N d'ordre : ..../23

Université Sultan Moulay Slimane
Faculté des Sciences et Techniques

Laboratoire de Mathématiques Appliquées et Calcul Scientifique

Centre d'Études Doctorales: Sciences et Techniques

Formation Doctorale: Mathématiques et Physique Appliquées

THÈSE

Présenté par

ARHRRABI EL HOUSSAIN

Pour l'obtention du grade de

DOCTEUR

Discipline: Mathématiques

Spécialité: Mathématiques

Contribution to fuzzy fractional stochastic differential equations

---

Soutenue le Samedi 17 Juin 2023 à 11h30 devant la commission d'examen :

Pr. Khalid HILAL	FST, USMS Béni Mellal	Président
Pr. Elhoussine AZROUL	FS, USMBA Fès	Rapporteur
Pr. Jaouad EL-MEKKAOUI	EST, USMBA Fès	Rapporteur
Pr. Chakir ALLALOU	FST, USMS Béni Mellal	Rapporteur
Pr. Jalila EL GHORDAF	CRMEF Béni Mellal	Examinateur
Pr. Dris BAKHADACH	ENSA, USMS Béni Mellal	Invité
Pr. M'hamed EL OMARI	FP, USMS Béni Mellal	Co-Encadrant
Pr. Lalla Saadia CHADLI	FST, USMS Béni Mellal	Encadrant

Dedication

This thesis is dedicated to

- My mother
- My father
- My sister
- My brothers

Acknowledgments

In the name of Allah, the Most Gracious and the Most Merciful Alhamdulillah, all praises to Allah for the strengths and His blessing in completing this thesis.

I want to thank especially my supervisor, Professor **Lalla Saadia CHADLI**; for his advice, his great availability and his generosity with which he shared with me his work, his ideas. The relevance of his remarks have always guided me in my research. I express my deepest gratitude to him for giving me the confidence to work with him, and for his constant support over the years.

My gratitude also goes to my co-supervisor, Professor **M'hamed ELOMARI**; who guided my work with competence and kindness during the years of collaboration with him. I learned a lot from him. It is a great pleasure to work with him. I also thank him for his encouragement and advice. I sincerely thank him.

I am very grateful to my three referees: Professor **Elhoussine AZROUL**, Professor **Jaouad EL-MEKKAOUI** and Professor **Chakir ALLALOU**. I thank them for agreeing to make the reports on the thesis and for their interest in my work. I would also thank Professor **Khalid HILAL**, Professor **Jalila EL GHORDAF** and Professor **Dris BAKHADACH** for agreeing to partecipate to the jury of my thesis.

I express my deepest gratitude to my family for supporting me during these years of during these years of study; especially to my parents, for their constant support and unconditional love that helped me to face the difficult moments I have encountered. No words will suffice to express my gratitude to them.

—ARHRRABI EL HOUSSAIN—

Publications and Conferences

List of Publications

❶ **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli (2021). Existence and stability of solutions of fuzzy fractional stochastic differential equations with fractional Brownian motions. **Adv Fuzzy Syst**, 3948493.

❷ **Elhoussain Arhrrabi**, Abdellah Taqbibt, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli (2021). Fuzzy fractional boundary value problem. **IEEE (ICOA)**, (pp. 1-6).

❸ Said Melliani, **Elhoussain Arhrrabi**, M'hamed Elomari & Lalla Saadia Chadli (2021). Ulam-Hyers-Rassias stability for fuzzy fractional integrodifferential equations under Caputo gH-differentiability. **International Journal On Optimization and Applications**, 51.

❹ **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli (2022). Averaging principle for fuzzy stochastic differential equations. **Contributions to Mathematics**, 5, 25–29.

❺ **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli (2022). Existence and stability of solutions for a coupled system of fuzzy fractional pantograph stochastic differential equations. **Asia Pacific Journal of Mathematics**, 9, 20.

❻ **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli (2022). Existence and controllability results for fuzzy neutral stochastic differential equations with impulses. **Boletim da Sociedade Paranaense de Matemática**. 2023 (41) : 1–14.

❼ **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli (2023). Existence and uniqueness results of fuzzy fractional stochastic differential equations with impulsive. **International Conference on Partial Differential Equations and Applications, Modeling and Simulation** (pp. 147-163). Springer, Cham.

❽ **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli (2023). Fuzzy Fractional Boundary Value Problems with Hilfer fractional derivatives. **Asia Pacific Journal of Mathematics**, 10(1):15.

❾ **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli. Fuzzy fractional pantograph stochastic differential equations: Existence, uniqueness and averaging principle. **Accepted**.

List of Submitted Articles

❶ **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli. Existence and stability results of fuzzy fractional impulsive stochastic differential equations with Granular derivative.

❷ **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli. Fuzzy fractional stochastic differential equations with time-delays: Existence and finite-time stability results.

❸ **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli. Existence and finite-time stability of solutions for a class of nonlinear Hilfer fuzzy fractional stochastic differential equations with time-delays.

❹ **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli. The averaging principle for nonlinear Hilfer fuzzy fractional stochastic differential equations.

❺ **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli. Existence and Ulam Hyers Mittag Leffler stability results of ψ-Hilfer fuzzy fractional differential equations.

❻ **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli. Existence and finite-time stability results of a class of nonlinear Hilfer fuzzy fractional differential equations with time-delays.

List of Conferences

❶ «Fuzzy fractional boundary value problem», co-authored with **Elhoussain Arhrrabi**, Abdellah Taqbibt, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli. The 7th International Conference on optimization and applications (ICOA 2021) . 26-27 March (2021), Wolfenbuttel, Germany.

❷ «Existence and uniqueness of solution to fuzzy stochastic differential equations with fractional Brownian motions», co-authored with **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli. The 3rd International Conference on Research in Applied Mathematics and Computer Science ICRAMCS (2021), Casablanca-Morocco.

❸ «Existence and uniqueness results of fuzzy fractional stochastic differential equations with impulsive», co-authored with **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli. International Conference on PDE and Applications, Modeling and Simulation ICPAMS, 2-3 June (2021), Beni Mellal- Morocco.

❹ «Existence and stability of solutions for a coupled system of fuzzy fractional pantograph stochastic differential equations», co-authored with **Elhoussain Arhrrabi**, M'hamed Elomari, Said Melliani & Lalla Saadia Chadli. International Conference on New Trends in Applied Mathematics (ICNTAM'22), 19-21 May (2022), Beni Mellal Morocco.

Abstract

Fractional stochastic differential equations (FSDEs) play an important role in the modeling of numerous complicated processes in several sectors of science and engineering. FSDEs theory and applications were examined. Furthermore, numerous academics have produced interest in systems with memory or after effects. There appears to be some difficulty in modeling a variety of modern-world systems, such as trying to characterize the physical system and differing viewpoints on its properties. The fuzzy set theory will be utilized to resolve this issue. It can handle linguistic claims like "large" and "cold" mathematically using this approach. A fuzzy set provides the ability to examine fuzzy differential equations (FDEs) in representing a variety of phenomena, including imprecision. For example, fuzzy stochastic differential equations (FSDEs) could be used to explore a wide range of economic and technical problems involving two types of uncertainty: randomness and fuzziness. Therefore, in this thesis we develop this concept of fuzzy fractional stochastic differential equations by studying some results about existence, uniqueness and stability of solution of several types of fuzzy FSDEs.

This thesis consists of two parts: In the first part we study three types of equations. Precisely, in Chapter 2, we study the existence, uniqueness and stability of solutions for fuzzy fractional stochastic differential equations (FFSDEs) driven by a fractional Brownian motion (fBm) under the Lipschitzian condition. Also, we investigate the exponential stability of solutions. In Chapter 3, we present results regarding the existence and controllability of the solutions of fuzzy neutral stochastic differential equations (FNSDEs) with impulses. For the last chapter of this part, we establish the existence and stability results of solutions for a coupled system of equations called fuzzy fractional pantograph stochastic differential equations.

The second part is devoted to the justification of the averaging technique in the setting of fuzzy stochastic and fuzzy fractional stochastic differential equations. Therefore, in Chapter 5 we prove that the solutions of fuzzy stochastic differential equations can be approximated by solutions of averaged fuzzy stochastic system under certain assumptions. In Chapter 6, we investigate the existence and averaging principle results for class of fuzzy fractional pantograph stochastic differential equations. Finally, in Chapter 7 we investigate a novel class of nonlinear Hilfer fuzzy fractional stochastic differential equations with time-delays. We discover an averaging principle for the system's solution under a few assumptions.

Keywords: Fuzzy fractional stochastic differential equations, stability of solution, pantograph stochastic differential equations, controllability result, Ulam-Hyers stability, Banach fixed point theorem, Cauchy–Schwarz inequality, Schauder fixed point theorem, Hölder inequality, averaging principle.

Classification AMS 2010: 34A07, 34A08, 60G22, 34K36, 03B52, 60H05.

Résumé

Les équations différentielles stochastiques fractionnaires (EDSF) jouent un rôle important dans la modélisation de nombreux complexes processus dans plusieurs secteurs de la science et de l'ingénierie. La théorie et les applications des EDSF ont été examinées. De plus, de nombreux universitaires sont intéressés aux systèmes avec mémoire ou séquelles. Il semble y avoir des difficultés à modéliser une variété de systèmes du monde moderne, comme essayer de caractériser le système physique et des points de vue divergents sur ses propriétés. La théorie des ensembles flous sera utilisée pour résoudre ce problème. Il peut gérer mathématiquement des revendications linguistiques telles que "grand" et "froid" en utilisant cette approche. Un ensemble flou offre la possibilité d'examiner des équations différentielles floues (EDF) en représentant une variété de phénomènes, y compris l'imprécision. Par exemple, les équations différentielles stochastiques floues (EDSF) pourrait être utilisé pour explorer un large éventail d'opportunités économiques et techniques des problèmes impliquant deux types d'incertitude : l'aléatoire et le flou. Ainsi, dans cette thèse, nous développons ce concept d'équations différentielles stochastiques fractionnaires floues (EDSFF) en étudiant certains résultats sur l'existence, l'unicité et la stabilité de la solution de plusieurs types d'EDSFF.

Cette thèse se compose de deux parties : Dans la première partie, nous étudions trois types d'équations. Précisément, dans le Chapitre 2, nous étudions l'existence, l'unicité et la stabilité de la solution d'une équations différentielles stochastiques fractionnaires floues (EDSFF) dirigées par un mouvement Brownien fractionnaire (mBf) sous la condition Lipschitzienne. De plus, nous étudions la stabilité exponentielle de la solution. Dans le Chapitre 3, nous présentons des résultats concernant l'existence et la contrôlabilité de la solution d'une équation différentielles stochastique neutre floue (EDSNF) avec impulsions. Pour le dernier chapitre de cette partie, nous établissons les résultats d'existence et de stabilité de la solution d'un système couplé d'équations différentielles stochastiques pantographe fractionnaires floues (EDSPFF).

La deuxième partie est consacrée à la justification de la technique du calcul de la moyenne dans le cadre des équations différentielles stochastiques floues et fractionnaires floues. Par conséquent, dans le Chapitre 5, nous prouvons que la solution d'équations différentielles stochastiques floues peut être approximée par la solution d'une équation différentielle stochastique floue moyennée sous certaines hypothèses. Dans le Chapitre 6, nous étudions les résultats d'existence et du principe de moyennisation pour une classe d'équations différentielles stochastiques pantographe fractionnaires floues. Enfin, dans le Chapitre 7, nous étudions une nouvelle classe d'équations différentielles stochastiques fractionnaires floues de Hilfer non linéaires avec des retards temporels. Nous découvrons un principe de moyennisation pour la solution du système sous quelques hypothèses.

Mots-clés: Équations différentielles stochastiques fractionnaires floues, stabilité de la solution, équations différentielles stochastiques pantographes, résultat de contrôlabilité, stabilité d'Ulam-Hyers, théorème du point fixe de Banach, inégalité de Cauchy-Schwarz, théorème du point fixe de Schauder, inégalité de Hölder, principe de moyennisation.

Classification AMS 2010: 34A07, 34A08, 60G22, 34K36, 03B52, 60H05.

Contents

General Introduction

This thesis mainly works on the existence and asymptotic approximation of solutions of fuzzy fractional stochastic differential equations. We attempt to investigate existence and uniqueness of solutions, controllability of solutions, averaging principle and stability properties such as asymptotic stability, exponential stability and Ulam-Hyers stability. We concentrate on various types of fuzzy stochastic and fuzzy fractional stochastic differential equations such as fuzzy stochastic differential equations, fuzzy neutral stochastic differential equations, fuzzy fractional stochastic differential equations with fractional Brownian motions and fuzzy fractional pantograph stochastic differential equations.

0.1 Organization of the thesis

This thesis is structured in 7 chapters. It is arranged as follows :

Chapter 1 : We recall some standard concepts of fuzzy sets and exhibit some extensions, definitions and other useful notations. Next, we define the fuzzy fractional differentiation and give some practical properties and lemmas. Finally, we introduce basic notations of probability theory and fuzzy stochastic calculus.

Chapter 2: The existence, uniqueness and stability of solutions for fuzzy fractional stochastic differential equations driven by a fractional Brownian motion under the Lipschitzian condition are studied. Finally, we investigate the exponential stability of solutions.

Chapter 3: By using the Banach fixed point analysis approach and using the fuzzy numbers whose values are normal, upper semicontinuous, convex and compact, the existence and controllability of solutions for fuzzy neutral stochastic differential equations with impulses are considered.

Chapter 4: We investigate a novel class of coupled system of fuzzy fractional pantograph stochastic differential equations, whose derivative is based on Caputo fractional derivative. Firstly, we convert the system under consideration into an analogous integral system. Secondly, using Banach fixed point theorem, the existence and uniqueness results of solutions for the system under consideration are then established. Additionally, we explore the Ulam-Hyers stability result.

Chapter 5: An averaging principle for fuzzy stochastic differential equations is studied.

By using Gronwall inequality, we prove that the solutions of fuzzy stochastic differential equations can be approximated in the sense of mean square by solutions of averaged fuzzy stochastic system under certain assumptions.

Chapter 6: Fuzzy fractional pantograph stochastic differential equations is investigated here. We first establish by using Banach fixed point theorem the existence and uniqueness of solutions. Then, under suitable conditions, we will prove that the solutions of the system under consideration can be approximated in the sense of mean square by the solutions of averaged fuzzy fractional stochastic system.

Chapter 7: We investigate a novel class of nonlinear Hilfer fuzzy fractional stochastic differential equations with time-delays. We discover an averaging principle for the system's solution under a few assumptions. Based on fuzzy fractional calculus, Cauchy-Schwarz inequality and Gronwall–Bellman inequality, we prove that the solution of averaged system nonlinear Hilfer fuzzy fractional stochastic differential equations converge to that of the standard one in the sense of mean square.

Chapter 1

Preliminaries

Contents

In this chapter, we give an introduction to the several concepts that are of a particular interest in what follows. First, we introduce fuzzy sets and exhibit some extensions, definitions, and other useful notations. Next, we define the fuzzy fractional differentiation and give some practical properties and lemmas. Finally, we introduce basic notations of probability theory and fuzzy stochastic calculus.

1.1 Fuzzy sets

In contrast to odd and even numbers which can be classified as belonging to exactly one class, in many engineering tasks, we are faced with classes such as ″tall‶, ″speed‶, ″age‶, etc. All of these convey a useful semantic meaning that is obvious to a certain community. However, a borderline between the belonging or not of a given object to such classes is not evident. Twovalued logic, used in describing these classes of situations, might be not well-suited,

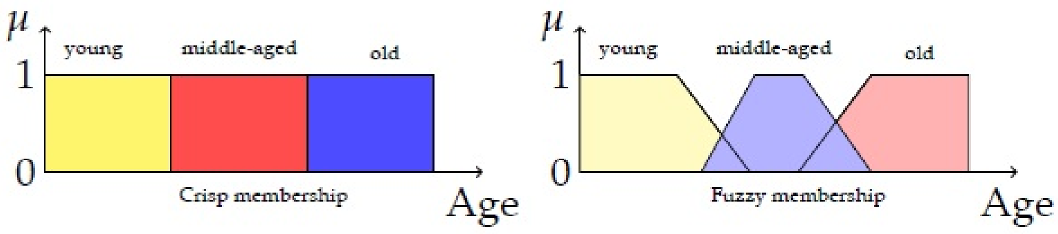

Figure 1.1: Crisp VS fuzzy membership logic.

1.1.1 Basic definitions and properties

Let's start by summarizing the fundamental concepts of fuzzy sets, fuzzy sets of numbers, and fuzzy logic. Additional support and applications for this theory can be found in [68, 121, 122].

Let $\mathbf{E^n}$ indicates the fuzzy subsets on $\mathbb{R}^n$, defined as $\zeta : \mathbb{R}^n \longrightarrow [0,1]$, which satisfies:

(1) ζ is normal, i.e $\exists z_0 \in \mathbb{R}^n$ such that $\zeta(z_0) = 1$,

(2) ζ is a convex fuzzy set, i.e for $0 \leq \beta \leq 1$

$$\min\{\zeta(z_1), \zeta(z_2)\} \leq \zeta\big(\beta z_1 + (1-\beta)z_2\big), \forall z_1, z_2 \in \mathbb{R}^n,$$

(3) ζ is upper semicontinous on $\mathbb{R}^n$,

(4) $[\zeta]^0 = cl\{z \in \mathbb{R}^n : \zeta(z) > 0\}$ is compact, where cl represents the closure of a set.

Let $\gamma \in (0,1]$, we define $[\zeta]^\gamma = \{z \in \mathbb{R}^n | \zeta(z) \geq \gamma\}$ and $[\zeta]^0 = \{z \in \mathbb{R}^n | \zeta(z) > 0\}$. From the conditions (1) to (4) the set $[\zeta]^\gamma$ are convex compact subsets of $\mathbb{R}^n$ for all $\gamma \in (0,1]$.

The notation $[\zeta]^\gamma = [\underline{\zeta}(\gamma), \overline{\zeta}(\gamma)]$, denote the γ-cut set of ζ, for all $\gamma \in [0,1]$. We denote by $\underline{\zeta}$ and $\overline{\zeta}$ as the left and right end point of ζ, respectively. For $\zeta \in \mathbf{E^n}$, we define the lengh of the γ-cut set of ζ as $len([\zeta]^\gamma) = \overline{\zeta}(\gamma) - \underline{\zeta}(\gamma)$.

Remark 1.1.1 *The fuzzy zero is defined by*

$$\hat{0}(z) = \begin{cases} 1 & if \quad z = 0, \\ 0 & if \quad z \neq 0. \end{cases}$$

Definition 1.1.2 *A fuzzy number ζ in the parametric form is a pair $\big(\underline{\zeta}(\gamma), \overline{\zeta}(\gamma)\big)$ of functions $\underline{\zeta}$, $\overline{\zeta}$ for $\gamma \in [0,1]$, which satisfy the following conditions*

- *$\underline{\zeta}$ is a bounded increasing left continuous function in $(0,1]$ and right continuous at 0,*

- *$\overline{\zeta}$ is a bounded decreasing left continuous function in $(0, 1]$ and right continuous at 0,*

- *we have $\underline{\zeta}(\gamma) \leq \overline{\zeta}(\gamma)$, for all $\gamma \in [0, 1]$.*

The characteristic function of a set A is denoted $\varphi : A \longrightarrow \{0, 1\}$ such that

$$\varphi(z) = \begin{cases} 1 & \text{if} \quad z \in A, \\ 0 & \text{if} \quad z \notin A. \end{cases}$$

A trapezoid fuzzy membership function $\mu : A \longrightarrow [0, 1]$ verifies

$$\mu(z) = \begin{cases} 0 & \text{if} \quad z \leq a, \\ \frac{z-a}{b-a} & \text{if} \quad a \leq z \leq b, \\ \frac{d-z}{d-c} & \text{if} \quad c \leq z \leq d, \\ 0 & \text{if} \quad z \geq d, \end{cases}$$

such as $[a, d] \subset A$. Below a crisp membership function is plot on Figure 1.2 and a fuzzy membership function is plot on Figure 1.3.

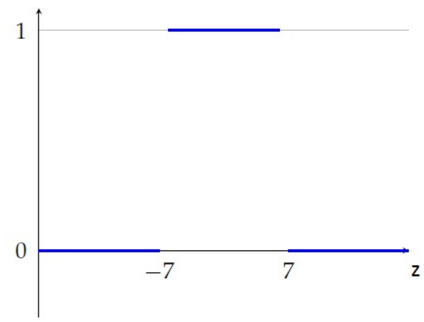

Figure 1.2: Example of a crisp membership function.

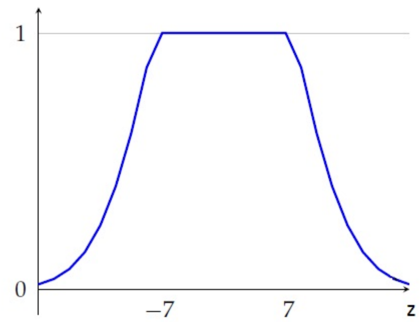

Figure 1.3: Example of a fuzzy membership function.

Operations on fuzzy numbers

The following standard operations, based on Zadeh's extension principle, define a semilinear structure on $\mathbf{E^n}$.

$$\big(u \oplus v\big)(x) = \sup_{y+z=x} \min\{u(y), v(z)\},$$

$$\lambda \odot u(x) = \begin{cases} u(\frac{x}{\lambda}) & \text{if} \quad \lambda \neq 0, \\ \hat{0} \quad \text{if} \quad \lambda = 0, \end{cases}$$

Hukuhara Difference

The Hausdorff distance between the fuzzy numbers is denoted by $\mathbf{D}_\infty : \mathbf{E^n} \times \mathbf{E^n} \longrightarrow [0, +\infty)$ such that

$$\begin{aligned} \mathbf{D}_\infty\big(\zeta_1, \zeta_2\big) &= \sup_{0\leq\gamma\leq 1} \left\{|\underline{\zeta}_1(\gamma) - \underline{\zeta}_2(\gamma)|, |\overline{\zeta}_1(\gamma) - \overline{\zeta}_2(\gamma)|\right\}, \\ &= \sup_{0\leq\gamma\leq 1} \mathbf{D}_H\big([\zeta_1]^\gamma, [\zeta_2]^\gamma\big). \end{aligned}$$

We know that $(\mathbf{E^n}, \mathbf{D}_\infty)$ is complet metric space and satisfies:

$$\mathbf{D}_\infty\big(\zeta_1 + \zeta_3, \zeta_2 + \zeta_3\big) = \mathbf{D}_\infty\big(\zeta_1, \zeta_2\big),$$

$$\mathbf{D}_\infty\big(a\zeta_1, a\zeta_2\big) = |a|\mathbf{D}_\infty\big(\zeta_1, \zeta_2\big),$$

$$\mathbf{D}_\infty\big(\zeta_1, \zeta_2\big) \leq \mathbf{D}_\infty\big(\zeta_1, \zeta_3\big) + \mathbf{D}_\infty\big(\zeta_3, \zeta_2\big),$$

for all $\zeta_1, \zeta_2, \zeta_3 \in \mathbf{E^n}$ and $a \in \mathbb{R}^n$.

Definition 1.1.3 *Let $u, v \in \mathbf{E^n}$, if there exists $w \in \mathbf{E^n}$ such that $u = v + w$ then w is called the Hukuhara difference of u and v and it is denoted by $u \ominus v$ and called H-difference.*

Remark 1.1.4 *Note that the sign $\ominus$ stands for H-difference and $u \ominus v \neq u + (-1)v$.*

Let $C\big([a, b], \mathbf{E^n}\big)$ denote the set of all continuous fuzzy functions and $AC\big([a, b], \mathbf{E^n}\big)$ the set of all absolutely continuous fuzzy functions on $[a, b]$ with value in $\mathbf{E^n}$ and $AC^{(n-1)}\big([a, b], \mathbf{E^n}\big)$ the set of all absolutely continuously H-derivatives fuzzy functions up to order $n - 1$ on the interval $[a, b]$ with value in $\mathbf{E^n}$. Also, we denote by $L\big([a, b], \mathbf{E^n}\big)$ the space of all Lebesgue integrable fuzzy valued functions on $[a, b]$.

1.1.2 Fuzzy differentiation and integration

Definition 1.1.5 *A function $f:(a,b)\longrightarrow \mathbf{E^n}$ is called H-differentiable on $u_0\in(a,b)$ if for $k>0$ sufficiently small there exist H-differences $f(u_0+k)\ominus f(u_0)$, $f(u_0)\ominus f(u_0-k)$ and an element $f'(u_0)\in\mathbf{E}^n$ such that*

$$\lim_{k\longrightarrow 0}\mathbf{D}_\infty\left(\frac{f(u_0+k)\ominus f(u_0)}{k},f'(u_0)\right)=\lim_{k\longrightarrow 0}\mathbf{D}_\infty\left(\frac{f(u_0)\ominus f(u_0-k)}{k},f'(u_0)\right)=0$$

This usual concept of differentiability has some shortcoming. Then, Bede et al [23], defined a generalised concept of differentiability.

Definition 1.1.6 *Let $f:(a,b)\longrightarrow\mathbf{E^n}$ and $u_0\in(a,b)$. We say that f is strongly generalised differentiable on u_0 if there exists an element $f'(u_0)\in\mathbf{E^n}$, such that for all $k>0$ sufficiently small, we have*

(1)

$$f'(u_0)=\lim_{k\longrightarrow 0}\frac{f(u_0+k)\ominus f(u_0)}{k}=\lim_{k\longrightarrow 0}\frac{f(u_0)\ominus f(u_0-k)}{k},$$

(2)

$$f'(u_0)=\lim_{k\longrightarrow 0}\frac{f(u_0)\ominus f(u_0+k)}{-k}=\lim_{k\longrightarrow 0}\frac{f(u_0-k)\ominus f(u_0)}{-k}.$$

Definition 1.1.7 *[12] Let $f\in C\big([0,1],\mathbf{E^n}\big)\cap L\big([0,1],\mathbf{E^n}\big)$. The fuzzy fractional integral of the fuzzy-valued function f is defined by*

$$\mathcal{I}^\alpha f(z;\gamma)=\Big[\mathcal{I}^\alpha \underline{f}(z;\gamma),\mathcal{I}^\alpha\overline{f}(z;\gamma)\Big],\quad \gamma\in[0,1],$$

where

$$\mathcal{I}^\alpha\underline{f}(z;\gamma)=\frac{1}{\Gamma(\alpha)}\int_0^z(z-s)^{\alpha-1}\underline{f}(s;\gamma)ds,$$
$$\mathcal{I}^\alpha\overline{f}(z;\gamma)=\frac{1}{\Gamma(\alpha)}\int_0^z(z-s)^{\alpha-1}\overline{f}(s;\gamma)ds.$$

Remark 1.1.8 *In [23], the authors considered four cases for derivative up to second order. Following [23] and for the sake of convenience, we only consider two cases. The other cases are trivial because they can be reduced to crisp elements.*

Definition 1.1.9 *[12] Let $f\in C^{(n)}\big([0,1],\mathbf{E^n}\big)\cap L\big([0,1],\mathbf{E^n}\big)$, $u_0\in(0,1)$ and $\phi(z)=\frac{1}{\Gamma(\alpha)}\int_0^z(z-s)^{\alpha-n+1}f(s)ds$ where $n=[\alpha]+1$. We say that f is fuzzy Riemann-Liouville fractional differentiable of order α at u_0, if there exists an element $(\mathcal{D}_0^\alpha)f(u_0)\in\mathbf{E^n}$, such that for all $k>0$ sufficiently small, we have*

(1)

$$(\mathcal{D}_0^\alpha f)(u_0)=\lim_{k\longrightarrow 0}\frac{\phi^{(n-1)}(u_0+k)\ominus\phi^{(n-1)}(u_0)}{k}=\lim_{k\longrightarrow 0}\frac{\phi^{(n-1)}(u_0)\ominus\phi^{(n-1)}(u_0-k)}{k},$$

(2)

$$(\mathcal{D}_0^\alpha f)(u_0)=\lim_{k\longrightarrow 0}\frac{\phi^{(n-1)}(u_0)\ominus\phi^{(n-1)}(u_0+k)}{-k}=\lim_{k\longrightarrow 0}\frac{\phi^{(n-1)}(u_0-k)\ominus\phi^{(n-1)}(u_0)}{-k}.$$

Let $C^{(n-1)}\big([0,a],\mathbf{E^n}\big)$ denote the space of fuzzy-value functions f on the interval $[0,a]$ which have continuous H-derivative up to order $n-2$ such that $f^{(n-1)} \in C\big([0,a],\mathbf{E^n}\big)$.

Remark 1.1.10 *The space* $C^{(n-1)}\big([0,a],\mathbf{E^n}\big)$ *is a complete metric space endowed by the metric* $\mathbf{D}$ *such that for every* $f,g \in C^{(n-1)}\big([0,a],\mathbf{E^n}\big)$

$$\mathbf{D}(f,g)=\sum_{i=0}^{n-1}\sup_{t\in[0,a]}\mathbf{D}_\infty\Big(f^{(i)}(t),g^{(i)}(t)\Big).$$

Notation: In the rest of this chapter, we say that a fuzzy-valued function f is $[(1)-\gamma]^{RL}$-differentiable if it is differentiable as in Definition 1.1.9 case (1) and is $[(2)-\gamma]^{RL}$-differentiable if it is differentiable as in Definition 1.1.9 case (2).

Definition 1.1.11 *[12] Let* $f\in C^{(n)}\big([0,1],\mathbf{E^n}\big)\cap L\big([0,1],\mathbf{E^n}\big)$, $u_0\in(0,1)$ *and* $\phi(z)=\frac{1}{\Gamma(\alpha)}\int_0^z(z-s)^{\alpha-n+1}f(s)ds$ *where* $n=[\gamma]+1$, *then*
(1) *if* f *is* $[(1)-\gamma]^{RL}$*-differentiable fuzzy valued function, then*

$$(\mathcal{D}_0^\alpha f)(u_0;\gamma)=\Big[(\mathcal{D}_0^\alpha \underline{f})(u_0;\gamma),(\mathcal{D}_0^\alpha \overline{f})(u_0;\gamma)\Big],$$

(2) *if* f *is* $[(2)-\gamma]^{RL}$*-differentiable fuzzy valued function, then*

$$(\mathcal{D}_0^\alpha f)(u_0;\gamma)=\Big[(\mathcal{D}_0^\alpha \overline{f})(u_0;\gamma),(\mathcal{D}_0^\alpha \underline{f})(u_0;\gamma)\Big],$$

where

$$(\mathcal{D}_0^\alpha \underline{f})(u_0;\gamma)=\left[\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dz}\right)^n\int_0^z(z-t)^{n-\alpha-1}\underline{f}(t;\gamma)dt\right]_{z=u_0},$$
$$(\mathcal{D}_0^\alpha \overline{f})(u_0;\gamma)=\left[\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dz}\right)^n\int_0^z(z-t)^{n-\alpha-1}\overline{f}(t;\gamma)dt\right]_{z=u_0}.$$

For $\delta\in(0,1)$, let $C_\delta\big([a,b],\mathbf{E^n}\big)$ the space of continuous functions defined by

$$C_\delta:=\{\mathbf{z}\in(a,b]\longrightarrow\mathbf{E^n}:\ \ (u-s)^{1-\delta}\mathbf{z}(u)\in C[a,b]\}.$$

Denote by $L\big([a,b],\mathbf{E^n}\big)$ the set of all fuzzy functions $\mathbf{z}:[a,b]\longrightarrow\mathbf{E^n}$ such that $u\mapsto\mathbf{D}_\infty[\mathbf{z}(u),\hat{0}]$ belong to $L^1[a,b]$.

Definition 1.1.12 *[57] The Riemann–Liouville fractional integral of order* $\gamma>0$ *of a continuous function is defined by*

$$\mathcal{I}_{a+}^\gamma\mathbf{z}(u)=\frac{1}{\Gamma(\gamma)}\int_a^u(u-s)^{\gamma-1}\mathbf{z}(s)ds.$$

Definition 1.1.13 *[57] The Caputo fractional derivative of order* $\gamma>0$ *of a absolutely continuous derivatives function up to order* $(n-1)$ *is define as follow:*

$${}^C\mathcal{D}^\gamma f(u)=\frac{1}{\Gamma(n-\gamma)}\int_0^u(u-\nu)^{n-\gamma-1}(f)^{(n)}(\nu)d\nu.$$

Definition 1.1.14 *[57] The Riemann–Liouville fractional derivative of order $\gamma > 0$ of a continuous function $\mathbf{z}$ is given by*

$$
\begin{aligned}
{}^{RL}\mathcal{D}^{\gamma}_{a^+}\mathbf{z}(u) &:= D^n \mathcal{I}^{n-\gamma}_{a^+}\mathbf{z}(u),\\
&= \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{du}\right)^n \int_a^u (u-s)^{n-\gamma-1}\mathbf{z}(s)ds,
\end{aligned}
$$

where $n = [\gamma] + 1$.

For $\mathbf{z} \in L\big([a,b], \mathbf{E^n}\big)$, we define the Riemann–Liouville fractional integral of order γ of the fuzzy function $\mathbf{z}$:

$$
\mathbf{z}_\gamma(u) := \mathcal{I}^{\gamma}_{a^+}\mathbf{z}(u) = \frac{1}{\Gamma(\gamma)}\int_a^u (u-s)^{\gamma-1}\mathbf{z}(s)ds, \ \ u \geq a.
$$

Since $[\mathbf{z}(u)]^\alpha = \big[\underline{\mathbf{z}}(u,\alpha), \overline{\mathbf{z}}(u,\alpha)\big]$, we can define the fuzzy Riemann–Liouville fractional integral of fuzzy function $\mathbf{z}$ based on lower and upper functions

$$
[\mathcal{I}^{\gamma}_{a^+}\mathbf{z}(u)]^\alpha = \big[\mathcal{I}^{\gamma}_{a^+}\underline{\mathbf{z}}(u,\alpha), \mathcal{I}^{\gamma}_{a^+}\overline{\mathbf{z}}(u,\alpha)\big], \qquad u \geq a.
$$

Where

$$
\mathcal{I}^{\gamma}_{a^+}\underline{\mathbf{z}}(u,\alpha) = \frac{1}{\Gamma(\gamma)}\int_a^u (u-s)^{\gamma-1}\underline{\mathbf{z}}(s,\alpha)ds,
$$

and

$$
\mathcal{I}^{\gamma}_{a^+}\overline{\mathbf{z}}(u,\alpha) = \frac{1}{\Gamma(\gamma)}\int_a^u (u-s)^{\gamma-1}\overline{\mathbf{z}}(s,\alpha)ds.
$$

It follows that the operator $\mathbf{z}_\gamma(u)$ is linear and bounded from $C\big([a,b], \mathbf{E^n}\big)$ to $C\big([a,b], \mathbf{E^n}\big)$.

Definition 1.1.15 *[99] Let $Df \in C(J, \mathbf{E}^n) \cap L(J, \mathbf{E}^n)$. The fuzzy fractional Caputo diffentiability of f is given by:*

$$
{}^{C}\mathcal{D}^{\gamma}f(u) = \mathcal{I}^{1-\gamma}_{c^+}(Df)(u) = \frac{1}{\Gamma(1-\gamma)}\int_0^u (u-v)^{-\gamma}(Df)(v)dv.
$$

Definition 1.1.16 *[69] The fuzzy Hilfer fractional derivative of order γ and parameter β of a function $\mathbf{z} \in C_{1-\delta}[a,b]$ is defined by*

$$
{}^{H}\mathcal{D}^{\gamma,\beta}_{a^+}\mathbf{z}(u) = \mathcal{I}^{\beta(n-\gamma)}_{a^+}\left(\frac{d}{du}\right)^n \mathcal{I}^{(1-\beta)(n-\gamma)}_{a^+}\mathbf{z}(u),
$$

if the gH-derivative $\mathbf{z}'_{1-\delta}(u)$ exists, where $n-1 < \gamma < n$, and $0 \leq \beta \leq 1$.

Definition 1.1.17 *[57] Mittag-Leffler function with two parameter is defined as*

$$
\mathfrak{M}_{\gamma,\beta}(u) = \sum_{j=0}^{\infty}\frac{u^j}{\Gamma(\gamma j + \beta)}, \quad \gamma, \beta > 0.
$$

Particularly, when $\beta = 1$, two parameter will degenerate into one parameter function, i.e $\mathfrak{M}_{\gamma,1}(u) = \mathfrak{M}_\gamma(u)$.

1.2 Fuzzy stochastic calculus

Fuzzy stochastic calculus has come to play an important role in many branches of science and technology where day by day more and more mathematician have encountered in this field. Fuzzy stochastic calculus is the area of mathematics that deals with processes containing a stochastic component and thus allows the modeling of random systems. Many stochastic processes are based on functions which are continuous, but nowhere differentiable. This rules out differential equations that require the use of derivative terms, since they are unable to be defined on non-smooth functions. Instead, a theory of integration is required where integral equations do not need the direct definition of derivative terms.

1.2.1 Basic Notations of Probability Theory

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the probability space [72]. Here Ω is the sample space and $\mathcal{A}$ is the σ-algebra of Ω and $\mathbb{P}$ is the probability measure defined on $\mathcal{A}$. A probability measure $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{A})$ is a function $\mathbb{P} : \mathcal{A} \longrightarrow [0, 1]$ such that
(1) $\mathbb{P}(\Omega) = 1$,
(2) for any disjoint sequence $A_i \subset \mathcal{A}$, that is, $A_i \cap A_j = \emptyset$, $i \neq j$,

$$\mathbb{P}(\cup A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Let $\mathbb{E}(X) = \int_{\Omega} X(\omega) d\mathbb{P}(\omega)$ denote the expectation of X.

1.2.2 Brownian Motions

The name of the Brownian motion is given to the irregular movement of pollen grains, suspended in water, observed by the Scottish botanist Robert Brown in 1828 [72]. The motion was later explained by the random collisions with the molecules of water. To describe the motion mathematically it is natural to use the concept of a stochastic process $B_t(\omega)$, interpreted as the position of the pollen grain ω at time t. Brownian motion is the actual physical motion of these particles, the Wiener process is the mathematical interpretation of this process. Let us now give the mathematical defition of Brownian motion [72].

Definition 1.2.1 *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{A}_t\}$. A standard one-dimensional Brownian motion is a real-valued continuous $\{\mathcal{A}_t\}$-adapted process noted by $\{B_t\}_{t \geq 0}$ with the following properties:*
(1) $B_0 = 0$ *a.s,*
(2) *for* $0 \leq s \leq t < \infty$ *the increment* $B_t - B_s$ *is normally distributed with mean zero and variance* $t - s$,
(3) *for* $0 \leq s \leq t < \infty$ *the increment* $B_t - B_s$ *is independent of* $\mathcal{A}_s$.

1.2.3 Fractional Brownian motion

In the modeling of many stochastic systems, the fractional Brownian motion (fBm) which shows a long-range dependence, is suggested to replace the Brownian motion as the driving process. The fBm with $H \in (0,1)$ as Hurst parameter is a Gaussian process with beneficial properties, long-range dependence, self-similarity and stationary of increments. This process is appropriate for the analysis of phenomena which present long-range and scale-invariant correlations. Nevertheless, when $H \neq \frac{1}{2}$, the fBm is not a semimartingale.

