Sultan Moulay Slimane University
Faculty of Sciences and Techniques
Beni-Mellal

Center for doctoral studies: Sciences and Techniques

Doctoral Formation: Mathematics and Physics Applied

THESIS

Presented by

MOUNIM EL OUARDY

For the award of the Degree of

DOCTOR

Discipline: Mathematics

Speciality: Applied Mathematics

==

Existence and regularity results for some nonlinear singular parabolic problems

==

Thesis publicly defended on October 22th, 2022
In front of the jury composed of

Pr. Nadia IDRISSI FATMI	ENSA-Khouribga	Chair
Pr. Imad HAFIDI	ENSA-Khouribga	Reviewer
Pr. Sidi Mohamed DOUIRI	FST-Errachidia	Reviewer
Pr. Mostafa EL MOUMNI	FS-El Jadida	Reviewer
Pr. Rachid FAKHAR	FP-khouribga	Examinator
Pr. Aziz IFZARNE	ENSA-Khouribga	Co-Supervisor
Pr. Youssef EL HADFI	ENSA-Khouribga	Supervisor

Acknowledgments

First and foremost, praises and thanks to Allah for his help and blessings throughout the research work to complete my thesis successfully.

I would like at first to express my immense gratitude to my supervisor **Professor Youssef El Hadfi**. I'am grateful to him for the trust he put on me as he accepted to supervise my thesis. Thanks for his wise guidance and his constant and precious advise throughout this work. I would also like to mention his personal qualities for he was very patient, humble and always encouraging. Also, I would like to thank **Professor Aziz Ifzarne** for his help during our work on this thesis. I thank him for his seriousness, openness and sincerity. I also thank him for accepting to be a member of my thesis's review committee. I specifically thank him for his guidance and his precious remarks. I would also like to thank **Professor Nadia Idrissi Fatmi** for accepting to be the president of the jury of my thesis. I would also like to thank **Professor Imad Hafidi** and **Professor Sidi Mohamed Douiri** and **Professor Mostafa El Moumni** for accepting to review my thesis. I thank **Professor Rachid Fakhar** for accepting to discuss my work. It is my honor to be judged by these dedicated and seasoned Professors of mathematics.

I extend a big thank you to all the Professors and research colleagues of the **LIPIM** Laboratory for all that they have brought me for the progress and the outcome of this work.

In this regard, I would like to thank all my friends I particularly thank my dear friends **Abdelaaziz Sbai** and **Hamid Ben yakou** for the support and the encouragements that have been given to me throughout my research and my doctoral training.

I would also sincerely thank from the button of my heart, my parents **Elghalia** and **Abdelaaziz** as well as my brothers **Mohamed, Abdeltif, Abdeslam, Ayoub,** and sisters **Jamila, Halima, and Ilham.** I would you like also like to add their spouses and children for their constant, continuous, abstract, and tangible help.

I will not forget to thank all teaching and administrative staff oh the mathematics departments of the Faculty of Sciences Dhar El Mahraz-Fes (during my master course) and the Faculty of Sciences Meknes (during the period of license) for their welcome and their help, gave me a lot, for beyond what i could have expected. Many thanks to them!

I also thank my teachers for teaching me to write and read and think throughout my life.

Contents

Abstract

This thesis is devoted to the study of some parabolic partial differential equations (PDEs) involving absorption term or singular natural growth or Hardy potential and singular lower order term. The thesis emphasizes mostly on the nonlinear evolutive PDEs. The main objective is to obtain the existence and regularity of solutions to the problem considered with certain Dirichlet boundary conditions in Sobolev space. Some of the key techniques employed in this thesis to guarantee the existence of solutions are the weak convergence method, Schauder fixed point theorem, etc. The regularity of the solutions is also established mostly by using the Gagliardo-Nirenberg inequality. One of the main difficulties that arises in this thesis (general in the parabolic case) is the proof of the strict positivity of the solution in the interior of the parabolic cylinder, in order to give sense to the weak formulation of the problems and also used in the convergence passages. The proof of this property use Harnack's inequality.

Keywords: Nonlinear parabolic equations; Singular parabolic equations ; Weak solution; positive solution; Existence; Regularity; Absorption term; Lower order term; Natural growth; Hardy potential.

Résumé

Cette thèse est consacrée à l'étude des équations aux dérivées partielles (EDP's) non linéaires. Plus précisément, nous étudions l'existence et la régularité des solutions pour certains problèmes paraboliques impliquant un terme d'absorption ou un terme singulier avec une croissance naturelle ou un potentiel de Hardy ou un terme d'ordre inférieur singulier. La thèse met l'accent principalement sur les EDP évolutives non linéaires. L'objectif principal est d'obtenir l'existence et la régularité des solutions aux problèmes considérés avec certaines conditions aux limites de Dirichlet dans les espaces de Sobolev et Lebesgue. Certaines des techniques clés utilisées dans cette thèse pour garantir l'existence de solutions sont la méthode de convergence faible, le théorème du point fixe de Schauder, etc. La régularité des solutions est également établie principalement en utilisant l'inégalité de Gagliardo-Nirenberg. L'une des principal difficultés qui se posent dans cette thèse (généralement dans le cas parabolique) est la preuve de la stricte positivité de la solution à l'intérieur du cylindre parabolique, afin de donner un sens à la formulation faible des problèmes, et ainsi que son utilisation dans les passages de convergence. La preuve de cette propriété est basé sur l'application de l'inégalité de Harnack.

Mots-clés: Équation parabolique non-linéaire; Équation parabolique singulier; Solution faible; Solution positive; Existence; Régularité; Terme d'absorption; Terme d'ordre inférieur; Croissance naturelle; Potentielle de Hardy.

Preface

Published:

1. Y. El Hadfi, M. El Ouardy, A. Ifzarne, A. Sbai: On nonlinear parabolic equations with singular lower order term. J Elliptic Parabol Equ, **8**, 49-75 (2021).

2. M. El Ouardy, Y. El Hadfi and A. Ifzarne, Existence and regularity results for a singular parabolic equations with degenerate coercivity, Disc. Conti. Dyn. Sys. Series S, **15**(1), 117-141 (2022). doi:10.3934/dcdss.2021012.

3. M. El Ouardy, Y. El Hadfi, Some nonlinear parabolic problems with singular natural growth term, Result in Mathematics, **77**, 95 (2022). doi:10.1007/s00025-022-01631-6.

4. M. El Ouardy, Y.El Hadfi and A. Sbai, Existence of positive solutions to nonlinear singular parabolic equations with Hardy potential, J. Pseudo-Differ. Oper. Appl. **13**, 28 (2022). doi:10.1007/s11868-022-00457-8.

Submitted:

5. M. El Ouardy, Y. El Hadfi, A. Sbai: On the regularity of solutions to singular parabolic p-Laplacian equations, (2022).

6. M. El Ouardy, Y. El Hadfi, A. Sbai: Degenerate parabolic equations with absorption terms and singular natural growth, (2022).

7. M. El Ouardy, Y. El Hadfi, A. Sbai: Existence and regularity results for some nonlinear singular parabolic problems with absorption terms, (2022).

8. M. El Ouardy, Y. El Hadfi, A. Sbai: Existence and regularity of parabolic problems with convection term and with singular nonlinearity, (2022).

9. A. Sbai, Y. El Hadfi and M. El Ouardy: Existence and regularity of positive solutions for Schrodinger-Maxwell system with singularity, (2021).

10. A. Sbai, Y. El Hadfi and M. El Ouardy, Singular elliptic problem involving a Hardy potential and lower order term, (2022).

Notations

Ω :	open set of $\mathbb{R}^N$, $N \in \mathbb{N}^*$
$\partial\Omega$:	boundary topological of Ω
Q	the parabolic cylinder $\Omega \times (0,T)$, $T > 0$
Γ	the lateral surface $\partial\Omega \times (0,T)$, $T > 0$
$x = (x_1, x_2, x_3, ..., x_N)$:	generic point of $\mathbb{R}^N$
$dx = dx_1 dx_2 dx_3 ... dx_N$:	Lebesgue measure on Ω
$d\sigma$:	area measure on $\partial\Omega$
∇u :	the gradient of u i.e $(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}, ..., \frac{\partial u}{\partial x_N})$
$\mathcal{D}(\Omega)$:	space of smooth functions with a support compact in Ω
$L^\infty(\Omega)$:	space of bounded functions in Ω
$L^p(\Omega)$:	space of power functions p-th integrable on Ω for the measure dx
$\|f\|_p =$	$(\int_\Omega \|f(x)\|^p dx)^{\frac{1}{p}}$
$W^{1,p}(\Omega) =$	$\{u \in L^p(\Omega);\ \nabla u \in (L^p(\Omega))^N\}$
$\|u\|_{1,p} =$	$(\|u\|_p^p + \|\nabla u\|_p^p)^{\frac{1}{p}}$
$W_0^{1,p}(\Omega)$:	adhesion of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$
$W^{-1,p'}(\Omega)$:	dual space of $W_0^{1,p}(\Omega)$:
$L^p(0,T;\Omega)$	the space of measurable functions $u : [0,T] \to \Omega$ such that $\|u\|_{L^p(0,T;\Omega)} = \left(\int_0^T \|u\|_\Omega^p \, dt\right)^{\frac{1}{p}} < +\infty$,
$L^\infty(0,T;\Omega)$:	the space of measurable functions such that $\|u\|_{L^\infty(0,T;\Omega)} = \sup_{[0,T]} \|u\|_\Omega < +\infty$.
$\operatorname{div} f =$	$\Sigma_{i=1}^N \frac{\partial f_i}{\partial x_i}$ where $f = (f_1, f_2, f_3, ..., f_{N-1}, f_N)$
$\|E\|$, $meas(E)$	the Lebesgue measure of subset E of $\mathbb{R}^N$
$s^+ =$	$\max(s, 0)$ the positive part of variable s
$s^- =$	$\min(0, s)$ the negative part of variable s
$q' =$	$\frac{q}{q-1}$, $q > 1$, the Hölder conjugate exponent of q
$q^* =$	$\frac{Nq}{N-q}$, $1 < q < N$, the Sobolev conjugate exponent of q
sign(s)	sign of variable s
T_k :	$T_k(s) = \max(-k, \min(s,k))$, $k > 0$, $s \in \mathbb{R}$ the truncation function of level k
G_k :	$G_k(s) = s - T_k(s) = (\|s\| - k)^+$
C, C_i, c_i, $i = \cdots$	several constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data but they will never depend on the indexes of the sequences we will introduce

Chapter 1

Introduction

This phD thesis provides contributions to the fields of Nonlinear Partial Differential Equations. More specifically, it is concerned with the existence and regularity of solutions to nonlinear parabolic boundary value problems of Dirichlet type. The general model, from which many interesting particular cases, is the following:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + g(x,t,u,\nabla u) = h(u)f & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Gamma, \\ u_0(x,t=0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$), Q is the cylinder $\Omega \times (0,T)$, $T > 0$, Γ the lateral surface $\partial\Omega \times (0,T)$, u_0 is a non-negative function belonging to $L^\infty(\Omega)$, and f is non-negative function which belongs to some Lebesgue space $L^m(Q)$, $m \geq 1$. The function $a(x,t,u,\nabla u) : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (i.e. it is continuous with respect to u and ∇u for almost $(x,t) \in Q$, and measurable with respect to (x,t) for every $u \in \mathbb{R}$ and $\nabla u \in \mathbb{R}^N$). The function $g(x,t,u,\nabla u)$: $\Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is Carathéodory possibly singular at 0, and $h : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous function possibly singular at $s = 0$.

In the recent years, there has been an increasing interest in the study of equations with singular lower order terms. On one hand, the interest in such equations is motivated by their connection in the study of non-Nowtonian fluids (in particular pseudoplastic fluids), boundary-layers phenomena for viscous fluids (see [67, 110, 117]), in the Langmuir–Hinshelwood model of chemical heterogeneous catalyst kinetics (see [12, 94]), in enzymatic kinetics models (see [46]), as well as in the theory of heat conduction in electrically conducting materials (see [133]), and in the study of guided modes of an electromagnetic field in the nonlinear medium (see [90]). In the context of laser beam propagation in plasmas, the corresponding equation involves a nonlinear term depending on ∇u and represents heat balance with reactant consumption ignored where u is a dimensionless temperature excess (see [90] for more details). In the particular case when $g \equiv 0$, $h(s) = s^{-1}$ appeared was in [80]; there the authors fall into the study problem as (1.1) while observing the temperature given by the solution $u(x,t)$ of an electrical conductor which occupies a three dimensional regions. Here $f(x,t)u^{-1}$ is thought as the rate of generation of heat where $h(s) = s^{-1}$ is the resistivity of the conductor.

From a purely mathematical, the problem as in (1.1) has been intensively studied by many authors. If $g \equiv 0$ and $h \equiv 1$ the problem was investigated in [108, 109] in the stationary varational case. For the variational parabolic case is treated in [108], with $u_0 \in L^2(\Omega)$.

Concerning the non variational elliptic-parabolic case i.e.: If $f \in L^1(Q)$ or f is a measure see [14, 16, 18, 19, 32, 104, 125, 126, 140].

If $h \equiv 1$ and the presence of the lower order term (i.e. $g \neq 0$) the problem (1.1) has been widely studied in the literature. More precisely, if g does not depend on the gradient, and f belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))$ existence results for problem (1.1) has been given in [33, 34, 100]. If g does depend on ∇u, an existence theorem has been proved by Landdes and Mustenov in [101]. Their results are obtained by means of an approximation of g with bounded functions proving the strong convergence of the solutions of the approximating problems. All these papers use a sign condition on g (namely g has the same sign of u), but assume no growth restrinction with respect to u. A different approach (see, e.g. [123, 17, 115]) to the existence of solutions can be done if f is more regular (for instance if $f \in L^\infty(0,T;W^{-1,r}(\Omega))$, with r large enough) and g is bounded with respect to u, in this case it is possible to prove the existence of bounded solutions of (1.1) without any sign condition on g. The authors in [59] have proved the existence of a weak solution to problem (1.1), when $f \in L^1(Q)$ and g having a natural growth with respect to the gradient and satisfies the sign condition.

Concerning the stationary singular case of problem (1.1), namely or $g(0) = +\infty$/or $h(0) = +\infty$ has been widely studied. When $g \equiv 0$ and $h(s) = s^{-\gamma}$ $(\gamma > 0)$ the following singular problem

$$\begin{cases} -\Delta u = \frac{f}{u^\gamma} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, with f is a non-negative function belonging to some Lebesgue space and $\gamma > 0$, has been investigated by many authors. More precisely, existence and uniqueness of a classical solution $u \in C^2(\bar\Omega) \cap C(\bar\Omega)$ of (1.2) are proved in [37, 141], when f is a positive Hölder continuous function in $\bar\Omega$ and Ω is a smooth domain. In the same framework, Lazer and Mackenna in [103] have proved that $u \in W_0^{1,2}(\Omega)$ if and only if $\gamma < 3$ and that $\gamma > 1$, the solution does not belong to $C_1(\bar\Omega)$, while in [45], under the weaker assumption that f is only non-negative and bounded, Del Pino has proved the existence and uniqueness of a positive distributional solution belonging to $C_1(\Omega)\cap C(\bar\Omega)$. These results are generalized by Lair and Shaker in [99].

Existence of positive distributional solution with data merely in $L^1(\Omega)$ has been proved by Boccardo and Orsina in [25]. The authors show that this solution, if $\gamma < 1$, belongs to an homogeneous Sobolev space larger than $W_0^{1,2}(\Omega)$, if $\gamma = 1$, it belongs to $W_0^{1,2}(\Omega)$ and, finally if $\gamma > 1$, it belongs to $W_{loc}^{1,2}(\Omega)$. In the last case, the boundary condition is assumed in a weaker sense i.e., $u^{\frac{\gamma+1}{2}} \in W_0^{1,2}(\Omega)$.

In the general case, many works study the existence, regularity and uniqueness of the following general singular elliptic problems

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = \frac{f}{u^\gamma} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

De Cave in [62] has proved that the problem (1.3) admit least one solution u, when $0 \leq f \in L^1(\Omega)$ satisfies the following regularity

i) if $\gamma = 1$ then $u \in W_0^{1,p}(\Omega)$;

ii) if $\gamma > 1$ then $u \in W_{loc}^{1,p}(\Omega)$ and $u^{(\tilde p)^*/p^*} \in W_0^{1,p}(\Omega)$ where $\tilde p = \frac{N(p+\gamma-1)}{N+\gamma-1}$;

iii) if $\gamma < 1$ then $u \in \mathcal{T}_0^{1,p}(\Omega)$ and $|\nabla u|^{\tilde{p}} \in L^1(\Omega)$;

iv) if $\gamma < 1$ and $2 - \gamma + \frac{\gamma - 1}{N} \leq p < N$ then $u \in W_0^{1,\tilde{p}}(\Omega)$.

Also the author has prove that if $f \in L^m(\Omega)$ with $m \geq 1$, then the solution u satisfies the following summability

v) if $\gamma > 0$ and $m > N/p$ then $u \in L^\infty(\Omega)$;

vi) if $\gamma \geq 1$ and $1 \leq m < N/p$ then $u \in L^s(\Omega)$, with $s = \frac{Nm(p+\gamma-1)}{N-pm}$;

vii) if $\gamma < 1$ and $(\frac{p^*}{1-\gamma})' < m < N/p$ then $u \in L^s(\Omega)$, with $s = \frac{Nm(p+\gamma-1)}{N-pm}$.

Concerning the uniqueness of solution to problem (1.3) has been addressed in [49].

On the other hand, for the uniform elliptic case, there is a great deal of literature about problems involving a lower order term, i.e. $g \neq 0$, we refer reader to see [81], when $h(s) = s^{-\gamma}$ $(\gamma > 0)$ and g does not depend on the gradient. More recently, in presence of general h, existence, regularity and uniqueness have been addressed in [120, 121]. For the case when the operator is not coercive, we refer the reader to see [135, 136] and references therein. In the case when the lower order term g exist and possibly singular in $u = 0$ (i.e. $g(x, t, u, \nabla u) \to +\infty$ as $u \to 0$) and having a natural growth with respect to the gradient, problem (1.1) has been studied by many authors, we refer reader to see [40, 89, 41, 42, 43, 144], when $h \equiv 1$.

Without the aim to be complet, we refer various works treating different aspects of the problems as (1.2) and (1.3) we refer the reader to see [8, 9, 10, 31, 38, 39, 50, 52, 63, 64, 70, 72, 73, 102, 118, 119] and reference therein.

Now, let us recall briefly the existing works in the literature and their influence directly in this thesis. Concerning the singular parabolic case as in the problem (1.1). In recent years, the existence, regularity and uniqueness of solutions to the nonlinear singular parabolic problems as in (1.1) have been studied extensively by many authors. When $g \equiv 0$, $p \geq 2$ and $h(s) = s^{-\gamma}(\gamma > 0)$, problem (1.1) is treated in [68]. Here, the authors have proved the existence of a weak solution via an approximation argument and one of the main tools is a suitable application of the Harnack inequality in order to deduce the positivity of the approximating sequence. More precisely, De Bonis and De Cave in [68] considered the following singular parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,\nabla u)) = \frac{f}{u^\gamma} & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Gamma, \\ u_0(x, t=0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$, $\gamma > 0$, $p \geq 2$, and f is a non-negative function which belongs to some Lebesgue space $L^m(Q)$, $m \geq 1$ and $u_0 \in L^\infty(\Omega)$ such that

$$\forall \, \omega \subset\subset \Omega, \; \exists \, D_\omega > 0 : \; u_0 \geq D_\omega \text{ in}\, \omega.$$

The authors proved that the problem admit a non-negative weak solution u satisfies the following regularity

i) if $\gamma < 1$ and $f \in L^{\frac{p(N+2)}{p(N+2)-N(1-\gamma)}}(Q)$, then $u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$;

ii) if $\gamma = 1$ and $f \in L^1(Q)$, then $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$;

iii) if $\gamma > 1$ and $f \in L^1(Q)$, then $u \in L^p(0,T;W_{loc}^{1,p}(\Omega)) \cap L^\infty(0,T;L^{\gamma+1}(\Omega))$.
Moreover, if $f \in L^m(Q)$ with $m \geq 1$, the solution u satisfies the following summability

iv) if $\gamma \geq 1$ and $m > N/p + 1$, then $u \in L^\infty(Q)$;

v) if $\gamma \geq 1$ and $m \in [1, N/p+1)$, then $u \in L^{\frac{m(N(p+\gamma-1)+p(\gamma+1)}{N-pm+p}}(Q)$;

vi) if $\gamma < 1$ and $m > N/p + 1$, then $u \in L^\infty(Q)$;

vii) if $\gamma < 1$ and $m \in [\frac{N(p+2)}{p(N+2)-N(1-\gamma)}, N/p+1)$, then $u \in L^{\frac{m(N(p+\gamma-1)+p(\gamma+1)}{N-pm+p}}(Q)$;

viii) if $\gamma < 1$ and $m \in [1, \frac{p(N+2)}{p(N+2)-N(1-\gamma)})$, then $u \in L^{q_m}(0,T;W_0^{1,q_m}(\Omega)) \cap L^{\frac{m(N(p+\gamma-1)+p(\gamma+1)}{N-pm+p}}(Q)$,
with $q_m = \frac{m[N(p+\gamma-1)+p(\gamma+1)]}{N+2-m(1-\gamma)}$.

In the same fashion, De Bonis and Giachetti in [69] have proved the existence of non-negative solution to the following singular parabolic problems involving p-Laplacian:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u = f(x,t)(\frac{1}{u^\theta} + 1) & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Gamma, \\ u_0(x, t=0) = u_0(x) & \text{in} \quad \Omega. \end{cases} \tag{1.5}$$

Here Ω is bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $0 < T < +\infty$, $\theta > 0$, $p > 1$, and f is non-negative function which belongs to $L^r(0,T;L^m(\Omega))$, with $\frac{1}{r} + \frac{N}{pm} < 1$, and $u_0(x) \geq 0$ a.e. in Ω. Also, the authors considered the case when the right-hand side of the above problem depends on the gradient. In this latest case the model of the right-hand side is $F(x,t,u,\nabla u) = \frac{f(x,t)+D|\nabla u|^q}{u^\theta}$, with $D > 0$, $1 < q < p$ and $f(x,t)$ as before.

More recently, if $g \equiv 0$ and in presence of a general h and measure data, existence and uniqueness have been addressed in [122], under suitable assumptions. In the same sense, Magliocca and Oliva in [112] have proved the existence of non-negative solutions to parabolic Cauchy-Dirichlet problems with superlinear gradient terms which are possibly singular. The model equation is

$$\frac{\partial u}{\partial t} - \Delta_p u = g(u)|\nabla u|^q + h(u)f \quad \text{in}\, Q$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with $N > 2$, $0 < T < +\infty, 1 < p < N$, and $q < p$ is superlinear. The functions g, h are continuous and possibly satisfying $g(0) = +\infty$ and/or $h(0) = +\infty$, with different rates, and finally f is a non-negative function which belongs to a suitable Lebesgue space.

When $h \equiv 1$ and the absorption terms does exist and appear in the problem (1.1) (i.e. $g(x,t,u,\nabla u) \neq 0$) and possibly singular at $u = 0$, the works studying the problems of this type is more limited. Martinez-Aparacio and Petitta in the first part of [113] have studied the problem (1.1) when $a(x,t,u,\nabla u) = M(x,t,u)\nabla u$ and g does not depend on the gradient. More precisely, the authors considered the following problems

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(M(x,t,u)\nabla u) + g(x,t,u) = f(x,t) & \text{in} \quad Q, \\ u(x,t) = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.6}$$

where Ω is an open bounded set of $\mathbb{R}^N$ $(N \geq 3)$ $M(x,t,s) := m_{ij}(x,t,s)$, $i,j = 1,...,N$ is a symetric matrix whose coefficient $m_{ij}(x,t,s) : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions (i.e., $m_{ij}(.,.,s)$ is measurable on Ω for every $s \in \mathbb{R}$, and $m_{ij}(x,t,.)$ is continuous on $\mathbb{R}$ for a.e. $(x,t) \in \Omega \times (0,T)$) such that there exist constants $0 < \alpha \leq \beta$ satisfying

$$\alpha|\xi|^2 \leq M(x,t,s)\xi \cdot \xi, \quad |M(x,t,s)| \leq \beta, \, \forall\, (s,\xi) \in \mathbb{R} \times \mathbb{R}^N, \text{ a.e. } x \in \Omega, \, \forall\, t \in (0,T).$$

f is non-negative function which belongs to $L^1(Q)$, $\kappa > 0$ and $g : \Omega \times (0,T) \times [0,\kappa) \to \mathbb{R}^+$ is a Carathéodory function such that

$$h(s) \leq g(x,t,s) \leq \rho(x,t)\delta(s), \, \forall s \in [0,\kappa), \text{ a.e. } x \in \Omega, \, \forall\, t \in (0,T),$$

where $\rho \in L^1(Q)$ and $\delta(s), h(s) : [0,\kappa) \to \mathbb{R}^+$ are continuous and increasing real function such that $\delta(0) = h(0) = 0$ and $\lim_{s \to \kappa^-} h(s) = +\infty$. Finally, u_0 is a measurable function such that $u_0(x) < \kappa$ for a.e. on Ω. Here, the authors have proved that the above problem admits a positive solution $u \in L^2(0,T; H_0^1(\Omega))$. In the second part of their work, the authors studied the problem (1.1), when the absorption term $g(x,t,u,\nabla u)$ possibly singular at $u = 0$ and possibly negative having a natural growth with respect to the gradient. More specifically, the authors have proved the existence of positive solution $u \in L^2(0,T; H_0^1(\Omega))$ to the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(M(x,t,u)\nabla u) + g(x,t,u)|\nabla u|^2 = f(x,t) & in \quad Q, \\ u(x,t) = 0 & on \quad \Gamma, \\ u(x,0) = u_0(x) & in \quad \Omega, \end{cases} \tag{1.7}$$

where M as before, and $f \in L^r(0,T; L^q(\Omega))$ with $\frac{1}{r} + \frac{2}{Nq} < 1$, $q \geq 1$, $r \geq 1$ satisfies

$$m_\omega(f) = essinf\{f(x,t) : \, x \in \omega, \, t \in (0,T)\} > 0, \;\; \forall \omega \subset\subset \Omega.$$

Moreover, the initial data $u_0 \in L^\infty(\Omega)$ such that

$$m_\omega(u_0) = essinf\{f(x,0) : \, x \in \omega\} > 0, \;\; \forall \omega \subset\subset \Omega.$$

Concerning the lower order term, $g(x,t,u)$ is Carathéodory function defined on $\Omega \times (0,T) \times (0,+\infty)$ satisfying for some $\mu > 0$

$$\frac{-\mu}{s} \leq g(x,t,s) \leq h(s), \quad \text{for } x \in \Omega, \, \forall\, s > 0, \;\; \forall t \in (0,T),$$

where $h : (0,+\infty) \to [0,+\infty)$ is a continuous non-negative function such that

$$\lim_{s \to 0^+} \int_s^1 \sqrt{h(t)}dt < +\infty,$$

and $h(s)$ is non increasing in a neighborhood of zero. In the same kinds Dall'Aglio et al in [60] have studied the problem (1.1), when $g(x,t,u,\nabla u) = B\frac{|\nabla u|^p}{u}$ and $f \equiv 0$, with $p > 1$, $B > 0$, and u_0 is a positive function in $L^\infty(\Omega)$ such that $u_0(x) \geq C > 0$ in Ω. The authors shown that the problem (1.1) admits a non-negative weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$. For the non-homogeneous case (when $f \neq 0$ and $h \equiv 1$), we refer the reader to see [15].

In the case when the problem (1.1) involving Hardy potential, there is an extensive literature which has studied the problems of this kind. If $p = 2$ and $f \equiv 0$ then we have a linear heat equation with potential

$$\frac{\partial u}{\partial t} = \Delta u + v(x)u.$$

If the potential v belongs to the kato class or L^p $(p > N/2)$ classes, then the Hamiltonian $\mathcal{H} = -\Delta + v$ has several good properties and so the linear heat equations with this potential is well understood. If the potential v does not belong to these classes, such as $v = c/|x|^2$ then the solutions of heat problem may have critical behavior. As an interesting results was obtained by Baras and Goldstein [13], they have shown that the following heat problem;

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \frac{c}{|x|^2}u & in \quad Q, \\ u(x,t) = 0 & on \quad \Gamma, \\ u(x,0) = u_0(x) & in \quad \Omega, \end{cases}$$

has no non-negative solutions except $u \equiv 0$ if $c > C^*(N) = ((N-2)/N)^2$ and positive weak solution does exist if $c < C^*(N)$. Thus $C^*(N) = ((N-2)/N)^2$ is the cut of point for existence of positive solutions for the heat equation with inverse square potential $c/|x|^2$, where $C^*(N)$ is also the sharp constant in Hardy's inequality. The results in [13] have been extended by Cabré and Martel [48], Goldstein and Zhang [87, 88], Goldstein and Kombe [85], Kombe [96] for the wide class of potentials. In the same context there are also some related works on nonlinear parabolic problems by Garcia and Peral [84], Aguilar and Peral [5], Goldstein and Kombe [86]. We refer the reader to see [4, 13, 82, 84, 130, 131, 132] and references therein.

Finally, Problems as (1.1) with degenerate coercivity and $h \equiv 1$ has been extensively studied in the past. See for instance [6, 7, 21, 24, 28, 53, 65] in the elliptic case and [29, 75, 66, 107, 124, 127, 129, 143] in the parabolic case.

Concerning the existence and regularity results for the problems as (1.1), when the operator A is non-coercive and the term h is singular at $s = 0$, the first contribution has been given by Croce [54], when $g \equiv 0$, $A(u) = -\operatorname{div}\left(\frac{a(x)\nabla u}{(1+|u|)^p}\right)$ and $h(u) = \frac{1}{u^\gamma}$. The author has proved the existence of non-negative solutions of problem (1.1) in the stationary case when $p > 1$, $p-1 \leq \gamma \leq p+1$ and $f \in L^m(\Omega)$ with $m \geq 1$. The regularity of the solutions also analyzed. More Recently, Sbai and El Hadfi [135] generalized the work [54]. The authors studied the problem (1.1) in the stationary case, when $g \equiv 0$ and $A(u) = -\operatorname{div}(a(x,u,\nabla u))$ such that $a(x,u,\nabla u)\cdot\nabla u \geq \frac{\alpha|\nabla u|^p}{(1+|u|)^{\theta(p-1)}}$, $p > 1$, $\alpha > 0$ and $0 < \theta < 1$, the function $h : [0,+\infty) \to [0,+\infty]$ is continuous bounded outside the origin with $h(0) \neq 0$ and possibly singular at $s = 0$ such that the following condition hold true:

$$\exists C > 0,\ 0 < \gamma < 1 \text{ s.t. } h(s) \leq \frac{C}{s^\gamma} \quad \forall\, s > 0.$$

They have proved the existence of weak non-negative solution to the problem (1.1) and the regularity of the solution also analyzed. Durastanti and Oliva [71] have considered the same problem studied in [135]. The authors have proved the existence and uniqueness of entropy solution in the stationary case of the problem (1.1). The regularity of the entropy solution also analyzed. In the presence of the lower order term (i.e. $g \neq 0$), but without any growth condition on the gradient i.e. $g(x,u,\nabla u) = g(x,u)$ and the operator A is non-coercive, the problem (1.1) has been studied by Sbai and El Hadfi in [136] in the stationary case.

This thesis is organized as follows. In chapter 2, we give some preliminaries (definitions of classical spaces, convergence theorems, and injection theorems, and classical integrability lemmas ...) which we will use later in the following chapters.

Chapter 3: On nonlinear parabolic equations with singular lower order term.

This chapter is devoted to study the problem (1.1), when $g \equiv 0$, $h(u) = u^{-\gamma}$ $(\gamma > 0)$ and $a(x,t,u,\nabla u) = (a(x,t) + |u|^q)\nabla u$. More precisely, we focus on the following nonlinear and singular parabolic problems

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}((a(x,t) + |u|^q)\nabla u) = \frac{f}{u^\gamma} & \text{in} \quad Q, \\ u(x,t) = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.8}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, Q is the cylinder $\Omega \times (0,T)$, $T > 0$, Γ the lateral surface $\partial\Omega \times (0,T)$, $q > 0$, $\gamma > 0$, and f is non-negative function wich belongs to some Lebesgue space $L^m(Q)$, $m \geq 1$,

$$u_0 \in L^\infty(\Omega) \text{ and } \forall\, \omega \subset\subset \Omega,\ \exists\, D_\omega > 0 :\ u_0 \geq D_\omega. \tag{1.9}$$

Moreover $a(x,t)$ is a measurable function satisfying

$$0 < \alpha \leq a(x,t) \leq \beta \quad \text{a.e. in } Q; \tag{1.10}$$

where α, β are fixed real numbers such that $\alpha < \beta$. we start by identifying the necessary conditions on the data in order to get existence of weak solutions of (1.8). Then, using the Schauder's fixed point Theorem, we shown the existence of non-negative solution for the non-singular problem, for every non-negative function f depending on the values of q and γ and by the application of Harnack inequality, we prove that this solution is strictly positive in the interior of the parabolic cylinder. Also, the regularity of solutions depending on the summability of the function f and the values of q and γ has been obtained.

This work is published in Journal of Elliptic and Parabolic Equations [76]

Chapter 4: Some nonlinear parabolic problems with singular natural growth term

In this chapter, we study the problem (1.1) when $a(x,t,u,\nabla u) = |\nabla u|^{p-2}\nabla u$, $g(x,t,u,\nabla u) = b(x,t)\frac{|\nabla u|^p}{u^\theta}$ and $h \equiv 1$. More precisely we study the following nonlinear parabolic problems:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + b(x,t)\frac{|\nabla u|^p}{u^\theta} = f & \text{in} \quad Q, \\ u(x,t) = 0 & \text{on} \quad \Gamma, \\ u(x,0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.11}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, and Q is the cylinder $\Omega \times (0,T)$, $T > 0$, Γ the lateral surface $\partial\Omega \times (0,T)$, $2 \leq p < N$, $0 < \theta < 1$, $b(x,t)$ is a measurable function satisfying

$$0 < \alpha \leq b(x,t) \leq \beta, \tag{1.12}$$

where α and β are fixed real numbers, and f belongs to some Lebesgue space $L^m(Q)$, $m \geq 1$, satisfying the condition

$$\operatorname{ess\,inf}\{f(x,t) : x \in \omega, t \in (0,t)\} > 0 \quad \forall\, \omega \subset\subset \Omega.$$

Our aim is to study the impact of the term $b(x,t)|\nabla u|^p u^{-\theta}$ ($\theta > 0$) (having a natural growth with respect to the gradient and singular at $u = 0$) in the existence of the weak solution of problem (1.11) for the largest possible classes of the data f. In order to obtain a weak solution, we approximate the problem (1.11) by another non-singular problem, we make some estimates that will allow us to prove that the solution of approximated problem converges to the solution of our problem. Also, this solution satisfies the property of the strict positivity in the interior of the parabolic cylinder. We use this important property to make a sense of the weak formulation of (1.11) and also in the convergence passage.

This work is published in Journal of Results in Mathematics [78]

Chaptre 5: Existence of positive solutions to nonlinear singular parabolic equations with Hardy potential

In this chapter, we focalize our attention on the studying of the problem (1.1) when $a(x,t,u,\nabla u) = a(x,t,\nabla u)$, $g(x,t,u,\nabla u) = -\mu\frac{u^{p-1}}{|x|^p}$, $\mu > 0$ and $h(u) = u^{-\gamma}$, $\gamma > 0$. More specifically, we study the following nonlinear parabolic problems involving Hardy potential with a singular lower order term:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,\nabla u)) - \mu\frac{u^{p-1}}{|x|^p} = \frac{f}{u^\gamma} & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.13}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $(N \geq 3)$, $2 \leq p < N$, $\gamma, \mu > 0$, $Q = \Omega \times (0,T)$, $\Gamma = \partial\Omega \times (0,T)$, with $T > 0$, f is a nononegative function belonging to suitable Lebesgue space, the initial datum $u_0 \in L^\infty(\Omega)$ and satisfies the following bound

$$\forall\, \omega \subset\subset \Omega, \quad \exists\, M_\omega > 0 \ : \ u_0 \geq M_\omega \text{ in } \omega. \tag{1.14}$$

Moreover, the function $a : \Omega \times (0,T) \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Caratheodory function satisfying the following conditions: there exist positive constants α, β such that

$$a(x,t,\xi) \cdot \xi \geq \alpha|\xi|^p, \tag{1.15}$$

$$|a(x,t,\xi)| \leq \beta|\xi|^{p-1}, \tag{1.16}$$

$$[a(x,t,\xi) - a(x,t,\xi')] \cdot [\xi - \xi'] > 0, \tag{1.17}$$

for almost every $x \in \Omega, t \in (0,T)$, for every $\xi, \xi' \in \mathbb{R}^N$, with $\xi \neq \xi'$. The main goal of this chapter is to analyze the interaction between the Hardy potential and the singular term $u^{-\gamma}$ in order to get a solution for the largest possible class of the datum f . The regularity of the solution is also analyzed.

