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Sujet:

ÉTUDE DE CERTAINES CONDITIONS ALGÉBRIQUES ET HOMOLOGIQUES SUR LES ANNEAUX COMMUTATIFS

Présentée par

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Étude de certaines conditions algébriques et homologiques sur les anneaux commutatifs

Dédicace

Je dédie ce travail à tous ceux qui ont contribué de prêt ou de loin à son réalisation:

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Contents

Author's papers involved in this thesis/Articles de l'auteur inclus dans cette thèse:

(1) N. Mahdou and S. Moussaoui, *About weak π-rings*, Boletim da Sociedade Paranaense de Matemática, Accepted for publication.

(2) N. Mahdou and S. Moussaoui, *On strongly $(*)$-rings*, submitted for publication.

(3) N. Mahdou, S. Moussaoui and S. Yassemi, *The divided, going-down, and Gaussian properties of amalgamation of rings*, Communications in Algebra, 49:5 (2021), 1938-1949.

(4) A. Y. Darani, N. Mahdou and S. Moussaoui, *Strongly ϕ-n-irreducible ideals*, Journal of Algebra and its Applications, Accepted for publication.

(5) N. Mahdou, S. Moussaoui and M.A.S. Moutui, *When every finitely projective ideal is projective*, Indian J Pure Appl Math (2021).

(6) A. El khalfi, N. Mahdou and S. Moussaoui and M.A.S. Moutui *On a weak version of S-Noetherianity*, Submitted for publication.

Other papers/Autres articles:

(1) N. Mahdou and S. Moussaoui, *weakly alpha-prime ideals*, Submitted for publication.

(2) N. Mahdou, S. Moussaoui and Y. Zahir, *Amalgamated algebra defined by radical factorization and related properties*, submitted for publication.

Summary (In French)

Cette thèse se compose de six chapitres recouvrant six articles [41, 49, 67, 68, 69, 70], à travers lesquels nous enrichissons la littérature par de nouvelles classes d'anneaux et de nouveaux exemples par le biais du transfert de certaines propriétés aux extensions d'un anneau commutatif, à savoir l'extension triviale et l'amalgamation d'anneaux le long d'un idéal.

Le premier chapitre porte sur l'étude de la factorisation des idéaux réguliers principaux en des idéaux premiers. Dans ce chapitre, nous établissons quelques caractérisations des faibles π-anneaux dans le cas général et nous étudions leur transfert et celui des presque faibles π-anneaux et des $(*)$-anneaux à l'extension triviale d'un anneau. Dans le deuxième chapitre nous introduisons une nouvelle classe d'anneaux appelée les fortement $(*)$-anneaux et nous étudions le transfert de cette propriété à l'extension triviale d'un anneau et à l'amalgamation des anneaux, ainsi que celui des propriétés mentionnées précédemment dans le premier chapitre. Le troisième chapitre est consacré à l'étude de la stabilité des propriétés anneau divisé, anneau localement divisé, anneau descendant et anneau Gaussien en terme de l'amalgamation des anneaux, ainsi que l'étude des conditions nécessaires et suffisantes pour avoir la forme des idéaux de ce fameux anneau. Ensuite, dans le quatrième chapitre, nous introduisons un nouveau type d'idéaux appelés les idéaux fortement ϕ-n-irréductibles et nous étudions leur extensions dans différents type de construction d'anneaux. Le cinquième chapitre a pour but d'étudier et d'établir le transfert de la propriété homologique des FPP-anneaux dans différentes extensions d'anneaux. Finalement, le sixième chapitre est consacré à l'étude d'une version faible de la propriété anneau S-Noethérien, appelée anneau w-S-Noethérien, ainsi que son transfert aux extensions d'anneaux. De plus, nous généralisons cette propriété dans le cas du domaine intègre, que nous appelons la propriété des anneaux non-nil w-S-Noethériens et nous caractérisons cette propriété grâce au produit fibré.

Quelques perspectives que nous voudrons aborder comme travaux futures, sont présentées à la fin de notre thèse.
Mots clés: faible π-anneau, $(*)$-anneau, fortement $(*)$-anneau, anneau divisé, anneau

localement divisé, anneau descendant, anneau Gaussien, idéal fortement ϕ-n-irréductible, FPP-anneau, anneau w-S-Noethérien, extension triviale, amalgamation d'anneaux le long d'un idéal, produit fibré.

Laboratoire où la thèse a été préparée:

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Summary

This thesis is consists of six chapters covering six papers [41, 49, 67, 68, 69, 70], through which we enrich the literature by new classes of rings and new examples by the ticket of the transfer of certain properties to the extensions of a commutative ring, namely the trivial ring extension and the amalgamation of rings along an ideal.

The first chapter relates to the study of the factorization of regular principal ideals into prime ideals. In this chapter, we establish some characterizations of weak π-rings in the general case and we study their transfer and that of the properties almost weak π-rings and $(*)$-rings to the trivial ring extension. In the second chapter, we introduce a new class of rings called the strongly $(*)$-rings and we study the transfer of this property to the trivial ring extension and to the amalgamation of rings, as well as that of the mentioned properties previously in the first chapter. The third chapter is devoted to study the stability of the properties divided rings, locally divided rings, going-down rings and Gaussian rings in terms of the amalgamation of rings, as well as the study of the necessary and sufficient conditions to have the form of ideals of this famous ring. Then, in the fourth chapter, we introduce a new type of ideals called strongly ϕ-n-irreducible ideals and we study their extensions in different type of ring constructions. The fifth chapter aims to study and establish the transfer of the homological property FPP-rings in different extensions of rings. Finally, the sixth chapter is devoted to the study of a weak version of the property S-Noetherian ring, called w-S-Noetherian ring, as well as its transfer to some ring extensions. Moreover, we generalize this property in the case of integral domains, which we call non-nil w-S-Noetherian ring property and we characterize this class of rings using pullbacks.

Some perspectives that we want to deal with as future work, are presented at the end of our thesis.
Keys words: strongly π-ring, $(*)$-ring, strongly $(*)$-ring, divided ring, localy divided ring, going-down ring, Gaussien ring, strongly ϕ-n-irreducible ideal, FPP-ring, w-S-Noetherian ring, trivial ring extension, amalgamation of rings along an ideal, pullbacks.
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Introduction (In French)

Tous les anneaux considérés dans cette thèse sont commutatifs et unitaires; les modules sont des modules sur des anneaux commutatifs et unitaires.

Cette recherche s'intéresse à l'étude de certaines propriétés algébriques et homologiques dans différents constructions d'anneaux. La factorisation en éléments premiers des idéaux d'un anneau commutatif, qui est le sujet du chapitre 1, est le moyen le plus pratique et le plus simple à utiliser pour étudier plusieurs propriétés liées à ces idéaux. Les π-anneaux, ce sont les anneaux où chaque idéal principal peut être présenter comme un produit fini d'idéaux premiers, ont toujours été un terrain de compétition entre les chercheurs, ce qui a donné naissance de nombreux résultats principaux dans la littérature. Le lecteur peut consulter les références suivantes [78] et [79], où Mori a caractérisé ces anneaux. En 1972, Levitz a montré qu'un π-domaine est un domaine de Krull dont tout idéal premier minimal est inversible. Dans [63], Kang a donné trois caractérisations de ces anneaux dans le cas intègre; parmi eux tout idéal premier non-nul contient un idéal premier inversible. En 2017, Jayaram a introduit deux généralisations des π-anneaux, la première concerne les faibles π-anneaux qui sont des anneaux où chaque idéal régulier principal est un produit fini d'idéaux premiers, et la deuxième généralisation concerne les anneaux vérifiant la condition $(*)$, qu'on va les appelés par la suite les $(*)$-anneaux, où tout idéal régulier principal a une décomposition primaire. L'auteur a caractérisé les faibles π-anneaux dans le cas des anneaux quasi-réguliers, c'est à dire les anneaux dont leurs anneaux de quotients sont réguliers au sens de Von Neumann. Restant dans le même article, l'auteur a caractérisé le fameux anneau de Dedekind, où chaque idéal a une factorisation en éléments premiers, par le fait qu'il soit un WI-anneau et un faible π-anneau. Un WI-anneau est un anneau où chaque idéal de type fini est faible inversible. Nous poursuivons ces recherches en prolongeons ces caractérisations au cas général et en étudiant le transfert des faibles π-anneaux, des presque faibles π-anneaux et des $(*)$-anneaux aux extensions triviales.

En 1976, Dobbs a introduit la notion d'un domaine divisé comme étant un domaine intègre où tout idéal premier P d'un anneau A satisfait la condition $P = PA_P$, i.e tout élément dans $A\backslash P$ divise tous les éléments de P. Ces domaines divisés ce sont les AV-domaines étudiés par Akiba dans [1]. Dans [45], l'auteur a introduit également la notion des domaines localement divisés, ce sont des domaines intègres dont A_M est divisé pour chaque idéal maximal M de A. Un peut plus tard, en 2001, Badawi et Dobbs ont généralisé ces deux notions au cas des anneaux avec diviseurs de zéro. Un anneau A est dit divisé si chaque idéal premier P est comparable avec tout idéal (idéal principal) de A. Un anneau A est dit localement divisé si A_M est divisé pour tout idéal maximal M de A. Les auteurs ont caractérisé ces anneaux en utilisant l'extension triviale. Citons par exemple les deux caractérisations suivantes: (1) Un anneau A est divisé si et seulement si $Nil(A) \in Spec(A)$ et $A \propto E$ est divisé pour chaque (ou certain) $A_{Nil(A)}$-module E; (2) un anneau A est localement divisé si et seulement si tout idéal maximal de A contient seulement un idéal premier minimal de A et $A \propto E$ est un anneau localement divisé pour tout A-module E vérifiant la condition suivante: pour tout $M \in Max(A)$, la structure des A-module sur E est induite par une structure des $(Nil(A_M))^{-1}A_M$-module sur E_M. Ce dernier type d'anneaux a une relation d'implication irréversible avec les anneaux descendant puisque chaque anneau localement divisé est descendant mais l'inverse n'est pas vrai en général. Rappelons qu'un anneau A est dit descendant, comme il est défini dans [74], si pour chaque sur-anneau T de A et tous $P \subseteq P'$ deux idéaux premiers de A et Q' un idéal premier de T avec $Q' \cap A = P'$, il existe un idéal premier Q de T tel que $Q \subseteq Q'$ et $Q \cap R = P$. Des travaux considérables ont été présentés dans les références suivantes [22, 44, 45, 46].

Une autre notion très importante que nous avons traité dans cette thèse est celle des anneaux Gaussiens. Un anneau A est dit Gaussien si pour tous $f,g \in A[X]$, on a $c(fg) = c(f)c(g)$ où $c(f)$ est l'idéal de A engendré par les coefficients de f. Pour plus d'informations, le lecteur peut consulter les articles suivants [32, 73, 82]. Un idéal I est dit irréductible (respectivement, fortement irréductible) si pour tous I_1, I_2 deux idéaux de A tels que $I_1 \cap I_2 = I$ (respectivement, $I_1 \cap I_2 \subseteq I$), alors $I = I_1$ ou $I = I_2$ (respectivement, $I_1 \subseteq I$ ou $I_2 \subseteq I$). Les idéaux premiers sont des exemples triviaux de ces deux notions. En 2019, Zeidi a défini un idéal n-irréductible (respectivement, fortement n-irréductible), où n est un entier positif, si chaque fois que $I_1 \cap ... \cap I_{n+1} = I$ (respectivement, $I_1 \cap ... \cap I_{n+1} \subseteq I$) pour tous $I_1, ..., I_{n+1}$ des idéaux de A, alors il existe n éléments des $(I_i)_i$ tels que leur intersection égale à I (respectivement, leur intersection est dans I). Selon [4], un idéal propre I d'un anneau A est appelé n-absorbant (respectivement, fortement n-absorbant) si chaque fois que $x_1...x_{n+1} \in I$ pour $x_1, ..., x_{n+1} \in A$ (respectivement, $I_1...I_{n+1} \subseteq I$ pour $I_1, ..., I_{n+1}$ des idéaux de A), alors il existe n éléments parmi ces $(x_i)_i$ (respectivement, n idéal parmi ces $(I_i)_i$) dont leur produit est dans I. D'après [77], notons qu'un idéal I de A est dit ϕ-n-absorbant primaire (respectivement, fortement ϕ-n-absorbant primaire) de A, si chaque fois que $x_1, ..., x_{n+1} \in I\backslash\phi(I)$ pour $x_1, ..., x_{n+1} \in A$ (respectivement, $I_1...I_{n+1} \subseteq I\backslash\phi(I)$ pour $I_1, ..., I_{n+1}$ des idéaux de A), alors soit $x_1...x_n \in I$ ou le produit de x_{n+1} avec $(n-1)$ éléments parmi $x_1, ..., x_n$ est dans $\sqrt{I}$ (respectivement, soit $I_1 ... I_n \subseteq I$ ou le produit de I_{n+1} avec $(n-1)$ idéals parmi $I_1, ..., I_n$ est dans $\sqrt{I}$). En s'inspirant de tous ces travaux pour définir la notion des idéaux fortement ϕ-n-irréductibles.

Dans le chapitre cinq, nous poursuivons l'étude des idéaux finiment projectifs en relation avec les idéaux projectifs. Rappelons qu'un idéal est dit projectif s'il est facteur direct d'un module libre. Un idéal I est dit finiment projectif si pour tout idéal $J \subseteq I$ de type fini, la fonction d'inclusion $J \to I$ se factorise à travers un module libre L. Il est bien connu qu'un idéal projectif est finiment projectif et tout idéal finiment projectif de type fini est projectif. Plusieurs résultats concernant les modules finiment projectifs sont donnés dans [57, 13]. La dernière notion traitée dans ce travail est la version faible des anneaux S-Noethériens. Soit S une partie multiplicative d'un anneau A. A est dit S-Noethérien si tout idéal de A est S-fini, c'est à dire pour tout idéal I de A ils existent $s \in S$ et un idéal J de A de type fini tels que $sI \subseteq J \subseteq I$. Les anneaux S-Noethériens généralise les anneaux Noethérien et notons que ces deux classes d'anneaux coïncident dans le cas où S est composée des éléments inversibles. Dans ce cas, les idéaux S-finis seront tout simplement les idéaux de type finis.

Notre recherche s'appuie entièrement sur l'étude du transfert de certaines propriétés aux extensions des anneaux commutatifs unitaires. Avant de citer les principaux résultats de chaque chapitre, nous présentons quelques définitions, terminologies et notations.

Définition 0.0.1 *Soient A un anneau et E un A-module. L'extension triviale de A par E (appelée aussi l'idéalisation de E), c'est l'anneau noté par $A \propto E$ dont le groupe abélien est $A \times E$ avec l'addition naturelle et une multiplication donnée par $(a,e)(b,f) = (ab, af + be)$. Cette construction d'anneau a été introduite par Nagata en 1956 [80, page 2]. Principalement, les extensions d'anneaux triviales ont été utiles pour générer des nouveaux exemples et résoudre de nombreux problèmes ouverts et conjectures dans la théorie des anneaux commutatifs et non commutatifs. Les idéaux de ce fameux anneau n'ont pas une forme précise, ce qu'il montre l'exemple donné par Mahdou et Kabbaj dans [61]. Particulièrement, Les idéaux premiers de $A \propto E$ ont toujours la forme $P \propto E$ où P est un idéal premier de A. Un idéal J de $A \propto E$ est dit homogène s'il a la forme $I \propto F$ où I est un idéal de A, F est un sous module de E et $IE \subseteq F$, nous avons aussi $I = \{a \in A \mid (a,e) \in J \text{ pour certain } e \in E\}$ et $F = \{e \in E \mid (a,e) \in J \text{ pour certain } a \in A\}$. Ces idéaux sont d'une grande importance dans l'étude du transfert d'une propriété de l'anneau A vers l'anneau $A \propto E$ et vice versa. Ces extensions d'anneaux ont été largement étudiées; et des travaux considérables, dont une partie est résumée dans le livre de Glaz[53] et celui de Huckaba [56], ont été effectués sur ces extensions, voir par exemple [11, 27, 56, 61].*

Dans [40], Marco D'Anna et Marco Fontana ont introduit la duplication amalgamée d'un anneau A le long d'un sous A-module E de $Q(A)$ (l'anneau total des fractions de A)

vérifiant $E^2 \subseteq E$. Notons que lorsque $E^2 = 0$, cette construction coïncide avec l'extension triviale de A par E. Dans le cas où E est un idéal de A, cette construction est définie par:

Définition 0.0.2 *Soient A un anneau et I un idéal de A.*

$$A \bowtie I := \{(a, a+i) | a \in A, i \in I\}$$

cet anneau, muni des opérations usuelles, est appelé la duplication amalgamée le long de l'idéal I.

Comme généralisation de cette duplication amalgamée, D'Anna, Finocchiaro et Fontana ont introduit en 2010 la construction suivante:

Définition 0.0.3 *Soient A et B deux anneaux, soit J un idéal de B et soit $f : A \to B$ un homomorphisme d'anneaux. Le sous-anneau de $A \times B$ suivant, muni de l'addition et la multiplication usuelles, appelée l'amalgamation de A avec B le long de J en respectant f est défini par:*

$$A \bowtie^f J := \{(a, f(a) + j) | a \in A, j \in J\}$$

L'intérêt de l'amalgamation réside, particulièrement, dans sa capacité à couvrir plusieurs constructions de base en algèbre commutative, y compris les produits fibrés et les extensions triviales d'anneaux. On rappelle que la construction $D+M$ peut être étudiée comme cas particulier de l'amalgamation. Des travaux considérables liés à ces célèbres anneaux ont été présentés dans les articles suivants [36, 37, 38, 40].

Soient A un anneau, I un idéal de A et E un A-module. Nous utiliserons les notations suivantes:

- $Z(A) := \{a \in A / ax = 0$ pour certain $x \in A \backslash \{0\}\}$, l'ensemble des diviseurs de zéro de A.
- $Ann(E) := \{a \in A / ax = 0$ pour tout $0 \neq x \in E\}$, l'annulateur de E.
- $T(A)$, l'anneau total des fractions de A.
- Si A est un anneau intègre, on note par $qf(A) = K$, le corps des fractions de A.
- $Nil(A)$, l'ensemble des éléments nilpotents de A.
- $Rad(A)$, le radial de Jacobson de A.
- $Spec(A)$, l'ensemble des idéaux premiers de A.
- $Max(A)$, l'ensemble des idéaux maximaux de A.
- $Reg(A)$, l'ensemble des éléments réguliers de A.
- $(I : J) := \{x \in T(A) / xJ \subseteq I\}$.
- $V(I) := \{P \in Spec(A) / P \supseteq I\}$.
- $\sqrt{I} := \{a \in A / a^n \in I \; pour \; certain \; entier \; positif \; n\}$, le radical de I.

Présentation des résultats

Dans ce qui suit, nous allons passer en revue les contributions majeures de chaque chapitre.

Dans **le premier chapitre,** nous étudions les faibles π-anneaux, les presque faibles π-anneaux et les $(*)$-anneaux et nous établissons leurs transferts aux extensions d'anneaux. Dans la première section, nous nous sommes spécialement intéressés à caractériser les faibles π-anneaux dans le cas général et à établir quelques résultats préliminaires de ces trois propriétés. Dans cette direction, nous avons examiné la stabilité de ces propriétés sous le produit direct fini d'anneaux, l'image homomorphe et la localisation d'un anneau. Notre résultat principal est présenté par le théorème suivant:

Théorème 0.0.4 *Soit R un anneau commutatif. Les assertions suivantes sont équivalentes:*

1. *R est un faible π-anneau;*
2. *Tout idéal premier régulier contient un idéal premier inversible;*
3. *Tout idéal inversible est un produit fini des idéaux premiers inversibles;*
4. *Tout idéal premier minimal sur un idéal inversible est inversible;*
5. *R est un presque faible π-anneau dans lequel tout idéal premier minimal sur un idéal inversible est de type fini.*

La deuxième section s'intéresse à l'étude du transfert des faibles π-anneaux, presque faibles π-anneaux et $(*)$-anneaux à l'extension triviale d'un anneau commutatif comme le présente le théorème suivant:

Théorème 0.0.5 *Soient A un anneau, E un A-module et $R = A \propto E$ l'extension triviale de A par E. Alors:*

1. *R est un faible π-anneau si et seulement si tout idéal principal non disjoint de S est un produit fini d'idéaux premiers et $sE = E$ pour tout $s \in S$ où $S = A - (Z(A) \cup Z(E))$;*
2. *Si R est un presque faible π-anneau, alors tout idéal principal I non disjoint de S, I_M est un produit fini d'idéaux premiers pour tout $M \in Max(A,I)$;*
3. *Supposons que $sE_M = E_M$ pour tout $M \in Max(A)$ et tout $s \in S$. Si A est un presque faible π-anneau, alors R l'est aussi;*
4. *Supposons que $E = aE$ pour tout $a \in S$. R est un $(*)$-anneau si et seulement si tout idéal régulier principal non disjoint de S a une décomposition primaire.*

Dans **le deuxième chapitre**, nous définissons un fortement $(*)$-anneau comme un anneau dans lequel tout idéal inversible a une décomposition primaire. La première section de ce chapitre présente quelques résultats généraux associés à cette nouvelle classe d'anneaux. La deuxième section est consacrée à l'étude du transfert de la propriété fortement $(*)$-anneau à l'extension triviale comme le montre le théorème suivant:

Théorème 0.0.6 *Soient A un anneau, E un A-module et $R = A \propto E$ tels que $E = aE$ pour tout $a \in S = A - (Z(A) \cup Z(E))$. Alors R est fortement $(*)$-anneau si et seulement si tout idéal inversible de A non disjoint de S a une décomposition primaire.*

Le résultat majeur de la troisième section présente le transfert sous certaines conditions des propriétés faible π-anneau, $(*)$-anneau et fortement $(*)$-anneau entre un anneau commutatif A et $R = A \bowtie^f J$.

Théorème 0.0.7 *Soient A et B deux anneaux, J un idéal de B, $f : A \to B$ un homomorphisme d'anneaux et $R = A \bowtie^f J$.*

1. *Supposons que $Ann(f(a)) \cap J = 0$ pour tout $a \in Reg(A)$. Si R est un faible π-anneau, alors A l'est aussi;*

2. *Supposons que $Ann(f^{-1}(J)) = 0$ et $J = (f(a) + j)J$ pour tout $(a, f(a) + j) \in Reg(R)$. Si A est un faible π-anneau, alors R l'est aussi;*

3. *Supposons que $J = (f(a) + j)J$ pour tout $(a, f(a) + j) \in Reg(R)$. Si R est un $(*)$-anneau, alors tout idéal régulier principal $I =< r >$ de A tel que $(r, f(r) + j)$ est régulier pour certain $j \in J$ a une décomposition primaire. De plus, si $Ann(f(a)) \cap J = 0$ pour tout $a \in Reg(A)$, alors A est un $(*)$-anneau;*

4. *Supposons que $Ann(f^{-1}(J)) = 0$ et $J = (f(a) + j)J$ pour tout $(a, f(a) + j) \in Reg(R)$. Si A est un $(*)$-anneau, alors R l'est aussi;*

5. *Supposons que $J = (f(a) + j)J$ pour tout $(a, f(a) + j) \in Reg(R)$. Si R est fortement $(*)$-anneau, alors tout idéal inversible I de A tel que $I \bowtie^f f(I)J$ est régulier, a une décomposition primaire. En plus de ça, si $Ann(f(a)) \cap J = 0$ pour tout $a \in Reg(A)$, alors A est fortement $(*)$-anneau;*

6. *Supposons que $Ann(f^{-1}(J)) = 0$ et $J = (f(a) + j)J$ pour tout $(a, f(a) + j) \in Reg(R)$. Si A est fortement $(*)$-anneau, alors R l'est aussi;*

7. *Si $f^{-1}(J) = 0$, alors R est un faible π-anneau (respectivement. $(*)$-anneau, fortement $(*)$-anneau) si et seulement si $f(A) + J$ l'est aussi.*

Le résultat principal de la première section du **troisième chapitre**, comme le présente le théorème suivant, est consacré à l'étude des conditions nécessaires et suffisantes pour que l'amalgamation des anneaux le long d'un idéal satisfait la propriété $(*)$. Soient A et B deux anneaux, J un idéal de B et $f : A \to B$ un homomorphisme d'anneaux. Nous disons que $A \bowtie^f J$ satisfait la propriété $(*)$ si tout idéal a l'une des trois formes suivantes:

- $I \times 0$ où $I \subseteq f^{-1}(J)$ est un idéal de A;
- $0 \times K$ où $K \subseteq J$ est un idéal de $f(A)+J$;
- $I \bowtie^f J$ où I est un idéal de A.

Théorème 0.0.8 *Soient A et B deux anneaux, J un idéal propre non nul de B et $f : A \to B$ un homomorphisme d'anneaux.*

(1) Si $A \bowtie^f J$ satisfait la propriété $(*)$, alors les conditions suivantes sont remplies:

(i) $f(A)$ est un domaine intègre;

(ii) $f(A) \cap J = 0$;

(iii) $0 \times J \subseteq ((a, f(a)+j))$ pour tout $a \in A - \{0\}$ et tout $j \in J$ tels que $f(a)+j \neq 0$.

(2) Si f est injectif et $A \bowtie^f J$ satisfait la propriété $(*)$, alors A est un domaine intègre;

(3) Si f n'est pas injectif et A est un anneau avec diviseurs de zéro tels que $A \bowtie^f J$ satisfait la propriété $(*)$, alors $Ann_{f(A)+J}(f(a)+j) \subseteq J$ pour tout $a \in A - \{0\}$ et $j \in J$ avec $f(a) \neq 0$. De plus, si $f^{-1}(J) \nsubseteq Z(A)$, alors $f(a)+j \in Reg(f(A)+J)$ pour tout $a \in Reg(A)$ et $j \in J$ avec $f(a) \neq 0$, et $Ann_{f(A)+J}(j) \subseteq f(Z(A) \backslash f^{-1}(J)) + J$ pour tout $j \in J$;

(4) Si f n'est pas injectif et A est un domaine intègre avec $A \bowtie^f J$ satisfait la propriété $(*)$, alors les conditions suivantes sont remplies:

(i) $f(A)+J$ est un domaine intègre;

(ii) J est idempotent.

(5) Si $0 \times J \subseteq ((a, f(a)+j))$ pour tout $a \in A - \{0\}$ et tout $j \in J$ tels que $f(a)+j \neq 0$, alors $A \bowtie^f J$ satisfait la propriété $(*)$.

Les autres sections de ce chapitre consistent à étudier la stabilité des propriétés anneaux divisés, anneaux localement divisés, anneaux descendants et anneaux Gaussiens dans l'amalgamation d'anneaux. Soient A et B deux anneaux, J un idéal non nul de B, $f : A \to B$ un homomorphisme d'anneaux et $R = A \bowtie^f J$. Dans ce qui suit, nous signifions par $J_{T_\mathfrak{p}}$ la localisation de J par l'idéal premier $T_\mathfrak{p} = f(A \backslash \mathfrak{p}) + J$ pour tout idéal premier $\mathfrak{p}$ contenant $f^{-1}(J)$.

Théorème 0.0.9 *Soient A et B deux anneaux, J un idéal non nul de B, $f : A \to B$ un homomorphisme d'anneaux et $R = A \bowtie^f J$.*

(1) Si A est un domaine intègre, alors R est un anneau divisé si et seulement si les conditions suivantes sont remplies:

(i) $f^{-1}(J) = 0$;

(ii) A *et* $f(A)+J$ *sont des anneaux divisés;*

(iii) $0 \times J \subseteq ((a, f(a)+j))$ *pour tout* $a \in A - \{0\}$ *et* $j \in J$.

(1') Si R est un anneau localement divisé, alors A l'est aussi et les conditions suivantes sont remplies:

(a) Pour tout idéal premier $\mathfrak{p}$ de A contenant $f^{-1}(J)$ tel que $A_{\mathfrak{p}}$ est un domaine intègre:

(i) $f_{\mathfrak{p}}^{-1}(J_{T_{\mathfrak{p}}}) = 0$;

(ii) $f_{\mathfrak{p}}(A_{\mathfrak{p}}) + J_{T_{\mathfrak{p}}}$ *est un anneau divisé;*

(iii) $0 \times J_{T_{\mathfrak{p}}} \subseteq ((\frac{a}{s}, f_{\mathfrak{p}}(\frac{a}{s}) + \frac{j}{t}))$ *pour tout* $\frac{a}{s} \in A_P - \{0\}$ *et* $\frac{j}{t} \in J_{T_{\mathfrak{p}}}$.

(b) Pour tout idéal premier $\mathfrak{p}$ de A contenant $f^{-1}(J)$ tel que $A_{\mathfrak{p}}$ est un anneau avec diviseurs de zéro:

(i) $J_{T_{\mathfrak{p}}} \subseteq Nil(B_{T_{\mathfrak{p}}})$;

(ii) $0 \times J_{T_{\mathfrak{p}}} \subseteq ((\frac{a}{s}, f_{\mathfrak{p}}(\frac{a}{s}) + \frac{j}{t}))$ *pour tout* $\mathfrak{q} \in V(\mathfrak{p})$, $\frac{a}{s} \in A_P \backslash \mathfrak{q}_{\mathfrak{p}}$ *et* $\frac{j}{t} \in J_{T_{\mathfrak{p}}}$.

(2) Si A est un anneau avec diviseurs de zéro, alors R est un anneau divisé si et seulement si les conditions suivantes sont remplies:

(i) $J \subseteq Nil(B)$;

(ii) A *est un anneau divisé;*

(iii) $0 \times J \subseteq ((a, f(a)+j))$ *pour tout* $\mathfrak{p} \in Spec(A)$, *pour tout* $a \in A \backslash \mathfrak{p}$ *et* $j \in J$.

(2') Si de plus B est localement divisé, alors R est localement divisé.

Théorème 0.0.10 *Soient A et B deux anneaux, J un idéal non nul de B, $f : A \to B$ un homomorphisme d'anneaux et $R = A \bowtie^f J$.*

1. *Si R est un anneau descendant, alors A est un anneau descendant;*

2. *Si $J \subseteq Nil(B)$, alors A est un anneau descendant si et seulement si R est un anneau descendant;*

3. *Supposons que* $\mathfrak{Q} \in Spec(f(A)+J)$ *pour tout* $\mathfrak{Q} \in Spec(B) \backslash V(J)$ *(en particulier:* f *est surjectif). Si* A *et* $f(A)+J$ *sont descendants, alors* R *est un anneau descendant;*

4. *Si* $f^{-1}(J) = 0$*, alors* R *est un anneau descendant si et seulement si* $f(A)+J$ *est un anneau descendant.*

Rappelons qu'un anneau local R est Gaussien si et seulement si pour tout idéal I engendré par deux éléments $a, b \in R$, les conditions suivantes sont remplies:

1. I^2 est engendré par a^2 ou b^2;

2. si I^2 est engendré par a^2 et $ab = 0$, alors $b^2 = 0$;

voir [82].
Soient A et B deux anneaux, J un idéal propre de B, $f : A \to B$ un homomorphisme d'anneaux et $R = A \bowtie^f J$. Notons que si $J = B$, alors l'amalgamation $A \bowtie^f J$ coïncide dans ce cas avec le produit direct $A \times B$ et si $J = 0$, alors $A \bowtie^f J \cong A$. Rappelons aussi que $f^{-1}(J) = 0$ si et seulement si $A \bowtie^f J$ et $f(A)+J$ sont isomorphe par la [37, Proposition 2.1]. D'où, pour éviter ces cas triviaux dans le prochain résultat, on peut supposer que "$f^{-1}(J) \neq 0$" et J est un idéal "propre non nul" de B. Par le [71, Lemme 2.2], nous rappelons que R est local si et seulement si A l'est aussi et $J \subseteq Jac(B)$.

Théorème 0.0.11 *Soient* (A, M) *un anneau local,* $J \subseteq Jac(B)$ *un idéal propre d'un anneau* B*,* $f : A \to B$ *un homomorphisme d'anneaux et* $R = A \bowtie^f J$*. Alors,* R *est Gaussien si et seulement si les conditions suivantes sont remplies:*

1. *A est Gaussien;*

2. $(f(a)+j)J = (f(a)+j)^2 J$ *pour tout* $a \in A$ *et tout* $j \in J$*;*

3. $J^2 = 0$.

ou,

(4) A *et* $f(A)+J$ *sont Gaussiens;*

(5) $x^2 = 0$ *pour tout* $x \in f^{-1}(J)$*;*

(6) $(f(a)+j)J = (f(a)+j)^2 J$ *pour tout* $a \in A$ *et tout* $j \in J$ *tel que* $a^2 \neq 0$*;*

(7) Pour tout $a \in A$ *et tout* $j \in J$ *tel que* $a^2 \neq 0$*, we have* $i^2 = k(f(a)+j)^2$ *for some* $k \in J$*;*

(8) Si $f(a^2) = 0$*, alors* $a^2 = 0$.

Corollaire 0.0.12 *Soient A et B deux anneaux, J un idéal propre de B, $f : A \to B$ un homomorphisme d'anneaux et $R = A \bowtie^f J$. Alors, R est Gaussien si et seulement si les conditions suivantes sont remplies:*

(1) A est Gaussien;

(2) B_Q est Gaussien pour tout $Q \in Max(B) \backslash V(J)$;

(3) Pour tout $a \in A_{\mathfrak{m}}$ et $j \in J_{T_{\mathfrak{m}}}$, $(f_{\mathfrak{m}}(a)+j)J_{T_{\mathfrak{m}}} = (f_{\mathfrak{m}}(a)+j)^2 J_{T_{\mathfrak{m}}}$ pour tout $\mathfrak{m} \in Max(A)$ contenant $f^{-1}(J)$;

(4) $J^2_{T_{\mathfrak{m}}} = 0$.

ou, pour tout $\mathfrak{m} \in Max(A)$ contenant $f^{-1}(J)$:

(5) A est Gaussien;

(6) $f_{\mathfrak{m}}(A_{\mathfrak{m}}) + J_{T_{\mathfrak{m}}}$ est Gaussien;

(7) B_Q est Gaussien pour tout $Q \in Max(B) \backslash V(J)$;

(8) $x^2 = 0$ pour tout $x \in f_{\mathfrak{m}}^{-1}(J_{T_{\mathfrak{m}}})$;

(9) Pour tout $a \in A_{\mathfrak{m}}$ et $j \in J_{T_{\mathfrak{m}}}$, $(f_{\mathfrak{m}}(a)+j)J_{T_{\mathfrak{m}}} = (f_{\mathfrak{m}}(a)+j)^2 J_{T_{\mathfrak{m}}}$;

(10) Si $f_{\mathfrak{m}}(a^2) = 0$, alors $a^2 = 0$ pour tout $a \in A_{\mathfrak{m}}$.