Definition 1.2.2 *[59] The fBm $B_H = \{B_H(t), t \in [0,T]\}$ with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process with the following covariance function:*

$$R_{B_H}(t,s) = \mathbb{E}\big(B_H(t)B_H(s)\big) = \frac{1}{2}\big(s^{2H} + t^{2H} - \mid t-s \mid^{2H}\big).$$

This process was introduced in [65] and studied in [83], where a stochastic integral representation was established in terms of the Brownian motion. The long-range dependence and self-similarity properties of this process, for $H > \frac{1}{2}$, yield a suitable driving noise in stochastic models, such as networks, finance and physics.

Remark 1.2.3 *In [83] a representation of $B_H(t)$ was given as follows:*

$$B_H(t) = \frac{1}{\Gamma(1+\alpha)}\Big(\int_{-\infty}^{0}[(t-s)^{\alpha} - (-s)^{\alpha}]dW(s) + B^H(t)\Big),$$

where W is a Brownian motion, $\alpha = H - \frac{1}{2}$ and $B^H(t) = \int_0^t (t-s)^{\alpha} dW(s)$. The process $B^H(t)$, with $H \in (0,1)$ is called the Liouville form of a fractional Brownian motion (LfBm) which holds many properties of the fBm except that it has non-stationary of increments.

Remark 1.2.4 *In [59] the Malliavin calculus technique was used to approximate $B_H(t)$ by a semi-martingale processas follows:*

$$B_{H,\varepsilon}(t) = \alpha \int_0^t \phi^{\varepsilon}(s)ds + \varepsilon^{\alpha}B(t), \quad (1.2.1)$$

where

$$\phi^{\varepsilon}(t) = \int_0^t (t-s+\varepsilon)^{\alpha-1}dB(s). \quad (1.2.2)$$

1.2.4 Stochastic Integral

Now we consider how to define the stochastic integral $\int_0^T f(t)dB_H(t)$ where $B_H(t) = B_H$ is the fBm. Here $f(t) = f(t,\omega) : [0,\infty) \times \Omega \longrightarrow \mathbb{R}$ is a measurable function. Since $B_H(t)$ is not of bounded variation, we cannot define $\int_0^T f(t)dB_H(t)$ by using usual Riemanm-Stieltjes integration method. We need to introduce other way to define the stochastic integral. Two approaches have been used to define stochastic integrals with respect to fBm. In the first one, the Riemann–Stieltjes stochastic integral can be defined using Young's integral [119] in the case of $H > \frac{1}{2}$. The second approach is based on the Malliavin calculus (see [31, 33]).

1.2.5 Fuzzy background

In this subsection, we provide some preliminaries on fuzzy random variable, fuzzy stochastic process and fuzzy stochastic integral (see [34, 70, 76]).
Let us denote by $\mathcal{K}(\mathbb{R}^n)$ the family of all nonempty, compact and convex subsets of $\mathbb{R}^n$. Or, the space $\mathcal{K}(\mathbb{R}^n)$ is a complete and separable metric space with respect to $\mathbf{D}_H$.

Definition 1.2.5 *[59] Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. The mapping $F : \Omega \longrightarrow \mathcal{K}(\mathbb{R}^n)$ is called $\mathcal{A}$-measurable if it satisfies $\{\omega \in \Omega : F(\omega) \cap C \neq \emptyset\} \in \mathcal{A}$, for every closed set $C \subset \mathbb{R}^n$.*

Let $\mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(\mathbb{R}^n))$ denote a family of $\mathcal{A}$-measurable multifunctions with values in $\mathcal{K}(\mathbb{R}^n)$.

Definition 1.2.6 *[59] A multifunction $F \in \mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(\mathbb{R}^n))$ is said to be L^p-integrably bounded, for $p \geq 1$, if $\exists h \in L^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$ such that $|||F||| \leq h$ $\mathbb{P}$-a.e and $|||F||| = \mathbf{D}_H(F, \hat{0})$.*

Remark 1.2.7 *It is well known that $F \in \mathcal{M}\big(\Omega, \mathcal{A}; \mathcal{K}(\mathbb{R}^n)\big)$ is L^p-integrably bounded if and only if $|||F||| \in L^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$ (see [59]).*

Let us denote $L^p\big(\Omega, \mathcal{A}, \mathbb{P}; \mathcal{K}(\mathbb{R}^n)\big) = \Big\{F \in \mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(\mathbb{R}^n)) : |||F||| \in L^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)\Big\}$. One can define the distance

$$d_p(A, B) := \big(\mathbf{D}_H^p(A, B)\big)^{\frac{1}{p}} \quad \text{for} \quad A, B \in L^p\big(\Omega, \mathcal{A}, \mathbb{P}; \mathcal{K}(\mathbb{R}^n)\big), p \geq 1.$$

In fact D_p is a metric in the set $L^p\big(\Omega, \mathcal{A}, \mathbb{P}; \mathcal{K}(\mathbb{R}^n)\big)$.

Proposition 1.2.8 *[81] For $p \geq 1$ the space $L^p\big(\Omega, \mathcal{A}, \mathbb{P}; \mathcal{K}(\mathbb{R}^n)\big)$ is a complete metric space with respect to the metric d_p.*

Definition 1.2.9 *[119] Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A fuzzy random variable is a function $X : \Omega \longrightarrow \mathbf{E^n}$, if the mapping $[X]^\gamma : \Omega \longrightarrow \mathcal{K}(\mathbb{R}^n)$ is an $\mathcal{A}$-measurable multifunction for all $\gamma \in [0, 1]$.*

Let us consider a metric ρ in the set $\mathbf{E^n}$, and σ-algebra $\mathcal{B}_\rho$ generated by the topology induced by ρ.

Remark 1.2.10 *[59] A fuzzy random variable can be viewed as a measurable mapping between two measurable spaces, namely $(\Omega, \mathcal{A})$ and $(\mathbf{E^n}, \mathcal{B}_\rho)$, we call X is $\mathcal{A}|\mathcal{B}_\rho$-measurable.*

Definition 1.2.11 *[59] A fuzzy random variable $X : \Omega \longrightarrow \mathbf{E^n}$, is L^p-integrably bounded, for $p \geq 1$, if $[X]^\gamma \in L^p\big(\Omega, \mathcal{A}, \mathbb{P}; \mathcal{K}(\mathbb{R}^n)\big)$ for every $\gamma \in [0, 1]$.*

Let us denote by $L^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbf{E^n})$ the set of all L^p-integrably bounded fuzzy random variables. We can define a metric in $L^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbf{E^n})$ in the following way

$$\mathbf{D}_p(x, y) := \sup_{0 \leq \gamma \leq 1} d_p\big([x]^\gamma, [y]^\gamma\big).$$

Proposition 1.2.12 *[81] For $p \geq 1$ the space $L^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbf{E^n})$ is a complete metric space with respect to the metric $\mathbf{D}_p$.*

Remark 1.2.13 *The random variables $X, Y \in L^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbf{E^n})$ are identical if $\mathbb{P}\big(\mathbf{D}_\infty(X, Y) = 0\big) = 1$.*

Proposition 1.2.14 *[59] For a fuzzy random variable $X : \Omega \longrightarrow \mathbf{E^n}$, and $p \geq 1$, the following conditions are equivalent:*
(i) $X \in L^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbf{E^n})$,
(ii) $[X]^0 \in L^p(\Omega, \mathcal{A}, \mathbb{P}; \mathcal{K}(\mathbb{R}^n))$,
(iii) $|||[X]^0||| \in L^p(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$.

Definition 1.2.15 *[59] If the mapping $X(t) : \Omega \longrightarrow \mathbf{E^n}$, for every $t \in J := [0, T]$, is a fuzzy random variable, then $X : J \times \Omega \longrightarrow \mathbf{E^n}$ is a fuzzy stochastic process.*

Definition 1.2.16 *[59] A fuzzy stochastic process X is $\mathbf{D}_\infty$-continuous, if almost all its trajectories, i.e. the mappings $X(., \omega) : J \times \Omega \longrightarrow \mathbf{E^n}$ are $\mathbf{D}_\infty$-continuous functions.*

Definition 1.2.17 *[59] A fuzzy stochastic process X is a measurable, if $[X]^\gamma : J \times \Omega \longrightarrow \mathcal{K}(\mathbb{R}^n)$ is $\mathcal{B}(J) \otimes \mathcal{A}$-measurable multifunction for all $\gamma \in [0, 1]$ where $\mathcal{B}(J)$ denotes the Borel σ-algebra of subsets of J.*

Definition 1.2.18 *[59] A process X is nonanticipating if and only if for every $\gamma \in [0, 1]$ the multifunction $[X]^\gamma$ is measurable with respect to the σ-algebra $\mathcal{N} := \Big\{A \in \mathcal{B}(J) \otimes \mathcal{A} : A^t \in \mathcal{A}_t \quad \forall t \in J\Big\}$, where $A^t = \{\omega : (t, \omega) \in A\}$.*

Definition 1.2.19 *[59] A fuzzy stochastic process X is called L^p-integrably bounded $(p \geq 1)$, if there exists a real-valued stochastic process $h \in L^p(J \times \Omega, \mathcal{N}; \mathbb{R}^n)$ such that*

$$|||[X(t, \omega)]^0||| \leq h(t, \omega),$$

for almost all $(t, \omega) \in J \times \Omega$.

Let us denote by $L^p\big(J\times\Omega,\mathcal{N};\mathbf{E^n}\big)$ the set of nonanticipating and L^p-integrably bounded fuzzy stochastic processes.

Notation: The set $\mathbb{R}^n$ can be embedded into $\mathbf{E^n}$ by using the following embedding $\langle . \rangle$: $\mathbb{R}^n \longrightarrow \mathbf{E^n}$ such that for $x \in \mathbb{R}^n$ we have

$$\langle x\rangle(b) = \begin{cases} 1\,, & b = x, \\ 0\,, & b \neq x. \end{cases}$$

If $X : \Omega \longrightarrow \mathbb{R}^n$ is a random variable on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, then $\langle X\rangle : \Omega \longrightarrow \mathbf{E^n}$ is a fuzzy random variable. For stochastic processes we have a similar property.

We define the fuzzy stochastic Itô integral by using the fuzzy random variable as

$$\Big\langle \int_0^T X(s)dB(s)\Big\rangle,$$

where B is a Wiener process.

Notations: Let $L^2(\Omega, \mathbf{E}^n)$ be the collection of all strongly measurable square integrable $(\Omega, \mathbf{E}^n)$-valued random variable, which is a complete metric space equipped with the following metric

$$D^2(\Lambda_1, \Lambda_2) = \mathbb{E}\mathbf{D}^2_\infty(\Lambda_1, \Lambda_2).$$

Let $C\big(J, L^2(\Omega, \mathbf{E}^n)\big)$ be the Banach space of all continuous process from J into $L^2(\Omega, \mathbf{E}^n)$ such that $\mathbb{E}\mathbf{D}^2_\infty(\Lambda_1, \Lambda_2) < \infty$. Denote by $\mathbb{K} := C(J, L^2(\Omega, \mathbf{E}^n))$ the closed bounded subspace of all continuous fuzzy process Λ in $L^2(\Omega, \mathbf{E}^n)$ consists of $\mathcal{A}_t$-adapted measurable process $\{\Lambda(t), t \in J\}$ equipped with the norm

$$\mathbf{D}^2_\infty(\Lambda_1, \Lambda_2) = \sup_{0\leq t\leq T} \mathbb{E}\mathbf{D}^2_\infty(\Lambda_1(t), \Lambda_2(t)).$$

Remark 1.2.20 *Note that* $(\mathbb{K}, \mathbf{D}_\infty)$ *is a complete metric space.*

Lemma 1.2.21 *[114] The nonnegative functions* M_γ *and* $M_{\gamma,\gamma}$ *given by*

$$M_\gamma(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k + 1)},$$

$$M_{\gamma,\gamma}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k + \gamma)},$$

have the following properties:

$1-$ $\forall A > 0$ *and* $t \in J$

$$M_\gamma(At^\gamma) \leq 1,$$

$$M_{\gamma,\gamma}(At^{\gamma}) \leq \frac{1}{\Gamma(\gamma)},$$

and $M_{\gamma}(0) = 1$, $M_{\gamma,\gamma}(0) = \frac{1}{\Gamma(\gamma)}$.

$2-$ $\forall A > 0$ *and* $t_1, t_2 \in J$ *such that* $t_1 \leq t_2$*:*

$$M_{\gamma}(At_2^{\gamma}) \leq M_{\gamma}(At_1^{\gamma}),$$

$$M_{\gamma,\gamma}(At_2^{\gamma}) \leq M_{\gamma,\gamma}(At_1^{\gamma}).$$

Definition 1.2.22 *[57] Mittag-Leffler function with two parameter is defined as*

$$\mathfrak{M}_{\gamma,\beta}(u) = \sum_{j=0}^{\infty} \frac{u^j}{\Gamma(\gamma j + \beta)}, \quad \gamma, \beta > 0.$$

Particularly, when $\beta = 1$*, two parameter will degenerate into one parameter function, i.e* $\mathfrak{M}_{\gamma,1}(u) = \mathfrak{M}_{\gamma}(u)$.

Proposition 1.2.23 *[59] Assume that the function* $X : J \longrightarrow \mathbb{R}^n$ *satisfies* $\int_0^T \|X(s)\|^2 ds < \infty$*. Then, the fuzzy stochastic Itô integral* $\left\langle \int_0^t X(s) dB^H(s) \right\rangle \in L^2(J \times \Omega, \mathcal{N}; \mathbf{E}^n)$.

Proposition 1.2.24 *[59] For* $f \in L^p(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$ *and* $p \geq 1$*. We have,* $J \times \Omega \ni (t, \upsilon) \longmapsto \int_0^t f(s, \upsilon) ds \in L^p(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$ *and* $\mathbf{D}_\infty$*-continuous.*

Proposition 1.2.25 *[59] For* $f, g \in L^p(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$ *and* $p \geq 1$*, we have*

$$\mathbb{E} \sup_{a \in [0,t]} \mathbf{D}^p_\infty \left(\int_0^a f(u) du, \int_0^a g(u) du \right) \leq t^{p-1} \int_0^t \mathbb{E}\mathbf{D}^p_\infty (f(u), g(u)) du.$$

Proposition 1.2.26 *[42] Let* $\psi : J \longrightarrow \mathbb{R}^n$*, then for* $t \in J$*;*

$$\sup_{a \in [0,t]} \mathbb{E} \left\| \int_0^a \psi(s) dB_H(s) \right\|^2 \leq \int_0^t \|\psi(s)\|^2 ds.$$

Proposition 1.2.27 *[59] Assume that the function* $\psi : J \longrightarrow \mathbb{R}^n$ *satisfies* $\int_0^T \|\psi(\nu)\|^2 d\nu < \infty$*. Then:*

(i) *The fuzzy stochastic Itô integral* $\left\langle \int_0^\nu \psi(u) dB_H(u) \right\rangle \in L^2(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$,

(ii) *For* $x \in L^2(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$ *we have, for* $u \leq \nu \in J$,

$$\mathbf{D}_\infty \left(\int_0^\nu x(s) ds + \int_0^\nu \psi(s) dB_H(s), \int_0^u x(s) ds + \int_0^u \psi(s) dB_H(s) \right)$$
$$= \mathbf{D}_\infty \left(\int_u^\nu x(s) ds + \int_u^\nu \psi(s) dB_H(s), \hat{0} \right).$$

Proposition 1.2.28 *[110] For* $f, g \in L^p(J, \mathbf{E^n})$ *and* $t \in J$ *we have*

$$\mathbb{E} \sup_{a\in[0,t]} \mathbf{D}_\infty^p \left(\Big\langle \int_0^a f(u)dW(u) \Big\rangle, \Big\langle \int_0^a g(u)dW(u) \Big\rangle \right) \leq \int_0^t \mathbb{E}\mathbf{D}_\infty^p(f(u), g(u))du.$$

Lemma 1.2.29 *Let* $\gamma \in (0,1)$ *and* $0 \leq u < s$, *we get*

$$u^{2\gamma-1} - s^{2\gamma-1} \leq (u-s)^{2\gamma-1}.$$

Proof. We have

$$u^{2\gamma-1} - s^{2\gamma-1} = (2\gamma-1)\int_s^u z^{2\gamma-2}dz,$$

and

$$(u-s)^{2\gamma-1} = (2\gamma-1)\int_s^u (z-s)^{2\gamma-2}dz.$$

Since $s \leq z \leq u$ and $-2 < 2\gamma - 2 < 0$, we have $z^{2\gamma-2} \leq (z-s)^{2\gamma-2}$. Then, $u^{2\gamma-1} - s^{2\gamma-1} \leq (u-s)^{2\gamma-1}$.

Part I

On the existence, uniqueness and stability of solutions for fuzzy fractional stochastic differential equations

Chapter 2

Existence and stability of solutions of fuzzy fractional stochastic differential equations with fractional Brownian motions

Contents

2.1 Introduction

A stochastic dynamical system can be frequently described by some stochastic differential equations. The equation of a given system specifies its behaviour overany given short period of time. To determine the system's behaviour for a long period, we often study the integration of the equations. When we investigate real world systems, we are interested in settling the system into its typical behaviour. The subset of the phase space of the dynamical system corresponding to the typical behaviour is the attractor, also known as the attracting set. In the mathematical field of dynamical systems, an attractor is a set of numerical values toward which a system tends to evolve, for a wide variety of starting conditions of the system.

There appears to be confusion of various kinds in the modeling of several real world systems, such as trying to characterize a physical system and opinions on its parameters. To deal with this ambiguity, the fuzzy set theory will be used [120]. It is able to handle such linguitic statements mathematically using this theory, such as, 'large' and 'less ' etc. The capacity to investigate fuzzy differential equations (FDEs) in modeling numerous phenom-

ena, including imprecision, is provided by a fuzzy set. In particular, the fuzzy stochastic differential equations (FSDEs), in instance, might be used to investigate a variety of economics and engineering problems that involve two types of uncertainty: randomness and fuzziness.
The fuzzy Itô stochastic integral was powered in [81, 82]. Therefore, in [78, 79] the fuzzy stochastic integral is driven by the Wiener process as a fuzzy adapted stochastic process. Also, in [38], Fei et al. studied the existence and uniqueness of solutions to the FSDEs with non-Lipschitzian condition. In [59] Jafari et al. study FSDEs driven by fBm. Jialu Zhu et al. in [124] prove existence of solutions to SDEs with fBm. Ding and Nieto [36] investigated analytical solutions of multi-time scale FSDEs driven by fBm, Vas'kovskii at al. [111] prove that the pth moments, $p \geq 1$, of strong solutions of a mixed-type SDEs driven by a standard Brownian motion and a fBm. Despite the fact that some research exists on the problem of the uniqueness and existence of solutions to SDEs and FSDEs which are disturbed by Brownian motions or semi martingales, ([39, 40, 41, 80, 89, 78]), a kind of the (FFSDEs) driven by a fBm has not been investigated. Agarwal et al. [3, 4] considered the concept of solution for FDEs with uncertainty and some results on FFDEs and optimal control nonlocal evolution equations. Recently, Zhou et al in [125, 58, 46] are given some important works on the stability analysis of such SFDEs. Motivated by the above discussions and based on the result in [43], we are interested by the existence, uniqueness and stability of solution of the FFSDEs defines in the next section. This chapter is organized as follows: In the first subsection, the existence and uniqueness of solution to the FFSDEs are proved. The second subsection will be devoted to the stability of solution.

2.2 Main results

In this section, we shall formulate and prove sufficient conditions for the existence, uniqueness and stability of solution of the FFSDEs defines by (2.2.1). We first prove the existence and uniqueness of solution. Then, we show that under certain assumptions, the solution of (2.2.1) exists and it is unique. Afterwards, we examine the stability of the solution with respect to initial values.

2.2.1 Existence and uniqueness results

Now, we are interested in the existence and uniqueness results of the following FFSDEs driven by an fBm

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}x(s) = f(s, x(s))ds + \Big\langle g(s, x(s))dB_H(s)\Big\rangle, \\ \\ x(0) = x_0, \end{cases} \tag{2.2.1}$$

where:

$$f : J \times \Omega \times \mathbf{E}^n \longrightarrow \mathbf{E}^n,$$

$$g : J \times \Omega \times \mathbf{E}^n \longrightarrow \mathbb{R}^n,$$

$$x_0 : \Omega \longrightarrow \mathbf{E}^n,$$

${}^{C}\mathcal{D}^{\alpha}$ is a Caputo fractional derivative of order $0 < \alpha < 1$ and $\{B_H(s)\}_{s\in J}$ is a fBm defined on $(\Omega, \mathcal{A}, \{\mathcal{A}_s^H\}_{s\in J}, \mathbb{P})$ with Hirst index $H \in (\frac{1}{2}, 1)$.

Definition 2.2.1 *A solution of problem (2.2.1) is a fuzzy process $x : J \times \Omega \longrightarrow \mathbf{E}^n$ satisfying the following conditions*

(i) $x \in \mathbf{L}^2(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$,

(ii) x *is* $\mathbf{D}_\infty$*-continuous,*

(iii) *we have*

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, x(s))}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \Big\langle \int_0^t \frac{g(s, x(s))}{(t-s)^{1-\alpha}} dB_H(s) \Big\rangle. \tag{2.2.2}$$

Remark 2.2.2 *We call a solution $x : J \times \Omega \longrightarrow \mathbf{E}^n$ of problem (2.2.1) is unique, if $x(t) = z(t)$, where a fuzzy process $z : J \times \Omega \longrightarrow \mathbf{E}^n$ is any solution of problem (2.2.1).*

We will assume that all through this chapter, $f : (J \times \Omega) \times \mathbf{E}^n \longrightarrow \mathbf{E}^n$ is $\mathcal{B}_{d_s} \otimes \mathbf{N} \mid \mathcal{B}_{\mathbf{D}_\infty}$-measurable.

We consider the following hypotheses:

$(\mathcal{H}1)$ x_0 is $\mathcal{A}_0$-measurable, such that

$$\mathbb{E}\mathbf{D}_\infty^2(x_0, \hat{0}) < \infty,$$

$(\mathcal{H}2)$ For all $u \in \mathbf{E}^n$ and $t \in J$, $\exists C > 0$ such that

$$\max\left\{\mathbb{E}\mathbf{D}_\infty^2\Big(f(t, u), \hat{0}\Big), \|g(t, u)\|^2\right\} \leq C.$$

$(\mathcal{H}3)$ For all $u, v \in \mathbf{E}^n$ and $t \in J$, $\exists L > 0$ such that

$$\max\left\{\mathbb{E}\mathbf{D}^2_\infty\Big(f(t,u), f(t,v)\Big), \mathbb{E}\big\|g(t,u) - g(t,v)\|^2\right\} \leq L\mathbb{E}\mathbf{D}^2_\infty(u,v),$$

Let us now introduce the main theorem in this part.

Theorem 2.2.3 *Under hypotheses* $(\mathcal{H}1)$-$(\mathcal{H}3)$ *and* $x_0 \in \mathbf{L}^2(\Omega, \mathcal{A}_0, \mathbb{P}; \mathbf{E}^n)$, *the problem* (2.2.1) *has a unique solution* $x(t)$.

Proof. Consider the FFSDE (2.2.2)

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s,x(s))}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\Big\langle \int_0^t \frac{g(s,x(s))}{(t-s)^{1-\alpha}}dB_H(s)\Big\rangle. \tag{2.2.3}$$

By Eq. (1.2.1), we can write

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s,x(s))}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\Big\langle \int_0^t \frac{\alpha\phi^\varepsilon(s)g(s,x(s))}{(t-s)^{1-\alpha}}ds + \int_0^t \frac{\varepsilon^\alpha g(s,x(s))}{(t-s)^{1-\alpha}}dW(s)\Big\rangle. \tag{2.2.4}$$

The method of successive approximations will be used to demonstrate the existence of solution of problem (2.2.1). Therefore, defined a sequence $x_n : J \times \Omega \longrightarrow \mathbf{E}^n$, as below:

$$x_0(t) = x_0, \tag{2.2.5}$$

and for $n = 1, ...,$

$$x_n(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s,x_{n-1}(s))}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\Big\langle \int_0^t \frac{\alpha\phi^\varepsilon(s)g(s,x_{n-1}(s))}{(t-s)^{1-\alpha}}ds + \int_0^t \frac{\varepsilon^\alpha g(s,x_{n-1}(s))}{(t-s)^{1-\alpha}}dW(s)\Big\rangle. \tag{2.2.6}$$

It's clear that x_n's are in $\mathbf{L}^2(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$ and $\mathbf{D}_\infty$-continuous. Indeed, we have $x_0 \in \mathbf{L}^2(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$ and x_0 is $\mathbf{D}_\infty$-continuous.

Let us define for $n \in \mathbb{N}$ and $t \in J$, $K_n(t) = \sup\limits_{0 \leq u \leq t} \mathbb{E}\mathbf{D}^2_\infty(x_n(u), x_{n-1}(u))$. Then, from Propositions 1.2.23-1.2.28, Cauchy–Schwarz inequality and hypotheses $(\mathcal{H}1)$-$(\mathcal{H}3)$, we have:

$$\begin{aligned}
K_1(t) &= \sup_{0\le u\le t} \mathbb{E}\mathbf{D}^2_\infty\left(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s,x_0)}{(u-s)^{1-\alpha}}ds + \left\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s,x_0)}{(u-s)^{1-\alpha}}dB_H(s)\right\rangle, \hat{0}\right),\\
&\le 3\sup_{0\le u\le t}\left[\mathbb{E}\mathbf{D}^2_\infty\left(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s,x_0)}{(u-s)^{1-\alpha}}ds,\hat{0}\right) + \mathbb{E}\mathbf{D}^2_\infty\left(\left\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{\alpha\varphi^\varepsilon(s)g(s,x_0)}{(u-s)^{1-\alpha}}ds\right\rangle,\hat{0}\right)\right]\\
&+3\sup_{0\le u\le t}\mathbb{E}\mathbf{D}^2_\infty\left(\left\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{\varepsilon^\alpha g(s,x_0)}{(u-s)^{1-\alpha}}dW(s)\right\rangle,\hat{0}\right),\\
&\le 6\sup_{0\le u\le t}\mathbb{E}\mathbf{D}^2_\infty\left(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s,x_0)}{(u-s)^{1-\alpha}}ds, \frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s,\hat{0})}{(u-s)^{1-\alpha}}ds\right)\\
&+6\sup_{0\le u\le t}\mathbb{E}\mathbf{D}^2_\infty\left(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s,\hat{0})}{(u-s)^{1-\alpha}}ds,\hat{0}\right) + 6\sup_{u\in[0,t]}\left[\frac{\varepsilon^{2\alpha}Ct^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^u \mathbb{E}\mathbf{D}^2_\infty(g(s,x_0),\hat{0})ds\right]\\
&+6\sup_{u\in[0,t]}\left[\frac{\alpha^2Ct^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\mathbf{D}^2_\infty\left(\int_0^u\left\langle\varphi^\varepsilon(s)g(s,x_0)ds\right\rangle,\hat{0}\right)\right],\\
&\le \frac{6T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t \mathbb{E}\mathbf{D}^2_\infty\left(f(s,x_0),f(s,\hat{0})\right)ds + \frac{6T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t \mathbb{E}\mathbf{D}^2_\infty\left(f(s,\hat{0}),\hat{0}\right)ds\\
&+\frac{6\varepsilon^{2\alpha}CT^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^u \mathbb{E}\mathbf{D}^2_\infty(g(s,x_0),\hat{0})ds + 6\left[\frac{\alpha^2Ct^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\mathbf{D}^2_\infty\left(\int_0^u\left\langle\varphi^\varepsilon(s)g(s,x_0)ds\right\rangle,\hat{0}\right)\right],\\
&\le \frac{6T^{2\alpha}L}{(2\alpha-1)\Gamma^2(\alpha)}\mathbf{D}^2_\infty(x_0,\hat{0}) + \frac{6T^{2\alpha}C}{(2\alpha-1)\Gamma^2(\alpha)} + \frac{6\varepsilon^{2\alpha}CT^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)} + \frac{6\alpha^2CLT^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)} := l_1T^{2\alpha},
\end{aligned}$$

where $l_1 = \frac{6L}{(2\alpha-1)\Gamma^2(\alpha)}\mathbf{D}^2_\infty(x_0,\hat{0}) + \frac{6C}{(2\alpha-1)\Gamma^2(\alpha)} + \frac{6\varepsilon^{2\alpha}C}{(2\alpha-1)\Gamma^2(\alpha)} + \frac{6\alpha^2CL}{(2\alpha-1)\Gamma^2(\alpha)}$. Moreover, similarly we have:

$$\begin{aligned}
K_{n+1}(t) &= \sup_{0\le u\le t}\mathbb{E}\mathbf{D}^2_\infty\Bigg(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s,x_n(s))}{(u-s)^{1-\alpha}}ds + \left\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s,x_n)}{(u-s)^{1-\alpha}}dB_H(s)\right\rangle,\\
&\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s,x_{n-1}(s))}{(u-s)^{1-\alpha}}ds + \left\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s,x_{n-1})}{(u-s)^{1-\alpha}}dB_H(s)\right\rangle\Bigg),\\
&\le 2\sup_{0\le u\le t}\mathbb{E}\mathbf{D}^2_\infty\left(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s,x_n(s))}{(u-s)^{1-\alpha}}ds, \int_0^u \frac{f(s,x_{n-1}(s))}{(u-s)^{1-\alpha}}ds\right)\\
&+2\sup_{0\le u\le t}\mathbb{E}\mathbf{D}^2_\infty\left(\left\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s,x_n)}{(u-s)^{1-\alpha}}dB_H(s)\right\rangle, \left\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s,x_{n-1})}{(u-s)^{1-\alpha}}dB_H(s)\right\rangle\right),\\
&\le l_2\int_0^t \sup_{u\in[0,s]}\mathbb{E}\mathbf{D}^2_\infty(x_n(s),x_{n-1}(s))ds,\\
&\le l_2\int_0^t K_n(s)ds,
\end{aligned}$$

where $l_2 = \frac{6L}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\mathbf{D}^2_\infty(x_0,\hat{0}) + \frac{6C}{(2\alpha-1)\Gamma^2(\alpha)} + \frac{6\varepsilon^{2\alpha}C}{(2\alpha-1)\Gamma^2(\alpha)} + \frac{6\alpha^2CL}{(2\alpha-1)\Gamma^2(\alpha)}$. Thus, we get

$$K_n(t) \le \frac{l_1}{l_2}\frac{(l_2t^\alpha)^n}{n!\Gamma(\alpha+1)}, \quad \forall t\in J, \quad n\in\mathbb{N}, \tag{2.2.7}$$

where $l_2 = 2Tc$.

Hence, from Chebyshev's inequality and (2.2.7), we get

$$\mathbb{P}\left(\sup_{u\in J}\mathbf{D}^2_\infty(x_n(u),x_{n-1}(u)) > \frac{1}{4^n}\right) \le \frac{l_1}{l_2}\frac{(4l_2T^{2\alpha})^n}{n!},$$

since the series $\sum_{n\geq 1}\frac{(4l_2T^{2\alpha})^n}{n!}$ is converge, according to Borel-Cantelli lemma we get

$$\mathbb{P}\left(\sup_{u\in J}\mathbf{D}_\infty(x_n(u),x_{n-1}(u))>\frac{1}{4^n}\right)=0.$$

Thus, the sequence $\{x_n(.,\upsilon)\}$ is uniformly convergent to $\tilde{x}(.,\upsilon):J\longrightarrow\mathbb{R}^n$ for $\upsilon\in\Omega_c$, where $\Omega_c\in\mathcal{A}$ and $\mathbb{P}(\Omega_c)=1$. Then

$$\lim_{n\longrightarrow\infty}\sup_{t\in J}\mathbb{E}\mathbf{D}^2_\infty(x_n(t),\tilde{x}(t))=0. \tag{2.2.8}$$

Let us define $x:J\times\Omega\longrightarrow\mathbf{E}^n$ as bellow:

$$x(.,\upsilon)=\begin{cases}\tilde{x}(.,\upsilon)\ , & \text{if}\quad \upsilon\in\Omega_c,\\ \text{freely chosen}\ , & \text{if}\quad \upsilon\in\upsilon\in\Omega\backslash\Omega_c.\end{cases}$$

We can observe that for each $0\leq\alpha\leq 1$ and $t\in J$, we have

$$\lim_{n\longrightarrow\infty}D_H([x_n(,\upsilon)]^\alpha,[x_{n-1}(,\upsilon)]^\alpha)=0.$$

Then $[x(t,.)]^\alpha:\Omega\longrightarrow\mathbf{K}(\mathbb{R}^n)$ is $\mathcal{A}_t$-measurable. Hence x is nonanticipating. By (2.2.8), we have

$$\lim_{n\longrightarrow\infty}\sup_{t\in J}\mathbb{E}\mathbf{D}^2_\infty(x_n(t),x(t))=0, \tag{2.2.9}$$

which show that $\exists\lambda>0$ independent of $n\in\mathbb{N}$ such that

$$\sup_{t\in J}\mathbb{E}\mathbf{D}^2_\infty(x_n(t),x(t))\leq\lambda. \tag{2.2.10}$$

Since $x_n\in\mathbf{L}^2(J\times\Omega,\mathbf{N};\mathbf{E}^n)$, we have $x_n(t)\in\mathbf{L}^2(\Omega,\mathcal{A},\mathbb{P};\mathbf{E}^n)$. In addition, we can prove that $x\in\mathbf{L}^2(J\times\Omega,\mathbf{N};\mathbf{E}^n)$. Indeed, for all $n\in\mathbb{N}$ and $t\in J$, let us denote

$$\psi_n(t)=\sup_{0\leq u\leq t}\mathbb{E}\mathbf{D}^2_\infty(x_n(u),\hat{0}).$$

Then, we get

$$\begin{aligned}\psi_n(t)\leq 3\mathbb{E}\mathbf{D}^2_\infty(x_0,\hat{0})&+3\sup_{0\leq u\leq t}\mathbb{E}\mathbf{D}^2_\infty\left(\frac{1}{\Gamma(\alpha)}\int_0^u(u-s)^{\alpha-1}f(s,x_{n-1}(s))ds,\hat{0}\right)\\&+3\sup_{0\leq u\leq t}\mathbb{E}\mathbf{D}^2_\infty\left(\left\langle\frac{1}{\Gamma(\alpha)}\int_0^u(u-s)^{\alpha-1}g(s,x_{n-1}(s))dB_H(s)\right\rangle,\hat{0}\right).\end{aligned}$$

By using Propositions 1.2.26, 1.2.27, Cauchy–Schwarz inequality and hypotheses $(\mathcal{H}1)$-$(\mathcal{H}3)$, we have

$$\begin{aligned}\psi_n(t)\leq 3\mathbb{E}\mathbf{D}^2_\infty(x_0,\hat{0})&+\frac{6t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t\left[\mathbb{E}\mathbf{D}^2_\infty\big(f(s,x_{n-1}(s)),f(s,\hat{0})\big)+\mathbb{E}\mathbf{D}^2_\infty\big(f(s,\hat{0}),\hat{0}\big)\right]ds\\&+3\mathbb{E}\mathbf{D}^2_\infty\left(\left\langle\frac{1}{\Gamma(\alpha)}\int_0^u\frac{\alpha\varphi^\varepsilon(s)g(s,x_0)}{(u-s)^{1-\alpha}}ds\right\rangle,\hat{0}\right)+3\sup_{0\leq u\leq t}\mathbb{E}\mathbf{D}^2_\infty\left(\left\langle\frac{1}{\Gamma(\alpha)}\int_0^u\frac{\varepsilon^\alpha g(s,x_0)}{(u-s)^{1-\alpha}}dW(s)\right\rangle,\hat{0}\right).\end{aligned}$$

We get

$$\psi_n(t) \leq A_1 + A_2 \int_0^t \psi_{n-1}(s)ds,$$

where $A_1 = \frac{6L}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\mathbf{D}^2_\infty(x_0, \hat{0}) + \frac{6C}{(2\alpha-1)\Gamma^2(\alpha)} + \frac{6\varepsilon^{2\alpha}C}{(2\alpha-1)\Gamma^2(\alpha)} + \frac{6\alpha^2 CL}{(2\alpha-1)\Gamma^2(\alpha)}$ and $A_2 = \frac{6t^{2\alpha-1}C}{(2\alpha-1)\Gamma^2(\alpha)}$.
According to Gronwall's inequality, we get

$$\psi_n(t) \leq A_1 e^{A_2 t}, \quad \forall t \in J. \tag{2.2.11}$$

Due to the hypothesis $(\mathcal{H}1)$, Eqs. (2.2.10) and (2.2.11), we get

$$\begin{aligned}\sup_{0\leq s\leq t} \mathbb{E}\mathbf{D}^2_\infty(x(s), \hat{0}) &\leq 2 \sup_{0\leq s\leq t} \mathbb{E}\mathbf{D}^2_\infty(x(s), x_n(s)) + 2 \sup_{0\leq s\leq t} \mathbb{E}\mathbf{D}^2_\infty(x_n(s), \hat{0}),\\ &\leq 2\lambda + 2A_1 e^{A_2 t} < \infty,\end{aligned}$$

which implies

$$\int_0^T \mathbb{E}\mathbf{D}^2_\infty(x(s), \hat{0})ds \leq T \sup_{t\in J} \mathbb{E}\mathbf{D}^2_\infty(x(t), \hat{0}) < \infty.$$

Thus, we get $x \in \mathbf{L}^2(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$.
On the other hand, we have

$$\sup_{t\in J} \mathbb{E}\mathbf{D}^2_\infty \left(x(t), x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, x(s))}{(t-s)^{1-\alpha}} ds + \left\langle \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s, x(s))}{(t-s)^{1-\alpha}} dB_H(s) \right\rangle \right) = 0. \tag{2.2.12}$$

Indeed, we observe that

$$\begin{aligned}&\sup_{t\in J} \mathbb{E}\mathbf{D}^2_\infty \left(x(t), x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, x(s))}{(t-s)^{1-\alpha}} ds + \left\langle \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s, x(s))}{(t-s)^{1-\alpha}} dB_H(s) \right\rangle \right)\\ &\leq 3\Bigg[\sup_{t\in J} \mathbb{E}\mathbf{D}^2_\infty (x(t), x_n(t))\\ &+ \sup_{t\in J} \mathbb{E}\mathbf{D}^2_\infty \left(x_n(t), x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, x_{n-1}(s))}{(t-s)^{1-\alpha}} ds + \left\langle \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s, x_{n-1}(s))}{(t-s)^{1-\alpha}} dB_H(s) \right\rangle \right)\\ &+ \sup_{t\in J} \mathbb{E}\mathbf{D}^2_\infty \Bigg(x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, x_{n-1}(s))}{(t-s)^{1-\alpha}} ds + \left\langle \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s, x_{n-1})}{(t-s)^{1-\alpha}} dB_H(s) \right\rangle, x_0 +\\ &\frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, x(s))}{(t-s)^{1-\alpha}} ds + \left\langle \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s, x(s))}{(t-s)^{1-\alpha}} dB_H(s) \right\rangle \Bigg) \Bigg] := I_1 + I_2 + I_3,\end{aligned}$$

where $\lim_{n\longrightarrow\infty} I_1 = 0$ and $I_2 = 0$.
For I_3, by using Propositions 1.2.26, 1.2.27, Cauchy–Schwarz inequality, hypothesis $(\mathcal{H}3)$ and Eq. (2.2.9), we have

$$\lim_{n\longrightarrow\infty} I_3 \leq \lim_{n\longrightarrow\infty} \left(\frac{T^{2\alpha}C}{(2\alpha-1)\Gamma^2(\alpha)} \sup_{t\in J} \mathbb{E}\mathbf{D}^2_\infty(x(t), x_{n-1}(t))du \right) = 0.$$

Hence, we get (2.2.12), which implies Eq. (2.2.2) holds. Therefore, from the definition 2.2.1, $x(t)$ is a solution to Eq. (2.2.1).