This work is published in Journal of Pseudo-Differential Operators and Applications [79]

Chapter 6: Existence and regularity results for a singular parabolic equations with degenerate coercivity

In this chapter, we are going to study the existence and regularity results of the problem (1.1) when $g(x,t,u,\nabla u) = |u|^{s-1}u$, $(s \geq 1)$, i.e. we consider the following singular parabolic problems with degenerate coercivity and absorption term

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u) + |u|^{s-1}u = h(u)f & \text{in} \quad Q, \\ u(x,0) = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma, \end{cases} \tag{1.18}$$

where

$$A(u) = -\mathrm{div}(a(x,t,u,\nabla u)).$$

Here Ω is a bounded open subset of $\mathbb{R}^N$, $(N > p \geq 2)$ and $0 < T < +\infty$, f is non-negative function that belongs to some Lebesgue space, $f \in L^m(Q)$, $m \geq 1$ $Q = \Omega \times (0,T)$, $\Gamma = \partial\Omega \times (0,T)$, $0 < \gamma < 1$ and $s \geq 1$. $a(x,t,\sigma,\xi) : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function (i.e it is continuous with respect to σ and ξ for almost $(x,t) \in Q$, and measurable with respect to (x,t) for every $\sigma \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$) satisfying for a.e $(x,t) \in Q$, $\forall\, \xi, \xi' \in \mathbb{R}^N$:

$$a(x,t,\sigma,\xi).\xi \geq \frac{\alpha|\xi|^p}{(1+|\sigma|)^{\theta(p-1)}}, \tag{1.19}$$

$$|a(x,t,\sigma,\xi)| \leq b(x,t) + |\sigma|^{p-1} + |\xi|^{p-1}, \tag{1.20}$$

$$(a(x,t,\sigma,\xi) - a(x,t,\sigma,\xi')).(\xi - \xi') > 0 \quad \xi \neq \xi', \tag{1.21}$$

where α is positive constant, $0 \leq \theta < 1$ and b is a non-negative function and belong to $L^{p'}(Q)$, $p' = \frac{p}{p-1}$. The function $h : [0,\infty) \longrightarrow \mathbb{R}^+$ is a continuous and bounded function and possibly singular at $s = 0$ such that

$$\exists\, c > 0 \text{ such that } \quad h(r) \leq \frac{c}{r^\gamma} \quad \forall r > 0. \tag{$h1$}$$

In the study of problem (1.18), there is one to two difficulties, the first one is the fact that, due to hypothesis (1.19) the differential operator $A(u) = -\mathrm{div}(a(x,t,u,\nabla u))$ is not coercive on $L^p(0,T;W_0^{1,p}(\Omega))$, when u is large. Due to the lack of coercivity, the classical theory for parabolic operators acting between spaces in duality (see [108]) cannot be applied. The second difficulty comes from the right-hand side is singular in variable u. We overcome these difficulties by replacing the operator A by another one defined by means of truncations, and approximating the singular term by non singular one. We will prove in Section 3 that problems admits a bounded $L^p(0,T;W_0^{1,p}(\Omega))$ solution by using Schauder's fixed point theorem and we prove some a prior estimates for the solution of the approximate problem and finally we pass to the limit.

This work is published in Journal of Discrete and Continuous Dynamical Systems-S [77]

Chapter 2

Preliminaries

In this chapter, we give some basic results that we will often use in the proofs of our results.

1 A review on some basic results of the theory of integration

1.1 Lebesgue and Sobolev spaces.

Let D be an open subset of $\mathbb{R}^N$. For $1 \leq p \leq +\infty$, we denote by $L^p(D)$ the space of Lebesgue measurable functions (in the fact, equivalence classes, since almost everywhere equal functions are identified) $u : D \to \mathbb{R}$ such that, if $p < +\infty$,

$$\|u\|_{L^p(D)} = \left(\int_D |u(x)|^p dx\right)^{\frac{1}{p}} < +\infty,$$

and if $p = +\infty$,

$$\|u\|_{L^\infty} = \text{ess-sup}_{x\in D}|u(x)|.$$

For the definition, the main properties and results on Lebesgue spaces we refer to [35, 97]. For a function u in a Lebesgue space, we set by $\frac{\partial u}{\partial x_i}$ (or simply u_{x_i}) its partial derivative in the direction x_i defined in the sense of distributions, that is

$$\langle u_{x_i}, \phi\rangle = -\int_D u\phi_{x_i} dx,$$

and we denote, in this way, by $\nabla u = (u_{x_1}; u_{x_2}; ...; u_{x_N})$ the gradient of the function u.

The Sobolev space $W^{1,p}(D)$, with $1 \leq p \leq +\infty$, is the space of functions u in $L^p(D)$ such that $\nabla u \in (L^p(D))^N$, endowed with its natural norm

$$||u||_{W^{1,p}(D)} = (\|u\|^p_{L^p(D)} + \|\nabla u\|^p_{L^p(D)})^{\frac{1}{p}},$$

while $W^{1,p}_0(D)$ will indicate the closure of $\mathcal{D}(D)$ (the space of $\mathcal{C}^\infty$ functions with compact support in D) with respect to this norm. For $1 \leq p \leq +\infty$ the dual space of $L^p(D)$ can be identified with $L^{p'}(D)$, where $p' = \frac{p}{p-1}$ is the Hölder conjugate exponent of p, and the dual space of $W^{1,p}_0(D)$ is denoted by $W^{-1,p'}(D)$. It is well known that if D is bounded, any element $T \in W^{-1,p'}(D)$ can be written, (see [35]), in the form $T = -\operatorname{div}(F)$ where $F \in (L^{p'}(D))^N$.

1.2 Basic tools of integration.

We recall here some useful results in the theory of integration.

Lemma 2.1 (Fatou's lemma [35]). *Let* $\{f_n\} \subset L^1(D)$ *be a sequence such that*

- *for each* n, $f_n(x) \geq 0$ *a.e. in* D,
- $sup_n \displaystyle\int_D f_n(x)\,dx < +\infty$.

Then $\liminf\limits_{n\to+\infty} f_n \in L^1(D)$, *and*

$$\int_D \liminf_{n\to+\infty} f_n(x)\,dx \leq \liminf_{n\to+\infty} \int_D f_n(x)\,dx.$$

Definition 2.2. (see [95]) We say that a sequence $\{f_n\} \subset L^1(D)$ is equi-integrable if for all $\varepsilon > 0$ there exist a measurable set $A \subset D$ of finite measure and a real $\delta > 0$ such that

- $\displaystyle\int_{D\setminus A} |f_n(x)|\,dx \leq \varepsilon$, for all $n \geq 1$,
- $\forall E \subset D$, $|E| < \delta \Rightarrow \sup_n \displaystyle\int_E |f_n(x)|\,dx \leq \varepsilon$.

Lemma 2.3. *Vitali's theorem (see [95]) Let* $\{f_n\} \subset L^1(D)$ *be a sequence such that* $f_n \to f$ *a.e. in* D. *Then, the two assertions are equivalent*

- $f_n \to f$ *strongly in* $L^1(D)$,
- $\{f_n\}$ *is equi-integrable*

We will also use the following technical lemma which can be found in [93].

Lemma 2.4. *Let* $\{f_n\} \subset L^1(D)$, *and let* $f \in L^1(D)$

- $f_n(x) \geq 0$ *a.e. in* D,
- $f_n \to f$ *a.e. in* D,
- $\displaystyle\int_D f_n(x)\,dx \to \int_D f(x)\,dx$.

Then $f_n \to f$ *strongly in* $L^1(D)$.

For an exhaustive treatment on Sbolev spaces we refer to [3] and [36]. We only racall the following fundamentals facts.
• **Sobolev's inequality**: there exists a positive constant S_0 depend only on N and p such that

$$\begin{cases} \|\phi\|_{L^\infty} \leq S_0|\Omega|^{\frac{1}{N}-\frac{1}{p}}\||\nabla\phi|\|_{L^p(\Omega)} & \text{if } p \in (N,\infty) \\ \|\phi\|_{L^{p^*}(\Omega)} \leq S_0\||\nabla\phi|\|_{L^p(\Omega)} & \text{if } p \in (1,N), \end{cases} \qquad \forall\phi \in W_0^{1,p}(\Omega)$$

where p^* is the Sobolev conjugate exponent of p, that is,

$$p^* = \frac{Np}{N-p} \quad \forall\, p \in [1, N).$$

In general, $W_0^{1,p}(\Omega)$ cannot be replaced by $W^{1,p}(\Omega)$ on the previous embedding result. However, this replacement can be made for a large class of open set Ω, which includes for exemple open sets with lipschitz boundary. More generally, if Ω satifies a uniform interior cone condition (that is there exists a fixed cone $U_\Omega(x)$ of height h an solid angle ω such that each $x \in \Omega$ is the vertex of a cone $U_\Omega(x) \subset \bar{\Omega}$ and congruent to U_Ω), then there exists a positive constants S which depends only on N and p, such that

$$\begin{cases} \|\phi\|_{L^\infty(\Omega)} \leq \frac{S}{\omega h^{\frac{N}{p}}} \left(\|\phi\|_{L^p(\Omega)} + \||\nabla\phi|\|_{L^p(\Omega)}\right) & \text{if } p \in (N, \infty) \\ \|\phi\|_{L^{p^*}(\Omega)} \leq \frac{S}{\omega} \left(\frac{1}{h}\|\phi\|_{L^p(\Omega)} + \||\nabla\phi|\|_{L^p(\Omega)}\right) & \text{if } p \in (1, N). \end{cases} \quad \forall \phi \in W^{1,p}(\Omega)$$

• **Rellich Kondrachov's theorems**: the embedding

$$W_0^{1,p}(\Omega) \subset \begin{cases} L^\infty(\Omega) & \text{if } p \in (N, \infty) \\ L^q(\Omega) \ \ \forall\, q \in [1, p^*) & \text{if } p \in (1, N), \end{cases}$$

is compact. Moreover, if Ω satifies a uniform interior cone condition, then also the embedding

$$W^{1,p}(\Omega) \subset \begin{cases} L^\infty(\Omega) & \text{if } p \in (N, \infty) \\ L^q(\Omega) \ \ \forall\, q \in [1, p^*) & \text{if } p \in (1, N), \end{cases}$$

is compact.
• **Poincaré's inequality**: there exists a positive constant $\mathcal{P}$ which depends only on N, p and Ω, such that

$$\|\phi\|_{L^p(\Omega)} \leq \mathcal{P}\||\nabla\phi|\|_{L^p(\Omega)} \quad \forall\, \phi \in W_0^{1,p}(\Omega).$$

Accordingly, the quantity $\||\nabla \cdot |\|_{L^p(\Omega)}$ defines as norm on $W_0^{1,p}(\Omega)$ which equivalent to $\| \cdot \|_{W^{1,p}(\Omega)}$.

We will often use the following result due to G. Stampachia.

Theorem 2.5. *(see [140]) Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $G(0) = 0$. Then for every $u \in W_0^{1,p}(D)$ we have $G(u) \in W_0^{1,p}(D)$ and $\nabla G(u) = G'(u)\nabla u$ almost everywhere in D.*

Theorem 2.6. *has an important consequence, that is*

$$\nabla u = 0 \ \ a.e \ in \ E_c = \{x : u(x) = c\},$$

for every $c > 0$. Hence, we are able to consider the composition of function in $W_0^{1,p}(D)$ with some useful auxiliary function. One of the most used will be the truncation function at level $k > 0$, that is $T_k(s) = \max(-k, \min(k, s))$.

Thus, if $u \in W_0^{1,p}(D)$, we have that $T_k(u) \in W_0^{1,p}(D)$, and

$$\nabla T_k(u) = \nabla u \chi_{\{u<k\}}, \ a.e. \ on \ D,$$

for every $k > 0$, where $\chi_{\{u<k\}}$ stands for the characteristic function of the set $\{x \in D : \quad |u(x)| < k\}$.

Remark 2.7. If u is such that its truncation belongs to $W_0^{1,p}(D)$, then we can define an approximated gradient of u defined as the a.e. unique measurable function $v : D \to \mathbb{R}^N$ such that

$$v = \nabla T_k(u)$$

almost everywhere on the set $\{|u| \leq k\}$, for every $k > 0$ (see for instance [47]).

1.3 Spaces of functions $L^p(a, b; V)$.

Given a real Banach space V, for $1 \leq p < +\infty$, for $a, b \in \mathbb{R}$, $L^p(a, b; V)$ is the space of measurable functions $u : [a, b] \to V$ such that

$$\|u\|_{L^p(a,b;V)} = \left(\int_a^b \|u\|_V^p \, dt \right)^{\frac{1}{p}} < +\infty,$$

while $L^\infty(a, b; V)$ is the space of measurable functions such that:

$$\|u\|_{L^\infty(a,b;V)} = \sup_{[a,b]} \|u\|_V < +\infty.$$

Of course both spaces are meant to be quotiented, as usual, with respect to the almost everywhere equivalence. The reader can find a presentation of these topics in [61]. Let us recall that, for $1 \leq p \leq +\infty$, $L^p(a, b; V)$ is a Banach space, moreover if for $1 \leq p < +\infty$ and V', the dual space of V, is separable, then the dual space of $L^p(a, b; V)$ can be identified with $L^{p'}(a, b; V')$.

For our purpose V will mainly be either the Lebesgue space L^p or the Sobolev space $W_0^{1,p}(\Omega)$, with $1 \leq p < +\infty$ and Ω is a bounded open set of $\mathbb{R}^N$. Since in this case V is separable we have that $L^p(a, b; L^p(\Omega)) = L^p((a, b) \times \Omega)$, the ordinary Lebesgue space defined in $(a, b) \times \Omega$ and $L^p(a, b; W_0^{1,p}(\Omega))$ consists of all functions $u : \Omega \to \mathbb{R}$ which belong to $L^p((a, b) \times \Omega)$ and such that $\nabla u = (u_{x_1}, \cdots, u_{x_N})$ belongs to $L^p((a, b) \times \Omega)^N$ (often, for simplicity, we will indicate this space only by $L^p((a, b) \times \Omega)$; moreover,

$$\left(\int_a^b \int_\Omega |\nabla u|^p \, dx \, dt \right)^{\frac{1}{p}}$$

defines an equivalent norm by Poincaré's inequality.

Given a function in $L^p(a, b; V)$ it is possible to define a time derivative of u in the space of vector valued distributions $\mathcal{D}'(a, b; V)$ which is the space of linear continuous functions from $C_0^\infty(a, b)$ into V (see [137]). In fact, the definition is the following:

$$\langle u_t, \varphi \rangle = -\int_a^b u \varphi_t \, dt, \quad \forall \varphi \in C_0^\infty(a, b),$$

where the equality is meant in V. In the following, we will also use sometimes the notation $\frac{\partial u}{\partial t}$ instead of u_t and $Q = (0, T) \times \Omega$. Now we state two embedding theorems that will play a central role in our work; the first one is an Aubin-Simon type result that we state in a form general enough to our purpose, while the second one is the well-known Gagliardo-Nirenberg embedding theorem followed by an important consequence of it for the evolution case.

Theorem 2.8. *(see [139]). Let $\{u_n\}$ be a sequence bounded in $L^m(0, T; W_0^{1,m}(\Omega))$ such that $\frac{\partial u_n}{\partial t}$ is bounded in $L^1(Q) + L^{s'}(0, T; W^{-1,s'}(\Omega))$ with $m, s > 1$, then $\{u_n\}$ is relatively strongly compact in $L^1(Q)$, that is, up to subsequences, $\{u_n\}$ strongly converges in $L^1(Q)$ to some function $u \in L^1(Q)$.*

Next, we will introduce the following Gagliardo-Nirenberg inequality and Stampcchia Lemma that will be used essentially throughout the memory.

Lemma 2.9. *[74, Proposition 3.1] Let v be a function in $W_0^{1,h}(\Omega) \cap L^\rho(\Omega)$, with $h \geq 1, \rho \geq 1$. Then there exists a positive constant C, depending on N, h, ρ, and σ such that*

$$\|v\|_{L^\sigma(\Omega)} \leq C\|\nabla v\|^\eta_{(L^h(\Omega))^N}\|v\|^{1-\eta}_{L^\rho(\Omega)} \tag{2.1}$$

for every η and σ satisfying

$$0 \leq \eta \leq 1, \quad 1 \leq \sigma < +\infty, \quad \frac{1}{\sigma} = \eta\Big(\frac{1}{h} - \frac{1}{N}\Big) + \frac{1-\eta}{\rho}.$$

An immediate consequence of the previous lemma is the following embedding result:

$$\int_Q |v|^\sigma \leq \|v\|^{\frac{\rho h}{N}}_{L^\infty(0,T;L^\rho(\Omega))} \int_Q |\nabla v|^h, \tag{2.2}$$

which holds for every function v in $L^h(0,T;W_0^{1,h}(\Omega)) \cap L^\infty(0,T;L^\rho(\Omega))$, with $h \geq 1, \rho > 1$ and $\sigma = \frac{h(N+\rho)}{N}$.

Lemma 2.10. *Let C, λ, k_0, μ be real positive numbers, where $\mu > 1$. Let $\varrho : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a decreasing function such that*

$$\varrho(h) \leq \frac{C}{(h-k)^\lambda}[\varrho(k)]^\mu, \ \ \forall h > k \geq k_0.$$

Then $\varrho(k_0 + d) = 0$, where $d^\lambda = C[\varrho(k_0)]^{\mu-1}2^{\frac{\mu\lambda}{\mu-1}}$.

Chapter 3

On nonlinear parabolic equations with singular lower order term

1 Introduction

In this chapter we prove existence and regularity results for a class of nonlinear singular parabolic equations. More precisely, we are interested in the following nonlinear problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{div}((a(x,t)+|u|^q)\nabla u) = \frac{f}{u^\gamma} & in \quad Q, \\ u(x,t) = 0 & on \quad \Gamma, \\ u(x,0) = u_0(x) & in \quad \Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, Q is the cylinder $\Omega \times (0,T)$, $T > 0$, Γ the lateral surface $\partial\Omega \times (0,T)$, $q > 0$, $\gamma > 0$, and f is non-negative function which belongs to some Lebesgue space $L^m(Q)$, $m \geq 1$, the data u_0 satisfies

$$u_0 \in L^\infty(\Omega) \text{ and } \forall\, \omega \subset\subset \Omega,\ \exists\, D_\omega > 0 :\ u_0 \geq D_\omega \text{ in } \omega. \tag{3.2}$$

Moreover $a(x,t)$ is a measurable function satisfying

$$0 < \alpha \leq a(x,t) \leq \beta \ \ a.e.\ Q; \tag{3.3}$$

where $\alpha,\ \beta$ are fixed real numbers.

If $\gamma = 0$ many works have appeared concerning the existence and regularity of elliptic equations. Boccardo In [26] has been studied the existence and regularity results of quasi linear elliptic problem

$$\begin{cases} -\mathrm{div}((a(x)+|u|^q)\nabla u) + b(x)u|u|^{p-2}|\nabla u|^2 = f(x) & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega, \end{cases}$$

where $a(x), b(x)$ are measurable bounded functions, $p, q \geq 0$ and $0 \leq f \in L^m(\Omega)$, $1 \leq m \leq \frac{N}{2}$, see also [114]. In the case parabolic the authors in [116] has been studied the existence and regularity results of nonlinear problems

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}((a(x,t)+|u|^q)\nabla u) + b(x,t)u|u|^{p-1}|\nabla u|^2 = f & in \quad Q, \\ u = 0 & on \quad \partial\Omega, \\ u(t=0) = 0 & in \quad \Omega, \end{cases}$$

where $a(x,t), b(x,t)$ are measurable positive bounded functions, $p, q > 0$ and f belongs to $L^m(Q)$ for some $m \geq 1$. If $q = 0$, then the operator $A(x,t,\xi) = b(x,t)\xi$ existing in [98] and [68]($p = 2$) is linear coercive, monotone and satisfying the growth condition $|A(x,t,\xi)| \leq C(d(x,t) + |\xi|)$ with C a positive constant and $d \in L^2(Q)$, we highlight that our case ($q > 0$) the required growth of $A(x,t,s,\xi) = (a(x,t) + s^q)\xi$ is more general, handling growths greater then linear case.

In the elliptic framework and when $\gamma > 0$ a rich amount of research has been conducted to prove the existence of solution to singular problems. For example Boccardo and Orsina in [25] proved the existence and regularity results to problem

$$\begin{cases} -\Delta u = \frac{f(x)}{u^\gamma} & in \quad \Omega, \\ u > 0 & on \quad \Omega, \\ u = 0 & in \quad \Omega, \end{cases}$$

where $\gamma > 0$ and f is a nonnegative function belonging to $L^m(Q), m \geq 1$. In the same concept the authors in [118] proved the existence of solution to problem

$$\begin{cases} -\Delta u = \frac{f(x)}{u^\gamma} + \mu & in \quad \Omega, \\ u > 0 & on \quad \Omega, \\ u = 0 & in \quad \Omega, \end{cases}$$

with $\gamma > 0, f$ is a nonnegative function on Ω, and μ is a nonnegative bounded Radon measures on Ω. Hence Charkaoui and Alaa [44] established the existence of weak periodic solution to singular parabolic problems

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \frac{f(x)}{u^\gamma} & in \quad Q, \\ u = 0 & on \quad \Gamma, \\ u(.,0) = u(.,T) & in \quad \Omega, \end{cases}$$

with $\gamma > 0$ and f is a nonnegative integrable function periodic in time with period T. Let us observe that we refer to [68, 69, 77, 112, 122] for more details on singular parabolic problems.

If $\gamma = 0$ and $q = 0$, the problem (3.1) has been studied in [98]. When $q = 0$ and $\gamma > 0$, the existence and regularity results of problem (3.1) has been obtained in [68]. The aim of this chapter is to prove the existence and regularity of solutions of problem (3.1) depending on the summability of the datum f and the parameters $q, \gamma > 0$. As we will see, our growth assumption on the function $a(x,t) + |u|^q$ has a regularization effect on the solution u and its gradient ∇u, allowing in some cases to have finite energy solution (i.e $u \in L^2(0,T; H_0^1(\Omega))$ even if $f \in L^1(Q)$.

We give now the definition of the weak solution of the problem (3.1) we will use throughout this chapter.

Definition 3.1. If $\gamma \leq 1$, a solution of (3.1) is a function $u \in L^1(0,T; W_0^{1,1}(\Omega))$ such that

$$\forall \omega \subset\subset \Omega \;\; \exists\, C_\omega > 0 : u \geq C_\omega \; in \;\; \omega \times (0,T), \tag{3.4}$$

$$(a(x,t) + u^q)\nabla u \in L^1(0,T; L^1_{loc}(\Omega)), \tag{3.5}$$

and

$$-\int_\Omega u_0(x)\varphi(x,0)-\int_0^T\int_\Omega u\frac{\partial\varphi}{\partial t}+\int_0^T\int_\Omega(a(x,t)+u^q)\nabla u\nabla\varphi=\int_0^T\int_\Omega\frac{f\varphi}{u^\gamma},\tag{3.6}$$

$\forall\varphi\in C_c^1(\Omega\times[0,T))$.
If $\gamma>1$, a solution of problem (3.1) is a function $u\in L^2(0,T;H^1_{loc}(\Omega))$, $u^r\in L^1(0,T;W_0^{1,1}(\Omega))$, for some $r>1$ and u satisfying (3.4)-(3.6).

2 The approximation scheme

Let f be a non-negative measurable function which belongs to some Lebesgue space, let $n\in\mathbb{N}$, $f_n=\frac{f}{1+\frac{1}{n}f}$, and let us consider the following approximation of problem (3.1)

$$\begin{cases}\dfrac{\partial u_n}{\partial t}-\operatorname{div}((a(x,t)+u_n^q)\nabla u_n)=\frac{f_n}{(u_n+\frac{1}{n})^\gamma} & \text{in}\quad Q,\\ u_n(x,t)=0 & \text{on}\quad \Gamma,\\ u_n(x,0)=u_0(x) & \text{in}\quad \Omega.\end{cases}\tag{3.7}$$

Lemma 3.2. *the problem* (3.7) *has a non-negative solution* $u_n\in L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q)$.

Proof. Let $k,n\in\mathbb{N}$, be fixed $v\in L^2(Q)$ and define $w:=S(v)$ to be the unique solution of (see [108])

$$\begin{cases}\dfrac{\partial w}{\partial t}-\operatorname{div}((a(x,t)+|T_k(v)|^q)\nabla w)=\frac{f_n}{(|v|+\frac{1}{n})^\gamma} & \text{in}\quad Q,\\ w=0 & \text{on}\quad \Gamma,\\ w(x,0)=u_0(x) & \text{in}\quad \Omega.\end{cases}$$

Using w as test function by (3.3) and dropping the non-negative terms, we have

$$\alpha\int_Q|\nabla w|^2\le n^{\gamma+1}\int_Q|w|+\frac{1}{2}\int_\Omega u_0^2,$$

an application of Poincaré inequality on the left hand side and Hölder inequality on the right hand side and the fact that $u_0\in L^\infty(\Omega)$ yields

$$\int_Q|w|^2\le Cn^{\gamma+1}\left(\int_Q|w|^2\right)^{\frac{1}{2}}+\frac{1}{2}||u_0||^2_{L^2(\Omega)},$$

this by Young inequality with ϵ, implies that

$$\int_Q|w|^2\le M,$$

where M is a positive constant independent of v . So that the ball of radius M is invariant under S.
• Now we prove that S is continuous.
Let us choose a sequence $v_r\to v$ strongly in $L^2(Q)$; then by Lebesgue convergence Theorem :

$$\frac{f_n}{(|v_r|+\frac{1}{n})^\gamma}\to\frac{f_n}{(|v|+\frac{1}{n})^\gamma}\quad\text{in } L^2(Q),$$

and the uniqueness of solution for linear problem yields that $w_r = S(v_r) \to w = S(v)$ strongly in $L^2(Q)$. Therefore, we proved that S is continuous.
As we proved before, we have that:

$$\int_Q |\nabla S(v)|^2 \leq C(n, \gamma, ||u_0||_{L^2(\Omega)}), \text{ for every } \ v \in L^2(Q).$$

Then, $S(v)$ is relatively compact in $L^2(Q)$, and by Shauder's fixed point Theorem, there exist $u_{n,k} \in L^2(0,T; H_0^1(\Omega))$ such that $S(u_{n,k}) = u_{n,k}$ for each n, k fixed. Moreover, $u_{n,k} \in L^\infty(Q)$, for all $k, n \in \mathbb{N}$. Indeed, for $h \geq 1$ fixed, using $G_h(u_{n,k})$ as test function, we obtain, since $u_{n,k} + \frac{1}{n} \geq h \geq 1$ on $\{u_{n,k} \geq h\}$

$$\frac{1}{2}\int_\Omega |G_h(u_{n,k})|^2 + \int_Q |\nabla G_h(u_{n,k})|^2 \leq \int_Q f_n G_h(u_{n,k}) + \frac{1}{2}\int_\Omega u_0^2.$$

From now, we can follow the standard technique used for the non-singular case in [11] to get $u_{n,k} \in L^\infty(Q)$. Furthermore, the estimate of $u_{n,k} \in L^\infty(Q)$ is independent from $k \in \mathbb{N}$, then for k large enough and for n fixed, $u_n \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(Q)$ is the solution of the following approximate problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div}((a(x,t) + u_n^q)\nabla u_n) = \frac{f_n}{(|u_n| + \frac{1}{n})^\gamma} & \text{in} \quad Q, \\ u_n(x,t) = 0 & \text{on} \quad \Gamma, \\ u_n(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}$$

Since $\frac{f_n}{(|u_n| + \frac{1}{n})^\gamma} \geq 0$. The maximum principle implies that $u_n \geq 0$, and this concludes the proof. □

Lemma 3.3. *Let u_n be a solution of (3.7). Then for every $\omega \subset\subset \Omega$ there exists $C_\omega > 0$ independent on n such that $u_n \geq C_\omega$ in $\omega \times (0,T)$, $\forall n \in \mathbb{N}$.*

Proof. Define for $s \geq 0$ the function

$$\psi_\delta(s) = \begin{cases} 1 & if \quad 0 \leq s \leq 1, \\ \frac{1}{\delta}(1 + \delta - s) & if \quad 1 \leq s \leq \delta + 1, \\ 0 & if \quad s > \delta + 1. \end{cases}$$

We choose $\psi_\delta(u_n)\varphi$ as test function in (3.7) with $\varphi \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(\Omega)$, $\varphi \geq 0$ then we have

$$\begin{aligned} &\int_0^T \int_\Omega \frac{\partial u_n}{\partial t}\psi_\delta(u_n)\varphi + \int_Q (a(x,t) + u_n^q)\nabla u_n \nabla\varphi\psi_\delta(u_n) \\ = \quad &\frac{1}{\delta}\int_{\{1 \leq u_n \leq \delta+1\}} (a(x,t) + u_n^q)|\nabla u_n|^2\varphi + \int_Q \frac{f_n}{(u_n + \frac{1}{n})^\gamma}\psi_\delta(u_n)\varphi, \end{aligned}$$

thus, dropping the non-negative term $\frac{1}{\delta}\int_{\{1 \leq u_n \leq \delta+1\}}(a(x,t) + u_n^q)|\nabla u_n|^2\varphi$, and letting δ goes to zero, we obtain

$$\begin{aligned} \int_0^T \int_\Omega \frac{\partial u_n}{\partial t}\chi_{\{0 \leq u_n < 1\}}\varphi + \int_Q (a(x,t) + u_n^q)\nabla u_n \cdot \nabla\varphi\chi_{\{0 \leq u_n < 1\}} \\ \geq \int_Q \frac{f_n}{(u_n + \frac{1}{n})^\gamma}\varphi\chi_{\{0 \leq u_n < 1\}}. \end{aligned}$$

Then for the last inequality we can write as follows

$$\int_0^T \int_\Omega \frac{\partial T_1(u_n)}{\partial t}\varphi + \int_Q (a(x,t) + T_1(u_n)^q)\nabla T_1(u_n)\nabla\varphi \geq \int_Q \frac{f}{2^\gamma(1+f)}\varphi\chi_{\{0\leq u_n<1\}},$$

for all $0 \leq \varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$. Since $\frac{f}{2^\gamma(1+f)}\chi_{\{0\leq u_n<1\}}$ not identically zero and $\alpha \leq a(x,t) + T_1(u_n)^q \leq \beta + 1$, then we have

$$\int_0^T \int_\Omega \frac{\partial T_1(u_n)}{\partial t}\varphi + (\beta+1)\int_Q \nabla T_1(u_n).\nabla\varphi \geq 0. \tag{3.8}$$

This yields that $v_n = T_1(u_n)$ is a weak solution of the variational inequality

$$\begin{cases} \frac{1}{\beta+1}\frac{\partial v_n}{\partial t} - \triangle v_n \geq 0 & \text{in} \quad Q, \\ v_n(x,t) = 0 & \text{on} \quad \Gamma, \\ v_n(x,0) = T_1(u_0(x)) & \text{in} \quad \Omega. \end{cases}$$

We are going to prove that

$$\forall\, \omega \subset\subset \Omega,\ \exists\, C_\omega > 0: \quad v_n(x,t) \geq C_\omega \text{ in } \ \omega \times (0,T),\ \forall n \in \mathbb{N}. \tag{3.9}$$

Let w_n be the solution of the following problem

$$\begin{cases} \frac{1}{\beta+1}\frac{\partial w_n}{\partial t} - \triangle w_n = 0 & \text{in} \quad Q, \\ w_n(x,t) = 0 & \text{on} \quad \Gamma, \\ w_n(x,0) = v_n(x,0) & \text{in} \quad \Omega. \end{cases} \tag{3.10}$$

From (3.8) v_n is a supersolution of (3.10), we have $v_n \geq w_n$, so that we only have to prove that

$$\forall\, \omega \subset\subset \Omega,\ \exists\, C_\omega > 0: \quad w_n(x,t) \geq C_\omega \text{ in } \ \omega \times (0,T),\ \forall n \in \mathbb{N}. \tag{3.11}$$

Since by (3.2)

$$\forall\, \omega \subset\subset \Omega,\ \exists\, d_\omega > 0: \quad w_n(x,0) = v_n(x,0) \geq d_\omega \text{ in } \ \omega \times (0,T),\ \forall n \in \mathbb{N}. \tag{3.12}$$

For the rest of the proof we can argue as Boccardo, Orsina and Porzio in [27] (see pp $414 - 416$), we deduce that there exists $C_\omega > 0$ such that $w_n \geq C_\omega$ in $\omega \times (0,T)$, $\forall \omega \subset\subset \Omega$, since $v_n \geq w_n$, then $T_1(u_n) = v_n \geq C_\omega$ in $\omega \times (0,T)$, $\forall \omega \subset\subset \Omega$. As $u_n \geq T_1(u_n) = v_n$, then we obtain

$$u_n \geq C_\omega \text{ in } \omega \times (0,T), \quad \forall \omega \subset\subset \Omega,\ \forall n \in \mathbb{N}.$$

□

3 A priori estimates and main results

Case $\gamma < 1$.

Lemma 3.4. *Let u_n be a solution of (3.7), with $\gamma < 1$ and $q > 1-\gamma$. Assume that $f \in L^1(Q)$, then u_n is bounded in $L^2(0,T;H_0^1(\Omega))$.*

Proof. For n fixed, we choose $\epsilon < \frac{1}{n}$ and using $\phi(u_n) = ((u_n+\epsilon)^\gamma - \epsilon^\gamma)(1-(1+u_n)^{1-(q+\gamma)})$ as test function, then we have

$$\begin{aligned}&\int_\Omega \Psi(u_n(x,t)) + \gamma\int_Q (u_n+\epsilon)^{\gamma-1}(1+u_n)^{1-(q+\gamma)}(a(x,t)+u_n^q)|\nabla u_n|^2\\&+(q+\gamma-1)\int_Q((u_n+\epsilon)^\gamma-\epsilon^\gamma)(a(x,t)+u_n^q)\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}}\\&=\int_Q\frac{f_n}{(u_n+\frac{1}{n})^\gamma}((u_n+\epsilon)^\gamma-\epsilon^\gamma)(1-(1+u_n)^{1-(q+\gamma)})+\int_\Omega\Psi(u_0),\end{aligned} \tag{3.13}$$

where $\Psi(s) = \displaystyle\int_0^s \phi(\ell)d\ell$. Dropping the first and second non-negative terms in the left hand side of (3.13), since $u_0 \in L^\infty(\Omega)$ and using (3.3), $\epsilon < \frac{1}{n}$ we have

$$\begin{aligned}&(q+\gamma-1)\int_Q((u_n+\epsilon)^\gamma-\epsilon^\gamma)(a(x,t)+u_n^q)\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}}\\&\le\int_Q\frac{f_n}{(u_n+\frac{1}{n})^\gamma}((u_n+\frac{1}{n})^\gamma-\epsilon^\gamma)(1-(1+u_n)^{1-(q+\gamma)})\le\int_Q f+C,\end{aligned} \tag{3.14}$$

and passing to the limit on ϵ, we get

$$\int_Q(\alpha u_n^\gamma+u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}}\le C\int_Q f+C. \tag{3.15}$$

By working in $\{u_n \ge 1\}$, we have

$$\int_{\{u_n\ge1\}}(\alpha+u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}}\le\int_Q(\alpha u_n^\gamma+u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}},$$

then it follows from (3.15) that

$$\frac{\min(\alpha,1)}{2^{q+\gamma-1}}\int_{\{u_n\ge1\}}|\nabla u_n|^2\le\min(\alpha,1)\int_{\{u_n\ge1\}}\frac{1+u_n^{q+\gamma}}{(1+u_n)^{q+\gamma}}|\nabla u_n|^2\le C\int_Q f+C.$$

we can deduce that

$$\int_{\{u_n\ge1\}}|\nabla u_n|^2\le C. \tag{3.16}$$

Now, we choose $(T_k(u_n)+\epsilon)^\gamma-\epsilon^\gamma$ as a test function with $\epsilon<\frac{1}{n}$ in (3.7), by (3.3) and dropping the nonnegative terms, we get

$$\begin{aligned}
&\alpha\int_Q \frac{|\nabla T_k(u_n)|^2}{(T_k(u_n)+\epsilon)^{1-\gamma}}\\
&\leq \int_Q \frac{f_n}{(u_n+\frac{1}{n})^\gamma}((T_k(u_n)+\epsilon)^\gamma-\epsilon^\gamma)+\frac{1}{\gamma+1}\int_\Omega (T_k(u_0)+\epsilon)^{\gamma+1}-\epsilon^\gamma\int_\Omega u_0 \qquad (3.17)\\
&\leq \int_Q f+\frac{1}{\gamma+1}\int_\Omega (T_k(u_0)+\epsilon)^{\gamma+1}-\epsilon^\gamma\int_\Omega u_0.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_Q |\nabla T_k(u_n)|^2 &= \int_Q \frac{|\nabla T_k(u_n)|^2}{(T_k(u_n)+\epsilon)^{1-\gamma}}(T_k(u_n)+\epsilon)^{1-\gamma}\\
&\leq (k+\epsilon)^{1-\gamma}\int_Q \frac{|\nabla T_k(u_n)|^2}{(T_k(u_n)+\epsilon)^{1-\gamma}}\\
&\leq (k+\epsilon)^{1-\gamma}\left[\int_Q f+\frac{1}{\gamma+1}\int_\Omega (T_k(u_0)+\epsilon)^{\gamma+1}-\epsilon^\gamma\int_\Omega u_0\right].
\end{aligned}$$

By the fact that $u_0\in L^\infty(\Omega)$ and letting ϵ goes to zero, implies that

$$\int_Q |\nabla T_k(u_n)|^2\leq Ck^{1-\gamma}. \qquad (3.18)$$

Combining (3.16) and (3.18) we obtain

$$\int_Q |\nabla u_n|^2=\int_{\{u_n\geq 1\}}|\nabla u_n|^2+\int_{\{u_n\leq 1\}}|\nabla u_n|^2\leq C.$$

Hence by last inequality we deduce that u_n is bounded in $L^2(0,T;H_0^1(\Omega))$ with respect to n. □

Lemma 3.5. *Let u_n be a solution of problem (3.7), with $\gamma<1$ and $q\leq 1-\gamma$. Suppose that f belong to $L^1(Q)$, then u_n is bounded in $L^r(0,T;W_0^{1,r}(\Omega))$; with $r=\frac{N(q+\gamma+1)}{N-(1-(q+\gamma))}$.*

Proof. For n fixed, we choose $\epsilon<\frac{1}{n}$ and using $\psi(u_n)=(u_n+\epsilon)^\gamma-\epsilon^\gamma$ as test function in (3.7), we obtain

$$\begin{aligned}
&\int_\Omega \Psi(u_n(x,t))+\gamma\int_Q (a(x,t)+u_n^q)(u_n+\epsilon)^{\gamma-1}|\nabla u_n|^2\\
&\qquad=\int_Q \frac{f_n}{(u_n+\frac{1}{n})^\gamma}((u_n+\epsilon)^\gamma-\epsilon^\gamma)+\int_\Omega \Psi(u_0),
\end{aligned}$$

where $\Psi(s)=\displaystyle\int_0^s \psi(\ell)d\ell$. By removing the first nonnegative terms and using (3.3), $u_0\in L^\infty(\Omega)$, since $q\leq 1-\gamma<1, \epsilon<\frac{1}{n}<1$ and by the fact that

$$\min(\alpha,1)(u_n+\epsilon)^q\leq \min(\alpha,1)(u_n+1)^q\leq \min(\alpha,1)(1+u_n^q)\leq \alpha+u_n^q\leq a(x,t)+u_n^q,$$

we have

$$\begin{aligned}\gamma\min(\alpha,1)\int_Q(u_n+\epsilon)^{q+\gamma-1}|\nabla u_n|^2 &\le \gamma\int_Q(\alpha+u_n^q)(u_n+1)^{\gamma-1}|\nabla u_n|^2\\ &\le \int_Q\frac{f_n}{(u_n+\frac{1}{n})^\gamma}((u_n+\epsilon)^\gamma-\epsilon^\gamma)\le\int_Q f+C.\end{aligned}$$

If $q=1-\gamma$, then u_n is bounded in $L^2(0,T;H_0^1(\Omega))$ with respect to n.
If $q<1-\gamma$, then applying Sobolev inequality, we have

$$\left(\int_Q((u_n+\epsilon)^{\frac{q+\gamma+1}{2}}-\epsilon^{\frac{q+\gamma+1}{2}})^{2^*}\right)^{\frac{2}{2^*}}\le C\int_Q|\nabla(u_n+\epsilon)^{\frac{q+\gamma+1}{2}}|^2\le C\int_Q f+C, \tag{3.19}$$

letting $\epsilon\to0$, then (3.19) implies

$$\int_Q u_n^{\frac{2^*(q+\gamma+1)}{2}}\le C. \tag{3.20}$$

Therefore, u_n is bounded in $L^{\frac{N(q+1+\gamma)}{N-2}}(Q)$ with respect to n.
Now, if $r<2$ as in the statement of Lemma 3.5, we have by the Hölder inequality

$$\begin{aligned}\int_Q|\nabla u_n|^r &= \int_Q\frac{|\nabla u_n|^r}{(u_n+\epsilon)^{(1-(q+\gamma))\frac{r}{2}}}(u_n+\epsilon)^{(1-(q+\gamma))\frac{r}{2}}\\ &\le\left(\int_Q\frac{|\nabla u_n|^2}{(u_n+\epsilon)^{1-(q+\gamma)}}\right)^{\frac{r}{2}}\left(\int_Q(u_n+\epsilon)^{(1-(q+\gamma))\frac{r}{2-r}}\right)^{1-\frac{r}{2}}\\ &\le C\left(\int_Q(u_n+\epsilon)^{(1-(q+\gamma))\frac{r}{2-r}}\right)^{1-\frac{r}{2}}.\end{aligned}$$

Thanks to (3.20), the value of r is such that $\frac{(1-(q+\gamma))r}{2-r}=\frac{N(q+\gamma+1)}{N-2}$, so that the right hand side of the above inequality is bounded, and then

$$\int_Q|\nabla u_n|^r\le M, \tag{3.21}$$

where M is a positive constant independent of n. Then u_n is bounded in $L^r(0,T;W_0^{1,r}(\Omega))$ with respect to n, with $r=\frac{N(q+\gamma+1)}{N-(1-(q+\gamma))}$ as desired. □

Remark 3.6. As consequence of both Lemma 3.5, there exists a sub-sequence (not relabeled) and a function u such that u_n converge weakly to u in $L^r(0,T;W_0^{1,r}(\Omega))$ (with $r=\frac{N(q+1+\gamma)}{N-(1-(q+\gamma))}$) and almost everywhere in Q as $n\to\infty$.

