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Regularity and approximation schemes for nonlinear parabolic equations with variable growth

JURY

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To my parents.

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Résumé

Dans cette thèse, on s'intéresse à la régularité et à l'étude de schémas d'approximation de certaines équations paraboliques dégénérées isotropiques et anisotropiques à exposant variable. En utilisant la moyenne de Steklov et l'inegalité de Young, on obtient les estimations énergétiques et logharithmiques des solutions à ces équations. Ensuite, par la méthode de DiBenedetto on montre que les solutions locales faibles sont localement continues. La deuxième partie est consacrée à l'étude de quelques schémas d'approximation appliquée aux certains équations non linéaires et doublement non linéaires à exposant variable. Concernant les schémas d'approximation on établit des résultats d'existence, d'unicité et on montre l'existence d'un attracteur global pour les solutions d'un schéma d'Euler semi-discrétisé en temps. On s'intéresse également à un schéma d'approximation par la méthode des éléments finis dans lequel on obtient des résultats de stabilité et d'estimation de l'erreur.

Mots-clefs: Problèmes paraboliques isotropiques. Problèmes paraboliques anisotropiques. Théorie de la régularité. La méthode de DiBenedetto. Attracteur compact. La méthode des éléments finis.

Abstract

In first part of this thesis, we discuss a class of degenerate isotropic and anisotropic parabolic equations with variable exponents. By using the Steklov average and Young's inequality, we establish energy and logarithmic estimates for solutions to these equations. Then based on the intrinsic scaling method, we prove that local weak solutions are locally continuous. In the second part, we study a time discretization for a doubly non linear parabolic equation with variable exponent by using Euler forward scheme. We investigate existence, uniqueness and stability questions and prove existence of the global compact attractor. Finally, we give a simple approach to a priori estimates for a singular parabolic equation with variable exponent.

keywords: Isotropic parabolic problems. Anisotropic parabolic problems. Regularity theory. Intrinsic scaling. Finite element method. Absorbing sets. Compact attractors.

Résumé de la thèse

Le but de ce travail est d'apporter une certaine contribution à l'étude de la régularité des solutions et à l'étude de certains schémas d'approximation (semi-discrétisation temporelle, semi-discrétisation spatiale, ou discrétisation totale) de certaines équations paraboliques non-linéaires à exposant variable isotropiques ou anisotropiques (voir [13,14,15,16]). La régularité des solutions joue un rôle essentiel dans l'approximation des solutions et l'estimation de l'erreur. Plus la solution est assez régulière, plus l'approximation est assez précise. Ce travail est divisé en deux parties.

1^{er} partie: La régularité

Dans la première partie, par la méthode de DiBenedetto on montre que les solutions locales faibles sont localement continues. Celle-ci s'articule autour de deux thèmes de recherche:

Premier thème: Problème parabolique dégénéré isotropique

Les équations aux dérivées partielles non linéaires isotropiques avec un exposant variable font l'objet d'une attention grandissante au cours de ces dernières années grâce à ses applications dans la modélisation du comportement de fluides électrorhéologiques [66,67] et dans beaucoup d'applications dans les domaines du traitement d'images [25], de l'élasticité [73], et des milieux poreux [7,9].

On s'intéresse à la continuité des solutions des équations paraboliques du type suivant:

$$u_t - \mathrm{div} A(x,t,u,\nabla u) = B(x,t,u,\nabla u) \text{ in } \Omega_T,$$

ou $\Omega_T \equiv \Omega \times (0,T]$, Ω est un domaine borné dans $\mathbb{R}^N$ avec $N \geq 2$ et $0 < T < +\infty$. Les applications $A : \Omega_T \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^N$ and $B : \Omega_T \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^N$ sont supposées mesurables et vérifient les conditions suivantes:

$$|A(x,t,u,\nabla u)| \leq C_1 \left(\phi(x,t) + |u|^{p(x,t)-1} + |\nabla u|^{p(x,t)-1}\right),$$

$$|B(x,t,u,\nabla u)| \leq C_2 \left(\phi(x,t) + |u|^{p(x,t)-1} + |\nabla u|^{p(x,t)-1}\right),$$

$$A(x,t,u,\nabla u)\nabla u \geq C_3 |\nabla u|^{p(x,t)},$$

avec $\phi(x,t) \in L^\infty(\Omega_T)$ et C_1, C_2, C_3 sont des constantes positives. Nous nous intéressons au cas dégénéré, c'est à dire lorsque $p(x,t)$ est une fonction continue et mesurable, telle que:

$$2 < p^- = \inf_{\Omega_T} p(x,t) \leq p(x,t) \leq \sup_{\Omega_T} p(x,t) = p^+ < +\infty.$$

Les résultats sur la continuité locale des solutions de notre problème dans le cas où $p(x,t) = p$ ont été largement étudiés et ils sont bien documentés (voir [29,30,70]).

Puisque la derivée par rapport au temps de la solution présumée n'est pas assez régulière, la formulation faible de notre équation impose d'utiliser la moyenne de Steklov pour surmonter cette difficulté. Donc, on obtient la formulation faible suivante:

$$\int_{K\times\{t\}} \{u_{h,t}\varphi + [A(x,t,u,\nabla u)]_h . \nabla\varphi - [B(x,t,u,\nabla u)]_h \varphi\} dx = 0,$$

pour tout compact $K \subset \Omega$ et $\varphi \in C_0^\infty(K)$.

Dans l'étape suivante, on va donner des estimations à priori sur la solution faible u. Une fois obtenues ces estimations, on peut négliger la formulation faible, et le problème devient un problème purement analytique: on montre que les fonctions qui vérifient certaines inégalités intégrales sont régulières. Ainsi, on obtient les estimations suivantes:
-Estimation énergétique locale

$$\begin{aligned}
&\sup_{-\tau<t<0} \int_{K_\rho} (u-k)_\pm^2 \xi^{p^+}(x,t) dx + C \int_{-\tau}^{t} \int_{K_\rho} |\nabla (u-k)_\pm|^{p^-} \xi^{p^+} dxdt \\
&\quad \leq \int_{K_\rho} (u-k)_\pm^2 \xi^{p^+}(x,-\tau) dx + C' \Bigg[\int_{-\tau}^{t} \int_{K_\rho} (u-k)_\pm^2 \xi^{p^+-1} \xi_t \, dxdt \\
&\quad + \int_{-\tau}^{t} \int_{K_\rho} (u-k)_\pm^{p^+} \left(|\nabla \xi|^{p^+} + \xi^{p^+} \right) dxdt + \int_{-\tau}^{t} \int_{K_\rho} \chi\left((u-k)_\pm > 0\right) dxdt \Bigg],
\end{aligned}$$

-Estimation logarithmique locale

$$\begin{aligned}
&ess \sup_{-\tau<t<0} \int_{K_\rho} \left[\psi^\pm(u)\right]^2 \xi^{p^+} dx \\
&\quad \leq \int_{K_\rho \times \{-\tau\}} \left[\psi^\pm(u)\right]^2 \xi^{p^+} dx + C \Bigg(\int_{-\tau}^{0} \int_{K_\rho} \psi^\pm(u) \left[(\psi^\pm)'(u)\right]^2 \xi^{p^+} dxdt \\
&\quad + \int_{-\tau}^{0} \int_{K_\rho} \psi^\pm(u) \left[(\psi^\pm)'(u)\right]^{2-p^-} \left(|\nabla u|^{p^+} + 1 + \xi^{p^+} \right) dxdt \\
&\quad + \int_{-\tau}^{0} \int_{K_\rho} \psi^\pm(u) \left(|\nabla u|^{p^+} + 1 + \xi^{p^+} \right) dxdt + \int_{-\tau}^{0} \int_{K_\rho} |u|^{p^+} \psi^\pm(u) \left[(\psi^\pm)'(u)\right]^2 \xi^{p^+} dxdt \Bigg).
\end{aligned}$$

La troisième étape sera consacrée à définir les cylindres dont les dimensions sont convenablement redimensionnées pour refléter la dégénérescence de notre équation. En effet, on prend un rayon R tel que $0 < R < 1$, suffisamment petit pour que

$$Q(R^2, R) = K_R \times (-R^2, 0) := \{x : \max_{1\leq i\leq N} |x_i| < R\} \times (-R^2, 0) \subset \Omega_T.$$

Soient

$$\mu^+ = \underset{Q(R^2,R)}{ess\sup}\, u, \quad \mu^- = \underset{Q(R^2,R)}{ess\inf}\, u \quad \text{and} \quad \omega = \underset{Q(R^2,R)}{ess\ osc}\, u = \mu^+ - \mu^-.$$

On construit ensuite le cylindre suivant

$$Q(a_0 R^{p^+}, R) = K_R \times (-a_0 R^{p^+}, 0), \text{ avec } \frac{1}{a_0} = \left(\frac{\omega}{2^\lambda}\right)^{2-p^-} \quad \text{tel que } \lambda > 1,$$

et on suppose que

$$R^{\frac{2-p^+}{2-p^-}} < \frac{\omega}{2^\lambda}.$$

Par conséquent, on obtient

$$Q(a_0 R^{p^+}, R) \subset Q(R^2, R),$$

et

$$\underset{Q(a_0 R^{p^+},R)}{ess\ osc}\ u \leq \omega.$$

Par la même méthode, on continue d'emboiter les cylindres dans le but d'avoir

$$\underset{Q_n}{ess\ osc}\, u \leq \omega_n \text{ avec } Q_{n+1} \subset Q_n.$$

Pour avoir l'inégalité précédente, on étudie le comportement de $\inf ess$ et $\sup ess$ de u dans le cylinder $(0, t^*) + Q(\theta R^{p^+}, R)$. Alors on a les deux cas suivants:

$$\left|\left\{(x,t) \in (0,t^*) + Q(\theta R^{p^+}, R) : \ u(x,t) < \mu^- + \frac{\omega}{2}\right\}\right| \leq \nu_0 \left|Q(\theta R^{p^+}, R)\right|,$$

ou bien

$$\begin{aligned}\left|\left\{(x,t) \in (0,t^*) + Q(\theta R^{p^+}, R) : \ u(x,t) > \mu^+ - \frac{\omega}{2}\right\}\right| \\ \leq (1-\nu_0)\left|Q(\theta R^{p^+}, R)\right|.\end{aligned}$$

Où t^*, θ et v_0 sont des paramètres spécifiques. Par conséquent, en utilisant les estimations énérgétique et logarithmique, le lemme de De Giorgi [chap 1, page 19](Lemme 1.4.1), l'inégalité de Poincaré [chap 1, page 20], et Lemme 1.4.3 [chap 1, page 19], on obtient

$$\underset{Q(\theta(\frac{R}{8})^{p^+}, \frac{R}{8})}{ess\ osc}\ u \leq \sigma\omega, \text{ avec } \sigma \in (0,1).$$

Ensuite, grâce à l'estimation précédente, on peut facilement déduire

$$\underset{Q_n}{ess\ osc}\, u \leq \omega_n \quad \text{avec} \qquad Q_{n+1} \subset Q_n.$$

Ce qui implique que pour tout cylindre $Q(a_0\rho^{p^+}, \rho)$ avec $0 < \rho \leq R$, on aura

$$\underset{Q(a_0\rho^{p^+},\rho)}{ess\ osc}\ u \leq \gamma\omega \left(\frac{\rho}{R}\right)^{\alpha},$$

avec $0 < \rho \leq R,\ \alpha \in (0, 1)$, et $\gamma > 0$.
Finalement, on définit la p-distance parabolique par

$$\text{p-dist}(K; \Gamma) := \inf_{\substack{(x,t)\in K \\ (y,s)\in\Gamma}} \left(|x - y| + M^{\frac{p^- -2}{p^+}} |t - s|^{\frac{1}{p^+}} \right),$$

et on déduit facilement que la solution faible u de notre problème est höldérienne.

Deuxième thème: Problème parabolique dégénéré anisotropique

Dans cette partie on s'intéresse au cas anisotropique. On se propose d'étudier la continuité des solutions de l'équation suivante:

$$u_t - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial u}{\partial x_i} \right] = 0 \text{ in } \Omega_T,$$

ou $\Omega_T \equiv \Omega \times (0, T]$, Ω est un domaine borné dans $\mathbb{R}^N$ avec $0 < T < +\infty$. On suppose que les exposants $p_i(x, t)$ avec $i = 1, .., N$ sont des fonctions continues et mesurables, telles que:

$$p_i(x, t) \subset \left(p_i^-, p_i^+\right) \subseteq \left[p^-, p^+\right] \subset (2, \infty),$$

pour tout $i = 1, .., N$.

Les équations paraboliques anisotropiques avec exposant variable ont beaucoup d'applications dans le domaine de la physique. Ils apparaissent, par exemple, dans l'étude de l'écoulement des fluides électrorhéologiques avec des conductivités différentes dans des directions différentes (voir [8,18]). Elles sont également présentes en biologie comme un modèle décrivant la propagation d'une maladie épidémique dans des environnements hétérogènes (voir [19,20]).

Comme dans le cas précédent, puisque la solution u ne possède pas assez de régularité concernant sa dérivée par rapport au temps, on aura besoin d'utiliser la moyenne de Steklov u_h pour surmonter cette difficulté. Donc, on obtient la formulation faible suivante:

$$\int_{K\times\{t\}} \left[u_{h,t}\varphi + \sum_{i=1}^{N} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial u}{\partial x_i} \right)_h . \frac{\partial \varphi}{\partial x_i} \right] dx = 0$$

pour tout $\varphi \in V_t(K)$.

On montre ensuite les estimations énergétique et logarithmique respectives suivantes:

$$\sup_{-\tau<t<0} \int_{K_\rho} (u-k)_\pm^2 \, \xi^{p^+}(x,t)dx + \sum_{i=1}^{N} \int_{-\tau}^{0} \int_{K_\rho} \left| \frac{\partial}{\partial x_i}(u-k)_\pm \right|^{p_i(x,t)} \xi^{p^+} dxdt$$
$$\leq \int_{K_\rho} (u-k)_\pm^2 \, \xi^{p^+}(x,-\tau)dx + p^+ \int_{-\tau}^{0} \int_{K_\rho} (u-k)_\pm^2 \, \xi^{p^+-1} \xi_t \, dxdt$$
$$+ C \sum_{i=1}^{N} \int_{-\tau}^{0} \int_{K_\rho} (u-k)_\pm^{p_i(x,t)} \left| \frac{\partial \xi}{\partial x_i} \right|^{p_i(x,t)} dxdt,$$

et

$$\underset{-\tau<t<0}{ess\sup} \int_{K_\rho \times \{t\}} \left[\psi^\pm(u)\right]^2 \xi^{p^+} dx \leq \int_{K_\rho \times \{-\tau\}} \left[\psi^\pm(u)\right]^2 \xi^{p^+} dx$$
$$+ C \sum_{i=1}^{N} \int_{-\tau}^{0} \int_{K_\rho} \psi^\pm(u) \left[(\psi^\pm)'(u)\right]^{2-p_i(x,t)} \left| \frac{\partial \xi}{\partial x_i} \right|^{p_i(x,t)} dxdt.$$

Par ailleurs, pour obtenir la continuité locale des solutions on aura besoin de considérer une géométrie qui reflète la dégénérescence de notre équation anisotropique. En effet, on prend un rayon R tel que $0 < R < 1$, suffisamment petit pour que

$$Q(R^2, R) = K_R \times (-R^2, 0) := \{x : \max_{1\leq i\leq N} |x_i| < R\} \times (-R^2, 0) \subset \Omega_T.$$

Soient

$$\mu^+ = \underset{Q(R^2,R)}{ess\sup} \, u, \quad \mu^- = \underset{Q(R^2,R)}{ess\inf} \, u \quad \text{and} \quad \omega = \underset{Q(R^2,R)}{ess\ osc} \, u = \mu^+ - \mu^-.$$

On construit le cylindre suivant

$$Q(a_0 R^{p^+}, R) = K_R \times (-a_0 R^{p^+}, 0), \text{ avec } \frac{1}{a_0} = \left(\frac{\omega}{2^\lambda}\right)^{2-p^-} \text{ tel que } \lambda > 1,$$

et on suppose que

$$R^{\frac{2-p^+}{2-p^-}} < \frac{\omega}{2^\lambda}.$$

Par conséquent, on obtient

$$Q(a_0 R^{p^+}, R) \subset Q(R^2, R),$$

et

$$\underset{Q(a_0 R^{p^+},R)}{ess\ osc} \, u \leq \omega.$$

Alors par la même méthode, on continue d'emboiter les cylindres dans le but d'avoir

$$\underset{Q_n}{ess\ osc}\ u \leq \omega_n \text{ avec } Q_{n+1} \subset Q_n.$$

Pour avoir le résultat désiré on va étudier le comportement de $inf\ ess$ et $sup\ ess$ de la solution u dans les deux cas possibles suivants:

$$\left|\left\{(x,t) \in (0,t^*) + Q(\theta R^{p^+}, R) :\ u(x,t) < \mu^- + \frac{\omega}{2}\right\}\right| \leq \nu_0 \left|Q(\theta R^{p^+}, R)\right|,$$

ou bien

$$\left|\left\{(x,t) \in (0,t^*) + Q(\theta R^{p^+}, R) :\ u(x,t) > \mu^+ - \frac{\omega}{2}\right\}\right| \\ \leq (1-\nu_0)\left|Q(\theta R^{p^+}, R)\right|.$$

Où t^*, θ et v_0 sont des paramétres spécifiques.

Malheureusement, on ne peut pas utiliser le lemme de De Giorgi dans le cas anisotropique. Pour surmonter cette difficulté, on va introduire une fonction doublement tronquée v_s telle que pour tout $t \in \left(-\frac{a_0}{2}R^{p^+}, 0\right)$

$$v_s = \begin{cases} 0 & \text{for} \quad u(x,t) < \mu^+ - \frac{\omega}{2^s}, \\ u - \left(\mu^+ - \frac{\omega}{2^s}\right) & \text{for} \quad \mu^+ - \frac{\omega}{2^s} \leq u(x,t) \leq \mu^+ - \frac{\omega}{2^{s+1}}, \\ \frac{\omega}{2^{s+1}} & \text{for} \quad \mu^+ - \frac{\omega}{2^{s+1}} \leq u(x,t). \end{cases}$$

Par un calcul élémentaire on obtient une estimation similaire à celle du lemme de De Giorgi. Ainsi, grâce aux estimations énergétique et logarithmique, l'inégalité de Poincaré (3.1.2), et Lemme 1.4.3, on obtient

$$\underset{Q(\theta(\frac{R}{8})^{p^+}, \frac{R}{8})}{ess\ osc}\ u \leq \sigma\omega, \text{ avec } \sigma \in (0,1),$$

ce qui est suffisant pour prouver la continuité locale des solutions de notre problème anisotropique.

2$^{\text{éme}}$ partie: Schémas d'approximation

La deuxième partie sera consacrée à l'étude de la semi-discrétisation locale et globale de quelques équations non linéaires et doublement non linéaires à exposant variable. Elle s'articule autour de deux thèmes de recherche:

Premier thème: Semidiscretisation d'une équation doublement non linéaire à exposant variable.

Dans ce thème, nous étudions une équation parabolique doublement non linéaire liée à l'opérateur p(x)-laplacien décrite dans [43]. Ce travail s'inspire, d'une part, des résultats d'El Hachimi et El Ouardi [43], Benzekri et El Hachimi [22], et, d'autre part, des travaux d'Eden, Michaux et Rakotoson [33]. Notre équation est sous la forme suivante:

$$\begin{cases} \frac{\partial \beta(u)}{\partial t} - \Delta_{p(x)} u + f(x,t,u) = 0 & \text{in } \Omega \times]0,\infty[, \\ u = 0 & \text{on } \partial\Omega \times]0,\infty[, \\ \beta(u) \mid_{t=0} = \beta(u_0) & \text{in } \Omega, \end{cases}$$

où Ω est un domaine borné dans $\mathbb{R}^N$ avec $1 \leq p(x) < \infty$.

Les résultats d'existence et certaines propriétés qualitatives concernant les solutions de notre problème et des problèmes plus généraux ont été obtenus par de nombreux auteurs au cours de la dernière décennie [5,7,24].

L'étude de cette équation est motivée par le fait qu'elle est considérée comme un modèle d'une classe importante de fluides non-Newtoniens, connus comme fluides électrorhéologiques (voir [67]).

En effet, nous considérons le schéma d'Euler associé au problème initial défini par:

$$\begin{cases} \beta(U^n) - \tau \Delta_{p(x)} U^n + \tau f(x, n\tau, U^n) = \beta(U^{n-1}) & \textit{dans } \Omega, \\ U^n = 0 & \textit{sur } \partial\Omega, \\ \beta(U^0) = \beta(u_0) & \textit{dans } \Omega, \end{cases}$$

où $N\tau = T$, T étant un réel positif fixé. Alors, 0n s'intéresse à l'étude de l'existence, de l'unicité, et de la stabilité de la solution dans les deux cas suivants:

-1$^{\text{er}}$ **cas**:$u_0 \in L^\infty(\Omega)$, avec les hypothèses suivantes:

(H_1) β est une fonction continue et croissante de $\mathbb{R}$ dans $\mathbb{R}$ telle que $\beta(u) \leq C|u|^{\alpha-1}$ pour tout $u \in \mathbb{R}$ avec $1 \leq \alpha < p^-$.

(H_2) pour $\xi \in \mathbb{R}$, l'application $(x,t) \longmapsto f(x,t,\xi)$ est mesurable et l'application $\xi \longmapsto f(x,t,\xi)$ est continue. De plus, on suppose qu'il existe $C_1 > 0$, telle que pour $(x,t) \in \Omega \times \mathbb{R}^+$, on a $sign(\xi).f(x,t,\xi) \geq -C_1$.

(H_3) il existe $C_2 > 0$, telle que pour tout $(x,t) \in \Omega \times \mathbb{R}^+$, l'application $\xi \longmapsto f(x,t,\xi) + C_2\beta(\xi)$ est croissante.

-2$^{\text{éme}}$ **cas**: $u_0 \in L^2(\Omega)$, avec (H_1), (H_2), et l'hypothèse suivante:

(H_5) $\forall \xi \in \mathbb{R}$, l'application $(x,t) \longmapsto f(x,t,\xi)$ est mesurable, et l'application $\xi \longmapsto f(x,t,\xi)$

est continue. De plus, on suppose qu'ils existent $r \in C_+(\Omega)$ avec $r(x) > \sup(2, p(x))$, et des constantes positives C_5 et C_6 telles que

$$signe(\xi) f(x,t,\xi) \geq C_5 \left|\xi\right|^{r(x)-1} - C_6.$$

En effet, dans le premier cas, on montre l'existence d'une solution unique $U^n \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ de notre équation semi-discrétisée. De plus, pour

$$\psi(t) = \int_0^t \beta(\tau)\ d\tau,$$

on montre que:

$$\|U^n\|_\infty \leq C(T, u_0),$$

$$\int_\Omega \psi^*(\beta(U^n))dx + \tau \sum_{k=1}^{n} \|U^k\|^\alpha_{1,p(x)} \leq C(T, u_0), \text{ où } \alpha \text{ dépend soit de } p^- \text{ ou } p^+,$$

et

$$\sum_{k=1}^{n} \|\beta(U^k) - \beta(U^{k-1})\|_2^2 \leq C(T, u_0),$$

où $C(T, u_0)$ est une constante positive.
Ensuite, on remplace l'hypothèse (H_3) par l'hypothèse suivante:
(H_4) $\forall M > 0,\ \exists C_M > 0$ telles que, si $|\xi| + |\xi'| \leq M$, on a

$$|f(x,t,\xi) - f(x,t,\xi')|^\theta \leq C_M(\beta(\xi) - \beta(\xi'))(\xi - \xi'),$$

où

$$\theta = \begin{cases} \sigma' & for\ 1 < p(x) < 2, \\ p'^- & for\ p(x) \geq 2. \end{cases}$$

Alors, sous les hypothèses (H_1), (H_2), et (H_4) on montre aussi qu'il existe une solution unique $U^n \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ de l'équation semi-discrétisée.

De même, dans le deuxième cas, on montre qu'il existe une unique solution $U^n \in W_0^{1,p(x)}(\Omega)$. De plus, on obtient les estimations suivantes:

$$\int_\Omega \psi^*(\beta(U^n))dx + \tau \sum_{k=1}^{n} \|U^k\|^\alpha_{1,p(x)} + C\tau \sum_{k=1}^{n} \|U^k\|^{\alpha'}_{q(x)} \leq C(T, u_0),$$

$$\max_{1\leq k\leq n} \sum_{k=1}^{n} \|\beta(U^k)\|_2^2 + \|\beta(U^k) - \beta(U^{k-1})\|_2^2 \leq C(T, u_0),$$

avec $\psi^*(\tau) = \sup\limits_{s\in\mathbb{R}}\{\tau s - \psi(s)\}$ et $C(T, u_0)$ est une constante positive.

Nous montrons ensuite l'existence d'un ensemble absorbant, c-à-d quelque soit $u_0 \in L^\infty(\Omega)$, il existe $n(\tau) \in \mathbb{N}$ tel que

$$\|U^n\|_{L^\infty(\Omega)} + \|U^n\|_{1,p(x)} \leq C, \quad \forall n \geq n(\tau).$$

Par conséquent, on peut montrer l'existence d'un attracteur compact. En effet, pour $U^0 = u_0 \in L^\infty(\Omega)$ et τ fixé de telle sorte que $\tau_2 = \min(1, \frac{1}{C_2})$, on considére le problème suivant:

$$\beta(U^n) - \tau\Delta_{p(x)}U^n + \tau f(x, n\tau, U^n) = \beta(U^{n-1}) \;\; dans\ \Omega,$$

$$U^n = 0 \quad sur\ \partial\Omega,$$

Sous les hypothèses (H_1), (H_2) et (H_3), et grâce à l'existence et l'unicité des solutions du problème semi-discrétisé, on peut définir l'application $S_\tau : L^\infty(\Omega) \longrightarrow L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$ telle que

$$S_\tau U^{n-1} = U^n.$$

Puisque S_τ est continue, on a aussi

$$S_\tau^n U^0 = U^n.$$

Alors, notre problème semi-discrétisé a un semi-groupe S_τ, qui, grâce aux résultats de Temam [68] a un attracteur compact

$$\mathfrak{A}_\tau = \cap_{n\geq 0}\overline{\cup_{m\geq n} S_\tau^m(\mathfrak{B}_\tau)},$$

où $\mathfrak{B}_\tau$ est une boule absorbante dans $L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$.

Deuxième thème: La Méthode des éléments finis pour le problème parabolique associé au p(x,t)-laplacien dans le cas singulier.

On va appliquer la méthode des éléments finis au problème suivant:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p(x,t)-2}\nabla u) = f(x,t) & \text{in } \Omega_T, \\ u = 0 & \text{on } \Gamma_T = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

avec f et l'exposant $p(x,t)$ sont höldériennes dans $\overline{\Omega_T}$. L'exposant $p(x,t)$ prend ses valeurs dans l'intervalle $[p^-, p^+] \subset \left(\dfrac{2N}{N+2}, 2\right)$.

Si l'exposant ne dépend pas de x et t, nous aurons l'équation parabolique p-laplacien classique. Dans ce cas, les approximations numériques utilisant la méthode des éléments finis ont déjà été étudiées par d'autres auteurs (voir[17,71]).

La difficulté que nous rencontrons ici est de montrer que les solutions sont assez régulières. Par conséquent, on introduit le problème régularisé suivant:

$$\begin{cases} u_{\varepsilon,t} - \operatorname{div}((|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p(x,t)-2}{2}} \nabla u_\varepsilon) = f & \text{in } \Omega_T = \Omega \times (0,T], \\ u_\varepsilon = 0 & \text{on } \Gamma_T = \partial\Omega \times (0,T], \\ u_\varepsilon(x,0) = u_{\varepsilon,0}(x) = u_0(x) & \text{in } \Omega, \end{cases}$$

avec $\varepsilon \in]0,1[$. Alors, grâce à [12] on a pour tout $s \in (0,T)$

$$\|\nabla u_\varepsilon\|_{C^{\delta,\frac{\delta}{2}}\left(\overline{\Omega}\times[s,T]\right)} \le C,$$

et

$$D^2_{ij} u_\varepsilon \in L^2(\Omega \times (s,t)),$$

où $\nabla p \in L^\infty(\Omega_T)$ et $f \in C^{\delta,\frac{\delta}{2}}(\overline{\Omega_T})$ avec $\delta \in (0,1)$. u_ε^h étant la solution du schéma obtenu par la semi-discrétisation en espace du probème régularisé. Ensuite, on obtient l'estimation d'erreur suivante

$$\|u_\varepsilon - u_\varepsilon^h\|^2_{C([0,T],L^2(\Omega^h))} + \|u_\varepsilon - u_\varepsilon^h\|^2_{L^2([0,T],H^1(\Omega^h))} \le Ch.$$

En effectuant une semi-discrétisation par rapport à la variable du temps, on obtient également l'estimation d'erreur suivante:

$$\|u_\varepsilon^h - U\|^2_{C([0,T],L^2(\Omega))} + \|u_\varepsilon^h - \widehat{U}\|^2_{L^2(0,T,H^1(\Omega))} \le C\Delta t.$$

avec

$$\begin{aligned} U(t) &= \frac{t - t_{n-1}}{\Delta t} U_\varepsilon^n + \frac{t_n - t}{\Delta t} U_\varepsilon^{n-1}, \quad t \in [t_{n-1}, t_n] \\ \widehat{U}(t) &= U_\varepsilon^n, \quad t \in (t_{n-1}, t_n] \\ \widehat{f}(t) &= f^n, \quad t \in (t_{n-1}, t_n], \end{aligned}$$

où $t_n = n\Delta t$ est une partition uniforme de $[0,T]$. Par conséquent, la discrétisation totale implique l'estimation suivante:

$$\|u_\varepsilon^h - U\|^2_{C([0,T],L^2(\Omega))} + \|u_\varepsilon^h - \widehat{U}\|^2_{L^2(0,T,H^1(\Omega))} \le C\Delta t.$$

Finalement, en comparant la solution de la première équation et la solution de l'équation régularisée nous obtenons ce qui suit:

$$\|u - U\|^2_{C([0,T],L^2(\Omega^h))} + \|u - \widehat{U}\|^2_{L^2(0,T,H^1(\Omega^h))} \le C(h + \Delta t + \varepsilon^2).$$

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Chapter 1

Preliminaries and definitions

This chapter is devoted to a brief exposition of the theory of function spaces that provide the analytic framework for the study of PDEs with variable nonlinearity. These are the Lebesgue and Sobolev spaces with variable exponents, which can be regarded as special cases of Orlicz's spaces [63], or semi-modular spaces [59-60]. The theory of such spaces is very interesting in itself, this challenging subject has been developing very rapidly in the last decades. A detailed discussion of the theory of Lebesgue and Sobolev spaces with variable exponents is beyond the scope of this thesis, for this reason i confine myself to outlining a minimal set of properties needed in the further proceeding.

1.1 Classical Lebesgue-Sobolev spaces

We will deal with bounded domains Ω contained in the Euclidean N-dimensional space $\mathbb{R}^N$. For $1 \leq p < \infty$, $L^p(\Omega)$ is the space of real functions on Ω which are L^p for the Lebesgue measure dx. It is a reflexive Banach space for the norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

For $p = \infty$, $L^\infty(\Omega)$ is the space of real functions on Ω which are measurable and essentially bounded. It is also a Banach space for the norm

$$\|u\|_{L^\infty(\Omega)} := \sup_{x \in \Omega} \text{ess}|u(x)|.$$

The space $L^2(\Omega)$ is a Hilbet space with the inner product

$$(u, v)_{2,\Omega} := \int_\Omega u \ v \ dx.$$

For $1 \leq q < \infty$,$k \geq 1$, $W^{k,q}(\Omega)$ is the Sobolev space of k times weakly differentiable functions with bounded norm

$$\|u\|_{W^{k,q}(\Omega)} := \left(\int_{\Omega} \sum_{0 \leq |\alpha| \leq k} |D^{\alpha} u|^q \right)^{\frac{1}{q}},$$

where, α is the multi-index, $\alpha := (\alpha_1, .., \alpha_N)$ with integer $\alpha_i \geq 0$, $|\alpha| = \sum_{i=1}^{N} \alpha_i$. $W^{k,q}(\Omega)$ endowed with the previous norm is a Banach space.
The space $W_0^{k,p}(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ (the set of smooth functions with compact support in Ω) in the norm of $W^{k,p}(\Omega)$. An equivalent norm of $W_0^{1,p}(\Omega)$ is given by

$$\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{p,\Omega}.$$

We denote by $L^q(0,T; L^p(\Omega))$, where T an arbitrary finite height, and $p, q \geq 1$, the space of measurable functions with the bounded norm

$$\|u\|_{L^q(0,T;L^p(\Omega))} := \left(\int_0^T \|u(.,t)\|_{p,\Omega}^q \, dt \right)^{\frac{1}{q}}.$$

The notation $L^q(0,T; W_0^{1,p}(\Omega))$ stands for the space of functions which are measurable, weakly differentiable, and bounded in the norm

$$\|u\|_{L^q(0,T;W_0^{1,p}(\Omega))} := \left(\int_0^T \|\nabla u(.,t)\|_{p,\Omega}^q \, dt \right)^{\frac{1}{q}}.$$

1.2 Generalized Lebesgue-Sobolev spaces

1.2.1 Lebesgue Spaces with Variable Exponents

Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain with Lipshitz continuous boundary $\partial\Omega$, and $p : \Omega \longrightarrow (1, \infty)$ be a measurable function such that

$$1 < p^- := \operatorname*{ess\,inf}_{x \in \Omega} p(x) \leq p(x) \leq p^+ := \operatorname*{ess\,sup}_{x \in \Omega} p(x) < \infty.$$

We start by the definition of the nonstandard $p(x)$-Lebesgue space. The space $L^{p(x)}(\Omega)$ is defined as the set of those measurable functions $u : \Omega \longrightarrow \mathbb{R}^N$, which satisfy $|u|^{p(x)} \in L^1(\Omega)$, i.e.