For the uniqueness of solution, we suppose that $x, z : J \times \Omega \longrightarrow \mathbf{E}^n$ are solutions to problem (2.2.1). Let $K(t) := \sup_{\nu \in J} \mathbb{E}\mathbf{D}^2_\infty(x(\nu), z(\nu))$. So for each $t \in J$, we have

$$K(t) = \sup_{\nu\in J} \mathbb{E}\mathbf{D}^2_\infty\Bigg(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s,x(s))}{(t-s)^{1-\alpha}}ds + \Big\langle \frac{1}{\Gamma(\alpha)}\int_0^t \frac{g(s,x(s))}{(t-s)^{1-\alpha}}dB_H(s)\Big\rangle, x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s,z(s))}{(t-s)^{1-\alpha}}ds + \Big\langle \frac{1}{\Gamma(\alpha)}\int_0^t \frac{g(s,z(s))}{(t-s)^{1-\alpha}}dB_H(s)\Big\rangle\Bigg),$$

then, by computations similar to the existence case, we have

$$K(t) \leq \frac{6t^{2\alpha-1}C}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t K(s)ds.$$

Thus, by Gronwall's inequality, we have for $t \in J$, $K(t) \equiv 0$. Which implies

$$\sup_{t\in J} \mathbf{D}_\infty(x(t), z(t)) = 0,$$

which completes the proof.□

2.2.2 Stability result

In this part, we examine the stability of the solution with respect to initial values. Namely, we explore the continuity (called the stability by Malinowski [82]) of the solution with respect to the initial value. This kind of continuity shows that in the case of replacement of x_0 by its approximate value z_0, the solution of equation with initial value z_0 does not differ much from the solution of equation with initial value x_0. Indeed, let x, z denote the solutions of the followings FFSDEs

$$\begin{cases} {}^C\mathcal{D}^\alpha x(s) = f(s,x(s))ds + \langle g(s,x(s))dB_H(s)\rangle, \\ x(0) = x_0, \end{cases} \tag{2.2.13}$$

$$\begin{cases} {}^C\mathcal{D}^\alpha z(s) = f(s,z(s))ds + \langle g(s,z(s))dB_H(s)\rangle, \\ z(0) = z_0, \end{cases} \tag{2.2.14}$$

respectively.

Proposition 2.2.4 *Suppose that* $x_0, z_0 \in \mathbf{L}^2(\Omega, \mathcal{A}_0, \mathbb{P}; \mathbf{E}^n)$, $f : J \times \Omega \times \mathbf{E}^n \longrightarrow \mathbf{E}^n$ *and* $g : J \times \Omega \times \mathbf{E}^n \longrightarrow \mathbb{R}^n$ *satisfy* $(\mathcal{H}1)$-$(\mathcal{H}3)$. *Then*

$$\sup_{0\leq u\leq t} \mathbb{E}\mathbf{D}^2_\infty(x(u), z(u)) \leq \lambda_0 e^{\lambda_1 t},$$

where $\lambda_0 = 3\mathbb{E}\mathbf{D}^2_\infty(x_0, z_0)$, $\lambda_1 = \frac{(3T^{2\alpha-1}C + 6\varepsilon^{2\alpha})}{(2\alpha-1)\Gamma^2(\alpha)}$. *Especially* $x(t) = z(t)$ *if* $x_0 = z_0$.

Proof. Suppose that $x, z : J \times \Omega \longrightarrow \mathbf{E}^n$ are solutions of problems (2.2.13) and (2.2.14) respectively. So let $K(t) := \sup\limits_{0\leq u\leq t} \mathbb{E}\mathbf{D}_\infty^2(x(u), z(u))$. By using Propositions 1.2.26, 1.2.27, Cauchy–Schwarz inequality and hypothesis $(\mathcal{H}3)$, we have

$$\begin{aligned}
K(t) &\leq 3\mathbb{E}\mathbf{D}_\infty^2(x_0, z_0) + \frac{3T^{2\alpha-1}C}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t \mathbb{E}\mathbf{D}_\infty^2\big(x(s), z(s)\big)ds \\
&\quad + 3\sup_{0\leq u\leq t} \mathbb{E}\mathbf{D}_\infty^2\left(\Big\langle \frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s, x_n(s))}{(u-s)^{1-\alpha}} dB_H(s)\Big\rangle, \Big\langle \frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s, x_{n-1}(s))}{(u-s)^{1-\alpha}} dB_H(s)\Big\rangle\right), \\
&\leq 3\mathbb{E}\mathbf{D}_\infty^2(x_0, z_0) + \frac{(3T^{2\alpha-1}C + 6\varepsilon^{2\alpha})}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t \sup_{0\leq u\leq t} \mathbb{E}\mathbf{D}_\infty^2\big(x(u), z(u)\big)du, \\
&= 3\mathbb{E}\mathbf{D}_\infty^2(x_0, z_0) + \frac{(3T^{2\alpha-1}C + 6\varepsilon^{2\alpha})}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t K(s)ds \\
&:= \lambda_0 + \lambda_1\int_0^t K(s)ds,
\end{aligned}$$

then, according to Gronwall's inequality, we have

$$K(t) \leq \lambda_0 e^{\lambda_1 t}, \quad \forall t \in J.$$

Then, $\lambda_0 = 0$ if $x_0 = z_0$. Therefor, we know that $x(t) = z(t)$.□

Finally, we examine the exponential stability of solutions to the FFSDEs disturbed an fBm with respect to f and g. Thus, let x, x_n denote solutions to the following FFSDEs

$$\begin{cases} {}^C\mathcal{D}^\alpha x(s) = f(s, x(s))ds + \big\langle g(s, x(s))dB_H(s)\big\rangle, \\ \\ x(0) = x_0, \end{cases} \tag{2.2.15}$$

$$\begin{cases} {}^C\mathcal{D}^\alpha x_n(s) = f_n(s, x_n(s))ds + \big\langle g_n(s, x_n(s))dB_H(s)\big\rangle, \\ \\ x_n(0) = x_0, \end{cases} \tag{2.2.16}$$

respectively.

Proposition 2.2.5 *Suppose that* $x_0 \in \mathbf{L}^2(\Omega, \mathcal{A}_0, \mathbb{P}; \mathbf{E}^n)$, $f, f_n : J \times \Omega \times \mathbf{E}^n \longrightarrow \mathbf{E}^n$ *and* $g, g_n : J \times \Omega \times \mathbf{E}^n \longrightarrow \mathbb{R}^n$ $(n \in \mathbb{N})$ *satisfy* $(\mathcal{H}1)$-$(\mathcal{H}3)$. *Furthermore, assume that*

$$\lim_{n\longrightarrow\infty}\left(\int_0^t \mathbb{E}\mathbf{D}_\infty^2((t, x), f_n(t, x))dt\right) = 0. \tag{2.2.17}$$

$$\lim_{n\longrightarrow\infty}\left(\int_0^t \mathbb{E}\|g_n(s, x) - g(s, x)\|^2 ds\right) = 0. \tag{2.2.18}$$

Then, we have

$$\lim_{n\longrightarrow\infty}\left(\sup_{t\in J} \mathbb{E}\mathbf{D}_\infty^2\big(x(t), x_n(t)\big)\right) = 0, \tag{2.2.19}$$

where $x, x_n : J \times \Omega \longrightarrow \mathbf{E}^n$ *are solutions of problems* (2.2.15) *and* (2.2.16), *respectively.*

Proof. According to Theorem 2.2.3, the solutions x and x_n are unique and exists. From Propositions 1.2.26, 1.2.27, Cauchy–Schwarz inequality and hypotheses $(\mathcal{H}1)$-$(\mathcal{H}3)$, we have that for every $t \in J$

$$\begin{aligned}
&\sup_{0\le u\le t} \mathbb{E}\mathbf{D}^2_\infty(x(u), x_n(u)) \\
&\quad\le 2\sup_{0\le u\le t} \mathbb{E}\mathbf{D}^2_\infty\left(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s,x(s))}{(u-s)^{1-\alpha}}ds, \frac{1}{\Gamma(\alpha)}\int_0^u \frac{f_n(s,x_n(s))}{(u-s)^{1-\alpha}}ds\right) \\
&\quad+2\sup_{0\le u\le t} \mathbb{E}\mathbf{D}^2_\infty\left(\left\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g(s,x(s))}{(u-s)^{1-\alpha}}dB_H(s)\right\rangle, \left\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{g_n(s,x_n(s))}{(u-s)^{1-\alpha}}dB_H(s)\right\rangle\right), \\
&\quad\le 4\sup_{0\le u\le t} \mathbb{E}\mathbf{D}^2_\infty\left(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f(s,x(s))}{(u-s)^{1-\alpha}}ds, \frac{1}{\Gamma(\alpha)}\int_0^u \frac{f_n(s,x(s))}{(u-s)^{1-\alpha}}ds\right) \\
&\quad+4\sup_{0\le u\le t} \mathbb{E}\mathbf{D}^2_\infty\left(\frac{1}{\Gamma(\alpha)}\int_0^u \frac{f_n(s,x(s))}{(u-s)^{1-\alpha}}ds, \frac{1}{\Gamma(\alpha)}\int_0^u \frac{f_n(s,x_n(s))}{(u-s)^{1-\alpha}}ds\right) \\
&\quad+4\sup_{0\le u\le t} \mathbb{E}\mathbf{D}^2_\infty\left(\left\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{\alpha\varphi^\varepsilon g(s,x(s))}{(u-s)^{1-\alpha}}ds\right\rangle, \left\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{\alpha\varphi^\varepsilon g_n(s,x_n(s))}{(u-s)^{1-\alpha}}ds\right\rangle\right) \\
&\quad+4\sup_{0\le u\le t} \mathbb{E}\mathbf{D}^2_\infty\left(\left\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{\varepsilon^\alpha g(s,x(s))}{(u-s)^{1-\alpha}}dW(s)\right\rangle, \left\langle\frac{1}{\Gamma(\alpha)}\int_0^u \frac{\varepsilon^\alpha g_n(s,x_n(s))}{(u-s)^{1-\alpha}}dW(s)\right\rangle\right), \\
&\quad\le \frac{4Ct^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t \mathbb{E}\mathbf{D}^2_\infty(x(s), x_n(s))ds + \frac{4(t^{2\alpha-1}+\alpha^2)}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t \mathbb{E}\mathbf{D}^2_\infty(f(s,x(s)), f_n(s,x(s)))ds \\
&\quad+\frac{4L\varepsilon^{2\alpha}t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t \|g(s,x(s)) - g_n(s,x(s))\|^2 ds, \\
&\quad\le \beta_1^n + \beta_2\int_0^t \sup_{0\le u\le s} \mathbb{E}\mathbf{D}^2_\infty(x(u), x_n(s))ds,
\end{aligned}$$

where

$$\beta_1^n := \frac{4(t^{2\alpha-1}+\alpha^2)}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t \mathbb{E}\mathbf{D}^2_\infty(f(s,x(s)), f_n(s,x(s)))ds + \frac{4L\varepsilon^{2\alpha}t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t \|g(s,x(s)) - g_n(s,x(s))\|^2 ds,$$

$\beta_2 := \frac{4Ct^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}$. Then, according to Gronwall's inequality, we get

$$\sup_{u\in[0,t]} \mathbb{E}\mathbf{D}^2_\infty(x(u), x_n(u)) \le \beta_1^n e^{\beta_2 T}.$$

Hence, from (2.2.17) and (2.2.18), we get $\lim_{n\longrightarrow\infty} \beta_1^n = 0$. Thus the proof is complete.□

Chapter 3

Existence and controllability results for fuzzy neutral stochastic differential equations with impulses

Contents

3.1 Introduction

In economics, chemistry, finance, and other fields, stochastic differential equations are utilized to create more realistic models. As a result, stochastic differential equations can be used to describe numerous real world problems. On the other hand, the fuzzy stochastic differential equations could be used to investigate a variety of technical and economic difficulties when phenomena are subjected to both randomness and fuzziness at the same time. The best type of equations with dispersed arguments is neutral differential equations, they naturally emerge in applied issues with a recurring property in their expression. Processes that are subjected to adrupt changes at particular points are described using impulsive differential equations. The impulsive effects can be found in a wide range of real world applications including electronics, neural network, economics and so on (see [21, 24, 25]). Then, a theory of impulsive differential equations has received a lot of attention in connection with this development.

Controllability is an important notion in both deterministic and stochastic control theory since it is one of the fundamental concepts in mathematical control theory (see [8, 9, 19, 29, 44, 90]). The notion "controllability" refers to ability to control a dynamical system from any

initial state to any final state using the set of admissible controls.
We wish to mention that the theory of fuzzy neutral stochastic differential equations with impulses have recently been the subject of important studies. As, for the controllability of fuzzy stochastic differential equations, even less has been done, with only a few works published in this topic as far as we know. In [103, 104] Sakthivel et al. studied the approximate controllability of nonlinear impulsive differential systems and stochastic systems with unbounded delay. Bouffoussi et al. [27] studied the existence, uniqueness and asymptotic behavior of mild solutions for the neutral stochastic differential equations with finite delay. In [19] Arhrrabi et al studied the existence and stability of solutions of fuzzy fractional stochastic differential equations with fractional brownian motions. Park et al. [94] proved the existence and uniqueness of fuzzy solutions and controllability for the impulsive semilinear fuzzy integrodifferential equations. In [28] Bouzahir et al. discussed the controllability of neutral functional differential equations with infinite delay. Ahmed [6, 7] studied the controllability of impulsive neutral stochastic differential equations with finite delay and fractional Browian motion in a Hilbert space. Achary et al. [1] studied the controllability of fuzzy solutions for first order nonlocal impulsive neutral functional differential equations by using the Banach fixed point theorem. Chalishajar et al. [29] proved the existence, uniqueness and controllability for impulsive fuzzy neutral functional integrodifferential equations. Our results are inspired by the one in [90] where the approximate controllability of impulsive neutral fuzzy stochastic differential equations with nonlocal condition in Banach space is studied.
This chapter is organized as follows: The existence and uniqueness results of solution for fuzzy neutral stochastic differential equations with impulses are proved in the first subsection, in the second subsection, the controllability result are discussed. Finally, an example is given to illustrate the results.

3.2 Main results

In this section, we investigate the existence, uniqueness and controllability results for FNSDEs with impulses given by

$$\begin{cases} d\Big[y(t) - f(t, y(t))\Big] = \Big[Ay(t) + g(t, y(t)) + \nu(t)\Big]dt + \Big\langle h(t)dB_H(t)\Big\rangle, t \in J = [0, T]. \\ \Delta y(t_k) = I_k(y(t_k^-)), \qquad k = 1, ..., m \qquad t \neq t_k. \\ y(t) = y_0, \end{cases} \tag{3.2.1}$$

where $A : J \longrightarrow \mathbf{E}^n$ is a fuzzy coefficient, $f, g : J \times \mathbf{E}^n \longrightarrow \mathbf{E}^n$ are nonlinear continuous functions, $h : J \longrightarrow \mathbb{R}^n$, $\nu : J \longrightarrow \mathbf{E}^n$ is an admissible control function and B_H is a fractional Browian motion defined on a filtered probability space $(\Omega, \mathcal{A}, \{\mathcal{A}_t^H\}_{0\leq t\leq T}, \mathbb{P})$, the initial data

$y_0 \in \mathbf{E}^n$ and $I_k \in C(\mathbf{E}^n, \mathbf{E}^n)$ are bounded functions, $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ represents the left and right limits of $y(t)$ at $t = t_k$ respectively, $k = 1, .., m$.

3.2.1 Existence and uniqueness results

In this subsection, we show the existence and uniqueness of fuzzy solution for FNSDEs with impulses given by (3.2.1) ($\nu \equiv 0$).

Definition 3.2.1 *A fuzzy process* $\{y(t),\ t \in J\}$ *is said to be solution of system* (3.2.1) *if:*

(i) $y(\cdot) \in C(J, \mathbf{E}^n)$,

(ii) $y(0) = y_0$,

(iii) *for* $t \in J$, *we have*

$$\begin{aligned} y(t) =& G(t)\Big[y_0 - f(0, y_0)\Big] + f(t, y(t)) + \int_0^t AG(t-s)f(s, y(s))ds + \int_0^t G(t-s)g(s, y(s))ds \\ &+ \left\langle \int_0^t G(t-s)h(s)dB_H(s) \right\rangle + \sum_{t_k=0}^{t} G(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{3.2.2}$$

Let us introduce the following hypotheses.

$(\mathcal{H}1)$ Let $G(t) \in \mathbf{E}^n$, such that

$$\big[G(t)\big]^\alpha = \big[\underline{G}^\alpha(t), \overline{G}^\alpha(t)\big], \qquad G(0) = I,$$

and $\underline{G}^\alpha, \overline{G}^\alpha$ are continuous such that $\max\{|\underline{G}^\alpha(t)|, |\overline{G}^\alpha(t)|\} \leq M$ and $|AG(t)| \leq N$, $\forall t \in J$.

$(\mathcal{H}2)$ The functions $f, g : J \times \mathbf{E}^n \longrightarrow \mathbf{E}^n$ are continuous and there exists a finite constants $\lambda, \gamma > 0$ such that

$$\mathbf{D}_H\Big([f(t, y(t))]^\alpha, [f(t, z(t))]^\alpha \Big) \leq \lambda \mathbf{D}_H\Big([y(t)]^\alpha, [z(t)]^\alpha\Big),$$

and

$$\mathbf{D}_H\Big([g(t, y(t))]^\alpha, [g(t, z(t))]^\alpha \Big) \leq \gamma \mathbf{D}_H\Big([y(t)]^\alpha, [z(t)]^\alpha\Big),$$

for all $y, z \in \mathbf{E}^n$ and $t \in J$.

$(\mathcal{H}3)$ For each $y, z \in \mathbf{E}^n$, there exist a constant $\mu > 0$ such that

$$\mathbf{D}_H\Big([I_k(y(t_k^-))]^\alpha, [I_k(z(t_k^-))]^\alpha\Big) \leq \mu \mathbf{D}_H\Big([y(t)]^\alpha, [z(t)]^\alpha\Big), \ \ k = 1, .., m.$$

$(\mathcal{H}4)$ $\lambda + \lambda NT + (\gamma + \mu)MT < 1$.

Theorem 3.2.2 *Suppose that the hypotheses* $(\mathcal{H}1)$*-*$(\mathcal{H}4)$ *holds, then, for all* $T > 0$*, the system* (3.2.1) $(\nu \equiv 0)$ *has a unique fuzzy solution on* J.

Proof. For each $y \in \mathbf{E}^n$ and $t \in J$, define $\phi : C(J, \mathbf{E}^n) \longrightarrow C(J, \mathbf{E}^n)$ by

$$\begin{aligned}\phi y(t) &= G(t)\Big[y_0 - f(0, y_0)\Big] + f(t, y(t)) + \int_0^t AG(t-s)f(s, y(s))ds + \int_0^t G(t-s)g(s, y(s))ds \\ &\quad + \Big\langle \int_0^t G(t-s)h(s)dB_H(s) \Big\rangle + \sum_{t_k=0}^{t} G(t-t_k)I_k(y(t_k^-)).\end{aligned}$$

Since $\underline{G}^\alpha, \overline{G}^\alpha$ are continuous, $\phi y : J \longrightarrow \mathbf{E}^n$ is continuous, so ϕ is a mapping from $C(J, \mathbf{E}^n)$ into itself. Now, for $y, z \in C(J, \mathbf{E}^n)$, we have:

$$\begin{aligned}\mathbf{D}_H\Big(\big[\phi y(t)\big]^\alpha, \big[\phi z(t)\big]^\alpha\Big) &= \mathbf{D}_H\Bigg(\Bigg[G(t)\{y_0 - f(0, y_0)\} + f(t, y(t)) + \int_0^t AG(t-s)f(s, y(s))ds \\ &\quad + \int_0^t G(t-s)g(s, y(s))ds + \Big\langle \int_0^t G(t-s)h(s)dB_H(s) \Big\rangle \\ &\quad + \sum_{t_k=0}^{t} G(t-t_k)I_k(y(t_k^-))\Bigg]^\alpha, \Bigg[G(t)\{y_0 - f(0, y_0)\} + f(t, z(t)) + \int_0^t AG(t-s)f(s, z(s))ds \\ &\quad + \int_0^t G(t-s)g(s, z(s))ds + \Big\langle \int_0^t G(t-s)h(s)dB_H(s) \Big\rangle + \sum_{t_k=0}^{t} G(t-t_k)I_k(z(t_k^-))\Bigg]^\alpha\Bigg), \\ &= \mathbf{D}_H\Bigg(\Big[G(t)\{y_0 - f(0, y_0)\}\Big]^\alpha + \Big[f(t, y(t))\Big]^\alpha + \Big[\int_0^t AG(t-s)f(s, y(s))ds\Big]^\alpha \\ &\quad + \Big[\int_0^t G(t-s)g(s, y(s))ds\Big]^\alpha + \Big[\Big\langle \int_0^t G(t-s)h(s)dB_H(s) \Big\rangle\Big]^\alpha \\ &\quad + \Big[\sum_{t_k=0}^{t} G\big(t-t_k\big)I_k\big(y(t_k^-)\big)\Big]^\alpha, \Big[G(t)\{y_0 - f(0, y_0)\}\Big]^\alpha + \Big[f(t, z(t))\Big]^\alpha \\ &\quad + \Big[\int_0^t AG(t-s)f(s, z(s))ds\Big]^\alpha + \Big[\int_0^t G(t-s)g(s, z(s))ds\Big]^\alpha \\ &\quad + \Big[\Big\langle \int_0^t G(t-s)h(s)dB_H(s) \Big\rangle\Big]^\alpha + \Big[\sum_{t_k=0}^{t} G(t-t_k)I_k(z(t_k^-))\Big]^\alpha\Bigg),\end{aligned}$$

then:

$$\begin{aligned}
\mathbf{D}_H\Big(\Big[\phi y(t)\Big]^\alpha, \Big[\phi z(t)\Big]^\alpha\Big) &\leq \int_0^t \mathbf{D}_H\Big(\Big[AG(t-s)f(s,y(s))\Big]^\alpha, \Big[AG(t-s)f(s,y(s))\Big]^\alpha\Big)ds \\
&+ \mathbf{D}_H\Big(\Big[f(t,y(t))\Big]^\alpha, \Big[f(t,z(t))\Big]^\alpha\Big) + \int_0^t \mathbf{D}_H\Big(\Big[G(t-s)g(s,y(s))\Big]^\alpha, \Big[G(t-s)g(s,z(s))\Big]^\alpha\Big)ds \\
&+ \mathbf{D}_H\Big(\Big[\sum_{t_k=0}^{t} G(t-t_k)I_k(y(t_k^-))\Big]^\alpha, \Big[\sum_{t_k=0}^{t} G(t-t_k)I_k(z(t_k^-))\Big]^\alpha\Big), \\
&\leq \int_0^t \Big|AG(t-s)\Big|\mathbf{D}_H\Big(\Big[f(s,y(s))\Big]^\alpha, \Big[f(s,z(s))\Big]^\alpha\Big)ds + \lambda\mathbf{D}_H\Big(\Big[y(t)\Big]^\alpha, \Big[z(t)\Big]^\alpha\Big) \\
&+ \int_0^t \Big|G(t-s)\Big|\mathbf{D}_H\Big(\Big[g(s,y(s))\Big]^\alpha, \Big[g(s,z(s))\Big]^\alpha\Big)ds \\
&+ \sum_{t_k=0}^{t} \Big|G(t-t_k)\Big|\mathbf{D}_H\Big(\Big[I_k(y(t_k^-))\Big]^\alpha, \Big[I_k(z(t_k^-))\Big]^\alpha\Big), \\
&\leq \lambda N \int_0^t \mathbf{D}_H\Big(\Big[y(s)\Big]^\alpha, \Big[z(s)\Big]^\alpha\Big)ds + \lambda\mathbf{D}_H\Big(\Big[y(t)\Big]^\alpha, \Big[z(t)\Big]^\alpha\Big) \\
&+ \gamma M \int_0^t \mathbf{D}_H\Big(\Big[y(s)\Big]^\alpha, \Big[z(s)\Big]^\alpha\Big)ds + \mu M\mathbf{D}_H\Big(\Big[y(t)\Big]^\alpha, \Big[z(t)\Big]^\alpha\Big),
\end{aligned}$$

therefore:

$$\begin{aligned}
\mathbf{D}_\infty\Big(\phi y(t), \phi z(t)\Big) &= \sup_{0\leq\alpha\leq1} \mathbf{D}_H\Big(\Big[\phi y(t)\Big]^\alpha, \Big[\phi z(t)\Big]^\alpha\Big) \leq \lambda \sup_{0\leq\alpha\leq1} \mathbf{D}_H\Big(\Big[y(t)\Big]^\alpha, \Big[z(t)\Big]^\alpha\Big) \\
&+ \lambda N \int_0^t \sup_{0\leq\alpha\leq1} \mathbf{D}_H\Big(\Big[y(s)\Big]^\alpha, \Big[z(s)\Big]^\alpha\Big)ds \\
&+ \gamma M \int_0^t \sup_{0\leq\alpha\leq1} \mathbf{D}_H\Big(\Big[y(s)\Big]^\alpha, \Big[z(s)\Big]^\alpha\Big)ds + \mu M \sup_{0\leq\alpha\leq1} \mathbf{D}_H\Big(\Big[y(t)\Big]^\alpha, \Big[z(t)\Big]^\alpha\Big), \\
&\leq \lambda\mathbf{D}_\infty\Big(y(t), z(t)\Big) + \lambda N \int_0^t \mathbf{D}_\infty\Big(y(s), z(s)\Big)ds \\
&+ \gamma M \int_0^t \mathbf{D}_\infty\Big(y(s), z(s)\Big)ds + \mu M\mathbf{D}_\infty\Big(y(t), z(t)\Big),
\end{aligned}$$

or we have

$$D\big(\phi y, \phi z\big) = \sup_{0\leq t\leq T} \mathbf{D}_\infty\big(\phi y(t), \phi z(t)\big),$$

then:

$$\begin{aligned}
D\big(\phi y, \phi z\big) &\leq \lambda \sup_{0\leq t\leq T} \mathbf{D}_\infty\Big(y(t), z(t)\Big) + \lambda N \int_0^t \sup_{0\leq s\leq T} \mathbf{D}_\infty\Big(y(s), z(s)\Big)ds \\
&+ \gamma M \int_0^t \sup_{0\leq s\leq T} \mathbf{D}_\infty\Big(y(s), z(s)\Big)ds + \mu M \sup_{0\leq t\leq T} \mathbf{D}_\infty\Big(y(t), z(t)\Big), \\
&\leq \Big(\lambda + \lambda NT + (\gamma+\mu)MT\Big)D\big(y, z\big).
\end{aligned}$$

So, by the hypothesis $(\mathcal{H}4)$, ϕ is a contraction mapping. Hence, by Banach fixed point Theorem, the fuzzy neutral stochastic differential equations (3.2.1) has a unique fixed point $y \in C(J, \mathbf{E}^n)$.□

3.2.2 Controllability result

In this subsection, we state the controllability result for fuzzy neutral stochastic differential equations with impulses (3.2.1) by using Banach fixed point theorem.

Definition 3.2.3 *[28] The system (3.2.1) is said to be controllable on J, if there exist a fuzzy control function $v(t)$ such that a fuzzy solution $y(t)$ of (3.2.1) satisfies $y(T) = y_1$, i.e. $[y(T)]^\alpha = [y_1]^\alpha$, where $y_1 \in \mathbf{E}^n$ is a target set.*

Defined the fuzzy mapping $\mathcal{W} : P(\mathbb{R}) \longrightarrow \mathbf{E}^n$ by

$$\mathcal{W}(v) = \begin{cases} \displaystyle\int_0^T G(T-s)v(s)ds\,, & v \subset \overline{\Gamma}_v, \\ 0\,, & \text{otherwise}, \end{cases}$$

where $\overline{\Gamma}_v$ is the closure of support v and $P(\mathbb{R})$ is a nonempty fuzzy subset of $\mathbb{R}$. Then, the α-level of $\mathcal{W}$ is given by

$$\underline{\mathcal{W}}^\alpha(\underline{v}) = \int_0^T \underline{G}^\alpha(T-s)\underline{v}(s)ds, \qquad \underline{v}(s) \in [\underline{v}^\alpha(s), v^1(s)],$$

$$\overline{\mathcal{W}}^\alpha(\overline{v}) = \int_0^T \overline{G}^\alpha(T-s)\overline{v}(s)ds, \qquad \overline{v}(s) \in [v^1(s), \overline{v}^\alpha(s)].$$

We assume that $\underline{\mathcal{W}}$,$\overline{\mathcal{W}}$ are bijective mappings. Hence, We can introduce α-level set of $v(s)$ given by:

$$\begin{aligned}
[v(s)]^\alpha &= \left[\underline{v}^\alpha(s), \overline{v}^\alpha(s)\right], \\
&= \Bigg[(\underline{\mathcal{W}}^\alpha)^{-1}\Big(\underline{y}_1^\alpha - \underline{G}^\alpha(T)\left\{\underline{y_0}^\alpha - \underline{f}^\alpha(0,y_0)\right\} - \int_0^T \underline{A}^\alpha \underline{G}^\alpha(T-s)\underline{f}^\alpha(s,\underline{y}^\alpha(s))ds \\
&\quad - \underline{f}^\alpha(T,\underline{y}^\alpha(T)) - \int_0^T \underline{G}^\alpha(T-s)\underline{g}^\alpha(s,\underline{y}^\alpha(s))ds - \int_0^T \underline{G}^\alpha(T-s)\underline{h}^\alpha(s)dB_H(s) \\
&\quad - \sum_{t_k=0}^{T} \underline{G}^\alpha(T-t_k)\underline{I}_k^\alpha(\underline{y}^\alpha(t_k^-))\Big), (\overline{\mathcal{W}}^\alpha)^{-1}\Big(\overline{y}_1^\alpha - \overline{G}^\alpha(T)\left\{\overline{y_0}^\alpha - \overline{f}^\alpha(0,y_0)\right\} \\
&\quad - \int_0^T \overline{A}^\alpha \overline{G}^\alpha(T-s)\overline{f}^\alpha(s,\overline{y}^\alpha(s))ds - \overline{f}^\alpha(T,\overline{y}^\alpha(T)) - \int_0^T \overline{G}^\alpha(T-s)\overline{g}^\alpha(s,\overline{y}^\alpha(s))ds \\
&\quad - \int_0^T \overline{G}^\alpha(T-s)\overline{h}^\alpha(s)dB_H(s) - \sum_{t_k=0}^{T} \overline{G}^\alpha(T-t_k)\overline{I}_k^\alpha(\overline{y}^\alpha(t_k^-))\Big)\Bigg].
\end{aligned}$$

Then, substitute this expression into the Eq. (3.2.2) yields α-level of $y(T)$, we get

$$
\begin{aligned}
\left[y(T)\right]^{\alpha} = &\Bigg[G(T)\left[y_0 - f(0,y_0)\right] + f(T,y(t)) + \int_0^T AG(T-s)f(s,y(s))ds \\
&+ \int_0^T G(T-s)g(s,y(s))ds + \left\langle \int_0^T G(T-s)h(s)dB_H(s) \right\rangle \\
&+ \sum_{t_k=0}^{T} G(T-t_k)I_k(y(t_k^-)) + \int_0^T G(T-s)\mathcal{W}^{-1}\Big(y_1 - G(T)[y_0 - f(0,y_0)] \\
&- f(T,y(t)) - \int_0^T AG(T-s)f(s,y(s))ds - \int_0^T G(T-s)g(s,y(s))ds \\
&- \left\langle \int_0^T G(T-s)h(s)dB_H(s) \right\rangle - \sum_{t_k=0}^{T} G(T-t_k)I_k(y(t_k^-))\Big) ds\Bigg]^{\alpha}, \\
= &\Bigg[\underline{G}^{\alpha}(T)\left\{\underline{y_0}^{\alpha} - \underline{f}^{\alpha}(0,y_0)\right\} + \underline{f}^{\alpha}(T,\underline{y}^{\alpha}(T)) + \int_0^T \underline{A}^{\alpha}\underline{G}^{\alpha}(T-s)\underline{f}^{\alpha}(s,\underline{y}^{\alpha}(s))ds \\
&+ \int_0^T \underline{G}^{\alpha}(T-s)\underline{g}^{\alpha}(s,\underline{y}^{\alpha}(s))ds + \int_0^T \underline{G}^{\alpha}(T-s)\underline{h}^{\alpha}(s)dB_H(s) \\
&+ \sum_{t_k=0}^{T} \underline{G}^{\alpha}(T-t_k)\underline{I}_k^{\alpha}(\underline{y}^{\alpha}(t_k^-)) + \int_0^T \underline{G}(T-s)(\underline{\mathcal{W}}^{\alpha})^{-1}\Big(\underline{y}_1^{\alpha} - \underline{G}^{\alpha}(T)\left\{\underline{y_0}^{\alpha} - \underline{f}^{\alpha}(0,y_0)\right\} \\
&- \underline{f}^{\alpha}(T,\underline{y}^{\alpha}(T)) - \int_0^T \underline{A}^{\alpha}\underline{G}^{\alpha}(T-s)\underline{f}^{\alpha}(s,\underline{y}^{\alpha}(s))ds - \int_0^T \underline{G}^{\alpha}(T-s)\underline{g}^{\alpha}(s,\underline{y}^{\alpha}(s))ds \\
&- \int_0^T \underline{G}^{\alpha}(T-s)\underline{h}^{\alpha}(s)dB_H(s) - \sum_{t_k=0}^{T} \underline{G}^{\alpha}(T-t_k)\underline{I}_k^{\alpha}(\underline{y}^{\alpha}(t_k^-))\Big) ds, \overline{G}^{\alpha}(T)\left\{\overline{y_0}^{\alpha} - \overline{f}^{\alpha}(0,y_0)\right\} \\
&+ \int_0^T \overline{A}^{\alpha}\overline{G}^{\alpha}(T-s)\overline{f}^{\alpha}(s,\overline{y}^{\alpha}(s))ds + \overline{f}^{\alpha}(T,\overline{y}^{\alpha}(T)) + \int_0^T \overline{G}^{\alpha}(T-s)\overline{g}^{\alpha}(s,\overline{y}^{\alpha}(s))ds \\
&+ \int_0^T \overline{G}^{\alpha}(T-s)\overline{h}^{\alpha}(s)dB_H(s) + \sum_{t_k=0}^{T} \overline{G}^{\alpha}(T-t_k)\overline{I}_k^{\alpha}(\overline{y}^{\alpha}(t_k^-)) \\
&+ \int_0^T \overline{G}((T-s))(\overline{\mathcal{W}}^{\alpha})^{-1}\Big(\overline{y}_1^{\alpha} - \overline{G}^{\alpha}(T)\left\{\overline{y_0}^{\alpha} - \overline{f}^{\alpha}(0,y_0)\right\} \\
&- \int_0^T \overline{A}^{\alpha}\overline{G}^{\alpha}(T-s)\overline{f}^{\alpha}(s,\overline{y}^{\alpha}(s))ds - \overline{f}^{\alpha}(T,\overline{y}^{\alpha}(T)) - \int_0^T \overline{G}^{\alpha}(T-s)\overline{g}^{\alpha}(s,\overline{y}^{\alpha}(s))ds \\
&- \int_0^T \overline{G}^{\alpha}(T-s)\overline{h}^{\alpha}(s)dB_H(s) - \sum_{t_k=0}^{T} \overline{G}^{\alpha}(T-t_k)\overline{I}_k^{\alpha}(\overline{y}^{\alpha}(t_k^-))\Big) ds\Bigg],
\end{aligned}
$$

then, we have

$$
\begin{aligned}
[y(T)]^{\alpha} = \Bigg[& \underline{G}^{\alpha}(T)\left\{\underline{y_0}^{\alpha} - \underline{f}^{\alpha}(0,y_0)\right\} + \underline{f}^{\alpha}(T,\underline{y}^{\alpha}(T)) + \int_0^T \underline{A}^{\alpha}\underline{G}^{\alpha}(T-s)\underline{f}^{\alpha}(s,\underline{y}^{\alpha}(s))ds \\
& + \int_0^T \underline{G}^{\alpha}(T-s)\underline{g}^{\alpha}(s,\underline{y}^{\alpha}(s))ds + \int_0^T \underline{G}^{\alpha}(T-s)\underline{h}^{\alpha}(s)dB_H(s) \\
& + \sum_{t_k=0}^{T} \underline{G}^{\alpha}(T-t_k)\underline{I}_k^{\alpha}(\underline{y}^{\alpha}(t_k^-)) + (\underline{\mathcal{W}}^{\alpha})(\underline{\mathcal{W}}^{\alpha})^{-1}\Bigg(\underline{y}_1^{\alpha} - \underline{G}^{\alpha}(T)\left\{\underline{y_0}^{\alpha} - \underline{f}^{\alpha}(0,y_0)\right\} \\
& - \underline{f}^{\alpha}(T,\underline{y}^{\alpha}(T)) - \int_0^T \underline{A}^{\alpha}\underline{G}^{\alpha}(T-s)\underline{f}^{\alpha}(s,\underline{y}^{\alpha}(s))ds - \int_0^T \underline{G}^{\alpha}(T-s)\underline{g}^{\alpha}(s,\underline{y}^{\alpha}(s))ds \\
& - \int_0^T \underline{G}^{\alpha}(T-s)\underline{h}^{\alpha}(s)dB_H(s) - \sum_{t_k=0}^{T} \underline{G}^{\alpha}(T-t_k)\underline{I}_k^{\alpha}(\underline{y}^{\alpha}(t_k^-))\Bigg), \overline{G}^{\alpha}(T)\left\{\overline{y_0}^{\alpha} - \overline{f}^{\alpha}(0,y_0)\right\} \\
& + \int_0^T \overline{A}^{\alpha}\overline{G}^{\alpha}(T-s)\overline{f}^{\alpha}(s,\overline{y}^{\alpha}(s))ds + \overline{f}^{\alpha}(T,\overline{y}^{\alpha}(T)) + \int_0^T \overline{G}^{\alpha}(T-s)\overline{g}^{\alpha}(s,\overline{y}^{\alpha}(s))ds \\
& + \int_0^T \overline{G}^{\alpha}(T-s)\overline{h}^{\alpha}(s)dB_H(s) + \sum_{t_k=0}^{T} \overline{G}^{\alpha}(T-t_k)\overline{I}_k^{\alpha}(\overline{y}^{\alpha}(t_k^-)) \\
& + (\overline{\mathcal{W}}^{\alpha})(\overline{\mathcal{W}}^{\alpha})^{-1}\Bigg(\overline{y}_1^{\alpha} - \overline{G}^{\alpha}(T)\left\{\overline{y_0}^{\alpha} - \overline{f}^{\alpha}(0,y_0)\right\} \\
& - \int_0^T \overline{A}^{\alpha}\overline{G}^{\alpha}(T-s)\overline{f}^{\alpha}(s,\overline{y}^{\alpha}(s))ds - \overline{f}^{\alpha}(T,\overline{y}^{\alpha}(T)) - \int_0^T \overline{G}^{\alpha}(T-s)\overline{g}^{\alpha}(s,\overline{y}^{\alpha}(s))ds \\
& - \int_0^T \overline{G}^{\alpha}(T-s)\overline{h}^{\alpha}(s)dB_H(s) - \sum_{t_k=0}^{T} \overline{G}^{\alpha}(T-t_k)\overline{I}_k^{\alpha}(\overline{y}^{\alpha}(t_k^-))\Bigg)\Bigg] = \left[\underline{y}_1^{\alpha}, \overline{y}_1^{\alpha}\right] = \left[y_1\right]^{\alpha}.
\end{aligned}
$$

Now, we set

$$
\begin{aligned}
(\Pi y)(t) = & G(t)\left[y_0 - f(0,y_0)\right] + f(t,y(t)) + \int_0^t AG(t-s)f(s,y(s))ds \\
& + \int_0^t G(t-s)g(s,y(s))ds + \left\langle \int_0^t G(t-s)h(s)dB_H(s) \right\rangle \\
& + \sum_{t_k=0}^{t} G(t-t_k)I_k(y(t_k^-)) + \int_0^t G(t-s)\mathcal{W}^{-1}\Bigg(y_1 - G(T)[y_0 - f(0,y_0)] \\
& - f(T,y(t)) - \int_0^T AG(T-s)f(s,y(s))ds - \int_0^T G(T-s)g(s,y(s))ds \\
& - \left\langle \int_0^T G(T-s)h(s)dB_H(s) \right\rangle - \sum_{t_k=0}^{T} G(T-t_k)I_k(y(t_k^-))\Bigg)ds, \qquad t \in J. \qquad (3.2.3)
\end{aligned}
$$

In the following theorem, the controllability result of fuzzy solutions for (3.2.1) is established.

