

5.2.2 Clustering Coefficient

To show the difference between the clustering coefficient of real-world networks and random networks, we use the models of networks mentioned in Table 5.1, we calculate their average clustering coefficient and compare it with those of random networks having a similar number of nodes and edges. We notice that the clustering coefficients of different complex networks mentioned in Table 5.1 are larger than that of random networks, except two networks: 2-Flower networks and 2-Mosaic networks, because the neighbors of any node of these networks are never neighbors of one another. In general, we can say that the ER networks do not have high clustering coefficient compared to real networks (See Figure 5.1).

N	Type of network	V	E	C_{real}	C_{random}
1	Zachary's karate club	34	78	0.570638	0.163772
2	Dolphin social network	62	159	0.258958	0.087321
3	Les Miserables	77	254	0.573136	0.083168
4	Watts-Strogatz Network	100	500	0.102503	0.098122
5	Books about US politics	105	441	0.487526	0.075720
6	Word adjacencies	112	425	0.172840	0.046981
7	American College football	115	613	0.403216	0.102725
8	Fractal Scale-Free Lattice	172	341	0.539960	0.006646
9	barabasi-albert Network	213	1040	0.128816	0.111455
10	2-Mosaic networks	343	1024	0	0.011570
11	Koch Network	513	768	0.818473	0.006092
12	Farey network	513	1023	0.692396	0.008893
13	2-Flower network	684	1024	0	0.003118
14	Small-World Exponential Network	729	1092	0.760371	0.002891

Table 5.1: Clustering coefficient of real and random networks.

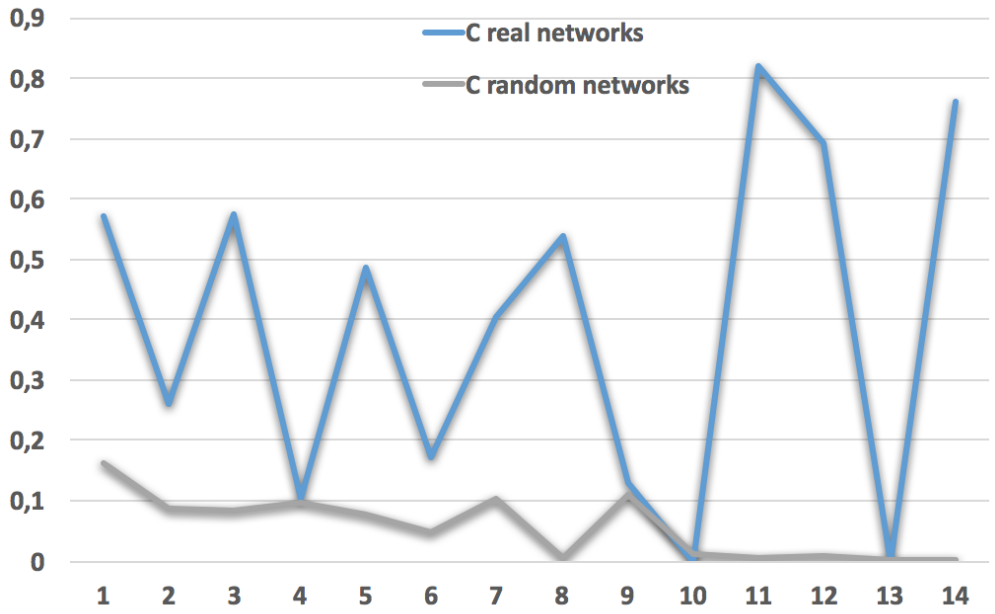


Figure 5.1: Comparison between the clustering coefficient of real and random networks

5.2.3 Degree Distribution

To show the different types of the degree distribution, we plot in Figure 5.2 the degree distribution of some real-world networks mentioned in Table 5.1 and random networks with the same number of nodes and edges. We see in Figure 5.2 **(b)** that the exact form of the degree distribution of random networks is the binomial distribution and in Figure 5.2 **(a)**, for the 2-Flower network, the Koch Network, the 2-Mosaic network, the Fractal Scale-Free Lattice and the Barabasi-albert Network, their degree distribution follows the scale-free power-law form. Whereas, the Small-World Exponential Network and the Farey network, their degree distribution follows an exponential distribution and for the Watts-Strogatz Network, Dolphin social network, Les Miserables, Zachary's karate club, Books about US politics, Word adjacencies and American College football, the shape of their degree distribution is similar to that of a random network.