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \longrightarrow \mathbb{R}^N \text{ is measurable in } \Omega : \int_{\Omega} |u|^{p(x)} dx < \infty \right\}.$$

The set $L^{p(x)}(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right| dx \leq 1\right\}$$

becomes a Banach spaces. Next, we make a note of the generalized $p(x)$-Hölder's inequality. This result is started by Kov$\acute{a}\check{c}$ik and R$\acute{a}$kosn$\acute{i}$k in [46, Theorem 2.1]. Therefore, we define the conjugate $p'(x)$ of $p(x)$ by

$$p'(x) := \begin{cases} \frac{p(x)}{p(x)-1} & \text{if } 1 < p(x) < \infty, \\ \infty & \text{if } p(x) = 1. \end{cases} \tag{1.2.1}$$

Then, for all $f \in L^{p(x)}(\Omega)$ and all $g \in L^{p'(x)}(\Omega)$, it holds the generalized $p(x)$-Hölder's inequality:

$$\int_\Omega |f(x)g(x)|\, dx \leq c_p \|f\|_{L^{p(x)}(\Omega)} \cdot \|g\|_{L^{p'(x)}(\Omega)}$$

where $1 \leq c_p \leq 3$.
Let us next consider modular version of Poincaré's inequality.

Lemma 1.2.1. *(see [38]) Let $p(x)$ an element of $L^\infty(\Omega)$ and let $u \in W^{1,p(x)}(\Omega)$. There exists a constant C depending only on Ω such that*

$$\rho_{p(x)}(u) \leq C\rho_{p(x)}(\nabla u),$$

with $\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx$

Next, we introduce the nonstandard $p(x,t)$-Lebesgue sapce. The space $L^{p(x,t)}(\Omega_T)$ where $\Omega_T = \Omega \times (0,T)$ is defined as the set of those measurable functions $u : \Omega_T \longrightarrow \mathbb{R}^N$, which satisfy $|u|^{p(x,t)} \in L^1(\Omega_T)$, i.e.

$$L^{p(x,t)}(\Omega_T) := \left\{u : \Omega_T \longrightarrow \mathbb{R}^N \text{ is measurable in } \Omega_T : \int_{\Omega_T} |u|^{p(x,t)} dxdt < \infty\right\}.$$

The set $L^{p(x,t)}(\Omega_T)$ equipped with the Luxemburg norm

$$\|u\|_{L^{p(x,t)}(\Omega_T)} := \inf\left\{\lambda > 0 : \int_{\Omega_T} \left|\frac{u}{\lambda}\right| dxdt \leq 1\right\}$$

becomes a Banach spaces. This Banach space is reflexive, see [7]. For the elements of $L^{p(x,t)}(\Omega_T)$ the generalized Hölder's inequality holds in the following form: if $f \in L^{p(x,t)}(\Omega_T)$ and $g \in L^{p'(x,t)}(\Omega_T)$, where $p'(x,t) = \frac{p(x,t)}{p(x,t)-1}$, we have

$$\left|\int_{\Omega_T} f\ g\ dxdt\right| \leq \left(\frac{1}{p^-} + \frac{p^+ - 1}{p^+}\right) \|f\|_{L^{p(x,t)}(\Omega_T)} \cdot \|g\|_{L^{p'(x,t)}(\Omega_T)}.$$

Notice that the norm $\|.\|_{L^{p(x,t)}(\Omega_T)}$ can be estimated as follows:

$$\min\left\{\|u\|_{L^{p(x,t)}(\Omega_T)}^{p_-}, \|u\|_{L^{p(x,t)}(\Omega_T)}^{p_+}\right\} \le \int_{\Omega_T} |u|^{p(x,t)} dxdt \le \max\left\{\|u\|_{L^{p(x,t)}(\Omega_T)}^{p_-}, \|u\|_{L^{p(x,t)}(\Omega_T)}^{p_+}\right\},$$

respectively

$$-1 + \|u\|_{L^{p(x,t)}(\Omega_T)}^{p_-} \le \int_{\Omega_T} |u|^{p(x,t)} dxdt \le \|u\|_{L^{p(x,t)}(\Omega_T)}^{p_+} + 1.$$

Note also that

$$\|u_\varepsilon - u\|_{L^{p(x,t)}(\Omega_T)} \longrightarrow 0 \text{ as } \varepsilon \downarrow 0$$

is equivalent to

$$\int_{\Omega_T} |u_\varepsilon - u|^{p(x,t)} \, dxdt \longrightarrow 0 \text{ as } \varepsilon \downarrow 0.$$

1.2.2 Sobolev Spaces with Variable Exponents

First, we introduce nonstandard parabolic Sobolev spaces $W^{1,p(x,t)}(\Omega)$. For every fixed $t \in (0, T)$, we define the Banach space $W^{1,p(x,t)}(\Omega)$ as follows:

$$W^{1,p(x,t)}(\Omega) := \left\{u \in L^{p(x,t)}(\Omega) | \ \nabla u \in L^{p(x,t)}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{W^{1,p(x,t)}(\Omega)} := \|u\|_{L^{p(x,t)}(\Omega)} + \|\nabla u\|_{L^{p(x,t)}(\Omega)}.$$

In addition, $W_0^{1,p(x,t)}(\Omega) \equiv$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x,t)}(\Omega)$ and denote $W^{1,p(x,t)}(\Omega)'$ its dual. For every $t \in (0, T)$ the inclusion

$$W_0^{1,p(x,t)}(\Omega) \subset W_0^{1,p^-}(\Omega)$$

holds. As a consequence, we have the following Poincaré inequality: For every fixed $t \in (0, T)$ and every $u \in W_0^{1,p(x,t)}(\Omega)$, there exists a positive constant C such that

$$\|u\|_{L^{p(x,t)}(\Omega)} \le C \|\nabla u\|_{L^{p(x,t)}(\Omega)}. \tag{1.2.2}$$

Now, we consider more general nonstandard parabolic Sobolev spaces without fixed t. Thus, by $W_0^{p(x,t)}(\Omega_T)$ we denote the Banach space

$$W_0^{p(x,t)}(\Omega_T) := \left\{u \in L^{p(x,t)}(\Omega_t) | \ \nabla u \in L^{p(x,t)}(\Omega_t); \ u \in L^1(0, T; W_0^{1,1}(\Omega))\right\}$$

equipped by the norm

$$\|u\|_{W^{p(x,t)}(\Omega_T)} := \|u\|_{L^{p(x,t)}(\Omega_T)} + \|\nabla u\|_{L^{p(x,t)}(\Omega_T)}.$$

1.3 Anisotropic Spaces of Functions Depending on x and t

Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain with Lipshitz continuous boundary $\partial\Omega$, $\Omega_T = \Omega \times (0,T]$ be the generic cylinder of an arbitrary finite height T. Recall the notation $\Omega_\tau = \Omega \times (0,\tau]$ is used whenever the height of the cylinder is of importance. The lateral boundary of Ω_T is denoted by $\Gamma = \partial\Omega \times [0,T]$ (or $\Gamma_\tau = \partial\Omega \times [0,\tau]$).

Let $p(x,t) = (p_1(x,t), .., p_N(x,t))$ be a vector-valued function defined in Ω_T. We assume that the components of $p(x,t)$ satisfy the conditions

$$\begin{cases} p_i(x,t) \text{ are measurable defined on } \Omega_T, \\ p_i(x,t) : \Omega_T \longrightarrow (1,\infty), \\ \text{there exits constants } p_i^\pm,\ p^\pm \text{ such that} \\ p_i(x,t) \in [p_i^-, p_i^+] \subseteq [p^-, p^+] \subset (1,\infty). \end{cases}$$

For a.e. $t \in (0,T)$ we introduce the anisotropic Banach space

$$V_t(\Omega) := \left\{ u(x) :\ u(x) \in L^2(\Omega) \cap W_0^{1,1}(\Omega),\ |D_i u(x)|^{p_i(x,t)} \in L^1(\Omega) \right\},$$

$$\|u\|_{V_t(\Omega)} := \|u\|_{2,\Omega} + \sum_{i=1}^{N} \|D_i u\|_{p_i(x,t),\Omega}.$$

The elements of the space $V_T(\Omega)$ depend on $t \in (0,T)$ as a parameter and the norms $\|u\|_{V_t(\Omega)}$ are functions of t. By $V_t^{'}()\Omega$ we denote the dual space to $V_t(\Omega)$ with respect to the scalar product in $L^2(\Omega)$. For a.e. $t \in (0,T)$ the inclusion

$$V_t(\Omega) \subset X = W_0^{1,p^+}(\Omega) \cap L^2(\Omega)$$

holds, which is why $V_t(\Omega)$ is reflexive and separable as a closed subspace of X.
Next, by $W(\Omega_T)$ we denote the Banach space

$$W(\Omega_T) := \left\{ u : (0,T) \longmapsto V_t(\Omega)|\ \ u \in L^2(\Omega_T),\ |D_i u|^{p_i(x,t)} \in L^1(\Omega_T),\ u = 0 \text{ on } \Gamma \right\},$$

$$\|u\|_{W(\Omega_T)} := \|u\|_{2,\Omega_T} + \sum_{i=1}^{N} \|D_i u\|_{p_i(x,t),\Omega_T}.$$

$W'(\Omega_T)$ is the dual of $W(\Omega_T)$ (the space of linear functionals over $W(\Omega_T)$):

$$\omega \in W'(\Omega_T) \Longleftrightarrow \begin{cases} \exists(\omega_0, .., \omega_N),\ \omega_0 \in L^2(\Omega_T),\ \ \omega_i \in L^{p_i'(x,t)}(\Omega_T), \\ \forall\phi \in W(\Omega_T), \quad << \omega, \phi >>= \int_{\Omega_T} \left(\omega_0\phi + \sum_{i=1}^{N} \omega_i D_i \phi \right) dxdt. \end{cases}$$

The norm in $W'(\Omega_T)$ is defined by

$$\|u\|_{W'(\Omega_T)} := \sup\left\{<< u, \phi >> \quad : \phi \in W(\Omega_T),\ \ \|\phi\|_{W(\Omega_T)} \leq 1\right\}.$$

As a consequence, the elements of $W(\Omega_T)$ satisfies the following inequality

$$\begin{aligned}
&\min\left\{\sum_{i=1}^{N} \|D_i u\|_{p_i(x,t),\Omega_T}^{p^-}, \sum_{i=1}^{N} \|D_i u\|_{p_i(x,t),\Omega_T}^{p^+}\right\} \\
&\quad \leq \sum_{i=1}^{N} \int_{\Omega_T} |D_i u|^{p_i(x,t)}\, dxdt \leq \max\left\{\sum_{i=1}^{N} \|D_i u\|_{p_i(x,t),\Omega_T}^{p^-}, \sum_{i=1}^{N} \|D_i u\|_{p_i(x,t),\Omega_T}^{p^+}\right\}.
\end{aligned}$$

To deal with anisotropic PDEs with variable nonlinearity we need specific embedding theorems. We adapt the known anisotropic inequalities of the Gagliargo-Nirenberg-Sobolev type. In [47,62,69], the theory of anisotropic Sobolev spaces is developed and, in particular, the corresponding Sobolev embedding theorems are studied. Define

$$p^* = \frac{N\overline{p}}{N - \overline{p}}, \text{ for } \overline{p} < N \text{ and } \frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i^-}. \tag{1.3.1}$$

In [69] it is proved that if $\overline{p} < N$, then

$$W_0^{1,p_i^-}(\Omega) \hookrightarrow L^r(\Omega), \quad \forall r \in [1, p^*].$$

This embedding is continuous and also compact if $r < p^*$. The following sobolev type inequality is also proved; there exists a positive constant C, depending only on $\Omega,\ p_i^-,\ r$ and N, such that

$$\|u\|_{r,\Omega} \leq C \prod_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i^-,\Omega}^{\frac{1}{N}}, \quad \forall r \in [1, p^*], \tag{1.3.2}$$

for any $u \in W_0^{1,p_i^-}(\Omega)$.

1.4 Some Technical Tools

We gather in this section a few technical facts that, although marginal to the theory, are essential in establishing its main results.

1.4.1 A Lemma of De Giorgi

Given a continuous function $u : \Omega \longrightarrow \mathbb{R}$ and two real numbers $k_1 < k_2$, we define

$$\begin{aligned} [u > k_2] &:= \{x \in \Omega : \ u(x) > k_2\}, \\ [u < k_1] &:= \{x \in \Omega : \ u(x) < k_1\}, \\ [k_1 < u < k_2] &:= \{x \in \Omega : \ k_1 < u(x) < k_2\}. \end{aligned} \tag{1.4.1}$$

Lemma 1.4.1. ***(De Giorgi,[42])****. Let $u \in W^{1,1}(B_\rho(x_0)) \cap C(B_\rho(x_0))$, with $\rho > 0$ and $x_0 \in \mathbb{R}^N$, and let $k_1 < k_2 \in \mathbb{R}^N$. There exists a constant C depending only on N such that*

$$(k_2 - k_1)\,|[u > k_2]| \leq C \frac{\rho^{N+1}}{|[u < k_1]|} \int_{[k_1<u<k_2]} |\nabla u|\, dx.$$

Remark 1.4.2. *The conclusion of the lemma remains valid, provided Ω is convex, for functions $u \in W^{1,1}(\Omega) \cap C(\Omega)$. We will use it in the case Ω is a cube. In fact, the continuity is not essential for the result to hold. For a function merely in $W^{1,1}(\Omega)$, we define the sets in (1.4.1) through any representative in the equivalence class. It can be shown that the conclusion of the lemma is independent of that choice.*

1.4.2 Geometric Convergence of Sequences

The following lemmas concern the geometric convergence of sequences and are instrumental in the iterative schemes that will be derived along the proofs.

Lemma 1.4.3. *Let (X_n), $n = 0, 1, 2, ...,$ be a sequence of positive real numbers satisfying the recurrence relation*

$$X_{n+1} \leq C b^n X_n^{1+\alpha}$$

where C, $b > 1$ and $\alpha > 0$ are given, if

$$X_0 \leq C^{\frac{-1}{\alpha}} b^{\frac{-1}{\alpha^2}}$$

then, $X_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Lemma 1.4.4. *Let (X_n) and (Z_n), $n = 0, 1, 2, ...,$ be sequences of positive real numbers satisfying the recurrence relations*

$$\begin{cases} X_{n+1} \leq C b^n \left(X_n^{1+\alpha} + X_n^{\alpha} Z_n^{1+k} \right), \\ Z_{n+1} \leq C b^n (X_n + Z_n^{1+k}) \end{cases} \tag{1.4.2}$$

where C, $b > 1$ and α, $k > 0$ are given, if

$$X_0 + Z_0^{1+k} \leq (2C)^{-\frac{1+k}{\sigma}} b^{-\frac{1+k}{\sigma^2}}, \ \textit{with } \sigma = \min\{\alpha, k\},$$

then, X_n, $Z_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Next, we are going to introduce the discrete version of the uniform Gronwall Lemma (see Lemma 7.5 of [33]).

Lemma 1.4.5. *Let $\{y^n\}_{n=0}^{\infty}$ be two sequences of real numbers, satisfying*

$$y^n \leq y^{n-1} + \tau h_n.$$

Assume further that there exists a positive integer l_0 such for all $l_1 \geq l_0$ and $N > 0$

$$\tau \sum_{n=l_1}^{l_1+N} h_n \leq a_1 \text{ and } \tau \sum_{n=l_1}^{l_1+N} y_n \leq a_2,$$

for some positive real numbers a_1 and a_2 that do not depend on l_1, then for all $l_1 \geq l_2$,

$$y^{l_1+N} \leq \frac{a_2}{N\tau} + a_1.$$

1.4.3 An Embedding Theorem

Let $V_0^p(\Omega_T)$ denote the space

$$V_0^p(\Omega_T) := L^\infty(0,T;L^p(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$$

endowed with the norm

$$\|u\|^p_{V_0^p(\Omega_T)} := \operatorname*{ess\,sup}_{0\leq t\leq T} \|u(x,t)\|^p_{p,\Omega} + \|\nabla u\|^p_{p,\Omega_T}.$$

The following embedding theorem holds (see [29])

Theorem 1.4.6. *Let $p, l > 1$. There exists a positive constant γ, depending only on N and p, such that for every $u \in V_0^p(\Omega_T)$,*

$$\int\int_{\Omega_T} |u|^{(N+l)\frac{p}{N}} \ dxdt \leq \gamma \left[\operatorname*{ess\,sup}_{t\in(0.T)} \int_\Omega |u|^p \, dx \right]^{\frac{p}{N}} \int\int_{\Omega_T} |\nabla u|^p \ dxdt.$$

Part I

Regularity

Introduction

Many relevant phenomena, not only in the natural sciences but also in engineering and economics, are modeled by (systems of) partial differential equations (PDEs) that exhibit some sort of degeneracy or singularity. Examples include the motion of multi-phase fluids in porous media, the melting of crushed ice (and phase transitions, in general), the behavior of composite materials or the pricing of assets in financial markets. Because of its significance in terms of the applications, but also due to the novel analytical techniques that it generates, the class of degenerate and singular parabolic equations is an important branch in the contemporary analysis of partial differential equations.

As a prototype, we will consider the degenerate parabolic p-Laplace equation

$$u_t - \operatorname{div} |\nabla u|^{p-2} \nabla u = 0, \quad p > 1.$$

If $p > 2$, the equation is degenerate in the space part since its modulus of ellipticity $|\nabla u|^{p-2}$ vanishes at points where $|\nabla u| = 0$. If $1 < p < 2$, the modulus of ellipticity becomes unbounded at points $|\nabla u| = 0$ and the equation is said to be singular. In this work we only consider the latter case when $p \geq 2$.

The purpose of these thesis is to describe intrintsic scaling method for obtaining continuity results for the weak solutions of degenerate isotropic and anisotropic parabolic equation with variable exponents. To understand what is at stake, let us start by placing the problem in its historical context.

As in any other mathematical journeys, it all started with one of Hilbert's problems. In 1900, in a then obscure and now legendary session of the International Congress of Mathematicians in Paris, David Hilbert presented his list of 23 problems that would shape the mathematics of the newborn century. Two of those problems were related to the Calculus of Variations:
-19th. are the solutions of regular problems in the Calculus of Variations always necessarily analytic?
-20th. do regular problems in the Calculus of Variations always possess a solution, (...) extending, if need be, the notion of solution?

in an other way, he asked whether minimizers to functionals of type

$$I(u) = \int F(x, u, \nabla u) dx \tag{1.4.3}$$

are necessarily analytic if the function $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ is assumed to satisfy certain regularity conditions.

Naturally, the problem was directly associated with the corresponding Euler-Lagrange equation. This resulted in the study of regularity for the solutions of the partial differential equation

$$- \operatorname{div} A(x, u, \nabla u) = 0. \tag{1.4.4}$$

Related to the assumptions on the kernel F in (1.4.3), the operator A was assumed to satisfy certain structure conditions.

It was soon found out that if a solution to this problem is at least twice continuously differentiable it is, indeed, analytic as well. However, for proving existence for nonlinear equations assuming this kind of a priori regularity is too restrictive. Moreover, the statement of the minimization problem (1.4.3) does not require such a regularity assumption. This lead to the study of so called weak solutions which were defined in Sobolev spaces. Then the aim was to prove that the solutions would still have nice behavior, a posteriori. The key point turned out to be proving the local Hölder continuity of the weak solutions which would then yield the higher regularity through a bootstrap argument.

The problem was finally settled in 1950's when Ennio De Giorgi [1.4.1] and John Nash [61] proved the continuity of weak solutions independently. Their work then inspired a lot of other research in the field. In 1960's Jürgen Moser used an iteration method for showing that subsolutions are locally bounded and, moreover, he proved the Harnack inequality for the weak solutions [56]. This in turn, provided a new way to obtain the continuity result of De Giorgi and Nash.

Let us now consider the parabolic analogues of equation (1.4.4), that is, with $\Omega_T = \Omega \times (0, T],\ 0 < T < \infty$

$$u_t - \operatorname{div} A(x, u, \nabla u) = 0 \quad \text{in} \quad \Omega_T, \tag{1.4.5}$$

with A satisfying structure assumptions analogous to (1.4.4). Moser [57] proved that weak solutions of (1.4.5) are locally Hölder continuous in Ω_T . Since the linearity is immaterial to the proof, one might expect, as in the elliptic case, an extension of these results to quasi-linear equations of the type (1.4.5). Surprisingly though, the methods of De Giorgi and Moser could not be extended. Ladyzhenskaya et als. [50] proved that solutions of (1.4.5) are Hölder continuous, provided the principal part has exactly a linear growth with respect to ∇u. Analogous results were established by Kruzkov [48, 49] and by Nash [61] using entirely different methods. Thus it appears that unlike the elliptic case, the degeneracy or singularity of the principal part plays a peculiar role, and for example, for the parabolic p-Laplace equation

$$u_t - \operatorname{div} |\nabla u|^{p-2} \nabla u = 0,$$

one could not establish whether a solution is locally Hölder continuous.

The issue remained open until the mid 1980's when DiBenedetto [32] showed that the solutions of general quasilinear equations of the type of (1.4.5) are locally Hölder continuous for $p > 2$. In the early 1990's, the theory was extended [26] to include also the case $1 < p < 2$. Surprisingly, the same techniques could be suitably modified to establish the local Hölder continuity of any local solution of quasilinear porous medium-type equations. These modified methods, in turn, were crucial in proving that weak solutions of the p-Laplace equation are of class $C^{1,\alpha}_{loc}$. Nevertheless, the problem remained open for equations with more general growth conditions.

These results follow, one way or another, from the single unifying idea of intrinsic scaling: the diffusion processes in the equations evolve in a time scale determined instant by instant by the solution itself, so that, loosely speaking, they can be regarded as the heat equation in their own intrinsic time configuration. A precise description of this fact, as well as its effectiveness, is linked to its technical implementation, which we will develop in this thesis.

The building blocks of the method of intrinsic scaling are a priori estimates for the weak solutions of the equation. Actually, there is more to it than that. Once these estimates are obtained, we can forget the equation and the problem becomes, purely, a problem in analysis: showing that functions that satisfy certain integral inequalities belong to a certain regularity class (e.g., are locally Hölder continuous). It does not really matter if these functions are solutions of an equation or extremals of a functional in the Calculus of Variations or neither of those; what counts is that they satisfy the integral estimates. These integral inequalities on level sets measure the behavior of the function near its infimum and its supremum in the interior of the cylinder. In the case of solutions of degenerate or singular equations, these estimates are not homogeneous since they involve integral norms corresponding to different powers. This lack of homogeneity precludes the use of certain functional inclusions because the appropriate norms are not disclosed in the analysis. Through intrinsic scaling, we are able to recover the homogeneity in the estimates, once we rewrite them over the intrinsically rescaled cylinders. The difficulty in the analysis is thus absorbed by the geometry.

After this short view on the history of the regularity theory, we give a short overview what we plan in this thesis. In Chapter I, we recall some definitions and basic proprieties of the generalized Lebesgue-Sobolev spaces. In chapter II, we give the precise definition of weak solution for a generalized isotropic parabolic equation with variable exponents, and we derive local energy and logarithmic estimates, and then we deal with the construction of the appropriate geometric setting, and finally we conclude with the proof of the Hölder continuity, after the analysis of an alternative aimed at reducing the oscillation of the solution. In chapter III, we generalize what we study in the previous chapter to an anisotropic parabolic equation with variable exponents.

Chapter 2

Hölder regularity for degenerate parabolic equations with variable exponents

2.1 Introduction

Partial differential equations with nonlinearities involving variable exponents have attracted an increasing amount of attention in recent years. The development, mainly by Ruzicka [66,67], of a theory modeling the behavior of electrorheological fluids, an important class of non-Newtonian fluids, seems to have boosted a still far from completed effort to study and understand this type of equations. Other important applications relate to image processing [25], elasticity [73] or flows in porous media [7,9].

We will consider the parabolic equation in divergence form

$$u_t - \mathrm{div} A(x,t,u,\nabla u) = B(x,t,u,\nabla u) \text{ in } \Omega_T, \tag{2.1.1}$$

where $\Omega_T \equiv \Omega \times (0,T]$, Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 2$ and $0 < T < +\infty$. The functions $A : \Omega_T \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^N$ and $B : \Omega_T \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^N$ are assumed to be measurable and satisfying the following structure conditions

$$|A(x,t,u,\nabla u)| \leq C_1 \left(\phi(x,t) + |u|^{p(x,t)-1} + |\nabla u|^{p(x,t)-1} \right), \tag{2.1.2}$$

$$|B(x,t,u,\nabla u)| \leq C_2 \left(\phi(x,t) + |u|^{p(x,t)-1} + |\nabla u|^{p(x,t)-1} \right), \tag{2.1.3}$$

$$A(x,t,u,\nabla u)\nabla u \geq C_3 |\nabla u|^{p(x,t)}, \tag{2.1.4}$$

where $\phi(x,t) \in L^\infty(\Omega_T)$ and C_1, C_2, C_3 are positive constant. Throughout this chapter, we will always suppose that $p(x,t)$ is a measurable function such that

$$2 < p^- = \inf_{\Omega} p(x,t) \leq p(x,t) \leq \sup_{\Omega} p(x,t) = p^+ < +\infty. \tag{2.1.5}$$

Moreover, we assume that p satisfies the following log-continuity condition:

$$|p(x,t)-p(y,\tau)| \leq \frac{c_0}{\ln \frac{1}{|t-\tau|+|x-y|}} \quad \text{for any} \;\; (x,t),\; (y,\tau) \in \Omega_T,\; |t-\tau|+|x-y| \leq \frac{1}{2}. \quad (2.1.6)$$

Results on the existence and uniqueness of weak solutions of (2.1.1), together with some localization properties, were obtained by P. Wittbold, A. Zimmermann [72], C. Zhang, S. Zhou [74], and recently S. Ouaro, A. Ouedraogo [64].

Our aim here is to obtain a local regularity result for local weak solutions of (2.1.1). In order to achieve this goal, and since the equation is degenerate (in fact, the diffusion coefficient vanishes when $|\nabla u| = 0$), the idea is to study the equation within a geometry that takes this feature into consideration.

The building blocks of Dibenedetto's intrinsic scaling is to show that the continuity of the solution at a point follows from measuring its oscillation in a sequence of nested and shrinking cylinders, with vertex at that point, and showing that the oscillation converge to zero as the cylinders shrink to the point. To fully understand the technical procedure, based on the study of an alternative argument which makes use of energy and logarithmic estimates, one has not only to be familiar with Dibenedetto's technique (see [29,30,70]) but also to overcome the difficulty of having a $(x,t)-$dependence on the exponent p.

2.2 Preliminary and main results

2.2.1 Mollification in time

Since weak solutions of parabolic equations, respectively inequalities possess only weak regularity properties with respect to time, it is in principle not possible to use the solution itself as a test-function in the weak formulation of the problem. In order to be nevertheless able to test properly, there are several possibilities to smooth the solution with respect to the time direction. To overcome these faculties, we consider the Friedrichs mollifier as was done in [6]. Indeed, taking the kernel

$$\rho \geq 0, \;\; \rho \in C_0^{\infty}(\mathbb{R}^N), \;\; \rho(x) \equiv 0 \quad \text{for} \;\; |x| \geq 1, \quad \int_{\mathbb{R}^N} \rho(x)\; dx = 1,$$

we introduce regularization of $f \in L^{p(x,t)}_{loc}(\Omega_T)$ by

$$I^h f = f_h(x,t) = h^{-1} \int_t^{t+h} \int_{|x-y|\leq h} f(y,\tau)\rho_h(x-y)\; dy d\tau, \;\; \rho_h(x) = h^{-N}\rho(h^{-1}x), \quad (2.2.1)$$

and consider these inside the cylinder Ω_T, i.e., in cylinders $\Omega'_T = \Omega' \times (T_1, T_2)$, where $\Omega' \subset \Omega,\; 0 < T_1 < T_2 < T$. The basic property of the mollification, which can be retrieved from [6, Lemma 2.1], is summarized in the following:

Lemma 2.2.1. *If the exponent p satisfies the condition (3.0.2), then $f_h \longrightarrow f$ in $L^{p(x,t)}_{loc}(\Omega_T)$ as $h \longrightarrow 0$, for any $f \in L^{p(x,t)}_{loc}(\Omega_T)$.*

2.2.2 Regularity result

The weak solutions of problem (2.1.1) is understood in the following way.

Definition 2.2.2. *A local weak solution of (2.1.1) is a measurable function $u(x,t)$ defined in Ω_T such that*

(i) $u \in L^\infty(0,T,L^\infty(\Omega))$ *with* $\nabla u \in L^{p(x,t)}(\Omega_T)$,

(ii) for every subset K of Ω and for every subinterval $[t_1,t_2]$ of $(0,T]$

$$\left[\int_K u\varphi\, dx\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \{-u\varphi_t + A(x,t,u,\nabla u).\nabla\varphi\} dxdt = \int_{t_1}^{t_2}\int_K B(x,t,u,\nabla u)\varphi dxdt, \tag{2.2.2}$$

for all locally bounded tested functions $\varphi \in C^1(0,T,C_0^\infty(\Omega))$.

We can write (ii) in a way that is technically more convenient and involves the discrete time derivative. This can be accomplished by using the Friedrichs mollifier of a function (see [6] for more details). Then, we get the following:

Lemma 2.2.3. *If u is a solution of equation (3.0.1) in the sense of Definition 3.1.2, then for every subset K of Ω, and for any $h < t_1 \le t_2 < T - h$, the following relation*

$$\int_{K\times\{t\}} \{u_{h,t}\varphi + [A(x,t,u,\nabla u)]_h .\nabla\varphi - [B(x,t,u,\nabla u)]_h\,\varphi\}dx = 0, \tag{2.2.3}$$

holds for all $\varphi \in C_0^\infty(K)$.

Consider a point $(x_0,t_0) \in \Omega_T$, by translation and to simplify assume $(x_0,t_0) = (0,0)$. Also, let $0 < R < 1$, be sufficiently small such that the cylinder

$$Q(R^2,R) = K_R \times (-R^2,0) := \{x : \max_{1\le i\le N} |x_i| < R\} \times (-R^2,0)$$

is a subset of Ω_T and define

$$\mu^+ = ess\sup_{Q(R^2,R)} u, \quad \mu^- = ess\inf_{Q(R^2,R)} u \quad \text{and} \quad \omega = ess\ osc_{Q(R^2,R)}\, u = \mu^+ - \mu^-.$$

Define the positive real number $a_0 = \left(\frac{\omega}{2^\lambda}\right)^{2-p^-}$ for some $\lambda > 1$ to be chosen later, and construct the cylinder

$$Q(a_0 R^{p^+}, R) = K_R \times (-a_0 R^{p^+}, 0).$$

Assuming that

$$R^{\frac{2-p^+}{2-p^-}} < \frac{\omega}{2^\lambda}, \tag{2.2.4}$$

consequently the inclusion $Q(a_0 R^{p^+}, R) \subset Q(R^2, R)$ holds, and so that

$$\underset{Q(a_0 R^{p^+}, R)}{ess\ osc}\ u \leq \omega.$$

Remark 2.2.4. *if (2.2.4) does not hold, then the essential oscillation ω goes to zero when the radius R goes to zero, and then there is nothing to prove.*

In order to begin our approach, inside $Q(a_0 R^{p^+}, R)$ consider subcylinders of small size constructed as follows

$$(0, t^*) + Q(\theta R^{p^+}, R), \quad \theta = \left(\frac{\omega}{2}\right)^{2-p^-}.$$

These are contained inside $Q(a_0 R^{p^+}, R)$ if

$$(2^{p^- -2} - 2^{\lambda(p^- -2)}) \frac{R^{p^+}}{\omega^{p^- -2}} < t^* < 0.$$

Now, given $\nu_0 \in (0, 1)$, to be determined in terms of the data, either

$$\left| \left\{ (x,t) \in (0, t^*) + Q(\theta R^{p^+}, R) : \ u(x,t) < \mu^- + \frac{\omega}{2} \right\} \right| \leq \nu_0 \left| Q(\theta R^{p^+}, R) \right| \tag{2.2.5}$$

or, nothing that $\mu^+ - \frac{\omega}{2} = \mu^- + \frac{\omega}{2}$

$$\begin{aligned} &\left| \left\{ (x,t) \in (0, t^*) + Q(\theta R^{p^+}, R) : \ u(x,t) > \mu^+ - \frac{\omega}{2} \right\} \right| \\ &\qquad \leq (1 - \nu_0) \left| Q(\theta R^{p^+}, R) \right|. \end{aligned} \tag{2.2.6}$$

The analysis of this alternative leads to the following result.

Proposition 2.2.5. *There exist positive constants ν_0, $\sigma \in (0,1)$ depending on the data, such that*

$$\underset{Q(\theta(\frac{R}{8})^{p^+}, \frac{R}{8})}{ess\ osc}\ u \leq \sigma \omega. \tag{2.2.7}$$

An immediate consequence we state the main result of this work.