Passons maintenant au **quatrième chapitre**. Soient A un anneau, I un idéal de A, n un entier positif et $\phi : \mathfrak{L}(A) \to \mathfrak{L}(A) \cup \{\emptyset\}$ où $\mathfrak{L}(A)$ est l'ensemble des idéaux de A. Nous introduisons dans ce chapitre une nouvelle généralisation des idéaux fortement n-irréductibles appelés les idéaux fortement ϕ-n-irréductibles, c'est-à-dire chaque fois que $I_1 \cap \ldots \cap I_{n+1} \subseteq I$ et $I_1 \cap \ldots \cap I_{n+1} \nsubseteq \phi(I)$ pour $I_1, \ldots, I_{n+1}$ des idéaux de A, alors il y a n des $(I_i)_i$ dont l'intersection est dans I. Dans la première section, nous nous somme intéressés à établir quelques résultats de base sur ces idéaux. Le théorème suivant donne une condition nécessaire pour avoir l'équivalence entre les deux notions suivantes: les idéaux fortement ϕ-irréductibles; ce sont les idéaux fortement ϕ-n-irréductibles avec $n = 1$; et les idéaux fortement irréductibles.
Par un corollaire de ce théorème, nous obtenons également celle du cas général.

Théorème 0.0.13 *Soient A un anneau, $\phi : \mathfrak{L}(A) \to \mathfrak{L}(A) \cup \{\emptyset\}$ une fonction et I un idéal propre de A. Si I est fortement ϕ-irréductible qui n'est pas fortement irréductible, alors $I^2 \subseteq \phi(I)$. Donc, un idéal I fortement ϕ-irréductible avec $I^2 \nsubseteq \phi(I)$ est fortement irréductible.*

Nous examinons la stabilité des idéaux fortement ϕ-n-irréductibles sous différents constructions de la théorie des anneaux.

Théorème 0.0.14 *Soient $f : A \to B$ un homomorphisme surjectif d'anneaux, $\phi : \mathfrak{L}(A) \to \mathfrak{L}(A) \cup \{\emptyset\}$ et $\psi : \mathfrak{L}(B) \to \mathfrak{L}(B) \cup \{\emptyset\}$ deux fonctions vérifiant $f(\phi(I)) \subseteq \psi(f(I))$ pour tout idéal I de A avec $\phi(I) \neq \emptyset$, et $\psi(f(I)) = \emptyset$ lorsque $\phi(I) = \emptyset$. Soit I un idéal de A. Si $f(I) \cap A$ est un idéal fortement ϕ-n-irréductible de A, alors $f(I)$ est un idéal fortement ψ-n-irréductibles de B.*

La deuxième section s'intéresse au transfert des idéaux fortement ϕ-n-irréductibles à l'extension triviale et à l'amalgamation d'anneaux. Soient A un anneau, E un A-module, n un entier positif non nul et $\beta : \mathfrak{S}(E) \to \mathfrak{S}(E) \cup \{\emptyset\}$ une fonction où $\mathfrak{S}(E)$ est l'ensemble des sous modules de E. Nous définissons un sous module F de E comme étant un module fortement β-n-irréductible si chaque fois que $F_1 \cap ... \cap F_{n+1} \subseteq F \backslash \beta(F)$ pour $F_1, ..., F_{n+1}$ des sous modules de E, alors il y a n des $(F_i)_i$ dont l'intersection est dans F, sans perte de généralité, on peut supposer que $F_1 \cap ... \cap F_n \subseteq F$.

Théorème 0.0.15 *Soient A un anneau, E un A-module, n un entier positif non null et $\phi : \mathfrak{L}(A) \to \mathfrak{L}(A) \cup \{\emptyset\}$ une fonction. Soit $\psi : \mathfrak{L}(A \propto E) \to \mathfrak{L}(A \propto E) \cup \{\emptyset\}$ une fonction vérifiant:*

$$\psi(I \propto F) = \begin{cases} \phi(I) \propto F & si\ \phi(I) \neq \emptyset \\ \emptyset & si\ \phi(I) = \emptyset \end{cases}$$

où I est un idéal de A et F un sous module de E vérifiant $IE \subseteq F$.

1. *Alors, $I \propto E$ est un idéal fortement ψ-n-irréductible de $A \propto E$ si et seulement si I est un idéal fortement ϕ-n-irréductible de A;*

2. *Si I est un idéal fortement ϕ-n_1-irréductible de A et F est un sous module fortement n_2-irréductible de E, alors $I \propto F$ est un idéal fortement ψ-n-irréductible de $A \propto E$ où $n = n_1 + n_2$.*

Théorème 0.0.16 *Soient A et B deux anneaux, $f : A \to B$ un homomorphisme d'anneaux, J un idéal de B, n un entier positif non null, $\phi : \mathfrak{L}(A) \to \mathfrak{L}(A) \cup \{\emptyset\}$ une fonction. Soit $\psi : \mathfrak{L}(A \bowtie^f J) \to \mathfrak{L}(A \bowtie^f J) \cup \{\emptyset\}$ une fonction vérifiant:*

$$\psi(I \bowtie^f K) = \begin{cases} (\phi(I) \bowtie^f K) & si\ \phi(I) \neq \emptyset \\ \emptyset & si\ \phi(I) = \emptyset \end{cases}$$

lorsque $f(I)J \subseteq K$ et pour $I \subseteq f^{-1}(J)$, nous avons:

$$\psi(I \times 0) = \begin{cases} (\phi(I) \times 0) & si\ \phi(I) \neq \emptyset \\ \emptyset & si\ \phi(I) = \emptyset \end{cases}$$

où I est un idéal de A et K un sous idéal de J. Alors:

1. *$I \bowtie^f J$ est un idéal fortement ψ-n-irréductible de $A \bowtie^f J$ si et seulement si I est un idéal fortement ϕ-n-irréductible de A;*

2. *Supposons que* $f(I)J \subseteq K$. *Si* I *est un idéal fortement* ϕ-n_1-*irréductible de* A *et* K *est fortement* n_2-*irréductible alors* $I \bowtie^f K$ *est un idéal fortement* ψ-n-*irréductible de* $A \bowtie^f J$ *où* $n = n_1 + n_2$;

3. *Supposons que* $I \subseteq f^{-1}(J)$. *Si* I *est un idéal fortement* ϕ-n_1-*irréductible de* A *et l'idéal nul de* B *est fortement* n_2-*irréductible alors* $I \times 0$ *est un idéal fortement* ψ-n-*irréductible de* $A \bowtie^f J$ *où* $n = n_1 + n_2$.

Dans **le cinquième chapitre**, nous étudions la notion de FPP-anneau comme étant un anneau où tout idéal finiment projectif est projectif. Nous examinons le transfert de cette propriété dans différents extensions d'anneaux. Maintenant, nous présentons le résultat majeur de la première section qui donne une condition nécessaire et suffisante pour que la propriété FPP-anneau descend dans un homomorphisme d'anneaux fidèlement plat.

Théorème 0.0.17 *Soit* $f : A \to B$ *un homomorphisme rendant* B *un* A-*module fidèlement plat de type fini.*

1. *Si* I *est un idéal de* A *et* $I \otimes_A B$ *est un idéal projectif de* B, *alors* I *est un idéal projectif de* A;

2. *Supposons que* $A \subseteq B$ *est une extension de domaines. Si* B *est un FPP-anneau, alors* A *l'est aussi.*

La deuxième et la troisième sections sont consacrées aux transferts de la propriété FPP-anneau à l'extension triviale et à l'amalgamation d'anneaux.

Théorème 0.0.18 *Soient* A *un anneau,* I *un idéal de* A, M *un* A-*module et* $R = A \propto M$.

1. *Supposons que tout idéal finiment projectif est de la forme* $I \propto IM$ *et* M *est un module plat. Si* A *est un FPP-anneau, alors* R *l'est aussi;*

2. *Supposons que* $(A, \mathscr{M})$ *est un anneau local tel que* $\mathscr{M}M = 0$. *Si* A *est un FPP-anneau, alors* R *l'est aussi.*

Théorème 0.0.19 *Soient* A *et* B *deux anneaux,* J *un idéal de* B, $f : A \to B$ *un homomorphisme d'anneaux et* $R = A \bowtie^f J$. *Alors, les assertions suivantes sont remplies:*

1. *Si* R *est un FPP-anneau, alors* A *l'est aussi;*

2. *Supposons que* f *est injectif,* J *est un* $f(A)$-*module projectif et tout idéal finiment projectif de* R *contient* $0 \times J$. *Alors* R *est un FPP-anneau si et seulement si* A *l'est aussi.*

Le dernier chapitre s'intéresse à l'étude d'une nouvelle notion appelée anneaux w-S-Noethériens qui est une version faible des anneaux S-Noethériens. Un anneau A est dit w-S-Noethérien si tout idéal premier S-fini est S-Noethérien où $S \subseteq A$ est une partie multiplicative. D'une façon équivalente, un anneau A est un anneau w-S-Noethérien si pour n'importe quelle paire d'idéaux I et P tels que $I \subseteq P$ et P est un idéal premier S-fini, alors I est S-fini. Rappelons qu'un anneau A est dit faiblement S-Noethérien si tout idéal propre S-fini de A est un A-module S-Noethérien. La première section est consacrée à l'étude de quelques résultats préliminaires et propriétés de base qui nous permettent de construire de nouveaux exemples originaux des anneaux w-S-Noethériens qui ne sont pas faiblement S-Noethériens. Les propositions suivantes sont tous deux parmi les résultats principaux de cette section.

Proposition 0.0.20 *Soient A et B deux anneaux, $f : A \to B$ un homomorphisme d'anneaux et S une partie multiplicative de A tel que $I^{ce} = I$ pour tout idéal I de B et J^c est un idéal S-fini de A pour tout idéal J de B $f(S)$-fini. Si A est w-S-Noethérien, alors B est w-$f(S)$-Noethérien.*

Proposition 0.0.21 *Soit $A \subseteq B$ une extension d'anneaux telle que $IB \cap A = I$ pour tout idéal I de A et PB est un idéal premier de B pour tout idéal premier P de A. Considérons une partie multiplicative $S \subseteq A$. Si B est un anneau w-S-Noethérien, alors A l'est aussi.*

La deuxième section de ce chapitre consiste à étudier le transfert de la propriété d'être un anneau w-S-Noethérien à l'extension triviale et à l'amalgamation d'anneaux.

Théorème 0.0.22 *Soient A un anneau, $S \subseteq A$ une partie multiplicative, et M un A-module S-fini. Posons $R = A \propto M$ et $S' = S \propto E$. Alors, R est un anneau w-S'-Noethérien si et seulement si l'une de ces assertions suivantes est remplie:*

1. *A ne contient pas un idéal premier S-fini;*
2. *A est un anneau w-S-Noethérien qui contient au moins un idéal premier S-fini et M est un module S-Noethérien.*

Théorème 0.0.23 *Soient A et B deux anneaux, J un idéal de B qui est S-fini comme A-module, $f : A \to B$ un homomorphisme d'anneaux, S une partie multiplicative de A, $S' = S \bowtie^f 0$ et $R = A \bowtie^f J$.*

1. *Si R est un anneau w-S'-Noethérien, alors A est un anneau w-S-Noethérien;*
2. *Supposons que $J \subseteq Nil(B)$. Alors R est un anneau w-S'-Noethérien si et seulement si A est un anneau w-S-Noethérien et J est S-Noethérien comme A-module.*

Dans la dernière section de ce chapitre nous définissons la notion des anneaux non-nil w-S Noethériens qui est une généralisation des anneaux w-S Noethériens dans le cas du domaine intègre. Posons $\mathscr{H} = \{A \mid A$ est un anneau commutatif et $Nil(A)$ est un idéal premier divisé de $A\}$. Soit $A \in \mathscr{H}$ un anneau et $S \subseteq A$ une partie multiplicative. A est appelé un anneau non-nil w-S-Noethérien si toute paire d'idéaux non-nil I et P de A telle que $I \subseteq P$ et P est un idéal premier S-fini, alors I est S-fini. Le premier résultat de cette section établit une caractérisation de la propriété d'être un anneau non-nil w-S-Noethérien.

Théorème 0.0.24 *Soient $A \in \mathscr{H}$ un anneau et $S \subseteq A$ une partie multiplicative. Posons $S' = \frac{S}{Nil(A)} = \{s + Nil(A) \mid s \in S\}$. Alors, A est un anneau non-nil w-S-Noethérien si et seulement si $\frac{A}{Nil(A)}$ est un domaine w-S'-Noethérien.*

Soient A un anneau avec T son anneau total de quotient tel que $Nil(A)$ est un idéal premier divisé de A. Comme dans [16], nous définissons $\phi : T \longrightarrow K := A_{Nil(A)}$ tel que $\phi(\frac{a}{b}) = \frac{a}{b}$ pour tout $a \in R$ et tout $b \in A \setminus Z(A)$. Alors, ϕ est un homomorphisme d'anneaux de T vers K et ϕ restreint à A est aussi un homomorphisme d'anneaux de A vers K donné par $\phi(x) = \frac{x}{1}$ pour tout $x \in A$.
Nous terminons cette section par la caractérisation des anneaux non-nil w-S-Noethériens en utilisant le produit fibré.
Rappelons la définition d'un produit fibré. Soit T un anneau et M un idéal de T et soit π la surjection naturelle $\pi : T \to T/M$ et D un sous-anneau de T/M. Alors, $R := \pi^{-1}(D)$ est un sous-anneau de T et M est un idéal commun de R et T tel que $D = R/M$. R est dit l'anneau produit fibré ou "pullback" associé au carré commutatif suivant:

$$\begin{tikzcd} R := \pi^{-1}(D) \arrow[r] \arrow[d] & D = R/M \arrow[d] \\ T \arrow[r] & T/M \end{tikzcd}$$

où les flèches verticales sont les injections naturelles.

Théorème 0.0.25 *Soient $A \in \mathscr{H}$ et $S \subseteq A$ une partie multiplicative. Alors, A est un anneau non-nil w-S-Noethérien si et seulement si $\phi(A)$ est un anneau isomorphe à un anneau R obtenu du diagramme du produit fibré suivant:*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & B \arrow[d] \\ T \arrow[r] & T/M \end{tikzcd}$$

où T est un anneau local zéro-dimensionnel avec M sont idéal maximal, $B := R/M$ est un sous-anneau w-S_1-Noethérien de T/M où $S_1 = \alpha(\phi(S))/M$ tel que α est un isomorphisme d'anneaux de $\phi(A)$ vers R, les flèches verticales sont les inclusions habituelles, et les flèches horizontales sont les surjections naturelles.

Introduction

All rings considered bellow are assumed to be commutative and unitary, and all modules are unitary.

This research is interested in the study of certain algebraic and homological properties in different construction of rings. The prime factorization of ideals of a commutative ring, which is the subject of Chapter 1, is the most practical and easy way to use for studying several properties related to these ideals. The π-rings, are rings in which every principal ideal can be presented as a finite product of prime ideals, have always been a field of competition between researchers, which has given rise to many interesting results in the literature. For more information about this class of rings, the reader may consult the following references [78] and [79], where Mori characterized these rings. In 2017, Jayaram introduced two generalizations of π-rings, the first concerns weak π-rings which are rings in which every regular principal ideal is a finite product of prime ideals, and the second generalization concerns rings satisfying the condition $(*)$, we will call these rings in what follows $(*)$-rings, where every regular principal ideal has a primary decomposition. The author has characterized weak π-rings in the case of quasi-regular rings, these are rings having Von Neumann regular quotient rings. In the same paper, the author characterized the famous Dedekind ring, where every ideal has a prime factorization, by the fact that it is a WI-ring and a weak π-ring. A WI-ring is a ring in which every finitely generated ideal is weak invertible. We pursue this research by extending these characterizations to the general case and by studying the transfer of weak π-rings, almost weak π-rings and $(*)$-rings to the trivial ring extension.

In 1976, Dobbs introduced the notion of a divided domain as being an integral domain in which any prime ideal P of a ring A satisfies the condition $P = PA_P$, that is, every element in $A \backslash P$ divides all the elements of P. These domains are the AV-domains studied by Akiba in [1]. In [45], the author introduced the notion of locally divided domains, they are integral domains of which A_M is divided for each maximal ideal M of A. Afterwards, in 2001, Badawi and Dobbs generalized these two notions to the case of rings with zero divisors. A ring A is said to be divided if each prime ideal P is comparable to each ideal (principal ideal) of A. A ring A is said to be locally divided if A_M is divided for each maximal ideal M of A. The authors characterized these rings using the trivial ring extensions. For example, we can cite the following two characterizations: (1) A ring A is divided if and only if $Nil(A) \in Spec(A)$ and $A \propto E$ is divided for each (or some) $A_{Nil(A)}$-module E; (2) a ring A is locally divided if and only if every maximal ideal of A contains only a minimal prime ideal of A and $A \propto E$ is a locally divided ring for each A-module E satisfying the following condition: for each $M \in Max(A)$, the structure of A-module on E is induced by a structure $(Nil(A_M))^{-1}A_M$-module on E_M. This last type of rings has an irreversible implication with going-down rings, since each locally divided ring is going-down but the converse is not true in general. Recall that a ring A is said to be going-down, as it is defined in [74], if for each overring T of A and all $P \subseteq P'$ two prime ideals of A and Q' a prime ideal of T with $Q' \cap A = P'$, there exists a prime ideal Q of T such that $Q \subseteq Q'$ and $Q \cap R = P$. Considerable works have been presented in the following references [22, 44, 45, 46]. Another very important notion that we have deal with in this thesis is that of Gaussian rings. A ring A is said to be Gaussian if for all $f, g \in A[X]$, we have $c(fg) = c(f)c(g)$ where $c(f)$ is the ideal of A generated by the coefficients of f. For more information, the reader may consult the following papers [32, 73, 82].

An ideal I is said to be irreducible (respectively, strongly irreducible) if for all I_1, I_2 two ideals of A such that $I_1 \cap I_2 = I$ (respectively, $I_1 \cap I_2 \subseteq I$), then $I = I_1$ or $I = I_2$ (respectively, $I_1 \subseteq I$ or $I_2 \subseteq I$). Prime ideals are trivial examples of these two notions. In 2019, Zeidi defined an ideal to be n-irreducible (respectively, strongly n-irreducible), where n is a positive integer, if whenever $I_1 \cap \ldots \cap I_{n+1} = I$ (respectively, $I_1 \cap \ldots \cap I_{n+1} \subseteq I$) for all ideals $I_1, \ldots, I_{n+1}$ of A, then there are n of the (I_i)'s such that their intersection equal to I (respectively, their intersection is in I). According to [4], a proper ideal I of a ring A is called n-absorbing (respectively, strongly n-absorbing) if whenever $x_1 \ldots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in A$ (respectively, $I_1 \ldots I_{n+1} \subseteq I$ for $I_1, \ldots, I_{n+1}$ ideals of A), then there are n of the (x_i)'s (respectively, n of the (I_i)'s) where their product is in I. By [77], an ideal I of A is said to be ϕ-n-absorbing primary (respectively, strongly ϕ-n-absorbing primary) ideal of A, if whenever $x_1, \ldots, x_{n+1} \in I \backslash \phi(I)$ for $x_1, \ldots, x_{n+1} \in A$ (respectively, $I_1 \ldots I_{n+1} \subseteq I \backslash \phi(I)$ for $I_1, \ldots, I_{n+1}$ ideals of A), then $x_1 \ldots x_n \in I$ or the product of x_{n+1} with $(n-1)$ elements among $x_1, \ldots, x_n$ is in $\sqrt{I}$ (respectively, $I_1 \ldots I_n \subseteq I$ or the product of I_{n+1} with $(n-1)$ ideals among $I_1, \ldots, I_n$ is in $\sqrt{I}$). We are inspired by all these works to define the notion of strongly ϕ-n-irreducible ideals.

In chapter five, we pursue the study of finitely projective ideals in relation to projective ideals. Let us recall that an ideal is said to be projective if it is a direct factor of a free module. An ideal I is said to be finitely projective if for any finitely generated ideal $J \subseteq I$, the inclusion map $J \to I$ factors through a free module L. It is well known that a projective ideal is finitely projective and any finitely projective ideal which is finitely generated is projective. Several results concerning finitely projective modules are given in [57, 13]. The last notion treated in this work is the weak version of the S-Noetherian rings. Let

S be a multiplicative set of a ring A. A is said to be S-Noetherian if every ideal of A is S-finite, that is, for each ideal I of A there exists $s \in S$ and a finitely generated ideal J of A such that $sI \subseteq J \subseteq I$. S-Noetherian rings generalize Noetherian rings, we note that these two classes of rings coincide in the case where S is a subset of units of A. In this case, the S-finite ideals of A are simply the finitely generated ideals of A.

Our research is based entirely on the study of the transfer of certain properties to the extensions of unitary commutative rings. In the sequel, we present some definitions, terminologies and notations.

Definition 0.0.1 *Let A be a ring and E an A-module. The trivial ring extension of A by E (also called the idealization of E), is the ring denoted by $A \propto E$ whose underlying abelian group is $A \times E$ with pairwise addition and multiplication given by $(a,e)(b,f) = (ab, af + be)$. This construction of ring was introduced by Nagata in 1956 [80, page 2]. Mainly, the trivial ring extensions have been useful to generate new examples and solve many open problems and conjectures in the theory of commutative and non-commutative rings. The ideals of this famous ring do not have a specific form, what is shown by the example given by Mahdou and Kabbaj [61]. Particularly, The prime ideals of $A \propto E$ always have the form $P \propto E$ where P is a prime ideal of A. An ideal J of $A \propto E$ is said to be homogeneous if it has the form $I \propto F$ where I is an ideal of A, F is a submodule of E and $IE \subseteq F$, also we have $I = \{a \in A \mid (a,e) \in J \; for\; some\; e \in E\}$ and $F = \{e \in E \mid (a,e) \in J \; for\; some\; a \in A\}$. These ideals are of great importance in the study of the transfer of some property from A to $A \propto E$ and vice versa.*
These ring extensions have been widely studied and considerable works, a part of which is summarized in Glaz's book [53] and Huckaba's book [56], have been done, see for instance [11, 27, 56, 61].

In [40], Marco D'Anna and Marco Fontana introduced the amalgamated duplication of a ring A along an A-module E of $Q(A)$ (the total ring of fractions of A) satisfying $E^2 \subseteq E$. We note that when $E^2 = 0$, this construction coincides with the trivial ring extension of A by E. In the case where E is an ideal of A, this construction is defined by:

Definition 0.0.2 *Let A be a ring and I an ideal of A.*

$$A \bowtie I := \{(a, a+i) | a \in A, i \in I\}$$

this ring, provided with the usual operations, is called the amalgamated duplication along the ideal I.

As a generalization of this amalgamated duplication, D'Anna, Finocchiaro and Fontana introduced in 2010 the following construction:

Definition 0.0.3 *Let A and B two rings, J an ideal of B and $f : A \to B$ a ring homomorphism. The following subring of $A \times B$, with usual addition and multiplication, called the amalgamation of A and B along J with respect to f defnined by:*

$$A \bowtie^f J := \{(a, f(a) + j) | a \in A, j \in J\}.$$

The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions. Recall that the construction $D + M$ can be studied as a particular case of amalgamation.
Considerable works related to this famous rings have been presented in the following papers [36, 37, 38, 40].

Let us fix some notations. For a ring A, I an ideal of A and E an A-module, we set:
• $Z(A) := \{a \in A/ax := 0$ for some $0 \neq x \in A\}$, denotes the set of zero divisors of A.
• $Ann(E) := \{a \in A/ax := 0$ for all $0 \neq x \in E\}$, denotes the annihilator of E
• $T(A)$ denotes the total ring of quotients of A, that is, the localization of A by the set of all its non zero divisors.
• $qf(A) = K$ denotes the quotient field of A.
• $Nilp(A)$, denotes the set of nilpotent elements of A.
• $Rad(A)$, denotes the jacobson radical of A.
• $Spec(A)$, denotes the set of prime ideals of A.
• $Max(A)$, denotes the set of maximal ideals of A.
• $Reg(A)$, denotes the set regular elements of A.
• $(I : J) := \{x \in T(A)/xJ \subseteq I\}$.
• $V(I) := \{P \in Spec(A)/P \supseteq I\}$.
• $\sqrt{I} := \{a \in A/a^n \in I\ for\ some\ positive\ integer\ n\}$, denotes the radical of I.

CHAPTER 1

ABOUT WEAK π-RINGS

Abstract.

As in [60], a ring is called a weak π-ring if every regular principal ideal is a finite product of prime ideals. In this chapter [*], we establish some characterizations for weak π-rings. Also, we translate the properties weak π-ring and $(*)$-ring of $A \propto E$ in terms of a commutative ring A and an A-module E.

1.1 Introduction

Dedekind domains are integral domains in which every ideal is a finite product of prime ideals. Dedekind rings are defined by the fact that every regular ideal is a finite product of prime ideals. More generally, a ring R with zero divisors is said to be general *ZPI*-ring if every ideal is a finite product of prime ideals. These rings has the property that every prime ideal (or equivalently, every ideal) is finitely generated and locally principal. General *ZPI*-rings are also characterized by the property that R is a finite direct product of Dedekind domains and special principal ideal rings (SPIRs), that are, local principal ideal rings, not a field, whose maximal ideal is nilpotent. A ring R has the property that every principal ideal is a finite product of prime ideals if and only if R is a finite direct product of (1) π-domains, (2) SPIRs, and (3) fields. In this case, R called a π-ring.

In [60], Jayaram defined two generalizations of the above-mentioned class of rings, namely, weak π-rings in which every regular principal ideal is a finite product of prime ideals, and rings satisfying the condition $(*)$ (in the sequel, they will be noted by $(*)$-rings) in which every regular principal ideal is a finite intersection of primary ideals and he proved that weak π-rings are $(*)$-rings. He also defined a ring R to be an almost weak

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π-ring if for each regular principal ideal I, I_M is a finite product of prime ideals in R_M for all maximal ideals M containing I. The author mention that weak π-rings are almost weak π-rings but the converse need not be true. By Theorem 1.2.1, we state a necessary and sufficient condition for an almost weak π-ring to be a weak π-ring. In the same paper, Jayaram characterizes weak π-rings inside the class of quasi-regular rings and gives necessary and sufficient conditions for a ring R to be a Dedekind ring. For more informations, the reader may consult [11, 51, 60].

The following diagram summarizes the relations between all these class of rings where the implications cannot be reversed in general.

$$
\begin{tikzcd}
& \textbf{General ZPI-ring} \arrow[d, shift right=1] & \\
& \textbf{Dedekind-ring} \arrow[u, shift right=1, "\text{Domain}"'] \arrow[d, shift right=1] & \\
& \pi\textbf{-ring} \arrow[u, shift right=1, "\text{Principal}"'] \arrow[d, shift right=1] & \\
& \textbf{Weak } \pi\textbf{-ring} \arrow[u, shift right=1, "\text{Domain}"'] \arrow[dl] \arrow[dr] & \\
(*)\textbf{-ring} & & \textbf{Almost weak } \pi\textbf{-ring}
\end{tikzcd}
$$

In this chapter, we provide an example of a weak π-ring which is neither a Dedekind ring nor a π-ring (see Example 1.3.5), and by Example 1.3.6, we show that $(*)$-rings are not necessary weak π-rings.

We will be using the following definitions. A regular element r of R is any element of $R\backslash Z(R)$; An ideal I of R is an invertible ideal if $II^{-1} = R$, where $I^{-1} = (R : I) = \{x \in qf(R) | xI \subseteq R\}$; I is called a multiplication ideal if for every ideal $J \subseteq I$, there exists an ideal K with $J = KI$ (invertible ideals are multiplication ideals); An R-module E is said to be divisible if, for each $e \in E$ and each regular element r of R, there exists $f \in E$ such that $e = rf$.

Our aims in this chapter is to extend the result of Jayaram about characterization of weak π-rings [60, Theorem 1] to the general case and to study the transfer of the weak π-ring property and the $(*)$-ring property to the trivial ring extension.

1.2 General results

We start this section by the following characterizations of a weak π-ring.

Theorem 1.2.1 *Let R be a commutative ring. The following statements are equivalent:*

1. *R is a weak π-ring;*
2. *Every regular prime ideal contains an invertible prime ideal;*
3. *Every invertible ideal is a finite product of invertible prime ideals;*
4. *Every prime ideal minimal over an invertible ideal is invertible;*
5. *R is an almost weak π-ring in which every prime ideal minimal over an invertible ideal is finitely generated.*

The proof of this theorem based on the following result. We are inspired by the proof of [60, Lemma 13].

Lemma 1.2.2 *Every regular prime ideal contains an invertible prime ideal if and only if every invertible ideal is a finite product of invertible prime ideals.*

Proof. Suppose that every invertible ideal is a finite product of invertible prime ideals. Let P be a regular prime ideal, so there exists $r \in P$ a regular element such that $< r >= P_1P_2...P_n$ where P_k's are invertible prime ideals. Since P is a prime ideal we have $P_k \subseteq P$ for some k. Conversely, assume that every regular prime ideal of R contains an invertible prime ideal and let I be an invertible ideal. We prove that I contains a finite product of invertible prime ideals. Let $S_0 = \{J \in L(R) | J\, is\, a\, finite\, product\, of\, invertible\, prime\, ideals\, of\, R\}$ and $S = \{J \in L(R) | I \subseteq J \text{ and } S_0 \cap (J] = \emptyset\}$ where $(J] = \{K \in L(R) | K \subseteq J\}$.. Suppose $S_0 \cap (I] = \emptyset$ then $S \neq \emptyset$. Since every element of S_0 is finitely generated, by Zorn's lemma there exists a prime ideal P such that $I \subseteq P$ and $S_0 \cap (P] = \emptyset$. Since $I \subseteq P$, it follows that P is regular, so $S_0 \cap (P] \neq \emptyset$, a contradiction. Then $S_0 \cap (I] \neq \emptyset$. Thus I contains a finite product of invertible prime ideal. Let $P_1P_2...P_n \subseteq I$ where P_k's are invertible prime ideals. Since I is finitely generated and locally principal, it follows that I is a multiplication ideal and so $IJ = P_1P_2...P_n$ for some $J \in L(R)$. By [60, Lemma 12], I is a finite product of invertible prime ideals.

Proof of Theorem 1.2.1.

$(1) \Rightarrow (2)$ Suppose that R is a weak π-ring and let P be a regular prime ideal of R. Let $r \in P$ be a regular element, we have $< r >= P_1P_2...P_n$ where every P_k is a regular, finitely generated and a locally principal ideal, so an invertible prime ideal. Hence P contains an invertible prime ideal.

$(2) \Rightarrow (3)$ follows from Lemma 1.2.2.

$(3) \Rightarrow (4)$ Suppose (3) holds. Let P be a prime ideal minimal over an invertible ideal I. By hypothesis $I = P_1P_2...P_n$ where P_k's are invertible prime ideals. As P is a prime ideal, it follows that $P_k \subseteq P$, then $P = P_k$ and hence P is invertible.

$(4) \Rightarrow (1)$. Suppose (4) holds. Let I be a regular principal ideal of R, so I is invertible. By hypothesis and [58, Lemma 5], I contains a finite product of invertible prime ideals minimal over I. Let $P_1P_2...P_n \subseteq I$, $IJ = P_1P_2...P_n$ for some $J \in L(R)$ as I is a multiplication ideal . By [60, Lemma 12], I is a finite product of invertible prime ideals.

$(4) \Rightarrow (5)$ is obvious since $(4) \Leftrightarrow (1)$.

$(5) \Rightarrow (2)$ Assume that (5) holds. Let P be a regular prime ideal and r a regular element of P. We may assume that P is not minimal over $< r >$, and let Q be a minimal prime over $< r >$ such that $Q \subseteq P$, so by hypothesis Q is finitely generated. Now remains to show that Q is locally principal. Let $Q \subseteq M$ for some maximal ideal M of R. Since R is an almost weak π-ring, $< r >_M = Q_{1M}Q_{2M}...Q_{nM}$ for some prime ideals $Q_1, Q_2, ..., Q_n$ of R. As Q is a prime minimal over $< r >$, it follows that $Q_M = Q_{iM}$ for some i. By [7, Lemma 2.3] and [75, Theorem 2], Q_M is principal in R_M, so a locally principal ideal. Therefore Q is invertible, as desired.

Proposition 1.2.3 *Let R be a weak π-ring, then:*

1. *Each invertible ideal is principal;*

2. *R is an almost weak π-ring in which every regular rank one prime ideal is invertible.*

Proof.

1. Let I be an invertible ideal of R. By [56, Lemma 18.1, p. 110], I contains a regular element r. Then $< r > = P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n}$ where P_k's are distinct invertible prime ideals by [7, Lemma 2.3], as R is a weak π-ring. By [60, Lemma 12], and since I is a multiplication ideal we obtain $I = P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n} = < r >$, as desired.

2. It's clear that R is an almost weak π-ring. So the result holds by hypothesis and [60, Lemma 5].

Next, we establish the transfer of weak π-ring, almost weak π-ring and $(*)$-ring properties to the direct product of rings.

Proposition 1.2.4 *Let $R = \prod_{i=1}^{n} R_i$ be a direct product of rings. Then the following results hold:*

1. *R is a weak π-ring if and only if so are R_i's;*

2. *R is an almost weak π-ring if and only if so are R_i's;*

3. *R is a $(*)$-ring if and only if so are R_i's.*

Proof. It suffices to study the case of a pairs of rings R and S.

1. Assume that $R \times S$ is a weak π-ring. Let $I =< i >$ and $J =< j >$ be two regular principal ideals respectively of R and S. By our assumption, $I \times J =< (i,j) >= (P_1 \times S)...(P_n \times S)(R \times Q_1)...(R \times Q_m)$ where P_l and Q_l are respectively prime ideals of R and S with $P_1 \neq R$ and $Q_1 \neq S$. Hence $I = P_1...P_n$ and $J = Q_1...Q_m$.
   Conversely, let $H =< (i,j) >$ be a regular principal ideal of $R \times S$. Set $I =< i >$ and $J =< j >$, clearly I(resp. J) is a regular principal ideal of R(resp. S). By hypothesis, $I = P_1...P_n$ and $J = Q_1...Q_m$, thus $H = (P_1 \times S)...(P_n \times S)(R \times Q_1)...(R \times Q_m)$.

2. The result holds since for all $I =< i >$ and $J =< j >$ be two regular principal ideals respectively of R and S we have $(I \times J)_{M \times S} \cong I_M$ for all $M \in Max(R,I)$ and $(I \times J)_{R \times N} \cong J_N$ for all $N \in Max(S,J)$.

3. Clearly, for a pairs of rings R and S, the primary ideals of $R \times S$ have the form $P \times S$ or the form $R \times Q$ where P and Q are primary ideals of R and S respectively. So assume that $R \times S$ is a $(*)$-ring. Let $I =< i >$ and $J =< j >$ be two regular principal ideals respectively of R and S. By our assumption, $I \times J =< (i,j) >= (P_1 \times S) \cap ... \cap (P_n \times S) \cap (R \times Q_1) \cap ... \cap (R \times Q_m)$ where P_l's and Q_l's are respectively primary ideals of R and S with $P_1 \neq R$ and $Q_1 \neq S$. Hence $I = P_1 \cap ... \cap P_n$ and $J = Q_1 \cap ... \cap Q_m$.
   Conversely, let $K =< (i,j) >$ be a regular principal ideal of $R \times S$. Set $I =< i >$ and $J =< j >$, clearly I(resp. J) is a regular principal ideal of R(resp. S). By hypothesis, $I = P_1 \cap ... \cap P_n$ and $J = Q_1 \cap ... \cap Q_m$ where P_l's and Q_l's are respectively primary ideals of R and S, thus $K = (P_1 \times S) \cap ... \cap (P_n \times S) \cap (R \times Q_1) \cap ... \cap (R \times Q_m)$.

By the following results, we identified a context in which one can see when a localization of a ring R is a weak π-ring, an almost weak π-ring and a $(*)$-ring.