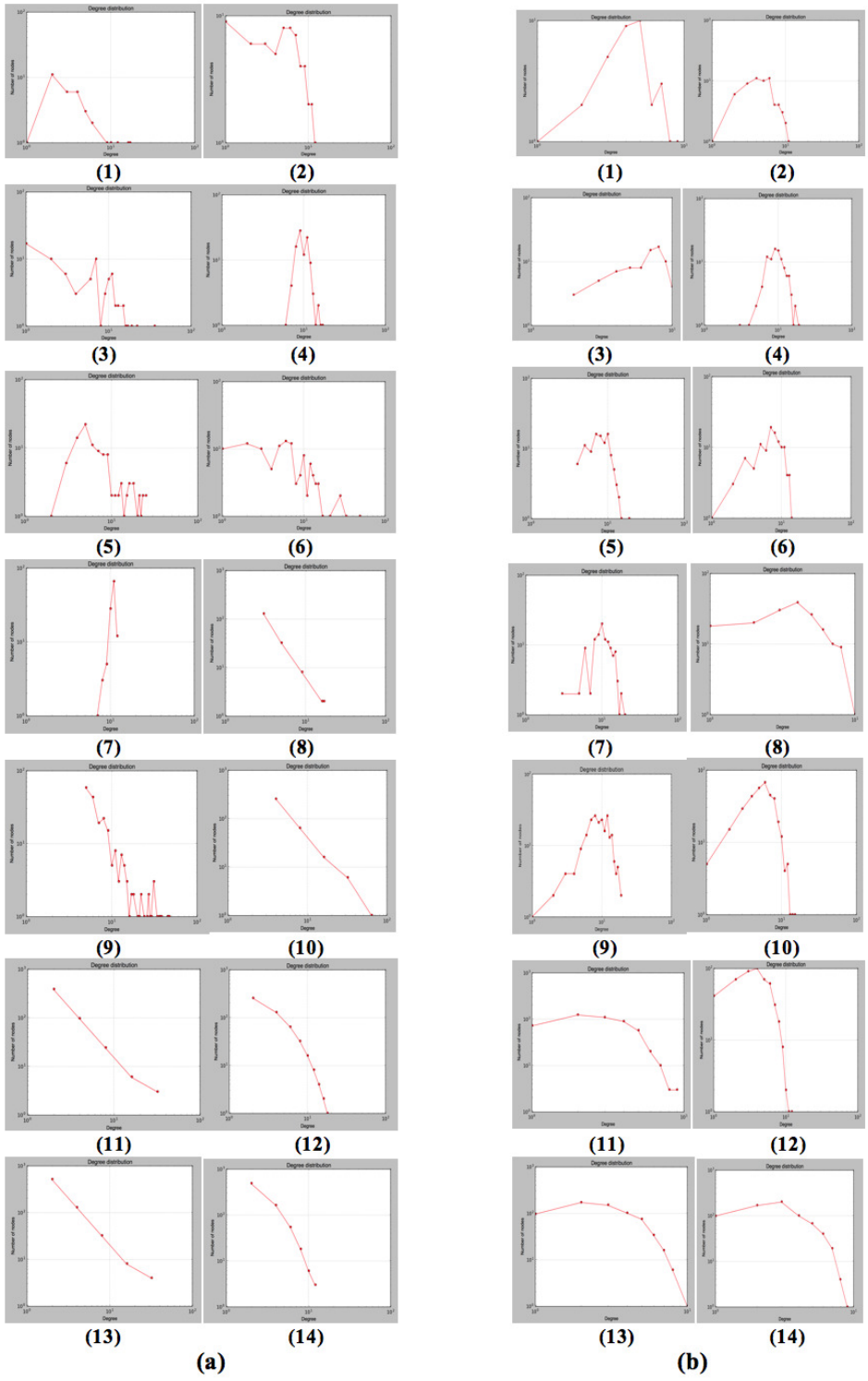


Figure 5.2: Comparaison between the degree distribution of real networks **(a)** and random networks **(b)**