In the next lemma we give an estimate of $u_n^q|\nabla u_n|$ in $L^\rho(Q)$ for any $\rho<\frac{N}{N-1}$.

Lemma 3.7. *Let u_n be a solution of problem (3.7), with $\gamma<1$. Suppose that $f\in L^1(Q)$, then $u_n^q|\nabla u_n|$ is bounded in $L^\rho(Q)$ for every $\rho<\frac{N}{N-1}$.*

Proof. For n fixed, we choose $\epsilon < \frac{1}{n}$ and we take as test function $\psi(u_n) = ((T_1(u_n)+\epsilon)^\gamma - \epsilon^\gamma)(1-(1+u_n)^{1-\lambda})$, with $\lambda > 1$, we have

$$\begin{aligned}
&\int_\Omega \Psi(u_n(x,t)) + \gamma \int_Q (T_1(u_n)+\epsilon)^{\gamma-1}(1-(1+u_n)^{1-\lambda})(a(x,t)+u_n^q)|\nabla T_1(u_n)|^2 \\
&+ (\lambda-1)\int_Q (T_1(u_n)+\epsilon)^\gamma - \epsilon^\gamma)(a(x,t)+u^q)\frac{|\nabla u_n|^2}{(1+u_n)^\lambda} \\
&= \int_Q \frac{f_n}{(u_n+\frac{1}{n})^\gamma}((T_1(u_n)+\epsilon)^\gamma - \epsilon^\gamma)(1-(1+u_n)^{1-\lambda}) + \int_\Omega \Psi(u_0),
\end{aligned} \tag{3.22}$$

where $\Psi(s) = \displaystyle\int_0^s \psi(\sigma)d\sigma$. In the following, we ignore the first and second non-negative terms in the left hand side of (3.22), using (3.3) and the fact that $\alpha + u_n^q \geq c_0(1+u_n)^q$ yield

$$\begin{aligned}
&(\lambda-1)c_0\int_Q ((T_1(u_n)+\epsilon)^\gamma - \epsilon^\gamma)(1+u_n)^{q-\lambda}|\nabla u_n|^2 \\
&\leq \int_Q \frac{f_n}{(u_n+\frac{1}{n})^\gamma}((T_1(u_n)+\epsilon)^\gamma - \epsilon^\gamma)(1-(1+u_n)^{1-\lambda}) + \int_\Omega \Psi(u_0).
\end{aligned} \tag{3.23}$$

Letting ϵ goes to zero and using the fact that $u_0 \in L^\infty(\Omega)$, then (3.23) becomes

$$\int_{\{u_n \geq 1\}} (1+u_n)^{q-\lambda}|\nabla u_n|^2 \leq \int_Q T_1(u_n)(1+u_n)^{q-\lambda}|\nabla u_n|^2 \leq C\int_Q f + C. \tag{3.24}$$

Combining (3.18) and (3.24) lead to

$$\begin{aligned}
\int_Q (1+u_n)^{q-\lambda}|\nabla u_n|^2 &= \int_{\{u_n \geq 1\}} (1+u_n)^{q-\lambda}|\nabla u_n|^2 \\
&+ \int_{\{u_n \leq 1\}} (1+u_n)^{q-\lambda}|\nabla u_n|^2 \leq C.
\end{aligned}$$

Now, let $\rho = \frac{N(2+q-\lambda)}{N(q+1)-(\lambda+q)}$ and using the previous result together with Hölder inequality, we have

$$\int_Q u_n^{q\rho}|\nabla u_n|^\rho \leq \int_Q (1+u_n)^{\frac{\rho(q+\lambda)}{2}} \frac{|\nabla u_n|^\rho}{(1+u_n)^{\frac{\rho(\lambda-q)}{2}}} \leq C\left(\int_Q (1+u_n)^{\frac{\rho(q+\lambda)}{2-\rho}}\right)^{\frac{2-\rho}{2}},$$

and by Sobolev inequality, we get

$$\left(\int_Q u_n^{\rho^*(q+1)}\right)^{\frac{\rho}{\rho^*}} \leq C\left(\int_Q u_n^{\frac{\rho(q+\lambda)}{2-\rho}}\right)^{\frac{2-\rho}{2}},$$

the previous choice of ρ implies that $\rho^*(q+1) = \rho(q+\lambda)/(2-\rho)$, and since $\lambda > 1$, we obtain an estimate of $u_n^q|\nabla u_n|$ in $L^\rho(Q)$ for every $\rho < N/(N-1)$, as desired. □

In order to pass to the limit in the approximate equations, the almost everywhere convergence of the ∇u_n to ∇u is required, this result will be proved following the same techniques as in [30] (see also [114]).

Lemma 3.8. *The sequence* $\{\nabla u_n\}$ *converges to* ∇u *a.e. in* Q.

Proof. Let $\varphi \in C_c^1(\Omega), \varphi \geq 0$ independent of $t \in [0,T]$ $\varphi \equiv 1$ on $w \subset Supp\varphi \subset\subset \Omega$ and using $T_h(u_n - T_k(u))\varphi$ as a test function in (3.7)

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial u_n}{\partial t} T_h(u_n - T_k(u))\varphi + \int_0^T \int_\Omega (a(x,t) + u_n^q)\varphi \nabla u_n \nabla T_h(u_n - T_k(u)) \\
&+ \int_0^T \int_\Omega (a(x,t) + u_n^q) T_h(u_n - T_k(u)) \nabla u_n \nabla \varphi \\
&= \int_0^T \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\gamma} T_h(u_n - T_k(u))\varphi.
\end{aligned} \tag{3.25}$$

Since $w = Supp\varphi \subset\subset \Omega$ and by Lemma 3.3 we have $u_n \geq C_{Supp\varphi}$, then we the above equality becomes

$$\begin{aligned}
&\frac{1}{2}\int_\Omega T_h^2(u_n - T_k(u))\varphi + \int_0^T \int_\Omega (a(x,t) + u_n^q)|\nabla T_h(u_n - T_k(u))|^2 \varphi \\
&\leq Ch||\nabla\varphi||_{L^\infty} + h||\varphi||_{L^\infty} \frac{1}{C_{Supp\varphi}^\gamma} \int_0^T \int_{Supp\varphi} f + \frac{1}{2}\int_\Omega T_h^2(u_0 - T_k(u_0))\varphi \\
&- \int_0^T \int_\Omega (a(x,t) + u_n^q) \nabla T_h(u) \nabla T_h(u_n - T_k(u))\varphi +,
\end{aligned} \tag{3.26}$$

by removing the first non-negative term, we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega (a(x,t) + u_n^q)|\nabla T_h(u_n - T_k(u))|^2 \varphi \\
&\leq Ch||\nabla\varphi||_{L^\infty} + h||\varphi||_{L^\infty} \frac{1}{C_{Supp\varphi}^\gamma} \int_0^T \int_{Supp\varphi} f + \frac{1}{2}h^2 \text{meas}(\Omega) \\
&- \int_0^T \int_\Omega (a(x,t) + u_n^q) \nabla T_h(u) \nabla T_h(u_n - T_k(u))\varphi.
\end{aligned} \tag{3.27}$$

Since $\nabla T_h(u_n - T_k(u)) \neq 0$ (which implies that $u_n \leq h + k$), we can easily to pass the limit as n tends to ∞, thanks to Remark 3.6, in the right hand side of the above inequality, so that

$$\alpha \limsup_{n\to\infty} \int_0^T \int_\Omega |\nabla T_h(u_n - T_k(u))|^2 \leq Ch. \tag{3.28}$$

Let now s be such that $s < r < 2$, where r is in the statement of Lemma 3.5

$$\begin{aligned}
&\int_0^T \int_w |\nabla u_n - \nabla u|^s \leq \int_0^T \int_\Omega |\nabla u_n - \nabla u|^s \varphi \\
&= \int_{\{|u_n - u| \leq h, u \leq k\}} |\nabla u_n - \nabla u|^s \varphi + \int_{\{|u_n - u| \leq h, u > k\}} |\nabla u_n - \nabla u|^s \varphi \\
&+ \int_{\{|u_n - u| > h\}} |\nabla u_n - \nabla u|^s \varphi.
\end{aligned} \tag{3.29}$$

From (3.21), we have

$$\begin{aligned}\int_0^T\int_\Omega |\nabla u_n-\nabla u|^s\varphi \le \int_0^T\int_\Omega |\nabla T_h(u_n-T_k(u))|^s\varphi \\ +||\varphi||_{L^\infty}\left(2^sM^s(meas\{u>k\})^{1-\frac{r}{s}}+2^sM^s(meas\{|u_n-u|>h\})^{1-\frac{r}{s}}\right).\end{aligned} \tag{3.30}$$

Thus, combining (3.28) and (3.29), we obtain for every $h>0$ and every $k>0$

$$\begin{aligned}\limsup_{n\to\infty}\int_0^T\int_\Omega |\nabla u_n-\nabla u|^s\varphi \le \left(\frac{2h}{\alpha}\int_0^T\int_\Omega\right)^{\frac{s}{2}}||\varphi||_{L^\infty}meas(Q)^{1-\frac{s}{2}} \\ +||\varphi||_{L^\infty}2^sM^s(meas\{u>k\})^{1-\frac{s}{r}}.\end{aligned} \tag{3.31}$$

Letting h tends to zero and k tends to infinity, we finally that

$$\limsup_{n\to\infty}\int_0^T\int_\Omega |\nabla u_n-\nabla u|^s\varphi=0,\quad \forall s<2.$$

Therefore, up to sub sequence, $\{\nabla u_n\}$ converges to ∇u a.e., and Lemma 3.8 is completely proved. □

Now we are in position to prove our existence result given by

Theorem 3.9. *Let $\gamma<1$ and f be nonnegative function in $L^1(Q)$, then there exists a nonnegative solution u of problem (3.1) in the sense of Definition 3.1. Moreover, u belong to $L^2(0,T;H_0^1(\Omega))$ if $q>1-\gamma$ and it belongs to $L^r(0,T;W_0^{1,r}(\Omega))$ (with r as in the statement of Lemma 3.5) if $q\le 1-\gamma$.*

Proof. As we have already said (see Remark 3.6), there exists a function $u\in L^r(0,T;W_0^{1,r}(\Omega))$, such that u_n converges weakly to u in $L^r(0,T;W_0^{1,r}(\Omega))$.
By Lemma 3.3, we have $\frac{f_n}{(u_n+\frac{1}{n})^\gamma}$ is bounded in $L^1(0,T;L^1_{loc}(\Omega))$ and Lemma 3.7 gives $(a(x,t)+u_n^q)|\nabla u_n|$ is bounded in $L^\rho(Q)$, $\rho<\frac{N}{N-1}<2$ then $\operatorname{div}((a(x,t)+u_n^q)\nabla u_n)$ is bounded $L^{\rho'}(Q)\subset L^2(Q)\subset L^2(0,T;H^{-1}(\Omega))$, then we deduce $\{\frac{\partial u_n}{\partial t}\}_n$ is bounded in $L^1(0,T;L^1_{loc}(\Omega))+L^2(0,T;H^{-1}(\Omega))$, using compactness argument in [139], we deduce that

$$u_n\longrightarrow u\quad \text{strongly in } L^1(Q). \tag{3.32}$$

On the other hand, Lemma 3.7, Lemma 3.8 and Remark 3.6 imply that the sequence $u_n^q|\nabla u_n|$ converges weakly to $u^q|\nabla u|$ in $L^\rho(Q)$ for every $\rho<\frac{N}{N-1}$. Hence for every $\varphi\in C_c^1(\Omega\times[0,T))$

$$\lim_{n\to\infty}\int_Q (a(x,t)+u_n^q)\nabla u_n\cdot\nabla\varphi=\int_Q(a(x,t)+u^q)\nabla u\cdot\nabla\varphi. \tag{3.33}$$

For the limit of the right hand of (3.7). Let $w=\{\varphi\neq 0\}$, then by Lemma 6.2, one has, for every $\varphi\in C_c^1(\Omega\times[0,T))$

$$\left|\frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}\right|\le\frac{||\varphi||_{L^\infty}}{C_w^\gamma}f, \tag{3.34}$$

then by Remark 3.6, (3.34) and dominated convergence theorem, we get

$$\frac{f_n}{(u_n+\frac{1}{n})^\gamma}\longrightarrow\frac{f}{u^\gamma}\quad \text{strongly in } L^1_{loc}(Q). \tag{3.35}$$

Let $\varphi \in C_c^1(\Omega \times [0,T))$ as test function in (3.7), by (3.32), (3.33), (3.34), (3.35) and letting $n \to +\infty$, we obtain

$$\int_\Omega u_0(x)\varphi(x,0) - \int_Q u\frac{\partial \varphi}{\partial t} + \int_Q (a(x,t)+u^q)\nabla u.\nabla\varphi = \int_Q \frac{f}{u^\gamma}\varphi. \tag{3.36}$$

Hence, we conclude that the solution u satisfies the conditions (3.4), (3.5) and (3.6) of Definition 3.1, so that the proof of Theorem 3.9 is now completed. □

Case $\gamma = 1$.

Lemma 3.10. *Let u_n be a solution of problem (3.7), with $\gamma = 1$. Suppose that f belongs to $L^1(Q)$. Then u_n is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^{\frac{N(q+2)}{N-2}}(Q)$.*

Proof. we use $u_n\chi_{(0,t)}$ as test function in (3.7) and by (3.3), we obtain

$$\frac{1}{2}\int_\Omega |u_n(x,t)|^2 + \alpha\int_0^t\int_\Omega |\nabla u_n|^2 + \int_0^t\int_\Omega u_n^q|\nabla u_n|^2 \leq \int_0^t\int_\Omega f_n + \frac{1}{2}\int_\Omega u_0^2,$$

as $f_n \leq f$ and $u_0 \in L^\infty(\Omega)$, passing to supremum for $t \in (0,T)$ in the above estimate, we get

$$\begin{aligned} &\frac{1}{2}||u_n||_{L^\infty(0,T;L^2(\Omega))} + \alpha\int_Q |\nabla u_n|^2 + \int_Q u_n^q|\nabla u_n|^2 \\ &\leq \int_Q f + \frac{1}{2}||u_0||^2_{L^2(\Omega)} \leq C. \end{aligned} \tag{3.37}$$

This implies that

$$||u_n||_{L^\infty(0,T;L^2(\Omega))} \leq C \quad \text{and} \quad ||u_n||_{L^2(0,T;H_0^1(\Omega))} \leq C. \tag{3.38}$$

In the other hand by Sobolev embedding Theorem and from (3.37), we can get

$$\int_Q u_n^{\frac{(q+2)2^*}{2}} \leq \frac{4S}{(q+2)^2}\int_Q |\nabla u_n^{\frac{q+2}{2}}|^2 \leq \int_Q f + \frac{1}{2}||u_0||^2_{L^2(\Omega)} \leq C.$$

where S the constant of Sobolev embedding, hence the above estimate implies that the boundedness of u_n in $L^{\frac{N(q+2)}{N-2}}(Q)$. Then the proof of Lemma 3.10 is completed. □

Lemma 3.11. *Let u_n be a solution of problem (3.7), with $\gamma = 1$. Suppose that $f \in L^1(Q)$, then $u_n^q|\nabla u_n|$ is bounded in $L^\rho(Q)$ for every $\rho < N/(N-1)$.*

Proof. We take $\varphi(u_n) = T_1(u_n)(1-(1+u_n)^{1-\lambda})$, with $\lambda > 1$, as test function in (3.7), we obtain

$$\begin{aligned} &\int_\Omega \psi(u_n) + \gamma\int_Q T_1(u_n)(1-(1+u_n)^{1-\lambda})(a(x,t)+u_n^q)|\nabla T_1(u_n)|^2 \\ &+ (\lambda-1)\int_Q T_1(u_n)(a(x,t)+u_n^q)\frac{|\nabla u_n|^2}{(1+u_n)^\lambda} \\ &= \int_Q \frac{f_n}{u_n+\frac{1}{n}}T_1(u_n)(1-(1+u_n)^{1-\lambda}) + \int_\Omega \psi(u_0), \end{aligned}$$

where $\psi(s) = \displaystyle\int_0^s \varphi(\ell)d\ell$. Dropping the non-negative terms, from (3.3) and by the fact that $u_0 \in L^\infty(\Omega)$, $\alpha + u_n^q \geq c_0(1+u_n)^q$, we have

$$\int_Q T_1(u_n)(1+u_n)^{q-\lambda}|\nabla u_n|^2 \leq C\int_Q f + C.$$

By working in the set $\{u_n \geq 1\}$ and using the above estimate, we get

$$\int_{\{u_n\geq 1\}} (1+u_n)^{q-\lambda}|\nabla u_n|^2 \leq \int_Q T_1(u_n)^\gamma(1+u_n)^{q-\lambda}|\nabla u_n|^2 \leq C\int_Q f + C. \tag{3.39}$$

The inequality (3.38) with (3.39), yields

$$\int_Q (1+u_n)^{q-\lambda}|\nabla u_n|^2 = \int_{\{u_n\geq 1\}} (1+u_n)^{q-\lambda}|\nabla u_n|^2 + \int_{\{u_n< 1\}} (1+u_n)^{q-\lambda}|\nabla u_n|^2 \leq C. \tag{3.40}$$

Now let us fix $\rho = \frac{N(2+q-\lambda)}{N(q+1)-(\lambda+q)}$, by Hölder's inequality and (3.40), we have

$$\int_Q u_n^{q\rho}|\nabla u_n|^\rho = \int_Q \frac{|\nabla u_n|^\rho}{(1+u_n)^{\frac{\rho(\lambda-q)}{2}}}(1+u_n)^{\frac{\rho(\lambda+q)}{2}} \leq C\left(\int_Q (1+u_n)^{\frac{\rho(q+\lambda)}{2-\lambda}}\right)^{\frac{2-\lambda}{2}},$$

applying Sobolev inequality and using the above estimate, we deduce

$$\left(\int_Q u_n^{\rho^*(q+1)}\right)^{\frac{\rho}{\rho^*}} \leq C\left(\int_Q u_n^{\frac{\rho(q+\lambda)}{2-\lambda}}\right)^{\frac{2-\lambda}{2}}.$$

The previous choice of ρ implies that $\rho^*(q+1) = \frac{\rho(q+\lambda)}{2-\rho}$, and since $\lambda > 1$, we obtain an estimate of $u_n^q|\nabla u_n|$ in $L^\rho(Q)$ for every $\rho < N/(N-1)$. □

Theorem 3.12. *Let $\gamma = 1$ and f be a function in $L^1(Q)$. Then there exists a solution u in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^{\frac{N(q+2)}{N-2}}(Q)$ of problem (3.1) in the sense of Definition 3.1.*

Proof. By Lemmas 3.3, 3.8, 3.10 and 3.11, the proof of Theorem 3.12 is identical to the of one Theorem 3.9. □

The strongly singular case $\gamma > 1$.

In this case we do not have an estimate on u_n in $L^2(0,T;H_0^1(\Omega))$, but we can prove that u_n is bounded in $L^2(0,T;H_{loc}^1(\Omega))$ such that $u^{\frac{q+\gamma+1}{2}} \in L^2(0,T;H_0^1(\Omega))$.

Lemma 3.13. *Let u_n be a solution of the problem (3.7), with $\gamma > 1$. Suppose that f belongs to $L^1(Q)$, then $u_n^{\frac{q+\gamma+1}{2}}$ is bounded in $L^2(0,T;H_0^1(\Omega))$, and u_n is bounded in $L^2(0,T;H_{loc}^1(\Omega)) \cap L^\infty(0,T;L^{\gamma+1}(\Omega))$. Moreover if $q \leq \gamma - 1$, then $u_n^q|\nabla u_n|$ is bounded in $L^2(w \times (0,T))$ for every $w \subset\subset \Omega$.*

Proof. Choosing $u_n^\gamma \chi_{(0,t)}$, as test function in (3.7) with $(0 < t \leq T)$. Since $0 \leq \frac{u_n^\gamma}{(u_n+\frac{1}{n})^\gamma} \leq 1$, recalling that (3.3), the fact that $0 \leq f_n \leq f$ and the dropping the non-negative term, we have;

$$\frac{1}{\gamma+1}\int_\Omega u_n(x,t)^{\gamma+1} + \gamma\int_0^t\int_\Omega u_n^{q+\gamma-1}|\nabla u_n|^2$$
$$\leq \int_0^t\int_\Omega \frac{f_n u_n^\gamma}{(u_n+\frac{1}{n})^\gamma} + \frac{1}{\gamma+1}\int_\Omega u_0^{\gamma+1} \leq \int_0^t\int_\Omega f + \frac{1}{\gamma+1}\int_\Omega u_0^{\gamma+1}.$$

Since $u_0 \in L^\infty(\Omega)$ and passing to supremum in $t \in [0,T]$, we obtain

$$\frac{1}{\gamma+1}||u_n||_{L^\infty(0,T;L^{\gamma+1}(\Omega))} + \gamma\int_Q u_n^{q+\gamma-1}|\nabla u_n|^2 \leq \int_Q f + \frac{1}{\gamma+1}||u_0||_{L^{\gamma+1}(\Omega)}^{\gamma+1}, \tag{3.41}$$

then we get

$$\frac{4}{(q+\gamma+1)^2}\int_Q |\nabla u_n^{\frac{q+\gamma+1}{2}}|^2 = \int_Q u_n^{q+\gamma-1}|\nabla u_n|^2 \leq \int_Q f + \frac{1}{\gamma+1}||u_0||_{L^{\gamma+1}(\Omega)}^{\gamma+1},$$

hence

$$\int_Q |\nabla u_n^{\frac{q+\gamma+1}{2}}|^2 \leq C.$$

The last inequality and (3.41), imply that $u_n^{\frac{q+\gamma+1}{2}}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and u_n is bounded in $L^\infty(0,T;L^{\gamma+1}(\Omega))$ with respect to n. We choose now $\varphi(u_n) = u_n^\gamma(1-(1+u_n)^{1-(q+\gamma)})$ as test function, dropping the non-negative terms, from (3.3), we have

$$(q+\gamma-1)\int_Q u_n^\gamma(\alpha+u_n^q)\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}} \leq \int_Q \frac{f_n u_n^\gamma}{(u_n+\frac{1}{n})^\gamma} + \int_\Omega \Psi(u_0)$$
$$\leq \int_Q f + \int_\Omega \Psi(u_0),$$

where $\Psi(s) = \displaystyle\int_0^s \varphi(\ell)d\ell$. By working in the set $\{u_n \geq 1\}$ and the fact that $u_0 \in L^\infty(\Omega)$, we get

$$\int_{\{u_n\geq 1\}}(\alpha+u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}} \leq \int_Q(\alpha u_n^\gamma + u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(u_n+1)^{q+\gamma}}$$
$$\leq \int_Q f + C,$$

the above estimate implies

$$\tfrac{\min(\alpha,1)}{2^{q+\gamma-1}}\int_{\{u_n\geq 1\}}|\nabla u_n|^2 \leq \min(\alpha,1)\int_{\{u_n\geq 1\}}\frac{1+u_n^{q+\gamma}}{(1+u_n)^{q+\gamma}}|\nabla u_n|^2$$
$$\leq C\int_Q f + C,$$

then we get

$$\int_{\{u_n \geq 1\}} |\nabla u_n|^2 \leq C. \tag{3.42}$$

Now we take $(T_k(u_n))^\gamma$ as test function in (3.7), by (3.3), Lemma 3.3 and the fact that $\frac{T_k(u_n)^\gamma}{(u_n+\frac{1}{n})^\gamma} \leq \frac{u_n^\gamma}{(u_n+\frac{1}{n})^\gamma} \leq 1$, $u_0 \in L^\infty(\Omega)$ and dropping the nonnegative terms, we obtain

$$\begin{aligned}\alpha C_w^{\gamma-1} \int_0^T \int_w |\nabla T_k(u_n)|^2 &\leq \alpha \int_Q T_k(u_n)^{\gamma-1} |\nabla T_k(u_n)|^2 \\ \leq \int_Q f + \frac{1}{\gamma+1} \int_\Omega T_k(u_0)^{\gamma+1} &\leq \int_Q f + \frac{1}{\gamma+1} ||u_0||_{L^{\gamma+1}(\Omega)}^{\gamma+1},\end{aligned}$$

then we get that

$$\int_0^T \int_w |\nabla T_k(u_n)|^2 \leq C \quad \forall w \subset\subset \Omega. \tag{3.43}$$

Combining (3.42) and (3.43), we can deduce that

$$\int_0^T \int_w |\nabla u_n|^2 \leq \int_0^T \int_{w \cap \{u_n \geq 1\}} |\nabla u_n|^2 + \int_0^T \int_w |\nabla T_1(u_n)|^2 \leq C \tag{3.44}$$

$\forall w \subset\subset \Omega$, so that u_n is bounded in $L^2(0, T, H^1_{loc}(\Omega))$, as achieved. Now going back to (3.41), we have

$$\int_{\{u_n \geq 1\}} u_n^{q+\gamma-1} |\nabla u_n|^2 \leq \int_Q u_n^{q+\gamma-1} |\nabla u_n|^2 \leq \frac{1}{\gamma} \int_Q f + \frac{1}{\gamma(\gamma+1)} ||u_0||_{L^{\gamma+1}(\Omega)}^{\gamma+1}.$$

Then we obtain since $2q \leq q + \gamma - 1$

$$\begin{aligned}\int_0^T \int_w u_n^{2q} |\nabla u_n|^2 &\leq \int_0^T \int_{w \cap \{u_n \geq 1\}} u_n^{q+\gamma-1} |\nabla u_n|^2 \\ &+ \int_0^T \int_w |\nabla T_1(u_n)|^2 \leq C, \quad \forall w \subset\subset \Omega,\end{aligned} \tag{3.45}$$

then the last inequality implies that $u_n^q |\nabla u_n|$ is bounded in $L^2(w \times (0, T))$ for every $w \subset\subset \Omega$. □

Remark 3.14. We note that by virtue of Lemma 3.13 we easily deduce the almost everywhere convergence of ∇u_n to ∇u following exactly the same proofs as the one of Lemma 3.8.

Theorem 3.15. *Let $\gamma > 1, q \leq \gamma - 1$ and f be a nonnegative function in $L^1(Q)$. Then there exists a nonnegative solution $u \in L^2(0, T; H^1_{loc}(\Omega))$ of problem* (3.1) *in the sense of Definition 3.1. Moreover* $u^{\frac{q+\gamma+1}{2}} \in L^2(0, T; H^1_0(\Omega))$.

Proof. Thanks to Lemmas 3.3, 3.8, 3.13, the proof of Theorem 3.15 is identical to the one of Theorem 3.9. □

4 Regularity results

In this section we study the regularity results of solution of problem (3.1) depending on $q, \gamma > 0$ and the summability of f.

Theorem 3.16. *Let $\gamma < 1$, f be a nonnegative function in $L^m(Q)$, $1 < m < \frac{N}{2} + 1$. Then the solution found in Theorem 3.9, satisfies the following summabilities:*

(i) If $\frac{2(N+2-q)}{N(q+\gamma+1)+2(2-q)} \leq m < \frac{N}{2} + 1$, $q \leq 1 - \gamma$ then u belongs to $L^2(0, T; H_0^1(\Omega)) \cap L^\sigma(Q)$, where

$$\sigma = m\frac{N(q + \gamma + 1) + 2(\gamma + 1)}{N - 2m + 2}$$

(ii) If $1 < m < \frac{2(N+2-q)}{N(q+\gamma+1)+2(2-q)}$, $q > 1 - \gamma$ then u belongs to $L^r(0, T; W_0^{1,r}(\Omega)) \cap L^\sigma(Q)$, where

$$r = m\frac{N(q + \gamma + 1) + 2(\gamma + 1)}{N + 2 - m(1 - \gamma) + q(m - 1)}, \quad \sigma = m\frac{N(q + \gamma + 1) + 2(\gamma + 1)}{N - 2m + 2}$$

Proof. Let u_n be a solution of (3.7) given by Lemma 3.2, such that u_n converges to a solution of (3.1). We choose $\varphi(u_n) = ((u_n + 1)^\lambda - 1)\chi_{(0,t)}$, $(\lambda > 0)$ as test function in (3.7), we have

$$\begin{aligned} \int_\Omega \Psi(u_n(x, t)) + \lambda \int_0^t \int_\Omega (1 + u_n)^{\lambda-1}(a(x, t) + u_n^q)|\nabla u_n|^2 \\ \leq C \int_0^t \int_\Omega |f_n| u_n^{\lambda-\gamma} + \int_\Omega \Psi(u_0), \end{aligned}$$

where $\Psi(s) = \displaystyle\int_0^s \varphi(\ell)d\ell$.

From the condition (3.3) and the fact that $u_0 \in L^\infty(\Omega)$, $c_0(1 + u_n)^q \leq \alpha + u_n^q$, and applying Hölder's inequality, we obtain

$$\begin{aligned} \int_\Omega \Psi(u_n(x, t)) + \lambda c_0 \int_0^t \int_\Omega (1 + u_n)^{\lambda+q-1}|\nabla u_n|^2 \\ \leq C \left(\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C. \end{aligned} \tag{3.46}$$

By the definition of $\Psi(s)$ and $\varphi(s)$, if $\gamma \leq 1 - q \leq \lambda$, we can write

$$\Psi(s) \geq \frac{|s|^{\lambda+1}}{\lambda + 1} \quad \forall s \in \mathbb{R}.$$

From the above estimate and some simplification the inequality (3.46), we can estimate as follows

$$\begin{aligned} \frac{1}{\lambda + 1} \int_\Omega [|u_n(x, t)|^{\frac{\lambda+q+1}{2}}]^{\frac{2(\lambda+1)}{\lambda+q+1}} + \frac{4\lambda c_0}{(\lambda + q + 1)^2} \int_0^t \int_\Omega |\nabla u_n^{\frac{\lambda+q+1}{2}}|^2 \\ \leq C \left(\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C. \end{aligned}$$

Now passing to supremum for $t \in [0,T]$, we get

$$\begin{aligned} \frac{1}{\lambda+1} \| |u_n|^{\frac{\lambda+q+1}{2}} \|^{\frac{2(\lambda+1)}{\lambda+q+1}}_{L^\infty(0,T;L^{\frac{2(\lambda+1)}{\lambda+q+1}}(\Omega))} &+ \frac{4\lambda c_0}{(\lambda+q+1)^2} \int_Q |\nabla u_n^{\frac{\lambda+q+1}{2}}|^2 \\ &\le C \left(\int_Q u_n^{(\lambda-\gamma)m'} \right)^{\frac{1}{m'}} + C. \end{aligned} \tag{3.47}$$

By Lemma 2.9 (where $v = u_n^{\frac{\lambda+q+1}{2}}$, $\rho = \frac{2(\lambda+1)}{\lambda+q+1}$, $h = 2$), (3.47), we have

$$\begin{aligned} \int_Q [|u_n|^{\frac{\lambda+q+1}{2}}]^{2\frac{N+\frac{2(\lambda+1)}{\lambda+q+1}}{N}} &\le \left(\| |u_n|^{\frac{\lambda+q+1}{2}} \|^{\frac{2(\lambda+1)}{\lambda+q+1}}_{L^\infty(0,T;L^{\frac{2(\lambda+1)}{\lambda+q+1}}(\Omega))} \right)^{\frac{2}{N}} \int_Q |\nabla u_n^{\frac{\lambda+q+1}{2}}|^2 \\ &\le \left[C \left(\int_Q u_n^{(\lambda-\gamma)m'} \right)^{\frac{1}{m'}} + C \right]^{\frac{2}{N}+1} \\ &\le C \left(\int_Q u_n^{(\lambda-\gamma)m'} \right)^{(\frac{2}{N}+1)\frac{1}{m'}} + C, \end{aligned}$$

then, we can obtain

$$\int_Q |u_n|^{\frac{N(\lambda+q+1)+2(\lambda+1)}{N}} \le C \left(\int_Q u_n^{(\lambda-\gamma)m'} \right)^{(\frac{2}{N}+1)\frac{1}{m'}} + C. \tag{3.48}$$

Now choosing λ such that

$$\sigma = \frac{N(\lambda+q+1)+2(\lambda+1)}{N} = (\lambda-\gamma)m', \tag{3.49}$$

then implies that

$$\lambda = \frac{N(q+1)+2+N\gamma m'}{Nm'-N-2}, \quad \sigma = m\frac{N(q+\gamma+1)+2(\gamma+1)}{N-2m+2}.$$

By virtue of $m < \frac{N}{2}+1$, then $(\frac{2}{N}+1)\frac{1}{m'} < 1$, and combining (3.48) and (3.49) with Young's inequality, we obtain

$$\int_Q |u_n|^\sigma \le C. \tag{3.50}$$

The condition $m \ge \frac{2(N+2-q)}{N(q+\gamma+1)+2(2-q)}$, ensure that $\lambda \ge 1-q \ge \gamma$ and going back to (3.46), from (3.49) and (3.50), we have

$$\begin{aligned} \int_Q |\nabla u_n|^2 &\le \int_Q (1+u_n)^{\lambda+q-1} |\nabla u_n|^2 \\ &\le C \left(\int_Q u_n^{(\lambda-\gamma)m'} \right)^{\frac{1}{m'}} + C \le C \left(\int_Q u_n^\sigma \right)^{\frac{1}{m'}} + C \le C. \end{aligned} \tag{3.51}$$

The estimate (3.50) and (3.51), implies that u_n is bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\sigma(Q)$ with respect to n, so $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\sigma(Q)$. Hence the proof of (i) is desired.

