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OLD AND NEW ORTHOGONAL POLYNOMIALS OF COMPLEX AND QUATERNIONIC VARIABLE : CONCRETE DESCRIPTION, ASSOCIATED FUNCTIONAL SPACES AND INTEGRAL TRANSFORMS.

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ABSRACT

We present a concrete and complete study of some classes of orthogonal polynomials of complex and quaternionic variable and provides different applications in the theory of integral transforms and spectral analysis of some special magnetic Landau Hamiltonian. Mainly, their basic properties are derived and then employed to introduce and study certain new integral transforms for some specific functional Hilbert spaces. Chapter 2 deals with the univariate polyanalytic complex Hermite polynomials generalizing the monomials. Chapter 3 is devoted to a novel class of orthogonal polyanalytic functions generalizing the holomorphic Hermite polynomials. The associated functional spaces are of Gelfand–Shilov type and generalize, somehow, the one introduced by van Eijndhoven and Meyers in 1990. In Chapter 5, we make use of the quaternionic Hermite polynomials to study different classes of slice polyregular Bargmann spaces. The explicit formulas of their reproducing kernels are given and associated Segal–Bargmann transforms are also studied.

Key-Words: Orthogonal polynomials; integral transforms; Spectral theory; Magnetic Laplacians; Generating functions; Polyanalytics functions; Segal–Bargmann transforms; Quaternionic Hilbert spaces.

RÉSUMÉ

Le présent travail porte sur l'analyse mathématique des polynômes orthogonaux à variables complexe et quaternionique et ses différentes applications, notamment à la théorie des transformées intégrales et l'analyse spectrale de certains Laplaciens. L'étude concrète de leurs propriétés nous permet d'introduire des nouvelles transformations intégrales pour certains espaces de Hilbert de fonctions polyanalytiques. Chapitre 2 traite les polynômes de Hermite polyanalytique généralisent les monômes . Chapitre 3 est consacré à une nouvelle classe de fonctions polyanalytiques orthogonales généralisant les polynômes holomorphes de Hermite. Les espaces fonctionnels associés sont de type Gelfand–Shilov et généralisent, en quelque sorte, celui introduit par van Eijndhoven et Meyers en 1990. Au Chapitre 5, nous utilisons les polynômes de Hermite quaternioniques pour étudier différentes classes d'espaces de Bargmann "S–polyregular". On donne les expressions explicites de leurs noyaux reproduisants et on étudie également les transformées de Segal–Bargmann associées. La description spectrale en tant que sous-espaces propres d'un opérateur S-différentiel de second ordre impliquant est étudiée. Le dernier chapitre est consacré à des transformations intégrales, pour les espaces de Bargmann-Fock holomorphes et hyperholomorphes, obtenue par la composition des transformées classiques.

Mots-clés: Polynômes orthogonaux ; Transformées intégrales ; Théorie spectral; Laplacien magnétique; Fonctions génératrices; Fonctions polyanalytique; Transformées de Segal-Bargmann ; Espaces de Hilbert quaternioniques.

RÉSUMÉ DE LA THÉSE

On étudie des classes de polynômes orthogonaux ainsi que ses différentes applications à la théorie des transformées intégrales et l'analyse spectrale de certains opérateurs différentielles de seconde degré. Pour y faire, on divisera ce travail en deux grandes parties

Partie I : Polynômes à variables complexes et applications

Dan un premier temps, on considère les polynômes de Hermite complexes

$$H_{m,n}(z,\overline{z}) = (-1)^{m+n} e^{|z|^2} \frac{\partial^{m+n}}{\partial \overline{z}^m \partial z^n} \left(e^{-|z|^2} \right), \tag{1}$$

et on établit les expressions explicites pour quelques sommations infinies (fonctions génératrices classiques et bilinéaires, Formules de Mehler, ...) associes a ces polynômes et qui jouent un rôle primordial dans la construction des transformées intégrales bidimensionnelle de type Segal–Bargmann $L^2(\mathbb{C}, e^{-\nu|z|^2} d\lambda(z))$ avec lui-même et avec l'espace de Bargmann–Fock généralisé. On caractérise l'image de ses transformées et on les utilisent ensuite avec ses propriétés pour obtenir les relation avec des transformées classiques telles que la transformée de Fourier fractionnaire et celle de Fourier–Wigner. On introduit ensuite (Chapitre 3), la classe

$$I_n^{\nu,\alpha}(z,\overline{z}|\xi) = e^{-\alpha z^2 - \xi z} \left(-\partial_z + \nu \overline{z} \right)^n \left(e^{\alpha z^2 + \xi z} \right) \tag{2}$$

généralisent ainsi les polynômes de Hermite classiques. Nous étudions les propriétés algébriques et analytiques, y compris les relations de récurrence, les équations différentielles qui vérifient, la formule de Rodrigues et la formule quadratique de type Nielsen ainsi que la formule explicite en termes de polynômes d'Hermite. On étudie leurs orthogonalité et nous fournissons également des fonctions génératrices et des représentations intégrales, y compris la réalisation de ces polynômes en fonction de la transformation de Fourier–Wigner avec comme fenêtre une fonctions de Hermite. Comme application directes des résultats obtenus, on démontre qu'une sous classe de ces polynômes est une base de l'espace des fonctions, $f : \mathbb{C} \to \mathbb{C}$, $\mathbb{Z}$-automorphes satisfaisant l'équation fonctionnelle

$$f(z+k) = e^{2i\pi\beta k} e^{2\alpha(z+\frac{k}{2})k} f(z),$$

pour tout $z \in \mathbb{C}$ and $k \in \mathbb{Z}$, et tels que

$$||f||^2_{\alpha,\mathbb{Z}} := \int_{\mathbb{C}/\mathbb{Z}} |f(z)|^2 e^{-2\alpha|z|^2} dxdy < +\infty.$$

Le troisième objectif dans cette partie est l'étude des fonctions spéciales généralisant celles

définie par

$$\phi_m^s(z) = \frac{1}{\sqrt{\pi m!}} \left(\frac{1-s^2}{2s} \right)^{(m+1)/2} e^{-\frac{1+s^2}{4s}z^2} z^m \tag{3}$$

introduites par Van Eijndhoven and Meyer en 1990 et qui forment une base orthogonal d'un espace de type Bargmann $\mathcal{X}_s = \mathcal{H}ol(\mathbb{C}) \cap L^2(\mathbb{C}, \omega_s d\lambda)$. Plus précisément, on propose une nouvelle base de l'espace $\mathcal{H}_s$ et on donne une décomposition orthogonal pour cette espace en terme des espaces $\mathcal{X}_{s,n}$ dont une base orthonormal est donnée par

$$\psi_{m,n}^s(z,\overline{z}) := \Gamma_{m,n}^s e^{-\frac{z^2}{2}} \left(\nabla^n_{\nu,\alpha-\frac{1}{2}} H_m \right)(z), \quad \nabla_{\nu,\alpha} := -\partial_z + \nu\overline{z} - 2\alpha z. \tag{4}$$

De plus on donne la forme explicite et compacte du noyau reproduisant de chaque espace $\mathcal{X}_{s,n}$ ainsi que la transformée de type Segal–Bargmann associée. On exploite ensuite les bases obtenues pour construire des transformées integrales de type Fourier–fractionaire.

Partie II : Espaces S-polyregular et polynômes à variable quaternionique

Dans le premier chapitre de cette partie (Chapitre 5), on décrit concrètement différents types des espaces S-polyregular quaternioniques qu'on introduit à l'aide de la dérivée slice. Il s'agit de $\mathcal{SR}^2_{1,n}$, $\mathcal{SR}^2_{2,n}$ et $\mathcal{SR}^2_{full,n}$. Dans cette description les polynômes de Hermite quaternioniques

$$H^Q_{m,n}(q,\overline{q}) = m!n! \sum_{j=0}^{\min(m,n)} \frac{(-1)^j}{j!} \frac{q^{m-j}\overline{q}^{n-j}}{(m-j)!(n-j)!} \tag{5}$$

joue un rôle crucial. On montre que $\mathcal{SR}^2_{1,n}$ et $\mathcal{SR}^2_{2,n}$ sont des espaces de Hilbert à noyau reproduisant qu'on donne explicitement. Son expression fait appeal au polynômes de Laguerre et le produit star pour les fonctions slice regular. On établit ensuite une décomposition orthogonal pour $L^2(\mathbb{C}_I; e^{-|\xi|^2}\lambda_I)$ en terme de $\mathcal{SR}^2_{2,n}$ Pour y arriver, on commence ce chapitre en étudiant ces dernières fonctions. A ces espaces, on associe des transformations de type Segal–Bargmann $B_{l,n}, l = 1,2$ définis sur l'espace de Hilbert (à gauche) des fonctions sur la droite réelle à à valeur quaternioniques. On close ce chapitre en donnant la réalisation spectrale de ces espaces en tant que sous-espaces spéciaux des espaces propres d'un opérateur différentiel du second ordre.

La discutons de quelques propriétés de certaines transformations intégrales associées à des espaces de Hilbert fonctionnels spécifiques sur $\mathbb{C}$ et $\mathbb{C}^2$ font l'objet du dernier chapitre. Il s'agit des espaces de Bargmann-Fock et sa version slice-hyperholomorphe. La plus importante et obtenue comme composition des transformations Segal-Bargmann unidimensionnelles et bidimensionnelles. On montre qu'elle se réduit à la transformée Segal-Bargmann unidimensionnelle avec un symbole spécifique ψ_1,

$$\mathcal{G}^\nu f(z,w) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \mathcal{C}_{\psi_1}(\mathcal{B}^{1,\nu} f)(z,w), \tag{6}$$

Entre autres propriétés discutées au Chapitre 6 de la transformation $\mathcal{G}^{\nu}$, on détermine son image ainsi que son inverse à gauche défini sur tout l'espace de Bargmann bidimensionnel. On étudie aussi la relation avec quelques transformées classiques. Comme conséquence, on établit la relation entre les espaces de Bargmann–Fock et l'espace slice-hyperholomorphe laissant invariant les slices.

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General description of different topics discussed in the present dissertation

A. The setting

The topics developed in this document are the author contributions to the mathematical analysis of complex and quaternionic orthogonal polynomials, as well as approximation theory, mathematical physics, integral transforms, hypercomplex analysis and concrete spectral theory of a specific magnetic Landau Hamiltonian, among others. The principal object is the study of special classes of old and new orthogonal polynomials on the complex plane $\mathbb{C}$ as well as on the (non-commutative) division algebra of quaternions $\mathbb{H}$ with concrete applications in complex and hypercomplex analyses, such as the theory of holomorphic, polyanalytic, slice regular and S-polyregular Bargmann spaces. Associated Segal–Bargmann and Fractional Fourier integral transforms. Accordingly, the content of this work can be divided in two main parts. The first one concerns different generalizations of complex (polyanalytic) Hermite polynomials and their application in complex analysis and integral transforms. The second part, is devoted to their quaternionic analogue. Interesting application is given in the context of slice hypercomplex analysis.

Throughout the work, we denote by $L^{2,\nu}(X)$; $\nu > 0$ the Hilbert space of all square integrable functions on $X = \mathbb{R}, \mathbb{R}^2 \cong \mathbb{C}, \mathbb{C}^2 \cong \mathbb{H}$ with respect to the Gaussian measure $d\lambda_\nu(s) := e^{-\nu|s|^2} d\lambda(s)$, where $d\lambda$ is the Lebesgue measure on X; $\lambda(s) = dx, \quad \lambda(s) = dx_z dy_z, \lambda(s) = dx_z dy_z dx_w dy_w$ for $s = x \in \mathbb{R}$, $s = z = x_z + iy_z \in \mathbb{C} \cong \mathbb{R}^2$, $s = (z, w) = z + wj \in \mathbb{C}^2 \cong \mathbb{H}$, respectively. When $\mathbb{H}$ is added as subscript, then $L^{2,\nu}_{\mathbb{H}}(X)$ means the considered functions are $\mathbb{H}$-valued. We denote by $\mathcal{F}^{2,\nu}(X)$ the Bargmann–Fock space constituted of all holomorphic functions on X, when $X = \mathbb{C}$ or $X = \mathbb{C}^2$, and belonging to $L^{2,\nu}(X)$,

$$\mathcal{F}^{2,\nu}(X) = \mathcal{H}ol(X) \cap L^{2,\nu}(X).$$

We will also use the notation $\mathcal{F}^2(X)$ to mean $\mathcal{F}^{2,\nu}(X)$ with $\nu = 1$.

B. Part I: Old and new orthogonal polynomials of complex variable: Basic properties and applications

B.1. Context, notations and tools

The well known Hermite polynomials (and their different generalizations) have been one of the most interesting fields for research, since their introduction by Lagrange and Chebyshev. They are extensively studied in the mathematics literature and appear in a wide spectrum of research domains including engineering sciences, pure and applied mathematics, and various branches of physics (see for examples [99,107,109] and references therein). The classical ones on the real line $\mathbb{R}$ are defined by ([70,99,107,109])

$$H_n^\nu(x) := (-1)^n e^{\nu x^2} \frac{d^n}{dx^n}\left(e^{-\nu x^2}\right) = \nu^m m! \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^k}{k!\nu^k} \frac{(2x)^{m-2k}}{(m-2k)!}, \tag{7}$$

Here and elsewhere after, we use ∂_x and ∂/∂_x to denote the partial differential operator with respect to x. Natural extensions to the two real variables can be obtained by considering the tensor product $H_{m,n}^\nu(x,y) = H_m^\nu(x)H_n^\nu(y)$ or by replacing the real variable x in $H_m^\nu(x)$ by the complex variable z, giving rise to the class of holomorphic Hermite polynomials (see e.g. [75,107])

$$H_n^\nu(z) = (-1)^n e^{\nu z^2} \frac{d^n}{dz^n}(e^{-\nu z^2}). \tag{8}$$

They inherit the most of algebraic properties of $H_m^\nu(x)$ by analytic continuation. Moreover, they possess further interesting analytic properties. The polynomials $H_n^\nu(z)$ (with $\nu = 1$) have been investigated in the study of some analytic function spaces [29,78,113] and showed to be useful for the coherent states theory [34,52]. Their combinatorics has been studied in [75]. It is shown in [113] that the associated functions

$$\psi_m^s(z) = \left(\frac{1-s}{\pi m!\sqrt{s}}\right)^{1/2} \left(\frac{1-s}{2(1+s)}\right)^{m/2} e^{-\frac{z^2}{2}} H_m(z), \tag{9}$$

for given fixed $0 < s < 1$, satisfy the orthogonal property ([113])

$$\int_{\mathbb{C}} \psi_n^s(z)\overline{\psi_m^s(z)} e^{-\frac{1-s^2}{2s}|z|^2} e^{\frac{1+s^2}{4s}(z^2+\bar{z}^2)} d\lambda(z) = \delta_{n,m}, \tag{10}$$

This is to say that the functions $\psi_m^s(z)$ form an orthonormal system in the Hilbert space

$$\mathscr{H}^{2,s}(\mathbb{C}) := L^2(\mathbb{C}, \omega_s d\lambda),$$

where the weight function ω_s is given by

$$\omega_s(z,\bar{z}) = e^{\frac{1+s^2}{4s}(z^2+\bar{z}^2) - \frac{1-s^2}{2s}|z|^2}.$$

Accordingly, we define the Hilbert subspace $\mathcal{X}_s(\mathbb{C})$ of holomorphic functions belonging to $\mathscr{H}^{2,s}(\mathbb{C})$ by

$$\mathcal{X}_s(\mathbb{C}) = \mathcal{H}ol(\mathbb{C}) \cap \mathscr{H}^{2,s}(\mathbb{C}). \tag{11}$$

Another generalization of $H^{\nu}_m(x)$ to the whole complex plane is given by the univariate complex Hermite polynomials (UCHP)

$$H^{\nu}_{m,n}(z,\overline{z}) = (-1)^{m+n} e^{\nu z\overline{z}} \frac{\partial^{m+n}}{\partial \overline{z}^m \partial z^n} \left(e^{-\nu z\overline{z}} \right) \tag{12}$$

and their specific generalization $G_{m,n}(z,\overline{z}|\xi)$ considered in [54]. The polynomials $H^{\nu}_{m,n}(z,\overline{z})$ has been introduced by Ito (1952) in the context of complex Markov process [76]. But, it is only in the last decades that they are widely used in many area of mathematics, like nonlinear analysis of traveling-wave tube amplifiers, signal processing, singular values of the Cauchy transform, coherent states theory, combinatorics as well as in distribution of zeros of the automorphic reproducing kernel function. The curious reader can refer to [7,41, 54–56,72,73,82,103] and references therein for their basic properties and their applications.

Both $H^{\nu}_{m,n}(z,\overline{z})$ and $G_{m,n}(z,\overline{z}|\xi)$ are special examples of polyanalytic polynomials of one complex variable for satisfying the generalized Cauchy equation

$$\partial^{n+1}_{\overline{z}} H^{\nu}_{m,n}(z,\overline{z}) = 0 \text{ and } \partial^{n+1}_{\overline{z}} G^{\nu}_{m,n}(z,\overline{z}) = 0$$

They play a crucial role in studying some basic properties of polyanalytic functions [4] and they appear as particular cases of the following general class of polyanalytic polynomials ([42])

$$G^{\nu,h}_{m,n}(z,\overline{z}) = (-1)^{m+n} e^{\nu|z|^2 - h(z)} \frac{\partial^{m+n}}{\partial \overline{z}^m \partial z^n} \left(e^{-\nu|z|^2 + h(z)} \right), \quad \nu > 0, \tag{13}$$

where $h(z)$ is a given holomorphic polynomial function. Such polynomials appear naturally, when dealing with the spectral theory of a special magnetic Laplacian leaving the space of mixed automorphic functions invariant [42]. As another special interesting class of (4.1) are the ones corresponding to the special holomorphic function $h^{\alpha,\xi}_0(z) = \alpha z^2 + \xi z$, for arbitrary real α and complex number ξ. In fact, we have to consider

$$I^{\nu,\alpha}_n (z,\overline{z}|\xi) = (-1)^n e^{\nu z\overline{z} - \alpha z^2 - \xi z} \frac{\partial^n}{\partial z^n} \left(e^{-\nu z\overline{z} + \alpha z^2 + \xi z} \right), \tag{14}$$

for varying $n = 0, 1, 2 \cdots$. Such class of functions can be seen as the polyanalytic generalization of the holomorphic Hermite polynomials $H_n(z) = I^{0,-1}_n (z,\overline{z}|0)$ as well as the monomials $I^{1,0}_n (z,\overline{z}|0) = \overline{z}^n$. The consideration of this class is motivated by their importance in the theory of the automorphic functions on the complex plane with respect to a given rank–one discrete subgroup $\Gamma = \mathbb{Z}$ of $(\mathbb{C}, +)$. More specifically, the particular case of

$\xi = 2i\pi(\beta + k)$, with $\beta \in \mathbb{R}$ and $k \in \mathbb{Z}$, leads to

$$I_{n,k}^{\nu,\alpha,\beta}(z, \overline{z}|\xi) := I_n^{\nu,\alpha}(z, \overline{z}|2i\pi(\beta + k)) \tag{15}$$

which for fixed nonnegative integer n gives rise to an orthogonal basis of the n^{th} L^2–eigenspace of a Schrödinger operator acting on some L^2–sections over the strip $\mathbb{C}/\mathbb{Z}$ of the L^2–line bundle $L = (\mathbb{C} \times \mathbb{C})/\mathbb{Z}$, constructed as the quotient of the trivial bundle over $\mathbb{C}$ by considering the $\mathbb{Z}$–action [58,106].

B.2. Main purposes of part I

- The first aim of this part is the obtainment of closed explicit expressions of some infinite sums involving the UCHP, including bilinear generating functions and Mehler's formulas. The acquired ones play crucial role in obtaining some integral kernel transforms. In fact, we employ them to introduce a special two–dimensional integral transform whose kernel function are the exponential generating functions of the UCHP. Moreover, we identify their images and investigate their basic properties such as the connection to the classical $2d$-Segal–Bargmann transform, to the Wigner transform, as well as to the $2d$-fractional Fourier transform ($2d$–FrFT) introduced recently by Zayed [115]. Accordingly, we are able to deduce some interesting properties of the $2d$–FrFT. Another class of $1d$ integral transforms connecting any two generalized Bargmann–Fock spaces $\mathcal{F}_n^{2,\nu}(\mathbb{C})$ is introduced.

- As second aim, we provide a concrete description of the basic properties of the polyanalytic polynomials $I_n^{\nu,\alpha}(z, \overline{z}|\xi)$ in (14). To this end, we begin by discussing their operational representations, recurrence relations, differential equations they satisfy, orthogonality relations, Rodrigues' formula and quadratic formula of Nielsen type as well as the explicit formula in terms of Hermite polynomials. We also provide generating functions and integral representations, including the one involving a Fourier–Wigner transform with a special window function close to the classical Mehler kernel. In the course of our investigation, we present two interesting applications. The first one is related to the concrete description of the spectral theory of some specific second order differential operator of Laplacian type acting on the Hilbert space $L^2(\mathbb{C}; e^{-\nu|z|^2}dxdy)$. The second application involves the subclass $I_{m,n}^{\alpha,\beta}(z, \overline{z}|\xi)$ in (3.5) and reproves the fact that they form a complete orthogonal system of the space $L^2(\mathbb{C}/\mathbb{Z}; e^{-\nu|z|^2}dxdy)$ of L^2–rank–one automorphic functions.

- The third aim is the study of the special holomorphic Hermite functions ψ_m^s, in (9), spanning the like-Bargmann Hilbert space $\mathcal{X}_s(\mathbb{C})$, defined through (11). We also propose an adequate orthogonal complement in $\mathscr{H}^{2,s}(\mathbb{C})$. More precisely, we determinate the Hilbertian decomposition of $\mathscr{H}^{2,s}(\mathbb{C})$ in terms of some reproducing kernel Hilbert subspaces $\mathcal{X}_{n,s}(\mathbb{C})$, and provide to each one an orthonormal basis of polyanalytic functions $\psi_{m,n}^s$, generalizing the ones in (9), so that $\psi_{m,0}^s = \psi_m^s$. We also compute the explicit expression of the corresponding reproducing kernel. As applications, we derive the associated Segal–Bargmann integral transforms for the configuration space $\mathcal{L}^2(\mathbb{R})$ with range the spaces $\mathcal{X}_{n,s}(\mathbb{C})$. Moreover, we provide two nontrivial $1d$- and $2d$-fractional like-Fourier transforms for the configuration space $\mathcal{L}^{2,\nu}(\mathbb{R})$ and $\mathscr{H}^{2,s}(\mathbb{C})$, respectively.

C. Part II: Quaternionic Hermite polynomials, S-polyregular Bargmann spaces and associated integral transforms

C.1. Notation and preliminaries

In order to generalize the obtained results to the quaternionic context, we begin by considering the natural quaternionic analogs of the polyanalytic Hermite polynomials $H_{m,n}(z,\overline{z})$. Indeed, we deal with the quaternionic Hermite polynomials

$$H^{Q}_{m,n}(q,\overline{q}) = m!n! \sum_{j=0}^{\min(m,n)} \frac{(-1)^j}{j!} \frac{q^{m-j}\overline{q}^{n-j}}{(m-j)!(n-j)!}. \tag{16}$$

For an systematic study of $H^{Q}_{m,n}(q,\overline{q})$ one can refer to [43]. We will use the polynomials $H^{Q}_{m,n}(q,\overline{q})$ to define new classes of generalized Bargmann spaces in the context of slice polyregular functions. To this end, let us recall that the classical Bargmann functional space $\mathcal{F}^2$ is defined as the phase space on the complex plane consisting of all $e^{-|z|^2}dxdy$-square integrable entire functions. As special generalizations, in the context of polyanalytic functions, are the generalized Bargmann spaces $\mathcal{F}^2_n$ of level $n = 0,1,2,\cdots$, (see for example [3,4,59,114]), so that $\mathcal{F}^2_0 = \mathcal{F}^2$. The corresponding theory has found remarkable applications in time-frequency analysis, analysis of higher Landau levels and in the multiplexing of signals (see [4] and references therein).

A quaternionic counterpart of $\mathcal{F}^2$ was introduced in [9],

$$\mathcal{F}^{2,\nu}_{slice}(\mathbb{H}) = \mathcal{SR}(\mathbb{H}) \cap L^{2,\nu}_{\mathbb{H}}(\mathbb{C}_I), \tag{17}$$

where $L^{2,\nu}_{\mathbb{H}}(\mathbb{C}_I)$ is the Hilbert space of $\mathbb{H}$-valued L^2 functions with respect to the Gaussian measure on an given slice $\mathbb{C}_I = \mathbb{R} + \mathbb{R}I$ and $\mathcal{SR}(\mathbb{H})$ denotes the space of (left) slice regular $\mathbb{H}$-valued functions on the quaternions, i.e., $\mathbb{H}$-valued real differentiable functions f on $\mathbb{H} \equiv \mathbb{R}^4$ such that

$$\overline{\partial_I} f(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f|_{\mathbb{C}_I}(x+yI) \tag{18}$$

vanishes identically on $\mathbb{C}_I$ for every $I \in \mathbb{S} = \{q \in \mathbb{H}; q^2 = -1\}$. Where, $f|_{\mathbb{C}_I}$ denotes the restriction of f to the slice $\mathbb{C}_I := \mathbb{R} + \mathbb{R}I$. More precisely,

$$\mathcal{F}^2_{slice} = \left\{ f(q) = \sum_{j=0}^{+\infty} q^j c_j;\ c_j \in \mathbb{H}, \quad \sum_{j=0}^{+\infty} j!|c_j|^2 < +\infty \right\}. \tag{19}$$

It is shown in [9] that $\mathcal{F}^2_{slice}$ is independent of I and is a reproducing kernel quaternionic Hilbert space (see also [40] for a quite different proof). The corresponding Segal–Bargmann transform $\mathcal{B}^{\nu}_{\mathbb{H}}$ is considered in [40] and maps the Hilbert space $L^{2,\nu}_{\mathbb{H}}(\mathbb{R})$ of $\mathbb{H}$-valued functions that are $e^{-\nu x^2}dx$-square integrable on the real line onto $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$. Its kernel function arises naturally as the unique extension of its holomorphic counterpart to a slice regular function. It can also be realized as the generating function of the rescaled real Hermite polynomials $H^{\nu}_n(x)$ in (2.2). This can be seen as special quaternionic analog of the standard Segal–Bargmann transform (with a slightly different convention from the classical one)

$$\mathcal{B}^{d,\nu} f(z) := \left(\frac{\nu}{\pi}\right)^{\frac{3d}{4}} \int_{\mathbb{R}^d} f(x) e^{-\nu\left(x-\frac{z}{\sqrt{2}}\right)^2} dx, \tag{20}$$

intertwining the Schrödinger representation and the complex wave representation of the quantum mechanical harmonic oscillator and plays an important role in quantum optics, in signal processing and in harmonic analysis on phase space [16,49,68,91,116]. It made the quantum mechanical configuration space $L^{2,\nu}_{\mathbb{C}}(\mathbb{R}^d)$, the Hilbert space of $\mathbb{C}$-valued $e^{-\nu x^2}dx$-square integrable functions on $\mathbb{R}^d$, unitarily isomorphic to the Bargmann–Fock space

$$\mathcal{F}^{2,\nu}(\mathbb{C}^d) = Hol(\mathbb{C}^d) \cap L^{2,\nu}(\mathbb{C}^d, \mathbb{C}).$$

Motivated by the works [3,4,15,59,114] studying and characterizing the polyaniliticity in the complex setting as well as by Brackx' works [23,24] studying the k-monogenic functions with respect to the Fueter operator, it is of interest to look for possible generalizations of $\mathcal{F}^2_{slice}$ and its associated Segal–Bargmann transform to the context of slice n-polyregular ($\mathcal{SR}_n$) functions with respect to the slice derivative. In fact, we consider two kinds of such generalizations. These spaces will be called here S-polyregular Bargmann space of level n of first and second kind, denoted by $\mathcal{SR}^2_{1,n}$ and $\mathcal{SR}^2_{2,n}$, respectively. They are natural extensions of $\mathcal{F}^2_{slice}$ to the setting of S-polyregular functions and appear as special subspaces of the Hilbert space

$$\mathcal{SR}^2_n := \mathcal{SR}_n \cap L^2(\mathbb{C}_I, e^{-|\xi|^2} d\lambda_I),$$

the space of all S-polyregular functions $f : \mathbb{H} \longrightarrow \mathbb{H}$ subject to the norm boundedness $\|f\|_{\mathbb{C}_I} < +\infty$, where $\|\cdot\|_{\mathbb{C}_I}$ is the norm induced by the inner product

$$\langle f, g\rangle_{\mathbb{C}_I} = \int_{\mathbb{C}_I} \overline{f|_{\mathbb{C}_I}(q)} g|_{\mathbb{C}_I}(q) e^{-|q|^2} d\lambda_I(q). \tag{21}$$

C.2. Main tasks of part II

- The first main aim in this second part is to introduce and give a concrete description of $\mathcal{SR}^2_{1,n}$ and $\mathcal{SR}^2_{2,n}$. This description invokes the quaternionic Hermite polynomials $H^Q_{m,n}(q,\overline{q})$, in (16). We prove that $\mathcal{SR}^2_{1,n}$ and $\mathcal{SR}^2_{2,n}$ are reproducing kernel quaternionic Hilbert spaces whose reproducing kernels are given explicitly in terms of stared-Laguerre polynomials. The proof is based essentially on a weak version of the Identity Principle for S-polyregular functions and on a natural extension of the left star product for S-polyregular functions. A basic properties of S-polyregular functions is also given. Moreover, a hilbertian decomposition of $L^2(\mathbb{C}_I; e^{-|\xi|^2} d\lambda_I)$ in terms of $\mathcal{SR}^2_{2,n}$ are described. It should be noted here that $\mathcal{SR}^2_{1,0}$ and $\mathcal{SR}^2_{2,0}$ reduce further to $\mathcal{F}^2_{slice}$ in (19).

- Associated Segal–Bargmann transforms $\mathcal{B}_{\ell,n}$, $\ell = 1,2$, are then introduced and studied in some details. They are defined on $L^2_{\mathbb{H}}(\mathbb{R})$. Their kernels involve the Hermite polynomials extended to the quaternions. It should be noted here that for $n = 0$, the transform $\mathcal{B}_{\ell,0}$ is equal to the one considered in [40].

- Another task of this part is to show that the constructed spaces are closely connected to the concrete L^2-spectral analysis of the semi-elliptic (slice) second-order differential operator
$$\square_q = -\partial_s \overline{\partial_s} + \overline{q}\,\overline{\partial_s}, \tag{22}$$
where
$$\overline{\partial_s} f(q) = \begin{cases} \overline{\partial_{I_q}} f(x + I_q y), & \text{if } q = x + I_q y \in \mathbb{H} \setminus \mathbb{R}; \\ \dfrac{df}{dx}(x), & \text{if } q = x \in \mathbb{R}, \end{cases} \tag{23}$$
which can seen as the conjugate of the left slice derivative ∂_s that we can define in a similar way in terms of ∂_{I_q}. In fact, such spaces are realized as special subspaces of the L^2-eigenspaces
$$\mathcal{F}^2_n = \left\{ f \in L^2(\mathbb{H}; e^{-|q|^2} d\lambda);\ \square_q f = nf \right\}, \tag{24}$$
where $n = 0, 1, 2, \cdots$. The L^2-spectral description of $\square_q f = nf$ was possible by dealing first with the $\mathcal{C}^\infty$ right-eigenvalue problem $\square_q f = f\mu$ on $\widetilde{\mathbb{H}} := \mathbb{H} \setminus \mathbb{R}$ and then by extending appropriately the obtained explicit solutions to the whole $\mathbb{H}$. Thereby, by manipulating the asymptotic behavior of such eigenfunctions, we show that the spectrum of $\square_q$ is purely discrete and consists of the eigenvalues $\mu = n$ which occur with infinite degeneracy.

- We consider the integral transform
$$\mathcal{G}^\nu f(z,w) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \mathcal{C}_{\psi_1}(\mathcal{B}^{1,\nu} f)(z,w), \tag{25}$$
obtained as the composition operator $\mathcal{C}_{\psi_1} f = f \circ \psi_1$ of the $1d$-Segal–Bargmann transform $\mathcal{B}^{1,\nu}$ with the specific symbol $\psi_1(z,w) = \frac{z+iw}{\sqrt{2}}$. We study its basic properties and characterize its image. Namely, we show that $\mathcal{G}^\nu$ is a special one-to-one transform mapping the standard Hilbert space $L^{2,\nu}(\mathbb{R}, \mathbb{C})$ on the real line into the $2d$-Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ onto
$$\mathcal{A}^{2,\nu}(\mathbb{C}^2) := \left\{ F \in \mathcal{F}^{2,\nu}(\mathbb{C}^2);\ \left(\frac{\partial}{\partial z} + i\frac{\partial}{\partial w}\right) F = 0 \right\}. \tag{26}$$
This was possible by realizing this transform in a natural way as the composition of

the $1d$- and $2d$-Segal–Bargmann transforms

$$\mathcal{G}^{\nu} = \mathcal{B}^{2,\nu} \circ \mathcal{B}^{1,\nu}. \tag{27}$$

It maps isometrically the standard Hilbert space on the real line into the two-dimensional Bargmann–Fock space. Moreover, if $Proj$ denotes the orthogonal projection on the one-dimensional Bargmann–Fock space, we show that the transform $\mathcal{R}^{\nu} := (\mathcal{B}^{1,\nu})^{-1} \circ Proj \circ (\mathcal{B}^{2,\nu})^{-1}$ defined on the whole $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ is a like-left inverse of $\mathcal{G}^{\nu}$ that can be expressed in terms of the inverse of $\mathcal{B}^{1,\nu}$ and a composition operator with a specific symbol $\psi_2 : \mathbb{C} \longrightarrow \mathbb{C}^2$. More explicitly, we have

$$\mathcal{R}^{\nu}F(x) = \left(\frac{\pi}{\nu}\right)^{\frac{1}{4}} \int_{\mathbb{C}} F\left(\frac{\xi}{\sqrt{2}}, -i\frac{\xi}{\sqrt{2}}\right) e^{-\frac{\nu}{2}\overline{\xi}^2 + \sqrt{2}\nu x\overline{\xi}} e^{-\nu|\xi|^2} d\lambda(\xi). \tag{28}$$

Further properties of the transform $\mathcal{G}^{\nu}$ when combined with the rescaled Fourier transform are also investigated. They give rise to two extremely integral operators connecting isometrically the Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C})$ to $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$.

The like-left inverse $\mathcal{R}^{\nu}$ in (28) as well as the quaternionic Segal–Bargmann transform $\mathcal{B}^{\nu}_{\mathbb{H}}$ are then employed to introduce and study the integral transform $\mathcal{I}^{\nu} := \mathcal{B}^{\nu}_{\mathbb{H}} \circ \mathcal{R}^{\nu}$. It is defined on the two-dimensional Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ with range in the slice hyperholomorphic Bargmann–Fock space $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$ in (17). We show that $\mathcal{I}^{\nu}$ reduces further to the integral operator

$$\mathcal{I}^{\nu}f(q) = \left(\frac{\nu}{\pi}\right) \int_{\mathbb{C}} f\left(\frac{\xi}{\sqrt{2}}, \frac{-i\xi}{\sqrt{2}}\right) K^{\nu}_{\mathbb{H}}(q,\xi) e^{-\nu|\xi|^2} d\lambda(\xi). \tag{29}$$

where $K^{\nu}_{\mathbb{H}}(q,\xi)$ is the reproducing kernel of $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$. The image $\mathcal{I}^{\nu}(\mathcal{F}^{2,\nu}(\mathbb{C}^2))$ is identified to be $\mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H})$ the space of slice (left) regular functions on the quaternions leaving invariant the slice $\mathbb{C}_i \simeq \mathbb{C}$. Added to $\mathcal{I}^{\nu}$, we consider the integral transform $\mathcal{J}^{\nu} := \mathcal{G}^{\nu} \circ (\mathcal{B}^{\nu}_{\mathbb{H}})^{-1}$ from $\mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H})$ into $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ with image coinciding with $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$. The action of $\mathcal{I}^{\nu}$ and $\mathcal{J}^{\nu}$ on the bases and the reproducing kernels are given. It turns out that these transforms connect the standard basis and the reproducing kernels of these two spaces.

D. Brief description of chapters

Chapter 1: is concerned with some preliminaries on

- Real and complex Hermite polynomials
- Polyanalytic Hermite polynomials
- Coherent states formalism
- Bargmann transform

- Fourier–Wigner and Wigner transforms
- Fractional Fourier transform
- Polyanalytic functions
- Slice hypercomplex analysis

In **Chapter 2**, special generating functions and Mehler's formula for the univariate complex Hermite polynomials (UCHP) are obtained and next employed to introduce and study some one– and two–dimensional integral transforms of Segal–Bargmann type in the framework of some specific functional Hilbert spaces; including the so-called generalized Bargmann–Fock spaces that are realized as L^2–eigenspaces of a special magnetic Schrödinger operator.

Chapter 3, discusses some algebraic and analytic properties of a general class of orthogonal polyanalytic polynomials, including their operational formulas, recurrence relations, generating functions, integral representations and different orthogonality identities. We establish their connection and rule in describing the L^2–spectral theory of some special second order differential operators of Laplacian type acting on the L^2–Gaussian Hilbert space on the whole complex plane. We will also show their importance in the theory of the so-called rank–one automorphic functions on the complex plane. In fact, a variant subclass leads to an orthogonal basis of the corresponding L^2–Gaussian Hilbert space on the strip $\mathbb{C}/\mathbb{Z}$.

In **chapter 4**, we study the orthogonal complement of the Hilbert subspace considered by van Eijndhoven and Meyers in [113] and associated to holomorphic Hermite polynomials. A polyanalytic orthonormal basis is given and the explicit expressions of the corresponding reproducing kernel functions and Segal–Bargmann integral transforms are given. The obtained basis are then used to provide a non-trivial $1d$ and 2-fractional like-Fourier transform.

Chapter 5, deals with two classes of right quaternionic Hilbert spaces in the context of slice polyregular functions, generalizing the so-called slice and full hyperholomorphic Bargmann spaces constricted by means of $H^Q_{m,n}$. Their basic properties are discussed, the explicit formulas of their reproducing kernels are given and associated Segal–Bargmann transforms are also introduced and studied. The spectral description as special subspaces of L^2-eigenspaces of a second order differential operator involving the slice derivative is investigated.

The last **Chapter**, we introduce and discuss some basic properties of some integral transforms in the framework of specific functional Hilbert spaces, the holomorphic Bargmann–Fock spaces on $\mathbb{C}$ and $\mathbb{C}^2$ and the slice hyperholomorphic Bargmann–Fock space on $\mathbb{H}$. We study the basic properties of $\mathcal{G}^\nu$ in (27), including the identification of its image and the determination of a like-left inverse defined on the whole two-dimensional Bargmann–Fock space. We examine their combination with the Fourier transform which lead to special integral transforms connecting the two-dimensional Bargmann–Fock space and its analogue on the complex plane. We also investigate the relationship between special subspaces of the two-dimensional Bargmann–Fock space and the slice-hyperholomorphic one on the quaternions by introducing appropriate integral transforms. We identify their image and their action on the reproducing kernel.

E. List of related publications

1. Non-trivial 1d and 2d integral transforms of Segal-Bargmann type. Integral Transforms Spec. Funct. 30 (2019), no. 7, 547–563. – Chapter 2.

   $\boxed{\text{Q2}}$ $\boxed{\text{IF=0.812}}$ $\boxed{\text{SJR=0.68}}$

2. On a Novel Class of Polyanalytic Hermite Polynomials. Results Math 74, 186 (2019) doi:10.1007/s00025-019-1110-z – Chapter 3.

   $\boxed{\text{Q2}}$ $\boxed{\text{IF=0.873}}$ $\boxed{\text{SJR=0.52}}$

3. A special orthogonal complement basis for holomorphic-Hermite functions and associated 1d - and 2d-fractional Fourier transforms. Integral Transforms Spec. Funct. 30 (2019), – Chapter 4.

   $\boxed{\text{Q2}}$ $\boxed{\text{IF=0.812}}$ $\boxed{\text{SJR=0.68}}$

4. S-polyregular Bargmann spaces. Adv. Appl. Clifford Algebr. 29 (2019), no. 4, Paper No. 84, 30 pp.– Chapter 5.

   $\boxed{\text{Q3}}$ $\boxed{\text{IF=0.857}}$ $\boxed{\text{SJR=0.4}}$

5. Composition of Segal-Bargmann transforms. Complex Var. Elliptic Equ. 64 (2019), no. 6, 950–964.– Chapter 6.

   $\boxed{\text{Q2}}$ $\boxed{\text{IF=0.806}}$ $\boxed{\text{SJR=0.75}}$

Chapter 1

Preliminaries

We review briefly in this chapter the basic concepts and fundamental background needed to develop the next chapters. Mainly, we are concerned with the real Hermite polynomials and their generalization to the complex plane, the holomrphic and poly-analytic Hermite polynomials (Subsections 1.1.1 and 1.1.2). The second ones lead to interesting classes of functional spaces like the generalized and true polyanalytic Bargmann spaces (Section 1.6), which can be realized as the phase spaces of the configuration space $L^2(\mathbb{R}, dx)$ via generalized Segal–Bargmann or Fourier–Wigner transforms (Sections 1.3 and 1.4). The definition of the fractional Fourier transform and their basic properties is recalled in Section 1.5. Section 1.7 is devoted to reviewing some concepts from slice hypercomplex analysis.

1.1 Real and complex orthogonal polynomials of Hermite type

Orthogonal polynomials have found wide application in various branches of mathematics, technology and physics. The basic example of single real Hermite polynomials [92,99,107, 109] as well as their different generalizations are well-known in the literature [26,28,66,97, 105], including the generalized Hermite polynomials

$$H_m^{\gamma}(x,\alpha,p) := (-1)^m x^{-\alpha} e^{px^{\gamma}} \frac{d^m}{dx^m}\left(x^{\alpha} e^{-px^{\gamma}}\right),$$

see [66]. Further kinds of generalizations to multi-index ones can be found in [17,74,104].

1.1.1 Real Hermite polynomials.

They are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{1.1}$$

or explicitly by

$$H_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{m!(n-2m)!}(2x)^{n-2m}. \tag{1.2}$$

In physics, these polynomials appear as solutions of the Schrödinger equation $y'' - 2xy' + 2my = 0$, so that the associated functions $h_m(x) = e^{x^2/2}H_m(x)$ (called Hermite functions) are eigensolutions of the quantum harmonic oscillator $H = x^2 - \partial_x^2$ and form an orthogonal basis of the functional Hilbert space $L^2(\mathbb{R}, e^{-x^2}dx)$ of square integrable functions on the real line (see for example [109]) endowed with the inner product is given by the integral

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}e^{-x^2}dx.$$

More precisely, we have

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)\, e^{-x^2}\, dx = \sqrt{\pi}\, 2^n n!\, \delta_{nm}.$$

Moreover, they enjoy a number of remarkable and interesting properties, like the Runge addition formula ([77,100])

$$H_n(x+y) = 2^{-\frac{n}{2}} \cdot \sum_{k=0}^{n} \binom{n}{k} H_{n-k}\left(x\sqrt{2}\right) H_k\left(y\sqrt{2}\right), \tag{1.3}$$

and the identity

$$H_{n+1}(x) = 2xH_n(x) - H_n'(x). \tag{1.4}$$

While, the exponential generating function reads

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}. \tag{1.5}$$

Both (1.3) and (1.5), as well as the the quadratic recurrence formula, (called also Nielsen's identity ([92]))

$$H_{m+n}(x) = m!n! \sum_{k=0}^{min(m,n)} \frac{(-2)^k}{k!} \frac{H_{m-k}(x)}{(m-n)!} \frac{H_{n-k}(x)}{(n-k)!}. \tag{1.6}$$

can be recovered in a easier way by utilizing the operational representation

$$H_m(x) = \left(-\frac{d}{dx} + 2x\right)^m \cdot (1) \tag{1.7}$$

which, can be seen, a special case of the Burchnall operational formula

$$\left(-\frac{d}{dx}+2x\right)^m (f) = m!\sum_{k=0}^{m}\frac{(-1)^k}{k!}\frac{H_{m-k}(x)}{(m-k)!}\frac{d^k}{dx^k}(f), \tag{1.8}$$

by taking $f = 1$. A fundamental tool for these polynomials is the Mehler's formula asserting

$$\sum_{n=0}^{+\infty}\frac{t^n H_n(x)H_n(y)}{2^n n!} = \frac{1}{\sqrt{1-t^2}}\exp\left(\frac{-t^2(x^2+y^2)+2txy}{1-t^2}\right) =: E_t(x,y), \tag{1.9}$$

and obtained by Mehler himself in 1866 [85] (see e.g. also [99]).