5.2.4 Small-World Property

The average path length l of a network characterizes the small-world property. In some cases, it is difficult to obtain the analytic solution of APL. Therefore, researchers adopt the "diameter D," as an alternative parameter to demonstrate the short distance between any two nodes of a network. If the diameter of a network grows logarithmically with the number of nodes V, i.e., small diameter and we have always $l \leq D$, then it is undoubtedly with a short APL: $l = \mathcal{O}(\log V)$. Then, the APL should increase more slowly. To show the relationship between the APL and the diameter more clearly, we calculate these two measures of some real-world networks and random networks with the same number of nodes and edges. The results are displayed in Table 5.2 and the simulation results are shown in Figures 5.3 and 5.4. We see for real-world networks in Figure 5.3 and random networks in Figure 5.4 that when the diameter changes (increases/decreases), the APL also changes (increases/decreases). We notice also that the values of the diameter and the APL of random networks (ER networks) are almost the same as the the real-world networks. Both the real network and the random network exhibit the same behaviour in the small-world property.

N	Type of network	V	E	D_{real}	D_{random}	l_{real}	l_{random}
1	Zachary's karate club	34	78	5	5	2.408199	2.399286
2	Dolphin social network	62	159	8	5	3.356953	2.693283
3	Les Miserables	77	254	5	4	2.641148	2.488380
4	Watts-Strogatz Network	100	500	4	4	2.235555	2.226060
5	Books about US politics	105	441	7	4	3.078754	2.394139
6	Word adjacencies	112	425	5	6	2.535553	2.558236
7	American College football	115	613	4	4	2.508161	2.250343
8	Fractal Scale-Free Lattice	172	341	16	7	6.622943	3.777373
9	barabasi-albert Network	213	1040	4	4	2.510541	2.595756
10	2-Mosaic networks	343	1024	32	7	6.240033	3.471007
11	Koch Network	513	768	10	13	5.157894	5.727294
12	Farey network	513	1023	9	11	5.449211	4.669691
13	2-Flower network	684	1024	32	15	11.142830	5.845276
14	Small-World Exponential Network	729	1092	11	14	7.010989	5.919793

Table 5.2: The diameter and the APL of real and random networks.