Now we prove (ii)
If $\gamma \le \lambda < 1-q$, by definition $\varphi(s),\ \Psi(s)$, we can get

$$\Psi(s) \ge C|s|^{\lambda+1} - C,$$

from the last inequality and going back to (3.46), we have

$$C\int_\Omega |u_n(x,t)|^{\lambda+1} + \lambda c_0 \int_0^t\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^{1-\lambda-q}} \le C\left(\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + \int_\Omega \Psi(u_0) + Cmeas(\Omega),$$

by the fact that $u_0 \in L^\infty(\Omega)$ and passing to supremum for $t\in[0,T]$, then we get

$$C||u_n||^{\lambda+1}_{L^\infty(0,T;L^{\lambda+1}(\Omega))} + \lambda c_0 \int_Q \frac{|\nabla u_n|^2}{(1+u_n)^{1-\lambda-q}} \le C\left(\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C. \tag{3.52}$$

Let $\delta \le 2$, applying Hölder's inequality, we have

$$\begin{aligned}\int_Q |\nabla u_n|^\delta &= \int_Q \frac{|\nabla u_n|^\delta}{(1+u_n)^{\frac{\delta(1-\lambda-q)}{2}}}(1+u_n)^{\frac{\delta(1-\lambda-q)}{2}} \\ &\le \left(\int_Q \frac{|\nabla u_n|^2}{(u_n+1)^{1-\lambda-q}}\right)^{\frac{\delta}{2}}\left(\int_Q (1+u_n)^{\frac{\delta(1-\lambda-q)}{2-\delta}}\right)^{\frac{2-\delta}{2}} \\ &\le C\left(1+\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{\delta}{2m'}}\left(1+\int_Q u_n^{\frac{\delta(1-\lambda-q)}{2-\delta}}\right)^{\frac{2-\delta}{2}}.\end{aligned} \tag{3.53}$$

Applying Lemma 2.9 (where $v=u_n,\ \rho=\lambda+1,\ h=\delta$) we get

$$\begin{aligned}\int_Q u_n^{\frac{\delta(N+\lambda+1)}{N}} &\le ||u_n||^{\frac{\delta(\lambda+1)}{N}}_{L^\infty(0,T;L^{\lambda+1}(\Omega))}\int_Q |\nabla u_n|^\delta \\ &\le C\left(1+\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{\delta}{Nm'}+\frac{\delta}{2m'}}\left(1+\int_Q u_n^{\frac{\delta(1-\lambda-q)}{2-\delta}}\right)^{\frac{2-\delta}{2}}.\end{aligned} \tag{3.54}$$

Let choose λ such that

$$\sigma = \frac{\delta(N+\lambda+1)}{N} = (\lambda-\gamma)m' = \frac{\delta(1-\lambda-q)}{2-\delta}, \tag{3.55}$$

then we deduce

$$\lambda = \frac{N(q+1)+2+N\gamma m'}{Nm'-N-2}, \quad \sigma = m\frac{N(q+\gamma+1)+2(\gamma+1)}{N-2m+2}.$$

$$r = m\frac{N(q+\gamma+1)+2(\gamma+1)}{N+2-m(1-\gamma)+q(m-1)}.$$

From (3.55), the inequality (3.54), becomes

$$\int_Q u_n^\sigma \leq C\left(1+\int_Q u_n^\sigma\right)^{\frac{\delta}{Nm'}+\frac{\delta}{2m'}+\frac{2-\delta}{2}}.$$

By virtue of $m < \frac{2(N+2-q)}{N(q+\gamma+1)+2(2-q)}$, ensure that $\frac{\delta}{Nm'}+\frac{\delta}{2m'}+\frac{2-\delta}{2} < 1$, then applying Young's inequality we can deduce that

$$\int_Q u_n^\sigma \leq C. \tag{3.56}$$

We combine (3.55) and (3.56) in (3.53), yields

$$\int_Q |\nabla u_n|^\delta \leq C. \tag{3.57}$$

Two last inequalities proved that the sequence u_n is bounded in $L^\delta(0,T;W_0^{1,\delta}(\Omega)) \cap L^\sigma(Q)$, and so $u \in L^\delta(0,T;W_0^{1,\delta}(\Omega)) \cap L^\sigma(Q)$. □

Theorem 3.17. *Let $\gamma = 1$, f be a nonnegative function in $L^m(Q)$, $1 \leq m < \frac{N}{2}+1$. Then the solution found in Theorem 3.12, satisfy the following summability $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\sigma(Q)$ with $\sigma = \frac{m(N(q+2)+4)}{N-2m+2}$.*

Proof. Let u_n be a solution of (3.7) given by Lemma 3.2, such that u_n converges to a solution of (3.1). Choosing $u_n^\lambda \chi_{(0,t)}$ as test function, with $\lambda \geq 1$, using (3.3) and applying Hölder's inequality, we have

$$\begin{aligned}&\frac{1}{\lambda+1}\int_\Omega |u_n(x,t)|^{\lambda+1} + \lambda\int_0^t\int_\Omega (\alpha+u_n^q)u_n^{\lambda-1}|\nabla u_n|^2\\ &\leq C\left(\int_Q u_n^{(\lambda-1)m'}\right)^{\frac{1}{m'}} + \frac{1}{\lambda+1}\int_\Omega u_0^{\lambda+1},\end{aligned}$$

thanks to $u_0 \in L^\infty(\Omega)$ and dropping the nonnegative term, we get

$$\begin{aligned}&\frac{1}{\lambda+1}\int_\Omega |u_n(x,t)|^{\lambda+1} + \lambda\int_0^t\int_\Omega u_n^{\lambda+q-1}|\nabla u_n|^2\\ &\leq C\left(\int_Q u_n^{(\lambda-1)m'}\right)^{\frac{1}{m'}} + \frac{1}{\lambda+1}||u_0||^{\lambda+1}_{L^{\lambda+1}(\Omega)} \leq C\left(\int_Q u_n^{(\lambda-1)m'}\right)^{\frac{1}{m'}} + C,\end{aligned}$$

by simple simplification the above estimate becomes

$$\begin{aligned}&\frac{1}{\lambda+1}\int_\Omega [|u_n(x,t)|^{\frac{\lambda+q+1}{2}}]^{\frac{2(\lambda+1)}{\lambda+q+1}} + \frac{4\lambda}{(\lambda+q+1)^2}\int_0^t\int_\Omega |\nabla u_n^{\frac{\lambda+q+1}{2}}|^2\\ &\leq C\left(\int_Q u_n^{(\lambda-1)m'}\right)^{\frac{1}{m'}} + C.\end{aligned}$$

Passing to supremum in $t \in [0,T]$, then we obtain

$$\begin{aligned}&\frac{1}{\lambda+1}||u_n^{\frac{\lambda+q+1}{2}}||^{\frac{2(\lambda+1)}{\lambda+q+1}}_{L^\infty(0,T;L^{\frac{2(\lambda+1)}{\lambda+q+1}}(\Omega))} + \frac{4\lambda}{(\lambda+q+1)^2}\int_Q |\nabla u_n^{\frac{\lambda+q+1}{2}}|^2\\ &\leq C\left(\int_Q u_n^{(\lambda-1)m'}\right)^{\frac{1}{m'}} + C.\end{aligned} \tag{3.58}$$

By Lemma 2.9 (where $v = u_n^{\frac{\lambda+q+1}{2}}$, $\rho = \frac{2(\lambda+1)}{\lambda+q+1}$, $h = 2$), we use the same proof as before, we get

$$\begin{aligned} \int_Q |u_n|^{\frac{N(\lambda+q+1)+2(\lambda+1)}{N}} &\leq \left[C \left(\int_Q u_n^{(\lambda-1)m'} \right)^{\frac{1}{m'}} + C \right]^{\frac{2}{N}+1} \\ &\leq C \left(\int_Q u_n^{(\lambda-1)m'} \right)^{\frac{1}{m'}(\frac{2}{N}+1)} + C. \end{aligned} \tag{3.59}$$

Choosing λ such that

$$\sigma = \frac{N(\lambda+q+1)+2(\lambda+1)}{N} = (\lambda-1)m', \tag{3.60}$$

then

$$\lambda = \frac{N(q+1)+2+Nm'}{Nm'-N-2}, \quad \sigma = \frac{m(N(q+2)+4)}{N-2m+2}.$$

Thanks to (3.60) and (3.59), implies that

$$\int_Q |u_n|^\sigma \leq C \left(\int_Q |u_n|^\sigma \right)^{\frac{1}{m'}(\frac{2}{N}+1)} + C.$$

The condition $m < \frac{N}{2}+1$ ensure that $\frac{1}{m'}(\frac{2}{N}+1) < 1$ and $\lambda \geq 1$ implies that $m \geq 1$, and using Young's inequality in the above estimate gives

$$\int_Q |u_n|^\sigma \leq C, \tag{3.61}$$

then we deduce that u_n is bounded in $L^\sigma(Q)$ and so u belong to $L^\sigma(Q)$. □

Theorem 3.18. *Let $\gamma > 1, q > \gamma - 1$ and f be a nonnegative function in $L^m(Q), m > 1$. then there exists a solution u of problem (3.1) such that if $\max(1, \frac{(N+2)(2q-\gamma+1)}{N(q+\gamma+1)+4(q+1)}) < m < \frac{N}{2}+1$, then u belong to $L^\sigma(Q)$ with*

$$\sigma = m\frac{N(q+\gamma+1)+2(\gamma+1)}{N-2m+2}.$$

Proof. We will take $u_n^\lambda \chi_{(0,t)} (\lambda > 1)$ as test function in (3.7), as in the case $\gamma = 1$ we will follow the proof of Theorem 3.17, repeating the same passage in order to arrive to the inequality

$$\int_Q |u_n|^{\frac{N(\lambda+q+1)+2(\lambda+1)}{N}} \leq C \left(\int_Q |u_n|^{(\lambda-\gamma)m'} \right)^{\frac{1}{m'}(\frac{2}{N}+1)} + C. \tag{3.62}$$

We now choose λ such that

$$\sigma = \frac{N(\lambda+q+1)+2(\lambda+1)}{N} = (\lambda-\gamma)m', \tag{3.63}$$

i.e $\lambda = \frac{N(q+1)+2+N\gamma m'}{Nm'-N-2}$, $\sigma = m\frac{N(q+\gamma+1)+2(\gamma+1)}{N-2m+2}$. Combining (3.62) and (3.63), implies that

$$\int_Q |u_n|^\sigma \leq C \left(\int_Q |u_n|^\sigma \right)^{\frac{1}{m'}(\frac{2}{N}+1)} + C,$$

by virtue of $m < \frac{N}{2}+1$, then we have $\frac{1}{m'}(\frac{2}{N}+1) < 1$ and $\lambda > 1$ ensure that $m > 1$, then by Young's inequality, we get

$$\int_Q |u_n|^\sigma \leq C. \tag{3.64}$$

Hence from (3.64) it follows that u_n is bounded in $L^\sigma(Q)$ so that $u \in L^\sigma(Q)$. Next we testing (3.7) by $u_n^\gamma T_1(u_n - T_k(u_n))$, we have

$$\begin{aligned}
&\int_Q \frac{\partial u_n}{\partial t} u_n^\gamma T_1(u_n - T_k(u_n)) + \gamma \int_Q u_n^{\gamma-1}(a(x,t)+u_n^q)|\nabla u_n|^2 T_1(u_n - T_k(u_n)) \\
&+ \int_{Q\cap\{k\leq u_n\leq k+1\}} u_n^\gamma (a(x,t)+u_n^q)|\nabla u_n|^2 = \int_Q \frac{f_n}{(u_n+\frac{1}{n})^\gamma} u_n^\gamma T_1(u_n - T_k(u_n)).
\end{aligned} \tag{3.65}$$

Dropping the first and second nonnegative terms in the left hand side of (3.65) and using the assumption (3.3), we obtain

$$\int_{Q\cap\{u_n\geq k\}} u_n^\gamma |\nabla u_n|^2 \leq \frac{1}{\gamma\alpha}\int_{Q\cap\{u_n\geq k\}} f + C. \tag{3.66}$$

Thus, thanks to the estimate (3.66), implies that

$$\begin{aligned}
\int_{Q\cap\{u_n>k\}} u_n^q |\nabla u_n| &\leq \left(\int_{Q\cap\{u_n>k\}} u_n^{2q-\gamma+1}\right)^{\frac{1}{2}} \left(\int_{Q\cap\{u_n>k\}} u_n^{\gamma-1}|\nabla u_n|^2\right)^{\frac{1}{2}} \\
&\leq \left(\int_{Q\cap\{u_n>k\}} u_n^{2q-\gamma+1}\right)^{\frac{1}{2}} \left(\frac{1}{\gamma\alpha}\int_{Q\cap\{u_n>k\}} f + C\right)^{\frac{1}{2}}.
\end{aligned}$$

Since u_n is bounded in $L^\sigma(Q)$, then $2q-\gamma+1 \leq \sigma$ is equivalent to $m \geq \frac{(N+2)(2q-\gamma+1)}{N(q+\gamma+1)+4(q+1)}$, hence we get

$$\int_{Q\cap\{u_n>k\}} u_n^q |\nabla u_n| \leq C\left(\frac{1}{\gamma\alpha}\int_{Q\cap\{u_n>k\}} f\right)^{\frac{1}{2}}. \tag{3.67}$$

Now let $\varphi \in C_c^1(\Omega\times[0,T))$, $\varphi \equiv 1$ on $w\times(0,T)$, $w \subset\subset \Omega$. and E be a measurable subset of Q, from (3.67) and Lemma 3.13, we can get

$$\begin{aligned}
&\int_{E\cap\{w\times(0,T)\}} u_n^q|\nabla u_n| \leq \int_E u_n^q |\nabla u_n|\varphi \leq \int_{Q\cap\{u_n>k\}} u_n^q|\nabla u_n|\varphi + k^q\int_E |\nabla u_n|\varphi \\
&\leq C||\varphi||_{L^\infty}\left(\int_{Q\cap\{u_n>k\}} f + C\right)^{\frac{1}{2}} + ||\varphi||_{L^\infty} k^q meas(E)^{\frac{1}{2}}\left(\int_{w\times(0,T)} |\nabla u_n|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Taking the limit as meas(E) tends to zero, k tend to infinity and since $u_n^q|\nabla u_n|$ converge to $u^q|\nabla u|$ almost everywhere, we easily verify thanks to Vitali's Theorem that

$$u_n^q|\nabla u_n| \to u^q|\nabla u| \quad \text{strongly in} \quad L^1(0,T;L^1_{loc}(\Omega)). \tag{3.68}$$

Therefore, putting together (3.68), Lemma 3.3 and Lemma 3.13, we conclude the proof of Theorem 3.18. □

Theorem 3.19. *Let $\gamma > 1, q \leq \gamma - 1$ and f be a non-negative function in $L^m(Q), 1 < m < \frac{N}{2}+1$. Then the solution found in Theorem 3.15, satisfies the following summability, $u \in L^\sigma(Q)$, with $\sigma = m\frac{N(q+\gamma+1)+2(\gamma+1)}{N-2m+2}$.*

Proof. The proof of Theorem 3.19 is similar to proof of item (i) of Theorem 3.16. □

Chapter 4

Some nonlinear parabolic problems with singular natural growth term

1 Introduction

This chapter is devoted to the study of the following nonlinear singular parabolic problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + b(x,t)\frac{|\nabla u|^p}{u^\theta} = f & \text{in} \quad Q, \\ u(x,t) = 0 & \text{on} \quad \Gamma, \\ u(x,0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, and Q is the cylinder $\Omega \times (0,T)$, $T > 0$, Γ the lateral surface $\partial\Omega \times (0,T)$, $2 \leq p < N$, $0 < \theta < 1$, $b(x,t)$ is a measurable function satisfying

$$0 < \alpha \leq b(x,t) \leq \beta, \tag{4.2}$$

where α and β are fixed real numbers, and f belongs to some Lebesgue space $L^m(Q)$, $m \geq 1$, satisfying the condition

$$\operatorname{ess\,inf}\{f(x,t) : x \in \omega, t \in (0,t)\} > 0 \quad \forall\, \omega \subset\subset \Omega.$$

When the singular lower-order term does not appear (i.e. $b(x,t) = 0$ in (4.1)), the existence and regularity results of solutions to problem (4.1) are proved in [92] under the hypothesis $f \in L^r(0,T;L^q(\Omega))$, $r \geq 1$, $q \geq 1$. If $\theta = 0$ and $b(x,t) \equiv cst$, the authors in [83] studied the existence and uniqueness of solution to nonlinear parabolic problems with natural growth with respect to the gradient

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) = H(x,t,u,\nabla u) - \operatorname{div}(g(x,t)) & \text{in} \quad D'(Q), \\ u(x,t) = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

where $|H(x,t,s,\xi)| \leq \nu|\xi|^p + f(x,t)$, ν is a positive constant, $f \geq 0$ belongs to $L^r(0,T;L^q(\Omega))$ with $q = r'N/p$ and $1 < r < \infty$, $|g|^{p'} \in L^r(0,T;L^q(\Omega))$, and the initial datum $u_0 \in L^\infty(\Omega)$ satisfies

$$\int_\Omega e^{pM|u_0(x)|}dx < +\infty,$$

for $M > 0$. In the same fashion, the authors shown in [57] the existence of solutions to problem parabolic

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u = d|\nabla u|^p + f(x,t) & \text{in} \quad Q, \\ u(x,t) = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

where $1 < p < N$, $f \in L^r(0,T;L^q(\Omega))$, with $q, r > 1$ are such that $q/r' \geq N/p$, and the initial datum $u_0 \in L^\infty(\Omega)$ satisfies

$$\int_\Omega \left(e^{l|u_0|} - 1\right)^2 dx < +\infty, \qquad \text{for every} \quad l \in I\!R.$$

See also [58, 77, 111]. When $b(x,t) = B$, $\theta = 1$ and $f \equiv 0$ the authors in [60] studied the existence of weak solutions to homogeneous nonlinear and singular parabolic problems as

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u + B\frac{|\nabla u|^p}{u} = 0 & \text{in} \quad Q, \\ u(x,t) = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

with $p > 1$, $B > 0$, and $0 \leq u_0$ belonging to $L^\infty(\Omega)$ such that $u_0 \geq c > 0$ a.e. on Ω. In the case $p = 2$, several works studied the existence of solutions for singular parabolic problems. For example, the authors in [113] proved the existence of solutions to the following parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(M(x,t,u)\nabla u) + g(x,t,u)|\nabla u|^2 = f(x,t) & \text{in} \quad Q, \\ u(x,t) = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

where $f \in L^r(0,T;L^q(\Omega))$ with $\frac{1}{r} + \frac{2}{Nq} < 1$, $q \geq 1$, $r > 1$, and $u_0 \in L^\infty(\Omega)$, and the function $g(x,t,s) : Q \times (0,+\infty) \to \mathbb{R}$ is a Carathéodory function which is singular at $s = 0$, and it possibly negative (see also [69, 112]). In the elliptic case, several works studied existence and regularity results for the singular case. In [144] the authors proved existence and non existence of solutions to problem

$$\begin{cases} -\Delta_p u + g(x,u)|\nabla u|^p = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $1 < p < +\infty$, $g(x,s)$ positive and singular at $s = 0$, $f \in L^q(\Omega)$ $(q \geq 1)$ satisfying the condition

$$\exists f_\omega > 0, \quad \text{such that} \quad f \geq f_\omega \ \text{ in } w, \quad \forall \omega \subset\subset \Omega.$$

In the case $p = 2$, Souilah [138] proved existence and regularity results of solutions to the problem

$$\begin{cases} -\operatorname{div}(M(x,u)\nabla u) + \frac{|\nabla u|^2}{u^\theta} = f + \lambda u^r & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

where $0 < \theta < 1$, $0 < r < 2 - \theta$, $\lambda > 0$, $f \in L^m(\Omega)$ $(m \geq 1)$. The author in [55] proved existence of solution $u \in H_0^1(\Omega)$ to the problem

$$\begin{cases} -\operatorname{div}\left(\frac{b(x)}{(1+|u|)^p}\nabla u\right) + B\frac{|\nabla u|^2}{u^\theta} = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $B, p > 0,\ 0 < \theta < 2;\ f \in L^m(\Omega)\ (m \geq 1)$. Here, the non existence of solutions $u \in H_0^1(\Omega)$ is proved for $\theta \geq 2$ (see also [27, 134] and references therein).

In the study of problem (4.1), the difficulty comes from the lower-order term: the natural growth dependence with respect to the gradient and the singular dependence with respect to u. To overcome this difficulty, we need to approximate the problem (4.1) by another non-singular one.

Now we give the definition of weak solution of problem (4.1).

Definition 4.1. A weak solution to problem (4.1) is a function u in $L^1(0,T;W_0^{1,1}(\Omega))$ such that, for every $\omega \subset\subset \Omega$, there exists c_ω such that $u \geq c_\omega > 0$ in $\omega \times (0,T)$, $\frac{|\nabla u|^p}{u^\theta} \in L^1(Q)$. Furthermore, we have that

$$-\int_Q u\frac{\partial \phi}{\partial t}\,dx\,dt + \int_Q |\nabla u|^{p-2}\nabla u \cdot \nabla\phi\,dx\,dt + \int_Q b(x,t)\frac{|\nabla u|^p}{u^\theta}\phi\,dx\,dt = \int_Q f\phi\,dx\,dt, \tag{4.3}$$

for every $\phi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

2 Main results

Now we will give our main results of this chapter.

Theorem 4.2. *Let $0 < \theta < 1$. Assume that f is a positive function belonging to $L^m(Q)$, with $m > \frac{N}{p}+1$. Then there exists a function*

$$u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$$

solution of problem (4.1) in the sense of Definition 4.1.

Theorem 4.3. *Let $0 < \theta < 1$. Assume that f is a positive function belonging to $L^m(Q)$, with $m = \frac{N}{p}+1$. Then there exists a function*

$$u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^{\frac{N(p-\theta)}{N-p}}(Q)$$

solution of problem (4.1) in the sense of Definition 4.1.

Theorem 4.4. *Let $0 < \theta < 1$. Assume that f is a positive function belonging to $L^m(Q)$, with*

$$\frac{p(N+2+\theta)}{p(N+2+\theta) - N(1+\theta)} \leq m < \frac{N}{p} + 1.$$

Then there exists a function

$$u \in L^p(0,T;W_0^{1,p}(\Omega))$$

solution of problem (4.1) in the sense of Definition 4.1. Moreover $u \in L^\sigma(Q)$, where

$$\sigma = \frac{m(N(p-1-\theta)+p)}{N - pm + p}.$$

Theorem 4.5. *Let $0 < \theta < 1$. Assume that f is a positive function belonging to $L^m(Q)$, with*

$$\max\left(1,\ \frac{(p-1)(N+2+\theta)}{(p-1)(N+2+\theta) - (N\theta - 1)}\right) < m < \frac{p(N+2+\theta)}{p(N+2+\theta) - N(1+\theta)}.$$

Then there exists a function

$$u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\sigma(Q)$$

solution of problem (4.1) *in the sense of Definition 4.1, where*

$$q = \frac{m(N(p-1-\theta)+p)}{N+1-(1+\theta)(m-1)} \text{ and } \sigma = \frac{m(N(p-1-\theta)+p)}{N-pm+p}.$$

Remark 4.6. The condition $m > \max\left(1, \frac{(p-1)(N+2+\theta)}{(p-1)(N+2+\theta)-(N\theta-1)}\right)$ is due to the fact that q must not be smaller than $p-1$ and the choice of $m>1$ in the above Theorem. Note that if $0<\theta<\frac{1}{N}$, then $\frac{(p-1)(N+2+\theta)}{(p-1)(N+2+\theta)-(N\theta-1)} < 1$.

Theorem 4.7. *Let $0<\theta<1$. Assume that f is a positive function belonging to $L^1(Q)$. Then there exists a function*

$$u \in L^\delta(0,T;W_0^{1,\delta}(\Omega))$$

solution of problem (4.1) *in the sense of Definition 4.1, where $\delta = \frac{N(p-\theta)}{N-\theta}$.*

Remark 4.8. If $p=2$, the results we that obtain are similar to the regularity ones concerning the elliptic case. More precisely, we refer to [138, Theorem 2.2] for Theorem 4.2, [55, Theorem 1.1] for Theorem 4.4, [138, Theorem 2.3] for Theorem 4.5 and [138, Theorem 2.4] for Theorem 4.7.

3 A priori estimate and preliminary facts

Let $n \in \mathbb{N}$. We approximate the problem (4.1) by the following nonlinear and non-singular problem

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) + b(x,t)\frac{u_n|\nabla u|^p}{(u_n+\frac{1}{n})^{\theta+1}} = f_n & \text{in} \quad Q, \\ u_n(x,t) = 0 & \text{on} \quad \Gamma, \\ u_n(x,0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{4.4}$$

where $f_n = \frac{f}{1+\frac{1}{n}f}$ and $f_n \in L^\infty(Q)$, such that

$$||f_n||_{L^m(Q)} \leq ||f||_{L^m(Q)} \text{ and } f_n \to f \text{ strongly in } L^m(Q),\ m \geq 1. \tag{4.5}$$

The problem (4.4) admits weak solutions u_n belonging to $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, see [11, 60, 108]. Since the right hand side of (4.4) is non-negative, this implies that u_n is non-negative.
We are now going to prove some a priori estimates. The next Lemma gives a control of the natural growth term.

Lemma 4.9. *Let u_n be solutions to problem* (4.4)*.Then it results*

$$\int_Q b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \leq \int_Q f. \tag{4.6}$$

Proof. For any fixed $h > 0$, let us consider $\frac{T_h(u_n)}{h}$ as a test function in the approximated problem (4.4). Then, we have

$$\int_0^T \int_\Omega \frac{\partial u_n}{\partial t} \frac{T_h(u_n)}{h} + \frac{1}{h} \int_Q |\nabla u_n|^{p-2} \nabla u_n \nabla T_h(u_n) + \int_Q b(x,t) \frac{u_n |\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} \frac{T_h(u_n)}{h} = \int_Q f_n \frac{T_h(u_n)}{h}.$$

Therefore

$$\int_\Omega S_k(u_n(x,T)) + \frac{1}{h} \int_{\{u_n \leq h\}} |\nabla T_h(u_n)|^p + \int_Q b(x,t) \frac{u_n |\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} \frac{T_h(u_n)}{h} = \int_Q f_n \frac{T_h(u_n)}{h},$$

where $S_k(y) = \int_0^y T_k(\ell)\, d\ell$. Observe that $S_k(y) \geq \frac{T_k(y)^2}{2}$ for every $y \geq 0$.

Now, dropping the first and second non-negative terms in the last equality and using (4.2), we obtain

$$0 \leq \int_Q b(x,t) \frac{u_n |\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} \frac{T_h(u_n)}{h} \leq \int_Q f_n \frac{T_h(u_n)}{h}.$$

Using the fact that $f_n \leq f$ and $\frac{T_h(u_n)}{h} \leq 1$, then

$$\int_Q b(x,t) \frac{u_n |\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} \frac{T_h(u_n)}{h} \leq \int_Q f.$$

Letting h tend to 0, we deduce (4.6) by Fatou's Lemma. □

Remark 4.10. In view of Lemma 4.9, from (4.2) and the fact $u_n \geq 0$, we have $\int_Q b(x,t) \frac{u_n |\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} \geq 0$, $f \in L^1(Q)$ one has that

$$\int_Q \left| b(x,t) \frac{u_n |\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} - f \right| \leq \int_Q b(x,t) \frac{u_n |\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} + \int_Q f \leq 2 \int_Q f < C,$$

where C not depending on n. Hence

$$b(x,t) \frac{u_n |\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} - f \in L^1(Q).$$

We now prove five a priori estimates on u_n, which are true for every $\theta \in (0,1)$.

Lemma 4.11. *Let the assumptions of Theorem 4.2 be in force. Then the solution u_n of (4.4) is uniformly bounded in $L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.*

Proof. For $k > 0$, choose $G_k(u_n)$ as test function in the approximate problem (4.4). We have

$$\int_0^t \int_\Omega \frac{\partial u_n}{\partial t} G_k(u_n) + \int_0^t \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla G_k(u_n)$$
$$+ \int_0^t \int_\Omega b(x,\tau) \frac{u_n |\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} G_k(u_n) = \int_0^t \int_\Omega f_n G_k(u_n),$$

for $t \in (0, T]$. Let as denoted by $A_{k,n}(t)$ the following set

$$A_{k,n}(t) = \{x \in \Omega : |u_n(x,t)| > k\}.$$

Dropping the third non-negative term, using integration by part and by Hölder's inequality in last equality, we get

$$\int_{A_{k,n}(t)} |G_k(u_n(t))|^2 + 2 \int_0^t \int_{A_{k,n}(\tau)} |\nabla G_k(u_n)|^p$$
$$\leq C \left(\int_0^t \int_{A_{k,n}(\tau)} |G_k(u_n)|^{m'} \right)^{\frac{1}{m'}}.$$

Then

$$\begin{aligned} &||G_k(u_n)||^2_{L^\infty(0,T;L^2(A_{k,n}(t))} + 2 \int_0^T \int_{A_{k,n}(t)} |\nabla G_k(u_n)|^p \\ &\leq C \left(\int_0^T \int_{A_{k,n}(t)} |G_k(u_n)|^{m'} \right)^{\frac{1}{m'}}. \end{aligned} \tag{4.7}$$

Applying Lemma 2.9 (here $\rho = 2$, $h = p$ and $v = G_k(u_n)$)

$$\int_0^T \int_{A_{k,n}(t)} |G_k(u_n)|^{\frac{p(N+2)}{N}}$$
$$\leq ||G_k(u_n)||^{\frac{2p}{N}}_{L^\infty(0,T;L^2(A_{k,n}(t)))} \int_0^T \int_{A_{k,n}(t)} |\nabla G_k(u_n)|^p.$$

Using (4.7) in last inequality, we deduce that

$$\int_0^T \int_{A_{k,n}(t)} |G_k(u_n)|^{\frac{p(N+2)}{N}} \leq C \left(\int_0^T \int_{A_{k,n}(t)} |G_k(u_n)|^{m'} \right)^{\frac{1}{m'}(\frac{p}{N}+1)}. \tag{4.8}$$

By virtue of $m > \frac{N}{p}+1$, then $\frac{p(N+2)}{Nm'} > 1$. Applying Hölder's inequality with indices $\left(\frac{p(N+2)}{Nm'}, \frac{p(N+2)}{p(N+2)-Nm'}\right)$ in (4.8), we get

$$\begin{aligned} \int_0^T \int_{A_{k,n}(t)} |G_k(u_n)|^{\frac{p(N+2)}{N}} &\leq C \left(\int_0^T \int_{A_{k,n}(t)} |G_k(u_n)|^{\frac{p(N+2)}{N}} \right)^{\frac{p+N}{p(N+2)}} \\ &\times \left(\int_0^T |A_{k,n}(t)| \right)^{\frac{1}{m'}(\frac{p}{N}+1)(1-\frac{Nm'}{p(N+2)})}. \end{aligned}$$

Thanks to Young's inequality with parameter ϵ, we obtain

$$\int_0^T \int_{A_{k,n}(t)} |G_k(u_n)|^{\frac{p(N+2)}{N}} \leq \epsilon \bar{C} \int_0^T \int_{A_{k,n}(t)} |G_k(u_n)|^{\frac{p(N+2)}{N}} + C_\epsilon \left(\int_0^T |A_{k,n}(t)| \right)^{\frac{1}{m'}(\frac{p}{N}+1)(1-\frac{Nm'}{p(N+2)})\frac{p(N+2)}{N(p-1)+p}},$$

where $\bar{C}$ is a positive constant independent on n. Taking $\epsilon = \frac{1}{2\bar{C}}$, we obtain that

$$\int_0^T \int_{A_{k,n}(t)} |G_k(u_n)|^{\frac{p(N+2)}{N}} \leq C \left(\int_0^T |A_{k,n}(t)| \right)^{\frac{1}{m'}(\frac{p}{N}+1)(1-\frac{Nm'}{p(N+2)})\frac{p(N+2)}{N(p-1)+p}}.$$

We not that, if $h > k$, we have $|G_k(u_n)| > h - k$ on $A_{k,n}(t)$ and $A_{h,n}(t) \subset A_{k,n}(t)$. Hence

$$\int_0^T A_{h,n}(t) \leq \frac{C}{(h-k)^{\frac{p(N+2)}{N}}} \left(\int_0^T |A_{k,n}(t)| \right)^{\frac{1}{m'}(\frac{p}{N}+1)(1-\frac{Nm'}{p(N+2)})\frac{p(N+2)}{N(p-1)+p}}.$$

Let $\varrho(k) = \int_0^T |A_{k,n}(t)|$, then

$$\varrho(h) \leq \frac{C}{(h-k)^\lambda} \left[\varrho(k)\right]^\mu, \tag{4.9}$$

where $\lambda = \frac{p(N+2)}{N} > 0$ and $\mu = \frac{1}{m'}\left(\frac{p}{N}+1\right)\left(1 - \frac{Nm'}{p(N+2)}\right)\frac{p(N+2)}{N(p-1)+p}$. By the fact that $m > \frac{N}{p} + 1$, then we have $\mu > 1$ and by Lemma 2.10, there exists $\gamma_1 > 1$ such that

$$\varrho(\gamma_1) = 0.$$

Hence there exists a constant $C > 0$, independent of n such that

$$\|u_n\|_{L^\infty(Q)} \leq C \ \ in \ \ Q. \tag{4.10}$$

Let us u_n test function in problem (4.4), obtaining

$$\frac{1}{2}\int_\Omega u_n(x,T)^2 + \int_Q |\nabla u_n|^p + \int_Q b(x,t)\frac{u_n^2|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} = \int_Q f_n u_n.$$

Since $0 < \alpha \leq b(x,t)$, then we can drop the first and third non-negative terms, we get

$$\int_Q |\nabla u_n|^p \leq \int_Q f_n u_n.$$

Applying Hölder's inequality twice and from (4.5), (4.10), it follows that

$$\int_Q |\nabla u_n|^p \leq \int_Q f_n u_n \leq \|u_n\|_{L^\infty(Q)}\|f\|_{L^m(Q)}|Q|^{\frac{1}{m'}} \leq C. \tag{4.11}$$

As consequence of estimate (4.10) and (4.11), u_n is uniformly bounded in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$. $\square$

Lemma 4.12. *Let the assumptions of Theorem 4.3 be in force. Then the solution u_n of (4.4) is uniformly bounded in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^{\frac{N(p+\theta)}{N-p}}(Q)$.*

Proof. We test (4.4) by $\varphi(u_n) = ((u_n+1)^{\theta+1}-1)$, obtaining

$$\int_\Omega \Psi(u_n(x,T)) + (\theta+1)\int_Q |\nabla u_n|^p (u_n+1)^\theta + \int_Q b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}(u_n+1)^{\theta+1}$$
$$= \int_Q f_n\left((u_n+1)^{\theta+1}-1\right) + \int_Q b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}},$$

where $\Psi(y) = \displaystyle\int_0^y \varphi(\ell)\,d\ell$. Observe that φ is increasing and positive on $[0,+\infty)$, we deduce that $\displaystyle\int_\Omega \Psi(u_n(x,T)) \geq 0$, and from (4.6), we have

$$\int_Q b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \leq \int_Q f.$$

Dropping the first non-negative term, recalling (4.2), and by the fact that $\frac{1}{(u_n+1)^{\theta+1}} \leq \frac{1}{(u_n+\frac{1}{n})^{\theta+1}}$, we deduce

$$(\theta+1)\int_Q |\nabla u_n|^p(u_n+1)^\theta + \alpha\int_Q u_n|\nabla u_n|^p \leq \int_Q f_n(u_n+1)^{\theta+1} + C. \tag{4.12}$$

Applying Hölder's inequality with indices $(m,\, m') = \left(\frac{N+p}{p},\, \frac{N+p}{N}\right)$, we get

$$\begin{aligned}\frac{(\theta+1)p^p}{(\theta+p)^p}\int_Q |\nabla(u_n+1)^{\frac{\theta+p}{p}}|^p + \alpha\int_Q u_n|\nabla u_n|^p \\ \leq C\left(\int_Q (u_n+1)^{\frac{(\theta+1)(N+p)}{N}}\right)^{\frac{N}{N+p}} + C.\end{aligned} \tag{4.13}$$

Thanks to the Sobolev inequality applied in (4.13), we have

$$\begin{aligned}\left(\int_Q (u_n+1)^{\frac{N(p+\theta)}{N-p}}\right)^{\frac{N-p}{p}} &= \left(\int_Q (u_n+1)^{\frac{p^*(\theta+p)}{p}}\right)^{\frac{N-p}{N}} \\ &\leq C\left(\int_Q (u_n+1)^{\frac{(\theta+1)(N+p)}{N}}\right)^{\frac{N}{N+p}} + C.\end{aligned} \tag{4.14}$$

Being $\frac{(\theta+1)(N+p)}{N} < \frac{N(p+\theta)}{N-p}$, we apply Hölder's inequality with indices $\left(\frac{N^2(p+\theta)}{(N^2-p^2)(\theta+1)},\, \frac{N^2(\theta+p)}{N^2(p+\theta)-(N^2-p^2)(\theta+1)}\right)$, we deduce

$$\left(\int_Q (u_n+1)^{\frac{N(p+\theta)}{N-p}}\right)^{\frac{N-p}{N}} \leq C\left(\int_Q (u_n+1)^{\frac{N(p+\theta)}{N-p}}\right)^{\frac{(N-p)(\theta+1)}{N(p+\theta)}} + C. \tag{4.15}$$

Note that $\frac{(N-p)(\theta+1)}{N(p+\theta)} < \frac{N-p}{N}$. Using Young's inequality in the above estimate, we get

$$\int_Q (1+u_n)^{\frac{N(p+\theta)}{N-p}} \leq C. \tag{4.16}$$

Therefore

$$\int_Q u_n^{\frac{N(p+\theta)}{N-p}} \leq C. \tag{4.17}$$

Let us suppose that $u_n \geq 1$. Then, we come back to (4.13), so we obtain that

$$\alpha \int_{\{u_n \geq 1\}} |\nabla u_n|^p \leq C \left(\int_Q (u_n+1)^{\frac{(\theta+1)(N+p)}{N}} \right)^{\frac{N}{N+p}} + C.$$

Being $\frac{(\theta+1)(N+p)}{N} < \frac{N(p+\theta)}{N-p}$. We apply again the Hölder inequality with the same indices used in (4.15), so we get

$$\alpha \int_{\{u_n \geq 1\}} |\nabla u_n|^p \leq C \left(\int_Q (u_n+1)^{\frac{N(p+\theta)}{N-p}} \right)^{\frac{(N-p)(\theta+1)}{N(p+\theta)}} + C.$$

Then, from (4.16), it follows that

$$\int_{\{u_n \geq 1\}} |\nabla u_n|^p \leq C. \tag{4.18}$$

It remains to analyse the behaviour of ∇u_n in $\{u_n < 1\}$. Taking $T_1(u_n)$ as a test function in (4.4), we get

$$\int_0^T \int_\Omega \frac{\partial u_n}{\partial t} T_1(u_n) + \int_Q |\nabla u_n|^{p-2} \nabla u_n \nabla T_1(u_n)$$
$$+ \int_Q b(x,t) \frac{u_n |\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} T_1(u_n) = \int_Q f_n T_1(u_n).$$

Therefore, we get from (4.2), that

$$\int_\Omega S_1(u_n(x,T)) + \int_{\{u_n<1\}} |\nabla T_1(u_n)|^p + \alpha \int_Q \frac{u_n |\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \leq \int_Q f_n T_1(u_n),$$

where $S_1(y) = \int_0^y T_1(\ell)\, d\ell$. Observe that $S_1(y) \geq \frac{T_1(y)^2}{2}$ for every $y \geq 0$.
Dropping the first and third non-negative terms and using (4.5), we obtain

$$\int_{\{u_n<1\}} |\nabla u_n|^p = \int_Q |\nabla T_1(u_n)|^p \leq \int_Q f_n T_1(u_n) \leq \int_Q f \leq C. \tag{4.19}$$