Theorem 2.2.6. *Under assumptions (2.1.2)-(2.1.5), any local bounded weak solution of (2.1.1) is locally Hölder continuous.*

2.3 Local energy and logarithmic estimates

Let τ and ρ be so small that $Q(\tau,\rho)\subset\Omega_T$. Let ξ denote a piecewise smooth cutoff function in $Q(\tau,\rho)$ such that

$$\xi\in[0,1],\quad |\nabla\xi|<\infty \text{ and } \xi(x,t)=0 \text{ for } x \text{ outside } K_\rho.$$

Proposition 2.3.1. *Let u be a local weak solution of (2.1.1) in Ω_T. There exist positive constants C and C' such that, for every cylinder $Q(\tau,\rho)\subset\Omega_T$ and for every $k\in\mathbb{R}$*

$$\begin{aligned}
&\sup_{-\tau<t<0}\int_{K_\rho}(u-k)_\pm^2\,\xi^{p^+}(x,t)dx + C\int_{-\tau}^{t}\int_{K_\rho}|\nabla(u-k)_\pm|^{p^-}\,\xi^{p^+}dxdt\\
&\qquad\le \int_{K_\rho}(u-k)_\pm^2\,\xi^{p^+}(x,-\tau)dx + C'\Bigg[\int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^2\,\xi^{p^+-1}\xi_t\;dxdt\\
&\qquad+\int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^{p^+}\left(|\nabla\xi|^{p^+}+\xi^{p^+}\right)dxdt\\
&\qquad+\int_{-\tau}^{t}\int_{K_\rho}\chi\left((u-k)_\pm>0\right)dxdt\Bigg].
\end{aligned}\tag{2.3.1}$$

Proof. In the weak formulation (2.2.3) take the testing function $\varphi=\pm(u_h-k)_\pm\xi^{p^+}$, where

$$(u_h-k)_-=(k-u_h)_+=\max\{k-u,0\}.$$

Integrate over $(-\tau,t)$, $t\in(-\tau,0)$, estimating the various terms separately, we have first

$$\begin{aligned}
\int_{-\tau}^{t}\int_{K_\rho}u_{h,t}\varphi\;dxdt &= \int_{-\tau}^{t}\int_{K_\rho}u_{h,t}\left(\pm(u_h-k)_\pm\xi^{p^+}\right)\;dxdt\\
&\underset{h\longrightarrow 0}{\longrightarrow} -\frac{p^+}{2}\int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^2\,\xi^{p^+-1}(x,t)\xi_t\;dxdt\\
&+\frac{1}{2}\int_{K_\rho}(u-k)_\pm^2\,\xi^{p^+}(x,t)dx\\
&-\frac{1}{2}\int_{K_\rho}(u-k)_\pm^2\,\xi^{p^+}(x,-\tau)dx.
\end{aligned}\tag{2.3.2}$$

For the remaining terms, let $h\longrightarrow 0$ and then use the structure conditions $(2.1.2)-(2.1.5)$, then

$$
\begin{aligned}
&\pm \int_{-\tau}^{t}\int_{K_\rho} [A(x,t,u,\nabla u)]_h \, \nabla \left((u_h - k)_\pm \xi^{p^+} \right) dxdt \underset{h\longrightarrow 0}{\longrightarrow} \\
&\int_{-\tau}^{t}\int_{K_\rho} A(x,t,u,\nabla u) \left[\pm \nabla (u-k)_\pm \xi^{p^+} \pm p^+ (u-k)_\pm \xi^{p^+-1} \nabla \xi \right] dxdt \\
&\geq C\Bigg(\int_{-\tau}^{t}\int_{K_\rho} |\nabla (u-k)_\pm|^{p(x,t)} \, \xi^{p^+} \; dxdt \\
&- p^+ \Bigg[\int_{-\tau}^{t}\int_{K_\rho} \phi(x,t)(u-k)_\pm \xi^{p^+-1} |\nabla \xi| dxdt \\
&\quad + \int_{-\tau}^{t}\int_{K_\rho} |\nabla (u-k)_\pm|^{p(x,t)-1} \, (u-k)_\pm \xi^{p^+-1} |\nabla \xi| dxdt \\
&\quad + \int_{-\tau}^{t}\int_{K_\rho} (u-k)_\pm^{p(x,t)} \xi^{p^+-1} |\nabla \xi| dxdt \Bigg] \Bigg).
\end{aligned} \tag{2.3.3}
$$

By Young's inequality and using the fact that $0 \leq \xi \leq 1$ and $\frac{p(x,t)}{p(x,t)-1} \geq \frac{p^+}{p^+-1}$ imply that $\xi^{\frac{p(x,t)(p^+-1)}{p(x,t)-1}} \leq \xi^{p^+}$, we get

$$
\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho} |\nabla (u-k)_\pm|^{p(x,t)-1} (u-k)_\pm \xi^{p^+-1} |\nabla \xi| dxdt \\
&\qquad \leq \varepsilon \int_{-\tau}^{t}\int_{K_\rho} |\nabla (u-k)_\pm|^{p(x,t)} \, \xi^{p^+} dxdt \\
&\qquad + C(\varepsilon) \int_{-\tau}^{t}\int_{K_\rho} (u-k)_\pm^{p(x,t)} |\nabla \xi|^{p(x,t)} dxdt,
\end{aligned} \tag{2.3.4}
$$

$$
\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho} \phi(x,t)(u-k)_\pm \xi^{p^+-1} |\nabla \xi| dxdt \\
&\qquad \leq \varepsilon' \int_{-\tau}^{t}\int_{K_\rho} (u-k)_\pm^{p^+} |\nabla \xi|^{p^+} dxdt \\
&\qquad + C(\varepsilon') \int_{-\tau}^{t}\int_{K_\rho} |\phi(x,t)|^{\frac{p^+}{p^+-1}} \, \xi^{p^+} \chi\left((u-k)_\pm > 0 \right) dxdt
\end{aligned} \tag{2.3.5}
$$

and

$$\int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^{p(x)}\xi^{p^+-1}|\nabla\xi|dxdt$$
$$\leq \varepsilon" \int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^{p(x)}|\nabla\xi|^{p(x,t)}dxdt \qquad (2.3.6)$$
$$+C(\varepsilon")\int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^{p(x,t)}\xi^{p^+}dxdt,$$

where χ_E denotes the characteristic function of the set E and ε, ε' and $\varepsilon"$ are positive constants. combining this in (2.3.3) we arrive at

$$\int_{-\tau}^{t}\int_{K_\rho}A(x,t,u,\nabla u)\left[\pm\nabla(u-k)_\pm\xi^{p^+}\pm p^+(u-k)_\pm\xi^{p^+-1}\nabla\xi\right]dxdt$$
$$\geq C_1\Bigg(\int_{-\tau}^{t}\int_{K_\rho}|\nabla(u-k)_\pm|^{p(x,t)}\,\xi^{p^+}\;dxdt$$
$$-\int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^{p^+}|\nabla\xi|^{p^+}dxdt \qquad (2.3.7)$$
$$-\int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^{p(x,t)}\left(|\nabla\xi|^{p(x,t)}+\xi^{p^+}\right)dxdt$$
$$-\int_{-\tau}^{t}\int_{K_\rho}|\phi(x,t)|^{\frac{p^+}{p^+-1}}\,\xi^{p^+}\chi\left((u-k)_\pm>0\right)dxdt\Bigg).$$

By the same method, the last term of (2.2.3) becomes

$$\int_{-\tau}^{t}\int_{K_\rho}\left|B(x,t,u,\nabla u)(u-k)_\pm\xi^{p^+}\right|dxdt$$
$$\leq C_2\Bigg(\int_{-\tau}^{t}\int_{K_\rho}|\nabla(u-k)_\pm|^{p(x,t)}\,\xi^{p^+}\;dxdt+\int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^{p^+}\xi^{p^+}dxdt$$
$$+\int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^{p(x,t)}\xi^{p^+}dxdt \qquad (2.3.8)$$
$$+\int_{-\tau}^{t}\int_{K_\rho}|\phi(x,t)|^{\frac{p^+}{p^+-1}}\,\xi^{p^+}\chi\left((u-k)_\pm>0\right)dxdt\Bigg),$$

where $0<C_2<C_1$.

Using Young's inequality once again we obtain

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^{p(x,t)}\left(|\nabla\xi|^{p(x,t)}+\xi^{p^+}\right)dxdt\\
&\qquad\leq C\Bigg(\int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^{p^+}\left(|\nabla\xi|^{p^+}+\xi^{p^+}\right)dxdt \qquad (2.3.9)\\
&\qquad+\int_{-\tau}^{t}\int_{K_\rho}(1+\xi^{p^+})\chi\left((u-k)_\pm>0\right)dxdt\Bigg),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho}|\nabla(u-k)_\pm|^{p^-}\xi^{p^+}\ dxdt\\
&\qquad\leq C\Bigg(\int_{-\tau}^{t}\int_{K_\rho}|\nabla(u-k)_\pm|^{p(x,t)}\,\xi^{p^+}dxdt \qquad (2.3.10)\\
&\qquad+\int_{-\tau}^{t}\int_{K_\rho}\chi\left((u-k)_\pm>0\right)\xi^{p^+}dxdt\Bigg).
\end{aligned}$$

Hence, by recalling that $\phi\in L^\infty(\Omega_T)$ and putting (2.3.7), (2.3.8), (2.3.9) and (2.3.10) into (2.2.3), we get the desired result. □

Now, introduce the logarithmic function

$$\psi^\pm(u)=\psi\left(H_k^\pm,(u-k)_\pm,c\right)=\left(\ln\left(\frac{H_k^\pm}{H_k^\pm-(u-k)_\pm+c}\right)\right)_+,$$

where $H_k^\pm = ess \sup_{Q(\tau,\rho)}|(u-k)_\pm|$ and $0<c<H_k^\pm$. To avoid the value zero of $\psi^\pm$ we will take our estimates in a smaller sets in K_R where $\psi^\pm$ is a positive function (see sets S_1 in the proof of Lemma 2.4.2 and S_2 in the proof of Lemma 2.4.7). In the cylinder $Q(\tau,\rho)$, we take a cutoff function satisfying $\xi\in[0,1]$, $|\nabla\xi|<\infty$ and ξ is independent of $t\in(-\tau,0)$.

Proposition 2.3.2. *Let u be local weak solution of (2.1.1) in Ω_T. There exists a positive* *constant C such that for every cylinder $Q(\tau,\rho)\in\Omega_T$ and for every level $k\in\mathbb{R}$,*

$$
\begin{aligned}
&\underset{-\tau<t<0}{ess\sup}\int_{K_\rho}\left[\psi^{\pm}(u)\right]^2\xi^{p^+}dx\\
&\qquad\le\int_{K_\rho\times\{-\tau\}}\left[\psi^{\pm}(u)\right]^2\xi^{p^+}dx\\
&\qquad+C\Bigg(\int_{-\tau}^{0}\int_{K_\rho}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^2\xi^{p^+}dxdt\\
&\qquad+\int_{-\tau}^{0}\int_{K_\rho}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^{2-p^-}\left(|\nabla u|^{p^+}+1+\xi^{p^+}\right)dxdt\\
&\qquad+\int_{-\tau}^{0}\int_{K_\rho}\psi^{\pm}(u)\left(|\nabla u|^{p^+}+1+\xi^{p^+}\right)dxdt\\
&\qquad+\int_{-\tau}^{0}\int_{K_\rho}|u|^{p^+}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^2\xi^{p^+}dxdt\Bigg).
\end{aligned}
\tag{2.3.11}
$$

Proof. In (2.2.3) we take the testing function $\varphi=2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u)\right]\xi^{p^+}$. By direct computation we get

$$\left(\psi^{\pm}(u)\right)''=\left\{\left(\psi^{\pm}(u)\right)'\right\}^2.$$

Therefore by integrating in time over $(-\tau,t)$ for $t\in(-\tau,0)$, estimate the various terms separately. The first term gives

$$
\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho}u_{h,t}\left\{2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u_h)\right]\xi^{p^+}\right\}dxdt\\
&\qquad=\int_{-\tau}^{t}\int_{K_\rho}\left(\psi^{\pm}(u_h)^2\right)_t\xi^{p^+}dxdt\\
&\qquad\underset{h\longrightarrow 0}{\longrightarrow}\int_{K_\rho\times\{t\}}\left[\psi^{\pm}(u)\right]^2\xi^{p^+}dx-\int_{K_\rho\times\{-\tau\}}\left[\psi^{\pm}(u)\right]^2\xi^{p^+}dx.
\end{aligned}
\tag{2.3.12}
$$

For the remaining term, when $h \longrightarrow 0$, we obtain

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho} A(x,t,u,\nabla u).\nabla\left(2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u)\right]\xi^{p^+}\right)dxdt \\
&\quad \geq C\Bigg(\int_{-\tau}^{t}\int_{K_\rho}|\nabla u|^{p(x,t)}\left(1+\psi^{\pm}(u)\right)\left(\left(\psi^{\pm}\right)'(u)\right)^2\xi^{p^+}dxdt \\
&\quad -\int_{-\tau}^{t}\int_{K_\rho}\phi(x,t)\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]\xi^{p^+-1}|\nabla\xi|dxdt \\
&\quad -\int_{-\tau}^{t}\int_{K_\rho}|u|^{p(x,t)}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]\xi^{p^+-1}|\nabla\xi|dxdt \\
&\quad -\int_{-\tau}^{t}\int_{K_\rho}|\nabla u|^{p(x,t)-1}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]\xi^{p^+-1}|\nabla\xi|dxdt\Bigg).
\end{aligned} \tag{2.3.13}$$

Therefore, by the same method we used in Proposition 2.3.1 and the fact that $\phi \in L^{\infty}(\Omega_T)$ we obtain that

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho} A(x,t,u,\nabla u).\nabla\left(2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u)\right]\xi^{p^+}\right)dxdt \\
&\quad \geq C_1\Bigg(\int_{-\tau}^{t}\int_{K_\rho}|\nabla u|^{p(x,t)}\left(1+\psi^{\pm}(u)\right)\left(\left(\psi^{\pm}\right)'(u)\right)^2\xi^{p^+}dxdt \\
&\quad -\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^2\xi^{p^+}dxdt \\
&\quad -\int_{-\tau}^{t}\int_{K_\rho}|u|^{p^+}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^2\xi^{p^+}dxdt \\
&\quad -\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^{2-p^-}\left(|\nabla\xi|^{p^+}+1\right)dxdt \\
&\quad -\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\left(|\nabla\xi|^{p^+}+1\right)dxdt\Bigg),
\end{aligned} \tag{2.3.14}$$

and

$$
\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho}\left|B(x,t,u,\nabla u)\left(2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u)\right]\xi^{p^+}\right)\right|dxdt\\
&\quad\le C_2\bigg(\int_{-\tau}^{t}\int_{K_\rho}|\nabla u|^{p(x,t)}\psi^{\pm}(u)\left[\left(\psi^{\pm}\right)'(u)\right]^2\xi^{p^+}dxdt\\
&\quad+\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^2\xi^{p^+}dxdt\\
&\quad+\int_{-\tau}^{t}\int_{K_\rho}|u|^{p^+}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^2\xi^{p^+}dxdt\\
&\quad+\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^{2-p^-}\xi^{p^+}dxdt+\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\xi^{p^+}dxdt\bigg),
\end{aligned}
\tag{2.3.15}
$$

where $0 < C_2 < C_1$. Hence, putting (2.3.14) and (2.3.15) into (2.2.3) we get the desired result. □

2.4 Continuity of the weak solutions

In this section we analyze the alternative and prove proposition 2.2.5. By assuming that (2.2.5) is verified, the following Lemma determine the number ν_0 and guarantee that the solution u is above a smaller level within a smaller cylinder.

Lemma 2.4.1. *There exists $\nu_0 \in (0,1)$ depending on the data, such that if (2.2.5) holds true then*

$$
u(x,t) > \mu^- + \frac{\omega}{4} \qquad a.e.\ in\ (0,t^*) + Q\left(\theta\left(\frac{R}{2}\right)^{p^+}, \frac{R}{2}\right). \tag{2.4.1}
$$

Proof. Up to translation we may assume that $(0,t^*) = (0,0)$. Define two decreasing sequences of positive numbers

$$
R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \quad k_n = \mu^- + \frac{\omega}{4} + \frac{\omega}{2^{n+2}}, \quad n = 0,1,...,
$$

construct the family of nested and shrinking cylinders $Q(\theta R_n^{p^+}, R_n)$, and let $0 \le \xi_n(x,t) \le 1$ be piecewise smooth functions in $Q(\theta R_n^{p^+}, R_n)$ such that

$$
\begin{cases}
\xi_n = 1 \quad \text{in } Q(\theta R_{n+1}^{p^+}, R_{n+1}), \quad \xi_n = 0 \quad \text{on } \partial_p Q(\theta R_n^{p^+}, R_n),\\
|\nabla \xi_n| \le \frac{2^{n+1}}{R}, \qquad 0 < (\xi_n)_t \le \frac{2^{p^+(n+1)}}{\theta R^{p^+}}.
\end{cases}
$$

Now, by using the energy inequality (2.3.1) for the functions $(u-k_n)_-$ we get

$$
\begin{aligned}
&\sup_{-\theta R_n^{p^+}<t<0} \int_{K_{R_n}} (u-k_n)_-^2 \, \xi_n^{p^+}(x,t)dx \\
&+ \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} |\nabla (u-k_n)_-|^{p^-} \, \xi_n^{p^+} dxdt \\
&\qquad \le C\Big[\int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^2 \, \xi_n^{p^+-1}(\xi_n)_t \, dxdt \\
&\qquad + \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \chi\left((u-k_n)_- > 0\right) dxdt \\
&\qquad\quad + \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^{p^+} \left(|\nabla \xi_n|^{p^+} + \xi_n^{p^+}\right) dxdt\Big] \\
&\qquad \le C\frac{2^{p^+(n+1)}}{R^{p^+}}\bigg(\frac{1}{\theta}\int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^2 \, dxdt \\
&\qquad + \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^{p^+} \, dxdt \\
&\qquad\quad + \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \chi\left((u-k_n)_- > 0\right) dxdt\bigg).
\end{aligned}
\tag{2.4.2}
$$

Using the fact that $(u-k_n)_- = 0$ or

$$
(u-k_n)_- = (\mu^- - u) + \frac{\omega}{4} + \frac{\omega}{2^{n+2}} \le \frac{\omega}{2}, \tag{2.4.3}
$$

we get

$$
(u-k_n)_-^2 \ge \theta (u-k_n)_-^{p^-}. \tag{2.4.4}
$$

Then the above estimates read

$$
\begin{aligned}
&\sup_{-\theta R_n^{p^+}<t<0} \int_{K_{R_n}} (u-k_n)_-^{p^-} \, \xi_n^{p^+}(x,t)dx \\
&+ \frac{1}{\theta}\int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} |\nabla (u-k_n)_-|^{p^-} \, \xi_n^{p^+} dxdt \\
&\le C\frac{2^{p^+(n+1)}}{\theta R^{p^+}}\left\{\left(\frac{\omega}{2}\right)^{p^+} + 1\right\}\left(\int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \chi\left((u-k_n)_- > 0\right) dxdt\right) \\
&\le C\frac{2^{np^+}}{\theta R^{p^+}}\left(\frac{\omega}{2}\right)^{p^+}\left(\int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \chi\left((u-k_n)_- > 0\right) dxdt\right).
\end{aligned}
\tag{2.4.5}
$$

Here we used Young's inequality for the right terms.

Let us now consider the change of variables $\overline{t} = \frac{t}{\theta}$ and define the functions

$$\overline{u}(.,\overline{t}) = u(.,t), \quad \overline{\xi_n}(.,\overline{t}) = \xi_n(.,t).$$

Then, for

$$A_n = \int_{-R_n^{p^+}}^{0} \int_{K_{R_n}} \chi\left((\overline{u} - k_n)_- > 0\right) dx d\overline{t},$$

the inequality (2.4.5) becomes

$$\begin{aligned} &\sup_{-R_n^{p^+} < t < 0} \int_{K_{R_n}} (\bar{u} - k_n)_-^{p^-} \bar{\xi_n}^{p^+}(x,t) dx \\ &\qquad + \int_{-R_n^{p^+}}^{0} \int_{K_{R_n}} |\nabla(\bar{u} - k_n)_-|^{p^-} \bar{\xi_n}^{p^+} dx d\bar{t} \leq C \frac{2^{np^+}}{R^{p^+}} \left(\frac{\omega}{2}\right)^{p^+} A_n. \end{aligned} \tag{2.4.6}$$

Next, by using Hölder's inequality, Theorem 1.4.6, we get

$$\begin{aligned} &\int_{-R_n^{p^+}}^{0} \int_{K_{R_n}} (\bar{u} - k_n)_-^{p^-} \bar{\xi_n}^{p^+} dx d\bar{t} \\ &\quad \leq \left\{ \int_{-R_n^{p^+}}^{0} \int_{K_{R_n}} \left[(\bar{u} - k_n)_- \bar{\xi_n}^{\frac{p^+}{p^-}} \right]^{\frac{p^-(N+p^-)}{N}} dx d\bar{t} \right\}^{\frac{N}{N+p^-}} A_n^{\frac{p^-}{N+p^-}} \\ &\quad \leq C \left[\sup_{-R_n^{p^+} < \bar{t} < 0} \int_{K_{R_n}} (\bar{u} - k_n)_-^{p^-} \bar{\xi_n}^{p^+}(x,\bar{t}) dx \right]^{\frac{p^-}{N+p^-}} \\ &\quad \times \left[\int_{-R_n^{p^+}}^{0} \int_{K_{R_n}} |\nabla(\bar{u} - k_n)_-|^{p^-} \bar{\xi_n}^{p^+} dx d\bar{t} \right. \\ &\quad \left. + \int_{-R_n^{p^+}}^{0} \int_{K_{R_n}} (\bar{u} - k_n)_-^{p^-} \left|\nabla \bar{\xi_n}\right|^{p^-} dx d\bar{t} \right]^{\frac{N}{N+p^-}} A_n^{\frac{p^-}{N+p^-}} \\ &\quad \leq C \frac{2^{np^+}}{R^{p^+}} \left(\frac{\omega}{2}\right)^{p^+} A_n^{1+\frac{p^-}{N+p^-}}. \end{aligned} \tag{2.4.7}$$

On the other hand

$$\begin{aligned} \int_{-R_n^{p^+}}^{0} \int_{K_{R_n}} (\bar{u} - k_n)_-^{p^-} \bar{\xi_n}^{p^+} dx d\bar{t} &\geq \int_{-R_{n+1}^{p^+}}^{0} \int_{K_{R_{n+1}}} (\bar{u} - k_n)_-^{p^-} dx d\bar{t} \\ &\geq |k_n - k_{n+1}|^{p^-} A_{n+1} \\ &= \frac{1}{2^{p^+(n+2)}} \left(\frac{\omega}{2}\right)^{p^+} A_{n+1}. \end{aligned} \tag{2.4.8}$$

Combining (2.4.7) and (2.4.8), we get that

$$A_{n+1} \leq C \frac{4^{np^+}}{R^{p^+}} A_n^{1+\frac{p^-}{N+p^-}}. \tag{2.4.9}$$

An direct computation leads to

$$\frac{\left|Q(R_n^{p^+}, R_n)\right|^{1+\frac{p^-}{N+p^-}}}{\left|Q(R_{n+1}^{p^+}, R_{n+1})\right|} \leq 2^{P^+ + N} R^{\frac{p^-(p^+ + N)}{N+p^-}}. \tag{2.4.10}$$

Next, define the numbers

$$X_n = \frac{A_n}{Q(R_n^{p^+}, R_n)},$$

dividing (2.4.9) by $Q(R_{n+1}^{p^+}, R_{n+1})$ and using (2.4.10), we obtain the following recursive relation

$$X_{n+1} \leq C 4^{np^+} X_n^{1+\frac{p^-}{N+p^-}}.$$

Therefore, Lemma 1.4.3 implies that if

$$X_0 \leq C^{-\frac{N+p^-}{p^-}} 4^{-p^+\left(\frac{p^- + N}{p^-}\right)^2} = \nu_0, \tag{2.4.11}$$

then

$$X_n \longrightarrow 0. \tag{2.4.12}$$

However, (2.4.11) is nothing but the assumption (2.2.5). Hence, the result easily follows from (2.4.12). □

Now consider the time level $-\hat{t} = t^* - \theta\left(\frac{R}{2}\right)^{p^+}$. From the conclusion of Lemma 2.4.1, we have

$$u(x, -\hat{t}) > \mu^- + \frac{\omega}{4} \quad \text{a.e. in } x \in K_{\frac{R}{2}},$$

we will use this time level as an initial condition to bring the information up to $t = 0$, and therefore to obtain an analogous inequality in a smaller cylinder. A first step in this direction is given by the following result.

Lemma 2.4.2. *For every $\nu_1 \in (0, 1)$, there exists a positive integer s_1 depending on the data, such that*

$$\left|x \in K_{\frac{R}{4}}, \quad u(x,t) < \mu^- + \frac{\omega}{2^{s_1}}\right| \leq \nu_1 |K_{\frac{R}{4}}|, \quad \forall t \in (-\hat{t}, 0). \tag{2.4.13}$$

Proof. Consider the cylinder $Q(\hat{t}, \frac{R}{2})$ and write the logarithmic estimate (2.3.11) over this cylinder for the function $(u-k)_-$ with

$$k = \mu^- + \frac{\omega}{4} \quad \text{and } c = \frac{\omega}{2^{n+2}},$$

where n is to be chosen later. Defining

$$k - u \le H_k^- = ess \sup_{Q(\hat{t}, \frac{R}{2})} \left| \left(u - \mu^- - \frac{\omega}{4} \right)_- \right| \le \frac{\omega}{4} \tag{2.4.14}$$

Assuming $H_k^- \le \frac{\omega}{8}$ (if not the result is trivial). Then the logarithmic function ψ^- is well defined and satisfies the inequalities

$$\psi^- \le n \ln(2), \qquad \text{since } \frac{H_k^-}{H_k^- + u - k + c} \le \frac{\frac{\omega}{4}}{c} = 2^n, \tag{2.4.15}$$

and for $u \ne -k + c$,

$$0 \le (\psi^-)' \le \frac{1}{H_k^- + u - k + c} \le \frac{1}{c}, \tag{2.4.16}$$

and

$$\left| \left(\psi^-\right)'(u) \right|^{2-p^-} = \left(H_k^- + u - k + c\right)^{p^- - 2} \le \left(\frac{\omega}{2}\right)^{p^- - 2}. \tag{2.4.17}$$

For $t = -\hat{t}$, by virtue of Lemma 2.4.1 we have $u(x, -\hat{t}) > k$, and therefore

$$\left[\psi^-(u)\right](x, -\hat{t}) = 0 \quad \text{for } x \in K_{\frac{R}{2}}.$$

Now, choose a cutoff function $0 < \xi(x) \le 1$, defined on $K_{\frac{R}{2}}$ such that

$$\xi = 1 \quad \text{in } K_{\frac{R}{4}} \quad \text{and} \quad |\nabla \xi| \le \frac{8}{R}.$$

From Definition 2.2.2, we know that if u is a weak solution of (2.1.1), then there exists a positive constant M such that

$$ess \sup_{\Omega_T} u \le M. \tag{2.4.18}$$

Gathering these estimates, and using the fact that

$$\hat{t} \le \left(\frac{\omega}{2^\lambda}\right)^{2-p^-} R^{p^+}, \tag{2.4.19}$$

we arrive at

$$
\begin{aligned}
&\underset{-\hat{t}<t<0}{ess\sup}\int_{K_{\frac{R}{2}}\times\{t\}}\left[\psi^-(u)\right]^2\xi^{p^+}dx\\
&\leq C\Bigg(\int_{-\hat{t}}^{0}\int_{K_{\frac{R}{2}}}\psi^-(u)\left[(\psi^-)'(u)\right]^2\xi^{p^+}dxdt\\
&\quad+\int_{-\hat{t}}^{0}\int_{K_{\frac{R}{2}}}\psi^-(u)\left[(\psi^-)'(u)\right]^{2-p^-}\left(|\nabla u|^{p^+}+1+\xi^{p^+}\right)dxdt\\
&\quad+\int_{-\hat{t}}^{0}\int_{K_{\frac{R}{2}}}\psi^-(u)\left(|\nabla u|^{p^+}+1+\xi^{p^+}\right)dxdt\\
&\quad+\int_{-\hat{t}}^{0}\int_{K_{\frac{R}{2}}}|u|^{p^+}\psi^-(u)\left[(\psi^-)'(u)\right]^2\xi^{p^+}dxdt\Bigg)\\
&\leq C\Bigg(n\ln(2)\left(\frac{\omega}{2^{n+2}}\right)^{-2}\left(\frac{\omega}{2^{\lambda}}\right)^{2-p^-}R^{p^+}\\
&+n\ln(2)\left(\frac{8}{R}\right)^{p^+}\left(\frac{\omega}{2^{n+2}}\right)^{p^--2}\left(\frac{\omega}{2^{\lambda}}\right)^{2-p^-}R^{p^+}\\
&\quad+n\ln(2)\left(\frac{8}{R}\right)^{p^+}\left(\frac{\omega}{2^{\lambda}}\right)^{2-p^-}R^{p^+}\\
&+M^{p^+}n\ln(2)\left(\frac{\omega}{2^{n+2}}\right)^{-2}\left(\frac{\omega}{2^{\lambda}}\right)^{2-p^-}R^{p^+}\Bigg)\left|K_{\frac{R}{4}}\right|.
\end{aligned}
\tag{2.4.20}
$$

Now, by virtue of Remark 2.2.4, we can estimate that

$$\left(\frac{\omega}{2^{n+2}}\right)^{-2}\omega^{2-p^-}R^{p^+}\leq 1\ \text{ and } \omega^{2-p^-}R^{p^+}\leq 1.$$

Consequently, we obtain

$$\underset{-\hat{t}<t<0}{ess\sup}\int_{K_{\frac{R}{2}}\times\{t\}}\left[\psi^-(u)\right]^2\xi^{p^+}dx\leq Cn2^{\lambda(p^--2)}\left|K_{\frac{R}{4}}\right|. \tag{2.4.21}$$

The left hand side of (2.4.20) is estimated from below considering integration over the smaller set

$$S_1=\left\{x\in K_{\frac{R}{4}},\ u(x,t)<\mu^-+\frac{\omega}{2^{n+2}}\right\}\subset K_{\frac{R}{2}},\quad t\in(-\hat{t},0).$$

On such a set

$$\left[\psi^-(u)\right]^2\geq\ln^2\left\{\frac{H_k^-}{H_k^-+u-k+c}\right\}\geq\ln^2\left\{\frac{\frac{\omega}{4}}{c}\right\}=(n-1)^2\ln^2(2). \tag{2.4.22}$$

Putting this into (2.4.21) gives that for all $t \in (-\hat{t}, 0)$

$$\left|\left\{x \in K_{\frac{R}{4}},\ u(x,t) < \mu^- + \frac{\omega}{2^{n+2}}\right\}\right| \leq C\frac{n}{(n-1)^2} 2^{\lambda(p^- - 2)} \left|K_{\frac{R}{4}}\right|.$$

The proof is complete once we choose $s_1 = n+2$ with $n > 1 + \frac{2C}{\nu_1} 2^{\lambda(p^- - 2)}$. □

The conclusion of Lemma 2.4.2 will be employed to deduce that, within the cylinder $Q\left(\hat{t}, \frac{R}{8}\right)$, the set where u is away from its infimum is arbitrarily small.