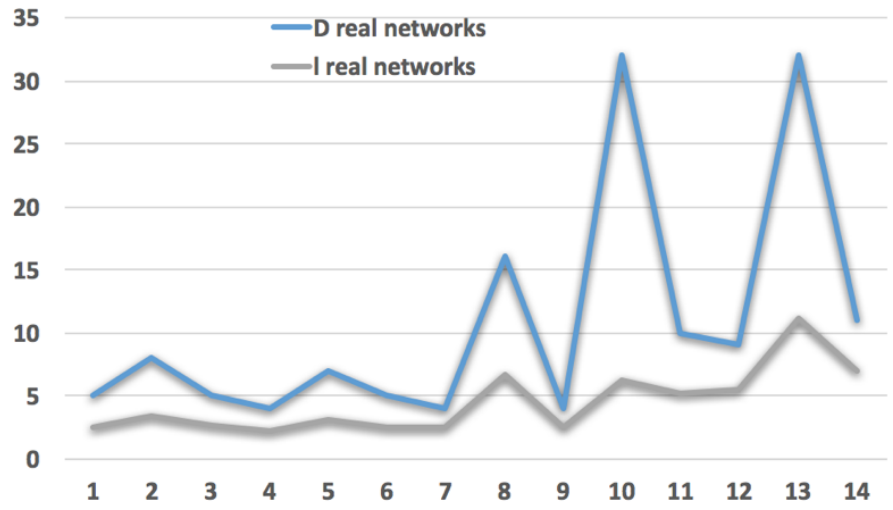


Figure 5.3: The diameter and the APL of real networks

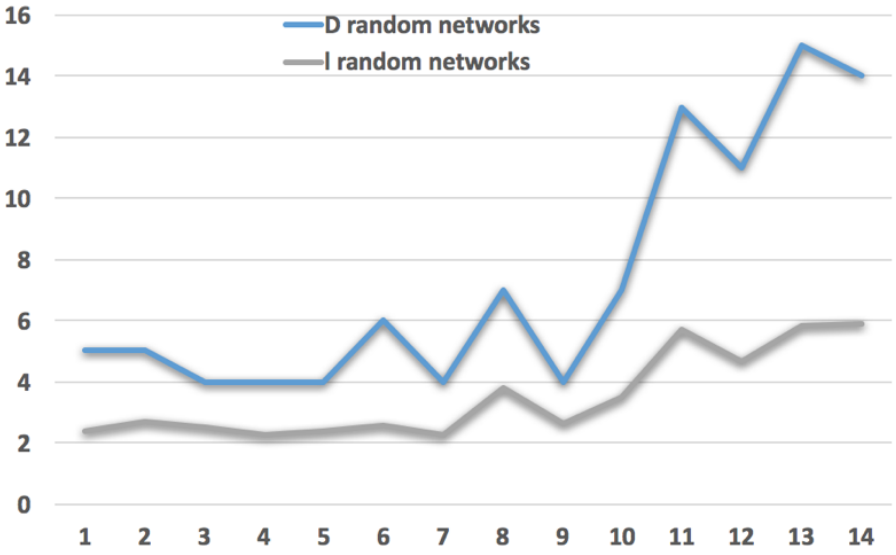


Figure 5.4: The diameter and the APL of random networks

5.3 Number and entropy of spanning trees of real and random Networks

In this section, we calculate the number of spanning trees for some real-world and random networks using the electrically equivalent technique without going through the calculation of Kirchhoff's matrix and without the need for the existence of articulation nodes and the planarity. Applying this technique for these types of networks, we get the exact value of

their number of spanning trees. So, no need to find the exact equation of this number and neither to verify the isomorphism. As an application of the number of spanning trees of a network, we compute the entropy of spanning trees to quantify the robustness. The most robust network is the network that has the highest entropy. In Table 5.3, applying the electrically equivalent technique, we give the results of the number and the entropy of spanning trees for some real-world networks. In Table 5.4, we give the results of the number and the entropy of spanning trees for Erdos-Renyi random networks having the same number of nodes and edges as the presented networks in Table 5.3. The only condition for calculating the number of spanning trees is that all these real and random networks must be connected. In Figure 5.5, we compare the entropy of spanning trees of some real and random networks. We can see that the entropy of random networks is almost larger than those of real-world networks due to their large number of spanning trees. Consequently, the random networks are robust and their structure is more homogeneous than the real-world networks having the same number of nodes and links.

N	Type of network	V	E	τ_{real}	ρ_{real}
1	Zachary's karate club	34	78	$5.09099632302 \times 10^{15}$	0.3616
2	Dolphin social network	62	159	$2.17175713551 \times 10^{32}$	0.7445
3	Les Miserables	77	254	$2.03974706969 \times 10^{42}$	0.9742
4	Watts-Strogatz Network	100	500	$9.16480112874 \times 10^{93}$	2.1635
5	Books about US politics	105	441	$5.55633429016 \times 10^{82}$	1.9052
6	Word adjacencies	112	425	$6.8594178138 \times 10^{76}$	1.7692
7	American College football	115	613	$7.74002476226 \times 10^{111}$	2.5763
8	Fractal Scale-Free Lattice	172	341	$5.78960446186581 \times 10^{76}$	1.0397
9	barabasi-albert Network	213	1040	$2.988403704743589 \times 10^{186}$	2.0158
10	2-Mosaic networks	343	1024	$2.006582604 \times 10^{205}$	1.3862
11	Koch Network	513	768	$1.39008452377145 \times 10^{122}$	0.5493
12	Farey network	513	1023	$9.90418468985189 \times 10^{209}$	0.9458
13	2-Flower network	684	1024	$2.006582604 \times 10^{205}$	0.6931
14	Small-World Exponential Network	729	1092	$4.70042056779386 \times 10^{173}$	0.5493

Table 5.3: The number and the entropy of spanning trees of real networks.