1.1.2 Holomorphic and poly-analytic Hermite polynomials

There is many ways to define univariate complex polynomials of Hermite type. The natural ones arise as the tensor product $H_m(x)H_n(y)$ of the real Hermite polynomials $H_m(x)$ or also the complex holomorphic Hermite polynomials $H_n(z)$, $z = x+iy$, $x,y \in \mathbb{R}$. Many algebraic properties of $H_n(x)$ remains valid for $H_n(z)$ by analytic continuation. This is the case of the identities (1.1), (1.4) and (1.5)-(1.9). However, their orthogonal property reads [113]

$$\int_{\mathbb{R}^2} H_m(z)\overline{H_n(z)}e^{-ax^2-by^2}dxdy = \frac{\pi}{\sqrt{ab}}2^n n!\left(\frac{a+b}{ab}\right)^n\delta_{m,n}$$

for given $0 < a < b$, $\frac{1}{a} = 1+\frac{1}{b}$. Moreover, the sequence $H_n(z)$; $n \geq 0$, form an orthogonal basis of the standard Bargmann–Fock space $\mathcal{F}^2_{a,b}(\mathbb{C})$ of holomorphic functions on belonging to $L^2(\mathbb{C}, e^{-ax^2-by^2}dxdy)$, to wit

$$\mathcal{F}^2_{a,b}(\mathbb{C}) := \mathcal{H}ol \cap L^2(\mathbb{C}, e^{-ax^2-by^2}dxdy).$$

The univariate poly-analytic Hermite polynomials, defined by their Rodrigues' formula

$$H_{m,n}(z,\overline{z}) = (-1)^{m+n}e^{|z|^2}\frac{\partial^{m+n}}{\partial\overline{z}^m\partial z^n}\left(e^{-|z|^2}\right), \tag{1.10}$$

is a nontrivial generalization of $H_n(x)$ to the complex plane. The relationship to the classical (physicist) univariate real Hermite polynomials $H_m^{real}(x)$ is given by [56,73]

$$H_{m,n}(z,\overline{z}) = \left(\frac{1}{2}\right)^{m+n} m!n!\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{(-1)^k(i)^{j+k}}{j!k!}\frac{H^{real}_{m+n-j-k}(x)H^{real}_{j+k}(y)}{(m-j)!(n-k)!}. \tag{1.11}$$

Equivalently, they can be defined by means of their exponential operational formula ([73, Theorem 2.1])

$$H_{m,n}(z,\bar{z}) = e^{-\Delta_{\mathbb{C}}}\left(z^m\bar{z}^n\right), \tag{1.12}$$

$\Delta_{\mathbb{C}} := \partial^2/\partial z \partial \bar{z}$ being the Laplace-Beltrami operator on $\mathbb{C}$. The explicit expression of $H_{m,n}(z,\bar{z})$ in terms of the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ reads

$$H_{p,q}(z,\bar{z}) = (-1)^{min(p,q)}(min(p,q))!|z|^{|p-q|}e^{i(p-q)\arg(z)}L^{(|p-q|)}_{min(p,q)}(|z|^2)$$

for $z = |z|e^{i\arg(z)}$ (see [72, Eq. (2.3)] where there $h_{m,p}$ means $H_{p,m}$ in ours), so that for $q = p+k$, we get

$$H_{p,p+k}(z,\bar{z}) = (-1)^p p! \bar{z}^k L_p^{(k)}(|z|^2). \tag{1.13}$$

We can reintroduce $H^{\nu}_{m,n}(z;\bar{z})$ by considering the integral representation ([20, Theorem 2.4])

$$H^{\nu}_{m,n}(z;\bar{z}) = \left(\frac{\mu}{\pi}\right)(-\alpha)^m(\beta)^n \int_{\mathbb{C}} \xi^m \overline{\xi}^n e^{\nu|z|^2 - \mu|\xi|^2 + \alpha\langle \xi, z\rangle - \beta\overline{\langle \xi, z\rangle}} d\lambda(\xi). \tag{1.14}$$

Here $\nu = \frac{\alpha\beta}{\mu}$ with $\mu > 0$ and $\alpha, \beta \in \mathbb{C}$ such that $\alpha\beta > 0$. By taking for example $\mu = 1$ and $\alpha = -\beta = i$, so that $\nu = \alpha\beta/\mu = 1$, the integral representation (3.5) reduces further to the one obtained by Ismail [73, Theorem 5.1]. Its proof lies on the following key result

$$\int_{\mathbb{C}} e^{-\mu|\xi|^2 + \alpha\xi + \beta\overline{\xi}} d\lambda(\xi) = \left(\frac{\pi}{\mu}\right) e^{\frac{\alpha\beta}{\mu}} \tag{1.15}$$

fulfilled for fixed positive real number $\mu > 0$ and two complex numbers $\alpha, \beta \in \mathbb{C}$. Using this integral representation one can obtain the following exponential generating function [54,76]

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!} H^{\nu}_{m,n}(z;\bar{z}) = e^{\nu(uz + v\bar{z} - uv)} \tag{1.16}$$

which can also be obtained by means of (a) of Proposition in [56], to wit

$$\sum_{n=0}^{+\infty} \frac{z^n}{n!} H^{\nu}_{n,m'}(w,\overline{w}) = \nu^{m'}(\overline{w} - z)^{m'} e^{\nu z w}. \tag{1.17}$$

We conclude this section by recalling the following identity

$$\sum_{n=0}^{+\infty} \frac{H^{\nu}_{m,n}(z,\overline{z})\overline{H^{\nu}_{m',n}(w,\bar{w})}}{\nu^n n!} = (-1)^{m'} H^{\nu}_{m,m'}(z-w, \overline{z-w}) e^{\nu\langle w,z\rangle} \tag{1.18}$$

which is exactly Proposition 3.6 in [56] (when $\nu = 1$). It appears as a particular case of [20, Theorem 3.1]

$$G^{\nu}_{m,m'}(t;z,w) := \sum_{n=0}^{+\infty} \frac{t^n}{n!\nu^n} H^{\nu}_{m,n}(z,\overline{z}) H^{\nu}_{n,m'}(w,\overline{w}) = (-t)^{m'} H^{\nu}_{m,m'}(z - tw, \overline{z} - \overline{t}\overline{w}) e^{\nu t w \overline{z}} \tag{1.19}$$

valid for every t in the unit circle and $z, w \in \mathbb{C}$. It should noted here that by taking $m = m'$

in (1.18), we recognize the explicit expression of the reproducing kernel of the generalized Bargmann space of level m, defined as the L^2-eigenspace of the self-adjoint magnetic Laplacian

$$\Delta_\nu = -\frac{\partial^2}{\partial z \partial \overline{z}} + \nu z \frac{\partial}{\partial z} \tag{1.20}$$

acting on $L^{2,\nu}(\mathbb{C}; e^{-\nu|z|^2} d\lambda)$ and associated to the eigenvalue νm.

1.1.3 Mehler formulas for the UCHP

In [55], two widest generalizations of the classical Mehler's formula are given for the univariate complex Hermite polynomials, by performing special double summations involving the special products of $H^\nu_{m,n}$, $H^{\nu'}_{m,n}$ and monomials. The first one is based on the generating function (1.18) and asserts that [55, Theorem 3.1]

$$\sum_{m,n=0}^{+\infty} \frac{u^m H^\nu_{m,n}(z,\overline{z})\overline{H^\nu_{m,n}(w,\overline{w})}}{\nu^{m+n} m! n!} = \frac{e^{\nu\langle w,z\rangle}}{(1-u)} \exp\left(\frac{-\nu u|z-w|^2}{1-u}\right) \tag{1.21}$$

valid for every $u, z \in \mathbb{C}$ such that $|u| < 1$. An interesting application is given when considering the Cauchy problem

$$(H) \qquad \begin{cases} \frac{\partial}{\partial t} u(t;z) = \Delta_\nu u(t;z); & (t;z) \in]0,+\infty[\times \mathbb{C}, \\ u(t;z) = f(z) \in \mathcal{C}^\infty_0(\mathbb{C}), \end{cases}$$

associated to the self-adjoint magnetic Laplacian Δ_ν in (1.20). In fact, the closed explicit expression of the Heat kernel function $K_\nu(t;z,z_0)$ for the heat solution of (H) is proved to be given by ([55, Theorem 3.3])

$$K_\nu(t;z,z_0) = \left(\frac{\nu}{\pi}\right) \frac{e^{\nu(t+\langle z_0,z\rangle)}}{1-e^{\nu t}} \exp\left(\frac{|z-z_0|^2}{e^{\nu t}-1}\right); \quad t > 0. \tag{1.22}$$

The second Mehler's formula concerns the kernel function

$$E^{\nu,\nu'}_{u,v}(z,w) := \frac{1}{1-\nu\nu' uv} \exp\left(-\frac{\nu\nu' \left[(\nu|z|^2 + \nu'|w|^2)uv - uzw - v\overline{z}\overline{w}\right]}{1-\nu\nu' uv}\right) \tag{1.23}$$

that can be seen as an analytic extension of the classical Poisson kernel (1.9). In fact, one proves that $E^{\nu,\nu'}_{u,v}(z,w)$ can be expanded in terms of $H^\nu_{m,n}$ as follows [55, Theorem 4.1]

$$E^{\nu,\nu'}_{u,v}(z,w) = \sum_{m,n=0}^{\infty} \frac{u^m v^n}{m! n!} H^\nu_{m,n}(z;\overline{z}) H^{\nu'}_{m,n}(w;\overline{w}) \tag{1.24}$$

valid for every $u, v \in \mathbb{C}$ such that $uv \in \mathbb{R}$ and arbitrary $\nu, \nu' \in \mathbb{R}$ such that $\nu\nu' uv < 1$. For the particular case $\nu = \nu' = 1$, it leads to the Mehler's formula for $H_{m,n}(z;\overline{z})$ given by

Wünsche [120] without proof and recovered by Ismail [73, Theorem 3.3] as a specific case of his Kibble-Slepian formula [73, Theorem 1.1]. As consequence of (1.24) combined with the integral representation (3.5), we can derive an interesting self-reciprocity property for the Hermite polynomials by a like Fourier transform [55, Theorem 4.2]

$$\begin{aligned}\int_{\mathbb{C}} \exp\left(\frac{-\nu'|w|^2 - \nu\nu'(uzw - v\overline{zw})}{1-\nu\nu' uv}\right) H^{\nu'}_{k,j}(w;\overline{w}) d\lambda(w) \qquad (1.25)\\ = \pi(\nu')^{j+k-1}(1-\nu\nu' uv)u^j v^k \exp\left(\frac{\nu^2\nu' uv}{1-\nu\nu' uv}|z|^2\right) H^{\nu}_{j,k}(z;\overline{z}).\end{aligned}$$

Added to (1.18), there are other additional interesting bilinear generating functions of Mehler type (see Chapter 2). As direct applications, we provide remarkable integral transforms connecting $L^2(\mathbb{R}; e^{-x^2}dx)$ to the $1d$-Bargmann-Fock space $\mathcal{F}^{2,\nu}(\mathbb{C})$ and more generally to the generalized Bargmann-Fock spaces that are L^2-eigenspaces of a magnetic Laplacian Δ_ν acting on $L^2(\mathbb{C}; e^{-\nu|z|^2}dxdy)$. An integral transform mapping $L^2(\mathbb{C}; e^{-\nu|z|^2}dxdy)$ to the two-dimensional Bargmann-Fock space $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ is also given. This will be studied in details in the next chapter.

1.2 Coherent states formalism

Most classical integral transforms in mathematical analysis are subject to a general principle issued from the reproducing kernel Hilbert space theory (known also as coherent states transform). Below, we present a brief description of this principle according to [53,69]. Let $(\mathcal{H}_X; \omega_X)$ be an infinite dimensional complex functional Hilbert space on X with an orthonormal basis $\{e_n\}_n$ with respect to the inner scaler product

$$\langle \phi, \psi \rangle_{\mathcal{H}_X} := \int_X \phi(x)\overline{\psi(x)}\omega_X(x)dx$$

for given weight measure ω_X. In a similar way we consider $(\mathcal{H}_Y; \omega_Y)$ with an orthonormal basis $\{f_n\}_n$ and assume that $\mathcal{H}_Y$ is in addition a reproducing kernel Hilbert space with reproducing kernel $K(y, y')$. Associated to the data $(\mathcal{H}_X; \omega_X; \{e_n\}_n)$ and $(\mathcal{H}_Y; \omega_Y; \{f_n\}_n)$, we perform the following kernel function $T : X \times Y \longrightarrow \mathbb{C}$ defined by

$$T(x, y) := \sum_{n=0}^{\infty} \overline{e_n(x)} f_n(y).$$

It is straightforward to check that $\langle T(\cdot, y), T(\cdot, y')\rangle_{\mathcal{H}_X}$ reduces further to $K(y, y')$.

Moreover, $T(\cdot, y) \in \mathcal{H}_X$ for every fixed $y \in Y$ and therefore the map $y \longmapsto T(\cdot, y)$ defines a quantization of Y into $\mathcal{H}_X$. Thus, we can consider the integral transform

$$\mathcal{T}(\phi)(y) := \int_X T(x, y)\phi(x)\omega_X(x)dx = \left\langle \phi, \overline{T(\cdot, y)} \right\rangle_{\mathcal{H}_X}$$

for every $\phi \in \mathcal{H}_X$. This transform maps $\mathcal{H}_X$ onto $\mathcal{H}_Y$ for $\|T(\cdot, y)\|^2_{\mathcal{H}_X} = K(y, y) < +\infty$, and satisfies

$$\mathcal{T}(e_k) = f_k.$$

Subsequently $\mathcal{T}(\phi) = \sum\limits_{n=0}^{\infty} \alpha_n f_n \in \mathcal{H}_Y$ for every $\phi = \sum\limits_{n=0}^{\infty} \alpha_n e_n \in \mathcal{H}_X$. Moreover, it is readily easy to see that

$$\|\phi\|^2_{\mathcal{H}_X} = \sum_{n=0}^{\infty} |\alpha_n|^2 = \sum_{n=0}^{\infty} |\beta_n|^2 = \|\mathcal{T}(\phi)\|^2_{\mathcal{H}_Y}.$$

Thereby, $\mathcal{T}$ defines an isometric linear transform from $\mathcal{H}_X$ onto $\mathcal{H}_Y$ and the function $x \longmapsto \langle \varphi, T(x,\cdot)\rangle_{\mathcal{H}_Y}$ belongs to $\mathcal{H}_X$ for every $\varphi \in \mathcal{H}_Y$, since

$$\langle \varphi, T(x,\cdot)\rangle_{\mathcal{H}_Y} = \left\langle \varphi, \sum_{n=0}^{\infty} \overline{e_n(x)} f_n \right\rangle_{\mathcal{H}_Y} = \sum_{n=0}^{\infty} \langle \varphi, f_n\rangle_{\mathcal{H}_Y} e_n(x) = \phi(x).$$

In addition, we have the following integral representation

$$\varphi(y) = \int_X T(x,y) \langle \varphi, T(x,\cdot)\rangle_{\mathcal{H}_Y} \omega_X(x) dx = \mathcal{T}\left(\langle \phi, T(x,\cdot)\rangle_{\mathcal{H}_Y} \right)(y)$$

for every $\varphi \in \mathcal{H}_Y$. This is equivalent to say that $\langle \varphi, T(x,\cdot)\rangle_{\mathcal{H}_Y}$ is the inverse transform of $\mathcal{T}$.

The next subsection gives as a concrete example of such Coherent states formalism.

1.3 Segal–Bargmann transform

It is a natural unitary operator introduced for the first time by Bargmann [16] and given through

$$[\mathcal{B}(\varphi)](z) := \pi^{-\frac{n}{4}} \int_{\mathbb{R}^n} \varphi(x) e^{-\frac{1}{2}(z^2+x^2)+\sqrt{2}zx} dx,$$

where $zx = \sum\limits_{j=1}^{n} z_j x_j$ for $x = (x_1, x_2, \cdots, x_n)$ and $z = (z_1, z_2, \cdots, z_n)$. The integral is taken on the n-real space $\mathbb{R}^n$ with respect to its standard Lebesgue measure $dx = dx_1 \cdots dx_n$. The transform $\mathcal{B}$ identifies the standard Hilbert space $L^2(\mathbb{R}^n)$ and the Fock space $\mathcal{F}^2(\mathbb{C}^n)$, the space of holomorphic functions F in the n-complex space $\mathbb{C}^n$ satisfying the square-integrability condition

$$\|F\|^2 := \pi^{-n} \int_{\mathbb{C}^n} |F(z)|^2 \exp(-|z|^2)\, d\lambda(z) < +\infty.$$

Here $d\lambda(z)$ denotes the $2n$-dimensional Lebesgue measure on $\mathbb{C}^n$ given by

$$d\lambda(z) = \prod_{k=1}^{n} d\lambda(z_k) = \prod_{k=1}^{n} dx_k dy_k; \quad z = (x_1 + iy_1, \cdots, x_n + iy_n).$$

The space $\mathcal{F}^2(\mathbb{C}^n)$ is a Hilbert space with respect to the associated inner product

$$\langle F \mid G\rangle = \pi^{-n}\int_{\mathbb{C}^n}\overline{F(z)}G(z)\exp(-|z|^2)\,d\lambda(z).$$

The kernel function of this integral transform is closely related to the generating function of the multi-dimensional Hermite functions

$$h_m(x) = \prod_{k=1}^{n} h_{m_k}(x_k)$$

for varying $m = (m_1, m_2, \cdots, m_n) \in \mathbb{N}^n$, where $h_{m_k}(x_k)$ the one-dimensional Hermite function which form an orthogonal basis of $L^2(\mathbb{R}^n)$.

1.4 The Fourier–Wigner and Wigner transforms.

The notion of Segal-Bargmann transform, introduced in the previous section, is closely connected to the so-called Fourier-Wigner transform $\mathcal{V} : (f, g) \longmapsto \mathcal{V}(f, g)$ for some specific fixed (window function) g [49,109]. This transform is important in harmonic analysis, signal analysis, engineering, and the physical sciences. In fact, it is a basic tool to study the Weyl transform [49,109,119] and to interpret quantum mechanics as a form of nondeterministic statistical dynamics [89]. It is also used to study the nonexisting joint probability distribution of positioned momentum in a given state [119]. It is defined as a windowed Fourier transform by

$$\mathcal{V}(f,g)(p,q) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}}\int_{\mathbb{R}^d} e^{i\left\langle x+\frac{p}{2},q\right\rangle} f(x+p)\overline{g(x)}dx \tag{1.1}$$

for every $(p, q) \in \mathbb{R}^d \times \mathbb{R}^d$ and every complex-valued functions f, g, or equivalently by

$$\mathcal{V}(f,g)(p,q) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}}\int_{\mathbb{R}^d} e^{i\langle y,q\rangle} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}dy. \tag{1.2}$$

Therefore, it can be seen as the Fourier transform,

$$\mathcal{V}(f,g)(p,q) = \mathcal{F}(K_{f,g}(\cdot|p))(-q),$$

of the function $y \longmapsto K_{f,g}(y|p)$ belonging to $L^1(\mathbb{R}^d)$ and defined on $\mathbb{R}^d$ by

$$K_{f,g}(y|p) = f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}. \tag{1.3}$$

It is then a well defined bilinear mapping on $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ thanks to the estimation

$$|\mathcal{V}(f,g)(p,q)| \leq \left(\frac{1}{2\pi}\right)^{\frac{d}{2}}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

An interesting result for $\mathcal{V}$ is the Moyal's formula

$$\langle \mathcal{V}(f,g), \mathcal{V}(\varphi,\psi)\rangle_{L^2(\mathbb{C}^d)} = \langle f,\varphi\rangle_{L^2(\mathbb{R}^d)} \langle \psi, g\rangle_{L^2(\mathbb{R}^d)}. \tag{1.4}$$

Subsequently, we have $\mathcal{V}(L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)) \subset L^2(\mathbb{C}^d)$. Moreover, it shown in [119] (see also [38, Proposition 5, p. 2101] that the Fourier-Wigner $\mathcal{V}$ can be used to construct orthogonal bases of $L^2(\mathbb{C}^d)$ from those of $L^2(\mathbb{R}^d)$. More precisely, if $\{\varphi_k, k\in\mathbb{N}\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$, then $\{\varphi_{jk} = \mathcal{V}(\varphi_j,\varphi_k);\, j,k\in\mathbb{N}\}$ is an orthonormal basis of $L^2(\mathbb{C}^d)$ with

$$\left\|\varphi_{jk}\right\|_{L^2(\mathbb{C}^d)} = \left\|\varphi_j\right\|_{L^2(\mathbb{R}^d)} \left\|\varphi_k\right\|_{L^2(\mathbb{R}^d)}. \tag{1.5}$$

It should be mentioned here that the univariate polyanalytic Hermite polynomials can be realized as the Fourier-Wigner transform ([7]) of the well-known real Hermite functions $h_n^{real}(x) = e^{-\frac{x^2}{2}} H_n^{real}(x)$ on the real line $\mathbb{R}$, namely

$$H_{m,n}(z,\overline{z}) = (-1)^n 2^{-(m+n-1)/2} e^{\frac{|z|^2}{2}} \mathcal{V}(h_m^{real}, h_n^{real})(\sqrt{2}x, \sqrt{2}y). \tag{1.6}$$

This follows using basically the generating functions of h_n and $h_{m,n}$. Subsequently, one re-derives the known fact that they constitute an orthogonal basis of the Hilbert space $L^2(\mathbb{C}; e^{-|z|^2}dxdy)$ ([72,76]). Another interesting result is the preservation of the tensor product [119]. More exactly, if $f_j, g_j \in L^2(\mathbb{R}); j = 1,\cdots,n$, then the functions $f(x) := f_1(x_1)\cdots f_n(x_n)$ and $g(x) := g_1(x_1)\cdots g_n(x_n)$ belong to $L^2(\mathbb{R}^n)$ and satisfy

$$\mathcal{V}(f,g)(p,q) = \prod_{j=1}^{n} \mathcal{V}(f_j,g_j)(p_j,q_j), \tag{1.7}$$

where $\mathcal{V}$ in the right-hand side denotes the one-dimensional Fourier-Wigner transform.

A close transformation to $\mathcal{V}$ is the Wigner transform defined for given $f,g\in L^2(\mathbb{R}^n)$ by $W(f,g)$. More explicitly,

$$W(f,g)(x,\xi) := \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi.p} f\left(x+\frac{p}{2}\right) \overline{g\left(x-\frac{p}{2}\right)} dp.$$

It was introduced by Eugene Wigner in 1932 as a probability quasi-distribution which allows expression of quantum mechanical expectation values in the same form as the averages of classical statistical mechanics [117]. It is also used in signal processing as a transform in time-frequency analysis. With close relating to the windowed Gabor transform [30,95]. The corresponding Moyal identity reads

$$\langle W(f_1,g_1), W(f_2,g_2)\rangle = \langle f_1,f_2\rangle\langle g_1,g_2\rangle$$

for all f_1, g_1, f_2, and g_2 in $L^2(\mathbb{R}^n)$.

1.5 Fractional Fourier transform

Fourier analysis is one of the oldest and special subjects in mathematical analysis that has great impact on different fields of mathematics, physics and engineering alike. The main tool in this theory is the Fourier transform on the real line

$$\mathcal{F}_f(w) = \hat{f}(w) := \int_{\mathbb{R}} f(x)\,\mathrm{e}^{-i\omega t} dt; \quad w \in \mathbb{R},$$

it enters in the resolution some differential equations. Moreover, it is essential for signal analysis and image processing. Notice for instance that the Hermite functions $H_n(x)e^{-x^2/2}$ are eigenfunctions of $\mathcal{F}$,

$$\mathcal{F}h_m = i^m h_m.$$

The so-called fractional Fourier transform, is a special generalization of $\mathcal{F}$. Such transform was first appeared in 1929 by Wiener paper [118] discussing the extension of certain results of Hermann Weyl, leading to Fourier developments of fractional order. Mainly, Wiener sets out to find a one–parameter family of unitary integral operators on $L^2(\mathbb{R})$,

$$\mathcal{K}_\theta \varphi(x) := \int_{-\infty}^{+\infty} K_\theta(x,y)\varphi(y)dy,$$

for which the n–th Hermite function $h_n(x) = H_n(x)e^{-x^2/2}$ is a eigenfunction with $e^{in\theta}$ as corresponding eigenvalue,

$$\mathcal{K}_\theta h_n(x) = e^{in\theta} h_n(x).$$

The explicit Wiener formula for the kernel function K_0 is a limiting case of Mehler's formula [85] for the Hermite functions as showed earlier by Hörmander [71]. This transform was rediscovered later in quantum mechanics by Namias [90] (who was the first to attribute such concept),

$$\mathcal{F}_\alpha(f)(\omega) = \int_{-\infty}^{\infty} K_\alpha(t,w) f(t)\, dt,$$

with

$$K_\alpha(t,w) = \begin{cases} \sqrt{\frac{1-i\cot(\alpha)}{2\pi}} e^{-i\csc(\alpha)\omega t + i\cot(\alpha)(\frac{w^2}{2}+\frac{t^2}{2})} \ if \ \alpha \neq p\pi, \\ \delta(t-w) \ if \ \alpha = 2p\pi, \\ \delta(t+w) \ if \ \alpha = (2p-1)\pi, \end{cases}$$

where $\alpha \in \mathbb{R}$; p is an integer, and δ is the Dirac delta function.

Namias was able to generalize many results of classical Fourier transform to FrFT, based on the properties of the Hermite orthogonal polynomials. He derived a number of operational formulas which he used to solve several types of Schrödinger equation. The fundamental mathematical foundation concerning the FrFT was developed later by McBride and Kerr in [84]. Applications of the FrFT are well–known in the context of signal processing [8,93,102] optics [5,6,93], and fractional differential equations [79]. For a new and brief introduction to the FrFT and its applications see [46]. A detailed overview of the theory of the fractional FT can be found in [93].

The kernel can be expanded in terms of Hermite functions as

$$K_\alpha(t,w) = \sum_{n\geq 0}(-i)^{\frac{2\alpha n}{\pi}}\frac{h_n(t)h_n(w)}{||h_n||^2}.$$

Therefore, one can the Hermite functions are eigenfunctions of the fractional Fourier transformation whose corresponding eigenvalues are $e^{in\alpha}$; that is

$$\mathcal{F}_\alpha(h_n)(w) = e^{in\alpha}h_n(w).$$

It is also no difficult to see that $\mathcal{F}_\alpha$ satisfies the semi-Group property

$$\mathcal{F}_\alpha\mathcal{F}_\theta = \mathcal{F}_{\alpha+\theta}.$$

So that the inverse-FrFT with respect to angle α is the FrFT with angle $-\alpha$,

$$(\mathcal{F}_\alpha)^{-1} = \mathcal{F}_{-\alpha}.$$

While the Parseval's relation reads

$$\langle\mathcal{F}_\alpha(f),\mathcal{F}_\alpha(g)\rangle = \langle f,g\rangle$$

and hence the $\mathcal{F}_\alpha$ defines an isometric transformation on $L^2(\mathbb{R})$.

1.6 Polyanalytic functions

In this section, we review from [15] the interesting results that we need on complex polyanalytic functions. A given complex valued function f on a domain Ω of $\mathbb{C}$, $f:\Omega\to\mathbb{C}$ is said to be polyanalytic of order n $(n\geq 1)$ if it satisfies the Cauchy equation

$$\left(\frac{\partial}{\partial\overline{z}}\right)^n f(z) = 0$$

for every $z\in\Omega$.
This is equivalent to the existence of some holomorphic functions f_k, $k=0,1,2,\cdots,n-1$ such that

$$f(z) = \sum_{k=0}^{n}\overline{z}^n f_k(z)$$

A fundamental tool in this theory is the uniqueness theorem. It states, if f and g are two polyanalytic functions of order n on a domain Ω that coincide on a sub-domain U of Ω, then f and g coincide everywhere in Ω.

The analogue of the Fock space in the context of n-analytic functions is the generalized Fock space defined by

$$\mathcal{F}_n^{2,\nu} := \{f \text{ polyanalytic of ordre } n \text{ such that } \int_{\mathbb{C}}|f(z)|^2e^{-\nu|z|^2}d\lambda(z)<\infty\}.$$

These spaces spaces are closely connected to the L^2-eigenspaces

$$\mathcal{E}_n^{2,\nu}(\Delta_\nu) = \{f \in L^{2,\nu}_{\mathbb{C}}(\mathbb{C}); \quad \Delta_\nu f = \nu n f\}, \tag{1.1}$$

where Δ_ν is the magnetic Schrödinger operator in (1.20).

1.7 Slice hypercomplex analysis

There was many attempt to generalize the rich theory of holomorphic functions to the quaternionic context, the first ones are due to Fueter [51]. However, we review below the recent one introduced by Gentili and Struppa in [62], by means of slice derivetive and leading to the new notion of slice hyperholomorphic or slice regular functions on quaternions. To this end recall, that the non-commutative field of quaternions is defined to be

$$\mathbb{H} = \{q = x_0 + x_1 i + x_2 j + x_3 k; \;\; x_0, \;\; x_1, \;\; x_2, \;\; x_3 \in \mathbb{R}\}$$

where i, j, k satisfied the multiplication rule: $ij = -ji = k, \;\; jk = -kj = i, \;\; ki = -ik = j$. It should be mentioned here that any quaternionic q formed by real part $Re(q)$ and imaginary part $Im(q)$, and its conjugate and norm respectively are given by $\overline{q} = Re(q) - Im(q)$ and

$$|q| = \sqrt{q\overline{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$$

. Notice also that the quaternionic conjugation satisfy the property $\overline{pq} = \overline{q}\;\overline{p}$ for any p, q $\in \mathbb{H}$. Moreover, the unit sphere

$$\{q = x_1 i + x_2 j + x_3 k; \;\; x_1^2 + x_2^2 + x_3^2 = 1\}$$

coincides with the set of all imaginary units given by

$$\mathbb{S} = \{q \in \mathbb{H}; q^2 = -1\}.$$

So that for any non-real quaternion $q \in \mathbb{H}$, there exist, and are unique, x, $y \in \mathbb{R}$ with $y > 0$ and $I \in \mathbb{S}$ such that $q = x + yI$, namely $x = Re(q)$, $I = \frac{q - Re(q)}{|q - Re(q)|}$ and $y = |q - Re(q)|$. Then $\mathbb{C}_I = \mathbb{R} + \mathbb{R}I$ for every given $I \in \mathbb{H}$, define slice $\mathbb{H}$ and can be considered as a complex plane in $\mathbb{H}$ passing through 0, 1 and I isomorphic to the complex plane $\mathbb{C}$.

Keep in mind the definition of slice regular function with respect to

$$\overline{\partial_I} f(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f|_{\mathbb{C}_I}(x + yI). \tag{1.1}$$

Then we assert the following:

Series expansion: An $\mathbb{H}$-valued function f is slice regular on $\mathbb{B}_R \subset \mathbb{H}$ if and only if it has a series expansion of the form:

$$f(q) = \sum_{n=0}^{\infty} \frac{q^n}{n!} \partial_S^n(f)(0)$$

where $\mathbb{B}_R = \{q \in \mathbb{H}; |q| < R; \ \ R > 0\}$.

Definition 1.7.1. A domain $\Omega \subset \mathbb{H}$ is said to be a slice domain (or just s-domain) if $\Omega \cap \mathbb{R}$ is nonempty and for all $I \in \mathbb{S}$, the set $\Omega_I := \Omega \cap \mathbb{C}_I$ is a domain of the complex plane $\mathbb{C}_I$. If moreover, for every $q = x + Iy \in \Omega$, the whole sphere $x + y\mathbb{S} := \{x + Jy; J \in \mathbb{S}\}$ is contained in Ω, we say that Ω is an axially symmetric slice domain.

Representation Formula: Let f be a slice regular function on an axially symmetric s-domain $\Omega \subseteq \mathbb{H}$. Choose any $J \in \mathbb{S}$. Then the following equality holds for all $q = x + Iy \in \Omega$:

$$f(x + yI) = \frac{1}{2}(1 - IJ)f(x + yJ) + \frac{1}{2}(1 + IJ)f(x - yJ).$$

Splitting lemma: If f is a slice regular function on U, then for every $I \in \mathbb{S}$, and every $J \in \mathbb{S}$, perpendicular to I, there are two holomorphic functions $F, G : U \cap \mathbb{C}_I \longrightarrow \mathbb{C}_I$ such that for any $z = x + Iy$

$$f_I(z) = F(z) + G(z)J.$$

Identity Principle: Let $f : U \longrightarrow \mathbb{H}$ be a slice regular function on a slice domain U. Denote by $Z_f = \{q \in U \ : f(q) = 0\}$ the zero set of f . If there exists $I \in \mathbb{S}$ such that $\mathbb{C}_I \cap Z_f$ has an accumulation point, then $f \equiv 0$ on U.

Chapter 2

Non-trivial 1d and 2d Segal–Bargmann transforms

Special generating functions and Mehler's formula for the univariate complex Hermite polynomials (UCHP) are obtained and next employed to introduce and study some one– and two–dimensional integral transforms of Segal–Bargmann type in the framework of some specific functional Hilbert spaces, such as the so-called generalized Bargmann–Fock spaces that are realized as L^2–eigenspaces of a special magnetic Schrödinger operator.

2.1 Special bilinear generating functions for UCHP

The Segal–Bargmann transform [16,101] has found many applications in quantum optics, in signal processing and in harmonic analysis on phase space. A nice overview of its properties and applications is given in [49,91]. Many generalizations have been considered in the literature such as the Hall's transforms for compact Lie groups [67,69,88] and the so–called generalized Segal–Bargmann transform of level n [87,114]. The two–dimensional one is specified with the formula (with a slightly different convention from the classical one)

$$\mathcal{B}^{2,\nu}\psi(z,w) = \left(\frac{\nu}{\pi}\right)^{\frac{3}{2}} \int_{\mathbb{R}^2} \rho_0^\nu\left(x - \frac{z}{\sqrt{2}}\right) \rho_0^\nu\left(y - \frac{w}{\sqrt{2}}\right) \psi(x,y)dxdy, \tag{2.1}$$

where $\rho_0^\nu(\xi) = e^{-\nu\xi^2}$ is the analytic continuation to $\mathbb{C}$ of the standard Gaussian density on $\mathbb{R}$. This transform makes the quantum mechanical configuration space $L^{2,\nu}(\mathbb{C})$ unitarily isomorphic to the phase space $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$. Its kernel function is the tensor product of two copies of the kernel function of the one–dimensional Segal–Bargmann transform $\mathcal{B}^{1,\nu}$. Therefore, it is the generating function of the tensor product $H_m^\nu(x)H_n^\nu(y)$ which form an orthogonal basis of $L^{2,\nu}(\mathbb{C})$. Here $H_n^\nu(x)$ denotes the nth rescaled real Hermite polynomial

$$H_n^\nu(x) := (-1)^n e^{\nu x^2} \frac{d^n}{dx^n}\left(e^{-\nu x^2}\right). \tag{2.2}$$

The construction of such integral transforms follows a general principle from the reproducing kernel Hilbert space theory (see Section 1.2 for a brief review). In fact, there exists a unique isometric isomorphism (also called coherent states transform) mapping a given orthonormal basis $e_n(x)$ to another orthonormal basis $f_n(y)$ and is given by an integral kernel transform whose kernel function is given by the un-summed

$$T(x;y) = \sum_n \overline{e_n(x)} f_n(y).$$

Having a summed formula for $T(x;y)$ will facilitate the study of the mapping properties of the integral kernel transform on the scale of L^p spaces. Thus, this is that hard part we deal with in the context of the UCHP. Namely, we prove the following main results.

Theorem 2.1.1. *We have the generating function of exponential type*

$$\sum_{m=0}^{+\infty} \frac{\xi^m H_m^\mu(x) \overline{H_{m,n}^\nu(z;\bar{z})}}{m! \nu^m} = e^{-\mu(\xi^2 \bar{z}^2 - 2x\xi\bar{z})} H_n^{\mu\xi^2} \left(\bar{z} + \frac{\nu}{2\mu\xi^2} z - \frac{x}{\xi} \right). \tag{2.3}$$

Proof. We begin by noting that the single real Hermite polynomials $H_m^\nu(x)$ in (2.2) form an orthogonal basis of $L^{2,\nu}(\mathbb{R})$ with norm given explicitly by

$$\|H_m^\nu\|^2_{L^{2,\nu}(\mathbb{R})} = \left(\frac{\pi}{\nu}\right)^{1/2} 2^m \nu^m m!. \tag{2.4}$$

While the UCHP $H_{m,n}^\nu(z;\bar{z})$ in (12), for fixed n and varying m, is an orthogonal basis of the generalized Bargmann–Fock space of level n, $\mathcal{F}_n^{2,\nu}(\mathbb{C})$. The square norm of $H_{m,n}^\nu(z;\bar{z})$ is given by

$$\|H_{m,n}^\nu\|^2_{L^{2,\nu}(\mathbb{C})} = \left(\frac{\pi}{\nu}\right) m! n! \nu^{m+n}. \tag{2.5}$$

Both $H_n^\nu(x)$ and $H_{m,n}^\nu(z,\bar{z})$ are the rescaled version of the real Hermite polynomials H_n and the complex Hermite polynomials $H_{m,n}$ (corresponding to $\nu = 1$), respectively. Moreover, we have

$$\sqrt{\nu}^m H_m(\sqrt{\nu}x) = H_m^\nu(x) \quad \text{and} \quad H_{m,n}^\nu(z,\bar{z}) = \nu^{\frac{m+n}{2}} H_{m,n}(\sqrt{\nu}z; \sqrt{\nu}\bar{z}). \tag{2.6}$$

Making use of $H_{m,n}(z,\bar{z}) = e^{-\Delta_{\mathbb{C}}}(z^m \bar{z}^n)$ as well as the well–known generating function for the real Hermite polynomials ([99, p.187]),

$$\sum_{n=0}^{+\infty} \frac{\xi^n H_n^\mu(x)}{n!} = e^{-\mu\xi^2 + 2\mu x\xi},$$

we obtain

$$\sum_{n=0}^{+\infty} \frac{t^n H_n^{\mu}(x) H_{m,n}(z,\overline{z})}{n!\nu^n} = e^{-\Delta_{\mathbb{C}}} \left(z^m \sum_{n=0}^{+\infty} \frac{(t\overline{z}/\nu)^n H_n^{\mu}(x)}{n!} \right)$$
$$= e^{\mu x^2} e^{-\Delta_{\mathbb{C}}} \left(z^m e^{-\mu[\frac{t\overline{z}}{\nu} - x]^2} \right).$$

Now, by utilizing the fact

$$\frac{\partial^j}{\partial \overline{z}^j} \left(e^{-(a\overline{z}-b)^2} \right) = (-1)^j a^j e^{-(a\overline{z}-b)^2} H_j(a\overline{z}-b)$$

as well as the well–known identity

$$\sum_{j=0}^{m} \binom{m}{j} H_j(x)(2\xi)^{m-j} = H_m(x+\xi),$$

we can rewrite the above sum as

$$\sum_{n=0}^{+\infty} \frac{t^n H_n^{\mu}(x) H_{m,n}(z,\overline{z})}{n!\nu^n} = e^{\mu x^2} \sum_{j=0}^{m} \binom{m}{j} z^{m-j} (-1)^j \frac{\partial^j}{\partial \overline{z}^j} \left(e^{-\mu[\frac{t\overline{z}}{\nu} - x]^2} \right)$$
$$= \left(\frac{\sqrt{\mu} t}{\nu} \right)^m e^{-\mu\left[\frac{t^2\overline{z}^2}{\nu^2} - 2x\frac{t\overline{z}}{\nu}\right]} \sum_{j=0}^{m} \binom{m}{j} \left(\frac{\nu}{\sqrt{\mu} t} z \right)^{m-j} H_j\left(\frac{\sqrt{\mu} t}{\nu} \overline{z} - \sqrt{\mu} x \right)$$
$$= \left(\frac{\sqrt{\mu} t}{\nu} \right)^m e^{-\mu\left[\frac{t^2\overline{z}^2}{\nu^2} - 2x\frac{t\overline{z}}{\nu}\right]} H_m\left(\frac{\sqrt{\mu} t}{\nu} \overline{z} + \frac{\nu}{2\sqrt{\mu} t} z - \sqrt{\mu} x \right).$$

Finally, the desired result follows thanks to (2.6). ∎

Remark 2.1.2. The right–hand side of (2.3) defines a new class of polyanalytic polynomials of Hermite type which form (for special values of ν, ξ and x) an orthogonal basis of the space of rank one automorphic functions (special sections on $\mathbb{C}/\mathbb{Z}$).

Theorem 2.1.3. *For every t such that $|t| = 1$, and $z, w \in \mathbb{C}$, we have the special partial Mehler's formula*

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!\nu^n} H_{m,n}^{\nu}(z,\overline{z}) H_{n,m'}^{\nu}(w,\overline{w}) = (-t)^{m'} H_{m,m'}^{\nu}(z - tw, \overline{z} - \overline{t}\overline{w}) e^{\nu t w \overline{z}}. \tag{2.7}$$

Proof. Notice first that Rodrigues' formula (12) infers

$$H_{m,n}^{\nu}(z,\overline{z}) = (-1)^m \nu^n e^{\nu z\overline{z}} \frac{\partial^m}{\partial \overline{z}^m} \left(\overline{z}^n e^{-\nu z\overline{z}} \right). \tag{2.8}$$

Subsequently, we can check that

$$H_{m,m'}^{\nu}(z-\xi, \overline{z}-\overline{\xi}) = (-1)^m e^{\nu|z-\xi|^2} \frac{\partial^m}{\partial \overline{z}^m} \left(\nu^{m'} (\overline{z}-\overline{\xi})^{m'} e^{-\nu|z-\xi|^2} \right). \tag{2.9}$$

Thus, using successively (2.8) as well as the generating function (2.20), one gets

$$\begin{aligned} G^{\nu}_{m,m'}(t;z,w) &= \sum_{n=0}^{+\infty} \frac{t^n}{\nu^n n!} \left[(-1)^m \nu^n e^{\nu z\overline{z}} \frac{\partial^m}{\partial \overline{z}^m} \left(\overline{z}^n e^{-\nu z\overline{z}}\right)\right] H^{\nu}_{n,m'}(w,\overline{w}) \\ &= (-1)^m e^{\nu z\overline{z}} \frac{\partial^m}{\partial \overline{z}^m} \left[\nu^{m'} (\overline{w} - t\overline{z})^{m'} e^{\nu t\overline{z} w} e^{-\nu z\overline{z}}\right]. \end{aligned}$$

Now, if t is assumed to belong to the unit circle, the above identity can be rewritten as

$$G^{\nu}_{m,m'}(t;z,w) = (-t)^{m'} e^{\nu z\overline{z}} e^{-\nu\overline{t}\overline{w}(z-tw)} (-1)^m \frac{\partial^m}{\partial \overline{z}^m} \left[\nu^{m'} (\overline{z} - \overline{t}\overline{w})^{m'} e^{-\nu|z-tw|^2}\right].$$

In the right–hand side of the previous equality we recognize (2.9). Thus, the expression of $G^{\nu}_{m,m'}(t;z,w)$ reduces further to the desired result (2.7). ∎

As a consequence of Theorem 2.1.3, we get

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!\nu^n} H^{\nu}_{m,n}(z,\overline{z}) H^{\nu}_{n,m}(w,\overline{w}) = (\nu t)^m m! L^{(0)}_m (\nu|z-tw|^2) e^{\nu t w\overline{z}}, \tag{2.10}$$

which follows readily by specifying $m = m'$ in (2.7) and making use of $H^{\nu}_{m,m}(\xi,\overline{\xi}) = (-\nu)^m m! L^{(0)}_m(\nu|\xi|^2)$, where $L^{(\gamma)}_m(x)$ denotes the generalized Laguerre polynomials. The particular case of $z = w$ yields the identity

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!\nu^n} |H^{\nu}_{m,n}(z,\overline{z})|^2 = m!(\nu t)^m L^{(0)}_m (\nu|1-t|^2|z|^2) e^{\nu t|z|^2} \tag{2.11}$$

for every t in the unit circle and $z \in \mathbb{C}$. More particularly, we have

$$\sum_{n=0}^{+\infty} \frac{|H^{\nu}_{m,n}(z,\overline{z})|^2}{n!\nu^n} = m!\nu^m e^{\nu|z|^2}. \tag{2.12}$$

Using similar arguments as the ones adopted above, we are able to establish the Mehler's formula involving the product $u^m t^n H^{\nu}_{m,n}(z,\overline{z}) H^{\nu}_{n,m'}(w,\overline{w})$.

Theorem 2.1.4. *For every $|t| = 1$ and complex numbers $u, z, w \in \mathbb{C}$, we have*

$$\sum_{m,n=0}^{+\infty} \frac{u^m t^n}{m!n!\nu^n} H^{\nu}_{m,n}(z,\overline{z}) H^{\nu}_{n,m'}(w,\overline{w}) = (-\nu t)^{m'} (\overline{z} - \overline{t}\overline{w} - u)^{m'} e^{\nu t\overline{z} w + \nu u(z-tw)}. \tag{2.13}$$

If in addition $\nu|u| < 1$, then we have

$$\sum_{m,n=0}^{+\infty} \frac{u^m t^n}{m!n!\nu^n} H^{\nu}_{m,n}(z,\overline{z}) H^{\nu}_{n,m}(w,\overline{w}) = \frac{1}{(1-\nu t u)} \exp\left(\frac{-\nu^2 t u|z-tw|^2}{1-\nu t u}\right) e^{\nu t w\overline{z}}. \tag{2.14}$$

Proof. Identity (2.13) follows by twice application of the generating function (2.20). We provide below a different simple one using (2.7) combined with (2.20). In fact, we have

$$\begin{aligned}
\sum_{m,n=0}^{+\infty} \frac{u^m t^n}{\nu^n m! n!} H^{\nu}_{m,n}(z,\bar{z}) H^{\nu}_{n,m'}(w,\overline{w}) &= \sum_{m=0}^{+\infty} \frac{u^m}{m!} \left(\sum_{n=0}^{+\infty} \frac{t^n}{n! \nu^n} H^{\nu}_{m,n}(z,\bar{z}) H^{\nu}_{n,m'}(w,\overline{w}) \right) \\
&= (-t)^{m'} \sum_{m=0}^{+\infty} \frac{u^m}{m!} H^{\nu}_{m,m'}(z - tw, \bar{z} - \overline{tw}) e^{\nu t w \bar{z}} \\
&= (-\nu t)^{m'} (\bar{z} - \overline{tw} - u)^{m'} e^{\nu u (z - tw)} e^{\nu t w \bar{z}} \\
&= (-\nu t)^{m'} (\bar{z} - \overline{tw} - u)^{m'} e^{\nu (uz + t\bar{z}w - tuw)}.
\end{aligned}$$

to get

$$\sum_{m,n=0}^{+\infty} \frac{u^m t^n}{m! n! \nu^n} H^{\nu}_{m,n}(z,\bar{z}) H^{\nu}_{n,m}(w,\overline{w}) = \sum_{m=0}^{+\infty} (\nu t u)^m L^{(0)}_m (\nu |z - tw|^2) e^{\nu t w \bar{z}}. \tag{2.15}$$

In the right–hand side of (2.15), we recognize the well–known generating function for the Laguerre polynomials, to wit ([99, p. 135]):

$$\sum_{n=0}^{\infty} z^n L^{(\alpha)}_n(x) = \frac{1}{(1-z)^{1+\alpha}} \exp\left(\frac{-xz}{1-z} \right); \ |z| < 1.$$

Thus, we get

$$\sum_{m,n=0}^{+\infty} \frac{u^m t^n}{\nu^n m! n!} H^{\nu}_{m,n}(z,\bar{z}) H^{\nu}_{n,m}(w,\overline{w}) = \frac{1}{(1 - \nu t u)} \exp\left(\frac{-\nu^2 t u |z - tw|^2}{1 - \nu t u} \right) e^{\nu t w \bar{z}}.$$

∎

We conclude this section by noting that the majority of obtained results in the framework of the UCHP, including the exponential generating function (2.2) as well as the generating function (2.20), can be re-derived making use of rescaled version of the integral representation of the UCHP.