Lemma 2.4.3. *There exists* $1 < s_2 \in \mathbb{N}$*, depending on the data, such that*

$$u(x,t) > \mu^- + \frac{\omega}{2^{s_2+1}} \qquad a.e.\ (x,t) \in Q\left(\hat{t}, \frac{R}{8}\right). \tag{2.4.23}$$

Proof. Define two decreasing sequences of positive numbers

$$R_n = \frac{R}{8} + \frac{R}{2^{n+1}}, \quad k_n = \mu^- + \frac{\omega}{2^{s_2+1}} + \frac{\omega}{2^{s_2+1+n}}, \quad n = 0, 1, ...$$

Construct the family of nested and shrinking cylinders $Q(\theta R_n^{p^+}, R_n)$, and let $0 \leq \xi_n(x) \leq 1$ be a piecewise smooth function in K_{R_n} that equals one on $K_{R_{n+1}}$ and $|\nabla \xi_n| \leq \frac{2^{n+4}}{R}$. Lemma 2.4.2 implies that

$$(u - k_n)_-(x, -\hat{t}) = 0 \text{ in } K_{R_n}.$$

Now, since $(u - k_n)_- \leq \frac{\omega}{2^{s_2}}$, using (2.4.19) and letting $s_2 > \lambda + \frac{p^+}{p^- - 2}$ we get

$$(u - k_n)_-^2 \geq \frac{\hat{t}}{\left(\frac{R}{2}\right)^{p^+}} (u - k_n)_-^{p^-}.$$

Therefore, with these choices and by dividing the local energy estimates (2.3.1) for $(u - k_n)_-$ by $\dfrac{\hat{t}}{\left(\frac{R}{2}\right)^{p^+}}$ we get

$$\begin{aligned}
&\sup_{-\hat{t}<t<0} \int_{K_{R_n} \times \{t\}} (u - k_n)_-^{p^-} \xi_n^{p^+} dx \\
&\qquad + \frac{\left(\frac{R}{2}\right)^{p^+}}{\hat{t}} \int_{-\hat{t}}^0 \int_{K_{R_n}} |\nabla (u - k_n)_-|^{p^-} \xi_n^{p^+} dxdt \\
&\leq C\frac{2^{np^+}}{\hat{t}} \left(\int_{-\hat{t}}^0 \int_{K_{R_n}} (u - k_n)_-^{p^+} dxdt + \int_{-\hat{t}}^0 \int_{K_{R_n}} \chi\left((u - k_n)_- > 0\right) dxdt \right) \\
&\leq C\frac{2^{np^+}}{\hat{t}} \left(\frac{\omega}{2^{s_2}}\right)^{p^+} \int_{-\hat{t}}^0 \int_{K_{R_n}} \chi\left((u - k_n)_- > 0\right) dxdt.
\end{aligned} \tag{2.4.24}$$

Introducing the change of variables $\bar{t} = t\frac{\left(\frac{R}{2}\right)^{p^+}}{\hat{t}}$ and defining the new function $\bar{u}(x,\bar{t}) = u(x,t)$. Accordingly, by using the same argument we used in the proof of Lemma 2.4.1, we get

$$\frac{1}{2^{p^+(n+2)}}\left(\frac{\omega}{2^{s_2}}\right)^{p^+} A_{n+1} \leq C\frac{2^{np^+}}{\left(\frac{R}{2}\right)^{p^+}}\left(\frac{\omega}{2^{s_2}}\right)^{p^+} A_n^{1+\frac{p^-}{n+p^-}}, \tag{2.4.25}$$

where

$$A_n = \int_{-\left(\frac{R}{2}\right)^{p^+}}^{0}\int_{K_{R_n}} \chi\left((\bar{u}-k_n)_- > 0\right) dx d\bar{t}.$$

Next, define the numbers

$$X_n = \frac{A_n}{Q\left(\left(\frac{R}{2}\right)^{p^+}, R_n\right)},$$

dividing (2.4.25) by $Q\left(\left(\frac{R}{2}\right)^{p^+}, R_{n+1}\right)$, we obtain the following recursive relation

$$X_{n+1} \leq C4^{np^+} X_n^{1+\frac{p^-}{N+p^-}}.$$

Therefore, Lemma 1.4.3 implies that if

$$X_0 \leq C^{-\frac{N+p^-}{p^-}} 4^{-p^+\left(\frac{p^-+N}{p^-}\right)^2} = \nu_1, \tag{2.4.26}$$

then

$$X_n \longrightarrow 0. \tag{2.4.27}$$

By applying Lemma 2.4.2 with $s_1 := s_2$ we get easily (2.4.26). Hence, the result follows from (2.4.27). □

As an immediate consequence we get the reduction of the oscillation of u.

Corollary 2.4.4. *There exists a constant $\sigma_0 \in (0,1)$ depending only on the data, such that if (2.2.5) holds then*

$$\operatorname*{ess\ osc}_{Q\left(\theta\left(\frac{R}{8}\right)^{p^+}, \frac{R}{8}\right)} u \leq \sigma_0\ \omega. \tag{2.4.28}$$

Proof. The proof follows since $Q\left(\theta\left(\frac{R}{8}\right)^{p^+}, \frac{R}{8}\right) \subset Q\left(\hat{t}, \frac{R}{8}\right)$, where we have $\sigma_0 = 1 - \frac{1}{2^{s_2+1}}$. □

Assume that (2.2.5) does not hold. Then, (2.2.6) is in force. Even in this case, we are able to deduce a result analogous to Corollary 2.4.4.

Lemma 2.4.5. *Assume that (2.2.6) holds true. there exists a time level*

$$t_0 \in \left[t^* - \theta R^{p^+}, t^* - \frac{\nu_0}{2}\theta R^{p^+}\right] \tag{2.4.29}$$

such that

$$\left|\left\{x \in K_R,\ u(x,t_0) > \mu^+ - \frac{\omega}{2}\right\}\right| \leq \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right)|K_R|. \tag{2.4.30}$$

Proof. In fact, if (2.4.30) does not hold, then also (2.2.6) does not hold. □

This lemma shows that at the time level t_0, the portion of the cube K_R where $u(x)$ is close to its supremum is small. The next lemma claims that this indeed occurs for all time levels near the top of the cylinder $(0,t^*) + Q(\theta R^{p^+}, R)$

Lemma 2.4.6. *There exists $1 < s_3 \in \mathbb{N}$ depending on the data such that, for all $t \in \left[t^* - \frac{\nu_0}{2}\theta R^{p^+}, t^*\right]$*

$$\left|\left\{x \in K_R,\ u(x,t) > \mu^+ - \frac{\omega}{2^{s_3}}\right\}\right| \leq \left(1 - \left(\frac{\nu_0}{2}\right)^2\right)|K_R|. \tag{2.4.31}$$

Proof. Consider the cylinder $K_R \times (t_0, t^*)$ and the level $k = \mu^+ - \frac{\omega}{2}$. Define

$$u - k \leq H_k^+ = \operatorname*{ess\,sup}_{K_R\times(t_0,t^*)} \left|(u - \mu^+ + \frac{\omega}{2})\right| \leq \frac{\omega}{2} \tag{2.4.32}$$

Assuming that $H_k^+ > \frac{\omega}{4}$ (otherwise there will be nothing to prove). Select $n \in \mathbb{N}$ big enough so that

$$0 < c = \frac{\omega}{2^{n+1}} < H_k^+.$$

Then the logarithmic function ψ^+ is well defined and satisfies the followings.

$$\psi^+ \leq n\ln(2) \ \text{ since } \frac{H_k^+}{H_k^+ - u + k + c} \leq \frac{\frac{\omega}{4}}{c} = 2^n, \tag{2.4.33}$$

and, for $u \neq k + c$,

$$0 \leq (\psi^+)' \leq \frac{1}{H_k^+ - u + k + c} \leq \frac{1}{c} \tag{2.4.34}$$

and,

$$\left|(\psi^+)'(u)\right|^{2-p^-} = \left(H_k^+ - u + k + c\right)^{p^- -2} \leq \left(\frac{\omega}{2}\right)^{p^- -2}. \tag{2.4.35}$$

In the logarithmic inequality (2.3.11) applied to the function $(u-k)_+$, let $x \longmapsto \xi(x)$ be a smooth cutoff function defined in K_R such that for some $\pi \in (0,1)$

$$\begin{cases} 0 \leq \xi \leq 1 \quad \text{in } K_R, \quad \xi = 0 \ \text{ on } K_{(1-\pi)R}, \\ |\nabla\xi| \leq (\pi R)^{-1}. \end{cases}$$

Gathering these estimates, using Lemma 2.4.5 and the fact that

$$t^* - t \le \theta R^{p^+}, \tag{2.4.36}$$

we arrive at

$$\begin{aligned} &\underset{t_0<t<t^*}{ess\sup} \int_{K_R\times\{t\}} \left[\psi^+(u)\right]^2 \xi^{p^+} dx \\ &\quad \le n^2(\ln 2)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right)|K_R| + C\Bigg[n\ln 2\left(\frac{\omega}{2^{n+1}}\right)^{-2}\theta R^{p^+} \\ &\quad + \frac{n\ln 2}{\pi^{p^+}} + n\ln 2\left(\frac{1}{\pi R}\right)^{p^+}\theta R^{p^+} \\ &\quad + M^{p^+} n\ln 2\left(\frac{\omega}{2^{n+1}}\right)^{-2}\theta R^{p^+}\Bigg]|K_R| \end{aligned} \tag{2.4.37}$$

Now, by virtue of Remark 2.2.4, we can estimate

$$\left(\frac{\omega}{2^{n+2}}\right)^{-2}\theta R^{p^+} \le 1 \quad \text{and } \theta R^{p^+} \le 1.$$

Consequently, we get

$$\underset{t_0<t<t^*}{ess\sup} \int_{K_R\times\{t\}} \left[\psi^+(u)\right]^2 \xi^{p^+} dx \le \left[n^2(\ln 2)^2\left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) + C\frac{n}{\pi^{p^+}}\right]|K_R|. \tag{2.4.38}$$

The left hand side is estimated below by integrating over the smaller set

$$S_2 = \left\{x \in K_{(1-\pi)R}: \quad u(x,t) > \mu^+ - \frac{\omega}{2^{n+1}}\right\} \subset K_R.$$

On such a set, $\xi = 1$ and $\psi^+ \ge (n-1)\ln 2$, because

$$\frac{H_k^+}{H_k^+ - u + k + c} \ge \frac{\frac{\omega}{2}}{\frac{\omega}{2} - u + k + \frac{\omega}{2^{n+1}}} \ge \frac{\frac{\omega}{2}}{\frac{\omega}{2^n}} \ge 2^{n-1},$$

since one has $-u + \mu^+ < \frac{\omega}{2^n}$. Therefore for all $t \in (t_0, t^*)$

$$|S_2| \le \left\{\left(\frac{n}{n-1}\right)^2\left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) + \frac{C}{n\pi^{p^+}}\right\}|K_R|.$$

Therefore, for all $t \in (t_0, t^*)$

$$\begin{aligned} &\left|\left\{x \in K_R,\ u(x,t) > \mu^+ - \frac{\omega}{2^{n+1}}\right\}\right| \le |S_2| + N\pi|K_R| \\ &\qquad \le \left\{\left(\frac{n}{n-1}\right)^2\left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) + \frac{c}{n\pi^{p^+}} + N\pi\right\}|K_R|. \end{aligned} \tag{2.4.39}$$

The proof is complete once we choose π so small that $N\pi \leq \frac{3}{8}\nu_0^2$, then n so large that

$$\frac{C}{n\pi^{p^+}} \leq \frac{3}{8}\nu_0^2 \quad \text{and} \quad \left(\frac{n}{n-1}\right)^2 \leq (1-\frac{\nu_0}{2})(1+\nu_0) > 1,$$

and finally take $s_3 = n+1$. □

Recalling that $t_0 \in \left[t^* - \theta R^{p^+}, t^* - \frac{\nu_0}{2}\theta R^{p^+}\right]$ and choosing λ such that $2^{(\lambda-1)(p^- -2)} \geq 2$, the previous lemma immediately implies the following lemma.

Lemma 2.4.7. *There exists* $1 < s_3 \in \mathbb{N}$ *depending on the data, such that for all* $t \in \left(-\frac{a_0}{2}R^{p^+}, 0\right)$,

$$\left|\left\{x \in K_R,\ u(x,t) > \mu^+ - \frac{\omega}{2^{s_3}}\right\}\right| \leq \left(1 - \left(\frac{\nu_0}{2}\right)^2\right)|K_R|. \tag{2.4.40}$$

From Lemma 2.4.7 we deduce that within the cylinder $Q(a_0 R^{p^+}, R)$, the set where u is close to its supremum is arbitrarily small.

Lemma 2.4.8. *For every* $v_1 \in (0,1)$, *there exists* $s_3 \leq \lambda \in \mathbb{N}$ *depending on the data, such that*

$$\left|\left\{(x,t) \in Q\left(\frac{a_0}{2}R^{p^+}, R\right),\ u(x,t) > \mu^+ - \frac{\omega}{2^\lambda}\right\}\right| \leq \nu_1 \left|Q\left(\frac{a_0}{2}R^{p^+}, R\right)\right|. \tag{2.4.41}$$

Proof. Consider the cylinder $Q(a_0 R^{p^+}, 2R)$ and the levels $k = \mu^+ - \frac{\omega}{2^s}$, for $s_3 \leq s \leq \lambda$. Next, consider the local energy estimates (2.3.1) for the functions $(u-k)_+$, where $0 \leq \xi(x,t) \leq 1$ is a smooth cutoff function defined in $Q(a_0 R^{p^+}, 2R)$ and satisfying

$$\begin{cases} \xi = 1 \quad \text{in } Q(\frac{a_0}{2}R^{p^+}, R), \quad \xi = 0 \qquad \text{on } \partial_p Q(a_0 R^{p^+}, 2R), \\ |\nabla \xi| \leq \frac{1}{R}, \qquad 0 < \xi_t \leq \frac{2}{a_0 R^{p^+}}. \end{cases}$$

Neglecting the first term on the left hand side of (2.3.1), and using the indicated choices, we obtain

$$\begin{aligned}
&\iint\limits_{Q(\frac{a_0}{2}R^{p^+},R)} |\nabla(u-k)_+|^{p^-}\,\xi^{p^+}dxdt \\
&\quad\le C\Bigg(\frac{2}{a_0R^{p^+}}\iint\limits_{Q(a_0R^{p^+},2R)}(u-k)_+^2dxdt \\
&\quad+\frac{1}{R^{p^+}}\iint\limits_{Q(a_0R^{p^+},2R)}(u-k)_+^{p^+}dxdt \\
&\quad\quad+\iint\limits_{Q(a_0R^{p^+},2R)}\chi\left((u-k)_+>0\right)dxdt\Bigg) \qquad (2.4.42)\\
&\quad\le C\Bigg(\frac{2}{a_0R^{p^+}}\left(\frac{\omega}{2^s}\right)^2+\frac{1}{R^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^+}+2^{N+1}\Bigg)\left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right| \\
&\quad\le C\Bigg(\frac{2}{R^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^--p^+}\left(\frac{\omega}{2^s}\right)^{p^+}+\frac{1}{R^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^+} \\
&\quad\quad+\frac{2^{N+1}}{R^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^+}\left(\frac{\omega}{2^s}\right)^{-p^+}R^{p^+}\Bigg)\left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right|,
\end{aligned}$$

here, we used the fact that $s\le\lambda$. Now, by virtue of Remark 2.2.4 we can estimate

$$\left(\frac{\omega}{2^s}\right)^{p^--p^+}R^{p^+}\le 1 \text{ and } \left(\frac{\omega}{2^s}\right)^{-p^+}R^{p^+}\le 1.$$

Consequently, we get

$$\iint\limits_{Q(\frac{a_0}{2}R^{p^+},R)}|\nabla u|^{p^-}\,\xi^{p^+}dxdt\le\frac{C}{R^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^+}\left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right|. \qquad (2.4.43)$$

Now, we consider the levels $k_1=\mu^+-\frac{\omega}{2^s}$, $k_2=\mu^+-\frac{\omega}{2^{s+1}}$, $k_2-k_1=\frac{\omega}{2^{s+1}}$, and define, for $t\in\left(-\frac{a_0}{2}R^{p^+},0\right)$

$$A_s(t)=\left\{x\in K_R,\ u(x,t)>\mu^+-\frac{\omega}{2^s}\right\} \text{ and } A_s=\int_{-\frac{a_0}{2}R^{p^+}}^0|A_s(t)|\,dt.$$

Using Lemma 1.4.1 and Remark 1.4.2 applied to the function $u(.,t)$ for all times $t\in$

$\left(-\frac{a_0}{2}R^{p^+},0\right)$, we get

$$\begin{aligned}\left(\frac{\omega}{2^{s+1}}\right)|A_{s+1}| &\leq C\frac{R^{N+1}}{|K_R-A_s(t)|}\int_{-\frac{a_0}{2}R^{p^+}}^{0}\int_{A_s(t)-A_{s+1}(t)}|\nabla u|dx\\ &\leq C\frac{R^{N+1}}{\left(\frac{\nu_0}{2}\right)^2|K_R|}\int_{-\frac{a_0}{2}R^{p^+}}^{0}\int_{A_s-A_{s+1}}|\nabla u|dx \qquad (2.4.44)\\ &\leq \frac{C}{\nu_0^2}\left(\frac{\omega}{2^s}\right)\left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right|^{\frac{1}{p^-}}|A_s(t)-A_{s+1}(t)|^{\frac{p^--1}{p^-}},\end{aligned}$$

here, we used Lemma 2.4.7, Hölder's inequality and (2.4.43). According to the previous energy estimates we get, for $s=s_3,\ s_3+1,...,\lambda-1$

$$|A_{s+1}|^{\frac{p^-}{p^--1}}\leq C(\nu_0)^{\frac{-2p^-}{p^--1}}\left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right|^{\frac{1}{p^--1}}|A_s-A_{s+1}|,$$

and we then add these inequalities for $s=s_3,\ s_3+1,...,\lambda-1$. Since $\mu^+-\frac{\omega}{2^{s+1}}\leq\mu^+-\frac{\omega}{2^\lambda}$, the quantities $A_{s+1}\geq A_\lambda$. we combine this fact to obtain

$$\sum_{s=s_3}^{\lambda-1}A_{s+1}^{\frac{p^-}{p^--1}}\geq(\lambda-s_3)A_\lambda^{\frac{p^-}{p^--1}}.$$

Note, also that $\sum\limits_{s=s_3}^{\lambda-1}|A_s-A_{s+1}|\leq\left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right|$. Collecting results, we arrive at

$$A_\lambda\leq\frac{C}{(\lambda-s_3)^{\frac{p^--1}{p^-}}}(\nu_0)^{-2}\left|Q\left(\frac{a_0}{2}R^{p^+},R\right)\right|$$

and the proof is complete once we choose $s_3<\lambda\in\mathbb{N}$ sufficiently large so that

$$\frac{C}{(\lambda-s_3)^{\frac{p^--1}{p^-}}}(\nu_0)^{-2}\leq\nu_1.$$

□

Lemma 2.4.9. *The number $v_1\in(0,1)$ can be chosen (and consequently, so λ), such that*

$$u(x,t)\leq\mu^+-\frac{\omega}{2^{\lambda+1}}\qquad a.e.\ (x,t)\in Q\left(\frac{a_0}{2}R^{p^+},R\right). \qquad (2.4.45)$$

Proof. Define two decreasing sequences of positive numbers

$$R_n=\frac{R}{2}+\frac{R}{2^{n+1}},\quad k_n=\mu^+-\frac{\omega}{2^{\lambda+1}}-\frac{\omega}{2^{\lambda+1+n}},\quad n=0,1,....$$

Now, consider the local energy estimates (2.3.1) for the functions $(u-k_n)_+$ over the constructed family of nested and shrinking cylinders $Q\left(\frac{a_0}{2}R_n^{p^+}, R_n\right)$, where $0 \leq \xi_n(x,t) \leq 1$ are smooth functions defined in $Q\left(\frac{a_0}{2}R_n^{p^+}, R_n\right)$ such that

$$\begin{cases} \xi_n = 1 \quad \text{in } Q\left(\frac{a_0}{2}R_{n+1}^{p^+}, R_{n+1}\right), \quad \xi_n = 0 \;\text{ on } \partial_p Q\left(\frac{a_0}{2}R_n^{p^+}, R_n\right), \\ |\nabla \xi_n| \leq \frac{2^{n+1}}{R}, \qquad 0 < (\xi_n)_t \leq \dfrac{2^{p^+(n+1)}}{\frac{a_0}{2}R^{p^+}}. \end{cases}$$

Once again, performing the same calculation used in the proof of Lemma 2.4.1, we get

$$\begin{aligned} &\operatorname*{ess\,sup}_{-\frac{a_0}{2}R_n^{p^+}<t<0} \int_{K_{R_n}\times\{t\}} (u-k_n)_+^{p^+} \xi_n^{p^+} dx \\ &\quad + \frac{1}{a_0} \iint_{Q(\frac{a_0}{2}R_n^{p^+}, R_n)} |\nabla (u-k_n)_+|^{p^-} \xi_n^{p^+} dxdt \\ &\leq C\frac{2^{np^+}}{a_0 R^{p^+}} \left(\frac{\omega}{2^\lambda}\right)^{p^+} \iint_{Q(\frac{a_0}{2}R_n^{p^+}, R_n)} \chi\left((u-k_n)_+ > 0\right) dxdt. \end{aligned} \tag{2.4.46}$$

Introducing the change of variables $\bar{t} = \dfrac{t}{\frac{a_0}{2}}$ and defining

$$\bar{u}(x,\bar{t}) = u(x,t) \quad \text{and} \quad \bar{\xi_n}(x,\bar{t}) = \xi_n(x,t).$$

Therefore, the previous estimates implies

$$\frac{1}{2^{p^+(n+2)}} \left(\frac{\omega}{2^\lambda}\right)^{p^+} A_{n+1} \leq C\frac{2^{np^+}}{R^{p^+}} \left(\frac{\omega}{2^\lambda}\right)^{p^+} A_n^{1+\frac{p^-}{N+p^-}}, \tag{2.4.47}$$

where, A_n is defined as

$$A_n = \iint_{Q(R_n^{p^+}, R_n)} \chi\left((\bar{u}-k_n)_+ > 0\right) dxd\bar{t}.$$

Next, defining $X_n = \frac{A_n}{\left|Q(R_n^{p^+}, R_n)\right|}$, we arrive at

$$X_{n+1} \leq C 4^{np^+} X_n^{1+\frac{p^-}{N+p^-}}.$$

Therefore, using Lemma 1.4.3, the result is proved if we can assume that

$$X_0 \leq C^{-\frac{N+p^-}{p^-}} 4^{-p^+\left(\frac{p^-+N}{p^-}\right)^2} = \nu_1. \tag{2.4.48}$$

For this value of ν_1, Lemma 2.4.8 implies that $X_0 \leq \nu_1$. Hence, we can conclude that $X_n \longrightarrow 0$ where $n \longrightarrow +\infty$ and the result follows. $\square$

As an immediate consequence we get the reduction of the oscillation of u in the second case

Corollary 2.4.10. *There exists a constant $\sigma_1 \in (0,1)$ depending only on the data, such that if (2.2.6) holds then*

$$\underset{Q\left(\frac{a_0}{2}\left(\frac{R}{2}\right)^{p^+},\frac{R}{2}\right)}{ess\ osc}\ u \leq \sigma_1\ \omega. \tag{2.4.49}$$

Proof. The proof follows by choosing $\sigma_1 = 1 - \frac{1}{2^{s_\lambda+1}}$. $\square$

Now, we are able to prove Proposition 2.2.5, recalling the conclusions of Corollaries 2.4.4 and 2.4.10 and since $\theta\left(\frac{R}{8}\right)^{p^+} \leq \frac{a_0}{2}\left(\frac{R}{2}\right)^{p^+}$, we get that

$$\underset{Q\left(\theta\left(\frac{R}{8}\right)^{p^+},\frac{R}{8}\right)}{ess\ osc}\ u \leq \sigma\ \omega,$$

where $\sigma = \max\{\sigma_0, \sigma_1\}$.

The proof of Theorem 2.2.6 follows from a slight modification of the arguments in Proposition 9 in [30]. Indeed, we have the following

Proposition 2.4.11. *There exists a positive constant C, depending only on the data, such that, defining the sequences*

$$R_n = C^{-1}R \ \textit{and}\ \ \omega_n = \sigma^n\omega$$

for $n = 0, 1, 2, ...$, where $\sigma \in (0,1)$ is already had been given in the proof of Proposition 2.2.5, and constructing the family of cylinders

$$Q_n = Q(a_n R_n^{p^+}, R) \ \textit{with}\ \ a_n = \left(\frac{\omega_n}{2^\lambda}\right)^{2-p^-}$$

where $\lambda > 0$ is already had been been given previously, we have

$$Q_{n+1} \subset Q_n \ \textit{and}\ \ \underset{Q_n}{ess\ osc}\ u \leq \omega_n,$$

for all $n = 0, 1, 2, ...$

Proof. Recall the definition of $a_0 = \left(\frac{\omega}{2^\lambda}\right)^{2-p^-}$ and the construction of the initial cylinder so that the starting relation

$$\underset{Q_0}{\text{ess osc}}\ u \leq \omega, \tag{2.4.50}$$

holds. we find

$$\begin{aligned}
\theta\left(\frac{R}{8}\right)^{p^+} &= \left(\frac{\omega}{2}\right)^{2-p^-}\frac{R^{p^+}}{8^{p^+}} \\
&= \left(\frac{\omega}{2}\right)^{2-p^-}\left(\frac{2^\lambda}{\omega_1}\right)^{2-p^-}\left(\frac{\omega_1}{2^\lambda}\right)^{2-p^-}\frac{R^{p^+}}{8^{p^+}} \\
&= \left(\frac{\omega}{\omega_1}\right)^{2-p^-}\left(\frac{2^\lambda}{2}\right)^{2-p^-}\left(\frac{\omega_1}{2^\lambda}\right)^{2-p^-}\frac{R^{p^+}}{8^{p^+}} \\
&= \sigma^{p^- -2}2^{(\lambda-1)(2-p^-)-3p^+}a_1 R^{p^+} \\
&= a_1 R_1
\end{aligned}$$

where $R_1 = c^{-1}R$, provided C is chosen from

$$C = \sigma^{\frac{2-p^-}{p^+}}2^{\frac{(\lambda-1)(p^- -2)}{p^+}+3} > 8.$$

From Proposition 2.2.5, we conclude

$$\operatorname*{ess\ osc}_{Q_1} u \le \operatorname*{ess\ osc}_{Q\left(\theta\left(\frac{R}{8}\right)^{p^+},\frac{R}{8}\right)} u \le \sigma\omega = \omega_1,$$

which puts us back to the setting of (2.4.50). The entire process can now be repeated inductively starting from Q_1. □

Lemma 2.4.12. *There exist constants $\gamma > 1$ and $\alpha \in (0,1)$, that can be determined a priori in terms of the data, such that*

$$\operatorname*{ess\ osc}_{Q\left(a_0\rho^{p^+},\rho\right)} u \le \gamma\omega\left(\frac{\rho}{R}\right)^\alpha,$$

for all the cylinders $Q\left(a_0\rho^{p^+},\rho\right)$, with $0 < \rho \le R$.

Proof. Let $0 < \rho \le R$ be fixed. there exixts a non-negative integer n such that

$$C^{-(n+1)}R \le \rho \le C^{-n}R.$$

Then, putting $\alpha = -\frac{\ln\sigma}{\ln C}$, we deduce

$$C^{-(n+1)} \le \frac{\rho}{R} \Longleftrightarrow \sigma^{\frac{n+1}{\alpha}} \le \frac{\rho}{R} \Longleftrightarrow \sigma^{n+1} \le \left(\frac{\rho}{R}\right)^\alpha.$$

Thus,

$$\omega_n = \sigma^n\omega \le \gamma\omega\left(\frac{\rho}{R}\right)^\alpha, \text{ with } \gamma = \sigma^{-1}.$$

To conclude the proof, observe that the cylinder $Q\left(a_0\rho^{p^+},\rho\right)$ is contained is contained in the cylinder $Q\left(a_n R_n^{p^+}, R_n\right)$, since $\omega_n \le \omega$ and $\rho \le C^{-n}R = R_n$. □

Let $\Gamma = \partial\Omega_T$ be the parabolic boundary of Ω_T and u be a bounded local weak solution of (2.1.1) i Ω_T, with $M = \|u\|_{\infty,\Omega_T}$. Introduce the degenerate intrinsic parabolic p-distance from a compact set $K \subset \Omega_T$ to Γ, by

$$p - \operatorname{dist}(K;\Gamma) := \inf_{\substack{(x,t)\in K\\(y,s)\in\Gamma}} \left(|x-y| + M^{\frac{p^- -2}{p^+}} |t-s|^{\frac{1}{p^+}} \right)$$

Now, we will give the final proof of Theorem 2.2.6.

Proof. Fix $(x_i, t_i) \in K$, $i = 1, 2$, such that $t_2 > t_1$ and construct the cylinder

$$(x_2, t_2) + Q\left(M^{2-p^-} R^{p^+}, R\right).$$

It is contained in Ω_T if we choose

$$R \leq \inf_{\substack{x\in K\\y\in\partial\Omega}} |x-y| \quad \text{and} \quad M^{\frac{2-p^-}{p^+}} R \leq \inf_{t\in K} t^{\frac{1}{p^+}}.$$

Thus, in particular, we may choose $2R = p - \operatorname{dist}(K;\Gamma)$. To prove the Hölder continuity in the t-variable assume first that

$$t_2 - t_1 < M^{2-p^-} R^{p^+}.$$

Then, there exists $\rho \in (0, R)$ such that $t_2 - t_1 = M^{2-p^-} \rho^{p^+}$, i.e.

$$\rho := M^{\frac{p^- -2}{p^+}} |t_2 - t_1|^{\frac{1}{p^+}}.$$

The oscillation inequality of Lemma 2.4.12, applied in the cylinder

$$(x_2, t_2) + Q(a_0 \rho^{p^+}, \rho)$$

implies

$$|u(x_2,t_2) - u(x_1,t_1)| \leq \gamma M \left(\frac{M^{\frac{p^- -2}{p^+}} |t_2 - t_1|^{\frac{1}{p^+}}}{p - \operatorname{dist}(K;\Gamma)} \right).$$

The Hölder continuity in the space variables is proved analogously. □

Chapter 3

Regularity for anisotropic quasi-linear parabolic equations with variable growth

In this chapter we are concerned with the anisotropic parabolic equations modeled by

$$u_t - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x,t)-2} \frac{\partial u}{\partial x_i} \right] = 0 \text{ in } \Omega_T, \tag{3.0.1}$$

where $\Omega_T \equiv \Omega \times (0,T]$, Ω is a bounded simple-connected domain in $\mathbb{R}^N$ and $0 < T < +\infty$. Throughout the paper we assume that the exponents $p_i(x,t)$ are given measurable functions in Ω_T such that for all $i = 1, .., N$

$$p_i(x,t) \subset \left(p_i^-, p_i^+\right) \subseteq \left[p^-, p^+\right] \subset (2, \infty).$$

with finite constants $p^{\pm}, p_i^{\pm} > 2$. Moreover, we assume that p_i satisfies the following log-continuity condition:

$$|p_i(x,t) - p_i(y,\tau)| \leq \frac{c_0}{\ln \frac{1}{|t-\tau|+|x-y|}} \text{ for any } (x,t),\ (y,\tau) \in \Omega_T,\ |t-\tau|+|x-y| \leq \frac{1}{2}. \tag{3.0.2}$$

In this framework, a particularly relevant class of interest is given by functionals with anisotropic structures, i.e. those whose energy sees each derivatives being penalized with a different exponent. Yet firstly studied by Marcellini [54], further contributions have been given by Leonetti [51,52], Acerbi and Fusco [1], Fusco and Sbordone [39,40].

Anisotropic equations like (3.0.1) have strong physical background. They emerge, for instance, from the mathematical description of the dynamics of fluids with different conductivities in different directions. We refer to the extensive books by Antontsev- Díaz-Shmarev [8] and Bear [18] for discussions in this direction. They also appear in biology, see Bendahmane-Karlsen [19] and Bendahmane-Langlais-Saad [20], as a model describing the spread of an epidemic disease in heterogeneous environments.

Our aim here is to obtain a local regularity result for local weak solutions of (3.0.1). In order to achieve this goal, and since the equation is degenerate (the diffusion coefficient vanishes when $\left|\frac{\partial u}{\partial x_i}\right| = 0$), the idea is to study the equation within a geometry that takes this feature into consideration. The building blocks of DiBenedetto's intrinsic scaling method is to show that the continuity of the solution at a point follows from measuring its oscillation in a sequence of nested and shrinking cylinders, with vertex at that point, and showing that the oscillation converge to zero as the cylinders shrink to the point. To fully understand the technical procedure, based on the study of an alternative argument which makes use of energy and logarithmic estimates, one has not only to be familiar with Dibenedetto's technique (see [29,30,70]) but also to overcome the difficulty of having an $(x_i, t)-$dependence on the exponents p_i for $i = 1, .., N$.

The local continuity of the anisotropic elliptic equation have been studied, and the results are well documented. We refer to [31,53] for the results and the references to the original papers. It is known that the local solutions of the isotropic parabolic case of equation (3.0.1) with variable growth are locally Hölder continuous [6]. To the best of the author's knowledge, no regularity result is known for the anisotropic parabolic equations with variable growth.

3.1 Preliminary and main results

3.1.1 The function spaces

We recall in what follows some definitions and basic proprieties of the generalized Lebesgue-Sobolev spaces. We begin by defining the variable exponent Lebesgue space as follows

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ is measurable with } \int_\Omega |u|^{p(x)}\, dxdt < +\infty \right\}.$$

The set $L^{p(x)}(\Omega)$ equipped with The Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \alpha > 0 : \int_\Omega \left|\frac{u}{\alpha}\right|^{p(x)} dxdt < 1 \right\}$$

becomes a reflexive Banach space. Next, we define the Sobolev space $W^{1,p(x)}(\Omega)$ as follows

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega), \nabla u \in L^{p(x)}(\Omega) \right\}$$

endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

In addition, if $p(x)$ is log-Hölder continuous then $W_0^{1,p(x)}(\Omega) \equiv$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Now, in connection with the anisotropic operators that we are considering, we need to recall the definitions of the anisotropic Sobolev spaces:

$$W^{1,(p_i)}(\Omega) = \left\{ u \in W^{1,1}(\Omega),\ \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega),\ \forall i = 1,..,N \right\},$$

and,

$$W_0^{1,(p_i)}(\Omega) = \left\{ u \in W_0^{1,1}(\Omega),\ \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega),\ \forall i = 1,..,N \right\}.$$

The space $W_0^{1,(p_i)}(\Omega)$ also denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,(p_i)} = \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}.$$

In [47,62,69], the theory of anisotropic Sobolev spaces is developed and, in particular, the corresponding Sobolev embedding theorems are studied. Define

$$p^* = \frac{N\overline{p}}{N - \overline{p}},\ \text{for } \overline{p} < N \text{ and } \frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i^-}. \tag{3.1.1}$$

In [69] it is proved that if $\overline{p} < N$, then

$$W_0^{1,(p_i^-)}(\Omega) \hookrightarrow L^r(\Omega), \quad \forall r \in [1, p^*].$$

This embedding is continuous and also compact if $r < p^*$. If $\overline{p} \geq N$, then

$$W_0^{1,(p_i^-)}(\Omega) \hookrightarrow L^r(\Omega), \quad \forall r \in [1, +\infty).$$

The following Sobolev type inequality is also proved; if $\overline{p} < N$, then there exists a positive constant C, depending only on Ω, p_i^-, r and N, such that

$$\|u\|_{r,\Omega} \leq C \prod_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i^-,\Omega}^{\frac{1}{N}}, \quad \forall r \in [1, p^*], \tag{3.1.2}$$

for any $u \in W_0^{1,(p_i^-)}(\Omega)$.