N	V	E	τ_{random}	ρ_{random}
1	34	78	$1.05860958932 \times 10^{18}$	0.4150
2	62	159	$1.45016355662 \times 10^{37}$	0.8556
3	77	254	$5.99585272075 \times 10^{55}$	1.2843
4	100	500	$7.052459640644547 \times 10^{93}$	2.1609
5	105	441	$1.43050550088 \times 10^{89}$	2.0528
6	112	425	$1.65635617438 \times 10^{89}$	2.0543
7	115	613	$4.62483134316 \times 10^{110}$	2.5481
8	172	341	$7.192292267420218 \times 10^{78}$	1.0556
9	213	1040	$1.0559617997499113 \times 10^{196}$	2.1190
10	343	1024	$1.6648092131229848 \times 10^{234}$	1.5723
11	513	768	$3.94135015033 \times 10^{159}$	0.7163
12	513	1023	$1.26835361025 \times 10^{236}$	1.0597
13	684	1024	$1.33756890122 \times 10^{215}$	0.7241
14	729	1092	$3.68796589012 \times 10^{229}$	0.7250

Table 5.4: The number and the entropy of spanning trees of random networks.

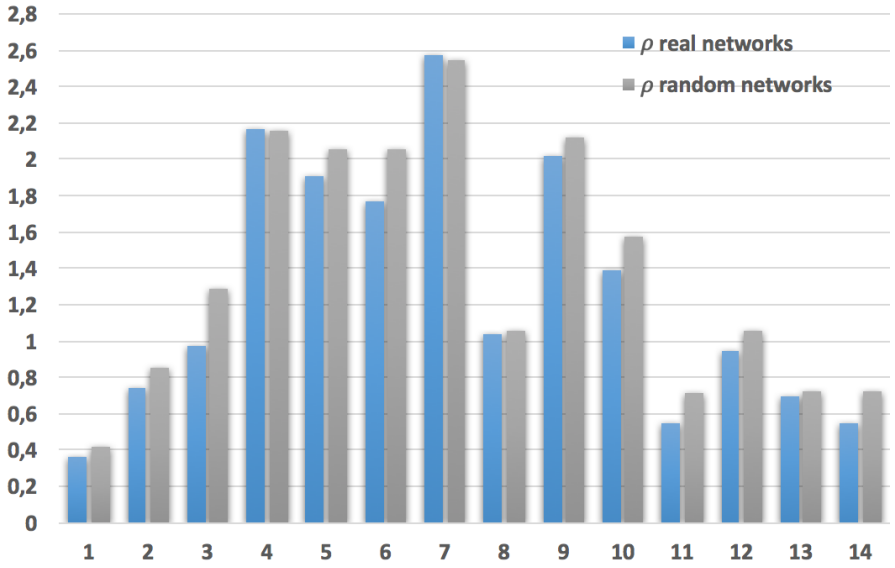


Figure 5.5: Comparison between the entropy of spanning trees of real and random networks.

- ***Discussion***

According to the above results, we deduce that the random network is the wrong model for most real networks because its structural properties are different from those of the real networks, but it is considered as the most robust model due to its high entropy and its large number of spanning trees.

5.4 Summary

In this chapter, we have given an overview of real networks, discussed their structural properties such as the small-world property, the clustering coefficient and the degree distribution. All these properties of real networks are compared with expected values for random networks, showing that the clustering coefficients in the real network are significantly larger than in the random networks. The low average distance and the diameter have been observed in the real and random network. For the degree distribution of random networks, it follows the binomial distribution. On the other hand, the degree distribution of some real networks either follows a form of power-law or it follows an exponential distribution or it is similar to that of the random network. Then, we have calculated their number of spanning trees by using the electrically equivalent technique, proving that this technique is general and can be applied to any type of networks. Finally, we have evaluated and compared their entropy of spanning trees in order to predict which network is more robust. As results, Although the random networks can not model the real networks, they are the most robust models due to its high entropy and its large number of spanning trees. .