Proposition 1.2.5 *Let R be a ring and S a multiplicative set of R such that $S \subseteq Reg(R)$.*

1. *If R is a weak π-ring, then so is $S^{-1}R$;*

2. *If R is an almost weak π-ring, then so is $S^{-1}R$;*

3. *If R is a $(*)$-ring, then so is $S^{-1}R$.*

Proof.

1. Let J be a regular principal ideal of $S^{-1}R$, so $J^c = \varphi^{-1}(J)$ is a regular ideal, where $\varphi : R \to S^{-1}R$ is a ring homomorphism defined by $\varphi(r) = r/1$, indeed:
   Let $\frac{r}{s}$ be a regular element of J, clearly $\frac{r}{1}$ is a regular element of J, then $r \in J^c$. Assume that r is not regular, so there exists $a \in R$ such that $ra = 0$, and so $(\frac{r}{s})(\frac{a}{1}) = \frac{0}{1}$, contradiction.
   As R is a weak π-ring, we deduced that $< r >= P_1...P_n$. Therefore, $S^{-1}(P_1...P_n) = S^{-1}P_1...S^{-1}P_n \subseteq S^{-1}(J^c) = J$. Since $r \in < r >= P_1...P_n$, it follows that $\frac{r}{s} \in S^{-1}P_1...S^{-1}P_n$. Hence $J = S^{-1}P_1...S^{-1}P_n$, as desired.

2. Straightforward.

3. Let J be a regular principal ideal of $S^{-1}R$. Similarly to the proof of Proposition 1.2.5, $J^c = \varphi^{-1}(J)$ is a regular ideal of R. Since R is a $(*)$-ring, we deduced that $< r >= P_1 \cap ... \cap P_n$ where P_l's are primary ideals of R. Therefore, $S^{-1}(P_1 \cap ... \cap P_n) = S^{-1}P_1 \cap ... \cap S^{-1}P_n \subseteq S^{-1}(J^c) = J$. Since $r \in < r >= P_1 \cap ... \cap P_n$, it follows that $\frac{r}{s} \in S^{-1}P_1 \cap ... \cap S^{-1}P_n$. Hence $J = S^{-1}P_1 \cap ... \cap S^{-1}P_n$. As P_l are primary ideals, then so are $S^{-1}P_l$. This completes the proof of the proposition.

By this proposition, we study the transfer of the above-mentioned classes to homomorphic image. We consider I an ideal of R and $f : R \to R/I$ the canonical surjection, we have then the next results:

Proposition 1.2.6 *Suppose that the following two conditions hold:*

1. *Each regular principal ideal of R contains I;*

2. *$f^{-1}\{\bar{r}\}$ contains a regular element where $\bar{r} \in Reg(R/I)$.*

Then:

(a) If R is a weak π-ring, then so is R/I;

(b) If R is an almost weak π-ring, then so is R/I;

(c) If R is a $(*)$-ring, then so is R/I.

Proof.

(a) Let $J =< \bar{r} >$ be a regular principal ideal of R/I. Along with the hypothesis that $f^{-1}\{\bar{r}\}$ contains a regular element, we may assume, without loss of generality, r is regular. We get then $K =< r >$ a regular principal ideal of R. As R is a weak π-ring, it follows that $K = P_1...P_n$ and hence $J = K/I = (P_1...P_n)/I = P_1/I...P_n/I$. Therefore J is a finite product of prime ideals.

(b) Let $J =< \bar{r} >$ be a regular principal ideal such that $f^{-1}\{\bar{r}\}$ contains a regular element. Thus, we may assume, without loss of generality, $K =< r >$ a regular principal ideal. Since R is an almost weak π-ring, we obtain $K_M = P_{1M}...P_{nM}$ for all $M \in Max(R,K)$, in particular for all $N \in Max(R/I,J)$ with $N = M/I$ for some $M \in Max(R,K)$, and hence $J_N = (K/I)_N \cong K_M/I_M = P_{1M}...P_{nM}/I_M = P_{1M}/I_M...P_{nM}/I_M$, which completes the proof.

(c) Let $J =< \bar{r} >$ be a regular principal ideal of R/I. Along with the hypothesis that $f^{-1}\{\bar{r}\}$ contains a regular element, without loss of generality, we may assume $K =< r >$ a regular principal ideal. As R is a $(*)$-ring, then $K = P_1 \cap ... \cap P_n$ and hence $J = K/I = (P_1 \cap ... \cap P_n)/I = P_1/I \cap ... \cap P_n/I$, as desired.

1.3 The transfer to the trivial ring extension

In the sequel, we study the possible transfer of the properties of being a weak π-ring, an almost weak π-ring and a $(*)$-ring between a commutative ring A and $A \propto E$.

A homogeneous ideal of $A \propto E$ is an ideal with the form $I \propto F$ where I is an ideal of A, F is a submodule of E, and $IE \subseteq F$, also we have $I = \{a \in A \mid (a,e) \in I \propto F \ for\ some\ e \in E\}$ and $F = \{e \in E \mid (a,e) \in I \propto F \ for\ some\ a \in A\}$.

Theorem 1.3.1 *Let A be a ring, E an A-module and $R = A \propto E$ be the trivial ring extension of A by E. Then:*

1. *R is a weak π-ring if and only if every principal ideal not disjoint of S is a finite product of prime ideals and $sE = E$ for all $s \in S$ where $S = A - (Z(A) \cup Z(E))$;*
2. *If R is an almost weak π-ring, then every principal ideal not disjoint of S, I_M is a finite product of prime ideals for all $M \in Max(A,I)$;*
3. *Suppose that $sE_M = E_M$ for all $M \in Max(A)$ and all $s \in S$. If A is an almost weak π-ring, then so is R;*
4. *Suppose that $E = aE$ for all $a \in S$. R is a $(*)$-ring if and only if every regular principal ideal not disjoint of S has a primary decomposition.*

Proof.

1. As R is a weak π-ring and by [11, Theorem 3.3] a product of homogeneous ideals is homogeneous, it follows that every regular principal ideal of R is homogeneous, so by [11, Theorem 3.9] $sE = E$ for all $s \in S$. Let $I =< a >$ be a principal ideal of A with $I \cap S \neq \emptyset$. Thus, $J =< (a,0) >= I \propto E$ is a regular ideal of R, hence $I \propto E = (P_1 \propto E)...(P_n \propto E)$ where P_i's are prime ideals of A. We conclude that

$I = P_1 ... P_n$.
Conversely, let J be a regular principal ideal of R. By hypothesis and [11, Theorem 3.9], $J = I \propto E$ for some ideal I of A with $I \cap S \neq \emptyset$. Again by hypothesis, I is a finite product of prime ideals of A, set $I = P_1 ... P_n$. Since $sE = E$ for all $s \in S$, we get $J = I \propto E = (P_1 \propto E)...(P_n \propto E)$, therefore R is a weak π-ring.

(2) and (3) are similar to the first statement.

(4) Suppose that R is a $(*)$-ring. Along with the hypothesis $E = aE$ for all $a \in S$, for every regular principal ideal I of A such that $I \cap S \neq \emptyset$, $I \propto E$ is a regular principal ideal of R. Then $I \propto E = (P_1 \propto E) \cap ... \cap (P_n \propto E) = (P_1 \cap ... \cap P_n) \propto E$ where P_k's are primary ideals of A, and hence $I = P_1 \cap ... \cap P_n$, as desired.
Conversely, let J be a regular principal ideal of R. By hypothesis, $J = I \propto E$ where I is a regular principal ideal of R such that $I \cap S \neq \emptyset$ and $I = P_1 \cap ... \cap P_n$ where P_k's are primary ideals of A. So, $J = (P_1 \cap ... \cap P_n) \propto E = (P_1 \propto E) \cap ... \cap (P_n \propto E)$. Therefore, R is a $(*)$-ring.

As a particular case of the previous theorem, we get:

Corollary 1.3.2 *Let A be a ring, E an A-module, $R = A \propto E$ be the trivial ring extension such that $Z(E) \subseteq Z(A)$ and $S = Reg(A)$. Then:*

1. *R is a weak π-ring if and only if A is a weak π-ring and $sE = E$ for all $s \in S$;*
2. *If R is an almost weak π-ring, then so is A;*
3. *Suppose that $sE_M = E_M$ for all $M \in Max(A)$ and all $s \in S$. If A is an almost weak π-ring, then so is R;*
4. *Suppose that $E = aE$ for all $a \in S$. Then R is a $(*)$-ring if and only if so is A.*

Emmy Noether proved that a Noetherian ring R is a Laskerian ring [52], hence a $(*)$-ring. The next corollary provide a non-Noetherian example of a $(*)$-ring.

Corollary 1.3.3 *Let D be a domain and E a divisible R-module. Then:*

1. *D is a weak π-ring if and only if so is $D \propto E$;*
2. *D is an almost weak π-ring if and only if so is $D \propto E$;*
3. *D is a $(*)$-ring if and only if so is $D \propto E$;*
4. *If E is a non-finitely generated D-module, then $D \propto E$ is a non-Noetherian ring and we have D is a $(*)$-ring if and only if so is $D \propto E$.*

Corollary 1.3.4 *Let (A, M) be a local ring and E an A/M-vector space. Then:*

1. $A \propto E$ is a weak π-ring;

2. $A \propto E$ is a $(*)$-ring.

As an application of our result, we construct the following example of a weak π-ring (a $(*)$-ring) which is neither a Dedekind ring nor a π-ring.

Example 1.3.5 *Let $R = \mathbb{Z} \propto \mathbb{Z}/4\mathbb{Z}$, it's clear that $\mathbb{Z}/4\mathbb{Z} = a\mathbb{Z}/4\mathbb{Z}$ for all $a \in S = \mathbb{Z} - Z(\mathbb{Z}/4\mathbb{Z})$, then:*

- *R is a weak π-ring;*
- *R is a $(*)$-ring;*
- *R not a Dedekind ring (by [59, Theorem 1]);*
- *R not a π-ring (by [11, Theorem 4.10]).*

By the next example, we show that $(*)$-rings are not necessary weak π-rings.

Example 1.3.6 *Let $R = \mathbb{Z}$ and $E = 4\mathbb{Z}$, we prove that:*

1. $R \propto E$ is not a weak π-ring (by theorem 1.3.1 (1));

2. $R \propto E$ is a $(*)$-ring.

The condition "$f^{-1}(r)$ contains a regular element for every regular element $\overline{r} \in R/I$" is necessary in Proposition 1.2.6 (1). To see this, consider the following example:

Example 1.3.7 *Let $A = \mathbb{Z}_{(2)} \propto \mathbb{Z}_{(2)}$, $M = 2\mathbb{Z}_{(2)} \propto \mathbb{Z}_{(2)}$, $E = A/M$ and $R = A \propto E$. Easily, we can see that $E = rE$ for all $r \in S = A - (Z(A) \cup Z(E))$. Thus every regular principal ideal J of R has the form $I \propto E$ where I is a regular principal ideal not disjoint with S. Hence the first condition in Proposition 1.2.6 is satisfied. Now, let $(a,0)$ be a regular element of A. For all $e \in E$ we have $((a,0),e)((0,0),e) = ((0,0),0)$.*
Clearly R is a weak π-ring, however A is not by corollary 1.3.4.

CHAPTER 2

ON STRONGLY $(*)$-RINGS

Abstract.

In this chapter *, we investigate strongly $(*)$-ring and we conjecture that this class of rings and the class of $(*)$-rings are equivalent. Also, we study the transfer of weak π-ring property, $(*)$-ring property and strongly $(*)$-ring property to some extensions of a ring.

2.1 Introduction

As defined in [60], a weak π-rings is a ring in which every regular principal ideal is a finite product of prime ideals. R is called a $(*)$-ring if each regular principal ideal has a primary decomposition, that is, a finite intersection of primary ideals. R is said to be an almost weak π-ring if for each regular principal ideal I, I_M is a finite product of prime ideals in R_M for all maximal ideals M containing I. A ring R is called arithmetical ring if every finitely generated ideal of R is locally principal.

We summariz the relationship that exists between the above-mentioned notions by the following implications noted in the following figure.

*Submitted for publication (in collaboration with N. Mahdou).

$$\begin{tikzcd}
& \textbf{General ZPI-ring} \arrow[d, shift right] & \\
& \textbf{Dedekind-ring} \arrow[u, shift right, "\text{Domain}"'] \arrow[d, shift right] & \\
& \pi\textbf{-ring} \arrow[u, shift right, "\text{Principal}"'] \arrow[d, shift right] & \\
& \textbf{Weak } \pi\textbf{-ring} \arrow[u, shift right, "\text{Domain}"'] \arrow[dl] \arrow[dr] & \\
\textbf{Strongly } (*)\textbf{-ring} \arrow[d] & & \textbf{Almost weak } \pi\textbf{-ring} \\
(*)\textbf{-ring} & &
\end{tikzcd}$$

I is said to be an invertible ideal if $II^{-1} = R$, where $I^{-1} = (R : I) = \{x \in qf(R) | xI \subseteq R\}$.

The purpose of this chapter is to introduce the strongly $(*)$-ring property and study its transfer to some extension of rings and the transfer of the two notions weak π-ring and $(*)$-ring to the amalgamated duplication along an ideal and the amalgamed algebra along an ideal.

2.2 General results

We start with the following definition:

Definition 2.2.1 *A ring R is called a strongly $(*)$-ring if every invertible ideal has a primary decomposition.*

Note that Laskerian rings [54] and Noetherian rings [52] are examples of strongly $(*)$-rings. Obviously, a strongly $(*)$-ring is a $(*)$-ring. We conjecture that these class of rings are equivalent. Easily, we can see that are equivalent when R is principal or local.

By the next proposition, we prove that a weak π-ring is a strongly $(*)$-ring.

Proposition 2.2.2 *If R is a weak π-ring, then R is a strongly $(*)$-ring.*

Proof. Let I be an invertible ideal of R. Thanks to [56, Lemma 18.1, p. 110], I contains a regular element, say r. Since R is a weak π-ring, $< r >= P_1^{\alpha_1} P_2^{\alpha_2} ... P_n^{\alpha_n}$ where P_k's are prime ideals of R. Then $I = P_1^{\alpha_1} P_2^{\alpha_2} ... P_n^{\alpha_n}$. This can be seen by applying [60, Lemma 12], as I is a multiplication ideal and P_k's are distinct invertible prime ideals by [7, Lemma 2.3]. As it now suffices to show that $P_k^{\alpha_k}$'s are invertible P_k-primary ideals, an appeal to [9, Theorem 3] completes the proof.

We note that the converse need not be true, as the following example shows.

Example 2.2.3 *Let $R = \mathbb{Z} \propto \mathbb{Z}$, we claim that R is not a weak π-ring.*
Deny, let (a,b) be a regular element of R, we get $< (a,b) >= (P_1 \propto \mathbb{Z}) ... (P_n \propto \mathbb{Z})$, so each regular principal ideal is homogenous and hence $\mathbb{Z}$ is divisible as a $\mathbb{Z}$-module by [11, Theorem 3.9], which is a contradiction. Finally, R is not a weak π-ring, while R is a strongly $(*)$-ring since it is Noetherian.

Proposition 2.2.4 *Let $R = \prod_{i=1}^{n} R_i$, then R is a strongly $(*)$-ring if and only if so are R_i's.*

Proof. It suffices to show that the result hold for a pairs of rings R and S. Suppose that $R \times S$ is a strongly $(*)$-ring. Let I and J be two invertible ideals respectively of R and S. By our assumption, $I \times J = (P_1 \times S) \cap ... \cap (P_n \times S) \cap (R \times Q_1) \cap ... \cap (R \times Q_m)$ where P_l's and Q_l's are respectively primary ideals of R and S with $P_1 \neq R$ and $Q_1 \neq S$. Hence $I = P_1 \cap ... \cap P_n$ and $J = Q_1 \cap ... \cap Q_m$.
Conversely, let K be an invertible ideal of $R \times S$. Then $K = I \times J$, clearly I(resp. J) is an invertible ideal of R(resp. S). By hypothesis, $I = P_1 \cap ... \cap P_n$ and $J = Q_1 \cap ... \cap Q_m$ where P_l's and Q_l's are respectively primary ideals of R and S, thus $K = (P_1 \times S) \cap ... \cap (P_n \times S) \cap (R \times Q_1) \cap ... \cap (R \times Q_m)$, as desired.

For a multiplicative set S of a ring R, we obtain the next result:

Proposition 2.2.5 *Let R be a ring and S a multiplicative set of R such that $S \subseteq Reg(R)$. If R is a strongly $(*)$-ring , then so is $S^{-1}R$.*

Proof. Let J be an invertible ideal of $S^{-1}R$. By [56, Lemma 18.1], $J = < \frac{a_1}{s_1}, ..., \frac{a_n}{s_n} >= S^{-1} < a_1, ..., a_n >= S^{-1}I$ where $I = < a_1, ..., a_n >$. Since $J_{MS^{-1}R} = (S^{-1}I)_{MS^{-1}R} = I_M$ for all $M \in Max(R, I)$, it follows that I is locally principal. Also I is regular since J is regular and $S \subseteq Reg(R)$, then I is invertible. As R is a strongly $(*)$-ring, we get $I = P_1 \cap ... \cap P_n$ where P_l's are primary ideals of R. Therefore, $J = S^{-1}I = S^{-1}(P_1 \cap ... \cap P_n) = S^{-1}P_1 \cap ... \cap S^{-1}P_n$. As P_l is a primary ideal, then so is $S^{-1}P_l$ for all l, as desired.

2.3 The transfer to the trivial ring extension

Now, we provide a result which translates the strongly $(*)$-ring property of $A \propto E$ in terms of A and E.

Theorem 2.3.1 *Let A be a ring, E an A-module and $R = A \propto E$ such that $E = aE$ for all $a \in S = A - (Z(A) \cup Z(E))$. Then R is a strongly $(*)$-ring if and only if every invertible ideal of A not disjoint from S has a primary decomposition.*

Proof. Suppose that R is a strongly $(*)$-ring and let I be an invertible ideal of A such that $I \cap S \neq \emptyset$. By hypothesis and [2, Theorem 7(1),(2)], $I \propto IE = I \propto E$ is an invertible ideal of R. As R is a strongly $(*)$-ring, then $I \propto E = (P_1 \propto E) \cap \ldots \cap (P_n \propto E) = (P_1 \cap \ldots \cap P_n) \propto E$ where P_k's are primary ideals of A, therefore $I = P_1 \cap \ldots \cap P_n$, as desired.
Conversely, let J be an invertible ideal of R. By hypothesis, $J = I \propto E$ where I is an invertible ideal not disjoint from S and $I = P_1 \cap \ldots \cap P_n$ such that P_k's are primary ideals of A. So, $J = (P_1 \cap \ldots \cap P_n) \propto E = (P_1 \propto E) \cap \ldots \cap (P_n \propto E)$, which completes the proof.

As an immediate consequence of the previous theorem, we provide the following corollary:

Corollary 2.3.2 *Let A be a ring, E an A-module and $R = A \propto E$ such that $Z(E) \subseteq Z(A)$ and $E = aE$ for all $a \in S = A - (Z(A) \cup Z(E))$. Then R is a strongly $(*)$-ring if and only if so is A.*

As mentioned above, Noetherian rings are strongly $(*)$-rings. By the next corollary we provide an example of a non-Noetherian ring which is a strongly $(*)$-ring.

Corollary 2.3.3 *Let D be a domain and E a divisible R-module. Then:*

1. *D is a strongly $(*)$-ring if and only if so is $D \propto E$;*
2. *If E is a non-finitely generated D-module, then $D \propto E$ is a non-Noetherian ring and D is a strongly $(*)$-ring if and only if so is $D \propto E$.*

2.4 The transfer to the amalgamation of rings and amalgamated duplication along an ideal

The proof of the first major result of this section (Theorem 2.4.3) relies on the following lemmas which are of independent interest. The next lemma investigate the form of regular ideals of $R \bowtie^f J$.

Lemma 2.4.1 *Let R and S be two rings, J an ideal of S and $f : R \to S$ be a ring homomorphism. Suppose that $Ann(f^{-1}(J)) = 0$, then the following statements are equivalent:*

1. *Every regular ideal has the form $I \bowtie^f J$ where I is regular;*
2. *$J = (f(a) + j)J$ for all $(a, f(a) + j) \in Reg(R \bowtie^f J)$.*

Proof.

$(1) \Rightarrow (2)$ Assume (1) holds and let $H =< (a, f(a)+j) >$, by hypothesis $H = I \bowtie^f J$ where $I =< a >$ is regular. Now let $k \in J$, we have $(0,k) = (0,k')(a, f(a)+j)$ for some $k' \in J$ since a is regular. Hence $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$.

$(2) \Rightarrow (1)$ Let H be a regular ideal of $R \bowtie^f J$, so there exists $(r, f(r)+j) \in H$. Now let $K =< (r, f(r)+j) >$. As $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$, so for all $k \in J$ there exists $k' \in J$ such that $k = (f(r)+j)k'$, thus $(0,k) = (0,k')(r, f(r)+j) \in K$, we get then $0 \times J \subseteq K \subseteq H$ and hence $H = I \bowtie^f J$ where $I =< r >$. We claim that I is regular. Deny, let $x \in R$ such that $rx = 0$, since $Ann(f^{-1}(J)) = 0$, there exists $y \in f^{-1}(J)$ such that $xy \neq 0$, then $(r, f(r)+j)(xy, 0) = (0,0)$, contradiction.

Lemma 2.4.2 *Let R and S be two rings, J an ideal of S, I an ideal of R and $f : R \to S$ be a ring homomorphism.*

1. *I is finitely generated (resp. a principal) if and only $I \bowtie^f f(I)J$ is finitely generated (resp. a principal) ideal of $R \bowtie^f J$;*

2. *If $I \bowtie^f f(I)J$ is a locally principal ideal of $R \bowtie^f J$, then I is a locally principal ideal of R;*

3. *Suppose that $J \subseteq Jac(S)$. If I is locally principal of R then $I \bowtie^f f(I)J$ is a locally principal ideal of $R \bowtie^f J$;*

4. *Suppose that $I \bowtie^f f(I)J$ is regular and $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$. If I is a locally principal ideal of R then $I \bowtie^f f(I)J$ is a locally principal ideal of $R \bowtie^f J$;*

5. *Suppose that K is a regular ideal and $I = aI$ for all $a \in Reg(R)$. If K is a locally principal ideal of R then $K \bowtie I$ is a locally principal ideal of $R \bowtie I$.*

Proof.

1. Let $I =< r_1, ..., r_n >$ be a finitely generated ideal of R. We claim that:

$$I \bowtie^f f(I)J =< (r_1, f(r_1)), ..., (r_n, f(r_n)) >$$

For that, let $(a, f(a) + \Sigma_{k=1}^m f(b_k) j_k) \in I \bowtie^f f(I)J$, then:

$$\begin{aligned}(a, f(a) + \Sigma_{k=1}^m f(b_k) j_k) &= (\Sigma_{i=1}^n \alpha_i r_i, f(\Sigma_{i=1}^n \alpha_i r_i) + \Sigma_{k=1}^m f(\Sigma_{i=1}^n \beta_{k,i} r_i) j_k) \\ &= (\Sigma_{i=1}^n \alpha_i r_i, \Sigma_{i=1}^n f(\alpha_i) f(r_i)) + (0, \Sigma_{k=1}^m \Sigma_{i=1}^n f(\beta_{k,i}) f(r_i) j_k) \\ &= \Sigma_{i=1}^n (\alpha_i, f(\alpha_i))(r_i, f(r_i)) + \Sigma_{i=1}^n (0, \Sigma_{k=1}^m f(\beta_k) j_k)(r_i, f(r_i)) \\ &= \Sigma_{i=1}^n [(\alpha_i, f(\alpha_i)) + (0, \Sigma_{k=1}^m f(\beta_k) j_k)](r_i, f(r_i))\end{aligned}$$

As desired.
Conversely, suppose that $I \bowtie^f f(I)J =< (r_1, f(r_1)+j_1), ..., (r_1, f(r_n)+j_n) >$, we show that $I =< r_1, ..., r_n >$. So let $a \in I$, $(a, f(a)) \in I \bowtie^f f(I)J$, we get then:

$(a, f(a)) = \Sigma_{i=1}^{n}(\alpha_i, f(\alpha_i)+\beta_i)(r_i, f(r_i)+j_i)$
$\quad\quad = (\Sigma_{i=1}^{n}\alpha_i r_i, \Sigma_{i=1}^{n}(f(\alpha_i)+\beta_i)(f(r_i)+j_i))$

Hence, $a = \Sigma_{i=1}^{n}\alpha_i r_i$, which completes the proof.
Similarly, we prove that I is a principal ideal of R if and only if so is $I \bowtie^f f(I)J$.

2. Suppose that $I \bowtie^f f(I)J$ is a locally principal ideal of $R \bowtie^f J$ and let $M \in Max(R,I)$. Using the ring homomorphism induced by φ defined in [62, Lemma 2.6.], we prove that:
If $M \in Max(R,I)\backslash V(f^{-1}(J))$, so $(I \bowtie^f f(I)J)_{M'^f} \cong I_M$ where $M'^f = M \bowtie^f J$, as desired.
If $M \in Max(R,I) \cap V(f^{-1}(J))$, so $(I \bowtie^f f(I)J)_{M'^f} \cong I_M \bowtie^{f_M} (f(I)J)_{T_M}$ where $T_M = f(R-M)+J$ is a multiplicative set of S and f_M is the ring homomorphism induced by f for all $M \in Max(R)$. We get the result by (1).

3. If $J \subseteq Jac(B)$, hence all maximal ideals contining $I \bowtie^f f(I)J$ have the form $M \bowtie^f J$ where $M \in Max(R,I)$. The remaining proof is straightforward since $(I \bowtie^f f(I)J)_{M'^f} \cong I_M$ if $M \in Max(R,I)\backslash V(f^{-1}(J))$ and $(I \bowtie^f f(I)J)_{M'^f} \cong I_M \bowtie^{f_M} (f(I)J)_{T_M}$ if $M \in Max(R,I) \cap V(f^{-1}(J))$.

4. It's obvious since $I \bowtie^f f(I)J = I \bowtie^f J$ by Lemma 2.4.1. The remaining proof is similar to (2).

5. The result holds by (1) and by the fact that $H_{M'} = (K \bowtie I)_{M'} \cong K_M$ if $M \in Max(R)\backslash V(I)$ where $M' = M \bowtie I$ and $H_{M'} = (K \bowtie I)_{M'} \cong K_M \bowtie I_M$ if $M \in Max(R) \cap V(I)$.

At present, we study the possible transfer of the properties of being a weak π-ring, a $(*)$-ring and a strongly $(*)$-ring between a commutative ring R and $R \bowtie^f J$.

Theorem 2.4.3 *Let R and S be two rings, J an ideal of S and $f : R \to S$ be a ring homomorphism.*

1. *Suppose that $Ann(f(a)) \cap J = 0$ for all $a \in Reg(R)$. If $R \bowtie^f J$ is a weak π-ring, then so is R;*

2. *Suppose that $Ann(f^{-1}(J)) = 0$ and $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$. If R is a weak π-ring, then so is $R \bowtie^f J$;*

3. *Suppose that $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$. If $R \bowtie^f J$ is a $(*)$-ring, then every regular principal ideal $I = < r >$ of R such that $(r, f(r)+j)$ is regular for some $j \in J$ has a primary decomposition. Moreover, if $Ann(f(a)) \cap J = 0$ for all $a \in Reg(R)$ then R is a $(*)$-ring;*

4. *Suppose that $Ann(f^{-1}(J)) = 0$ and $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$. If R is a $(*)$-ring, then so is $R \bowtie^f J$;*

5. *Suppose that $J=(f(a)+j)J$ for all $(a,f(a)+j)\in Reg(R\bowtie^f J)$. If $R\bowtie^f J$ is a strongly $(*)$-ring, then every invertible ideal I of R such that $I\bowtie^f f(I)J$ is regular has a primary decomposition. In addition to that, if $Ann(f(a))\cap J=0$ for all $a\in Reg(R)$ then R is a strongly $(*)$-ring;*

6. *Suppose that $Ann(f^{-1}(J))=0$ and $J=(f(a)+j)J$ for all $(a,f(a)+j)\in Reg(R\bowtie^f J)$. If R is a strongly $(*)$-ring, then so is $R\bowtie^f J$;*

7. *If $f^{-1}(J)=0$, then $R\bowtie^f J$ is a weak π-ring (resp. $(*)$-ring, strongly $(*)$-ring) if and only if so is $f(R)+J$.*

Proof.

1. Let $I=<r>$ be a regular ideal of R. As $Ann(f(a))\cap J=0$ for all $a\in Reg(R)$, $H=<(r,f(r))>$ is a regular ideal of $R\bowtie^f J$, and so $H=P_1'^f...P_n'^f\overline{Q}_1^f...\overline{Q}_m^f$ where $P_k'^f$'s and $\overline{Q}_l^f$'s are prime ideals of $R\bowtie^f J$. We show that $I=P_1...P_nf^{-1}(Q_1)...f^{-1}(Q_m)$. For that, let $\alpha\in R$:
   $(\alpha,f(\alpha))(r,f(r))=\Sigma_{l=1}^d(p_1^l,f(p_1^l))...(p_n^l,f(p_n^l))(q_1^l,f(q_1^l))...(q_m^l,f(q_m^l))$ where for all $l=1,...,d$ we have $p_t^l\in P_t$ for all $t=1,...,n$ and $q_h^l\in Q_h$ for all $h=1,...,m$. Then:
   $(\alpha,f(\alpha))(r,f(r))=\Sigma_{l=1}^d(p_1^l...p_n^lq_1^l...q_m^l,f(p_1^l...p_n^lq_1^l...q_m^l))$, so $\alpha r=\Sigma_{l=1}^d p_1^l...p_n^lq_1^l...q_m^l$, and thus $I\subseteq P_1...P_nf^{-1}(Q_1)...f^{-1}(Q_m)$.
   Now, let $p_1...p_nq_1...q_m\in P_1...P_nf^{-1}(Q_1)...f^{-1}(Q_m)$, then;
   $(p_1...p_nq_1...q_m,f(p_1...p_nq_1...q_m))=(p_1,f(p_1))...(p_n,f(p_n))(q_1,f(q_1))$ $...(q_m,f(q_m))\in P_1'^f...P_n'^f\overline{Q}_1^f...\overline{Q}_m^f=<(r,f(r))>$, so there exists $\alpha\in R$ such that $p_1...p_nq_1...q_m=\alpha r$, consequently $I=P_1...P_nf^{-1}(Q_1)...f^{-1}(Q_m)$.

2. Let $H=<(r,f(r)+j)>$ be a regular ideal. As $J=(f(a)+j)J$ for all $(a,f(a)+j)\in Reg(R\bowtie^f J)$ and $Ann(f^{-1}(J))=0$, by Lemma 2.4.1 we get $H=I\bowtie^f J$ where $I=<r>$ is a regular ideal of R. As R is a weak π-ring, we get $I=P_1...P_n$ where P_l's are prime ideals of R, so $H=I\bowtie^f J=(P_1...P_n)\bowtie^f J\supseteq(P_1\bowtie^f J)...(P_n\bowtie^f J)$. Now let $(a,f(a)+k)\in(P_1...P_n)\bowtie^f J$, so $(a,f(a)+k)=(\Sigma_{l=1}^m p_1^l...p_n^l,f(\Sigma_{l=1}^m p_1^l...p_n^l)+k)=\Sigma_{l=1}^m(p_1^l,f(p_1^l))...(p_n^l,f(p_n^l))+(0,k)=\Sigma_{l=1}^m(p_1^l,f(p_1^l))...(p_n^l,f(p_n^l)+(a,f(a)+k)...(a,f(a)+k)(0,k')$ since $(a^{n-1},(f(a)+k)^{n-1})$ is regular and $k=(f(a)+k)^{n-1}k'$ for some $k'\in J$, hence $(P_1...P_n)\bowtie^f J\subseteq(P_1\bowtie^f J)...(P_n\bowtie^f J)$. Consequently, $R\bowtie^f J$ is a weak π-ring.

3. Let $I=<r>$ be a regular ideal of R. By hypothesis, $<(r,f(r)+j)>=I\bowtie^f J$ is regular for some $j\in J$. Thus $I\bowtie^f J=(P_1\bowtie^f J)\cap...\cap(P_n\bowtie^f J)=(P_1\cap...\cap P_n)\bowtie^f J$ where P_l's are primary ideals, since $R\bowtie^f J$ is a $(*)$-ring. Hence I has a primary decomposition.
   The "moreover" statement is clear since $(r,f(r))$ is regular for each $r\in Reg(R)$, and the remaining proof is similar to the last one.

4. Suppose R is a $(*)$-ring, and let $H = < (r, f(r)+j) >$ be a regular ideal. As $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$ and $Ann(f^{-1}(J)) = 0$, by Lemma 2.4.1 we get $H = I \bowtie^f J$ where $I = < r >$ is a regular ideal of R. Since R is a $(*)$-ring, it follows that $I = P_1 \cap ... \cap P_n$ where P_l's are primary ideals of R, so $H = I \bowtie^f J = (P_1 \cap ... \cap P_n) \bowtie^f J = (P_1 \bowtie^f J) \cap ... \cap (P_n \bowtie^f J)$ where $P_l \bowtie^f J$ is a primary ideal for all $l = 1, ..., n$. Consequently, $R \bowtie^f J$ is a $(*)$-ring.

5. Let I be an invertible ideal of R such that $I \bowtie^f f(I)J$ is regular. By hypothesis and lemma 2.4.2, we get $I \bowtie^f f(I)J = I \bowtie^f J$ an invertible ideal of $R \bowtie^f J$ which is strongly $(*)$-ring. Thus $I \bowtie^f J = (P_1 \bowtie^f J) \cap ... \cap (P_n \bowtie^f J) = (P_1 \cap ... \cap P_n) \bowtie^f J$ and hence I has a primary decomposition.
   The "moreover" statement is similar to the one of (3).

6. Suppose that R is a strongly $(*)$-ring, and let H be an invertible ideal. As $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$ and $Ann(f^{-1}(J)) = 0$, by Lemma 2.4.2 and Lemma 2.4.1, we get $H = I \bowtie^f J$ where I is an invertible ideal of R. Since R is a strongly $(*)$-ring, it follows that $I = P_1 \cap ... \cap P_n$ where P_l's are primary ideals of R, so $H = I \bowtie^f J = (P_1 \cap ... \cap P_n) \bowtie^f J = (P_1 \bowtie^f J) \cap ... \cap (P_n \bowtie^f J)$. Consequently, $R \bowtie^f J$ is a strongly $(*)$-ring.

7. If $f^{-1}(J) = 0$, then $R \bowtie^f J \cong f(R) + J$, as desired.

Corollary 2.4.4 *Let R and S be two rings, J an ideal of S and $f : R \to S$ be a ring homomorphism.*

1. *Suppose that $Ann(f(a)) \cap J = 0$ for all $a \in Reg(R)$. If $R \bowtie^f J$ is an almost weak π-ring, then so is R;*

2. *Suppose that $Ann(f^{-1}(J)) = 0$ and $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$. If R is an almost weak π-ring, then so is $R \bowtie^f J$.*

Proof.