For a.e. $t \in (0,T)$ we introduce the anisotropic Banach space

$$V_t(\Omega) = \left\{ u(x) :\ u(x) \in L^2(\Omega) \cap W_0^{1,1}(\Omega),\ \left| \frac{\partial u(x)}{\partial x_i} \right|^{p_i(x,t)} \in L^1(\Omega) \right\},$$

$$\|u\|_{V_t(\Omega)} = \|u\|_{2,\Omega} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i(.,t),\Omega}.$$

The elements of the space $V_t(\Omega)$ depend on $t \in (0,T)$ as a parameter and the norms $\|u\|_{V_t(\Omega)}$ are functions of t. By $W(\Omega_T)$ we denote the Banach space

$$W(\Omega_T) = \left\{ u : (0,T) \longmapsto V_t(\Omega) | \ u \in L^2(\Omega_T), \ \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x,t)} \in L^1(\Omega_T), \ u = 0 \text{ on } \Gamma \right\},$$

$$\|u\|_{W(\Omega_T)} = \|u\|_{2,\Omega_T} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i(.),\Omega_T}.$$

3.1.2 Mollification in time

Since weak solutions of parabolic equations, respectively inequalities possess only weak regularity properties with respect to time, it is in principle not possible to use the solution itself as a test-function in the weak formulation of the problem. In order to be nevertheless able to test properly, there are several possibilities to smooth the solution with respect to the time direction. To overcome these faculties, we consider the Friedrichs mollifier as was done in [6]. Indeed, taking the kernel

$$\rho \geq 0, \quad \rho \in C_0^\infty(\mathbb{R}^N), \quad \rho(x) \equiv 0 \quad \text{for} \quad |x| \geq 1, \quad \int_{\mathbb{R}^N} \rho(x)\ dx = 1,$$

we introduce regularization of $f \in L^{p(x,t)}_{loc}(\Omega_T)$ by

$$I^h f = f_h(x,t) = h^{-1} \int_t^{t+h} \int_{|x-y| \leq h} f(y,\tau) \rho_h(x-y)\ dy d\tau, \quad \rho_h(x) = h^{-N} \rho(h^{-1}x), \quad (3.1.3)$$

and consider these inside the cylinder Ω_T, i.e., in cylinders $\Omega'_T = \Omega' \times (T_1, T_2)$, where $\Omega' \subset \Omega,\ 0 < T_1 < T_2 < T$. The basic property of the mollification, which can be retrieved from [6, Lemma 2.1], is summarized in the following:

Lemma 3.1.1. *If the exponent p satisfies the condition (3.0.2), then $f_h \longrightarrow f$ in $L^{p(x,t)}_{loc}(\Omega_T)$ as $h \longrightarrow 0$, for any $f \in L^{p(x,t)}_{loc}(\Omega_T)$.*

3.1.3 Formulation of the problem

We will consider here local weak solutions of equation (3.0.1), the existence of such solutions is guaranteed by [10,11].

Definition 3.1.2. *A local weak solution of (3.0.1) is a measurable function $u(x,t)$ defined in Ω_T, such that*

(i) $u \in W(\Omega_T) \cap C([0,T]; L^2(\Omega))$;

(ii) for every subset K of Ω and for every subinterval $[t_1, t_2]$ of $(0,T]$, we have

$$\left[\int_K u\phi \, dx\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \left\{-u\phi_t + \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}.\frac{\partial \phi}{\partial x_i}\right\} dxdt = 0, \qquad (3.1.4)$$

for any locally bounded tested function $\phi \in W_{loc}(\Omega_T) \cap W^{1,2}_{loc}(0,T; W_0^{1,2}(K))$.

We can write (ii) in a way that is technically more convenient and involves the discrete time derivative. This can be accomplished by using the Friedrichs mollifier of a function (see [6] for more details). Then, we get the following:

Lemma 3.1.3. *If u is a solution of equation (3.0.1) in the sense of Definition 3.1.2, then for every subset K of Ω, and for any $h < t_1 \leq t_2 < T - h$, the following relation*

$$\int_{t_1}^{t_2}\int_K \left[u_{h,t}\varphi + \sum_{i=1}^{N}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\right)_h .\frac{\partial \varphi}{\partial x_i}\right] dxdt = 0 \qquad (3.1.5)$$

holds for any locally bounded tested function $\varphi \in W_{loc}(\Omega_T) \cap W^{1,2}_{loc}(0,T; W_0^{1,2}(K))$.

Proof. As in [6], we introduce the following regularization operator:

$$I^{-h} f = f_{-h}(x,t) = h^{-1}\int_{t-h}^{t}\int_{|x-y|\leq h} f(y,\tau)\rho_h(x-y) \, dyd\tau. \qquad (3.1.6)$$

Consider equation (3.1.4) with

$$\phi = I^{-h}(\varphi\chi), \quad \varphi \in W_{loc}(\Omega_T) \cap W^{1,2}_{loc}(0,T; W_0^{1,2}(K)).$$

Since

$$-\int_{t_1}^{t_2}\int_K u\frac{\partial I^{-h}(\varphi\chi)}{\partial t} \, dxdt = \int_{t_1}^{t_2}\int_K u_{h,t}\varphi\chi \, dxdt,$$

it follows that

$$\int_{t_1}^{t_2}\int_K \left[u_{h,t}\varphi\chi + \sum_{i=1}^{N}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\right)_h .\frac{\partial(\varphi\chi)}{\partial x_i}\right] dxdt = 0.$$

Passing here from $\chi \in C_0^\infty(t_1, t_2)$ to characteristic function of the segment $[t_1, t_2]$, we obtain the desired relation (3.1.5).

□

3.1.4 Regularity result

In order to obtain the interior continuity of the solutions by means of intrinsic scaling, we need to consider a geometry that accommodates the degeneracy of the anisotropic parabolic equation (3.0.1). For this purpose let (x_0, t_0) be an interior point of the space time domain Ω_T, by translation and to simplify, assume $(x_0, t_0) = (0, 0)$. Also, let $0 < R < 1$, be sufficiently small such that the cylinder

$$Q(R^2, R) = K_R \times (-R^2, 0) := \{x : \max_{1 \le i \le N} |x_i| < R\} \times (-R^2, 0)$$

is a subset of Ω_T, and define

$$\mu^+ = \underset{Q(R^2,R)}{ess \sup}\, u, \quad \mu^- = \underset{Q(R^2,R)}{ess \inf}\, u \quad \text{and} \quad \omega = \underset{Q(R^2,R)}{ess\ osc}\, u = \mu^+ - \mu^-.$$

To this data, let $a_0 = \left(\frac{\omega}{2^\lambda}\right)^{2-p^-}$ be a positive real number , for some $\lambda > 1$ to be chosen later. We construct the cylinder

$$Q(a_0 R^{p^+}, R) = K_R \times (-a_0 R^{p^+}, 0).$$

Under the assumption

$$R^{\frac{2-p^+}{2-p^-}} < \frac{\omega}{2^\lambda}, \tag{3.1.7}$$

the inclusion $Q(a_0 R^{p^+}, R) \subset Q(R^2, R)$ holds, and consequently we have

$$\underset{Q(a_0 R^{p^+},R)}{ess\ osc}\ u \le \omega.$$

Remark 3.1.4. *If (3.1.7) does not hold, then the essential oscillation ω goes to zero when the radius R goes to zero, and then there is nothing to prove.*

In order to begin our approach, inside $Q(a_0 R^{p^+}, R)$ consider subcylinders of small size constructed as follows

$$(0, t^*) + Q(\theta R^{p^+}, R), \quad \theta = \left(\frac{\omega}{2}\right)^{2-p^-}.$$

These are contained in $Q(a_0 R^{p^+}, R)$ if

$$(2^{p^- - 2} - 2^{\lambda(p^- - 2)}) \frac{R^{p^+}}{\omega^{p^- - 2}} < t^* < 0.$$

For a given $\nu_0 \in (0, 1)$, to be determined in terms of the data and ω, either

$$\left| \left\{ (x, t) \in (0, t^*) + Q(\theta R^{p^+}, R) : \ u(x, t) < \mu^- + \frac{\omega}{2} \right\} \right| \le \nu_0 \left| Q(\theta R^{p^+}, R) \right| \tag{3.1.8}$$

or, nothing that $\mu^+ - \frac{\omega}{2} = \mu^- + \frac{\omega}{2}$

$$\left| \left\{ (x, t) \in (0, t^*) + Q(\theta R^{p^+}, R) : \ u(x, t) > \mu^+ - \frac{\omega}{2} \right\} \right| \le (1 - \nu_0) \left| Q(\theta R^{p^+}, R) \right|. \tag{3.1.9}$$

The analysis of this alternative leads to the following result.

Proposition 3.1.5. *Assume that $\overline{p} < N$, then there exist positive numbers ν_0, $\sigma \in (0,1)$, depending on the data and ω, such that*

$$\underset{Q\left(\theta\left(\frac{R}{8}\right)^{p^+},\frac{R}{8}\right)}{ess\ osc} u \le \sigma\omega. \tag{3.1.10}$$

An immediate consequence is the following.

Theorem 3.1.6. *Under the assumption that $\overline{p} < N$, any locally bounded weak solution of (3.0.1) is locally continuous in Ω_T.*

Remark 3.1.7. *The proof of Theorem 3.1.6 follows from a slight modification of the arguments in Proposition 9 in [30]. From (3.1.10) one defines recursively a sequence Q_n of nested and shrinking cylinders and a sequence ω_n converging to zero, such that*

$$\underset{Q_n}{ess\ osc}\, u \le \omega_n.$$

This is enough to obtain the continuity of u but we are unable to derive a modulus since the constant σ appearing in Proposition 3.1.5 depends on the oscillation ω.

3.2 Local energy and logarithmic estimates

Let τ and ρ be small such that $Q(\tau,\rho) \subset \Omega_T$, in addition let ξ be a piecewise smooth cutoff function in $Q(\tau,\rho)$ such that

$$\xi \in [0,1], \quad \left|\frac{\partial \xi}{\partial x_i}\right| < \infty \ \forall i = 1, ..N, \text{ and } \xi(x,t) = 0 \text{ for } x \text{ outside } K_\rho.$$

Proposition 3.2.1. *Let u be a local weak solution of (3.0.1) in Ω_T, then there exists a positive constant C such that, for every cylinder $Q(\tau,\rho) \subset \Omega_T$ and for every $k \in \mathbb{R}$, we have*

$$\begin{aligned} \sup_{-\tau<t<0} \int_{K_\rho} (u-k)_\pm^2\, \xi^{p^+}(x,t)dx + \sum_{i=1}^N \int_{-\tau}^0 \int_{K_\rho} \left|\frac{\partial}{\partial x_i}(u-k)_\pm\right|^{p_i(x,t)} \xi^{p^+} dxdt \\ \le \int_{K_\rho} (u-k)_\pm^2\, \xi^{p^+}(x,-\tau)dx + p^+ \int_{-\tau}^0 \int_{K_\rho} (u-k)_\pm^2\, \xi^{p^+-1}\xi_t\ dxdt \\ + C \sum_{i=1}^N \int_{-\tau}^0 \int_{K_\rho} (u-k)_\pm^{p_i(x,t)} \left|\frac{\partial \xi}{\partial x_i}\right|^{p_i(x,t)} dxdt. \end{aligned} \tag{3.2.1}$$

Proof. In the weak formulation (3.1.5) take the testing function $\varphi = \pm(u_h - k)_\pm \xi^{p^+}$, where

$$(u_h - k)_- = (k - u_h)_+ = \max\{k - u, 0\},$$

and u_h are regularizations of the form (3.1.3), integrate over $(-\tau, t)$; $t \in (-\tau, 0)$, and use Lemma 3.1.1. Estimating the various terms separately. The first term gives

$$\int_{-\tau}^{t}\int_{K_\rho} u_{h,t}\varphi\, dxdt = \int_{-\tau}^{t}\int_{K_\rho} u_{h,t}\left(\pm(u_h-k)_\pm\xi^{p^+}\right)\, dxdt$$
$$\underset{h\longrightarrow 0}{\longrightarrow} -\frac{p^+}{2}\int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^2\,\xi^{p^+-1}\xi_t\, dxdt + \frac{1}{2}\int_{K_\rho}(u-k)_\pm^2\,\xi^{p^+}(x,t)dx$$
$$-\frac{1}{2}\int_{K_\rho}(u-k)_\pm^2\,\xi^{p^+}(x,-\tau)dx.$$

For the remaining term, when $h \longrightarrow 0$, we get

$$\sum_{i=1}^{N}\int_{-\tau}^{t}\int_{K_\rho}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\left[\frac{\partial}{\partial x_i}\left(\pm(u-k)_\pm\right)\xi^{p^+} \pm p^+(u-k)_\pm\xi^{p^+-1}\frac{\partial\xi}{\partial x_i}\right]dxdt$$
$$\geq \sum_{i=1}^{N}\int_{-\tau}^{t}\int_{K_\rho}\left|\frac{\partial}{\partial x_i}(u-k)_\pm\right|^{p_i(x,t)}\xi^{p^+}\, dxdt$$
$$-p^+\sum_{i=1}^{N}\int_{-\tau}^{t}\int_{K_\rho}\left|\frac{\partial}{\partial x_i}(u-k)_\pm\right|^{p_i(x,t)-1}(u-k)_\pm\xi^{p^+-1}\left|\frac{\partial\xi}{\partial x_i}\right|\, dxdt$$
$$\geq \frac{1}{2}\sum_{i=1}^{N}\int_{-\tau}^{t}\int_{K_\rho}\left|\frac{\partial}{\partial x_i}(u-k)_\pm\right|^{p_i(x,t)}\xi^{p^+}\, dxdt - C\sum_{i=1}^{N}\int_{-\tau}^{t}\int_{K_\rho}(u-k)_\pm^{p_i(x,t)}\left|\frac{\partial\xi}{\partial x_i}\right|^{p_i(x,t)}\, dxdt.$$

Here we have used Young's inequality, and the fact that $0 \leq \xi \leq 1$ and $\frac{p_i(x,t)}{p_i(x,t)-1} \geq \frac{p^+}{p^+-1}$ imply that $\xi^{\frac{p_i(x,t)(p^+-1)}{p_i(x,t)-1}} \leq \xi^{p^+}$, $\forall i = 1, ..., N$. Hence, since $t \in (-\tau, 0)$ is arbitrary, we can combine both estimates to obtain (3.2.1). □

Now, introduce the logarithmic function

$$\psi^\pm(u) = \psi\left(H_k^\pm, (u-k)_\pm, c\right) = \left(\ln\left(\frac{H_k^\pm}{H_k^\pm - (u-k)_\pm + c}\right)\right)_+,$$

where $H_k^\pm = \operatorname*{ess\,sup}_{Q(\tau,\rho)} |(u-k)_\pm|$ and $0 < c < H_k^\pm$. In the cylinder $Q(\tau, \rho)$, we take a cutoff function satisfying $\xi \in [0, 1]$, $\left|\frac{\partial\xi}{\partial x_i}\right| < \infty$ $\forall i = 1, .., N$ and ξ is independent of $t \in (-\tau, 0)$.

Proposition 3.2.2. *Let u be a local weak solution of (3.0.1) in Ω_T, then there exists a positive constant C such that for every cylinder $Q(\tau, \rho) \subset \Omega_T$ and for every level $k \in \mathbb{R}$,*

$$\operatorname*{ess\,sup}_{-\tau<t<0}\int_{K_\rho\times\{t\}}\left[\psi^\pm(u)\right]^2\xi^{p^+}dx \leq \int_{K_\rho\times\{-\tau\}}\left[\psi^\pm(u)\right]^2\xi^{p^+}dx$$
$$+C\sum_{i=1}^{N}\int_{-\tau}^{0}\int_{K_\rho}\psi^\pm(u)\left[(\psi^\pm)'(u)\right]^{2-p_i(x,t)}\left|\frac{\partial\xi}{\partial x_i}\right|^{p_i(x,t)}dxdt. \tag{3.2.2}$$

Proof. In (3.1.5) take the testing function $\varphi = 2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u)\right]\xi^{p^+}$, and by direct computation we obtain

$$\left(\psi^{\pm}(u)\right)'' = \left\{\left(\psi^{\pm}(u)\right)'\right\}^2.$$

Therefore, we estimate the various terms separately, integrate in time over $(-\tau, t)$ for $t \in (-\tau, 0)$, and use Lemma 3.1.1. The first term gives

$$\begin{aligned}\int_{-\tau}^{t}\int_{K_\rho} u_{h,t}\left\{2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u_h)\right]\xi^{p^+}\right\}dxdt &= \int_{-\tau}^{t}\int_{K_\rho}\left(\psi^{\pm}(u_h)^2\right)_t \xi^{p^+}dxdt \\ \underset{h\longrightarrow 0}{\longrightarrow}\int_{K_\rho\times\{t\}}\left[\psi^{\pm}(u)\right]^2\xi^{p^+}dx &- \int_{K_\rho\times\{-\tau\}}\left[\psi^{\pm}(u)\right]^2\xi^{p^+}dx.\end{aligned} \tag{3.2.3}$$

As for the remaining term, we first let $h \longrightarrow 0$, to obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{-\tau}^{t}\int_{K_\rho}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}.\frac{\partial \varphi}{\partial x_i}\,dxdt \\ &= \sum_{i=1}^{N}\int_{-\tau}^{t}\int_{K_\rho}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\left[2\frac{\partial u}{\partial x_i}\left[(\psi^{\pm})'(u)\right]^2\xi^{p^+} + 2\frac{\partial u}{\partial x_i}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^2\xi^{p^+}\right]dxdt \\ &+ \sum_{i=1}^{N}\int_{-\tau}^{t}\int_{K_\rho}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}2p^+\psi^{\pm}(u)\left[(\psi^{\pm})'(u_h)\right]\xi^{p^+-1}\frac{\partial \xi}{\partial x_i}\,dxdt \\ &\geq \sum_{i=1}^{N}\int_{-\tau}^{t}\int_{K_\rho}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x,t)}2\left[(\psi^{\pm})'(u)\right]^2\left(1+\psi^{\pm}(u)-\psi^{\pm}(u)\right)\xi^{p^+}dxdt \\ &- C\sum_{i=1}^{N}\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^{2-p_i(x,t)}\left|\frac{\partial \xi}{\partial x_i}\right|^{p_i(x,t)}dxdt \\ &\geq -C\sum_{i=1}^{N}\int_{-\tau}^{t}\int_{K_\rho}\psi^{\pm}(u)\left[(\psi^{\pm})'(u)\right]^{2-p_i(x,t)}\left|\frac{\partial \xi}{\partial x_i}\right|^{p_i(x,t)}dxdt.\end{aligned}$$

Hence, since $t \in (-\tau, 0)$ is arbitrary, we can combine both estimates to obtain (3.2.2). □

3.3 Continuity of the weak solutions

In this section we analyze the alternative and prove Proposition 3.1.5. Assume that (3.1.8) is verified. The following lemma determine the number ν_0 and guarantees that the solution u is above a smaller level within a smaller cylinder.

Lemma 3.3.1. *There exists $\nu_0 \in (0,1)$, depending on the data and ω, such that if (3.1.8) holds true then*

$$u(x,t) > \mu^- + \frac{\omega}{4} \qquad a.e. \ in \ (0,t^*) + Q\left(\theta\left(\frac{R}{2}\right)^{p^+}, \frac{R}{2}\right). \tag{3.3.1}$$

Proof. Up to translation we can assume that $(0,t^*) = (0,0)$. Define two decreasing sequences of positive numbers

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \quad k_n = \mu^- + \frac{\omega}{4} + \frac{\omega}{2^{n+2}}, \quad n = 0,1,...$$

We construct the family of nested and shrinking cylinders $Q(\theta R_n^{p^+}, R_n)$, and let $0 \leq \xi_n(x,t) \leq 1$ be piecewise smooth functions in $Q(\theta R_n^{p^+}, R_n)$ such that

$$\begin{cases} \xi_n = 1 \quad \text{in } Q(\theta R_{n+1}^{p^+}, R_{n+1}), \quad \xi_n = 0 \quad \text{on } \partial_p Q(\theta R_n^{p^+}, R_n), \\ \left|\frac{\partial \xi_n}{\partial x_i}\right| \leq \dfrac{2^{\frac{(n+1)p^+}{p_i^+}}}{R^{\frac{p^+}{p_i^+}}}, \quad 0 < (\xi_n)_t \leq \dfrac{2^{p^+(n+1)}}{\theta R^{p^+}}, \ \forall i = 1,..,N. \end{cases}$$

Now, by using the energy inequality (3.2.1) for the functions $(u-k_n)_-$ we get

$$\begin{aligned}
&\sup_{-\theta R_n^{p^+} < t < 0} \int_{K_{R_n}} (u-k_n)_-^2 \, \xi_n^{p^+}(x,t) dx + \sum_{i=1}^N \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \left|\frac{\partial}{\partial x_i}(u-k_n)_-\right|^{p_i^-} \xi_n^{p^+} dxdt \\
&\leq C\bigg(\int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^2 \, \xi_n^{p^+-1} (\xi_n)_t \, dxdt + \sum_{i=1}^N \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^{p_i(x,t)} \left|\frac{\partial \xi_n}{\partial x_i}\right|^{p_i(x,t)} dxdt \\
&\quad + \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \chi\left((u-k_n)_- > 0\right) dxdt\bigg) \\
&\leq C\frac{2^{p^+(n+1)}}{R^{p^+}}\bigg(\frac{1}{\theta}\int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^2 \ dxdt + \sum_{i=1}^N \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} (u-k_n)_-^{p_i(x,t)} dxdt \\
&\quad + \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \chi\left((u-k_n)_- > 0\right) dxdt\bigg) \\
&\leq C\frac{2^{p^+(n+1)}}{R^{p^+}}\left(\left(\frac{\omega}{2}\right)^{p^-} + \left(\frac{\omega}{2}\right)^{p^+} + 1\right) \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \chi\left((u-k_n)_- > 0\right) dxdt \\
&\leq C\frac{2^{p^+(n+2)}}{R^{p^+}}\left(\left(\frac{\omega}{2}\right)^{p^+} + 1\right) \int_{-\theta R_n^{p^+}}^0 \int_{K_{R_n}} \chi\left((u-k_n)_- > 0\right) dxdt
\end{aligned}$$

By means of (3.1.7), this implies that

$$\sup_{-\theta R_n^{p^+}<t<0}\int_{K_{R_n}}(u-k_n)_-^2\,\xi_n^{p^+}(x,t)dx+\sum_{i=1}^N\int_{-\theta R_n^{p^+}}^0\int_{K_{R_n}}\left|\frac{\partial}{\partial x_i}(u-k_n)_-\right|^{p_i^-}\xi_n^{p^+}dxdt$$
$$\le C\frac{2^{p^+(n+2)}}{R^{p^+}}\left(\frac{\omega}{2}\right)^{p^+}\int_{-\theta R_n^{p^+}}^0\int_{K_{R_n}}\chi\left((u-k_n)_->0\right)dxdt,$$

where χ_E denotes the characteristic function of the set E. Using the fact that $(u-k_n)_-=0$ or

$$(u-k_n)_-=(\mu^--u)+\frac{\omega}{4}+\frac{\omega}{2^{n+2}}\le\frac{\omega}{2},\tag{3.3.2}$$

we get

$$(u-k_n)_-^2\ge\theta(u-k_n)_-^{p^-}.\tag{3.3.3}$$

Then the above estimates read

$$\sup_{-\theta R_n^{p^+}<t<0}\int_{K_{R_n}}(u-k_n)_-^{p^-}\,\xi_n^{p^+}(x,t)dx+\frac{1}{\theta}\sum_{i=1}^N\int_{-\theta R_n^{p^+}}^0\int_{K_{R_n}}\left|\frac{\partial}{\partial x_i}(u-k_n)_-\right|^{p_i^-}\xi_n^{p^+}dxdt$$
$$\le C\frac{2^{p^+(n+2)}}{R^{p^+}}\left(\frac{\omega}{2}\right)^{p^+}\frac{1}{\theta}\int_{-\theta R_n^{p^+}}^0\int_{K_{R_n}}\chi\left((u-k_n)_->0\right)dxdt,\tag{3.3.4}$$

Let us now consider the change of variable $\tilde{t}=\frac{t}{\theta}$ and define the functions

$$\tilde{u}(.,\tilde{t})=u(.,t),\qquad \tilde{\xi_n}(.,\tilde{t})=\xi_n(.,t).$$

Then, for

$$A_n=\int_{-R_n^{p^+}}^0\int_{K_{R_n}}\chi\left((\tilde{u}-k_n)_->0\right)dxd\tilde{t},$$

the inequality (3.3.4) becomes

$$\sup_{-R_n^{p^+}<\tilde{t}<0}\int_{K_{R_n}}(\tilde{u}-k_n)_-^{p^-}\,\tilde{\xi_n}^{p^+}(x,\tilde{t})dx+\sum_{i=1}^N\int_{-R_n^{p^+}}^0\int_{K_{R_n}}\left|\frac{\partial}{\partial x_i}(\tilde{u}-k_n)_-\right|^{p_i^-}\tilde{\xi_n}^{p^+}dxd\tilde{t}$$
$$\le C\frac{2^{p^+(n+2)}}{R^{p^+}}\left(\frac{\omega}{2}\right)^{p^+}A_n.\tag{3.3.5}$$

By the definition of k_n, we have

$$\left(\frac{\omega}{2^{n+3}}\right)^{\overline{p}}A_{n+1}=|k_n-k_{n+1}|^{\overline{p}}A_{n+1}\le\int_{-R_n^{p^+}}^0\int_{K_{R_n}}(\tilde{u}-k_n)_-^{\overline{p}}\tilde{\xi_n}^{\beta}dxd\tilde{t}.$$

Now we use Hölder's inequality with exponents $\frac{N}{N-\overline{p}}$ and $\frac{N}{\overline{p}}$ to obtain

$$\left(\frac{\omega}{2^{n+3}}\right)^{\overline{p}} A_{n+1} \leq C \int_{-R_n^{p^+}}^{0} \left(\int_{K_{R_n}} \left((\tilde{u} - k_n)_- \tilde{\xi_n}^{\frac{\beta}{\overline{p}}} \right)^{p^*} dx \right)^{\frac{\overline{p}}{p^*}} d\tilde{t}\; A_n^{\frac{\overline{p}}{N}},$$

where p^* is defined in (3.1.1). So, by the anisotropic Sobolev inequality (3.1.2), we have

$$\begin{aligned}
\left(\frac{\omega}{2^{n+3}}\right)^{\overline{p}} A_{n+1} &\leq C \int_{-R_n^{p^+}}^{0} \prod_{i=1}^{N} \left\{ \int_{K_{R_n}} \left| \frac{\partial}{\partial x_i} \left[(\tilde{u} - k_n)_- \tilde{\xi_n}^{\frac{\beta}{\overline{p}}} \right] \right|^{p_i^-} dx \right\}^{\frac{\overline{p}}{N p_i^-}} d\tilde{t}\; A_n^{\frac{\overline{p}}{N}} \\
&\leq C \prod_{i=1}^{N} \left\{ \int_{-R_n^{p^+}}^{0} \int_{K_{R_n}} \left| \frac{\partial}{\partial x_i} (\tilde{u} - k_n)_- \right|^{p_i^-} \tilde{\xi_n}^{\frac{\beta p_i^-}{\overline{p}}} dx d\tilde{t} \right. \\
&\quad \left. + \int_{-R_n^{p^+}}^{0} \int_{K_{R_n}} (\tilde{u} - k_n)_-^{p_i^-} \left| \frac{\partial \tilde{\xi_n}}{\partial x_i} \right|^{p_i^-} dx d\tilde{t} \right\}^{\frac{\overline{p}}{N p_i^-}} A_n^{\frac{\overline{p}}{N}}.
\end{aligned}$$

Since $0 \leq \xi_n(x,t) \leq 1$, we choose β such that $p^+ \leq \frac{\beta p_i^-}{\overline{p}}$, $\forall i = 1, 2, .., N$. Therefore, by using (3.3.5) we obtain

$$\left(\frac{\omega}{2^{n+3}}\right)^{\overline{p}} A_{n+1} \leq C \frac{2^{p^+(n+2)}}{R^{p^+}} \left(\frac{\omega}{2}\right)^{p^+} A_n^{1+\frac{\overline{p}}{N}}. \tag{3.3.6}$$

A direct calculation leads to

$$\frac{\left| Q(R_n^{p^+}, R_n) \right|^{1+\frac{\overline{p}}{N}}}{\left| Q(R_{n+1}^{p^+}, R_{n+1}) \right|} \leq 2^{P^+ + N} R^{\overline{p}(1+\frac{\overline{p}}{N})}. \tag{3.3.7}$$

Next, if we define the numbers X_n by

$$X_n = \frac{A_n}{Q(R_n^{p^+}, R_n)},$$

we obtain the following recursive relation

$$X_{n+1} \leq C 4^{np^+} \left(\frac{\omega}{2}\right)^{p^+ - \overline{p}} X_n^{1+\frac{\overline{p}}{N}}.$$

Thus, Lemma 4.1 of Chapter I in [29] implies that if

$$X_0 \leq \left[C \left(\frac{\omega}{2}\right)^{p^+ - \overline{p}} \right]^{-\frac{N}{\overline{p}}} 4^{-p^+ (\frac{N}{\overline{p}})^2} = \nu_0, \tag{3.3.8}$$

then

$$X_n \longrightarrow 0. \tag{3.3.9}$$

But (3.3.8) is nothing but the assumption (3.1.8). Hence, the result easily follows from (3.3.9). ☐

Now consider the time level $-\hat{t} = t^* - \theta\left(\frac{R}{2}\right)^{p^+}$, then from the conclusion of Lemma 3.3.1, we have

$$u(x, -\hat{t}) > \mu^- + \frac{\omega}{4} \quad \text{a.e. in } x \in K_{\frac{R}{2}}.$$

We will use this time level as an initial condition to bring the information up to $t = 0$, and therefore to obtain an analogous inequality in a in a smaller cylinder. The first step in this direction is given by the following lemma.

Lemma 3.3.2. *For every $\nu_1 \in (0,1)$, there exists a positive integer s_1 depending on the data and ω, such that*

$$\left| x \in K_{\frac{R}{4}}, \quad u(x,t) < \mu^- + \frac{\omega}{2^{s_1}} \right| \leq \nu_1 \left| K_{\frac{R}{4}} \right|, \quad \forall t \in (-\hat{t}, 0). \tag{3.3.10}$$

Proof. Consider the cylinder $Q(\hat{t}, \frac{R}{2})$ and write the logarithmic estimate (3.2.2) over this cylinder, for the function $(u-k)_-$, with

$$k = \mu^- + \frac{\omega}{4} \quad \text{and } c = \frac{\omega}{2^{n+2}},$$

where n is to be chosen later. We define H_k^- such that

$$k - u \leq H_k^- = \underset{Q(\hat{t}, \frac{R}{2})}{ess\sup} \left| \left(u - \mu^- - \frac{\omega}{4} \right)_- \right| \leq \frac{\omega}{4}. \tag{3.3.11}$$

Assuming $H_k^- \leq \frac{\omega}{8}$ (else the result is trivial). Then the logarithmic function ψ^- is well defined and satisfies the inequalities

$$\psi^- \leq n \ln(2) \text{ since } \frac{H_k^-}{H_k^- + u - k + c} \leq \frac{\frac{\omega}{4}}{c} = 2^n, \tag{3.3.12}$$

and, for $u \neq -k + c$,

$$0 \leq (\psi^-)' \leq \frac{1}{H_k^- + u - k + c} \leq \frac{1}{c} \tag{3.3.13}$$

and,

$$\left| \left(\psi^-\right)'(u) \right|^{2-p^-} = \left(H_k^- + u - k + c\right)^{p^- - 2} \leq \left(\frac{\omega}{2}\right)^{p^- - 2}. \tag{3.3.14}$$

For $t=-\hat{t}$, by virtue of Lemma 3.3.1, we have $u(x,-\hat{t})>k$, and therefore

$$\left[\psi^-(u)\right](x,-\hat{t})=0 \ \text{ for } x\in K_{\frac{R}{2}}.$$

To obtain the estimate, we choose a cutoff function $0<\xi(x)\leq 1$, defined on $K_{\frac{R}{2}}$, such that

$$\xi=1 \ \text{ in } K_{\frac{R}{2}} \ \text{ and } \ \left|\frac{\partial\xi}{\partial x_i}\right|\leq\left(\frac{8}{R}\right)^{\frac{p^+}{p_i^+}}, \ \forall i=1,2,..N.$$

Gathering these estimates in (3.2.2), and using the fact that

$$\hat{t}\leq\left(\frac{\omega}{2^\lambda}\right)^{2-p^-}R^{p^+}, \tag{3.3.15}$$

we arrive at

$$\begin{aligned}
&\underset{-\hat{t}<t<0}{ess\sup}\int_{K_{\frac{R}{2}}\times\{t\}}\left[\psi^-(u)\right]^2\xi^{p^+}dx\\
&\leq C\sum_{i=1}^{N}\int_{-\hat{t}}^{0}\int_{K_{\frac{R}{2}}}\psi^-(u)\left[(\psi^-)'(u)\right]^{2-p_i(x,t)}\left|\frac{\partial\xi}{\partial x_i}\right|^{p_i(x,t)}dxdt\\
&\leq C\,n\ln(2)\left(\frac{\omega}{2}\right)^{p^--2}\left(\frac{8}{R}\right)^{p^+}\widehat{t}\left|K_{\frac{R}{4}}\right|\\
&\leq C\,n\left(\frac{\omega}{2}\right)^{p^--2}\left(\frac{8}{R}\right)^{p^+}\left(\frac{\omega}{2^\lambda}\right)^{2-p^-}R^{p^+}\left|K_{\frac{R}{4}}\right|\\
&\leq C\,n\,2^{\lambda(p^--2)}\left|K_{\frac{R}{4}}\right|.
\end{aligned} \tag{3.3.16}$$

The left hand side of (3.3.16) is estimated from below integrating over the smaller set

$$S=\left\{x\in K_{\frac{R}{4}},\ u(x,t)<\mu^-+\frac{\omega}{2^{n+2}}\right\}\subset K_{\frac{R}{2}},\quad t\in(-\hat{t},0).$$

On such set, $\xi=1$ and $\psi^-\geq((n-1)\ln(2))$, because

$$\frac{H_k^-}{H_k^-+u-k+\frac{\omega}{2^{n+2}}}\geq\frac{\frac{\omega}{4}}{\frac{\omega}{4}+u-k+\frac{\omega}{2^{n+2}}}=\frac{\frac{\omega}{4}}{u-\mu^-+\frac{\omega}{2^{n+2}}}\geq\frac{\frac{\omega}{4}}{2^{n+2}}=2^{n-1}.$$

Putting this in (3.3.16), we obtain that for all $t\in(-\hat{t},0)$

$$|S|\leq C\frac{n}{(n-1)^2}2^{\lambda(p^--2)}\left|K_{\frac{R}{4}}\right|.$$

The proof is complete once we choose $s_1=n+2$ with $n>1+\dfrac{2C}{\nu_1}2^{\lambda(p^--2)}$. □

The conclusion of Lemma 3.3.2 will be employed to deduce that, within the cylinder $Q\left(\hat{t}, \frac{R}{8}\right)$, the set where u is away from its infimum is arbitrarily small.