General Conclusion and Perspectives

In this general conclusion, we summarize the main results of our research on the modelling and the analysis of complex networks and the evaluation of their complexity. Furthermore, we propose and discuss horizons for future research.

During this thesis, we have investigated three categories of complex networks: Scale-free networks, small-world networks and random networks. We have proposed new models dynamically evolving in time for each category. In one hand, we have discussed their iterative construction and their structural properties to understand their mechanism and behaviour. On the other hand, we have focused on developing some combinatorial methods and geometric approaches that facilitate the calculation of their number of spanning trees. As an application, we have used a measure of the entropy to quantify their robustness and characterize their structures. In **chapter 1**, we have introduced an overview of complex networks and some basic concepts of graph theory. In particular, spanning trees and some matrices to represent a graph. Then, we have studied some important structural properties based on graph theory to analyze complex networks. Besides, we have quoted the different models proposed by scientists. Then, we have moved to **chapter 2**, which presents the different methods of calculating the number of spanning trees of a network, classed as algebraic, combinatorial and geometric methods, especially the Kirchhoff theorem, Fussner's formula, the decomposition method, the suppression and contraction method and the duality, etc. We have examined their main advantages and drawbacks and we have also proposed some examples of their applications, such as the evaluation of the complexity of Book networks, etc. In addition, we have suggested two real applications of

the number of spanning trees, which are the reliability and the robustness of a network. Our main result in this chapter was to develop some combinatorial methods and geometric approaches as the generalization of the decomposition method following one articulation node, the bipartition and the reduction approaches and the electrically equivalent technique to facilitate the calculation of the number of spanning trees of complex networks without using the algebraic methods as Kirchhof theorem. To show the efficiency of the proposed methods, we have applied them to the three categories of complex networks. We have started by the small-world networks in **chapter 3**. We have examined three models of small-world networks: Small-World Exponential network, Koch Network and Farey network. we have proposed the generalization of each model. We have analyzed their iterative construction and their structural properties. Then, we have calculated their number of spanning trees: For the Small-World Exponential network and the Koch Network, we have used the generalized decomposition method following one node and for the Farey network, we have applied the electrically equivalent transformations. Finally, we have evaluated their entropy of spanning trees to quantify their robustness and compare them with other networks having the same average degree to estimate the robust model. The main result of this chapter is that the generalization of our models affects the small-world properties and the generalizations of the two first proposed models have the same robustness, meaning the same entropy, although their structures, properties and complexities are different. After that, we have moved to scale-free networks, which are presented in **chapter 4**. We have proposed three models of scale-free networks and their generalization: Flower network, Mosaic network and Fractal Scale-Free Lattice. We have analyzed their construction and their structural properties such as the degree distribution, the clustering coefficient, the diameter, the average path length, etc. Then, we have enumerated their spanning trees: For the Flower network and the Mosaic network, we have used the reduction and the bipartition approaches and for the Fractal Scale-Free Lattice, we have applied the electrically equivalent transformations. Finally, we have evaluated their robustness by calculating their entropy and compared it with other networks having the same average degree. The important result of this chapter is that the proposed generalization does not affect the scale-free property and the combinations of the reduction

and the bipartition approaches lead to the same complexity with two different entropies in spite of the difference in the structure and the properties of the two first proposed models. In **chapter 5**, we have analyzed the real and random networks. We have performed a comparative study between real-world networks, all proposed models in the previous chapters and random networks having the same number of nodes and links. We have discussed their structural properties. Then, we have calculated their number of spanning trees by applying the electrically equivalent transformations. Finally, we have evaluated and compared their entropy of spanning trees to predict which network is more robust. The crucial result of this chapter is that the proposed technique is efficient and more general compared to the classical ones and we have deduced that the random network can not model the real networks, but it is considered as the most robust model due to its high entropy and its large number of spanning trees.