1. Let $I = < r >$ be a regular ideal of R, so $I \bowtie^f f(I)J$ is a regular principal ideal of $R \bowtie^f J$ by hypothesis and by Lemma 2.4.2. Now let $M \in Max(R, I)$, two cases are then possible:
   If $M \in Max(R, I) \backslash V(f^{-1}(J))$, so $(I \bowtie^f f(I)J)_{M'^f} \cong I_M$ where $M'^f = M \bowtie^f J$, as desired.
   If $M \in Max(R, I) \cap V(f^{-1}(J))$, so $(I \bowtie^f f(I)J)_{M'^f} \cong I_M \bowtie^{f_M} (f(I)J)_{T_M}$ where $T_M = f(R-M) + J$ is a multiplicative set of S and f_M is the ring homomorphism induced by f for all $M \in Max(R)$. By applying the same reasoning of the proof of Theorem 2.4.3(1), we obtain the desired result.

2. Let $H = <(r, f(r)+j)>$ be a regular ideal. As $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$ and $Ann(f^{-1}(J)) = 0$, by Lemma 2.4.2 we get $H = I \bowtie^f J$ where $I = <r>$ is a regular ideal of R. As R is almost weak π-ring, we get $I_M = P_{1M}...P_{nM}$ for all $M \in Max(R, I)$ where P_l's are prime ideals of R. Two cases are then possible:
   If $M \in Max(R) \setminus V(f^{-1}(J))$, so $H_{M'^f} = (I \bowtie^f J)_{M'^f} \cong I_M$, as desired.
   If $M \in Max(R) \cap V(f^{-1}(J))$, so $H_{M'^f} = (I \bowtie^f J)_{M'^f} \cong I_M \bowtie^{f_M} J_{T_M}$. Moreover we have $I_M \bowtie^{f_M} J_{T_M} = P_{1M}...P_{nM} \bowtie^{f_M} J_{T_M}$. By the same way of the proof of theorem 2.4.3(2), we get $P_{1M}...P_{nM} \bowtie^{f_M} J_{T_M} = P_{1M} \bowtie^{f_M} J_{T_M}...P_{nM} \bowtie^{f_M} J_{T_M}$.
   Finally, $R \bowtie^f J$ is an almost weak π-ring.

We next present necessary and sufficient conditions for $A \bowtie I$ to be a weak π-ring, a $(*)$-ring and a strongly $(*)$-ring.

Theorem 2.4.5 *Let R be a ring and I be an ideal of R.*

1. *If $R \bowtie I$ is a weak π-ring, then so is R;*

2. *Suppose that $I = aI$ for all $a \in Reg(R)$. If R is a weak π-ring, then so is $R \bowtie I$;*

3. *If R is an arithmetical weak π-ring and I a finitely generated ideal of R satisfying $I_M = 0$ for all $M \in Max(R, I)$, then $R \bowtie I$ is a weak π-ring;*

4. *Suppose that $I = aI$ for all $a \in Reg(R)$. $R \bowtie I$ is a $(*)$-ring if and only if so is R;*

5. *Suppose that $I = aI$ for all $a \in Reg(R)$. $R \bowtie I$ is a strongly $(*)$-ring if and only if so is R.*

Proof.

1. Holds by Theorem 2.4.3(1).

2. Suppose that R is a weak π-ring.
   If I is regular, then there exists $x \in I$ a regular element of R, since $I = aI$ for all $a \in Reg(R)$, we get $x = xk$ for some $k \in I$ which implies that $k = 1$ and hence $I = R$. Finally, we obtain $R \bowtie I = R \times R$ which is a weak π-ring if and only if so is R.
   If I is not regular, let $H = <(r, r+i)>$ be a regular principal ideal of $R \bowtie I$. We claim that r and $r+i$ are regular elements.
   Assume that there exists $x \in R - \{0\}$ such that $rx = 0$:

   - If $x \in Ann(I)$, then $(r, r+i)(x, x) = (0, 0)$, contradiction since $(r, r+j)$ is regular.
   - If is not, then there exists $k \in I$ such that $xk \neq 0$, so $(r, r+i)(xk, 0) = (0, 0)$, contradiction since $(r, r+j)$ is regular.

Assume that there exists $y \in R - \{0\}$ such that $(r+i)y = 0$:

- If $y \in Ann(I)$, then $(r+i)y = 0 \Rightarrow ry = 0$, contradiction since r is regular.
- If is not, then there exists $k \in I$ such that $yk \neq 0$, so $(r, r+i)(0, yk) = (0,0)$, contradiction since $(r, r+j)$ is regular.

As $I = aI$ for all $a \in Reg(R)$, we get for all $k \in I$ there exists $k' \in I$ such that $k = (r+i)k'$, so $(0,k) = (0,k')(r,r+i) \in H$ and $0 \times I \subseteq H$ and hence $H = J \bowtie I$ where $J =< r >$ is a regular principal ideal of R. As R is weak π-ring, we get $J = P_1...P_n$ where P_l's are prime ideals, so $H = J \bowtie I = P_1...P_n \bowtie I \supseteq (P_1 \bowtie I)...(P_n \bowtie I)$. Now let $(a, a+j) \in P_1...P_n \bowtie I$, so $(a, a+j) = (\Sigma_{l=1}^{m} p_1^l...p_n^l, \Sigma_{l=1}^{m} p_1^l...p_n^l + j) = \Sigma_{l=1}^{m}(p_1^l, p_1^l)...(p_n^l, p_n^l) + (0, j) = \Sigma_{l=1}^{m}(p_1^l, p_1^l)...(p_n^l, p_n^l) + (a,a)...(a,a)(0, j')$ since a^{n-1} is a regular element and $j = a^{n-1}j'$ for some $j' \in I$, hence $P_1...P_n \bowtie I \subseteq (P_1 \bowtie I)...(P_n \bowtie I)$. Consequently, $R \bowtie I$ is a weak π-ring.

3. Let H be a regular prime ideal of $R \bowtie I$.
Case 1: $H = P' = P \bowtie I$, so P is a regular prime. Since R is a weak π-ring, it follows that $K \subseteq P$ for some invertible prime ideal K of R. Now we show that the prime ideal $K \bowtie I$ is invertible. Since I is finitely generated, we have $K \bowtie I$ is a finitely generated regular ideal. By [32, Corollary 3.8] $R \bowtie I$ is an arithmetical ring, thus $K \bowtie I$ is locally principal. Consequently, $K \bowtie I$ is an invertible prime ideal.
Case 2: $H = \overline{P}$, so P is a regular prime. Since R is a weak π-ring, it follows that $K \subseteq P$ for some invertible prime ideal K of R. It's clear that $\overline{K}$ is regular. Since K and I are finitely generated, it follows that $\overline{K}$ is finitely generated, indeed :
Let $K =< (k_j)_{j=1}^{n} >$, $I =< (i_l)_{l=1}^{m} >$ and $(r, r+i) \in \overline{K}$, we have $r+i = \Sigma_{j=1}^{n} a_j k_j$ and $i = \Sigma_{l=1}^{m} b_l i_l$, so $r = \Sigma_{j=1}^{n} a_j k_j - \Sigma_{l=1}^{m} b_l i_l$, then:
$(r, r+i) = (\Sigma_{j=1}^{n} a_j k_j - \Sigma_{l=1}^{m} b_l i_l, \Sigma_{j=1}^{n} a_j k_j) = \Sigma_{j=1}^{n}(a_j, a_j)(k_j, k_j) - \Sigma_{l=1}^{m}(b_l, b_l)(i_l, 0)$.
Since R is an arithmetical ring, it follows that $\overline{K}$ is locally principal, and hence an invertible prime ideal. Finaly, $R \bowtie I$ is a weak π-ring.

4. Suppose $R \bowtie I$ a $(*)$-ring and let $J =< r >$ be a regular ideal of R. By the same way in the proof of statement 2, $< (r,r) >= J \bowtie I$ is regular. Hence by hypothesis, $J \bowtie I = (P_1 \bowtie^f J) \cap ... \cap (P_n \bowtie^f J) = (P_1 \cap ... \cap P_n) \bowtie^f J$ where P_l's are primary ideals. Therefore, J has a primary decomposition.
Conversely, suppose R is a $(*)$-ring, and let $H =< (r, r+i) >$ be a regular ideal. Similarly to the proof of statement 2, $H = J \bowtie I$ where $J =< r >$ is a regular ideal of R. We get then $J = P_1 \cap ... \cap P_n$ where P_l's are primary ideals of R, so $H = (P_1 \cap ... \cap P_n) \bowtie I = (P_1 \bowtie I) \cap ... \cap (P_n \bowtie I)$ where $P_l \bowtie I$ are primary ideals for all $l = 1, ..., n$. Consequently, $R \bowtie I$ is a $(*)$-ring.

5. Assume that $R \bowtie I$ is a strongly $(*)$-ring and let J be an invertible ideal of R. By applying lemma 2.4.2, we get $J \bowtie I$ an invertible ideal, and so $J \bowtie I = (P_1 \bowtie I) \cap$

$... \cap (P_n \bowtie I) = (P_1 \cap ... \cap P_n) \bowtie I$, as desired.
Conversely, suppose R is a strongly $(*)$-ring, and let H be an invertible ideal of $R \bowtie I$. Similarly to the proof of Theorem 2.4.3, $H = J \bowtie I$ where J is an invertible ideal of R. We get then $J = P_1 \cap ... \cap P_n$ where P_l's are primary ideals of R, so $H = (P_1 \cap ... \cap P_n) \bowtie I = (P_1 \bowtie I) \cap ... \cap (P_n \bowtie I)$ where $P_l \bowtie I$ are primary ideals for all $l = 1, ..., n$, which completes the proof.

The result in the next corollary follows at once from Theorem 2.4.5.

Corollary 2.4.6 *Let R be a ring and I an ideal of R.*

1. *If $R \bowtie I$ is an almost weak π-ring, then so is R;*
2. *Suppose that $I = aI$ for all $a \in Reg(R)$. If R is an almost weak π-ring, then so is $R \bowtie I$.*

Proof.

1. Follows from Corollary 2.4.4(1).
2. Let $H = < (r, r+i) >$ be a regular ideal. Similarly to the previous proof we have $H = J \bowtie I$ where $J = < r >$ is a regular ideal of R. Since R is almost weak π-ring, we get $J_M = P_{1M} ... P_{nM}$ for all $M \in Max(R, J)$ where P_l's are prime ideals of R. Two cases are then possible:
If $M \in Max(R) \setminus V(I)$, so $H_{M'} = (J \bowtie I)_{M'} \cong J_M$ where $M' = M \bowtie I$, as desired.
If $M \in Max(R) \cap V(I)$, so $H_{M'} = (J \bowtie I)_{M'} \cong J_M \bowtie I_M$. As R is almost weak π-ring, we have $J_M \bowtie I_M = P_{1M} ... P_{nM} \bowtie I_M = P_{1M} \bowtie I_M ... P_{nM} \bowtie I_M$, that is to say $R \bowtie^f J$ is an almost weak π-ring.

We close this section with some examples of application of our results. We construct the following examples of a weak π-ring (a strongly $(*)$-ring) for the amalgamated duplication along an ideal and the amalgamation of rings along an ideal with respect to f.

Example 2.4.7 *Let $R = \mathbb{Z} \propto \mathbb{Z}/4\mathbb{Z}$ and $I = 0 \propto 2\mathbb{Z}/4\mathbb{Z}$. Thus $R \bowtie I$ is a weak π-ring and hence a strongly $(*)$-ring.*

Proof. By [67, Example 3.1], R is a weak π-ring. We claim that $I = aI$ for all $a \in Reg(R)$. Indeed, since $Reg(R) = \{(2k+1, \overline{s}^*) \in \mathbb{Z} \propto \mathbb{Z}/4\mathbb{Z} | k \in \mathbb{Z}\}$ and $I = < (0, \overline{2}) >$, we get $(0, \overline{2}) = (0, \overline{4k} + \overline{2}) = (2k+1, \overline{z})(0, \overline{2})$ for all $k, z \in \mathbb{Z}$. Hence $R \bowtie I$ is a weak π-ring and hence a strongly $(*)$-ring.

Example 2.4.8 *Let $R = \mathbb{Z} \propto \mathbb{Z}/4\mathbb{Z}$, $S = \mathbb{Z}/6\mathbb{Z}$, f a ring homomorphism defined by $f((r, \overline{s}^*)) = \overline{r}$ and $J = 3\mathbb{Z}/6\mathbb{Z}$. Therefore $R \bowtie^f J$ is a weak π-ring and thus a strongly $(*)$-ring.*

Proof. By [67, Example 3.1], R is a weak π-ring. We claim that $J = (f(a)+j)J$ for all $(a, f(a)+j) \in Reg(R \bowtie^f J)$. Indeed, let $(a, f(a)+j) \in Reg(R \bowtie^f J)$, since $(3, \overline{0}^*) \in f^{-1}(J)$, it follows that $Ann(f^{-1}(J)) = 0$, by Lemma 2.4.2 a is a regular element of R. Now we prove that if $(a, f(a)+j)$ is regular, then $j = \overline{0}$. Indeed, as $Reg(R) = \{(2k+1, \overline{s}^*) \in \mathbb{Z} \propto \mathbb{Z}/4\mathbb{Z} | k \in \mathbb{Z}\}$, it follows that $f(Reg(R)) = \{\overline{1}, \overline{3}, \overline{5}\}$. Clearly $J = \{\overline{0}, \overline{3}\}$, so three cases are possible:
$f(a)+j = \overline{1}+\overline{3} = \overline{4}$, so $(a, f(a)+j)(0, \overline{3}) = (0, \overline{0})$, contradiction.
$f(a)+j = \overline{3}+\overline{3} = \overline{0}$, so $(a, f(a)+j)(0, \overline{3}) = (0, \overline{0})$, contradiction.
$f(a)+j = \overline{5}+\overline{3} = \overline{8}$, so $(a, f(a)+j)(0, \overline{3}) = (0, \overline{0})$, contradiction.
We get then, $\overline{3} = \overline{1} \times \overline{3}$, $\overline{3} = \overline{3} \times \overline{3}$ and $\overline{3} = \overline{5} \times \overline{3}$. Hence $R \bowtie^f J$ is a weak π-ring and thus a strongly $(*)$-ring.

Example 2.4.9 *Let $R = \mathbb{Z}/8\mathbb{Z}$ and $I = 2\mathbb{Z}/8\mathbb{Z}$. R is a weak π-ring since R is a total quotient ring. Clearly I which is not regular satisfies $I = \overline{a}I$ for all $\overline{a} \in Reg(R)$. Therefore, $R \bowtie I$ is a weak π-ring and thus a strongly $(*)$-ring.*

CHAPTER 3

THE DIVIDED, GOING-DOWN, AND GAUSSIAN PROPERTIES OF AMALGAMATION OF RINGS

Abstract.

In this chapter *, we provide necessary and sufficient conditions for the amalgamation of rings $A \bowtie^f J$ to be a divided ring, locally divided ring, going-down ring and Gaussian ring.

3.1 Introduction

An R-module E is said to be divisible if, for each $e \in E$ and each regular element r of R, there exists $f \in E$ such that $e = rf$.

In [22], Badawi and Dobbs studied divided rings, that is, rings in which every prime ideal $\mathfrak{p}$ is divided, i.e. $\mathfrak{p}$ is comparable under inclusion to each ideal (principal ideal) of R. Using idealization, they characterized a ring to be divided if and only if $\mathrm{Nil}(R) \in \mathrm{Spec}(R)$ and $R \propto E$ is a divided ring for each (or some) $R_{\mathrm{Nil}(R)}$-module E [22, Proposition 2.14]. The authors generalized locally divided domains, namely, locally divided rings for which $R_{\mathfrak{p}}$ is a divided ring for each $\mathfrak{p} \in \mathrm{Spec}(R)$. They also characterized R to be a locally divided ring by the fact that each maximal ideal of R contains only one minimal prime ideal of R and $R \propto E$ is a locally divided ring for each R-module E satisfying the following condition: for each $\mathfrak{m} \in Max(R)$, the $R_{\mathfrak{m}}$-module structure on $E_{\mathfrak{m}}$ is induced by a $Nil(R_{\mathfrak{m}})^{-1}(R_{\mathfrak{m}})$-module structure on $E_{\mathfrak{m}}$.

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In [22], the authors showed that $A \propto E$ is a locally divided ring for each (or some) A-module E satisfying the following condition: for each $\mathfrak{m} \in \mathrm{Max}(A)$; the $A_{\mathfrak{m}}$-module structure on $E_{\mathfrak{m}}$ is induced by a $(\mathrm{Nil}(A_{\mathfrak{m}})^{-1}(A_{\mathfrak{m}})$-module structure on $E_{\mathfrak{m}}$ [22, Proposition 2.16]. They also studied going-down rings, that is, rings satisfying $R/\mathfrak{p}$ is going-down domain for each $\mathfrak{p} \in \mathrm{Spec}(R)$. Recall from [44], that a domain R is called a going-down domain if $R \subseteq T$ satisfies the going-down property GD for each domain T containing R. A ring R is going-down ring if and only if $R_{\mathfrak{m}}$ is going-down ring for each $\mathfrak{m} \in \mathrm{Max}(R)$ [46, Proposition 2.1(b)]. As divided rings and locally divided rings, they proved that a ring R is a going-down ring if and only if $R \propto E$ is a going-down ring for each (or some) R-module E [22, Proposition 3.8].

For a ring R and $f \in R[X]$, we denote by $c(f)$ (called the content of f) the ideal of R generated by the coefficients of f. R is a Gaussian ring if for every $f,g \in R[X]$, one has the content ideal equation $c(fg) = c(f)c(g)$. Considerable works related to this property were introduced in these following papers [32, 73, 82].

Let A and B be two rings, J an ideal of B and $f : A \to B$ a ring homomorphism. We say that $A \bowtie^f J$ satisfies the property $(*)$ if every ideal has one of the following three forms:

- $I \times 0$ where $I \subseteq f^{-1}(J)$ is an ideal of A;
- $0 \times K$ where $K \subseteq J$ is an ideal of $f(A) + J$;
- $I \bowtie^f J$ where I is an ideal of A.

In Section 3.2, we state necessary and sufficient conditions for the amalgamation of rings $A \bowtie^f J$ to satisfy the property $(*)$. In Sections 3.3, 3.4 and 3.5, we study the stability of the divided ring, locally divided ring, going-down ring and Gaussian ring properties in terms of amalgamation of rings, and as corollaries we get characterizations of some of these class of rings using idealization and amalgamated duplication.

3.2 Form of ideals of amalgamated algebra

The following Theorem, which is one of the main result of this paper, provides necessary and sufficient conditions for the ring $A \bowtie^f J$ to satisfy the property $(*)$.

Theorem 3.2.1 *Let A and B be two rings, J be a non-zero proper ideal of B and $f : A \to B$ be a ring homomorphism.*

(1) If $A \bowtie^f J$ satisfies the property $(*)$, then the following conditions hold:

(i) $f(A)$ is an integral domain;

(ii) $f(A) \cap J = 0$;

(iii) $0 \times J \subseteq ((a, f(a) + j))$ *for all* $a \in A - \{0\}$ *and all* $j \in J$ *such that* $f(a) + j \neq 0$.

(2) If f *is injective and* $A \bowtie^f J$ *satisfies the property* $(*)$, *then* A *is an integral domain;*

(3) If f *is not injective and* A *is a ring with zero-divisors with* $A \bowtie^f J$ *satisfies the property* $(*)$, *then* $Ann_{f(A)+J}(f(a) + j) \subseteq J$ *for all* $a \in A - \{0\}$ *and* $j \in J$ *with* $f(a) \neq 0$. *Moreover, if* $f^{-1}(J) \nsubseteq Z(A)$, *then* $f(a) + j \in Reg(f(A) + J)$ *for all* $a \in Reg(A)$ *and* $j \in J$ *with* $f(a) \neq 0$, *and* $Ann_{f(A)+J}(j) \subseteq f(Z(A) \backslash f^{-1}(J)) + J$ *for all* $j \in J$;

(4) If f *is not injective and* A *is an integral domain with* $A \bowtie^f J$ *satisfies the property* $(*)$, *then the following conditions hold:*

(i) $f(A) + J$ *is an integral domain;*

(ii) J *is idempotent.*

(5) If $0 \times J \subseteq ((a, f(a) + j))$ *for all* $a \in A - \{0\}$ *and all* $j \in J$ *such that* $f(a) + j \neq 0$, *then* $A \bowtie^f J$ *satisfies the property* $(*)$.

Proof. (1) Assume that $A \bowtie^f J$ satisfies the property $(*)$. We claim that $f(A)$ is an integral domain. Indeed, let $(a, f(a)) \in A \bowtie^f J$ such that $f(a) \neq 0$, by hypothesis $((a, f(a))) = (a) \bowtie^f J$ then for all $i \in J$ we get $(0, i) = (a, f(a))(b, f(b) + k)$ for some $k \in J$ and $b \in A$, that is $J = f(a)J$ for all $a \in A$ such that $f(a) \neq 0$. Now, suppose that there exists $f(a), f(b) \in f(A) - \{0\}$ such that $f(a)f(b) = 0$, then $J = f(a)J = f(a)f(b)J = 0$, contradiction. Then (i) holds. Let $(a, f(a) + j) \in A \bowtie^f J$ such that $a \neq 0$ and $f(a) + j \neq 0$, by our assumption $((a, f(a) + j)) = (a) \bowtie^f J$, so $0 \times J \subseteq ((a, f(a) + j))$ for all $a \in A - \{0\}$ and all $j \in J$ such that $f(a) + j \neq 0$. Therefore (iii) holds. Now we suppose that $f(A) \cap J \neq 0$, then there exist $0 \neq f(a) \in J$. We claim that $f(a) \in Reg(f(A) + J)$. Deny, let $0 \neq i \in J$ such that $f(a)i = 0$. Since $A \bowtie^f J$ satisfies the property $(*)$, then for each $k \in J$ we have:
(1) $(0, k) = (a, f(a))(b, f(b) + h)$ hence $k = hf(a)$ for some $b \in A$ and $h \in J$, since $f(A)$ is a domain we have $f(b) = 0$.
(2) $(0, k) = (a, i)(c, f(c) + t)$ hence $k = i(f(c) + t)$ for some $c \in A$ and $t \in J$
From (1) and (2), we get for all $k \in J$ $k = hf(a) = i(f(c) + t)f(a) = 0$, a contradiction. Hence $i = 0$.
Now, let $f(b) \neq 0$ and $i \neq 0$ such that $f(a)(f(b) + i) = 0$ and suppose that $f(b) + i \neq 0$. As $A \bowtie^f J$ satisfies the property $(*)$, then for each $k \in J$ we have:
$(0, k) = (b, f(b) + i)(c, f(c) + t)$ for some $c \in A$ and $t \in J$. As $f(A)$ is a domain, we get $f(c) = 0$, hence $k = t(f(b) + i)$. Since $J = f(a)J$, then $k = t(f(b) + i) = hf(a)(f(b) + i) = 0$ for some $h \in J$, a contradiction. Thus $f(b) + i = 0$ and so $f(a)$ is regular in $f(A) + J$.

Since $f(a)\in J$ and $J=f(b)J$ for all $f(b)\neq 0$, we get, $f(a)=f(a)j$ for some $j\in J$, so $j=1$, a contradiction. Hence (ii) holds.

(2) Assume that f is injective and $A\bowtie^f J$ satisfies the property $(*)$. Let $a,b\in A-\{0\}$ such that $ab=0$. By hypothesis, $f(a)\neq 0$ and $f(b)\neq 0$, then $((a,f(a)))=(a)\bowtie^f J$ and $((b,f(b)))=(b)\bowtie^f J$, that is $J=f(a)J=f(b)J$, and so $J=f(a)J=f(a)f(b)J=0$, a contradiction. Hence A is an integral domain.

(3) Assume that f is not injective and A is a ring with zero-divisors with $A\bowtie^f J$ satisfies the property $(*)$. Let $a\in A-\{0\}$ and $j\in J$ such that $f(a)\neq 0$. Suppose that there exist $b\in A$ and $i\in J$ such that $(f(a)+j)(f(b)+i)=0$ with $f(b)\neq 0$ and so $f(ab)\in J$, as $A\bowtie^f J$ satisfies the property $(*)$, we get $f(A)\cap J=0$ and then $f(a)f(b)=0$. This is a contradiction since $f(A)$ is a domain. Therefore $\mathrm{Ann}_{f(A)+J}(f(a)+j)\subseteq J$.

For the second part, suppose $f^{-1}(J)\nsubseteq \mathrm{Z}(A)$ and let $a\in \mathrm{Reg}(A)$ and $j\in J$ such that $f(a)\neq 0$, by the previous statement $\mathrm{Ann}(f(a)+j)\subseteq J$. Assume that there exists $i\in J$ such that $i(f(a)+j)=0$, by hypothesis $J=iJ=(f(a)+j)J$ and so $J=iJ=i(f(a)+j)J=0$, a contradiction. Thus $f(a)+j\in \mathrm{Reg}(f(A)+J)$. Since $J=jJ$ and $f(a)+j\in \mathrm{Reg}(f(A)+J)$ for all $j\in J$ and $a\in \mathrm{Reg}(A)$ with $f(a)\neq 0$, then $\mathrm{Ann}_{f(A)+J}(j)\subseteq f(\mathrm{Z}(A)\backslash f^{-1}(J))+J$ for all $j\in J$.

(4) Suppose that f is not injective and A is an integral domain with $A\bowtie^f J$ satisfies the property $(*)$. Similarly to the second part of the previous statement, we prove that $f(A)+J$ is an integral domain. Then (i) holds. As f is not injective, we get $f(a)=0$ with $a\neq 0$ so $J=jJ$ for all $j\in J$ and hence J is idempotent. Then (ii) holds.

(5) Suppose that $0\times J\subseteq ((a,f(a)+j))$ for all $a\in A-\{0\}$ and all $j\in J$ such that $f(a)+j\neq 0$. Let K be an ideal of $A\bowtie^f J$. Three cases are then possible:

- If for all $(a,f(a)+j)\in K$, $a=0$, then $K\subseteq 0\times J$ and hence $K=0\times I$ where $I\subseteq J$ is an ideal of $f(A)+J$.

- If for all $(a,f(a)+j)\in K$, $f(a)+j=0$, then $K\subseteq f^{-1}(J)\times 0$ and hence $K=I\times 0$ where $I\subseteq f^{-1}(J)$ is an ideal of A.

- If there exists $(a,f(a)+j)\in K$ such that $a\neq 0$ and $f(a)+j\neq 0$, then by hypothesis $0\times J\subseteq ((a,f(a)+j))$ and so $0\times J\subseteq K$. Therefore $A\bowtie^f J$ satisfies the property $(*)$.

Corollary 3.2.2 *Let A be a ring, let I be a non-zero ideal of A and let E be an A-module. Then:*

1. *$R=A\bowtie I$ does not satisfy the property $(*)$;*

2. *Every ideal of $R=A\propto E$ has the form $0\propto F$ where F is a submodule of E or the form $I\propto E$ where I is an ideal of A if and only if A is a domain and E is a divisible module.*

Proof. (1) This is clear.

(2) Let $J = 0 \propto E$ and $f : A \to R$, clearly $A \bowtie^f J \cong R$. Since f is injective, we apply Theorem 3.2.1(2) and we get the desired result.

As an application of our results, we construct the following examples.

Examples 3.2.3 *Let K be a field.*

1. *$A = K$, $B = K \propto K$, $J = 0 \propto K$ and $f : A \to B$ defined by $f(k) = (k, 0)$ for all $k \in A$. It's clear that $J = (f(k) + j)J$, then by Theorem 2.1(2) $A \bowtie^f J$ satisfies the property $(*)$;*

2. *$A = K[X]$, $B = K \times K$, $J = 0 \times K$ and $f : A \to B$ defined by $f(P) = (P(0), P(0))$ for all $P \in A$. Clearly $J = (f(P) + j)J$, then by Theorem 2.1(4) $A \bowtie^f J$ satisfies the property $(*)$;*

3. *$A = K[X] \propto K[X]$, $B = K \times K$, $J = 0 \times K$ and $f : A \to B$ defined by $f((P,Q)) = (P(0), P(0))$ for all $(P,Q) \in A$. Clearly $0 \times J \subseteq (((P,Q), f((P,Q)) + j))$, then by Theorem 2.1(3) $A \bowtie^f J$ satisfies the property $(*)$.*

3.3 Divided and Locally divided rings

In this section, we study the transfer of the properties of being a divided ring and a locally divided ring between two commutative ring A and B and their amalgamation. We mean by $J_{T_{\mathfrak{p}}}$ the localization of J at the prime ideal $T_{\mathfrak{p}} = f(A \backslash \mathfrak{p}) + J$ for each prime ideal $\mathfrak{p}$ containing $f^{-1}(J)$.

Theorem 3.3.1 *Let A and B be two rings, J a non-zero ideal of B, $f : A \to B$ a ring homomorphism and $R = A \bowtie^f J$.*

(1) If A is an integral domain, then R is a divided ring if and only if the following conditions hold:

(i) $f^{-1}(J) = 0$;

(ii) A and $f(A) + J$ are divided rings;

(iii) $0 \times J \subseteq ((a, f(a) + j))$ for all $a \in A - \{0\}$ and $j \in J$.

(1') If R is a locally divided ring, then so is A and the following conditions hold:

(a) For each prime $\mathfrak{p}$ of A containing $f^{-1}(J)$ such that $A_{\mathfrak{p}}$ is an integral domain:

(i) $f_{\mathfrak{p}}^{-1}(J_{T_{\mathfrak{p}}}) = 0$*;*

(ii) $f_{\mathfrak{p}}(A_{\mathfrak{p}}) + J_{T_{\mathfrak{p}}}$ *is a divided ring;*

(iii) $0 \times J_{T_{\mathfrak{p}}} \subseteq ((\frac{a}{s}, f_{\mathfrak{p}}(\frac{a}{s}) + \frac{j}{t}))$ *for all* $\frac{a}{s} \in A_P - \{0\}$ *and* $\frac{j}{t} \in J_{T_{\mathfrak{p}}}$.

(b) For each prime $\mathfrak{p}$ *of* A *containing* $f^{-1}(J)$ *such that* $A_{\mathfrak{p}}$ *is a ring with zero-divisors:*

(i) $J_{T_{\mathfrak{p}}} \subseteq Nil(B_{T_{\mathfrak{p}}})$*;*

(ii) $0 \times J_{T_{\mathfrak{p}}} \subseteq ((\frac{a}{s}, f_{\mathfrak{p}}(\frac{a}{s}) + \frac{j}{t}))$ *for all* $\mathfrak{q} \in V(\mathfrak{p})$, $\frac{a}{s} \in A_P \backslash \mathfrak{q}_{\mathfrak{p}}$ *and* $\frac{j}{t} \in J_{T_{\mathfrak{p}}}$.

(2) If A *is a ring with zero-divisors, then* R *is a divided ring if and only if the following conditions hold:*

(i) $J \subseteq Nil(B)$*;*

(ii) A *is a divided ring;*

(iii) $0 \times J \subseteq ((a, f(a) + j))$ *for all* $\mathfrak{p} \in Spec(A)$*, for all* $a \in A \backslash \mathfrak{p}$ *and* $j \in J$.

(2') If moreover B *is locally divided, then* R *is locally divided.*

Proof. (1) Let A be an integral domain, and assume that R is a divided ring. Since $0 \times J$ is a prime ideal then it is comparable with $f^{-1}(J) \times 0$. As J is a non-zero ideal; it follows that $f^{-1}(J) = 0$. Then (i) holds. Now, let $P \in Spec(A)$ and $a \in A \backslash P$. By hypothesis $P \bowtie^f J \subseteq ((a, f(a)))$, so $((a, f(a))) = (a) \bowtie^f J$, then $P \subseteq (a)$ and hence A is divided. As $f^{-1}(J) = 0$, we get $R \cong f(A) + J$. Hence (ii) holds. We claim that $0 \times J \subseteq ((a, f(a) + j))$ for all $a \in A - \{0\}$ and $j \in J$. Indeed, since $0 \times J$ is a prime ideal, we get $0 \times J \subseteq ((a, f(a) + j))$ for all $((a, f(a) + j)) \in R \backslash 0 \times J$, as desired.
Conversely, let $\mathfrak{P} \in \mathrm{Spec}(R)$. Two cases are then possible:
Case 1: $\mathfrak{P} = \mathfrak{p} \bowtie^f J$, let $(a, f(a) + j) \in R \backslash \mathfrak{P}$. By hypothesis, we get $((a, f(a) + j)) = (a) \bowtie^f J$ and $\mathfrak{p} \subseteq (a)$, hence $\mathfrak{P} \subseteq ((a, f(a) + j))$.
Case 2: $\mathfrak{P} = \overline{\mathfrak{p}}^f = 0 \times \mathfrak{p}$ where $\mathfrak{p} \in \mathrm{Spec}(B)$ and $\mathfrak{p} \subseteq J$ since $0 \times J \subseteq ((a, f(a) + j))$ for all $a \in A - \{0\}$ and $j \in J$. Now, let $(a, f(a) + j) \in R \backslash \mathfrak{P}$. As $f(A) + J$ is divided, we get $\mathfrak{p} \subseteq (f(a) + j)$ as $f(A) + J$-ideal, thus $\mathfrak{P} \subseteq ((a, f(a) + j))$

(1') Suppose that R is locally divided and let $\mathfrak{p} \in \mathrm{Spec}(A)$. Two cases are then possible:

- If $f^{-1}(J) \not\subseteq \mathfrak{p}$, then $R_{\mathfrak{p} \bowtie^f J} \cong A_{\mathfrak{p}}$;
- If $f^{-1}(J) \subseteq \mathfrak{p}$, then $R_{\mathfrak{p} \bowtie^f J} \cong A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} J_{T_{\mathfrak{p}}}$.

Applying statements (1) and (2), we get A a locally divided ring satisfying the above conditions.

(2) Let A be a ring with zero-divisors. Suppose that R is divided. We claim that $J \subseteq \mathrm{Nil}(B)$. Deny, there exists $\mathfrak{p} \in \mathrm{Spec}(B)$ such that $J \not\subseteq \mathfrak{p}$, so $(0, j) \in R \backslash \overline{\mathfrak{p}}^f$ for some

$j \in J$. By hypothesis, $\overline{\mathfrak{p}}^f \subseteq ((0,j))$, and so $\overline{p}^f = 0 \times I$ where $I \in Spec(f(A)+J)$ and $I \subseteq J$. Now, let $a,b \in A-\{0\}$ such that $ab=0$, we get $(a,f(a))(b,f(b)) = (0,0) \in \overline{\mathfrak{p}}^f$ however $(a,f(a)) \notin \overline{\mathfrak{p}}^f$ and $(b,f(b)) \notin \overline{\mathfrak{p}}^f$, a contradiction. So (i) holds. Let $\mathfrak{p} \in \mathrm{Spec}(A)$ and $a \in A \backslash \mathfrak{p}$, we have $(a,f(a)) \in R\backslash \mathfrak{p} \bowtie^f J$, by hypothesis $\mathfrak{p} \bowtie^f J \subseteq ((a,f(a))) = (a) \bowtie^f J$ (clearly $f(a) \neq 0$ since $f^{-1}(J) \times 0 \subseteq \mathrm{Nil}(A) \bowtie^f J$ and $a \notin \mathrm{Nil}(A)$). Hence $\mathfrak{p} \subseteq (a)$, therefore A is divided and so (ii) holds. Let $\mathfrak{p} \in \mathrm{Spec}(A)$, $a \in A\backslash\mathfrak{p}$ and $i,j \in J$, we have $\mathfrak{p} \bowtie^f J \subseteq ((a,f(a)+j))$, and hence $0 \times J \subseteq ((a,f(a)+j))$. Therefore (iii) holds.
Conversely, since $J \subseteq \mathrm{Nil}(B)$, we get $\mathrm{Spec}(R) = \{\mathfrak{p} \bowtie^f J, \mathfrak{p} \in \mathrm{Spec}(A)\}$. So, let $(a,f(a)+j) \in R\backslash \mathfrak{p} \bowtie^f J$, as A is divided and $0 \times J \subseteq ((a,f(a)+j))$, we get $\mathfrak{p} \bowtie^f J \subseteq ((a,f(a)+j))$ and hence R is divided.