Lemma 3.3.3. *There exists $1 < s_2 \in \mathbb{N}$, depending on the data and ω, such that*

$$u(x,t) > \mu^- + \frac{\omega}{2^{s_2+1}} \qquad a.e.\ (x,t) \in Q\left(\hat{t}, \frac{R}{8}\right). \tag{3.3.17}$$

Proof. Define two decreasing sequences of positive numbers

$$R_n = \frac{R}{8} + \frac{R}{2^{n+3}}, \quad k_n = \mu^- + \frac{\omega}{2^{s_2+1}} + \frac{\omega}{2^{s_2+1+n}}, \quad n = 0, 1, \dots.$$

We construct the family of nested and shrinking cylinders $Q(\hat{t}, R_n)$, and let $0 \leq \xi_n(x) \leq 1$ be piecewise smooth functions in K_{R_n} that equal one on $K_{R_{n+1}}$ and

$$\left|\frac{\partial \xi_n}{\partial x_i}\right| \leq \frac{2^{(n+4)\frac{p^+}{p_i^+}}}{R^{\frac{p^+}{p_i^+}}} \ \forall i = 1, .., N.$$

Lemma 3.3.1 implies that

$$(u - k_n)_-(x, -\hat{t}) = 0 \text{ in } K_{R_n}.$$

Now, since $(u - k_n)_- \leq \frac{\omega}{2^{s_2}}$, by using (3.3.15) and letting $s_2 > \lambda + \frac{p^+}{p^- - 2}$ we get

$$(u - k_n)_-^2 \geq \frac{\hat{t}}{\left(\frac{R}{2}\right)^{p^+}} (u - k_n)_-^{p^-}.$$

Therefore, with these choices, and by applying the local energy inequalities (3.2.1) on the functions $(u - k_n)_-$, we get

$$\begin{aligned}
&\frac{\hat{t}}{\left(\frac{R}{2}\right)^{p^+}} \sup_{-\hat{t}<t<0} \int_{K_{R_n}\times\{t\}} (u - k_n)_-^{p^-} \xi_n^{p^+} dx + \sum_{i=1}^{N} \int_{-\hat{t}}^{0} \int_{K_{R_n}} \left|\frac{\partial}{\partial x_i}(u - k_n)_-\right|^{p_i^-} \xi_n^{p^+} dxdt \\
&\leq C\left(\sum_{i=1}^{N} \int_{-\hat{t}}^{0} \int_{K_{R_n}} (u - k_n)_-^{p_i^+} \left|\frac{\partial \xi_n}{\partial x_i}\right|^{p_i^+} dxdt + \int_{-\hat{t}}^{0} \int_{K_{R_n}} \chi\left((u - k_n)_- > 0\right) dxdt\right) \\
&\leq C \frac{2^{np^+}}{R^{p^+}} \left(\frac{\omega}{2^{s_2}}\right)^{p^+} \int_{-\hat{t}}^{0} \int_{K_{R_n}} \chi\left((u - k_n)_- > 0\right) dxdt.
\end{aligned} \tag{3.3.18}$$

We divide by $\frac{\hat{t}}{\left(\frac{R}{2}\right)^{p^+}}$ throughout (3.3.18), introduce the change of variable $\widetilde{t} = t\frac{\left(\frac{R}{2}\right)^{p^+}}{\hat{t}}$. Similarly to the proof of Lemma 3.3.1, we arrive at

$$\left(\frac{\omega}{2^{s_2+2+n}}\right)^{\overline{p}} A_{n+1} \leq C\frac{2^{np^+}}{R^{p^+}}\left(\frac{\omega}{2^{s_2}}\right)^{p^+} A_n^{1+\frac{\overline{p}}{N}}, \tag{3.3.19}$$

where

$$A_n = \int_{-\left(\frac{R}{2}\right)^{p^+}}^{0}\int_{K_{R_n}} \chi\left((\widetilde{u}-k_n)_- > 0\right) dx d\widetilde{t}.$$

Here we have considered

$$\widetilde{u}(x,\overline{t}) = u(x,t).$$

Next, we define the numbers X_n

$$X_n = \frac{A_n}{Q\left(\left(\frac{R}{2}\right)^{p^+}, R_n\right)}.$$

Dividing (3.3.19) by $Q\left(\left(\frac{R}{2}\right)^{p^+}, R_{n+1}\right)$, we obtain the following recursive relation

$$X_{n+1} \leq C4^{np^+}\left(\frac{\omega}{2^{s_2}}\right)^{p^+-\overline{p}} X_n^{1+\frac{\overline{p}}{N}}.$$

Therefore, Lemma 4.1 of Chapter I in [29] implies that if

$$X_0 \leq \left[C\left(\frac{\omega}{2^{s_2}}\right)^{p^+-\overline{p}}\right]^{-\frac{N}{\overline{p}}} 4^{-p^+\left(\frac{N}{\overline{p}}\right)^2} = \nu_1, \tag{3.3.20}$$

then

$$X_n \longrightarrow 0. \tag{3.3.21}$$

By applying Lemma 3.3.2 with $s_1 := s_2$ we get easily (3.3.20). Hence, the result easily follows from (3.3.21). □

As an immediate consequence we get the reduction of the oscillation of u.

Corollary 3.3.4. *There exists a constant $\sigma_0 \in (0,1)$, depending only on the data and ω, such that if (3.1.8) holds then*

$$\underset{Q\left(\theta\left(\frac{R}{8}\right)^{p^+}, \frac{R}{8}\right)}{ess\ osc}\ u \leq \sigma_0\ \omega. \tag{3.3.22}$$

Proof. The proof follows since $Q\left(\theta\left(\frac{R}{8}\right)^{p^+}, \frac{R}{8}\right) \subset Q\left(\hat{t}, \frac{R}{8}\right)$, where we have $\sigma_0 = 1 - \frac{1}{2^{s_2+1}}$. □

Assume that (3.1.8) does not hold, then (3.1.9) is in force. Even in this case, we are able to deduce a result analogous to Corollary 3.3.4.

Lemma 3.3.5. *Assume that (3.1.9) holds true, then there exists a time level*

$$t_0 \in \left[t^* - \theta R^{p^+}, t^* - \frac{\nu_0}{2}\theta R^{p^+}\right], \tag{3.3.23}$$

such that

$$\left|\left\{x \in K_R,\ u(x,t_0) > \mu^+ - \frac{\omega}{2}\right\}\right| \leq \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right)|K_R|. \tag{3.3.24}$$

Proof. In fact, if (3.3.24) does not hold, then also (3.1.9) does not hold. □

This lemma shows that at the time level t_0, the portion of the cube K_R where $u(x)$ is close to its supremum is small. The next lemma claims that this indeed occurs for all time levels near the top of the cylinder $(0,t^*) + Q(\theta R^{p^+}, R)$.

Lemma 3.3.6. *There exists* $1 < s_3 \in \mathbb{N}$*, depending on the data and* ω*, such that, for all* $t \in \left[t^* - \frac{\nu_0}{2}\theta R^{p^+}, t^*\right]$,

$$\left|\left\{x \in K_R,\ u(x,t) > \mu^+ - \frac{\omega}{2^{s_3}}\right\}\right| \leq \left(1 - \left(\frac{\nu_0}{2}\right)^2\right)|K_R|. \tag{3.3.25}$$

Proof. Consider the cylinder $K_R \times (t_0, t^*)$, the level $k = \mu^+ - \frac{\omega}{2}$ and define

$$u - k \leq H_k^+ = \underset{K_R\times(t_0,t^*)}{ess\sup}\left|\left(u - \mu^+ + \frac{\omega}{2}\right)_+\right| \leq \frac{\omega}{2}. \tag{3.3.26}$$

Assuming that $H_k^+ > \frac{\omega}{4}$ (otherwise there is nothing to prove). Select $n \in \mathbb{N}$ big enough so that

$$0 < c = \frac{\omega}{2^{n+1}} < H_k^+.$$

Then the logarithmic function ψ^+ is well defined and satisfies the inequalities

$$\psi^+ \leq n\ln(2) \ \text{ since } \frac{H_k^+}{H_k^+ - u + k + c} \leq \frac{\frac{\omega}{4}}{c} = 2^n, \tag{3.3.27}$$

and, for $u \neq k + c$

$$0 \leq (\psi^+)' \leq \frac{1}{H_k^+ - u + k + c} \leq \frac{1}{c}, \tag{3.3.28}$$

and

$$\left|\left(\psi^+\right)'(u)\right|^{2-p^-} = \left(H_k^+ - u + k + c\right)^{p^- - 2} \leq \left(\frac{\omega}{2}\right)^{p^- - 2}. \tag{3.3.29}$$

In the logarithmic inequality (3.2.2) applied to the function $(u-k)_+$, let $x \longmapsto \xi(x)$ be a smooth cutoff function defined in K_R such that for some $\pi \in (0,1)$

$$\begin{cases} 0 \le \xi \le 1 \quad \text{in } K_R, \quad \xi = 0 \ \text{ on } K_{(1-\pi)R}, \\ \left|\dfrac{\partial \xi}{\partial x_i}\right| \le (\pi R)^{-\frac{p^+}{p_i^+}}, \ \forall i = 1,..,N. \end{cases}$$

Gathering these estimates in (3.2.2), using Lemma 3.3.5 and the fact that

$$t^* - t \le \theta R^{p^+}, \tag{3.3.30}$$

we arrive at

$$\begin{aligned} ess \sup_{t_0<t<t^*} \int_{K_R\times\{t\}} \left[\psi^+(u)\right]^2 \xi^{p^+} dx &\le \int_{K_R\times\{t_0\}} \left[\psi^+(u)\right]^2 \xi^{p^+} dx \\ &+ C \sum_{i=1}^{N} \int_{t_0}^{t*} \int_{K_R} \psi^+(u) \left[(\psi^+)'(u)\right]^{2-p_i(x,t)} \left|\frac{\partial \xi}{\partial x_i}\right|^{p_i(x,t)} dxdt \\ &\le n^2(\ln 2)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) |K_R| + C\ n \ln(2) \left(\frac{\omega}{2}\right)^{p^- - 2} (\pi R)^{-p^+} (t^* - t_0)|K_R| \\ &\le n^2(\ln 2)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) |K_R| + C\ n \ln(2) \left(\frac{\omega}{2}\right)^{p^- - 2} (\pi R)^{-p^+} \theta R^{p^+} |K_R| \\ &\le n^2(\ln 2)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) |K_R| + C \frac{n}{\pi^{p^+}} |K_R|. \end{aligned} \tag{3.3.31}$$

The left hand side is estimated below by integrating over the smaller set

$$S = \left\{ x \in K_{(1-\pi)R} : \quad u(x,t) > \mu^+ - \frac{\omega}{2^{n+1}} \right\} \subset K_R.$$

On such set, $\xi = 1$ and $\psi^+ \ge (n-1)\ln 2$, because

$$\frac{H_k^+}{H_k^+ - u + k + c} \ge \frac{\frac{\omega}{2}}{\frac{\omega}{2} - u + k + \frac{\omega}{2^{n+1}}} \ge \frac{\frac{\omega}{2}}{\frac{\omega}{2^n}} \ge 2^{n-1},$$

since one has $-u + \mu^+ < \frac{\omega}{2^n}$. Therefore for all $t \in (t_0, t^*)$

$$|S| \le \left\{ \left(\frac{n}{n-1}\right)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) + \frac{C}{n\pi^{p^+}} \right\} |K_R|.$$

Consequently, for all $t \in (t_0, t^*)$,

$$
\begin{aligned}
\left|\left\{x \in K_R,\ u(x,t) > \mu^+ - \frac{\omega}{2^{n+1}}\right\}\right| &\le |S| + N\pi|K_R| \\
&\le \left\{\left(\frac{n}{n-1}\right)^2\left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) + \frac{c}{n\pi^{p^+}} + N\pi\right\}|K_R|.
\end{aligned} \tag{3.3.32}
$$

The proof is complete once we choose π so small that $N\pi \le \frac{3}{8}\nu_0^2$, then n so large that

$$
\frac{C}{n\pi^{p^+}} \le \frac{3}{8}\nu_0^2 \quad \text{and} \quad \left(\frac{n}{n-1}\right)^2 \le (1-\frac{\nu_0}{2})(1+\nu_0) < 1,
$$

and finally take $s_3 = n+1$. □

Recalling that $t_0 \in \left[t^* - \theta R^{p^+}, t^* - \frac{\nu_0}{2}\theta R^{p^+}\right]$ and choosing λ such that $2^{(\lambda-1)(p^- - 2)} \ge 2$, the previous lemma immediately implies

Lemma 3.3.7. *There exists $1 < s_3 \in \mathbb{N}$, depending on the data and ω, such that for all $t \in \left(-\frac{a_0}{2}R^{p^+}, 0\right)$,*

$$
\left|\left\{x \in K_R,\ u(x,t) > \mu^+ - \frac{\omega}{2^{s_3}}\right\}\right| \le \left(1 - \left(\frac{\nu_0}{2}\right)^2\right)|K_R|. \tag{3.3.33}
$$

From Lemma 3.3.7 we deduce that within the cylinder $Q(a_0R^{p^+}, R)$, the set where u is close to its supremum is arbitrarily small.

Lemma 3.3.8. *For every $v_1 \in (0,1)$, there exists $s_3 \le \lambda \in \mathbb{N}$ depending on the data and ω, such that*

$$
\left|\left\{(x,t) \in Q\left(\frac{a_0}{2}R^{p^+}, R\right),\ u(x,t) > \mu^+ - \frac{\omega}{2^\lambda}\right\}\right| \le \nu_1 \left|Q\left(\frac{a_0}{2}R^{p^+}, R\right)\right|. \tag{3.3.34}
$$

Proof. Consider the cylinder $Q(a_0R^{p^+}, 2R)$ and the level $k = \mu^+ - \frac{\omega}{2^s}$, for $s_3 \le s \le \lambda$, consider also the local energy estimates (3.2.1) for the functions $(u-k)_+$, where $0 \le \xi(x,t) \le 1$ is a smooth cutoff function defined in $Q(a_0R^{p^+}, 2R)$ and satisfying

$$
\begin{cases}
\xi = 1 \quad \text{in } Q(\frac{a_0}{2}R^{p^+}, R), \quad \xi = 0 \quad \text{on } \partial_p Q(a_0R^{p^+}, 2R), \\
\left|\frac{\partial \xi}{\partial x_i}\right| \le \frac{1}{R^{\frac{p^+}{p_i^+}}}, \ \forall i = 1,..,N. \qquad 0 < \xi_t \le \frac{2}{a_0R^{p^+}}.
\end{cases}
$$

Neglecting the first term on the left hand side of (3.2.1), and using the indicated choices, we obtain

$$\sum_{i=1}^{N} \iint_{Q(\frac{a_0}{2}R^{p^+},R)} \left|\frac{\partial}{\partial x_i}(u-k)_+\right|^{p^-} dxdt$$

$$\leq C\Big(\sum_{i=1}^{N} \iint_{Q(a_0R^{p^+},2R)} (u-k)_+^{p_i^+} \left|\frac{\partial \xi}{\partial x_i}\right|^{p_i^+} + \iint_{Q(a_0R^{p^+},2R)} (u-k)_+^2 \xi_t \ dxdt$$

$$+ \iint_{Q(a_0R^{p^+},2R)} \chi\left((u-k)_+ > 0\right) dxdt\Big)$$

$$\leq C\Big(\frac{1}{R^{p^+}} \sum_{i=1}^{N} \iint_{Q(\frac{a_0}{2}R^{p^+},R)} (u-k)_+^{p_i^+} \ dxdt + \frac{1}{a_0R^{p^+}} \iint_{Q(\frac{a_0}{2}R^{p^+},R)} (u-k)_+^2 \ dxdt$$

$$+ \iint_{Q(\frac{a_0}{2}R^{p^+},R)} \chi\left((u-k)_+ > 0\right) dxdt\Big)$$

$$\leq C\Big(\frac{1}{R^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^+} + \frac{1}{R^{p^+}}\left(\frac{\omega}{2^\lambda}\right)^{p^- -2}\left(\frac{\omega}{2^s}\right)^2 + 1\Big)\left|Q(\frac{a_0}{2}R^{p^+},R)\right|$$

$$\leq \frac{C}{R^{p^+}}\left(\frac{\omega}{2^s}\right)^{p^+}\left|Q(\frac{a_0}{2}R^{p^+},R)\right|. \tag{3.3.35}$$

Here, we have used (3.1.7) and the fact that $s \leq \lambda$.

Next, for each $s \leq \lambda$, introduce the two complimentary sets

$$A_s(t) = \Big\{x \in K_R,\ u(x,t) > \mu^+ - \frac{\omega}{2^s}\Big\} \ , \quad K_R - A_s(t) = \Big\{x \in K_R,\ u(x,t) \leq \mu^+ - \frac{\omega}{2^s}\Big\},$$

and let

$$A_s = \int_{\frac{-a_0}{2}R^{p^+}}^{0} A_s(t)\ dt.$$

Now, consider the doubly truncated function such that for all $t \in \Big(-\frac{a_0}{2}R^{p^+}, 0\Big)$

$$v_s = \begin{cases} 0 & \text{for} \quad u(x,t) < \mu^+ - \frac{\omega}{2^s}, \\ u - \left(\mu^+ - \frac{\omega}{2^s}\right) & \text{for} \quad \mu^+ - \frac{\omega}{2^s} \leq u(x,t) \leq \mu^+ - \frac{\omega}{2^{s+1}}, \\ \frac{\omega}{2^{s+1}} & \text{for} \quad \mu^+ - \frac{\omega}{2^{s+1}} \leq u(x,t). \end{cases} \tag{3.3.36}$$

By construction v_s vanishes on $K_R - A_s(t)$. Pick any two points $x = (x_1, .., x_N, t) \in A_s$ and $y = (y_1, .., y_N, t) \in K_R - A_s(t)$, and construct a polygonal joining x and y and sides parallel to the coordinate axes, say for example $P_N = x$ and

$$P_{N-1} = (x_1, .., x_{N-1}, y_N);\ P_{N-2} = (x_1, .., y_{N-1}, y_N), ...P_1 = (x_1, y_2.., y_N);\ P_0 = (y_1, .., y_N).$$

By elementary computation, we have

$$\begin{aligned} v_s(x,t) &= [v_s(P_N,t) - v_s(P_{N-1},t)] + ... + [v_s(P_1,t) - v_s(P_0,t)] \\ &= \int_{y_N}^{x_N} \frac{\partial}{\partial x_N} v_s(x_1, .., x_{N-1}, \zeta, t) d\zeta + \int_{y_{N-1}}^{x_{N-1}} \frac{\partial}{\partial x_{N-1}} v_s(x_1, .., x_{N-2}, \zeta, y_N, t) d\zeta \\ &+ .... + \int_{y_1}^{x_1} \frac{\partial}{\partial x_1} v_s(\zeta, y_2, .., y_N, t) d\zeta \\ &\leq \sum_{i=1}^{N} \int_{-R}^{R} \left| \frac{\partial}{\partial x_i} v_s(x_1, ...,\ \ \zeta\ \ , ..., y_N, t) \right| d\zeta. \end{aligned}$$

Integrate in dx over $A_s(t)$ and in dy over $K_R - A_s(t)$, and take into account Lemma 3.3.7 to get

$$\left(\frac{\nu_0}{2}\right)^2 |K_R| \int_{K_R} v_s dx \leq 2R|K_R| \sum_{i=1}^{N} \int_{K_R} \left| \frac{\partial v_s}{\partial x_i} \right| dx.$$

Therefore, by the definitions of $A_s(t)$ and v_s, we have

$$\frac{\omega}{2^{s+1}} |A_{s+1}(t)| \leq \frac{C\ R}{\nu_0^2} \sum_{i=1}^{N} \int_{A_s(t) - A_{s+1}(t)} \left| \frac{\partial u}{\partial x_i} \right| dx.$$

Integrating over $t \in \left(-\frac{a_0}{2} R^{p^+}, 0\right)$, and using (3.3.35) we conclude that

$$\begin{aligned} \frac{\omega}{2^{s+1}} |A_{s+1}| &\leq \frac{C\ R}{\nu_0^2} \sum_{i=1}^{N} \iint_{A_s - A_{s+1}} \left| \frac{\partial u}{\partial x_i} \right| dxdt \\ &\leq \frac{C\ R}{\nu_0^2} \sum_{i=1}^{N} \left(\iint_{A_s} \left| \frac{\partial u}{\partial x_i} \right|^{p^-} dxdt \right)^{\frac{1}{p^-}} |A_s - A_{s+1}|^{\frac{p^- - 1}{p^-}} \qquad (3.3.37) \\ &\leq \frac{C}{\nu_0^2} \left(\frac{\omega}{2^s}\right)^{\frac{p^+}{p^-}} \left| Q(\frac{a_0}{2} R^{p^+}, R) \right|^{\frac{1}{p^-}} |A_s - A_{s+1}|^{\frac{p^- - 1}{p^-}}. \end{aligned}$$

If s is large enough so that $\left(\frac{\omega}{2^s}\right)^{\frac{p^+}{p^-}} \frac{2^{s+1}}{\omega} < 1$, from (3.3.37) we get

$$|A_{s+1}| \leq \frac{C}{\nu_0^2} \left| Q(\frac{a_0}{2} R^{p^+}, R) \right|^{\frac{1}{p^-}} |A_s - A_{s+1}|^{\frac{p^- - 1}{p^-}}, \qquad (3.3.38)$$

for all $s_3 \le s \le \lambda$. According to the previous energy estimates we get, for $s = s_3, s_3 + 1, ..., \lambda - 1$

$$|A_{s+1}|^{\frac{p^-}{p^--1}} \le C(\nu_0)^{\frac{-2p^-}{p^--1}} \left|Q\left(\frac{a_0}{2}R^{p^+}, R\right)\right|^{\frac{1}{p^--1}} |A_s - A_{s+1}|,$$

and we then add these inequalities for $s = s_3, s_3 + 1, ..., \lambda - 1$.

Since $\mu^+ - \frac{\omega}{2^{s+1}} \le \mu^+ - \frac{\omega}{2^\lambda}$, $A_{s+1} \ge A_\lambda$, and therefore

$$\sum_{s=s_3}^{\lambda-1} A_{s+1}^{\frac{p^-}{p^--1}} \ge (\lambda - s_3) A_\lambda^{\frac{p^-}{p^--1}}.$$

Note also that $\sum_{s=s_3}^{\lambda-1} |A_s - A_{s+1}| \le \left|Q\left(\frac{a_0}{2}R^{p^+}, R\right)\right|$. Collecting these results, we arrive at

$$A_\lambda \le \frac{C}{(\lambda - s_3)^{\frac{p^--1}{p^-}}} (\nu_0)^{-2} \left|Q\left(\frac{a_0}{2}R^{p^+}, R\right)\right|.$$

The proof is complete once we choose $s_3 < \lambda \in \mathbb{N}$ sufficiently large so that

$$\frac{C}{(\lambda - s_3)^{\frac{p^--1}{p^-}}} (\nu_0)^{-2} \le \nu_1.$$

□

Lemma 3.3.9. *The number $v_1 \in (0,1)$ can be chosen (and consequently, so can λ), such that*

$$u(x,t) \le \mu^+ - \frac{\omega}{2^{\lambda+1}} \qquad a.e.\ (x,t) \in Q\left(\frac{a_0}{2}\left(\frac{R}{2}\right)^{p^+}, R\right). \tag{3.3.39}$$

Proof. Define two decreasing sequences of positive numbers

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \quad k_n = \mu^+ - \frac{\omega}{2^{\lambda+1}} - \frac{\omega}{2^{\lambda+1+n}}, \quad n = 0, 1, ...$$

Now, consider the local energy estimates (3.2.1) for the functions $(u - k_n)_+$ over the constructed family of nested and shrinking cylinders $Q\left(\frac{a_0}{2}R_n^{p^+}, R_n\right)$, where $0 \le \xi_n(x,t) \le 1$ are smooth functions defined in $Q\left(\frac{a_0}{2}R_n^{p^+}, R_n\right)$ such that

$$\begin{cases} \xi_n = 1 \quad \text{in } Q\left(\frac{a_0}{2}R_{n+1}^{p^+}, R_{n+1}\right), \qquad \xi_n = 0 \ \text{ on } \partial_p Q\left(\frac{a_0}{2}R_n^{p^+}, R_n\right), \\ \left|\frac{\partial \xi_n}{\partial x_i}\right| \le \left(\frac{2^{n+1}}{R}\right)^{\frac{p^+}{p_i^+}}, \ \forall i = 1, .., N, \quad \text{and } \ 0 < (\xi_n)_t \le \dfrac{2^{p^+(n+1)}}{\frac{a_0}{2}R^{p^+}}. \end{cases}$$

Then, since

$$(u-k_n)_+^2 \geq a_0 (u-k_n)_+^{p^-},$$

we get

$$\begin{aligned}
&a_0 \underset{-\frac{a_0}{2}R_n^{p^+}<t<0}{ess\sup} \int_{K_{R_n}\times\{t\}} (u-k_n)_+^{p^-}\, \xi_n^{p^+} dx + \sum_{i=1}^{N} \int_{-\frac{a_0}{2}R_n^{p^+}}^{0} \int_{K_{R_n}} \left| \frac{\partial}{\partial x_i}(u-k_n)_+ \right|^{p_i^-} \xi_n^{p^+} dxdt \\
&\leq C \frac{2^{p^+(n+1)}}{R^{p^+}} \Bigg(\frac{1}{a_0} \int_{-\frac{a_0}{2}R_n^{p^+}}^{0} \int_{K_{R_n}} (u-k_n)_+^2 \; dxdt + \sum_{i=1}^{N} \int_{-\frac{a_0}{2}R_n^{p^+}}^{0} \int_{K_{R_n}} (u-k_n)_+^{p_i(x,t)}\, dxdt \\
&\qquad + \int_{-\frac{a_0}{2}R_n^{p^+}}^{0} \int_{K_{R_n}} \chi\left((u-k_n)_+ > 0\right) dxdt \Bigg) \\
&\leq C \frac{2^{p^+(n+1)}}{R^{p^+}} \left(\frac{\omega}{2^\lambda}\right)^{p^+} \int_{-\frac{a_0}{2}R_n^{p^+}}^{0} \int_{K_{R_n}} \chi\left((u-k_n)_+ > 0\right) dxdt.
\end{aligned}$$

Divide by a_0 throughout the above inequality, and we introduce the change of variable $\widetilde{t} = \dfrac{t}{\frac{a_0}{2}}$. Using the same tools of Lemma 3.3.1, we arrive at the following inequality

$$\left(\frac{\omega}{2^{\lambda+2+n}}\right)^{\overline{p}} A_{n+1} \leq C \frac{2^{np^+}}{R^{p^+}} \left(\frac{\omega}{2^\lambda}\right)^{p^+} A_n^{1+\frac{\overline{p}}{N}} \tag{3.3.40}$$

where

$$A_n = \iint\limits_{Q(R_n^{p^+}, R_n)} \chi\left((\widetilde{u}-k_n)_+ > 0\right) dx d\widetilde{t},$$

here we have considered

$$\widetilde{u}(x,\bar{t}) = u(x,t) \text{ and } \widetilde{\xi_n}(x,\widetilde{t}) = \xi_n(x,t).$$

Next, if we denote $X_n = \dfrac{A_n}{\left|Q(R_n^{p^+}, R_n)\right|}$, we get

$$X_{n+1} \leq C 4^{np^+} \left(\frac{\omega}{2^\lambda}\right)^{p^+-\overline{p}} X_n^{1+\frac{\overline{p}}{N}}.$$

Therefore, by using Lemma 4.1 of Chapter I in [29], the result is proved if we can assume that

$$X_0 \leq \left[C \left(\frac{\omega}{2^\lambda}\right)^{p^+-\overline{p}} \right]^{-\frac{N}{\overline{p}}} 4^{-p^+\left(\frac{N}{\overline{p}}\right)^2} = \nu_1. \tag{3.3.41}$$

For this value of ν_1, Lemma 3.3.8 implies that $X_0 \leq \nu_1$. Hence, we can conclude that $X_n \longrightarrow 0$ when $n \longrightarrow +\infty$ and the result follows. □

As an immediate consequence we get the reduction of the oscillation of u in the second case.

Corollary 3.3.10. *There exists a constant $\sigma_1 \in (0,1)$, depending only on the data and ω, such that if (3.1.9) holds then*

$$\operatorname*{ess\ osc}_{Q\left(\frac{a_0}{2}\left(\frac{R}{2}\right)^{p^+}, \frac{R}{2}\right)} u \leq \sigma_1\ \omega. \tag{3.3.42}$$

Proof. The proof follows by choosing $\sigma_1 = 1 - \frac{1}{2^{\lambda+1}}$. □

Now, we are able to prove Proposition 3.1.5, recalling the conclusions of Corollaries 3.3.4 and 3.3.10 and since $\theta\left(\frac{R}{8}\right)^{p^+} \leq \frac{a_0}{2}\left(\frac{R}{2}\right)^{p^+}$, we get that

$$\operatorname*{ess\ osc}_{Q\left(\theta\left(\frac{R}{8}\right)^{p^+}, \frac{R}{8}\right)} u \leq \sigma\ \omega,$$

where $\sigma = \max\{\sigma_0, \sigma_1\}$.

Part II

Approximation schemes

Chapter 4

Introduction

All interesting problems are difficult to solve. This observation in particular holds true in algorithm oriented areas like combinatorial optimization, mathematical programming, operations research, and theoretical computer science where researchers often face computationally intractable problems. Since solving an intractable problem to optimality is a tough goal, these researchers usually resort to simpler suboptimal approaches that yield decent solutions, while hoping that those decent solutions come at least close to the true optimum. An approximation scheme is a suboptimal approach that provably works fast and that provably yields solutions of very high quality.

While it is difficult to quote a date of the invention of approximation schemes like the finite element method, finite difference method, or finite volume method, these methods originated from the need to solve complex elasticity and structural analysis problems in civil and aeronautical engineering. Its development can be traced back to the work by A. Hrennikoff [45] and R. Courant [28] in the early 1940s. In the USSR, the introduction of the practical application of the method is usually connected with name of Leonard Oganesyan. In China, in the later 1950s and early 1960s, based on the computations of dam constructions, K. Feng proposed a systematic numerical method for solving partial differential equations. The method was called the finite difference method based on variation principle, which was another independent invention of the finite element method. Although the approaches used by these pioneers are different, they share one essential characteristic: mesh discretization of a continuous domain into a set of discrete sub-domains, usually called elements.

Hrennikoff's work discretizes the domain by using a lattice analogy, while Courant's approach divides the domain into finite triangular subregions to solve second order elliptic partial differential equations (PDEs) that arise from the problem of torsion of a cylinder. Courant's contribution was evolutionary, drawing on a large body of earlier results for PDEs developed by Rayleigh, Ritz, and Galerkin.

After this short view on the history of approximation schemes theories, we give a short overview what we plan in this part. In chapter V, we study a time discretization for a doubly non linear parabolic equation related to the p(x)-Laplacian by using Euler forward scheme. We investigate existence, uniqueness and stability questions and prove existence of the global compact attractor. In chapter VI, we give a simple approach to a priori estimates for the singular parabolic p(x,t)-Laplace Dirichlet problem in domain Ω with the boundary $\partial\Omega \in C^{1+\beta}$ such that $\beta \in (0, 1)$.