The inequality (4.18) combined with (4.19), implies that

$$\int_Q |\nabla u_n|^p = \int_{\{u_n \geq 1\}} |\nabla u_n|^p + \int_{\{u_n<1\}} |\nabla u_n|^p \leq C. \tag{4.20}$$

Then (4.17) and (4.20) imply that u_n is uniformly bounded in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^{\frac{N(p+\theta)}{N-p}}(Q)$ This completes the proof of Lemma 4.12. □

Lemma 4.13. *Let the assumptions of Theorem 4.4 be in force. Then the solution u_n of (4.4) is uniformly bounded in $L^p(0,T;W_0^{1,p}(\Omega))\cap L^\sigma(Q)$, where*

$$\sigma=\frac{m(N(p-1-\theta)+p)}{N-pm+p}.$$

Proof. Taking $\psi(u_n)=((1+u_n)^\lambda-1)\chi_{(0,t)}$, (with $\lambda\geq 1+\theta$) as a test function in problem (4.4), we have

$$\begin{aligned}&\int_\Omega \Psi(u_n(x,t))+\lambda\int_0^t\int_\Omega|\nabla u_n|^p(1+u_n)^{\lambda-1}\\&+\int_0^t\int_\Omega b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}(u_n+1)^\lambda\\&\leq\int_Q f((u_n+1)^\lambda-1)+\int_Q b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}},\end{aligned}\tag{4.21}$$

where

$$\Psi(s)=\int_0^s\psi(z)dz.\tag{4.22}$$

By using (4.6) in the right hand side and (4.2) in the left hand side in (4.21), we get

$$\begin{aligned}&\int_\Omega \Psi(u_n(x,t))+\lambda\int_0^t\int_\Omega|\nabla u_n|^p(1+u_n)^{\lambda-1}\\&+\alpha\int_0^t\int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}(1+u_n)^\lambda\\&\leq\int_Q f((1+u_n)^\lambda-1)+C.\end{aligned}\tag{4.23}$$

By the definitions of $\Psi(s)$ and $\psi(s)$, we can get whenever $\lambda>1$

$$\Psi(s)\geq\frac{s^{\lambda+1}}{\lambda+1},\qquad\forall s\in\mathbb{R}^+.\tag{4.24}$$

Combining (4.23) and (4.24) and applying Hölder's inequality with indices (m,m'), we have

$$\begin{aligned}&\frac{1}{\lambda+1}\int_\Omega u_n^{\lambda+1}+\lambda\int_Q|\nabla u_n|^p(1+u_n)^{\lambda-1}\\&+\alpha\int_0^t\int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}(1+u_n)^\lambda\\&\leq C\left(\int_Q((1+u_n)^\lambda-1)^{m'}\right)^{\frac{1}{m'}}+C.\end{aligned}\tag{4.25}$$

By easy simplifications we can write (4.25) as follows

$$\begin{aligned}\frac{1}{\lambda+1}\int_\Omega u_n^{\lambda+1}+\int_0^t\int_\Omega|\nabla u_n|^p(1+u_n)^{\lambda-\theta-1}\left[\lambda(1+u_n)^\theta+\alpha u_n\right]\\\leq C\left(\int_Q(1+u_n)^{\lambda m'}\right)^{\frac{1}{m'}}+C.\end{aligned}$$

Since λ, α, $u_n \geq 0$, we have $\lambda(1+u_n)^\theta + \alpha u_n \geq \lambda$. Furthermore, recalling that $\lambda \geq 1+\theta$, we can estimate the last inequality as follows

$$\frac{1}{\lambda+1}\int_\Omega [u_n^{\frac{\lambda-1-\theta+p}{p}}]^{\frac{p(\lambda+1)}{\lambda-1-\theta+p}} + \frac{p^p\lambda}{(\lambda-1-\theta+p)^p}\int_0^t\int_\Omega |\nabla u_n^{\frac{\lambda-1-\theta+p}{p}}|^p \leq C\left(\int_Q (1+u_n)^{\lambda m'}\right)^{\frac{1}{m'}} + C.$$

Now passing to the supremum in time for $t \in (0,T)$ in the last inequality, we deduce

$$\frac{1}{\lambda+1}\|u_n^{\frac{\lambda-1-\theta+p}{p}}\|^{\frac{p(\lambda+1)}{\lambda-1-\theta+p}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda-1-\theta+1}}(\Omega))} + \frac{p^p\lambda}{(\lambda-1-\theta+p)^p}\int_Q |\nabla u_n^{\frac{\lambda-1-\theta+p}{p}}|^p \leq C\left(\int_Q (1+u_n)^{\lambda m'}\right)^{\frac{1}{m'}} + C. \tag{4.26}$$

Applying Lemma 2.9 (here $v = u_n^{\frac{\lambda-1-\theta+p}{p}}$, $\rho = \frac{p(\lambda+1)}{\lambda-1-\theta}$, $h = p$) and from (4.26), we have

$$\int_Q [u_n^{\frac{\lambda-1-\theta+p}{p}}]^{p\frac{N+\frac{p(\lambda+1)}{\lambda-1-\theta+p}}{N}} \leq C\left(\|u_n^{\frac{\lambda-1-\theta+p}{p}}\|^{\frac{p(\lambda+1)}{\lambda-1-\theta+p}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda-1-\theta+1}}(\Omega))}\right)^{\frac{p}{N}} \times \int_Q |\nabla u_n^{\frac{\lambda-1-\theta+p}{p}}|^p \leq C\left(\int_Q (1+u_n)^{\lambda m'}\right)^{(\frac{p}{N}+1)\frac{1}{m'}} + C.$$

Then, we can write the last inequality as follows

$$\int_Q u_n^{\frac{N(\lambda-1-\theta+p)+p(\lambda+1)}{N}} \leq C\left(\int_Q (1+u_n)^{\lambda m'}\right)^{(\frac{p}{N}+1)\frac{1}{m'}} + C. \tag{4.27}$$

Choose now λ such that

$$\sigma = \frac{N(\lambda-1-\theta+p)+p(\lambda+1)}{N} = \lambda m', \tag{4.28}$$

that is

$$\lambda = \frac{(m-1)(N(p-1-\theta)+p)}{N-pm+p}, \quad \sigma = \frac{m(N(p-1-\theta)+p)}{N-pm+p}.$$

Combining (4.27) and (4.28), we get

$$\int_Q u_n^\sigma \leq C\left(\int_Q (1+u_n)^\sigma\right)^{(\frac{p}{N}+1)\frac{1}{m'}} + C. \tag{4.29}$$

By virtue of $m < \frac{N}{p}+1$, we have $\left(\frac{p}{N}+1\right)\frac{1}{m'} < 1$ and applying Young's inequality with indices $\left(\frac{Nm'}{N+p}, \frac{Nm'}{Nm'-(N+p)}\right)$ in (4.29), we deduce that

$$\int_Q u_n^\sigma \leq C. \tag{4.30}$$

The condition $m \geq \frac{p(N+2+\theta)}{p(N+2+\theta)-N(1+\theta)}$ ensures that $\lambda \geq 1+\theta$. By the fact that $(1+u_n)^{\lambda-1-\theta} \geq 1$ and combining (4.26), (4.30), we get

$$\int_Q |\nabla u_n|^p \leq \int_Q |\nabla u_n|^p (1+u_n)^{\lambda-1-\theta} \leq C\left(\int_Q (1+u_n)^{\lambda m'}\right)^{\frac{1}{m'}} + C$$
$$\leq C\left(\int_Q u_n^{\lambda m'}\right)^{\frac{1}{m'}} + C = C\left(\int_Q u_n^{\sigma}\right)^{\frac{1}{m'}} + C \leq C.$$

This implies

$$\int_Q |\nabla u_n|^p \leq C. \tag{4.31}$$

□

Lemma 4.14. *Let the assumptions of Theorem 4.5 be in force. Then the solution u_n of (4.4) is uniformly bounded in $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\sigma(Q)$, where*

$$q = \frac{m(N(p-1-\theta)+p)}{N+1-(1+\theta)(m-1)} \text{ and } \sigma = \frac{m(N(p-1-\theta)+p)}{N-pm+p}.$$

Proof. By the definitions of $\Psi(s)$ and $\psi(s)$ in the proof of Lemma 4.13, we also have

$$\Psi(s) \geq Cs^{\lambda+1} - C, \quad \forall s \in \mathbb{R}^+, \tag{4.32}$$

assuming $0 < \lambda < 1+\theta$. Going back to (4.23) and from (4.32), we get

$$C\int_\Omega u_n(x,t)^{\lambda+1} + \lambda \int_0^t \int_\Omega |\nabla u_n|^p (1+u_n)^{\lambda-1}$$
$$+\alpha \int_0^t \int_\Omega u_n |\nabla u_n|^p (1+u_n)^{\lambda-1-\theta}$$
$$\leq \int_Q f(1+u_n)^\lambda + C meas(\Omega) + C.$$

By the fact that $\lambda(1+u_n)^\theta + \alpha u_n \geq \lambda$, and applying Hölder's inequality with indices $(m,\, m')$, the last inequality can be estimate as follows

$$C\int_\Omega u_n(x,t)^{\lambda+1} + \lambda \int_0^t \int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-\lambda}}$$
$$\leq C\left(\int_Q (1+u_n)^{\lambda m'}\right)^{\frac{1}{m'}} + C.$$

Passing to the supremum in time for $t \in (0,T)$, we have

$$\begin{aligned} &C\|u_n\|^{\lambda+1}_{L^\infty(0,T;L^{\lambda+1}(\Omega))} + \lambda \int_Q \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-\lambda}} \\ &\leq C\left(\int_Q (1+u_n)^{\lambda m'}\right)^{\frac{1}{m'}} + C. \end{aligned} \tag{4.33}$$

Let $1 < q < p$. Applying Hölder's inequality with indices $\left(\frac{p}{q}, \frac{p}{p-q}\right)$, we get

$$\begin{aligned}\int_Q |\nabla u_n|^q &= \int_Q \frac{|\nabla u_n|^q}{(1+u_n)^{\frac{(1+\theta-\lambda)q}{p}}}(1+u_n)^{\frac{(1+\theta-\lambda)q}{p}} \\ &\leq \left(\int_Q \frac{|\nabla u_n|^p}{(1+u_n)^{1+\theta-\lambda}}\right)^{\frac{q}{p}} \left(\int_Q (1+u_n)^{\frac{(1+\theta-\lambda)q}{p-q}}\right)^{\frac{p-q}{p}}. \qquad (4.34)\end{aligned}$$

The inequality (4.33), combined with (4.34), implies that

$$\int_Q |\nabla u_n|^q \leq C\left(\left(\int_Q (1+u_n)^{\lambda m'}\right)^{\frac{1}{m'}} + 1\right)^{\frac{q}{p}} \left(\int_Q (1+u_n)^{\frac{(1+\theta-\lambda)q}{p-q}}\right)^{\frac{p-q}{p}}. \qquad (4.35)$$

Applying Lemma 2.9 (here $v = u_n$, $\rho = \lambda + 1$, $h = q$), we have

$$\int_Q u_n^{\frac{q(N+\lambda+1)}{N}} \leq ||u_n||^{\frac{q(\lambda+1)}{N}}_{L^\infty(0,T;L^{\lambda+1}(\Omega))} \int_Q |\nabla u_n|^q.$$

We improve the above estimate using (4.33) and (4.35), obtaining

$$\int_Q u_n^{\frac{q(N+\lambda+1)}{N}} \leq C\left(\left(\int_Q (1+u_n)^{\lambda m'}\right)^{\frac{1}{m'}} + 1\right)^{\frac{q}{p}+\frac{q}{N}} \times \left(\int_Q (1+u_n)^{\frac{(1+\theta-\lambda)q}{p-q}}\right)^{\frac{p-q}{p}}. \qquad (4.36)$$

Choose now λ such that

$$\sigma = \frac{q(N+\lambda+1)}{N} = \lambda m' = \frac{(1+\theta-\lambda)q}{p-q}, \qquad (4.37)$$

that is equivalent to

$$\begin{aligned}\lambda = \frac{(m-1)(N(p-1-\theta)+p)}{N-pm+p},\ q &= \frac{m(N(p-1-\theta)+p)}{N+1-(1+\theta)(m-1)} \\ \text{and } \sigma = \frac{m(N(p-1-\theta)+p)}{N-pm+p}&. \end{aligned} \qquad (4.38)$$

By using (4.37) in (4.36), we deduce

$$\int_Q u_n^\sigma \leq C\left(\int_Q (1+u_n)^\sigma\right)^{\frac{1}{m'}(\frac{q}{p}+\frac{q}{N})+\frac{p-q}{p}} + C. \qquad (4.39)$$

By virtue of $m < \frac{N}{p} + 1$, then $\frac{1}{m'}\left(\frac{q}{p} + \frac{q}{N}\right) + \frac{p-q}{p} < 1$. Applying Young's inequality, we deduce

$$\int_Q u_n^\sigma \leq C. \qquad (4.40)$$

Since $\lambda < 1 + \theta$ (i.e $m < \frac{p(N+2+\theta)}{p(N+2+\theta)-N(1+\theta)}$), and using (4.37) in (4.35), we get

$$\int_Q |\nabla u_n|^q \leq \left(C\left(\int_Q (1+u_n)^\sigma\right)^{\frac{1}{m'}} + C\right)^{\frac{q}{p}} \left(\int_Q (1+u_n)^\sigma\right)^{\frac{p-q}{p}}.$$

The above estimate and (4.40) allow to conclude

$$\int_Q |\nabla u_n|^q \leq C. \tag{4.41}$$

The estimates (4.40) and (4.41) completed the proof of Lemma 4.14. □

Lemma 4.15. *Let the assumptions of Theorem 4.7 be in force. Then the solution u_n of (4.4) is bounded in $L^\delta(0,T;W_0^{1,\delta}(\Omega))$, where $\delta = \frac{N(p-\theta)}{N-\theta}$. Moreover, the sequence $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$.*

Proof. By (4.6) and using (4.2), we have

$$\frac{\alpha}{2^{\theta+1}} \int_{\{u_n\geq 1\}} \frac{|\nabla u_n|^p}{u_n^\theta} \leq \int_{\{u_n\geq 1\}} b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \leq \int_Q f. \tag{4.42}$$

Let δ any positive real number such that $1<\delta<p$. Using Hölder's inequality with indices $\left(\frac{p}{\delta}, \frac{p}{p-\delta}\right)$, we obtain

$$\int_Q |\nabla G_1(u_n)|^\delta = \int_{\{u_n\geq 1\}} \frac{|\nabla u_n|^\delta}{u_n^{\frac{\theta\delta}{p}}} u_n^{\frac{\theta\delta}{p}} \leq C\left[\int_{\{u_n\geq 1\}} \frac{|\nabla u_n|^p}{u_n^\theta}\right]^{\frac{\delta}{p}} \left[\int_{\{u_n\geq 1\}} u_n^{\frac{\theta\delta}{p-\delta}}\right]^{\frac{p-\delta}{p}}.$$

Using (4.42) in the last inequality, we get

$$\int_Q |\nabla G_1(u_n)|^\delta \leq C\left[\int_{\{u_n\geq 1\}} u_n^{\frac{\theta\delta}{p-\delta}}\right]^{\frac{p-\delta}{p}}. \tag{4.43}$$

The choice of $\delta = \frac{N(p-\theta)}{N-\theta}$ implies that $\delta^* = \frac{\delta\theta}{p-\delta}$. By Sobolev's inequality on the first term of (4.43), we have

$$\begin{aligned}\left(\int_Q G_1(u_n)^{\delta^*}\right)^{\frac{\delta}{\delta^*}} &\leq C_0\int_Q |\nabla G_1(u_n)|^\delta \leq C\left[\int_{\{u_n\geq 1\}} u_n^{\frac{\theta\delta}{p-\delta}}\right]^{\frac{p-\delta}{p}} \\ &= C\left[\int_{\{u_n\geq 1\}} u_n^{\delta^*}\right]^{\frac{\theta}{\delta^*}} \leq C\left[\int_{\{u_n\geq 1\}} G_1(u_n)^{\delta^*}\right]^{\frac{\theta}{\delta^*}} + C,\end{aligned} \tag{4.44}$$

where C_0 is the Sobolev constant. Since $\theta<1$, the inequality (4.44) implies that $G_1(u_n)$, hence u_n, is bounded in $L^{\delta^*}(Q)$. From (4.43), it follows the boundedness of $G_1(u_n)$ in $L^\delta(0,T;W_0^{1,\delta}(\Omega))$. Using $T_1(u_n)$ as test function in (4.4), we have

$$\begin{aligned}&\int_0^T\int_\Omega \frac{\partial u_n}{\partial t}T_1(u_n) + \int_Q |\nabla u_n|^{p-2}\nabla u_n \nabla T_1(u_n) + \int_Q b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}T_1(u_n) \\ &= \int_Q f_n T_1(u_n) \leq \int_Q f_n \leq \int_Q f.\end{aligned}$$

Therfore

$$\int_{\Omega} S_1(u_n(T)) + \int_Q |\nabla T_1(u_n)|^p + \alpha \int_Q \frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} T_1(u_n) \leq \int_Q f_n T_1(u_n),$$

where $S_1(u_n(T)) = \int_0^{u_n(T)} T_1(s)\,ds$. Since $u_n \geq 0$, it easy to se that $S_1(u_n(T)) \geq 0$ a.e. in Ω. After dropping the first and third non-negative terms and using (4.5), the last inequality becomes

$$\int_Q |\nabla T_1(u_n)|^p \leq \int_Q f \leq C.$$

We deduce that $T_1(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, hence in $L^\delta(0,T;W_0^{1,\delta}(\Omega))$. Since $u_n = G_1(u_n) + T_1(u_n)$, then we deduce that u_n is bounded in $L^\delta(0,T;W_0^{1,\delta}(\Omega))$. Moreover, testing (4.4) by $T_k(u_n)$, it is follows that $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$. □

Lemma 4.16. *Let u_n be a solution of (4.4). Then for every $\omega \subset\subset \Omega$, there exists a positive constant c_ω such that*

$$u_n \geq c_\omega > 0, \quad \textit{in } \omega \times (0,T), \textit{ for every } n \in \mathbb{N}.$$

Proof. For $s > 0$, we define the non decreasing function

$$H(s) = \int_s^1 \tilde{h}(\sigma)d\sigma = \int_s^1 h(\sigma)d\sigma - (p-1)\log s,$$

where $\tilde{h}(s) = h(s) + \frac{p-1}{s}$, $h(s) = \frac{1}{s^\theta}$, and we then consider the non increasing function

$$\psi(s) = \int_s^1 e^{-\beta H(\ell)}d\ell.$$

Observe that $\lim_{s\to 0^+}\psi(s) = +\infty$ and $\lim_{s\to+\infty}\psi(s) = \psi_\infty \in [-\infty, 0)$.
Let $0 < \phi \in C_c^\infty(\Omega)$, and take $e^{-\beta H(u_n)}\phi \in L^p(0,T;W_0^{1,p}(\Omega))$ as a test function in (4.4). Then, we have

$$\begin{aligned}\int_0^T\int_\Omega \frac{\partial u_n}{\partial t}\phi e^{-\beta H(u_n)} + \int_Q |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\phi e^{-\beta H(u_n)} \\ -\beta\int_Q |\nabla u_n|^p \tilde{h}(u_n)\phi e^{-\beta H(u_n)} + \int_Q b(x,t)\frac{u_n|\nabla u_n|^2}{(u_n+\frac{1}{n})^{\theta+1}}\phi e^{-\beta H(u_n)} \\ = \int_Q f_n\phi e^{-\beta H(u_n)}.\end{aligned}$$

Thanks to easy simplification in the last equality, we can write as follows

$$\begin{aligned}\int_0^T\int_\Omega \frac{\partial u_n}{\partial t}\phi e^{-\beta H(u_n)} + \int_Q |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\phi e^{-\beta H(u_n)} \\ + \int_Q |\nabla u_n|^p\phi e^{-\beta H(u_n)}\left[\frac{b(x,t)u_n}{(u_n+\frac{1}{n})^{\theta+1}} - \beta\tilde{h}(u_n)\right] = \int_Q f_n\phi e^{-\beta H(u_n)}.\end{aligned}$$

By the fact that $\frac{s}{(s+\epsilon)^{\theta+1}} \leq h(s) \leq \tilde{h}(s)$, with $0 < \epsilon < 1$ and using (4.2), we get

$$\int_0^T \int_\Omega \frac{\partial u_n}{\partial t} \phi e^{-\beta H(u_n)} + \int_Q |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \phi e^{-\beta H(u_n)} \geq \int_Q f_n e^{-\beta H(u_n)} \phi \geq \int_Q f_n (e^{-\beta H(u_n)} - 1)\phi.$$

Let $v_n := \psi(u_n)$, then $\nabla v_n = -e^{\beta H(u_n)} \nabla u_n$, and so we can write the last inequality as follows

$$-\int_0^T \int_\Omega \frac{\partial v_n}{\partial t} \phi - \int_Q |\nabla v_n|^{p-2} \nabla v_n \cdot \nabla \phi \geq \int_Q f_n (e^{-\beta H(\psi^{-1}(v_n))} - 1)\phi.$$

Thus, we deduce that v_n is subsolution of

$$\frac{\partial z}{\partial t} - \Delta_p z + f(x,t) g(z) = 0, \quad \text{in } Q,$$

with $g(s) = e^{-\beta H(\psi^{-1}(s))} - 1$ for every $s \in (\psi_\infty, +\infty)$. The function $g(s)$ satisfies:

(1) $\frac{g(s)}{s^{p-1}}$ is increasing for $s > 0$ large.

(2) The Keller-Osserman condition, i.e.,

$$\int_{\sigma_0}^{+\infty} \left(\int_0^\sigma g(s)\, ds \right)^{\frac{-1}{p}} d\sigma < +\infty \text{ for some } \sigma_0 > 0.$$

For the proof of (1) and (2) see [144]. Since f satisfies

$$\operatorname{ess\,inf}\{f(x,t) : x \in \omega, t \in (0,t)\} > 0 \quad \forall\, \omega \subset\subset \Omega,$$

we can apply Lemma 3.12 in [105] to the previous equation to obtain the existence of $C_{\omega,T} > 0$ such that

$$v_n \leq C_{\omega,T} \quad \forall x \in \omega \ \textit{and}\ t \in (0,T).$$

Therefore, there exists $c_\omega > 0$ (independent of n) such that

$$u_n \geq \psi(C_0) = c_\omega, \ \textit{in}\ \omega \times (0,T).$$

□

4 Proof of main results

Because the proofs of Theorem 4.2 and Theorem 4.3 are similar too that of Theorem 4.4, and the proof of Theorem 4.5 is also similar to that of Theorem 4.7, here we only detail the proofs of Theorem 4.4 and Theorem 4.5.

Proof. of Theorem 4.4. By Lemma 4.13, there exist a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and a measurable function u such that

$$u_n \rightharpoonup u \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{4.45}$$

$$u_n \rightharpoonup u \text{ weakly in } L^\sigma(Q). \tag{4.46}$$

In view of Lemma 4.13 and Remark 4.10, we have that $\{\frac{\partial u_n}{\partial t}\}$ is bounded in the space $L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$. Then, using compactness results (see [139]), we obtain

$$u_n \longrightarrow u \text{ strongly in } L^1(Q) \text{ and a.e. in } Q. \tag{4.47}$$

Let $z_n = f_n - b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}$. From (4.5) and (4.47), we have z_n converges to $f + b(x,t)\frac{|\nabla u|^p}{u^\theta}$ a.e. in Q. By (4.6), we get

$$\int_Q |z_n| \le \int_Q f_n + \int_Q b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \le 2\int_Q f.$$

Then by (4.5) and (4.47), and the Dominated Convergence Theorem, we obtain z_n strongly converges in $L^1(Q)$. Since u_n is solution of

$$\begin{cases} \frac{\partial u_n}{\partial t} - \Delta_p u_n = z_n & \text{in} \quad Q, \\ u_n(x,t) = 0 & \text{on} \quad \Gamma, \\ u_n(x,0) = 0 & \text{in} \quad \Omega, \end{cases}$$

then we can be adopting the approach of [22, Theorem 3.1], we deduce that there exist a subsequence, still denoted u_n, such that

$$\nabla u_n \longrightarrow \nabla u \text{ a.e. in } Q. \tag{4.48}$$

From (4.45) we obtain

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u \text{ weakly in } \left(L^{p'}(Q)\right)^N. \tag{4.49}$$

□

Now we prove that

$$b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \longrightarrow b(x,t)\frac{|\nabla u|^p}{u^\theta} \text{ strongly in } L^1(Q).$$

Let E be a compact subset in Q, we have

$$\begin{aligned} \int_E \frac{b(x,t)u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} &= \int_{E\cap\{u_n\le k\}} \frac{b(x,t)u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} + \int_{E\cap\{u_n>k\}} \frac{b(x,t)u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \\ &\le \int_{E\cap\{u_n\le k\}} b(x,t)\frac{|\nabla u_n|^p}{u_n^\theta} + \int_{E\cap\{u_n>k\}} b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}. \end{aligned}$$

By Lemma 4.16, we get

$$\int_E b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \le \frac{1}{c_w^\theta}\int_E b(x,t)|\nabla T_k(u_n)|^p + \int_{E\cap\{u_n>k\}} b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}.$$

Let $\epsilon > 0$ be fixed. For $k > 1$, we use $T_1(u_n - T_{k-1}(u_n))$ as test function in (4.4), obtaining

$$\int_0^T \int_\Omega \frac{\partial u_n}{\partial t} T_1(u_n - T_{k-1}(u_n)) + \int_Q |\nabla u_n|^{p-2}\nabla u_n \nabla T_1(u_n - T_{k-1}(u_n))$$
$$+ \int_Q b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} T_1(u_n - T_{k-1}(u_n)) = \int_Q f_n T_1(u_n - T_{k-1}(u_n)).$$

Therefore

$$\int_\Omega S_1(u_n(T)) + \int_{\{k-1\le u_n\le k\}} |\nabla u_n|^p + \int_Q b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} T_1(u_n - T_{k-1}(u_n))$$
$$= \int_Q f_n T_1(u_n - T_{k-1}(u_n)),$$

where $S_1(u_n(T)) = \displaystyle\int_0^{u_n(T)} T_1(s - T_{k-1}(s))\, ds$. It easy to see that $S_1(u_n(T)) \ge 0$ a.e. in Ω. Dropping the first and second non-negative terms, the last equality becomes

$$\int_Q b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} T_1(u_n - T_{k-1}(u_n)) \le \int_Q f_n T_1(u_n - T_{k-1}(u_n)). \tag{4.50}$$

Since $T_1(u_n - T_{k-1}(u_n)) \ge 0$, $T_1(u_n - T_{k-1}(u_n)) = 0$ if $u_n \le k-1$, and $T_1(u_n - T_{k-1}(u_n)) = 1$ if $u_n > k$, we have

$$\int_Q b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} T_1(u_n - T_{k-1}(u_n))$$
$$= \int_{Q\cap\{u_n>k\}} b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} T_1(u_n - T_{k-1}(u_n))$$
$$+ \int_{Q\cap\{u_n\le k\}} b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} T_1(u_n - T_{k-1}(u_n))$$
$$= \int_{Q\cap\{u_n>k\}} b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}$$
$$+ \int_{Q\cap\{u_n\le k\}} b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} T_1(u_n - T_{k-1}(u_n))$$
$$\ge \int_{E\cap\{u_n>k\}} b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}},$$

and

$$\int_Q f_n T_1(u_n - T_{k-1}(u_n)) = \int_{Q\cap\{u_n\le k-1\}} f_n T_1(u_n - T_{k-1}(u_n))$$
$$+ \int_{Q\cap\{k-1<u_n\le k\}} f_n T_1(u_n - T_{k-1}(u_n)) + \int_{Q\cap\{u_n>k\}} f_n T_1(u_n - T_{k-1}(u_n))$$
$$= \int_{Q\cap\{k-1<u_n\le k\}} f_n T_1(u_n - (k-1)) + \int_{Q\cap\{u_n>k\}} f_n$$
$$\le \int_{Q\cap\{k-1<u_n\le k\}} f_+ \int_{Q\cap\{u_n>k\}} f = \int_{Q\cap\{u_n\ge k-1\}} f.$$

Therefore, from (4.50) and the two later inequalities we obtain

$$\int_{E\cap\{u_n>k\}} b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \leq \int_{Q\cap\{u_n\geq k-1\}} f.$$

It follows from $f \in L^1(Q)$ that

$$\int_{E\cap\{u_n>k\}} b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \longrightarrow 0 \ \text{ as } \ k \longrightarrow \infty.$$

Then, there exist $k_0 > 1$ such that

$$\int_{E\cap\{u_n>k\}} b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \leq \frac{\epsilon}{2}, \ \ \forall k \geq k_0, \ \forall n \in \mathbb{N}. \tag{4.51}$$

Moreover, similar to the proof of [60, Proposition 3.4] we obtain $T_k(u_n) \longrightarrow T_k(u)$ strongly in $L^p(0,T;W^{1,p}_{loc}(\Omega))$. Then, there exits $n_\epsilon, \delta_\epsilon$ such that $meas(E) \leq \delta_\epsilon$ we have

$$\frac{1}{c_w^\theta}\int_E b(x,t)|\nabla T_k(u_n)|^p \leq \frac{\epsilon}{2} \quad \forall n \geq n_\epsilon. \tag{4.52}$$

The estimates (4.51) and (4.52), implies that $b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}$ is equi-integrable. This fact, together with a.e. convergence of this term to $b(x,t)\frac{|\nabla u|^p}{u^\theta}$, implies by Vitali's Theorem that

$$b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \longrightarrow b(x,t)\frac{|\nabla u|^p}{u^\theta}, \ \text{ strongly in } L^1(Q). \tag{4.53}$$

Let $\varphi \in C^\infty(\overline{Q})$ which is zero in a neighborhood of $\Gamma \cup (\Omega \times \{T\})$. Taking φ as a test function in problem (4.4), by (4.5), (4.47), (4.49) and (4.53), we can let $n \to +\infty$ obtaining

$$-\int_Q u\frac{\partial \varphi}{\partial t} + \int_Q |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \int_Q b(x,t)\frac{|\nabla u|^p}{u^\theta}\varphi = \int_Q f\varphi.$$

Thus Theorem 4.4 is proved.

Proof. of Theorem 4.5. By Lemma 4.14, there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and a measurable function u such that

$$u_n \rightharpoonup u \text{ weakly in } \ L^q(0,T;W_0^{1,q}(\Omega)), \tag{4.54}$$

$$u_n \rightharpoonup u \text{ weakly in } \ L^\sigma(Q). \tag{4.55}$$

In view of Lemma 4.14 and Remark 4.10, we have that $\{\frac{\partial u_n}{\partial t}\}$ is bounded in the space $L^s(0,T;W^{-1,s}(\Omega))$ $+ L^1(Q)$ with $s = \frac{q}{p-1}$, which is sufficient to apply [139, Corollary 4] in order to deduce that

$$u_n \longrightarrow u \ \text{ strongly in } \ L^1(Q) \text{ and a.e. in } Q. \tag{4.56}$$

We repeat the same proof as in Theorem 4.4, obtaining

$$\nabla u_n \longrightarrow \nabla u \text{ a.e. in } \ Q. \tag{4.57}$$

Using the same proof as in Theorem 4.4, we obtain

$$b(x,t)\frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} \longrightarrow b(x,t)\frac{|\nabla u|^p}{u^\theta} \text{ strongly in } L^1(Q). \tag{4.58}$$

Since $m > \max\left(1, \frac{(p-1)(N+2+\theta)}{(p-1)(N+2+\theta)-(N\theta-1)}\right)$, then $q > p-1$. By Lemma 4.14, (4.57) and using Vitali's Theorem, we can show

$$|\nabla u_n|^{p-1} \longrightarrow |\nabla u|^{p-1} \text{ strongly in } L^1(Q). \tag{4.59}$$

Let $\varphi \in C^\infty(\overline{Q})$ which is zero in a neighborhood of $\Gamma \cup (\Omega \times \{T\})$. Taking φ as a test function in problem (4.4), by (4.5), (4.56), (4.58) and (4.59), we can let $n \to +\infty$ obtaining

$$-\int_Q u\frac{\partial\varphi}{\partial t} + \int_Q |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \int_Q b(x,t)\frac{|\nabla u|^p}{u^\theta}\varphi = \int_Q f\varphi.$$

□

Chapter 5

Existence of positive solutions to nonlinear singular parabolic equations with Hardy potential

1 Introduction

In this chapter, we are interested to prove existence and regularity results for a class of nonlinear singular parabolic equations involving Hardy potential, as following model

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,\nabla u)) - \mu \dfrac{u^{p-1}}{|x|^p} = \dfrac{f}{u^\gamma} & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{5.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $(N \geq 3)$, $2 \leq p < N$, $\gamma, \mu > 0$, $Q = \Omega \times (0,T)$, $\Gamma = \partial\Omega \times (0,T)$, with $T > 0$, f is a nonnegative function belonging a suitable Lebesgue space, the initial datum $u_0 \in L^\infty(\Omega)$ and satisfies the following bound

$$\forall\, \omega \subset\subset \Omega, \quad \exists\, M_\omega > 0 \;:\; u_0 \geq M_\omega \text{ in } \omega. \tag{5.2}$$

Moreover, the function $a : \Omega \times (0,T) \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfying the following conditions: there exist positive constants α, β such that

$$a(x,t,\xi) \cdot \xi \geq \alpha|\xi|^p, \tag{5.3}$$

$$|a(x,t,\xi)| \leq \beta|\xi|^{p-1}, \tag{5.4}$$

$$[a(x,t,\xi) - a(x,t,\xi')] \cdot [\xi - \xi'] > 0, \tag{5.5}$$

for almost every $x \in \Omega, t \in (0,T)$, for every $\xi, \xi' \in \mathbb{R}^N$, with $\xi \neq \xi'$.
Under assumptions (5.3), (5.4) and (5.5), the differential operator defined by

$$A(u) = -\operatorname{div}(a(x,t,\nabla u)), \quad u \in L^p(0,T; W_0^{1,p}(\Omega))$$

is coercive and monotone operator acting from the space $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$. The simplest example is the one in which the operator A is the p-Laplacian: $A(u) = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

From a purely mathematical point of view the literature is wide. In the case $\mu = 0$ and $\gamma = 0$, the existence and regularity of problem (5.1) has been studied in [16, 22, 75, 98, 108, 128] under the different assumptions on the data. If $\gamma = 0, f = 0$ and $\mu > 0$ the existence and nonexistence of solution of problem (5.1) depending the value of μ has been studied by the authors in [84, 131]. When $\gamma = 0, f \neq 0$ and $\mu > 0$, the authors in [4] has been studied the existence and summability of elliptic problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + b|u|^{r-2}u = \mu\dfrac{u}{|x|^2} + f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $b > 0, \mu > 0, r > 2^*$ and $f \in L^m(\Omega)$, $m > 1$, $M(x)$ is a matrix satisfies $M(x)\xi\cdot\xi \geq \alpha|\xi|^2; |M(x)| \leq \beta$ with $\alpha, \beta \geq 0$ for all $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$. Baras in [13] studied the existence and nonexistence of problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = v(x)u + f(x,t) & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

where $v(x) = c/|x|^2, c > 0, v \in L^\infty(\Omega\backslash B_\epsilon)$ (where $B_\epsilon = \{x : |x| < \epsilon\}$), the function v is singular at the origin and $u_0, f \geq 0$ satisfies some conditions. In the same contexts Porzio [130] showed that the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) = \mu\dfrac{u}{|x|^2} + f & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

admits a solution for $0 < \mu < \varrho_1\left(\frac{N-2}{2}\right)^2$, where ϱ_1 is the coercivity constant of $a(x,t,u,\nabla u)$, $f \in L^r(0,T;L^q(\Omega))$, with $r > 1, q > 1$, and the summability of solution also obtained (See also [82, 132]). When $\mu = 0$ and $\gamma > 0$, the problem of existence, regularity and uniqueness (sometimes partial uniqueness) results of (5.1) have been investigated in different contexts by several authors (see [68, 122, 135, 136] and references therein). The authors in [68] proved the existence, regularity and uniqueness of solution to singular parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,\nabla u)) = \dfrac{f}{u^\gamma} & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

where $\gamma > 0$ and $0 \leq f \in L^m(Q)$, $m \geq 1$ and $u_0 \in L^\infty(\Omega)$ satisfies

$$\forall \omega \subset\subset \Omega, \exists\, d_\omega > 0 : \; u_0 \geq d_\omega.$$

Finally in the elliptic framework when $\mu > 0, \gamma > 0$ the author in [142] proved the existence of one positive solution to singular problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) - \mu\frac{u}{|x|^2} = \frac{f}{u^\gamma} & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $0 \in \Omega, \gamma > 0, 0 \leq f \in L^m(\Omega)$, $1 < m < \frac{N}{2}$ and $0 < \mu < \left(\frac{N-2}{2}\right)^2$. The stationary problem associated to problem (5.1) has been studied in [2]; the authors proved the existence and regularity (and partial uniqueness) results of solution to singular problem

$$\begin{cases} \Delta_p u = \mu \frac{u^{p-1}}{|x|^p} + \frac{f}{u^\gamma} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $0 \in \Omega, \gamma > 0, 0 < \mu \leq \left(\frac{N-p}{p}\right)^p$ and $0 \leq f \in L^m(\Omega), m \geq 1$. If $\mu > \left(\frac{N-p}{p}\right)^p$, then the problem has non solution (see [1]), and also the authors proved that if f is a singular measure with respect to the p-Capacity associated to $W_0^{1,p}(\Omega)$ the problem has a non-negative solution in suitable sense.

The aim of this chapter1101011 is to analyze the interaction between the Hardy potential and the singular term $u^{-\gamma}$ in order to get a solution for largest possible class of the datum f.