(2') We claim that R is locally divided. Indeed, let $\mathfrak{P} \in \mathrm{Spec}(R)$. Three cases are then possible:
Case 1: $\mathfrak{P} = \overline{\mathfrak{p}}^f$, so $R_{\mathfrak{P}} \cong B_{\mathfrak{p}}$.
Case 2: $\mathfrak{P} = \mathfrak{p} \bowtie^f J$ where $\mathfrak{p}$ is a prime ideal of A not containing $f^{-1}(J)$, so $R_{\mathfrak{P}} \cong A_{\mathfrak{p}}$.
Case 3: $\mathfrak{P} = \mathfrak{p} \bowtie^f J$ where $\mathfrak{p}$ is a prime ideal of A containing $f^{-1}(J)$, so $R_{\mathfrak{P}} \cong A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} J_{T_{\mathfrak{p}}}$.

For the three cases, we get the desired result by hypothesis and statements (1) and (2).

Corollary 3.3.2 *Let A be a ring, I a non-zero ideal of A and $R = A \bowtie I$.*

(1) If A is an integral domain, then R is never a (locally) divided ring;

(2) If A is a ring with zero-divisors, then R is a divided ring if and only if the following conditions hold:

(i) A is a divided ring;

(ii) $I \subseteq Nil(A)$;

(iii) $0 \times I \subseteq ((a, a+i))$ for all $\mathfrak{p} \in Spec(A)$, $a \in A\backslash\mathfrak{p}$ and $i \in I$.

(2') If A is a ring with zero-divisors, then R is a locally divided ring if and only if A is locally divided and the following conditions hold:

(a) For each prime $\mathfrak{p}$ of A containing I such that $A_{\mathfrak{p}}$ is an integral domain, $I_{\mathfrak{p}} = 0$;

(b) For each prime $\mathfrak{p}$ of A containing I such that $A_{\mathfrak{p}}$ is a ring with zero-divisors:

(i) $I_{\mathfrak{p}} \subseteq Nil(A_{\mathfrak{p}})$;
(ii) $0 \times I_{\mathfrak{p}} \subseteq ((\frac{a}{s}, \frac{a}{s} + \frac{i}{t}))$ for all $\mathfrak{q} \in V(\mathfrak{p})$, $\frac{a}{s} \in A_{\mathfrak{p}}\backslash\mathfrak{q}_{\mathfrak{p}}$ and $\frac{i}{t} \in I_{\mathfrak{p}}$.

Theorem 3.3.1 gives the necessary conditions for $A \propto E$ to be (locally) divided as the following result shows.

Corollary 3.3.3 *Let A be a ring, E an A-module and* $R = A \propto E$.

(1) If A is an integral domain, then R is a (locally) divided ring if and only if A is a (locally) divided ring and E is (locally) divisible (E is called locally divisible if $E_{\mathfrak{p}}$ *is divisible for each* $\mathfrak{p} \in Spec(A)$*);*

(2) If A is a ring with zero-divisors, then R is a divided ring if and only if the following conditions hold:

(i) *A is a divided ring;*

(ii) $0 \propto E \subseteq ((a,e))$ *for all* $\mathfrak{p} \in Spec(A)$, $a \in A \backslash \mathfrak{p}$ *and* $e \in E$.

(2') If A is a ring with zero-divisors, then R is a locally divided ring if and only if A is locally divided and the following conditions hold:

(a) For each prime $\mathfrak{p}$ *of A such that* $A_{\mathfrak{p}}$ *is an integral domain:*

(i) $E_{\mathfrak{p}} \subseteq ((\frac{a}{s}, \frac{e}{t}))$ *for all* $\frac{a}{s} \in A_{\mathfrak{p}} - \{\frac{0}{1}\}$ *and* $\frac{e}{t} \in E_{\mathfrak{p}}$.

(b) For each prime $\mathfrak{p}$ *of A such that* $A_{\mathfrak{p}}$ *is a ring with zero-divisors:*

(i) $E_{\mathfrak{p}} \subseteq ((\frac{a}{s}, \frac{e}{t}))$ *for all* $\mathfrak{q} \in V(\mathfrak{p})$, $\frac{a}{s} \in A_{\mathfrak{p}} \backslash \mathfrak{q}_{\mathfrak{p}}$ *and* $\frac{e}{t} \in E_{\mathfrak{p}}$.

In [22, Corollary 2.17], Badawi and Dobbs characterize a domain A to be (locally) divided using its trivial ring extension with a vector space E over its quotiont field. By our previous result, we have come to characterize a (locally) divided ring with zero divisors A using its amalgamated duplication along its ideal $Nil(A)$.

Proposition 3.3.4 *Let A be a ring. If A is a ring with zero divisors. Then the following assertions are equivalent:*

1. $A \bowtie Nil(A)$ *is a (locally) divided ring;*
2. $A \propto Nil(A)$ *is a (locally) divided ring;*
3. *A is a (locally) divided ring.*

3.4 Going-down rings

Now, we investigate the transfer of the notion of going-down rings to the amalgamation of rings $A \bowtie^f J$ in order to characterize this class of rings.

Theorem 3.4.1 *Let A and B be two rings, J a non-zero ideal of B,* $f : A \to B$ *a ring homomorphism and* $R = A \bowtie^f J$.

1. *If R is a going-down ring, then A is a going-down ring;*

2. *If $J \subseteq Nil(B)$, then A is a going-down ring if and only if R is a going-down ring;*

3. *Suppose that $\mathfrak{Q} \in Spec(f(A)+J)$ for each $\mathfrak{Q} \in Spec(B)\backslash V(J)$ (in particular: f is surjective). If A and $f(A)+J$ are going-down rings, then R is a going-down ring;*

4. *If $f^{-1}(J)=0$, then R is a going-down ring if and only if $f(A)+J$ is a going-down ring.*

The proof of this theorem is based on the following result.

Lemma 3.4.2 *Let A and B be two rings, J a non-zero ideal of B, $f : A \to B$ a ring homomorphism and $R = A \bowtie^f J$. Suppose that $\mathfrak{Q} \in Spec(f(A)+J)$ for each $\mathfrak{Q} \in Spec(B)\backslash V(J)$ (in particular: f is surjective), then $R/\overline{\mathfrak{Q}}^f \cong (f(A)+J)/\mathfrak{Q}$.*

Proof. Let $\mathfrak{Q} \in \mathrm{Spec}(B)\backslash V(J)$ and consider the following ring homomorphism $\varphi : R \to (f(A)+J)/\mathfrak{Q}$. Clearly, φ is surjective and $Ker(\varphi) = \overline{\mathfrak{Q}}^f$. So, by the first theorem of homomorphism $R/\overline{\mathfrak{Q}}^f \cong (f(A)+J)/\mathfrak{Q}$.

Proof of Theorem 3.4.1.

1. Suppose that R is a going-down ring and let $\mathfrak{p} \in \mathrm{Spec}(A)$. Clearly $R/(\mathfrak{p} \bowtie^f J) \cong A/\mathfrak{p}$, so A is a going-down ring.

2. Suppose that $J \subseteq \mathrm{Nil}(B)$ and A is a going-down ring. Let $\mathfrak{P} \in \mathrm{Spec}(R)$, by hypothesis $\mathfrak{P} = \mathfrak{p} \bowtie^f J$ where $\mathfrak{p} \in \mathrm{Spec}(A)$ and so $R/\mathfrak{P} \cong A/\mathfrak{p}$, as desired.

3. Suppose that $\mathfrak{Q} \in \mathrm{Spec}(f(A)+J)$ for each $\mathfrak{Q} \in \mathrm{Spec}(B)\backslash V(J)$, A and $f(A)+J$ are going-down ring. Let $\mathfrak{P} \in Spec(R)$, two cases are then possible:
   Case 1: $\mathfrak{P} = \mathfrak{p} \bowtie^f J$ where $\mathfrak{p} \in Spec(A)$, we get $R/\mathfrak{P} \cong A/\mathfrak{p}$.
   Case 2: $\mathfrak{P} = \overline{\mathfrak{p}}^f$ where $p \in Spec(B)\backslash V(J)$, by Lemma 3.4.2 we get $R/\mathfrak{P} \cong (f(A)+J)/\mathfrak{p}$. By our assumption, we conclude that R is a going-down ring.

4. Suppose $f^{-1}(J)=0$, we get then $R \cong f(A)+J$, as desired.

Similarly to the result of [22, Propostion 3.8], we get the following characterizations of going-down rings in terms of idealization and amalgamated duplication by the last corollaries.

Corollary 3.4.3 *Let A be a ring. Then the following assertions are equivalent:*

1. *$A \bowtie I$ is a going-down ring for each ideal I of A;*

2. *$A \bowtie I$ is a going-down ring for some ideal I of A;*

3. *$A \propto E$ is a going-down ring for each A-module E;*

4. *$A \propto E$ is a going-down ring for some A-module E;*

5. *A is a going-down ring.*

3.5 Gaussian rings

We recall that a local ring R is Gaussian if and only if, for each ideal I generated by two elements $a,b \in R$, the following conditions hold:

1. I^2 is generated by a^2 or b^2;

2. if I^2 is generated by a^2 and $ab=0$, then $b^2=0$.

see [82].
Let A and B be two rings, J a proper ideal of B, $f: A \to B$ a ring homomorphism and $R = A \bowtie^f J$. We note that if $J=B$, then the amalgamation degenerates in the direct product $A \bowtie^f J = A \times B$ and if $J=0$, then $A \bowtie^f J \cong A$. Also recall that $f^{-1}(J)=0$ if and only if $A \bowtie^f J$ and $f(A)+J$ are isomorphic by [37, Proposition 2.1]. Hence, to avoid these trivial cases in the next result, we may assume that "$f^{-1}(J) \neq 0$" and J is "a nonzero proper" ideal of B. By [71, Lemma 2.2], we recall that R is local if and only if so is A and $J \subseteq Jac(B)$.

Theorem 3.5.1 *Let (A,M) be a local ring, $J \subseteq Jac(B)$ a proper ideal of a ring B, $f: A \to B$ a ring homomorphism and $R = A \bowtie^f J$. Then R is Gaussian if and only if the following conditions hold:*

1. *A is Gaussian;*

2. *$(f(a)+j)J = (f(a)+j)^2J$ for each $a \in A$ and each $j \in J$;*

3. *$J^2 = 0$.*

or,

(4) A and $f(A)+J$ are Gaussian;

(5) $x^2 = 0$ for each $x \in f^{-1}(J)$;

(6) $(f(a)+j)J = (f(a)+j)^2J$ for each $a \in A$ and each $j \in J$ with $a^2 \neq 0$;

(7) For each $a \in A$ and $i,j \in J$ such that $a^2 \neq 0$ we have $i^2 = k(f(a)+j)^2$ for some $k \in J$;

(8) If $f(a^2)=0$, then $a^2=0$.

Proof. For the "if" part, A and $f(A)+J$ are Gaussian since they are factor rings of R. So, (1) and (4) hold. Now, as $f^{-1}(J) \neq 0$, two cases are possible:
Case 1: if $j^2=0$ for each $j \in J$.
Let $a \in A$ and $i,j \in J$ such that $(f(a)+j)^2 \neq 0$. We have $(0,i)^2 = (0,0)$ and $(a, f(a)+$

$j)^2 \neq (0,0)$. Then $(a, f(a)+j)(0,i) = (b, f(b)+k)(a^2, (f(a)+j)^2)$ for some $b \in A$ and $k \in J$. For some $t \in J$, we get:

$$\begin{aligned}
(f(a)+j)i &= (f(a)+j)^2(f(b)+k) \\
&= (f(a)+j)^2k + f(b)f(a^2) + 2jf(a)f(b) + f(b)j^2 \\
&= (f(a)+j)^2k + 2jf(a)f(b) \\
&= (f(a)+j)^2k + 2tf(a^2)f(b) \\
&= (f(a)+j)^2k
\end{aligned}$$

as desired. Now, if $(f(a)+j)^2 = 0$, we get $(a, f(a)+j)(0,i) = (0,0)$. It follows that $(f(a)+j)i = 0$. Finally we obtain $(f(a)+j)J = (f(a)+j)^2J$ for each $a \in A$ and $j \in J$. Thus (2) holds.

Let $i, j \in J$. By (2) and since $j^2 = 0$, we obtain $ij = tj^2 = 0$ for some $t \in J$, so (3) holds.

Case 2: there exists $x \in J$ such that $x^2 \neq 0$.

As R is a local Gaussian ring and $(a,0)(0,x) = (0,0)$ for each $a \in f^{-1}(J)$, we get $(a^2, 0) = (0,0)$ and so $a^2 = 0$ for each $a \in f^{-1}(J)$ and hence (5) holds. Now, let $a \in A$ and $i, j \in J$. By our assumption, we get:

(i) $(a, f(a)+j)(0,i) = (c, f(c)+k)(a^2, (f(a)+j)^2)$ *for some* $c \in A$ *and* $k \in J$;

(ii) $(0,i)^2 = (d, f(d)+h)(a^2, (f(a)+j)^2)$ *for some* $d \in A$ *and* $h \in J$.

and

(i') $(a, f(a))(0, j) = (c', f(c')+k')(a^2, f(a^2))$ *for some* $c' \in A$ *and* $k' \in J$;

(ii') $(0, j)^2 = (d', f(d')+h')(a^2, f(a^2))$ *for some* $d' \in A$ *and* $h' \in J$.

By (i), we get:

$$\begin{aligned}
(f(a)+j)i &= (f(a)+j)^2(f(c)+k) \\
&= (f(a)+j)^2k + f(c)f(a^2) + 2jf(a)f(c) + j^2f(c) \\
&= (f(a)+j)^2k + 2(f(c')+k')f(a^2)f(c) + (f(d')+h')f(a^2)f(c) \\
&= (f(a)+j)^2k
\end{aligned}$$

We have then $(f(a)+j)J = (f(a)+j)^2J$ for $a^2 \neq 0$. Thus (6) holds.

By (ii), (i') and (ii'), we get:

$$\begin{aligned}
i^2 &= (f(a)+j)^2(f(d)+h) \\
&= (f(a)+j)^2h + f(d)f(a^2) + 2jf(a)f(d) + j^2f(d) \\
&= (f(a)+j)^2k + 2k'f(a^2)f(d) + h'f(a^2)f(d) \\
&= (f(a)+j)^2k
\end{aligned}$$

Hence, for each $a\in A$ and $i,j\in J$ such that $a^2\neq 0$ we have $i^2=k(f(a)+j)^2$ for some $k\in J$, thus (7) holds. For (8), let $a\in A$ such that $f(a^2)=0$. Since R is Gaussian and $x^2\neq 0$, we get $(a,f(a))(0,x)=(b,f(b)+i)(0,x^2)$ and hence $a^2=0$, as desired.
Conversely, two cases arise:
Case 1: we suppose that A is Gaussian, $J^2=0$ and $(f(a)+j)J=(f(a)+j)^2J$ for each $a\in A$ and $j\in J$ and we prove that R is Gaussian. For, let $(a,f(a)+j),(b,f(b)+i)\in R$. By hypothesis, we may assume that $ab=ca^2$ and $b^2=da^2$ for some $c,d\in A$. We claim that:

(i) $(a,f(a)+j)(b,f(b)+i)=(c,f(c)+k)(a,f(a)+j)^2$

(ii) $(b,f(b)+i)^2=(d,f(d)+h)(a,f(a)+j)^2$

Indeed, for (i) we need only to show that $(f(a)+j)^2k=if(a)+jf(b)+ij-2jf(c)f(a)-j^2f(c)$. We have $if(a)+j(f(b)+i)-2jf(c)f(a)-j^2f(c)=i(f(a)+j)-jf(c)(f(a)+j)+jf(b)-jf(c)f(a)=(i-jf(c))(f(a)+j)+jf(b-ac)$. Using the fact that $(f(x)+y)^2J=(f(x)+y)J$ for each $x\in A$ and $y\in J$, we get for some $t,l\in J$:

$$\begin{array}{rcl} jf(b-ac) & = & tf((b-ac)^2) \\ & = & tf(b)f(b-ac)-tf(c)f(ab-a^2c) \\ & = & lf(b^2)f(b-ac) \\ & = & lf(da^2)f(b-ac) \\ & = & lf(da)f(ab-a^2c) \\ & = & 0 \end{array}$$

hence $if(a)+j(f(b)+i)-2jf(c)f(a)-j^2f(c)=(i-jf(c))(f(a)+j)=k(f(a)+j)^2$ for some $k\in J$. We obtain then $(f(a)+j)(f(b)+i)=(f(c)+k)(f(a)+j)^2$, as desired.

For (ii), we show that $(f(a)+j)^2h=2if(b)+i^2-2jf(d)f(a)-j^2f(d)$. By hypothesis, we need only to show that $(f(a)+j)^2h=2if(b)-2jf(d)f(a)$. We use the equality $(f(x)+y)^2J=(f(x)+y)J$ for each $x\in A$ and $y\in J$ and we obtain for some $t,l\in J$:

$$\begin{array}{rcl} 2if(b)-2jf(d)f(a) & = & 2tf(b^2)-2lf(d)f(a^2) \\ & = & 2tf(d)f(a^2)-2lf(d)f(a^2) \\ & = & 2tf(d)f(a^2)-2tf(d)j^2-2lf(d)f(a^2)+2lf(d)j^2 \\ & = & 2tf(d)(f(a^2)-j^2)-2lf(d)(f(a^2)-j^2) \\ & = & 2tf(d)(f(a)-j)(f(a)+j)-2lf(d)(f(a)-j)(f(a)+j) \\ & = & (2tf(d)(f(a)-j)-2lf(d)(f(a)-j))(f(a)+j) \end{array}$$

then $2if(b)-2jf(d)f(a)=(2tf(d)(f(a)-j)-2lf(d)(f(a)-j))(f(a)+j)$, we use the equality $(f(a)+j)J=(f(a)+j)^2J$ to get the desired result.

If $(a,f(a)+j)(b,f(b)+i)=(0,0)$, then $ab=0$ and we get $b^2=0$ since A is Gaussian. As $f(b)J=f(b^2)J$ and $J^2=0$, we obtain $(f(b)+i)^2=f(b^2)+2if(b)+i^2=f(b^2)+2tf(b^2)+i^2=0$ for some $t\in J$, thus $(b,f(b)+i)^2=0$, whence R is Gaussian.
Case 2: we suppose that the following conditions hold:

(4) A and $f(A)+J$ are Gaussian;

(5) $x^2=0$ for each $x\in f^{-1}(J)$;

(6) $(f(a)+j)J=(f(a)+j)^2J$ for each $a\in A$ and each $j\in J$ with $a^2\neq 0$;

(7) For each $a\in A$ and $i,j\in J$ such that $a^2\neq 0$ we have $i^2=k(f(a)+j)^2$ for some $k\in J$

(8) If $f(a^2)=0$, then $a^2=0$.

We show that R is Gaussian. For, let $(a,f(a)+j),(b,f(b)+i)\in R$.

-If $f(a^2)\neq 0$. Since A is Gaussian, we may assume that $ab=ca^2$ and $b^2=da^2$ for some $c,d\in A$.
So, we claim that for some $k,h\in J$ we have:

(i) $(a,f(a)+j)(b,f(b)+i)=(c,f(c)+k)(a,f(a)+j)^2$

(ii) $(b,f(b)+i)^2=(d,f(d)+h)(a,f(a)+j)^2$

For (i), we need only to show that $(f(a)+j)^2k=if(a)+jf(b)+ij-2jf(c)f(a)-j^2f(c)=i(f(a)+j)-jf(c)(f(a)+j)+jf(b)-jf(c)f(a)=(i-jf(c))(f(a)+j)+jf(b-ac)$ for some $k\in J$.
Case 2.1: if $b^2=0$, by hypothesis we have for some $p\in A$ and $q,t_1,t_2,t\in J$:

$$\begin{aligned}
jf(b-ac) &= jf(b)-jf(ac)\\
&= (f(p)+q)j^2-t_1f(a^2)f(c)\\
&= (f(p)+q)t_2(f(a)+j)^2-t_1f(a^2)f(c)+t_1f(c)j^2-t_1f(c)j^2\\
&= t_2(f(p)+q)(f(a)+j)^2-t_1f(c)(f(a)-j)(f(a)+j)-t_1f(c)t_2(f(a)+j)^2\\
&= t(f(a)+j)
\end{aligned}$$

Case 2.2: if $b^2\neq 0$, by hypothesis we have for some $t_1,t_2,t_3,t\in J$:

$$\begin{aligned}
jf(b-ac) &= jf(b)-jf(ac)\\
&= t_1f(b)^2-t_2f(a^2)f(c)\\
&= t_1f(d)f(a^2)-t_1f(d)j^2-t_2f(a^2)f(c)+t_2f(c)j^2+(t_1f(d)-t_2f(c))j^2\\
&= t_1f(d)(f(a)-j)(f(a)+j)-t_2f(c)(f(a)-j)(f(a)+j)+(t_1f(d)-t_2f(c))t_3(f(a)+\\
&= t(f(a)+j)
\end{aligned}$$

In both cases, we get $(i - jf(c) + t)(f(a) + j) = k(f(a) + j)^2$ for some $k \in J$, as desired.

For (ii), we need only to show that $(f(a) + j)^2 h = 2if(b) + i^2 - 2jf(d)f(a) - j^2 f(d)$. By hypothesis, we obtain:
$2if(b) + i^2 - 2jf(d)f(a) - j^2 f(d) = 2if(b) + i^2 - jf(d)f(a) - jf(d)(f(a) + j)$ for some $u \in J$, we have:

$$i^2 = u(f(a) + j)^2$$

for some $v, y, z \in J$, we have:

$$\begin{aligned} jf(d)f(a) &= vf(d)f(a^2) \\ &= vf(d)f(a^2) - vf(d)j^2 + vf(d)j^2 \\ &= vf(d)(f(a) - j)(f(a) + j) + vf(d)y(f(a) + j)^2 \\ &= z(f(a) + j) \end{aligned}$$

for the term $2if(b)$, two cases are possible:
Case 1.1: if $b^2 = 0$, then for some $p \in A$ and $q, t_1 \in J$:

$$\begin{aligned} 2if(b) &= 2(f(p) + q)i^2 \\ &= 2(f(p) + q)t_1(f(a) + j)^2 \end{aligned} \quad = \quad t(f(a) + j)^2$$

Case 1.2: if $b^2 \neq 0$, then for some $t_1, t, y \in J$:

$$\begin{aligned} 2if(b) &= 2t_1 f(b^2) \\ &= 2t_1 f(d)f(a^2) - 2t_1 f(d)j^2 + 2t_1 f(d)j^2 \\ &= 2t_1 f(d)(f(a) - j)(f(a) + j) + 2t_1 f(d)y(f(a) + j)^2 \\ &= t(f(a) + j) \end{aligned}$$

In both cases, $2if(b) + i^2 - 2jf(d)f(a) - j^2 f(d) = h(f(a) + j)^2$ for some $h \in J$.

-If $f(a^2) = 0$, then $f(b^2) = 0$ and so by hypothesis $a^2 = b^2 = 0$. Hence, for some $c \in A$ and $k \in J$ we have:

$$\begin{aligned} (a, f(a) + j)(b, f(b) + i) &= (ab, (f(a) + j)(f(b) + i)) \\ &= (0, (f(c) + k)(f(a) + j)^2) \\ &= (c, f(c) + k)(a, f(a) + j)^2 \end{aligned}$$

and for some $d \in A$ and $h \in J$ we have:

$$\begin{aligned} (b, f(b) + i)^2 &= (b^2, (f(b) + i)^2) \\ &= (0, (f(d) + h)(f(a) + j)^2) \\ &= (d, f(d) + h)(a, f(a) + j)^2 \end{aligned}$$

Therefore, R is Gaussian.
If $((a,f(a)+j),(b,f(b)+i)) = ((a,f(a)+j)^2)$ and $(a,f(a)+j)(b,f(b)+i) = (0,0)$, then $ab = 0$ and we get $b^2 = 0$ since A is Gaussian and $(f(b)+i)^2 = 0$ since $f(A)+J$ is Gaussian, we obtain then $(b,f(b)+i)^2 = 0$, which complete the proof.

In the sequel, we denote by $J_{T_\mathfrak{m}}$ the localization of J at the maximal ideal $T_\mathfrak{m} = f(A\backslash\mathfrak{m})+J$ for each maximal ideal $\mathfrak{m}$ containing $f^{-1}(J)$.

Corollary 3.5.2 *Let A and B be two rings, J a proper ideal of B, $f : A \to B$ a ring homomorphism and $R = A \bowtie^f J$. Then R is Gaussian if and only if the following conditions hold:*

(1) A is Gaussian;

(2) B_Q is Gaussian for each $Q \in Max(B)\backslash V(J)$;

(3) For each $a \in A_\mathfrak{m}$ and $j \in J_{T_\mathfrak{m}}$, $(f_\mathfrak{m}(a)+j)J_{T_\mathfrak{m}} = (f_\mathfrak{m}(a)+j)^2 J_{T_\mathfrak{m}}$ for each $\mathfrak{m} \in Max(A)$ containing $f^{-1}(J)$;

(4) $J^2_{T_\mathfrak{m}} = 0$.

or, for each $\mathfrak{m} \in Max(A)$ containing $f^{-1}(J)$:

(5) A is Gaussian;

(6) $f_\mathfrak{m}(A_\mathfrak{m})+J_{T_\mathfrak{m}}$ is Gaussian;

(7) B_Q is Gaussian for each $Q \in Max(B)\backslash V(J)$;

(8) $x^2 = 0$ for each $x \in f_\mathfrak{m}^{-1}(J_{T_\mathfrak{m}})$;

(9) For each $a \in A_\mathfrak{m}$ and $j \in J_{T_\mathfrak{m}}$, $(f_\mathfrak{m}(a)+j)J_{T_\mathfrak{m}} = (f_\mathfrak{m}(a)+j)^2 J_{T_\mathfrak{m}}$;

(10) For each $a \in A_\mathfrak{m}$ and $i,j \in J_{T_\mathfrak{m}}$ such that $a^2 \neq 0$ we have $i^2 = k(f(a)+j)^2$ for some $k \in J_{T_\mathfrak{m}}$

(11) If $f_\mathfrak{m}(a^2) = 0$, then $a^2 = 0$ for each $a \in A_\mathfrak{m}$.

Proof. Let $\mathfrak{M} \in Max(R)$. Three cases are then possible:
Case 1: $\mathfrak{M} = \overline{Q}^f$ where $Q \in Max(B)\backslash V(J)$, so $R_\mathfrak{M} \cong B_Q$.
Case 2: $\mathfrak{M} = \mathfrak{m} \bowtie^f J$ where $\mathfrak{m}$ is a maximal ideal of A not containing $f^{-1}(J)$, so $R_\mathfrak{M} \cong A_\mathfrak{m}$.
Case 3: $\mathfrak{M} = \mathfrak{m} \bowtie^f J$ where $\mathfrak{m}$ is a maximal ideal of A containing $f^{-1}(J)$, so $R_\mathfrak{M} \cong A_\mathfrak{m} \bowtie^{f_\mathfrak{m}} J_{T_\mathfrak{m}}$.
Now, we apply Theorem 3.5.1 to conclude since R is Gaussian if and only if $R_\mathfrak{M}$ is Gaussian for each $\mathfrak{M} \in Max(R)$

Corollary 3.5.3 *[32, Corollary 3.8] Let A be a ring, I a proper ideal of A and $R = A \bowtie I$. Then R is Gaussian if and only if A is Gaussian, $I_{\mathfrak{m}}^2 = 0$ and $aI_{\mathfrak{m}} = a^2 I_{\mathfrak{m}}$ for each $a \in m$ and for each $\mathfrak{m} \in Max(A)$ containing I.*

We close this section by giving the following example.

Example 3.5.4 *Let $A = \mathbb{Z}$, $B = \mathbb{Z}/4\mathbb{Z}$ and $J = 2\mathbb{Z}/4\mathbb{Z}$, then $R = A \bowtie^f J$ is Gaussian.*

Proof. We claim that $(f(a)+j)J = (f(a)+j)^2 J$. Indeed, clearly f is surjective then $f(a)+j \in \{\overline{0},\overline{1},\overline{2},\overline{3}\}$ for each $a \in A$ and $j \in J = \{\overline{0},\overline{2}\}$. Thus:

$$\overline{2}.\overline{2} = \overline{0} = \overline{2}^2.\overline{2}$$

$$\overline{3}.\overline{2} = \overline{6} = \overline{2} = \overline{3}^2.\overline{2}$$

as desired. We can easily check that $\mathfrak{m} = f^{-1}(J) = 2\mathbb{Z}$ which is a maximal ideal. Therefore $(f_{\mathfrak{m}}(\frac{a}{u}) + \frac{j}{t})J_{T_{\mathfrak{m}}} = (f_{\mathfrak{m}}(\frac{a}{u}) + \frac{j}{v})^2 J_{T_{\mathfrak{m}}}$. We use then Corollary 3.5.2(2) to conclude.

CHAPTER 4

STRONGLY ϕ-n-IRREDUCIBLE IDEALS

Abstract.

Let R be a commutative ring, I be an ideal of R, n be a non-null positive integer and $\phi : \mathfrak{L}(R) \to \mathfrak{L}(R) \cup \{\emptyset\}$ be a function where $\mathfrak{L}(R)$ is the set of ideals of R. In this chapter [*], we define a new generalization of strongly n-irreducible ideals called strongly ϕ-n-irreducible ideal, that are, whenever $I_1 \cap ... \cap I_{n+1} \subseteq I$ and $I_1 \cap ... \cap I_{n+1} \nsubseteq \phi(I)$ for $I_1, ..., I_{n+1}$ ideals of R, then there are n of the I_i's whose intersection is in I. We study the stability of this new concept with respect to various ring-theoretic constructions such as the trivial ring extension and the amalgamation of rings along an ideal.

4.1 Introduction

We mean by a proper ideal I of R an ideal $I \in \mathfrak{L}(R)$ with $I \neq R$. We denote the set of proper ideals of R by $\mathfrak{L}^*(R)$. Let n be a non-null positive integer and $\phi : \mathfrak{L}(R) \to \mathfrak{L}(R) \cup \{\emptyset\}$ be a function.

In [10], Anderson and Smith called a proper ideal I of R to be weakly prime if whenever $a, b \in R$ and $0 \neq ab \in I$, either $a \in I$ or $b \in I$. In [28], Bhatwadekar and Sharma defined a proper ideal I of an integral domain R to be almost prime (resp. m-almost prime) if for $a, b \in R$ with $ab \in I \setminus I^2$, (resp. $ab \in I \setminus I^m$, $m \geq 3$) either $a \in I$ or $b \in I$. In [5], Anderson and Batanieh gave a generalization of prime ideals. A proper ideal I of R is said to be ϕ-prime if for $a, b \in R$ with $ab \in I \setminus \phi(I)$, $a \in I$ or $b \in I$. Considerable work related to ϕ-prime ideals was introduced in [48]. Recall from [77] that a proper ideal I of R is a ϕ-n-absorbing primary (respectively, strongly ϕ-n-absorbing primary) ideal of R if whenever $x_1...x_{n+1} \in I \setminus \phi(I)$ for $x_1, ..., x_{n+1} \in R$ (respectively, $I_1...I_{n+1} \subseteq I \setminus \phi(I)$ for

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ideals $I_1,...,I_{n+1}$ of R) implies that either $x_1...x_n \in I$ or the product of x_{n+1} with $(n-1)$ of $x_1,...,x_n$ is in $\sqrt{I}$ (respectively, either $I_1...I_n \subseteq I$ or the product of I_{n+1} with $(n-1)$ of $I_1,...,I_n$ is in $\sqrt{I}$). If I is ϕ-n-absorbing primary (respectively, strongly ϕ-n-absorbing primary) and $\phi(I)=\emptyset$, then I is called n-absorbing primary (respectively, strongly n-absorbing primary).

In [83], Zeidi define an ideal I to be an n-irreducible ideal (respectively, a strongly n-irreducible ideal) of R if whenever $I_1 \cap I_2 \cap ... \cap I_{n+1} = I$ (respectively, $I_1 \cap I_2 \cap ... \cap I_{n+1} \subseteq I$) for each $I_1, I_2, ..., I_{n+1}$ ideals of R, then there are n of the I_i's whose intersection is I , (respectively, is contained in I). In this work, we define I to be a strongly ϕ-n-irreducible ideal if whenever $I_1 \cap ... \cap I_{n+1} \subseteq I$ and $I_1 \cap ... \cap I_{n+1} \nsubseteq \phi(I)$ for $I_1,...,I_{n+1}$ ideals of R, then there are n of the I_i's whose intersection is in I, without loss of generality, we may assume that $I_1 \cap ... \cap I_n \subseteq I$. There is no loss of generality to assume that $\phi(I) \subseteq I$ since $I - \phi(I) = I - (\phi(I) \cap I)$. In section 4.2, we show that the concepts of strongly n-irreducible ideals and of strongly ϕ-n-irreducible ideals are different in general.

4.2 Strongly ϕ-n-irreducible ideals

We start this section by the following definitions.

Definition 4.2.1 *Let R be a ring, n be a non-null positive integer and $\phi : \mathfrak{L}(R) \to \mathfrak{L}(R) \cup \{\emptyset\}$ be a function. We call a proper ideal I of R a strongly ϕ-n-irreducible ideal if whenever $I_1 \cap ... \cap I_{n+1} \subseteq I$ and $I_1 \cap ... \cap I_{n+1} \nsubseteq \phi(I)$ for $I_1,...,I_{n+1}$ ideals of R, then there are n of the I_i's whose intersection is in I, without loss of generality, we may assume that $I_1 \cap ... \cap I_n \subseteq I$.*

It is obvious that any strongly n-irreducible ideal of a ring R is a strongly ϕ-n-irreducible ideal of R. In the sequel, we show that these concepts are not equivalent in general. Also it is evident that if $\phi(0) \neq \emptyset$ then the zero ideal is a strongly ϕ-n-irreducible ideal of R for each positive integer $n \geq 1$. We use the term strongly ϕ-irreducible instead of stongly ϕ-1-irreducible.

Throughout this work, we consider $\phi(I) \subsetneqq I$ for each ideal I of a ring R.

Definition 4.2.2 *Let R be a ring and n,m two non-null positive integers. A proper ideal I is said to be almost (m,n)-irreducible ideal if whenever $I_1 \cap ... \cap I_{n+1} \subseteq I$ and $I_1 \cap ... \cap I_{n+1} \nsubseteq I^{m+1}$ for $I_1,...,I_{n+1}$ ideals of R, then there are n of the I_i's whose intersection is in I, there is no loss of generality to assume that $I_1 \cap ... \cap I_n \subseteq I$.*
Simply, an almost (m,n)-irreducible ideal is a strongly ϕ-n-irreducible ideal with $\phi(I) = I^{m+1}$.