Theorem 2.1.5. *For the scalar parameters $\mu > 0$ and $\alpha, \beta \in \mathbb{C}$ such that $\alpha\beta > 0$ and $\nu = \frac{\alpha\beta}{\mu}$, we have*

$$H^{\nu}_{m,n}(z;\bar{z}) = \left(\frac{\mu}{\pi}\right) (-\alpha)^m (\beta)^n \int_{\mathbb{C}} \xi^m \bar{\xi}^n e^{\frac{\alpha\beta}{\mu}|z|^2 - \mu|\xi|^2 + \alpha\xi\bar{z} - \beta\bar{\xi}z} d\lambda(\xi). \tag{2.16}$$

Proof. The proof we present here is direct and uses the integral representation of the Gaussian function $e^{-\frac{\alpha\beta}{\mu}|z|^2}$,

$$\int_{\mathbb{C}} e^{-\mu|\xi|^2 + \alpha\xi\bar{z} - \beta\bar{\xi}z} d\lambda(\xi) = \left(\frac{\pi}{\mu}\right) e^{-\frac{\alpha\beta}{\mu}|z|^2}. \tag{2.17}$$

The integral involved in left–hand side of (2.17) converges uniformly in z on every disc $D(0,r)$ of $\mathbb{C}$. Thus, by differentiating repeatedly both sides of (2.17), with respect to z and $\bar{z}$, we obtain the integral representation for the $H^{\nu}_{m,n}(z;\overline{z})$ given by (3.5). ∎

Remark 2.1.6. The particular case of $\alpha = -\beta = i$ in the identity (2.17) reads simply

$$e^{-\frac{|z|^2}{\mu}} = \left(\frac{\mu}{\pi}\right) \int_{\mathbb{C}} e^{-\mu|\xi|^2 + 2i\Re\xi\overline{z}} d\lambda(\xi), \tag{2.18}$$

and leads to the well–known fact that the Fourier transform reproduces the Gaussian function. If in addition $\mu = 1$, the integral representation (3.5) reduces further to the one obtained in [73, Theorem 5.1] by making use of the exponential generating function (2.2).

Remark 2.1.7. The right–hand side of (2.14) is not to be confused with the one (Poisson kernel) obtained in [73, Eq. (3.3)],

$$\sum_{m,n=0}^{+\infty} \frac{u^m v^n}{m!n!} H_{m,n}(z,\overline{z}) H_{n,m}(w,\overline{w}) = \frac{1}{1-uv} \exp\left(\frac{-uv(|z|^2 + |w|^2) + uz\overline{w} + v\overline{z}w}{1-uv} \right). \tag{2.19}$$

valid for $\max\{|u|,|v|\} < 1$. Notice also that the formula (2.19) was first given in [120] without proof and recovered by Ismail as a particular case of the Kibble–Slepian formula [73, Theorem 1.1]. A simple and direct proof can be found in [55]. In our formula t belongs to the unit circle, $|t| = 1$, so that it can be seen as a special extension of (2.19). We show in Section 4 that (2.19) remains valid to a large class of parameters $u, v \in \mathbb{C}$. Thus, we claim that (2.19) is valid for $\Re(uv) < 1$.

Remark 2.1.8. The same observation holds true for (2.13) compared to (2.19). In fact, identity (2.13) is completely different from (2.19) for $|t| = 1$ and the non-symmetry in the indices. This formula can be seen as a special generalization of Proposition 3.4 (a) in [56], to wit

$$\sum_{k=0}^{+\infty} \frac{u^k}{k!} H^{\nu}_{k,n}(z,\overline{z}) = \nu^n (\overline{z} - u)^n e^{\nu u z}, \tag{2.20}$$

which readily follows from (2.13) by taking there $u = 0$.

Applications of the obtained results are given in the context of the theory of integral transforms.

2.2 On the transform $\mathcal{T}^{\nu}$

Motivated by the fact that the UCHP form another "non-trivial" orthogonal basis of $L^{2,\nu}(\mathbb{C})$ [7,54,72], one can reproduce a two–dimensional integral transform $\mathcal{T}^{\nu}$ of Segal–Bargmann type which is closely connected to $\mathcal{B}^{2,\nu}$ but with a non-trivial kernel function (different from the one of $\mathcal{B}^{2,\nu}$). In fact, it is defined by

$$\mathcal{T}^{\nu}(\psi)(z,w) := \left(\frac{\nu}{\pi}\right)^{3/2} \int_{\mathbb{C}} e^{-\nu(z-\xi)(w-\overline{\xi})} \psi(\xi) d\lambda(\xi). \tag{2.1}$$

Its associated kernel function involves the exponential generating function [56],

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!} H^{\nu}_{m,n}(z;\overline{z}) = e^{\nu(uz+v\overline{z}-uv)}. \tag{2.2}$$

Accordingly, one obtains the following

Theorem 2.2.1. *The integral operator $\mathcal{T}^{\nu}$ in* (2.1) *defines an isometric isomorphism from $L^{2,\nu}(\mathbb{C})$ onto the two–dimensional Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$. Its inverse is given by*

$$\left(\mathcal{T}^{\nu}\right)^{-1}(\varphi)(\xi) = \left(\frac{\nu}{\pi}\right)^{3/2} \int_{\mathbb{C}^2} e^{-\nu(|z|^2+|w|^2)+\nu(\overline{z}\xi+\overline{w}\overline{\xi}-\overline{z}\overline{w})} \varphi(z,w) d\lambda(z,w). \tag{2.3}$$

Moreover, we have

$$\int_{\mathbb{C}} e^{-\nu(z-\xi)(w-\overline{\xi})} H^{\nu}_{m,n}(\xi,\overline{\xi}) d\lambda(\xi) = \left(\frac{\pi}{\nu}\right) \nu^{m+n} z^m w^n. \tag{2.4}$$

Proof. The kernel function of the integral operator $\mathcal{T}^{\nu}$ in (2.1) is related to the exponential generating function involving the product of $e_{m,n}(z,w) = z^m w^n$ and $H^{\nu}_{m,n}(\xi;\overline{\xi})$. Indeed, we have

$$T^{\nu}(\xi|z,w) = \left(\frac{\nu}{\pi}\right)^{3/2} \sum_{m,n=0}^{\infty} \frac{\nu^{m+n} z^m w^n}{m!n!} \overline{H^{\nu}_{m,n}(\xi;\overline{\xi})}. \tag{2.5}$$

for the functions $e_{m,n}(z,w) = z^m w^n$ being an orthogonal basis of the two–dimensional Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ with square norm

$$\|e_{m,n}\|^2_{L^{2,\nu}} = \left(\frac{\pi}{\nu}\right)^2 \frac{m!n!}{\nu^{m+n}},$$

while the polynomials $H^{\nu}_{m,n}(\xi;\overline{\xi})$, for varying m and n, constitute an orthogonal basis of the Hilbert space $L^{2,\nu}(\mathbb{C})$ with square norm given by (2.5). The closed formula of $T^{\nu}(\xi|z,w)$ in (2.5) is then obtained by the exponential generating function (2.2) combined with the fact $\overline{H^{\nu}_{m,n}(\xi;\overline{\xi})} = H^{\nu}_{m,n}(\overline{\xi};\xi)$. Accordingly, the proof of Theorem 2.2.1 immediately follows in virtue of the general principle described in Section 1.2. ∎

Remark 2.2.2. By considering the integral transform $\mathcal{T}^{\nu}$, there is non-evidence in asserting if it is closely connected to the $2d$–Bargmann transform $B^{2,\nu}$ or not. This is the subject of Theorem 2.2.5. In fact, we show that the action of the transforms $\mathcal{T}^{\nu}$ and $\mathcal{B}^{2,\nu}$ are the same by making a specific linear change of variable, to wit

$$\mathcal{T}^{\nu}\varphi(z,w) = \mathcal{B}^{2,\nu}\varphi\left(\frac{z+w}{\sqrt{2}}, \frac{z-w}{i\sqrt{2}}\right).$$

Consequently, all properties of $\mathcal{T}^{\nu}$ can directly be read off from those of $\mathcal{B}^{2,\nu}$. However, the explicit expression of its kernel function is so important to obtain, in an easy way, many interesting properties of $\mathcal{T}^{\nu}$ and therefore of $\mathcal{B}^{2,\nu}$, which seems to be hard to handle directly using $\mathcal{B}^{2,\nu}$. This is clear from Theorem 2.4.1, Proposition 2.2.4 and Corollary 2.4.5.

Remark 2.2.3. By non-trivial, we mean the way of constructing the kernel function for our integral transform $\mathcal{T}^{\nu}$. It is constructed using the UCHP, while the one for the standard $2d$-Segal–Bargmann transform $B^{2,\nu}$ is based on the simple tensor product of the real Hermite polynomials $H^{\nu}_{m}(x)H^{\nu}_{m}(y)$ leading to the tensor product of two copies of the one-dimensional kernel function.

Consider the L^2–eigenspaces

$$\mathcal{F}^{2,\nu}_{n}(\mathbb{C}) = \{f \in L^{2,\nu}(\mathbb{C}); \quad \Delta_{\nu} f = \nu n f\} \tag{2.6}$$

of the magnetic Schrödinger operator ([59,72,87])

$$\Delta_{\nu} := -\frac{\partial^2}{\partial z \partial \overline{z}} + \nu \overline{z} \frac{\partial}{\partial \overline{z}}.$$

Taking into account the orthogonal Hilbertian decomposition

$$L^{2,\nu}(\mathbb{C}) = \bigoplus_{n=0}^{\infty} \mathcal{F}^{2,\nu}_{n}(\mathbb{C}),$$

a quiet natural question arises of whether the image of the nth true–poly–Fock space $\mathcal{F}^{2,\nu}_{n}(\mathbb{C})$ in (2.6) by the transform $\mathcal{T}^{\nu}$ can be characterized.

Proposition 2.2.4. *We have*

$$\mathcal{T}^{\nu}(\mathcal{F}^{2,\nu}_{n}(\mathbb{C})) = \{f(z,w) = w^n h(z); \, h \in \mathcal{F}^{2,\nu}(\mathbb{C})\}. \tag{2.7}$$

Proof. The assertion of Proposition 2.2.4 readily follows making use of (2.4). ∎

The following result shows that the integral operator $\mathcal{T}^{\nu}$ is closely related to the two–dimensional Segal–Bargmann transform in (2.1) as well as to the Wigner transform

$$\mathcal{W}^{\nu}(f)(x,y) = \left(\frac{1}{2\pi}\right)^{1/2} \int_{\mathbb{R}} e^{-i\nu x t} f\left(y + \frac{t}{2}, y - \frac{t}{2}\right) dt; \quad f \in L^{2,0}(\mathbb{R}^2). \tag{2.8}$$

The transform $\mathcal{W}^{\nu}$ is connected with the phase space formulation of quantum mechanics and Weyl correspondence [49,96,110]. The exact statement make appeal to the standard action of the group of 2×2 matrices $M_2(\mathbb{C})$ defined by

$$g \cdot (z,w) = (az + bw, cz + dw); \, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$$

on $\mathbb{C}^2$ that we extend to functions on $\mathbb{C}^2$ by considering $\Gamma_g f(z,w) := f(g \cdot (z,w))$. Below, we use g_i to denote the special 2×2 matrix

$$g_i := \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \tag{2.9}$$

Notice for instance that $g_i \in \sqrt{2}U(2)$, where $U(2)$ is the subgroup of unitary matrices in $M_2(\mathbb{C})$.

Theorem 2.2.5. *We have the identities*

$$\mathcal{B}^{2,\nu} = \Gamma_{\frac{g_i}{\sqrt{2}}}\mathcal{T}^{\nu} \tag{2.10}$$

and

$$\mathcal{T}^{\nu}\mathcal{W}^{\nu} = \left(\frac{1}{2\nu}\right)^{1/2} e^{-\frac{\nu}{4}(z+w)^2}\Gamma_{-ig_i}\mathcal{T}^{\frac{\nu}{2}}. \tag{2.11}$$

Proof. The proof of $\mathcal{B}^{2,\nu} = \Gamma_{g_i}\mathcal{T}^{\nu}$ follows by direct computation starting from

$$\Gamma_{g_i}\mathcal{T}^{\nu}\psi(z,w) = \mathcal{T}^{\nu}\psi\left(z+iw, z-iw\right)$$

and next using the identity

$$e^{-\nu(U-\xi)(V-\overline{\xi})} = \rho_0^{\nu}\left(\Re\xi - \frac{U+V}{2}\right)\rho_0^{\nu}\left(\Im\xi - \frac{U-V}{2i}\right).$$

Indeed, we obtain

$$\begin{aligned}\Gamma_{g_i}\mathcal{T}^{\nu}\psi(z,w) &= \left(\frac{\nu}{\pi}\right)^{3/2}\int_{\mathbb{C}} e^{-\nu(z+iw-\xi)\left(z-iw-\overline{\xi}\right)}\psi(\xi)d\lambda(\xi)\\ &= \left(\frac{\nu}{\pi}\right)^{3/2}\int_{\mathbb{C}} \rho_0^{\nu}\left(\Re\xi - z\right)\rho_0^{\nu}\left(\Im\xi - w\right)\psi(\xi)d\lambda(\xi),\end{aligned}$$

which gives rise to the integral transform $\mathcal{B}^{2,\nu}$ given by (2.1). To prove (2.11), we rewrite $\mathcal{T}^{\nu}\mathcal{W}^{\nu}$ as

$$\begin{aligned}\mathcal{T}^{\nu}(\mathcal{W}^{\nu}(f))(z,w) &= \left(\frac{\nu}{\pi}\right)^{3/2}\int_{\mathbb{C}} e^{-\nu(z+iw-\xi)\left(z-iw-\overline{\xi}\right)}\mathcal{W}^{\nu}(f)(\xi)d\lambda(\xi)\\ &= \left(\frac{\nu}{\pi}\right)^{3/2}\int_{\mathbb{C}} e^{-\nu x^2+\nu(z+w)x-\nu(z-iy)(w+iy)}\mathcal{W}^{\nu}(f)(\xi)d\lambda(\xi)\end{aligned}$$

with $\xi = x+iy \sim (x,y)$. By the definition (2.8) of $\mathcal{W}^{\nu}$ and the Gaussian integral formula, we get

$$\begin{aligned}\mathcal{T}^{\nu}(\mathcal{W}^{\nu}(f))(z,w) &= \left(\frac{1}{2\nu}\right)^{1/2}\left(\frac{\nu}{\pi}\right)^{3/2}\int_{\mathbb{R}^2} f\left(y-\frac{t}{2}, y+\frac{t}{2}\right)e^{-\nu(z-iy)(w+iy)+\frac{\nu}{4}(z+w-it)^2}dydt\\ &= \left(\frac{1}{2\nu}\right)^{1/2}\left(\frac{\nu}{\pi}\right)^{3/2} e^{\frac{\nu}{4}(z+w)^2}\int_{\mathbb{C}} f(\xi)e^{-\frac{\nu}{2}(-iz-w-\xi)(-iz+w-\overline{\xi})}d\lambda(\xi)\\ &= \left(\frac{1}{2\nu}\right)^{1/2} e^{\frac{\nu}{4}(z+w)^2}\mathcal{T}^{\frac{\nu}{2}}(f)(-iz-w,-iz+w)\end{aligned}$$

thanks to the change of variables $X = y + \frac{t}{2}$ and $Y = y - \frac{t}{2}$ and the key observation that

$$e^{-\nu\left(z-i\frac{X+Y}{2}\right)\left(w+i\frac{X+Y}{2}\right)+\frac{\nu}{4}(z+w-i(X-Y))^2} = e^{\frac{\nu}{4}(z+w)^2}e^{-\frac{\nu}{2}(-iz-w-\xi)(-iz+w-\overline{\xi})}.$$

To conclude, it suffices to see that $\mathcal{T}^{\frac{\nu}{2}}(f)(-iz-w,-iz+w) = \Gamma_{-i\sqrt{2}g_i}\mathcal{T}^{\frac{\nu}{2}}(f)(z,w)$, where g_i is the matrix in (2.9).

∎

Remark 2.2.6. Notice that the formula (2.11) is in some sort the analogue of the one proved by Shun-Long Luo in [81, Proposition 2].

Remark 2.2.7. Further interesting integral transforms can be derived as special cases of $\mathcal{T}^{\nu}$, such as those obtained by restriction to the diagonal and to the "anti-diagonal" of $\mathbb{C}^2$, to wit

$$\mathcal{R}^{\nu}_{+}\psi(z) := \mathcal{T}^{\nu}\psi(z,z) = \left(\frac{\nu}{\pi}\right)^{3/2}\int_{\mathbb{C}} e^{-\nu(z-\xi)(z-\overline{\xi})}\psi(\xi)d\lambda(\xi) \tag{2.12}$$

and

$$\mathcal{R}^{\nu}_{-}\psi(z) := \mathcal{T}^{\nu}\psi(z,\overline{z}) = \left(\frac{\nu}{\pi}\right)^{3/2}\int_{\mathbb{C}} e^{-\nu|\zeta|^2}\psi(z-\zeta)d\lambda(\zeta). \tag{2.13}$$

2.3 Integral transforms for the true–poly–Fock spaces

Following the same scheme as above, we use Theorem 2.1.1 to provide a direct and simpler proof of the fact that the generalized Segal–Bargmann transform of level n [114, Theorem 2.5] (see also [87]),

$$\mathcal{B}^{1,\nu}_{n}(\phi)(z) := \left(\frac{\nu}{\pi}\right)^{3/4}\left(\frac{1}{2^n\nu^n n!}\right)^{1/2}\int_{\mathbb{R}}\rho^{\nu}_{0}\left(x-\frac{z}{\sqrt{2}}\right)H^{\nu}_{n}\left(\frac{z+\overline{z}}{\sqrt{2}}-x\right)\phi(x)dx \tag{2.1}$$

is an isometric operator linking the space of square integrable functions on the real line with the so–called true–poly–Fock spaces according to the terminology of Vasilevski [2,114]. Theorem 2.5 in [114] is reproved in [87] and can be reworded as follows

Theorem 2.3.1. *The integral operator in* (2.1) *defines an isometric isomorphism from* $L^{2,\nu}(\mathbb{R})$ *onto the generalized Bargmann–Fock space* $\mathcal{F}^{2,\nu}_{n}(\mathbb{C})$ *defined by* (2.6)*. Moreover, we have*

$$\mathcal{B}^{1,\nu}_{n}(H^{\nu}_{m})(z) = \left(\frac{\nu}{\pi}\right)^{1/4}\left(\frac{2^m}{n!\nu^n}\right)^{1/2}H^{\nu}_{m,n}(z,\overline{z}). \tag{2.2}$$

Proof. The kernel function associated to the Hilbert space $L^{2,\nu}(\mathbb{R})$ and the generalized Bargmann–Fock space $\mathcal{F}^{2,\nu}_{n}(\mathbb{C})$ is given by

$$\begin{aligned} T^{\nu}_{n}(x;z) &:= \left(\frac{\nu}{\pi}\right)^{3/4}\left(\frac{1}{n!\nu^n}\right)^{1/2}\sum_{m=0}^{+\infty}\frac{H^{\nu}_{m}(x)H^{\nu}_{m,n}(z,\overline{z})}{\sqrt{2^m}\nu^m m!} \\ &= \left(\frac{\nu}{\pi}\right)^{3/4}\left(\frac{1}{n!\nu^n}\right)^{1/2}\sum_{m=0}^{+\infty}\frac{H^{\nu}_{m}(x)\overline{H^{\nu}_{m,n}(\overline{z},z)}}{\sqrt{2^m}\nu^m m!}. \end{aligned}$$

According to the general principle (see Section 1.2) and using the expression of the norms of $H^{\nu}_{m}(x)$ and $H^{\nu}_{m,n}(z;\overline{z})$ given explicitly by (2.4) and (2.5), we recognize the generating function (2.3). Thus, we get

$$T^{\nu}_{n}(x;z) = \left(\frac{\nu}{\pi}\right)^{3/4}\left(\frac{1}{2^n\nu^n n!}\right)^{1/2} e^{-\frac{\nu}{2}z^2+\sqrt{2}\nu xz} H^{\nu}_{n}\left(\frac{z+\overline{z}}{\sqrt{2}}-x\right). \tag{2.3}$$

This completes the proof of Theorem 2.3.1. ∎

The next application corresponds to Theorem 2.1.3 and concerns the integral kernel transform

$$\mathcal{T}^{\nu}_{n,n'}(\psi)(z) := \frac{(-1)^{n'}\nu}{\pi\sqrt{n!n'!\nu^{n+n'}}}\int_{\mathbb{C}} e^{-\nu|\xi|^2+\nu\overline{\xi}z} H^{\nu}_{n,n'}(\xi-z,\overline{\xi}-\overline{z})\psi(\xi)d\lambda(\xi). \tag{2.4}$$

Namely, we have

Theorem 2.3.2. *The integral transform $\mathcal{T}^{\nu}_{n,n'}$ is a unitary operator from $\mathcal{F}^{2,\nu}_{n}(\mathbb{C})$ onto $\mathcal{F}^{2,\nu}_{n'}(\mathbb{C})$ and its inverse is given by $\left(\mathcal{T}^{\nu}_{n,n'}\right)^{-1} = \mathcal{T}^{\nu}_{n',n}$. Moreover, we have the following integral reproducing property for the UCHP*

$$\mathcal{T}^{\nu}_{n,n'}(H^{\nu}_{m,n})(z) = \left(\frac{n!\nu^n}{n'!\nu^{n'}}\right)^{1/2} H^{\nu}_{m,n'}(z,\overline{z}). \tag{2.5}$$

Proof. We apply the general principle described in Section 1.2. Indeed, the kernel function in the integral transform $\mathcal{T}^{\nu}_{n,n'}$ defined by (2.4),

$$\mathcal{T}^{\nu}_{n,n'}(\psi)(z) := \left(\frac{(-1)^{n'}\nu}{\pi\sqrt{n!n'!\nu^{n+n'}}}\right)\int_{\mathbb{C}} e^{-\nu|\xi|^2+\nu\overline{\xi}z} H^{\nu}_{n,n'}(\xi-z,\overline{\xi}-\overline{z})\psi(\xi)d\lambda(\xi)$$

is in fact the exponential generating function involving the product of $H^{\nu}_{m,n}$ and $H^{\nu}_{m,n'}$, which for varying m, are special orthogonal bases of the generalized Bargmann–Fock spaces $\mathcal{F}^{2,\nu}_{n}(\mathbb{C})$ and $\mathcal{F}^{2,\nu}_{n'}(\mathbb{C})$, respectively. To conclude, we make use of Theorem 2.1.3. ∎

Remark 2.3.3. The particular integral operator $\mathcal{T}^{\nu}_{0,n}$ maps isometrically the standard Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C})$ onto $\mathcal{F}^{2,\nu}_{n}(\mathbb{C})$. Its inverse is given by

$$\mathcal{T}^{\nu}_{n,0}(\psi)(z) := \left(\frac{\nu}{\pi}\right)\left(\frac{\nu^n}{n!}\right)^{1/2}\int_{\mathbb{C}} e^{-\nu|\xi|^2+\nu\overline{\xi}z}(\xi-z)^n\psi(\xi)d\lambda(\xi). \tag{2.6}$$

Remark 2.3.4. By taking $n = n'$ in (2.5), one sees that the univariate complex Hermite polynomials $H^{\nu}_{m,n}$, for varying m and fixed n, is a common set of L^2–eigenfunctions of Δ_ν and the integral operator $\mathcal{T}^{\nu}_{n} := \mathcal{T}^{\nu}_{n,n}$.

2.4 Associated 2-d fractional Fourier transform

Using the nature of the kernel function in $\mathcal{T}^\nu$, one can show that $\mathcal{T}^\nu$ is closely connected to the $2d$–fractional Fourier transform ($2d$–FrFT) $\widetilde{\mathcal{F}}^\nu_{u,v}$ in [115],

$$\widetilde{\mathcal{F}}^\nu_{u,v}\psi(\xi) = \frac{\nu e^{\frac{-\nu uv}{1-uv}|\xi|^2}}{\pi(1-uv)} \int_{\mathbb{C}} \exp\left(-\frac{\nu\left(|\zeta|^2 - u\overline{\zeta}\xi - v\zeta\overline{\xi}\right)}{1-uv}\right)\psi(\zeta)d\lambda(\zeta) \tag{2.1}$$

for given complex–valued function on the real line provided that the integral exists. More precisely, we have

Theorem 2.4.1. *The transform $\widetilde{\mathcal{F}}^\nu_{u,v}$ is well–defined from $L^{2,\nu}(\mathbb{C})$ onto $L^{2,\nu}(\mathbb{C})$ for every $|u| = |v| = 1$ with $\Re(uv) < 1$. Moreover, it satisfies*

$$\widetilde{\mathcal{F}}^\nu_{u,v}\psi(\xi) = [\mathcal{T}^\nu]^{-1}(\Gamma_{u,v}\mathcal{T}^\nu\psi)(\xi) \tag{2.2}$$

for every $\psi \in L^{2,\nu}(\mathbb{C})$ and $\xi \in \mathbb{C}$, where $\Gamma_{u,v}\varphi(z,w) := \varphi(uz, vw)$.

Remark 2.4.2. Notice that the identity (2.2) will facilitate further the study of the basic properties of $\widetilde{\mathcal{F}}^\nu_{u,v}$, including its Plancherel theorem and its inversion formula. It should be noticed here that $2d$–FrFT in (2.1) is completely different from the standard $2d$–fractional Fourier transform based on the product $H_m(x)H_m(y)$ (see [115]). Its kernel is related to the Mehler?s formula (2.19) for the UCHP by formally taking $u = e^{i\alpha}$ and $v = e^{i\beta}$ for given reals α, β.

We present below a sketched proof of Theorem 2.4.1 and discuss some of its immediate consequences.

Proof of Theorem 2.4.1. We begin by noticing that $[\mathcal{T}^\nu]^{-1} \circ \Gamma_{u,v} \circ \mathcal{T}^\nu$ is well–defined from $L^{2,\nu}(\mathbb{C})$ onto $L^{2,\nu}(\mathbb{C})$ if and only if $(u,v) \in S^1 \times S^1$, $S^1 = \{u \in \mathbb{C}; |u| = 1\}$. This is in fact is equivalent to say that $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ is invariant by the Γ–action. Moreover, direct computation shows that $[\mathcal{T}^\nu]^{-1} \circ \Gamma_{u,v} \circ \mathcal{T}^\nu$ is an integral kernel transform

$$[\mathcal{T}^\nu]^{-1}(\Gamma_{u,v}\mathcal{T}^\nu\psi)(\xi) = \left\langle \psi, \overline{K_{u,v}(\cdot;\xi)} \right\rangle_{L^{2,\nu}(\mathbb{C})} = \int_{\mathbb{C}} \psi(\zeta)K_{u,v}(\zeta;\xi)e^{-\nu|\zeta|^2}d\lambda(\zeta)$$

with kernel function given by

$$K_{u,v}(\zeta;\xi) := \langle G(\zeta;\Gamma_{u,v}\cdot\cdot), G(\xi;\cdot\cdot)\rangle_{L^{2,\nu}(\mathbb{C}^2)},$$

where $G(\xi; z, w)$ denotes the generating function of the complex Hermite polynomials defined by

$$\begin{aligned} G(\xi; z, w) &:= \left(\frac{\nu}{\pi}\right)^{3/2} \sum_{m,n=0}^{\infty} \frac{z^m w^n \overline{H^\nu_{m,n}(\xi,\overline{\xi})}}{m!n!} \\ &= \left(\frac{\nu}{\pi}\right)^{3/2} e^{\nu(z\overline{\xi}+w\xi-zw)}. \end{aligned} \tag{2.3}$$

Thus, using the orthogonality of the monomials $e_{m,n}(z,w) = z^m w^n$ in $L^{2,\nu}(\mathbb{C}^2)$, we formally get

$$K_{u,v}(\zeta;\xi) = \left(\frac{\nu}{\pi}\right) \sum_{m,n=0}^{\infty} \left(\frac{u}{\nu}\right)^m \left(\frac{v}{\nu}\right)^n \frac{\overline{H^\nu_{m,n}(\zeta,\overline{\zeta})} H^\nu_{m,n}(\xi,\overline{\xi})}{m!n!}. \tag{2.4}$$

At this stage, one can not use directly the Mehler's formula in (2.19) (valid for $\max|u|,|v| < 1$) to recover the $2d$–FrFT $\widetilde{\mathcal{F}}^\nu_{u,v}$, unless one can prove an extended Mehler's formula for $|u| = |v| = 1$, except for some spacial values. However, a direct computation, using the exponential generating function (2.3) as well as the specific integral representation of the kernel function $e^{\nu zw}$ (the reproducing property),

$$\int_{\mathbb{C}} e^{-\nu|\xi|^2 + \nu(z\overline{\xi} + w\xi)} d\lambda(\xi) = \left(\frac{\pi}{\nu}\right) e^{\nu zw}, \tag{2.5}$$

shows that under the condition that $uv \in \mathbb{R}$ such that $\Re(uv) < 1$, we have

$$\begin{aligned} K_{u,v}(\zeta;\xi) &= \left(\frac{\nu}{\pi}\right)^3 \int_{\mathbb{C}} e^{-\nu|w|^2 + \nu(v\zeta w + \overline{\xi}\overline{w})} \left(\int_{\mathbb{C}} e^{-\nu|z|^2 + \nu\left(u[\overline{\zeta} - vw]z + [\xi - \overline{w}]\right)} d\lambda(z) \right) d\lambda(w) \\ &= \left(\frac{\nu}{\pi}\right)^2 \int_{\mathbb{C}} e^{-\nu(1-uv)|w|^2 + \nu(1-uv)\left\{\frac{v(\zeta - u\xi)}{(1-uv)} w + \frac{(\overline{\xi} - u\overline{\zeta})}{(1-uv)} \overline{w}\right\}} e^{\nu u \overline{\zeta}\xi} d\lambda(w). \end{aligned}$$

Finally, we get

$$K_{u,v}(\zeta;\xi) = \frac{\nu}{\pi(1-uv)} \exp\left(\frac{\nu}{1-uv} \left\{ -uv(|\zeta|^2 + |\xi|^2) + u\overline{\zeta}\xi + v\zeta\overline{\xi} \right\} \right). \tag{2.6}$$

This completes the proof. ∎

Remark 2.4.3. Starting from the fact that the transform $(\mathcal{T}^\nu)^{-1} \circ \Gamma_{u,v} \circ \mathcal{T}^\nu$ is well–defined for $|u| = |v| = 1$ and taking into account the explicit computation we provide above, we claim that the Poisson kernel (2.19), initially valid for $\max|u|,|v| < 1$, can be extended to the case $|u| = |v| = 1$ with $\Re(uv) < 1$ by equating the right–hand sides of (2.4) and (2.6).

Remark 2.4.4. The transform $\widetilde{\mathcal{F}}^\nu_{u,v}\psi(\xi)$ is a special kind of generalization of the rescaled Fourier transform in two dimensions $\widetilde{\mathcal{F}}^\nu$ defined on $L^{2,\nu}(\mathbb{C})$ by

$$\widetilde{\mathcal{F}}^\nu(\varphi)(\xi) := \left(\frac{\nu}{2\pi}\right) \int_{\mathbb{C}} e^{\frac{\nu}{2}(\xi - iu)(\overline{\xi} - i\overline{u})} \varphi(u) d\lambda(u), \tag{2.7}$$

which is the $L^{2,\nu}(\mathbb{C})$–version of the standard Fourier transform $\mathcal{F}^\nu$ on $L^{2,0}(\mathbb{C})$ with $\widetilde{\mathcal{F}}^\nu = \mathcal{M}_{\frac{\nu}{2}} \mathcal{F}^\nu \mathcal{M}_{-\frac{\nu}{2}}$. Here $\mathcal{M}_\alpha$ denotes the multiplication operator (ground state transform) $\mathcal{M}_\alpha f := e^{-\alpha|z|^2} f$. In fact, we have

$$(\mathcal{T}^\nu)^{-1} \circ \Gamma_{-i,-i} \circ \mathcal{T}^\nu = \widetilde{\mathcal{F}}^\nu, \tag{2.8}$$

which is exactly the classical result obtained by V. Bargmann in [16] for the standard Segal–Bargmann and the Fourier transforms, thanks to Theorem 2.2.5.

As immediate consequence, we conclude from (2.4) and (2.2) the following

Corollary 2.4.5. *The UCHP form an orthogonal eigenfunction basis of $\widetilde{\mathcal{F}}^{\nu}_{u,v}$ with*

$$\widetilde{\mathcal{F}}^{\nu}_{u,v}(H^{\nu}_{m,n}) = u^m v^n H^{\nu}_{m,n}. \tag{2.9}$$

Theorem 2.4.1 is interesting in itself since it allows one to deduce easily some of the basic properties of $\widetilde{\mathcal{F}}^{\nu}_{u,v}$ using those of $\mathcal{T}^{\nu}$, including the uniqueness theorem, the Plancherel theorem and the inversion formula for the $2d$–FrFT, $\widetilde{\mathcal{F}}^{\nu}_{u,v}$.

Corollary 2.4.6. *If for given $\varphi, \psi \in L^{2,\nu}(\mathbb{C})$ we have $\widetilde{\mathcal{F}}^{\nu}_{u,v}\varphi = \widetilde{\mathcal{F}}^{\nu}_{u,v}\psi$, then $\varphi = \psi$.*

Corollary 2.4.7. *For every $\psi \in L^{2,\nu}(\mathbb{C})$ we have $\left\|\widetilde{\mathcal{F}}^{\nu}_{u,v}\psi\right\|_{L^{2,\nu}(\mathbb{C}^2)} = \|\psi\|_{L^{2,\nu}(\mathbb{C})}$.*

Corollary 2.4.8. *The inversion formula for $\widetilde{\mathcal{F}}^{\nu}_{u,v}$ is the $2d$–FrFT with the parameters $\{\frac{1}{u}, \frac{1}{v}\}$, to wit*

$$[\widetilde{\mathcal{F}}^{\nu}_{u,v}]^{-1}\psi = \widetilde{\mathcal{F}}^{\nu}_{\frac{1}{u},\frac{1}{v}}\psi$$

for every fixed $u, v \in S^1$ and every $\psi \in L^{2,\nu}(\mathbb{C}^2)$.

Remark 2.4.9. The inversion formula can also be seen as immediate consequence of the semi–group property $\widetilde{\mathcal{F}}^{\nu}_{u,v} \circ \widetilde{\mathcal{F}}^{\nu}_{u',v'} = \widetilde{\mathcal{F}}^{\nu}_{uu',vv'}$ that readily follows from Theorem 2.4.1 since $\Gamma_{u,v} \circ \Gamma_{u',v'} = \Gamma_{uu',vv'}$.

Chapter 3

On a novel class of polyanalytic Hermite polynomials

We discuss some algebraic and analytic properties of a general class of orthogonal polyanalytic polynomials, including their operational formulas, recurrence relations, generating functions, integral representations and different orthogonality identities. We establish their connection and rule in describing the L^2–spectral theory of some special second order differential operators of Laplacian type acting on the L^2–Gaussian Hilbert space on the whole complex plane. We will also show their importance in the theory of the so-called rank–one automorphic functions on the complex plane. In fact, a variant subclass leads to an orthogonal basis of the corresponding L^2–Gaussian Hilbert space on the strip $\mathbb{C}/\mathbb{Z}$.

3.1 Preliminary results

This section incorporates a preliminary study of the polynomials $I_n^{\nu,\alpha}(z,\overline{z}|\xi)$ (abbreviated sometimes as $I_n^{\nu,\alpha}$). For the unity of the formulation, we put $I_n^{\nu,\alpha} = 0$ whenever $n < 0$. Notice for instance that $I_0^{\nu,\alpha}(z,\overline{z}|\xi) = 1$ and

$$I_1^{\nu,\alpha}(z,\overline{z}|\xi) = \nu\overline{z} - 2\alpha z - \xi. \tag{3.1}$$

The first result concerns useful operational formulas for $I_n^{\nu,\alpha}(z,\overline{z}|\xi)$.

Proposition 3.1.1. *The polynomials $I_n^{\nu,\alpha}(z,\overline{z}|\xi)$ can be realized as*

$$I_n^{\nu,\alpha}(z,\overline{z}|\xi) = e^{-\alpha z^2-\xi z}\left(-\partial_z + \nu\overline{z}\right)^n\left(e^{\alpha z^2+\xi z}\right) \tag{3.2}$$

$$= e^{-\alpha z^2}(-\partial_z + \nu\overline{z} - \xi)^n e^{\alpha z^2}. \tag{3.3}$$

Moreover, the first order differential operators $-\partial_z + I_1^{\nu,\alpha}$ and $\partial_{\overline{z}}$ are respectively the corresponding raising and lowering operators in the sense that we have

$$\left(-\partial_z I_n^{\nu,\alpha} + I_1^{\nu,\alpha}\right) I_n^{\nu,\alpha} = I_{n+1}^{\nu,\alpha} \tag{3.4}$$

and

$$\partial_{\bar{z}} I_n^{\nu,\alpha} = \nu n I_{n-1}^{\nu,\alpha}. \tag{3.5}$$

Proof. The representation (3.2) as well as (3.3) follow from the Rodrigues' type formula (14) making use of the fact

$$(\partial_z - \nu\bar{z} + \xi)^n (f) = e^{-\xi z} (\partial_z - \nu\bar{z})^n (e^{\xi z} f) = e^{\nu z\bar{z} - \xi z} \partial_z^n \left(e^{-\nu z\bar{z} + \xi z} f \right)$$

for sufficiently differentiable function f. Both (14) and (3.2) can be used to establish (3.4). A direct calculation gives

$$\partial_z I_n^{\nu,\alpha} = (\nu\bar{z} - 2\alpha z - \xi) I_n^{\nu,\alpha} - I_{n+1}^{\nu,\alpha} = I_1^{\nu,\alpha} I_n^{\nu,\alpha} - I_{n+1}^{\nu,\alpha}.$$

This proves (3.4). To establish (3.5), we make use of

$$\partial_{\bar{z}} (\partial_z - \nu\bar{z})^n h = -\nu n (\partial_z - \nu\bar{z})^{n-1} h, \tag{3.6}$$

which holds true for any holomorphic function h and in particular for $h(z) = e^{\alpha z^2 + \xi z}$. Therefore, we obtain

$$\partial_{\bar{z}} I_n^{\nu,\alpha} = (-1)^n e^{-\alpha z^2 - \xi z} \partial_{\bar{z}} \left[(\partial_z - \nu\bar{z})^n e^{\alpha z^2 + \xi z} \right] \overset{(3.6)}{=} \nu n I_{n-1}^{\nu,\alpha}.$$

∎

The following result gives another interesting representation of the polynomials $I_n^{\nu,\alpha}$.

Proposition 3.1.2. *The polynomials $I_n^{\nu,\alpha}$ can be represented as*

$$I_n^{\nu,\alpha} = \left(-\partial_z + I_1^{\nu,\alpha} \right)^n \cdot (1). \tag{3.7}$$

Subsequently, we have

$$\partial_z I_n^{\nu,\alpha} = -2\alpha n I_{n-1}^{\nu,\alpha}. \tag{3.8}$$

Proof. Notice first that (3.4) can be rewritten as $\left(-\partial_z + I_1^{\nu,\alpha} \right) I_k^{\nu,\alpha} = I_{k+1}^{\nu,\alpha}$. Therefore, we get

$$\left(-\partial_z + I_1^{\nu,\alpha} \right)^n I_k^{\nu,\alpha} = I_{k+n}^{\nu,\alpha} \tag{3.9}$$

for any arbitrary nonnegative integers n and k. Hence, for $k = 0$, we obtain $\left(-\partial_z + I_1^{\nu,\alpha} \right)^n \cdot (1) = I_n^{\nu,\alpha}$. This proves (3.7). The proof of (3.8) lies essentially in the fact that

$$\partial_z \left(-\partial_z + I_1^{\nu,\alpha} \right)^n \cdot (1) = \left(-\partial_z I_n^{\nu,\alpha} + I_1^{\nu,\alpha} \right)^{n-1} \partial_z I_1^{\nu,\alpha} = I_n^{\nu,\alpha} - 2\alpha n \left(-\partial_z + I_1^{\nu,\alpha} \right)^{n-1} \cdot (1). \tag{3.10}$$

∎

Remark 3.1.3. By comparing (3.5) and (3.8), we get $\partial_z I_n^{\nu,\alpha} = -\frac{2\alpha}{\nu}\partial_{\overline{z}} I_n^{\nu,\alpha}$. Thus for $\alpha > 0$ and $\nu = 2\alpha$, the corresponding polynomials $I_n^{\nu,\nu/2}$ are antisymmetric in the sense that $I_n^{\nu,\nu/2}(z,\overline{z}) = -I_n^{\nu,\nu/2}(\overline{z},z)$. Moreover, they depend only on the imaginary part of z.

Combination of (3.4) and (3.8) yields the following

Corollary 3.1.4. *The polynomials $I_n^{\nu,\alpha}$ satisfy the three term recurrence formula*

$$I_{n+1}^{\nu,\alpha} = I_1^{\nu,\alpha} I_n^{\nu,\alpha} + 2\alpha n I_{n-1}^{\nu,\alpha}. \tag{3.11}$$

Remark 3.1.5. In the proof of Proposition 3.1.1 (resp. Proposition 3.1.2), we made use of the identity (3.6) (resp. (3.10)). These identities can be handled by induction. They also are particular cases of the well–established algebraic identity $AB^{n+1} - B^{n+1}A = \lambda n B^n$ whenever $AB - BA = \lambda Id$.

It may be of interest to point out that $I_n^{\nu,\alpha}$ are also polynomials in ξ with degree n. This can be seen easily in virtue of the following

Lemma 3.1.6. *We have*

$$2\alpha\partial_\xi I_n^{\nu,\alpha} = \partial_z I_n^{\nu,\alpha} = -2\alpha n I_{n-1}^{\nu,\alpha} \tag{3.12}$$

and consequently, the following recurrence formula

$$I_{n+1}^{\nu,\alpha} = I_1^{\nu,\alpha} I_n^{\nu,\alpha} - 2\alpha\partial_\xi I_n^{\nu,\alpha} \tag{3.13}$$

holds true.

Proof. Direct computation, using the fact that ∂_z and $\partial_z - \nu\overline{z}$ commute, entails

$$\begin{aligned}
\partial_z I_n^{\nu,\alpha} &= -(2\alpha z + \xi) I_n^{\nu,\alpha} + (-1)^n e^{-\alpha z^2 - \xi z} (\partial_z - \nu\overline{z})^n \left(\partial_z e^{\alpha z^2 + \xi z}\right) \\
&= 2\alpha \left\{ -z I_n^{\nu,\alpha} + (-1)^n e^{-\alpha z^2 - \xi z} (\partial_z - \nu\overline{z})^n \left(z e^{\alpha z^2 + \xi z}\right) \right\} \\
&= 2\alpha (-1)^n \partial_\xi \left\{ e^{-\alpha z^2 - \xi z} (\partial_z - \nu\overline{z})^n \left(z e^{\alpha z^2 + \xi z}\right) \right\} \\
&= 2\alpha\partial_\xi I_n^{\nu,\alpha}.
\end{aligned}$$

Insertion of (3.12) in (3.4) yields (3.13). ∎

Remark 3.1.7. The property (3.4) in Proposition 3.1.1 (resp. (3.5) in Proposition 3.1.1 and (3.12) in Lemma 3.1.6) shows that the considered polynomials $I_n^{\nu,\alpha}$ constitute an Appell sequence with respect to z (resp. $\overline{z}$ and ξ).

Added to the Rodrigues' formula (14) defining $I_n^{\nu,\alpha}$, these polynomials admit a second useful Rodrigues' formula.

Theorem 3.1.8. *We have*

$$I_n^{\nu,\alpha}(z,\overline{z}|\xi) = (-1)^n e^{\frac{-\left(I_1^{\nu,\alpha}(z,\overline{z}|\xi)\right)^2}{4\alpha}} \frac{\partial^n}{\partial z^n} \left(e^{\frac{\left(I_1^{\nu,\alpha}(z,\overline{z}|\xi)\right)^2}{4\alpha}} \right) \tag{3.14}$$

Proof. We proceed by induction. Obviously, (3.14) holds true for $n = 0$ and $n = 1$. In fact $\partial_z\left(e^{\frac{(I_1^{\nu,\alpha})^2}{4\alpha}}\right) = -I_1^{\nu,\alpha} e^{\frac{(I_1^{\nu,\alpha})^2}{4\alpha}}$. Now, assume that (3.14) holds true for every nonnegative integer $k \leq n$, for given fixed n. Since $\partial_z^j\left(I_1^{\nu,\alpha}\right) = 0$ for $j = 2, 3, \cdots$, we get

$$\partial_z^{n+1}\left(e^{\frac{(I_1^{\nu,\alpha})^2}{4\alpha}}\right) = -I_1^{\nu,\alpha}\partial_z^n\left(e^{\frac{(I_1^{\nu,\alpha})^2}{4\alpha}}\right) + n\partial_z\left(-I_1^{\nu,\alpha}\right)\partial_z^{n-1}\left(e^{\frac{(I_1^{\nu,\alpha})^2}{4\alpha}}\right),$$

so that

$$\begin{aligned}(-1)^{n+1}e^{\frac{-(I_1^{\nu,\alpha})^2}{4\alpha}}\partial_z^{n+1}\left(e^{\frac{(I_1^{\nu,\alpha})^2}{4\alpha}}\right) &= I_1^{\nu,\alpha}(-1)^n e^{\frac{-(I_1^{\nu,\alpha})^2}{4\alpha}}\partial_z^n\left(e^{\frac{(I_1^{\nu,\alpha})^2}{4\alpha}}\right)\\ &\quad + 2\alpha n(-1)^{n-1}e^{\frac{-(I_1^{\nu,\alpha})^2}{4\alpha}}\partial_z^{n-1}\left(e^{\frac{(I_1^{\nu,\alpha})^2}{4\alpha}}\right)\\ &= I_1^{\nu,\alpha}I_n^{\nu,\alpha} + 2\alpha n I_{n-1}^{\nu,\alpha}.\end{aligned}$$

Thus, one arrives at the desired result by means of the recurrence formula (3.11). ∎

The previous result shows in particular that the polynomials $I_n^{\nu,\alpha}$ should be closely connected to the univariate Hermite polynomials $H_n(x)$. In fact, the following result asserts that they are essentially the H_n in the variable $I_1^{\nu,\alpha}$.

Corollary 3.1.9. *The explicit expression of $I_n^{\nu,\alpha}$ in terms of the classical Hermite polynomials is given by*

$$I_n^{\nu,\alpha}(z,\bar{z}|\xi) = (-i)^n\alpha^{n/2}H_n\left(\frac{iI_1^{\nu,\alpha}}{2\alpha^{1/2}}\right) = (-i)^n\alpha^{n/2}H_n\left(\frac{2\alpha z - \nu\bar{z} + \xi}{2i\alpha^{1/2}}\right), \tag{3.15}$$

with $\alpha \neq 0$ and the convention that $\alpha^{1/2} = i\sqrt{|\alpha|}$ if $\alpha < 0$.