Chapter 5

Semi-discretization for a doubly nonlinear parabolic equation related to the p(x)-Laplacian

5.1 Introduction

In this chapter, our purpose is to study the time discretization for a doubly nonlinear parabolic associated with the p(x)-Laplacian; where in addition to the standard existence, uniqueness and stability questions, we shall prove existence of absorbing sets and global compact attractor also. The problem under consideration is of the form

$$\begin{cases} \frac{\partial \beta(u)}{\partial t} - \Delta_{p(x)} u + f(x,t,u) = 0 & \text{in } \Omega\times]0,\infty[, \\ u = 0 & \text{on } \partial\Omega\times]0,\infty[, \\ \beta(u)\mid_{t=0} = \beta(u_0) & \text{in } \Omega. \end{cases} \tag{5.1.1}$$

where $\Delta_{p(x)} u = div(|\nabla u|^{p(x)-2}\nabla u)$, $p \in C(\overline{\Omega})$ with $1 < p(x) < +\infty$, β is a nonlinearity of porous medium type, f a nonlinearity of reaction-absorbtion type and Ω is an open bounded set of $\mathbb{R}^N$ with smooth boundary. Also, we assume that the exponent p satisfies the following log-Hölder continuity condition:

$$\forall x, y \in \Omega, \quad |x-y| < 1, \quad |p(x) - p(y)| \leq \omega(|x-y|),$$

where ω satisfies

$$\limsup_{\tau \longrightarrow 0^+} \omega(\tau) \ln(\frac{1}{\tau}) < \infty.$$

5.2 The semi-discretized problem: existence, uniqueness and stability

Let β be a continuous increasing function with $\beta(0)=0$. For $t \in \mathbb{R}$, we set

$$\psi(t) = \int_0^t \beta(\tau)d\tau.$$

We consider the following Euler forward scheme associated to (5.1.1):

$$\begin{cases} \beta(U^n) - \tau\Delta_{p(x)}U^n + \tau f(x, n\tau, U^n) = \beta(U^{n-1}) & in\ \Omega, \\ U^n = 0 & on\ \partial\Omega, \\ \beta(U^0) = \beta(u_0) & in\ \Omega, \end{cases} \tag{5.2.1}$$

where $N\tau = T$, T being a fixed positive real, and $1 \leq n \leq N$. We shall be concerned with one of the following two cases: $u_0 \in L^\infty(\Omega)$ and $u_0 \in L^2(\Omega)$.

5.2.1 Case 1: $u_0 \in L^\infty(\Omega)$.

We shall assume the following assumptions:
-(H_1) the function β is an increasing and continuous from $\mathbb{R}$ to $\mathbb{R}$ such that $\beta(u) \leq C|u|^{\alpha-1}$ for any $u \in \mathbb{R}$ with $1 \leq \alpha < p^-$.
-(H_2) for $\xi \in \mathbb{R}$, the map $(x,t) \longmapsto f(x,t,\xi)$ is measurable and, a.e. in $\Omega \times \mathbb{R}^+$, $\xi \longmapsto f(x,t,\xi)$ is continuous. Furthermore, we assume that there exists $C_1 > 0$, such that for a.e. $(x,t) \in \Omega \times \mathbb{R}^+$, we have $sign(\xi).f(x,t,\xi) \geq -C_1$.
-(H_3) there exists $C_2 > 0$, such that for almost $(x,t) \in \Omega \times \mathbb{R}^+$, $\xi \longmapsto f(x,t,\xi) + C_2\beta(\xi)$ is increasing.

Lemma 5.2.1. *Assume (H_1) and (H_2). Then, for all $n \in \{0,..,N\}$, we have $U^n \in L^\infty(\Omega)$.*

Proof. To show that $U^1 \in L^\infty(\Omega)$ we can write (5.2.1) as:

$$\begin{cases} -\tau\Delta_{p(x)}U^1 = \beta(u_0) - \beta(U^1) - \tau f(x,\tau,U^1), \\ \\ U^1 \in W_0^{1,p(x)}, \end{cases}$$

Then by (H_1), (H_2) and theorem 4.1 of [35], we can conclude that $U^1 \in L^\infty(\Omega)$. Then, by a simple induction, we deduce that $U^n \in L^\infty(\Omega)$ for all $n = 0,..,N$. □

Theorem 5.2.2. *Assume (H_1), (H_2) and (H_3). For $n = 0,...,N$. Then, there exists a unique solution U^n of (5.2.1) in $W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ provided that $0 < \tau < \frac{1}{C_2}$.*

Proof. We can write (5.2.1) as

$$-\tau\Delta_{p(x)}U^n = \beta(U^{n-1}) - \beta(U^n) - \tau f(x, n\tau, U^n)$$
$$U^n \in W_0^{1,p(x)}(\Omega).$$

By using (H_1), (H_2) and applying Theorem (4.3) of [34] and Lemma (5.2.1), there exists at least one solution $U^n \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ for $n = 1, ..., N$.
Let us now prove the uniqueness. For simplicity, we set

$$\omega = U^n, \quad \overline{f}(x,\omega) = f(x, n\tau, U^n), \ \textit{and} \ g(x) = \beta(U^{n-1}).$$

Then problem (5.2.1) reads

$$-\tau\Delta_{p(x)}\omega + \tau\overline{f}(x,\omega) + \beta(\omega) = g(x) \quad ,\omega \in W_0^{1,p(x)}. \tag{5.2.2}$$

If ω_1 and ω_2 are two solutions of (5.2.1), then

$$-\tau\Delta_{p(x)}\omega_1 + \tau\Delta_{p(x)}\omega_2 + \tau(\overline{f}(x,\omega_1) - \overline{f}(x,\omega_2)) + \beta(\omega_1) - \beta(\omega_2) = 0 \tag{5.2.3}$$

Multiplying (5.2.3) by $\omega_1 - \omega_2$ and integrating over Ω, gives

$$\left\langle -\tau\Delta_{p(x)}\omega_1 + \tau\Delta_{p(x)}\omega_2, \omega_1 - \omega_2 \right\rangle + \tau\int_\Omega (\overline{f}(x,\omega_1) - \overline{f}(x,\omega_2))(\omega_1 - \omega_2)dx$$
$$+ \int_\Omega (\beta(\omega_1) - \beta(\omega_2))(\omega_1 - \omega_2)dx = 0 \tag{5.2.4}$$

Then , applying (H_3) yields

$$\int_\Omega (\overline{f}(x,\omega_1) - \overline{f}(x,\omega_2))(\omega_1 - \omega_2)dx \geq -C_2\int_\Omega (\beta(\omega_1) - \beta(\omega_2))(\omega_1 - \omega_2)dx. \tag{5.2.5}$$

Now by using (5.2.5) and the monotonicity of the p(x)-Laplacian operator, (5.2.4) reduces to

$$(1 - \tau C_2)\int_\Omega (\beta(\omega_1) - \beta(\omega_2))(\omega_1 - \omega_2)dx \leq 0 \tag{5.2.6}$$

Then by (H_1), we get $\omega_1 = \omega_2$ for $\tau < \frac{1}{C_2}$. □

Theorem 5.2.3. *Assume (H_1) and (H_2). Then, there exists a constant $C(T, u_0) > 0$, depending on T, u_0, β and Ω, but not on N, such that for all $n = 1, .., N$:*

i)

$$\|U^n\|_\infty \leq C(T, u_0). \tag{5.2.7}$$

ii)

$$\int_\Omega \psi^*(\beta(U^n))dx + \tau \sum_{k=1}^{n} \|U^k\|_{1,p(x)}^{\alpha} \le C(T, u_0), \tag{5.2.8}$$

where α depends either onp^- or p^+. iii)

$$\sum_{k=1}^{n} \|\beta(U^k) - \beta(U^{k-1})\|_2^2 \le C(T, u_0). \tag{5.2.9}$$

Proof. i) From Lemma 5.2.1, we have $U^n \in L^\infty(\Omega)$. Then, multiplying (5.1.1) by $|\beta(U^n)|^k\beta(U^n)$, and integrating over Ω, we get

$$\begin{aligned} &\int_\Omega |\beta(U^n)|^{k+2}dx - \tau \int_\Omega \Delta_{p(x)}U^n|\beta(U^n)|^k\beta(U^n)dx \\ &\quad + \tau \int_\Omega |\beta(U^n)|^k\beta(U^n)f(x, n\tau, U^n)dx = \int_\Omega |\beta(U^n)|^k\beta(U^n)\beta(U^{n-1})dx. \end{aligned} \tag{5.2.10}$$

Since $\beta(0) = 0$ and β and $-\Delta_{p(x)}$ are monotone, then we have

$$-\tau \int_\Omega \Delta_{p(x)}U^n|\beta(U^n)|^k\beta(U^n)dx \ge 0.$$

Therefore, we have

$$\|\beta(U^n)\|_{k+2}^{k+2} \le \|\beta(U^n)\|_{k+2}^{k+1}\|\beta(U^{n-1})\|_{k+2} + C\tau\|\beta(U^n)\|_{k+2}^{k+1}. \tag{5.2.11}$$

Hence,

$$\|\beta(U^n)\|_{k+2} \le \|\beta(U^{n-1})\|_{k+2} + C\tau\|\beta(U^n)\|_{k+2}^{k+1}. \tag{5.2.12}$$

By simple induction, we get

$$\|\beta(U^n)\|_{k+2} \le \|\beta(U_0)\|_{k+2} + NC\tau. \tag{5.2.13}$$

Finally, as $k \longrightarrow \infty$, we obtain (5.2.7).

ii) In order to prove (5.2.8), we multiply (5.1.1) by U^k (with k instead of n). By using (H_2), we get

$$\int_\Omega (\beta(U^k) - \beta(U^{k-1}))U^k dx + \tau\rho_{p(x)}(\nabla U^k) \le \tau C_1\|U^k\|_1.$$

Thanks to the properties of the Legendre transformation, we get

$$\int_\Omega \psi^*(\beta(U^k))dx - \int_\Omega \psi^*(\beta(U^{k-1}))dx \le \int_\Omega (\beta(U^k) - \beta(U^{k-1}))U^k dx.$$

Then, we have

$$\int_{\Omega} \psi^*(\beta(U^k))dx - \int_{\Omega} \psi^*(\beta(U^{k-1}))dx + \tau\rho_{p(x)}(\nabla U^k) \leq \tau C_1 \|U^k\|_{L^1(\Omega)}, \tag{5.2.14}$$

Finally, after summing (5.2.14) from k=1 to n, we deduce that

$$\begin{aligned} \int_{\Omega} \psi^*(\beta(U^n))dx + \tau \sum_{k=1}^{n} \min(\|U^k\|_{1,p(x)}^{p^+}, \|U^k\|_{1,p(x)}^{p^-}) \\ \leq \tau C_1 \sum_{k=1}^{n} \|U^k\|_{L^1(\Omega)} + \int_{\Omega} \psi^*(\beta(u_0))dx. \end{aligned} \tag{5.2.15}$$

We set

$$\|U^k\|_{p(x)}^{\alpha} = \min(\|U^k\|_{1,p(x)}^{p^+}, \|U^k\|_{1,p(x)}^{p^-}).$$

Then, the continuity of β and Lemma 5.2.1 allow us to conclude to the proof of point (5.2.8).

iii) To prove point (5.2.9), we multiply the first equation of (5.1.1) by $\beta(U^k)$. By using (H_2) we get

$$\int_{\Omega} (\beta(U^k) - \beta(U^{k-1}))\beta(U^k)dx - \tau \int_{\Omega} \Delta_{p(x)} U^k \beta(U^k)dx \leq C_1\tau \|\beta(U^k)\|_{L^1(\Omega)}. \tag{5.2.16}$$

With the aid of the elementary identity :

$$2a(a-b) = a^2 - b^2 + (a-b)^2,$$

we get from (5.2.16) that

$$\|\beta(U^k)\|_{L^2(\Omega)}^2 - \|\beta(U^{k-1})\|_{L^2(\Omega)}^2 + \|\beta(U^k) - \beta(U^{k-1})\|_{L^2(\Omega)}^2 \leq C\tau \|\beta(U^k)\|_{L^1(\Omega)}. \tag{5.2.17}$$

We now sum (5.2.17) from $k = 1$ to n to obtain

$$\|\beta(U^n)\|_{L^2(\Omega)}^2 + \sum_{k=1}^{n} \|\beta(U^k) - \beta(U^{k-1})\|_{L^2(\Omega)}^2 \leq C\tau \|\beta(U^k)\|_{L^1(\Omega)} + \|\beta(u_0)\|_{L^2(\Omega)}^2. \tag{5.2.18}$$

Thus, by (H_1) and Lemma 5.2.1 we deduce point (5.2.9). □

Lemma 5.2.4. *For all $u, v \in W_0^{1,p(x)}(\Omega)$, there exists a positive constant α depending either on p^+ or p^- such that for $1 < p(x) < 2$, we have*

$$\rho_{p(x)}(\nabla(u-v))^{\frac{2}{\alpha}} \leq C < -\Delta_{p(x)} u + \Delta_{p(x)} v, u - v > . \tag{5.2.19}$$

Proof. If $1 < p(x) < 2$ then, for any $x \in \Omega$ we have the following inequality for any $\xi, \eta \in \mathbb{R}^N$

$$(|\xi| + |\eta|)^{2-p(x)}(|\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta)(\xi - \eta) \geq \delta|\xi - \eta|^2$$

By setting $\xi = \nabla u$ and $\eta = \nabla v$ and integrating over Ω, we get

$$\delta^{\frac{p^+}{2}} \int_\Omega |\nabla u - \nabla v|^{p(x)} dx \leq \int_\Omega \left((|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v).\nabla(u-v)\right)^{\frac{p(x)}{2}} \times (|\nabla u| + |\nabla v|)^{\frac{(2-p(x))p(x)}{2}} dx.$$

Then by Hölder's inequality we get

$$\delta^{\frac{p^+}{2}} \int_\Omega |\nabla u - \nabla v|^{p(x)} dx \leq \left\| \left((|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v).\nabla(u-v)\right)^{\frac{p(x)}{2}} \right\|_{\frac{2}{p(x)}} \times \left\| (|\nabla u| + |\nabla v|)^{\frac{(2-p(x))p(x)}{2}} \right\|_{\frac{2}{2-p(x)}}.$$

Let α and ξ be such that

$$\left(\int_\Omega (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v).\nabla(u-v)\right)^{\frac{\alpha}{2}} = \max\left(\left(\int_\Omega (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v).\nabla(u-v)\right)^{\frac{p^-}{2}}, \left(\int_\Omega (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v).\nabla(u-v)\right)^{\frac{p^+}{2}}\right)$$

and

$$\left(\int_\Omega (|\nabla u| + |\nabla v|)^{p(x)}\right)^{\xi} = \max\left(\left(\int_\Omega (|\nabla u| + |\nabla v|)^{p(x)}\right)^{\frac{2-p^-}{2}}, \left(\int_\Omega (|\nabla u| + |\nabla v|)^{p(x)}\right)^{\frac{2-p^+}{2}}\right)$$

then, we get

$$\delta^{\frac{p^+}{2}} \int_\Omega |\nabla u - \nabla v|^{p(x)} dx \leq 2^{\xi p^+} \left\langle -\Delta_{p(x)} u + \Delta_{p(x)} v, u - v \right\rangle^{\frac{\alpha}{2}} \left(\rho_{p(x)}(\nabla u) + \rho_{p(x)}(\nabla v)\right)^{\xi}.$$

Hence, by (5.2.8) of theorem 5.2.3 we get the desired result. □

Lemma 5.2.5. *Assume $p(x) \geq 2$. Then, for all $u, v \in W_0^{1,p(x)}(\Omega)$, we have*

$$(\frac{1}{2})^{p^-} \rho_{p(x)}(\nabla(u-v)) \leq < -\Delta_{p(x)} u + \Delta_{p(x)} v, u - v > . \qquad (5.2.20)$$

Proof. As $p(x) \geq 2$, for any $x \in \Omega$, then we have the following inequality for any $\xi, \eta \in \mathbb{R}^N$

$$(|\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta)(\xi - \eta) \geq (\frac{1}{2})^{p-}|\xi - \eta|^{p(x)}$$

By setting $\xi = \nabla u$ and $\eta = \nabla v$ and integrating over Ω, we get

$$(\frac{1}{2})^{p-} \int_\Omega |\nabla u - \nabla v|^{p(x)} dx \leq \int_\Omega (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)(\nabla(u-v))dx$$

Hence

$$(\frac{1}{2})^{p^-} \rho_{p(x)}(\nabla(u-v)) \leq < -\Delta_{p(x)}u + \Delta_{p(x)}v, u - v > .$$

□

We can also have a uniqueness result for problem (5.2.1) by replacing H_3 by the following hypothesis
-(H_4) for all $M > 0$, there exists $C_M > 0$ such that, if $|\xi| + |\xi'| \leq M$ then

$$|f(t,x,\xi) - f(t,x,\xi')|^\theta \leq C_M(\beta(\xi) - \beta(\xi'))(\xi - \xi'),$$

where $\theta = \begin{cases} \sigma' & for\ 1 < p(x) < 2, \\ p'^- & for\ p(x) \geq 2, \end{cases}$ whith σ' a positive constant to be prescribed below .

Proposition 5.2.6. *Assume (H_1), (H_2) and (H_4). Then, problem (5.2.1) has a unique solution for all $0 < \tau < \eta$, where η is a prescribed constant.*

Proof. Let ω_1 and ω_2 two solutions of (5.2.1).
First case:$1 < p(x) < 2$. From (5.2.4) and by using Lemma 5.2.4 and Hölder's inequality, we get

$$\begin{aligned} &\tau C \rho_{p(x)}(\nabla(\omega_1 - \omega_2))^{\frac{2}{\alpha}} + \int_\Omega (\beta(\omega_1) - \beta(\omega_2))(\omega_1 - \omega_2)dx \\ &\leq \tau \|\overline{f}(x,\omega_1) - \overline{f}(x,\omega_2)\|_{L^\infty(\Omega)} \|\omega_1 - \omega_2\|_{L^1(\Omega)}. \end{aligned} \tag{5.2.21}$$

Let λ be such that

$$\rho_{p(x)}(\omega_1 - \omega_2)^{\frac{1}{\lambda}} = \max\left(\rho_{p(x)}(\omega_1 - \omega_2)^{\frac{1}{p^-}}, \rho_{p(x)}(\omega_1 - \omega_2)^{\frac{1}{p^+}}\right),$$

and,

$$\sigma = \frac{2\lambda}{\alpha}, \quad \frac{1}{\sigma} + \frac{1}{\sigma'} = 1.$$

Then, by (H_1), (H_1) and (H_4), Lemma 1.2.1 and Young's inequality, we get

$$\begin{aligned}&\tau C\rho_{p(x)}(\nabla(\omega_1-\omega_2))^{\frac{2}{\alpha}}\\&\le(\frac{1}{\sigma'}-1)\int_\Omega(\beta(\omega_1)-\beta(\omega_2))(\omega_1-\omega_2)dx+C^{'}\tau^{\sigma}\rho_{p(x)}(\nabla(\omega_1-\omega_2))^{\frac{2}{\alpha}}.\end{aligned}\tag{5.2.22}$$

Therefore, for $0<\tau<\left(\frac{C}{C'}\right)^{\frac{1}{\sigma-1}}$, we get $\omega_1=\omega_2$.
Second case: $p(x)\ge 2$. From (5.2.4) and by using lemma 5.2.5 and Young's inequality, we get

$$\begin{aligned}&\tau(\frac{1}{2})^{p^-}\rho_{p(x)}(\nabla(\omega_1-\omega_2))+\int_\Omega(\beta(\omega_1)-\beta(\omega_2))(\omega_1-\omega_2)dx\\&\le\frac{1}{p'^-C_M}\int_\Omega|(\overline{f}(x,\omega_1)-\overline{f}(x,\omega_2))|^{p'(x)}dx+\frac{\tau^{p^+}C_M^{\frac{p^+}{p'^-}}}{p^-}\int_\Omega|\omega_1-\omega_2|^{p(x)}dx.\end{aligned}\tag{5.2.23}$$

Then, by using (H_1) and (H_4), we get

$$\tau(\frac{1}{2})^{p^-}\rho_{p(x)}(\nabla(\omega_1-\omega_2))\le(\frac{1}{p'^-}-1)\int_\Omega(\beta(\omega_1)-\beta(\omega_2))(\omega_1-\omega_2)dx+\frac{\tau^{p^+}C_M^{\frac{p^+}{p'^-}}}{p^-}\int_\Omega|\omega_1-\omega_2|^{p(x)}dx.$$

Thus, from Lemma 1.2.1, we get

$$\tau(\frac{1}{2})^{p^-}\rho_{p(x)}(\nabla(\omega_1-\omega_2))\le\frac{\tau^{p^+}C_M^{\frac{p^+}{p'^-}}}{p^-}\rho_{p(x)}(\nabla(\omega_1-\omega_2)).$$

Hence, when $0<\tau<\left(\frac{(\frac{1}{2})^{p^-}p^-}{C.C_M^{\frac{p^+}{p'^-}}}\right)^{\frac{1}{p^+-1}}$, we have $\omega_1=\omega_2$. □

5.2.2 Case 2: $u_0\in L^2(\Omega)$.

Theorem 5.2.7. *Assume that (H_1), (H_2) and (H_3) hold. Then for $n=0,...,N$, there exists a unique solution U^n of (5.2.1) in $W_0^{1,p(x)}(\Omega)$ provided that $0<\tau<\frac{1}{C}$ where C is some positive constant.*

Proof. The proofs of existence and uniqueness are the same as those of Theorem 5.2.7. Therefore, we omit them. □

Now, we consider the following assumption:
-(H_5) for any $\xi\in\mathbb{R}$, the map $(x,t)\longmapsto f(x,t,\xi)$ is measurable and, a.e. in $\Omega\times\mathbb{R}^+$,

$\xi \longmapsto f(x,t,\xi)$ is continuous. Furthermore, we assume that there exist $q(x) > \sup(2, p(x))$ and positive constants C_5 and C_6 such that

$$sign\xi f(x,t,\xi) \geq C_5|\xi|^{q(x)-1} - C_6.$$

Then, we have the following stability theorem.

Theorem 5.2.8. *Assume that (H_1) , (H_5) are fulfilled and take $p(x) \in L^\infty(\Omega)$. Then, there exists a constant $C(T,u_0) > 0$ such that for all $n = 1,..,N$:*

$$\int_\Omega \psi^*(\beta(U^n))dx + \tau\sum_{k=1}^{n} \|U^k\|^{\alpha}_{1,p(x)} + C\tau\sum_{k=1}^{n} \|U^k\|^{\alpha'}_{q(x)} \leq C(T,u_0), \tag{5.2.24}$$

$$\max_{1\leq k\leq n} \sum_{k=1}^{n} \|\beta(U^k)\|_2^2 + \|\beta(U^k) - \beta(U^{k-1})\|_2^2 \leq C(T,u_0). \tag{5.2.25}$$

Where α and α' are two constants each depending either on p^+ or p^-.

Proof. Since the proof is nearly the same as that of theorem 5.2.3, we just sketch it. The argument that allowed us to get (5.2.15), with (H_5) allows also to write

$$\begin{aligned} \int_\Omega \psi^*(\beta(U^n))dx + \tau\sum_{k=1}^{N} \rho_{p(x)}(\nabla U^k) + \tau\sum_{k=1}^{N} \rho_{q(x)}(U^k) \\ \leq \tau C_6 \sum_{k=1}^{N} \|U^k\|_{L^1(\Omega)} + \int_\Omega \psi^*(\beta(u_0))dx, \end{aligned} \tag{5.2.26}$$

By using lemma 5.2.1, Lemma 1.2.1 , (H_1) and Young's inequality, we get for all $\eta > 0$ that there exists $C_\eta(T,u_0) > 0$ such that

$$\begin{aligned} \int_\Omega \psi^*(\beta(U^n))dx + \tau\sum_{k=1}^{n} \rho_{p(x)}(\nabla U^k) + \tau\sum_{k=1}^{n} \rho_{q(x)}(U^k) \\ \leq \eta\tau \sum_{k=1}^{N} \rho_{p(x)}(\nabla U^k) + C_\eta(T,u_0). \end{aligned} \tag{5.2.27}$$

Since $\psi^*(\beta(u))$ is positive then, for a suitable choice of η we infer from (5.2.27) that

$$\tau\sum_{k=1}^{n} \rho_{p(x)}(U^k) \leq \widetilde{C}_\eta(T,u_0). \tag{5.2.28}$$

By taking α and α' such that ,

$$\begin{cases} \|U^k\|^{\alpha}_{1,p(x)} = \min(\|U^k\|^{p^+}_{1,p(x)}, \|U^k\|^{p^-}_{1,p(x)}), \\ \\ \|U^k\|^{\alpha'}_{q(x)} = \min(\|U^k\|^{q^+}_{q(x)}, \|U^k\|^{q^-}_{q(x)}), \end{cases}$$

and using (5.2.28) and (5.2.2), we deduce that

$$\int_\Omega \psi^*(\beta(U^n))dx + \tau \sum_{k=1}^{n} \|U^k\|^{\alpha}_{1,p(x)} + C\tau \sum_{k=1}^{n} \|U^k\|^{\alpha'}_{q(x)} \leq C(T, u_0).$$

As in (5.2.18), by using (H_5), we get

$$\|\beta(U^n)\|^2_{L^2(\Omega)} + \sum_{k=1}^{n} \|\beta(U^k) - \beta(U^{k-1})\|^2_{L^2(\Omega)} \leq C_1\tau\|\beta(U^k)\|_{L^1(\Omega)} + \|\beta(u_0)\|^2_{L^2(\Omega)}.$$

Hence, by (H_1) , (5.2.28) and lemma 5.2.1, we get (5.2.25) . □

5.3 Absorbing sets in $W_0^{1,p(x)}(\Omega)$, existence of the attractor.

In this section we consider the following systems: for all integer $n > 0$

$$\begin{cases} \beta(U^n) - \tau\Delta_{p(x)}U^n + \tau f(x, n\tau, U^n) = \beta(U^{n-1}) & in\ \Omega, \\ U^n = 0 & in\ \partial\Omega, \end{cases} \tag{5.3.1}$$

with $U^0 = u_0$ and τ fixed such that $0 < \tau < \tau_2$ where $\tau_2 = \min(1, \frac{1}{C_2})$.
We assume (H_1), (H_2) and (H_3) in all the remaining section. the result of Theorem 5.2.2 on the existence and uniqueness of the solution of (5.2.1) allows us to define a map S_τ on $L^\infty(\Omega) \cap W^{1,p(x)}(\Omega)$ by setting

$$S_\tau U^{n-1} = U^n.$$

Since S_τ is continuous, we have

$$S_\tau^n U^0 = U^n.$$

The existence of an absorbing set in $W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ allows us to prove the existence of a global compact attractor (see [68]). This will be done next in Theorem 5.3.2.

Proposition 5.3.1. *If τ satisfies $\tau < \frac{1}{C_2}$ then, there is an absorbing set $\mathfrak{B}$ in $W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. Namely for any $u_0 \in L^\infty(\Omega)$ there exists $n(\tau)$ such that*

$$\|U^n\|_{L^\infty(\Omega)} + \|U^n\|_{1,p(x)} \leq C, \quad \forall n \geq n(\tau).$$

Proof. We multiply the equation (5.2.1) by $\Delta_n = (U^n - U^{n-1})$ and obtain

$$\left\langle \frac{\beta(U^n) - \beta(U^{n-1})}{\tau}, U^n - U^{n-1} \right\rangle + \langle \nabla U^n, \nabla \Delta_n \rangle + \int_\Omega f(x, n\tau, U^n)\Delta_n dx = 0. \quad (5.3.2)$$

Denote by

$$F_\beta(u) = \int_0^u (f(x, n\tau, \omega) + C_2\beta(\omega)) .$$

By (H_3), $F_\beta(u)$ is a convex function and hence satisfies the standard inequality

$$F'_\beta(u)(u - v) \geq F_\beta(u) - F_\beta(v),$$

Consequently,

$$\begin{aligned}\langle f(x, n\tau, U^n), \Delta_n \rangle &= \langle f(x, n\tau, U^n) + C_2\beta(U^n), \Delta_n \rangle - C_2 \langle \beta(U^n), \Delta_n \rangle \\ &\geq \int_\Omega (F_\beta(U^n) - F_\beta(U^{n-1}))dx - C_2 \langle \beta(U^n), \Delta_n \rangle .\end{aligned}$$

Now, by using (H_1), we get that $\psi(u)$ is a convex function and hence have

$$\psi'(v)(u - v) \geq \psi(u) - \psi(v).$$

Thus, we obtain

$$\begin{aligned}\int_\Omega \beta(U^n)(U^n - U^{n-1})dx &= \int_\Omega (\beta(U^n) - \beta(U^{n-1}))(U^n - U^{n-1})dx + \int_\Omega \beta(U^{n-1})(U^n - U^{n-1})dx \\ &\leq \int_\Omega (\beta(U^n) - \beta(U^{n-1}))(U^n - U^{n-1})dx + \int_\Omega (\psi(U^n) - \psi(U^{n-1}))dx.\end{aligned}$$

The following inequality holds

$$\frac{1}{p^+}|a|^{p(x)} - \frac{1}{p^-}|b|^{p(x)} \leq C^*|a|^{p(x)-2}a(a - b).$$

By setting $a = \nabla U^n$ and $b = \nabla U^{n-1}$ and integrating over Ω ,we get

$$\frac{1}{p^+}\rho_{p(x)}(U^n) - \frac{1}{p^-}\rho_{p(x)}(U^{n-1})) \leq \int_\Omega |\nabla U^n|^{p(x)-2}\nabla U^n(\nabla U^n - \nabla U^{n-1}).$$

Now, since $\tau < \frac{1}{C_2}$, then from (5.3.2), we deduce that

$$\frac{1}{p^+}\rho_{p(x)}(\nabla U^n)+\int_\Omega F_\beta(U^n)dx \leq \frac{1}{p^-}\rho_{p(x)}(\nabla U^{n-1})+\int_\Omega F_\beta(U^{n-1})dx+C_2\int_\Omega(\psi(U^n)-\psi(U^{n-1}))dx$$

On the other hand, by writing

$$\int_\Omega F_\beta(u)dx = \int_\Omega F(u)dx + C_2\int_\Omega \psi(u)dx,$$

where $F(u) = \int_0^u f(x,t,\omega)d\omega$, we have

$$\frac{1}{p^+}\rho_{p(x)}(\nabla U^n) + \int_\Omega F(U^n)dx \leq \frac{1}{p^-}\rho_{p(x)}(\nabla U^{n-1}) + \int_\Omega F(U^{n-1})dx. \tag{5.3.3}$$

Denote the left hand side of (5.3.3) by y^n. By using (H_1), relations (5.2.7) and (5.2.8) of Theorem 5.2.3 and taking $N\tau = 1$, we deduce that there exists n_τ such that

$$\tau\sum_{n=n_0}^{n_0+N} y^n \leq a_1 \quad for\ all\ n_0 \geq n_\tau$$

Then, by applying Lemma 1.4.5 with $h_n = 0$, we obtain

$$\frac{1}{C^*}\rho_{p(x)}(\nabla U^n) + \int_\Omega F(U^n)dx \leq C \quad for\ all\ n \geq n_\tau. \tag{5.3.4}$$

Thus by Lemma 5.2.1, we deduce that

$$\|U^n\|_{1,p(x)} \leq C \quad for\ all\ n \geq n_\tau, \tag{5.3.5}$$

Therefore, from (5.3.5) and Theorem 5.2.3, we conclude to the desired relation

$$\|U^n\|_{L^\infty(\Omega)} + \|U^n\|_{1,p(x)} \leq C, \quad \forall n \geq n(\tau).$$

□

Now we are able to show the existence of the compact attractor.

Theorem 5.3.2. *Suppose that $f(x,t,\xi) = f(x,\xi)$. Then, for $u_0 \in L^\infty(\Omega)$, the discretized problem (4.1) has an associated semigroup solution S_τ that maps $L^\infty(\Omega)$ into $L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$. This semigroup has a compact attractor $\mathfrak{A}_\tau$ which is bounded in $L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$.*

Proof. The nonlinear map S_τ defines a semigroup from $L^\infty(\Omega)$ into $L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$. By the Proposition 5.3.1, the existence of an absorbing ball $\mathfrak{B}_\tau$ in $L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$ is guaranteed.
We define the $\omega - limit$ set of $\mathfrak{B}_\tau$ as

$$\mathfrak{A}_\tau = \omega(\mathfrak{B}_\tau) = \cap_{n\geq 0}\overline{\cup_{m\geq n} S_\tau^m(\mathfrak{B}_\tau)}.$$

Then, by Theorem 1.1 of [68], $\mathfrak{A}_\tau = \omega(\mathfrak{B}_\tau)$ is a compact attractor which attracts all bounded sets of $L^\infty(\Omega)$. That means that for all $u_0 \in L^\infty(\Omega)$, we have

$$dist(\mathfrak{A}_\tau, S_\tau^n u_0) \longmapsto 0 \ \ as\ n \longrightarrow +\infty.$$

□

Chapter 6

Finite element approximation of singular parabolic $p(x,t)$-Laplacian equation

6.1 Introduction

We shall study the Dirichlet problem for a class of singular nonlinear parabolic equations with variable exponent. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a domain with the boundary $\partial\Omega \in C^{1+\beta}$ with $\beta \in (0,1)$ and $\Omega_T = \Omega \times (0,T)$ a cylinder of height $T < \infty$. Let us consider the following problem:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p(x,t)-2}\nabla u) = f(x,t) & \text{in } \Omega_T, \\ u = 0 & \text{on } \Gamma_T = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{6.1.1}$$

It is assumed that f and the exponent $p(x,t)$ are Hölder-continuous in $\overline{\Omega_T}$. The exponent $p(x,t)$ takes values in an interval $[p^-, p^+] \subset \left(\frac{2N}{N+2}, 2\right)$.