Theorem 3.2.4 *If the hypotheses* $(\mathcal{H}1)$-$(\mathcal{H}3)$ *are satisfied and* $\lambda(1+NT) + MT\big(\gamma + \mu + \lambda - \lambda NT - \gamma MT - \mu M\big) < 1$, *then for all* $T > 0$, *the system* (3.2.1) *is controllable on* J.

Proof. We can easily check that Π is continuous mapping from $C(J, \mathbf{E}^n)$ to itself. For $y, z \in C(J, \mathbf{E}^n)$, we have

$$\begin{aligned}
\mathbf{D}_H\big([\Pi y(t)]^\alpha, [\Pi z(t)]^\alpha\big) &= \mathbf{D}_H\Bigg(\Big[G(t)\{y_0 - f(0, y_0)\}\Big]^\alpha + \Big[\int_0^t AG(t-s)f(s, y(s))ds\Big]^\alpha \\
&+ \Big[f(t, y(t))\Big]^\alpha + \Big[\int_0^t G(t-s)g(s, y(s))ds\Big]^\alpha + \Big[\Big\langle \int_0^t G(t-s)h(s)dB_H(s)\Big\rangle\Big]^\alpha \\
&+ \Big[\sum_{t_k=0}^{t} G(t-t_k)I_k(y(t_k^-))\Big]^\alpha + \Big[\int_0^t G(t-s)\mathcal{W}^{-1}\Big(y_1 - G(T)\{y_0 - f(0, y_0)\} \\
&- f(T, y(t)) - \int_0^T AG(T-s)f(s, y(s))ds - \int_0^T G(T-s)g(s, y(s))ds \\
&- \Big\langle \int_0^T G(T-s)h(s)dB_H(s)\Big\rangle - \sum_{t_k=0}^{t} G(T-t_k)I_k(y(t_k^-))\Big)ds\Big]^\alpha, \Big[G(t)\{y_0 \\
&- f(0, y_0)\}\Big]^\alpha + \Big[\int_0^t AG(t-s)f(s, z(s))ds\Big]^\alpha + \Big[\int_0^t G(t-s)g(s, z(s))ds\Big]^\alpha \\
&+ \Big[f(t, z(t))\Big]^\alpha + \Big[\Big\langle \int_0^t G(t-s)h(s)dB_H(s)\Big\rangle\Big]^\alpha + \Big[\sum_{t_k=0}^{t} G(t-t_k)I_k(z(t_k^-))\Big]^\alpha \\
&+ \Big[\int_0^t G(t-s)\mathcal{W}^{-1}\Big(y_1 - G(T)\{y_0 - f(0, y_0)\} - f(T, z(t)) - \int_0^T AG(T-s)f(s, z(s))ds \\
&- \int_0^T G(T-s)g(s, z(s))ds - \Big\langle \int_0^T G(T-s)h(s)dB_H(s)\Big\rangle - \sum_{t_k=0}^{t} G(T-t_k)I_k(y(t_k^-))\Big)ds\Big]^\alpha\Bigg), \\
&\le \lambda \mathbf{D}_H\Big([y(t)]^\alpha, [z(t)]^\alpha\Big) + \lambda N \int_0^t \mathbf{D}_H\Big([y(s)]^\alpha, [z(s)]^\alpha\Big)ds \\
&+ \gamma M \int_0^t \mathbf{D}_H\Big([y(s)]^\alpha, [z(s)]^\alpha\Big)ds + \mu M \mathbf{D}_H\Big([y(t)]^\alpha, [z(t)]^\alpha\Big) \\
&+ M \int_0^t \Big(\lambda \mathbf{D}_H\Big([y(t)]^\alpha, [z(t)]^\alpha\Big) - \lambda N \int_0^T \mathbf{D}_H\Big([y(s)]^\alpha, [z(s)]^\alpha\Big)ds \\
&- \gamma M \int_0^T \mathbf{D}_H\Big([y(s)]^\alpha, [z(s)]^\alpha\Big)ds - \mu M \mathbf{D}_H\Big([y(s)]^\alpha, [z(s)]^\alpha\Big)\Big)ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{D}_\infty\Big(\Pi y(t),\Pi z(t)\Big) &= \sup_{0\le\alpha\le1}\mathbf{D}_H\Big([\Pi y(t)]^\alpha,[\Pi z(t)]^\alpha\Big),\\
&\le \lambda\sup_{0\le\alpha\le1}\mathbf{D}_H\Big([y(t)]^\alpha,[z(t)]^\alpha\Big)+\lambda N\int_0^t\sup_{0\le\alpha\le1}\mathbf{D}_H\Big([y(s)]^\alpha,[z(s)]^\alpha\Big)ds\\
&+\gamma M\int_0^t\sup_{0\le\alpha\le1}\mathbf{D}_H\Big([y(s)]^\alpha,[z(s)]^\alpha\Big)ds+\mu M\sup_{0\le\alpha\le1}\mathbf{D}_H\Big([y(t)]^\alpha,[z(t)]^\alpha\Big)\\
&+M\int_0^t\Bigg(\lambda\sup_{0\le\alpha\le1}\mathbf{D}_H\Big([y(t)]^\alpha,[z(t)]^\alpha\Big)-\mu M\sup_{0\le\alpha\le1}\mathbf{D}_H\Big([y(s)]^\alpha,[z(s)]^\alpha\Big)\\
&-\lambda N\int_0^T\sup_{0\le\alpha\le1}\mathbf{D}_H\Big([y(s)]^\alpha,[z(s)]^\alpha\Big)ds\\
&-\gamma M\int_0^T\sup_{0\le\alpha\le1}\mathbf{D}_H\Big([y(s)]^\alpha,[z(s)]^\alpha\Big)ds\Bigg)ds,\\
&\le \lambda\mathbf{D}_\infty\Big(y(t),z(t)\Big)+\lambda N\int_0^t\mathbf{D}_\infty\Big(y(s),z(s)\Big)ds\\
&+\gamma M\int_0^t\mathbf{D}_\infty\Big(y(s),z(s)\Big)ds+\mu M\mathbf{D}_\infty\Big(y(t),z(t)\Big)\\
&+M\int_0^t\Bigg(\lambda\mathbf{D}_\infty\Big(y(t),z(t)\Big)-\lambda N\int_0^T\mathbf{D}_\infty\Big(y(s),z(s)\Big)ds\\
&-\gamma M\int_0^T\mathbf{D}_\infty\Big(y(s),z(s)\Big)ds-\mu M\mathbf{D}_\infty\Big(y(s),z(s)\Big)\Bigg)ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
D\Big(\Pi y,\Pi z\Big) &= \sup_{0\le t\le T}\mathbf{D}_\infty\Big(\Pi y(t),\Pi z(t)\Big)\\
&\le \lambda\sup_{0\le t\le T}\mathbf{D}_\infty\Big(y(t),z(t)\Big)+\lambda N\int_0^t\sup_{0\le s\le T}\mathbf{D}_\infty\Big(y(s),z(s)\Big)ds\\
&+\gamma M\int_0^t\sup_{0\le s\le T}\mathbf{D}_\infty\Big(y(s),z(s)\Big)ds+\mu M\sup_{0\le t\le T}\mathbf{D}_\infty\Big(y(t),z(t)\Big)\\
&+M\int_0^t\Bigg(\lambda\sup_{0\le s\le T}\mathbf{D}_\infty\Big(y(t),z(t)\Big)-\lambda N\int_0^T\sup_{0\le s\le T}\mathbf{D}_\infty\Big(y(s),z(s)\Big)ds\\
&-\gamma M\int_0^T\sup_{0\le s\le T}\mathbf{D}_\infty(y(s),z(s))ds-\mu M\sup_{0\le s\le T}\mathbf{D}_\infty\Big(y(s),z(s)\Big)\Bigg)ds,\\
&\le \lambda D(y,z)+\lambda NTD(y,z)+\gamma MTD(y,z)+\mu MD(y,z)+MT\Bigg(\lambda D(y,z)\\
&-\lambda NTD(y,z)-\gamma MTD(y,z)-\mu MD(y,z)\Bigg),\\
&\le\Bigg(\lambda(1+NT)+MT(\gamma+\mu+\lambda-\lambda NT-\gamma MT-\mu M)\Bigg)D(y,z).
\end{aligned}$$

Then, Π is a contraction mapping. So, by Banach fixed point Theorem, Eq. (3.2.3) has a unique fixed point $x\in C(J,\mathbf{E}^n)$. Thus, the system (3.2.1) is controllable on J.□

3.3 Example

In this section, we give an example to illustrate our results. let

$$\begin{cases} \frac{d}{dt}\left[y(t) - 3ty^3(t)\right] = 2y(t) + 2ty^3(t) + \nu(t) + \left\langle h(t)dB_H(t)\right\rangle, t \in J, \\ I_k(y(t_k^-)) = \frac{2}{2+y(t_k)}, \\ y(0) = 0 \in \mathbf{E}^n. \end{cases} \tag{3.3.1}$$

Then, we set $f(t, y(t)) = 3ty^3(t)$ and $g(t, y(t)) = 2ty^3(t)$. Hence, α-level of f is

$$\begin{aligned}\Big[f(t, y(t))\Big]^{\alpha} &= \Big[3ty^3(t)\Big]^{\alpha}, \\ &= \Big[t(\alpha+2)(\underline{y}^{\alpha}(t))^3, t(4-\alpha)(\overline{y}^{\alpha}(t))^3\Big], \qquad \forall\alpha \in [0,1].\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathbf{D}_H\Big(\left[f(t,y(t))\right]^{\alpha}, \left[f(t,z(t))\right]^{\alpha}\Big) &= \mathbf{D}_H\Big(\left[t(\alpha+2)(\underline{y}^{\alpha}(t))^3, t(4-\alpha)\left(\overline{y}^{\alpha}(t)\right)^3\right]^{\alpha}, \left[t(\alpha+2)\right. \\ &\left.\left(\underline{z}^{\alpha}(t)\right)^3, t(4-\alpha)\left(\overline{z}^{\alpha}(t)\right)^3\right]^{\alpha}\Big), \\ &\le t\max\left\{(\alpha+2)|(\underline{y}^{\alpha}(t))^3 - (\underline{z}^{\alpha}(t))^3|, (4-\alpha)|(\overline{y}^{\alpha}(t))^3 - (\overline{z}^{\alpha}(t))^3|\right\}, \\ &\le 4t\Big((\overline{y}^{\alpha}(t))^2 + \overline{y}^{\alpha}(t)\overline{z}^{\alpha}(t) + (\overline{z}^{\alpha}(t))^2\Big)\max\left\{|\underline{y}^{\alpha}(t) - \underline{z}^{\alpha}(t)|, |\overline{y}^{\alpha}(t) - \overline{z}^{\alpha}(t)|\right\}, \\ &\le K_1\mathbf{D}_H\Big([y(t)]^{\alpha}, [z(t)]^{\alpha}\Big),\end{aligned}$$

where $K_1 = 4t\Big((\overline{y}^{\alpha}(t))^2 + \overline{y}^{\alpha}(t)\overline{z}^{\alpha}(t) + (\overline{z}^{\alpha}(t))^2\Big) > 0$.
On the other hand, the α-level of g is given by:

$$\begin{aligned}\Big[g(t, y(t))\Big]^{\alpha} &= \Big[2ty^3(t)\Big]^{\alpha} \\ &= \Big[t(\alpha+1)(\underline{y}^{\alpha}(t))^3, t(3-\alpha)(\overline{y}^{\alpha}(t))^3\Big], \qquad \forall\alpha \in [0,1].\end{aligned}$$

Therefore, we get:

$$\begin{aligned}\mathbf{D}_H\Big(\left[g(t,y(t))\right]^{\alpha}, \left[g(t,z(t))\right]^{\alpha}\Big) &= \mathbf{D}_H\Big(\left[t(\alpha+1)(\underline{y}^{\alpha}(t))^3, t(3-\alpha)(\overline{y}^{\alpha}(t))^3\right], \left[t(\alpha+1)\right. \\ &\left.(\underline{z}^{\alpha}(t))^3, t(3-\alpha)(\overline{z}^{\alpha}(t))^3\right]\Big), \\ &\le t\max\left\{(\alpha+1)|(\underline{y}^{\alpha}(t))^3 - (\underline{z}^{\alpha}(t))^3|, (3-\alpha)|(\overline{y}^{\alpha}(t))^3 - (\overline{z}^{\alpha}(t))^3|\right\}, \\ &\le 3t\Big((\overline{y}^{\alpha}(t))^2 + \overline{y}^{\alpha}(t)\overline{z}^{\alpha}(t) + (\overline{z}^{\alpha}(t))^2\Big)\max\left\{|\underline{y}^{\alpha}(t) - \underline{z}^{\alpha}(t)|, |\overline{y}^{\alpha}(t) - \overline{z}^{\alpha}(t)|\right\}, \\ &\le K_2\mathbf{D}_H\Big([y(t)]^{\alpha}, [z(t)]^{\alpha}\Big),\end{aligned}$$

where $K_2 = 3t\Big((\overline{y}^{\alpha}(t))^2 + \overline{y}^{\alpha}(t)\overline{z}^{\alpha}(t) + (\overline{z}^{\alpha}(t))^2\Big) > 0$. And, the α-level of I is given by:

$$\Big[I_k(y(t_k^-))\Big]^{\alpha} = \left[\frac{2}{2+\underline{y}^{\alpha}(t_k)}, \frac{2}{2+\overline{y}^{\alpha}(t_k)}\right], \qquad \forall\alpha \in [0,1].$$

Then, we have:

$$\begin{aligned}\mathbf{D}_H\Big(\big[I_k(y(t_k^-))\big]^\alpha,\big[I_k(z(t_k^-))\big]^\alpha\Big) &= \mathbf{D}_H\Big(\Big[\frac{2}{2+\underline{y}^\alpha(t_k)},\frac{2}{2+\overline{y}^\alpha(t_k)}\Big],\Big[\frac{2}{2+\underline{z}^\alpha(t_k)},\frac{2}{2+\overline{z}^\alpha(t_k)}\Big]\Big),\\ &\le \max\Big\{\Big|\frac{2}{2+\underline{y}^\alpha(t_k^-)}-\frac{2}{2+\underline{z}^\alpha(t_k^-)}\Big|,\Big|\frac{2}{2+\overline{y}^\alpha(t_k^-)}-\frac{2}{2+\overline{z}^\alpha(t_k^-)}\Big|\Big\},\\ &\le K_3\mathbf{D}_H\Big(\big[y(t)\big]^\alpha,\big[z(t)\big]^\alpha\Big),\end{aligned}$$

where $K_3=\frac{2}{(2+\overline{y}^\alpha(t_k^-))(2+\overline{z}^\alpha(t_k^-))}>0$. Hence, the constants k_1, K_2, K_3 are satisfied the hypotheses $(\mathcal{H}1)$-$(\mathcal{H}4)$. Thus, the conditions of theorem 3.2.2 are satisfied. Therefore, systeme (3.3.1) has a unique fuzzy solution.

Now, we examine the controllability result, then, let's consider the target state $y_1=3\in\mathbf{E}^n$.

$$\begin{aligned}\big[v(s)\big]^\alpha &= \big[\underline{v}^\alpha(s),\overline{v}^\alpha(s)\big],\\ &= \Big[(\underline{\mathcal{W}}^\alpha)^{-1}\Big((\alpha+2)-T(\alpha+2)\underline{y}^\alpha(T)-\int_0^T G(\alpha+1)\underline{G}^\alpha(T-s)(\alpha+2)\underline{y}^\alpha(s)ds\\ &-\int_0^T s\underline{G}^\alpha(T-s)(\alpha+1)\underline{y}^\alpha(s)ds-\int_0^T\underline{G}^\alpha(T-s)\underline{h}^\alpha(s)dB_H(s)\\ &-2\sum_{t_k=0}^{t}\frac{\underline{G}^\alpha(T-t_k)}{2+\underline{y}^\alpha(t_k^-)}\Big),(\overline{\mathcal{W}}^\alpha)^{-1}\Big((4-\alpha)-T(4-\alpha)\overline{y}^\alpha(T)\\ &-\int_0^T G(3-\alpha)\overline{G}^\alpha(T-s)(4-\alpha)\overline{y}^\alpha(s)ds-\int_0^T s\overline{G}^\alpha(T-s)(3-\alpha)\overline{y}^\alpha(s)ds\\ &-\int_0^T\overline{G}^\alpha(T-s)\overline{h}^\alpha(s)dB_H(s)-2\sum_{t_k=0}^{t}\frac{\overline{G}^\alpha(T-t_k)}{2+\overline{y}^\alpha(t_k^-)}\Big)\Big].\end{aligned}$$

Further, by remplacing the above derived values into the integral equation with respect to system (3.3.1), we obtain $[y(T)]^\alpha=[3]^\alpha$. Hence, all conditions of theorem 3.2.4 are fulfilled. Therefore, the system (3.3.1) is controllable.

Chapter 4

Existence and stability of solutions for a coupled system of fuzzy fractional pantograph stochastic differential equations

Contents

4.1 Introduction

The most significant area of practical mathematics that converts all currently available integer-order operators to arbitrary-order ones is fractional calculus. This significance results from the excellent precision of fractional operators in the simulation of numerous real-world occurrences in the context of various fractional boundary value problems.

Pantograph differential equations are functional differential equations with proportional delays. Because these equations have numerous real-world applications in areas like electrodynamics, astrophysics, and cell development, many researchers have explored them numerically and analytically.

On the other hand, one essential qualitative theory for dynamical systems is the notion of stability. As a result, the theory of stability characteristics has attracted significant interest through applications in a number of study domains.

Particularly, many scholars have looked into the Ulam-Hyers (UH) stability analysis and its applicability to many kinds of differential equations. We wish to mention that the theory of fuzzy fractional pantograph stochastic differential equations (FFPSDEs) have recently been

the subject of important studies. As, for the UH stability of this type of system, even less has been done, with only a few works published in this topic as for as we know.
In [60], Jameel et al. studied approximate solution fuzzy pantograph equation by using homotopy perturbation method. Mikaeilvand et al. [88] introduced a numerical method based on the Taylor polynomials for the approximate solution of fuzzy pantograph equation. Hoseinzadet et al. [53] studied the pantograph volterra fuzzy integrodifferential equation. In [5] Agilan et al. proved the existence of solutions of fuzzy fractional panto-graph equations. Recently, Priyadharsini et al. [58] proposed a new type of equation nemely fuzzy fractional stochastic pantograph delay differential system. Then, Arhrrabi et al. [19, 16, 17, 18] studied the existence and stability of solutions of fuzzy fractional stochastic differential equations with fractional Brownian motions, averaging principle for fuzzy stochastic differential equations, fuzzy fractional boundary value problem and existence and uniqueness results of fuzzy fractional stochastic differential equations with impulsive. Melliani et al. [87] studied Ulam-Hyers-Rassias stability for fuzzy fractional integrodifferential equations under Caputo gH-differentiability. On the other hand, UH stability is another interesting topic for mathematicians researchers. Then, stability theory research is currently fairly common and many articles have been published. Sajedi et al. [101] investigate the existence, uniqueness and UH stability of solutions of an impulsive coupled system of fractional differential equation with Caputo-Katugampola fuzzy fractional derivative. For more details, references [97, 98, 99, 100] are some of the studies on the UH stability. Chalishajar et al. [30] studied existence and stability of solutions for a coupled system of fractional differential equation. Nabil [91] established the existence and Ulam stability of nonlinear coupled system of fractional differential equation including generalized Caputo fractional derivative. For more details, references [63, 13, 64, 92, 123, 47] are some of important studies on the coupled system of FDEs.
Motivated by the above mentioned works and its importance in many applied fields, it is interesting to study the coupled system of FFPSDEs. In this chapter we have two results: The first one is the existence and uniqueness of solutions, this result is given in the first subsection. In the second subsection, UH stability result is proved.

4.2 Main results

In this section, we will study the existence, uniqueness and UH stability of the following coupled system

$$\begin{cases} {}^{C}\mathcal{D}^{\gamma_1}x(t) = Ax(t) + f_1\big(t, x(t), x(\lambda_1 t), y(t)\big) + \Big\langle \int_0^t g_1\big(s, x(s), x(\lambda_1 s), y(s)\big)dB(s)\Big\rangle, \\ {}^{C}\mathcal{D}^{\gamma_2}y(t) = Ay(t) + f_2\big(t, y(t), y(\lambda_2 t), x(t)\big) + \Big\langle \int_0^t g_2\big(s, y(s), y(\lambda_2 s), x(s)\big)dB(s)\Big\rangle, \\ x(0) = x_0 \in \mathbf{E}^n \qquad \text{and} \qquad y(0) = y_0 \in \mathbf{E}^n, \end{cases} \tag{4.2.1}$$

where ${}^{C}\mathcal{D}^{\gamma_1}$, ${}^{C}\mathcal{D}^{\gamma_2}$ denote the Caputo fractional derivative of order γ_1 and γ_2 respectively, A is $n\times n$ matrix, the functions $f_1, f_2 : J\times\mathbf{E}^n\times\mathbf{E}^n\times\mathbf{E}^n \longrightarrow \mathbf{E}^n$ and $g_1, g_2 : J\times\mathbf{E}^n\times\mathbf{E}^n\times\mathbf{E}^n \longrightarrow \mathbb{R}^{n\times m}$ are continuous on J, $B(t)$ is standard Brownian motion with n-dimensional.

4.2.1 Existence and uniqueness results

In this subsection, we show the existence and uniqueness of fuzzy solution for a coupled system (4.2.1).

Definition 4.2.1 *A couple (x, y) is a solution of problem (4.2.1) if it satisfies*

$$\begin{cases} {}^{C}\mathcal{D}^{\gamma_1}x(t) = Ax(t) + f_1\big(t, x(t), x(\lambda_1 t), y(t)\big) + \Big\langle \int_0^t g_1\big(s, x(s), x(\lambda_1 s), y(s)\big)dB(s)\Big\rangle, \\ \\ {}^{C}\mathcal{D}^{\gamma_2}y(t) = Ay(t) + f_2\big(t, y(t), y(\lambda_2 t), x(t)\big) + \Big\langle \int_0^t g_2\big(s, y(s), y(\lambda_2 s), x(s)\big)dB(s)\Big\rangle, \end{cases}$$

and the conditions initial $x(0) = x_0$, $y(0) = y_0$.

It follows from [95], that the solution of (4.2.1) can be expressed as follows:

$$\begin{aligned} x(t) &= M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t (t-s)^{\gamma_1-1}M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)f_1\big(s, x(s), x(\lambda_1 s), y(s)\big)ds \\ &\quad + \int_0^t (t-s)^{\gamma_1-1}M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)\Big\langle \int_0^s g_1\big(u, x(u), x(\lambda_1 u), y(u)\big)dB(u)\Big\rangle ds. \end{aligned}$$

$$\begin{aligned} y(t) &= M_{\gamma_2}(At^{\gamma_2})y_0 + \int_0^t (t-s)^{\gamma_2-1}M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big)f_2\big(s, y(s), y(\lambda_2 s), x(s)\big)ds \\ &\quad + \int_0^t (t-s)^{\gamma_2-1}M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big)\Big\langle \int_0^s g_2\big(u, y(u), y(\lambda_2 u), x(u)\big)dB(u)\Big\rangle ds. \end{aligned}$$

We will study the proposed system under the following assumptions:

($\mathcal{A}1$) The functions f_1, f_2 are continuous and $\exists L_1, L_2 > 0$ such that

$$\mathbb{E}\mathbf{D}^2_\infty\Big(f_1(t, x, y, z), f_1(t, u, v, w)\Big) \leq L_1\Big(\mathbb{E}\mathbf{D}^2_\infty(x, u) + \mathbb{E}\mathbf{D}^2_\infty(y, v) + \mathbb{E}\mathbf{D}^2_\infty(z, w)\Big).$$

$$\mathbb{E}\mathbf{D}_{\infty}^{2}\Big(f_2(t,x,y,z),f_2(t,u,v,w)\Big)\leq L_2\Big(\mathbb{E}\mathbf{D}_{\infty}^{2}(x,u)+\mathbb{E}\mathbf{D}_{\infty}^{2}(y,v)+\mathbb{E}\mathbf{D}_{\infty}^{2}(z,w)\Big).$$

$(\mathcal{A}2)$ The functions g_1, g_2 are continuous and $\exists N_1, N_2 > 0$ such that

$$\mathbb{E}\Big\|g_1(t,x,y,z)-g_1(t,u,v,w)\Big\|^2\leq N_1\Big(\mathbb{E}\mathbf{D}_{\infty}^{2}(x,u)+\mathbb{E}\mathbf{D}_{\infty}^{2}(y,v)+\mathbb{E}\mathbf{D}_{\infty}^{2}(z,w)\Big).$$

$$\mathbb{E}\Big\|g_2(t,x,y,z)-g_2(t,u,v,w)\Big\|^2\leq N_2\Big(\mathbb{E}\mathbf{D}_{\infty}^{2}(x,u)+\mathbb{E}\mathbf{D}_{\infty}^{2}(y,v)+\mathbb{E}\mathbf{D}_{\infty}^{2}(z,w)\Big).$$

$(\mathcal{A}3)$ For all $t\in J$, $\exists Q_1, Q_2 > 0$ such that

$$\mathbb{E}\mathbf{D}_{\infty}^{2}\Big(f_1(t,\hat{0},\hat{0},\hat{0}),\hat{0}\Big)\leq Q_1 \qquad \text{and} \qquad \mathbb{E}\mathbf{D}_{\infty}^{2}\Big(f_2(t,\hat{0},\hat{0},\hat{0}),\hat{0}\Big)\leq Q_2.$$

$(\mathcal{A}4)$ For all $t\in J$, $\exists P_1, P_2 > 0$ such that

$$\mathbb{E}\Big\|g_1(t,\hat{0},\hat{0},\hat{0})\Big\|^2\leq P_1 \qquad \text{and} \qquad \mathbb{E}\Big\|g_2(t,\hat{0},\hat{0},\hat{0})\Big\|^2\leq P_2.$$

Theorem 4.2.2 *Assume that the assumptions* $(\mathcal{A}1)$*-*$(\mathcal{A}4)$ *holds. Then the system* (4.2.1) *has a unique solution provided that*

$$\frac{6(L_1+L_2)T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}+\frac{6(N_1+N_2)T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}<1.$$

Proof. We define the operator $F:\mathbb{K}\times\mathbb{K}\longrightarrow\mathbb{K}\times\mathbb{K}$ by

$$F(x,y)(t)=\Big(T_1(x,y)(t),T_2(x,y)(t)\Big),$$

where

$$\begin{aligned}T_1(x,y)(t)&=M_{\gamma_1}(At^{\gamma_1})x_0+\int_0^t(t-s)^{\gamma_1-1}M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)f_1\big(s,x(s),x(\lambda_1 s),y(s)\big)ds\\&+\int_0^t(t-s)^{\gamma_1-1}M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)\Big\langle\int_0^s g_1\big(u,x(u),x(\lambda_1 u),y(u)\big)dB(u)\Big\rangle ds.\end{aligned}$$

$$\begin{aligned}T_2(x,y)(t)&=M_{\gamma_2}(At^{\gamma_2})y_0+\int_0^t(t-s)^{\gamma_2-1}M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big)f_2\big(s,y(s),y(\lambda_2 s),x(s)\big)ds\\&+\int_0^t(t-s)^{\gamma_2-1}M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big)\Big\langle\int_0^s g_2\big(u,y(u),y(\lambda_2 u),x(u)\big)dB(u)\Big\rangle ds.\end{aligned}$$

Then, the fixed point of the operator F coincides with a solution of coupled system (4.2.1). For each positive number r, we define

$$\mathcal{B}_r=\Big\{(x,y)\in\mathbb{K}\times\mathbb{K}:\mathbb{E}\mathbf{D}_{\infty}^{2}\big((x,y),\hat{0}\big)\leq r\Big\}.$$

We divide the subsequent proof into two steps.

Step 1: We prove that $F(\mathcal{B}_r)\subseteq\mathcal{B}_r$. We choose

$$r\geq\frac{3\mathbb{E}\mathbf{D}_{\infty}^{2}(x_0,\hat{0})+3\mathbb{E}\mathbf{D}_{\infty}^{2}(y_0,\hat{0})+J_1+K_1}{1-J_2-K_2}.$$

By using the assumptions $(\mathcal{A}1)$-$(\mathcal{A}4)$, Cauchy–Schwarz inequality and Itô isometry, we get

$$\begin{aligned}
\mathbb{E}\mathbf{D}^2_\infty\big(T_1(x,y)(t),\hat{0}\big) &= \mathbb{E}\mathbf{D}^2_\infty\Bigg(M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}} f_1\big(s,x(s),x(\lambda_1 s),y(s)\big)ds \\
&+ \int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}}\Big\langle \int_0^s g_1\big(u,x(u),x(\lambda_1 u),y(u)\big)dB(u)\Big\rangle ds, \hat{0}\Bigg), \\
&\leq 3\mathbb{E}\mathbf{D}^2_\infty\Big(M_{\gamma_1}(At^{\gamma_1})x_0,\hat{0}\Big) + 3\mathbb{E}\mathbf{D}^2_\infty\Bigg(\int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}} f_1\big(s,x(s),x(\lambda_1 s),y(s)\big)ds,\hat{0}\Bigg) \\
&+ 3\mathbb{E}\mathbf{D}^2_\infty\Bigg(\int_0^t \frac{M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)}{(t-s)^{1-\gamma_1}}\Big\langle \int_0^s g_1\big(u,x(u),x(\lambda_1 u),y(u)\big)dB(u)\Big\rangle ds,\hat{0}\Bigg), \\
&\leq 3\mathbb{E}\mathbf{D}^2_\infty(x_0,\hat{0}) + \frac{6T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \mathbb{E}\mathbf{D}^2_\infty\Big(f_1\big(s,x(s),x(\lambda_1 s),y(s)\big), f_1\big(s,\hat{0},\hat{0},\hat{0}\big)\Big)ds \\
&+ \frac{6T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \mathbb{E}\mathbf{D}^2_\infty\Big(f_1\big(s,\hat{0},\hat{0},\hat{0}\big),\hat{0}\Big)ds \\
&+ \frac{6T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t\Bigg(\int_0^s \mathbb{E}\Big\| g_1\big(u,x(u),x(\lambda_1 u),y(u)\big) - g_1\big(u,\hat{0},\hat{0},\hat{0}\big)\Big\|^2 du\Bigg)ds \\
&+ \frac{6C_T T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t\Bigg(\int_0^s \mathbb{E}\Big\| g_1\big(u,\hat{0},\hat{0},\hat{0}\big)\Big\|^2 du\Bigg)ds, \\
&\leq 3\mathbb{E}\mathbf{D}^2_\infty(x_0,\hat{0}) + \frac{6L_1T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t\Big[\mathbb{E}\mathbf{D}^2_\infty(x(s),\hat{0}) + \mathbb{E}\mathbf{D}^2_\infty(x(\lambda_1 s),\hat{0}) + \mathbb{E}\mathbf{D}^2_\infty(y(s),\hat{0})\Big]ds \\
&+ \frac{6N_1T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t (t-s)^{\gamma_1-1}\Bigg(\int_0^s\Big[\mathbb{E}\mathbf{D}^2_\infty(x(u),\hat{0}) + \mathbb{E}\mathbf{D}^2_\infty(x(\lambda_1 u),\hat{0}) + \mathbb{E}\mathbf{D}^2_\infty(y(u),\hat{0})\Big]du\Bigg)ds \\
&+ \frac{6Q_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{3P_1C_TT^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}, \\
&\leq 3\mathbb{E}\mathbf{D}^2_\infty(x_0,\hat{0}) + \frac{6L_1T^{2\gamma_1}r}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6Q_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6N_1T^{2\gamma_1+1}r}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{3P_1C_TT^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}, \\
&\leq 3\mathbb{E}\mathbf{D}^2_\infty(x_0,\hat{0}) + \frac{6Q_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{3P_1C_TT^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \Bigg(\frac{6L_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6N_1T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\Bigg)r, \\
&\leq 3\mathbb{E}\mathbf{D}^2_\infty(x_0,\hat{0}) + J_1 + J_2 r,
\end{aligned}$$

where

$$J_1 = \frac{6Q_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{3P_1C_TT^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} \quad \text{and} \quad J_2 = \frac{6L_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6N_1T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}.$$

In the same way, we can obtain that:

$$\mathbb{E}\mathbf{D}^2_\infty\big(T_2(x,y)(t),\hat{0}\big) \leq 3\mathbb{E}\mathbf{D}^2_\infty(y_0,\hat{0}) + K_1 + K_2 r,$$

where

$$K_1 = \frac{6Q_2T^{2\gamma_2}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2} + \frac{3P_2C_TT^{2\gamma_2+1}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2} \quad \text{and} \quad K_2 = \frac{6L_2T^{2\gamma_2}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2} + \frac{6N_2T^{2\gamma_2+1}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2}.$$

Finally, we have

$$\mathbb{E}\mathbf{D}^2_\infty\big(F(x,y)(t),\hat{0}\big) \leq \mathbb{E}\mathbf{D}^2_\infty\big(T_1(x,y)(t),\hat{0}\big) + \mathbb{E}\mathbf{D}^2_\infty\big(T_2(x,y)(t),\hat{0}\big) \leq r,$$

which implies that $F(\mathcal{B}_r) \subseteq \mathcal{B}_r$.
Step 2: We show that F is a contraction operator. For $(x,y),(x',y') \in \mathbb{K}\times\mathbb{K}$ and $t \in J$, using

the assumptions $(\mathcal{A}1)$-$(\mathcal{A}4)$, Cauchy–Schwarz inequality and Itô isometry, we have

$$\begin{aligned}
&\mathbb{E}\mathbf{D}^2_\infty(T_1(x,y)(t),T_1(x',y')(t)) = \mathbb{E}\mathbf{D}^2_\infty\Big(M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} f_1(s,x(s),\\
&x(\lambda_1 s),y(s))ds + \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} \Big\langle \int_0^s g_1(u,x(u),x(\lambda_1 u),y(u))dB(u)\Big\rangle ds, M_{\gamma_1}(At^{\gamma_1})x_0\\
&+ \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} f_1(s,x'(s),x'(\lambda_1 s),y'(s))ds\\
&+ \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} \Big\langle \int_0^s g_1(u,x'(u),x'(\lambda_1 u),y'(u))dB(u)\Big\rangle ds\Big),\\
&\leq 2\mathbb{E}\mathbf{D}^2_\infty\Big(\int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} f_1(s,x(s),x(\lambda_1 s),y(s))ds, \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}}\\
&f_1(s,x'(s),x'(\lambda_1 s),y'(s))ds\Big) + 2\mathbb{E}\mathbf{D}^2_\infty\Big(\int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} \Big\langle \int_0^s g_1(u,x(u),\\
&x(\lambda_1 u),y(u))dB(u)\Big\rangle ds, \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} \Big\langle \int_0^s g_1(u,x'(u),x'(\lambda_1 u),y'(u))dB(u)\Big\rangle ds\Big),\\
&\leq \frac{2T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \mathbb{E}\mathbf{D}^2_\infty\Big(f_1(s,x(s),x(\lambda_1(s)),y(s)), f_1(s,x'(s),x'(\lambda_1(s)),y'(s))\Big)ds\\
&+ \frac{2T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t\Big(\int_0^s \mathbb{E}\Big\| g_1(u,x(u),x(\lambda_1 u),y(u)) - g_1(u,x'(u),x'(\lambda_1 u),y'(u))\Big\|^2 du\Big)ds,\\
&\leq \frac{2L_1T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t\Big(\mathbb{E}\mathbf{D}^2_\infty(x(s),x'(s)) + \mathbb{E}\mathbf{D}^2_\infty(x(\lambda_1 s),x'(\lambda_1 s)) + \mathbb{E}\mathbf{D}^2_\infty(y(s),y'(s))\Big)ds,\\
&+ \frac{2N_1T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t\Big(\int_0^s\Big[\mathbb{E}\mathbf{D}^2_\infty(x(s),x'(s)) + \mathbb{E}\mathbf{D}^2_\infty(x(\lambda_1 s),x'(\lambda_1 s)) + \mathbb{E}\mathbf{D}^2_\infty(y(s),y'(s))\Big]du\Big)ds,\\
&\leq \frac{6L_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\Big(\mathbb{E}\mathbf{D}^2_\infty(x,x') + \mathbb{E}\mathbf{D}^2_\infty(y,y')\Big) + \frac{6N_1T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\Big(\mathbb{E}\mathbf{D}^2_\infty(x,x') + \mathbb{E}\mathbf{D}^2_\infty(y,y')\Big),\\
&\leq \Big(\frac{6L_1T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6N_1T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\Big)\Big(\mathbb{E}\mathbf{D}^2_\infty(x,x') + \mathbb{E}\mathbf{D}^2_\infty(y,y')\Big).
\end{aligned}$$

With a similar method, we also get:

$$\mathbb{E}\mathbf{D}^2_\infty(T_2(x,y)(t),T_2(x',y')(t)) \leq \Big(\frac{6L_2T^{2\gamma_2}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2} + \frac{6N_2T^{2\gamma_2+1}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2}\Big)\Big(\mathbb{E}\mathbf{D}^2_\infty(x,x') + \mathbb{E}\mathbf{D}^2_\infty(y,y')\Big).$$

Finally, we can get:

$$\mathbb{E}\mathbf{D}^2_\infty(F(x,y)(t),F(x',y')(t)) \leq \Big(\frac{6(L_1+L_2)T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6(N_1+N_2)T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\Big)\Big(\mathbb{E}\mathbf{D}^2_\infty(x,x') + \mathbb{E}\mathbf{D}^2_\infty(y,y')\Big).$$

So, since $\frac{6(L_1+L_2)T^{2\gamma_1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6(N_1+N_2)T^{2\gamma_1+1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} < 1$, then F is a contraction operator. Therefore, by using Banach's contraction mapping principle, we conclude that F has a fixed point which is the unique solution of (4.2.1).□

4.2.2 Stability result

In this subsection, we study Ulam's type stability for the coupled system (4.2.1). First, we recall the definition of those types of Ulam stability.