Remark 3.1.10. For the particular case of $\nu = 2\alpha > 0$, the result of Corollary 3.1.9 shows that $I_m^{\nu,\nu/2}(z,\bar{z}|\xi)$ are polynomials in $\Im(z)$ and simply reads

$$I_n^{\nu,\nu/2}(z,\bar{z}|\xi) = (-i)^n\left(\frac{\nu}{2}\right)^{n/2}H_n\left(\frac{2\nu\Im(z) + \xi}{i(2\nu)^{1/2}}\right). \tag{3.16}$$

This is in accordance with Remark 3.1.3 The special case of adequate ξ ($\xi = 2i\pi(\beta + k)$) will be reconsidered in Section 7 when dealing with rank–one automorphic functions.

The following result gives the expression of $I_n^{\nu,\alpha}(z,\bar{z}|\xi)$ in terms of the tensor product $H_j^\tau(x)H_k^\mu(y)$ of the rescaled real Hermite polynomials,

$$H_k^\tau(t) = (-1)^n e^{\tau t^2}\frac{d^n}{dt^n}(e^{-\tau t^2}), \quad \tau > 0. \tag{3.17}$$

Proposition 3.1.11. *For ν and α such that $2|\alpha| < \nu$, we have*

$$I_n^{\nu,\alpha}(z,\overline{z}|\xi) = \frac{1}{2^n}\sum_{k=0}^{n}(-i)^k\binom{n}{k}H_{n-k}^{\nu-2\alpha}\left(x-\frac{\Re(\xi)}{\nu-2\alpha}\right)H_k^{\nu+2\alpha}\left(y+\frac{\Im(\xi)}{\nu+2\alpha}\right). \tag{3.18}$$

Proof. Notice first that by considering the first order differential operators

$$A_x^{\nu,\alpha,\xi}f = -\frac{1}{2}\left(\partial_x - 2(\nu-2\alpha)x + 2\Re(\xi)\right)f$$

and

$$B_y^{\nu,\alpha,\xi}f = -\frac{1}{2}\left(\partial_y - 2(\nu+2\alpha)y - 2\Im(\xi)\right)f,$$

we clearly have $[A_x^{\nu,\alpha,\xi}, B_y^{\nu,\alpha,\xi}] = 0$. Moreover,

$$(A_x^{\nu,\alpha,\xi})^n\cdot(1) = \frac{1}{2^n}H_n^{\nu-2\alpha}\left(x-\frac{\Re(\xi)}{\nu-2\alpha}\right)$$

and

$$(B_y^{\nu,\alpha,\xi})^n\cdot(1) = \frac{1}{2^n}H_n^{\nu+2\alpha}\left(y+\frac{\Im(\xi)}{\nu+2\alpha}\right)$$

which readily follows by induction from the fact

$$\left(\partial_t - 2\tau t + \mu\right)f = e^{\tau\left(t-\frac{\mu}{2\tau}\right)^2}\partial_t\left(e^{-\tau\left(t-\frac{\mu}{2\tau}\right)^2}f\right).$$

Now from Proposition 3.1.2, we have

$$\begin{aligned}
I_n^{\nu,\alpha} &= \left(-\partial_z + I_1^{\nu,\alpha}\right)^n\cdot(1)\\
&= \left(A_x^{\nu,\alpha,\xi} - iB_y^{\nu,\alpha,\xi}\right)^n\cdot(1)\\
&= \sum_{k=0}^{n}(-i)^k\binom{n}{k}\left(A_x^{\nu,\alpha,\xi}\right)^{n-k}\cdot(1)\left(B_y^{\nu,\alpha,\xi}\right)^k\cdot(1)\\
&= \frac{1}{2^n}\sum_{k=0}^{n}(-i)^k\binom{n}{k}H_{n-k}^{\nu-2\alpha}\left(x-\frac{\Re(\xi)}{\nu-2\alpha}\right)H_k^{\nu+2\alpha}\left(y+\frac{\Im(\xi)}{\nu+2\alpha}\right).
\end{aligned}$$

∎

We conclude this section by proving a Nielsen identity for these polynomials, which consists of expressing $I_n^{\nu,\alpha}$ as a weighted sum of a product of the same polynomials. Namely, we have

Theorem 3.1.12. *Nielsen identity for the polynomials $I_n^{\nu,\alpha}$ reads*

$$I_{m+n}^{\nu,\alpha} = m!n!\sum_{k=0}^{\min(m,n)}\frac{(2\alpha)^k}{k!}\frac{I_{m-k}^{\nu,\alpha}}{(m-k)!}\frac{I_{n-k}^{\nu,\alpha}}{(n-k)!}. \tag{3.19}$$

Proof. Starting from the Rodrigues formula (14), we can easily see that $I^{\nu,\alpha}_{m+n}$ takes the form

$$I^{\nu,\alpha}_{m+n} = (-1)^m e^{\nu z\bar{z} - \alpha z^2 - \xi z} \partial_z^m \left(e^{-\nu z\bar{z} + \alpha z^2 + \xi z} I^{\nu,\alpha}_n \right).$$

Now, by means of the Leibniz formula combined with (14) and

$$\partial_z^k I^{\nu,\alpha}_m(z,\bar{z}|\xi) = \frac{m!(-2\alpha)^k}{(m-k)!} I^{\nu,\alpha}_{m-k}, \tag{3.20}$$

which follows by induction starting from (3.8), we obtain

$$I^{\nu,\alpha}_{m+n} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} I^{\nu,\alpha}_{m-k} \partial_z^k \left(I^{\nu,\alpha}_n \right) = m!n! \sum_{k=0}^{\min(m,n)} \frac{(2\alpha)^k}{k!} \frac{I^{\nu,\alpha}_{m-k}}{(m-k)!} \frac{I^{\nu,\alpha}_{n-k}}{(n-k)!}.$$

This completes the proof. ∎

Remark 3.1.13. We recover from (3.19) the three term recurrence formula (3.11) satisfied by the polynomials $I^{\nu,\alpha}_n$ by taking $m = 1$ with $n \geq 1$.

Remark 3.1.14. The most discussed algebraic results concerning the polynomials $I^{\nu,\alpha}_n$, can be recovered easily by means of the well–known properties of the real Hermite polynomials $H_n(x)$ thanks to Corollary 3.1.9 or also Proposition 3.1.11. This is the case of the identity (3.12) as well as Theorem 3.1.12, whose proof can also be handled by means of Corollary 3.1.9 above combined with the standard Nielsen identity for the single real Hermite polynomials, or also using (3.2). The same observation holds true for Theorem 3.2.1 below. However, the analytic properties of these polynomials are far from to be derived by employing Corollary 3.1.9 as will be clarified in the following sections (see Sections 3.4,3.5 and 3.6).

3.2 Generating functions

The first generating function we deal with is a exponential one one.

Theorem 3.2.1. *The polynomials $I^{\nu,\alpha}_n$ satisfy the generating function identity*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} I^{\nu,\alpha}_n = e^{\alpha t^2 + t I^{\nu,\alpha}_1}. \tag{3.1}$$

Proof. Notice first that we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!} I^{\nu,\alpha}_n &= e^{-\alpha z^2 - \xi z} \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} (\partial_z - \nu\bar{z})^n \left(e^{\alpha z^2 + \xi z} \right) \\
&= e^{-\alpha z^2 - \xi z} e^{-t\partial_z + \nu t\bar{z}} \left(e^{\alpha z^2 + \xi z} \right) \\
&= e^{-\alpha z^2 - \xi z} e^{\nu t\bar{z}} \exp(-t\partial_z) \left(e^{\alpha z^2 + \xi z} \right).
\end{aligned}$$

Now, in view of Lemma 3.2.5 and making appeal to the usual generating function of the Hermite polynomials ([99,107]):

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x) = e^{-t^2+2tx},$$

it follows

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} I_n^{\nu,\alpha} = e^{\nu t\bar{z}} \left(\sum_{k=0}^{\infty} \frac{(i\alpha^{1/2}t)^k}{k!} H_k \left(i\alpha^{1/2}z + \frac{i\xi}{2\alpha^{1/2}} \right) \right) = e^{\alpha t^2 - (2\alpha z - \nu\bar{z} + \xi)t}.$$

This ends the proof. ∎

The next generating function generalizes the previous one. Its proof is based essentially on Nielsen identity. Namely, we assert

Theorem 3.2.2. *We have*

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!} I_{m+n}^{\nu,\alpha} = e^{\alpha(u+v)^2 + (u+v)I_1^{\nu,\alpha}}. \tag{3.2}$$

Proof. In view of (3.19), we can write the right hand–side of (3.2) as follows

$$\begin{aligned}
\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!} I_{m+n}^{\nu,\alpha} &= \sum_{m,n=0}^{\infty} u^m v^n \sum_{k=0}^{m} \frac{(2\alpha)^k}{k!} \frac{I_{m-k}^{\nu,\alpha}}{(m-k)!} \frac{I_{n-k}^{\nu,\alpha}}{(n-k)!} \\
&= \sum_{m=0}^{\infty} u^m \sum_{k=0}^{m} \frac{(2\alpha v)^k}{k!} \frac{I_{m-k}^{\nu,\alpha}}{(m-k)!} \left(\sum_{n=0}^{\infty} \frac{v^{n-k} I_{n-k}^{\nu,\alpha}}{(n-k)!} \right) \\
&= \sum_{m=0}^{\infty} u^m \sum_{k=0}^{m} \frac{(2\alpha v)^k}{k!} \frac{I_{m-k}^{\nu,\alpha}}{(m-k)!} \left(\sum_{j=0}^{\infty} v^j \frac{I_j^{\nu,\alpha}}{j!} \right).
\end{aligned}$$

According to (3.1), this leads to

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!} I_{m+n}^{\nu,\alpha} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(2\alpha v)^k}{k!} u^m \frac{I_{m-k}^{\nu,\alpha}}{(m-k)!} e^{\alpha v^2 + v I_1^{\nu,\alpha}}.$$

Now, by interchanging the order of summation in the double sum,

$$\sum_{m=0}^{\infty} \sum_{j=0}^{m} T_{j,m} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} T_{j,j+k},$$

it follows

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!} I_{m+n}^{\nu,\alpha} = \sum_{k=0}^{\infty} \frac{(2\alpha uv)^k}{k!} \sum_{j=0}^{\infty} \frac{u^j I_j^{\nu,\alpha}}{j!} e^{\alpha v^2 + v I_1^{\nu,\alpha}}.$$

Using again (3.1), we obtain

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!} I^{\nu,\alpha}_{m+n} = e^{2\alpha uv} e^{\alpha u^2 + uI^{\nu,\alpha}_1} e^{\alpha v^2 + vI^{\nu,\alpha}_1} = e^{\alpha(u+v)^2 + (u+v)I^{\nu,\alpha}_1}.$$

∎

Remark 3.2.3. For $u = 0$ or $v = 0$, the identity (3.2) reduces further to (3.1).

The third generating function in this section is the following

Theorem 3.2.4. *We have the following identity*

$$\sum_{j=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} (i\alpha^{1/2})^{j-k} \frac{\xi^j}{\nu^j j!} H_{n-k}(i\alpha^{1/2} z) H^{\nu}_{j,k}(z,\overline{z}) = e^{\xi z} I^{\alpha,\xi}_n(z). \tag{3.3}$$

Proof. Direct computation making use the Leibniz formula infers

$$\frac{\partial^n}{\partial z^n}(e^{-\nu|z|^2+\xi z} e^{\alpha z^2}) = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k}{\partial z^k}(e^{-\nu|z|^2+\xi z}) \frac{\partial^{n-k}}{\partial z^{n-k}}(e^{\alpha z^2}).$$

By expanding $e^{\xi z}$ in power series and making use of Definition 8 of the holomorphic Hermite polynomials, we get

$$\begin{aligned}
\frac{\partial^n}{\partial z^n}(e^{-\nu|z|^2+\xi z} e^{\alpha z^2}) &= \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k}{\partial z^k}\left(\sum_{j=0}^{\infty} \frac{\xi^j}{j!} z^j e^{-\nu|z|^2}\right) (-i\alpha^{1/2})^{n-k} H_{n-k}(i\alpha^{1/2} z) e^{\alpha z^2} \\
&= e^{\alpha z^2} \sum_{j=0}^{\infty} \frac{\xi^j}{j!\nu^j} \left(\sum_{k=0}^{n} \binom{n}{k} (-i\alpha^{1/2})^{n-k} \frac{\partial^k}{\partial z^k}(\nu^j z^j e^{-\nu|z|^2})\right) H_{n-k}(i\alpha^{1/2} z) \\
&= (-1)^n e^{-\nu|z|^2+\alpha z^2} \sum_{j=0}^{\infty} \frac{\xi^j}{j!\nu^j} \sum_{k=0}^{n} \binom{n}{k} (i\alpha^{1/2})^{n-k} e^{\nu|z|^2} H^{\nu}_{j,k}(z,\overline{z}) H_{n-k}(i\alpha^{1/2} z).
\end{aligned}$$

Therefore,

$$I^{\alpha,\xi}_n(z) = e^{-\xi z} \sum_{j=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} (i\alpha^{1/2})^{n-k} \frac{(\xi)^j}{\nu^j j!} H_{n-k}(i\alpha^{1/2} z) H^{\nu}_{j,k}(z,\overline{z}).$$

∎

The last generating function in this section shows that the polynomials $I^{\nu,\alpha}_n(z,\overline{z}|\xi)$ can be generated from the ξ–holomorphic Hermite polynomials $H_n(\xi)$ and the polyanalytic Hermite polynomials $H_{m,n}(z,\overline{z})$. To this end, we use a variant (analytic continuation) of the generating function of the real Hermite polynomials whichcan be stated as follows.

Lemma 3.2.5. *The explicit expression of the k–th z–derivative of $e^{\alpha z^2+\xi z}$ in terms of the usual*

Hermite polynomials $H_k(z)$ is given by

$$\partial_z^k e^{\alpha z^2+\xi z} = (-i)^k \alpha^{k/2} H_k\left(i\alpha^{1/2} z + \frac{i\xi}{2\alpha^{1/2}}\right) e^{\alpha z^2+\xi z}. \tag{3.4}$$

Subsequently, we have

$$e^{\alpha z^2+\xi z} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \alpha^{\frac{n}{2}} H_n\left(\frac{i\xi}{2\alpha^{1/2}}\right) z^n. \tag{3.5}$$

Theorem 3.2.6. *We have the generating function*

$$\sum_{k=0}^{\infty} \frac{(-i)^k \alpha^{k/2}}{\nu^k k!} H_k\left(\frac{i\xi}{2\alpha^{1/2}}\right) H^{\nu}_{k,m}(z,\overline{z}) = I^{\nu,\alpha}_m(z,\overline{z}|\xi) e^{\alpha z^2+\xi z}. \tag{3.6}$$

Proof. The proof follows easily starting from the definition of $I^{\nu,\alpha}_n$ and using the expansion series of the entire function $e^{\alpha z^2+\xi z}$ as in (3.5). Indeed, we get

$$\begin{aligned} I^{\nu,\alpha}_n(z,\overline{z}|\xi) &= e^{-\alpha z^2-\xi z} \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \alpha^{\frac{m}{2}} H_m\left(\frac{i\xi}{2\alpha^{1/2}}\right) (-1)^n e^{\nu|z|^2} \partial_z^n (z^m e^{-\nu|z|^2}) \\ &= e^{-\alpha z^2-\xi z} \sum_{m=0}^{\infty} \frac{\left(-\frac{i\alpha^{1/2}}{\nu}\right)^m}{m!} H_m\left(\frac{i\xi}{2\alpha^{1/2}}\right) H^{\nu}_{m,n}(z,\overline{z}). \end{aligned} \tag{3.7}$$

The last equality follows by observing that the rescaled complex Hermite polynomials $H^{\nu}_{m,k}(z,\overline{z}) = G^{\nu,0}_{m,n}(z,\overline{z})$ (see (4.1)) can be represented also as $H^{\nu}_{k,m}(z,\overline{z}) = (-1)^m e^{\nu|z|^2} \partial_z^m (z^k e^{-\nu|z|^2})$. ∎

Remark 3.2.7. The identity (3.6) states that the polynomials $I^{\nu,\alpha}_n(z,\overline{z}|\xi)$ appear as the bilinear generating function of the polynomials $H^{\nu}_{m,n}$ and H_n. This fact can be used to recover the result of Corollary 3.1.9 giving the explicit expression of $I^{\nu,\alpha}_n(z,\overline{z}|\xi)$ to Theorem 2.1 in [20].

3.3 Orthogonality

We begin by considering the case of $\xi = 0$.

Theorem 3.3.1. *Let $\nu > 0$ and $\alpha \in \mathbb{R}$ such that $2|\alpha| < \nu$. Then, the polynomials $I^{\nu,\alpha}_m(z,\overline{z}|0)$ satisfy the orthogonality property*

$$\int_{\mathbb{C}} I^{\nu,\alpha}_m(z,\overline{z}|0) \overline{I^{\nu,\alpha}_n(z,\overline{z}|0)} e^{-\nu|z|^2+\alpha(z^2+\overline{z}^2)} d\lambda(z) = \frac{\pi \nu^n n!}{\sqrt{\nu^2-4\alpha^2}} \delta_{n,m}. \tag{3.1}$$

Proof. Under the assumption $2|\alpha| < \nu$ and keeping in mind the result of Proposition 3.1.11 as well as the orthogonality of the rescaled real Hermite polynomials H^{τ}_k from (3.17) in the

Hilbert space $L^2(\mathbb{R}, e^{-\tau t^2}dt)$,

$$\int_{\mathbb{R}} H_j^\tau(t)H_k^\tau(t)e^{-\tau t^2}dt = \left(\frac{\pi}{\tau}\right)^{1/2} 2^k\tau^k k!,$$

we get

$$\begin{aligned}
&\int_{\mathbb{C}} I_m^{\nu,\alpha}(z,\overline{z}|0)\overline{I_n^{\nu,\alpha}(z,\overline{z}|0)}e^{-\nu|z|^2+\alpha(z^2+\overline{z}^2)}d\lambda(z)\\
&= \int_{\mathbb{C}} I_m^{\nu,\alpha}(z,\overline{z}|0)\overline{I_n^{\nu,\alpha}(z,\overline{z}|0)}e^{-(\nu-2\alpha)x^2-(\nu+2\alpha)y^2}dxdy\\
&= \sum_{j=0}^{m}\sum_{k=0}^{n}\frac{(-i)^j(i)^k}{2^{m+n}}\binom{m}{j}\binom{n}{k}\left\|H_{n-k}^{\nu-2\alpha}\right\|^2_{L^{2,\nu-2\alpha}(\mathbb{R})}\left\|H_k^{\nu+2\alpha}\right\|^2_{L^{2,\nu+2\alpha}(\mathbb{R})}\delta_{m-j,n-k}\delta_{j,k}\\
&= \frac{1}{2^{m+n}}\sum_{k=0}^{\min(m,n)}\binom{m}{k}\binom{n}{k}\left\|H_{n-k}^{\nu-2\alpha}\right\|^2_{L^{2,\nu-2\alpha}(\mathbb{R})}\left\|H_k^{\nu+2\alpha}\right\|^2_{L^{2,\nu+2\alpha}(\mathbb{R})}\delta_{m-k,n-k}\\
&= \frac{\pi}{\sqrt{\nu^2-4\alpha^2}}\frac{n!}{2^n}\sum_{k=0}^{n}\binom{n}{k}(\nu-2\alpha)^{n-k}(\nu+2\alpha)^k\delta_{m,n}\\
&= \frac{\pi\nu^n n!}{\sqrt{\nu^2-4\alpha^2}}\delta_{m,n}.
\end{aligned}$$

This completes the proof. ∎

Remark 3.3.2. For $\alpha = 0$ with $\nu > 0$, we recover the classical orthogonality for the monomials $I_n^{\nu,0}(z,\overline{z}|0) = \nu^n\overline{z}^n$.

Remark 3.3.3. The proof we have furnished above is also valid for the general case of arbitrary ξ under the assumption that $2|\alpha| < \nu$, but it is more tedious.

Based on the orthogonal property obtained in [113] for the holomorphic Hermite polynomials $H_n(z)$, to wit

$$\int_{\mathbb{R}^2} H_m(x+iy)H_n(x-iy)e^{-(1-\theta)x^2-\left(\frac{1}{\theta}-1\right)y^2}dxdy = \frac{\theta^{1/2}\pi}{1-\theta}\left(\frac{2(1+\theta)}{1-\theta}\right)^n n!\delta_{m,n}, \quad (3.2)$$

where $0 < \theta < 1$, we can deduce two orthogonality relations for the polynomials $I_n^{0,\alpha}(z,\overline{z}|0)$ corresponding to $\nu = 0 = \xi$ according to $\alpha > 0$ or $\alpha < 0$. The one for $\alpha > 0$ reads

$$\int_{\mathbb{R}^2} I_n^{0,\alpha}(z,\overline{z}|0)\overline{I_m^{0,\alpha}(z,\overline{z}|0)}e^{-\alpha\left(\frac{1}{\theta}-1\right)x^2-\alpha(1-\theta)y^2}dxdy = \frac{\theta^{1/2}\pi}{\alpha(1-\theta)}\left(\frac{2\alpha(1+\theta)}{1-\theta}\right)^n n!\delta_{m,n}, \quad (3.3)$$

since in this case $I_n^{0,\alpha}(z,\overline{z}|0) = (i\sqrt{\alpha})^n H_n(i\sqrt{\alpha}z)$.

We establish below an orthogonal property for $I_n^{\nu,\alpha}(z,\overline{z}|\xi)$ for arbitrary $\nu > 0$ and $\xi \in \mathbb{C}$, generalizing (3.1) as well as (3.2). To this end, for given reals $a, b > 0$, we consider the weight function

$$\omega_{\nu,\alpha,\xi}^{a,b}(z,\overline{z}) = e^{-A_{\nu,\alpha}^{a,b}|z|^2-B_{\nu,\alpha}^{a,b}\left(z^2+\overline{z}^2\right)+2\Re(C_{\nu,\alpha,\xi}^{a,b}z)}e^{-a\Re(\xi)^2-b\Im(\xi)^2}$$

where the quantities $A^{a,b}_{\nu,\alpha}$, $B^{a,b}_{\nu,\alpha}$ and $C^{\nu,\alpha,\xi}_{a,b}$ are given by

$$\begin{aligned} A^{a,b}_{\nu,\alpha} &:= \frac{(\nu-2\alpha)^2 a + (\nu+2\alpha)^2 b}{2} \\ B^{a,b}_{\nu,\alpha} &:= \frac{(\nu-2\alpha)^2 a - (\nu+2\alpha)^2 b}{4} \\ C^{\nu,\alpha,\xi}_{a,b} &:= a(\nu-2\alpha)\Re(\xi) + ib(\nu+2\alpha)\Im(\xi). \end{aligned}$$

Theorem 3.3.4. *Let $a,b>0$ such that $4\alpha ab = a-b$. Then, the polynomials $I^{\nu,\alpha}_n(z,\bar{z}|\xi)$ satisfy the orthogonality property*

$$\int_{\mathbb{C}} I^{\nu,\alpha}_m(z,\bar{z}|\xi)\overline{I^{\nu,\alpha}_n(z,\bar{z}|\xi)}\omega^{a,b}_{\nu,\alpha,\xi}(z,\bar{z})d\lambda(z) = \frac{\pi}{\sqrt{ab}|\nu^2-4\alpha^2|}\left(\frac{a+b}{2ab}\right)^n n!\delta_{m,n}. \tag{3.4}$$

Proof. Theorem 3.2.1 yields

$$\begin{aligned} S^{\nu,\alpha,\xi}_{m,n}(u,v|z,\bar{z}) &= \sum_{m,n}^{\infty} \frac{u^m v^n}{m!n!} I^{\nu,\alpha}_m(z,\bar{z}|\xi)\overline{I^{\nu,\alpha}_n(z,\bar{z}|\xi)} \\ &= e^{\alpha(u^2+v^2)+uI^{\nu,\alpha}_1(z,\bar{z}|\xi)+v\overline{I^{\nu,\alpha}_1(z,\bar{z}|\xi)}} \\ &= e^{\alpha(u^2+v^2)+[(\nu-2\alpha)x-\Re(\xi)](u+v)-i[(\nu+2\alpha)y+\Im(\xi)](u-v)} \\ &= e^{\alpha(u^2+v^2)+(u+v)X-i(u-v)Y}, \end{aligned}$$

where we have set $X=(\nu-2\alpha)x-\Re(\xi)$ and $Y=(\nu+2\alpha)y+\Im(\xi)$ for given $z=x+iy$. Now, if we denote the left–hand side of (3.4) by $T^{\nu,\alpha}_{m,n}(\xi)$, then

$$T^{\nu,\alpha}_{m,n}(\xi) = \frac{1}{|\nu^2-4\alpha^2|}\int_{\mathbb{R}^2} I^{\nu,\alpha}_m(z,\bar{z}|\xi)\overline{I^{\nu,\alpha}_n(z,\bar{z}|\xi)}\omega^{a,b}_{\nu,\alpha,\xi}(z,\bar{z})dXdY$$

with $z=z(X,Y)$ and $\bar{z}=\overline{z(X,Y)}$. Subsequently, we have

$$\begin{aligned} \sum_{m,n=0}^{\infty}\frac{u^m v^n}{m!n!}T^{\nu,\alpha}_{m,n}(\xi) &= \frac{1}{|\nu^2-4\alpha^2|}\int_{\mathbb{R}^2} S^{\nu,\alpha,\xi}_{m,n}(u,v|z,\bar{z})\omega^{a,b}_{\nu,\alpha,\xi}(z,\bar{z})dXdY \\ &= \frac{e^{\alpha(u^2+v^2)}}{|\nu^2-4\alpha^2|}\int_{\mathbb{R}^2} e^{-aX^2+(u+v)X-bY^2-i(u-v)Y}dXdY \\ &= \frac{\pi}{\sqrt{ab}|\nu^2-4\alpha^2|}e^{\frac{4\alpha ab+b-a}{4ab}(u^2+v^2)}e^{\frac{a+b}{2ab}uv} \\ &= \frac{\pi}{\sqrt{ab}|\nu^2-4\alpha^2|}.e^{\frac{a+b}{2ab}uv}, \end{aligned}$$

The third equality is obtained making use of the well–known Gaussian integral

$$\int_{\mathbb{R}} e^{-\tau y^2+\zeta y}dy = \left(\frac{\pi}{\tau}\right)^{1/2} e^{\frac{\zeta^2}{4\tau}}; \qquad \tau>0,\ \zeta\in\mathbb{C}, \tag{3.5}$$

while the last equality readily follows since $4\alpha ab + b - a = 0$. Subsequently, we obtain

$$S_{m,n}^{\nu,\alpha}(\xi) = \frac{\pi}{\sqrt{ab}|\nu^2 - 4\alpha^2|}\left(\frac{a+b}{2ab}\right)^n n!\delta_{m,n}.$$

This completes our check of (3.4). ∎

Remark 3.3.5. As example of pairs (a,b); $a,b > 0$ satisfying the condition $4\alpha ab = a - b$, we can consider $a = (\nu - 2\alpha)^{-1}$ and $b = (\nu + 2\alpha)^{-1}$. Therefore, we have $\nu > 2|\alpha|$ and the corresponding $A_{\nu,\alpha}^{a,b}$, $B_{\nu,\alpha}^{a,b}$ and $C_{a,b}^{\nu,\alpha,\xi}$ are given by $A_{\nu,\alpha}^{a,b} = \nu$, $B_{\nu,\alpha}^{a,b} = -\alpha$ and $C_{a,b}^{\nu,\alpha,\xi} = \xi$, so that the orthogonality (3.4) reduces to

$$\int_{\mathbb{C}} I_m^{\nu,\alpha}(z,\overline{z}|\xi)\overline{I_n^{\nu,\alpha}(z,\overline{z}|\xi)}e^{-\nu|z|^2+\alpha\left(z^2+\overline{z}^2\right)+2\Re(\xi z)}d\lambda(z) = \frac{\pi\nu^n n!}{\sqrt{\nu^2-4\alpha^2}}e^{\nu|\xi|^2-\alpha\left(\xi^2+\overline{\xi}^2\right)}\delta_{m,n}.$$

which for $\xi = 0$ leads to the one obtained in Theorem 3.3.1.

Remark 3.3.6. For $\nu = 0 = \xi$ and $\alpha > 0$, the identity (3.4) reduces further to (3.3) by taking

$$a = \frac{1}{4\alpha}\left(\frac{1}{\theta} - 1\right) \quad \text{and} \quad b = \frac{1}{4\alpha}(1-\theta)$$

with $0 < \theta < 1$.

3.4 Integral representations

In virtue of Theorem 3.1.8, we obtain the following integral representation of the polynomials $I_n^{\nu,\alpha}(z,\overline{z}|\xi)$.

Proposition 3.4.1. *For every $\nu > 0$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, we have*

$$I_n^{\nu,\alpha}(z,\overline{z}|\xi) = \left(\frac{1}{\alpha\pi}\right)^{1/2} 2^n \int_{\mathbb{R}} t^n e^{-\frac{1}{4\alpha}\left(2t - I_1^{\nu,\alpha}(z,\overline{z}|\xi)\right)^2} dt. \tag{3.1}$$

Proof. By means of the explicit formula for the Gaussian integral (3.5), we can write

$$e^{\frac{\left(I_1^{\nu,\alpha}(z,\overline{z}|\xi)\right)^2}{4\alpha}} = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} e^{-\alpha t^2 + tI_1^{\nu,\alpha}(z,\overline{z}|\xi)} dt. \tag{3.2}$$

The integral in the right–hand side converges uniformly on every disc $D(0,r) \subset \mathbb{C}$ and one can repeatedly differentiate it with respect to z. Hence, by (3.14) we obtain

$$\begin{aligned} I_n^{\nu,\alpha}(z,\bar{z}|\xi) &= (-1)^n \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\left(I_1^{\nu,\alpha}(z,\bar{z}|\xi)\right)^2}{4\alpha}} \int_{\mathbb{R}} \frac{\partial^n}{\partial z^n} e^{-\alpha t^2 + t I_1^{\nu,\alpha}(z,\bar{z}|\xi)} dt \\ &= (-1)^n \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} (-2\alpha t)^n e^{-\frac{1}{\alpha}\left((\alpha t)^2 - \alpha t I_1^{\nu,\alpha}(z,\bar{z}|\xi) + \frac{\left(I_1^{\nu,\alpha}(z,\bar{z}|\xi)\right)^2}{4}\right)} dt \\ &= \left(\frac{1}{\alpha\pi}\right)^{\frac{1}{2}} 2^n \int_{\mathbb{R}} u^n e^{-\frac{1}{\alpha}\left(u - \frac{I_1^{\nu,\alpha}(z,\bar{z}|\xi)}{2}\right)^2} du. \end{aligned}$$

This completes our check of (3.1). ∎

The next result is a consequence of Theorem 3.2.6 combined with the integral representation of the complex Hermite polynomials.

Theorem 3.4.2. *For $\nu > 0$ and $a, b \in \mathbb{C}$ such that $ab > 0$, we have*

$$I_n^{\nu,\alpha}(z,\bar{z}|\xi) = \left(\frac{ab}{\nu\pi}\right) e^{\nu|z|^2 - \alpha z^2 - \xi z} \int_{\mathbb{C}} (b\bar{\zeta})^n e^{-\frac{ab}{\nu}|\zeta|^2 + \frac{a^2\alpha}{\nu^2}\zeta^2 - \frac{a\xi}{\nu}\zeta} e^{a\zeta\bar{z} - b\bar{\zeta}z} d\lambda(\zeta). \tag{3.3}$$

More particularly, we have

$$I_n^{\nu,\alpha}(z,\bar{z}|\xi) = \left(\frac{\nu}{\pi}\right) e^{\nu|z|^2 - \alpha z^2 - \xi z} \int_{\mathbb{C}} (-\nu\bar{\zeta})^n e^{-\nu|\zeta|^2 + \alpha\zeta^2 + \xi\zeta} e^{2i\nu\Im\langle z,\zeta\rangle} d\lambda(\zeta). \tag{3.4}$$

Proof. The result follows by a tedious but straightforward computation. In fact, starting from Theorem 3.2.6 and using the integral representation of $H^{\nu}_{m,n}(z,\bar{z})$ given by Theorem 3.1 in [20], to wit

$$H^{\nu}_{m,n}(z;\bar{z}) = \left(\frac{ab}{\nu\pi}\right) (-a)^m (b)^n \int_{\mathbb{C}} \zeta^m \bar{\zeta}^n e^{\nu|z|^2 - \frac{ab}{\nu}|\zeta|^2 + a\zeta\bar{z} - b\bar{\zeta}z} d\lambda(\zeta) \tag{3.5}$$

(valid for $\nu > 0$ and $a, b \in \mathbb{C}$ such that $ab > 0$), we obtain

$$\begin{aligned} I_n^{\nu,\alpha}(z,\bar{z}|\xi) &= \left(\frac{ab}{\nu\pi}\right) e^{-\alpha z^2 - \xi z} \int_{\mathbb{C}} (b\bar{\zeta})^n e^{\nu|z|^2 - \frac{ab}{\nu}|\zeta|^2 + a\zeta\bar{z} - b\bar{\zeta}z} \left(\sum_{m=0}^{\infty} \frac{\left(\frac{-ia\alpha^{1/2}\zeta}{\nu}\right)^m}{m!} H_m\left(-\frac{i\xi}{2\alpha^{1/2}}\right)\right) d\lambda(\zeta) \\ &= \left(\frac{ab}{\nu\pi}\right) e^{\nu|z|^2 - \alpha z^2 - \xi z} \int_{\mathbb{C}} (b\bar{\zeta})^n e^{-\frac{ab}{\nu}|\zeta|^2 + \frac{a^2\alpha}{\nu^2}\zeta^2 - \frac{a\xi}{\nu}\zeta} e^{a\zeta\bar{z} - b\bar{\zeta}z} d\lambda(\zeta). \end{aligned}$$

The particular case of $a = b = -\nu$ gives rise to (3.4). This completes the proof. ∎

Remark 3.4.3. The obtained result (3.4) can also be reproved directly. Indeed, by rewriting $e^{-\nu|z|^2 + \alpha z^2 + \xi z}$ as

$$e^{-\nu|z|^2 + \alpha z^2 + \xi z} = e^{-\frac{\nu^2}{\nu+\alpha}\left(x - \frac{\xi}{2\nu}\right)^2 + (\nu+\alpha)\left(iy + \frac{2\alpha x + \xi}{2(\nu+\alpha)}\right)^2}$$

and next using twice the integral representation of the Gaussian function (3.5), we obtain the integral representation of $e^{-\nu|z|^2+\alpha z^2+\beta z}$, to wit

$$e^{-\nu|z|^2+\alpha z^2+\xi z} = \frac{1}{4\pi\nu}\int_{\mathbb{R}^2} e^{-\frac{1}{4\nu^2}\left((\nu-\alpha)t^2+(\nu+\alpha)s^2\right)+i(yt+xs)+\frac{\xi}{2\nu}(t-is)-\frac{i\alpha}{2\nu^2}ts} dtds$$

under the assumption that $\nu+\alpha>0$ with $z=x+iy$. It can be rewritten in the form

$$e^{-\nu|z|^2+\alpha z^2+\xi z} = \frac{1}{2\pi}\int_{\mathbb{C}} e^{-\nu|\zeta|^2+\alpha\zeta^2+\xi\zeta+2i\nu\Im\langle z,\zeta\rangle} d\lambda(\zeta) \tag{3.6}$$

making the change $\zeta=\frac{t-is}{2\nu}$. Thus (3.4) follows readily by derivation of (3.6).

We conclude this section by realizing the polynomials $I_n^{\nu,\alpha}(z,\overline{z}|\xi)$ as the image of the real Hermite function $h_n^\nu(t)=\sqrt{\nu}^n e^{-\frac{\nu}{2}t^2}H_n(\sqrt{\nu}t)$ by rescaled Fourier–Wigner transform defined on $L^2(\mathbb{R})$ by [39,50,110,119]

$$\mathcal{V}^\nu(f,g)(x,y) = \left(\frac{\nu}{2\pi}\right)^{1/2}\int_{\mathbb{R}} e^{i\nu\left(t-\frac{x}{2}\right)y} f(t)\overline{g(t-x)}dt$$

with respect to a special window function g that we determine explicitly. Thus, for $f\in L^2(\mathbb{R})$ and) $z=x+iy$ we define

$$\mathcal{W}^\nu_{\alpha,\xi}(f)(z,\overline{z}) := \frac{(-1)^n}{2^n}\left(\frac{2\nu}{\nu+2\alpha}\right)^{1/2} e^{\frac{\nu}{2}|z|^2}\mathcal{V}^{2\nu}\left(M^\nu_\alpha, f\right)(x,y),$$

where we have set

$$M^\nu_\alpha(y) := e^{-\frac{\xi^2}{2(\nu+2\alpha)}}\exp\left(-\frac{\nu}{\nu+2\alpha}\left((\nu-2\alpha)y^2-2\xi y\right)\right). \tag{3.7}$$

More explicitly,

$$\begin{aligned}\mathcal{W}^\nu_{\alpha,\xi}(f)(z,\overline{z}) = {} & \frac{(-1)^n}{2^n}\left(\frac{2\nu^2}{(\nu+2\alpha)\pi}\right)^{1/2} e^{\frac{\nu}{2}|z|^2-\alpha z^2-\xi z}e^{-\frac{\xi^2}{2(\nu+2\alpha)}} \\ & \times\int_{\mathbb{R}} e^{2i\nu\left(t-\frac{x}{2}\right)y}\exp\left(-\frac{\nu}{\nu+2\alpha}\left((\nu-2\alpha)t^2-2t\xi\right)\right)f(t-x)dt.\end{aligned} \tag{3.8}$$

Theorem 3.4.4. *Let ν and α be such that $2|\alpha|<\nu$. Then, for every z, we have*

$$\mathcal{W}^\nu_{\alpha,\xi}(h_n^{2\nu})(z,\overline{z}) = I_n^{\nu,\alpha}(z,\overline{z}|\xi).$$

Proof. Observe first that the polynomials $H^\nu_{m,n}(z,\overline{z})$ in can be realized as

$$H^\nu_{m,n}(z,\overline{z}) = (-1)^n\frac{\sqrt{2}}{2^{m+n}}e^{\frac{\nu}{2}|z|^2}\mathcal{V}^{2\nu}(h_m^{2\nu},h_n^{2\nu})(x,y);\quad z=x+iy. \tag{3.9}$$

This follows by straightforward computation using [7, Theorem 3.1] as well as the fact that

$h^{\tau}_{m,n}(z,\overline{z}) := \tau^{(m+n)/2}h_{m,n}(\tau^{1/2}z,\tau^{1/2}\overline{z})$. Now, by Theorem 3.2.6, we get

$$\begin{aligned}
I^{\nu,\alpha}_n(z,\overline{z}|\xi)e^{\alpha z^2+\xi z} &= (-1)^n\frac{\sqrt{2}}{2^n}e^{\frac{\nu}{2}|z|^2}\sum_{k=0}^{\infty}\frac{(-i)^k\alpha^{k/2}}{2^k\nu^k k!}H_k\left(\frac{i\xi}{2\alpha^{1/2}}\right)\mathcal{V}^{2\nu}(h^{2\nu}_k,h^{2\nu}_n)(x,y)\\
&= (-1)^n\frac{\sqrt{2}}{2^n}e^{\frac{\nu}{2}|z|^2}\mathcal{V}^{2\nu}\left(\sum_{k=0}^{\infty}\frac{(-i)^k\alpha^{k/2}}{2^k\nu^k k!}H_k\left(\frac{i\xi}{2\alpha^{1/2}}\right)h^{2\nu}_k,h^{2\nu}_n\right)(x,y).
\end{aligned}$$

Making use of Mehler formula ([85,99]) for the rescaled Hermite functions h^{τ}_n, to wit

$$\sum_{k=0}^{\infty}\frac{\lambda^k h^{\tau}_k(X)h^{\tau}_k(Y)}{2^k\tau^k k!} = \frac{1}{\sqrt{1-\lambda^2}}\exp\left(-\frac{\tau(1+\lambda^2)}{2(1-\lambda^2)}(X^2+Y^2)+\frac{2\tau\lambda}{1-\lambda^2}XY\right) \tag{3.10}$$

valid for $|\lambda|<1$, with $\tau=2\nu$, $X=\frac{i\xi}{2(2\nu\alpha)^{1/2}}$ and $\lambda=-i\left(\frac{2\alpha}{\nu}\right)^{1/2}$, we get

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{(-i)^k\alpha^{k/2}}{2^k\nu^k k!}H_k\left(\frac{i\xi}{2\alpha^{1/2}}\right)h^{2\nu}_k(Y) &= e^{-\frac{\xi^2}{2.4\alpha}}\sum_{k=0}^{\infty}\frac{\left(-i\left(\frac{2\alpha}{\nu}\right)^{1/2}\right)^k}{2^k(2\nu)^k k!}h^{2\nu}_k\left(\frac{i\xi}{2(2\nu\alpha)^{1/2}}\right)h^{2\nu}_k(Y)\\
&= \left(\frac{\nu}{\nu+2\alpha}\right)^{1/2}e^{-\frac{\xi^2}{2(\nu+2\alpha)}}\exp\left(-\frac{\nu\left((\nu-2\alpha)Y^2-2\xi Y\right)}{\nu+2\alpha}\right)\\
&= \left(\frac{\nu}{\nu+2\alpha}\right)^{1/2}M^{\nu}_{\alpha}(Y),
\end{aligned}$$

where M^{ν}_{α} is exactly the function given through (3.7) under the assumption that $2|\alpha|<\nu$. Therefore, we arrive at

$$I^{\nu,\alpha}_n(z,\overline{z}|\xi)e^{\alpha z^2+\xi z} = \frac{(-1)^n}{2^n}\left(\frac{2\nu}{\nu+2\alpha}\right)^{1/2}e^{\frac{\nu}{2}|z|^2}\mathcal{V}^{2\nu}\left(M^{\nu}_{\alpha},h^{2\nu}_n\right)(x,y).$$

The obtained expression reads equivalently as (3.8). This completes the proof. ∎

Remark 3.4.5. For the particular case of $\alpha=0=\xi$ and $\nu=1/2$, the transform $\mathcal{W}^{\nu}_{\alpha,\xi}$ in (3.8) reduces further to the Segal–Bargmann transform $\mathcal{B}$ from $L^2(\mathbb{R})$ onto the Bargmann space $\mathcal{F}^{2,1/2}(\mathbb{C})=\mathcal{H}ol(\mathbb{C})\cap L^2(\mathbb{C};e^{-\frac{|z|^2}{2}}dxdy)$. In fact, we have

$$\begin{aligned}
\mathcal{W}^{1/2}_{0,0}(h_n)(z,\overline{z}) &= \frac{(-1)^n}{2^n\sqrt{\pi}}e^{\frac{\nu}{4}|z|^2}\int_{\mathbb{R}}e^{\frac{i}{2}(2t+x)y}e^{-\frac{1}{2}(t+x)^2}h_n(t)dt\\
&= \frac{(-1)^n}{2^n\sqrt{\pi}}\int_{\mathbb{R}}e^{-\frac{1}{2}\left(t^2+2t\overline{z}+\frac{\overline{z}^2}{2}\right)}h_n(t)dt\\
&= \frac{(-1)^n}{2^n}\mathcal{B}(h_n)(-\overline{z}),
\end{aligned}$$

so that the result of our Theorem which reads $\mathcal{W}^{1/2}_{0,0}(h_n)(z,\overline{z})=I^{1/2,0}_n(z,\overline{z}|0)=(1/2)^n\overline{z}^n$ is exactly the reproducing property for the monomials by $\mathcal{B}$, $\mathcal{B}(h_n)(-\overline{z})=(-1)^n\overline{z}^n$ (see e.g. [50]).

3.5 Polyanalyticity and partial differential equations

The introduced polynomials are a special subclass of polyanalytic functions on the complex plane. As counterpart to the $H^{\nu}_{m,n}(z,\bar{z})$ which are polyanalytic of order $n+1$ and anti–polyanalytic of order $m+1$, the polyanlyticity and the anti–polyanalyticity of the polynomials $I^{\nu,\alpha}_n(z,\bar{z}|\xi)$ have the same order. This is due to the fact that

$$\partial^{n+1}_{\bar{z}} I^{\nu,\alpha}_n(z,\bar{z}|\xi) = 0 = \partial^{n+1}_{z} I^{\nu,\alpha}_n(z,\bar{z}|\xi),$$

which can be handled easily using (3.5) and (3.8) keeping in mind the fact $I^{\nu,\alpha}_0(z,\bar{z}|\xi) = 1$. Indeed, by induction we have

$$\partial^{k}_{\bar{z}} I^{\nu,\alpha}_n(z,\bar{z}|\xi) = \frac{n!\nu^k}{(n-k)!} I^{\nu,\alpha}_{n-k} \quad \text{and} \quad \partial^{k}_{z} I^{\nu,\alpha}_n = \frac{(-2\alpha)^k n!}{(n-k)!} I^{\nu,\alpha}_{n-k} \tag{3.1}$$

for every nonnegative integer $k \leq n$. This can also be recovered from (14), since the polyanalyticity of a complex–valued function f is equivalent to f being of the form

$$f(z,\bar{z}) = (-1)^n e^{\nu z\bar{z}} \partial^n_z \left(e^{-\nu z\bar{z}} h \right)$$

for some nonnegative integer n and holomorphic function h (see e.g. [4,15]). Subsequently, by means of [25] there exist certain holomorphic functions h_k; $k = 0, 1, \cdots, n$ such that

$$I^{\nu,\alpha}_n(z,\bar{z}|\xi) = \bar{z}^n h_n + \cdots + \bar{z} h_1 + h_0. \tag{3.2}$$

The next result gives the explicit expressions of the holomorphic component h_k of $I^{\nu,\alpha}_n$.