Let I be an ideal of a ring R, n and m two non-null positive integers. The following diagram summarizes the relationship between the above concepts.

$$\begin{tikzcd}
& I\ is\ strongly\ irreducible \arrow[d] & \\
& I\ is\ strongly\ n-irreducible \arrow[dl] \arrow[dr] & \\
I\ is\ strongly\ \phi-n-irreducible & & I\ is\ almost\ (m,n)-irreducible
\end{tikzcd}$$

Next, we give an example of a strongly ϕ-n-irreducible ideal that is not strongly n-irreducible.

Example 4.2.3 *Let $R=\prod_{i=1}^{\infty}\mathbb{Z}_2$ and $I=\{(x_i)\in R|x_{2i-1}=0\ for\ all\ i\in\mathbb{N}\}$. Clearly, R is a von Neumann regular ring. Then, by [4, Example 2.3] I is not n-absorbing ideal and by [83, Proposition 3.5] I is not n-irreducible so it is not strongly n-irreducible for any positive integer n. Also, since R is von Neumann regular ring, we get $I^m=I$ for any positive integer m, hence I is an almost (m,n)-irreducible ideal.*

In what follows, we mean by $\phi\leq\psi$ where $\phi,\psi:\mathfrak{L}(R)\to\mathfrak{L}(R)\cup\{\emptyset\}$ two functions, if ϕ and ψ satisfy $\phi(I)\subseteq\psi(I)$ for each $I\in\mathfrak{L}(R)$.

Proposition 4.2.4 *Let R be a ring, I be an ideal of R, n be a non-null positive integer and $\phi:\mathfrak{L}(R)\to\mathfrak{L}(R)\cup\{\emptyset\}$ be a function.*

1. *If I is strongly ϕ-n-irreducible, then I is strongly ψ-n-irreducible for each function $\psi:\mathfrak{L}(R)\to\mathfrak{L}(R)\cup\{\emptyset\}$ such that $\phi\leq\psi$;*
2. *If I is strongly ϕ-n-irreducible, then I is strongly ϕ-p-irreducible for each positive integer $p>n$;*
3. *If I is strongly ϕ-n-irreducible for some $n\geq 1$, then there exists the least $n_0\geq 1$ such that I is strongly ϕ-n_0 -irreducible. In this case, I is strongly ϕ-n-irreducible for all $n\geq n_0$ and it is not strongly ϕ-i-irreducible for $n_0>i>0$.*

Proof. (1) Suppose that I is stronly ϕ-n-irreducible and let $\psi:\mathfrak{L}(R)\to\mathfrak{L}(R)\cup\{\emptyset\}$ be a function such that $\phi\leq\psi$. Consider $I_1\cap...\cap I_{n+1}\subseteq I$ and $I_1\cap...\cap I_{n+1}\nsubseteq\psi(I)$ for $I_1,...,I_{n+1}$ ideals of R. Clearly, $I_1\cap...\cap I_{n+1}\nsubseteq\phi(I)$. By hypothesis, there are n of the I_i's whose intersection is in I, we may assume that $I_1\cap...\cap I_n\subseteq I$, hence I is a strongly ψ-n-irreducible ideal.

(2) Suppose that I is stronly ϕ-n-irreducible, we claim that I is strongly ϕ-p-irreducible for each positive integer $p\geq n$. Indeed, let $p>n$ and let $I_1\cap...\cap I_{p+1}\subseteq I$ and $I_1\cap$

$\ldots \cap I_{p+1} \nsubseteq \phi(I)$ for $I_1,\ldots,I_{p+1}$ ideals of R. Consider $J_i = I_i$ for each $i = 1,\ldots,n$ and $J_{n+1} = I_{n+1} \cap \ldots \cap I_{p+1}$, so $J_1 \cap \ldots \cap J_{n+1} \subseteq I$ and $J_1 \cap \ldots \cap J_{n+1} \nsubseteq \phi(I)$. By our assumption, there are n of the J_i's whose intersection is in I, we may assume that $J_1 \cap \ldots \cap J_n \subseteq I$ and hence $I_1 \cap \ldots \cap I_p \subseteq J_1 \cap \ldots \cap J_n \subseteq I$, as desired.

(3) Straightforward.

Corollary 4.2.5 *Let I be an ideal of a ring R and n,m two non-null positive integers.*

1. *If I is almost (m,n)-irreducible, then I is almost (p,n)-irreducible for each $1 \leq p \leq m$;*

2. *If I is almost (m,n)-irreducible, then I is almost (m,p)-irreducible for each positive integer $p > n$.*

As follows, we present a sufficient condition to have the equivalence between the two concepts strongly irreducible ideal and strongly ϕ-irreducible ideal.

Theorem 4.2.6 *Let R be a ring, $\phi : \mathfrak{L}(R) \rightarrow \mathfrak{L}(R) \cup \{\emptyset\}$ be a function and I a proper ideal of R. If I is a strongly ϕ-irreducible ideal that is not strongly irreducible, then $I^2 \subseteq \phi(I)$. Therefore, a strongly ϕ-irreducible ideal I with $I^2 \nsubseteq \phi(I)$ is strongly irreducible.*

Proof. Assume that I is strongly ϕ-irreducible with $I^2 \nsubseteq \phi(I)$. We claim that I is strongly irreducible. Indeed, let $I_1 \cap I_2 \subseteq I$ if $I_1 \cap I_2 \nsubseteq \phi(I)$ then $I_1 \subseteq I$ or $I_2 \subseteq I$ by hypothesis. So assume that $I_1 \cap I_2 \subseteq \phi(I)$, suppose $I_1 \cap I^2 \nsubseteq \phi(I)$. Then $I_1 \cap (I_2 + I^2) \subseteq I$ and $I_1 \cap (I_2 + I^2) \nsubseteq \phi(I)$ since $(I_1 \cap I_2) + (I_1 \cap I^2) \subseteq I_1 \cap (I_2 + I^2)$. So $I_1 \subseteq I$ or $I_2 + I^2 \subseteq I$ thus $I_1 \subseteq I$ or $I_2 \subseteq I$. Hence, we may assume that $I_1 \cap I^2 \subseteq \phi(I)$. By the same way, we assume $I_2 \cap I^2 \subseteq \phi(I)$. We have $(I_1 + I^2) \cap (I_2 + I^2) \subseteq I$ and $(I_1 + I^2) \cap (I_2 + I^2) \nsubseteq \phi(I)$ since $(I_1 \cap I_2) + (I_1 \cap I^2) + (I_2 \cap I^2) + I^2 \subseteq (I_1 + I^2) \cap (I_2 + I^2)$, then $I_1 + I^2 \subseteq I$ or $I_2 + I^2 \subseteq I$ thus $I_1 \subseteq I$ or $I_2 \subseteq I$.

Corollary 4.2.7 *Let R be a ring, n be a non-null positive integer, $\phi : \mathfrak{L}(R) \rightarrow \mathfrak{L}(R) \cup \{\emptyset\}$ be a function and I a proper ideal of R. If I is a strongly ϕ-n-irreducible ideal that is not strongly n-irreducible, then $I^{n+1} \subseteq \phi(I)$. Therefore a strongly ϕ-n-irreducible ideal I with $I^{n+1} \nsubseteq \phi(I)$ is strongly n-irreducible.*

Proof. Similar to the proof of Theorem 4.2.6.

Proposition 4.2.8 *Let R be a ring and $\phi : \mathfrak{L}(R) \rightarrow \mathfrak{L}(R) \cup \{\emptyset\}$ be a function such that $\phi(J) \subseteq \phi(I)$ for each $I \subseteq J$ ideals of R. If I_i is a strongly ϕ-n_i-irreducible ideal of R for each $1 \leq i \leq m$, then $I_1 \cap \ldots \cap I_m$ is a strongly ϕ-n-irreducible ideal of R for $n = n_1 + \ldots + n_m$.*

Proof. By induction on m, it suffices to proving the result for $m = 2$. Suppose that I_1 is a strongly ϕ-n_1-irreducible ideal of R and I_2 is a strongly ϕ-n_2-irreducible ideal of R. Set $I = I_1 \cap I_2$ and $n = n_1 + n_2$. Let $J_1 \cap ... \cap J_{n+1} \subseteq I$ and $J_1 \cap ... \cap J_{n+1} \nsubseteq \phi(I)$. Clearly, $J_1 \cap ... \cap J_{n+1} \subseteq I_1$ and $J_1 \cap ... \cap J_{n+1} \nsubseteq \phi(I_1)$, likewise for the ideal I_2. By our assumption, there are n_1 (respectively, n_2) of the J_j's whose intersection is in I_1 (respectively, I_2), without loss of generality, we may assume that $J_1 \cap ... \cap J_{n_1}$ is in I_1 (respectively, $J_{n_1+1} \cap ... \cap J_{n_1+n_2}$ I_2). Therefore, $J_1 \cap ... \cap J_n \subseteq I$, which achieve the proof.

In what follows, we investigate strongly ϕ-n-irreducible ideals for various ring-theoretic constructions.

Theorem 4.2.9 *Let $f : R \to T$ be a surjective homomorphism of rings, $\phi : \mathfrak{L}(R) \to \mathfrak{L}(R) \cup \{\emptyset\}$ and $\psi : \mathfrak{L}(T) \to \mathfrak{L}(T) \cup \{\emptyset\}$ be two functions satisfy $f(\phi(I)) \subseteq \psi(f(I))$ for each ideal I of R with $\phi(I) \neq \emptyset$ and $\psi(f(I)) = \emptyset$ when $\phi(I) = \emptyset$. Let I be an ideal of R. If $f(I) \cap R$ is a strongly ϕ-n-irreducible ideal of R then $f(I)$ is a strongly ψ-n-irreducible ideal of T.*

Proof. Since f is surjective, we get then $f(J \cap R) = J$ for each ideal J of T. Suppose that $f(I) \cap R$ is a strongly ϕ-n-irreducible ideal of R where $\phi(I) \neq \emptyset$ and suppose that $J_1 \cap ... \cap J_{n+1} \subseteq f(I)$ and $J_1 \cap ... \cap J_{n+1} \nsubseteq \psi(f(I))$ for $J_1, J_2, ..., J_{n+1}$ ideals of T. We have then $(J_1 \cap R) \cap ... \cap (J_{n+1} \cap R) \subseteq f(I) \cap R$ and we claim that $(J_1 \cap R) \cap ... \cap (J_{n+1} \cap R) \nsubseteq \phi(f(I) \cap R)$. Deny, let $y \in J_1 \cap ... \cap J_{n+1}$, as f is surjective there exists $x \in J_1 \cap ... \cap J_{n+1} \cap R$ such that $y = f(x)$ so $x \in \phi(f(I) \cap R)$ and hence $y \in f(\phi(f(I) \cap R)) = \psi(f(f(I) \cap R)) = \psi(f(I))$, a contradiction. By hypothesis, we get $(J_1 \cap R) \cap ... \cap (J_n \cap R) \subseteq f(I) \cap R$. Therefore, $f((J_1 \cap R) \cap ... \cap (J_n \cap R)) \subseteq f(f(I) \cap R)$ and so $J_1 \cap ... \cap J_n \subseteq f(I)$, as desired.
For the other case, when $\phi(I) = \emptyset$, then it comes to showing that $f(I)$ is strongly n-irreducible by the same reasoning.

Remark 4.2.10 *In the case where I containing $ker(f)$ we get the same result as in Theorem 4.2.9 if I is strongly ϕ-n-irreducible since $f(I) \cap R = I$.*

Corollary 4.2.11 *Let $f : R \to T$ be a surjective homomorphism of rings. Let I be an ideal of R. If $f(I) \cap R$ is an almost (m,n)-irreducible ideal of R, then $f(I)$ is an almost (m,n)-irreducible ideal of T.*

Consider J an ideal of R and $\phi : \mathfrak{L}(R) \to \mathfrak{L}(R) \cup \{\emptyset\}$ a function. We define $\phi_J : \mathfrak{L}(R/J) \to \mathfrak{L}(R/J) \cup \{\emptyset\}$ by $\phi_J(I/J) = (\phi(I) + J)/J$ for every ideal $I \in \mathfrak{L}(R)$ contains J with $\phi(I) \neq \emptyset$ and $\phi_J(I/J) = \emptyset$ if $\phi(I) = \emptyset$.

Corollary 4.2.12 *Let I, J two ideals of a ring R where $J \subseteq I$, n be a non-null positive integer and $\phi : \mathfrak{L}(R) \to \mathfrak{L}(R) \cup \{\emptyset\}$ be a function.*

1. *If I is a strongly ϕ-n-irreducible ideal of R, then I/J is a strongly ϕ_J-n-irreducible ideal of R/J;*

2. *Suppose that $J \subseteq \phi(I)$. If I/J is a strongly ϕ_J-n-irreducible ideal of R/J, then I is a strongly ϕ-n-irreducible ideal of R.*

Proof. (1) Holds by Theorem 4.2.9.

(2) Let $I_1 \cap ... \cap I_{n+1} \subseteq I$ and $I_1 \cap ... \cap I_{n+1} \nsubseteq \phi(I)$ for some $I_1, ..., I_{n+1}$ ideals of R, then $I_1/J \cap ... \cap I_{n+1}/J \subseteq I/J$ and $I_1/J \cap ... \cap I_{n+1}/J \nsubseteq \phi(I)/J = \phi_J(I/J)$ since $J \subseteq \phi(I)$. As I/J is assumed to be strongly ϕ_J-n-irreducible ideal we get, without loss of generality, $I_1/J \cap ... \cap I_n/J \subseteq I/J$ and hence $I_1 \cap ... \cap I_n \subseteq I$, as desired.

Corollary 4.2.13 *Let I and J two ideals of a ring R, m and n two non-null positive integers where $J \subseteq I^{m+1}$. Then I is an almost (m,n)-irreducible ideal of R if and only if I/J is an almost (m,n)-irreducible ideal of R/J.*

4.3 strongly ϕ-n-irreducible ideals in trivial ring extension and amalgamation of rings

For a ring R and E an R-module, we denote by $\mathfrak{S}(E)$ the set of all submodules of E.

Definition 4.3.1 *Let R be a ring, F a submodule of an R-module E, n a non-null positive integer and $\beta : \mathfrak{S}(E) \to \mathfrak{S}(E) \cup \{\emptyset\}$. F is said to be strongly β-n-irreducible if whenever $F_1 \cap ... \cap F_{n+1} \subseteq F \backslash \beta(F)$ for $F_1, ..., F_{n+1}$ submodules of E, then there are n of the F_i's whose intersection is in F, without loss of generality, we may assume that $F_1 \cap ... \cap F_n \subseteq F$. Also, we may assume that $\beta(F) \subseteq F$ since $F \backslash \beta(F) = F \backslash (\beta(F) \cap F)$.*

In this part, we study the extension of strongly ϕ-n-irreducible ideals to the trivial ring extension. For an ideal H of the trivial ring extension of a ring R by an R-module E, we set $I_H = \{a \in R | (a,e) \in H \; for\, some\; e \in E\}$ and $F_H = \{e \in E | (a,e) \in H \; for\, some\; a \in R\}$.

Proposition 4.3.2 *Let R be a ring, E an R-module, n a non-null positive integer, $\phi : \mathfrak{L}(R) \to \mathfrak{L}(R) \cup \{\emptyset\}$ and $\beta : \mathfrak{S}(E) \to \mathfrak{S}(E) \cup \{\emptyset\}$ two functions with $\phi(0) \in \{\emptyset, 0\}$. Let $\psi : \mathfrak{L}(R \propto E) \to \mathfrak{L}(R \propto E) \cup \{\emptyset\}$ be a function satisfies:*

$$\psi(0 \propto F) = \begin{cases} 0 \propto \beta(F) & if\, \phi(0) = 0 \\ \emptyset & if\, \phi(0) = \emptyset \end{cases}$$

Then:

1. *If $0 \propto F$ is a strongly ψ-n-irreducible ideal then F is a strongly β-n-irreducible submodule of E;*

2. *If R is a domain, E is a divisible module and $\phi(0) = 0$. Then $0 \propto F$ is strongly ψ-n-irreducible if and only if F is a strongly β-n-irreducible submodule of E;*

Proof. (1) We consider $F_1 \cap ... \cap F_{n+1} \subseteq F \backslash \beta(F)$ for some submodules $F_1, ..., F_{n+1}$ of E. Then $(0 \propto F_1) \cap ... \cap (0 \propto F_{n+1}) \subseteq 0 \propto F \backslash \psi(0 \propto F)$. By hypothesis, we obtain $(0 \propto F_1) \cap ... \cap (0 \propto F_n) \subseteq 0 \propto F$ and hence $F_1 \cap ... \cap F_n \subseteq F$, as desired.

(2) The direct sense holds by the previous statement. Conversely, let $H_1 \cap ... \cap H_{n+1} \subseteq 0 \propto F \backslash \psi(0 \propto F)$. If there exists i such that $I_{H_i} \neq 0$ then $H_i = I_{H_i} \propto E$ so it's clear that $H_1 \cap ... \cap H_{i-1} \cap H_{i+1} \cap ... \cap H_{n+1} = H_1 \cap ... \cap H_{n+1} \subseteq 0 \propto F$. Now, if for all $i = 1, ..., n+1$ we have $I_{H_i} = 0$ then $H_i = 0 \propto F_{H_i}$, then $(0 \propto F_{H_1}) \cap ... \cap (0 \propto F_{H_{n+1}}) \subseteq 0 \propto F \backslash 0 \propto \beta(F)$. So $F_{H_1} \cap ... \cap F_{H_{n+1}} \subseteq F \backslash \beta(F)$, we get then $F_{H_1} \cap ... \cap F_{H_n} \subseteq F$. Therefore $(0 \propto F_{H_1}) \cap ... \cap (0 \propto F_{H_n}) \subseteq 0 \propto F$, which achieve the proof.

Example 4.3.3 *Let $R = \prod_{i=1}^{\infty} \mathbb{Z}_2$ and $I = \{(x_i) \in R | x_{2i-1} = 0 \; for \; all \; i \in \mathbb{N}\}$. We consider $\phi : \mathfrak{L}(R) \to \mathfrak{L}(R) \cup \{\emptyset\}$ a function satisfies $\phi(0) = \emptyset$ and $R \propto R$ the trivial ring extension. We set $\psi : \mathfrak{L}(R \propto R) \to \mathfrak{L}(R \propto R) \cup \{\emptyset\}$ a function satisfies $\psi(0 \propto J) = \emptyset$ for all $J \in \mathfrak{L}(R)$. As we already show in Example 4.2.3, I is not a strongly n-irreducible ideal. Therefore $O \propto F$ is not strongly n-irreducible.*

Theorem 4.3.4 *Let R be a ring, E an R-module, n a non-null positive integer and $\phi : \mathfrak{L}(R) \to \mathfrak{L}(R) \cup \{\emptyset\}$ a function. Let $\psi : \mathfrak{L}(R \propto E) \to \mathfrak{L}(R \propto E) \cup \{\emptyset\}$ be a function satisfies:*

$$\psi(I \propto F) = \begin{cases} \phi(I) \propto F & if \; \phi(I) \neq \emptyset \\ \emptyset & if \; \phi(I) = \emptyset \end{cases}$$

where I is an ideal of R and F a submodule of E satisfy $IE \subseteq F$.

1. *Then, $I \propto E$ is a strongly ψ-n-irreducible ideal of $R \propto E$ if and only if I is a strongly ϕ-n-irreducible ideal of R;*

2. *If I is a strongly ϕ-n_1-irreducible ideal of R and F is a strongly n_2-irreducible submodule of E, then $I \propto F$ is a strongly ψ-n-irreducible ideal of $R \propto E$ where $n = n_1 + n_2$.*

Proof. (1) ($\Rightarrow$) Suppose that $I \propto E$ is a strongly ψ-n-irreducible ideal of $R \propto E$ and let $I_1, ..., I_{n+1}$ ideals of R such that $I_1 \cap ... \cap I_{n+1} \subseteq I \backslash \phi(I)$. So $(I_1 \propto E) \cap ... \cap (I_{n+1} \propto E) \subseteq I \propto E \backslash \psi(I \propto E)$. By hypothesis, we get $(I_1 \propto E) \cap ... \cap (I_n \propto E) \subseteq I \propto E$ and hence $I_1 \cap ... \cap I_n \subseteq I$, as desired.

($\Leftarrow$) Suppose that I is a strongly ϕ-n-irreducible ideal of R. Let $H_1 \cap ... \cap H_{n+1} \subseteq I \propto E \backslash \psi(I \propto E)$ where $H_1, ..., H_{n+1}$ are ideals of $R \propto E$, so $I_{H_1} \cap ... \cap I_{H_{n+1}} \subseteq I \backslash \phi(I)$. Therefore, by our assumption, we obtain $I_{H_1} \cap ... \cap I_{H_n} \subseteq I$ so $H_1 \cap ... \cap H_n \subseteq (I_{H_1} \propto E) \cap ... \cap (I_{H_n} \propto E) \subseteq I \propto E$, as desired.

(2) Suppose that I is a strongly ϕ-n_1-irreducible ideal of R and F is a strongly n_2-irreducible submodule of E. Consider $n = n_1 + n_2$ and $H_1 \cap ... \cap H_{n+1} \subseteq I \propto F \backslash \psi(I \propto F)$ where $H_1, ..., H_{n+1}$ are ideals of $R \propto E$, so $I_{H_1} \cap ... \cap I_{H_{n+1}} \subseteq I \backslash \phi(I)$ and $F_{H_1} \cap ... \cap F_{H_{n+1}} \subseteq F$. Thus, by our assumption we obtain $I_{H_1} \cap ... \cap I_{H_{n_1}} \subseteq I$ and $F_{H_{n_1+1}} \cap ... \cap F_{H_n} \subseteq F$ so it's

easy to see that $H_1 \cap ... \cap H_n \subseteq I \propto F$, which complete the proof.

Our next results establishes the transfer of strongly ϕ-n-irreducible ideals in amalgamation of rings. Let H be an ideal of $R \bowtie^f J$ and set $I_H = \{a \in R | (a, f(a) + j) \in H\ for\ some\ j \in J\}$ and $J_H = \{j \in J | (a, f(a) + j) \in H\ for\ some\ a \in R\}$.

Theorem 4.3.5 *Let R and T be two rings, $f : R \to T$ a ring homomorphism, J an ideal of T, n a non-null positive integer, $\phi : \mathfrak{L}(R) \to \mathfrak{L}(R) \cup \{\emptyset\}$ a function. Let $\psi : \mathfrak{L}(R \bowtie^f J) \to \mathfrak{L}(R \bowtie^f J) \cup \{\emptyset\}$ be a function satisfies:*

$$\psi(I \bowtie^f K) = \begin{cases} (\phi(I) \bowtie^f K) & if\ \phi(I) \neq \emptyset \\ \emptyset & if\ \phi(I) = \emptyset \end{cases}$$

when $f(I)J \subseteq K$ and for $I \subseteq f^{-1}(J)$, we have:

$$\psi(I \times 0) = \begin{cases} (\phi(I) \times 0) & if\ \phi(I) \neq \emptyset \\ \emptyset & if\ \phi(I) = \emptyset \end{cases}$$

where I is an ideal of R and K a sub-ideal of J. Then:

1. *$I \bowtie^f J$ is a strongly ψ-n-irreducible ideal of $R \bowtie^f J$ if and only if I is a strongly ϕ-n-irreducible ideal of R;*
2. *Suppose that $f(I)J \subseteq K$. If I is a strongly ϕ-n_1-irreducible ideal of R and K is strongly n_2-irreducible then $I \bowtie^f K$ is a strongly ψ-n-irreducible ideal of $R \bowtie^f J$ where $n = n_1 + n_2$;*
3. *Suppose that $I \subseteq f^{-1}(J)$. If I is a strongly ϕ-n_1-irreducible ideal of R and the zero ideal of T is strongly n_2-irreducible then $I \times 0$ is a strongly ψ-n-irreducible ideal of $R \bowtie^f J$ where $n = n_1 + n_2$.*

Proof. (1) Assume that $I \bowtie^f J$ is a strongly ψ-n-irreducible ideal of $R \bowtie^f J$ and let $I_1, ..., I_{n+1}$ ideals of R satisfy $I_1 \cap ... \cap I_{n+1} \subseteq I \backslash \phi(I)$, so $(I_1 \bowtie^f J) \cap ... \cap (I_{n+1} \bowtie^f J) \subseteq I \bowtie^f J \backslash \psi(I \bowtie^f J)$. By our assumption, we obtain $(I_1 \bowtie^f J) \cap ... \cap (I_n \bowtie^f J) \subseteq I \bowtie^f J$ and hence we get the desired result $I_1 \cap ... \cap I_n \subseteq I$. Conversely, assume that I is strongly ϕ-n-irreducible and let $H_1 \cap ... \cap H_{n+1} \subseteq I \bowtie^f J \backslash \psi(I \bowtie^f J)$. Clearly, $I_{H_1} \cap ... \cap I_{H_{n+1}} \subseteq I \backslash \phi(I)$. So, by our assumption, we get $I_{H_1} \cap ... \cap I_{H_n} \subseteq I$ and hence $H_1 \cap ... \cap H_n \subseteq (I_{H_1} \bowtie^f J) \cap ... \cap (I_{H_n} \bowtie^f J) \subseteq I \bowtie^f J$, as desired.

(2) Assume that I is a strongly ϕ-n_1-irreducible ideal of R and K is a strongly n_2-irreducible ideal of T. Let $H_1 \cap ... \cap H_{n+1} \subseteq I \bowtie^f K \backslash \psi(I \bowtie^f K)$ where $n = n_1 + n_2$ and $H_1, ..., H_{n+1}$ are ideals of $R \bowtie^f J$, so $I_{H_1} \cap ... \cap I_{H_{n+1}} \subseteq I \backslash \phi(I)$ and $J_{H_1} \cap ... \cap J_{H_{n+1}} \subseteq K$. Thus, by hypothesis we get $I_{H_1} \cap ... \cap I_{H_{n_1}} \subseteq I$ and $J_{H_{n_1+1}} \cap ... \cap J_{H_n} \subseteq K$. Easily, we can show that $H_1 \cap ... \cap H_{n_1+n_2} \subseteq I \bowtie^f K$, as desired.

(3) The proof is similar to the proof of statement (2).

Consider S a multiplicative set of a ring R and $f : R \to S^{-1}R$ the natural homomorphism defined by $f(r) = \frac{r}{1}$. For each ideal I of the ring $S^{-1}R$, we consider $I^c = \{r \in R | \frac{r}{1} \in I\} = I \cap R$ and $C = \{I^c | I \textit{ is an ideal of } S^{-1}R\}$.

Proposition 4.3.6 *Let R be a ring, S be a multiplicative set of R consists of units, n a non-null positive integer, $\phi : \mathfrak{L}(R) \to \mathfrak{L}(R) \cup \{\emptyset\}$ a function. Consider $\psi : \mathfrak{L}(S^{-1}R) \to \mathfrak{L}(S^{-1}R) \cup \{\emptyset\}$ a function satisfies:*

$$\psi(S^{-1}I) = \begin{cases} S^{-1}(\phi(I)) & \textit{if } \phi(I) \neq \emptyset \\ \emptyset & \textit{if } \phi(I) = \emptyset \end{cases}$$

and $(\psi(I))^c = \phi(I^c)$.
Then there is a one-to-one correspondence between the strongly ψ-n-irreducible ideals of $S^{-1}R$ and strongly ϕ-n-irreducible ideals of R contained in C which do not meet S.

Proof. Assume that I is a strongly ψ-n-irreducible ideal of $S^{-1}R$. Let $I_1,...,I_{n+1}$ be ideals of R such that $I_1 \cap ... \cap I_{n+1} \subseteq I^c$ and $I_1 \cap ... \cap I_{n+1} \nsubseteq \phi(I^c) = (\psi(I))^c$. Then $(S^{-1}I_1) \cap ... \cap (S^{-1}I_{n+1}) = S^{-1}(I_1 \cap ... \cap I_{n+1}) \subseteq S^{-1}(I^c) = I$. Clearly, $(S^{-1}I_1) \cap ... \cap (S^{-1}I_{n+1}) \nsubseteq S^{-1}\phi(I^c) = \psi(S^{-1}I^c) = \psi(I)$. Deny, let $i \in I_1 \cap ... \cap I_{n+1}$ then $\frac{i}{1} \in (S^{-1}I_1) \cap ... \cap (S^{-1}I_{n+1}) \subseteq \psi(I)$ so $i \in (\psi(I))^c$, a contradiction. Hence, as I is strongly ψ-n-irreducible, then $(S^{-1}I_1) \cap ... \cap (S^{-1}I_n) \subseteq I$. Therefore, $I_1 \cap ... \cap I_n \subseteq I^c$. Consequently, I^c is a strongly ϕ-n-irreducible ideal of R.

Conversely, let I be a strongly ϕ-n-irreducible ideal of R such that $I \cap S = \emptyset$, so $S^{-1}I \neq S^{-1}R$. Let $I_1 \cap ... \cap I_{n+1} \subseteq S^{-1}I$ and $I_1 \cap ... \cap I_{n+1} \nsubseteq \psi(S^{-1}I)$ where $I_1,...,I_{n+1}$ are ideals of $S^{-1}R$. Hence $(I_1^c) \cap ... \cap (I_{n+1})^c = (I_1 \cap ... \cap I_{n+1})^c \subseteq (S^{-1}I)^c$ and $(I_1^c) \cap ... \cap (I_{n+1})^c \nsubseteq (\psi(S^{-1}I))^c$. Deny, let $\frac{i}{t} \in I_1 \cap ... \cap I_{n+1}$ then $\frac{i}{t} = \frac{r}{s}$ where $r \in I_1^c \cap ... \cap I_{n+1}^c \subseteq (\psi(S^{-1}I))^c$. Since $I \in C$, then $(S^{-1}I)^c = I$ we get then $(\psi(S^{-1}I))^c = \phi(I)$ so $\frac{i}{t} \in S^{-1}(\phi(I)) = \psi(S^{-1}I)$, a contradiction. Thus $(I_1)^c \cap ... \cap (I_{n+1})^c \subseteq I \backslash \phi(I)$. Therefore, $(I_1)^c \cap ... \cap (I_n)^c \subseteq I$. Hence, $I_1 \cap ... \cap I_n = S^{-1}((I_1)^c) \cap ... \cap S^{-1}((I_n)^c) \subseteq S^{-1}I$. Therefore, $S^{-1}I$ is a strongly ψ-n-irreducible ideal of $S^{-1}R$.

Now, we determine the strongly ϕ-n-irreducible ideals in the product of two, and hence any finite number of rings.

Proposition 4.3.7 *Let I_i be an ideal of a ring R_i for each $i = 1,2$ and set $R = R_1 \times R_2$. Let $\phi_i : \mathfrak{L}(R_i) \to \mathfrak{L}(R_i) \cup \{\emptyset\}$ be a function for each $i = 1,2$. Set $\phi : \mathfrak{L}(R) \to \mathfrak{L}(R) \cup \{\emptyset\}$ a function defined by:*

$$\phi(I \times J) = \begin{cases} \phi_1(I) \times \phi_2(J) & \textit{if } \phi_1(I) \neq \emptyset \textit{ and } \phi_2(J) \neq \emptyset \\ \emptyset & \textit{if } \phi_1(I) = \emptyset \textit{ or } \phi_2(J) = \emptyset \end{cases}$$

If $I\times J$ is a strongly ϕ-n-irreducible ideal of R, then I (respectively, J) is a strongly n-irreducible ideal of R_1 (respectively, R_2).

Proof. Suppose that $I\times J$ is a strongly ϕ-n-irreducible ideal of R. We claim that I (respectively, J) is a strongly n-irreducible ideal of R_1 (respectively, R_2). Deny, we may suppose that I is not strongly n-irreducible, so there are $n+1$ ideal of R_1 such that $I_1\cap\ldots\cap I_{n+1}\subseteq I$ and every intersection of n ideal among these ideals is not in I. Thus, $(I_1\times J)\cap\ldots\cap(I_{n+1}\times J)\subseteq I\times J\backslash\phi(I\times J)$ since $\phi_2(J)\subsetneq J$. As $I\times J$ is strongly ϕ-n-irreducible then $(I_1\times J)\cap\ldots\cap(I_n\times J)\subseteq I\times J$, a contradiction. Hence I is a strongly n-irreducible ideal of R_1. Likewise, we show that J is a strongly n-irreducible ideal of R_2.

Remark 4.3.8 *The other sense of the previous proposition holds by [83, Proposition 2.18].*

In view of the above proposition, we have the following corollary.

Corollary 4.3.9 *Let I_i be an ideal of a ring R_i and $\phi_i:\mathfrak{L}(R_i)\to\mathfrak{L}(R_i)\cup\{\emptyset\}$ be a function for each $1\leq i\leq m$. Consider $R=R_1\times\ldots\times R_m$.*
Let $\phi:\mathfrak{L}(R)\to\mathfrak{L}(R)\cup\{\emptyset\}$ be a function defined by:

$$\phi(I_1\times\ldots\times I_m)=\begin{cases}\prod_1^m\phi_i(I_i) & \text{if every } \phi_i(I_i)\neq\emptyset\\ \emptyset & \text{if } \phi_i(I_i)=\emptyset \text{ for some } i=1,\ldots,m\end{cases}$$

If $I_1\times\ldots\times I_m$ is a strongly ϕ-n-irreducible ideal of R, then I_i is a strongly n-irreducible ideal of R_i for each $1\leq i\leq m$.

CHAPTER 5

WHEN EVERY FINITELY PROJECTIVE IDEAL IS PROJECTIVE

Abstract.

This chapter *, studies the class of rings in which every finitely projective ideal is projective (FPP-ring for short). We examine the transfer of this property to various context of commutative ring extensions such as direct product, homomorphic image, trivial ring extension and amalgamation of rings. Our work is motivated by an attempt to generate new original classes of rings possessing this property.

5.1 Introduction

Let R be a ring and M be an R-module. Recall that M is finitely projective if, for any finitely generated sub-module N, the inclusion map $N \to M$ factors through a free module F. The notions of finitely projective modules is due to Jones who uses the term f-projective [60]. An interesting study of finitely projective modules is also done by Azumaya in [13]. It is well known that every projective module is finitely projective and any finitely generated finitely projective module is projective and also every finitely projective module is flat. The following diagram summarizes the relations between the above notions of R-modules:

$$M \text{ is projective} \Rightarrow M \text{ is finitely projective} \Rightarrow M \text{ is flat.}$$

These implications are not generally reversible, for instance the $\mathbb{Z}$-module $\mathbb{Q}$ is finitely projective that is not projective. Let F be any field, $R = \prod_{n\in\mathbb{N}} F$ and $K = \oplus_{n\in\mathbb{N}} F$, R/K is

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R-flat since R is a von Neumann regular, but R/K is not finitely projective by [60, page 1611].

In this study, we deal with rings in which every finitely projective ideal is projective and which will be called FPP-ring. Clearly, every Noetherian ring, FF-ring (that is a ring in which every flat ideal is finitely generated) and FP-ring (that is a ring in which every flat ideal is projective) are FPP-rings. FFP-rings are rings in which every flat ideal is finitely projective. The following diagram of implications summarizes the relation between the above class of rings.

$$\begin{tikzcd}
& Noetherian \arrow[d] & \\
& FF-ring \arrow[d] & \\
& FP-ring \arrow[dl] \arrow[dr] & \\
FPP-ring & & FFP-ring
\end{tikzcd}$$

For an ideal I of R, we denote by $pd_R(I)$ the usual projective dimension of I.