Definition 4.2.3 *[30] The coupled system (4.2.1) is said to be UH stable if there exists a constant $\omega = (\omega_1, \omega_2) > 0$ such that for each $\epsilon = (\epsilon_1, \epsilon_2) > 0$ and solution $(x, y) \in \mathbb{K} \times \mathbb{K}$ of the following inequality*

$$\begin{cases} \mathbb{E}\mathbf{D}^2_\infty\Big({}^C\mathcal{D}^{\gamma_1}x(t), Ax(t) + f_1\big(t, x(t), x(\lambda_1 t), y(t)\big) + \Big\langle \int_0^t g_1\big(s, x(s), x(\lambda_1 s), y(s)\big)dB(s)\Big\rangle\Big) \leq \epsilon_1, \\ \mathbb{E}\mathbf{D}^2_\infty\Big({}^C\mathcal{D}^{\gamma_2}y(t), Ay(t) + f_2\big(t, y(t), y(\lambda_2 t), x(t)\big) + \Big\langle \int_0^t g_2\big(s, y(s), y(\lambda_2 s), x(s)\big)dB(s)\Big\rangle\Big) \leq \epsilon_2, \end{cases} \tag{4.2.2}$$

there exists a solution $(v, k) \in \mathbb{K} \times \mathbb{K}$ of system (4.2.1), such that

$$\mathbb{E}\mathbf{D}^2_\infty\Big((x, y)(t), (v, k)(t)\Big) \leq \omega\epsilon, \qquad t \in J.$$

Definition 4.2.4 *[30] The system (4.2.1) is said to be generalized UH (GUH) stable if there exists $\varphi \in C^1\big(J, \mathbf{E}^n\big)$, $\varphi(0) = 0$ such that for each solution $(x, y) \in \mathbb{K} \times \mathbb{K}$ of (4.2.2), there exists a solution $(v, k) \in \mathbb{K} \times \mathbb{K}$ of system (4.2.1) such that*

$$\mathbb{E}\mathbf{D}^2_\infty\Big((x, y)(t), (v, k)(t)\Big) \leq \varphi(\epsilon), \qquad t \in J.$$

Remark 4.2.5 *A couple $(x, y) \in \mathbb{K} \times \mathbb{K}$ is a solution of (4.2.2) if and only if $\exists(\phi, \psi) \in \mathbb{K} \times \mathbb{K}$ such that*

(i)- $\mathbb{E}\mathbf{D}^2_\infty\big(\phi(t), \hat{0}\big) \leq \epsilon_1,$ *and* $\mathbb{E}\mathbf{D}^2_\infty\big(\psi(t), \hat{0}\big) \leq \epsilon_2,$ $t \in J.$

(ii)- *For* $t \in J$,

$${}^C\mathcal{D}^{\gamma_1}x(t) = Ax(t) + f_1\big(t, x(t), x(\lambda_1 t), y(t)\big) + \Big\langle \int_0^t g_1\big(s, x(s), x(\lambda_1 s), y(s)\big)dB(s)\Big\rangle + \phi(t),$$

$${}^C\mathcal{D}^{\gamma_2}y(t) = Ay(t) + f_2\big(t, y(t), y(\lambda_2 t), x(t)\big) + \Big\langle \int_0^t g_2\big(s, y(s), y(\lambda_2 s), x(s)\big)dB(s)\Big\rangle + \psi(t).$$

Lemma 4.2.6 *Suppose that $(x, y) \in \mathbb{K} \times \mathbb{K}$ is a solution of the inequality system (4.2.2). So, we have*

$$\mathbb{E}\mathbf{D}^2_\infty\big[x(t), L(t)\big] \leq \frac{T^{2\gamma_1 - 1}}{(2\gamma_1 - 1)(\Gamma(\gamma_1))^2}\epsilon_1 \quad \text{and} \quad \mathbb{E}\mathbf{D}^2_\infty\big[y(t), P(t)\big] \leq \frac{T^{2\gamma_2 - 1}}{(2\gamma_2 - 1)(\Gamma(\gamma_2))^2}\epsilon_2$$

Proof. By remark 4.2.5, we have

$$\begin{cases} {}^C\mathcal{D}^{\gamma_1}x(t) = Ax(t) + f_1\big(t, x(t), x(\lambda_1 t), y(t)\big) + \Big\langle \int_0^t g_1\big(s, x(s), x(\lambda_1 s), y(s)\big)dB(s)\Big\rangle + \phi(t), \\ {}^C\mathcal{D}^{\gamma_2}y(t) = Ay(t) + f_2\big(t, y(t), y(\lambda_2 t), x(t)\big) + \Big\langle \int_0^t g_2\big(s, y(s), y(\lambda_2 s), x(s)\big)dB(s)\Big\rangle + \psi(t). \end{cases} \tag{4.2.3}$$

Thanks to Definition 4.2.1, the solution of system (4.2.3) can be reformulated immediately as

$$\begin{aligned}
x(t) &= M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t (t-s)^{\gamma_1-1} M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big) f_1\big(s, x(s), x(\lambda_1 s), y(s)\big)\,ds \\
&+ \int_0^t (t-s)^{\gamma_1-1} M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big) \Big\langle \int_0^s g_1\big(u, x(u), x(\lambda_1 u), y(u)\big)\,dB(u) \Big\rangle ds \\
&+ \int_0^t (t-s)^{\gamma_1-1} M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big) \phi(s)\,ds, \\
&:= L(t) + \int_0^t (t-s)^{\gamma_1-1} M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big) \phi(s)\,ds.
\end{aligned}$$

$$\begin{aligned}
y(t) &= M_{\gamma_2}(At^{\gamma_2})y_0 + \int_0^t (t-s)^{\gamma_2-1} M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big) f_2\big(s, y(s), y(\lambda_2 s), x(s)\big)\,ds \\
&+ \int_0^t (t-s)^{\gamma_2-1} M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big) \Big\langle \int_0^s g_2\big(u, y(u), y(\lambda_2 u), x(u)\big)\,dB(u) \Big\rangle ds \\
&+ \int_0^t (t-s)^{\gamma_2-1} M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big) \psi(s)\,ds, \\
&:= P(t) + \int_0^t (t-s)^{\gamma_2-1} M_{\gamma_2,\gamma_2}\big(A(t-s)^{\gamma_2}\big) \psi(s)\,ds.
\end{aligned}$$

Then, using Cauchy–Schwarz inequality and Itô isometry, we get

$$\begin{aligned}
\mathbb{E}\mathbf{D}_\infty^2\big[x(t), L(t)\big] &= \mathbb{E}\mathbf{D}_\infty^2\Big[\int_0^t (t-s)^{\gamma_1-1} M_{\gamma_1,\gamma_1}\big(A(t-s)^{\gamma_1}\big)\phi(s)\,ds, \hat{0}\Big], \\
&\leq \frac{T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\mathbb{E}\mathbf{D}_\infty^2[\phi, \hat{0}] \leq \frac{T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\epsilon_1.
\end{aligned}$$

Similarly, we find

$$\mathbb{E}\mathbf{D}_\infty^2\big[y(t), P(t)\big] \leq \frac{T^{2\gamma_2-1}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2}\epsilon_2.$$

Now, we prove the UH stability for the system (4.2.1).

Theorem 4.2.7 *Assume that the assumptions* (A1) *and* (A2) *holds. Then, the system* (4.2.1) *will be UH stable and consequently GUH stable.*

Proof. Let (x, y) be the solution of the system (4.2.2) and (v, k) be the solution of the proposed system (4.2.1). Using Cauchy–Schwarz inequality, Lemma 4.2.6 and Itô isometry, we get

$$\begin{aligned}
\mathbb{E}\mathbf{D}^2_\infty\big(x(t),v(t)\big) &= \mathbb{E}\mathbf{D}^2_\infty\Big(x(t), M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} f_1\big(s,v(s),v(\lambda_1 s),k(s)\big)ds \\
&+ \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} \Big\langle \int_0^s g_1\big(u,v(u),v(\lambda_1 u),k(u)\big)dB(u)\Big\rangle ds\Big), \\
&\le 2\mathbb{E}\mathbf{D}^2_\infty\Big(x(t), M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} f_1\big(s,x(s),x(\lambda_1 s),y(s)\big)ds \\
&+ \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} \Big\langle \int_0^s g_1\big(u,x(u),x(\lambda_1 u),y(u)\big)dB(u)\Big\rangle ds\Big) \\
&+ 2\mathbb{E}\mathbf{D}^2_\infty\Big(M_{\gamma_1}(At^{\gamma_1})x_0 + \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} f_1\big(s,x(s),x(\lambda_1 s),y(s)\big)ds \\
&+ \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} \Big\langle \int_0^s g_1\big(u,x(u),x(\lambda_1 u),y(u)\big)dB(u)\Big\rangle ds, M_{\gamma_1}(At^{\gamma_1})x_0 \\
&+ \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} f_1\big(s,v(s),v(\lambda_1 s),k(s)\big)ds + \int_0^t \frac{M_{\gamma_1,\gamma_1}(A(t-s)^{\gamma_1})}{(t-s)^{1-\gamma_1}} \Big\langle \int_0^s g_1\big(u,v(u),v(\lambda_1 u),k(u)\big)dB(u)\Big\rangle ds\Big), \\
&\le \frac{T^{2\gamma_1-1}\epsilon_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{2T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \mathbb{E}\mathbf{D}^2_\infty\Big(f_1\big(x(s),x(\lambda_1 s),y(s)\big), f_1\big(v(s),v(\lambda_1 s),k(s)\big)\Big)ds \\
&+ \frac{2T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \Big(\int_0^s \mathbb{E}\Big\| g_1\big(x(s),x(\lambda_1 s),y(s)\big) - g_1\big(v(s),v(\lambda_1 s),k(s)\big)\Big\|^2 du\Big)ds, \\
&\le \frac{T^{2\gamma_1-1}\epsilon_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{2T^{2\gamma_1-1}L_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \Big(\mathbb{E}\mathbf{D}^2_\infty\big(x(s),v(s)\big) + \mathbb{E}\mathbf{D}^2_\infty\big(x(\lambda_1 s),v(\lambda_1 s)\big) + \mathbb{E}\mathbf{D}^2_\infty\big(y(s),k(s)\big)\Big)ds \\
&+ \frac{2T^{2\gamma_1-1}N_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\int_0^t \Big[\int_0^s \Big(\mathbb{E}\mathbf{D}^2_\infty\big(x(u),v(u)\big) + \mathbb{E}\mathbf{D}^2_\infty\big(x(\lambda_1 u),v(\lambda_1 u)\big) + \mathbb{E}\mathbf{D}^2_\infty\big(y(u),k(u)\big)\Big)du\Big]ds, \\
&\le \frac{T^{2\gamma_1-1}\epsilon_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6T^{2\gamma_1}L_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\Big(\mathbb{E}\mathbf{D}^2_\infty(x,v) + \mathbb{E}\mathbf{D}^2_\infty(y,k)\Big) \\
&+ \frac{6T^{2\gamma_1+1}N_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\Big(\mathbb{E}\mathbf{D}^2_\infty(x,v) + \mathbb{E}\mathbf{D}^2_\infty(y,k)\Big), \\
&\le \frac{T^{2\gamma_1-1}\epsilon_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \Big(\frac{6T^{2\gamma_1}L_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6T^{2\gamma_1+1}N_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}\Big)\Big(\mathbb{E}\mathbf{D}^2_\infty(x,v) + \mathbb{E}\mathbf{D}^2_\infty(y,k)\Big), \\
&\le \frac{T^{2\gamma_1-1}\epsilon_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \xi_1\Big(\mathbb{E}\mathbf{D}^2_\infty(x,v) + \mathbb{E}\mathbf{D}^2_\infty(y,k)\Big), \text{where} \quad \xi_1 := \frac{6T^{2\gamma_1}L_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{6T^{2\gamma_1+1}N_1}{(2\gamma_1-1)(\Gamma(\gamma_1))^2}.
\end{aligned}$$

With a similar method, we get

$$\mathbb{E}\mathbf{D}^2_\infty\big(y(t),k(t)\big) \le \frac{T^{2\gamma_2-1}\epsilon_2}{(2\gamma_2-1)(\Gamma(\gamma_2))^2} + \xi_2\Big(\mathbb{E}\mathbf{D}^2_\infty(x,v) + \mathbb{E}\mathbf{D}^2_\infty(y,k)\Big),$$

where $\xi_2 := \frac{6T^{2\gamma_2}L_2}{(2\gamma_2-1)(\Gamma(\gamma_2))^2} + \frac{6T^{2\gamma_2+1}N_2}{(2\gamma_2-1)(\Gamma(\gamma_2))^2}$.

Let $\epsilon = \max(\epsilon_1,\epsilon_2)$ and $\xi = \max(\xi_1,\xi_2)$, hence

$$\mathbb{E}\mathbf{D}^2_\infty\big((x,y)(t),(v,k)(t)\big) \le R\epsilon + \xi\mathbb{E}\mathbf{D}^2_\infty\Big((x,y)(t),(v,k)(t)\Big),$$

where $R := \frac{T^{2\gamma_1-1}}{(2\gamma_1-1)(\Gamma(\gamma_1))^2} + \frac{T^{2\gamma_2-1}}{(2\gamma_2-1)(\Gamma(\gamma_2))^2}$. Then

$$\mathbb{E}\mathbf{D}^2_\infty\big((x,y)(t),(v,k)(t)\big) \le \frac{R}{1-\xi}\epsilon := \omega\epsilon.$$

Hence the system (4.2.1) is UH stable. Therefore, if we put $\varphi(\epsilon) = \omega\epsilon$, we have $\varphi(0) = 0$ and $\mathbb{E}\mathbf{D}^2_\infty\big((x,y)(t),(v,k)(t)\big) \le \varphi(\epsilon)$. Then, the system (4.2.1) is GUH stable.□

Part II

On averaging principle for fuzzy fractional stochastic differential equations

Chapter 5

The averaging principle for fuzzy stochastic differential equations

Contents

5.1 Introduction

The averaging concept is a powerful tool for investigating the qualitative property of dynamical system, physics and variety of other fields. This method shows the connection between the solutions of averaged systems and the solutions of a standard form [117, 118]. Nevertheless, up to now, the averaging principle for fuzzy stochastic differential equations (FSDEs) still has a big challenge to face. So, we will make the first attempt to study this method for this type of equations, therefore, this result will be enriched the averaging results for FSDEs. For crisp stochastic differential equations (SDEs), the first results on averaging principle are [61, 62, 107]. Tan et al. [109] established an averaging method for stochastic differential delay equations (SDDEs) under non-Lipschitz conditions. In [85, 116] the authors investigated the averaging principle for SDDEs with Jumps and with fractional Brownian motion.

Recently, Guo et al. [48] established the averaging method for a class of SDEs with nonlinear terms satisfying the monotone condition, and Luo et al. [73, 74] investigated an averaging principle for a class of stochastic fractional differential equations (SFDEs) with time-delays. Ahmed et al. [10] established the averaging principle of Hilfer fractional stochastic delay differential equations with Poisson jumps.

On the other hand, FSDEs are utilised is real world system where the phenomena are connected to randomness and fuzziness as two types of uncertainty. There are several articles on FSDEs, each of which takes a different approach. For example, in [59, 19] the authors presented a definition of the fuzzy stochastic Itô integral using a method that allows to embedding a crisp Itô stochastic integral into fuzzy space to build a fuzzy random variable. Motivated by the above mentioned works and its importance in many applied fields, it is interesting to extend the averaging principle to the FSDEs. This chapter is organized as follows: In the first subsection, the existence and uniqueness of solution of problem (5.2.1) are proved. The second subsection will be devoted to an averaging principle for the considered system is studied.

5.2 Main results

In this section, we investigate the existence, uniqueness and averaging principle results for FSDEs given by

$$\begin{cases} dx(s) = f(t, x(t))dt + \langle g(t, x(t))dB(t)\rangle, \\ \\ x(0) = x_0 \in \mathbf{E}^n, \end{cases} \tag{5.2.1}$$

where

$$f : J \times \mathbf{E}^n \longrightarrow \mathbf{E}^n,$$

$$g : J \times \mathbf{E}^n \longrightarrow \mathbb{R}^{n\times m}$$

$$x_0 : \Omega \longrightarrow \mathbf{E}^n$$

is a fuzzy random variable and $B(t)$ is standard Brownian motion with n-dimensional..

5.2.1 Existence and uniqueness results

In this subsection, we are interested in the existence and uniqueness results of system (5.2.1).

Definition 5.2.1 *A fuzzy process* $x : J \times \Omega \longrightarrow \mathbf{E}^n$ *is said to be a solution of system* (5.2.1) *if the following holds:*

(i) $x \in \mathbf{L}^2(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$,

(ii) x *is* $\mathbf{D}_\infty$*-continuous,*

(iii) *we have*

$$x(t) = x_0 + \int_0^t f(s, x(s))ds + \left\langle \int_0^t g(s, x(s))dB(s) \right\rangle, \qquad t \in J. \tag{5.2.2}$$

Remark 5.2.2 *We call a solution* $x : J \times \Omega \longrightarrow \mathbf{E}^n$ *of system* (5.2.1) *is unique, if* $x(t) \stackrel{\mathbb{JP}.1}{=} z(t)$, *where a fuzzy process* $z : J \times \Omega \longrightarrow \mathbf{E}^n$ *is any solution of* (5.2.1).

We will assume that all through this chapter, $f : (J \times \Omega) \times \mathbf{E}^n \longrightarrow \mathbf{E}^n$ is $\mathcal{B}_{d_S} \otimes \mathbf{N} \mid \mathcal{B}_{\mathbf{D}_\infty}$-measurable.
In order, we apply a conditions on the coefficient functions to ensure that the solution of (5.2.1) exists and is unique.

$(\mathcal{A}1)$ There exists a constant $C_1 > 0$ such that for all $t \in J$ and $x \in \mathbf{E}^n$ we have

$$\mathbb{E}\mathbf{D}_\infty^2\big(f(t,x),\hat{0}\big) \leq C_1^2\big(1 + \mathbb{E}\mathbf{D}_\infty^2(x,\hat{0})\big).$$

$$\|g(t,x)\|^2 := \mathbb{E}\mathbf{D}_\infty^2\big(\langle g(t,x)\rangle,\hat{0}\big) \leq C_1^2\big(1 + \mathbb{E}\mathbf{D}_\infty^2(x,\hat{0})\big).$$

$(\mathcal{A}2)$ There exists a constant $C_2 > 0$ such that for all $t \in I$ and for all $x, y \in \mathbf{E}^n$ we have

$$\mathbf{D}_\infty^2\big(f(t,x), f(t,y)\big) \leq C_2^2\mathbf{D}_\infty^2(x,y).$$

$$\|g(t,x) - g(t,y)\|^2 = \mathbb{E}\mathbf{D}_\infty^2\big(\langle g(t,x)\rangle, \langle g(t,y)\rangle\big) \leq C_2^2\mathbb{E}\mathbf{D}_\infty^2(x,y).$$

Theorem 5.2.3 *[81] Under assumptions* $(\mathcal{A}1)$-$(\mathcal{A}2)$ *and* $x_0 \in \mathbf{L}^2(\Omega, \mathcal{A}_0, \mathbb{P}; \mathbf{E}^n)$, *the system* (5.2.1) *has a unique solution* $x(t)$.

Proof. let us define a Picard type sequence $x_n : J \times \Omega \longrightarrow \mathbf{E}^n$ as follows
for $n = 0$

$$x_0(t) = x_0,$$

and for $n = 1, 2, ...$

$$x_n(t) = x_0 + \int_0^t f(s, x_{n-1}(s))ds + \left\langle \int_0^t g(s, x_{n-1}(s))dB(s) \right\rangle.$$

It is easy to known that x_n's are the well defined $\mathbf{D}_\infty$-continuous fuzzy stochastic processes from $\mathbf{L}^2(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$. In fact, we have that $x_0 \in \mathbf{L}^2(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$ and x_0 is $\mathbf{D}_\infty$-continuous. Let us define $K_n(t) = \mathbb{E}\mathbf{D}_\infty^2\big(x_n(t), x_{n-1}(t)\big)$ for $n \in \mathbb{N}$ and $t \in J$. Thus, by Propositions

1.2.25,1.2.27 and assumptions $(\mathcal{A}1)$-$(\mathcal{A}2)$, we obtain for $t \in J$

$$\begin{aligned}
K_1(t) &= \mathbb{E}\mathbf{D}^2_\infty\left(x_0 + \int_0^t f(s,x_0)ds + \left\langle \int_0^t g(s,x_0)dB(s)\right\rangle, x_0\right),\\
&= \mathbb{E}\mathbf{D}^2_\infty\left(\int_0^t f(s,x_0)ds + \left\langle \int_0^t g(s,x_0)dB(s)\right\rangle, \hat{0}\right),\\
&\le 2\mathbb{E}\mathbf{D}^2_\infty\left(\int_0^t f(s,x_0)ds, \hat{0}\right) + 2\mathbb{E}\mathbf{D}^2_\infty\left(\left\langle \int_0^t g(s,x_0)dB(s)\right\rangle, \hat{0}\right),\\
&\le 2t\int_0^t \mathbb{E}\mathbf{D}^2_\infty\big(f(s,x_0),\hat{0}\big)ds + 2\int_0^t \|g(s,x_0)\|^2 ds,\\
&\le 2TC_1^2(1+T)\mathbb{E}\mathbf{D}^2_\infty\big(x_0,\hat{0}\big) := l_1,\\
&< \infty.
\end{aligned}$$

Moreover, similarly we have

$$\begin{aligned}
K_n(t) = \mathbb{E}\mathbf{D}^2_\infty&\left(x_0 + \int_0^t f(s,x_{n-1}(s))ds + \left\langle \int_0^t g(s,x_{n-1}(s))dB(s)\right\rangle, x_0 + \int_0^t f(s,x_{n-2}(s))ds\right.\\
&\left. + \left\langle \int_0^t g(s,x_{n-2}(s))dB(s)\right\rangle\right),\\
&\le 2\mathbb{E}\mathbf{D}^2_\infty\left(\int_0^t f(s,x_{n-1}(s))ds, \int_0^t f(s,x_{n-2}(s))ds\right)\\
&+ 2\mathbb{E}\mathbf{D}^2_\infty\left(\left\langle \int_0^t g(s,x_{n-1}(s))dB(s)\right\rangle, \left\langle \int_0^t g(s,x_{n-2}(s))dB(s)\right\rangle\right),\\
&\le 2C_2^2(1+T)\int_0^t \mathbb{E}\mathbf{D}^2_\infty\Big(f(s,x_{n-1}), f(s,x_{n-2})\Big)ds := l_2\int_0^t \mathbb{E}\mathbf{D}^2_\infty\Big(f(s,x_{n-1}), f(s,x_{n-2})\Big)ds.
\end{aligned}$$

Thus, we have

$$K_n(t) \le \frac{l_1}{l_2}\frac{(l_2T)^n}{n!} < \infty.$$

Therefore $x_n(t) \in \mathbf{L}^2(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$ for every n and every t. Moreover, for every n the mapping $x_n(\cdot)$ is continuous with respect to the metric $\mathbf{D}_\infty$.
In the sequel we will show that the sequence $(x_n(t))_{n=0}^\infty$ satisfies Cauchy condition uniformly in t. Notice that

$$\mathbb{E}\mathbf{D}^2_\infty\big(x_n(t), x_m(t)\big) \le \left(\frac{l_1}{l_2}\right)^{\frac{1}{2}}\sum_{k=m+1}^{n}\left(\frac{(l_2T)^k}{k!}\right)^{\frac{1}{2}},$$

and the series $\sum_{k=0}^\infty\left(\frac{x^k}{k!}\right)^{\frac{1}{2}}$ is convergent for every $x \in \mathbb{R}$. Hence for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n, m \ge n_0$ it holds

$$\sup_{t\in[0,T]} \mathbb{E}\mathbf{D}^2_\infty\big(x_n(t), x_m(t)\big) \le \epsilon.$$

Therefore, Thus $(x_n(t))_{n=0}^\infty$ is uniformly convergent to some fuzzy stochastic process $x : J \times \Omega \longrightarrow \mathbb{E}^n$ which is $\{\mathcal{A}_t\}$-adapted and $\mathbf{D}_\infty$-continuous. Thus, we have $x \in \mathbf{L}^2(J \times \Omega, \mathbf{N}; \mathbf{E}^n)$.

Finally, we want to show that this limit process is a solution to (5.2.1). In order to do this, we show that x satisfies (5.2.2). Indeed, for every $t \in J$, we have

$$\begin{aligned}
&\mathbb{E}\mathbf{D}^2_\infty\Big(x(t), x_0 + \int_0^t f(s,x(s))ds + \Big\langle \int_0^t g(s,x(s))dB(s)\Big\rangle\Big)\\
&\quad \le 3\mathbb{E}\mathbf{D}^2_\infty\big(x(t),x_n(t)\big) + 3\mathbb{E}\mathbf{D}^2_\infty\Big(x_n(t), x_0 + \int_0^t f(s,x_{n-1}(s))ds + \Big\langle \int_0^t g(s,x_{n-1}(s))dB(s)\Big\rangle\Big)\\
&\quad + 3\mathbb{E}\mathbf{D}^2_\infty\Big(x_0 + \int_0^t f(s,x_{n-1}(s))ds + \Big\langle \int_0^t g(s,x_{n-1}(s))dB(s)\Big\rangle, x_0 + \int_0^t f(s,x(s))ds + \Big\langle \int_0^t g(s,x(s))dB(s)\Big\rangle\Big),\\
&\quad := S_1 + S_2 + S_3,
\end{aligned}$$

where S_1 converge to zero and S_2 is equal to zero. For the third term S_3, according to Propositions 1.2.25,1.2.27 and assumption $(\mathcal{A}2)$, we get

$$\begin{aligned}
S_3 \le & 2\mathbf{D}^2_\infty\Big(\int_0^t f(s,x_{n-1}(s))ds, \int_0^t f(s,x(s))ds\Big) + 2\mathbf{D}^2_\infty\Big(\Big\langle \int_0^t g(s,x_{n-1}(s))dB(s)\Big\rangle, \Big\langle \int_0^t g(s,x(s))dB(s)\Big\rangle\Big),\\
&\le 2C_2^2(1+T)\int_0^t \mathbf{D}^2_\infty\big(x_{n-1}(s),x(s)\big)ds,\\
&\le 2C_2^2(1+T)T \sup_{0\le t\le T} \mathbb{E}\mathbf{D}^2_\infty\big(x_{n-1}(t),x(t)\big) \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.
\end{aligned}$$

Therefore,

$$\mathbb{E}\mathbf{D}^2_\infty\Big(x(t), x_0 + \int_0^t f(s,x(s))ds + \Big\langle \int_0^t g(s,x(s))dB(s)\Big\rangle\Big) = 0.$$

Hence, (5.2.2) holds. Then, due to the Definition 5.2.1, $x(t)$ is a solution of (5.2.1).

Finally, we will show the uniqueness of solution. Assume that $x : J \times \Omega \longrightarrow \mathbb{E}^n$ and $y : J \times \Omega \longrightarrow \mathbb{E}^n$ are two solutions to (5.2.1). Then, let us notice that

$$\mathbb{E}\mathbf{D}^2_\infty\big(x(t),y(t)\big) \le 2C_2^2(1+T)\int_0^t \mathbb{E}\mathbf{D}^2_\infty\big(x(s),y(s)\big)ds.$$

Thus, by Gronwall's inequality, we obtain $\mathbb{E}\mathbf{D}^2_\infty\big(x(t),y(t)\big) \equiv 0$ for every $t \in J$. This implies that for every $t \in J$ it holds $\mathbb{P}\Big([x(t)]^\alpha = [y(t)]^\alpha \ \forall \alpha \in [0,1]\Big) = 1.$□

5.2.2 Averaging result

The construction of an averaging principle for system (5.2.1) is the focus of this subsection. First, we look at the standard form of Eq. (5.2.2)

$$x_\epsilon(t) = x_0 + \epsilon\int_0^t f(s,x_\epsilon(s))ds + \sqrt{\epsilon}\Big\langle \int_0^t g(s,x_\epsilon(s))dB(s)\Big\rangle, \tag{5.2.3}$$

where the initial value x_0, functions f and g have the same conditions as in system (5.2.1) and $\epsilon \in (0,\epsilon_0)$ is a positive small parameter with ϵ_0 a fixed number.

Based on the existence and uniqueness results, the Eq. (5.2.3) also has a unique solution $x_\epsilon(t)$ for every fixed $\epsilon \in (0,\epsilon_0)$ and $t \in J$.

In order, we set certain assumptions on the coefficients to see if the solution $x_\epsilon(t)$ can be approximated by a small process to a simple process.

Let $\tilde{f} : \mathbf{E}^n \longrightarrow \mathbf{E}^n$ and $\tilde{g} : \mathbf{E}^n \longrightarrow \mathbb{R}^n$ be measurable functions satisfying $(\mathcal{A}1)$-$(\mathcal{A}2)$ and the following additional inequalities:

($\mathcal{A}3$) For $x \in \mathbf{E}^n$ and $T' \in J$ we have

$$\frac{1}{T'}\int_0^{T'} \mathbb{E}\mathbf{D}_\infty^2\Big(f(s,x),\tilde{f}(x)\Big)ds \le \beta_1(T')\big(1+\mathbb{E}\mathbf{D}_\infty^2(x,\hat{0})\big),$$

$$\frac{1}{T'}\int_0^{T'} \mathbb{E}\|g(s,x)-\tilde{g}(x)\|^2 ds \le \beta_2(T')\big(1+\mathbb{E}\mathbf{D}_\infty^2(x,\hat{0})\big),$$

where $\lim_{T'\longrightarrow\infty}\beta_i(T')=0,\ i=1,2.$

With the appropriate preparations above, we will show that the solution x_ϵ converge as $\epsilon \longrightarrow 0$, to the solution y_ϵ of the following averaged FSDEs

$$y_\epsilon(t) = x_0 + \epsilon\int_0^t \tilde{f}(y_\epsilon(s))ds + \sqrt{\epsilon}\Big\langle \int_0^t \tilde{g}(y_\epsilon(s))dB(s)\Big\rangle. \tag{5.2.4}$$

Clearly, under similar assumptions as Eq. (5.2.3), the Eq. (5.2.4) also has a unique solution y_ϵ.

The main result of this subsection is now presented, in which we consider the connections between the process x_ϵ and y_ϵ.

Theorem 5.2.4 *Assume that the assumptions* ($\mathcal{A}1$)-($\mathcal{A}3$) *are satisfied. For a given arbitrarily small number* $\Delta > 0$ *and a constant* $k > 0$, $\alpha \in (0,1)$, *there exist* $\epsilon_1 \in (0,\epsilon_0]$ *such that* $\forall\epsilon \in (0,\epsilon_1]$, *we have*

$$\sup_{t\in[0,k\epsilon^{-\alpha}]} \mathbb{E}\mathbf{D}_\infty^2\big(x_\epsilon(t),y_\epsilon(t)\big) \le \Delta.$$

Proof. For any $t \in [0,u] \subset J$, we have

$$\begin{aligned}
\sup_{t\in[0,u]} \mathbb{E}\mathbf{D}_\infty^2\big(x_\epsilon(t),y_\epsilon(t)\big) &= \sup_{t\in[0,u]} \mathbb{E}\mathbf{D}_\infty^2\Big(x_0+\epsilon\int_0^t f(s,x_\epsilon(s))ds+\sqrt{\epsilon}\Big\langle\int_0^t g(s,x_\epsilon(s))dB(s)\Big\rangle,\\
&\qquad x_0+\epsilon\int_0^t \tilde{f}(y_\epsilon(s))ds+\sqrt{\epsilon}\Big\langle\int_0^t \tilde{g}(y_\epsilon(s))dB(s)\Big\rangle\Big),\\
&\le 2\epsilon^2\sup_{t\in[0,u]} \mathbb{E}\mathbf{D}_\infty^2\Big(\int_0^t f(s,x_\epsilon(s))ds,\int_0^t \tilde{f}(y_\epsilon(s))ds\Big)\\
&\quad + 2\epsilon\sup_{t\in[0,u]} \mathbb{E}\mathbf{D}_\infty^2\Big(\Big\langle\int_0^t g(s,x_\epsilon(s))dB(s)\Big\rangle,\Big\langle\int_0^t \tilde{g}(y_\epsilon(s))dB(s)\Big\rangle\Big).
\end{aligned}$$

Denote by

$$J_1 = 2\epsilon^2\sup_{t\in[0,u]} \mathbb{E}\mathbf{D}_\infty^2\Big(\int_0^t f(s,x_\epsilon(s))ds,\int_0^t \tilde{f}(y_\epsilon(s))ds\Big),$$

$$J_2 = 2\epsilon\sup_{t\in[0,u]} \mathbb{E}\mathbf{D}_\infty^2\Big(\Big\langle\int_0^t g(s,x_\epsilon(s))dB(s)\Big\rangle,\Big\langle\int_0^t \tilde{g}(y_\epsilon(s))dB(s)\Big\rangle\Big).$$

Then, using the properties of the metric $\mathbf{D}_\infty$, we get

$$\begin{aligned}
J_1 &\leq 4\epsilon^2 \sup_{t\in[0,u]} \mathbb{E}\mathbf{D}_\infty^2\Big(\int_0^t f(s,x_\epsilon(s))ds, \int_0^t f(s,y_\epsilon(s))ds,\Big)\\
&+ 4\epsilon^2 \sup_{t\in[0,u]} \mathbb{E}\mathbf{D}_\infty^2\Big(\int_0^t f(s,y_\epsilon(s))ds, \int_0^t \tilde{f}(y_\epsilon(s))ds,\Big),\\
&:= J_{11} + J_{12}.
\end{aligned}$$

By using the Proposition 1.2.25 and the assumption $(\mathcal{A}2)$, we have

$$\begin{aligned}
J_{11} &\leq 4\epsilon^2 \sup_{t\in[0,u]} \Big(t\int_0^t \mathbb{E}\mathbf{D}_\infty^2\big(f(s,x_\epsilon(s)), f(s,y_\epsilon(s))\big)ds\Big),\\
&\leq 4\epsilon^2 C_2 u \int_0^u \mathbb{E}\mathbf{D}_\infty^2\big(x_\epsilon(s), y_\epsilon(s)\big)ds.
\end{aligned}$$

For J_{12}, we use the Proposition 1.2.25 and the assumption $(\mathcal{A}3)$, we get

$$\begin{aligned}
J_{12} &\leq 4\epsilon^2 \sup_{t\in[0,u]} \Big(t\int_0^t \mathbb{E}\mathbf{D}_\infty^2\big(f(s,y_\epsilon(s)), \tilde{f}(y_\epsilon(s))\big)ds\Big),\\
&\leq 4\epsilon^2 \sup_{t\in[0,u]} \Big(t^2\frac{1}{t}\int_0^t \mathbb{E}\mathbf{D}_\infty^2\big(f(s,y_\epsilon(s)), \tilde{f}(y_\epsilon(s))\big)ds\Big),\\
&\leq 4\epsilon^2 u^2 \beta_1(u)\big[1 + \sup_{t\in[0,u]} \mathbb{E}\mathbf{D}_\infty^2\big(y_\epsilon(t), \hat{0}\big)\big] := 4\epsilon^2 u^2 \lambda_1.
\end{aligned}$$

Therefore

$$J_1 \leq 4\epsilon^2 C_2 u \int_0^u \mathbb{E}\mathbf{D}_\infty^2\big(x_\epsilon(s), y_\epsilon(s)\big)ds + 4\epsilon^2 u^2 \lambda_1. \tag{5.2.5}$$

For the second term J_2, by using the Proposition 1.2.26, we have

$$\begin{aligned}
J_2 &\leq 2\epsilon \sup_{t\in[0,u]} \int_0^t \mathbb{E}\big\|g(s,x_\epsilon(s)) - \tilde{g}(y_\epsilon(s))\big\|^2 ds,\\
&\leq 4\epsilon \sup_{t\in[0,u]} \int_0^t \mathbb{E}\big\|g(s,x_\epsilon(s)) - g(s,y_\epsilon(s))\big\|^2 ds\\
&+ 4\epsilon \sup_{t\in[0,u]} \int_0^t \mathbb{E}\big\|g(s,y_\epsilon(s)) - \tilde{g}(y_\epsilon(s))\big\|^2 ds,\\
&:= J_{21} + J_{22}.
\end{aligned}$$

Using the assumption $(\mathcal{A}2)$, we get

$$J_{21} \leq 4\epsilon C_2 \int_0^u \mathbb{E}\mathbf{D}_\infty^2\big(x_\epsilon(s), y_\epsilon(s)\big)ds.$$

Also, we use the assumption $(\mathcal{A}3)$, we have

$$\begin{aligned}
J_{22} &\leq 4\epsilon \sup_{t\in[0,u]} \Big(t\frac{1}{t}\int_0^t \mathbb{E}\big\|g(s,y_\epsilon(s)) - \tilde{g}(y_\epsilon(s))\big\|^2 ds\Big),\\
&\leq 4\epsilon u \beta_2(u)\big[1 + \sup_{t\in[0,u]} \mathbb{E}\mathbf{D}_\infty^2\big(y_\epsilon(t), \hat{0}\big)\big] := 4\epsilon u \lambda_2.
\end{aligned}$$

Therefore

$$J_2 \leq 4\epsilon C_2 \int_0^u \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(s), y_\epsilon(s)\big)\,ds + 4\epsilon u \lambda_2. \tag{5.2.6}$$

So, combining (5.2.5) and (5.2.6) together, we get

$$\begin{aligned}\sup_{t\in[0,u]} \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(t), y_\epsilon(t)\big) &\leq 4\epsilon u\big(\lambda_2 + \epsilon u \lambda_1\big) + 4\epsilon C_2\big(1+\epsilon u\big)\int_0^u \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(s), y_\epsilon(s)\big)\,ds,\\ &\leq 4\epsilon u\big(\lambda_2 + \epsilon u \lambda_1\big) + 4\epsilon C_2\big(1+\epsilon u\big)\int_0^u \sup_{v\in[0,s]} \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(v), y_\epsilon(v)\big)\,ds.\end{aligned}$$

Hence, by using the Gronwall inequality, we get

$$\sup_{t\in[0,u]} \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(t), y_\epsilon(t)\big) \leq 4\epsilon u\big(\lambda_2 + \epsilon u\lambda_1\big)e^{4\epsilon C_2(1+\epsilon u)}.$$

Choose $\alpha \in (0,1)$ and $L > 0$ such that for every $t \in [0, L\epsilon^{-\alpha}] \subseteq J$, we have

$$\sup_{t\in[0,L\epsilon^{-\alpha}]} \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(t), y_\epsilon(t)\big) \leq kL\epsilon^{1-\alpha},$$

where $k = 4\big(\lambda_2 + L\epsilon^{1-\alpha}\lambda_1\big)\exp\Big\{4\epsilon C_2\big(1+L\epsilon^{1-\alpha}\big)\Big\}$ is a constant. Therefore, for any given number Δ, $\exists \epsilon_1 \in (0, \epsilon_0]$ such that for each $\epsilon \in (0, \epsilon_1]$ and $t \in [0, L\epsilon^{-\alpha}]$, we have

$$\sup_{t\in[0,L\epsilon^{-\alpha}]} \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(t), y_\epsilon(t)\big) \leq \Delta.$$

Thus the proof is complete.□

5.3 Example

In this section, we give an example to illustrate our result.