Theorem 3.5.1. *The polynomials $I^{\nu,\alpha}_n(z,\bar{z}|\xi)$ are connected to the holomorphic Hermite polynomials by*

$$I^{\nu,\alpha}_n(z,\bar{z}|\xi) = n! \sum_{k=0}^{n} \frac{\nu^k}{k!} \frac{(i)^{n-k}\alpha^{n-k/2}}{(n-k)!} H_{n-k}\left(i\alpha^{1/2} z + \frac{i\xi}{2\alpha^{1/2}} \right) \bar{z}^k \tag{3.3}$$

Proof. By applying the binomial formula to (3.2) and taking into account (3.4), we obtain

$$\begin{aligned} I^{\nu,\alpha}_n(z,\bar{z}|\xi) &= (-1)^n e^{-\alpha z^2 - \xi z} \sum_{k=0}^{n} \binom{n}{k} \left(\partial^k_z (e^{\alpha z^2 + \xi z}) \right) (-\nu\bar{z})^{n-k} \\ &= n! \sum_{k=0}^{n} \frac{(i)^k \alpha^{k/2}}{k!} H_k\left(i\alpha^{1/2} z + \frac{i\xi}{2\alpha^{1/2}} \right) \frac{(\nu\bar{z})^{n-k}}{(n-k)!}. \end{aligned}$$

∎

Remark 3.5.2. The k–th holomorphic component of $I^{\nu,\alpha}_n$ in (3.1) is given by

$$h_k(z) = \frac{(i)^{n-k}\alpha^{n-k/2}}{(n-k)!n!} H_{n-k}\left(i\alpha^{1/2} z + \frac{i\xi}{2\alpha^{1/2}} \right).$$

Added to the generalized Cauchy equation $\partial_{\bar{z}}^{n+1} = 0$ satisfied by $I_n^{\nu,\alpha}(z,\bar{z}|\xi)$, we can show that these polynomials are common eigenfunctions of the partial differential operators of Laplacian type defined by

$$\Delta^{\nu}_{\alpha,\xi} := -\partial_z\partial_{\bar{z}} + I_1^{\nu,\alpha}\partial_{\bar{z}} \quad \text{and} \quad \widetilde{\Delta}^{\nu}_{\alpha,\xi} := -\partial_z\partial_{\bar{z}} + I_1^{\nu,\alpha}\partial_z. \tag{3.4}$$

Theorem 3.5.3. *The polynomials $I_n^{\nu,\alpha}(z,\bar{z}|\xi)$ satisfy the partial differential equations*

$$\Delta^{\nu}_{\alpha,\xi} I_n^{\nu,\alpha}(z,\bar{z}|\xi) = \nu n I_n^{\nu,\alpha} \tag{3.5}$$

and

$$\widetilde{\Delta}^{\nu}_{\alpha,\xi} I_n^{\nu,\alpha}(z,\bar{z}|\xi) = -2\alpha n I_n^{\nu,\alpha}. \tag{3.6}$$

Proof. Using (3.5) and (3.9), we get

$$\Delta^{\nu}_{\alpha,\xi} I_n^{\nu,\alpha}(z,\bar{z}|\xi) = \left(-\partial_z + I_1^{\nu,\alpha}\right)\partial_{\bar{z}} I_n^{\nu,\alpha} \overset{(3.5)}{=} \nu n\left(-\partial_z + I_1^{\nu,\alpha}\right) I_{n-1}^{\nu,\alpha} \overset{(3.9)}{=} \nu n I_n^{\nu,\alpha}.$$

The identity (3.5) can be handled by applying $\partial_{\bar{z}}$ to both sides of the recurrence relation (3.4) involving ∂_z, and next use (3.5) in Proposition 3.1.1. The identity (3.6) follows by proceeding in a similar way using (3.8) and (3.9). ∎

Remark 3.5.4. According to the above result, the polynomials $I_n^{\nu,\alpha}$ are also eigenfunctions of

$$-\partial_z^2 \mp \partial_z\partial_{\bar{z}} + I_1^{\nu,\alpha}\partial_z \pm I_1^{\nu,\alpha}\partial_z$$

associated to the eigenvalue $(\nu \mp 2\alpha)n$. In fact, the first order differential operators $\partial_z \pm \partial_{\bar{z}}$ are lowering operators for the polynomials $I_n^{\nu,\alpha}$, satisfying

$$(\partial_z \pm \partial_{\bar{z}}) I_n^{\nu,\alpha}(z,\bar{z}|\xi) = (\nu \mp 2\alpha) n I_{n-1}^{\nu,\alpha} \tag{3.7}$$

Remark 3.5.5. The polynomials $I_n^{\nu,\alpha}$ belong to the kernel of the operator $\nu\partial_z + 2\alpha\partial_{\bar{z}}$.

3.6 Connection to rank–one automorphic functions

In this section, we present an application in the context of the so–called automorphic functions of Landau type with respect to the $\mathbb{Z}$-character $\chi_\beta(k) = e^{2i\pi\beta k}$, i.e., the space of all complex–valued functions satisfying the functional equation

$$f(z+k) = e^{2i\pi\beta k} e^{2\alpha(z+\frac{k}{2})k} f(z) \tag{3.1}$$

for all $k \in \mathbb{Z}$ and $z \in \mathbb{C}$. To this end, let $L^2(\mathbb{C}/\mathbb{Z}, e^{-2\alpha|z|^2}dxdy)$ denote the space of $f : \mathbb{C} \longrightarrow \mathbb{C}$ satisfying (3.1) and subject to the norm boundedness on the strip $\mathbb{C}/\mathbb{Z}$ with respect to the

Gaussian measure

$$||f||^2_{\alpha,\mathbb{Z}} = \int_{\mathbb{C}/\mathbb{Z}} |f(z)|^2 e^{-2\alpha|z|^2} dxdy < +\infty. \tag{3.2}$$

We denote by $\langle\cdot,\cdot\rangle_{\alpha,\mathbb{Z}}$ the associated Hermitian scalar product. Then, it is proved in [58] that the Hermite-like functions

$$\varphi^{\nu,\alpha,\beta}_{m,n}(z,\overline{z}) = (-i)^m H^{\alpha}_m\left(2\Im m(z) + \frac{\pi(\beta+n)}{\alpha}\right) e^{\alpha,\beta}_n(z) \tag{3.3}$$

with $\Im m(z) = \frac{z-\overline{z}}{2i}$, form an orthogonal basis of $L^2(\mathbb{C}/\mathbb{Z}, e^{-2\alpha|z|^2}dxdy)$. A different proof of this result can be given using the first order differential operator $A^*_{2\alpha} = -\partial_z + \nu\overline{z}$ and the corresponding functions

$$\psi^{\nu,\alpha,\beta}_{m,n}(z,\overline{z}) := (A^*_{2\alpha})^m(e^{\alpha,\beta}_n(z)),$$

for $(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}$, where $e^{\alpha,\beta}_n(z) = e^{\alpha z^2 + 2i\pi(\beta+n)z}$ It is contained in the following lemmas

Lemma 3.6.1. *We have*

$$\psi^{\nu,\alpha,\beta}_{m,n}(z,\overline{z}) = I^{\frac{\nu}{2},\beta}_{m,n}(z,\overline{z}) e^{\alpha,\nu}_n = \varphi^{\nu,\alpha,\beta}_{m,n}(z,\overline{z}), \tag{3.4}$$

where

$$I^{\alpha,\beta}_{m,n}(z,\overline{z}) := I^{2\alpha,\alpha}_m(z,\overline{z}|2i\pi(\beta+n)), \tag{3.5}$$

with $\alpha >$, $\beta \in \mathbb{R}$, $m = 0,1,2,\cdots$, and $n \in \mathbb{Z}^+$.

Proof. Since $A^*_{2\alpha}f = -e^{2\alpha|z|^2}\partial_z\left(e^{-2\alpha|z|^2}f\right)$, we get $(A^*)^m_{2\alpha}f = (-1)^m e^{2\alpha|z|^2}\partial^m_z\left(e^{2\alpha|z|^2}f\right)$. Therefore,

$$(A^*_{2\alpha})^m(e^{\alpha,\beta}_n(z)) = (-1)^m e^{2\alpha|z|^2}\partial^m_z\left(e^{-2\alpha|z|^2}e^{\alpha,\beta}_n(z)\right) = I^{2\alpha,\alpha,\beta}_{m,n}(z,\overline{z})\, e^{\alpha,\beta}_n(z)$$

∎

Lemma 3.6.2 ([58])**.** *The functions $e^{\alpha,\beta}_n(z) = e^{\alpha z^2+2i\pi(\beta+n)z}$; $n = 0,1,\cdots$, form a complete system of the theta Bargmann-Fock space $\mathcal{F}^{2,2\alpha}_{\mathbb{Z},\beta}(\mathbb{C})$ of all complex-valued entire functions satisfying* (3.1)*) and belonging to $L^2(\mathbb{C}/\mathbb{Z}, e^{-2\alpha|z|^2}dxdy)$.*

Lemma 3.6.3. *The functions $\psi^{\nu,\alpha,\beta}_{m,n}$ are eigenfunctions of $\Delta_{2\alpha}$ associated to the eigenvalue $2\alpha m$.*

Proof. The result readily follows by induction. It is clear for $m = 0$. Next, if $\Delta_{2\alpha}\psi^{\nu,\alpha,\beta}_{k,n} = 2\alpha k\psi^{\nu,\alpha,\beta}_{k,n}$ is verified for $k \leq m$, we use the fact $\Delta_\nu = A^*_{2\alpha}A = AA^*_{2\alpha}$ to get

$$\Delta_{2\alpha}\psi^{\nu,\alpha,\beta}_{m+1,n} = (A^*A)A^*(A^*)^m(e^{\alpha,\nu}_n) = A^*(\nu+\Delta_\nu)(A^*)^m(e^{\alpha,\nu}_n) = \nu(m+1)\psi^{\nu,\alpha,\beta}_{m+1,n}$$

∎

Lemma 3.6.4. *The functions* $\psi_{m,n}^{\nu,\alpha,\beta}$ *satisfy the functional equation* (3.1) *and are orthogonal in* $L^2(\mathbb{C}/\mathbb{Z}, e^{-2\alpha|z|^2}dxdy)$.

Proof. Notice that for $f = e^{\alpha z^2+2i\pi\beta z}F$ and $g = e^{\alpha z^2+2i\pi\beta z}G$ satisfying the autoumorphic equation (3.1), the functions F and G are $\mathbb{Z}$–periodic and we have

$$\langle f, g\rangle_{\alpha,\mathbb{Z}} = \int_{[0,1]\times\mathbb{R}} |g(z)|^2 e^{-4\alpha y^2-4\pi\beta y} dxdy.$$

Then, the result follows by a tedious but straightforward computations using the explicit expression of $\psi_{m,n}^{\nu,\alpha,\beta}$ given by Lemma 3.6.1 combined with the orthogonality of $e_n^{\alpha,\beta}$ (see Lemma 3.6.2). ∎

Remark 3.6.5. The above discussion shows that the polynomials $I_{m,n}^{\alpha,\beta}(z,\overline{z})$ in (3.5) characterize the orthogonal complement of $\mathcal{F}^{2,2\alpha}_{\mathbb{Z},\beta}(\mathbb{C})$ in the Hilbert space $L^2(\mathbb{C}/\mathbb{Z}; e^{-2\alpha|z|^2}dxdy)$.

Chapter 4

The orthogonal complement of the Hilbert space associated to holomorphic Hermite polynomials

We study the orthogonal complement of the Hilbert subspace considered by van Eijndhoven and Meyers in [113] and associated to holomorphic Hermite polynomials. A polyanalytic orthonormal basis is given and the explicit expressions of the corresponding reproducing kernel functions and Segal–Bargmann integral transforms are provided. The obtained basis are then used to provide a non-trivial $1d$-fractional like-Fourier transform.

4.1 Complements on $\mathcal{X}_s(\mathbb{C})$

We begin with the following

Proposition 4.1.1 ([113]). *The functions*

$$\psi^s_m = \frac{1}{\sqrt{\pi m!}}\left(\frac{1-s^2}{2s}\right)^{(m+1)/2} e^{-\frac{1+s^2}{4s}z^2} z^m$$

constitute an orthonormal basis of the reproducing kernel Hilbert space $\mathcal{X}_s(\mathbb{C})$ with kernel given explicitly by

$$K^s(z,w) = \frac{1-s^2}{2\pi s} e^{-\frac{1+s^2}{4s}(z^2+\overline{w}^2)+\frac{1-s^2}{2s}z\overline{w}}. \tag{4.1}$$

Proof. The proof of (4.1) can be handled by invoking the unitary operator $M_\alpha f = e^{\alpha z^2} f$ and observing that the functions

$$\phi^s_m(z) = \frac{1}{\sqrt{\pi m!}}\left(\frac{1-s^2}{2s}\right)^{(m+1)/2} e^{-\frac{1+s^2}{4s}z^2} z^m \tag{4.2}$$

form an orthonormal basis of $\mathcal{X}_s(\mathbb{C})$, so that one concludes for the explicit expression of $K^s(z,w)$ by performing $K^s(z,w) = \sum\limits_{m=0}^{+\infty} \phi_m^s(z)\overline{\phi_m^s(w)}$ and next using the generating function of the Hermite polynomials $H_n(z)$ ([99, p. 130]). ∎

Remark 4.1.2. The expression of the reproducing kernel can also be proved in an easy way, by making appeal to the following general principle. Let $\mathcal{H}$ be a separable reproducing kernel Hilbert space (RKHS) on the complex plane and denotes by $K^{\mathcal{H}}$ its reproducing kernel function. If M is a multiplication operator by a function $M(z) := e^{\psi(z)}$. Then, $\mathcal{H}' = M(\mathcal{H})$ is a RKHS whose kernel function is given by

$$K^{\mathcal{H}'}(z,w) = e^{\psi(z)} K^{\mathcal{H}}(z,w) e^{\overline{\psi(w)}}. \tag{4.3}$$

Remark 4.1.3. The space $\bigcup\limits_{0<s<1} \mathcal{X}_s(\mathbb{C}) = S_{1/2}^{1/2}(\mathbb{C})$ is the holomorphic Gelfand–Shilov space extended to $\mathbb{C}$ (see [113]).

In the sequel, we consider the integral transform of Segal–Bargmann type

$$[\mathscr{B}_s f](z) := \int_{\mathbb{R}} B_s(t,z) f(t) dt \tag{4.4}$$

associated to the kernel function

$$B_s(t,z) := \left(\frac{1-s^2}{2\pi s \sqrt{s\pi}} \right)^{1/2} \exp\left(-\frac{1}{2s} t^2 - \frac{1}{2s} z^2 + \frac{\sqrt{1-s^2}}{s} tz \right). \tag{4.5}$$

Then, we assert

Theorem 4.1.4. *The transform $\mathscr{B}_s$ defines a unitary isometric integral transform from the configuration Hilbert space $\mathcal{L}^2(\mathbb{R})$ onto $\mathcal{X}_s(\mathbb{C})$.*

Proof. The kernel function $B_s(t,z)$ in (4.5) can be rewritten as

$$B_s(t,z) := \sum_{m=0}^{\infty} f_m(t) \psi_m^s(z), \tag{4.6}$$

where

$$f_m(t) = \frac{e^{-\frac{t^2}{2}}}{\sqrt{2^m m! \sqrt{\pi}}} H_m(t) \tag{4.7}$$

is an orthonormal basis of $\mathcal{L}^2(\mathbb{R})$. Indeed, we have

$$\sum_{m=0}^{\infty} f_m(t) \psi_m^s(z) = \left(\frac{1-s}{\pi\sqrt{s\pi}} \right)^{1/2} e^{-\frac{1}{2}(t^2+z^2)} \sum_{m=0}^{\infty} \left(\frac{1-s}{1+s} \right)^{m/2} \frac{H_m(t) H_m(z)}{2^m m!}.$$

The rest of the proof is straightforward making use of Mehler formula for the Hermite polynomials extended to the complex plane, to wit ([85, p.174, Eq. (18)], see also [99, p.198, Eq. (2)])

$$\sum_{m=0}^{\infty} \frac{\lambda^m}{2^m m!} H_m(t) H_m(z) = \frac{1}{\sqrt{1-\lambda^2}} \exp\left(\frac{-\lambda^2(t^2+z^2)+2\lambda tz}{1-\lambda^2} \right) \tag{4.8}$$

valid for every fixed $|\lambda| < 1$. ∎

Remark 4.1.5. By means of (4.4) and (4.6), we have $[\mathscr{B}_s f_m](z) = \psi_m^s(z)$. Moreover, the inversion formula of $\mathscr{B}_s$ is given by

$$[\mathscr{B}_s^{-1}\varphi](t) = \int_{\mathbb{C}} \varphi(z) B_s(t,\overline{z}) \omega_s(z,\overline{z}) d\lambda(z).$$

Remark 4.1.6. By considering $\widetilde{B}_s(t,z) := s^{1/4} B_s(s^{1/2}t, z)$, we define an integral transform $\widetilde{\mathscr{B}}_s$ from $\mathcal{L}^2(\mathbb{R})$ onto $\mathcal{X}_s(\mathbb{C})$ such that $[\widetilde{\mathscr{B}}_s f_m n](z) = \phi_m^s(z)$, where ϕ_m^s are as in (4.2), since

$$\widetilde{B}_s(t,z) = \sum_{m=0}^{\infty} f_m(t) \phi_m^s(z).$$

4.2 A special orthonormal basis for $\mathscr{H}^{2,s}(\mathbb{C})$

The multiplication operator $M_\alpha : f \longmapsto M_\alpha f = e^{\alpha z^2} f$ defines a unitary operator from $\mathscr{H}^{2,s}(\mathbb{C})$ onto $\mathcal{L}^{2,\nu}(\mathbb{C})$. Moreover, it maps isometrically the Hilbert subspace $\mathcal{X}_s(\mathbb{C})$ onto the Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C})$. Therefore, an orthogonal decomposition of $\mathscr{H}^{2,s}(\mathbb{C})$ can be deduced easily from the one of $\mathcal{L}^{2,\nu}(\mathbb{C})$,

$$\mathcal{L}^{2,\nu}(\mathbb{C}) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n^{2,\nu}(\mathbb{C}),$$

given in terms of the polyanalytic Hilbert spaces

$$\mathcal{F}_n^{2,\nu}(\mathbb{C}) = Ker|_{\mathscr{H}^{2,s}(\mathbb{C})} (\Delta_\nu - n\nu Id)$$

where $\Delta_\nu := -\partial_z \partial_{\overline{z}} + \nu \overline{z} \partial_{\overline{z}}$ and with $\mathcal{F}_0^{2,\nu}(\mathbb{C}) = \mathcal{F}^{2,\nu}(\mathbb{C})$. See, e.g. [59] for details. In fact, the consideration of $\mathcal{X}_{n,s}(\mathbb{C}) := M_{-\alpha} \mathcal{F}_n^{2,\nu}(\mathbb{C})$ leads to the orthogonal decomposition

$$\mathscr{H}^{2,s}(\mathbb{C}) = \bigoplus_{n=0}^{\infty} \mathcal{X}_{n,s}(\mathbb{C}).$$

An immediate orthonormal basis of $\mathcal{X}_{n,s}(\mathbb{C})$ is then given by $e^{-\alpha z^2} H_{m,n}^{\nu}(z,\overline{z})$ for varying $m,n = 0,1,2,\cdots$, where

$$H_{m,n}^{\nu}(z,\overline{z}) := (-1)^{m+n} e^{\nu|z|^2} \partial_{\overline{z}}^m \partial_z^n \left(e^{-\nu|z|^2} \right) \tag{4.1}$$

denotes the rescaled polyanalytic complex Hermite polynomials [55,56,76], generalizing the monomials $\nu^m z^m = H_{m,0}^{\nu}(z,\overline{z})$.

The main aim in this section is to provide another non-trivial orthonormal basis $\psi^s_{m,n}(z,\overline{z})$ of $\mathscr{H}^{2,s}(\mathbb{C})$, consisting of polyanalytic functions generalizing ψ^s_m and such that whose first elements are the holomorphic functions $\psi^s_m(z)$ in (4.2), i.e., $\psi^s_{m,0}(z,\overline{z}) = \psi^s_m(z)$, leading to an appropriate basis of the space $\mathcal{X}_{n,s}(\mathbb{C})$, for fixed n. The introduction of $\mathcal{X}_{n,s}(\mathbb{C})$ entails the consideration of the integral transform

$$[\mathscr{W}^s_n f](z,\overline{z}) = \left(\frac{\nu}{\pi}\right)\left(\frac{\nu^n}{n!}\right)^{1/2} e^{-\alpha z^2}\int_{\mathbb{C}} e^{-\nu|\xi|^2+\alpha\xi^2+\nu\overline{\xi}z}(\overline{z}-\overline{\xi})^n\psi(\xi)d\lambda(\xi)$$

as well as of the functions

$$\psi^s_{m,n}(z,\overline{z}) := \Gamma^s_{m,n} e^{-\frac{z^2}{2}}\left(\nabla^n_{\nu,\alpha-\frac{1}{2}} H_m\right)(z), \tag{4.2}$$

where $\nabla_{\nu,\alpha} := -\partial_z + \nu\overline{z} - 2\alpha z$ and

$$\Gamma^s_{m,n} := \left(\frac{1-s}{\pi\nu^n n! m!\sqrt{s}}\right)^{1/2}\left(\frac{1-s}{2(1+s)}\right)^{m/2}, \tag{4.3}$$

Then, we can prove the following

Theorem 4.2.1. *The transform $\mathscr{W}^s_n$ is a unitary integral transform from $\mathcal{X}_s(\mathbb{C})$ onto $\mathcal{X}_{n,s}(\mathbb{C})$. Moreover, we have*

$$\psi^s_{m,n}(z,\overline{z}) = \mathscr{W}^s_n(\psi^s_m)$$

and hence $\psi^s_{m,n}(z,\overline{z})$, for varying $m = 0,1,2,\cdots$, form an orthonormal basis of $\mathcal{X}_{n,s}(\mathbb{C})$.

Proof. The proof lies essentially on the observation that the unitary operator $\mathscr{W}^s_n$ can be rewritten as $\mathscr{W}^s_n = M_{-\alpha}\mathscr{T}^\nu_{0,n}M_\alpha$, where $\mathscr{T}^\nu_{k,n}$ is the integral transform considered in [20, Eq. (2.17)] and given by

$$[\mathscr{T}^\nu_{k,n}\psi](z,\overline{z}) = \left(\frac{(-1)^n\nu}{\pi\sqrt{k!n!\nu^{k+n}}}\right)\int_{\mathbb{C}} e^{-\nu|\xi|^2+\nu\overline{\xi}z} H^\nu_{k,n}(\xi - z, \overline{\xi}-\overline{z})\psi(\xi)d\lambda(\xi),$$

so that $\mathscr{W}^s_n(\psi^s_m) = M_{-\alpha}\mathscr{T}^\nu_{0,n}(M_\alpha\psi^s_m)$. Therefore, by means of [20, Theorem 2.12], keeping in mind the fact that the polynomials $H^\nu_{m,n}(z,\overline{z}) =: \nabla^n_{\nu,0}(z^m)$ is an orthogonal basis of $\mathcal{L}^{2,\nu}(\mathbb{C})$ [56,76], the following

$$[\mathscr{T}^\nu_{0,n}\psi](z,\overline{z}) = \left(\frac{1}{\nu^n n!}\right)^{1/2}\nabla^n_{\nu,0}\psi,$$

holds true for every non-negative integer n and any $\psi \in \mathcal{L}^{2,\nu}(\mathbb{C}) \cap \mathcal{C}^n(\mathbb{C})$, where $\mathcal{C}^n(\mathbb{C})$ denotes the set of all n-fold differentiable functions whose n-th derivative is continuous. The rest of the second assertion is straightforward since the functions ψ^s_m form an orthonormal basis of $\mathcal{X}_s(\mathbb{C})$. The explicit expression of $\psi^s_{m,n}(z,\overline{z})$ follows by direct computation. Indeed, we have

$$\begin{aligned}M_{-\alpha}\mathscr{T}^{\nu}_{0,n}M_{\alpha}(\psi^s_m) &= \left(\frac{1}{\nu^n n!}\right)^{1/2} M_{-\alpha}\nabla^n_{\nu,0}\left(M_\alpha\psi^s_m\right)\\ &= \left(\frac{1}{\nu^n n!}\right)^{1/2} \nabla^n_{\nu,\alpha}\left(\psi^s_m\right)\\ &= \left(\frac{1}{\nu^n n!}\right)^{1/2} e^{\frac{-z^2}{2}}\nabla^n_{\nu,\alpha-\frac{1}{2}}\left(e^{\frac{z^2}{2}}\psi^s_m\right),\end{aligned}$$

since

$$\nabla_{\nu,a}\left(M_\gamma\psi\right) = M_\gamma\nabla_{\nu,a+\gamma}\psi \text{ and } \nabla^n_{\nu,0}\left(M_\gamma\psi\right) = M_\gamma\nabla^n_{\nu,\gamma}\psi.$$

The last equality shows that $\mathscr{W}^s_n(\psi^s_m) = M_{-\alpha}\mathscr{T}^{\nu}_{0,n}M_\alpha(\psi^s_m) = \psi^s_{m,n}(z,\overline{z})$. ∎

Remark 4.2.2. The inverse of $\mathscr{W}^s_n : \mathcal{X}_s(\mathbb{C}) \longrightarrow \mathcal{X}_{n,s}(\mathbb{C})$ is given by $[\mathscr{W}^s_n]^{-1} = M_{-\alpha}\mathscr{T}^{\nu}_{n,0}M_\alpha$, More explicitly

$$[\mathscr{W}^s_n]^{-1}\psi(z) = \left(\frac{\nu}{\pi}\right)\left(\frac{\nu^n}{n!}\right)^{1/2} e^{-\alpha z^2}\int_{\mathbb{C}} e^{-\nu|\xi|^2+\alpha\xi^2+\nu\overline{\xi}z}(\xi-z)^n\psi(\xi)d\lambda(\xi).$$

The new class of functions in (4.2) generalizes the one studied in [21], and the previous theorem provides an integral representation of the special functions $\psi^s_{m,n}(z,\overline{z})$. Moreover, such functions can be expressed as a special finite sum of the holomorphic Hermite polynomials $H_j(z)$ and the polyanalytic polynomials $I^{a,b}_n(z,\overline{z}|c)$ defined in [21] by

$$I^{a,b}_n(z,\overline{z}|c) := (-1)^n e^{a|z|^2-bz^2-cz}\partial^n_z\left(e^{-a|z|^2+bz^2+cz}\right) \tag{4.4}$$

for given $a>0$, $b\in\mathbb{R}$ and $c\in\mathbb{C}$. More exactly, we assert

Corollary 4.2.3. *We have*

$$\psi^s_{m,n}(z,\overline{z}) = m!n!\Gamma^s_{m,n}e^{-\frac{1}{2}z^2}\sum_{k=0}^{n}\frac{(-2)^k}{k!}\frac{I^{\nu,\alpha-\frac{1}{2}}_{n-k}(z,\overline{z}|0)}{(n-k)!}\frac{H_{m-k}(z)}{(m-k)!}, \tag{4.5}$$

where $\Gamma^s_{m,n}$ *is as in* (4.3).

Proof. Notice first that using the fact $\nabla_{\nu,\gamma}f = -e^{\nu|z|^2-\gamma z^2}\partial_z\left(e^{-\nu|z|^2+\gamma z^2}f\right)$, we get

$$\nabla^n_{\nu,\gamma}f = (-1)^n e^{\nu|z|^2-\gamma z^2}\partial^n_z\left(e^{-\nu|z|^2+\gamma z^2}f\right) \tag{4.6}$$

by mathematical induction. Therefore, identities (4.2) or (4.6) combine with the Leibniz formula entail

$$\begin{aligned}\psi^s_{m,n}(z,\overline{z}) &= (-1)^n \Gamma^s_{m,n} e^{\nu|z|^2-\alpha z^2} \partial_z^n \left(e^{-\nu|z|^2+\left(\alpha-\frac{1}{2}\right)z^2} H_m \right)(z) \\ &= (-1)^n \Gamma^s_{m,n} e^{\nu|z|^2-\alpha z^2} \sum_{k=0}^{n} \binom{n}{k} \partial_z^{n-k} \left(e^{-\nu|z|^2+\left(\alpha-\frac{1}{2}\right)z^2} \right) \partial_z^k \left(H_m\right)(z).\end{aligned}$$

Thus, one concludes for (4.5) since $\partial_z^k (H_m) = 0$ for $k > m$ and

$$\partial_z^k (H_m) = \frac{2^k m!}{(m-k)!} H_{m-k}(z)$$

when $k \leq m$. ∎

Corollary 4.2.4. *The connection to the polyanalytic Hermite polynomials $H^\nu_{m,n}(z,\overline{z})$ in* (4.1) *is given by*

$$\psi^s_{m,n}(z,\overline{z}) = m! \Gamma^s_{m,n} e^{-\alpha z^2} \sum_{k=0}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^j \left(\frac{2}{\nu}\right)^{m-2j} \left(\frac{2\alpha-1}{2\nu^2}\right)^k}{j!(m-2j)!k!} H^\nu_{2k+m-2j,n}(z,\overline{z}). \tag{4.7}$$

Proof. Direct computation, keeping in mind (4.6), shows that

$$\begin{aligned}\nabla^n_{\nu,\alpha-\frac{1}{2}} H_m &= e^{-\left(\alpha-\frac{1}{2}\right)z^2} \sum_{k=0}^{\infty} \frac{\left(\alpha-\frac{1}{2}\right)^k}{k!} (-1)^n e^{\nu|z|^2} \partial_z^n \left(z^{2k} e^{-\nu|z|^2} H_m \right) \\ &= m! e^{-\left(\alpha-\frac{1}{2}\right)z^2} \sum_{k=0}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^j \left(\frac{2}{\nu}\right)^{m-2j} \left(\frac{2\alpha-1}{2\nu^2}\right)^k}{j!(m-2j)!k!} H^\nu_{2k+m-2j,n}.\end{aligned}$$

This completes the proof since $\psi^s_{m,n}(z,\overline{z}) = \Gamma^s_{m,n} e^{-\frac{1}{2}z^2} \nabla^n_{\nu,\alpha-\frac{1}{2}} H_m$. ∎

The considered space $\mathcal{X}_{n,s}(\mathbb{C})$ is a reproducing kernel Hilbert space for the point evaluation map in $\mathcal{X}_{n,s}(\mathbb{C})$ being continuous. This, can be recovered easily by means of Remark 4.1.2. Thus, we assert

Theorem 4.2.5. *The explicit expression of the reproducing kernel of $\mathcal{X}_{n,s}(\mathbb{C})$ is given by*

$$K^s_n(z,w) = \left(\frac{1-s^2}{2\pi s} \right) \frac{(-1)^n}{n! \nu^n} e^{\nu z \overline{w} - \alpha(z^2+\overline{w}^2)} H^\nu_{n,n}(z-w, \overline{z}-\overline{w}).$$

Proof. By means of Remark 4.1.2, the reproducing kernel $K^s_n(z,w)$ of $\mathcal{X}_{n,s}(\mathbb{C})$ obeys (4.3). Hence, we have $K^s_n(z,w) = M_\alpha(z) K^{\mathcal{F}^{2,\nu}(\mathbb{C})}(z,w) \overline{M_\alpha(w)}$, where $K^{\mathcal{F}^{2,\nu}_n(\mathbb{C})}$ is the reproducing kernel of the generalized Bargmann space $\mathcal{F}^{2,\nu}_n(\mathbb{C})$ given by [59]

$$K_n^{\mathcal{F}^{2,\nu}_n(\mathbb{C})}(z,w) = \left(\frac{\nu}{\pi} \right) \frac{(-1)^n}{n! \nu^n} e^{\nu z \overline{w}} H^\nu_{n,n}(z-w, \overline{z}-\overline{w}).$$

$\blacksquare$

Remark 4.2.6. For $n = 0$, we recover the reproducing kernel of the Hilbert space $\mathcal{X}_s(\mathbb{C})$ in Proposition 4.1.1.

Corollary 4.2.7. *The identity*

$$H^{\nu}_{n,n}(z-w,\overline{z}-\overline{w}) = (-1)^n e^{\left(\alpha-\frac{1}{2}\right)(z^2+\overline{w}^2)-\nu z\overline{w}} \nabla^{n,z}_{\nu,\alpha-\frac{1}{2}} \overline{\nabla^{n,w}_{\nu,\alpha-\frac{1}{2}}} e^{-\left(\alpha-\frac{1}{2}\right)(z^2+\overline{w}^2)+\nu z\overline{w}},$$

or equivalently

$$H^{\nu}_{n,n}(z-w,\overline{z}-\overline{w}) = (-1)^n e^{\nu(|z|^2+|w|^2-z\overline{w})} \partial^n_z \partial^n_{\overline{w}} e^{-\nu(|z|^2+|w|^2+z\overline{w})},$$

holds true. The z-variable in $\nabla^{n,z}_{\nu,\alpha-\frac{1}{2}}$ *is to mean that the derivation is done with respect to z.*

Proof. The assertion of Corollary 4.2.7 follows by comparing the result of Theorem 4.2.5 to the fact that the reproducing kernel K^s_n can be rewritten as $K^s_n(z,w) = \sum\limits_{m=0}^{+\infty} \psi^s_{m,n}(z)\overline{\psi^s_{m,n}(w)}$, for $\{\psi^s_{m,n}(z,\overline{z}), m = 0,1,2,\cdots\}$, in (4.2), being an orthonormal basis of $\mathcal{X}_{n,s}(\mathbb{C})$. $\blacksquare$

We conclude this section by giving the explicit expression of the generalized Segal–Bargmann integral transform for the spaces $\mathcal{X}_{n,s}(\mathbb{C})$. We have to consider the weighted configuration Hilbert space $\mathcal{L}^{2,\nu}(\mathbb{R})$, instead of $\mathcal{L}^2(\mathbb{R})$, of all square integrable $\mathbb{C}$-valued functions on $\mathbb{R}$ with respect to the Gaussian measure $e^{-\nu x^2}dx$; $\nu > 0$, for which the rescaled Hermite polynomials

$$g^{\nu}_m(x) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{4}} \frac{H_m(\sqrt{\nu}x)}{\sqrt{2^m m!}} \tag{4.8}$$

form an orthonormal basis. The associated coherent states transform from $\mathcal{L}^{2,\nu}(\mathbb{R})$ onto $\mathcal{X}_{n,s}(\mathbb{C})$ mapping g^{ν}_m to $\psi^s_{m,n}$ is given by

$$\mathscr{S}^s_n f(z) := \left\langle f, \overline{S^s_n(.,z)} \right\rangle_{\mathcal{L}^{2,\nu}(\mathbb{R})} = \int_{\mathbb{R}} f(x) S^s_n(x,z) e^{-\nu x^2} dx,$$

where the kernel function $S^s_n(x,z)$ is given by

$$S^s_n(x,z) = \sum_{m=0}^{+\infty} g^{\nu}_m(x)\psi^s_{m,n}(z,\overline{z}).$$

Theorem 4.2.8. *We have*

$$S^s_n(x,z) = \left(\frac{\nu}{\pi s}\right)^{\frac{1}{4}} \left(\frac{1-s^2}{2\pi s \nu^n n!}\right)^{1/2} e^{-\frac{1}{2s}z^2 - \frac{\nu(1-s)}{2s}x^2 + \frac{\nu\sqrt{2s}}{s}xz} I^{\nu,-\frac{\nu}{2}}_n \left(z,\overline{z} \middle| \frac{\nu\sqrt{2s}}{s}x \right). \tag{4.9}$$

Moreover, the transform $\mathscr{S}^s_n$ *defines an isometric transform from* $\mathcal{L}^{2,\nu}(\mathbb{R})$ *onto* $\mathcal{X}_{n,s}(\mathbb{C})$.

Proof. We need only to prove the closed formula (4.9) for $S_n^s(x,z)$. The rest holds true for general coherent states transformation on the reproducing kernel Hilbert spaces, like $\mathcal{X}_{n,s}(\mathbb{C})$. Indeed, starting from (4.8) and (4.2), and applying Mehler formula (4.8), the expression of $S_n^s(x,z)$ reduces further to

$$S_n^s(x,z) = \left(\frac{\nu}{\pi s}\right)^{\frac{1}{4}} \left(\frac{1-s^2}{2\pi s \nu^n n!}\right)^{1/2} e^{-\frac{z^2}{2} - \frac{\nu(1-s)}{2s}x^2} \nabla^{n_z}_{\nu,\alpha-\frac{1}{2}} \exp\left(-\frac{1-s}{2s}z^2 + \frac{\nu\sqrt{2s}}{s}xz\right).$$

By applying the identity (4.6), we get

$$S_n^s(x,z) = \left(\frac{\nu}{\pi s}\right)^{\frac{1}{4}} \left(\frac{1-s^2}{2\pi s \nu^n n!}\right)^{1/2} e^{\nu|z|^2 - \alpha z^2 - \frac{\nu(1-s)}{2s}x^2} (-1)^n \partial_z^n \left(e^{-\nu|z|^2 - \frac{\nu}{2}z^2 + \frac{\nu\sqrt{2s}}{s}xz}\right).$$

Subsequently,

$$S_n^s(x,z) = \left(\frac{\nu}{\pi s}\right)^{\frac{1}{4}} \left(\frac{1-s^2}{2\pi s \nu^n n!}\right)^{1/2} e^{-\frac{1}{2s}z^2 - \frac{\nu(1-s)}{2s}x^2 + \frac{\nu\sqrt{2s}}{s}xz} I_n^{\nu,-\frac{\nu}{2}}\left(z,\bar{z}\middle|\frac{\nu\sqrt{2s}}{s}x\right).$$

∎

4.3 A $1d$-fractional like-Fourier transform for $\mathcal{L}^{2,\nu}(\mathbb{R})$

In the previous section the space $\mathcal{X}_{n,s}(\mathbb{C})$ is realized as the image of $\mathcal{X}_s(\mathbb{C})$ by the integral transform $\mathcal{W}_n^s$ or also as the image of $\mathcal{L}^{2,\nu}(\mathbb{R})$ by the generalized Segal–Bargmann transform $\mathscr{S}_n^s$. Another realization of $\mathcal{X}_{n,s}(\mathbb{C})$ is by considering the n-th standard Segal–Bargmann transform [21]

$$\mathscr{B}_n^\nu\varphi(z) = \frac{\left(\frac{\nu}{\pi}\right)^{\frac{3}{4}}}{\sqrt{2^n\nu^n n!}} \int_{\mathbb{R}} e^{-\nu(x-\frac{z}{\sqrt{2}})^2} H_n^\nu\left(\frac{z+\bar{z}}{\sqrt{2}} - x\right)\varphi(x)dx$$

from $\mathcal{L}^{2,\nu}(\mathbb{R})$ onto $\mathcal{F}^{2,\nu}(\mathbb{C})$. Indeed, one has to deal with $\widetilde{\mathscr{B}_n^\nu} : \mathcal{L}^{2,\nu}(\mathbb{R}) \longrightarrow \mathcal{X}_{n,s}(\mathbb{C})$,

$$\widetilde{\mathscr{B}_n^\nu} f(z,\bar{z}) = \left(M_{-\alpha}\mathscr{B}_n^\nu f\right)(z,\bar{z}).$$

It is clear that for every fixed n, the functions $[\widetilde{\mathscr{B}_n^\nu}]^{-1}\psi_{m,n}^s$ form an orthonormal basis of $\mathcal{L}^{2,\nu}(\mathbb{R})$. But, there is no clear evidence if they are the same or not. The next result provides a positive answer. To this end, we give the explicit expression of $[\widetilde{\mathscr{B}_n^\nu}]^{-1}\psi_{m,n}^s$ in terms of the generalized Hermite polynomials of $H_m(x,y)$ Gould-Hopper type defined in [36, Eq. (5b), p. 72] by

$$H_m(x,y) = m! \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^j y^j}{j!} \frac{(2x)^{m-2j}}{(m-2j)!}. \tag{4.1}$$

Namely, if we let Φ^s_m stands for

$$\Phi^s_m(x) = \left(\frac{\nu s}{\pi m!^2}\right)^{1/4}\left(\frac{s}{1+s}\right)^{m/2} e^{\frac{\nu(1-s)}{2}x^2} H_m\left(\left(\frac{\nu(1+s)}{2}\right)^{1/2} x, \frac{1+s}{2s}\right). \tag{4.2}$$

then, we can prove the following.

Theorem 4.3.1. *The functions $[\widetilde{\mathscr{B}^\nu_n}]^{-1}\psi^s_{m,n}$, for varying non-negative integers m and n, are independents of n. Moreover, we have $[\widetilde{\mathscr{B}^\nu_n}]^{-1}\psi^s_{m,n} = \Phi^s_m$, and they form an orthonormal basis for the configuration space $\mathcal{L}^{2,\nu}(\mathbb{R})$.*

Remark 4.3.2. Using $H_n(x,y) = y^{\frac{n}{2}} H_n(xy^{-\frac{1}{2}})$, we can rewrite $[\widetilde{\mathscr{B}^\nu_n}]^{-1}\psi^s_{m,n}$ as

$$[\widetilde{\mathscr{B}^\nu_n}]^{-1}(\psi^s_{m,n})(x) = \left(\frac{\nu s}{\pi m!^2}\right)^{1/4}\left(\frac{1}{2\nu s}\right)^{m/2} e^{\frac{\nu(1-s)}{2}x^2} H^{\nu s}_m(x). \tag{4.3}$$

Remark 4.3.3. The standard basis of $\mathcal{L}^{2,\nu}(\mathbb{R})$ is the classical polynomials H^ν_m. Here, we have provide a non-trivial orthogonal basis of $\mathcal{L}^{2,\nu}(\mathbb{R})$ constituted of the special Hermite functions $e^{\frac{\nu(1-s)}{2}x^2} H^{\nu s}_m(x)$. This is conform with the general fact that an orthogonal basis of $\mathcal{L}^{2,a}(\mathbb{R})$ can be obtained from the one of $\mathcal{L}^{2,b}(\mathbb{R})$, $a,b > 0$, by the multiplication operator $e^{\frac{b-a}{2}x^2}$.

For the proof of Theorem 4.3.1 we need to the following lemma.

Lemma 4.3.4. *We have*

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}H^\nu_{2k+m}(x) = \frac{1}{\sqrt{1+4\nu t}} e^{\frac{4\nu^2 t}{1+4\nu t}x^2} H_m^{\frac{\nu}{1+4\nu t}}(x).$$

Proof. Using the fact $H^\nu_{m+n} = D^n_\nu(H^\nu_m)$, where $D_\nu := -\partial_x + 2\nu x$, we get

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}H^\nu_{2k+m}(x) = \sum_{k=0}^{\infty}\frac{t^k}{k!}D^m_\nu H^\nu_{2k}(x) = D^m_\nu\left(\frac{e^{\frac{4\nu^2 t}{1+4\nu t}x^2}}{\sqrt{1+4\nu t}}\right).$$

The last equality follows by means of the "generating function"

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{2n}(x) = \frac{e^{\frac{4tx^2}{1+4t}}}{\sqrt{1+4t}}.$$

Finally, we use the operational formula $D^m_\nu(e^{\gamma x^2}) = e^{\gamma x^2} H^{\nu-\gamma}_m(x)$ to obtain

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}H^\nu_{2k+m}(x) = \frac{1}{\sqrt{1+4\nu t}} e^{\frac{4\nu^2 t}{1+4\nu t}x^2} H_m^{\frac{\nu}{1+4\nu t}}(x).$$

∎

Proof of Theorem 4.3.1. By definition of $\widetilde{\mathscr{B}}_n^\nu = M_{-\alpha}\mathscr{B}_n^\nu$, we have

$$[\widetilde{\mathscr{B}}_n^\nu]^{-1}(\psi^s_{m,n}) = [\mathscr{B}_n^\nu]^{-1}\left(M_\alpha \psi^s_{m,n}\right).$$

On the other hand, from Corollary 4.2.4, we have

$$\left(M_\alpha \psi^s_{m,n}\right)(z) = \Gamma^s_{m,n} m! \sum_{k=0}^{\infty}\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor} \frac{(-1)^j \left(\frac{2}{\nu}\right)^{m-2j}\left(\frac{2\alpha-1}{2\nu^2}\right)^k}{j!(m-2j)!k!} H^\nu_{2k+m-2j,n}(z,\overline{z}).$$

Subsequently, one can apply [20, Theorem 5.1],

$$[\mathscr{B}_n^\nu]^{-1}(H^\nu_{m,n})(x) = [\gamma^s_{m,n}]^{-1} H^\nu_m(x), \quad \gamma^s_{m,n} := \left(\frac{\nu}{\pi}\right)^{1/4}\left(\frac{2^m}{\nu^n n!}\right)^{1/2},$$

as well as Lemma 4.3.4 to get

$$\begin{aligned}[\widetilde{\mathscr{B}}_n^\nu]^{-1}(\psi^s_{m,n})(x) &= \frac{2^m\Gamma^s_{m,n}}{\nu^m\gamma^s_{m,n}} m! \sum_{j=0}^{\lfloor \frac{m}{2}\rfloor} \frac{\left(-\frac{\nu^2}{2}\right)^j}{j!(m-2j)!}\left(\sum_{k=0}^{\infty}\frac{\left(\frac{2\alpha-1}{4\nu^2}\right)^k}{k!} H^\nu_{2k+m-2j}(x)\right)\\ &= \left(\frac{1+s}{2}\right)^{1/2}\frac{2^m\Gamma^s_{m,n}}{\nu^m\gamma^s_{m,n}} e^{\frac{\nu(2\alpha-1)}{\nu+2\alpha-1}x^2} m! \sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\frac{\left(-\frac{\nu^2}{2}\right)^j}{j!(m-2j)!} H^{\frac{\nu^2}{\nu+2\alpha-1}}_{m-2j}(x).\end{aligned}$$

In the last equality we recognize the identity [36, Eq. (6), p. 72]

$$H_m(x,y) = m!\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor}\frac{(1-y)^j}{j!}\frac{H_{m-2j}(x)}{(m-2j)!}. \tag{4.4}$$

Thus, we obtain

$$[\widetilde{\mathscr{B}}_n^\nu]^{-1}(\psi^s_{m,n})(x) = \left(\frac{\nu s}{\pi m!^2}\right)^{1/4}\left(\frac{s}{1+s}\right)^{m/2} e^{\frac{\nu(1-s)}{2}x^2} H_m\left(\left(\frac{\nu(1+s)}{2}\right)^{1/2} x, \frac{1+s}{2s}\right).$$

This completes the proof. ∎

Remark 4.3.5. The first assertion in Theorem 4.3.1 can be reworded as follows: for every non-negative integers m, j and k, we have

$$\psi^s_{m,j} = \widetilde{\mathscr{B}}_j^\nu \circ [\widetilde{\mathscr{B}}_k^\nu]^{-1}(\psi^s_{m,k}).$$

We conclude this section by considering a $1d$-fractional like-Fourier transform for $\mathcal{L}^{2,\nu}(\mathbb{R})$.

To this end, we set

$$P_{s,t}(x,y) = \left(\frac{\nu s}{\pi(1-t^2)}\right)^{1/2} e^{\frac{\nu}{2}(x^2+y^2)} e^{-\nu s\left(\frac{(1+t^2)(x^2+y^2)-4xy}{2(1-t^2)}\right)} \tag{4.5}$$

for $0 < s < 1$ and $|t| < 1$.