The aim of this chapter is to investigate the transfer of the FPP-ring property to various contexts of constructions such as direct product, homomorphic image, trivial ring extensions and amalgamation of rings.

5.2 General results

Definition 5.2.1 *An FPP-ring is a ring in which every finitely projective ideal is projective.*

Remark 5.2.2 *Let R be a ring.*

1. *R is an FPP-domain if and only if every ideal finitely projective is finitely generated;*
2. *R is a local FPP-ring if and only if every ideal finitely projective is principal.*

Our first result in this section examines the stability of FPP-ring property under direct product.

Proposition 5.2.3 *Let $(R_i)_{i=1,\ldots,n}$ be a family of commutative rings. Then $R = \prod_{i=1}^{n} R_i$ is an FPP-ring if and only if so are R_i's.*

The proof of this proposition requires the following lemma.

Lemma 5.2.4 *[65, Lemma 2.5] Let $(R_i)_{i=1;2}$ be a family of rings and $(M_i)_{i=1;2}$ be an R_i-module for $i = 1;2$. Then,*

$$pd_{R_1\times R_2}(M_1 \times M_2) = sup\{pd_{R_1}(M_1); pd_{R_2}(M_2)\}.$$

Proof of Proposition 5.2.3. the proof is done by induction on n and it suffices to check it for $n = 2$. Assume that R is an FPP-ring and let I be a finitely projective ideal of R_1, we claim that $I \times R_2$ is a finitely projective ideal of R. Indeed, let $J \times R_2$ be a finitely generated subideal of $I \times R_2$. Clearly J is a finitely generated subideal of I, then there exists a free R_1-module $F \cong R_1^p$ for some positive integer p, a morphism $h : J \to R_1^p$ and a morphism $g : R_1^p \to I$ such that $id_J = g \circ h$. Consider the morphisms $h' : J \times R_2 \to (R_1 \times R_2)^p$ defined by $h'((j,r)) = (h(j),(r)_{i=1,\dots,p})$ for every $(j,r) \in J \times R_2$ and $g' : (R_1 \times R_2)^p \to I \times R_2$ where $g'(((r_i,s_i))_{i=1,\dots,p}) = (g((r_i)_{i=1,\dots,p}),s_1)$, so for every $(j,r) \in J \times R_2$ we have:

$$\begin{aligned}(j,r) &= (g \circ h(j),r)\\ &= g'((h(j),(r)_{i=1,\dots,p}))\\ &= g' \circ h'((j,r))\end{aligned}$$

thus $id_{J\times R_2} = g' \circ h'$. From assumption, $I \times R_2$ is projective. Since $0 = pd(I \times R_2) = sup(pd(I), pd(R_2)) = pd(I)$, that is, I is a projective ideal and so R_1 is an FPP-ring. With similar arguments as previously, we show that R_2 is also an FPP-ring.

Conversely, suppose that each R_i is an FPP-ring and let $I \times J$ be a finitely projective ideal of R. We claim that I (resp.,J) is a finitely projective ideal of R_1 (resp., R_2). Indeed, let I_1 be a finitely generated subideal of I. Clearly, $I_1 \times 0$ is a finitely generated subideal of $I \times J$, so there exists a free module $F \cong (R_1 \times R_2)^p$ for some finite index p, a morphism $h : I_1 \times 0 \to (R_1 \times R_2)^p$ and a morphism $g : (R_1 \times R_2)^p \to I \times J$ such that $id_{I_1\times 0} = g \circ h$. Consider the morphisms $h' : I_1 \to R_1^p$ defined by $h'(i_1) = \pi(h(inj(i_1)))$ for every $i_1 \in I_1$ and $g' : R_1^p \to I$ where $g'((r_k)_{k=1,..,p}) = \pi'(g(inj'((r_k)_{k=1,\dots,p})))$, for every $(r_k)_{k=1,\dots,p} \in R_1^p$ where inj, inj', π and π' are defined by the following diagram:

$$\begin{tikzcd}
I_1 \arrow[r, "inj"] \arrow[ddrr, "h'"'] & I_1 \times 0 \arrow[rr, "id_{I_1\times 0}"] \arrow[dr, "h"] & & I \times J \arrow[r, "\pi'"] & I \\
& & (R_1 \times R_2)^p \arrow[ur, "g"'] \arrow[d, shift left, "\pi"] & & \\
& & (R_1)^p \arrow[u, shift left, "inj'"] \arrow[uurr, "g'"'] & &
\end{tikzcd}$$

thus $g' \circ h' = g' \circ \pi \circ h \circ inj = \pi' \circ g \circ h \circ inj = \pi \circ id_{I_1 \times 0} \circ inj = id_{I_1}$, as desired. Similarly, we show that J is also finitely projective. Therefore, I and J are projectives (as R_1 and R_2 are FPP-rings). So, by Lemma 5.2.4,

$$pd_{R_1 \times R_2}(I \times J) = sup\{pd_{R_1}(I); pd_{R_2}(J)\} = 0.$$

It follows that $I \times J$ is projective. Thus, R is an FPP-ring, as desired.

Lemma 5.2.5 *[34, Lemma 5] Let R be a subring of a ring S and let M be a flat left R-module. Assume that $S \otimes_R M$ is finitely projective over S. Then M is finitely projective.*

Our next result establishes the transfer of FPP-ring property to a particular homomorphic image.

Proposition 5.2.6 *Let R be a ring and I be a pure ideal of R such that R/I is isomorphic to a subring of R. If R is an FPP-ring, then so is R/I.*

Proof. Let J/I be a finitely projective ideal of R/I. Using the fact that the following sequence $0 \to I \to J \to J/I \to 0$ is exact and the fact that J/I and I are R-flat ideals (as I is pure), it follows that J is flat. By Lemma 5.2.5, J is a finitely projective ideal of R and so J is projective since R is an FPP-ring. Therefore, $J/I = J \otimes_R R/I$ is projective. Hence, R/I is an FPP-ring.

Remark 5.2.7 *It is worthwhile to mention that the converse of Proposition 5.2.6 does not hold even if I is finitely generated. Indeed, if I is a nonzero proper ideal of R such that R/I isomorphic to a subring R_1 of R, then R is the product of two rings. The identity element e of R_1 is a non trivial idempotent since $Ie = 0$ and I is the annihilator of e. So, $I = R(1-e) := R_2$. Consequently, if R/I is an FPP-ring and R_2 is not an FPP-ring, then R is not an FPP-ring by Proposition 5.2.3.*

5.3 Transfer of FPP-ring property to the trivial ring extension

In this section, we study the transfer of FPP-property to the trivial ring extension. Let A be a ring, I be an ideal of A, M be an A-module. All along this section, $R = A \propto M$ will denote the trivial ring extension of A by the A-module M. Recall that an ideal H of R is homogeneous if $H = I \propto N$, where I is an ideal of A and N a submodule of M. In this case $IM \subseteq N$, $I = \{a \in A | (a,m) \in H \ for\ some\ m \in M\}$ and $N = \{m \in M | (a,m) \in H \ for\ some\ a \in A\}$. A principal ideal $R(a,m)$ is homogeneous if and only if $R(a,m) = Aa \propto (Am + aM)$. In [11, Theorem 3.3], Anderson and Winders show that every ideal of R is homogeneous if and only if every principal ideal of R is homogeneous and if A is an integral domain, then every ideal of R is homogeneous if and only if M is divisible, that is $M = aM$ for all $a \in A - \{0\}$. The first result of this section establishes conditions under which some trivial extensions of rings inherit the FPP-ring property.

Theorem 5.3.1 *Let A be a ring, I be an ideal of A, M be an A-module and $R = A \propto M$. Assume that every finitely projective ideal has the form $I \propto IM$ and M is a flat module. If A is an FPP-ring, then so is R.*

The proof of the previous theorem involves the following lemmas.

Lemma 5.3.2 *[3, Theorem 9(1)] Let R be a ring, M an R-module, $I \propto N$ a homogeneous ideal of $R = A \propto M$. If I is a finitely generated ideal of A and N a finitely generated submodule of M then $I \propto N$ is finitely generated. Conversely, if $I \propto N$ is finitely generated then I is finitely generated. Assuming further that M is finitely generated, then N is finitely generated.*

Lemma 5.3.3 *Let A be a ring, I an ideal of A, M an A-module, N a submodule of M and $R = A \propto M$. If $I \propto N$ is a finitely projective ideal of R, then I is a finitely projective ideal of A.*

Proof. Suppose that $I \propto N$ is a finitely projective ideal and let K be a finitely generated subideal of I. Clearly, $K \propto KM$ is a finitely generated subideal of $I \propto N$ which is a finitely projective ideal of R. So, there exists a free R-module $F \cong R^p$ for some integer p, a morphism $f : K \propto KM \to R^p$ and a morphism $g : R^p \to I \propto N$ such that $id_{K \propto KM} = g \circ f$. Consider $h' : K \to A^p$ defined by $h'(a) = \pi(h((a,0)))$ and $g' : A^p \to I$ defined by $g'((a_q)_{q=1,\dots,p}) = \pi'(g(inj'((a_q)_{q=1,\dots,p})))$, where inj', π and π' are defined by the following diagram:

$$\begin{tikzcd}
K \arrow[r, "inj"] \arrow[ddrr, "h'"'] & K \propto KM \arrow[rr, "id_{K\propto KM}"] \arrow[dr, "h"] & & I \propto N \arrow[r, "\pi'"] & I \\
 & & R^p \arrow[ur, "g"] \arrow[d, shift left, "\pi"] & & \\
 & & A^p \arrow[u, shift left, "inj'"] \arrow[uurr, "g'"'] & &
\end{tikzcd}$$

thus $g' \circ h' = g' \circ \pi \circ h \circ inj = \pi' \circ g \circ h \circ inj = \pi' \circ id_{K \propto KM} \circ inj = id_K$, as desired.

Lemma 5.3.4 *[3, Theorem 8(2)] Let A be a ring, M an A-module, I an ideal of A and $R = A \propto M$. If $I \propto IM$ is a projective ideal of R, then I is projective. The converse is true if M is flat.*

The next lemma show that if $R = A \propto M$ is an FPP-ring, then each homogeneous finitely projective ideal has necessary the form $I \propto IM$.

Lemma 5.3.5 *Let A be a ring, M an A-module, I an ideal of A and* $R = A \propto M$*. Assume that R is an FPP-ring. If* $I \propto N$ *is a finitely projective ideal of R, then* $N = IM$.

Proof. This follows from [3, Lemma 6(1)] and the fact that R-is an FPP-ring.
Proof of Theorem 5.3.1. Assume that every finitely projective ideal has the form $I \propto IM$ and M is a flat module and A is an FPP-ring. let $I \propto IM$ be a finitely projective ideal of R. By Lemma 5.3.3, I is a finitely projective ideal of A. Since A is an FPP-ring, then I is a projective ideal of A and so $I \propto IM$ is projective by Lemma 5.3.4. Hence, R is an FPP-ring, as desired.

Remark 5.3.6 *Let* $(A, \mathscr{M})$ *be a local ring which contains at least one proper finitely projective ideal I, M an A-module such that* $\mathscr{M}M = 0$ *et* $R = A \propto M$*. Then R is never an FPP-ring since R contains no proper projective ideal and* $I \propto 0 \cong I$ *is a finitely projective ideal of R.*

As an immediate consequence of Theorem 5.3.1, we obtain the next corollary.

Corollary 5.3.7 *Let D be an integral domain, M a divisible flat module and* $R = D \propto M$*. If D is an FPP-ring, then so is R.*

As applications of Corollary 5.3.7, we provide new original examples of FPP-rings which are not Noetherian.

Example 5.3.8 *Let A be an FPP-domain,* $K = qf(A)$*, M be a nonzero K-vector space such that* $dim(M) = \infty$ *and* $R = A \propto M$*. Then:*

1. *R is an FPP-ring;*
2. *R is not Noetherian.*

Proof. (1) By Corollary 5.3.7, R is an FPP-ring.
(2) With similar arguments as [61, Theorem 3.1(1)], we show that R is not coherent. Indeed, let $0 \neq f \in M$ and $L = (A \propto M)(0, f)$. Then it is easy to see that the principal ideal L is not finitely presented; and therefore $A \propto M$ is not coherent and so $A \propto M$ is not Noetherian.

Example 5.3.9 *Let* $D = \mathbb{Z}$*,* $M = \mathbb{Q}^{\infty}$ *(that is M has an infinite dimension) and* $R = D \propto M$*. Then by Corollary 5.3.7 R is an FPP-ring which is not Noetherian (as M is a D-module that is not finitely generated).*

Remark 5.3.10 *Let A be a ring, M an A-module, N a nonzero submodule of M and $R = A \propto M$. If A is a domain, then $0 \propto N$ is a non finitely projective ideal since it is not P-flat by [31, Lemma 2.8]. Recall that M is said to be P-flat if for any $(r,m) \in R \times M$ such that $rm = 0$, $m \in (0 : r)M$. If M is flat, then M is P-flat. In the case R is an FPP-ring, $0 \propto N$ is a non finitely projective ideal since it is not projective.*

5.4 FPP-ring property in amalgamated algebra

To avoid unnecessary repetition, let us fix notation for the rest of the paper. Let (A,B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be an ideal of B. All along this section, $R := A \bowtie^f J$ will denote the amalgamation of A and B along J with respect to f. Let I be an ideal of A. Notice that $I \bowtie^f J := \{(i, f(i) + j)/i \in I, j \in J\}$ is an ideal of R. Our main result of this section establishes the transfer of FPP-ring property to $A \bowtie^f J$.

Theorem 5.4.1 *Assume that $R = A \bowtie^f J$. Then the following statements hold:*

1. *If R is an FPP-ring, then so is A;*
2. *Assume that f is injective, J is $f(A)$-projective module and every finitely projective ideal of R has the form $I \bowtie^f J$ with $f(I)J = 0$. Then R is an FPP-ring if and only if so is A.*

The proof of this theorem is based on the following lemmas which are of independent interest.

Lemma 5.4.2 *With the notation introduced at the beginning of the present section, I is a finitely generated ideal of A if and only if $I \bowtie^f f(I)J$ is a finitely generated ideal of R.*

Proof. Assume that $I =< i_1, ..., i_n >$ is a finitely generated ideal of A and let $(a, f(a) + \Sigma_{k=1}^m f(b_k)j_k) \in I \bowtie^f f(I)J$. Then:

$$\begin{aligned}(a, f(a) + \Sigma_{k=1}^m f(b_k)j_k) &= (\Sigma_{i=1}^n \alpha_i i_l, f(\Sigma_{l=1}^n \alpha_l i_l) + \Sigma_{k=1}^m f(\Sigma_{l=1}^n \beta_{k,l} i_l)j_k) \\ &= (\Sigma_{l=1}^n \alpha_l i_l, \Sigma_{l=1}^n f(\alpha_l)f(i_l)) + (0, \Sigma_{k=1}^m \Sigma_{l=1}^n f(\beta_{k,l})f(i_l)j_k) \\ &= \Sigma_{l=1}^n (\alpha_l, f(\alpha_l))(i_l, f(i_l)) + \Sigma_{l=1}^n (0, \Sigma_{k=1}^m f(\beta_k)j_k)(i_l, f(i_l)) \\ &= \Sigma_{l=1}^n [(\alpha_l, f(\alpha_l)) + (0, \Sigma_{k=1}^m f(\beta_k)j_k)](i_l, f(i_l)),\end{aligned}$$

As desired.
Conversely, suppose that $I \bowtie^f f(I)J =< (i_1, f(i_1) + j_1), ..., (i_n, f(i_n) + j_n) >$, we claim that $I =< i_1, ..., i_n >$. Indeed, let $a \in I$. Then $(a, f(a)) \in I \bowtie^f f(I)J$ and so:

$$\begin{aligned}(a, f(a)) &= \Sigma_{l=1}^n (\alpha_l, f(\alpha_l) + \beta_l)(i_l, f(i_l) + j_l) \\ &= (\Sigma_{l=1}^n \alpha_l i_l, \Sigma_{l=1}^n (f(\alpha_l) + \beta_l)(f(i_l) + j_l))\end{aligned}$$

Hence, $a = \Sigma_{l=1}^n \alpha_l i_l$, which completes the proof of the lemma.

Lemma 5.4.3 *I is a finitely projective ideal of A if and only if $I \bowtie^f J$ is finitely projective ideal of R.*

Proof. Suppose that I is a finitely projective ideal and let K be a finitely generated subideal of $I \bowtie^f J$. Clearly, $T = \{a \in A | (a, f(a) + j) \in K \; for \; some \; j \in J\}$ is a finitely generated subideal of I. From assumption, there exist a free A-module $F \cong A^n$ for some integer n and two morphisms $h : T \to A^n$ and $g : A^n \to I$ such that $id_T = g \circ h$. Consider $h' : K \to R^n$ defined by $h'((a, f(a) + j)) = (h(a), f^n(h(a)) + (j, 0, ..., 0))$ and $g' : R^n \to I \bowtie^f J$ defined by $g'(((a_i)_{i=1,...,n}, f^n((a_i)_{i=1,...,n} + (j_i)_{i=1,...,n})) = (g((a_i)_i), f(g((a_i)_i)) + \sum_i j_i)$. Now, let $(a, f(a) + j) \in K$, we have:

$$\begin{aligned}(a, f(a) + j) &= (g \circ h(a), f(g \circ h(a)) + j) \\ &= (g(h(a)), f(g(h(a))) + j) \\ &= g'((h(a), f^n(h(a)) + (j, 0, ..., 0)) \\ &= g' \circ h'((a, f(a) + j)),\end{aligned}$$

as desired.

Conversely, suppose that $I \bowtie^f J$ is a finitely projective ideal and let K be a finitely generated subideal of I. By Lemma 5.4.2, $K \bowtie^f f(K)J$ is a finitely generated subideal of $I \bowtie^f J$. So, there exist a free R-module $F \cong R^p$ for some integer p and two morphisms $h : K \bowtie^f f(K)J \to R^p$ and $g : R^p \to I \bowtie^f J$ such that $id_{K \bowtie^f f(K)J} = g \circ h$. Consider $h' : K \to A^p$ defined by $h'(a) = \pi(h(inj(a)))$ and $g' : A^n \to I$ defined by $g'((a_i)_{i=1,...,n}) = \pi'(g(inj'((a_i)_{i=1,...,n})))$ where inj, inj', π and π' are defined by the following diagram:

$$\begin{tikzcd}
K \arrow[r, "inj"] \arrow[ddrr, "h'"'] & K \bowtie^f f(K)J \arrow[rr, "id_{K \bowtie^f f(K)J}"] \arrow[dr, "h"] & & I \bowtie^f J \arrow[r, "\pi'"] & I \\
& & R^p \arrow[ur, "g"'] \arrow[d, shift left, "\pi"] & & \\
& & A^p \arrow[u, shift left, "inj'"] \arrow[uurr, "g'"'] & &
\end{tikzcd}$$

thus $g' \circ h' = g' \circ \pi \circ h \circ inj = \pi' \circ g \circ h \circ inj = \pi' \circ id_{K \bowtie^f f(K)J} \circ inj = id_K$, as desired.

Lemma 5.4.4 *The following assertions hold:*

1. *If $I \bowtie^f J$ is a projective ideal of R, then I is a projective ideal of A;*

2. *Suppose that f is injective and J is $f(A)$-projective module. If I is a projective ideal of A and $f(I)J = 0$, then so is $I \bowtie^f J$.*

Proof.

1. Let $I \bowtie^f J$ be a projective ideal of R. By the Dual Basis Lemma [50, Lemma 3.23], there exist a family of R-morphisms $(\alpha_q)_{q\in Q} \subseteq Hom(I \bowtie^f J, R)$ and a family of elements $((i_q, f(i_q)+j_q))_{q\in Q} \subseteq I \bowtie^f J$ such that for every $(i, f(i)+j) \in I \bowtie^f J$ we have:

   * $\alpha_q((i, f(i)+j)) = 0$ except for some finite number of $q \in Q$;
   * $(i, f(i)+j) = \sum_{q\in Q} \alpha_q((i, f(i)+j))(i_q, f(i_q)+j_q)$.

   For each q we consider $\psi_q : I \to A$ such that for all $i \in I$, $\psi_q(i) = \pi(\alpha_q(inj(i)))$ where $\pi : R \to A$ is the canonical surjection and $inj : A \to R$ is the natural injection. It is easy to show that ψ_q's are A-morphisms. Thus, for every $i \in I$ we obtain:

   $$\begin{aligned}(i, f(i)) &= \sum_{q\in Q} \alpha_q((i, f(i)))(i_q, f(i_q)+j_q)\\ &= \sum_{q\in Q} \alpha_q(inj(i))(i_q, f(i_q)+j_q)\end{aligned}$$

   which implies that $i = \sum_{q\in Q} \pi(\alpha_q(inj(i)))i_q = \sum_{q\in Q} \psi_q(i)i_q$. Clearly, $\psi_q(i) = 0$ except for some finite number of $q \in Q$. Hence, I is projective, as desired.

2. Let I be a projective ideal of A. By the Dual Basis Lemma [50, Lemma 3.23], there exist a family of A-homomorphisms $(\alpha_q)_{q\in Q} \subseteq Hom(I, A)$ and a family of elements $(i_q)_{q\in Q} \subseteq I$ such that for every $i \in I$ we have:

   * $\alpha_q(i) = 0$ except for some finite number of $q \in Q$;
   * $i = \sum_{q\in Q} \alpha_q(i)i_q$.

   For each q we define $\psi_q : I \bowtie^f J \to R$ such that for all $(i, f(i)+j) \in I \bowtie^f J$, $\psi_q((i, f(i)+j)) = inj(\alpha_q(\pi((i, f(i)+j))))$ where $\pi : R \to A$ is the canonical surjection and $inj : A \to R$ is the natural injection. Obviously, ψ_q's are R-homomorphisms since α_q's, π and inj are A-homomorphisms. Since J is $f(A)$-projective, there exist a family of $f(A)$-homomorphisms $(\beta_p)_{p\in P} \subseteq Hom(J, f(A))$ and a family of elements $(j_p)_{p\in P} \subseteq J$ such that for every $j \in J$ we have:

   * $\beta_p(j) = 0$ except for some finite number of $p \in P$;
   * $j = \sum_{p\in P} \beta_p(j)j_p$.

   Consider $\psi_p : I \bowtie^f J \to R$ such that for all $(i, f(i)+j) \in I \bowtie^f J$, $\psi_p((i, f(i)+j)) = inj'(\beta_p(\pi'((i, f(i)+j))))$ for every $p \in P$ where $\pi' : R \to A$ defined by $\pi'((a, f(a)+j)) = j$ for all $(a, f(a)+j) \in R$ and $inj' : f(A) \to R$ defined by $inj'(f(a)) = (a, f(a))$

which is well defined since f is injective. Clearly, for each $p \in P$ and $j \in J$ there exists $a_p^j \in A$ such that $\beta_p(j) = f(a_p^j)$. Now, let $(i, f(i)+j) \in I \bowtie^f J$. Then:

$$
\begin{aligned}
(i, f(i)+j) &= (i, f(i)) + (0, j) \\
&= (\sum_{q\in Q} \alpha_q(i) i_q, \sum_{q\in Q} f(\alpha_q(i)) f(i_q) + (0, \sum_{p\in P} \beta_p(j) j_p) \\
&= \sum_{q\in Q} (\alpha_q(i), f(\alpha_q(i)))(i_q, f(i_q)) + \sum_{p\in P} (a_p^j, \beta_p(j))(0, j_p) \\
&= \sum_{q\in Q} inj(\alpha_q(i))(i_q, f(i_q)) + \sum_{p\in P} (a_p^j, f(a_p^j))(0, j_p) \\
&= \sum_{q\in Q} inj(\alpha_q(\pi((i, f(i)+j))))(i_q, f(i_q)) + \sum_{p\in P} inj'(f(a_p^j)(0, j_p) \\
&= \sum_{q\in Q} \psi_q((i, f(i)+j))(i_q, f(i_q)) + \sum_{p\in P} inj'(\beta_p(\pi'((i, f(i)+j))))(0, j_p) \\
&= \sum_{q\in Q} \psi_q((i, f(i)+j))(i_q, f(i_q)) + \sum_{p\in P} \psi_p((i, f(i)+j))(0, j_p).
\end{aligned}
$$

One can easily check that $\psi_k((i, f(i)+j)) = 0$ except for some finite number of $k \in P \cup Q$ since so are α_q's and β_p's and the fact that f is injective. Hence, $I \bowtie^f J$ is projective, as desired.

Proof of Theorem 5.4.1.

1. Let I be a finitely projective ideal of A. By Lemma 5.4.3, $I \bowtie^f J$ is a finitely projective ideal of R. Since R is an FPP-ring, then $I \bowtie^f J$ is a projective ideal of R and so by assertion (1) of Lemma 5.4.4, I is a projective ideal of A, as desired.

2. It suffices to prove the if part. Let K be a finitely projective ideal of R. From assumption, $K = I \bowtie^f J$. Applying Lemma 5.4.3, I is a finitely projective ideal of A. Along with the hypothesis A is an FPP-ring, we get I a projective ideal. Since J is a projective module of $f(A)$, we obtain $I \bowtie^f J$ is a projective ideal by Lemma 5.4.4(2) and so R is an FPP-ring.

The next corollary is an immediate consequence of Theorem 5.4.1 on the transfer of FPP-ring property to duplications.

Corollary 5.4.5 *Let A be a ring, I an ideal of A and $R = A \bowtie I$.*

1. *If R is an FPP-ring, then so is A;*

2. *Assume that I is projective and every finitely projective ideal of R contains $0 \times I$. Then, R is an FPP-ring if and only if so is A.*

As another application of Theorem 5.4.1, we complete Theorem 5.3.1 of Section 5.3.

Corollary 5.4.6 *Let A be a ring, I be an ideal of A, M be an A-module and $R := A \propto M$ be the trivial ring extension of A by M.*

1. *If R is an FPP-ring, then so is A;*

2. *Assume that every finitely projective ideal has the form $I \propto IM$ and M is a flat module. Then R is an FPP-ring if and only if so is A.*

Proof. (1) Consider $f : A \hookrightarrow R$ the injective ring homomorphism defined by $f(a) = (a, 0)$, for every $a \in A$, $J := 0 \propto E$ be an ideal of R. Clearly, $f^{-1}(J) = 0$. Therefore, by [40, Proposition 5.1 (3)], $f(A) + J = A \propto 0 + 0 \propto E = A \propto E = B \simeq A \bowtie^f J$. Hence, by application to Theorem 5.4.1(1), we have the desired result.
(2) The result follows directly by combining assertion (1) above and Theorem 5.3.1(1).

CHAPTER 6

ON A WEAK VERSION OF S-NOETHERIANITY

Abstract.

In this chapter *, we introduce a new class of ring called w-S-Noetherian ring, which is a weak version of S-Noetherian ring property and study the transfer of this notion to various context of commutative ring extensions such as direct product, trivial ring extensions and amalgamation of rings. Furthermore, we define the concept of nonnil w-S-Noetherian ring property which is a generalization of the w-S-Noetherian domain property and establish a characterization of this notion using pullbacks.

6.1 Introduction

In [66], the authors raised the following question: If every finitely generated proper ideal of R is a Noetherian R-module, can we conclude that R is a Noetherian ring. To this purpose, they introduced the concept of weakly Noetherian ring. They defined a ring R to be a weakly Noetherian if every finitely generated proper ideal of R is a Noetherian R-module. Observe that every Noetherian ring is weakly Noetherian.

Let M be an R-module and S a multiplicative subset of R. In [8], the authors introduced the concept of "almost finitely generated" to study Querre's characterization of divisorial ideals in integrally closed polynomial rings. Later, in [6], Anderson and Dumitrescu abstracted this previous concept to any commutative ring and defined a general concept of Noetherian rings. They call R an S-Noetherian ring if each ideal of R is S-finite, i.e., for each ideal I of R, there exist an $s \in S$ and a finitely generated ideal J of R such that

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$sI \subseteq J \subseteq I$. They defined M to be S-finite if there exist an $s \in S$ and a finitely generated R-submodule F of M such that $sM \subseteq F$. Also, M is called S-Noetherian if each submodule of M is S-finite. Notice that if S is a subset of units of R, then the notions of S-Noetherian and Noetherian rings collapse and an S-finite R-module is a finitely generated R-module. Recall from [64] that R is said to be a weakly S-Noetherian ring if every S-finite proper ideal of R is an S-Noetherian R-module.

Recall from [15, 45] that a prime ideal P of R is said to be *divided* if it is comparable to every ideal of R, equivalently, if $P \subseteq (x)$ for any $x \in R \setminus P$. A ring R is called a *divided ring* if every prime ideal of R is divided. Recently, A. Badawi, in [14, 16, 18, 17, 19], has studied the following class of rings: $\mathscr{H} = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime ideal of $R\}$. If $R \in \mathscr{H}$, then R is called a *ϕ-ring*. It is easy to see that every integral domain is a ϕ-ring. An ideal I of R is said to be a *nonnil ideal* if $I \nsubseteq Nil(R)$. If I is a nonnil ideal of a ϕ-ring R, then $Nil(R) \subseteq I$. Let R be a ring with total quotient ring T such that $Nil(R)$ is a divided prime ideal of R. As in [16], we define $\phi : T(R) \longrightarrow K := R_{Nil(R)}$ such that $\phi(\frac{a}{b}) = \frac{a}{b}$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into K, and ϕ restricted to R is also a ring homomorphism from R into K given by $\phi(x) = \frac{x}{1}$ for every $x \in R$. Observe that if $R \in \mathscr{H}$, then $\phi(R) \in \mathscr{H}$, $Ker(\phi) \subseteq Nil(R)$, $Nil(T(R)) = Nil(R)$, $Nil(R_{Nil(R)}) = \phi(Nil(R)) = Nil(\phi(R)) = Z(\phi(R))$, $T(\phi(R)) = R_{Nil(R)}$ is quasilocal with maximal ideal $Nil(\phi(R))$, and $R_{Nil(R)}/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R))$ is the quotient field of $\phi(R)/Nil(\phi(R))$.

In section 6.2, we introduce the concept of "w-S-Noetherian ring" and study some basic properties which allow us to construct new original examples of w-S-Noetherian ring which are not weakly S-Noetherian. Section 6.3 is devoted to the transfer of w-S-Noetherian to trivial ring extensions and amalgamation of rings. In section 6.4, we are interested to extend the w-S-Noetherian ring property to ϕ-rings, called nonnil w-S-Noetherian ring property and we use the trivial ring extension to show that these two properties are not equivalent.

6.2 w-S-Noetherian rings

We start this section by introducing the definition of a w-S-Noetherian ring.

Definition 6.2.1 *A ring A is said to be a w-S-Noetherian ring if every S-finite prime ideal is S-Noetherian where $S \subseteq A$ is a multiplicative set. Equivalently, a ring A is a w-S-Noetherian ring if for any pair of ideals I and P such that $I \subseteq P$ and P is a S-finite prime ideal, then I is S-finite.*

Our first proposition of this section establishes some basic facts of w-S-Noetherian rings.

Proposition 6.2.2 *Let A be a ring and $S \subseteq A$ be a multiplicative set. Then:*

1. *If A is a weakly S-Noetherian ring, then A is a w-S-Noetherian ring;*

2. *Assume that A contains a regular prime element. Then A is S-Noetherian ring if and only if A is a w-S-Noetherian ring;*

3. *A is S-Noetherian if and only if A is w-S-Noetherian and each maximal ideal of A is S-finite;*

4. *A is a w-S-Noetherian ring if and only if the sub-ideals of the form PQ are S-finite for every S-finite proper prime ideal P and each prime ideal Q of A such that $Q \cap S = \emptyset$.*

Proof. (1)Straightforward.

(2) If A is S-Noetherian, then it is clear that A is w-S-Noetherian. Conversely, assume that A contains a regular prime element p and let I be a propre ideal of A. Our aim is to show that I is S-finite. Clearly, $pI \subseteq pA$ where pA is an S-finite prime ideal of A. As A is a w-S-Noetherian ring, we get pI is S-finite. Therefore, I is S-finite since $pI \cong I$, making A, an S-Noetherian ring.

(3) The direct sense holds by the definition of S-Noetherian rings and by statement (1) above. Conversely, let I be a proper ideal of A, then there exists a maximal ideal M containing I. From assumption, M is S-Noetherian and so I is S-finite, as desired.

(4) Straightforward by [6, Proposition 4].

As consequences of the previous proposition, we establish the following corollaries.

Corollary 6.2.3 *Let A be a ring, $S \subseteq A$ be a multiplicative set and X be an indeterminate over A. Then:*

1. *Assume that S is an anti-archimedean multiplicative set and A is an integral domain. Then the polynomial ring $A[X]$ is w-S-Noetherian if and only if $A[X]$ is S-Noetherian if and only if A is S-Noetherian;*

2. *If A is local with maximal ideal M such that $M^2 = 0$, then A is w-S-Noetherian.*

Proof.

1. Assume that S is an anti-archimedean multiplicative set and A is an integral domain. By Proposition 6.2.2(2) and since X is a regular prime element of $A[X]$, we have $A[X]$ is w-S-Noetherian if and only if $A[X]$ is S-Noetherian. Suppose that $A[X]$ is S-Noetherian, so by [6, Proposition 9] we have A is S-Noetherian. Conversely, suppose that $A[X]$ is S-Noetherian and let I be an ideal of A. Then $I[X]$ is an ideal of $A[X]$ and so by hypothesis $I[X]$ is S-finite, that is $sI[X] \subseteq J \subseteq I[X]$ for some

$s \in S$ and $J =< (g_j(X))_{j=1,...,n} >$ a finitely generated ideal of $A[X]$. Let $i \in I$, we have $si \in K =< (g_j(0))_{j=1,...,n} > \subseteq I$. Therefore, $sI \subseteq K \subseteq I$ and so I is S-finite, as desired.

2. Holds immediately from Proposition 6.2.2(4).

Corollary 6.2.4 *Let A be a local ring with maximal ideal M and $S \subseteq A$ a multiplicative set such that $S \cap M \neq \emptyset$. Then A is S-Noetherian if and only if A is w-S-Noetherian.*

Proof. Let $s \in S \cap M$. Thus, $sM \subseteq sR \subseteq M$ which implies that M is S-finite. Hence, the result follows immediately from Proposition 6.2.2(3).

Let A and B be two rings, $f : A \to B$ be a ring homomorphism and I be an ideal of A. We denote by I^e, the extension of I defined by the ideal of B generated by $f(I)$; and by J^c the contraction of J defined by the set of antecedents of elements of $f(A) \cap J$, that is, $J^c = \{a \in A | f(a) \in J\}$.

Proposition 6.2.5 *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and S be a multiplicative set of A such that $I^{ce} = I$ for each ideal I of B and J^c is an S-finite ideal of A for each $f(S)$-finite ideal J of B. If A is w-S-Noetherian, then B is w-$f(S)$-Noetherian.*

Proof. Assume that A is w-S-Noetherian. Let P be a $f(S)$-finite prime ideal of B. Then P^c is an S-finite prime ideal of A. Along with the hypothesis that A is a w-S-Noetherian ring, we get P^c is S-Noetherian. Now, let Q be a prime ideal disjoint from $f(S)$, so $sQ^cP^c \subseteq J \subseteq Q^cP^c$ for some $s \in S$ and some finitely generated ideal J of A. Thus $f(s)QP = f(s)Q^{ce}P^{ce} \subseteq J^e \subseteq Q^{ce}P^{ce} = QP$, and so QP is $f(S)$-finite. Finally, B is w-$f(S)$-Noetherian, as desired.