Consider the following FSDEs

$$\begin{cases} dX(t) = 4\cos^2(t)X(t)dt + \big\langle X(t)dB(t)\big\rangle, \\ X(0) = 0, \end{cases} \tag{5.3.1}$$

So, the corresponding standard form of the above FSDEs is

$$dX^\epsilon = 4\epsilon\cos^2(t)X^\epsilon dt + \sqrt{\epsilon}\big\langle X^\epsilon dB(t)\big\rangle.$$

Then, $f(t, X^\epsilon) = 4\cos^2(t)X^\epsilon$ and $g(t, X^\epsilon) = X^\epsilon$. Hence

$$\tilde{f}(X^\epsilon) = \frac{1}{\pi}\int_0^\pi 4\cos^2(t)X^\epsilon dt = 2X^\epsilon \quad \text{and} \quad \tilde{g}(X^\epsilon) = \frac{1}{\pi}\int_0^\pi g(t, X^\epsilon)dt = X^\epsilon.$$

Therefore, the averaging form of (5.3.1) is

$$dY^\epsilon = 2\epsilon Y^\epsilon dt + \sqrt{\epsilon}\big\langle Y^\epsilon dB(t)\big\rangle. \tag{5.3.2}$$

The coefficients $f(t, X^\epsilon)$ and $g(t, X^\epsilon)$ satisfy the assumptions $(\mathcal{A}1)$-$(\mathcal{A}2)$, then the FSDEs (5.3.1) has a unique fuzzy solution. On the other hand, we can naturally see that the coefficient $\tilde{f}(X^\epsilon)$ and $\tilde{g}(X^\epsilon)$ satisfy the assumption $(\mathcal{A}3)$, then, according to Theorem 5.2.4, as $\epsilon \longrightarrow 0$, the solution X^ϵ and Y^ϵ to Eqs. (5.3.1) and (5.3.2) are equivalent in the sense of mean square.

Chapter 6

Fuzzy fractional pantograph stochastic differential equations: Existence, uniqueness and averaging principle

Contents

6.1 Introduction

Pantograph equations are a sort of delay differential equation that may be encountered in physics, medicine, biology, and other domains. The word pantograph come from the work of Ockendon and Taylor [93]. Many academics have recently explored pantograph differential equations (PDEs) of fractional order, for example, we recommend the reader to [22, 37, 49, 50, 55, 5]. Furthermore, multiple writers have demonstrated the existence and uniqueness of solutions for various fractional pantograph differential equations (FPDEs) with distinct fractional derivatives, for example see [2, 54, 11, 112, 113].

Recently, PDEs have also been extended to pantograph stochastic differential equations (PSDEs), see [86, 84], in this context, Priyadharsini et al. [95], extended PSDEs to fuzzy setting, they proposed a new type of equation nemely fuzzy fractional stochastic pantograph differential equations (FFSPDEs).

On the other hand, the notion of averaging principle has a long history. It's a great way to look at the qualitative properties of a dynamical system. Then, the study of this method for stochastic differential equations (SDEs) has received a lot of attention as theory has progressed [118, 75, 96, 73, 10] and [74, 45]. Arhrrabi et al. [15] initiated the study of averaging principle of fuzzy SDEs, also, arhrrabi et al. [19, 17, 18] studied the existence and stability of solutions for fuzzy fractional SDEs (FFSDEs) with Brownian motions, existence and uniqueness results of FFSDEs with impulsive and fuzzy fractional boundary value problem. To our knowledge, no publication has looked at the averaging principle of fuzzy fractional PSDEs, instead, numerous studies have looked at the averaging principle of fractional PSDEs in a crisp case. To close this gap, in this chapter, we will investigate the existence, uniqueness and averaging principle of solutions for a class of fuzzy fractional PSDEs defined in the next section.

6.2 Main results

In this section, we investigate the existence, uniqueness and averaging principle results for FFPSDEs given by

$$\begin{cases} {}^{C}\mathcal{D}^{\gamma}x(u) = f\big(u, x(u), x(\lambda t)\big) + \left\langle \int_0^u g\big(s, x(s), x(\lambda s)\big)\,dB(s) \right\rangle, u \in J. \\ \\ x(0) = x_0, \end{cases} \tag{6.2.1}$$

where ${}^{C}\mathcal{D}^{\gamma}$ is the Caputo fractional derivative of order $\gamma \in (0, 1)$ and $\lambda \in (0, 1)$. The functions $f : J \times \mathbf{E}^n \times \mathbf{E}^n \longrightarrow \mathbf{E}^n$ and $g : J \times \mathbf{E}^n \times \mathbf{E}^n \longrightarrow \mathbb{R}^{n\times m}$ are continuous on J.

6.2.1 Existence and uniqueness results

In this part, by using Banach's contraction mapping principle, we will show the existence and uniqueness of solution for FFPSDEs (6.2.1).

Definition 6.2.1 *We say that* $\{x(u), u \in J\}$ *is a solution of problem* (6.2.1) *if*

(i) $x(\cdot) \in C\big(J, \mathbf{E}^n\big)$,

(ii) $x(0) = x_0$,

(iii) *for* $0 \le u \le T$, *we have*

$$\begin{aligned} x(u) = x_0 &+ \frac{1}{\Gamma(\gamma)} \int_0^u (u-s)^{\gamma-1} f\big(s, x(s), x(\lambda s)\big)\,ds \\ &+ \frac{1}{\Gamma(\gamma)} \int_0^u (u-s)^{\gamma-1} \left\langle \int_0^s g\big(v, x(v), x(\lambda v)\big)\,dB(v) \right\rangle ds. \end{aligned} \tag{6.2.2}$$

The following assumptions are being prepared in order to get the primary conclusion in this subsection:

($\mathcal{A}1$) f is continuous and $\exists L_1 > 0$ such that

$$\mathbb{E}\mathbf{D}^2_\infty\big(f(u,x,w), f(u,\mathbf{z}',\mathbf{w}')\big) \leq L_1\Big(\mathbb{E}\mathbf{D}^2_\infty(x,\mathbf{z}') + \mathbb{E}\mathbf{D}^2_\infty(w,\mathbf{w}')\Big).$$

($\mathcal{A}2$) g is continuous and $\exists L_2 > 0$ such that

$$\mathbb{E}\big\|g(u,x,w) - g(u,\mathbf{z}',\mathbf{w}')\big\|^2 \leq L_2\Big(\mathbb{E}\mathbf{D}^2_\infty(x,\mathbf{z}') + \mathbb{E}\mathbf{D}^2_\infty(w,\mathbf{w}')\Big).$$

($\mathcal{A}3$) We have

$$\mathbb{E}\mathbf{D}^2_\infty\big(f(u,\hat{0},\hat{0}),\hat{0}\big) \leq q_1 \qquad \text{and} \qquad \mathbb{E}\|g(u,\hat{0},\hat{0})\|^2 \leq q_2.$$

Theorem 6.2.2 *Suppose that the assumptions* ($\mathcal{A}1$)-($\mathcal{A}3$) *holds, then problem* (6.2.1) *has a unique solution provided that*

$$\frac{4L_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{2L_2T^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2} < 1.$$

Proof. We define the operator $\mathbf{T} : \mathbb{K} \longrightarrow \mathbb{K}$ by

$$(\mathbf{T}x)(u) = x_0 + \frac{1}{\Gamma(\gamma)}\int_0^u \frac{f\big(s,x(s),x(\lambda s)\big)}{(u-s)^{1-\gamma}}ds + \frac{1}{\Gamma(\gamma)}\int_0^u \frac{\Big\langle \int_0^s g\big(u,x(u),x(\lambda u)\big)dB(u)\Big\rangle}{(u-s)^{1-\gamma}}ds.$$

For each positive number r, we define

$$\mathcal{B}_r = \Big\{x \in \mathbb{K} : \mathbb{E}\mathbf{D}^2_\infty\big(x,\hat{0}\big) \leq r\Big\}.$$

We divide the subsequent proof into two steps.

Step 1: T is well defined. For this, we prove that $\mathbf{T}(\mathcal{B}_r) \subseteq \mathcal{B}_r$. We choose

$$r \geq \frac{3\mathbb{E}\mathbf{D}^2_\infty(x_0,\hat{0}) + J_1}{1 - J_2}.$$

By using the assumptions above, Propositions 1.2.26,1.2.27, Hölder inequality and Itô isometric, that for $x \in \mathcal{B}_r$, we get

$$\begin{aligned}
\mathbb{E}\mathbf{D}_\infty^2\big((\mathbf{T}x)(u),\hat{0}\big) &= \mathbb{E}\mathbf{D}_\infty^2\Big(x_0 + \frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1} f\big(s,x(s),x(\lambda s)\big)\,ds \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\Big\langle \int_0^s g\big(u,x(u),x(\lambda u)\big)\,dB(u)\Big\rangle ds, \hat{0}\Big), \\
&\le 3\mathbb{E}\mathbf{D}_\infty^2\big(x_0,\hat{0}\big) + 3\mathbb{E}\mathbf{D}_\infty^2\Big(\frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1} f\big(s,x(s),x(\lambda s)\big)\,ds, \hat{0}\Big) \\
&\quad + 3\mathbb{E}\mathbf{D}_\infty^2\Big(\frac{1}{\Gamma(\gamma)}\int_0^u \Big\langle \int_0^s (u-s)^{\gamma-1} g\big(u,x(u),x(\lambda u)\big)\,dB(u)\Big\rangle ds, \hat{0}\Big), \\
&\le 3\mathbb{E}\mathbf{D}_\infty^2\big(x_0,\hat{0}\big) + \frac{6T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \mathbb{E}\mathbf{D}_\infty^2\Big(f\big(s,x(s),x(\lambda s)\big), f\big(s,\hat{0},\hat{0}\big)\Big)\,ds \\
&\quad + \frac{6T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \mathbb{E}\mathbf{D}_\infty^2\Big(f\big(s,\hat{0},\hat{0}\big),\hat{0}\Big)\,ds \\
&\quad + \frac{6T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \Big(\int_0^s \mathbb{E}\Big\| g\big(u,x(u),x(\lambda u)\big) - g\big(u,\hat{0},\hat{0}\big)\Big\|^2 du\Big)\,ds \\
&\quad + \frac{6C_T T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \Big(\int_0^s \mathbb{E}\Big\| g\big(u,\hat{0},\hat{0}\big)\Big\|^2 du\Big)\,ds, \\
&\le 3\mathbb{E}\mathbf{D}_\infty^2\big(x_0,\hat{0}\big) + \frac{6L_1T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \Big[\mathbb{E}\mathbf{D}_\infty^2(x(s),\hat{0}) + \mathbb{E}\mathbf{D}_\infty^2(x(\lambda s),\hat{0})\Big]\,ds + \frac{6q_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} \\
&\quad + \frac{6T^{2\gamma-1}L_2}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \Big(\int_0^s \Big[\mathbb{E}\mathbf{D}_\infty^2(x(u),\hat{0}) + \mathbb{E}\mathbf{D}_\infty^2(x(\lambda u),\hat{0})\Big]\,du\Big)\,ds + \frac{3C_T T^{2\gamma+1}q_2}{(2\gamma-1)(\Gamma(\gamma))^2} \\
&\le 3\mathbb{E}\mathbf{D}_\infty^2\big(x_0,\hat{0}\big) + \frac{12L_1T^{2\gamma}r}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{6q_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{6T^{2\gamma+1}L_2r}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{3C_T T^{2\gamma+1}q_2}{(2\gamma-1)(\Gamma(\gamma))^2} \\
&\le 3\mathbb{E}\mathbf{D}_\infty^2\big(x_0,\hat{0}\big) + J_1 + J_2 r,
\end{aligned}$$

where

$$J_1 = \frac{6q_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{3C_T T^{2\gamma+1}q_2}{(2\gamma-1)(\Gamma(\gamma))^2} \quad \text{and} \quad J_2 = \frac{12L_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{6T^{2\gamma+1}L_2}{(2\gamma-1)(\Gamma(\gamma))^2}.$$

Finally, we have

$$\mathbb{E}\mathbf{D}_\infty^2\big((\mathbf{T}x)(u),\hat{0}\big) \le r,$$

which implies that $\mathbf{T}(\mathcal{B}_r) \subseteq \mathcal{B}_r$.

Step 2: In this step, we will prove that $\mathbf{T}$ is a contraction operator. Using the assumptions $(\mathcal{A}1)$-$(\mathcal{A}3)$, Proposition 1.2.26, 1.2.27, Hölder inequality and Itô isometric, we have for $x, x' \in$

$\mathcal{B}_r$ and $u \in J$

$$\begin{aligned}
\mathbb{E}\mathbf{D}_\infty^2\big((\mathbf{T}x)(u),(\mathbf{T}x')(u)\big) &= \mathbb{E}\mathbf{D}_\infty^2\bigg(x_0 + \frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1} f\big(s,x(s),x(\lambda s)\big)\,ds \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\Big\langle \int_0^s g_1\big(u,x(u),x(\lambda u)\big)\,dB(u)\Big\rangle ds, x_0 \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1} f\big(s,x'(s),x'(\lambda s)\big)\,ds \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\Big\langle \int_0^s g_1\big(u,x'(u),x'(\lambda u)\big)\,dB(u)\Big\rangle ds\bigg), \\
&\le 2\mathbb{E}\mathbf{D}_\infty^2\bigg(\frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1} f\big(s,x(s),x(\lambda s)\big)\,ds, \frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1} f\big(s,x'(s),x'(\lambda s)\big)\,ds\bigg) \\
&\quad + 2\mathbb{E}\mathbf{D}_\infty^2\bigg(\frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\Big\langle \int_0^s g\big(u,x(u),x(\lambda u)\big)\,dB(u)\Big\rangle ds, \\
&\quad \frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\Big\langle \int_0^s g\big(u,x'(u),x'(\lambda u)\big)\,dB(u)\Big\rangle ds\bigg), \\
&\le \frac{2T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \mathbb{E}\mathbf{D}_\infty^2\Big(f\big(s,x(s),x(\lambda s)\big), f\big(s,x'(s),x'(\lambda s)\big)\Big)\,ds \\
&\quad + \frac{2T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \bigg(\int_0^s \mathbb{E}\Big\| g\big(u,x(u),x(\lambda u)\big) - g\big(u,x'(u),x'(\lambda u)\big)\Big\|^2 du\bigg)\,ds, \\
&\le \frac{2L_1T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \Big(\mathbb{E}\mathbf{D}_\infty^2\big(x(s),x'(s)\big) + \mathbb{E}\mathbf{D}_\infty^2\big(x(\lambda s),x'(\lambda s)\big)\Big)\,ds \\
&\quad + \frac{2L_2T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \bigg(\int_0^s \Big[\mathbb{E}\mathbf{D}_\infty^2\big(x(s),x'(s)\big) + \mathbb{E}\mathbf{D}_\infty^2\big(x(\lambda s),x'(\lambda s)\big)\Big]\,du\bigg)\,ds, \\
&\le \frac{4L_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2}\mathbb{E}\mathbf{D}_\infty^2(x,x') + \frac{2L_2T^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2}\mathbb{E}\mathbf{D}_\infty^2(x,x'), \\
&\le \bigg(\frac{4L_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{2L_2T^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2}\bigg)\mathbb{E}\mathbf{D}_\infty^2(x,x').
\end{aligned}$$

Finally, we can get

$$\mathbb{E}\mathbf{D}_\infty^2\big((\mathbf{T}x)(u),(\mathbf{T}x')(u)\big) \le \bigg(\frac{4L_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{2L_2T^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2}\bigg)\mathbb{E}\mathbf{D}_\infty^2(x,x').$$

So, since $\frac{4L_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{2L_2T^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2} < 1$, then $\mathbf{T}$ is a contraction operator. Consequently, using Banach's contraction mapping principle, we get to the conclusion that $\mathbf{T}$ has a fixed point, which is the unique solution of (6.2.1).□

6.2.2 Averaging result

The construction of an averaging concept for FFPSDEs is the focus of this subsection. First, we look at the standard form of Eq. (6.2.2).

$$\begin{aligned}
x_\epsilon(u) &= x_0 + \frac{\epsilon}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1} f\big(s,x_\epsilon(s),x_\epsilon(\lambda s)\big)\,ds \\
&\quad + \frac{\sqrt{\epsilon}}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\Big\langle \int_0^s g\big(u,x_\epsilon(u),x_\epsilon(\lambda u)\big)\,dB(u)\Big\rangle ds, \qquad (6.2.3)
\end{aligned}$$

where $0 < \epsilon < \epsilon_0$ and ϵ_0 is a fixed integer. Moreover x_0, f and g have the same requirements as in Eq. (6.2.2).

For every fixed $0 < \epsilon < \epsilon_0$ and $u \in J$, according to the existence and uniqueness findings, the Eq. (6.2.3) has a unique solution $x_\epsilon(u)$.
In order to determine if $x_\epsilon(u)$ can be approximated by a small process to a simple process, we make certain assumptions about the coefficients.
Let $\tilde{f} : \mathbf{E}^n \times \mathbf{E}^n \longrightarrow \mathbf{E}^n$ and $\tilde{g} : \mathbf{E}^n \times \mathbf{E}^n \longrightarrow \mathbb{R}^{n\times m}$ be measurable functions satisfying $(\mathcal{A}1)$-$(\mathcal{A}3)$ and the following inequalities:

$(\mathcal{A}4)$ For $K \in J$ and $x, w \in \mathbf{E}^n$, we have

$$\frac{1}{K}\int_0^K \mathbb{E}\mathbf{D}^2_\infty\Big(f(s,x,w),\tilde{f}(x,w)\Big)ds \leq \gamma_1(K)\big(1+\mathbb{E}\mathbf{D}^2_\infty(x,\hat{0})+\mathbb{E}\mathbf{D}^2_\infty(w,\hat{0})\big),$$

$$\frac{1}{K}\int_0^K \mathbb{E}\parallel g(s,x,w)-\tilde{g}(x,w)\parallel^2 ds \leq \gamma_2(K)\big(1+\mathbb{E}\mathbf{D}^2_\infty(x,\hat{0})+\mathbb{E}\mathbf{D}^2_\infty(w,\hat{0})\big),$$

where $\lim\limits_{K\longrightarrow\infty}\gamma_i(K)=0,\ i=1,2.$
After making the necessary preparations, we will demonstrate that when the time scale ϵ approaches zero, the original solution x_ϵ converges to solution of the following averaged FFPSDEs

$$\begin{aligned} y_\epsilon(u) = x_0 &+ \frac{\epsilon}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\tilde{f}\big(y_\epsilon(s),y_\epsilon(\lambda s)\big)ds \\ &+ \frac{\sqrt{\epsilon}}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\Big\langle \int_0^s \tilde{g}\big(y_\epsilon(u),y_\epsilon(\lambda u)\big)dB(u)\Big\rangle ds, \end{aligned} \tag{6.2.4}$$

Under the same presumptions as Eq. (6.2.3), it is obvious that Eq. (6.2.4) likewise has a unique solution y_ϵ.
As the main outcome of this section, we now examine the connections between the processes x_ϵ and y_ϵ.

Theorem 6.2.3 *If the conditions $(\mathcal{A}1)$-$(\mathcal{A}4)$ are verified. Then, for a given random tiny number $\delta > 0$ and a constant $k > 0$, $0 < \gamma < 1$, there exist $0 < \epsilon_1 \leq \epsilon_0 \mid \forall\epsilon \in (0,\epsilon_1]$, we have*

$$\sup_{u\in[0,k\epsilon^{-\gamma}]} \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(u),y_\epsilon(u)\big) \leq \delta.$$

Proof. For $0 < u \leq v$, we have

$$\begin{aligned} \sup_{u\in[0,v]} \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(u),y_\epsilon(u)\big) &= \sup_{u\in[0,v]} \mathbb{E}\mathbf{D}^2_\infty\Big(x_0+\frac{\epsilon}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}f(s,x_\epsilon(s),x_\epsilon(\lambda s))ds \\ &+\frac{\sqrt{\epsilon}}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\Big\langle\int_0^s g(s,x_\epsilon(s),x_\epsilon(\lambda s))dB(s)\Big\rangle ds, x_0+\frac{\epsilon}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}f(y_\epsilon(s),y_\epsilon(\lambda s))ds \\ &+\frac{\sqrt{\epsilon}}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\Big\langle\int_0^s g(y_\epsilon(s),y_\epsilon(\lambda s))dB(s)\Big\rangle ds\Big), \\ &\leq \frac{2\epsilon^2}{(\Gamma(\gamma))^2}\sup_{u\in[0,v]} \mathbb{E}\mathbf{D}^2_\infty\Big(\int_0^u (u-s)^{\gamma-1}f(s,x_\epsilon(s),x_\epsilon(\lambda s))ds, \int_0^u (u-s)^{\gamma-1}\tilde{f}(y_\epsilon(s),y_\epsilon(\lambda s))ds\Big)+\frac{2\epsilon}{(\Gamma(\gamma))^2} \\ &\sup_{u\in[0,v]} \mathbb{E}\mathbf{D}^2_\infty\Big(\int_0^u (u-s)^{\gamma-1}\Big\langle\int_0^s g(s,x_\epsilon(s),x_\epsilon(\lambda s))dB(s)\Big\rangle ds, \int_0^u (u-s)^{\gamma-1}\Big\langle\int_0^s g(y_\epsilon(s),y_\epsilon(\lambda s))dB(s)\Big\rangle ds\Big). \end{aligned}$$

Denote by

$$J_1 = \frac{2\epsilon^2}{(\Gamma(\gamma))^2} \sup_{u\in[0,v]} \mathbb{E}\mathbf{D}^2_\infty\Big(\int_0^u (u-s)^{\gamma-1} f(s,x_\epsilon(s),x_\epsilon(\lambda s))ds, \int_0^u (u-s)^{\gamma-1}\tilde{f}(y_\epsilon(s),y_\epsilon(\lambda s))ds\Big),$$

$$J_2 = \frac{2\epsilon}{(\Gamma(\gamma))^2} \sup_{u\in[0,v]} \mathbb{E}\mathbf{D}^2_\infty\Big(\int_0^u (u-s)^{\gamma-1}\Big\langle\int_0^s g(s,x_\epsilon(s),x_\epsilon(\lambda s))dB(s)\Big\rangle ds, \int_0^u (u-s)^{\gamma-1}\Big\langle\int_0^s g(y_\epsilon(s),y_\epsilon(\lambda s))dB(s)\Big\rangle ds\Big).$$

Then, by utilizing the attributes of the metric $\mathbf{D}_\infty$, we obtain

$$\begin{aligned}
J_1 &\le \frac{4\epsilon^2}{(\Gamma(\gamma))^2} \sup_{u\in[0,v]} \mathbb{E}\mathbf{D}^2_\infty\Big(\int_0^u (u-s)^{\gamma-1} f(s,x_\epsilon(s),x_\epsilon(\lambda s))ds, \int_0^u (u-s)^{\gamma-1} f(s,y_\epsilon(s),y_\epsilon(\lambda s))ds\Big)\\
&+ \frac{4\epsilon^2}{(\Gamma(\gamma))^2} \sup_{u\in[0,v]} \mathbb{E}\mathbf{D}^2_\infty\Big(\int_0^u (u-s)^{\gamma-1} f(s,y_\epsilon(s),y_\epsilon(\lambda s))ds, \int_0^u (u-s)^{\gamma-1}\tilde{f}(y_\epsilon(s),y_\epsilon(\lambda s))ds,\Big),\\
&:= J_{11} + J_{12}.
\end{aligned}$$

By using Hölder inequality and assumption $(\mathcal{A}1)$, we get

$$\begin{aligned}
J_{11} &\le \frac{4\epsilon^2 v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} \sup_{u\in[0,v]} \Big(\int_0^u \mathbb{E}\mathbf{D}^2_\infty\Big(f(s,x_\epsilon(s),x_\epsilon(\lambda s)), f(s,y_\epsilon(s),y_\epsilon(\lambda s))\Big)ds\Big),\\
&\le \frac{4\epsilon^2 L_1 v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} \int_0^u \Big(\mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(s),y_\epsilon(s)\big) + \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(\lambda s),y_\epsilon(\lambda s)\big)\Big)ds,\\
&\le \frac{8\epsilon^2 L_1 v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} \int_0^u \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(s),y_\epsilon(s)\big)ds.
\end{aligned}$$

For J_{12}, we use Hölder inequality and assumption $(\mathcal{A}4)$, we get

$$\begin{aligned}
J_{12} &\le \frac{4\epsilon^2 v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} \sup_{u\in[0,v]} \Big(\int_0^u \mathbb{E}\mathbf{D}^2_\infty\Big(f(s,y_\epsilon(s),y_\epsilon(\lambda s)), \tilde{f}(y_\epsilon(s),y_\epsilon(\lambda s))\Big)ds\Big),\\
&\le \frac{4\epsilon^2 v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\gamma_1(v)\Big[1 + \sup_{u\in[0,v]} \mathbb{E}\mathbf{D}^2_\infty\big(y_\epsilon(u),\hat{0}\big) + \sup_{u\in[0,v]} \mathbb{E}\mathbf{D}^2_\infty\big(y_\epsilon(\lambda t),\hat{0}\big)\Big],\\
&\le \frac{4\epsilon^2 v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} v\gamma_1(v)\big[1 + 2\sup_{u\in[0,v]} \mathbb{E}\mathbf{D}^2_\infty\big(y_\epsilon(u),\hat{0}\big)\big],\\
&:= 4\epsilon^2 v^{2\gamma}\beta_1,
\end{aligned}$$

where $\beta_1 = \frac{\gamma_1(v)}{(2\gamma-1)(\Gamma(\gamma))^2}\big[1 + 2\sup_{u\in[0,u]} \mathbb{E}\mathbf{D}^2_\infty\big(y_\epsilon(u),\hat{0}\big)\big]$.

Therefore

$$J_1 \le \frac{8\epsilon^2 L_1 v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} \int_0^u \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(s),y_\epsilon(s)\big)ds + 4\epsilon^2 v^{2\gamma}\beta_1. \tag{6.2.5}$$

For the second term J_2, by using Proposition 1.2.26, 1.2.27 and Hölder inequality, we have

$$\begin{aligned}
J_2 &\le \frac{2\epsilon v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} \sup_{u\in[0,v]} \int_0^u \Big(\int_0^s \mathbb{E}\big\| g(v,x_\epsilon(v),x_\epsilon(\lambda v)) - \tilde{g}(y_\epsilon(v),y_\epsilon(\lambda v))\big\|^2 dv\Big)ds,\\
&\le \frac{4\epsilon v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} \sup_{u\in[0,v]} \int_0^u \Big(\int_0^s \mathbb{E}\big\| g(v,x_\epsilon(v),x_\epsilon(\lambda v)) - g(v,y_\epsilon(v),y_\epsilon(\lambda v))\big\|^2 dv\Big)ds\\
&+ \frac{4\epsilon v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} \sup_{u\in[0,v]} \int_0^u \Big(\int_0^s \mathbb{E}\big\| g(v,y_\epsilon(v),y_\epsilon(\lambda v)) - \tilde{g}(y_\epsilon(v),y_\epsilon(\lambda v))\big\|^2 dv\Big)ds,\\
&:= J_{21} + J_{22}.
\end{aligned}$$

Using assumption $(\mathcal{A}2)$, we get

$$
\begin{aligned}
J_{21} &\leq \frac{4\epsilon L_2 \nu^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u\left(\int_0^s \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(\nu),y_\epsilon(\nu)\big)+\mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(\lambda\nu),y_\epsilon(\lambda\nu)\big)d\nu\right)ds,\\
&\leq \frac{4\epsilon L_2 \nu^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(s),y_\epsilon(s)\big)ds.
\end{aligned}
$$

Also, we use assumption $(\mathcal{A}4)$, we have

$$
\begin{aligned}
J_{22} &\leq \frac{4\epsilon \nu^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\sup_{u\in[0,\nu]}\left(\int_0^u\left(s\frac{1}{s}\int_0^\nu \mathbb{E}\big\|g(\nu,y_\epsilon(\nu),y_\epsilon(\lambda\nu))-\tilde{g}(y_\epsilon(\nu),y_\epsilon(\lambda\nu))\big\|^2 d\nu\right)ds,\\
&\leq \frac{4\epsilon \nu^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2}\gamma_2(\nu)\Big[1+\sup_{u\in[0,\nu]}\mathbb{E}\mathbf{D}^2_\infty\big(y_\epsilon(u),\hat{0}\big)+\sup_{u\in[0,\nu]}\mathbb{E}\mathbf{D}^2_\infty\big(y_\epsilon(\lambda t),\hat{0}\big)\Big],\\
&\leq \frac{4\epsilon \nu^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2}\gamma_2(\nu)\big[1+2\sup_{u\in[0,\nu]}\mathbb{E}\mathbf{D}^2_\infty\big(y_\epsilon(u),\hat{0}\big)\big],\\
&:= 4\epsilon\nu^{2\gamma+1}\beta_2,
\end{aligned}
$$

where $\beta_2=\frac{\gamma_2(\nu)}{(2\gamma-1)(\Gamma(\gamma))^2}\big[1+2\sup_{u\in[0,u]}\mathbb{E}\mathbf{D}^2_\infty\big(y_\epsilon(u),\hat{0}\big)\big]$.

Therefore

$$
J_2 \leq \frac{4\epsilon L_2 \nu^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(s),y_\epsilon(s)\big)ds + 4\epsilon\nu^{2\gamma+1}\beta_2. \tag{6.2.6}
$$

So, combining (6.2.5) and (6.2.6) together, we get

$$
\begin{aligned}
\sup_{u\in[0,\nu]}\mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(u),y_\epsilon(u)\big) &\leq 4\epsilon\nu^{2\gamma}\big(\epsilon\beta_1+\nu\beta_2\big)+\frac{4\epsilon\nu^{2\gamma}\big(\epsilon L_1\nu^{-1}+L_2\big)}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^\nu \mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(s),y_\epsilon(s)\big)ds,\\
&\leq 4\epsilon\nu^{2\gamma}\big(\epsilon\beta_1+\nu\beta_2\big)+\frac{4\epsilon\nu^{2\gamma}\big(\epsilon L_1\nu^{-1}+L_2\big)}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^\nu \sup_{\nu'\in[0,s]}\mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(\nu'),y_\epsilon(\nu')\big)d\nu'.
\end{aligned}
$$

Thus, using Gronwall inequality, we obtain

$$
\sup_{u\in[0,\nu]}\mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(u),y_\epsilon(u)\big)\leq 4\epsilon\nu^{2\gamma}\big(\epsilon\beta_1+\nu\beta_2)\big)\exp\left(\frac{4\epsilon\nu^{2\gamma}\big(\epsilon L_1\nu^{-1}+L_2\big)}{(2\gamma-1)(\Gamma(\gamma))^2}\right).
$$

Choose $0<\gamma<1$ and $L>0$ such that for every $u\in[0,L\epsilon^{-\gamma}]\subseteq J$, we get

$$
\sup_{u\in[0,L\epsilon^{-\gamma}]}\mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(u),y_\epsilon(u)\big)\leq k\epsilon^{1-\gamma},
$$

where

$$
k=4L^{2\gamma}\epsilon^{1-2\gamma\gamma}\big(\epsilon\beta_1+L\epsilon^{-\gamma}\beta_2\big)\exp\left(\frac{4L^{2\gamma}\epsilon^{1-2\gamma\gamma}(L_1L^{-1}\epsilon^{1+\gamma}+L_2)}{(2\gamma-1)(\Gamma(\gamma))^2}\right)
$$

is a constant. Therefore, for any given number δ, $\exists\epsilon_1\in(0,\epsilon_0]$ such that for each $\epsilon\in(0,\epsilon_1]$ and $u\in[0,L\epsilon^{-\gamma}]$, we get

$$
\sup_{u\in[0,L\epsilon^{-\gamma}]}\mathbb{E}\mathbf{D}^2_\infty\big(x_\epsilon(u),y_\epsilon(u)\big)\leq\delta.
$$

Thus the proof is complete.□

6.3 Example

We give an example to illustrate our findings in this section. Consider the following FFPSDEs

$$\begin{cases} {}^{C}\mathcal{D}^{\gamma}x(u) = x(u) + x(u)(\frac{u}{2} - 1)^2 + \langle x(u)dB(u)\rangle, & 0 \le u \le 1, \quad \frac{1}{2} < \gamma < 1. \\ x(0) = 0, \end{cases} \tag{6.3.1}$$

Thus, the appropriate standard form of the FFPSDEs mentioned above is

$${}^{C}\mathbf{D}^{\gamma}x^{\epsilon} = x^{\epsilon} + x^{\epsilon}(\frac{u}{2} - 1)^2 + \langle x^{\epsilon}dB(u)\rangle.$$

Then, $f(u, x^{\epsilon}(u), x^{\epsilon}(\lambda u)) = x^{\epsilon} + x^{\epsilon}(\frac{u}{2} - 1)^2$ and $g(u, x^{\epsilon}(u), x^{\epsilon}(\lambda u)) = x^{\epsilon}$. Hence

$$\begin{aligned} \tilde{f}(x^{\epsilon}(u), x^{\epsilon}(\lambda u)) &= \int_0^1 f(s, x^{\epsilon}(s), x^{\epsilon}(\lambda s))ds, \\ &= \frac{19x^{\epsilon}}{12}, \end{aligned}$$

and

$$\tilde{g}(x^{\epsilon}(u), x^{\epsilon}(\lambda u)) = \int_0^1 g(s, x^{\epsilon}(s), x^{\epsilon}(\lambda s))ds = x^{\epsilon}.$$

As a result, the average form of (6.3.1) may be expressed as

$${}^{C}\mathbf{D}^{\gamma}w^{\epsilon} = \frac{19x^{\epsilon}}{12}du + \sqrt{\epsilon}\langle x^{\epsilon}dB(u)\rangle. \tag{6.3.2}$$

We can see that the coefficients f and g satisfy the assumptions $(\mathcal{A}1)$-$(\mathcal{A}3)$. Then, according to Theorem 6.2.2 the FFPSDEs (6.3.1) has a unique fuzzy solution. On the other hand, we can naturally see that the coefficient $\tilde{f}$ and $\tilde{g}$ satisfy the assumption $(\mathcal{A}4)$, then, according to Theorem 6.2.3, as $\epsilon \longrightarrow 0$, the solution x^{ϵ} and w^{ϵ} to Eqs. (6.3.1) and (6.3.2) are equivalent in the sense of mean square.

Clearly, the reduced system (6.3.2) is much easier to understand than the standard system (6.3.1). Even better, Theorem 6.2.3 ensures that just a minor mistake is introduced throughout the substitution procedure.

Chapter 7

The averaging principle for nonlinear Hilfer fuzzy fractional stochastic differential equations

Contents

7.1 Introduction

Due to its extensive use in the fields of technology and applied analysis, fractional calculus has recently gained assimilated abundant flow and considerable importance. Mathematical models of thermodynamics, fluid flow, and mathematical biology have all seen fresh growth thanks to fractional derivative operators [20, 106, 32]. Fractional derivatives were discovered by Leibnitz in 1695, and since then, a growing number of scientists have dedicated themselves to study of fractional calculus.

The most often used fractional calculus definitions are Riemann-Liouiville definition and Caputo definition [69, 35, 126]. When Hilfer [57] studied fractional time development in physical phenomena, he gave a generalization of both Riemann-Liouiville and Caputo derivative, and many authors call it the Hilfer fractional derivative [56]. Fuzzy analysis and fuzzy differential equations (FDEs) have recently been proposed as solutions to the uncertainty imposed by insufficient information in various mathematical models that prodict real-world situations [52, 108].

The averaging method, however, is an effective tool for studying different types of nonlinear dynamical systems because it makes it possible the replacement of the complex original

time-varying system with a simpler averaged autonomous system, resulting in a reasonable reduction in complexity. Then, the first step in creating an averaging principle is to identify the conditions in which the averaged system's solutions can be sufficiently close to the original system's solution.
The averaging concept has garnered increased attention recently in the field of fractional stochastic dynamical systems. For example, Liu and Xu [71] derived the averaging principle for impulsive fractional neutral stochastic differential equations. Under a non-Lipschitz condition, the averaging principle for the general stochastic differential delay equations is established by Tan and Lei [109]. Xu et al. [116] derived an averaging principle for a class of stochastic differential delay equations driven by fractional Brownian motion. Under suitable non-Lipschitz conditions, Gao et al. [45] proved the averaging principle for stochastic pantograph equations. Ahmed and Zhu [10] established an averaging principle for Hilfer fractional stochastic delay differential equations with Poisson jumps. Arhrrabi et al. [15] initiated the study of averaging principle of fuzzy stochastic differential equations. To our knowledge, no publication has looked at the averaging principle of fuzzy fractional stochastic differential equations, instead, numerous studies have looked at the averaging principle of fractional stochastic differential equations in a crisp case. To close this gap, we will investigate an averaging principle for a class of fuzzy fractional stochastic differential delay equations with Hilfer fractional derivative defined by the system (7.2.1).

7.2 Main results

The following nonlinear Hilfer fuzzy fractional stochastic differential equations with time-delays are discussed in this section.

$$\begin{cases} {}^{H}\mathcal{D}_{0^+}^{\xi,\sigma} y(u) = Ay(u) + f(u, y(u), y(u-\nu))du + \left\langle \int_0^u g(s, y(s), y(s-\nu))dB(s) \right\rangle, u \in J \\ \\ y(u) = \phi(u), \qquad u \in [-\nu, 0], \\ \\ \mathcal{I}_{0^+}^{(1-\xi)(1-\sigma)} y(0) = y_0 \in \mathbf{E}^n, \end{cases} \tag{7.2.1}$$

where ${}^{H}\mathcal{D}_{0^+}^{\xi,\sigma}$ is the Hilfer fractional derivative with $0 \leq \xi \leq 1$, $\frac{1}{2} < \sigma < 1$, A is an n-dimensional matrix and $f : J \times \mathbf{E}^n \times \mathbf{E}^n \longrightarrow \mathbf{E}^n$, $g : J \times \mathbf{E}^n \times \mathbf{E}^n \longrightarrow \mathbb{R}^{n\times m}$ are continuous functions, $y(u)$ is fuzzy random variable, $\nu \in \mathbb{R}^+$ represents the delay, $B(u)$ is standard Brownian motion with n-dimensional, $\phi : [-\nu, 0] \longrightarrow \mathbf{E}^n$ is a continuous function satisfying $\mathbb{E}\mathbf{D}^2_\infty[\phi(0), \hat{0}] < \infty$.