Theorem 4.3.6. *The integral transform*

$$(\mathcal{F}_{s,t}f)(y) = \int_{\mathbb{R}} f(x)P_{s,t}(x,y)e^{-\nu x^2}dx$$

defines a $1d$*-fractional like-Fourier transform for* $\mathcal{L}^{2,\nu}(\mathbb{R})$*. Moreover, we have*

$$\mathcal{F}_{s,t}\Phi^s_m = t^m\Phi^s_m \quad \textit{and} \quad \mathcal{F}_{s,tt'} = \mathcal{F}_{s,t} \circ \mathcal{F}_{s,t'}.$$

Proof. Recall from Theorem 4.3.1 that the functions Φ^s_m in (4.2) form an orthonormal basis of $\mathcal{L}^{2,\nu}(\mathbb{R})$. Using (4.3), the corresponding Poisson kernel $\sum\limits_{m=0}^{+\infty} t^m\Phi^s_m(x)\overline{\Phi^s_m}(y)$ can be shown to be given by (4.5),

$$\sum_{m=0}^{+\infty} t^m\Phi^s_m(x)\overline{\Phi^s_m}(y) = P_{s,t}(x,y). \tag{4.6}$$

This readily follows making appeal to the classical Mehler formula (4.8). Therefore, the identity $\mathcal{F}_{s,t}\Phi^s_m = t^m\Phi^s_m$ is immediate from (4.6). Now, using $\mathcal{F}_{s,t}\Phi^s_m = t^m\Phi^s_m$, we can easily prove that $\mathcal{F}_{s,tt'} = \mathcal{F}_{s,t} \circ \mathcal{F}_{s,t'}$. ∎

Chapter 5

S-polyregular Bargmann spaces

We introduce two classes of right quaternionic Hilbert spaces in the context of slice polyregular functions, generalizing the so-called slice and full hyperholomorphic Bargmann spaces. Their basic properties are discussed, the explicit formulas for their reproducing kernels are given and associated Segal–Bargmann transforms are also introduced and studied. The spectral description as special subspaces of L^2-eigenspaces of a second order differential operator involving the slice derivative is investigated.

5.1 S-polyregular functions

5.1.1 The real skew algebra of quaternions

The elements of the division algebra of quaternions $\mathbb{H}$ are 4-component extended complex numbers of the form $q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and the imaginary components $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy the Hamiltonian computation rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$; $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. According to this algebraic representation, the quaternionic conjugate is defined to be $x_0 - x_1 i - x_2 j - x_3 k = \Re(q) - \Im(q)$, where $\Re(q) = x_0$ and $\Im(q) = x_1 i + x_2 j + x_3 k$. Here and elsewhere after $\overline{q}$ denotes the algebraic conjugate of the quaternion $q \in \mathbb{H}$. Then, we have $\overline{pq} = \overline{q}\,\overline{p}$ for $p, q \in \mathbb{H}$, and the modulus of q is defined to be $|q| = \sqrt{q\overline{q}}$. The polar representation is given by $q = re^{I\theta}$, where $r = |q| \geq 0$, $\theta \in [0, 2\pi[$, and I belongs to the set of imaginary units $\mathbb{S}$, which can be identified with the unit sphere $S^2 = \{q \in \Im\mathbb{H}; |\Im(q)| = 1\}$ in $\Im\mathbb{H} = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$. The representation $q = re^{I\theta}$ is not unique unless q is not real. Another interesting representation of $q \in \mathbb{H}$ is given by $q = x + Iy$ for some real numbers x and y and imaginary unit $I \in \mathbb{S}$. It is unique for any $q \in \widetilde{\mathbb{H}} = \mathbb{H} \setminus \mathbb{R}$ by requiring $y > 0$. Thus, $\mathbb{H}$ can be seen as the infinite union of the slices $\mathbb{C}_I := \mathbb{R} + \mathbb{R}I$. The last representation was crucial in developing the theory of quaternionic slice regular functions that has been introduced by Gentili and Struppa in their seminal work [62]. Since then, they have been object of intensive research and the corresponding hypercomplex analysis has been developed. It has found many interesting applications in operator theory, quantum physics, Schur analysis and different branches of differential geometry. See for instance [11,12,61,63] and references therein.

5.1.2 S-polyregular functions and first properties.

The solution of the Cauchy–Riemann equation $\overline{\partial_I} f_{|\mathbb{C}_I} = 0$ on $\mathbb{H}$, involving the slice derivative

$$\overline{\partial_I} f(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f|_{\mathbb{C}_I}(x+yI) \tag{5.1}$$

leads to the left power series

$$\varphi(x+Iy) = \sum_{j=0}^{+\infty}(x+Iy)^j \alpha_j(I), \tag{5.2}$$

with infinite convergent radius, where α_j are seen as functions $\alpha_j : I \longmapsto \alpha_j(I)$ on $\mathbb{S}$ with values in $\mathbb{H}$. If in addition $\alpha_j(I)$ are constants on $\mathbb{S}$, we recover the standard space of slice regular functions [61,62]. A natural generalization is that of S-polyregular functions.

Definition 5.1.1. A quaternionic-valued function f on a domain $\Omega \subset \mathbb{H}$ such that $\Omega \cap \mathbb{R} \neq \emptyset$ is said to be (left) slice polyregular (S-polyregular) of level n (order $n+1$), if it is a real differentiable in Ω and its restriction f_{Ω_I} is polyanalytic in $\Omega_I := \Omega \cap \mathbb{C}_I$ for every $I \in \mathbb{S}$, in the sense that the function $\overline{\partial_I}^{n+1} f : \Omega_I \longrightarrow \mathbb{H}$ vanishes identically on Ω_I. We denote by $\mathcal{SR}_n(\Omega)$ the corresponding right quaternionic vector space.

Topologically, the space $\mathcal{SR}_n(\Omega)$ is endowed with the natural topology of uniform convergence on compact sets in Ω, so that it turns out to be a right vector space over the non-commutative field $\mathbb{H}$. We provide below some of their basic properties that we need to develop the rest of the present chapter for the case $\Omega = \mathbb{H}$. Thus, one can easily prove the following elementary characterization for the elements in $\mathcal{SR}_n := \mathcal{SR}_n(\mathbb{H})$ in terms of the elements of $\mathcal{SR}$. Whose proof is immediate and lies essentially on the characterization of polyanalytic functions in complex setting [4,15].

Proposition 5.1.2. *For every $f \in \mathcal{SR}_n$, there exist some $\varphi_k \in \mathcal{SR}$, $k = 0, 1, \cdots, n$, such that*

$$f(q, \overline{q}) = \sum_{k=0}^{n} \overline{q}^k \varphi_k(q).$$

The following result is a second characterization of S-polyregular functions.

Theorem 5.1.3. *A function f belongs to $\mathcal{SR}_n$ if and only if there exists $\varphi_0 \in \mathcal{SR}$ such that*

$$f(q, \overline{q}) = \varphi_0(q) + \sum_{j=1}^{n}\sum_{k=0}^{n-j}(-1)^k \frac{\overline{q}^{j+k}}{j!k!}\overline{\partial_s}^{j+k} f(q).$$

Proof. By Proposition 5.1.2, any $f \in \mathcal{SR}_n$ is of the form $f(q, \overline{q}) = \sum\limits_{k=0}^{n} \overline{q}^k \varphi_k(q)$ for some

$\varphi_k \in \mathcal{SR}$, $k = 0, 1, \cdots, n$. Therefore, $\overline{\partial_s}^k f = 0$ whenever $k > n$, and

$$\overline{\partial_s}^k f = \sum_{j=0}^{n} \overline{\partial_s}^k (\overline{q}^j)\varphi_j = \sum_{j=k}^{n} \frac{j!}{(j-k)!}\overline{q}^{j-k}\varphi_j$$

when $k \leq n$. By considering the particular cases $k = n$, $k = n-1$, $k = n-2$ and $k = n-3$, one can claim the following

$$(n-k)!\varphi_{n-k} = \sum_{s=0}^{k}(-1)^s \frac{\overline{q}^s}{s!}\overline{\partial_s}^{n-k+s} f$$

for $k < n$, which can be proved by induction. Equivalently, we write

$$\varphi_j = \frac{1}{j!}\sum_{s=0}^{n-j}(-1)^s \frac{\overline{q}^s}{s!}\overline{\partial_s}^{j+s} f(q); \quad j \geq 1. \tag{5.3}$$

Therefore, the expression of f becomes

$$f(q, \overline{q}) = \varphi_0(q) + \sum_{j=1}^{n}\sum_{k=0}^{n-j}(-1)^k \frac{\overline{q}^{j+k}}{j!k!}\overline{\partial_s}^{j+k} f(q).$$

∎

Remark 5.1.4. The component functions in Proposition 5.1.2, of a given S-polyregular function f, are given in terms of f and its successive derivatives (see Equation (5.3)).

Thanks to these characterizations (Proposition 5.1.2 and Theorem 5.1.3) many interesting analytic properties of S-polyregular functions can be derived from their analogs of the slice regular functions. However, one must be careful since (as is the case for complex polyanalytic functions) several known properties for $\mathcal{SR}$ prove false when applied to $\mathcal{SR}_n$. For example, S-polyregular functions may even vanish on an accumulation set. This is the case of $1 - q\overline{q}$ which is a nonzero S-polyregular on $\mathbb{H}$ but vanishes on the closed set $\{q \in \mathbb{H}, |q| = 1\}$.

Similarly to the complex setting, the first order differential operator $\partial_s - \overline{q}$, will play a crucial rule in this theory. By considering the differential transformation

$$[\mathcal{H}_n(F)](q) := (\partial_s - \overline{q})^n(F)(q),$$

one proves the following.

Theorem 5.1.5. *Let F be a given S-regular function. Then, the functions $\mathcal{H}_n(F)$, $n = 0, 1, 2, \cdots$, are S-polyregular and form an orthogonal system in $L^2(\mathbb{C}_I; e^{-|q|^2}d\lambda_I)$.*

Proof. Notice first that

$$\mathcal{H}_n(F) = \sum_{j=0}^{n}(-1)^j \binom{n}{j}\overline{q}^j \partial_s^{n-j} F. \tag{5.4}$$

Therefore, we have

$$\overline{\partial}_s^n \mathcal{H}_n F = (-1)^n n! F.$$

Hence, $\mathcal{H}_n F$ is clearly S-polyregular of order n, for F being slice regular. This is also clear from (5.4) according to Proposition 5.1.2. Consequently, using the fact that $\partial_s - \overline{q}$ is the formal adjoint operator of $\overline{\partial}_s$ when acting on the Hilbert space $L^2(\mathbb{C}_I; e^{-|q|^2} d\lambda_I)$, one can prove that $(\partial_s - \overline{q})F$ is orthogonal to F when F is slice regular in $L^2(\mathbb{C}_I; e^{-|q|^2} d\lambda_I)$. More generally, if $n > m$, we have $\overline{\partial}_s^{n-m}(F) = 0$ and therefore

$$\langle \mathcal{H}_n(F), \mathcal{H}_m(F) \rangle = \left\langle F, \overline{\partial}_s^n \mathcal{H}_m(F) \right\rangle = (-1)^m m! \left\langle F, \overline{\partial}_s^{n-m}(F) \right\rangle = 0.$$

Thus, $\mathcal{H}_n(F)$, $n = 0, 1, 2, \cdots$, form an orthogonal system in $L^2(\mathbb{C}_I; e^{-|q|^2} d\lambda_I)$. ∎

Remark 5.1.6. By specifying $F(q) = F_m(q) = q^m$, we recover the quaternionic Hermite polynomials $H^Q_{m,n}$. Indeed,

$$[\mathcal{H}_n(F_m)](q) = (-1)^m e^{|q|^2} \partial_s^n (e^{-|q|^2} q^m) = H^Q_{m,n}(q, \overline{q}).$$

Theorem 5.1.7. *The following assertions hold true.*

(i) The space $\mathcal{SR}_n^2 := \mathcal{SR}_n \cap L^2(\mathbb{C}_I; e^{-|q|^2} d\lambda_I)$ is spanned by the polynomials $H^Q_{j,n}$, $j = 0, 1, 2, \cdots$. Moreover, we have

$$\mathcal{SR}_n^2 = \sum_{k=0}^{n} \mathcal{H}_k(\mathcal{SR}_0^2).$$

(ii) A function f belongs to $\mathcal{SR}_n^2 \cap Ker(\Box_q - nId)$ if and only if there exists some $F \in \mathcal{SR}_0^2$ such that $f = \mathcal{H}_n(F)$.

Proof. Let $f \in \mathcal{SR}_n^2$ and recall that $H^Q_{j,k}(x + Iy, x - Iy)$ is an orthogonal basis of $L^2(\mathbb{C}_I; e^{-|q|^2} d\lambda_I)$ (see [43, Theorem 4.2]). Thus, we can expand $f|_{\mathbb{C}_I}$ as

$$f|_{\mathbb{C}_I}(x + Iy) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} H^Q_{j,k}(x + Iy, x - Iy) \alpha_{jk}(I)$$

for some quaternionic sequence $\alpha_{jk}(I) \in \mathbb{H}$ satisfying the growth condition

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} j! k! |\alpha_{jk}(I)|^2 < +\infty.$$

Now since f is a polynomial in $\overline{q}$ of degree n (for f being in $\mathcal{SR}_n$), we conclude that $\alpha_{jk}(I) = 0$ for every $k > n$, so that

$$f|_{\mathbb{C}_I} = \sum_{k=0}^{n} \sum_{j=0}^{\infty} H^Q_{j,k}(q, \overline{q}) \alpha_{jk}(I). \tag{5.5}$$

Therefore,

$$f|_{\mathbb{C}_I} = \sum_{k=0}^{n}\sum_{j=0}^{\infty} \mathcal{H}_k(q^j)\alpha_{jk}(I) = \sum_{k=0}^{n} \mathcal{H}_k\left(\sum_{j=0}^{\infty} q^j\alpha_{jk}(I)\right) = \sum_{k=0}^{n} \mathcal{H}_k(F_k),$$

where F_k stands for $F_k = \sum_{j=0}^{\infty} q^j\alpha_{jk}(I)$, which clearly belongs to $\mathcal{SR}_0^2$. This completes the proof of (i). To prove (ii), we need only to establish the "only if". Thus, for $f \in \mathcal{SR}_n^2 \cap Ker(\Box_q - n)$, we assert that $\Box_q f = nf$ is equivalent to have

$$\sum_{k=0}^{n}\sum_{j=0}^{\infty} kH_{j,k}^{Q}(q,\overline{q})\alpha_{jk}(I) = \sum_{k=0}^{n}\sum_{j=0}^{\infty} nH_{j,k}^{Q}(q,\overline{q})\alpha_{jk}(I)$$

thanks to (5.5) combined with $\Box_q H_{j,k}^{Q}(q,\overline{q}) = kH_{j,k}^{Q}(q,\overline{q})$ (see [43]). By identification, we get $\alpha_{jk}(I) = 0$ for every $k \neq n$. This completes the proof. ∎

The following result is a Splitting Lemma for the S-polyregular functions generalizing the standard one for the slice regular functions.

Proposition 5.1.8 (Splitting lemma for S-polyregular functions). *If f is a S-polyregular function, then for every $I \in \mathbb{S}$, and every $J \in \mathbb{S}$ perpendicular to I, there are two polyanalytic functions $F, G : \mathbb{C}_I \longrightarrow \mathbb{C}_I$ such that for any $q = x + Iy$, we have*

$$f|_{\mathbb{C}_I}(q) = F(q) + G(q)J.$$

Remark 5.1.9. The proof of Proposition (5.1.8) readily follows from Proposition 5.1.2 and the standard Splitting Lemma ([61]) for the slice regular functions applied to each component function φ_k. It can also be handled using sliceness. In fact, each slice function f on $\mathbb{H}$ (not necessarily regular) can be split as

$$f|_{\mathbb{C}_I}(x + Iy) = F(x + Iy) + G(x + Iy)J$$

, where $J \perp I$ (see e.g. [64]). Then, f is polyregular of order n if and only if F and G are polyanalytic of order n_F and n_G, respectively, with $n = \max\{n_F, n_G\}$.

An analogue of the Identity Principle for the S-polyregular functions can also be obtained. To this end, we begin by recalling the standard one for the slice regular functions on slice domains.

Definition 5.1.10 ([61]). A domain $U \subset \mathbb{H}$ such that $U \cap \mathbb{R} \neq \emptyset$ is said to be slice, if for every arbitrary $I \in \mathbb{S}$ the set $U_I := U \cap L_I$ is a domain of the complex plane $\mathbb{C}_I := \mathbb{R} + \mathbb{R}I$.

Lemma 5.1.11 ([61]). *Let $f : U \longrightarrow \mathbb{H}$ be a slice regular function on a slice domain U. Denote by $Z_f = \{q \in U; f(q) = 0\}$ the zero set of f. If there exists $I \in \mathbb{S}$ such that $\mathbb{C}_I \cap Z_f$ has an accumulation point, then $f \equiv 0$ on U.*

This principle is no longer valid for S-polyregular functions as shown by the counterexample $1 - q\overline{q}$. However, we can provide a weak version of such uniqueness theorem.

Proposition 5.1.12 (Identity Principle for S-polyregular functions). *Let f be a S-polyregular function in $\mathcal{SR}_n$ such that f is identically zero on a sub-domain $\Omega \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$. Then f is identically zero on the whole $\mathbb{H}$.*

Proof. By Proposition 5.1.2, we can write $f \in \mathcal{SR}_n$ as $f(q) = \sum_{k=0}^{n} \overline{q}^k \varphi_k(q)$ with $\varphi_k \in \mathcal{SR}$. Now, by the assumption that $f|_\Omega \equiv 0$ with Ω is a subdomain of some slice $\mathbb{C}_I$, we obtain

$$n!\varphi_n|_\Omega(x+Iy) = \overline{\partial_I}^n \left(\sum_{k=0}^{n} (x-Iy)^k \varphi_k|_\Omega(x+Iy) \right) \equiv 0.$$

Repeating this procedure, we conclude that $\varphi_k|_\Omega \equiv 0$ for every $k = n, n-1, \cdots, 1, 0$. Therefore, $\varphi_k \equiv 0$ on the whole $\mathbb{H}$ by Lemma 5.1.11. This implies that $f \equiv 0$ on $\mathbb{H}$. ∎

Remark 5.1.13. Other powerful uniqueness theorems as well as additional properties for the *S*-polyregular functions can be obtained. They will be the subject of a forthcoming investigation.

5.1.3 Star product for S-polyregular functions.

Recall first that the left $\star_s^L$-product for left slice regular functions is defined by

$$(f \star_s^L g)(q) = \sum_{n=0}^{\infty} q^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right) \tag{5.6}$$

for given convergent series $f(q) = \sum_{n=0}^{\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{\infty} q^n b_n$ on $\mathbb{H}$. This is in fact the product of two formal series with coefficients in a ring [48]. The performed series in (5.6) is convergent on $\mathbb{H}$ and is a slice regular function [60]. This product is introduced to overcome the fact that the pointwise product of left slice regular functions is not necessarily a left slice regular function, but it is a S-polyregular function under further assumptions (see [57] for details). For interesting results on the left $\star_s^L$-product, one can refer to [13,61] and references therein. To solve analogue problem in the context of left S-polyregular functions, a natural extension of the $\star_s^L$-product can be defined by considering

$$(f \star_{sp}^L g)(q, \overline{q}) = \sum_{\substack{j=0,1,\cdots,m \\ k=0,1,\cdots,n}} \overline{q}^{j+k} (\varphi_j \star_s^L \psi_k)(q) \tag{5.7}$$

for given $f(q,\overline{q}) = \sum_{j=0}^{m} \overline{q}^j \varphi_j(q) \in \mathcal{SR}_m$ and $g(q,\overline{q}) = \sum_{k=0}^{n} \overline{q}^k \psi_k(q) \in \mathcal{SR}_n$. We define in a similar way the right star product for right S-polyregular functions $f(q,\overline{q}) = \sum_{j=0}^{m} \varphi_j(q)\overline{q}^j$

and $g(q,\overline{q}) = \sum\limits_{k=0}^{n} \psi_k(q)\overline{q}^k$ as follows

$$(f \star_{sp}^{R} g)(q,\overline{q}) = \sum_{\substack{j=0,1,\cdots,m \\ k=0,1,\cdots,n}} (\varphi_j \star_s^R \psi_k)(q) q^{j+k}. \tag{5.8}$$

Thus, one can easily check the following

Lemma 5.1.14. *For every $f \in \mathcal{SR}_m$ and $g \in \mathcal{SR}_n$, we have*

(i) $\overline{f \star_{sp}^{L} g} = \overline{g} \star_{sp}^{R} \overline{f}$, *where $\overline{f}(q) = \overline{f(q)}$ denotes the algebraic conjugation.*

(ii) $f \star_{sp}^{L} g = g \star_{sp}^{L} f$ *if the coefficients of any components slice regular functions φ_j and ψ_k commute.*

Proof. Assertion (i) follows by taking the algebraic conjugate in (5.7) and next using the well-established fact $\overline{\varphi_j \star_s^L \psi_k} = \overline{\psi_k} \star_s^R \overline{\varphi_j}$ for slice regular functions φ_j and ψ_k. The second assertion is immediate by comparing $f \star_{sp}^{L} g$ and $g \star_{sp}^{L} f$. ∎

A characterization for two S-polyregular functions to commute with respect to the $\star_{sp}^{L}$-product can be obtained, generalizing the one given in [13] for $\mathbb{C}_J$-preserving slice regular functions.

Definition 5.1.15 ([13,32]). Let $J \in \mathbb{S}$. A slice regular function φ is said to be $\mathbb{C}_J$-preserving if both F and G in its stem function, $\varphi = \mathcal{I}(F + iG)$ are $\mathbb{C}_J$-valued.

Definition 5.1.16. A S-polyregular function $f(q,\overline{q}) = \sum\limits_{k=0}^{n} \overline{q}^k \varphi_k(q)$ is said to be $\mathbb{C}_J$-preserving, for given $J \in \mathbb{S}$, if their components slice regular functions φ_k are $\mathbb{C}_J$-preserving.

Lemma 5.1.17. *If f and g are two S-polyregular $\mathbb{C}_J$-preserving functions for given $J \in \mathbb{S}$, then $f \star_{sp}^{L} g = g \star_{sp}^{L} f$.*

Proof. The proof follows by making use of the fact that for $\mathbb{C}_J$-preserving functions φ and ψ, the $\star_s^L$-product satisfies $\varphi \star_s^L \psi = \psi \star_s^L \varphi$ (see [13]). ∎

As basic example of computation with such $\star_{sp}^{L}$-product, we explicit the one of the following function

$$S_k(\overline{p}, p; q, \overline{q}) := \left(|p-q|^2_{\star_{sp}^{L}} \right)^{k\star_{sp}^{L}},$$

with $|p-q|^2_{\star_{sp}^{L}} := (p-q) \star_{sp}^{L} \overline{(p-q)} = h_q(p) \star_{sp}^{L} \overline{h_q(p)}$, where we have set $h_q(p) = p - q$. Namely, we assert the following.

Lemma 5.1.18. *For every $k = 1, 2, \cdots$, and $p, q \in \mathbb{H}$, we have*

$$S_k(\overline{p}, p; q, \overline{q}) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \overline{p}^{k-j} h_q^{k\star_s^L}(p) \overline{q}^j. \tag{5.9}$$

Proof. The proof can be handled by induction. Indeed, direct computation shows that for $k = 1, 2$, we have

$$S_1(\overline{p}, p; q, \overline{q}) = \overline{p}(p-q) - (p-q)\overline{q} = \overline{p}h_q(p) - h_q(p)\overline{q}$$

and

$$\begin{aligned} S_2(\overline{p}, p; q, \overline{q}) &= \left(\overline{p}h_q(p) - h_q(p)\overline{q}\right) \star^L_{sp} \left(\overline{p}h_q(p) - h_q(p)\overline{q}\right) \\ &= \overline{p}^2 h_q^{2\star^L_s}(p) - \overline{p}h_q^{2\star^L_s}(p)\overline{q} - \overline{p}h_q^{2\star^L_s}(p)\overline{q} + h_q^{2\star^L_s}(p)\overline{q}^2 \\ &= \overline{p}^2 h_q^{2\star^L_s}(p) - 2\overline{p}h_q^{2\star^L_s}(p)\overline{q} + h_q^{2\star^L_s}(p)\overline{q}^2. \end{aligned}$$

Now, assume that (5.1.19) holds true for fixed k. Then, we have

$$\begin{aligned} S_{k+1}(\overline{p}, p; q, \overline{q}) &= S_k(\overline{p}, p; q, \overline{q}) \star^L_{sp} S_1(\overline{p}, p; q, \overline{q}) \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\overline{p}^{k-j} h_q^{k\star^L_s}(p)\overline{q}^j\right) \star^L_{sp} \left(\overline{p}h_q(p) - h_q(p)\overline{q}\right) \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \overline{p}^{k+1-j} h_q^{(k+1)\star^L_s}(p)\overline{q}^j \\ &\quad - \sum_{j=0}^{k} (-1)^j \binom{k}{j} \overline{p}^{k-j} h_q^{(k+1)\star^L_s}(p)\overline{q}^{j+1}. \end{aligned}$$

Making the change of indices in the second sum in the right-hand side and using the well-known identity $\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}$, we get

$$\begin{aligned} S_{k+1}(\overline{p}, p; q, \overline{q}) &= \overline{p}^{k+1} h_q^{(k+1)\star^L_s}(p) + (-1)^{k+1} h_q^{(k+1)\star^L_s}(p)\overline{q}^{k+1} \\ &\quad + \sum_{j=1}^{k} (-1)^j \left(\binom{k}{j} + \binom{k}{j-1}\right) \overline{p}^{k+1-j} h_q^{(k+1)\star^L_s}(p)\overline{q}^j \\ &= \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} \overline{p}^{k+1-j} h_q^{(k+1)\star^L_s}(p)\overline{q}^j. \end{aligned}$$

This competes the proof. ∎

Accordingly, it is clear that the following assertions hold true.

(i) The function $p \longmapsto S_k(\overline{p}, p; q, \overline{q})$ is left S-polyregular for every fixed q.

(ii) The function $q \longmapsto S_k(\overline{p}, p; q, \overline{q})$ is right S-polyregular for every fixed p.

(iii) We have $\overline{S_k(\overline{p}, p; q, \overline{q})} = S_k(q, \overline{q}; \overline{p}, p)$ for every $p, q \in \mathbb{H}$.

The next result concerns the function on $\mathbb{H} \times \mathbb{H}$ defined by

$$L^{(\gamma,S_1)}_{\star n}(p,q) := L^{(\gamma)}_{\star n}(S_1(\overline{p},p;q,\overline{q})) = L^{(\gamma)}_{\star n}\left(|p-q|^2_{\star^L_{sp}}\right), \tag{5.10}$$

where $L^{(\gamma)}_{\star n}$ is essentially the generalized Laguerre polynomial $L^{(\gamma)}_{n}$ ([99]) but with respect to the $\star^L_{sp}$-product. It will be used to obtain the explicit expression of the reproducing kernels for the S-polyregular Bargmann spaces (see Section 5.2).

Lemma 5.1.19. *The function* $L^{(\gamma,S_1)}_{\star n}$ *in* (5.10) *satisfies the properties* (i), (ii) *and* (iii) *above.*

Proof. The proof readily follows since $L^{(\gamma)}_{\star n}$ is a finite linear expansion of the functions $S_k(\overline{p},p;q,\overline{q})$ with real coefficients. More precisely, we have

$$L^{(\gamma)}_{\star n}(S_1(\overline{p},p;q,\overline{q})) = \sum_{k=0}^{n} \frac{\Gamma(\gamma+n+1)}{\Gamma(n-k+1)\Gamma(\gamma+k+1)} \frac{(-1)^k}{k!} S_k(\overline{p},p;q,\overline{q}),$$

where Γ denotes the classical gamma function. ∎

In the next sections, we introduce two classes of infinite dimensional right quaternionic reproducing kernels Hilbert spaces that will be considered as the quaternionic analog of complex polyanalytic Bargmann spaces.

5.2 S-polyregular Bargmann spaces

The well-known analytic Hilbert spaces on the complex plane have been generalized to various contexts such as the quaternion setting (see for example [9,37,64**?**]). Thus, the idea of generalizing the true-polyanalytic Bargmann spaces ([4,59,114]) to the slice polyregular case is rather natural. This is the aim of the present section. To this end, let $\mathcal{SR}^2_{1,n}$ denote the space of all convergent series

$$f(q,\overline{q}) = \sum_{k=0}^{n}\sum_{j=0}^{\infty} \overline{q}^k q^j \alpha_{jk}; \quad \alpha_{j,k} \in \mathbb{H},$$

on $\mathbb{H}$, belonging to the right $\mathbb{H}$-vector space $\mathcal{SR}^2_n := \mathcal{SR}_n \cap L^2(\mathbb{C}_{I_0}, e^{-|\xi|^2}d\lambda)$, for some $I_0 \in \mathbb{S}$.

The particular case of $n = 0$ gives rise to the slice Bargmann space $\mathcal{F}^2_{slice}$ considered in [9], for which the monomials $e_m(q) := q^m$ constitute an orthogonal basis. In contrast to what one can think, the monomials $e_{j,k}(q,\overline{q}) := q^j\overline{q}^k$ does not form an orthogonal system in $\mathcal{SR}^2_n$ as showed by

$$\langle e_{j,0}, e_{j+k,k}\rangle_{\mathbb{C}_I} = \|e_{j+k}\|^2_{\mathbb{C}_I} = \pi(j+k)!.$$

Thus, motivated by Theorem 5.1.7, we will make use of the univariate quaternionic Hermite polynomials $H^Q_{j,k}$, instead of monomials $e_{j,k}$, to describe $\mathcal{SR}^2_n$.

Proposition 5.2.1. *A function f belongs to $\mathcal{SR}^2_{1,n}$ if and only it can be expanded as follows*

$$f(q,\overline{q}) = \sum_{k=0}^{n} \sum_{j=0}^{+\infty} H^Q_{j,k}(q,\overline{q})\alpha_{j,k}$$

for some quaternionic constants $\alpha_{j,k}$ satisfying the growth condition

$$\sum_{j=0}^{+\infty} j!|\alpha_{j,k}|^2 < +\infty$$

for every $k = 0, 1, \cdots, n$.

Proof. The direct implication follows making use of [43, Proposition 3.8], expressing the monomials $\overline{q}^k q^j$ in terms of $H^Q_{r,s}$,

$$q^m\overline{q}^n = m!n! \sum_{k=0}^{min(m,n)} \frac{H^Q_{m-k,n-k}(q,\overline{q})}{k!(m-k)!(n-k)!}. \tag{5.1}$$

The orthogonality

$$\left\langle H^Q_{m,n}, H^Q_{j,k} \right\rangle_{\mathbb{C}_I} = \pi m!n!\delta_{m,j}\delta_{n,k} \tag{5.2}$$

of $H^Q_{r,s}$ shows that the condition $\|f\|_{\mathbb{C}_I} < +\infty$ becomes equivalent to

$$\begin{aligned} \|f\|^2_{\mathbb{C}_I} &= \sum_{k,k'=0}^{n} \sum_{j,j'=0}^{+\infty} \overline{\alpha_{j,k}} \left\langle H^Q_{j,k}, H^Q_{j',k'} \right\rangle_{\mathbb{C}_I} \alpha_{j',k'} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{+\infty} |\alpha_{j,k}|^2 \left\| H^Q_{j,k} \right\|^2 < +\infty. \end{aligned}$$

The argument for obtaining the inverse implication is Theorem 5.1.7. ∎

Definition 5.2.2. The right quaternionic vector space $\mathcal{SR}^2_{1,n}$, generalizing the slice hyperholomorphic Bargmann space $\mathcal{F}^2_{slice}$, is called S-polyregular Bargmann space of first kind and level n (ordre n).

Another interesting subspace to deal with is the following

$$\mathcal{SR}^2_{2,k} := \left\{ \sum_{j=0}^{+\infty} H^Q_{j,k}(q,\overline{q})c_j; c_j \in \mathbb{H}, \quad \sum_{j=0}^{+\infty} j!|c_j|^2 < +\infty \right\}.$$

Definition 5.2.3. The right quaternionic vector space $\mathcal{SR}^2_{2,k}$ is called here S-polyregular Bargmann space of second kind and (exact) level k.

Theorem 5.2.4. *The spaces $\mathcal{SR}^2_{1,n}$ and $\mathcal{SR}^2_{2,k}$ are Hilbert spaces with orthogonal basis $\{H^Q_{j,k}; k = 0,1,\cdots,n;\ j = 0,1,\cdots\}$ and $\{H^Q_{j,k};, j = 0,1,\cdots\}$, respectively. Moreover, we have*

$$\mathcal{SR}^2_{1,n} = \bigoplus_{k=0}^{n} \mathcal{SR}^2_{2,k}. \tag{5.3}$$

Proof. As for $n = 0$, it is not difficult to see that the considered spaces are closed subspaces of the Hilbert space $L^2(\mathbb{C}_I; e^{-|q|^2}d\lambda_I)$, and therefore they are right quaternionic Hilbert spaces. Now, for fixed nonnegative integer k, the polynomials $H^Q_{j,k}$, $j = 0,1,2,\cdots$, form an orthogonal system with respect to the gaussian measure and generate $\mathcal{SR}^2_{2,k}$. Their linear independence is equivalent to their completion. In fact, for a given $g = \sum\limits_{j=0}^{+\infty} H^Q_{j,k}c_j \in \mathcal{SR}^2_{2,k}$, the condition that $\left\langle f, H^Q_{\ell,k} \right\rangle = 0$, for every $\ell = 0,1,2,\cdots$, implies that $c_\ell = 0$ and therefore g is identically zero on $\mathbb{H}$, for $\left\langle f, H^Q_{\ell,k} \right\rangle_{\mathbb{C}_I} = \overline{c_j} \left\| H^Q_{\ell,k} \right\|^2_{\mathbb{C}_I}$. Thus, $\{H^Q_{j,k}, j = 0,1,\cdots\}$ is orthogonal basis of $\mathcal{SR}^2_{2,k}$. The assertion that $\{H^Q_{j,k}, k = 0,1,\cdots,n,\ j = 0,1,\cdots\}$ form an orthogonal basis of $\mathcal{SR}^2_{1,n}$ follows in a similar way. It is also an immediate consequence decomposition which (5.3) readily follows since for given $f \in \mathcal{SR}^2_{1,n}$, we have

$$f = \sum_{k=0}^{n} \sum_{j=0}^{+\infty} H^Q_{j,k}\alpha_{j,k} = \sum_{k=0}^{n} g_k,$$

where

$$g_k := \sum_{j=0}^{+\infty} H^Q_{j,k}\alpha_{j,k}.$$

are clear in $\mathcal{SR}^2_{2,k}$. In addition, the family $\{g_k, k = 0,1,\cdots,n\}$ is orthogonal, since for $k \neq k'$, we have

$$\langle g_k, g_{k'} \rangle_{\mathbb{C}_I} = \left(\sum_{j,j'=0}^{+\infty} \overline{\alpha_{j,k}}\alpha_{j',k'}\delta_{j,j'} \left\| H^Q_{j,k} \right\|^2_{\mathbb{C}_I} \right) \delta_{k,k'} = 0.$$

Moreover,

$$\|f\|^2_{\mathbb{C}_I} = \sum_{k=0}^{n} \|g_k\|^2_{\mathbb{C}_I} = \pi \sum_{k=0}^{n} \sum_{j=0}^{+\infty} j!k!|\alpha_{j,k}|^2. \tag{5.4}$$

∎

In order to show that the considered Hilbert spaces $\mathcal{SR}^2_{1,n}$ and $\mathcal{SR}^2_{2,k}$ possess reproducing kernels, we need the following.

Lemma 5.2.5. *For every fixed $q \in \mathbb{H}$, the evaluation map $\delta_q f = f(q,\overline{q})$ is a continuous linear form* *on the Hilbert spaces $\mathcal{SR}^2_{1,n}$ and $\mathcal{SR}^2_{2,k}$. Moreover, we have*

$$|f(q,\overline{q})| \leq \frac{1}{\sqrt{\pi}} e^{\frac{|q|^2}{2}} \|f\|_{\mathbb{C}_I}. \tag{5.5}$$

for every $f \in \mathcal{SR}^2_{1,n}$ and therefore for every $f \in \mathcal{SR}^2_{2,k}$.

Proof. Let $g \in \mathcal{SR}^2_{2,k}$ such that $g = \sum\limits_{j=0}^{+\infty} H^Q_{j,k} c_j$. Using the Cauchy-Schwartz inequality and the expression of the square norm of g, $\|g\|^2_{\mathbb{C}_I} = \pi k! \sum\limits_{j=0}^{+\infty} j! |c_j|^2$, we get

$$|g(q,\overline{q})| \leq \left(\sum_{j=0}^{+\infty} \frac{|H^Q_{j,k}(q,\overline{q})|^2}{\pi j! k!} \right)^{\frac{1}{2}} \|g\|_{\mathbb{C}_I}. \tag{5.6}$$

The series in the right-hand side of (5.6) is absolutely convergent on $B(0,r_0)$ for every fixed r_0 and is independent of g. This follows making use of the following upper bound (see [43, Corollary 4.3]):

$$\left| H^Q_{n+k,n}(q,\overline{q}) \right| \leq \frac{(n+k)!}{k!} |q|^k e^{\frac{|q|^2}{2}}. \tag{5.7}$$

More explicitly, by means of [20, Eq. (3.8)], we have

$$\sum_{j=0}^{+\infty} \frac{|H^Q_{j,k}(q,\overline{q})|^2}{\pi j! k!} = \frac{e^{|q|^2}}{\pi}. \tag{5.8}$$

This proves that

$$|g(q,\overline{q})| \leq \frac{1}{\sqrt{\pi}} e^{\frac{|q|^2}{2}} \|g\|_{\mathbb{C}_I}. \tag{5.9}$$

Now, for $f \in \mathcal{SR}^2_{1,n}$, we have $f = \sum\limits_{k=0}^{n} g_k$ with $g_k \in \mathcal{SR}^2_{2,k}$. Therefore, we obtain

$$|f(q,\overline{q})|^2 \leq \sum_{k=0}^{n} |g_k(q,\overline{q})|^2 \leq \sum_{k=0}^{n} \frac{1}{\pi} e^{|q|^2} \|g_k\|^2 \leq \frac{1}{\pi} e^{|q|^2} \|f\|^2_{\mathbb{C}_I}$$

by means of (5.4) and (5.9). This completes the proof. ∎

The previous Lemma combined with the quaternionic version of the Riesz' representation theorem [111, Theorem 1] ensures the existence of the reproducing kernels of $\mathcal{SR}^2_{1,n}$ and $\mathcal{SR}^2_{2,k}$. The next result gives their explicit expressions in terms of the Laguerre polynomial

$L^{(\gamma)}_{\star n}$ and the special convergent series

$$e_*^{[a,b]} := \sum_{n=0}^{+\infty} \frac{a^k b^k}{k!}.$$

Theorem 5.2.6. *The reproducing kernels of $\mathcal{SR}^2_{1,n}$ and $\mathcal{SR}^2_{2,k}$ are given respectively by*

$$\mathcal{K}_{1,n}(p,q) = \frac{1}{\pi} e_*^{[\overline{p},q]} \quad \overset{L}{\star}_{sp} \quad L^{(1)}_{\star n}(|p-q|^2_{\overset{L}{\star}_{sp}}) \tag{5.10}$$

and

$$\mathcal{K}_{2,k}(p,q) = \frac{1}{\pi} e_*^{[\overline{p},q]} \quad \overset{L}{\star}_{sp} \quad L_{\star k}(|p-q|^2_{\overset{L}{\star}_{sp}}). \tag{5.11}$$

Shortly, we have

$$f(p,\overline{p}) = \langle \mathcal{K}_{1,n}(p,\cdot), f\rangle_{\mathbb{C}_I} \quad \text{and} \quad g(p,\overline{p}) = \langle \mathcal{K}_{2,k}(p,\cdot), g\rangle_{\mathbb{C}_I}$$

for every $f \in \mathcal{SR}^2_{1,n}$ and $g \in \mathcal{SR}^2_{2,k}$.

Proof. For $\mathcal{SR}^2_{2,k}$, the computation of $\mathcal{K}_{2,k}(p,q)$ can be done by performing

$$\mathcal{K}_{2,k}(p,q) = \frac{1}{\pi k!} \sum_{j=0}^{+\infty} \frac{H^Q_{j,k}(q,\overline{q}) H^Q_{k,j}(p,\overline{p})}{j!},$$

since $\{H^Q_{j,k}; j = 0, 1, \cdots\}$ is an orthogonal basis of $\mathcal{SR}^2_{2,k}$ (see Theorem 5.2.4) and $\overline{H^Q_{k,j}} = H^Q_{j,k}$. For real $q = x$ or for p, q belonging to the same slice $\mathbb{C}_I$, the result follows by means of

$$\begin{aligned} \frac{1}{\pi j!} \sum_{k=0}^{+\infty} \frac{H^Q_{j,k}(z,\overline{z}) H^Q_{k,j}(w,\overline{w})}{k!} &= \frac{(-1)^j}{\pi j!} e^{\overline{z}w} H^Q_{j,j}(z-w, \overline{z}-\overline{w}) \\ &= \frac{1}{\pi} e^{\overline{z}w} L_j(|z-w|^2) \end{aligned} \tag{5.12}$$

which is an immediate consequence of Theorem 2.3 in [20], stating that

$$\sum_{j=0}^{+\infty} \frac{t^j}{j!} H_{k,j}(z,\overline{z}) H_{j,k'}(w,\overline{w}) = (-t)^{k'} H_{kk'}(z - tw, \overline{z} - \overline{t}\overline{w}) e^{t\overline{z}w}$$

is true for every $|t| = 1$ and $z, w \in \mathbb{C}$, combined with the fact that $H^Q_{k,k}(\xi,\bar{\xi}) = (-1)^k k! L_k(|\xi|^2)$, where $L_k = L^{(1)}_k$ is the classical Laguerre polynomial of degree k.

Now, for given fixed non-real q, let I_q be in $\mathbb{S}$ such that $q \in \mathbb{C}_{I_q}$. The functions

$$\varphi : p \longmapsto \mathcal{K}_{2,k}(p,q) \tag{5.13}$$

and

$$\psi : p \longmapsto \frac{1}{\pi} e_*^{[\overline{p},q]} \overset{L}{\star_{sp}} L_{\star k}(|p-q|^2_{\overset{L}{\star_{sp}}}) \tag{5.14}$$

are clearly S-polyregular by the definition of the $\star_{sp}$-product (see (5.7)) and Lemma 5.1.19. Moreover, they coincide on the slice $\mathbb{C}_{I_q}$ by means of (5.12). Thus, by invoking the Identity Principle for S-polyregular functions (Proposition 5.1.12), we conclude that $\phi \equiv \psi$ on the whole $\mathbb{H}$, for arbitrary $q \in \mathbb{H}$. Therefore, we have

$$\mathcal{K}_{2,k}(p,q) = \frac{1}{\pi} e_*^{[\overline{p},q]} \overset{L}{\star_{sp}} L_{\star k}(|p-q|^2_{\overset{L}{\star_{sp}}})$$

for $p, q \in \mathbb{H}$. This completes our check for (5.11).

To conclude for Theorem 5.2.6, it suffices to observe that since $\mathcal{SR}^2_{1,n} = \bigoplus_{k=0}^{n} \mathcal{SR}^2_{2,k}$, we have

$$\mathcal{K}_{1,n}(p,q) = \sum_{k=0}^{n} \mathcal{K}_{2,k}(p,q).$$

Hence, in virtue of $\sum_{k=0}^{n} L_k^{(\gamma)}(x) = L_n^{(\gamma+1)}(x)$ (see [99, Eq. 12, p. 203]), we get

$$\begin{aligned}\mathcal{K}_{1,n}(p,q) &= \sum_{k=0}^{n} \frac{1}{\pi} e_*^{[\overline{p},q]} \overset{L}{\star_{sp}} L_{\star k}(|p-q|^2_{\overset{L}{\star_{sp}}}) \\ &= \frac{1}{\pi} e_*^{[\overline{p},q]} \overset{L}{\star_{sp}} L^{(1)}_{\star n}(|p-q|^2_{\overset{L}{\star_{sp}}}).\end{aligned}$$

∎

Remark 5.2.7. The restriction of $\mathcal{K}_{2,k}$ to $\mathbb{C}_i \times \mathbb{C}_i$ coincides with the reproducing kernel of the true-polyanalytic Bargmann space [14,20,59]. Indeed, we have

$$\mathcal{K}_{2,k}|_{\mathbb{C}_i\times\mathbb{C}_i}(z,w) = \frac{1}{\pi} e^{\bar{z}w} L_k(|z-w|^2),$$

so that for $k = 0$, we recover the one of the classical Bargmann space $\frac{1}{\pi} e^{\bar{z}w}$.

Remark 5.2.8. The expression of $\mathcal{K}_{1,n}(p,q)$ can be rewritten in the equivalent form

$$\mathcal{K}_{1,n}(p,q) = \frac{1}{\pi} L^{(1)}_{\star n}(|p-q|^2_{\overset{L}{\star_{sp}}}) \overset{L}{\star_{sp}} e_*^{[\overline{p},q]}, \tag{5.15}$$

thanks to (ii) of Lemma 5.1.14. The same observation holds true for $\mathcal{K}_{2,k}(p,q)$.

Remark 5.2.9. The operator $f \longmapsto P_k f$ given by $P_k f(p,\overline{p}) = \langle \mathcal{K}_{2,k}(p,\cdot), f \rangle_{\mathbb{C}_I}$, defined on the whole $\mathbb{H}$, defines a sort of extended orthogonal projection of $L^2(\mathbb{C}_I; e^{-|q|^2} d\lambda_I)$ onto $\mathcal{SR}^2_{2,k}$.

More explicitly, it reads

$$P_k f(p,\overline{p}) = \frac{1}{\pi}\int_{\mathbb{C}_I} \overline{e_*^{[\overline{p},q]} \overset{L}{\star_{sp}} L_{\star k}(|p-q|^2_{\overset{L}{\star_{sp}}})} f(q,\overline{q}) e^{-|q|^2} d\lambda_I(q) \tag{5.16}$$

for arbitrary $p \in \mathbb{H}$, which we can rewrite also as

$$P_k f(p,\overline{p}) = \frac{1}{\pi}\int_{\mathbb{C}_I} L_{\star k}(|p-q|^2_{\overset{R}{\star_{sp}}}) \overset{R}{\star_{sp}} \overline{e_*^{[\overline{q},p]}} f(q,\overline{q}) e^{-|q|^2} d\lambda_I(q) \tag{5.17}$$

by means of (ii) in Lemma 5.1.14.