The problem (5.1) is related to the following classical Hardy inequality (see [84])

$$C_{N,p} \int_{\mathbb{R}^N} \frac{|\psi|^p}{|x|^p} dx \leq \int_{\mathbb{R}^N} |\nabla \psi|^p dx, \quad \text{for all } \psi \in W^{1,p}(\mathbb{R}^N),$$

where $C_{N,p} = \left(\frac{N-p}{p}\right)^p$ is optimal and is not attained. Let's now give the meaning of the weak solution to the problem (5.1) we will use throughout this chapter.

Definition 5.1. We will say that a function $u \in L^1(0,T; W^{1,1}_{loc}(\Omega))$ is a distributional solution of (5.1) if

$$|\nabla u|^{p-1} \in L^1(0,T; L^1_{loc}(\Omega)), \quad \frac{|u|^{p-1}}{|x|^p} \in L^1(0,T; L^1_{loc}(\Omega)) \tag{5.6}$$

$$u = 0 \text{ on } \partial\Omega \times (0,T) \quad \text{in weak sense,} \tag{5.7}$$

i.e., some positive power of u belongs to a Sobolev space $L^r(0,T; W_0^{1,r}(\Omega))$, $r > 1$. Moreover, we require that

$$\forall \omega \subset\subset \Omega \ \exists c_\omega > 0 : \quad u \geq c_\omega \ \text{ in } \ \omega \times (0,T), \tag{5.8}$$

and that

$$\begin{aligned} -\int_\Omega u_0(x)\varphi(x,0) - \iint_Q u \frac{\partial \varphi}{\partial t} + \iint_Q a(x,t,\nabla u)\nabla\varphi \\ = \iint_Q \frac{u^{p-1}}{|x|^p}\varphi + \iint_Q \frac{f}{u^\gamma}\varphi, \end{aligned} \tag{5.9}$$

for all $\varphi \in C_c^1(\Omega \times [0,T))$.

2 The approximation scheme

Let $n \in \mathbb{N}$ and $f_n(x,t)$ be defined by $f_n(x,t) = \min(f(x,t), n)$; we will consider the following approximation of (5.1)

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div}(a(x,t,\nabla u_n)) - \mu \dfrac{u_n^{p-1}}{|x|^p + \frac{1}{n}} = \dfrac{f_n}{(|u_n| + \frac{1}{n})^\gamma} & \text{in} \quad Q, \\ u_n = 0 & \text{on} \quad \Gamma, \\ u_n(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases} \tag{5.10}$$

Lemma 5.2. *The problem* (5.10) *has a non-negative solution belonging to* $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ *for all* $\mu < \alpha C_{N,p}$ *and* $2 \leq p < N$.

Proof. Let $v \in L^p(Q)$ and we define $S : L^p(Q) \longrightarrow L^p(Q)$ such that $S(v) = w$, with $w \in L^p(0,T;W_0^{1,p}(\Omega)) \cap C([0,T];L^2(\Omega))$ the unique solution of problem

$$\begin{cases} \dfrac{\partial w}{\partial t} - \operatorname{div}(a(x,t,\nabla w)) - \mu \dfrac{w^{p-1}}{|x|^p + \frac{1}{n}} = \dfrac{f_n}{(|v| + \frac{1}{n})^\gamma} & \text{in} \quad Q, \\ w = 0 & \text{on} \quad \Gamma, \\ w(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}$$

The existence of solution of above problem assured by [108]. Let us take w as a test function in the above problem, from (5.3), we have

$$\frac{1}{2}\int_\Omega w^2(x,t) + \alpha \iint_Q |\nabla w|^p - \mu \iint_Q \frac{w^p}{|x|^p} \leq \iint_Q \frac{f_n w}{(|v| + \frac{1}{n})^\gamma} + \frac{1}{2}\int_\Omega u_0^2,$$

since $u_0 \in L^\infty(\Omega)$ and by Hardy inequality implies

$$\frac{1}{2}\int_\Omega w^2(x,t) + \left(\alpha - \frac{\mu}{C_{N,p}}\right) \iint_Q |\nabla w|^p \leq \iint_Q \frac{f_n w}{(|v| + \frac{1}{n})^\gamma} + \frac{1}{2}||u_0||^2_{L^2(\Omega)}. \tag{5.11}$$

Dropping the first non-negative term and thanks to Hölder's inequality, we have

$$\left(\alpha - \frac{\mu}{C_{N,p}}\right) \iint_Q |\nabla w|^p \leq |Q|^{\frac{1}{p'}} n^{\gamma+1} \left(\iint_Q |w|^p\right)^{\frac{1}{p}} + \frac{1}{2}||u_0||^2_{L^2(\Omega)}.$$

By application of Poincaré inequality in the right hand side, it hold that

$$||\nabla w||^p_{L^p(Q)} \leq \frac{|Q|^{p'} n^{\gamma+1} C_p}{(\alpha - \frac{\mu}{C_{N,p}})} ||\nabla w||_{L^p(Q)} + \frac{1}{2(\alpha - \frac{\mu}{C_{N,p}})} ||u_0||^2_{L^2(\Omega)}, \tag{5.12}$$

where C_p is the Poincaré constant. This implies that

$$||w||_{L^p(Q)} \leq R, \tag{5.13}$$

for some constant R independent of v. So that the ball of radius R is invariant under S. Using Sobolev embedding Theorem, it is easy to prove that S is both continuous and compact on $L^p(Q)$, so that by Shauder's fixed point Theorem there exist $u_n \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that $S(u_n) = u_n$, for all $n \in \mathbb{N}$, $2 \leq p < N$, i.e. u_n solves

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div}(a(x,t,\nabla u_n)) - \mu \dfrac{u_n^{p-1}}{|x|^p + \frac{1}{n}} = \dfrac{f_n}{(|u_n| + \frac{1}{n})^\gamma} & \text{in} \quad Q, \\ u_n = 0 & \text{on} \quad \Gamma, \\ u_n(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}$$

Moreover, since $\frac{f_n}{(|u_n| + \frac{1}{n})^\gamma} \geq 0$, taking $u_n^- = \min(u_n, 0)$ test function in (5.10) and using (5.3), then we have

$$\frac{1}{2}\int_\Omega |u_n^-|^2 + \alpha \iint_Q |\nabla u_n^-|^p - \mu \iint_Q \frac{u_n^{-p}}{|x|^p} \leq 0,$$

dropping the first nonnegative term and by Hardy inequality, we can get

$$(\alpha - \frac{\mu}{C_{N,p}}) \iint_Q |\nabla u_n^-|^p \leq 0,$$

as $\alpha - \frac{\mu}{C_{N,p}} > 0$, then we deduce that

$$\iint_Q |\nabla u_n^-|^p \leq 0,$$

that implies that $u_n^- = 0$ a.e and hence $u_n \geq 0$. a.e.. □

Lemma 5.3. *Let u_n be a solution of* (5.10). *Then*

$$\forall \omega \subset\subset \Omega,\ \exists\, c_\omega > 0 \ \textit{(independent of n)}: \ \ u_n \geq c_\omega \ \textit{in} \ \omega \times [0,T],\ \forall n \in \mathbb{N}.$$

Proof. Since u_n solution of (5.10), then

$$\frac{\partial u_n}{\partial t} - \operatorname{div}(a(x,t,\nabla u_n)) - \mu \frac{u_n^{p-1}}{|x|^p + \frac{1}{n}} = \frac{f_n}{(|u_n| + \frac{1}{n})^\gamma},$$

as $\mu > 0$, then we obtain

$$\frac{\partial u_n}{\partial t} - \operatorname{div}(a(x,t,\nabla u_n)) \geq \frac{f_n}{(|u_n| + \frac{1}{n})^\gamma},$$

this implies that the sequence u_n is a sub-solution to problem

$$\begin{cases} \dfrac{\partial v}{\partial t} - \operatorname{div}(a(x,t,\nabla v)) = \dfrac{f_n}{(|v| + \frac{1}{n})^\gamma} & \text{in} \quad Q, \\ v = 0 & \text{on} \quad \Gamma, \\ v(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}$$

Thanks to Proposition 2.2 in [68], $\exists\, c_\omega > 0$ (independent of n) such that

$$v \geq c_w \ \text{ in } \ \omega \times (0,T),\ \forall n \in \mathbb{N},\ \forall\, \omega \subset\subset \Omega,$$

since $u_n \geq v$, so

$$u_n \geq c_\omega \ \text{ in } \ \omega \times (0,T),\ \forall n \in \mathbb{N},\ \forall\, \omega \subset\subset \Omega.$$

□

3 A priori estimate and main results

Now, we prove some a priori estimates on the sequence of approximated solutions u_n.

Lemma 5.4. *Assume that* (5.3)-(5.5) *hold true,* $f \in L^{\frac{p(N+2)}{p(N+2)-N(1-\gamma)}}(Q)$. *If* $\gamma < 1$ *and* $\mu < \alpha C_{N,p}$, *then the sequence* u_n *is uniformly bounded in* $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

Proof. Take $u_n\chi_{(0,t)}$ as a test function in (5.10) (with $0 < t \le T$), from (5.3) and $f_n \le f$ we have

$$\frac{1}{2}\int_\Omega u_n^2(x,t) + \alpha\int_0^t\int_\Omega |\nabla u_n|^p - \mu\int_0^t\int_\Omega \frac{u_n^p}{|x|^p}$$
$$\le \int_0^t\int_\Omega f_n u_n^{1-\gamma} + \frac{1}{2}\int_\Omega u_0^2 \le \iint_Q f u_n^{1-\gamma} + \frac{1}{2}\int_\Omega u_0^2,$$

since $u_0 \in L^\infty(\Omega)$, thanks to Hölder's and Hardy inequalities imply that

$$\frac{1}{2}\int_\Omega u_n^2(x,t) + \left(\alpha - \frac{\mu}{C_{N,p}}\right)\int_0^t\int_\Omega |\nabla u_n|^p$$
$$\le ||f||_{L^{\frac{p(N+2)}{p(N+2)-N(1-\gamma)}}(Q)}\left(\iint_Q u_n^{\frac{p(N+2)}{N}}\right)^{\frac{N(1-\gamma)}{p(N+2)}} + \frac{1}{2}||u_0||^2_{L^2(\Omega)}.$$

Passing to the supremum for $t \in [0,T]$

$$\frac{1}{2}||u_n||^2_{L^\infty(0,T;L^2(\Omega))} + \left(\alpha - \frac{\mu}{C_{N,p}}\right)\iint_Q |\nabla u_n|^p$$
$$\le ||f||_{L^{\frac{p(N+2)}{p(N+2)-N(1-\gamma)}}(Q)}\left(\iint_Q u_n^{\frac{p(N+2)}{N}}\right)^{\frac{N(1-\gamma)}{p(N+2)}} + \frac{1}{2}||u_0||^2_{L^2(\Omega)}.$$

By Lemma 2.9 (here $v = u_n,\ \rho = 2,\ h = p$), we can write

$$\iint_Q |u_n|^{\frac{p(N+2)}{N}} \le C_G||u_n||^{\frac{2p}{N}}_{L^\infty(0,T;L^2(\Omega))}\iint_Q |\nabla u_n|^p$$
$$\le C\left(\iint_Q |u_n|^{\frac{p(N+2)}{N}}\right)^{\frac{(p+N)(1-\gamma)}{p(N+2)}} + C(u_0),$$

since $0 < \gamma < 1$ then $\frac{(p+N)(1-\gamma)}{p(N+2)} < 1$, this implies the sequence u_n is bounded in $L^{\frac{p(N+2)}{N}}(Q)$, hence u_n is bounded in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ with respect to n. □

Lemma 5.5. *Assume that* (5.3)-(5.5) *hold true,* $\gamma \ge 1$, $\mu < \alpha C_{N,p}$ *and* $f \in L^1(Q)$, *then*

i) If $\gamma = 1$, *then* u_n *is bounded in* $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

ii) If $\gamma > 1$, *then* u_n *is bounded in* $L^p(0,T;W_{loc}^{1,p}(\Omega))$ *and* $T_k(u_n)^{\frac{\gamma+p-1}{p}}$ *is bounded in* $L^p(0,T;W_0^{1,p}(\Omega))$. *Moreover if* $\alpha\left(\frac{p}{\gamma+p-1}\right)^p - \frac{\mu}{C_{N,p}} > 0$, *then* $u_n^{\frac{\gamma+p-1}{p}}$ *is bounded in* $L^p(0,T;W_0^{1,p}(\Omega))$ *and* u_n *is bounded in* $L^\infty(0,T;L^{\gamma+1}(\Omega))$.

Proof. **First case:** $\gamma = 1$
Choosing $u_n\chi_{(0,t)}$ as a test function in (5.10) (with $0 < t \le T$), by (5.3) and the fact that $0 \le \frac{u_n}{u_n+\frac{1}{n}} \le 1$, $f_n \le f$, we have

$$\frac{1}{2}\int_\Omega u_n^2(x,t) + \alpha\int_0^t\int_\Omega |\nabla u_n|^p - \mu\int_0^t\int_\Omega \frac{u_n^p}{|x|^p}$$
$$\le \int_0^t\int_\Omega f_n\frac{u_n}{u_n+\frac{1}{n}} + \frac{1}{2}\int_\Omega u_0^2 \le \int_0^t\int_\Omega f + \frac{1}{2}||u_0||^2_{L^2(\Omega)},$$

thanks to Hardy inequality, there result that

$$\frac{1}{2}\int_\Omega u_n^2(x,t)+\left(\alpha-\frac{\mu}{C_{N,p}}\right)\int_0^t\int_\Omega|\nabla u_n|^p\leq\int_0^t\int_\Omega f+\frac{1}{2}||u_0||^2_{L^2(\Omega)}.$$

Passing to the supremum for $t\in[0,T]$ and the fact that $u_0\in L^\infty(\Omega)$, we get

$$\frac{1}{2}||u_n||^2_{L^\infty(0,T;L^2(\Omega))}+\left(\alpha-\frac{\mu}{C_{N,p}}\right)\iint_Q|\nabla u_n|^p\leq\iint_Q f+\frac{1}{2}||u_0||^2_{L^2(\Omega)}\leq C,$$

since $\alpha-\frac{\mu}{C_{N,p}}>0$, then the sequence u_n is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and in $L^\infty(0,T;L^2(\Omega))$ with respect to n. Hence the proof of item $i)$ is achieved.

Second case: $\gamma>1$

Now taking $G_k(u_n)$ as test function in (5.10), from (5.3) we arrive to

$$\begin{aligned}&\frac{1}{2}\int_\Omega|G_k(u_n(x,T))|^2+\alpha\iint_Q|\nabla G_k(u_n)|^p-\mu\iint_Q\frac{u_n^{p-1}G_k(u_n)}{|x|^p}\\&\leq\iint_Q\frac{f_nG_k(u_n)}{(u_n+\frac{1}{n})^\gamma}+\frac{1}{2}\int_\Omega|G_k(u_0(x))|^2,\end{aligned}\tag{5.14}$$

dropping the first nonnegative term and as $G_k(u_n)=0$ if $u_n\leq k$ and the fact that $G_k(u_0(x))\leq u_0(x)$, then

$$\begin{aligned}&\alpha\iint_Q|\nabla G_k(u_n)|^p-\mu\iint_Q\frac{u_n^{p-1}G_k(u_n)}{|x|^p}\\&\leq\iint_{Q\cap\{u_n>k\}}\frac{f_nG_k(u_n)}{(u_n+\frac{1}{n})^\gamma}+\frac{1}{2}\int_\Omega|u_0(x)|^2\\&\leq\frac{1}{k^{\gamma-1}}\iint_Q f+\frac{1}{2}\int_\Omega|u_0(x)|^2\leq\frac{1}{k^{\gamma-1}}\iint_Q f+\frac{1}{2}||u_0||^2_{L^2(\Omega)}.\end{aligned}\tag{5.15}$$

Notice that for all $a,b\geq0$ and for all $\epsilon>0$, we have

$$(a+b)^r\leq(1+\epsilon)^{r-1}a^r+(1+\frac{1}{\epsilon})^{r-1}b^r,\quad\text{if }\ r>1.$$

For $u_n>k$, we have $u_n^{p-1}G_k(u_n)=(G_k(u_n)+k)^{p-1}G_k(u_n)$ and $p\geq2$, then from the previous estimate we reach that

$$u_n^{p-1}G_k(u_n)\leq(1+\epsilon)^{p-2}(G_k(u_n))^p+(1+\frac{1}{\epsilon})^{p-2}k^{p-1}G_k(u_n).\tag{5.16}$$

In view of (5.15) and (5.16), it follows that

$$\begin{aligned}&\alpha\iint_Q|\nabla G_k(u_n)|^p-\mu(1+\epsilon)^{p-2}\iint_Q\frac{(G_k(u_n))^p}{|x|^p}\\&\leq\mu(1+\frac{1}{\epsilon})^{p-2}k^{p-1}\iint_Q\frac{G_k(u_n)}{|x|^p}+\frac{1}{k^{\gamma-1}}\iint_Q f+\frac{1}{2}||u_0||^2_{L^2(\Omega)},\end{aligned}\tag{5.17}$$

as $\mu < \alpha C_{N,p}$, choosing ϵ small enough and by Hardy inequality, we get

$$C(\alpha, \epsilon, \mu, C_{N,p}) \iint_Q |\nabla G_k(u_n)|^p \leq \mu k^{p-1} \iint_Q \frac{G_k(u_n)}{|x|^p} + C(k, f, ||u_0||_{L^2(\Omega)}). \tag{5.18}$$

Applying Hölder, Young and Hardy inequalities we conclude that

$$\iint_Q |\nabla G_k(u_n)|^p \leq C(\alpha, \epsilon, \mu, k^{p-1}, C_{N,p,}, f, ||u_0||_{L^2(\Omega)}). \tag{5.19}$$

Testing now (5.10) by $(T_k(u_n))^\gamma$, so that, from (5.3) and (5.19)

$$\iint_Q T_k(u_n)^{\gamma-1} |\nabla T_k(u_n)|^p \leq C(\alpha, k, \mu, f, ||u_0||_{L^2(\Omega)}). \tag{5.20}$$

There hold

$$\frac{p^p}{(\gamma+p-1)^p} \iint_Q |\nabla T_k(u_n)^{\frac{\gamma+p-1}{p}}|^p \leq C(\alpha, k, \mu, f, ||u_0||_{L^2(\Omega)}),$$

this implies that the sequence $T_k(u_n)^{\frac{\gamma+p-1}{p}}$ is bounden in $L^p(0,T; W_0^{1,p}(\Omega))$. By Lemma 5.3 and (5.20), yields that $T_k(u_n)$ is bounded in $L^p(0,T; W_{loc}^{1,p}(\Omega))$. Collecting the last affirmation with (5.19), assume that the sequence u_n is bounded in $L^p(0,T; W_{loc}^{1,p}(\Omega))$. Using $u_n^\gamma \chi_{(0,t)}$ as test function in (5.10) (with $0 < t \leq T$), from (5.3), $u_0 \in L^\infty(\Omega)$ and applying Hardy inequality, we get

$$\begin{aligned} &\frac{1}{\gamma+1} \int_\Omega u_n^{\gamma+1}(x,t) + \left(\alpha \left(\frac{p}{\gamma+p-1}\right)^p - \frac{\mu}{C_{N,p}}\right) \int_0^t \int_\Omega |\nabla u_n^{\frac{\gamma+p-1}{p}}|^p \\ &\leq \iint_Q f_n + \frac{1}{\gamma+1} \int_\Omega |u_0(x)|^{\gamma+1} \leq \iint_Q f + \frac{1}{\gamma+1} ||u_0||_{L^\infty(\Omega)}^{\gamma+1} \leq C, \end{aligned}$$

since $\alpha \left(\frac{p}{\gamma+p-1}\right)^p - \frac{\mu}{C_{N,p}} > 0$, passing to the supremum for $t \in [0,T]$, we deduce that

$$\frac{1}{\gamma+1} ||u_n||_{L^\infty(0,T;L^{\gamma+1}(\Omega))}^{\gamma+1} + \left(\alpha \left(\frac{p}{\gamma+p-1}\right)^p - \frac{\mu}{C_{N,p}}\right) \iint_Q |\nabla u_n^{\frac{\gamma+p-1}{p}}|^p \leq C,$$

this implies that $u_n^{\frac{\gamma+p-1}{p}}$ is bounded in $L^p(0,T; W_0^{1,p}(\Omega))$ and u_n is bounded in $L^\infty(0,T; L^{\gamma+1}(\Omega))$ with respect to n. Since the proof of item *ii)* is achieved. □

Theorem 5.6. *Assume that* (5.3)-(5.5) *holds true. If* $\gamma < 1, \mu < \alpha C_{N,p}$ *and* $f \in L^{\frac{p(N+2)}{p(N+2)-N(1-\gamma)}}(Q)$. *Then, there exists a solution u to problem* (5.1) *in the sense of Definition 5.1. Moreover* $u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ *and* $\frac{\partial u}{\partial t} \in L^{p'}(0,T; W^{-1,p'}(\Omega)) + L^1(0,T; L^1_{loc}(\Omega))$.

Remark 5.7. If $\mu = 0$, then the result of Theorem 5.6 coincide with result of Theorem 1.3 in [68].

Theorem 5.8. *Suppose that* (5.3)-(5.5) *holds true. If* $\gamma \geq 1$, $\mu < \alpha C_{N,p}$ *and* $f \in L^1(Q)$. *Then, there exists a solution u to problem* (5.1) *in the sense of Definition 5.1 with the following regularity:*

a) If $\gamma = 1$, *then* $u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ *and*

$$\frac{\partial u}{\partial t} \in L^{p'}(0,T; W^{-1,p'}(\Omega)) + L^1(0,T; L^1_{loc}(\Omega)).$$

b) If $\gamma > 1$*, then* $u \in L^p(0,T; W^{1,p}_{loc}(\Omega))$ *and* $T_k(u)^{\frac{p+\gamma-1}{p}} \in L^p(0,T; W^{1,p}_0(\Omega))$*. If* $\alpha \left(\frac{p}{p+\gamma-1}\right)^p - \frac{\mu}{C_{N,p}} > 0$*, then* $u^{\frac{p+\gamma-1}{p}} \in L^p(0,T; W^{1,p}_0(\Omega))$ *and* $u \in L^\infty(0,T; L^{\gamma+1}(\Omega))$ *and* $\frac{\partial u}{\partial t} \in L^{p'}(0,T; W^{-1,p'}(\omega)) + L^1(0,T; L^1(\omega))$ *for all* $\omega \subset\subset \Omega$*.*

Remark 5.9. If $\mu = 0$, then the result of Theorem 5.8 coincide with result of Theorem 1.3 in [68].

Before giving the proof of Theorems 5.6 and 5.8, we need the following results:

Proposition 5.10. *Under the assumptions of Lemmas 5.4 and 5.5 there exists* $u \in L^p(0,T; W^{1,p}_{loc}(\Omega))$ *such that, up to a subsequence,* u_n *converges to* u *a.e. on* Q*, weakly in* $L^p(0,T; W^{1,p}_{loc}(\Omega))$ *and strongly in* $L^1(0,T; L^1_{loc}(\Omega))$*.*

Proof. From Lemmas 5.4 and 5.5 we know that u_n is bounded in the space $L^p(0,T; W^{1,p}_{loc}(\Omega))$. The last affirmation and Lemma 5.3 imply the sequence $\left\{\mu \frac{u_n^{p-1}}{|x|^p + \frac{1}{n}} + \frac{f_n}{(u_n + \frac{1}{n})^\gamma}\right\}$ is bounded in $L^1(0,T; L^1_{loc}(\Omega))$. Hence, let $\varphi \in C^1_c(\Omega)$ then one has that $\{\frac{\partial(u_n\varphi)}{\partial t}\}$ is bounded in $L^{p'}(0,T; W^{-1,p'}(\Omega)) + L^1(Q)$, which is sufficient to apply [139, Corollary 4] in order to deduce that u_n converges to a function $u \in L^1(0,T; L^1_{loc}(\Omega))$ and u_n converges to u a.e. in Q. □

In the following proposition, we will prove the almost everywhere convergence of the gradient of u_n.

Proposition 5.11. *Let* u_n *be a solution of problem* (5.10) *and assume that* $f \in L^{\frac{p(N+2)}{p(N+2)-N(1-\gamma)}}(Q)$ *if* $\gamma < 1$ *and* $f \in L^1(Q)$ *if* $\gamma \geq 1$ *respectively. Then the sequence* $T_k(u_n)$ *strongly converges to* $T_k(u)$ *in* $L^p(0,T; W^{1,p}_{loc}(\Omega))$ *and so, in particular,* ∇u_n *converges to* ∇u *almost everywhere in* Q*.*

Proof. Let $n, m \in \mathbb{N}$ denote two value of the parameter describing the approximation. Since (5.10) is non-singular problem, we can take $T_{2k}(u_n - u_m)\varphi \in L^p(0,T; W^{1,p}_0(\Omega)) \cap L^\infty(Q)$ as a test function in the difference of the approximating equations solved by u_n and u_m, with $\varphi \in C^1_c(\Omega)$ independent of $t \in [0,T]$ and such that $0 \leq \varphi \leq 1$, obtaining

$$\begin{aligned} &\int_0^T \int_\Omega \frac{\partial(u_n - u_m)}{\partial t} T_{2k}(u_n - u_m)\varphi(x) \\ &+ \int_0^T \int_\Omega (a(x,t,\nabla u_n) - a(x,t,\nabla u_m))\nabla(T_{2k}(u_n - u_m)\varphi(x)) \\ &= \int_0^T \int_\Omega \left(\frac{f_n}{(u_n + \frac{1}{n})^\gamma} - \frac{f_m}{(u_m + \frac{1}{m})^m}\right) T_{2k}(u_n - u_m)\varphi(x) \\ &+ \int_0^T \int_\Omega \mu \left[\frac{u_n^{p-1}}{|x|^p + \frac{1}{n}} - \frac{u_m^{p-1}}{|x|^p + \frac{1}{m}}\right] T_{2k}(u_n - u_m)\varphi(x). \end{aligned}$$

Observe that

$$\begin{aligned} \int_0^T \int_\Omega \frac{\partial(u_n - u_m)}{\partial t} T_{2k}(u_n - u_m)\varphi(x) &= \int_\Omega \int_0^T \frac{d}{dt}(\theta_{2k}(u_n - u_m))\varphi(x) \\ &= \int_\Omega \theta_{2k}(u_n - u_m)(T)\varphi(x), \end{aligned} \tag{5.21}$$

where $\theta_{2k}(t)$ is the primitive of $T_{2k}(t)$ which vanishes for $t = 0$, and so we can drop the parabolic term (5.21) (since it is nonnegative) obtaining

$$\begin{aligned}
&\int_0^T \int_\Omega (a(x,t,\nabla u_n) - a(x,t,\nabla u_m))\nabla(T_{2k}(u_n - u_m))\varphi(x) \\
&+ \int_0^T \int_\Omega (a(x,t,\nabla u_n) - a(x,t,\nabla u_m))\nabla\varphi T_{2k}(u_n - u_m) \\
&\leq 2k \int_{Q\cap supp(\varphi)} \left| \frac{f_n}{(u_n + \frac{1}{n})^\gamma} - \frac{f_m}{(u_m + \frac{1}{m})^\gamma} \right| \\
&+ \int_0^T \int_\Omega \mu \left[\frac{u_n^{p-1}}{|x|^p + \frac{1}{n}} - \frac{u_m^{p-1}}{|x|^p + \frac{1}{m}} \right] T_{2k}(u_n - u_m)\varphi(x).
\end{aligned}$$

We denote by

$$A_{k,n} = \{(x,t) \in Q : u_n \leq k\} \quad \text{and} \quad A_{k,n,m} = \{(x,t) \in Q : u_n \leq k,\, u_m \leq k\},$$

since $A_{k,n,m} \subset \{(x,t) \in Q : |u_n - u_m| \leq 2k\}$, we have

$$\begin{aligned}
&\iint_Q (a(x,t,\nabla u_n) - a(x,t,\nabla u_m)\nabla(T_{2k}(u_n - u_m))\varphi \\
&= \iint_{\{(x,t)\in Q:|u_n - u_m|\leq 2k\}} (a(x,t,\nabla u_n) - a(x,t,\nabla u_m))\nabla(u_n - u_m)\varphi \\
&\geq \iint_{A_{k,n,m}} (a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u_m)))(\nabla T_k(u_n) - \nabla T_k(u_m))\varphi \\
&= \iint_{A_{k,n}} a(x,t,\nabla T_k(u_n))\nabla T_k(u_n)\varphi - \iint_{A_{k,n,m}} a(x,t,\nabla T_k(u_n))\nabla T_k(u_m)\varphi \\
&- \iint_{A_{k,n,m}} a(x,t,\nabla T_k(u_m))\nabla T_k(u_n)\varphi + \iint_{A_{k,m}} a(x,t,\nabla T_k(u_m))\nabla T_k(u_m)\varphi.
\end{aligned}$$

In conclusion, we found that

$$\begin{aligned}
&\iint_{A_{k,n}} a(x,t,\nabla T_k(u_n))\nabla T_k(u_n)\varphi - \iint_{A_{k,n,m}} a(x,t,\nabla T_k(u_n))\nabla T_k(u_m)\varphi \\
&- \iint_{A_{k,n,m}} a(x,t,\nabla T_k(u_m))\nabla T_k(u_n)\varphi + \iint_{A_{k,m}} a(x,t,\nabla T_k(u_m))\nabla T_k(u_m)\varphi \\
&\leq 2k \int_{Q\cap supp(\varphi)} \left| \frac{f_n}{(u_n + \frac{1}{n})^\gamma} - \frac{f_m}{(u_m + \frac{1}{m})^\gamma} \right| \\
&+ \int_0^T \int_\Omega \mu \left[\frac{u_n^{p-1}}{|x|^p + \frac{1}{n}} - \frac{u_m^{p-1}}{|x|^p + \frac{1}{m}} \right] T_{2k}(u_n - u_m)\varphi(x) \\
&- \iint_Q (a(x,t,\nabla u_n) - a(x,t,\nabla u_m))\nabla\varphi(x)T_{2k}(u_n - u_m).
\end{aligned} \tag{5.22}$$

The right-hand side of the previous inequality is infinitesimal for $n, m \to +\infty$ and we denote by $r(n,m)$ a quantity that goes to zero from $n, m \to +\infty$.

By using the same proof as Proposition 3.2 in [68], we have

$$\int_{Q\cap supp(\varphi)} \left| \frac{f_n}{(u_n+\frac{1}{n})^\gamma} - \frac{f_m}{(u_m+\frac{1}{m})^m} \right| = r(n,m),$$

and

$$\iint_{Q\cap supp(\varphi)} (a(x,t,\nabla u_n) - a(x,t,\nabla u_m))\nabla\varphi(x) T_{2k}(u_n-u_m) = r(n,m).$$

Now, we prove

$$\int_0^T \int_\Omega \mu \left[\frac{u_n^{p-1}}{|x|^p+\frac{1}{n}} - \frac{u_m^{p-1}}{|x|^p+\frac{1}{m}} \right] T_{2k}(u_n-u_m)\varphi(x) = r(n,m). \tag{5.23}$$

First of all we prove that

$$\frac{u_n^{p-1}}{|x|^p+\frac{1}{n}} \text{ is bounded in } L^{\bar h}(0,T;L^{\bar h}_{loc}\Omega), \text{ for every } 1<\bar h<\frac{pN}{p+(p-1)N}. \tag{5.24}$$

Notice that it results $1<\frac{pN}{p+(p-1)N}<p'$. As matter of fact, for every compact $\omega\subset\Omega$ it results (thanks to Hardy inequality and Lemmas 5.4 and 5.5)

$$\int_0^T\int_\omega \left|\frac{u_n^{(p-1)}}{|x|^p+\frac{1}{n}}\right|^{\bar h} \le \int_0^T\int_\omega \frac{|u_n|^{\bar h(p-1)}}{|x|^{p\bar h}} = \int_0^T\int_\omega \frac{|u_n|^{\bar h(p-1)}}{|x|^{\bar h(p-1)}}\frac{1}{|x|^{\bar h}}$$

$$\le \left(\int_0^T\int_\omega \frac{|u_n|^p}{|x|^p}\right)^{\frac{\bar h(p-1)}{p}} \left(\int_0^T\int_\omega \frac{1}{|x|^{\bar h\left(\frac{p}{\bar h(p-1)}\right)'}}\right)^{1-\frac{\bar h(p-1)}{p}} \le C,$$

where the last integral in the right-hand side is finite since it results

$$\bar h\left(\frac{p}{\bar h(p-1)}\right)' < N \Leftrightarrow \bar h < \frac{pN}{p+(p-1)N}.$$

Hence, by (5.24) and the convergence a.e. of u_n to u in Q we deduce that

$$\left[\frac{u_n^{p-1}}{|x|^p+\frac{1}{n}} - \frac{u_m^{p-1}}{|x|^p+\frac{1}{m}}\right]\varphi(x) \rightharpoonup 0 \text{ weakly in } L^{\bar h}(Q).$$

Notice that, thanks to the Lebesgue Theorem, it results

$$T_{2k}(u_n-u_m)\to 0 \text{ strongly in } L^s(Q), \text{ for every } 1<s<+\infty,$$

and thus it convergences also in $L^{\bar h'}(Q)$ and (5.23) follows.

Then, the rest of the proof, we proceed as Proposition 3.2 in [68], we obtain up to subsequences, $T_k(u_n)\to T_k(u)$ in $L^p(0,T;W^{1,p}_{loc}(\Omega))$, and so $\nabla u_n\to\nabla u$ a.e. in Q. □

Proof of Theorems 5.6 and 5.8. If $\gamma<1$, by Lemma 5.4, we have u_n is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and in $L^\infty(0,T;L^2(\Omega))$. Then, by Lemma 5.3, Proposition 5.10 and Fatou's Lemma $u\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(0,T;L^2(\Omega))$, and moreover $\frac{u^{p-1}}{|x|^p}, \frac{f}{u^\gamma}\in L^1(0,T;L^1_{loc}(\Omega))$ since u satisfies (5.8), in particular

$$\frac{\partial u}{\partial t}\in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(0,T;L^1_{loc}(\Omega)).$$

If $\gamma = 1$, thanks to Lemma 5.5, we have u_n is bounded in

$$L^p(0,T;W_0^{1,p}(\Omega)) \text{ and in } L^\infty(0,T;L^2(\Omega)),$$

as before, we get $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and u satisfies (5.8); Moreover

$$\frac{\partial u}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(0,T;L^1_{loc}(\Omega)).$$

In the case $\gamma > 1$, in view of Lemma 5.5, we have that $u_n^{\frac{p+\gamma-1}{p}}$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, while u_n is bounded in

$$L^p(0,T;W^{1,p}_{loc}(\Omega)) \text{ and in } L^\infty(0,T;L^{\gamma+1}(\Omega)).$$

Then

$$u \in L^p(0,T;W^{1,p}_{loc}(\Omega)) \text{ and } u^{\frac{p+\gamma-1}{p}} \in L^p(0,T;W_0^{1,p}(\Omega)),$$

in particular, $u = 0$ on $\partial\Omega \times (0,T)$ in weak-sense and

$$\frac{\partial u}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(w)) + L^1(0,T;L^1_{loc}(\Omega)), \text{ for all } w \subset\subset \Omega.$$

Using Lemma 5.3, Proposition 5.10 and Fatou's Lemma deduce that u satisfies the condition (5.8). Now we fix $\varphi \in C_c^1(\Omega \times [0,T))$, by Lemma 5.4 and Lemma 5.5, we have the boundedness of the sequence u_n in the space $L^p(0,T;W^{1,p}_{loc}(\Omega))$ and from (5.4), implies that the sequence $a(x,t,\nabla u_n)$ is bounded in $L^{p'}(\omega \times (0,T))$ for all $\omega \subset\subset \Omega$. As $supp(\varphi)$ is a compact subset of $\Omega \times [0,T)$, then $a(x,t,\nabla u_n)$ is bounded in $L^{p'}(supp(\varphi))$ and $\frac{u_n^{p-1}}{|x|^p+\frac{1}{n}}$ is bounded in $L^1(supp(\varphi))$. From Propositions 5.10 and 5.11, we have $u_n \to u$ a.e. in Q and $\nabla u_n \to \nabla u$ a.e. in Q and by Vitali's Theorem we obtain

$$\lim_{n\to+\infty} \iint_Q a(x,t,\nabla u_n)\nabla\varphi = \iint_Q a(x,t,\nabla u)\nabla\varphi \quad \forall\varphi \in C_c^1(\Omega \times [0,T)), \tag{5.25}$$

and

$$\lim_{n\to+\infty} \iint_Q \frac{u_n^{p-1}}{|x|^p + \frac{1}{n}}\varphi = \iint_Q \frac{u^{p-1}}{|x|^p}\varphi \quad \forall\varphi \in C_c^1(\Omega \times [0,T)). \tag{5.26}$$

Concerning the passage of limit of term in the right of the approximating problem (5.10), since $supp(\varphi)$ is a compact subset of $\Omega \times [0,T)$, thanks to Lemma 5.3, there exists a constant $c_{supp(\varphi)} > 0$ such that $u_n \geq c_{supp(\varphi)}$, then

$$\left|\frac{f_n}{(u_n + \frac{1}{n})^\gamma}\varphi\right| \leq \frac{f}{c^\gamma_{supp(\varphi)}}||\varphi||_{L^\infty(Q)},$$

for every $(x,t) \in supp(\varphi)$, since it is a.e. convergent to $\frac{f}{u^\gamma}\varphi$ for $n \longrightarrow +\infty$, by Lebesgue Theorem, implies that

$$\lim_{n\to+\infty} \iint_Q \frac{f_n}{(u_n + \frac{1}{n})^\gamma}\varphi = \iint_Q \frac{f}{u^\gamma}\varphi \quad \forall\varphi \in C_c^1(\Omega \times [0,T)). \tag{5.27}$$

By Proposition 5.10, we have

$$\lim_{n\to+\infty} \iint_Q u_n\frac{\partial\varphi}{\partial t} = \iint_Q u\frac{\partial\varphi}{\partial t}, \quad \forall\varphi \in C_c^1(\Omega \times [0,T)). \tag{5.28}$$

Take now $\varphi \in C_c^1(\Omega \times [0,T))$ as a test function in problem (5.10), by the convergences results (5.25), (5.26), (5.27), (5.28) and letting $n \longrightarrow +\infty$, we get

$$-\int_\Omega u_0(x)\varphi(x,0) - \iint_Q u\frac{\partial \varphi}{\partial t} + \iint_Q a(x,t,\nabla u)\nabla\varphi - \mu \iint_Q \frac{u^{p-1}}{|x|^p}\varphi = \iint_Q \frac{f}{u^\gamma}\varphi.$$

□

4 Regularity results

In this section we study the regularity of solutions of problem (5.1) depending on $\mu, \gamma > 0$ and the summability of f.