Regarding future work, we intend to analyze other new models of complex networks and apply the electrically equivalent transformations to calculate their number of spanning trees. We would like also to investigate the second application of the number of spanning trees, which is the reliability of a network. In addition to the robustness of a network, we will analyze the reliability of all the proposed networks in this thesis using our methods to calculate the number of spanning trees and perform a comparative study between them to predict the most reliable model. From a practical point of view, we would like to bring theory into practice, especially in social networks. So, we can use the number of spanning trees of a network to calculate two important measures, which are spanning edge centrality and spanning edge betweenness. These measures are used to study how information propagates or how traffic flows in a social network. They are considered as useful tools for the analysis of very large networks with millions of nodes.

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Résumé

L'analyse des réseaux complexes a été largement stimulée par les ressources de données massives et leur étude a été initiée pour une volonté de comprendre le comportement de divers systèmes réels. Pour comprendre leur mécanisme, de nombreuses propriétés structurelles sont utilisées, telles que la distance moyenne, le coefficient de clustering, la distribution des degrés, etc. Ces caractéristiques définissent trois modèles de réseaux complexes notamment les réseaux sans échelle, les réseaux petit monde et les réseaux aléatoires. Ces trois modèles affichent un comportement riche, observé dans une grande variété de systèmes réels, y compris Internet, WWW et les réseaux sociaux. Pour caractériser et analyser leur structure, la théorie des graphes dispose d'un outil puissant, nommé le nombre d'arbres couvrants d'un réseau, également appelé la complexité d'un réseau. Il est défini comme le nombre total des arbres contenant tous les sommets du réseau avec le plus petit nombre possible d'arêtes. L'objectif principal de ce travail est le calcul du nombre d'arbres couvrants d'un réseau, qui permet de prédire sa fiabilité et sa robustesse. En effet, le calcul de ce nombre reste un défi, en particulier pour les réseaux complexes. Nous nous sommes intéressés à la recherche des méthodes efficaces pour obtenir la formule exacte du nombre d'arbres couvrants pour les réseaux complexes. Le premier but de cette étude est de créer de nouveaux modèles évoluant de manière dynamique dans le temps pour chaque catégorie de réseaux complexes. Ensuite, calculer leurs propriétés structurelles pertinentes pour comprendre leurs mécanismes et leurs comportements. De plus, évaluer leur complexité en utilisant et améliorant des méthodes combinatoires et géométriques. Finalement, comme application, nous calculons leur entropie afin de quantifier leur robustesse et la comparer avec d'autres réseaux ayant le même degré moyen.

Mots-clefs : Réseaux Complexes, Théorie des Graphes, Distance Moyenne, Diamètre, Coefficient de Clustering, Distribution des Degrés, Degré Moyen, Réseau Sans échelle, Réseau Petit Monde, Réseau Aléatoire, Arbre Couvrant, Complexité, Fiabilité, Robustesse, Entropie.

Abstract

Many real-world networks are modelled as complex networks due to their large structure and their dynamical behaviour. Graph theory provides efficient tools to understand and to analyze their mechanism. In fact, many structural properties are used such as the average path length, the diameter, the clustering coefficient, the degree distribution, the average degree, etc. Based on these features, three categories of complex networks are defined, namely scale-free networks, small-world networks and random networks. One of the important invariants to characterize their structures is the number of spanning trees of a network, which is defined as the total number of connected and acyclic subgraphs of a network having all its vertices and some or all its edges. In this work, the main objective is the calculation of the number of spanning trees of a network also known as the complexity of a network, which provides the prediction of its reliability and its robustness. However, the enumeration of spanning trees remains a challenge, particularly for complex networks. Recently, there has been much interest in finding efficient methods to obtain exact expressions of the number of spanning trees for complex networks. The primary interest of this study is to create new models for each category of complex networks based on real-networks that grow by the gradual addition of vertices and edges. Then, find their relevant structural properties to understand their mechanism. Furthermore, evaluate their complexity using combinatorial and geometric approaches. In the end, as an application, we calculate their entropy of spanning trees to quantify their robustness and compare them with other networks having the same average degree.

Key Words: Complex Networks, Graph theory, Average Path Length, Diameter, Clustering Coefficient, Degree Distribution, Average Degree, Scale-Free Network, Small-World Network, Random Network, Spanning Tree, Complexity, Reliability, Robustness, Entropy.