We conclude this section with the following result giving an orthogonal hilbertian decomposition of the Hilbert space $L^2(\mathbb{C}_I; e^{-|q|^2} d\lambda_I)$.

Theorem 5.2.10. *We have*

$$L^2(\mathbb{C}_I; e^{-|q|^2} d\lambda_I) = \bigoplus_{k=0}^{+\infty} \mathcal{SR}^2_{2,k}.$$

Proof. Notice first that such decomposition is equivalent to say that the orthogonal complement of $\bigoplus\limits_{k\geq 0} \mathcal{SR}^2_{2,k}$ in $L^2(\mathbb{C}_I; e^{-|q|^2} d\lambda_I)$ is $\{0\}$. To this end, we claim that

$$T(t|q) := \int_{\mathbb{C}_I} \frac{1}{(1-t)} \overline{e_*^{[\overline{q},\xi]} \overset{L}{\star_{sp}} \exp\left(-\frac{t}{1-t}|q-\xi|^2_{\overset{L}{\star_{sp}}}\right)} f(\xi) e^{-|\xi|^2} d\lambda(\xi) = 0 \tag{5.18}$$

holds for every $f \in \left(\bigoplus\limits_{k\geq 0} \mathcal{SR}^2_{2,k}\right)^{\perp}$, every $t \in]0,1[$ and every fixed $q \in \mathbb{H}$. In fact, this follows readily making use of the generating function for the Laguerre polynomials ([99, Eq. (14), p. 135])

$$\sum_{k=0}^{\infty} t^k L_k^{(\alpha)}(\xi) = \frac{1}{(1-t)^{\alpha+1}} \exp\left(\frac{t\xi}{t-1}\right).$$

Indeed,

$$\begin{aligned} T(t|q) &= \int_{\mathbb{C}_I} \overline{e_*^{[\overline{q},\xi]} \overset{L}{\star_{sp}} \left(\sum_{k=0}^{+\infty} t^k L_{\star k}(|w-q|^2_{\overset{L}{\star_{sp}}})\right)} f(\xi) e^{-|\xi|^2} d\lambda(\xi) \\ &= \sum_{k=0}^{\infty} t^k \int_{\mathbb{C}_I} \overline{e_*^{[\overline{q},\xi]} \overset{L}{\star_{sp}} L_{\star k}(|\xi-q|^2_{\overset{L}{\star_{sp}}})} e^{-|\xi|^2} f(\xi) d\lambda(\xi) \\ &= 0. \end{aligned}$$

The limit $t \longrightarrow 1^-$ in (5.18) yields an integral involving the Dirac δ-function at the point $q \in \mathbb{H}$. From that, the left-hand side of (5.18) reduces further to $e_*^{[\overline{q},\xi]} f(\xi) e^{-|\xi|^2}$. Therefore, $f(q) = 0$ for every $q \in \mathbb{H}$. ∎

5.3 Segal–Bargmann transforms for S-polyregular Bargmann spaces

In this section, we introduce a family of suitable Bargmann's type transforms defined on the right quaternionic Hilbert space $L^2_{\mathbb{H}}(\mathbb{R}; dt)$, consisting of all square integrable $\mathbb{H}$-valued functions with respect to the inner product

$$\langle f, g\rangle_{\mathbb{R}} := \int_{\mathbb{R}} \overline{f(t)} g(t) dt.$$

Their images will be the S-polyregular Bargmann spaces defined and studied in the previous section. To this end, we define the kernel functions $B_{\ell,n}(x;q)$, $\ell = 1, 2$, on $\mathbb{R} \times \mathbb{H}$ to be the bilinear generating functions

$$B_{2,k}(x;q) = \sum_{j=0}^{+\infty} \frac{h_j(t)\overline{H^Q_{j,k}(q,\overline{q})}}{\|h_j\|_{\mathbb{R}} \left\|H^Q_{j,k}\right\|_{\mathbb{C}_I}} \tag{5.1}$$

and

$$B_{1,n}(x;q) = \sum_{k=0}^{n} B_{2,k}(x;q), \tag{5.2}$$

where $h_j(t)$ denotes the j-th real Hermite function associated to the real Hermite polynomial $H_j(t)$,

$$h_j(t) = (-1)^j e^{\frac{t^2}{2}} \frac{d^j}{dt^j}(e^{-t^2}) = e^{-\frac{t^2}{2}} H_j(t). \tag{5.3}$$

We recall that the real Hermite functions form an orthogonal basis of $L^2_{\mathbb{H}}(\mathbb{R}; dt)$, with square norm given by

$$\|h_j\|^2_{\mathbb{R}} = 2^j j! \sqrt{\pi}. \tag{5.4}$$

Thus, we assert

Theorem 5.3.1. *For every $t \in \mathbb{R}$ and $q \in \mathbb{H}$, the kernel function $B_{2,k}$ is given bythe closed formula*

$$B_{2,k}(t;q) := \left(\frac{1}{\pi}\right)^{\frac{3}{4}} \frac{1}{\sqrt{2^k k!}} \exp\left(-\frac{t^2+\overline{q}^2}{2} + \sqrt{2}\overline{q}t\right) H_k\left(\frac{q+\overline{q}}{\sqrt{2}} - t\right). \tag{5.5}$$

Moreover, the function $B_{2,k;q} : t \longmapsto B_{2,k}(t;q)$ belongs to $L^2_{\mathbb{H}}(\mathbb{R}; dt)$ for every fixed $q \in \mathbb{H}$, and we have

$$\|B_{2,k;q}\|_{\mathbb{R}} = \frac{1}{\sqrt{\pi}} e^{\frac{|q|^2}{2}}. \tag{5.6}$$

Proof. The explicit expression of the kernel function $B_{2,k}(t;q)$ can be obtained by [43, Theorem 5.7]. For the second assertion, fix $q = x + Iy$ in $\mathbb{H}$ and write the modulus of the kernel function $B_{2,k}(t;q)$ as

$$\begin{aligned}|B_{2,k}(t;q)|^2 &= \left(\frac{1}{\pi}\right)^{\frac{3}{2}} \frac{1}{2^k k!} \left| e^{-\frac{t^2}{2}-\frac{x^2}{2}+\frac{y^2}{2}+Ixy+\sqrt{2}t\overline{q}} \right|^2 \left| H_k(\sqrt{2}x-t) \right|^2 \\ &= \left(\frac{1}{\pi}\right)^{\frac{3}{2}} \frac{1}{2^k k!} e^{-t^2-x^2+y^2+2\sqrt{2}xt} \left| H_k(\sqrt{2}x-t) \right|^2.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\|B_{2,k;q}\|_{\mathbb{R}}^2 &= \left(\frac{1}{\pi}\right)^{\frac{3}{2}} \frac{1}{2^k k!} e^{x^2+y^2} \int_{\mathbb{R}} e^{-(t-\sqrt{2}x)^2} |H_k(t-\sqrt{2}x)|^2 dt \\ &= \left(\frac{1}{\pi}\right)^{\frac{3}{2}} \frac{1}{2^k k!} e^{|q|^2} \int_{\mathbb{R}} e^{-u^2} |H_k(u)|^2 du \\ &= \left(\frac{1}{\pi}\right)^{\frac{3}{2}} \frac{1}{2^k k!} e^{|q|^2} \int_{\mathbb{R}} |h_k(u)|^2 du \\ &= \left(\frac{1}{\pi}\right)^{\frac{3}{2}} \frac{1}{2^k k!} e^{|q|^2} \|h_k\|_{\mathbb{R}}^2 \\ &= \frac{1}{\pi} e^{|q|^2}.\end{aligned}$$

∎

Remark 5.3.2. By comparing (5.6) to (5.8), we conclude that $\|B_{2,k;q}\|_{\mathbb{R}} = \sqrt{K_k(q,q)}$ for every $q \in \mathbb{H}$.

Associated to the kernel function $B_{2,k}$ given through (5.1) (or also (5.5)), we are able to introduce a unitary integral transform (of Bargmann type) mapping isometrically the configuration space $L^2_{\mathbb{H}}(\mathbb{R};dt)$ onto the constructed S-polyregular Bargmann space $\mathcal{SR}^2_{2,k}$. In fact, we have to consider

$$[\mathcal{B}_{2,k}\phi](q) := \langle B_{2,k}(\cdot;q), \phi \rangle_{\mathbb{R}}.$$

More explicitly,

$$[\mathcal{B}_{2,k}\phi](q) := \left(\frac{1}{\pi}\right)^{\frac{3}{4}} \frac{1}{\sqrt{2^k k!}} \int_{\mathbb{R}} e^{-\frac{t^2+q^2}{2}+\sqrt{2}qt} H_k\left(\frac{q+\overline{q}}{\sqrt{2}} - t\right) \phi(t) dt$$

for a given function $\phi : \mathbb{R} \to \mathbb{H}$, provided that the integral exists. The following result shows that $\mathcal{B}_{2,k}$ is well-defined on $L^2_{\mathbb{H}}(\mathbb{R};dt)$. Namely, we have

Lemma 5.3.3. *For every quaternion $q \in \mathbb{H}$ and every $\phi \in L^2_{\mathbb{H}}(\mathbb{R};dt)$, we have*

$$|[\mathcal{B}_{2,k}\phi](q,\overline{q})| \leq \frac{1}{\sqrt{\pi}} e^{\frac{|q|^2}{2}} \|\phi\|_{\mathbb{R}}.$$

Proof. The proof readily follows by applying the Cauchy-Schwartz inequality. In fact, we obtain

$$|\mathcal{B}_{2,k}\phi(q,\overline{q})| \leq \int_{\mathbb{R}} |B_{2,k}(t;q)||\phi(t)|dt \leq \left\|B_{2,k;q}\right\|_{\mathbb{R}} \|\phi\|_{\mathbb{R}} . \tag{5.7}$$

In view of (5.6), the inequality (5.7) reduces further to

$$|\mathcal{B}_{2,k}\phi(q,\overline{q})| \leq \frac{e^{\frac{|q|^2}{2}}}{\sqrt{\pi}} \|\phi\|_{\mathbb{R}} .$$

∎

Theorem 5.3.4. *The transform $\mathcal{B}_{2,k}$ defines a Hilbert space isomorphism from $L^2_{\mathbb{H}}(\mathbb{R};dt)$ onto $\mathcal{SR}^2_{2,k}$.*

Proof. Notice first that the Segal–Bargmann transform $\mathcal{B}_{2,k}$ maps the orthogonal basis h_j of $L^2_{\mathbb{H}}(\mathbb{R};dt)$ to the orthogonal basis $H^Q_{j,k}(q,\overline{q})$ of the S-polyregular Bargmann space $\mathcal{SR}^2_{2,k}$. More precisely, we have

$$[\mathcal{B}_{2,k}(h_j)](q,\overline{q}) = \left(\frac{1}{\pi}\right)^{\frac{1}{4}} \frac{\sqrt{2^j}}{\sqrt{k!}} H^Q_{j,k}(q,\overline{q}).$$

Then, one can conclude since $\mathcal{B}_{2,k}$ is continuous (by Lemma 5.3.4). ∎

5.4 Spectral realization of the S-polyregular Bargmann spaces

5.4.1 Discussion.

In this section, we show that the S-polyregular Bargmann space $\mathcal{SR}^2_{2,n}$ (and therefore $\mathcal{SR}^2_{1,n}$) is closely connected to the concrete L^2-spectral analysis of the slice differential operator $\square_q$ in (22). To this end, we begin by considering the $\mathcal{C}^\infty$-spectral properties of $\square_q$ which requires to solve two problems. The first one is connected to the uniqueness problem of the polar representation $q = re^{I\theta}$ of the slice representation $q = x + Iy$, of given $q \in \mathbb{H}$. This can be resolved by restricting q to $\widetilde{\mathbb{H}} = \mathbb{H} \setminus \mathbb{R}$ and next extend, somehow, the obtained results to the whole $\mathbb{H}$. The second problem is related to the notion of the slice derivative given by (23) which makes $\square_q$ not necessarily elliptic. To see this, notice that ∂_s can be rewritten in the following unified form

$$\partial_s = \frac{1}{2}\left(\left(1 + \chi_{\mathbb{R}}(q)\right)\frac{\partial}{\partial x} - \left(1 - \chi_{\mathbb{R}}(q)\right) I_q \frac{\partial}{\partial y}\right), \tag{5.1}$$

so that the operator $\Box_q$ reads

$$\Box_q = -\frac{1}{4}\left\{(1+\chi_{\mathbb{R}}(q))^2\frac{\partial^2}{\partial x^2} + (1-\chi_{\mathbb{R}}(q))^2\frac{\partial^2}{\partial y^2}\right\} \tag{5.2}$$
$$+\frac{1}{2}(1+\chi_{\mathbb{R}}(q))\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\frac{I_q}{2}(1-\chi_{\mathbb{R}}(q))\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right).$$

It can be seen as a family of second order differential operators on $\mathbb{R}^2$ labeled by $\mathbb{S}$. Accordingly, for every fixed $I_q \in \mathbb{S}$, the operator $\Box_q$ is not elliptic nor uniform elliptic. However, it is semi-elliptic since the eigenvalues of the corresponding matrix

$$-\frac{1}{4}\begin{pmatrix} (1+\chi_{\mathbb{R}}(q))^2 & 0 \\ 0 & (1-\chi_{\mathbb{R}}(q))^2 \end{pmatrix}$$

are clearly non-positives (but not necessary negatives).

In ordre to provide, a spectral realization of the S-polyregular Bargmann spaces, introduced in Section 5.2, we begin by studying the right eigenvalue problem of $\Box_q$ in (5.2) when acting on the $\mathcal{C}^\infty$- as well as on the L^2-quaternionic-valued functions on $\widetilde{\mathbb{H}}$, and next extend the obtained expressions to the real line.

Notice that $\mathbb{R}$ is a negligible Borel measurable set with respect to the gaussian measure on $\mathbb{H}$, and therefore

$$\int_{\mathbb{H}} f(q)e^{-|q|^2}d\lambda(q) = \int_{\widetilde{\mathbb{H}}} f(q)e^{-|q|^2}d\lambda(q)$$
$$= \int_{\mathbb{R}^{+*}\times]0,2\pi[\times\mathbb{S}} f(re^{I\theta})e^{-r^2}rdrd\theta d\sigma(I_q), \tag{5.3}$$

where dr (resp. $d\theta$) denotes the Lebesgue measure on positive real line (the unit circle) and $d\sigma(I)$ stands for the standard area element on $\mathbb{S}$. This observation will be used systematically when discussing square integrability of the appropriate extension on the whole $\mathbb{H}$.

5.4.2 $\mathcal{C}^\infty$-right-eigenvalue problem.

Let μ be a fixed quaternionic number and consider the right eigenvalue problem $\Box_q f = f\mu$ for f belonging to the right quaternionic vector space $\mathcal{C}^\infty(\mathbb{H})$ of all quaternionic-valued functions that are $\mathcal{C}^\infty$ on the whole $\mathbb{H} \simeq \mathbb{R}^4$. Thus, associated to μ, we define the $\mathcal{C}^\infty$-eigenspace

$$\mathcal{E}^\infty_\mu(\mathbb{H},\Box_q) := \{f \in \mathcal{C}^\infty(\mathbb{H});\ \Box_q f = f\mu\}. \tag{5.4}$$

Notice for instance that $\mathcal{E}^\infty_\mu(\mathbb{H},\Box_q)$ is not necessarily a quaternionic right vector space. But, it is a $\mathcal{C}_\mu$-right vector space, where $\mathcal{C}_\mu := \{p \in \mathbb{H},\ p\mu = \mu p\}$ is the set of all quaternion numbers commuting with μ. We have $\mathcal{C}_\mu = \mathbb{H}$ when μ is real and $\mathcal{C}_\mu$ is $\mathbb{C}_\mu$ otherwise.

The first main result of this section concerns the explicit characterization of the elements of $\mathcal{E}^{\infty}_{\mu}(\mathbb{H}, \Box_q)$. Such description involves the Kummer's function defined by

$$M\left(\begin{array}{c} a \\ c \end{array}\middle| x\right) = \sum_{j=0}^{\infty} \frac{(a)_j}{(c)_j}\frac{x^j}{j!}, \tag{5.5}$$

for given $a \in \mathbb{H}$ and $x, c \in \mathbb{R}$, $c \neq 0, -1, -2, \cdots$, where $(a)_j$ denotes the rising factorial $(a)_j = a(a+1)\cdots(a+j-1)$ with $(a)_0 = 1$. Namely, we have

Theorem 5.4.1. *A $\mathcal{C}^{\infty}$-quaternionic-valued function f on $\mathbb{H}$ is a solution of $\Box_q f = f\mu$ on $\widetilde{\mathbb{H}}$ if and only if it can be expanded as*

$$f(|q|e^{I_q\theta}) = \sum_{j\in\mathbb{Z}} q^{(1+\mathrm{sgn}(j))\frac{|j|}{2}} \overline{q}^{(1-\mathrm{sgn}(j))\frac{|j|}{2}} M\left(\begin{array}{c} -\mu - (1-\mathrm{sgn}(j))\frac{j}{2} \\ |j|+1 \end{array}\middle| |q|^2\right)\gamma^I_{\mu,j} \tag{5.6}$$

for some quaternionic-valued functions $I_q \mapsto \gamma^I_{\mu,j}$ on $\mathbb{S}$ with values in $\mathcal{C}_{\mu}$. Here sgn *denotes the signum function.*

Proof. Let $f : \mathbb{H} \longrightarrow \mathbb{H}$ be a $\mathcal{C}^{\infty}$-quaternionic-valued function which is solution of $\Box_q f = f\mu$ on $\widetilde{\mathbb{H}}$. Then, $\widetilde{f} = f|_{\widetilde{\mathbb{H}}}$ satisfies $\Delta_q\widetilde{f} = \widetilde{f}\mu$, where Δ_q denotes the restriction of the slice differential operator $\Box_q$ in (22) to $\widetilde{\mathbb{H}}$. It takes the form

$$\Delta_q = -\frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + \frac{I}{2}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right). \tag{5.7}$$

Its expression in polar coordinates $q = re^{I\theta}$, with $r > 0$, $0 \leq \theta \leq 2\pi$ and $I \in \mathbb{S}$, is given by the following

$$\Delta_q = -\frac{1}{4}\left(\frac{\partial^2}{\partial r^2} + \left[\frac{1}{r} - 2r\right]\frac{\partial}{\partial r} + \frac{\partial^2}{r^2\partial\theta^2} - 2I\frac{\partial}{\partial\theta}\right)$$

and its action, on any $\mathcal{C}^{\infty}$ function $(r,\theta) \longrightarrow e^{Ij\theta}a^I_j(r)$ on $[0, 2\pi[\times[0, +\infty[$, is given by

$$\Delta_q(e^{Ij\theta}a^I_j(r)) = -\frac{e^{In\theta}}{4r^2}\left[r^2\frac{\partial^2}{\partial r^2} + (1-2r^2)r\frac{\partial}{\partial r} + (2jr^2 - j^2)\right]a^I_j(r). \tag{5.8}$$

Now, since $\widetilde{f} \in \mathcal{C}^{\infty}(\widetilde{\mathbb{H}})$ and its restriction $\widetilde{f}(re^{I\theta})$ to $\mathbb{C}_I$ is in addition periodic with respect to θ, one can expand it in Fourier series as

$$\widetilde{f}(re^{I\theta}) = \sum_{j\in\mathbb{Z}} e^{Ij\theta}a^I_j(r), \tag{5.9}$$

where the functions $(r, I) \longmapsto a^I_j(r)$ are $\mathcal{C}^{\infty}$ on $[0, +\infty[\times\mathbb{S}$. Therefore, by inserting (5.9) in the right-eigenvalue problem $\Delta_q\widetilde{f} = \widetilde{f}\mu$ taking into account (5.8), we see that

$$\left[r^2\frac{\partial^2}{\partial r^2} + (1-2r^2)r\frac{\partial}{\partial r} + (2jr^2 - j^2)\right]a^I_j(r) = -4r^2a^I_j(r)\mu \tag{5.10}$$

holds for every integer j and fixed r and I. Now, by the changes of variable $t = r^2 > 0$ and of function $a_j^I(r) = t^\alpha b_j(t, I)$, we get

$$tb_j''(\cdot, I) + (2\alpha + 1 - t)b_j'(\cdot, I) + \frac{1}{t}\left(\alpha - \frac{j}{2}\right)\left(\alpha + \frac{j}{2} - t\right) b_j(\cdot, I) = -b_j(\cdot, I)\mu. \tag{5.11}$$

For the ansatz $\alpha = |j|/2$, we recognize the left-quaternionic version of the confluent hypergeometric differential equation

$$tb_j''(\cdot, I) + (c - t)b_j'(\cdot, I) = b_j(\cdot, I)a \tag{5.12}$$

satisfied by $b_j(\cdot, I)$ on $]0, +\infty[$, with $c = |j| + 1$ and $a = -\mu - j\chi_{\mathbb{Z}^-}(j) = -\mu - (1 - \operatorname{sgn}(j))\frac{j}{2} \in \mathbb{H}$. Its first solution is given by Kummer's function $M\left(\begin{array}{c} a \\ c \end{array}\middle| t\right)$ for $c = |j| + 1$ being a positive integer, the second (linearly independent) solution is given by Tricomi's logarithmic function [112, p. 21] (see also [1, p. 504])

$$\begin{aligned} U\left(\begin{array}{c} a \\ |j|+1 \end{array}\middle| t\right) &:= \frac{(|j|-1)!}{\Gamma(a)} S_j^a(t) + \frac{(-1)^{|j|+1}}{|j|!\Gamma(a-|j|)} \left\{ M\left(\begin{array}{c} a \\ |j|+1 \end{array}\middle| t\right) \ln t \right. \\ &\quad \left. + \sum_{k=0}^{+\infty} \frac{(a)_k}{(|j|+1)_k} (\psi(a+k) - \psi(1+k) - \psi(|j|+1+k)) \frac{t^k}{k!} \right\}, \end{aligned}$$

where $\psi(x)$ denotes the logarithmic derivative of the gamma function, $\psi(x) = \Gamma'(x)/\Gamma(x)$, and $S_j^a(t)$ is the finite sum given by

$$S_j^a(t) := \sum_{k=0}^{+\infty} \frac{(a-|j|)_k}{(1-|j|)_k} \frac{t^{k-|j|}}{k!}$$

and interpreted as 0 when $j = 0$. Thus, the only solution of (5.12) that can be extended to a A $\mathcal{C}^\infty$ function at $t = 0$ is given by

$$b_j(t, I) = M\left(\begin{array}{c} -\mu - (1 - \operatorname{sgn}(j))\frac{j}{2} \\ |j| + 1 \end{array}\middle| t\right) \gamma_{\mu,j}^I$$

for some quaternionic constants $\gamma_{\mu,j}^I \in \mathcal{C}_\mu$ (viewed as functions on $\mathbb{S}$). Therefore, the corresponding f, whose restriction to $\widetilde{\mathbb{H}}$ are solutions of the right-eigenvalue problem $\Delta_q \widetilde{f} = \widetilde{f}\mu$, are given by

$$f(re^{I\theta}) = \sum_{j\in\mathbb{Z}} r^{|j|} e^{jI\theta} M\left(\begin{array}{c} -\mu - (1 - \operatorname{sgn}(j))\frac{j}{2} \\ |j| + 1 \end{array}\middle| r^2\right) \gamma_{\mu,j}^I.$$

They can rewritten as in (5.6). Such expression is well-defined as a $\mathcal{C}^\infty$ function on the whole $\mathbb{H}$. ∎

Remark 5.4.2. The extension of the solution of differential equation (5.12) at the regular singular point 0 corresponds to the extension of the solution of the right-eigenvalue problem $\Delta_q f = f\mu$ on $\widetilde{\mathbb{H}}$ to the whole $\mathbb{H}$.

Remark 5.4.3. The quaternionic $\mathcal{C}_\mu$-right-vector space $\mathcal{E}^\infty_\mu(\mathbb{H}, \Box_q)$ is generated by the functions

$$\psi_{\mu,j}(q) := q^{(1+\mathrm{sgn}(j))\frac{|j|}{2}} \overline{q}^{(1-\mathrm{sgn}(j))\frac{|j|}{2}} M \left(\begin{matrix} -\mu - (1-\mathrm{sgn}(j))\frac{j}{2} \\ |j|+1 \end{matrix} \middle| |q|^2 \right) \tag{5.13}$$

for varying $j \in \mathbb{Z}$. The expansion of any $f \in \mathcal{E}^\infty_\mu(\mathbb{H}, \Box_q)$ in terms of $\psi_{\mu,j}(q)$ involves slice right coefficients $\gamma^I_{\mu,j} \in \mathcal{C}_\mu$.

5.4.3 L^2-right-eigenvalue problem.

In the sequel, we are interested in giving a concrete description of L^2-eigenspaces of the right-eigenvalue problem $\Box_q f = f\mu$. To this end, we define

$$\mathcal{F}^2_\mu := \left\{ f \in L^2(\mathbb{H}; e^{-|q|^2} d\lambda);\ \Box_q f = f\mu \right\}, \tag{5.14}$$

as well as

$$\widetilde{\mathcal{F}^2_\mu} := \left\{ f \in L^2(\widetilde{\mathbb{H}}; e^{-|q|^2} d\lambda);\ \Delta_q \widetilde{f} = \widetilde{f}\mu \right\}, \tag{5.15}$$

where $L^2(\mathbb{H}; e^{-|q|^2} d\lambda)$ denotes the right quaternionic Hilbert space of all quaternionic-valued square integrable functions on $\mathbb{H}$ with respect to the inner product

$$\langle f, g \rangle_{\mathbb{H}} := \int_{\mathbb{H}} \overline{f(q)} g(q) e^{-|q|^2} d\lambda(q) \tag{5.16}$$

with $d\lambda(q) = dx_0 dx_1 dx_2 dx_3$ being the Lebesgue measure on $\mathbb{H} \simeq \mathbb{R}^4$. We define in a similar way $L^2(\widetilde{\mathbb{H}}; e^{-|q|^2} d\lambda)$ and $\left\langle \widetilde{f}, \widetilde{g} \right\rangle_{\widetilde{\mathbb{H}}}$. Thus, the following lemmas are fundamentals for our investigation of the L^2-eigenspaces $\mathcal{F}^2_\mu$.

Lemma 5.4.4. *With the same notations as above, we have the following results.*

(i) It holds

$$Spec_{L^2(\mathbb{H}; e^{-|q|^2} d\lambda)}(\Box_q) \subset Spec_{L^2(\widetilde{\mathbb{H}}; e^{-|q|^2} d\lambda)}(\Delta_q),$$

where $Spec$ denotes the spectrum of the prescribed operator.

(ii) The space $\mathcal{F}^2_\mu$ is a L^2-subspace of $\mathcal{E}^\infty_\mu(\widetilde{\mathbb{H}}, \Delta_q)$ and we have

$$\mathcal{F}^2_\mu \subset \widetilde{\mathcal{F}^2_\mu} = L^2(\widetilde{\mathbb{H}}; e^{-|q|^2} d\lambda) \cap \mathcal{E}^\infty_\mu(\widetilde{\mathbb{H}}, \Delta_q). \tag{5.17}$$

Proof. The first assertion holds true since for every $f \in L^2(\mathbb{H}; e^{-|q|^2} d\lambda)$, we have $\widetilde{f} \in L^2(\widetilde{\mathbb{H}}; e^{-|q|^2} d\lambda)$ with $\|f\|_{\mathbb{H}} = \left\| \widetilde{f} \right\|_{\widetilde{\mathbb{H}}}$. The second assertion is an immediate consequence of the ellipticity of Δ_q seen as a second order differential operator on $\mathbb{R} \times \mathbb{R}^*$ (see [45,86]). ∎

The second key lemma concerns the elementary functions

$$\varphi_{\mu,j}(q) := \psi_{\mu,j}(q)\alpha_j^I$$

associated to given $\alpha_j^I \in \mathcal{C}_\mu$, where $q = x + Iy \in \mathbb{H}$ and $\psi_{\mu,j}$ are as in (5.13).

Lemma 5.4.5. *The following results hold true.*

(i) The functions $\varphi_{\mu,j}$ are pairwise orthogonal in the sense that $\langle \varphi_{\mu,j}, \varphi_{\mu,k} \rangle = 0$ whenever $j \neq k$.

(ii) The functions $\varphi_{\mu,j}$ belong to $L^2(\mathbb{H}; e^{-|q|^2} d\lambda)$ if and only if $\mu_j = \mu + j$ is a nonnegative integer.

(iii) Let $\mu_j = 0, 1, 2, \cdots$. Then, the square norm of $\varphi_{\mu_j,j}$ in $L^2(\mathbb{H}; e^{-|q|^2} d\lambda)$ is given by

$$\left\| \varphi_{\mu_j,j} \right\|_{\mathbb{H}}^2 = \pi \frac{\mu_j!(|j|!)^2}{(\mu_j + j)!} \int_{\mathbb{S}} |\alpha_j^I|^2 d\sigma(I). \tag{5.18}$$

Proof. The first assertion follows by direct computation using polar coordinates, $q = re^{I\theta}$. Indeed, in these coordinates, the Lebesgue measure $d\lambda$ becomes the product of the standard measures rdr on $\mathbb{R}^+$ and the Lebesgue measure $d\theta$ on the unit circle times the standard area element $d\sigma(I)$ on $\mathbb{S}$, the two-dimensional sphere of imaginary units in $\mathbb{H}$. Therefore, for every $\alpha_j^I \in \mathbb{H}$, we have

$$\begin{aligned} \left\langle \varphi_{\mu_j,j}, \varphi_{\mu_k,k} \right\rangle &= \int_{\widetilde{\mathbb{H}}} \overline{\psi_{\mu_j,j}(q)\alpha_j^I} \psi_{\mu_k,k}(q)\alpha_k^I e^{-|q|^2} d\lambda(q) \\ &= \int_0^\infty r^{|j|+|k|+1} \int_{\mathbb{S}} \overline{\alpha_j^I} R_{j,k}(I) \alpha_k^I e^{-r^2} d\sigma(I) dr, \end{aligned} \tag{5.19}$$

where $R_{j,k}(I)$ stands for

$$R_{j,k}(I) := \overline{M\left(\begin{matrix} -\mu_j \\ |j|+1 \end{matrix} \middle| r^2 \right)} \left(\int_0^{2\pi} e^{(k-j)I\theta} d\theta \right) M\left(\begin{matrix} -\mu_k \\ |k|+1 \end{matrix} \middle| r^2 \right).$$

The use of the well-known fact $\int_0^{2\pi} e^{(k-j)I\theta} d\theta = 2\pi\delta_{j,k}$ completes our check of (i). Now, by the change of variable $t = r^2$ we obtain

$$\left\langle \varphi_{\mu_j,j}, \varphi_{\mu_k,k} \right\rangle = \pi \left(\int_{\mathbb{S}} |\alpha_j^I|^2 d\sigma(I) \right) \left(\int_0^\infty t^j \left| M\left(\begin{matrix} -\mu_j \\ |j|+1 \end{matrix} \middle| t \right) \right|^2 e^{-t} dt \right) \delta_{j,k}. \tag{5.20}$$

Therefore, to prove the second assertion, we make use of the asymptotic behavior

$$M\left(\begin{matrix} a \\ c \end{matrix} \middle| t \right) \sim \frac{e^t t^{a-c}}{\Gamma(a)}$$

for t large enough and $a \neq 0, -1, -2, \cdots$, that follows from the Poincaré-type expansion

[108, Section 7.2]

$$M\left(\begin{array}{c} a \\ c \end{array} \middle| t\right) \sim \frac{e^t t^{a-c}}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(1-a)_k (c-a)_k}{k!} t^{-k}.$$

Indeed, if $\mu_j \neq 0, 1, 2, \cdots$, then the nature of the integral involved in the right-hand side of (5.20) is equivalent to

$$\frac{1}{|\Gamma(-\mu_j)|^2} \int_0^{\infty} t^{-(2\Re(\mu_j)+|j|+2)} e^t dt$$

which is clearly divergent. Thus, we necessarily have $\mu_j = 0, 1, 2, \cdots$. In this case, the involved Kummer's function is the generalized Laguerre polynomial ([99, Eq. (1), p. 200])

$$M\left(\begin{array}{c} -\mu_j \\ |j|+1 \end{array} \middle| t\right) = \frac{\mu_j!}{(|n|+1)_{\mu_j}} L_{\mu_j}^{(j)}(t) \tag{5.21}$$

which satisfies the following orthogonality property [99, Eq. (4), p. 205 - Eq. (7), p. 206]

$$\int_{\mathbb{R}^+} L_j^{(\alpha)}(t) L_k^{(\alpha)}(t) t^{\alpha} e^{-t} dt = \frac{\Gamma(\alpha+j+1)}{\Gamma(j+1)} \delta_{j,k}. \tag{5.22}$$

More precisely, starting from (5.20), the explicit computation yields

$$\begin{aligned} \left\|\varphi_{\mu_j,j}\right\|_{\mathbb{H}}^2 &= \pi \left(\frac{\mu_j!}{(|j|+1)_{\mu_j}}\right)^2 \left(\int_0^{\infty} (L_{\mu_j}^{(j)}(t))^2 t^j e^{-t} dt\right) \times \left(\int_{\mathbb{S}} |\alpha_j^I|^2 d\sigma(I)\right) \\ &= \pi \frac{\mu_j!(j!)^2}{(\mu_j+j)!} \left(\int_{\mathbb{S}} |\alpha_j^I|^2 d\sigma(I)\right). \end{aligned}$$

This completes the proof of (ii) and (iii). ∎

Remark 5.4.6. If μ is a fixed nonnegative integer $\mu = n$, then $\psi_{\mu_j,j}\alpha_j^I$ belongs to $L^2(\mathbb{H}; e^{-|q|^2} d\lambda)$ if and only if $j \geq -n$, unless the corresponding α_j^I is zero. In this case, the square norm of $\psi_{n,j}$ (in (5.13)) is given by

$$\|\psi_{n,j}\|_{\mathbb{H}}^2 = \pi \frac{n!(j!)^2}{(n+j)!} Area(\mathbb{S}), \tag{5.23}$$

where $Area(\mathbb{S})$ denotes the surface area of $\mathbb{S}$.

The next result shows in particular that the spectrum of $\square_q$ acting $L^2(\mathbb{H}; e^{-|q|^2} d\lambda)$ is purely discrete and reduces to the quantized eigenvalues known as Landau levels.

Theorem 5.4.7. *The space $\mathcal{F}_\mu^2$ is nontrivial if and only if $\mu = n = 0, 1, 2, \cdots$. In this case, a nonzero quaternionic-valued function f belongs to $\mathcal{F}_n^2(\mathbb{H})$ if and only if it can be expanded as*

$$f(q) = \sum_{j=-n}^{+\infty} q^j M\left(\begin{array}{c} -n \\ |j|+1 \end{array} \middle| |q|^2\right) C_j(I), \tag{5.24}$$

where the quaternionic constants $C_j(I)$ satisfy the growth condition

$$\|f\|^2_{\mathbb{H}} = \pi \sum_{j=-n}^{+\infty} \frac{n!(j!)^2}{(n+j)!} \left(\int_{\mathbb{S}} |C_j(I)|^2 d\sigma(I) \right) < +\infty. \tag{5.25}$$

Proof. Fix $\mu \in \mathbb{H}$ and assume that there is a nonzero $f \in L^2(\mathbb{H}; e^{-|q|^2} d\lambda)$ solution of $\square_q f = f\mu$. Then, the realization (5.17) and the proof of Theorem 5.4.1 show that $\widetilde{f} := f|_{\widetilde{\mathbb{H}}}$ admits the expansion

$$\widetilde{f}(|q|e^{I_q\theta}) = \sum_{j\in\mathbb{Z}} \psi_{\mu_j,j}(q)\gamma^I_{\mu,j}.$$

The orthogonality of the $(\psi_{\mu_j,j})_j$ (see (i) of Lemma 5.4.5) infers that

$$\begin{aligned}
\left\|\widetilde{f}\right\|^2_{\widetilde{\mathbb{H}}} &= \sum_{j\in\mathbb{Z}} \left\|\psi_{\mu_j,j}\gamma^I_{\mu,j}\right\|^2_{\mathbb{H}} \\
&= \frac{\pi}{Area(\mathbb{S})} \sum_{j\in\mathbb{Z}} \left(\int_{\mathbb{S}} |\gamma^I_{\mu_j,j}|^2 d\sigma(I) \right) \left\|\psi_{\mu_j,j}\right\|^2_{\mathbb{H}}.
\end{aligned}$$

Therefore, since the nonzero function f belongs to $\mathcal{F}^2_\mu$, we have necessarily $\left\|\psi_{\mu_j,j}\right\|^2_{\mathbb{H}}$ is finite for every j such that

$$\int_{\mathbb{S}} |\gamma^I_{\mu,j}|^2 d\sigma(I) \neq 0.$$

Now, (ii) of Lemma 5.4.5 readily implies that μ is necessary of the form $\mu = n = 0, 1, 2, \cdots$, and $j \geq -n$. In such case, the $\gamma^I_{\mu,j} =: C_j(I)$ are arbitrary in $\mathbb{H} = \mathcal{C}_\mu$ for μ being real. Moreover, we have

$$\|f\|^2_{\mathbb{H}} = \pi \sum_{j=-n}^{+\infty} \frac{n!(j!)^2}{(n+j)!} \int_{\mathbb{S}} |C_j(I)|^2 d\sigma(I).$$

This yields the growth condition (5.25) and the proof is completed. ∎

The following result describes the fact that the elements of $\mathcal{F}^2_n$ can be expanded as series of the quaternionic Hermite polynomials $H^Q_{j,n}(q,\overline{q})$.

Corollary 5.4.8. *The space $\mathcal{F}^2_n$ contains the quaternionic Hermite polynomials $(H^Q_{n+j,n})_j$ defined by* (16). *Moreover, every element f belonging to $\mathcal{F}^2_n$ can be expanded as*

$$f(q) = \sum_{j=-n}^{+\infty} \frac{(-1)^j j!}{(n+j)!} H^Q_{n+j,n}(q,\overline{q}) C_j(I) \tag{5.26}$$

for some slice quaternionic constants $C_j(I)$ displaying the growth condition (5.25).

Proof. This lies in the fact that the involved confluent hypergeometric function is connected to the quaternionic Hermite polynomials through

$$q^j M\left(\begin{array}{c} -n \\ |j|+1 \end{array} \middle| |q|^2 \right) = \frac{(-1)^n j!}{(n+j)!} H^Q_{n+j,n}(q,\overline{q}), \tag{5.27}$$

see Lemma 3.2 in [43]. Therefore, the expression of $f(q)$ given through (5.24) reduces further to (5.26) with the same growth condition (5.25). ∎

5.4.4 Connection to S-polyregular Bargmann spaces of first kind.

By Corollary 5.4.8, the space $\mathcal{F}^2_n$ can be realized as the space of the convergent series

$$\sum_{j=0}^{+\infty} \frac{(-1)^n j!}{j!} H^Q_{j,n}(q,\overline{q}) C_j(I_q)$$

on $\mathbb{H}$, where $(C_j(I_q))_j$ are certain quantities in $\in \mathbb{H}$ such that

$$\pi \sum_{j=-n}^{+\infty} \frac{n!(j!)^2}{(n+j)!} \left(\int_{\mathbb{S}} |C_j(I_q)^2 d\sigma(I) \right) < +\infty.$$

It reduces further to $\mathcal{SR}^2_{2,n}$ when the $C_j(I_q)$ are assumed to be constant functions on $\mathbb{S}$, $C_j(I_q) = C_j$. In particular, by taking $n = 0$, the previous growth condition reads simply as

$$\pi Area(\mathbb{S}) \sum_{j=0}^{+\infty} j! |C_j|^2 < +\infty.$$

Comparing this to the sequential characterization of the slice hyperholomorphic Bargmann space $\mathcal{F}^2_{slice}$ given by Proposition 3.11 in [9], we see that $\mathcal{F}^2_{slice} \subset \mathcal{F}^2_0$.

5.5 Full S-polyregular Bargmann spaces

Motivated by Theorem 4.2 in [43] asserting that the quaternionic Hermite polynomials $(H^Q_{j,k})_{j,k}$ form a slice basis of the Hilbert space $L^2(\mathbb{H}; e^{-|q|^2} d\lambda)$, equipped with the scalar product

$$\langle f,g \rangle_{\mathbb{H}} = \int_{\mathbb{H}} \overline{f(q)} g(q) e^{-|q|^2} d\lambda(q), \tag{5.1}$$

we define $\mathcal{SR}^2_{n,full}$ to be the space of S-polyregular functions (of level n) spanned by the quaternionic Hermite polynomials $H^Q_{j,n}$, for varying $j = 0,1,2,\cdots$, and belonging to $L^2(\mathbb{H}; e^{-|q|^2} d\lambda)$. Then, we have

$$\langle f,g \rangle_{\mathbb{H}} = \int_{\mathbb{S}} \langle f,g \rangle_{\mathbb{C}_I} d\sigma(I)$$

and subsequently, the space $\mathcal{SR}^2_{n,full}$ can be described as the right quaternionic vector space consisting of the convergent series

$$\sum_{j=0}^{+\infty} H^Q_{j,n}(q,\overline{q})C_j(I_q)$$

on $\mathbb{H}$, with $C_j : \mathbb{S} \longrightarrow \mathbb{H}$, and such that

$$\pi n! \sum_{j=0}^{\infty} \int_{\mathbb{S}} |C_j(I)|^2 d\sigma(I) < +\infty.$$

This is exactly the sequential characterization of L^2-eigenspace $\mathcal{F}^2_n$. The particular case of $n = 0$ corresponds to the full hyperholomorphic Bargmann space

$$\mathcal{F}^2_{full} := \mathcal{SR} \cap L^2(\mathbb{H}; e^{-|q|^2} d\lambda) \tag{5.2}$$

defined as the right quaternionic Hilbert space of all slice regular functions that are $e^{-|q|^2}d\lambda$-square integrable on $\mathbb{H}$. This lies on the fact that $\mathcal{F}^2_{full}$ is the space of functions $f(q) = \sum\limits_{j=0}^{\infty} q^j C_j(I)$ satisfying

$$\|f\|^2_{\mathbb{H}} = \pi \sum_{j=0}^{+\infty} j! \left(\int_{\mathbb{S}} |C_j(I)|^2 d\sigma(I) \right) < +\infty.$$

More generally, it is not difficult to prove that the spaces $\mathcal{SR}^2_{n,full}$ are right quaternionic Hilbert spaces. We call them here the full S-polyregular Bargmann spaces of second kind of level n. The quaternionic Hermite polynomials $H^Q_{j,n}$, for varying $j = 0, 1, 2, \cdots$, constitute an orthogonal "slice" basis of it.

Chapter 6

Composition of Segal–Bargmann transforms

We introduce and discuss some basic properties of some integral transforms in the framework of specific functional Hilbert spaces, the holomorphic Bargmann–Fock spaces on $\mathbb{C}$ and $\mathbb{C}^2$ and the slice hyperholomorphic Bargmann–Fock space on $\mathbb{H}$. The first one is a natural integral transform mapping isometrically the standard Hilbert space on the real line into the two-dimensional Bargmann–Fock space. It is obtained as composition of the one and two dimensional Segal–Bargmann transforms and reduces further to an extremely integral operator that looks like a composition operator of the one-dimensional Segal–Bargmann transform with a specific symbol. We study its basic properties, including the identification of its image and the determination of a like-left inverse defined on the whole two-dimensional Bargmann–Fock space. We examine their combination with the Fourier transform which lead to special integral transforms connecting the two-dimensional Bargmann–Fock space and its analogue on the complex plane. We also investigate the relationship between special subspaces of the two-dimensional Bargmann–Fock space and the slice-hyperholomorphic one on the quaternions by introducing appropriate integral transforms. We identify their image and their action on the reproducing kernel.

6.1 On composition of Segal–Bargmann transforms

The kernel function of the d-dimensional Segal–Bargmann transform $\mathcal{B}^{d,\nu}$ in (20) is the analytic continuation to $\mathbb{C}^d$ of the standard Gaussian density on $\mathbb{R}^d$. It is given by

$$A_d^\nu(z,x) = c_d^\nu e^{-\nu\left(x-\frac{z}{\sqrt{2}}\right)^2} \tag{6.1}$$

with $z^2 := z_1^2 + z_2^2 + \cdots + z_d^2$ for $z = (z_1, \cdots, z_d) \in \mathbb{C}^d$. Then, the integral transform in (25) acts on $L^{2,\nu}(\mathbb{R}, \mathbb{C})$ by

$$\mathcal{G}^\nu f(z,w) := \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} f(x) A_1^\nu \left(\frac{z+iw}{\sqrt{2}}, x\right) dx. \tag{6.2}$$

The following result shows that the transform $\mathcal{G}^\nu$ can be realized in a natural way by means of the Segal–Bargmann transforms $\mathcal{B}^{d,\nu}$; $d = 1, 2$, according to the following diagram

$$\begin{tikzcd}
L^{2,\nu}(\mathbb{R}, \mathbb{C}) \arrow[r, "\mathcal{B}^{1,\nu}"] \arrow[dr, "\mathcal{G}^\nu"'] & \mathcal{F}^{2,\nu}(\mathbb{C}) \arrow[d, "\mathcal{B}^{2,\nu}"] \\
& \mathcal{F}^{2,\nu}(\mathbb{C}^2)
\end{tikzcd}$$

Theorem 6.1.1. *The above diagram is commutative, in the sense that we have $\mathcal{G}^\nu = \mathcal{B}^{2,\nu} \circ \mathcal{B}^{1,\nu}$ on $L^{2,\nu}(\mathbb{R}, \mathbb{C})$. Moreover, $\mathcal{G}^\nu$ defines an isometric operator mapping the Hilbert space $L^{2,\nu}(\mathbb{R}, \mathbb{C})$ into the Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$.*

Proof. For every given $\varphi \in L^{2,\nu}(\mathbb{R}, \mathbb{C})$, the function $\mathcal{B}^{2,\nu} \circ \mathcal{B}^{1,\nu}(\varphi)$ is clearly a holomorphic function on $\mathbb{C}^2$ and belongs to $L^{2,\nu}(\mathbb{C}^2, \mathbb{C})$. Moreover, $\mathcal{B}^{2,\nu} \circ \mathcal{B}^{1,\nu}$ defines an isometric operator from $L^{2,\nu}(\mathbb{R}, \mathbb{C})$ into $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ since $\mathcal{B}^{2,\nu}$ and $\mathcal{B}^{1,\nu}$ are. To conclude for the proof of Theorem 6.1.1, we only need to show that the diagram is commutative. Thus, for every given $z, w \in \mathbb{C}$ and $x, y \in \mathbb{R}$, we have

$$\begin{aligned}
\mathcal{B}^{2,\nu} \circ \mathcal{B}^{1,\nu} f(z,w) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}} f(t) A_2^\nu((z,w),(x,y)) A_1^\nu(x+iy, t) dt dx dy \\
&= c_2^\nu c_1^\nu \int_{\mathbb{R}^2} \int_{\mathbb{R}} f(t) e^{-\nu\left\{ \left(x - \frac{z}{\sqrt{2}}\right)^2 + \left(y - \frac{w}{\sqrt{2}}\right)^2 + \left(t - \frac{x+iy}{\sqrt{2}}\right)^2 \right\}} dt dx dy \\
&\overset{(*)}{=} \left(\frac{\pi}{\nu}\right) c_2^\nu \int_{\mathbb{R}} f(t) A_1^\nu \left(\frac{z+iw}{\sqrt{2}}, t\right) dt.
\end{aligned} \tag{6.3}$$

The transition $(*)$ follows by direct computation, making appeal of the Fubini's Theorem as well as the explicit formula for the Gaussian integral. The proof of the theorem is completed by comparing the right-hand side of (6.3) to (6.2). ∎

The next result identifies the image of $L^{2,\nu}(\mathbb{R}, \mathbb{C})$ by the one-to-one transform $\mathcal{G}^\nu$, and characterizes it as the kernel $\ker_{\mathcal{F}^{2,\nu}(\mathbb{C}^2)}(D_{z,w})$ of the first-order differential operator

$$D_{z,w} := \frac{\partial}{\partial z} + i\frac{\partial}{\partial w}$$

acting on $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$. More precisely, we assert the following

Theorem 6.1.2. *Keep notations as above and define $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$ as in* (26)*, $\mathcal{A}^{2,\nu}(\mathbb{C}^2) := \ker_{\mathcal{F}^{2,\nu}(\mathbb{C}^2)}(D_{z,w})$. Then, we have*

(i) $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$ is a closed subspace of $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$.