The case $\gamma \geq 1$

Theorem 5.12. *Let $\gamma \geq 1$ and suppose that f belongs to $L^m(Q)$ with $1 < m < \frac{N}{p} + 1$. If*

$$0 < \mu < \alpha C_{N,p}\frac{(Np - N + p)(m-1) + N\gamma m}{N - pm + p}\times$$
$$\left(\frac{p(N - pm + p)}{N(p+\gamma-1)m - p(p-2)(m-1)}\right)^p,$$

then the solution u of (5.1) found in Theorem 5.8 satisfies the following summability $u \in L^\sigma(Q)$, where

$$\sigma = m\frac{N(p+\gamma-1) + p(\gamma+1)}{N - pm + p}.$$

Proof. Let now choosing $u_n^{p\delta-p+1}\chi_{(0,t)}$ as test function in (5.10), $\delta > \frac{p+\gamma-1}{p}$ and $0 < t < T$, then we get

$$\int_0^t \int_\Omega \frac{\partial u_n}{\partial t}u_n^{p\delta-p+1} + (p\delta - p + 1)\int_0^t \int_\Omega u_n^{p\delta-p} a(x,t,\nabla u_n).\nabla u_n$$
$$\leq \mu \int_0^t \int_\Omega \frac{u_n^{p\delta}}{|x|^p} + \int_0^t \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\gamma}u_n^{p\delta-p+1},$$

from (5.3), it follows that

$$\frac{1}{p\delta - p + 2}\int_\Omega u_n^{p\delta-p+2}(x,t) + \alpha(p\delta - p + 1)\int_0^t \int_\Omega u_n^{p\delta-p}|\nabla u_n|^p$$
$$\leq \mu \int_0^t \int_\Omega \frac{u_n^{p\delta}}{|x|^p} + \int_0^t \int_\Omega f_n u_n^{p\delta-p+1-\gamma} + \frac{1}{p\delta - p + 2}\int_\Omega u_0^{p\delta-p+2}.$$

Thanks to $u_0 \in L^\infty(\Omega)$ and $u_n^{p\delta-p}|\nabla u_n|^p = \frac{1}{\delta^p}|\nabla u_n^\delta|^p$, the last inequality becomes

$$\frac{1}{p\delta - p + 2}\int_\Omega [u_n^\delta]^{\frac{p\delta-p+2}{\delta}} + \frac{\alpha(p\delta - p + 1)}{\delta^p}\int_0^t \int_\Omega |\nabla u_n^\delta|^p$$
$$\leq \mu \int_0^t \int_\Omega \frac{(u_n^\delta)^p}{|x|^p} + \int_0^t \int_\Omega f_n u_n^{p\delta-p+1-\gamma} + \frac{1}{p\delta - p + 2}||u_0||_{L^\infty(\Omega)}^{p\delta-p+2},$$

applying Hardy and Hölder's inequalities, yields

$$\frac{1}{p\delta-p+2}\int_\Omega [u_n^\delta]^{\frac{p\delta-p+2}{\delta}}+\left(\frac{\alpha(p\delta-p+1)}{\delta^p}-\frac{\mu}{C_{N,p}}\right)\int_0^t\int_\Omega |\nabla u_n^\delta|^p$$
$$\leq ||f||_{L^m(Q)}\left(\iint_Q u_n^{(p\delta-p+1-\gamma)m'}\right)^{\frac{1}{m'}}+C.$$

Passing to supremum for $t\in(0,T)$ we have

$$\begin{aligned}&\frac{1}{p\delta-p+2}||u_n^\delta||^{\frac{p\delta-p+2}{\delta}}_{L^\infty(0,T;L^{\frac{p\delta-p+2}{\delta}}(\Omega))}+\left(\frac{\alpha(p\delta-p+1)}{\delta^p}-\frac{\mu}{C_{N,p}}\right)\iint_Q |\nabla u_n^\delta|^p\\&\leq ||f||_{L^m(Q)}\left(\iint_Q u_n^{(p\delta-p+1-\gamma)m'}\right)^{\frac{1}{m'}}+C.\end{aligned}\tag{5.29}$$

Since $u_n\in L^\infty(Q)\cap L^p(0,T;W_0^{1,p}(\Omega))$, then in view to Lemma 2.9 (with $\rho=\frac{p\delta-p+2}{\delta}$, $h=p$, $v=u_n^\delta$) and by (5.29), we get

$$\begin{aligned}\iint_Q (u_n^\delta)^{p\frac{N+\frac{p\delta-p+2}{\delta}}{N}}&\leq C_G||u_n^\delta||^{\frac{p(p\delta-p+2)}{N\delta}}_{L^\infty(0,T;L^{\frac{p\delta-p+2}{\delta}}(\Omega))}\iint_Q|\nabla u_n^\delta|^p\\&\leq C\left(\iint_Q u_n^{(p\delta-p+1-\gamma)m'}\right)^{(\frac{p}{N}+1)\frac{1}{m'}}+C,\end{aligned}$$

hence

$$\iint_Q u_n^{\frac{p(N\delta+p\delta-p+2)}{N}}\leq C\left(\iint_Q u_n^{(p\delta-p+1-\gamma)m'}\right)^{(\frac{p}{N}+1)\frac{1}{m'}}+C.\tag{5.30}$$

Choosing now δ such that

$$\sigma=\frac{p(N\delta+p\delta-p+2)}{N}=(p\delta-p+1-\gamma)m',\tag{5.31}$$

this equivalent to

$$\delta=\frac{Nm(p+\gamma-1)-p(p-2)(m-1)}{p(N-pm+p)},\quad \sigma=m\frac{N(p+\gamma-1)+p(\gamma+1)}{N-pm+p}.$$

Collecting (5.30) with (5.31), we conclude that

$$\iint_Q u_n^\sigma\leq C\left(\iint_Q u_n^\sigma\right)^{(\frac{p}{N}+1)\frac{1}{m'}}+C.\tag{5.32}$$

By virtue of $m<\frac{N}{p}+1$, then $(\frac{p}{N}+1)\frac{1}{m'}<1$, since $\delta>\frac{p+\gamma-1}{p}$ gives $m>1$ and applying Young's inequality implies that

$$\iint_Q u_n^\sigma\leq C,\tag{5.33}$$

this last estimate yields that the sequence u_n is bounded in $L^\sigma(Q)$, and so $u\in L^\sigma(Q)$. □

Theorem 5.13. *Let $\gamma \geq 1$ and $f \in L^m(Q)$ with $m \geq \frac{N}{p} + 1$. Then the solution of problem (5.1) found in Theorem (5.8) satisfies the following regularity:*
If $\lambda \geq \gamma$ and $\frac{\alpha\lambda p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}} > 0$, then $u^{\frac{\lambda+p-1}{p}} \in L^p(0,T;W_0^{1,p}(\Omega))$ and $u \in L^\infty(0,T;L^{\lambda+1}(\Omega))$.

Proof. Choosing $u_n^\lambda \chi_{(0,t)}$ with $\lambda > 0$ as test function in (5.10)

$$\begin{aligned}&\frac{1}{\lambda+1}\int_\Omega u_n^{\lambda+1}(x,t) + \lambda\int_0^t\int_\Omega u_n^{\lambda-1}a(x,t,\nabla u_n)\cdot\nabla u_n\\&= \mu\int_0^t\int_\Omega \frac{u_n^{\lambda+p-1}}{|x|^p} + \int_0^t\int_\Omega \frac{f_n u_n^\lambda}{(u_n+1)^\gamma} + \frac{1}{\lambda+1}\int_\Omega |u_0(x)|^{\lambda+1}.\end{aligned}$$

From (5.3) and the fact that $\frac{1}{(u_n+1)^\gamma} \leq \frac{1}{u_n^\gamma}, u_0 \in L^\infty(\Omega)$ we have

$$\begin{aligned}&\frac{1}{\lambda+1}\int_\Omega u_n^{\lambda+1}(x,t) + \lambda\alpha\int_0^t\int_\Omega |\nabla u_n|^p u_n^{\lambda-1}\\&\leq \mu\int_0^t\int_\Omega \frac{u_n^{\lambda+p-1}}{|x|^p} + \int_0^t\int_\Omega f_n u_n^{\lambda-\gamma} + \frac{|\Omega|}{\lambda+1}||u_0||_{L^\infty(\Omega)}^{\lambda+1}.\end{aligned}$$

By Hardy inequality the later inequality implies

$$\begin{aligned}&\frac{1}{\lambda+1}\int_\Omega u_n^{\lambda+1}(x,t) + \left(\frac{\alpha\lambda p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right)\int_0^t\int_\Omega |\nabla u_n^{\frac{\lambda+p-1}{p}}|^p\\&\leq \int_0^t\int_\Omega f_n u_n^{\lambda-\gamma} + C.\end{aligned}$$

Passing to supremum for $t \in [0,T]$ we get

$$\begin{aligned}\frac{1}{\lambda+1}||u_n||_{L^\infty(0,T;L^{\lambda+1}(\Omega))}^{\lambda+1} + \left(\frac{\alpha\lambda p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right)\iint_Q |\nabla u_n^{\frac{\lambda+p-1}{p}}|^p\\ \leq \iint_Q f_n u_n^{\lambda-\gamma} + C,\end{aligned}$$

applying Hölder inequality we conclude that

$$\begin{aligned}\frac{1}{\lambda+1}||u_n||_{L^\infty(0,T;L^{\lambda+1}(\Omega))}^{\lambda+1} + \left(\frac{\alpha\lambda p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right)\iint_Q |\nabla u_n^{\frac{\lambda+p-1}{p}}|^p\\ \leq C\left(\iint_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C.\end{aligned} \tag{5.34}$$

Using Sobolev inequality and by the above estimate, we get

$$\left(\iint_Q u_n^{\frac{N(\lambda+p-1)}{N-p}}\right)^{\frac{p}{p^*}} \leq C\iint_Q |\nabla u_n^{\frac{\lambda+p-1}{p}}|^p \leq C\left(\iint_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C.$$

By $m \geq \frac{N}{p}+1$ we have $m' \leq \frac{N+p}{N}$, then for all $\lambda \geq \gamma$, we get $(\lambda-\gamma)m' \leq \frac{(\lambda-\gamma)(N+p)}{N} \leq \frac{N(\lambda+p-1)}{N-p}$. Thus choosing λ such that $\frac{\alpha\lambda p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}} > 0$. Using Hölder's inequality in the later estimate, we have

$$\left(\iint_Q u_n^{\frac{N(\lambda+p-1)}{N-p}}\right)^{\frac{p}{p^*}} \leq C\left(\iint_Q u_n^{\frac{N(\lambda+p-1)}{N-p}}\right)^{\frac{(N-p)(\lambda-\gamma)}{N(\lambda+p-1)}} + C. \tag{5.35}$$

Since $\frac{p}{p^*} = \frac{N-p}{N} > \frac{(N-p)(\lambda-\gamma)}{N(\lambda+p-1)}$, then by Young inequality we deduce that

$$\iint_Q u_n^{\frac{N(\lambda+p-1)}{N-p}} \leq C. \tag{5.36}$$

By the fact that $(\lambda-\gamma)m' < \frac{N(\lambda+p-1)}{N-p}$, (5.36) and using Hölder inequality in (5.34), we obtain

$$\begin{aligned} &\frac{1}{\lambda+1}||u_n||^{\lambda+1}_{L^\infty(0,T;L^{\lambda+1}(\Omega))} + \left(\frac{\alpha\lambda p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right)\iint_Q |\nabla u_n^{\frac{\lambda+p-1}{p}}|^p \\ &\leq C\left(\iint_Q u_n^{\frac{N(\lambda+p-1)}{N-p}}\right)^{\frac{(N-p)(\lambda-\gamma)}{N(\lambda+p-1)}} + C \leq C. \end{aligned}$$

Since $\lambda \geq \gamma$, $\left(\frac{\alpha\lambda p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right) > 0$ and the later estimate we deduce that the sequence $u_n^{\frac{\lambda+p-1}{p}}$ is uniformly bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and u_n is bounded $L^\infty(0,T;L^{\lambda+1}(\Omega))$ with respect to n for all $\lambda \geq \gamma$, so $u^{\frac{\lambda+p-1}{p}} \in L^p(0,T;W_0^{1,p}(\Omega))$ and $u \in L^\infty(0,T;L^{\lambda+1}(\Omega))$ for all $\lambda \geq \gamma$. This completed the proof of Theorem 5.13. □

The case $\gamma < 1$

Theorem 5.14. *Let $\gamma < 1$, and suppose that $f \in L^m(Q), m \geq 1$ and*

$$\begin{aligned} 0 \leq \mu < \alpha C_{N,p}\frac{(m-1)[N(p-1)+p]+Nm\gamma}{N-pm+p} \times \\ \left(\frac{p(N-pm+p)}{(m-1)[(N-p)(p-1)+p]+N(m\gamma+p-1)}\right)^p. \end{aligned} \tag{5.37}$$

Then

(*i*) *If $\frac{p(N+2)}{p(N+2)-N(1-\gamma)} \leq m < \frac{N}{p}+1$, then the solution u of (5.1) found in Theorem 5.6, satisfies the following regularity $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\sigma(Q)$, with $\sigma = m\frac{N(p+\gamma-1)+p(\gamma+1)}{N-pm+p}$.*

(*ii*) *If $1 \leq m < \frac{p(N+2)}{p(N+2)-N(1-\gamma)}$, then there exists a weak solution u of problem (5.1) such that $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\sigma(Q)$, with*

$$q = m\frac{N(p+\gamma-1)+p(\gamma+1)}{N+2-m(1-\gamma)} \text{ and } \sigma = m\frac{N(p+\gamma-1)+p(\gamma+1)}{N-pm+p}.$$

(iii) If $m \geq \frac{N}{p}+1$ *and* $0<\mu<\alpha C_{N,p}$, *then the solution* u *of* (5.1) *found in Theorem 5.6 satisfies the following regularity* $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

Proof. Taking $\varphi(u_n)=((u_n+a)^\lambda - a^\lambda)\chi_{(0,t)}$ as a test function in (5.10), $0<a<\frac{1}{n}$, $\lambda>0$ and using the ellipticity condition (5.3) we have

$$\int_0^t\int_\Omega \frac{\partial u_n}{\partial t}\varphi(u_n) + \lambda\alpha\int_0^t\int_\Omega (u_n+a)^{\lambda-1}|\nabla u_n|^p \leq \mu\int_0^t\int_\Omega \frac{u_n^{p-1}(u_n+a)^\lambda}{|x|^p} + \int_0^t\int_\Omega \frac{f_n|(u_n+a)^\lambda - a^\lambda|}{(u_n+\frac{1}{n})^\gamma}.$$

By the fact that $\frac{1}{(u_n+\frac{1}{n})^\gamma} \leq \frac{1}{(u_n+a)^\gamma}$ and $u_n^{p-1}(u_n+a)^\lambda \leq (u_n+a)^{\lambda+p-1}$, we obtain

$$\int_\Omega \Psi(u_n(x,t)) + \lambda\alpha\int_0^t\int_\Omega (u_n+a)^{\lambda-1}|\nabla u_n|^p \leq \mu\int_0^t\int_\Omega \frac{(u_n+a)^{\lambda+p-1}}{|x|^p} + \int_0^t\int_\Omega f_n(u_n+a)^{\lambda-\gamma} + \int_\Omega \Psi(u_0),$$

where $\Psi(s)=\displaystyle\int_0^s \varphi(\ell)d\ell$. Since $(u_n+a)^{\lambda-1}|\nabla u_n|^p = \frac{p^p}{(\lambda+p-1)^p}|\nabla(u_n+a)^{\frac{\lambda+p-1}{p}}|^p$, then the last estimate becomes

$$\begin{aligned}&\int_\Omega \Psi(u_n(x,t)) + \frac{\lambda\alpha p^p}{(\lambda+p-1)^p}\int_0^t\int_\Omega |\nabla(u_n+a)^{\frac{\lambda+p-1}{p}}|^p \\ &\leq \mu\int_0^t\int_\Omega \frac{((u_n+a)^{\frac{\lambda+p-1}{p}})^p}{|x|^p} + \int_0^t\int_\Omega f(u_n+a)^{\lambda-\gamma} + \int_\Omega \Psi(u_0).\end{aligned} \tag{5.38}$$

Since $u_0 \in L^\infty(\Omega)$, applying Hölder and Hardy inequalities, we find that

$$\int_\Omega \Psi(u_n(x,t)) + \left(\frac{\lambda\alpha p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right)\int_0^t\int_\Omega |\nabla(u_n+a)^{\frac{\lambda+p-1}{p}}|^p \leq ||f||_{L^m(Q)}\left(\iint_Q (u_n+a)^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C.$$

If $\lambda \geq 1$, by definition of $\varphi(u_n)$ and $\Psi(u_n)$, we reach that

$$\Psi(s) \geq \frac{|s|^{\lambda+1}}{\lambda+1}, \qquad \forall s \in \mathbb{R}. \tag{5.39}$$

Therefore we obtain that

$$\frac{1}{\lambda+1}\int_\Omega u_n^{\lambda+1}(x,t) + \left(\frac{\lambda\alpha p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right)\int_0^t\int_\Omega |\nabla(u_n+a)^{\frac{\lambda+p-1}{p}}|^p \leq ||f||_{L^m(Q)}\left(\iint_Q (u_n+a)^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C.$$

Observing that $u_n^{\lambda+1}(x,t) = \left(u_n^{\frac{\lambda+p-1}{p}}(x,t)\right)^{\frac{p(\lambda+1)}{\lambda+p-1}}$, then the last inequality becomes

$$\begin{aligned}&\frac{1}{\lambda+1}\int_\Omega [u_n^{\frac{\lambda+p-1}{p}}]^{\frac{p(\lambda+1)}{\lambda+p-1}} + \left(\frac{\lambda\alpha p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right)\int_0^t\int_\Omega |\nabla(u_n+a)^{\frac{\lambda+p-1}{p}}|^p \\ &\leq ||f||_{L^m(Q)}\left(\iint_Q (u_n+a)^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C.\end{aligned}$$

Now passing to the supremum for $t \in (0,T)$, we obtain

$$\begin{aligned}&||u_n^{\frac{\lambda+p-1}{p}}||^{\frac{p(\lambda+1)}{\lambda+p+1}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda+p+1}}(\Omega))} + \iint_Q |\nabla(u_n+a)^{\frac{\lambda+p-1}{p}}|^p \\ &\leq C\left(\iint_Q (u_n+a)^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C. \qquad (5.40)\end{aligned}$$

From (5.40) and applying Lemma 2.9 (with $\rho = \frac{p(\lambda+1)}{\lambda+p-1}$, $h = p$, $v = u_n^{\frac{\lambda+p-1}{p}}$), we have

$$\begin{aligned}&\iint_Q [u_n^{\frac{\lambda+p-1}{p}}]^{p\frac{N+\frac{p(\lambda+1)}{\lambda+p-1}}{N}} \leq C_G ||u_n^{\frac{\lambda+p-1}{p}}||^{\frac{p}{N}\times\frac{p(\lambda+1)}{\lambda+p-1}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda+p-1}}(\Omega))}\iint_Q |\nabla u_n^{\frac{\lambda+p-1}{p}}|^p \\ &\leq C\left(\iint_Q (u_n+a)^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C,\end{aligned}$$

where $C = C(\alpha, \lambda, m, p, \mu, C_{N,p}, C_G, ||u_0||_{L^\infty(\Omega)})$. Thus we get

$$\iint_Q u_n^{\frac{N(\lambda+p-1)+p(\lambda+1)}{N}} \leq C\left(\iint_Q (u_n+a)^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C.$$

Letting $a \to 0$, we reach that

$$\iint_Q u_n^{\frac{N(\lambda+p-1)+p(\lambda+1)}{N}} \leq C\left(\iint_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C, \qquad (5.41)$$

choosing λ such that

$$\sigma = \frac{N(\lambda+p-1)+p(\lambda+1)}{N} = (\lambda-\gamma)m', \qquad (5.42)$$

this equivalent to

$$\lambda = \frac{(m-1)(N(p-1)+p)+Nm\gamma}{N-pm+p} \quad \text{and} \quad \sigma = m\frac{N(p+\gamma-1)+p(\gamma+1)}{N-pm+p}.$$

From (5.42), the estimate (5.41) becomes

$$\iint_Q u_n^\sigma \leq C\left(\iint_Q u_n^\sigma\right)^{(\frac{p}{N}+1)\frac{1}{m'}} + C.$$

The condition $\frac{p(N+2)}{p(N+2)-N(1-\gamma)} \leq m < \frac{N}{p}+1$, ensure that $\lambda \geq 1$ and $(\frac{p}{N}+1)\frac{1}{m'} < 1$, and thanks to Young inequality we deduce that

$$\iint_Q u_n^\sigma \leq C. \tag{5.43}$$

From (5.37) and, by the fact that and $|\nabla u_n|^p \leq (u_n+a)^{\lambda-1}|\nabla u_n|^p$ $(a>0,\ \lambda \geq 1)$ going back to (5.40) and using (5.42), (5.43) yield that

$$\iint_Q |\nabla u_n|^p \leq \iint_Q (u_n+a)^{\lambda-1}|\nabla u_n|^p \leq C\left(\iint_Q u_n^\sigma\right)^{(\frac{p}{N}+1)\frac{1}{m'}} + C \leq C. \tag{5.44}$$

Then by estimates (5.43) and (5.44) we deduce that the sequence u_n is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and in $L^\sigma(Q)$ with respect to n, and so $u \in L^p(0,T;W_0^{1,p}(\Omega) \cap L^\sigma(Q)$. Hence the proof of item (i) is achieved.

Now we prove item (ii). Let now taking $\gamma < \lambda < 1$ and by definition of $\varphi(u_n)$ and $\Psi(u_n)$, we can get

$$\Psi(s) \geq C|s|^{\lambda+1} - C \qquad \forall s \in \mathbb{R}. \tag{5.45}$$

From (5.45) and going back to (5.38), we have

$$\begin{aligned} &C\int_\Omega \Psi(u_n(x,t)) + \alpha\lambda \int_0^t \int_\Omega (u_n+a)^{\lambda-1}|\nabla u_n|^p \\ &\leq \mu \int_0^t \int_\Omega \frac{(u_n+a)^{\lambda+p-1}}{|x|^p} + \int_0^t \int_\Omega f_n(u_n+a)^{\lambda-\gamma} + \int_\Omega \Psi(u_0) + C|\Omega|. \end{aligned}$$

We proceed as before, we obtain that

$$\begin{aligned} &C||u_n^{\frac{\lambda+p-1}{p}}||^{\frac{p(\lambda+1)}{\lambda+p-1}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda+p-1}}(\Omega))} \\ &+ \left(\frac{\lambda\alpha p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right)\iint_Q |\nabla(u_n+a)^{\frac{\lambda+p-1}{p}}|^p \\ &\leq C\left(\iint_Q (u_n+a)^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C. \end{aligned} \tag{5.46}$$

Thanks to Lemma 2.9 and repeat the above process, it hold that

$$\iint_Q u_n^{\frac{N(\lambda+p-1)+p(\lambda+1)}{N}} \leq C\left(\iint_Q u_n^{(\lambda-\gamma)m'}\right)^{(\frac{p}{N}+1)\frac{1}{m'}} + C. \tag{5.47}$$

Let now choosing λ such that

$$\sigma = \frac{N(\lambda+p-1)+p(\lambda+1)}{N} = (\lambda-\gamma)m', \tag{5.48}$$

this yields that

$$\sigma = m\frac{N(p+\gamma-1)+p(\gamma+1)}{N-pm+p} \ \text{ and } \ \lambda = \frac{(m-1)(N(p-1)+p)+Nm\gamma}{N-pm+p}.$$

Since $\lambda < 1$, then $m < \frac{p(N+2)}{p(N+2)-N(1-\gamma)} < \frac{N}{p} + 1$, and $(\frac{p}{N}+1)\frac{1}{m'} < 1$, from (5.47), (5.48) and thanks to Young inequality it hold that

$$\iint_Q u_n^\sigma \leq C. \tag{5.49}$$

By (5.37), then we have $\frac{\lambda\alpha p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}} > 0$. Let $1 < q < p$, applying Hölder's inequality and from (5.46), we get

$$\begin{aligned} \iint_Q |\nabla u_n|^q &= \iint_Q \frac{|\nabla u_n|^q}{(u_n+a)^{\frac{q(1-\lambda)}{p}}}(u_n+a)^{\frac{q(1-\lambda)}{p}} \\ &\leq \left(\iint_Q \frac{|\nabla u_n|^p}{(u_n+a)^{1-\lambda}}\right)^{\frac{q}{p}} \left(\iint_Q (u_n+a)^{\frac{q(1-\lambda)}{p-q}}\right)^{\frac{p-q}{p}} \\ &\leq \left[C\left(\iint_Q (u_n+a)^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C\right]^{\frac{q}{p}} \left(\iint_Q (u_n+a)^{\frac{q(1-\lambda)}{p-q}}\right)^{\frac{p-q}{p}}, \end{aligned} \tag{5.50}$$

we take q such that

$$\frac{q(1-\lambda)}{p-q} = (\lambda-\gamma)m', \tag{5.51}$$

this equivalent to $q = m\frac{N(p+\gamma-1)+p(\gamma+1)}{N+2-m(1-\gamma)}$. Using (5.51) in (5.50) and letting $a \to 0$, we hold that

$$\iint_Q |\nabla u_n|^q \leq \left(C\left(\iint_Q u_n^\sigma\right)^{\frac{1}{m'}} + C\right)^{\frac{q}{p}} \left(\iint_Q u_n^\sigma\right)^{\frac{p-q}{p}}.$$

From (5.49) it follows that

$$\iint_Q |\nabla u_n|^q \leq C. \tag{5.52}$$

Therefore estimates (5.49) and (5.52) imply that the sequence u_n is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$ and in $L^\sigma(Q)$ with respect to n, and so $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\sigma(Q)$.

Now we give the proof of item (iii). Taking $G_k(u_n)\chi_{(0,t)}$ as a test function in (5.10) for $t \in (0,T)$, we have

$$\begin{aligned} &\int_0^t \int_\Omega \frac{\partial u_n}{\partial t} G_k(u_n) + \int_0^t \int_\Omega a(x,t,\nabla u_n)\nabla G_k(u_n) \\ &-\mu \int_0^t \int_\Omega \frac{u_n^{p-1}}{|x|^p + \frac{1}{n}} G_k(u_n) \leq \int_0^t \int_\Omega \frac{f_n G_k(u_n)}{(u_n+\frac{1}{n})^\gamma}. \end{aligned} \tag{5.53}$$

We observe that the function $G_k(u_n)$ is different from zero only on the set $A_{k,n} = \{(x,t) \in Q : u_n(x,t) > k\}$, and that, on this set, we have $u_n + \frac{1}{n} \geq k \geq 1$. Note that

$$\begin{aligned} \int_0^t \int_\Omega a(x,t,\nabla u_n)\nabla G_k(u_n) &= \iint_{A_{k,n}} a(x,t,\nabla u_n)\nabla u_n \\ &\geq \alpha \iint_{A_{k,n}} |\nabla u_n|^p = \alpha \int_0^t \int_\Omega |\nabla G_k(u_n)|^p \end{aligned}$$

and

$$\int_0^t \int_\Omega \frac{\partial u_n}{\partial t} G_k(u_n) = \frac{1}{2} \iint_{A_{k,n}} \frac{\partial}{\partial t}(u_n - k)^2 = \frac{1}{2} \iint_{A_{k,n}} \frac{\partial}{\partial t} \left((u_n - k)^+\right)^2$$
$$= \frac{1}{2} \int_\Omega G_k(u_n(x,t))^2 - \frac{1}{2} \int_\Omega G_k^2(u_0),$$

applying Hardy inequality and using the fact that $G_k(u_n) \leq u_n$ in the set $A_{k,n}$, we can write

$$\int_0^t \int_\Omega \frac{u_n^{p-1} G_k(u_n)}{|x|^p + \frac{1}{n}} = \iint_{A_{k,n}} \frac{u_n^{p-1} G_k(u_n)}{|x|^p + \frac{1}{n}} \leq \iint_{A_{k,n}} \frac{u_n^p}{|x|^p}$$
$$\leq \frac{1}{C_{N,p}} \iint_{A_{k,n}} |\nabla u_n|^p = \frac{1}{C_{N,p}} \iint_{A_{k,n}} |\nabla G_k(u_n)|^p \leq \frac{1}{C_{N,p}} \int_0^t \int_\Omega |\nabla G_k(u_n)|^p.$$

Inequality (5.53) becomes

$$\frac{1}{2} \int_\Omega G_k^2(u_n) + \left(\alpha - \frac{\mu}{C_{N,p}}\right) \int_0^t \int_\Omega |\nabla G_k(u_n)|^p$$
$$\leq \int_0^t \int_\Omega f G_k(u_n) + \frac{1}{2} \int_\Omega G_k^2(u_0).$$

Passing to the supremum in $t \in (0,T)$, we get

$$\frac{1}{2} ||G_k(u_n)||^2_{L^\infty(0,T;L^2(\Omega))} + \left(\alpha - \frac{\mu}{C_{N,p}}\right) \iint_Q |\nabla G_k(u_n)|^p$$
$$\leq \iint_Q f G_k(u_n) + \frac{1}{2} \int_\Omega G_k^2(u_0).$$

From now on, we can follow the standard technique used for the non-singular case in [11], we deduce there exist a constant C_∞ independent of n such that

$$||u_n||_{L^\infty(Q)} \leq C_\infty. \tag{5.54}$$

Now taking u_n as a test function in (5.10), by (5.3) and Hardy inequality, we have

$$\frac{1}{2} \int_\Omega u_n^2(x,T) + \left(\alpha - \frac{\mu}{C_{N,p}}\right) \iint_Q |\nabla u_n|^p \leq \iint_Q f u_n^{1-\gamma} + \frac{1}{2} \int_\Omega u_0^2.$$

Since $u_0 \in L^\infty(\Omega)$ and by (5.54) and Hölder's inequality,we have

$$\frac{1}{2} \int_\Omega u_n^2(x,T) + \left(\alpha - \frac{\mu}{C_{N,p}}\right) \iint_Q |\nabla u_n|^p$$
$$\leq ||u_n||^{1-\gamma}_{L^\infty(Q)} \iint_Q f + \frac{1}{2} ||u_0||^2_{L^2(\Omega)}$$
$$\leq ||u_n||^{1-\gamma}_{L^\infty(Q)} ||f||_{L^m(Q)} |Q|^{1-\frac{1}{m}} + \frac{1}{2} ||u_0||^2_{L^2(\Omega)} \leq C.$$

As $0 \leq \mu < \alpha C_{N,p}$, and by the last estimate, we obtain

$$\iint_Q |\nabla u_n|^p \leq C, \tag{5.55}$$

where C is a positive constant independent of n. Hence the proof of Theorem 5.14 is completed. □

In the following Theorem we are interesting to prove regularity of u solution of (5.1) when the datum f belong to $L^r(0,T;L^q(\Omega))$, with $r, q > 1$.

Theorem 5.15. *Under the hypothesis* (5.3)-(5.5)*, if* $0 < \gamma < 1$ *and* $0 \leq \mu < \alpha M C_{N,p}$*, with*

$$M = \frac{Nq(r(p+\gamma-1)-(p-2)) - Nr + pq(r-1)}{Nr - pq(r-1)} \times$$
$$\left(\frac{p[Nr - pq(r-1)]}{Nqr(p+\gamma-1) + (p-2)[N(r-q) - pq(r-1)]}\right)^p,$$

$f \in L^r(0,T;L^q(\Omega))$ *with* q *and* r *be real numbers such that*

$$r > 1,\ q > 1; \quad p \leq \frac{N}{q} + \frac{p}{r} \leq \min\{\theta_1, \theta_2\},$$

where

$$\theta_1 = \frac{N}{r} + p \quad \text{and} \quad \theta_2 = \frac{N}{r}\left(1 - \frac{p}{2}\right) + \frac{Np + 2p + N(\gamma-1)}{2}.$$

Then there exists a weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\delta(Q)$ *to problem* (5.1) *with*

$$\delta = \frac{qr(N+p)(\gamma+1) + N(p-2)(pr-q+r)}{Nr - pq(r-1)}.$$

Remark 5.16. If $\gamma, \mu \to 0$, then the result of Theorem 5.15 coincides with classical regularity results for parabolic problems with coercivity (see [22, Theorem 1.1]).