7.2.1 Existence and uniqueness results

In this part, by using Schauder's fixed point Theorem, we will show the existence and uniqueness of solution for Hilfer FFSDEs (7.2.1).

Lemma 7.2.1 *According to [110], the system (7.2.1) is equivalent to the following integral equation:*

$$x(u) = \begin{cases} \phi(u), \quad -\tau \leq u \leq 0, \\ \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)f(s,x(s),x(s-\tau))ds \\ + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)\Big\langle \int_0^s g(v,x(v),x(v-\tau))dB(v)\Big\rangle ds, \quad u \in J. \end{cases} \tag{7.2.2}$$

We make the following hypotheses concerning the coefficients of the system under consideration:

$(\mathcal{H}1)$ For all $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathbf{E^n}$ and for all $u \in J$, we have

$$\mathbb{E}\mathbf{D}^2_\infty\left[f(u,\varphi_1,\psi_1), f(u,\varphi_2,\psi_2)\right] \vee \mathbb{E}\|g(u,\varphi_1,\psi_1) - g(u,\varphi_2,\psi_2)\|^2 \leq H\Big(u, \mathbb{E}\mathbf{D}^2_\infty[\varphi_1,\varphi_2], \mathbb{E}\mathbf{D}^2_\infty[\psi_1,\psi_2]\Big),$$

where $H : J \times \mathbf{E^n} \times \mathbf{E^n} \longrightarrow \mathbf{E^n}$ is a monotone increasing, continuous and concave function with $H(u,0,0) = 0$ and $H(u,x(u),x(u)) = kH(u,x(u))$, k is a constant.

$(\mathcal{H}2)$ For all $u \geq 0$, $\exists \lambda > 0$ such that

$$\mathbb{E}\mathbf{D}^2_\infty[f(u,\widehat{0},\widehat{0}),\widehat{0}] \vee \mathbb{E}\|g(u,\widehat{0},\widehat{0})\|^2 \leq \lambda.$$

$(\mathcal{H}3)$ For any $\varphi, \psi \in \mathbf{E^n}$, we suppose that there exists a function $h \in C(J,\mathbf{E^n})$ such that

$$H\Big(u, \mathbb{E}\mathbf{D}^2_\infty[\varphi,\psi]\Big) \leq h(u)\mathbb{E}\mathbf{D}^2_\infty[\varphi,\psi].$$

We will now use Schauder's fixed point Theorem to demonstrate our result.

Theorem 7.2.2 *Suppose that $f : J \times \mathbf{E^n} \times \mathbf{E^n} \longrightarrow \mathbf{E^n}$ and $g : J \times \mathbf{E^n} \times \mathbf{E^n} \longrightarrow \mathbb{R}^{n\times m}$ are continuous and satisfying the hypotheses $(\mathcal{H}1)$-$(\mathcal{H}3)$. Then, there exist at least a solution to the system (7.2.1).*

Proof. Consider the operator $\mathfrak{L}$ on $\mathbb{K}' := C\big([-\tau,T], L^2(\Omega,\mathbf{E^n})\big)$ defined as follows

$$\mathfrak{L}(x(u)) = \begin{cases} \phi(u), \quad u \in [-\tau,0], \\ \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)f(s,x(s),x(s-\tau))ds \\ + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)\Big\langle \int_0^s g(v,x(v),x(v-\tau))dB(v)\Big\rangle ds, \quad u \in J. \end{cases} \tag{7.2.3}$$

To prove this result, we divide the subsequent proof into two steps.

Step 1: $\mathfrak{L}$ is completely continuous. For this, let us prove that:

ⓐ- $\mathfrak{L}$ is continuous. Indeed, for any integer $n \geq 1$, define $x_n(u) = \phi(0)$ for all $u \in [-\tau, 0]$. For all $u \in J$

$$x_n(u) = \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)f(s,x(s),x(s-\tau))ds \qquad (7.2.4)$$
$$+ \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)\Big\langle \int_0^s g(v,x(v),x(v-\tau))dB(v)\Big\rangle ds.$$

Let $\overline{\mathfrak{M}}_1 = \sup\limits_{0\leq u\leq T} \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)$, with the aid of the hypotheses $(\mathcal{H}1)$-$(\mathcal{H}3)$, Propositions 1.2.25-1.2.26 and Cauchy–Schwarz inequality, we get

$$\begin{aligned}
\mathbb{E}\mathbf{D}^2_\infty\big[\mathfrak{L}(x_n(u)),\mathfrak{L}(x(u))\big] &= \Big[\mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)f(s,x_n(s),x_n(s-\tau))ds \\
&\quad + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)\Big\langle\int_0^s g(v,x_n(v),x_n(v-\tau))dB(v)\Big\rangle ds, \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) \\
&\quad + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)f(s,x(s),x(s-\tau))ds \\
&\quad + \int_0^u (u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)\Big\langle\int_0^s g(v,x(v),x(v-\tau))dB(v)\Big\rangle ds\Big], \\
&\leq 2\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}}f(s,x_n(s),x_n(s-\tau))ds, \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}}f(s,x(s),x(s-\tau))ds\Big] \\
&\quad + 2\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}}\Big\langle\int_0^s g(v,x_n(v),x_n(v-\tau))dB(v)\Big\rangle ds, \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}} \\
&\quad \Big\langle\int_0^s g(v,x(v),x(v-\tau))dB(v)\Big\rangle ds\Big], \\
&\leq \frac{2T^{2\beta-1}\overline{\mathfrak{M}}_1^2}{2\beta-1}\int_0^u \mathbb{E}D^2_\infty\big[f(s,x_n(s),x_n(s-\tau)), f(s,x(s),x(s-\tau))\big]ds \\
&\quad + \frac{2T^{2\beta-1}\overline{\mathfrak{M}}_1^2}{2\beta-1}\int_0^u\int_0^s \mathbb{E}\|g(v,x_n(v),x_n(v-\tau)) - g(v,x(v),x(v-\tau))\|^2 dvds, \\
&\leq \frac{2T^{2\beta-1}\overline{\mathfrak{M}}_1^2}{2\beta-1}\int_0^u H\Big(s,\mathbb{E}\mathbf{D}^2_\infty\big[x_n(s),x(s)\big],\mathbb{E}\mathbf{D}^2_\infty\big[x_n(s-\tau),x(s-\tau)\big]\Big)ds \\
&\quad + \frac{2T^{2\beta-1}\overline{\mathfrak{M}}_1^2}{2\beta-1}\int_0^u\int_0^s H\Big(v,\mathbb{E}\mathbf{D}^2_\infty\big[x_n(v),x(v)\big],\mathbb{E}\mathbf{D}^2_\infty\big[x_n(v-\tau),x(v-\tau)\big]\Big)dvds,
\end{aligned}$$

where, by using the definition of $\mathbf{D}_\infty$, we have

$$\begin{aligned}
\mathbf{D}_\infty[x_n(s-\tau),x(s-\tau)] &= \sup_{0\leq r\leq 1}\max_{0\leq s\leq u}\{|\,\underline{x_n}(s-\tau,r) - \underline{x}(s-\tau,r)\,|,|\,\overline{x_n}(s-\tau,r) - \overline{x}(s-\tau,r)\,|\}, \\
&= \sup_{0\leq r\leq 1}\max_{-\tau\leq\mu\leq u-\tau}\{|\,\underline{x_n}(\mu,r) - \underline{x}(\mu,r)\,|,|\,\overline{x_n}(\mu,r) - \overline{x}(\mu,r)\,|\}, \\
&\leq \sup_{0\leq r\leq 1}\max_{-\tau\leq\mu\leq 0}\{|\,\underline{x_n}(\mu,r) - \underline{x}(\mu,r)\,|,|\,\overline{x_n}(\mu,r) - \overline{x}(\mu,r)\,|\} \\
&\quad + \sup_{0\leq r\leq 1}\max_{0\leq\mu\leq u-\tau}\{|\,\underline{x_n}(\mu,r) - \underline{x}(\mu,r)\,|,|\,\overline{x_n}(\mu,r) - \overline{x}(\mu,r)\,|\}, \\
&\leq \sup_{0\leq r\leq 1}\max_{0\leq s\leq \tau}\{|\,\underline{x_n}(s,r) - \underline{x}(s,r)\,|,|\,\overline{x_n}(s,r) - \overline{x}(s,r)\,|\} \\
&\quad + \sup_{0\leq r\leq 1}\max_{\tau\leq s\leq u}\{|\,\underline{x_n}(s,r) - \underline{x}(s,r)\,|,|\,\overline{x_n}(s,r) - \overline{x}(s,r)\,|\}, \\
&\leq \sup_{0\leq r\leq 1}\max_{0\leq s\leq u}\{|\,\underline{x_n}(s,r) - \underline{x}(s,r)\,|,|\,\overline{x_n}(s,r) - \overline{x}(s,r)\,|\} = \mathbf{D}_\infty[x_n(s),x(s)].
\end{aligned}$$

Then, using the hypothesis $(\mathcal{H}1)$, we have

$$\begin{aligned}\mathbb{E}\mathbf{D}_\infty^2\left[\mathfrak{L}(x_n(u)),\mathfrak{L}(x(u))\right] &\leq \frac{2T^{2\beta-1}\overline{\mathfrak{M}}_1^2}{2\beta-1}\int_0^u H\Big(s,\mathbb{E}\mathbf{D}_\infty^2\left[x_n(s),x(s)\right],\mathbb{E}\mathbf{D}_\infty^2\left[x_n(s),x(s)\right]\Big)ds\\&\quad+\frac{2T^{2\beta-1}\overline{\mathfrak{M}}_1^2}{2\beta-1}\int_0^u\int_0^s H\Big(v,\mathbb{E}\mathbf{D}_\infty^2\left[x_n(v),x(v)\right],\mathbb{E}\mathbf{D}_\infty^2\left[x_n(v),x(v)\right]\Big)dvds,\\&\leq \frac{2T^{2\beta-1}\overline{\mathfrak{M}}_1^2k}{2\beta-1}\int_0^u H\Big(s,\mathbb{E}\mathbf{D}_\infty^2\left[x_n(s),x(s)\right]\Big)ds\\&\quad+\frac{2T^{2\beta-1}\overline{\mathfrak{M}}_1^2k}{2\beta-1}\int_0^u\int_0^s H\Big(v,\mathbb{E}\mathbf{D}_\infty^2\left[x_n(v),x(v)\right]\Big)dvds,\\&\leq\left(\frac{2T^{2\beta}\overline{\mathfrak{M}}_1^2k}{2\beta-1}+\frac{2T^{2\beta+1}\overline{\mathfrak{M}}_1^2k}{2\beta-1}\right)H\Big(u,\mathbb{E}\mathbf{D}_\infty^2\left[x_n(u),x(u)\right]\Big).\end{aligned}$$

Since H is continuous, we can conclude that $\mathbb{E}\mathbf{D}_\infty^2\left[\mathfrak{L}(x_n(u)),\mathfrak{L}(x(u))\right]\longrightarrow 0$ as $n\longrightarrow\infty$. Hence, $\mathfrak{L}$ is continuous.
ⓑ- We prove that there exists a positive constant ξ_1 and for all $\sigma_1>0$ satisfying for all $x(u)\in\mathcal{B}_{\sigma_1}:=\Big\{x(u)\in C\big([-\tau,T],L^2(\Omega,\mathbf{E^n})\big)|\mathbb{E}\mathbf{D}_\infty^2[x(u),\hat{0}]\leq\sigma_1\Big\}$ one has $\mathbb{E}\mathbf{D}_\infty^2\left[\mathfrak{L}\big(x(u)\big),\hat{0}\right]\leq\xi_1$. So, we let $\overline{\mathfrak{M}}_2=\sup_{0\leq u\leq T}\mathfrak{M}_{\beta,\beta}(A(u-s)^\beta)$ and for all $u\in J$ and $x(u)\in\mathcal{B}_{\sigma_1}$, we have

$$\begin{aligned}\mathbb{E}\mathbf{D}_\infty^2\left[\mathfrak{L}\big(x(u)\big),\hat{0}\right]&=\mathbb{E}\mathbf{D}_\infty^2\bigg[\mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0)+\int_0^u(u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)f(s,x(s),x(s-\tau))ds\\&\quad+\int_0^u(u-s)^{\beta-1}\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)\Big\langle\int_0^s g(v,x(v),x(v-\tau))dB(v)\Big\rangle ds,\hat{0}\bigg],\\&\leq 3\mathbb{E}\mathbf{D}_\infty^2\left[\mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0),\hat{0}\right]+3\mathbb{E}\mathbf{D}_\infty^2\left[\int_0^u\frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}}f(s,x(s),x(s-\tau))ds,\hat{0}\right]\\&\quad+3\mathbb{E}\mathbf{D}_\infty^2\left[\int_0^u\frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}}\Big\langle\int_0^s g(v,x(v),x(v-\tau))dB(v)\Big\rangle ds,\hat{0}\right],\\&\leq 3\overline{\mathfrak{M}}_1^2\mathbb{E}\mathbf{D}_\infty^2\left[\phi(0),\hat{0}\right]+\frac{3T^{2\beta-1}\overline{\mathfrak{M}}_2^2}{2\beta-1}\int_0^u\mathbb{E}\mathbf{D}_\infty^2\left[f(s,x(s),x(s-\tau)),\hat{0}\right]ds\\&\quad+\frac{3T^{2\beta-1}\overline{\mathfrak{M}}_2^2}{2\beta-1}\int_0^u\int_0^s\mathbb{E}\|g(v,x(v),x(v-\tau))\|^2dvds.\end{aligned}$$

Since the functions f and g are continuous, there is a constant $N_{f,g}>0$ such that

$$\mathbb{E}\mathbf{D}_\infty^2\left[f(u,\varphi,\psi),\hat{0}\right]\vee\mathbb{E}\|g(u,\varphi,\psi)\|^2\leq N_{f,g}.$$

Then,

$$\mathbb{E}\mathbf{D}_\infty^2\left[\mathfrak{L}\big(x(u)\big),\hat{0}\right]\leq 3\overline{\mathfrak{M}}_1^2\mathbb{E}\mathbf{D}_\infty^2\left[\phi(0),\hat{0}\right]+\left(\frac{3T^{2\beta}\overline{\mathfrak{M}}_2^2}{2\beta-1}+\frac{3T^{2\beta+1}\overline{\mathfrak{M}}_2^2}{2\beta-1}\right)N_{f,g}:=\xi_1.$$

Therefore, for every $x(u)\in\mathcal{B}_{\sigma_1}$, we have $\mathbb{E}\mathbf{D}_\infty^2\left[\mathfrak{L}\big(x(u)\big),\hat{0}\right]\leq\xi_1$, this implies that $\mathfrak{L}(\mathcal{B}_{\sigma_1})\subseteq\mathcal{B}_{\xi_1}$.
ⓒ- $\mathfrak{L}$ maps bounded set into equi-continuous set. Indeed, for each $x(u)\in\mathcal{B}_{\sigma_2}$ and $u_1,u_2\in J$

such that $0 \leq u_1 < u_2 \leq T$, we have

$$\begin{aligned}
\mathbb{E}\mathbf{D}^2_\infty[\mathfrak{L}(x(u_1)), \mathfrak{L}(x(u_2))] &= \mathbb{E}\mathbf{D}^2_\infty\Big[\mathfrak{M}_{\beta,\gamma(1-\beta)}(Au_1)\phi(0) + \int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}(A(u_1-s)^\beta)}{(u_1-s)^{1-\beta}} f(s,x(s),x(s-\tau))ds \\
&+ \int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}(A(u_1-s)^\beta)}{(u_1-s)^{1-\beta}} \Big\langle \int_0^s g(v,x(v),x(v-\tau))dB(v)\Big\rangle ds, \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au_2)\phi(0) \\
&+ \int_0^{u_2} \frac{\mathfrak{M}_{\beta,\beta}(A(u_2-s)^\beta)}{(u_2-s)^{1-\beta}} f(s,x(s),x(s-\tau))ds \\
&+ \int_0^{u_2} \frac{\mathfrak{M}_{\beta,\beta}(A(u_2-s)^\beta)}{(u_2-s)^{1-\beta}} \Big\langle \int_0^s g(v,x(v),x(v-\tau))dB(v)\Big\rangle ds\Big], \\
&\leq 2\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}(A(u_1-s)^\beta)}{(u_1-s)^{1-\beta}} f(s,x(s),x(s-\tau))ds, \int_0^{u_2} \frac{\mathfrak{M}_{\beta,\beta}(A(u_2-s)^\beta)}{(u_2-s)^{1-\beta}} f(s,x(s),x(s-\tau))ds\Big] \\
&+ 2\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}(A(u_1-s)^\beta)}{(u_1-s)^{1-\beta}} \Big\langle \int_0^s g(v,x(v),x(v-\tau))dB(v)\Big\rangle ds, \int_0^{u_2} \frac{\mathfrak{M}_{\beta,\beta}(A(u_2-s)^\beta)}{(u_2-s)^{1-\beta}} \Big\langle \int_0^s g(v, \\
&x(v),x(v-\tau))dB(v)\Big\rangle ds\Big] := P_1 + P_2,
\end{aligned}$$

Applying the hypotheses $(\mathcal{H}1)$-$(\mathcal{H}3)$, Propositions 1.2.25-1.2.26, Lemma 1.2.29 and Cauchy–Schwarz inequality, we get

$$\begin{aligned}
P_1 &\leq 2\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}(A(u_1-s)^\beta)}{(u_1-s)^{1-\beta}} f(s,x(s),x(s-\tau))ds, \int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}(A(u_2-s)^\beta)}{(u_2-s)^{1-\beta}} f(s,x(s),x(s-\tau))ds \\
&+ \int_{u_1}^{u_2} \frac{\mathfrak{M}_{\beta,\beta}(A(u_2-s)^\beta)}{(u_2-s)^{1-\beta}} f(s,x(s),x(s-\tau))ds\Big], \\
&\leq 4\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}(A(u_1-s)^\beta)}{(u_1-s)^{1-\beta}} f(s,x(s),x(s-\tau))ds, \int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}(A(u_2-s)^\beta)}{(u_2-s)^{1-\beta}} f(s,x(s),x(s-\tau))ds\Big] \\
&+ 4\mathbb{E}\mathbf{D}^2_\infty\Big[\int_{u_1}^{u_2} \frac{\mathfrak{M}_{\beta,\beta}(A(u_2-s)^\beta)}{(u_2-s)^{1-\beta}} f(s,x(s),x(s-\tau))ds, \hat{0}\Big], \\
&\leq \int_0^{u_1} 4\,|\,(u_1-s)^{2\beta-2}(\mathfrak{M}_{\beta,\beta}(A(u_1-s)^\beta)^2 - (u_2-s)^{2\beta-2}(A(u_2-s)^\beta)^2\,|\,ds \cdot \int_0^{u_1} \mathbb{E}\mathbf{D}^2_\infty[f(s,x(s),x(s-\tau)), \hat{0}]ds \\
&+ \frac{4(u_2-u_1)^{2\beta-1}\overline{\mathfrak{M}}_2^2}{2\beta-1}\int_{u_1}^{u_2} \mathbb{E}\mathbf{D}^2_\infty[f(s,x(s),x(s-\tau)), \hat{0}]ds, \\
&\leq \frac{4\Big(u_1^{2\beta-1} - u_2^{2\beta-1} + (u_2-u_1)^{2\beta-1}\Big)\overline{\mathfrak{M}}_2^2 T N_{f,g}}{2\beta-1} + \frac{4(u_2-u_1)^{2\beta-1}\overline{\mathfrak{M}}_2^2 T N_{f,g}}{2\beta-1}, \\
&\leq \frac{12(u_2-u_1)^{2\beta-1}\overline{\mathfrak{M}}_2^2 T N_{f,g}}{2\beta-1}.
\end{aligned}$$

In the same way, we get

$$\begin{aligned}
P_2 &\leq 2\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}(A(u_1-s)^\beta)}{(u_1-s)^{1-\beta}} \Big\langle \int_0^s g(v,x(v),x(v-\tau))dB(v)\Big\rangle ds, \int_0^{u_1} \frac{\mathfrak{M}_{\beta,\beta}(A(u_2-s)^\beta)}{(u_2-s)^{1-\beta}} \Big\langle \int_0^s g(v, \\
&x(v),x(v-\tau))dB(v)\Big\rangle ds + \int_{u_1}^{u_2} \frac{\mathfrak{M}_{\beta,\beta}(A(u_2-s)^\beta)}{(u_2-s)^{1-\beta}} \Big\langle \int_0^s g(v,x(v),x(v-\tau))dB(v)\Big\rangle ds\Big], \\
&\leq \int_0^{u_1} 4\,|\,(u_1-s)^{2\beta-2}(\mathfrak{M}_{\beta,\beta}(A(u_1-s)^\beta)^2 - (u_2-s)^{2\beta-2}(A(u_2-s)^\beta)^2\,|\,ds \cdot \int_0^{u_1}\int_0^s \mathbb{E}\|g(v,x(v), \\
&x(v-\tau))\|^2 dvds + \frac{4(u_2-u_1)^{2\beta-1}\overline{\mathfrak{M}}_2^2}{2\beta-1}\int_{u_1}^{u_2}\int_0^s \mathbb{E}\|g(v,x(v),x(v-\tau))\|^2 dvds, \\
&\leq \frac{12(u_2-u_1)^{2\beta-1}\overline{\mathfrak{M}}_2^2 T^2 N_{f,g}}{2\beta-1}.
\end{aligned}$$

Therefore, we get

$$\mathbb{E}\mathbf{D}^2_\infty[\mathfrak{L}(x(u_1)), \mathfrak{L}(x(u_2))] \leq \frac{12(u_2-u_1)^{2\beta-1}\overline{\mathfrak{M}}_2^2 T N_{f,g}(T+1)}{2\beta-1}.$$

We have $\frac{12(u_2-u_1)^{2\beta-1}\overline{\mathfrak{M}}_2^2 T N_{f,g}(T+1)}{2\beta-1}$ is independent of $x(u)$ and $\frac{12(u_2-u_1)^{2\beta-1}\overline{\mathfrak{M}}_2^2 T N_{f,g}(T+1)}{2\beta-1} \longrightarrow 0$ as $u_2 \longrightarrow u_1$. Then, we obtain

$$\mathbb{E}\mathbf{D}_\infty^2\left[\mathfrak{L}\big(x(u_1)\big),\mathfrak{L}\big(x(u_2)\big)\right] \longrightarrow 0.$$

It means that $\mathfrak{L}(\mathcal{B}_{\sigma_2})$ is equi-continuous. Then, according to Arzela-Ascoli Theorem, $\mathfrak{L}$ is completely continuous.

Step 2: In this step, we will demonstrate that there is a closed, convex and bounded subset $\mathcal{B}_\xi = \Big\{x(u) \in C\big([-\tau,T],L^2(\Omega,\mathbf{E}^n)\big)|\mathbb{E}\mathbf{D}_\infty^2[x(u),\hat{0}] \leq \xi\Big\}$ such that $\mathfrak{L}(\mathcal{B}_\xi) \subseteq \mathcal{B}_\xi$. We know that $\mathcal{B}_\xi$ is a closed, convex and bounded subset of $C\big([-\tau,T],L^2(\Omega,\mathbf{E}^n)\big)$ for all $\xi > 0$. Suppose that for all $\xi > 0$, $\exists x_\xi(u) \in \mathcal{B}_\xi$ such that $\mathfrak{L}(x_\xi(u)) \notin \mathcal{B}_\xi$, that is $\mathbb{E}\mathbf{D}_\infty^2\big[\mathfrak{L}(x_\xi(u)),\hat{0}\big] > \xi$. Then

$$\begin{aligned}
\xi < \mathbb{E}\mathbf{D}_\infty^2\left[\mathfrak{L}(x_\xi(u)),\hat{0}\right] &= \mathbb{E}\mathbf{D}_\infty^2\Bigg[\mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}} f(s,x_\xi(s),x_\xi(s-\tau))ds \\
&\quad + \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}}\Big\langle \int_0^s g(v,x_\xi(v),x_\xi(v-\tau))dB(v)\Big\rangle ds,\hat{0}\Bigg],\\
&\leq 3\overline{\mathfrak{M}}_1^2\mathbb{E}\mathbf{D}_\infty^2\left[\phi(t),\hat{0}\right] + \frac{3T^{2\beta-1}\overline{\mathfrak{M}}_2^2}{2\beta-1}\int_0^u \mathbb{E}\mathbf{D}_\infty^2\left[f(s,x_\xi(s),x_\xi(s-\tau)),\hat{0}\right]ds \\
&\quad + \frac{3T^{2\beta-1}\overline{\mathfrak{M}}_2^2}{2\beta-1}\int_0^u\int_0^s \mathbb{E}\|g(v,x_\xi(v),x_\xi(v-\tau))\|^2 dvds,\\
&\leq 3\overline{\mathfrak{M}}_1^2\mathbb{E}\mathbf{D}_\infty^2\left[\phi(t),\hat{0}\right] + \left(\frac{3T^{2\beta}\overline{\mathfrak{M}}_2^2}{2\beta-1} + \frac{3T^{2\beta+1}\overline{\mathfrak{M}}_2^2}{2(2\beta-1)}\right)N_{f,g}.
\end{aligned}$$

Taking limit as $\xi \longrightarrow +\infty$, we obtain that $3\overline{\mathfrak{M}}_1^2\mathbb{E}\mathbf{D}_\infty^2\left[\phi(t),\hat{0}\right] + \big(\frac{3T^{2\beta}\overline{\mathfrak{M}}_2^2}{2\beta-1} + \frac{3T^{2\beta+1}\overline{\mathfrak{M}}_2^2}{2(2\beta-1)}\big)N_{f,g} \longrightarrow +\infty$ which is in contradiction with $3\overline{\mathfrak{M}}_1^2\mathbb{E}\mathbf{D}_\infty^2\left[\phi(t),\hat{0}\right] + \big(\frac{3T^{2\beta}\overline{\mathfrak{M}}_2^2}{2\beta-1} + \frac{3T^{2\beta+1}\overline{\mathfrak{M}}_2^2}{2(2\beta-1)}\big)N_{f,g}$ is bounded. Therefore, for every positive constant ξ, we obtain $\mathfrak{L}(\mathcal{B}_\xi) \subseteq \mathcal{B}_\xi$. By means of Schauder's fixed point Theorem implying that there is at least one solution of the system (7.2.1).□

For the uniqueness result, we have the following theorem:

Theorem 7.2.3 *Assume that the hypotheses* $(\mathcal{H}1)$-$(\mathcal{H}3)$ *holds. Then, if*

$$\sup_{0\leq u\leq T} h(u)\frac{2T^{2\beta}\overline{\mathfrak{M}}_2^2(1+T)}{2\beta-1} < 1,$$

then the solution of system (7.2.1) *is unique.*

Proof. We know that $x(u)$ is a solution of system (7.2.1) if

$$\begin{aligned}
x(u) = \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) &+ \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}} f(s,x(s),x(s-\tau))ds \\
&+ \int_0^u \frac{\mathfrak{M}_{\beta,\beta}\big(A(u-s)^\beta\big)}{(u-s)^{1-\beta}}\Big\langle \int_0^s g(v,x(v),x(v-\tau))dB(v)\Big\rangle ds,
\end{aligned}$$

hold. If $x(u) \in C\big([-\tau,T],L^2(\Omega,\mathbf{E}^n)\big)$ is a fixed point of $\mathfrak{L}$ which define as in Theorem 7.2.2, therefore $x(u)$ is the solution of system (7.2.1). Let $x_1(u),x_2(u) \in C\big([-\tau,T],L^2(\Omega,\mathbf{E}^n)\big)$ and for $u \in [-\tau, 0]$, $x_1(u) = x_2(u) = \phi(u)$. For all $u \in J$, we have

$$\begin{aligned}
\mathbb{E}\mathbf{D}^2_\infty[\mathfrak{L}(x_1(u)), \mathfrak{L}(x_2(u))] &= \mathbb{E}\mathbf{D}^2_\infty\Big[\mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) + \int_0^u \frac{\mathfrak{M}_{\beta,\beta}(A(u-s)^\beta)}{(u-s)^{1-\beta}} f(s, x_1(s), x_1(s-\tau))ds \\
&+ \int_0^u \frac{\mathfrak{M}_{\beta,\beta}(A(u-s)^\beta)}{(u-s)^{1-\beta}} \Big\langle \int_0^s g(v, x_1(v), x_1(v-\tau))dB(v)\Big\rangle ds, \mathfrak{M}_{\beta,\gamma(1-\beta)}(Au)\phi(0) \\
&+ \int_0^u \frac{\mathfrak{M}_{\beta,\beta}(A(u-s)^\beta)}{(u-s)^{1-\beta}} f(s, x_2(s), x_2(s-\tau))ds + \int_0^u \frac{\mathfrak{M}_{\beta,\beta}(A(u-s)^\beta)}{(u-s)^{1-\beta}} \Big\langle \int_0^s g(v, x_2(v), x_2(v-\tau))dB(v)\Big\rangle ds\Big], \\
&\leq 2\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^u \frac{\mathfrak{M}_{\beta,\beta}(A(u-s)^\beta)}{(u-s)^{1-\beta}} f(s, x_1(s), x_1(s-\tau))ds, \int_0^u \frac{\mathfrak{M}_{\beta,\beta}(A(u-s)^\beta)}{(u-s)^{1-\beta}} f(s, x_2(s), x_2(s-\tau))ds\Big] \\
&+ 2\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^u \frac{\mathfrak{M}_{\beta,\beta}(A(u-s)^\beta)}{(u-s)^{1-\beta}} \Big\langle \int_0^s g(v, x_1(v), x_1(v-\tau))dB(v)\Big\rangle ds, \int_0^u \frac{\mathfrak{M}_{\beta,\beta}(A(u-s)^\beta)}{(u-s)^{1-\beta}} \Big\langle \int_0^s g(v, x_2(v), \\
&\quad x_2(v-\tau))dB(v)\Big\rangle ds\Big], \\
&\leq \frac{2T^{2\beta-1}\overline{\mathfrak{M}}_2^2}{2\beta-1}\int_0^u H\Big(s, \mathbb{E}\mathbf{D}^2_\infty[x_1(s), x_2(s)], \mathbb{E}\mathbf{D}^2_\infty[x_1(s-\tau), x_2(s-\tau)]\Big)ds \\
&+ \frac{2T^{2\beta-1}\overline{\mathfrak{M}}_2^2}{2\beta-1}\int_0^u\int_0^s H\Big(v, \mathbb{E}\mathbf{D}^2_\infty[x_1(v), x_2(v)], \mathbb{E}\mathbf{D}^2_\infty[x_1(v-\tau), x_2(v-\tau)]\Big)dvds, \\
&\leq \left(\frac{2T^{2\beta}\overline{\mathfrak{M}}_2^2}{2\beta-1} + \frac{2T^{2\beta+1}\overline{\mathfrak{M}}_2^2}{2\beta-1}\right)\sup_{0\leq u\leq T} H(u, \mathbb{E}\mathbf{D}^2_\infty[x_1(u), x_2(u)]), \\
&\leq \left(\frac{2T^{2\beta}\overline{\mathfrak{M}}_2^2(1+T)}{2\beta-1}\right)\sup_{0\leq u\leq T} h(u)\mathbb{E}\mathbf{D}^2_\infty[x_1(u), x_2(u)],
\end{aligned}$$

since $\sup_{0\leq u\leq T} h(u) \leq \frac{2\beta-1}{2T^{2\beta}\overline{\mathfrak{M}}_2^2(1+T)}$, we have

$$\mathbb{E}\mathbf{D}^2_\infty\big[\mathfrak{L}(x_1(u)), \mathfrak{L}(x_2(u))\big] \leq \mathbb{E}\mathbf{D}^2_\infty\big[x_1(u), x_2(u)\big].$$

Based on the Banach contraction principle, $\mathfrak{L}$ has an unique fixed point $x(u)$.□

7.2.2 Averaging result

The construction of an averaging concept for Hilfer FFSDEs (7.2.1) is the focus of this subsection. Let us consider the standard form of Eq. (7.2.2)

$$\begin{aligned}
y_\varepsilon(u) =&\mathfrak{M}_{\sigma,\xi(1-\sigma)}(Au)\phi(0) + \varepsilon\int_0^u (u-s)^{\sigma-1}\mathfrak{M}_{\sigma,\sigma}(A(u-s)^\sigma)f(s, y_\varepsilon(s), y_\varepsilon(s-\nu))ds \qquad (7.2.5)\\
&+ \sqrt{\varepsilon}\int_0^u (u-s)^{\sigma-1}\mathfrak{M}_{\sigma,\sigma}(A(u-s)^\sigma)\Big\langle \int_0^s g(v, y_\varepsilon(v), y_\varepsilon(v-\nu))dB(v)\Big\rangle ds, \qquad u \in J.
\end{aligned}$$

where $\varepsilon \in (0, \varepsilon_0)$ is a positive small parameter with ε_0 a fixed number.

In order, we set certain hypothesis on the coefficients to see if the solution $y_\varepsilon(u)$ can be approximated by a small process to a simple process.

Let $\overline{f} : \mathbf{E}^n \times \mathbf{E}^n \longrightarrow \mathbf{E}^n$ and $\overline{g} : \mathbf{E}^n \times \mathbf{E}^n \longrightarrow \mathbb{R}^{n\times m}$ be measurable functions satisfying $(\mathcal{H}1)$-$(\mathcal{H}3)$ and the additional inequalities:

$(\mathcal{H}4)$ For $\psi, \varphi \in \mathbf{E}^n$ and $T' \in J$, there exists a positive bounded functions $\eta_i(T')$, $i = 1, 2$ such that

$$\frac{1}{T'}\int_0^{T'} \mathbb{E}\mathbf{D}^2_\infty\Big[f(u, \varphi, \psi), \overline{f}(\varphi, \psi)\Big]du \leq \eta_1(T')\Big(1 + \mathbb{E}\mathbf{D}^2_\infty[\varphi, \hat{0}] + \mathbb{E}\mathbf{D}^2_\infty[\psi, \hat{0}]\Big),$$

$$\frac{1}{T'}\int_0^{T'}\mathbb{E}\left\| g(u,\varphi,\psi)-\overline{g}(\varphi,\psi)\right\|^2du\le\eta_2(T')\Big(1+\mathbb{E}\mathbf{D}^2_\infty[\varphi,\hat{0}]+\mathbb{E}\mathbf{D}^2_\infty[\psi,\hat{0}]\Big),$$

where $\lim\limits_{T'\longrightarrow\infty}\eta_i(T')=0,\ i=1,2.$
With the appropriate preparations above, we will show that the solution y_ε converge as $\varepsilon\longrightarrow 0$, to the solution x_ε of the following averaged equation

$$\begin{aligned}x_\varepsilon(u)=&\mathfrak{M}_{\sigma,\xi(1-\sigma)}(Au)\phi(0)+\varepsilon\int_0^u(u-s)^{\sigma-1}\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)\overline{f}(x_\varepsilon(s),x_\varepsilon(s-\nu))ds \\ &+\sqrt{\varepsilon}\int_0^u(u-s)^{\sigma-1}\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)\Big\langle\int_0^s\overline{g}(x_\varepsilon(v),x_\varepsilon(v-\nu))dB(v)\Big\rangle ds,\qquad u\in J.\end{aligned}\tag{7.2.6}$$

The main result of this subsection is now presented, in which we consider the connections between the process y_ε and x_ε.