(ii) The functions $e^{\nu}_m(z,w) := (z+iw)^m$ *form an orthogonal basis of the Hilbert space* $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$.

(iii) $\mathcal{A}^{2,\nu}(\mathbb{C}^2) = \mathcal{G}^{\nu}(L^{2,\nu}(\mathbb{R},\mathbb{C}))$.

Proof. Notice first that by the definition in (25) and the fact that

$$\mathcal{B}^{1,\nu}(H^{\nu}_m)(\xi) = \left(\frac{\nu}{\pi}\right)^{1/4} \sqrt{2}^m \nu^m \xi^m,$$

the action of $\mathcal{G}^{\nu}$ on the rescaled Hermite polynomials in (2.2) is given by

$$\begin{aligned} \mathcal{G}^{\nu}(H^{\nu}_m)(z,w) &= \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \mathcal{B}^{1,\nu}(H^{\nu}_m)\left(\frac{z+iw}{\sqrt{2}}\right) \\ &= c^{\nu}_1 \nu^m (z+iw)^m. \end{aligned} \tag{6.4}$$

Subsequently, the image $\mathcal{G}^{\nu}(L^{2,\nu}(\mathbb{R},\mathbb{C}))$ is then spanned by the functions $e^{\nu}_m(z,w) := (z+iw)^m$, since the polynomials H^{ν}_k form an orthogonal basis of $L^{2,\nu}(\mathbb{R},\mathbb{C})$. Accordingly, the proof of (i) readily follows and then $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$ is a Hilbert space for the scalar product induced from $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$, while (iii) is an immediate consequence of (ii). Moreover, the functions $e^{\nu}_k(z,w)$ satisfy $D_{z,w}e^{\nu}_k(z,w) = 0$ and form an orthogonal system in the Hilbert space $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$. To conclude for (ii), we should prove completeness of $e^{\nu}_k(z,w)$ in $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$. To this end, let $F \in \mathcal{F}^{2,\nu}(\mathbb{C}^2)$ such that $D_{z,w}F = 0$ and $\langle F, e^{\nu}_k\rangle_{L^{2,\nu}(\mathbb{C}^2,\mathbb{C})} = 0$ for all k and show that F is then identically zero on $\mathbb{C}^2$. Indeed, by expanding F as series $F(z,w) = \sum\limits_{m,n=0}^{+\infty} a_{m,n} z^m w^n \in \mathcal{F}^{2,\nu}(\mathbb{C}^2)$, we show that

$$\langle F, e^{\nu}_k\rangle_{L^{2,\nu}(\mathbb{C}^2,\mathbb{C})} = \sum_{j=0}^{k} \binom{k}{j} (-i)^j a_{k-j,j} \left\|e_{k-j}\right\|^2_{L^{2,\nu}_{\mathbb{C}}(\mathbb{C})} \left\|e_j\right\|^2_{L^{2,\nu}_{\mathbb{C}}(\mathbb{C})},$$

where $e_j(\xi) = \xi^j$. Hence $\langle F, e^{\nu}_k\rangle_{L^{2,\nu}(\mathbb{C}^2,\mathbb{C})} = 0$, for every $k = 0, 1, \cdots$, implies that

$$\left(\frac{\pi}{\nu}\right)^2 \frac{k!}{\nu^k} \sum_{j=0}^{k} (-i)^j \, a_{k-j,j} = 0. \tag{6.5}$$

Moreover, we can show that the condition $D_{z,w}F = 0$ is equivalent to that

$$a_{m+1,n} = -i\left(\frac{n+1}{m+1}\right) a_{m,n+1}$$

for all $m, n = 0, 1, \cdots$, which by induction infers

$$a_{m,n} = i^n \left(\frac{(m+n)!}{m!n!}\right) a_{m+n,0}, \quad m = 0, 1, \cdots; n = 1, \cdots. \tag{6.6}$$

Inserting this in (6.5), it yields $a_{k,0} = 0$ for all k and therefore $a_{m,n} = 0$ for all m, n by means of (6.6). This infers the required result. ∎

Remark 6.1. The space $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$ is of interest in itself. It is the phase space in 2-complex dimension that is unitary isomorphic to the configuration space $L^{2,\nu}(\mathbb{R},\mathbb{C})$. Moreover, the transform $\mathcal{G}^\nu$ is a coherent state transform from $L^{2,\nu}(\mathbb{R},\mathbb{C})$ onto $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$, in the sense that its kernel function can be recovered as a bilinear generating function of the orthonormal bases of the Hilbert spaces $L^{2,\nu}(\mathbb{R},\mathbb{C})$ and $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$.

Remark 6.2. The assertion (iii) in Theorem 6.1.2 shows that $\mathcal{B}^{2,\nu}(\mathcal{F}^{2,\nu}(\mathbb{C})) = \mathcal{A}^{2,\nu}(\mathbb{C}^2)$. This is in fact contained in (6.4). Indeed, for $e_m(\xi) = \xi^m$, we have

$$\mathcal{B}^{2,\nu}\left(e_m\right)(z,w) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}}\left(\frac{1}{2}\right)^{\frac{m}{2}} e^\nu_m(z,w).$$

Remark 6.3. The inverse transform of $\mathcal{G}^\nu$ is defined from $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$ onto $L^{2,\nu}(\mathbb{R},\mathbb{C})$ and is clearly given by $(\mathcal{B}^{1,\nu})^{-1}\circ(\mathcal{B}^{2,\nu})^{-1}$ and coincides with the restriction to $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$ of $\mathcal{R}^\nu$ introduced below.

Now, let us consider the transform $\mathcal{R}^\nu$ from $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ into $L^{2,\nu}(\mathbb{R},\mathbb{C})$ defined by the following commutative diagram

$$\begin{tikzcd}
\mathcal{F}^{2,\nu}(\mathbb{C}^2) \arrow[r, "\mathcal{R}^\nu"] \arrow[d, "(\mathcal{B}^{2,\nu})^{-1}"'] & L^{2,\nu}(\mathbb{R},\mathbb{C}) \\
L^{2,\nu}(\mathbb{R}^2,\mathbb{C}) \arrow[r, "Proj"'] & \mathcal{F}^{2,\nu}(\mathbb{C}), \arrow[u, "(\mathcal{B}^{1,\nu})^{-1}"']
\end{tikzcd}$$

where $Proj$ stands for the orthogonal projection from $L^{2,\nu}_{\mathbb{C}}(\mathbb{C})$ onto the standard Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C})$ and given by (see e.g. [116])

$$Projf(\xi) = \left(\frac{\nu}{\pi}\right)\int_{\mathbb{C}} f(\zeta)e^{\nu\xi\overline{\zeta}}e^{-\nu|\zeta|^2}d\lambda(\zeta). \tag{6.7}$$

The following result gives an integral representation of the operator $\mathcal{R}^\nu := (\mathcal{B}^{1,\nu})^{-1}\circ Proj\circ(\mathcal{B}^{2,\nu})^{-1}$. It involves of the inverse of $\mathcal{B}^{1,\nu}$ and the composition operator $\mathcal{C}_{\psi_2}F = F\circ\psi_2$ with the symbol function $\psi_2 : \mathbb{C}\longrightarrow\mathbb{C}^2$ given by

$$\psi_2(\xi) := \left(\frac{\xi}{\sqrt{2}}, -i\frac{\xi}{\sqrt{2}}\right). \tag{6.8}$$

Theorem 6.1.3. *The transform $\mathcal{R}^\nu$ defined on the whole $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ looks like a left inverse of $\mathcal{G}^\nu$. Moreover, we have*

$$\mathcal{R}^\nu F = \left(\frac{\pi}{\nu}\right)^{\frac{1}{2}}(\mathcal{B}^{1,\nu})^{-1}(\mathcal{C}_{\psi_2}F) \tag{6.9}$$

for every $F\in\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ which explicitly reads,

$$\mathcal{R}^\nu F(x) = \left(\frac{\pi}{\nu}\right)^{\frac{1}{4}}\int_{\mathbb{C}} F\left(\frac{\xi}{\sqrt{2}}, -i\frac{\xi}{\sqrt{2}}\right)e^{-\frac{\nu}{2}\overline{\xi}^2+\sqrt{2}\nu x\overline{\xi}}e^{-\nu|\xi|^2}d\lambda(\xi). \tag{6.10}$$

Proof. For every $f \in L^{2,\nu}(\mathbb{R},\mathbb{C})$, the function $\mathcal{B}^{1,\nu} f$ belongs to the one-dimensional Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C})$ and therefore $Proj(\mathcal{B}^{1,\nu} f) = \mathcal{B}^{1,\nu} f$, so that $\mathcal{R}^\nu \circ \mathcal{G}^\nu = id_{L^{2,\nu}(\mathbb{R},\mathbb{C})}$. This shows that $\mathcal{R}^\nu$ is a left inverse of $\mathcal{G}^\nu$. Moreover, making use of the integral representation of the orthogonal projection (6.7) and of

$$(\mathcal{B}^{2,\nu})^{-1}F(\zeta) = c_2^\nu \int_{\mathbb{C}^2} F(z,w) e^{-\frac{\nu}{2}(\overline{z}^2+\overline{w}^2)+\frac{\nu}{\sqrt{2}}(\zeta[\overline{z}-i\overline{w}]+\overline{\zeta}[\overline{z}+i\overline{w}])} e^{-\nu(|z|^2+|w|^2)} d\lambda(z,w)$$

for $\zeta \in \mathbb{C}$ and $F \in \mathcal{F}^{2,\nu}(\mathbb{C}^2)$, we get

$$Proj(\mathcal{B}^{2,\nu})^{-1}F(\xi) = c_2^\nu \int_{\mathbb{C}^2} e^{-\frac{\nu}{2}(\overline{z}^2+\overline{w}^2)} F(z,w) I(\xi,\overline{z},\overline{w}) e^{-\nu(|z|^2+|w|^2)} d\lambda(z,w),$$

where for $\xi, z, w \in \mathbb{C}$ we have

$$\begin{aligned} I(\xi,\overline{z},\overline{w}) &:= \left(\frac{\nu}{\pi}\right) \int_{\mathbb{C}} e^{-\nu|\zeta|^2+\frac{\nu}{\sqrt{2}}(\zeta[\overline{z}-i\overline{w}]+\overline{\zeta}[\overline{z}+i\overline{w}+\sqrt{2}\xi])} d\lambda(\zeta) \\ &= e^{\frac{\nu}{2}(\overline{z}^2+\overline{w}^2)+\nu\xi\frac{(\overline{z}-i\overline{w})}{\sqrt{2}}}. \end{aligned}$$

Therefore, by the reproducing property for the two-dimensional Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$, we obtain

$$\begin{aligned} Proj(\mathcal{B}^{2,\nu})^{-1}F(\xi) &= c_2^\nu \int_{\mathbb{C}^2} F(z,w) e^{\nu\left(\frac{\xi}{\sqrt{2}}\overline{z}-\frac{i\xi}{\sqrt{2}}\overline{w}\right)} e^{-\nu(|z|^2+|w|^2)} d\lambda(z,w) \\ &= \left(\frac{\pi}{\nu}\right)^{\frac{1}{2}} F\left(\frac{\xi}{\sqrt{2}}, -i\frac{\xi}{\sqrt{2}}\right) \end{aligned}$$

for

$$K_2^\nu((u,v),(z,w)) = \left(\frac{\nu}{\pi}\right)^2 e^{\nu(u\overline{z}+v\overline{w})} \tag{6.11}$$

being the reproducing kernel of $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$. ∎

6.2 Connecting holomorphic and slice hyperholomorphic Bargmann–Fock spaces

The slice hyperholomorphic quaternionic Bargmann–Fock space $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$, considered in [9], is a quaternionic counterpart of the holomorphic Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C})$. It is defined to be the right $\mathbb{H}$-vector space of all slice left regular functions on $\mathbb{H}$, $F \in \mathcal{SR}(\mathbb{H})$, subject to the norm boundedness $\|F\|^2_{\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})} < +\infty$. This norm is associated with the inner product

$$\langle F, G\rangle_{\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})} = \int_{\mathbb{C}_I} \overline{G_I(q)} F_I(q) e^{-\nu|q|^2} d\lambda_I(q), \tag{6.1}$$

where for given $I \in \mathbb{S} = \{I \in \mathbb{H}; I^2 = -1\}$, the function $F_I = F|_{\mathbb{C}_I}$ denotes the restriction of F to the slice $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$ and $d\lambda_I(q) = dxdy$ for $q = x + yI$. It was shown in [9] that

$\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$ does not depend on the choice of the imaginary unit I and is a reproducing kernel Hilbert space, whose the reproducing kernel is given by

$$K^{\nu}_{\mathbb{H}}(q,p) = \left(\frac{\nu}{\pi}\right) e_*^{\nu[q,\overline{p}]} := \left(\frac{\nu}{\pi}\right) \sum_{m=0}^{+\infty} \frac{\nu^m q^m \overline{p}^m}{m!}; \quad p,q \in \mathbb{H}. \tag{6.2}$$

This space is closely connected to $L^{2,\nu}_{\mathbb{H}}(\mathbb{R})$, the Hilbert space of all L^2 and $\mathbb{H}$-valued functions on the real line with respect to the Gaussian measure. In fact, $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$ can be realized as the image of $L^{2,\nu}_{\mathbb{H}}(\mathbb{R})$ by considering the quaterenionic Segal–Bargmann transform [40]

$$\mathcal{B}^{\nu}_{\mathbb{H}} f(q) := c_1^{\nu} \int_{\mathbb{R}} f(x) e^{-\nu\left(x - \frac{q}{\sqrt{2}}\right)^2} dx. \tag{6.3}$$

Its inverse transform mapping $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$ onto $L^{2,\nu}_{\mathbb{H}}(\mathbb{R})$ is given by

$$(\mathcal{B}^{\nu}_{\mathbb{H}})^{-1} F(x) = c_1^{\nu} \int_{\mathbb{C}_I} F_I(q) e^{-\frac{\nu}{2}\overline{q}^2 + \sqrt{2}\nu x \overline{q}} e^{-\nu|q|^2} d\lambda_I(q). \tag{6.4}$$

Examples of slice hyperholomorphic functions in $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$ can also be obtained from the one of the standard Bargmann–Fock space on $\mathbb{C}$ by the extension lemma below.

Lemma 6.1 ([33,61]). Let $\Omega_I = \Omega \cap \mathbb{C}_I$; $I \in \mathbb{S}$, be a symmetric domain in $\mathbb{C}_I$ with respect to the real axis such that $\Omega_I \cap \mathbb{R}$ is not empty and $\widetilde{\Omega} = \bigcup\limits_{x+yJ\in\Omega} x + y\mathbb{S}$ be the symmetric completion of Ω_I. For every holomorphic function $F : \Omega_I \longrightarrow \mathbb{H}$, the function $Ext(F)$ defined by

$$Ext(F)(x+yJ) := \frac{1}{2}[F(x+yI) + F(x-yI)] + \frac{JI}{2}[F(x-yI) - F(x+yI)]; \quad J \in \mathbb{S},$$

extends F to a regular function on $\widetilde{\Omega}$. Moreover, $Ext(F)$ is the unique slice regular extension of F.

This lemma can be extended to the context of the two-dimensional Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ on $\mathbb{C}^2$. This lies on the simple idea that consists of considering an appropriate restriction operator from $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ into $\mathcal{F}^{2,\nu}(\mathbb{C})$ and next apply the extension Lemma 6.1. For example, one can consider

$$\mathcal{I}^{\nu} : F \longmapsto F \circ \psi_2 \longmapsto Ext(F \circ \psi_2)$$

from $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ into a specific subspace of $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$, where $\psi_2 : \mathbb{C} \longrightarrow \mathbb{C}^2$ is the one defined in (6.8). The following result shows that the transform $\mathcal{I}^{\nu}$ is in fact realized by the following commutative diagram

$$\begin{tikzcd}
\mathcal{F}^{2,\nu}(\mathbb{C}^2) \arrow[r, "\mathcal{I}^{\nu}"] \arrow[d, "\mathcal{R}^{\nu}"'] & \mathcal{F}^{2,\nu}_{slice}(\mathbb{H}) \\
L^{2,\nu}(\mathbb{R},\mathbb{C}) \arrow[r, "inj"'] & L^{2,\nu}_{\mathbb{H}}(\mathbb{R}) \arrow[u, "\mathcal{B}^{\nu}_{\mathbb{H}}"']
\end{tikzcd}$$

where $\mathcal{B}^{\nu}_{\mathbb{H}}$ is the quaternionic Segal–Bargmann transform in (6.3) and $\mathcal{R}^{\nu}$ is the transform given by (6.9).

Theorem 6.2.1. *The transform $\mathcal{B}^{\nu}_{\mathbb{H}} \circ \mathcal{R}^{\nu}$ coincides with $\mathcal{I}^{\nu}$ and acts on $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ by*

$$\mathcal{B}^{\nu}_{\mathbb{H}} \circ \mathcal{R}^{\nu} F(q) = \left(\frac{\nu}{\pi}\right) \int_{\mathbb{C}} F\left(\frac{\xi}{\sqrt{2}}, \frac{-i\xi}{\sqrt{2}}\right) K^{\nu}_{\mathbb{H}}(q,\xi) e^{-\nu|\xi|^2} d\lambda(\xi), \tag{6.5}$$

where $K^{\nu}_{\mathbb{H}}(q,\xi)$ is the reproducing kernel of $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$ as given by (6.2).

For the proof, we will make use of the identity principle for slice regular functions

Lemma 6.2 ([33,61]). Let F be a slice regular function on a slice domain Ω and denote by $\mathcal{Z}_F$ its zero set. If $\mathcal{Z}_F \cap \mathbb{C}_I$ has an accumulation point in Ω_I for some $I \in \mathbb{S}$, then F vanishes identically on Ω.

Proof. On the one hand, the function $\mathcal{B}^{\nu}_{\mathbb{H}} \circ \mathcal{R}^{\nu} F$ is slice regular by construction. On the other hand, one can show easily that the function $\xi \longmapsto F\left(\frac{\xi}{\sqrt{2}}, \frac{-i\xi}{\sqrt{2}}\right)$ belongs to the one-dimensional Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C})$ for every $F \in \mathcal{F}^{2,\nu}(\mathbb{C}^2)$, therefore, its extension given by Lemma 6.1, is slice regular and belongs to $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$. Moreover, by means of the reproducing property for the elements in $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$, we obtain the following identity

$$\mathcal{I}^{\nu} F(q) = \left(\frac{\nu}{\pi}\right) \int_{\mathbb{C}} F\left(\frac{\xi}{\sqrt{2}}, \frac{-i\xi}{\sqrt{2}}\right) K^{\nu}_{\mathbb{H}}(q,\xi) e^{-\nu|\xi|^2} d\lambda(\xi). \tag{6.6}$$

To conclude that $\mathcal{B}^{\nu}_{\mathbb{H}} \circ \mathcal{R}^{\nu} F$ and $\mathcal{I}^{\nu} F$ are identically the same, we need only to prove it for their restrictions on $\mathbb{C}_i \simeq \mathbb{C}$ and then apply the identity principle for the slice left regular functions (Lemma 6.2). To this end, we begin by rewriting the transforms $\mathcal{B}^{\nu}_{\mathbb{H}}$ and $\mathcal{R}^{\nu}$ as

$$\mathcal{B}^{\nu}_{\mathbb{H}} f(q) = \left\langle f, \overline{S^{\nu}(q,\cdot)} \right\rangle_{L^{2,\nu}(\mathbb{R},\mathbb{C})} = \int_{\mathbb{R}} f(x) S^{\nu}(q,x) e^{-\nu x^2} dx$$

and

$$\mathcal{R}^{\nu} F(x) = \langle \mathcal{C}_{\psi_2} F, S^{\nu}(\cdot,x) \rangle_{L^{2,\nu}_{\mathbb{C}}(\mathbb{C})} = \int_{\mathbb{C}} F(\xi) S^{\nu}(\overline{\xi},x) e^{-\nu|\xi|^2} d\lambda(\xi),$$

where S^{ν} denotes the generating function of the rescaled Hermite polynomials H^{ν}_m given by

$$S^{\nu}(q,x) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \sum_{m=0}^{+\infty} \left(\frac{\nu^m}{m!}\right)^{\frac{1}{2}} \frac{q^m H^{\nu}_m(x)}{\|H^{\nu}_m\|_{L^{2,\nu}(\mathbb{R},\mathbb{C})}} = \left(\frac{\nu}{\pi}\right)^{\frac{3}{4}} e^{-\frac{\nu}{2}q^2 + \sqrt{2}\nu q x}. \tag{6.7}$$

Such kernel function satisfies

$$\langle S^{\nu}(q,\cdot), S^{\nu}(\xi,\cdot) \rangle_{L^{2,\nu}(\mathbb{R},\mathbb{C})} = \left(\frac{\nu}{\pi}\right) \sum_{m=0}^{+\infty} \frac{\nu^m q^m \overline{\xi}^m}{m!} =: \left(\frac{\nu}{\pi}\right) e_*^{\nu[q,\overline{\xi}]} = K^{\nu}_{\mathbb{H}}(q,\xi).$$

Thus, for every $F \in \mathcal{F}^{2,\nu}(\mathbb{C}^2)$ and $q \in \mathbb{C}_i \simeq \mathbb{C}$, we have

$$\begin{aligned} \mathcal{B}^{\nu}_{\mathbb{H}} \circ \mathcal{R}^{\nu} F(q) &= \left\langle \mathcal{C}_{\psi_2} F, \langle S^{\nu}(q,\cdot), S^{\nu}(\cdot\cdot,\cdot)\rangle_{L^{2,\nu}(\mathbb{R},\mathbb{C})} \right\rangle_{L^{2,\nu}_{\mathbb{C}}(\mathbb{C})} \\ &= \left(\frac{\nu}{\pi}\right) \int_{\mathbb{C}} F\left(\frac{\xi}{\sqrt{2}}, \frac{-i\xi}{\sqrt{2}}\right) e_*^{\nu[q,\overline{\xi}]} e^{-\nu|\xi|^2} d\lambda(\xi) \qquad (6.8) \\ &= \left(\frac{\nu}{\pi}\right) \int_{\mathbb{C}} F\left(\frac{\xi}{\sqrt{2}}, \frac{-i\xi}{\sqrt{2}}\right) e^{\nu q\overline{\xi}} e^{-\nu|\xi|^2} d\lambda(\xi) \\ &= F\left(\frac{q}{\sqrt{2}}, \frac{-iq}{\sqrt{2}}\right) \\ &=: \mathcal{I}^{\nu} F(q), \end{aligned}$$

since $(\nu/\pi)e^{\nu q\overline{\xi}}$ is the reproducing kernel of $\mathcal{F}^{2,\nu}(\mathbb{C})$ and $\xi \longmapsto F\left(\frac{\xi}{\sqrt{2}}, \frac{-i\xi}{\sqrt{2}}\right) \in \mathcal{F}^{2,\nu}(\mathbb{C})$. The proof is completed. ∎

The following result identifies $\mathcal{I}^{\nu}(\mathcal{F}^{2,\nu}(\mathbb{C}^2))$ as the specific subspace of slice regular functions in $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$ leaving the slice $\mathbb{C}_i$ invariant,

$$\mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H}) := \{F \in \mathcal{F}^{2,\nu}_{slice}(\mathbb{H});\ F(\mathbb{C}_i) \subset \mathbb{C}_i\}.$$

Its sequential characterization reads

$$\mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H}) = \left\{ F(q) = \sum_{m=0}^{+\infty} q^m c_m;\ c_m \in \mathbb{C}_i,\ \sum_{m=0}^{+\infty} \frac{m!}{\nu^m} |c_m|^2 < +\infty \right\}.$$

Theorem 6.2.2. *The transform $\mathcal{I}^{\nu}$ maps $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ onto $\mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H})$ and its action on the reproducing kernel $K^{\nu}_2((u,v),(z,w))$ in (6.11) is given by*

$$\mathcal{I}^{\nu}(K^{\nu}_2(\cdot,(z,w)))(q) = K^{\nu}_{\mathbb{H}}\left(q, \frac{z+iw}{\sqrt{2}}\right). \qquad (6.9)$$

Proof. Let $F(z,w) = \sum\limits_{m,n=0}^{+\infty} a_{m,n} e_{m,n}(z,w) \in \mathcal{F}^{2,\nu}(\mathbb{C}^2)$, where $e_{m,n}(z,w) = z^m w^n$. By means of Theorem 6.2.1, we have $\mathcal{I}^{\nu} F = \mathcal{B}^{\nu}_{\mathbb{H}} \circ \mathcal{R}^{\nu} F \in \mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$. Moreover, for every $q \in \mathbb{H}$, we have

$$\mathcal{I}^{\nu}(e_{m,n})(q) = Ext(\mathcal{C}_{\psi_2} e_{m,n})(q) = q^{m+n}(-i)^n 2^{-\frac{m+n}{2}}$$

since $\mathcal{C}_{\psi_2} e_{m,n}(\xi) = (-i)^n 2^{-\frac{m+n}{2}} \xi^{m+n}$. Therefore

$$\mathcal{I}^{\nu}(f)(q) = \sum_{j=0}^{+\infty} q^j \left(\sum_{k=0}^{j} (-i)^k 2^{-\frac{j}{2}} a_{j-k,k} \right) = \sum_{j=0}^{+\infty} q^j b_j,$$

where the coefficients $b_j = \sum\limits_{k=0}^{j} (-i)^k 2^{-\frac{j}{2}} a_{j-k,k}$ belong to $\mathbb{C}_i$. This shows that $\mathcal{I}^{\nu}(\mathcal{F}^{2,\nu}(\mathbb{C}^2)) \subset$

$\mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H})$. For the inverse inclusion, let $F \in \mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H})$ and let $f \in L^{2,\nu}_{\mathbb{H}}(\mathbb{R})$ such that $F = \mathcal{B}^{\nu}_{\mathbb{H}} f$. Now, since $F(\mathbb{C}_i) \subset \mathbb{C}_i$ we get $f_0 \in L^{2,\nu}(\mathbb{R},\mathbb{C})$ and therefore $f_0 = \mathcal{R}^{\nu} F_0$ for some $F_0 \in \mathcal{F}^{2,\nu}(\mathbb{C}^2)$. Thus, $F = \mathcal{B}^{\nu}_{\mathbb{H}} \circ \mathcal{R}^{\nu} F_0 = \mathcal{I}^{\nu} F_0$.

Formula (6.9) for arbitrary fixed $(z,w) \in \mathbb{C}^2$ immediately follows from the identity principle (Lemma 6.2) for the left slice regular functions. Indeed, the left slice regular functions

$$q \longmapsto \mathcal{I}^{\nu}(K^{\nu}_2(\cdot,(z,w)))(q) = Ext\left(\xi \longmapsto K^{\nu}_2\left(\left(\frac{\xi}{\sqrt{2}}, -\frac{i\xi}{\sqrt{2}}\right),(z,w)\right)\right)(q) \tag{6.10}$$

and

$$q \longmapsto K^{\nu}_{\mathbb{H}}\left(q, \frac{z+iw}{\sqrt{2}}\right) = \left(\frac{\nu}{\pi}\right) e_*^{\nu\left[q,\frac{\overline{z+iw}}{\sqrt{2}}\right]} \tag{6.11}$$

coincide on the slice $\mathbb{C}_i$, therefore, their difference is identically zero on the whole $\mathbb{H}$. ∎

Remark 6.4. For $F(q) = \sum_{m=0}^{+\infty} q^m c_m \in \mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H})$, i.e., with $c_m \in \mathbb{C}_i$ and $\sum_{m=0}^{+\infty} \frac{m!}{\nu^m}|c_m|^2 < +\infty$, then the function $f_0 = \mathcal{R}^{\nu} F_0$ involved in the above proof is given by

$$f_0(x) = \sum_{m=0}^{+\infty} \frac{\|e_m\|_{L^{2,\nu}(\mathbb{C},\mathbb{C})}}{\|H^{\nu}_m\|_{L^{2,\nu}(\mathbb{R},\mathbb{C})}} c_m H^{\nu}_m(x) \in L^{2,\nu}(\mathbb{R},\mathbb{C}).$$

Moreover, we have $\|f_0\|_{L^{2,\nu}(\mathbb{R},\mathbb{C})} = \|F\|_{L^{2,\nu}(\mathbb{C},\mathbb{C})}$.

The last result of this section concerns the following integral transform

$$\mathcal{J}^{\nu} := \mathcal{G}^{\nu} \circ (\mathcal{B}^{\nu}_{\mathbb{H}})^{-1}$$

mapping $\mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H})$ into the two-dimensional Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ and suggested by the commutative diagram

$$\begin{tikzcd}
\mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H}) \arrow[r, "\mathcal{J}^{\nu}"] \arrow[d, "(\mathcal{B}^{\nu}_{\mathbb{H}})^{-1}"'] & \mathcal{A}^{2,\nu}(\mathbb{C}^2) . \\
L^{2,\nu}(\mathbb{R},\mathbb{C}) \arrow[ur, "\mathcal{G}^{\nu}"'] &
\end{tikzcd}$$

Theorem 6.2.3. *The image of $\mathcal{J}^{\nu}$ coincides with $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$ in* (26)*, and its action on any $f \in \mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H})$ is given by*

$$\mathcal{J}^{\nu} F(z,w) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} F\left(\frac{z+iw}{\sqrt{2}}\right). \tag{6.12}$$

Moreover, for every fixed $\xi \in \mathbb{C}$, we have

$$\mathcal{J}^{\nu}\left(K^{\nu}_{\mathbb{H}}(\cdot,\xi)\right)(z,w) = K^{\nu}_2\left(\left(\frac{\xi}{\sqrt{2}}, \frac{-i\xi}{\sqrt{2}}\right),(z,w)\right), \tag{6.13}$$

where $K^{\nu}_{\mathbb{H}}(q,\xi)$ *and* $K^{\nu}_{2}((u,v),(z,w))$ *are the reproducing kernel of* $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$ *and* $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ *given by* (6.2) *and* (6.11) *respectively.*

Proof. Below, we identify $\mathbb{C}$ and $\mathbb{C}_i$. The restriction of $(\mathcal{B}^{\nu}_{\mathbb{H}})^{-1}$ to $\mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H})$ has as image $L^{2,\nu}(\mathbb{R},\mathbb{C})$ which is contained in $L^{2,\nu}_{\mathbb{H}}(\mathbb{R})$. This readily follows by proceeding in a similar way as in Theorem 6.1.2 since the rescaled Hermite polynomials H^{ν}_{m} is an orthogonal basis of $L^{2,\nu}(\mathbb{R},\mathbb{C})$. Thus, by Theorem 6.1.2, we obtain

$$\mathcal{G}^{\nu}\circ(\mathcal{B}^{\nu}_{\mathbb{H}})^{-1}(\mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H})) = \mathcal{G}^{\nu}(L^{2,\nu}(\mathbb{R},\mathbb{C})) = \mathcal{A}^{2,\nu}(\mathbb{C}^2).$$

This can also be reproved since

$$\mathcal{J}^{\nu}(e_m)(z,w) = \left(\frac{z+iw}{\sqrt{2}}\right)^{m} = e_m(z,w) \tag{6.14}$$

which immediately follows from the formula (6.13), whose the proof can be handled by direct computation. Indeed, for given $F\in\mathcal{F}^{2,\nu}_{slice,i}(\mathbb{H})$, we have $F(\mathbb{C}_i)\subset\mathbb{C}_i$ and $(\mathcal{B}^{\nu}_{\mathbb{H}})^{-1}F = (\mathcal{B}^{1,\nu})^{-1}F_i$, where $(\mathcal{B}^{1,\nu})^{-1}$ is the inverse of the one-dimensional Segal–Bargmann transform and F_i is the restriction of F to the slice $\mathbb{C}_i$. Then, the proof is completed making use of the definition of $\mathcal{G}^{\nu}f(z,w) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}}\mathcal{C}_{\psi_1}(\mathcal{B}^{1,\nu}f)(z,w)$. ∎

Remark 6.5. The restriction of $\mathcal{I}^{\nu} = \mathcal{B}^{\nu}_{\mathbb{H}}\circ\mathcal{R}^{\nu}$ to $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$ is the inverse of $\mathcal{J}^{\nu} := \mathcal{G}^{\nu}\circ(\mathcal{B}^{\nu}_{\mathbb{H}})^{-1}$ for satisfying $\mathcal{J}^{\nu}\circ\mathcal{I}^{\nu} = Id_{\mathcal{A}^{2,\nu}(\mathbb{C}^2)}$.

In the section, we investigate further properties of the integral transform $\mathcal{G}^{\nu}$ when combined with the Fourier transform and connecting one and two-dimensional Bargmann–Fock spaces. We also discuss possible generalization to d-complex space $\mathbb{C}^d$.

6.3 Further new integral transforms

We consider the rescaled Fourier transform $\widetilde{\mathcal{F}}^{\nu}_{\mp}$ defined on $L^{2,\nu}(\mathbb{R},\mathbb{C})$ by $\widetilde{\mathcal{F}}^{\nu}_{\mp} = \mathcal{M}_{\nu/2}\mathcal{F}^{\nu}_{\mp}\mathcal{M}_{-\nu/2}$, where $\mathcal{M}_{\alpha}$ denotes the ground state transform $\mathcal{M}_{\alpha}f := e^{-\alpha|z|^2}f$, and $\mathcal{F}^{\nu}$ is the standard Fourier transform on $L^{2,0}(\mathbb{R},\mathbb{C}) = L^2(\mathbb{R},dx)$ with

$$\mathcal{F}^{\nu}_{\mp}(\varphi)(x) := \left(\frac{\nu}{2\pi}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\varphi(u)e^{\mp\nu ixu}dx.$$

More explicitly, $\widetilde{\mathcal{F}}^{\nu}_{\mp}$ acts on $L^{2,\nu}(\mathbb{R},\mathbb{C})$ as a bounded linear operator by

$$\widetilde{\mathcal{F}}^{\nu}_{\mp}(\varphi)(x) := \left(\frac{\nu}{2\pi}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\varphi(u)e^{\frac{\nu}{2}(x\mp iu)^2}d\lambda(u). \tag{6.1}$$

Thanks to the well-known Plancherel's Theorem, it turns out that the Fourier transform $\widetilde{\mathcal{F}}^{\nu}_{\mp}$ maps unitary $L^{2,\nu}(\mathbb{R},\mathbb{C})$ onto itself. Accordingly, we can consider the following commutative diagrams

$$\begin{array}{ccc} \mathcal{F}^{2,\nu}(\mathbb{C}) & \xrightarrow{\mathcal{T}^{\nu}_{1,\mp}} & \mathcal{A}^{2,\nu}(\mathbb{C}^2) \\ {\scriptstyle (\mathcal{B}^{1,\nu})^{-1}}\downarrow & & \uparrow{\scriptstyle \mathcal{G}^{\nu}} \\ L^{2,\nu}(\mathbb{R},\mathbb{C}) & \xrightarrow[\widetilde{\mathcal{F}}^{\nu}_{\mp}]{} & L^{2,\nu}(\mathbb{R},\mathbb{C}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}^{2,\nu}(\mathbb{C}^2) & \xrightarrow{\mathcal{T}^{\nu}_{2,\mp}} & \mathcal{F}^{2,\nu}(\mathbb{C}) \\ {\scriptstyle \mathcal{R}^{\nu}}\downarrow & & \uparrow{\scriptstyle \mathcal{B}^{1,\nu}} \\ L^{2,\nu}(\mathbb{R},\mathbb{C}) & \xrightarrow[\widetilde{\mathcal{F}}^{\nu}_{\mp}]{} & L^{2,\nu}(\mathbb{R},\mathbb{C}) \end{array} .$$

The transform $\mathcal{T}^{\nu}_{1,\mp} := \mathcal{G}^{\nu} \circ \widetilde{\mathcal{F}}^{\nu}_{\mp} \circ (\mathcal{B}^{1,\nu})^{-1}$ (resp. $\mathcal{T}^{\nu}_{2,\mp} := \mathcal{B}^{1,\nu} \circ \widetilde{\mathcal{F}}^{\nu}_{\mp} \circ \mathcal{R}^{\nu}$) maps $\mathcal{F}^{2,\nu}(\mathbb{C})$ onto $\mathcal{A}^{2,\nu}(\mathbb{C}^2)$ (resp. $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ onto $\mathcal{F}^{2,\nu}(\mathbb{C})$). Their explicit formulas reduced further to elementary composition operators involving the symbols $\psi_1(z,w) = \frac{z+iw}{\sqrt{2}}$ and $\psi_2(\xi) = \frac{1}{\sqrt{2}}(\xi, -i\xi)$, and the reducible representation of the unitary group $U(1) := \{\theta \in \mathbb{C}; |\theta| = 1\}$ defined by $\Gamma_\theta \varphi(\xi) := \varphi(\theta\xi)$.

Theorem 6.3.1. *The action of $\mathcal{T}^{\nu}_{1,\mp}$ and $\mathcal{T}^{\nu}_{2,\mp}$ are given, respectively, by*

$$\mathcal{T}^{\nu}_{1,\mp} = \mathcal{B}^{2,\nu}|_{\mathcal{F}^{2,\nu}(\mathbb{C})} \circ \Gamma_{\mp i} = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \mathcal{C}_{\mp i\psi_1} \tag{6.2}$$

on $\mathcal{F}^{2,\nu}(\mathbb{C})$, and

$$\mathcal{T}^{\nu}_{2,\mp} = \Gamma_{\mp i} \circ Proj \circ (\mathcal{B}^{2,\nu})^{-1} = \left(\frac{\pi}{\nu}\right)^{\frac{1}{2}} \mathcal{C}_{\mp i\psi_2} \tag{6.3}$$

on $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$. Moreover, we have $\mathcal{T}^{\nu}_{2,\mp} \circ \mathcal{T}^{\nu}_{1,\pm} = Id_{\mathcal{F}^{2,\nu}(\mathbb{C})}$ and $\mathcal{T}^{\nu}_{2,\mp} \circ \mathcal{T}^{\nu}_{1,\mp} = \Gamma_{-1} Id_{\mathcal{F}^{2,\nu}(\mathbb{C})}$.

Proof. Recall first that the expression of $(\mathcal{B}^{1,\nu})^{-1}$ given by

$$(\mathcal{B}^{1,\nu})^{-1} f(x) = \langle f, S^{\nu}(\cdot, x)\rangle_{L^{2,\nu}_{\mathbb{C}}(\mathbb{C})},$$

where S^{ν} is the kernel function associated to the rescaled Hermite polynomials H^{ν}_m and given by (6.7). Therefore, by Fubini's Theorem, we get

$$\widetilde{\mathcal{F}}^{\nu} \circ (\mathcal{B}^{1,\nu})^{-1}(f)(x) = \left(\frac{\nu}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{C}} f(\xi) \left(\int_{\mathbb{R}} e^{\frac{\nu}{2}(x \mp iu)^2} S^{\nu}(\overline{\xi}, u) du \right) e^{-\nu|\xi|^2} d\lambda(u).$$

Straightforwardly, we obtain

$$\left(\frac{\nu}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} e^{\frac{\nu}{2}(x \mp iu)^2} S^{\nu}(\zeta, u) du = S^{\nu}(\mp i\zeta, x).$$

Hence

$$\widetilde{\mathcal{F}}^{\nu} \circ (\mathcal{B}^{1,\nu})^{-1} f(x) = \langle \Gamma_{\mp i} f, S^{\nu}(\cdot, x)\rangle_{L^{2,\nu}_{\mathbb{C}}(\mathbb{C})} = (\mathcal{B}^{1,\nu})^{-1} \circ \Gamma_{\mp i} f(x). \tag{6.4}$$

Consequently, the transform $\mathcal{T}^{\nu}_{1,\mp} = \mathcal{G}^{\nu} \circ \widetilde{\mathcal{F}}^{\nu}_{\mp} \circ (\mathcal{B}^{1,\nu})^{-1}$ reduces further to

$$\mathcal{T}^{\nu}_{1,\mp} f(z,w) = \mathcal{B}^{2,\nu} \circ \mathcal{B}^{1,\nu}((\mathcal{B}^{1,\nu})^{-1} \circ \Gamma_{\mp i} f)(z,w) = \mathcal{B}^{2,\nu} f(\mp iz, \mp iw)$$

by means of Theorem 6.1.1, as well as to

$$\mathcal{T}^{\nu}_{1,\mp} = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \mathcal{C}_{\psi_1} \circ \mathcal{B}^{1,\nu}(\mathcal{B}^{1,\nu})^{-1} \circ \Gamma_{\mp i} = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \Gamma_{\mp i} \circ \mathcal{C}_{\psi_1} = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \mathcal{C}_{\mp i\psi_1}$$

on $\mathcal{F}^{2,\nu}(\mathbb{C})$. Moreover, by Theorem 6.1.3 and (6.4), the action of $\mathcal{T}^{\nu}_{2,\mp} := \mathcal{B}^{1,\nu} \circ \widetilde{\mathcal{F}}^{\nu} \circ \mathcal{R}^{\nu}$ on $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ reads

$$\mathcal{T}^{\nu}_{2,\mp} = \left(\frac{\pi}{\nu}\right)^{\frac{1}{2}} \mathcal{B}^{1,\nu} \circ \widetilde{\mathcal{F}}^{\nu}_{\mp} \circ (\mathcal{B}^{1,\nu})^{-1} \mathcal{C}_{\psi_2} = \left(\frac{\pi}{\nu}\right)^{\frac{1}{2}} \Gamma_{\mp i} \circ \mathcal{C}_{\psi_2} = \left(\frac{\pi}{\nu}\right)^{\frac{1}{2}} \mathcal{C}_{\mp i\psi_2}.$$

We also have

$$\begin{aligned}
\mathcal{T}^{\nu}_{2,\mp} &:= \mathcal{B}^{1,\nu} \circ \widetilde{\mathcal{F}}^{\nu}_{\mp} \circ \mathcal{R}^{\nu} \\
&= \mathcal{B}^{1,\nu} \circ \widetilde{\mathcal{F}}^{\nu}_{\mp} \circ (\mathcal{B}^{1,\nu})^{-1} \circ Proj \circ (\mathcal{B}^{2,\nu})^{-1} \\
&= \Gamma_{\mp i} \circ Proj \circ (\mathcal{B}^{2,\nu})^{-1}.
\end{aligned}$$

Finally, from (6.2) and (6.3), we obtain

$$\begin{aligned}
\mathcal{T}^{\nu}_{2,\mp}(\mathcal{T}^{\nu}_{1,\mp} f)(\xi) &= \mathcal{C}_{\mp i\psi_2}(\mathcal{C}_{\mp i\psi_1} f)(\xi) \\
&= \mathcal{C}_{\mp i\psi_1} f\left(\frac{\mp i\xi}{\sqrt{2}}, \frac{\mp \xi}{\sqrt{2}}\right) \\
&= f(-\xi)
\end{aligned}$$

as well as

$$\begin{aligned}
\mathcal{T}^{\nu}_{2,\mp}(\mathcal{T}^{\nu}_{1,\pm} f)(\xi) &= \mathcal{C}_{\mp i\psi_2}(\mathcal{C}_{\pm i\psi_1} f)(\xi) \\
&= \mathcal{C}_{\pm i\psi_1} f\left(\frac{\mp i\xi}{\sqrt{2}}, \frac{\mp \xi}{\sqrt{2}}\right) \\
&= f(\xi).
\end{aligned}$$

∎

6.4 The case of high dimensions

We conclude this chapter by discussing the generalization to d-complex space $\mathbb{C}^d$. This is possible for $d = 2^k$ by considering the integral transform $\mathcal{G}^{\nu}_k$ mapping isometrically the standard Hilbert space $L^{2,\nu}(\mathbb{R},\mathbb{C}))$ into the Bargmann–Fock space $\mathcal{F}^{2,\nu}(\mathbb{C}^{2^k})$ defined by induction

$$\mathcal{G}^{\nu}_k := \mathcal{B}^{2^k,\nu} \circ \mathcal{B}^{2^{k-1},\nu} \circ \cdots \circ \mathcal{B}^{2,\nu} \circ \mathcal{B}^{1,\nu}.$$

We claim that for every $f \in L^{2,\nu}(\mathbb{R}, \mathbb{C})$ and $Z = (z_1, \cdots, z_{2^k}) \in \mathbb{C}^{2^k}$ we have

$$\mathcal{G}_k^\nu f(Z) = c_{2^k}^\nu \mathcal{C}_{\psi_k} \mathcal{B}^{1,\nu} f(Z) = c_{2^k}^\nu \mathcal{B}^{1,\nu} f(\psi_k(Z))$$

where $\mathcal{C}_{\psi_k}$ denotes the composition operator with the special symbol

$$\psi_k(Z) := \frac{1}{2^{\frac{k}{2}}} \sum_{m=0}^{2^{k-1}-1} i^m (z_{2m+1} + i z_{2m+2}).$$

The computations hold true for $k = 1$ and $k = 2$.

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