The increasing interest of problems of type (6.1.1) was motivated by the model for electrorheological fluids [3,66,67]. These are smart materials whose viscosity depends on the applied electric field. Electrorheological fluids can, for example, be used in the construction of clutches and shock absorbers.

Further applications of (6.1.1) can be found in the area of image processing [25], elasticity [73], the processes of filtration in complex media [7], stratigraphy problems [41] and also mathematical biology [37].

If the exponent does not depend on x and t, we have classical parabolic p-Laplacian equation. In this case, numerical approximations using the finite element method in space and Euler schemes in time have been studied previously by other authors in [17,71]. More recently, Breit, Diening and Schwarzacher in [23] and Del Pezzo and Martinez in [65] studied the variable exponent case for the stationary problem.

6.2 Preliminaries and main result

We begin with a review of some basic results that will be needed in the following sections. The known results are generally stated without proofs, but we provide references where the proofs can be found. We introduce some of our notational conventions also.

Let us denote $\Omega_T = \Omega \times (0, T)$ with $0 < T < \infty$. We introduce the function spaces

$$V_t(\Omega) = \left\{ u : \Omega \longmapsto \mathbb{R} |\ u \in L^2(\Omega) \cap W_0^{1,1}(\Omega),\ \ |\nabla u|^{p(x,t)} \in L^1(\Omega) \right\},$$
$$W(\Omega_T) = \left\{ u : (0, T) \longmapsto V_t(\Omega) |\ u \in L^2(\Omega_T),\ \ |\nabla u|^{p(x,t)} \in L^1(\Omega_T) \right\}.$$

Problem (6.1.1) does not, in general, admit classical solutions. A weak solution of problem (6.1.1) is understood as follows.

Definition 6.2.1. *A local weak solution of (6.1.1) is a measurable function $u(x, t)$ defined in Ω_T, such that*

(i) $u \in W(\Omega_T) \cap C([0, T], L^2(\Omega))$, $u_t \in L^2(0, T, L^2(\Omega))$;

(ii) for every test function $\phi \in L^2(\Omega_T) \cap W(\Omega_T)$ *with* $\phi_t \in L^2(0, T, L^2(\Omega))$

$$\int_{\Omega_T} \left(u_t \phi + |\nabla u|^{p(x,t)-2} \nabla u . \nabla \phi - f \phi \right) dx dt = 0;$$

(iii) for every $\phi \in C_0^1(\Omega)$

$$\int_{\Omega} (u(x, t) - u_0(x))\, \phi(x) dx = 0 \ \text{as}\ t \longrightarrow 0.$$

The existence and uniqueness of weak solutions, in the sense of this definition, was proved in [11,74]. To the best of the author's knowledge, there are no results concerning the convergence of the finite element method when applied to problems of this type with variable exponent. Our main result is the derivation of error estimates for numerical approximations to solutions of problem (6.1.1). To be precise we show in Theorem 6.6.4 that

$$\|u - U\|^2_{C([0,T],L^2(\Omega^h))} + \left\| u - \widehat{U} \right\|^2_{L^2(0,T,W^{1,2}(\Omega^h))} \leq C(h + \Delta t + \varepsilon^2).$$

6.3 Regularization of the problem.

One source of difficulty in deriving error estimates for degenerate parabolic problems is the roughness of the solutions. In order to obtain a parabolic boundary value problem with a smooth solution, we must perturb problem (6.1.1) and this can be done in several ways. We use here the following approximate problem, regularized by introducing a parameter $\varepsilon \in]0,1[$ as follows

$$\begin{cases} u_{\varepsilon,t} - \operatorname{div}((|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p(x,t)-2}{2}} \nabla u_\varepsilon) = f & \text{in } \Omega_T = \Omega \times (0,T], \\ u_\varepsilon = 0 & \text{on } \Gamma_T = \partial\Omega \times (0,T], \\ u_\varepsilon(x,0) = u_{\varepsilon,0}(x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{6.3.1}$$

The definition of a weak solution to this problem is similar to the previous one.

Definition 6.3.1. *A local weak solution of (6.3.1) is a measurable function $u_\varepsilon(x,t)$ defined in Ω_T, such that*

(i) $u_\varepsilon \in W(\Omega_T) \cap C([0,T], L^2(\Omega))$, $u_{\varepsilon,t} \in L^2(0,T,L^2(\Omega))$;

(ii) for every test function $\phi \in L^2(\Omega_T) \cap W(\Omega_T)$ *with* $\phi_t \in L^2(0,T,L^2(\Omega))$

$$\int_{\Omega_T} \left(u_{\varepsilon,t}\phi + \left(|\nabla u_\varepsilon|^2 + \varepsilon^2\right)^{\frac{p(x,t)-2}{2}} \nabla u_\varepsilon . \nabla \phi - f\phi \right) dxdt = 0;$$

(iii) for every $\phi \in C_0^1(\Omega)$

$$\int_\Omega \left(u_\varepsilon(x,t) - u_{\varepsilon,0}(x)\right)\phi(x)dx = 0 \text{ as } t \longrightarrow 0.$$

The regularized problem is introduced to efficiently deal with the nonlinear diffusion term. The weak solutions of this regularized problem are obtained by using Schauder's fixed-point theorem; see [12] and it's easy to verify that the weak solutions to (6.3.1) are unique. Since the influence of the parameter ε on the regularity of the solutions of problem (6.3.1) is not clear, we begin with a collection of basic regularity results. In what follows, C will represent a constant, but not always the same value.

Lemma 6.3.2. *Assume that* $\nabla p \in L^\infty(\Omega_T)$ *and* $f \in C^{\delta,\frac{\delta}{2}}(\overline{\Omega_T})$ *with* $\delta \in (0,1)$. *If* u_ε *is a weak solution of (6.3.1), then for every* $s \in (0,T)$

$$\|\nabla u_\varepsilon\|_{C^{\delta,\frac{\delta}{2}}\left(\overline{\Omega}\times[s,T]\right)} \leq C,$$

where C is a positive constant depending on s and the problem data, but independent of ε.

Theorem 6.3.3. *Let the conditions of Lemma 6.3.2 be fulfilled. If u_ε is a weak solution of (6.3.1) then* $D^2_{ij}u_\varepsilon \in L^2(\Omega \times (s,t))$ *for any* $s \in (0,T)$.

See the proofs of Lemma 6.3.2 and Theorem 6.3.3 in [12].

6.4 Semidiscrete approximation

Let V^h be a finite dimensional subspace of $W_0^{1,p(x,t)}(\Omega)$ for every $t \in (0,T)$. Then, the corresponding semidiscrete approximation to (6.3.1) is

Find $u_\varepsilon^h \in W(\Omega_T) \cap C([0,T]; L^2(\Omega)) \cap L^{p(x,t)}(0,T; V^h)$ and $u_{\varepsilon,t}^h \in L^2(0,T,L^2(\Omega))$, such that for almost every $t \in (0,T)$, we have

$$\int_\Omega u_{\varepsilon,t}^h \chi^h dx + \int_\Omega \left(|\nabla u_\varepsilon^h|^2 + \varepsilon^2\right)^{\frac{p(x,t)-2}{2}} \nabla u_\varepsilon^h . \nabla \chi^h dx = \int_\Omega f \chi^h dx, \qquad \forall \chi^h \in V^h. \tag{6.4.1}$$

$$u_\varepsilon^h(x,0) = u_{\varepsilon,0}^h,$$

where $u_{\varepsilon,0}^h$ is an approximation to $u_{\varepsilon,0}$.

It is easy to establish the existence of a unique weak solution to (6.4.1) and that $u_\varepsilon^h \in W(\Omega_T) \cap C([0,T]; L^2(\Omega))$ and $u_{\varepsilon,t}^h \in L^2(0,T,L^2(\Omega))$ by adapting the argument for (6.3.1), i.e., the argument in [12]; and we have also the following analogue of Lemma 6.3.2 and Theorem 6.3.3.

Lemma 6.4.1. *Assume that $\nabla p \in L^\infty(\Omega_T)$ and $f \in C^{\delta,\frac{\delta}{2}}(\overline{\Omega_T})$ with $\delta \in (0,1)$. If u_ε^h is a weak solution of (6.4.1), then for every $s \in (0,T)$*

$$\left\|\nabla u_\varepsilon^h\right\|_{C^{\delta,\frac{\delta}{2}}\left(\overline{\Omega}\times[s,T]\right)} \leq C,$$

where C is a positive constant depending on s and the problem data, but independent of ε.

Theorem 6.4.2. *Let the conditions of Lemma 6.4.1 be fulfilled. If u_ε^h is a weak solution of (6.4.1) then $D_{ij}^2 u_\varepsilon^h \in L^2(\Omega \times (s,t))$ for any $s \in (0,T)$.*

Theorem 6.4.3. *Let u_ε and u_ε^h be the unique weak solutions of (6.3.1) and (6.4.1) respectively, then, for any $v^h \in L^2(0,T,V^h)$*

$$\begin{aligned} &\left\|u_\varepsilon - u_\varepsilon^h\right\|_{C([0,T],L^2(\Omega))}^2 + \left\|u_\varepsilon - u_\varepsilon^h\right\|_{L^2(0,T,H^1(\Omega))}^2 \\ &\qquad \leq C\left(\left\|u_\varepsilon - v^h\right\|_{L^2(0,T,H^1(\Omega))} + \left\|u_{\varepsilon,0} - u_{\varepsilon,0}^h\right\|_{L^2(\Omega)}^2\right). \end{aligned} \tag{6.4.2}$$

Proof. For any $v^h \in L^2(0,T,V^h)$ and any $s \in (0,T)$, we have

$$\begin{aligned} &\frac{1}{2}\left\|(u_\varepsilon - u_\varepsilon^h)(s)\right\|_{L^2(\Omega)}^2 + \int_0^s \int_\Omega \left[a_\varepsilon(x,t,\nabla u_\varepsilon) - a_\varepsilon(x,t,\nabla u_\varepsilon^h)\right].\nabla(u_\varepsilon - u_\varepsilon^h) dxdt \\ &= \int_0^s \int_\Omega \left[a_\varepsilon(x,t,\nabla u_\varepsilon) - a_\varepsilon(x,t,\nabla u_\varepsilon^h)\right].\nabla(u_\varepsilon - v^h) dxdt \\ &+ \int_0^s \int_\Omega (u_\varepsilon - u_\varepsilon^h)_t (u_\varepsilon - v^h) dxdt + \frac{1}{2}\left\|u_{\varepsilon,0} - u_{\varepsilon,0}^h\right\|_{L^2(\Omega)}^2, \end{aligned} \tag{6.4.3}$$

where $a_\varepsilon\left(x,t,\nabla u_\varepsilon\right)=\left(|\nabla u_\varepsilon|^2+\varepsilon^2\right)^{\frac{p(x,t)-2}{2}}\nabla u_\varepsilon$.

Now, we are going to estimate various terms of (6.4.3) separately. By the mean value theorem there exists $\theta\in(0,1)$ such that

$$\begin{aligned}
&a_\varepsilon(x,t,\nabla u_\varepsilon)-a_\varepsilon(x,t,\nabla u_\varepsilon^h)=a'_\varepsilon(x,t,\nabla u_\varepsilon^h+\theta\left(\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right)).\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\\
&=\left|\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right|\left(\varepsilon^2+\left|\nabla u_\varepsilon^h+\theta\left(\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right)\right|^2\right)^{\frac{p(x,t)-2}{2}}\left[(p(x,t)-2)(A,B).A+B\right],
\end{aligned}\tag{6.4.4}$$

where

$$A=\frac{\nabla u_\varepsilon^h+\theta\left(\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right)}{|\nabla u_\varepsilon^h+\theta\left(\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right)|},\quad B=\frac{\nabla\left(u_\varepsilon-u_\varepsilon^h\right)}{|\nabla\left(u_\varepsilon-u_\varepsilon^h\right)|}.$$

Then,

$$\begin{aligned}
&\left(a_\varepsilon(x,t,\nabla u_\varepsilon)-a_\varepsilon(x,t,\nabla u_\varepsilon^h),\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right)\\
&=\left|\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right|^2\left(\varepsilon^2+\left|\nabla u_\varepsilon^h+\theta\left(\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right)\right|^2\right)^{\frac{p(x,t)-2}{2}}\left[(p(x,t)-2)(A,B)^2+|B|^2\right]\\
&=\left|\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right|^2\left(\varepsilon^2+\left|\nabla u_\varepsilon^h+\theta\left(\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right)\right|^2\right)^{\frac{p(x,t)-2}{2}}\left[(p(x,t)-2)\cos^2(\widehat{A,B})+1\right]\\
&\geq(p(x,t)-1)\left(\varepsilon^2+\left|\nabla u_\varepsilon^h+\theta\left(\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right)\right|^2\right)^{\frac{p(x,t)-2}{2}}\left|\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right|^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left|\nabla u_\varepsilon^h+\theta\left(\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right)\right|^2&\leq\theta^2|\nabla u_\varepsilon|^2+2\theta(1-\theta)\left(\nabla u_\varepsilon,\nabla u_\varepsilon^h\right)+(1-\theta)^2|\nabla u_\varepsilon^h|^2\\
&\leq2\left(|\nabla u_\varepsilon|^2+|\nabla u_\varepsilon^h|^2\right),
\end{aligned}$$

whence,

$$\begin{aligned}
&\left(a_\varepsilon(x,t,\nabla u_\varepsilon)-a_\varepsilon(x,t,\nabla u_\varepsilon^h),\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right)\\
&\geq(p(x,t)-1)\left(\varepsilon^2+2\left|\nabla u_\varepsilon\right|^2+2\left|\nabla u_\varepsilon^h\right|^2\right)^{\frac{p(x,t)-2}{2}}\left|\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right|^2\\
&\geq(p(x,t)-1)\left(1+2\left|\nabla u_\varepsilon\right|^2+2\left|\nabla u_\varepsilon^h\right|^2\right)^{\frac{p(x,t)-2}{2}}\left|\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right|^2\\
&\geq C\left|\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right|^2,
\end{aligned}$$

with C is a positive constant independent of ε. Also, here we use the fact that ∇u_ε and ∇u_ε^h are bounded by means of Lemmas 6.3.2 and 6.4.1.

Now, we can easily deduce that

$$\int_0^s\int_\Omega\left[a_\varepsilon(x,t,\nabla u_\varepsilon)-a_\varepsilon(x,t,\nabla u_\varepsilon^h)\right].\nabla\left(u_\varepsilon-u_\varepsilon^h\right)dxdt\geq C\int_0^s\int_\Omega\left|\nabla\left(u_\varepsilon-u_\varepsilon^h\right)\right|^2dxdt.$$

Next, we have that

$$
\begin{aligned}
&\left|a_\varepsilon(x,t,\nabla u_\varepsilon) - a_\varepsilon(x,t,\nabla u_\varepsilon^h)\right| \le |a_\varepsilon(x,t,\nabla u_\varepsilon)| + \left|a_\varepsilon(x,t,\nabla u_\varepsilon^h)\right| \\
&= \frac{|\nabla u_\varepsilon|}{(\varepsilon^2 + |\nabla u_\varepsilon|^2)^{\frac{2-p(x,t)}{2}}} + \frac{|\nabla u_\varepsilon^h|}{(\varepsilon^2 + |\nabla u_\varepsilon^h|^2)^{\frac{2-p(x,t)}{2}}} \le \frac{|\nabla u_\varepsilon|}{(|\nabla u_\varepsilon|^2)^{\frac{2-p(x,t)}{2}}} + \frac{|\nabla u_\varepsilon^h|}{(|\nabla u_\varepsilon^h|^2)^{\frac{2-p(x,t)}{2}}} \\
&= |\nabla u_\varepsilon|^{p(x,t)-1} + \left|\nabla u_\varepsilon^h\right|^{p(x,t)-1} \le C,
\end{aligned}
$$

where, by virtue of Lemmas 6.3.2 and 6.4.1, C is a positive constant independent of ε. Therefore, we get that

$$
\begin{aligned}
\int_0^s \int_\Omega \left[a_\varepsilon(x,t,\nabla u_\varepsilon) - a_\varepsilon(x,t,\nabla u_\varepsilon^h)\right].\nabla\left(u_\varepsilon - v^h\right) dxdt &\le C \int_0^s \int_\Omega \left|\nabla\left(u_\varepsilon - v^h\right)\right|^2 dxdt \\
&\le C\|u_\varepsilon - v^h\|_{L^2(0,T,H^1(\Omega))}.
\end{aligned}
$$

Finally, by using Hölder's inequality and since $u_{\varepsilon,t},\ u_{\varepsilon,t}^h \in L^2(0,T,L^2(\Omega))$, it follows that

$$
\begin{aligned}
&\left\|u_\varepsilon - u_\varepsilon^h\right\|^2_{C([0,T],L^2(\Omega))} + \left\|u_\varepsilon - u_\varepsilon^h\right\|^2_{L^2(0,T,H^1(\Omega))} \\
&\qquad \le C\Bigg(\left\|u_\varepsilon - v^h\right\|_{L^2(0,T,H^1(\Omega))} + \left\|u_{\varepsilon,0} - u_{\varepsilon,0}^h\right\|^2_{L^2(\Omega)}\Bigg).
\end{aligned}
$$

□

Before proving explicit error bounds for the semidiscrete approximation, we prove an abstract error bound for a fully discrete approximation.

6.5 Fully discrete approximation

We consider the following fully discrete approximation of (6.3.1), the backward Euler discretisation applied to (6.4.1):

Let $\Delta t = \frac{T}{N}$, then for $n = 1,..,N$, $U_\varepsilon^n \in V^h$ such that

$$
\int_\Omega \frac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t}\chi^h dx + \int_\Omega \left(|\nabla U_\varepsilon^n|^2 + \varepsilon^2\right)^{\frac{p(x,t)-2}{2}} \nabla U_\varepsilon^n.\nabla\chi^h dx = \int_\Omega f^n \chi^h dx,\ \forall \chi^h \in V^h, \tag{6.5.1}
$$

where $f^n(.) = f(.,n\Delta t)$ and $U_\varepsilon^0 = u_{\varepsilon,0}^h$. It what follows we assume that $U_\varepsilon^0 \in V^h \cap L^\infty(\Omega)$.

Lemma 6.5.1. *If $\{U_\varepsilon^n\}_{n=0}^N$ is a weak solution to (6.5.1), then $U_\varepsilon^n \in C_{loc}^{1,\alpha}(\Omega)$ with $\alpha \in (0,1)$ for any n=1,..,N.*

Proof. First, we are going to prove that $U_\varepsilon^1 \in L^\infty(\Omega)$. From (6.5.1), we have

$$-\text{div}\left(\left(\left|\nabla U_\varepsilon^1\right|^2+\varepsilon^2\right)^{\frac{p(x,t)-2}{2}}\nabla U_\varepsilon^1\right) = f^1 - \frac{U_\varepsilon^1 - U_\varepsilon^0}{\Delta t}.$$

Since $U_\varepsilon^0 \in L^\infty(\Omega)$ and $f \in C^{\delta,\frac{\delta}{2}}(\overline{\Omega_T})$, then by Theorem 4.1 of [35] we get that $U_\varepsilon^1 \in L^\infty(\Omega)$. Therefore, by a simple induction we deduce that $U_\varepsilon^n \in L^\infty(\Omega)$ for all n=0,..,N.
Hence, by using Theorem 1.1 of [36] we get that $U_\varepsilon^n \in C_{loc}^{1,\alpha}(\Omega)$ with $\alpha \in (0,1)$ for any n=1,..,N. □

Proposition 6.5.2. *If $U_\varepsilon^{n-1} \in L^2(\Omega)$, then the problem (6.5.1) has a unique weak solution.*

Proof. Let $n \geq 1$ be fixed. For each $\Delta t > 0$, h we define the continuous mapping

$$F : V^h \longrightarrow V^h,$$

by

$$\langle F(U_\varepsilon), V\rangle = \frac{1}{\Delta t}\left\langle U_\varepsilon - U_\varepsilon^{n-1}, V\right\rangle + \left\langle \left(|\nabla U_\varepsilon|^2+\varepsilon^2\right)^{\frac{p(x,t)-2}{2}}\nabla U_\varepsilon, \nabla V\right\rangle - \langle f^n, V\rangle\,, \quad \forall V \in V^h.$$

Choosing $V = U_\varepsilon$ and using Young's inequality, we get

$$\langle F(U_\varepsilon), U_\varepsilon\rangle = \frac{1}{\Delta t}\left\langle U_\varepsilon - U_\varepsilon^{n-1}, U_\varepsilon\right\rangle + \left\langle \left(|\nabla U_\varepsilon|^2+\varepsilon^2\right)^{\frac{p(x,t)-2}{2}}\nabla U_\varepsilon, \nabla U_\varepsilon\right\rangle - \langle f^n, U_\varepsilon\rangle$$

By using Lemma 6.5.1, we obtain

$$\varepsilon^{p^- -2} \geq \left(|\nabla U_\varepsilon|^2+\varepsilon^2\right)^{\frac{p(x,t)-2}{2}} \geq C, \tag{6.5.2}$$

where C is a positive constant independent of ε. Therefore, using (6.5.2) and Young's inequality we get

$$\langle F(U_\varepsilon), U_\varepsilon\rangle \geq \frac{1}{\Delta t}\left(\eta\|U_\varepsilon\|^2_{L^2(\Omega)} - C_\eta\|U_\varepsilon^{n-1}\|^2_{L^2(\Omega)} - C_\eta(\Delta t)^2\|f^n\|^2_{L^2(\Omega)}\right) + C\|\nabla U_\varepsilon\|^2_{L^2(\Omega)},$$

for suitable $\eta \in (0,1),\ C_\eta > 1$.

We consider $U_\varepsilon \in B = \left\{v \in V^h,\ \|V\|_{L^2(\Omega)} = R_n\right\}$, where

$$R_n = R_n\left(\Delta t, \|U_\varepsilon^{n-1}\|_{L^2(\Omega)}, \|f^n\|^2_{L^2(\Omega)}\right) > 0$$

is chosen such that

$$\|V\|^2_{L^2(\Omega)} \geq \frac{C_\eta}{\eta}\left(\|U_\varepsilon^{n-1}\|^2_{L^2(\Omega)} + (\Delta t)^2\|f^n\|^2_{L^2(\Omega)}\right), \quad \text{for all } \|V\|_{L^2(\Omega)} = R_n.$$

Then for any $U_\varepsilon \in B$ we have $\langle F(U_\varepsilon), U_\varepsilon\rangle \geq 0$. A corollary to Brouwer's fixed-point theorem implies the existence of $U^* \in B$ such that $F(U^*) = 0$. The claim is proved with $U^* = U_\varepsilon$. □

For the purposes of the error analysis it's convenient to introduce

$$U(t) = \frac{t - t_{n-1}}{\Delta t} U_\varepsilon^n + \frac{t_n - t}{\Delta t} U_\varepsilon^{n-1}, \qquad t \in [t_{n-1}, t_n]$$
$$\widehat{U}(t) = U_\varepsilon^n, \qquad t \in (t_{n-1}, t_n]$$
$$\widehat{f}(t) = f^n, \qquad t \in (t_{n-1}, t_n].$$

where $t_n = n\Delta t$ be a uniform partition of $[0, T]$. Then (6.5.1) can be restated: for almost every $t \in [0, T]$

$$\int_\Omega U_t \chi^h dx + \int_\Omega \left(|\nabla \widehat{U}|^2 + \varepsilon^2 \right)^{\frac{p(x,t)-2}{2}} \nabla \widehat{U} . \nabla \chi^h dx = \int_\Omega \widehat{f} \chi^h dx, \qquad \forall \chi^h \in V^h, \tag{6.5.3}$$
$$U(0) = u_{\varepsilon,0}^h.$$

It follows from (6.5.2), Young's inequality and the following inequality

$$\forall a, b \in \mathbb{R}^N \qquad |a|^2 - |b|^2 \leq a(a - b),$$

that

$$\|U_t\|_{L^2(0,T,L^2(\Omega))}^2 + \|U\|_{C([0,T],H^1(\Omega))}^2 + \|\widehat{U}\|_{L^\infty(0,T,H^1(\Omega))}^2 \leq C \left(\Delta t \sum_{n=1}^N \|f^n\|_{L^2(\Omega)}^2 + \|u_{\varepsilon,0}^h\|_{H^1(\Omega)}^2 \right). \tag{6.5.4}$$

and hence

$$\|U - \widehat{U}\|_{L^2(0,T,L^2(\Omega^h))}^2 = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (t_n - t)^2 \|U\|_{L^2(\Omega)}^2 dt \leq C \left(\Delta t \right)^2. \tag{6.5.5}$$

Theorem 6.5.3. *The unique weak solutions u_ε^h and U of (6.4.1) and (6.5.3) respectively, are such that*

$$\|u_\varepsilon^h - U\|_{C([0,T],L^2(\Omega))}^2 + \|u_\varepsilon^h - \widehat{U}\|_{L^2(0,T,H^1(\Omega))}^2 \leq C\Delta t. \tag{6.5.6}$$

Proof. For all $\chi^h \in V^h$, we have for any $s \in (0, T]$

$$\int_0^s \int_\Omega (u_{\varepsilon,t}^h - U_t) \chi^h dx dt + \int_0^s \int_\Omega \left(a_\varepsilon \left(x, t, \nabla u_\varepsilon^h \right) - a_\varepsilon \left(x, t, \nabla \widehat{U} \right) \right) \nabla \chi^h dx dt = \int_0^s \int_\Omega (f - \widehat{f}) \chi^h dx dt.$$

By setting $\chi^h = u_\varepsilon^h - \widehat{U}$, using Lemmas 6.4.1 and 6.5.1, inequality (6.5.2) and Hölder's inequality we obtain

$$\frac{1}{2} \|(u_\varepsilon^h - U)(s)\|_{L^2(\Omega)}^2 + C \|u_\varepsilon^h - \widehat{U}\|_{L^2(0,T,H^1(\Omega))}^2$$
$$\leq \|(u_\varepsilon^h - U)_t\|_{L^2(0,T,L^2(\Omega))} \|\widehat{U} - U\|_{L^2(0,T,L^2(\Omega))}$$
$$+ \|f - \widehat{f}\|_{L^2(0,T,L^2(\Omega))} \|u_\varepsilon^h - \widehat{U}\|_{L^2(0,T,L^2(\Omega))}.$$

Hence, by using (6.5.4) and (6.5.5) we get the desired result. □

6.6 Error estimation

In this section we deduce explicit error bounds from the abstract bounds (6.4.2) and (6.5.6). Let Ω^h be a polygonal approximation to Ω defined by $\overline{\Omega^h} = \cup_{k \in T^h} \overline{k}$ where T^h is a partitioning of Ω^h into a finite number of disjoint open regular N-simplices. Let $\{P_j\}_{j=1}^{J}$ be the vertices associated with this triangulation. We assume that (i) $P_j \in \partial\Omega^h \Longrightarrow P_j \in \partial\Omega$ and (ii) Ω is convex so that $\Omega^h \subseteq \Omega$.

Associated with T^h the finite dimensional spaces

$$S^h \equiv \left\{ \chi \in C(\overline{\Omega^h}) : \ \chi|_k \text{ is linear } \forall k \in T^h \right\}$$

and

$$S_0^h \equiv \left\{ \chi \in C(\overline{\Omega}) : \ \chi|_{\overline{\Omega^h}} \in S^h \text{ and } \chi|_{\overline{\Omega} \setminus \overline{\Omega^h}} \equiv 0 \right\}.$$

Let $\pi_h : \ C(\overline{\Omega^h}) \longrightarrow S^h$ denote the interpolation operator such that for any $\omega \in C(\overline{\Omega^h})$, $\pi_h \omega \in S^h$ satisfies $(\pi_h \omega)(P_j) = \omega(P_j)$ $j = 1, .., J$. As $S_0^h \subset W^{1,\infty}(\Omega)$, we can choose $V^h \equiv S_h^0$. For ease of exposition we assume that $u_{\varepsilon,0} \in W^{2,p(x,t)}(\Omega)$.

Theorem 6.6.1. *Let $u_{\varepsilon,0}^h = \pi_h u_{\varepsilon,0}$, then the unique weak solutions u_ε and u_ε^h of* (6.3.1) *and* (6.4.1)*, respectively, are such that*

$$\|u_\varepsilon - u_\varepsilon^h\|^2_{C([0,T],L^2(\Omega^h))} + \|u_\varepsilon - u_\varepsilon^h\|^2_{L^2([0,T],H^1(\Omega^h))} \le Ch.$$

Proof. Choosing $v^h = \pi_h u_\varepsilon$ in (6.4.2). Since $v^h \in S_0^h \Longrightarrow v^h = 0$ on $\Omega \setminus \Omega^h$. it also follows that (6.4.2) hold with Ω replaced by Ω^h. Finally, taking into consideration Theorems 6.3.3 and 6.4.2, we apply Theorem 3.1.6 of [27] and we obtain the following

$$\begin{aligned}
&\|u_\varepsilon - u_\varepsilon^h\|^2_{C([0,T],L^2(\Omega^h))} + \|u_\varepsilon - u_\varepsilon^h\|^2_{L^2([0,T],H^1(\Omega^h))} \\
&\le C\left(\|u_\varepsilon - \pi_h u_\varepsilon^h\|_{L^2(0,T,L^2(\Omega^h))} + \|u_\varepsilon - \pi_h u_\varepsilon^h\|_{L^2(0,T,H^1(\Omega^h))} + \|u_{\varepsilon,0} - \pi_h u_{\varepsilon,0}^h\|^2_{L^2(\Omega^h)} \right) \\
&\le Ch.
\end{aligned}$$

□

Theorem 6.6.2. *Let u_ε and U be the unique weak solutions of (6.3.1) and (6.5.3) respectively, then we have the following*

$$\|u_\varepsilon - U\|^2_{C([0,T],L^2(\Omega^h))} + \|u_\varepsilon - \widehat{U}\|^2_{L^2([0,T],H^1(\Omega^h))} \le C(h + \Delta t).$$

Proof. It's an immediate consequence of Theorems 6.5.3 and 6.6.1. □

By Lemma 6.3.2, we have for all $s \in (0, T]$

$$\|\nabla u_\varepsilon\|_{C^{\delta,\frac{\delta}{2}}\left(\overline{\Omega}\times[s,T]\right)} \leq C,$$

with C independent of ε, whence we get that

$$u_\varepsilon \longrightarrow u \text{ in } C([0,T], L^2(\Omega^h))$$

and

$$\nabla u_\varepsilon \longrightarrow \nabla u \text{ in } L^2([0,T], L^2(\Omega^h)),$$

as ε tends to zero. Therefore, we get the following theorem

Theorem 6.6.3. *Let u and u_ε be the unique weak solutions of (6.1.1) and (6.3.1) respectively, then we have the following estimate*

$$\|u - u_\varepsilon\|^2_{C([0,T],L^2(\Omega^h))} + \|u - u_\varepsilon\|^2_{L^2(0,T,H^1(\Omega^h))} \leq C\varepsilon^2. \tag{6.6.1}$$

Theorem 6.6.4. *Let u and U be the unique weak solutions of (6.1.1) and (6.5.3) respectively, then we have the following*

$$\|u - U\|^2_{C([0,T],L^2(\Omega^h))} + \|u - \widehat{U}\|^2_{L^2(0,T,H^1(\Omega^h))} \leq C(h + \Delta t + \varepsilon^2). \tag{6.6.2}$$

Proof. This is an application of Theorems 6.6.2 and 6.6.3 . □

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Abstract:

In first part of this thesis, we discuss a class of degenerate isotropic and anisotropic parabolic equations with variable exponents. By using the Steklov average and Young's inequality, we establish energy and logarithmic estimates for solutions to these equations. Then based on the intrinsic scaling method, we prove that local weak solutions are locally continuous. In the second part, we study a time discretization for a doubly non linear parabolic equation with variable exponent by using Euler forward scheme. We investigate existence, uniqueness and stability questions and prove existence of the global compact attractor. Finally, we give a simple approach to a priori estimates for a singular parabolic equation with variable exponent.

Keywords: *Isotropic parabolic problems - Anisotropic parabolic problems - Regularity theory - Intrinsic scaling - Finite element method - Absorbing sets - Compact attractors.*

Résumé:

Dans cette thèse, on s'intéresse à la régularité et à l'étude de schémas d'approximation de certaines équations paraboliques dégénérées isotropiques et anisotropiques à exposant variable. En utilisant la moyenne de Steklov et l'inégalité de Young, on obtient les estimations énergétiques et logarithmiques des solutions à ces équations. Ensuite, par la méthode de DiBenedetto on montre que les solutions locales faibles sont localement continues. La deuxième partie est consacrée à l'étude de quelques schémas d'approximation appliquée aux certains équations non linéaires et doublement non linéaires à exposant variable. Concernant les schémas d'approximation on établit des résultats d'existence, d'unicité et on montre l'existence d'un attracteur global pour les solutions d'un schéma d'Euler semi-discrétisé en temps. On s'intéresse également à un schéma d'approximation par une méthode d'éléments finis pour lequel on obtient des résultats de stabilité et d'estimation de l'erreur.

Mots-clés: *Problèmes paraboliques isotropiques - Problèmes paraboliques anisotropiques - Théorie de la régularité - La méthode de DiBenedetto - Attracteur compact - La méthode des éléments finis .*

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