Proposition 6.2.6 *Let $A \subseteq B$ be a ring extension such that $IB \cap A = I$ for each ideal I of A and PB is a prime ideal of B for every prime ideal P of A. Consider a multiplicative set $S \subseteq A$. If B is a w-S-Noetherian ring, then so is A.*

Proof. Let P be an S-finite prime ideal of A and Q a prime ideal of A disjoint from S. We have $sP \subseteq J \subseteq P$ for some $s \in S$ and some finitely generated ideal J of A. Then $sPB \subseteq JB \subseteq PB$ and so PB is an S-finite prime ideal of B since JB is a finitely generated ideal of B. From assumption, it follows that PB is an S-Noetherian ideal and so QPB is an S-finite ideal of B. Thus,

$$wQPB \subseteq (a_1, a_2, ..., a_n)B \quad \textit{where} \;\; (a_1, a_2, ..., a_n) \subseteq QP$$

. Consequently, $wQP = wQPB \cap A \subseteq (a_1, a_2, ..., a_n)B \cap A = (a_1, a_2, ..., a_n)A$ and $(a_1, a_2, ..., a_n)A \subseteq QP$. Finally, QP is an S-finite ideal of A and so P is an S-Noetherian ideal of A, that is, A is a w-S-Noetherian ring.

We say that a ring A is a w-Noetherian ring if every finitely generated prime ideal of A is Noetherian as A-module. Observe that if S is the set of unit elements of A then A is w-S-Noetherian if and only if A is w-Noetherian. For a prime ideal M of A, we define A to be a w-M-Noetherian ring if A is a w-$A\setminus M$-Noetherian ring. The next proposition shows that every w-M-Noetherian ring is w-Noetherian.

Proposition 6.2.7 *Let A be a ring. If A is a w-M-Noetherian ring for every maximal ideal M of A, then A is a w-Noetherian ring.*

Proof. Assume that A is a w-M-Noetherian ring for every maximal ideal M of A. Let $I \subseteq P$ be two ideals of A such that P is a finitely generated prime ideal of A. Then for each maximal ideal M of A, there exist an element $s_M \in A\setminus M$ and a finitely generated subideal F_M of I such that $s_M I \subseteq F_M$. The elements s_M generate the unit ideal, thus the same is true for some finite subset $\{s_{M_1},...,s_{M_n}\}$. So, $I = (s_{M_1},...,s_{M_n})I \subseteq F_{M_1}+...+F_{M_n} \subseteq I$, and therefore $I = F_{M_1}+...+F_{M_n}$. Hence, I is finitely generated and so A is a w-Noetherian ring.

Our next proposition examines the w-S-Noetherian property under direct product.

Proposition 6.2.8 *Let $(A_i)_{i=1,..,n}$ be a family of rings, and $(S_i)_{i=1,..,n}$ be a family of multiplicative set such that $S_i \subseteq R_i$ for each $i = 1,...,n$. Set $S = \prod_{i=1}^n S_i$. Then $\prod_{i=1}^n A_i$ is a w-S-Noetherian ring if and only if A_i is a w-S_i-Noetherian ring for each $i = 1,...,n$.*

Before proving this result, we state the following lemma.

Lemma 6.2.9 *Let A_1 and A_2 be two rings, $A = A_1 \times A_2$, $(S_i)_{i=1,2}$ be two multiplicative sets such that $S_i \subseteq A_i$ for $i = 1,2$, $S = S_1 \times S_2$ and $I \times J$ be a proper ideal of A. Then:*

1. *I and J are respectively S_i-finite for $i = 1,2$ if and only if $I \times J$ is S-finite;*
2. *I and J are respectively S_i-Noetherian for $i = 1,2$ if and only if $I \times J$ is S-Noetherian.*

Proof. $(1)(\Rightarrow)$ Suppose that I and J are respectively S_i-finite for $i = 1,2$. Then:

$$sI \subseteq H \subseteq I \quad for\ some\ s \in S_1\ and\ some\ finitely\ generated\ ideal\ H\ of\ A_1$$

$$tJ \subseteq K \subseteq J \quad for\ some\ t \in S_2\ and\ some\ finitely\ generated\ ideal\ K\ of\ A_2$$

hence $(s,t)I \times J \subseteq H \times K \subseteq I \times J$ and $H \times K$ is finitely generated, that is $I \times J$ is S-finite.

$(\Leftarrow)$ Suppose that $I \times J$ is S-finite, that is, $(s,m)I \times J \subseteq H \times K \subseteq I \times J$ for some $(s,m) \in S$ and some finitely generated ideals H and K. Therefore, $sI \subseteq H \subseteq I$ and $tJ \subseteq K \subseteq J$, as desired.

$(2)(\Rightarrow)$ Assume that I and J are respectively S_i-Noetherian for $i = 1,2$. Let $H \times K \subseteq I \times J$. Then H and K are respectively S_i-finite for $i = 1,2$. Using assertion (1) above, we get $H \times K$ is S-finite, as desired.

$(\Leftarrow)$ Let $H \subseteq I$ and $K \subseteq J$. From assumption, $H \times K$ is S-finite. It follows that H and K are respectively S_i-finite for $i = 1, 2$ by assertion (1) above, which completes the proof of the lemma.

Proof of Proposition 6.2.8. By induction on n, it suffices to prove the assertion for $n = 2$. Assume that $A_1 \times A_2$ is a w-S-Noetherian ring. Without loss of generality, let P be an S_1-finite prime ideal of A_1. Then by Lemma 6.2.9(1), $P \times A_2$ is an S-finite prime ideal of $A_1 \times A_2$, thus $P \times A_2$ is S-Noetherian and so P is S_1-Noetherian by Lemma 6.2.9(2). Conversely, without loss of generality, let $P \times A_2$ be S-finite prime ideal. Hence, P is an S_1-finite prime ideal of A_1 by Lemma 6.2.9(1). Along with the hypothesis that each A_i is a w-S_i-Noetherian ring, we get P is S_1-Noetherian and so $P \times A_2$ is S-Noetherian by Lemma 6.2.9(2).

The next proposition investigates the transfer of weakly S-Noetherian property to trivial ring extension.

Proposition 6.2.10 *Let $(A, \mathscr{M})$ be a local ring, $S \subseteq A$ be a multiplicative set, $S' = S \propto M$, M an A-module such that $\mathscr{M}M = 0$ and $R = A \propto M$. Then R is a weakly S'-Noetherian ring if and only if A is a weakly S-Noetherian ring.*

The proof of the previous proposition requires the following lemma.

Lemma 6.2.11 *Let A be a ring, I an ideal of A, S be a multiplicative set of A, M an A-module and N a sub-module of M. Let $R = A \propto E$ be the trivial ring extension of A by E. Then:*

1. *If I and N are S-finite and $IM \subseteq N$, then $I \propto N$ is S'-finite;*
2. *If K is an S'-finite ideal of R, then so is $I = \{a \in A | (a, m) \in K \ for\ some\ m \in M\}$;*
3. *I is S-finite if and only if $I \propto IM$ is S'-finite;*
4. *If K is an S'-Noetherian ideal of R, then so is $I = \{a \in A | (a, m) \in K \ for\ some\ m \in M\}$;*
5. *I and N are S-Noetherian and $IM \subseteq N$ if and only if $I \propto N$ is S'-Noetherian.*

Proof. (1) Assume that I and N are S-finite. Then:

$$sI \subseteq J \subseteq I \quad where\ s \in S\ \ and\ J\ is\ finitely\ generated\ ideal$$

$$tN \subseteq F \subseteq N \quad where\ t \in S\ \ and\ F\ is\ finitely\ generated\ module$$

As $IM \subseteq N$, we get then:

$$(st, 0)I \propto N \subseteq tJ \propto F \subseteq I \propto N$$

$tJ \propto F$ is an ideal since $tJM \subseteq tIM \subseteq tN \subseteq F$ and it is finitely generated since so are tJ and F, as desired.

(2) Assume that K is an S'-finite ideal of R and let $I = \{a \in A | (a,m) \in K \ for\, some\, m \in M\}$. Then there exist $(s,m) \in S'$ and H a finitely generated ideal of R such that $(s,m)K \subseteq H \subseteq K$. Let $i \in I$, there exists $n \in M$ such that $(i,n) \in K$, we have $(s,m)(i,n) = (si, sn + im) \in (s,m)K \subseteq H$ then $si \in J = \{a \in A | (a,w) \in H \ for\, some\, w \in M\}$. Therefore, $sI \subseteq J \subseteq I$ and hence I is S-finite.

(3) The "if" part holds from the assertion (2) above. For the "only if" part, we may assume that I is S-finite. Then there exist $s \in S$ and a finitely generated ideal J of A such that $sI \subseteq J \subseteq I$. So, $(s,0)I \propto IM \subseteq J \propto JM \subseteq I \propto IM$ where $J \propto JM$ is finitely generated ideal of R. Therefore, $I \propto IM$ is S'-finite.

(4) Assume that K is an S'-Noetherian ideal of R. Let $I = \{a \in A | (a,m) \in K \ for\, some\, m \in M\}$ and P be a prime ideal disjoint from S. Then $K(P \propto M)$ is S'-finite, that is, $(s,m)K(P \propto M) \subseteq H \subseteq K(P \propto M)$ for some $(s,m) \in S'$ and some finitely generated ideal H of R. Let $i \in I$ and $p \in P$, there exists $n \in M$ such that $(i,n) \in K$, then $(s,m)(i,n)(p,0) = (sip, snp + mip) \in (s,m)K(P \propto M)$, so $sIP \subseteq J \subseteq IP$ where $J = \{a \in A | (a,w) \in H \ for\, some\, w \in M\}$. One can easily check that J is finitely generated, making IP, an S-finite ideal of A, as desired.

(5) Assume that I and N are S-Noetherian and $IM \subseteq N$. We claim that $I \propto N$ is S'-Noetherian. Indeed, let $P \propto M$ be a prime ideal disjoint from S'. We have $(I \propto N)(P \propto M) = IP \propto (IM + PN)$. By our assumption, IP and $IM + PN$ are S-finite. So, by assertion (1) above, $(I \propto N)(P \propto M)$ is S'-finite, as desired. Conversely, assume that $I \propto N$ is S'-Noetherian. Then I is S-Noetherian by assertion (4) above. Since $I \propto N$ is S'-Noetherian, then $0 \propto F$ is S'-finite where F is a sub-module of N, that is, $(s,m)0 \propto F \subseteq 0 \propto E \subseteq 0 \propto F$ for some $(s,m) \in S'$ and some finitely generated sub-module E of M, thus $sF \subseteq E \subseteq F$. Hence, F is S-finite, making N, an S-Noetherian ring.

Proof of Proposition 6.2.10. Assume that R is a weakly S'-Noetherian ring and let $I \subseteq J$ be two ideals of A such that J is S-finite. Then $I \propto 0 \subseteq J \propto 0$ such $J \propto 0$ is S'-finite by Lemma 6.2.11(1). Using the fact that R is weakly S'-Noetherian, we get $I \propto 0$ is S'-finite and so I is an S-finite ideal of A. Conversely, let J be an S'-finite ideal of R and $Q = P \propto M$ is a prime ideal of R such $Q \cap S' = \emptyset$, where P is a prime ideal of A. Our aim is to prove that QJ is a S'-finite ideal of R by Theorem 6.4.2. Set $I = \{a \in A/(a,m) \in J$, for some $m \in M\}$. By Lemma 6.2.11(2), I is an S-finite ideal of A. One can easily check that $QJ = IP \propto 0$. As A is weakly S-Noetherian, we get IP is S-finite and so $IP \propto 0$ is an S'-finite ideal of R. Finally, QJ is S'-finite, as desired.

Proposition 6.2.10 provides a new original classes of w-S'-Noetherian rings that are not weakly S'-Noetherian.

Example 6.2.12 *Let $(A, \mathscr{M})$ be a non-Noetherian local domain, $S \subseteq A \setminus \mathscr{M}$ a multiplica-*

tive subset of A, $M = (A/\mathscr{M})^{\infty}$, $S' = S \propto M$ and $R = A \propto M$.

1. *R is a w-S'-Noetherian ring;*
2. *R is not a weakly S'-Noetherian ring.*

Proof. Notice first that S' is a subset of unit elements of R. So, an S'-finite ideal is a finitely generated ideal and in this case we talk about weakly Noetherian rings.

(1) We claim that R contains no finitely generated prime ideal. Deny. Let $P \propto M$ be a finitely generated prime ideal of R such that $P \propto M = ((a_i, m_i)_{i=1,\dots,q}$ and let $m \in M$. Then $(0,m) = \sum_{i=1,\dots,q}(b_i, n_i)(a_i, m_i)$. Consequently, $m = \sum_{i=1,\dots,q}(b_i m_i + a_i n_i) = 0$ since M is non-finitely generated A-module. Therefore, it follows that R is a w-S'-Noetherian ring.

(2) Since A is a non-Noetherian domain, then by [66, Theorem 1(2)], A is a non-weakly Noetherian domain. Using Proposition 6.2.10, it follows that R is a non-weakly Noetherian ring.

Example 6.2.13 *Let R be a ring defined as in the previous example, S_1 be the multiplicative set consists of unit elements of R, T be an S_2-Noetherian ring for a multiplicative subset $S_2 \subseteq T$ and $S_3 = S_1 \times S_2$. Then $R \times T$ is a w-S_3-Noetherian ring which is not a weakly S_3-Noetherian ring (hence not S_3-Noetherian).*

Proof. This follows by Proposition 6.2.2 and [64, Proposition 5].

6.3 w-S-Noetherian ring property in trivial ring extensions and amalgamation of rings

Our first result of this section examines the transfer of w-S-Noetherian property to trivial ring extensions.

Theorem 6.3.1 *Let A be a ring, $S \subseteq A$ be a multiplicative set, and M an S-finite A-module. Set $R = A \propto M$ and $S' = S \propto E$. Then R is a w-S'-Noetherian ring if and only if one of the following assertions holds:*

1. *A does not contain S-finite prime ideal;*
2. *A is a w-S-Noetherian ring which contains at least one S-finite prime ideal and M is an S-Noetherian module.*

Proof. Suppose that R is a w-S'-Noetherian ring such that A contains at least one S-finite prime ideal and M is S-finite. First, we show that M is an S-Noetherian module. Since R contains at least one S-finite prime ideal by Lemma 6.2.11(1), we obtain $P \propto M$ is an S'-Noetherian prime ideal for some S-finite prime ideal P of A. Next, let N be a submodule of M. Clearly, $0 \propto N$ is an S'-finite ideal, that is, $(s,m)0 \propto N \subseteq 0 \propto F \subseteq 0 \propto N$ for some $s \in S$ and some finitely generated sub-module F of M, and so $sN \subseteq F \subseteq N$. Thus N is an S-finite sub-module of M. Now, let P be an S-finite prime ideal of A. As M is S-finite, $P \propto M$ is an S'-finite prime ideal by Lemma 6.2.11(1), so $P \propto M$ is S'-Noetherian and hence P is S-Noetherian by Lemma 6.2.11(5). Conversely, if A contains no S-finite prime ideal then so is R by Lemma 6.2.11(2). Now, assume that A is a w-S-Noetherian ring which contains at least one S-finite prime ideal and M is an S-Noetherian module. Let $P \propto M$ be an S'-finite prime ideal of R. By Lemma 6.2.11(2), P is S-finite and so P is S-Noetherian. By Lemma 6.2.11(5), $P \propto M$ is S'-Noetherian. It follows that R is a w-S'-Noetherian ring.

Remark 6.3.2 *Under the notations of Theorem 6.3.1. Observe that if R is a w-S'-Noetherian ring which contains at least one S'-finite prime ideal, then M is necessary an S-finite A-module. Indeed, let $P \propto M$ be an S'-finite prime ideal of R. Thus $P \propto M$ is S'-Noetherian since R is a w-S'-Noetherian ring. As $0 \propto M \subseteq P \propto M$, we get that $0 \propto M$ is S'-finite and so M is S-finite.*

The next corollary is an immediate consequence of Theorem 6.3.1 which gives a sufficient and necessary condition for a ring A to be S-Noetherian.

Corollary 6.3.3 *Let A be a non-trivial w-S-Noetherian ring and $S \subseteq A$ be a multiplicative set. Then A is S-Noetherian if and only if $A \propto A$ is w-S-Noetherian.*

The next result establishes the transfer of the w-S-Noetherian property to trivial ring extension in the special case of local setting.

Proposition 6.3.4 *Let A be a local ring with maximal ideal $\mathscr{M}$, $S \subseteq A$ a multiplicative set, and M a nonzero A-module such that $\mathscr{M}M = 0$. Set $R = A \propto M$ and $S' = S \propto E$. Then the following assertions hold:*

1. *If M is not S-finite, then R is a w-S'-Noetherian ring;*
2. *Assume that M is S-finite. Then R is a w-S'-Noetherian ring if and only if A is a w-S-Noetherian ring.*

Proof. (1) Observe that S consists of unit elements since M is not S-finite. Our aim is to prove that R contains no S-finite prime ideal. Deny. let $P \propto M$ be an S'-finite prime ideal of R. So, $P \propto M$ is a finitely generated ideal of R since S' consists of unit elements of R. Hence $P \propto M = ((a_i, m_i)_{i=1,\dots,n}$ and let $m \in M$. Then, $(0, m) = \sum_{i=1,\dots,n}(b_i, n_i)(a_i, m_i)$ which implies that $m = \sum_{i=1,\dots,n}(b_i m_i + a_i n_i)$. As $a_i \in \mathscr{M}$ for $i = 1, \dots, n$ we get that $m = \sum_{i=1,\dots,n} b_i m_i$ which is a contradiction since M is a nonzero non-finitely generated A-module.
(2) By Theorem 6.3.1, we only need to prove that if A is a w-S-Noetherian ring, along with the hypothesis that $\mathscr{M}M = 0$, then R is a w-S'-Noetherian ring. Let $P \propto M$ be an

S'-finite prime ideal of R. By Lemma 6.2.11(2), P is an S-finite prime ideal of A. Let $Q \propto M$ be a prime ideal disjoint from S', by hypothesis P is S-Noetherian and so PQ is S-finite. As $(P \propto M)(Q \propto M) = PQ \propto 0$, $(P \propto M)(Q \propto M)$ is S'-finite by Lemma 6.2.11(1) and so $P \propto M$ is S'-Noetherian, as desired.

The class of w-S-Noetherian rings is not always stable under localization, as shown by the next example:

Example 6.3.5 *Let $A := \mathbb{Z}_{(2)} + X\mathbb{R}[[X]]$ be a local ring with maximal ideal $\mathscr{M} = 2\mathbb{Z}_{(2)} + X\mathbb{R}[[X]]$, M be an $A/\mathscr{M}$-vector space with infinite rank and let $R := A \propto M$ be the trivial ring extension of A by M. Let S_1 be the multiplicative subset of A given by $S_1 := \{2^n/n \in \mathbb{N}\}$, the multiplicative subset of R given by $S_2 := \{(2,0)^n \mid n \in \mathbb{N}\}$ and $S = S_1 \propto E$. Then:*

1. *R is a w-Noetherian ring;*
2. *R is not a w-S-Noetherian ring;*
3. *$S_2^{-1}(R)$ is not a w-Noetherian ring.*

Proof. (1) This follows immediately from Proposition 6.3.4 since M is S-finite.
(2) Since X is a regular prime element of A, by Proposition 6.2.2(2), A is w-S_1-Noetherian if and only if A is S_1-Noetherian. But A is not an S_1-Noetherian ring by [26, Theorem 3.8] since $\mathbb{R}$ is not S_1-finite as $\mathbb{Z}_{(2)}$-module. Moreover, M is an S_1-finite. Indeed, since $2 \in M \cap \mathscr{M}$, then $2M = 0 \subseteq M$. Thus, we conclude that R is not an w-S-Noetherian by Proposition 6.3.4.
(3) Clearly, $S_1^{-1}(M) = 0$ since $2M = 0$ and $2 \in S_1$. Hence $S_2^{-1}(0 \propto M) = 0$ and so $S_2^{-1}(R) = \{\frac{(x,0)}{((s,0)} \mid x \in A, s \in S_1\}$ which is isomorphic to the ring $S_1^{-1}(A)$. But $S_1^{-1}(A) = S_1^{-1}(\mathbb{Z}_{(2)} + X\mathbb{R}[[X]]) = \mathbb{Q} + X\mathbb{R}[[X]]$ which is not a Noetherian domain by [36, Corollary 5.9] since R is not finitely generated as $\mathbb{Q}$-module. Finally, $S_2^{-1}(R)$ is not a w-Noetherian ring by Proposition 6.2.2(2).

Let A and B be two rings, $f : A \to B$ a ring homomorphism and $S \subseteq A$ a multiplicative set. In what follows, without loss of generality, we may assume that A contains at least one S-finite prime ideal. Next, we study the transfer of the w-S-Noetherian property to amalgamation of rings.

Theorem 6.3.6 *Let A, B, f and S defined as above, and let J be an S-finite A-module, $S' = S \bowtie^f 0$ and $R = A \bowtie^f J$.*

1. *If R is a w-S'-Noetherian ring, then A is a w-S-Noetherian ring;*
2. *Suppose that $J \subseteq Nil(B)$. Then R is a w-S'-Noetherian ring if and only if A is a w-S-Noetherian ring and J is S-Noetherian as A-module.*

The proof of this previous theorem requires the following lemma.

Lemma 6.3.7 *Let A, B, f and S defined as above and let J be an ideal of B, $S' = S \bowtie^f 0$ and $R = A \bowtie^f J$.*

1. *If I is an S-finite ideal of A and J is an S-finite A-module, then $I \bowtie^f J$ is S'-finite;*
2. *If K is an S'-finite ideal of R, then $I = \{a \in A | (a, f(a) + j) \in K \; for \; some \; j \in J\}$ is an S-finite ideal of A;*
3. *I is an S-finite ideal of A if and only if $I \bowtie^f f(I)J$ is S'-finite of R;*
4. *If K is an S'-Noetherian ideal of R, then $I = \{a \in A | (a, f(a) + j) \in K \; for \; some \; j \in J\}$ is S-Noetherian;*
5. *If $I \bowtie^f J$ is S'-Noetherian, then I is S-Noetherian and J is S-Noetherian A-module. The converse holds if $J \subseteq Nil(B)$.*

Proof. (1) Assume that I is S-finite and J is S-finite as A-module. Then:

$$sI \subseteq T \subseteq I \;\; for \; some \; s \in S \; and \; some \; finitely \; generated \; ideal \; T \; of \; A$$

and

$$f(t)J \subseteq H \subseteq J \;\; for \; some \; t \in S \; and \; some \; finitely \; generated \; ideal \; H \; of \; B.$$

Hence,

$$(st, f(st))I \bowtie^f J \subseteq tT \bowtie^f H \subseteq I \bowtie^f J.$$

Thus, $I \bowtie^f J$ is S'-finite.

(2) Assume that K is S'-finite. So, $(s, f(s))K \subseteq F \subseteq K$ for some $s \in S$ and some finitely generated ideal F of R. One can easily show that $sI \subseteq T \subseteq I$ where $I = \{a \in A | (a, f(a) + j) \in K \; for \; some \; j \in J\}$ and $T = \{a \in A | (a, f(a) + j) \in F \; for \; some \; j \in J\}$, making I, an S-finite ideal of A.

(3) The "if" part holds from assertion (2) above. For the "only if" part, assume that I is S-finite. Then there exist $s \in S$ and a finitely generated ideal T of A such that $sI \subseteq T \subseteq I$. So, $(s, f(s))I \bowtie^f f(I)J \subseteq T \bowtie^f f(T)J \subseteq I \bowtie^f f(I)J$ where $T \bowtie^f f(T)J$ is finitely generated ideal of R. Therefore, $I \bowtie^f f(I)J$ is S'-finite.

(4) Assume that K is S'-Noetherian. We claim that $I = \{a \in A | (a, f(a) + j) \in K \; for \; some \; j \in J\}$ is S-Noetherian. Indeed, let P be a prime ideal disjoint from S, we have $(P \bowtie^f J)K$ is S'-finite. By assertion (2) above, we get PI is S-finite, as desired.

(5) Assume that $I \bowtie^f J$ is S'-Noetherian. By assertion (4) above, I is an S-Noetherian ideal of A. We claim that J is S-Noetherian. Indeed, let $G \subseteq J$. From assumption, $0 \times G$ is S'-finite, so $(s, f(s))0 \times G \subseteq 0 \times F \subseteq 0 \times G$ for some $s \in S$ and some finitely generated ideal F of B. Thus, $f(s)G \subseteq F \subseteq G$. Conversely, assume that I is S-Noetherian, $J \subseteq Nil(B)$ and J is $f(S)$-Noetherian. We prove that $I \bowtie^f J$ is S'-Noetherian. Let $(P \bowtie^f J)$ be a prime ideal such that $P \bowtie^f J \cap S' = \emptyset$. We have $(I \bowtie^f J)(P \bowtie^f J) = IP \bowtie^f (f(I)J + f(P)J + J^2)$. By assertion (1) above, we have the desired result.

Proof of Theorem 6.3.6. (1) Assume that R is a w-S'-Noetherian ring and let P be an S-finite prime ideal of A. As J is S-finite, we get $P \bowtie^f J$ is an S'-finite prime ideal by Lemma 6.3.7(1). Therefore, $P \bowtie^f J$ is S'-Noetherian and so P is S-Noetherian by Lemma 6.3.7(5).

(2) Assume that R is a w-S'-Noetherian ring and $J \subseteq Nil(B)$. Then by assertion (1) above, A is a w-S-Noetherian ring. Since A contains at least one S-finite prime ideal P and J is an S-finite A-module, then $P \bowtie^f J$ is S'-finite by Lemma 6.3.7(1), and so it is an S'-Noetherian ideal of R. Hence, by Lemma 6.3.7(5), J is an S-Noetherian A-module. Conversely, assume that A is a w-S-Noetherian ring, $J \subseteq Nil(B)$ is an S-Noetherian A-module and let $P \bowtie^f J$ be an S'-finite prime ideal of R. So, P is S-finite by Lemma 6.3.7(2), we obtain then P is S-Noetherian. Using the fact that J is S-Noetherian A-module and by Lemma 6.3.7(5), it follows that $P \bowtie^f J$ is an S'-Noetherian prime ideal of R, making R, a w-S'-Noetherian ring.

Theorem 6.3.6 enriches the current literature with a new original class of w-S-Noetherian rings that are not Noetherian.

Example 6.3.8 *Let (A_1, m_1) be a local ring such that $m_1^2 = 0$ (for instance take $A_1, m_1 := \mathbb{Z}_4, 2\mathbb{Z}_4$), $E = (A_1/m_1)^\infty = (\mathbb{Z}_2)^\infty$ be an (A_1/m_1)-vector space. Consider $A := A_1 \propto E$ be the trivial ring extension of A_1 by E, $f = id_A$ and $J := m_1 \propto E$ be an ideal of A. Observe that $J \subseteq Nil(A)$. Let $S := \{(1,0)^n\}$ be a multiplicative subset of A. Then:*

1. *$A \bowtie J$ is w-S-Noetherian;*
2. *$A \bowtie J$ is not Noetherian.*

Proof. (1) Notice first that A is local with maximal ideal $M := m_1 \propto E$ such that $M^2 = 0$. Then by Corollary 6.2.3(2), A is w-S-Noetherian. On the other hand, one can easily check that J is an S-Noetherian A-module. Hence, by Theorem 6.3.6, it follows that $A \bowtie J$ is w-S-Noetherian.
(2) We claim that $A \bowtie J$ is not Noetherian. Indeed, A is not Noetherian since E is not a finitely generated A-module. Therefore, by [36, Proposition 5.6], $A \bowtie J$ is not Noetherian.

6.4 nonnil w-S-Noetherian rings

In this section, we introduce the concept of nonnil w-S-Noetherian rings and study this property to various context of commutative extensions.

Definition 6.4.1 *Let $A \in \mathscr{H}$ be a ring and $S \subseteq A$ a multiplicative set. A is called a nonnil w-S-Noetherian ring if for any pair of nonnil ideals I and P of A such that $I \subseteq P$ and P is an S-finite prime ideal, we have I is S-finite.*

The first result of this section establishes a characterization of nonnil w-S-Noetherian property.

Theorem 6.4.2 *Let $A \in \mathscr{H}$ be a ring and $S \subseteq A$ a multiplicative set. Set $S' = \frac{S}{Nil(A)} = \{s + Nil(A) \mid s \in S\}$. Then A is a nonnil w-S-Noetherian ring if and only if $\frac{A}{Nil(A)}$ is a w-S'-Noetherian domain.*

Proof. Assume that A is a nonnil w-S-Noetherian ring and let $\frac{I}{Nil(A)} \subseteq \frac{P}{Nil(A)}$ be a pair of ideals of A such that $\frac{P}{Nil(A)}$ is an S'-finite prime ideal of $\frac{A}{Nil(A)}$. We may assume that $\frac{I}{Nil(A)} \neq 0$. Hence, $I \subseteq P$ is a pair of nonnil ideals of A. Moreover, there exist $s' \in S'$ and a finitely generated ideal $\frac{J}{Nil(A)}$ of $\frac{A}{Nil(A)}$ such that $s'\frac{P}{Nil(A)} \subseteq \frac{J}{Nil(A)} \subseteq \frac{P}{Nil(A)}$ and $s' = s + Nil(A)$ for some $s \in S$. Therefore, $\frac{J}{Nil(A)} = (a_1 + Nil(A), ..., a_m + Nil(A))$ for some $a_1, ..., a_m$ elements of J. Let a be a nonnilpotent element of J. Then $a + Nil(A) = b_1a_1 + ... + b_ma_m + Nil(A)$ in $\frac{A}{Nil(A)}$ for some b_n's in A. Thus, there exists $t \in Nil(A)$ such that $a + t = b_1a_1 + ... + b_ma_m$ in A. Since a is nonnilpotent, $t = ad$ for some $d \in Nil(A)$. So, $a + t = a + ad = a(1 + d) = b_1a_1 + ... + b_ma_m$. Using the fact that $d \in Nil(A)$, $1 + d$ is a unit of A. Hence, $a \in (a_1, ..., a_m)$ and so J is a finitely generated ideal of A. One can easily check that $sP \subseteq J \subseteq P$ and so P is an S-finite prime ideal of A. Consequenlty, I is S-finite since A is a nonnil w-S-Noetherian ring. It follows that $\frac{I}{Nil(A)}$ is an S'-finite ideal of $\frac{A}{Nil(A)}$. Therefore, $\frac{A}{Nil(A)}$ is a w-S'-Noetherian domain. Conversely, let $I \subseteq P$ be a pair of nonnil ideal of A such that P is an S-finite prime ideal of A. Then $\frac{I}{Nil(A)} \subseteq \frac{P}{Nil(A)}$ is a pair of nonzero ideals of $\frac{A}{Nil(A)}$ and clearly $\frac{P}{Nil(A)}$ is an S'-finite prime ideal of $\frac{A}{Nil(A)}$. Therefore, $\frac{I}{Nil(A)}$ is S'-finite since $\frac{A}{Nil(A)}$ is a w-S'-Noetherian domain. Finally, I is S-finite, making A, a nonnil w-S-Noetherian ring.

The following example shows that the class of nonnil w-S-Noetherian rings and the class of w-S-Noetherian rings are not equivalent.

Example 6.4.3 *Let $(A, Nil(A))$ be a non-trivial local w-Noetherian ring which is not Noetherian. Then $R = A \propto A$ is a non-nil w-Noetherian ring which is not w-Noetherian.*

Proof. R is a non-nil w-Noetherian ring since it contains no non-nil prime ideal. By Corollary 6.3.3, R is not w-Noetherian.

Our next proposition gives a characterization of nonnil w-S-Noetherian rings.

Proposition 6.4.4 *Let $A \in \mathscr{H}$ be a ring and $S \subseteq A$ be a multiplicative set. Then A is a nonnil w-S-Noetherian ring if and only if $\phi(A)$ is a nonnil w-$\phi(S)$-Noetherian ring.*

Proof. Set $D = \frac{A}{Nil(A)}$. Assume that A is a nonnil w-S-Noetherian ring. By Theorem 6.4.2, D is a w-S'-Noetherian domain with $S' = \frac{S}{Nil(A)}$. From [19, Lemma 1.1], D is ring-isomorphic to $\frac{\phi(A)}{Nil(\phi(A))}$ and therefore $\frac{\phi(A)}{Nil(\phi(A))}$ is a w-$\frac{\phi(S)}{Nil(\phi(S))}$-Noetherian domain. Hence, $\phi(A)$ is a nonnil w-$\phi(S)$-Noetherian ring by Theorem 6.4.2. The converse holds using similar argument as above.

Now, we give a necessary and sufficient condition for a w-S-Noetherian ring to be nonnil w-S-Noetherian.

Proposition 6.4.5 *Let $A \in \mathscr{H}$ be a ring which has a nonnil S-finite prime ideal and $S \subseteq A$ be a multiplicative set. Then A is a w-S-Noetherian ring if and only if A is a nonnil w-S-Noetherian ring and every nil ideal is S-finite.*

Proof. Straightforward by the definitions of a w-S-Noetherian ring and a nonnil w-S-Noetherian ring.

We close this section by the following pullback characterization of nonnil w-S-Noetherian rings.

Theorem 6.4.6 *Let $A \in \mathscr{H}$ and $S \subseteq A$ a multiplicative set. Then A is a nonnil w-S-Noetherian ring if and only if $\phi(A)$ is ring-isomorphic to a ring R obtained from the following pullback diagram:*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & B \arrow[d] \\ T \arrow[r] & T/M \end{tikzcd}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, $B := R/M$ is a w-S_1-Noetherian subring of T/M where $S_1 = \alpha(\phi(S))/M$ such that α is the ring isomorphism from $\phi(A)$ to R, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Proof. Suppose that $\phi(A)$ is ring-isomorphic to a ring R obtained from the given diagram. Then $R \in \mathscr{H}$ and $Nil(R) = Z(R) = M$. Since R/M is a w-S_1-Noetherian domain, R is a nonnil w-S_2-Noetherian ring by Theorem 6.4.2, where $S_2 = \alpha(\phi(S))$, and thus $\phi(A)$ is a nonnil w-$\phi(S)$-Noetherian ring. Hence A is a nonnil w-S-Noetherian ring. Conversely, assume that A is a nonnil w-S-Noetherian ring. Set $T = A_{Nil(A)}$, $M = Nil(A_{Nil(A)})$, and $R = \phi(A)$ yields the desired pullback diagram.

Perspectives

To end this thesis, we introduce the following perspectives:

1. We look for proving the equivalence between the class of $(*)$-rings and strongly $(*)$-rings or finding an example of a $(*)$-ring which is not a strongly $(*)$-ring.

2. We aim to study and characterize the class of rings in which every strongly ϕ-n-irreducible ideal is strongly ϕ-$(n-1)$-irreducible.

3. We aim to charateriz the amalgamation of rings in the case where each ideal is homogeneous or has the form $\overline{K}^f$.

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