Theorem 7.2.4 *Assume that the hypotheses $(\mathcal{H}1)$-$(\mathcal{H}4)$ are satisfied. For a given arbitrarily small number $\Delta>0$ and a constant $k>0$, $\alpha\in(0,1)$, there exist $\varepsilon_1\in(0,\varepsilon_0]$ such that $\forall\varepsilon\in(0,\varepsilon_1]$, we have*

$$\sup_{u\in[-\nu,k\varepsilon^{-\alpha}]}\mathbb{E}\mathbf{D}^2_\infty\big[y_\varepsilon(u),x_\varepsilon(u)\big]\le\Delta.$$

Proof. For any $u\in[0,u_1]\subset J$, we have

$$\begin{aligned}\sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty\big[y_\epsilon(u),x_\epsilon(u)\big]=&\sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty\Big[\mathfrak{M}_{\sigma,\xi(1-\sigma)}(Au)\phi(0)+\varepsilon\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}f(s,y_\varepsilon(s),y_\varepsilon(s-\nu))ds \\ &+\sqrt{\varepsilon}\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}\Big\langle\int_0^s g(v,y_\varepsilon(v),y_\varepsilon(v-\nu))dB(v)\Big\rangle ds,\mathfrak{M}_{\sigma,\xi(1-\sigma)}(Au)\phi(0) \\ &+\varepsilon\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}\overline{f}(x_\varepsilon(s),x_\varepsilon(s-\nu))ds+\sqrt{\varepsilon}\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}\Big\langle\int_0^s\overline{g}(x_\varepsilon(v),x_\varepsilon(v-\nu))dB(v)\Big\rangle ds\Big], \\ \le&2\varepsilon^2\sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}f(s,y_\varepsilon(s),y_\varepsilon(s-\nu))ds,\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}\overline{f}(x_\varepsilon(s),x_\varepsilon(s-\nu))ds\Big] \\ &+2\epsilon\sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}\Big\langle\int_0^s g(v,y_\varepsilon(v),y_\varepsilon(v-\nu))dB(v)\Big\rangle ds,\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}} \\ &\Big\langle\int_0^s\overline{g}(x_\varepsilon(v),x_\varepsilon(v-\nu))dB(v)\Big\rangle ds\Big].\end{aligned}$$

Denote by

$$J_1=2\varepsilon^2\sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}f(s,y_\varepsilon(s),y_\varepsilon(s-\nu))ds,\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}\overline{f}(x_\varepsilon(s),x_\varepsilon(s-\nu))ds\Big],$$

$$\begin{aligned}J_2=2\epsilon\sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty\Big[&\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}\Big\langle\int_0^s g(v,y_\varepsilon(v),y_\varepsilon(v-\nu))dB(v)\Big\rangle ds,\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}} \\ &\Big\langle\int_0^s\overline{g}(x_\varepsilon(v),x_\varepsilon(v-\nu))dB(v)\Big\rangle ds\Big].\end{aligned}$$

Then, using the properties of the metric $\mathbf{D}_\infty$, we get

$$\begin{aligned}J_1\le&4\varepsilon^2\sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}f(s,y_\varepsilon(s),y_\varepsilon(s-\nu))ds,\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}f(s,x_\varepsilon(s),x_\varepsilon(s-\nu))ds\Big] \\ &+4\varepsilon^2\sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}f(s,x_\varepsilon(s),x_\varepsilon(s-\nu))ds,\int_0^u\frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}}\overline{f}(x_\varepsilon(s),x_\varepsilon(s-\nu))ds\Big], \\ :=&J_{11}+J_{12}.\end{aligned}$$

By using the Cauchy–Schwarz inequality and the hypothesis $(\mathcal{H}1)$, we have

$$J_{11} \leq 4\varepsilon^2 \sup_{u\in[0,u_1]} \left(\int_0^u \left| \frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}} \right|^2 ds \cdot \int_0^u \mathbb{E}\mathbf{D}_\infty^2 \Big[f(s, y_\varepsilon(s), y_\varepsilon(s-\nu)), f(s, x_\varepsilon(s), x_\varepsilon(s-\nu)) \Big] ds \right),$$
$$\leq \frac{4\varepsilon^2 u_1^{2\sigma-1}\Theta}{2\sigma-1} \int_0^{u_1} F\Big(\mathbb{E}\mathbf{D}_\infty^2[y(s), x(s)] + \mathbb{E}\mathbf{D}_\infty^2[y(s-\nu), x(s-\nu)] \Big) ds,$$

where $\Theta := \sup_{u\in[0,u_1]} |\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)|^2$. Or, by using the definition of $\mathbf{D}_\infty$, we have

$$\begin{aligned}
\mathbf{D}_\infty[y(s-\nu), x(s-\nu)] &= \sup_{0\leq r\leq 1} \max_{0\leq s\leq u} \{| \underline{w}(s-\nu, r) - \underline{x}(s-\nu, r) |, | \overline{w}(s-\nu, r) - \overline{x}(s-\nu, r) |\}, \\
&= \sup_{0\leq r\leq 1} \max_{-\nu\leq \mu\leq u-\nu} \{| \underline{w}(\mu, r) - \underline{x}(\mu, r) |, | \overline{w}(\mu, r) - \overline{x}(\mu, r) |\}, \\
&\leq \sup_{0\leq r\leq 1} \max_{-\nu\leq \mu\leq 0} \{| \underline{w}(\mu, r) - \underline{x}(\mu, r) |, | \overline{w}(\mu, r) - \overline{x}(\mu, r) |\} \\
&\quad + \sup_{0\leq r\leq 1} \max_{0\leq \mu\leq u-\nu} \{| \underline{w}(\mu, r) - \underline{x}(\mu, r) |, | \overline{w}(\mu, r) - \overline{x}(\mu, r) |\}, \\
&\leq \sup_{0\leq r\leq 1} \max_{0\leq s\leq \nu} \{| \underline{w}(s, r) - \underline{x}(s, r) |, | \overline{w}(s, r) - \overline{x}(s, r) |\} \\
&\quad + \sup_{0\leq r\leq 1} \max_{\nu\leq s\leq u} \{| \underline{w}(s, r) - \underline{x}(s, r) |, | \overline{w}(s, r) - \overline{x}(s, r) |\}, \\
&\leq \sup_{0\leq r\leq 1} \max_{0\leq s\leq u} \{| \underline{w}(s, r) - \underline{x}(s, r) |, | \overline{w}(s, r) - \overline{x}(s, r) |\} = \mathbf{D}_\infty[y(s), x(s)].
\end{aligned}$$

Then

$$J_{11} \leq \frac{8\varepsilon^2 u_1^{2\sigma-1}\Theta F}{2\sigma-1} \int_0^{u_1} \mathbb{E}\mathbf{D}_\infty^2[y(s), x(s)] ds,$$

For J_{12}, we use the Cauchy–Schwarz inequality and the hypothesis $(\mathcal{H}4)$, we get

$$J_{12} \leq 4\varepsilon^2 \sup_{u\in[0,u_1]} \left(\int_0^u \left| \frac{\mathfrak{M}_{\sigma,\sigma}\big(A(u-s)^\sigma\big)}{(u-s)^{1-\sigma}} \right|^2 ds \cdot \int_0^u \mathbb{E}\mathbf{D}_\infty^2 \Big[f(s, x_\varepsilon(s), x_\varepsilon(s-\nu)), \overline{f}(x_\varepsilon(s), x_\varepsilon(s-\nu)) \Big] ds \right),$$
$$\leq \frac{4\varepsilon^2 u_1^{2\sigma}\Theta}{2\sigma-1} \eta_1(u_1) \Big(1 + \sup_{u\in[0,u_1]} \mathbb{E}\mathbf{D}_\infty^2[x(u), \hat{0}] + \sup_{u\in[0,u_1]} \mathbb{E}\mathbf{D}_\infty^2[x(u-\nu), \hat{0}] \Big),$$

where, by using the definition of $\mathbf{D}_\infty$, we get

$$\begin{aligned}
\mathbf{D}_\infty[x(s-\tau), \hat{0}] &= \sup_{0\leq \ \leq 1} \max_{0\leq s\leq u} \{| \underline{x}(s-\tau, r) |, | \overline{x}(s-\tau, r) |\}, \\
&= \sup_{0\leq r\leq 1} \max_{-\tau\leq \mu\leq u-\tau} \{| \underline{x}(\mu, r) |, | \overline{x}(\mu, r) |\}, \\
&= \sup_{0\leq r\leq 1} \max_{-\tau\leq \mu\leq 0} \{| \underline{x}(\mu, r) |, | \overline{x}(\mu, r) |\} + \sup_{0\leq r\leq 1} \max_{0\leq \mu\leq u-\tau} \{| \underline{x}(\mu, r) |, | \overline{x}(\mu, r) |\}, \\
&= \sup_{0\leq r\leq 1} \max_{0\leq s\leq \tau} \{| \underline{x}(s, r) |, | \overline{x}(s, r) |\} + \sup_{0\leq r\leq 1} \max_{\tau\leq s\leq u} \{| \underline{x}(s, r) |, | \overline{x}(s, r) |\}, \\
&= \sup_{0\leq r\leq 1} \max_{0\leq s\leq u} \{| \underline{x}(s, r) |, | \overline{x}(s, r) |\} = \mathbf{D}_\infty[x(s), \hat{0}].
\end{aligned}$$

Therefore

$$J_{12} \leq \frac{4\varepsilon^2 u_1^{2\sigma}\Theta}{2\sigma-1} \eta_2(u_1) \Big(1 + 2 \sup_{u\in[0,u_1]} \mathbb{E}\mathbf{D}_\infty^2[x(u), \hat{0}] \Big) := 4\varepsilon^2 u_1^{2\sigma} \Lambda_1,$$

where $\Lambda_1 = \frac{\eta_1(u_1)\Theta}{2\sigma-1} \Big(1 + 2 \sup_{u\in[0,u_1]} \mathbb{E}\mathbf{D}_\infty^2[x(u), \hat{0}] \Big)$. Ainsi

$$J_1 \leq \frac{8\varepsilon^2 u_1^{2\sigma-1}\Theta F}{2\sigma-1} \int_0^{u_1} \mathbb{E}\mathbf{D}_\infty^2[y(s), x(s)] ds + 4\varepsilon^2 u_1^{2\sigma} \Lambda_1. \tag{7.2.7}$$

For the second term J_2, using the properties of the metric $\mathbf{D}_\infty$, we have

$$\begin{aligned}J_2 \le 4\varepsilon \sup_{u\in[0,u_1]} \mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^u \frac{\mathfrak{M}_{\sigma,\sigma}(A(u-s)^\sigma)}{(u-s)^{1-\sigma}}\Big\langle\int_0^s g(\nu,y_\varepsilon(\nu),y_\varepsilon(\nu-\nu))dB(\nu)\Big\rangle ds, \int_0^u \frac{\mathfrak{M}_{\sigma,\sigma}(A(u-s)^\sigma)}{(u-s)^{1-\sigma}}\\ \Big\langle\int_0^s g(\nu,x_\varepsilon(\nu),x_\varepsilon(\nu-\nu))dB(\nu)\Big\rangle ds\Big]\\ +4\varepsilon \sup_{u\in[0,u_1]} \mathbb{E}\mathbf{D}^2_\infty\Big[\int_0^u \frac{\mathfrak{M}_{\sigma,\sigma}(A(u-s)^\sigma)}{(u-s)^{1-\sigma}}\Big\langle\int_0^s g(\nu,x_\varepsilon(\nu),x_\varepsilon(\nu-\nu))dB(\nu)\Big\rangle ds,\\ \int_0^u \frac{\mathfrak{M}_{\sigma,\sigma}(A(u-s)^\sigma)}{(u-s)^{1-\sigma}}\Big\langle\int_0^s \overline{g}(x_\varepsilon(\nu),x_\varepsilon(\nu-\nu))dB(\nu)\Big\rangle ds\Big],\\ := J_{21}+J_{22}.\end{aligned}$$

For J_{21}, we use the Cauchy–Schwarz inequality and the hypothesis $(\mathcal{H}1)$, we get

$$\begin{aligned}J_{21} &\le 4\varepsilon \sup_{u\in[0,u_1]}\left(\int_0^u\left|\frac{\mathfrak{M}_{\sigma,\sigma}(A(u-s)^\sigma)}{(u-s)^{1-\sigma}}\right|^2 ds\cdot\int_0^u\int_0^s \mathbb{E}\big\|g(\nu,y_\varepsilon(\nu),y_\varepsilon(\nu-\nu))-g(\nu,x_\varepsilon(\nu),x_\varepsilon(\nu-\nu))\big\|^2 d\nu ds\right),\\ &\le \frac{8\varepsilon u_1^{2\sigma}\Theta F}{2\sigma-1}\int_0^{u_1}\mathbb{E}\mathbf{D}^2_\infty[y(s),x(s)]ds,\end{aligned}$$

Also, we use the Cauchy–Schwarz inequality and the hypothesis $(\mathcal{H}4)$, we get

$$\begin{aligned}J_{22} &\le 4\varepsilon \sup_{u\in[0,u_1]}\left(\int_0^u\left|\frac{\mathfrak{M}_{\sigma,\sigma}(A(u-s)^\sigma)}{(u-s)^{1-\sigma}}\right|^2 ds\cdot\int_0^u\int_0^s \|g(s,x_\varepsilon(s),x_\varepsilon(s-\nu))-\overline{g}(x_\varepsilon(s),x_\varepsilon(s-\nu))\|^2 d\nu ds\right),\\ &\le \frac{4\varepsilon u_1^{2\sigma}\Theta}{2\sigma-1}\eta_2(u_1)\Big(1+\sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty[x(u),\hat{0}]+\sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty[x(u-\nu),\hat{0}]\Big)\\ &\le \frac{4\varepsilon u_1^{2\sigma}\Theta}{2\sigma-1}\eta_2(u_1)\Big(1+2\sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty[x(u),\hat{0}]\Big) := 4\varepsilon u_1^{2\sigma}\Lambda_2,\end{aligned}$$

where $\Lambda_2 = \frac{\eta_2(u_1)\Theta}{2\sigma-1}\Big(1+2\sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty[x(u),\hat{0}]\Big)$. Therefore

$$J_2 \le \frac{8\varepsilon u_1^{2\sigma}\Theta F}{2\sigma-1}\int_0^{u_1}\mathbb{E}\mathbf{D}^2_\infty[y(s),x(s)]ds + 4\varepsilon u_1^{2\sigma}\Lambda_2. \tag{7.2.8}$$

Therefore, combining (7.2.7) and (7.2.8) together, we get

$$\sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty\big[y_\epsilon(u),x_\epsilon(u)\big] \le 4\varepsilon u_1^{2\sigma}(\varepsilon\Lambda_1+\Lambda_2)+\frac{8\varepsilon u_1^{2\sigma-1}\Theta F(\varepsilon+u_1)}{2\sigma-1}\int_0^{u_1}\mathbb{E}\mathbf{D}^2_\infty[y(s),x(s)]ds.$$

Let $\Upsilon(u_1) := \sup_{u\in[0,u_1]}\mathbb{E}\mathbf{D}^2_\infty\big[y_\epsilon(u),x_\epsilon(u)\big]$ and $\sup_{u\in[-\nu,0]}\mathbb{E}\mathbf{D}^2_\infty\big[y_\epsilon(u),x_\epsilon(u)\big]=0$. Therefore

$$\Upsilon(u_1) \le 4\varepsilon u_1^{2\sigma}(\varepsilon\Lambda_1+\Lambda_2)+\frac{8\varepsilon u_1^{2\sigma-1}\Theta F(\varepsilon+u_1)}{2\sigma-1}\int_0^{u_1}\Upsilon(s)ds.$$

So, let $\Phi(u_1) := \sup_{u\in[-\nu,u_1]}\Upsilon(u), \forall u_1\in J$. Hence

$$\begin{aligned}\Phi(u_1) &= \sup_{u\in[-\nu,u_1]}\Upsilon(u) = \sup_{u\in[0,u_1]}\Upsilon(u)\\ &\le 4\varepsilon u_1^{2\sigma}(\varepsilon\Lambda_1+\Lambda_2)+\frac{8\varepsilon u_1^{2\sigma-1}\Theta F(\varepsilon+u_1)}{2\sigma-1}\int_0^{u_1}\Phi(s)ds.\end{aligned}$$

Then, by using the Gronwall–Bellman inequality, we have

$$\Phi(u_1) \leq 4\varepsilon u_1^{2\sigma}(\varepsilon\Lambda_1 + \Lambda_2)\exp\left(\frac{8\varepsilon u_1^{2\sigma-1}\Theta F(\varepsilon + u_1)}{2\sigma - 1}\right).$$

Therefore

$$\sup_{u\in[0,u_1]} \mathbb{E}\mathbf{D}^2_\infty\big[y_\varepsilon(u), x_\varepsilon(u)\big] \leq 4\varepsilon u_1^{2\sigma}(\varepsilon\Lambda_1 + \Lambda_2)\exp\left(\frac{8\varepsilon u_1^{2\sigma-1}\Theta F(\varepsilon + u_1)}{2\sigma - 1}\right).$$

Choose $\beta \in (0,1)$ and $L > 0$ such that for every $u \in [0, L\epsilon^{-\beta}] \subseteq J$, we have

$$\sup_{u\in[0,L\epsilon^{-\beta}]} \mathbb{E}\mathbf{D}^2_\infty\big[y_\varepsilon(u), x_\varepsilon(u)\big] \leq \kappa L\epsilon^{1-\beta},$$

where $\kappa = 4L^{2\sigma}\varepsilon^{\beta-2\beta\sigma}(\varepsilon\Lambda_1 + \Lambda_2)\exp\left(\frac{8L^{2\sigma-1}\varepsilon^{1-\beta\sigma+\beta}\Theta F(\varepsilon+L\varepsilon^{-\beta})}{2\sigma-1}\right)$ is a constant. Therefore, for any given number Δ, $\exists\varepsilon_1 \in (0,\varepsilon_0]$ such that for each $\varepsilon \in (0,\varepsilon_1]$ and $u \in [-\nu, L\varepsilon^{-\beta}]$, we have

$$\sup_{u\in[-\nu,L\epsilon^{-\beta}]} \mathbb{E}\mathbf{D}^2_\infty\big[y_\varepsilon(u), x_\varepsilon(u)\big] \leq \Delta.$$

Thus the proof is complete.□

Conclusion and Perspectives

At the end of this thesis, we estimate that the presented results will contribute to the development of the study of fuzzy fractional stochastic differential equations, by opening new horizons to scientific research on this new topic.

After having presented the preliminary notions useful for a good understanding of the present work, we presented results of existence, uniqueness, stability and averaging principle of certain fuzzy fractional stochastic differential problems relating to different fractional derivatives. First, we have established existence, uniqueness, stability and controllability results for different fuzzy fractional stochastic differential problems. The results obtained in this part are published in [19], [16] and [14].

Furthermore, we have presented existence, uniqueness and averaging principle results for different types of fuzzy stochastic and fuzzy fractional stochastic differential problems. These results is obtained using the fixed point technique and stochastic calculus. The results obtained in this part are published in [15] and submitted for possible publication.
The results presented in this thesis naturally offer many perspectives:

• The first is the study the existence and finite-time stability results of fuzzy fractional stochastic differential equations with time-delays.

• The second perspective would be the study the existence and finite-time stability of solutions for a class of nonlinear ψ-Hilfer fuzzy fractional stochastic differential equations with time-delays.

• The third possible perspective would be the study of the averaging principle for nonlinear Hilfer fuzzy fractional stochastic differential equations.

Bibliography

[1] Acharya F., Kushawaha V., Panchal J., Chalishajar D. : Controllability of Fuzzy Solutions for Neutral Impulsive Functional Differential Equations with Nonlocal Conditions. Axioms 2021, 10, 84.

[2] Afshari H., Marasi H. R. and Alzabut J. : Applications of new contraction mappings on existence and uniqueness results for implicit ϕ-Hilfer fractional pantograph differential equations. Journal of Inequalities and Applications, (1), 1-14 (2021).

[3] Agarwal R. P., Baleanu D., Nieto J. J., Torres D. F. and Zhou Y. : A survey on fuzzy fractional differential and optimal control nonlocal evolution equations. Journal of Computational and Applied Mathematics, 339, 3-29, (2018).

[4] Agarwal R. P., Lakshmikantham V., and Nieto J. J., : On the concept of solution for fractional differential equations with uncertainty. Nonlinear Analysis: Theory, Methods and Applications, 72(6), 2859-2862, (2010).

[5] Agilan K., Parthiban V. : Existence of Solutions of Fuzzy Fractional panto-graph Equations. Computer Science, 15(4), 1117-1122, 2020.

[6] Ahmed H. M. : Approximate controllability of impulsive neutral stochastic differential equations with fractional Brownian motion in a Hilbert space. Advances in Difference Equations 2014, 2014:113.

[7] Ahmed H. M. : Controllability of impulsive neutral stochastic differential equations with fractional Brownian motion. IMA Journal of Mathematical Control and Information (2014) Page 1 of 14.

[8] Ahmed H. M. : Boundary controllability of nonlinear fractional integrodifferential systems. Adv. Differ. Equ. 2010, Article ID 279493 (2010).

[9] Ahmed H. M. : Controllability of fractional stochastic delay equations. Lobachevskii J. Math. 30, 195-202 (2009).

[10] Ahmed H., Zhu Q. : The averaging principle of Hilfer fractional stochastic delay differential equations with Poisson jumps. Appl. Math. Lett. 112 (2021), Paper No. 106755, 7 pp.

[11] Ahmed I., Kumam P., Abubaka J., Borisut P. and Sitthithakerngkiet K. : Solutions for impulsive fractional pantograph differential equation via generalized anti-periodic boundary condition. Advances in Difference Equations, (1), 1-15 (2020).

[12] Allahviranloo T., Abbasbandy S., Balooch Shahryari M. R., Salahshour S. and Baleanu D. : On solutions of linear fractional differential equations with uncertainty. Abstr. Appl. Anal., pages Art. ID 178378, 13, 2013.

[13] Ali A., Samet B., Shah K., Khan R. A. : Existence and stability of solution to a toppled systems of differential equations of non-integer order. Boundary Value Problems, 2017(1), 1-13, 2017.

[14] Arhrrabi E., Elomari M., Melliani S., Chadli L. S. : Existence and Stability of Solutions for a Coupled System of Fuzzy Fractional Pantograph Stochastic Differential Equations, Asia Pac. J. Math., 9 (2022), 20. doi:10.28924/APJM/9-20.

[15] Arhrrabi E., Elomari M., Melliani S. : Averaging principle for fuzzy stochastic differential equations, Contrib. Math. 5, 25–29 (2022).

[16] Arhrrabi E., Elomari M. H., Melliani S. and Chadli L. S. : Existence and Controllability results for fuzzy neutral stochastic differential equations with impulses. Boletim da Sociedade Paranaense de Matemática, 41, 1-14, 2023.

[17] Arhrrabi E., Taqbibt A., Elomari M. H., Melliani S., and Chadli, L. S. : Fuzzy fractional boundary value problem. In 2021 7th International Conference on Optimization and Applications (ICOA) (pp. 1-6) 2021, IEEE.

[18] Arhrrabi E., Elomari M., Melliani S. and Chadli L. S. : Existence and Uniqueness Results of Fuzzy Fractional Stochastic Differential Equations with Impulsive. In International Conference on Partial Differential Equations and Applications, Modeling and Simulation (pp. 147-163) 2023. Springer, Cham.

[19] Arhrrabi E., Elomari M., Melliani S., Chadli L. S. : "Existence and Stability of Solutions of Fuzzy Fractional Stochastic Differential Equations with Fractional Brownian Motions", Advances in Fuzzy Systems, vol. 2021, Article ID 3948493, 9 pages, 2021.

[20] Atangana A. : Modelling the spread of COVID-19 with new fractal-fractional operators: Can the lockdown save mankind before vaccination, Chaos Soliton. Fract., 136 (2020), 109860.

[21] Balachandran K., Sakthivel R. : Controllability of integrodifferential systems in Banach spaces. Appl. Math. Comput. 118, 63-71 (2001).

[22] Balachandran K., Kiruthika S. and Trujillo J. J. : Existence of solutions of Nonlinear fractional pantograph equations, Acta Mathematica Scientia, 33B, 1–9 (2013).

[23] Bede B. and Gal S. G. : Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. Fuzzy Sets and Systems, 151(3):581–599, 2005.

[24] Benchohra M., Henderson J., Ntouyas S. K. : Existence results for impulsive multivalued semilinear neutral functional inclusions in Banach spaces. J. Math. Anal. Appl. 263, 763-780 (2001).

[25] Bogdan P., Marculescu R. : Towards a science of cyber-physical systems design. In: Proc. 2nd ACM/IEEE Int'l Conf. Cyber-Physical Systems, pp. 99-108. IEEE Comput. Soc., Los Alamitos (2011).

[26] Bogdan P. : Implantable pacemakers control and optimization via fractional calculus approaches: a cyber-physical systems perspective. In: Proceedings of the IEEE/ACM Third International Conference on Cyber-Physical Systems (ICCPS'12) (2012).

[27] Boufoussi B., Hajji S. : Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space. Statistics and Probability Letters 82 (2012) 1549–1558.

[28] Bouzahir H., Fu X. : Controllability of neutral functional differential equations with infinite delay, Acta Mathematica Scientia, Volume 31, Issue 1, 2011, Pages 73-80.

[29] Chalishajar D. N., and Ramesh R. : Controllability for impulsive fuzzy neutral functional integrodifferential equations. AIP Conference Proceedings 2159, 030007 (2019).

[30] Chalishajar D., Kumar A. : Existence, uniqueness and Ulam's stability of solutions for a coupled system of fractional differential equations with integral boundary conditions. Mathematics, 6(6), 96, 2018.

[31] Carmona P., Coutin L., Montseny G. : Stochastic integration with respect to fractional Brownian motion. Ann. Inst. Henri Poincaré 39, 27–68 (2003).

[32] Danane J., Allali K., Hammouch Z. : Mathematical analysis of a fractional differential model of HBV infection with antibody immune response, Chaos Soliton. Fract., 136 (2020), 109787.

[33] Decreusefond L., Üstünel A. S. : Stochastic analysis of the fractional Brownian motion. Potential Anal. 10, 177–214 (1998).

[34] Diamond P., Kloeden P. E. : Metric Spaces of Fuzzy Sets: Theory and Applications. World Scientific, Singapore (1994).

[35] Diethelm K. : The Analysis of Fractional Differential Equations, Lecture Notes in Mathematics, 2004, Springer-Verlag, Berlin, 2010.

[36] Ding X. L., and Nieto J. J. : Analytical solutions for multi-time scale fractional stochastic differential equations driven by fractional Brownian motion and their applications. Entropy, 20(1), 63 (2018).

[37] Elsayed E. M., Harikrishnan S. and Kanagarajan K. : Analysis of nonlinear neutral pantograph differential equations with Hilfer fractional derivative, MathLAB 1(2) , 231–240 (2018).

[38] Fei W.Y. : Existence and uniqueness for solutions to fuzzy stochastic differential equations driven by local martingales under the non-Lipschitzian condition, Nonlinear Anal. TMA 76 (2013) 202-214.

[39] Fei W. Y., Li Y. H. and Fei C. : Properties of solutions to stochastic set differential equations under non-Lipschitzian coefficients. Abstract and Applied Analysis, 2014(2), 1-8.

[40] Fei W. Y. and Liang Y. : Stochastic set differential equations driven by local martingales under the non-Lipschitzian condition. Acta Mathematica Sinica, Chinese Series, 56(4), 561-574.

[41] Fei W. Y. and Xia D. F. : On solutions to stochastic set differential equations of Itô type under the non-Lipschitzian condition. Dynamic Systems and Applications, 22, 137-156.

[42] Fei W. Y. : Existence and uniqueness of the solution to SDDEs for fractional Brownian motion. Chinese Kournal of Contemporary Mathematics, 28(3), 309-323.

[43] Fei W., Liu H., Zhang W. : On solutions to fuzzy stochastic differential equations with local martingales, Systems and Control Letters, Volume 65, 2014, Pages 96-105.

[44] Fu X. : Controllability of abstract neutral functional differential systems with unbounded delay. Appl. Math. Comput. 151, 299-314 (2004).

[45] Gao D., Li J., Luo Z. and Luo D. : The averaging principle for stochastic pantograph equations with non-Lipschitz conditions. Math. Probl. Eng. Art. ID 5578936, 2021, 7 pp.

[46] Guo Y., Zhu Q., Wang F. : Stability analysis of impulsive stochastic functional differential equations, Communications in Nonlinear Science and Numerical Simulation, Volume 82, 2020, 105013.

[47] Guo Y., Shu X. B., Li Y., Xu F. : The existence and Hyers–Ulam stability of solution for an impulsive Riemann–Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$. Boundary Value Problems, 2019(1), 1-18.

[48] Guo Z., Xu Y., Wang W, Hu J. : Averaging principle for stochastic differential equations with monotone condition. Applied Mathematics Letters, 125, (2022) 107705.

[49] Harikrishnan S., Kanagarajan K. and Vivek D. : Solutions of nonlocal initial value problems for fractional pantograph equation, Journal Nonlinear Analysis and Application, (2), 136–144 (2018).

[50] Harikrishnan S., Ibrahim R. and Kanagarajan K. : Establishing the existence of Hilfer fractional pantograph equations with impulses, Fundam. J. Math. Appl., 1(1) , 36–42 (2018).

[51] Hoa V. : Fuzzy fractional functional differential equations under Caputo gH-differentiability, Communications in Nonlinear Science and Numerical Simulation (2014).

[52] Hoa N. V. : On the initial value problem for fuzzy differential equations of non-integer order $\alpha \in (1, 2)$ Soft Comput. 24, 935-954 (2020).

[53] Hosseinzadeh R., Zarebnia M. : Application and comparison of the two efficient algorithms to solve the pantograph Volterra fuzzy integro-differential equation. Soft Computing, 25(10), 6851-6863, 2021.

[54] Houas M. : Existence and stability of fractional pantograph differential equations with Caputo-Hadamard type derivative. Turkish J. Ineq. 4(1), 1-10 (2020).

[55] Hosseinzadeh R., Zarebnia M. : Application and comparison of the two effcient algorithms to solve the pantograph Volterra fuzzy integro-differential equation. Soft Computing, 25(10), (2021), 6851-6863.

[56] Hilfer R., Luchko Y. and Tomovski J. : Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives, Fract. Calc. Appl. Anal. 12 (2009), no. 3, 299-318.

[57] Hilfer R. : Applications of Fractional Calculus in Physics, World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[58] Hu W., Zhu Q. and Karimi H. R. : "Some Improved Razumikhin Stability Criteria for Impulsive Stochastic Delay Differential Systems" in IEEE Transactions on Automatic Control, vol. 64, no. 12, pp. 5207-5213, Dec. 2019, doi: 10.1109/TAC.2019.2911182.

[59] Jafari H., Malinowski M. T. and Ebadi M. J. : Fuzzy stochastic differential equations driven by fractional Brownian motion, Advances in Difference Equations, 16 (2021).

[60] Jameel A. F., Saaban A., Ahadkulov H., Alipiah F. M. : Approximate solution fuzzy pantograph equation by using homotopy perturbation method. In Journal of Physics: Conference Series (Vol. 890, No. 1, p. 012023). IOP Publishing, 2017.

[61] Khasminskii R. Z. : Principle of averaging of parabolic and elliptic differential equations for Markov process with small diffusion. Theory Probab. Appl. 8, 1-21 (1963).

[62] Khasminskii R. Z. : On the averaging principle for Itô stochastic differential equations. Kybernetika 4, 260-279 (1968).

[63] Khan H., Gomez-Aguilar J. F., Abdeljawad T., Khan A. : Existence results and stability criteria for ABC-fuzzy-Volterra integro-differential equation. Fractals, 28(08), 2040048, 2021.

[64] Khan H., Li Y., Chen W., Baleanu D., Khan A. : Existence theorems and Hyers-Ulam stability for a coupled system of fractional differential equations with p-Laplacian operator. Boundary Value Problems, 2017(1), 1-16, 2017.

[65] Kolmogorov A. N.: Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. (German) C. R. (Doklady) Acad. URSS (N.S.) 26, 115–118 (1940).

[66] Kim Y.K. : Measurability for fuzzy valued functions, Fuzzy Sets Syst., vol. 129, pp. 105-109, 2002.

[67] Klamka J. : Stochastic controllability of systems with multiple delays in control. Int. J. Appl. Math. Comput. Sci. 19(1), 39-47 (2009).

[68] Klir G. J. and Yuan B. : Fuzzy sets and fuzzy logic. Prentice Hall PTR, Upper Saddle River, NJ, 1995. ISBN 0-13-101171-5. xvi+574 pp. Theory and applications, With a foreword by Lotfi A. Zadeh.

[69] Lakshmikantham V., Leela S., Devi J. V. : Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers (2009).

[70] Lakshmikantham V., Mohapatra R. N. : Theory of Fuzzy Differential Equations and Inclusions. Taylor and Francis, London (2003).

[71] Liu J. and Xu W. : An averaging result for impulsive fractional neutral stochastic differential equations, Applied Mathematics Letters (2020).

[72] Lord G. J., Powell C. E., Shardlow T. : An Introduction to Computational Stochastic PDEs, Cambridge University Press, Cambridge, 2014.

[73] Luo D., Zhu Q., Luo Z. : An averaging principle for stochastic fractional differential equations with time-delays. Applied Mathematics Letters, 105, (2020) 106290.

[74] Luo D., Zhu Q., Luo Z. : A novel result on averaging principle of stochastic Hilfer-type fractional system involving non-Lipschitz coefficients. Appl. Math. Lett. 122 (2021), Paper No. 107549, 7 pp.

[75] Ma S. and Kang Y. : Periodic averaging method for impulsive stochastic differential equations with Levy noise, Applied Mathematics Letters, vol. 93, pp. 91–97 (2019).

[76] Malinowski M. T. : Strong solutions to stochastic fuzzy differential equations of Itô type. Math. Comput. Model. 55, 918–928 (2012).

[77] Malinowski M. T. : 'Strong solutions to stochastic fuzzy differential equations of Itô type', Mathematical and Computer Modelling, vol. 55, no. 3-4, pp. 918-928, 2012.

[78] Malinowski M.T. : Strong solutions to stochastic fuzzy differential equations of Itô type, Math. Comput. Modelling 55 (2012) 918-928.

[79] Malinowski M.T. : Itô type stochastic fuzzy differential equations with delay, Systems Control Lett. 61 (2012) 692-701.

[80] Malinowski M.T. and Michta M. : Stochastic set differential equations. Nonlinear Analysis: Theory, Methods and Applications, 72(3-4), 1247-1256.

[81] Malinowski Stochastic fuzzy differential equations with an application. Kybernetika 47, 123-143 (2011).

[82] Malinowski M.T. : Some properties of strong solutions to stochastic fuzzy differential equations. Information Sciences 252, 62-80 (2013).

[83] Mandelbrot B. B., Van Ness J. W. : Fractional Brownian motions, fractional noises and applications. SIAM Rev. 10, 422–437 (1968).

[84] Mao W., Hu L., Xu X. : The existence and asymptotic estimations of solutions to stochastic pantograph equations with diffusion and Lévy jumps, Appl. Math. Comput. 268 883–896 (2015).

[85] Mao W., You S., Wu X., Mao X. : On the averaging principle for stochastic delay differential equations with jumps. Advances in Difference Equations, 2015(1), 1-19.

[86] Meng X., Hu S., Wu P. : Pathwise estimation of stochastic differential equations with unbounded delay and its application to stochastic pantograph equations, Acta Appl. Math. 113 231–246 (2011).

[87] Melliani S., Arhrrabi E., Elomari M. H., Chadli L. S. : Ulam-Hyers-Rassias stability for fuzzy fractional integrodifferential equations under Caputo gH-differentiability. International Journal On Optimization and Applications, 51, 2021.

[88] Mikaeilvand N., Hossieni L. : The Taylor method for numerical solution of fuzzy generalized pantograph equations with linear functional argument, 2010.

[89] Michta M. : On set-valued stochastic integrals and fuzzy stochastic equations. Fuzzy Sets and Systems, 177(1), 1-19.

[90] Narayanamoorthy S., and Sowmiya S. : Approximate controllability result for nonlinear impulsive neutral fuzzy stochastic differential equations with nonlocal conditions, Advances in Difference Equations (2015) 2015:121.

[91] Nabil T. : Ulam stabilities of nonlinear coupled system of fractional differential equations including generalized Caputo fractional derivative. AIMS Math, 6, 5088-5105, 2021.

[92] Nabil T. : Existence results for nonlinear coupled system of integral equations of Urysohn Volterra-Chandrasekhar mixed type. Demonstratio Mathematica, 53(1), 236-248, 2020.

[93] Ockendon J. R., Taylor A. B. : The dynamics of a current collection system for an electric locomotive, Proc. R. Soc. Lond. Ser. A 322 447–468 (1971).

[94] Park J. H., Park J. S., Ahn Y. C., Kwun Y. C. : Controllability for the impulsive semilinear fuzzy integrodifferential equations, Advances in Soft Computing. 40, 704–713, 2007.

[95] Priyadharsini J., Balasubramaniam P. : (2010-6217) Solvability of fuzzy fractional stochastic Pantograph differential system. Iranian Journal of Fuzzy Systems, 2021.

[96] Pei B., Xu Y. and Wu J. L. : Stochastic averaging for stochastic differential equations driven by fractional Brownian motion and standard Brownian motion, Applied Mathematics Letters, (2020) vol. 100, Article ID 106006.

[97] Rassias T. M. : The problem of SM Ulam for approximately multiplicative mappings. Journal of Mathematical Analysis and Applications, 246(2), 352-378, 2000.

[98] Rassias T. M. : On the stability of functional equations in Banach spaces. Journal of Mathematical Analysis and Applications, 251(1), 264-284, 2000.

[99] Rassias T. M. : On the stability of functional equations and a problem of Ulam. Acta Applicandae Mathematica, 62(1), 23-130, 2000.

[100] Rassias T. M., Rassias T. M. : Functional equations, inequalities, and applications (pp. x+-224). Boston and London: Kluwer Academic Publishers, 2003.

[101] Sajedi L., Eghbali N., Aydi H. : Impulsive Coupled System of Fractional Differential Equations with Caputo–Katugampola Fuzzy Fractional Derivative. Journal of Mathematics, 2021.

[102] Sakthivel R., Ganesh R., Suganya S. : Approximate controllability of fractional neutral stochastic system with infinite delay. Rep. Math. Phys. 70, 291-311 (2012).

[103] Sakthivel R., Mahmudov N. I., Kim J. H. : Approximate controllability of nonlinear impulsive differential systems. Rep. Math. Phys. 60, 85-96 (2007).

[104] Sakthivel R., Nieto J., Mahmudov N. I. : Approximate controllability of nonlinear deterministic and stochastic systems with unbounded delay. Taiwan. J. Math. 14(5), 1777-1797 (2010).

[105] Shiryaev A. : Essentials of stochastic finance. World Scientific.

[106] Singh J., Kumar D., Hammouch Z., Atangana A. : A fractional epidemiological model for computer viruses pertaining to a new fractional derivative, Appl. Math. Comput, 316 (2018), 504–515.

[107] Stoyanov I. M., Bainov D. D. : The averaging method for a class of stochastic differential equations. Ukr. Math. J. 26(2), 186-194 (1974).

[108] Stefanini L. : A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. Fuzzy Sets Syst. 161, 1564-1584 (2010).

[109] Tan L., Lei D. : The averaging method for stochastic differential delay equations under non-Lipschitz conditions. Advances in Difference Equations, 2013(1), 1-12.

[110] Tian M., Luo D., and Abro K. A. : Caratheodory Approximation and Finite-Time Stability Results for Fuzzy Fractional Stochastic Delay System. Available at SSRN 4123072.

[111] Vas'kovskii M. M., Karpovich A.A. : Finiteness of Moments of Solutions to Mixed-Type Stochastic Differential Equations Driven by Standard and Fractional Brownian Motions. Diff Equat 57, 148-154 (2021).

[112] Vivek D., Kanagarajan K. and Sivasundaram S. : Dynamics and stability of pantograph equations via Hilfer fractional derivative, Nonlinear Studies, 23(4), 685–698 (2016).

[113] Vivek D., Kanagarajan K. and Sivasundaram S. : On the behavior of solutions of Hilfer-Hadamard type fractional neutral pantograph equations with boundary conditions, Communications in Applied Analysis, 22(2), 211–232 (2018).

[114] Wang J., Feckan M., Zhou Y. : Presentation of solutions of impulsive fractional Langevin equations and existence results. The European Physical Journal Special Topics, 222(8), 1857-1874, 2013.

[115] Wang J., Zhou Y. : Ulam stability and data dependence for fractional differential equations with Caputo derivative, Electronic Journal of Qualitative Theory of Differential Equations, 2011, No. 63, 1-10.

[116] Xu Y., Pei B., Li Y. : An averaging principle for stochastic differential delay equations with fractional Brownian motion. In Abstract and Applied Analysis (Vol. 2014). Hindawi.

[117] Xu W., Xu W. and Lu K. : "An averaging principle for stochastic differential equations of fractional order $0 < \alpha < 1$" Fractional Calculus and Applied Analysis, vol. 23, no. 3, 2020, pp. 908-919.

[118] Xu W., Xu W. and Zhang S. : The averaging principle for stochastic differential equations with Caputo fractional derivative, Applied Mathematics Letters, vol. 93, pp. 79–84 (2019).

[119] Young L. C. : An inequality of the Hôlder type connected with Stieltjes integration. Acta Math. 67, 251–282 (1936).

[120] Zadeh L.A. : Fuzzy sets, Inform Control 8 (1965), 338-353.

[121] Zadeh L. A. : Fuzzy sets, fuzzy logic, and fuzzy systems, volume 6 of Advances in Fuzzy Systems—Applications and Theory. World Scientific Publishing Co., Inc., River Edge, NJ, 1996. ISBN 981-02-2421-4; 981- 02-2422-2. xiv+826 pp. Selected papers by Lotfi A. Zadeh, Edited and with a preface by George J. Klir and Bo Yuan.

[122] Zimmermann H. J. : Fuzzy set theory—and its applications. Kluwer Academic Publishers, Boston, MA, second edition, 1992. ISBN 0-7923- 9075-X. xxii+399 pp. With a foreword by L. A. Zadeh.

[123] Zeeshan A. L. I., Akbar Z. A. D. A., Kamal S. H. A. H. : Ulam stability results for the solutions of nonlinear implicit fractional order differential equations. Hacettepe Journal of Mathematics and Statistics, 48(4), 1092-1109, 2018.

[124] Zhu J., Liang Y. and Fei W. : On uniqueness and existence of solutions to stochastic set-valued differential equations with fractional Brownian motions, Systems Science and Control Engineering, 8:1, 618-627, (2020).

[125] Zhu Q. : "Stabilization of Stochastic Nonlinear Delay Systems With Exogenous Disturbances and the Event-Triggered Feedback Control" in IEEE Transactions on Automatic Control, vol. 64, no. 9, pp. 3764-3771, Sept. 2019, doi: 10.1109/TAC.2018.2882067.

[126] Zhou Y. : Basic Theory of Fractional Differential Equations, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ,2014.