Proof. Let now testing (5.3) by $\varphi(u_n) = ((u_n+a)^\lambda - a^\lambda)\chi_{(0,t)}$, $0 < a < \frac{1}{n}$, $\lambda > 0$ and repeating the same passage of proof of item (i) of Theorem 5.14 in order to arrive to the following inequality

$$\frac{1}{\lambda+1}\int_\Omega u_n^{\lambda+1}(x,t) + \left(\frac{\alpha\lambda p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right)\int_0^t\int_\Omega |\nabla(u_n+a)^{\frac{\lambda+p-1}{p}}|^p$$
$$\leq \int_0^t\int_\Omega f(u_n+a)^{\lambda-\gamma} + C.$$

Passing to supremum for $t \in [0,T]$, we obtain

$$c_0||u_n||^{\lambda+1}_{L^\infty(0,T;L^{\lambda+1}(\Omega))} + \left(\frac{\alpha\lambda p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right)\iint_Q |\nabla(u_n+a)^{\frac{\lambda+p-1}{p}}|^p$$
$$\leq \iint_Q f(u_n+a)^{\lambda-\gamma} + C. \tag{5.56}$$

Setting $v_n = u_n^{\frac{\lambda+p-1}{p}}$ and $I = \iint_Q f(u_n+a)^{\lambda-\gamma}$, formula (5.56) can be rewritten as

$$c_0 ||v_n||^{\frac{p(\lambda+1)}{\lambda+p-1}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda+p-1}}(\Omega))} + \left(\frac{\alpha\lambda p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right) \iint_Q |\nabla v_n|^p \leq I + C. \tag{5.57}$$

Using Hölder's inequality twice, for all $q > 1$ and $r > 1$ we get

$$\begin{aligned} I &\leq \int_0^T \left(\int_\Omega f^q\right)^{\frac{1}{q}} \left(\int_\Omega v_n^{\frac{p(\lambda-\gamma)}{\lambda+p-1}\frac{q}{q-1}}\right)^{\frac{q-1}{q}} \\ &\leq ||f||_{L^r(0,T;L^q(\Omega))} \left[\int_0^T \left(\int_\Omega v_n^{\frac{p(\lambda-\gamma)}{\lambda+p-1}\frac{q}{q-1}}\right)^{\frac{q-1}{q}\frac{r}{r-1}}\right]^{\frac{r-1}{r}} \\ &= C_f \left[\int_0^T ||v_n||^{\frac{p(\lambda-\gamma)r}{(\lambda+p-1)(r-1)}}_{L^{\frac{p(\lambda-\gamma)q}{(\lambda+p-1)(q-1)}}(\Omega)}\right]^{\frac{r-1}{r}}. \end{aligned} \tag{5.58}$$

Let us define $\eta \in (0,1)$ such that

$$\frac{(\lambda+p-1)(q-1)}{p(\lambda-\gamma)q} = \eta\left(\frac{1}{p} - \frac{1}{N}\right) + (1-\eta)\frac{\lambda+p-1}{p(\lambda+1)}. \tag{5.59}$$

Thus, by the Lemma 2.9, applied to

$$\sigma = \frac{p(\lambda-\gamma)q}{(\lambda+p-1)(q-1)} \quad \text{and} \quad \rho = \frac{p(\lambda+1)}{\lambda+p-1}, \tag{5.60}$$

we have

$$||v_n||^{\frac{p(\lambda-\gamma)r}{(\lambda+p-1)(r-1)}}_{L^{\frac{p(\lambda-\gamma)q}{(\lambda+p-1)(q-1)}}(\Omega)} \leq C_G ||v_n||^{(1-\eta)\frac{p(\lambda-\gamma)r}{(\lambda+p-1)(r-1)}}_{L^{\frac{p(\lambda+1)}{\lambda+p-1}}(\Omega)} ||\nabla v_n||^{\eta\frac{p(\lambda-\gamma)r}{(\lambda+p-1)(r-1)}}_{L^p(\Omega)}. \tag{5.61}$$

Integrating on time we obtain

$$\begin{aligned} &\left[\int_0^T ||v_n||^{\frac{p(\lambda-\gamma)r}{(\lambda+p-1)(r-1)}}_{L^{\frac{p(\lambda-\gamma)q}{(\lambda+p-1)(q-1)}}(\Omega)}\right]^{\frac{r-1}{r}} \\ &\leq ||v_n||^{(1-\eta)\frac{p(\lambda-\gamma)}{(\lambda+p-1)}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda+p-1}}(\Omega))} \left[\int_0^T ||\nabla v_n||^{\eta\frac{p(\lambda-\gamma)r}{(\lambda+p-1)(r-1)}}_{L^p(\Omega)}\right]^{\frac{r-1}{r}}. \end{aligned} \tag{5.62}$$

If $\eta < 1$, applying the Young inequality with exponents

$$\frac{\lambda+1}{(1-\eta)(\lambda-\gamma)} \quad \text{and} \quad \frac{\lambda+1}{1+\gamma+\eta(\lambda-\gamma)},$$

we deduce

$$\begin{aligned} &\left[\int_0^T ||v_n||^{\frac{p(\lambda-\gamma)r}{(\lambda+p-1)(r-1)}}_{L^{\frac{p(\lambda-\gamma)q}{(\lambda+p-1)(q-1)}}(\Omega)}\right]^{\frac{r-1}{r}} \\ &\leq \epsilon ||v_n||^{\frac{p(\lambda+1)}{\lambda+p-1}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda+p-1}}(\Omega))} + C_\epsilon \left[\int_0^T ||\nabla v_n||^{\eta\frac{p(\lambda-\gamma)r}{(\lambda+p-1)(r-1)}}_{L^p(\Omega)}\right]^{\frac{r-1}{r}\frac{\lambda+1}{1+\gamma+\eta(\lambda-\gamma)}}. \end{aligned} \tag{5.63}$$

Letting $\epsilon = \frac{c_0}{2C_f}$ and collecting (5.57), (5.58) and (5.63), we have

$$\begin{aligned} & c_0||v_n||^{\frac{p(\lambda+1)}{\lambda+p-1}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda+p-1}}(\Omega))} + \left(\frac{\alpha\lambda p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right)||\nabla v_n||^p_{L^p(Q)} \\ & \leq \frac{c_0}{2}||v_n||^{\frac{p(\lambda+1)}{\lambda+p-1}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda+p-1}}(\Omega))} \\ & + C_f C_\epsilon \left[\int_0^T ||\nabla v_n||^{\eta\frac{p(\lambda-\gamma)r}{(\lambda+p-1)(r-1)}}_{L^p(\Omega)}\right]^{\frac{r-1}{r}\frac{\lambda+1}{1+\gamma+\eta(\lambda-\gamma)}} + c_1. \end{aligned} \tag{5.64}$$

Now we choose λ satisfying

$$\frac{\eta p(\lambda-\gamma)r}{(\lambda+p-1)(r-1)} = p, \tag{5.65}$$

such that $\lambda > \gamma > 0,\ r > 1$ and $0 \leq \mu < \frac{\alpha\lambda p^p C_{N,p}}{(\lambda+p-1)^p}$. From (5.65), it hold that

$$\begin{aligned} & \frac{c_0}{2}||v_n||^{\frac{p(\lambda+1)}{\lambda+p-1}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda+p-1}}(\Omega))} + \left(\frac{\alpha\lambda p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right)||\nabla v_n||^p_{L^p(Q)} \\ & \qquad \leq C_f C_\epsilon ||\nabla v_n||^{\frac{p(r-1)}{r}\frac{\lambda+1}{1+\gamma+\eta(\lambda-\gamma)}}_{L^p(Q)} + c_1. \end{aligned} \tag{5.66}$$

Since, from (5.65)

$$r\eta(\lambda-\gamma) = (\lambda+p-1)(r-1)$$

we have

$$\begin{aligned} \beta = \tfrac{r-1}{r} \times \tfrac{\lambda+1}{1+\gamma+\eta(\lambda-\gamma)} &= \tfrac{(r-1)(\lambda+1)}{r(1+\gamma)+r\eta(\lambda-\gamma)} = \tfrac{(r-1)(\lambda+1)}{r(1+\gamma)+(\lambda+p-1)(r-1)} \\ &= \tfrac{(r-1)(\lambda+1)}{(r-1)(\lambda+1)+r(1+\gamma)+(r-1)(p-2)} < 1, \end{aligned}$$

and so

$$\begin{aligned} & \frac{c_0}{2}||v_n||^{\frac{p(\lambda+1)}{\lambda+p-1}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda+p-1}}(\Omega))} + \left(\frac{\alpha\lambda p^p}{(\lambda+p-1)^p} - \frac{\mu}{C_{N,p}}\right)||\nabla v_n||^p_{L^p(Q)} \\ & \qquad \leq C_f C_\epsilon ||\nabla v_n||^{p\beta}_{L^p(Q)} + c_1, \end{aligned} \tag{5.67}$$

with $\beta < 1$. If $\eta = 1$, choosing λ as in (5.65), (5.63) becomes (5.67) with $\beta = \frac{r-1}{r} < 1$. Thus from (5.67) immediately follows

$$||v_n||^{\frac{p(\lambda+1)}{\lambda+p-1}}_{L^\infty(0,T;L^{\frac{p(\lambda+1)}{\lambda+p-1}}(\Omega))} + ||\nabla v_n||^p_{L^p(Q)} \leq c_2. \tag{5.68}$$

Thanks to Lemma 2.9, we obtain

$$||v_n||_{L^\sigma(Q)} \leq c_3,$$

where $\sigma = p\frac{N+\frac{p(\lambda+1)}{\lambda+p-1}}{N}$. Recalling the definition of v_n we thus have proved that

$$||u_n||_{L^\delta(Q)} \leq c_3, \tag{5.69}$$

where c_3 is a positive constant independent of n, and

$$\delta = \sigma\frac{\lambda+p-1}{p} = \frac{N(\lambda+p-1)+p(\lambda+1)}{N}. \tag{5.70}$$

From (5.59) and (5.65), we deduce that

$$\lambda+1 = \frac{Nq[r(p+\gamma-1)-(p-2)]}{Nr-pq(r-1)}, \tag{5.71}$$

which implies, by (5.71)

$$\delta = \frac{qr(N+p)(\gamma+1)+N(p-2)(qr-q+r)}{Nr-pq(r-1)}.$$

we now have to check that $\lambda \geq 1$ and that η, defined in (5.59), belong to $(0,1)$. After easy calculations, we obtain that $\lambda \geq 1$ if and only if

$$p < \frac{N}{q} + \frac{p}{r} \leq \frac{N}{r}\left(1-\frac{p}{2}\right) + \frac{Np+2p+N(\gamma-1)}{2}$$

while the condition $\eta \leq 1$ hold is satisfied and only if

$$\frac{N}{q} + \frac{p}{r} < \frac{N}{r} + p.$$

The condition $\eta \geq 0$ is automatically satisfied if $\lambda \geq 1$.
It remains to prove the estimate in $L^p(0,T;W_0^{1,p}(\Omega))$. By (5.56), (5.58), (5.68) and $\lambda \geq 1$, we obtain

$$\iint_Q |\nabla u_n|^p \leq \iint_Q |\nabla u_n|^p (u_n+a)^{\lambda-1} \leq c_2, \tag{5.72}$$

then the sequence u_n bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, and so $u \in L^p(0,T;W_0^{1,p}(\Omega))$. The estimates (5.69) and (5.72) completed the proof of Theorem 5.15. □

Chapter 6

Existence and regularity results for a singular parabolic equations with degenerate coercivity

1 Introduction

In this final chapter, we study the existence and regularity results of double nonlinear parabolic problems with absorption term and with singular lower order term, whose the model example is the following:

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u) + |u|^{s-1}u = h(u)f & \text{in} \quad Q, \\ u(x,0) = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma, \end{cases} \tag{6.1}$$

where

$$A(u) = -\mathrm{div}(a(x,t,u,\nabla u)).$$

Here Ω is a bounded open subset of $\mathbb{R}^N$, $(N > p \geq 2)$ and $0 < T < +\infty$, f is a non-negative function that belong to some Lebesgue space, $f \in L^m(Q)$, $m \geq 1$ $Q = \Omega \times (0,T)$, $\Gamma = \partial\Omega \times (0,T)$, $0 < \gamma < 1$ and $s \geq 1$. $a(x,t,\sigma,\xi) : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \longrightarrow I\!R^N$ is a Carathéodory function (i.e it is continuous with respect to σ and ξ for almost $(x,t) \in Q$, and measurable with respect to (x,t) for every $\sigma \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$) satisfying for a.e $(x,t) \in Q$, $\forall\, \xi, \xi' \in \mathbb{R}^N$:

$$a(x,t,\sigma,\xi).\xi \geq \frac{\alpha|\xi|^p}{(1+|\sigma|)^{\theta(p-1)}}, \tag{6.2}$$

$$|a(x,t,\sigma,\xi)| \leq b(x,t) + |\sigma|^{p-1} + |\xi|^{p-1}, \tag{6.3}$$

$$(a(x,t,\sigma,\xi) - a(x,t,\sigma,\xi')).(\xi - \xi') > 0 \quad \xi \neq \xi', \tag{6.4}$$

where α is positive constant, $0 \leq \theta < 1$ and b is a non-negative function and belong to $L^{p'}(Q)$, $p' = \frac{p}{p-1}$. The singular sourcing term $h : [0,\infty) \longrightarrow [0,\infty]$ is a continuous, bounded outside the origin with $h(0) \neq 0$ and such that the following propertied hold true

$$(h1) \qquad \exists c > 0 \text{ such that } \quad h(s) \leq \frac{c}{s^\gamma} \quad \forall s > 0.$$

In the non-singular case (i.e. $h \equiv 1$) in [75, 143] existence and regularity results for nonlinear parabolic equations in divergence form depending on the summability of f have been proved when the absorption term $|u|^{s-1}u\,(s \geq 1)$ doesn't appear, we recall that under uniform ellipticity, that is when $\theta = 0$, the existence and regularity solutions was obtained in [20, 56, 91]. When the term $|u|^{s-1}u$ exists, several works study the existence and regularity of solution of problem (6.1) (see [23, 106], and reference therein).

Finally, concerning the singular model case the authors in [68] studied existence and regularity of problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = \frac{f(x,t)}{u^\gamma} & \text{in} \quad Q, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma, \end{cases}$$

with $\gamma > 0$, $p \geq 2$, $f > 0$, $f \in L^m(Q)$, $m \geq 1$ and $u_0 \in L^\infty(\Omega)$. In [122] the authors studied the existence and uniqueness solution of problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = h(u)f + \mu & \text{in} \quad \Omega \times (0,T), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u = u_0 & \text{in} \quad \Omega \times \{0\}, \end{cases}$$

where $p > 2 - \frac{1}{N+1}$, u_0 is a non-negative function, μ is a non-negative bounded Radon measure on $\Omega \times (0,T)$, f is a non-negative function in $L^1(\Omega \times (0,T))$, and h is a positive continuous function possibly blowing up at the origin.

In the elliptic case the authors in [63], studied existence a solution of (6.1), where $A(u) = -\text{div}(a(x, \nabla u))$, $f = \mu$ and h continuous positive function outside the origin such that $\lim_{s \to 0^+} h(s) = +\infty$. In [62] the authors proving existence and regularity of (6.1), where $A(u) = -\Delta_p u$ and $h(u) = \frac{1}{u^\gamma}$ with $\gamma > 0$. See as well [81, 118, 136]. If (6.2) hold true, the differential operator $A(u)$ is not coercive as u becomes large. This shows that the classical methods (see [108]) can't be applied to prove the existence of solution to problem (6.1) even if the data $h(u)f$ is sufficiently regular.

We overcome this difficulty by replacing operator $A(u)$ by another one defined by means of truncations and using Shouder's fixed point Theorem, our objective is to look for the existence of solution to problem (6.1), for different summabilities of the datum.

The main tool we use is an a prior estimate for solutions of approximate equations with non degenerate coercivity (which thus have solution) and then we pass to the limit to find a solutions.
We first define the notion of a weak solution to (6.1) as follows:

Definition 6.1. We say that $u \in L^p(0,T; W_0^{1,p}(\Omega))$ is a weak solution to problem (6.1), if $a(x,t,u,\nabla u) \in L^1(Q)$, $|u|^{s-1}u \in L^1(Q)$ and $h(u)f \in L^1(Q)$, and the equality

$$\begin{aligned} -\int_0^T \int_\Omega u \frac{\partial \psi}{\partial t} dxdt + \int_0^T \int_\Omega a(x,t,u,\nabla u)\nabla \psi dxdt \\ + \int_0^T \int_\Omega |u|^{s-1}u\psi dxdt = \int_0^T \int_\Omega h(u)f\psi dxdt, \end{aligned} \tag{6.5}$$

holds for every $\psi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

2 Some technical lemmas and main results

In order to prove the main results of this chapter, we need to the following lemmas.
Let a, b, λ and p be a positive real numbers with $p > 1$. Let us define

$$\varphi(s) = \begin{cases} e^{\lambda s} - 1 & \text{if } s \geq 0, \\ -e^{-\lambda s} + 1 & \text{if } s < 0. \end{cases} \tag{6.6}$$

Note that the function φ has the same sign as its argument. Furthermore, we have

Lemma 6.2. *[51, Lemma 2.1] If $\lambda > \left(\frac{2a}{b}\right) + p$, then we have*

$$a\varphi'(s) - b|\varphi(s)| \geq \frac{a}{2}e^{\lambda s} \qquad \forall s \geq 0, \tag{6.7}$$

$$\varphi(s) \geq \left[\varphi\left(\frac{s}{p}\right)\right]^p \qquad \forall s \geq 0, \tag{6.8}$$

$\exists d \geq 0$ and $M > 0$ such that

$$\varphi(s) \leq M\left[\varphi\left(\frac{s}{p}\right)\right]^p, \quad \varphi'(s) \leq M\left[\varphi\left(\frac{s}{p}\right)\right]^p \quad \forall s \geq d, \tag{6.9}$$

$$|\varphi(s)| \geq |s| \qquad \forall s \in I\!R. \tag{6.10}$$

Lemma 6.3. *[51, Lemma 6.1] Let ϕ be the function defined by*

$$\phi(\sigma) = \int_0^\sigma \varphi(s)ds, \tag{6.11}$$

where φ defined in (6.6). Then there exist a constant $C_0 > 0$ such that

$$\phi(s) \geq C_0\left[\varphi\left(\frac{s}{p}\right)\right]^p \quad \forall s \geq 0, \quad p \geq 2. \tag{6.12}$$

Now we state the mains results.

Theorem 6.4. *Under the assumptions (6.2)- (6.4) and h satisfies $(h1)$. If $f \in L^m(Q)$ with $m > \frac{N}{p} + 1$, then there exists a bounded weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ to problem (6.1).*

Remark 6.5. The results of Theorem 6.4 coincide with regularity results of [143].

Theorem 6.6. *Under the assumptions (6.2)-(6.4) and h satisfies $(h1)$. If $f \in L^m(Q)$ with $m = \frac{N}{p} + 1$, Then there exists a solution u to problem (6.1) such that $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^r(Q)$. With $2 \leq r < +\infty$.*

Remark 6.7. The result of Theorem 6.6 has been obtained in [107].

Theorem 6.8. *Under the assumptions* (6.2)-(6.4) *and* h *satisfied* $(h1)$. *If* $f \in L^m(Q)$ *with*

$$\frac{p(N+2+\theta(p-1))}{p(N+2+\theta(p-1)) - N[\theta(p-1)+1-\gamma]} \leq m < \frac{N}{p} + 1$$

and $s \geq 1$, *then there exist a solution* u *to problem* (6.1) *such that* $u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^r(Q)$, *where*

$$r = \begin{cases} \frac{(m-1)[p(N+1-s)-N\theta(p-1)]+N(s+1-m(1-\gamma))}{N-pm+p} & s > \frac{p(1+m\gamma)+N(p-1)(1-\theta)}{N-pm+p}, \\ \frac{m[N(p+\gamma-1)+p(\gamma+1)-N\theta(p-1)]}{N-pm+p} & s \leq \frac{p(1+m\gamma)+N(p-1)(1-\theta)}{N-pm+p}. \end{cases}$$

Theorem 6.9. *Under the assumptions* (6.2)-(6.4) *and* h *satisfied* $(h1)$. *If* $f \in L^m(Q)$ *with*

$$1 \leq m < \frac{p(N+2+\theta(p-1))}{p(N+2+\theta(p-1)) - N[\theta(p-1)+1-\gamma]}$$

and $s \geq 1$, *then there exist a solution* u *to problem* (6.1) *such that* $L^q(0,T; W_0^{1,q}(\Omega)) \cap L^r(Q)$, *where*

$$q = \frac{m[N(p+\gamma-1)+p(\gamma+1)-N\theta(p-1)]}{N+2-m(1-\gamma)-\theta(p-1)(m-1)}$$

and r *is defined in Theorem 6.8.*

Remark 6.10. If $\theta = 0$, then the result of Theorem 6.8 coincide with result case (b) of item (iii) of Theorem 4.1 in [68], and the result of Theorem 6.9 coincide with result of Theorem 4.2 in [68].

In the following theorem we will see the impact of the term $|u|^{s-1}u$ on the regularity of solution u of problem (6.1) when the data $f \in L^m(Q)$, with $1 < m < \frac{p(N+2+\theta(p-1))}{p(N+2+\theta(p-1))-N[\theta(p-1)+1-\gamma]}$.

Theorem 6.11. *Under the assumptions* (6.2)-(6.4) *and* h *satisfied* $(h1)$, $f \in L^m(Q)$ *with*

$$1 < m < \frac{p(N+2+\theta(p-1))}{p(N+2+\theta(p-1)) - N[\theta(p-1)+1-\gamma]}.$$

(i) *If* $s \geq \frac{1+\theta(p-1)-m\gamma}{m-1}$, *then there exist a solution* u *of problem* (6.1) *with* $u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^{(s+\gamma)m}(Q)$.

(ii) *If* $\frac{1+\theta(p-1)-mp\gamma}{mp-1} < s < \frac{1+\theta(p-1)-m\gamma}{m-1}$, *then there exist a solution* u *of problem* (6.1) *with the regularity* $u \in L^q(0,T; W_0^{1,q}(\Omega))$, *where* $q = \frac{pm(s+\gamma)}{1+\theta(p-1)+s}$, *moreover* $u \in L^r(Q)$, *where*

$$r = \begin{cases} (s+\gamma)m & s \geq \frac{p(N+1+m\gamma)-N(1+\theta(p-1))}{N-pm+p}, \\ \frac{pm(s+\gamma)[N+1+(m-1)s+m\gamma]}{N[1+\theta(p-1)+s]} & s < \frac{p(N+1+m\gamma)-N(1+\theta(p-1))}{N-pm+p}. \end{cases}$$

Remark 6.12. If $\theta = 0$ and $\gamma \to 0$, then the result of Theorem 6.11 coincide with result of Theorem 2.2 and Theorem 2.3 in [106].

3 A priori estimates

For $n \in I\!N$, let $T_k(s) = \max(-k, \min(s,k))$, we will consider the following approximation of (6.1)

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div} a(x,t,T_n(u_n),\nabla u_n) + |u_n|^{s-1}u_n = h_n(u_n)f_n & \text{in} \quad Q, \\ u_n(x,0) = u_0(x) = 0 & \text{in} \quad \Omega, \\ u_n = 0 & \text{on} \quad \Gamma. \end{cases} \tag{6.13}$$

where $f_n = T_n(f)$ and $f_n \in C_0^\infty(\overline{Q})$, such that

$$||f_n||_{L^m(Q)} \leq ||f||_{L^m(Q)} \text{ and } f_n \longrightarrow f \text{ strongly in } L^m(Q).$$

Moreover, define $h(0) = \lim_{s\to 0} h(s)$, we set

$$h_n(s) = \begin{cases} T_n(h(s)) & \text{for} \quad s > 0, \\ \min(n, h(0)) & \text{otherwise.} \end{cases}$$

Lemma 6.13. *Let a satisfy* (6.2), (6.3) *and* (6.4). *Then the approximating problem* (6.13) *has a non-negative solution* $u_n \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ *for all* $n \in I\!N$ *fixed and* $2 \leq p < N$.

Proof. Let $n \in I\!N$ and $v \in L^p(Q)$ be fixed. We know that the following class of doubly degenerate nonlinear singular parabolic problem

$$\begin{cases} \dfrac{\partial w}{\partial t} - \operatorname{div} a(x,t,T_n(w),\nabla w) + |w|^{s-1}w = h_n(v)f_n & \text{in} \quad Q, \\ w(x,0) = u_0(x) = 0 & \text{in} \quad \Omega, \\ w(x,t) = 0 & \text{on} \quad \Gamma, \end{cases} \tag{6.14}$$

has a unique solution $w \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that $w_t \in L^1(Q)+ L^{p'}(0,T;W^{-1,p'}(\Omega))$, see [108]. Moreover, since the datum $h_n(v)f_n$ bounded, we have that $w \in L^\infty(Q)$ and there exists a positive constant d, independents of v and w (but possibly depending in n), such that $||w||_{L^\infty(Q)} \leq d$. Our aim is to prove the existence of fixed point of the map $S : L^p(Q) \longrightarrow L^p(Q)$, where $S(v) = w$, and w the weak solution of problem (6.14). Again, thanks to the regularity of the datum $h_n(v)f_n$, we can take $((1 + |w|)^{\theta(p-1)+1} - 1)\text{sign}(w)$ as test function in (6.14), we obtain

$$\begin{aligned} &\int_0^T \int_\Omega w_t((1+|w|)^{\theta(p-1)+1} - 1)\text{sign}(w)dxdt \\ &+(\theta(p-1)+1)\int_Q a(x,t,T_n(w),\nabla w)\cdot \nabla w(1+|w|)^{\theta(p-1)}dxdt \\ &+\int_Q |w|^s((1+|w|)^{\theta(p-1)+1} - 1)dxdt \\ &= \int_0^T \int_\Omega h_n(u_n)f_n((1+|w|)^{\theta(p-1)+1} - 1)\text{sign}(w)dxdt \end{aligned}$$

By (6.2) and by classical integration by parts formulas, we have

$$\begin{aligned}&\frac{1}{\theta(p-1)+2}\int_{\Omega}((1+|w(x,T)|)^{\theta(p-1)+2}-|w(x,T))dx\\&-\frac{|\Omega|}{\theta(p-1)+2}+\alpha(\theta(p-1)+1)\int_{Q}|\nabla w|^{p}dxdt\\&\leq\int_{0}^{T}\int_{\Omega}w_{t}(1+|w|)^{\theta(p-1)+1}\text{sign}(w)dxdt\\&+(\theta(p-1)+1)\int_{Q}a(x,t,T_{n}(w),\nabla w)\cdot\nabla w(1+|w|)^{\theta(p-1)}dxdt.\end{aligned}$$

The term on the right of the last equality is estimated as follows

$$\begin{aligned}&\int_{Q}h_{n}(u_{n})f_{n}(1+|w|)^{\theta(p-1)+1}\text{sign(w)}dxdt\leq n^{2}\int_{Q}(1+|w|)^{\theta(p-1)+1}dxdt\\&\leq n^{2}2^{\theta(p-1)}|Q|+n^{2}2^{\theta(p-1)}\int_{Q}|w|^{\theta(p-1)+1}dxdt,\end{aligned}$$

and so, dropping the positive term and using Hölder's inequality, we obtain

$$\begin{aligned}&\int_{Q}|\nabla w|^{p}dxdt\leq n^{2}2^{\theta(p-1)}|Q|+n^{2}2^{\theta(p-1)}\int_{Q}|w|^{\theta(p-1)+1}dxdt\\&\leq\frac{1}{\alpha(\theta(p-1)+1)}\left(n^{2}2^{\theta(p-1)}|Q|+\frac{|\Omega|}{\theta(p-1)+2}\right)\\&+\frac{n^{2}2^{\theta(p-1)}|Q|^{1-\frac{\theta(p-1)+1}{p}}}{\alpha(\theta(p-1)+1)}\left(\int_{Q}|w|^{p}dxdt\right)^{\frac{\theta(p-1+1)}{p}}\\&\leq C_{1}+C_{2}\left(\int_{Q}|w|^{p}dxdt\right)^{\frac{\theta(p-1)+1}{p}},\end{aligned}$$

where $C_1=\frac{1}{\alpha(\theta(p-1)+1)}\left(n^2 2^{\theta(p-1)}|Q|+\frac{|\Omega|}{\theta(p-1)+2}\right)$, $C_2=\frac{n^2 2^{\theta(p-1)}|Q|^{1-\frac{\theta(p-1)+1}{p}}}{\alpha(\theta(p-1)+1)}$.

By Poincaré inequality and applying Young's inequality with ϵ, we obtain

$$\frac{1}{C_{p}^{p}}\int_{Q}|w|^{p}dxdt\leq C_{1}+\epsilon C_{2}\int_{Q}|w|^{p}dxdt+C_{\epsilon},$$

take $\epsilon=\frac{1}{2C_2C_p^p}$ in last inequality, we have

$$\int_{Q}|w|^{p}dxdt\leq 2C_{p}^{p}(C_{1}+C_{\epsilon}),$$

where C_p the constant of Poincaré. Which implies

$$\left(\int_{Q}|w|^{p}dxdt\right)^{\frac{1}{p}}\leq C_{3},\tag{6.15}$$

where $C_3 = \left(2C_p^p(C_1 + C_\epsilon)\right)^{\frac{1}{p}}$, for some constant C_3 independent of v and w (possible depending on n).

We are going to prove that S is both continuous and compact on B. (B is a ball of $L^p(Q)$ of radius C_3). B is invariant for S. Let v_r be a bounded sequence in B. We will show that there exists a subsequence of w_r that is strongly convergent in $L^p(Q)$. Taking $((1+|w_r|)^{\theta(p-1)+1} - 1)\text{sign}(w_r)$ as a test function in the problem solved by w_r, that is the following

$$\begin{cases} \dfrac{\partial w_r}{\partial t} - \ diva(x,t,T_n(w_r),\nabla w_r) + |w_r|^{s-1}w_r = h_n(v_r)f_n & \text{in} \quad Q, \\ w_r(x,t) = 0 & \text{on} \quad \Gamma, \\ w_r(x,0) = u_0(x) = 0 & \text{in} \quad \Omega. \end{cases}$$

We have

$$\int_0^T ||\nabla w_r||_{L^p(\Omega)} dt \le C_1 + C_2 \left(\int_Q |w_r|^p dxdt\right)^{\frac{\theta(p-1)+1}{p}},$$

with C_1 is defined before and independent of r. Since the ball of $L^p(Q)$ is invariant for S, we have w_r belong to B and so, from the last inequality, we obtain that w_r is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$.

$$\begin{aligned} &|a(x,t,T_n(w_r),\nabla w_r)|^{p'} \le (b(x,t) + |w_r|^{p-1} + |\nabla w_r|^{p-1})^{p'} \\ &\le 2^{2p'-2}|b(x,t)|^{p'} + 2^{2p'-2}|w_r|^p + 2^{p-1}|\nabla w_r|^p \\ &\Rightarrow \int_0^T \int_\Omega |a(x,t,w_r,\nabla w_r)|^{p'} \le 2^{2p'-2}||b(x,t)||^{p'}_{L^{p'}(Q)} + 2^{2p'-2}||w_r||^p_{L^p(Q)} \\ &+2^{p-1}||w_r||^p_{L^p(0,T;W_0^{1,p}(\Omega))} < +\infty. \end{aligned}$$

We have w_r bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, then $(w_r)_t$ is bounded in dual space $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ see [139] implies that w_r is relatively strongly compact in $L^1(Q)$; thus, there exists a subsequence of w_r that almost everywhere converge to some limit function $w \in L^1(Q)$.

Now, we recall that w_r is bounded in $L^\infty(Q)$ with $||w||_{L^\infty(Q)} \le d$, where d is a positive constant independent of r. Thus, since there exists a subsequence of w_r that converge a.e. to w, this allows to use Lebesgue Theorem to ensure that this subsequence of w_r converges strongly to w in $L^p(Q)$, and so S is compact. Now we prove that S is continuous. Let v_r be a sequence of functions converging to v in $L^p(Q)$, and let $w_r := S(v_r)$. $v_r \longrightarrow v$ strongly in $L^p(Q)$, implies that $v_r \longrightarrow v$ a.e in Q, hence $h_n(v_r)f_n$ converges to $h_n(v)f_n$ a.e in Q and by the dominated convergence theorem one has that $h_n(v_r)f_n$ converge strongly to $h_n(v)f_n$ in $L^p(Q)$. Hence, by uniqueness, one deduce that $w_r := S(v_r)$ converges to $w := S(v)$ in $L^p(Q)$. This gives the continuity of S. So that by Shouder's fixed point Theorem, u_n will exist in B such that $u_n = S(u_n)$, i.e., such that u_n solves (6.13). In particular, we will have that $u_n \in L^p(0,T;W_0^{1,p}(\Omega))$ with $(u_n)_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ for all $n \in I\!N$ and $2 \le p < N$ and, since the right hand side of (6.13) is non-negative, that u_n is non-negative. □

Lemma 6.14. *Assume that the hypotheses* (6.2)-(6.4) *hold true and the datum f is a function in $L^m(Q)$. If $m > \frac{N}{p} + 1$, then for every solution u_n of* (6.13) *there exists a positives constants C_∞, C_0 independent of n, such that*

$$\begin{aligned} &||u_n||_{L^\infty(Q)} \le C_\infty, \\ &||u_n||_{L^p(0,T;W_0^{1,p}(\Omega))} \le C_0. \end{aligned}$$

Proof. Let $G_k(s) = s - T_k(s)$, for all $s \in I\!R$ and $k > 0$. We define the following function

$$H(s) = \int_0^s \frac{1}{(1+|\sigma|)^\theta} d\sigma, \;\; s \in I\!R.$$

For a solution u_n of problem (6.13) we set $v = \varphi(G_k(H(u_n)))$, where $k > 0$ and φ is defined by (6.6). Observe that v has the same sign as u_n and belong to $L^p(0,T;W_0^{1,p}(\Omega))$. Let as denoted by $A_{k,n}(t)$ the following set

$$A_{k,n}(t) = \{x \in \Omega : |H(u_n(x,t))| > k\}.$$

A straight forward computation gives

$$\nabla v = \varphi'(G_k(H(u_n))) \frac{\nabla u_n}{(1+|u_n|)^\theta} \chi_{A_{k,n}},$$

where $\chi_{A_{k,n}}$ stands for the characteristic function of the set $A_{k,n}$.

By definitions of H and $A_{k,n}(t)$, we deduce that $u_n \geq H(u_n)$ and $u_n \geq k$ in the set $A_{k,n}(t)$.

Now, choosing v as a test function in (6.13), we have for all $\tau \in (0,T]$

$$\int_0^\tau \int_\Omega \frac{\partial u_n}{\partial t} \varphi(G_k(H(u_n))) dxdt + \int_0^\tau \int_\Omega a(x,t,T_n(u_n),\nabla u_n) \cdot \nabla v dxdt$$
$$+ \int_0^\tau \int_\Omega |u_n|^{s-1} u_n v dxdt \leq \int_0^\tau \int_\Omega h_n(u_n) f_n v dxdt.$$

Hence

$$\int_0^\tau \int_{A_{k,n}(t)} \frac{\partial \phi}{\partial t}(G_k(H(u_n)))(1+|u_n|)^\theta dxdt$$
$$+ \int_0^\tau \int_{A_{k,n}(t)} a(x,t,T_n(u_n),\nabla u_n) \cdot \frac{\nabla u_n}{(1+|u_n)^\theta} \varphi' dxdt$$
$$+ \int_0^\tau \int_{A_{k,n}(t)} |u_n|^{s-1} u_n v dxdt \leq \int_0^\tau \int_{A_{k,n}(t)} h(u_n) f_n |v| dxdt,$$

where $\varphi' = \varphi'(G_k(H(u_n)))$. In the set $A_{k,n}(t)$, v has the same sign as u_n i.e $\int_0^\tau \int_{A_{k,n}(t)} |u_n|^{s-1} u_n v dxdt \geq 0$ and the fact of h is bounded in $(0,+\infty)$, then

$$\int_0^\tau \int_{A_{k,n}(t)} \frac{\partial \phi}{\partial t}(G_k(H(u_n)))(1+|u_n|)^\theta dxdt$$
$$+ \int_0^\tau \int_{A_{k,n}(t)} a(x,t,T_n(u_n),\nabla u_n) \cdot \frac{\nabla u_n}{(1+|u_n)^\theta} \varphi' dxdt$$
$$\leq ||h||_{L^\infty((0,+\infty))} \int_0^\tau \int_{A_{k,n}(t)} |f_n||v| dxdt.$$

Note that on the set $A_{k,n}(t)$ one has $(1+|u_n(x,t)|) > (k(1-\theta)+1)^{\frac{1}{1-\theta}}$. thus by (6.2) we obtain

$$
\begin{aligned}
&(k(1-\theta)+1)^{\frac{\theta}{1-\theta}} \int_0^\tau \int_{A_{k,n}(t)} \frac{\partial \phi}{\partial t}(G_k((1+|u_n|)^{1-\theta}))dxdt \\
&+ \alpha \int_0^\tau \int_{A_{k,n}(t)} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta p}} \varphi'(G_k(H(u_n)))dxdt \\
&\leq ||h||_{L^\infty((0,+\infty))} \int_0^\tau \int_{A_{k,n}(t)} |f_n||v|dxdt.
\end{aligned}
$$

Observe that since $k > 0$, we have

$$
\begin{aligned}
&\int_0^\tau \frac{\partial \phi}{\partial t}(G_k(H(u_n(x,t))))dxdt = \int_{A_{k,n}(\tau)} \phi(G_k(H(u_n(x,\tau))))dx \\
&- \int_{A_{k,n}(\tau)} \phi(G_k(H(u_n(x,0))))dx = \int_{A_{k,n}(\tau)} \phi(G_k(H(u_n(x,\tau))))dx.
\end{aligned}
$$

Using (6.12) we obtain

$$
\begin{aligned}
&C_0(k(1-\theta)+1)^{\frac{\theta}{1-\theta}} \int_{A_{k,n}(\tau)} |w_k|^p dx \\
&+ \alpha \int_0^\tau \int_{A_{k,n}(t)} \frac{|\nabla u_n|^p}{(1+|u_n|)^{p\theta}} \varphi'(G_k(H(u_n)))dxdt \\
&\leq ||h||_{L^\infty((0,+\infty))} \int_0^\tau \int_{A_{k,n}(t)} |f_n||v|dxdt,
\end{aligned}
$$

where $w_k = \varphi\left(\frac{|G_k(H(u_n))|}{p}\right)$. Now, for all $s \geq 0$ we have

$$
\left|\varphi'\left(\frac{s}{p}\right)\right|^p = |\lambda e^{\lambda \frac{s}{p}}|^p = \lambda^{p-1}|\lambda e^{\lambda s}| = \lambda^{p-1}|\varphi'(s)|,
$$

which implies

$$
|\nabla w_k|^p = \lambda^{p-1}\left(\frac{1}{p}\right)^p \frac{|\nabla u_n|^p}{(1+|u_n|)^{p\theta}} \varphi'(|G_k((1+|u_n|)^{1-\theta})|).
$$

Therefore, we can write

$$
C_4 \int_{A_{k,n}(\tau)} |w_k|^p dx + C_5 \int_0^\tau \int_{A_{k,n}(t)} |\nabla w_k|^p dxdt \leq C_h \int_0^\tau \int_{A_{k,n}(t)} |f_n||v|dxdt,
$$

where $C_4 = C_0(k(1-\theta)+1)^{\frac{\theta}{1-\theta}}$, $C_5 = \frac{\alpha p^p}{\lambda^{p-1}}$ and $C_h = ||h||_{L^\infty((0,+\infty))}$.

Let $t_1 \in (0,T]$ be arbitrary and which will be chosen later. For all $t \in (0,t_1]$, we have

$$
C_6\left(||w_k||^p_{L^p(0,t_1;L^p(A_{k,n}(t)))} + ||\nabla w_k||^p_{L^p(A_{k,n}(t))}\right) \leq \int_0^{t_1} \int_{A_{k,n}(t)} |f_n||v|dxdt, \tag{6.16}
$$

with $c_6 = \frac{\min\{C_4,C_5\}}{C_h}$. Now we estimate the integral in the right hand side of (6.16). By (6.9) we have

$$\begin{aligned}
&\int_0^{t_1}\int_{A_{k,n}(t)} |f_n||v|dxdt \\
&= \int_0^{t_1}\int_{A_{k+d,n}(t)} |f_n||v|dxdt + \int_0^{t_1}\int_{A_{k,n}(t)\setminus A_{k+d,n}(t)} |f_n||v|dxdt \\
&\leq M\int_0^{t_1}\int_{A_{k+d,n}(t)} |f_n||w_k|^p dxdt + \int_0^{t_1}\int_{A_{k,n}(t)\setminus A_{k+d,n}(t)} |f_n||v|dxdt \\
&\leq M\int_0^{t_1}\int_{A_{k,n}(t)} |f_n||w_k|^p dxdt + \varphi(d)\int_0^{t_1}\int_{A_{k,n}(t)} |f_n|dxdt.
\end{aligned} \tag{6.17}$$

Applying Hölder inequality twice, we obtain

$$\begin{aligned}
&\int_0^{t_1}\int_{A_{k,n}(t)} |f_n||w_k|^p dxdt \\
&\leq \left(\int_0^{t_1}\int_{A_{k,n}(t)} |f_n|^m dxdt\right)^{\frac{1}{m}} \left(\int_0^{t_1}\int_{A_{k,n}(t)} |w_k|^{\frac{pm}{m-1}} dxdt\right)^{\frac{m-1}{m}} \\
&\leq \left(\int_0^{t_1}\int_{A_{k,n}(t)} |f|^m dxdt\right)^{\frac{1}{m}} \left(\int_0^{t_1}\int_{A_{k,n}(t)} |w_k|^{\frac{pm}{m-1}} dxdt\right)^{\frac{m-1}{m}} \\
&\leq ||f||_{L^m(Q)} \left(\int_0^{t_1}\int_{A_{k,n}(t)} |w_k|^{\frac{pm}{m-1}} dxdt\right)^{\frac{m-1}{m}}.
\end{aligned} \tag{6.18}$$

Since we are going to chose m large enough, we can define $\nu_1 \in (0,1)$ as

$$\frac{1}{m} + \frac{N}{pm} = 1 - \nu_1.$$

Let as also define

$$\bar{m} = \frac{pm}{m-1},\ \nu = \frac{p\nu_1}{N}, \text{ and } \hat{m} = \bar{m}(1+\nu). \tag{6.19}$$

Combining (6.17) and (6.18), we obtain

$$\begin{aligned}
&\int_0^{t_1}\int_{A_{k,n}(t)} |f_n||w_k|^p dxdt \leq ||f||_{L^m(Q)} \left(\int_0^{t_1}\int_{A_{k,n}(t)} |w_k|^{\frac{pm}{m-1}} dxdt\right)^{\frac{m-1}{m}} \\
&= ||f||_{L^m(Q)} \left(\int_0^{t_1}\int_{A_{k,n}(t)} |w_k|^{\bar{m}} dxdt\right)^{\frac{p}{\bar{m}}} \\
&= ||f||_{L^m(Q)} \left(\int_0^{t_1}\int_{A_{k,n}(t)} |w_k|^{\frac{\hat{m}}{1+\nu}} dxdt\right)^{\frac{p(1+\nu)}{\hat{m}}} \\
&= C_f \left(\int_0^{t_1}\int_{A_{k,n}(t)} |w_k|^{\frac{\hat{m}}{1+\nu}} dxdt\right)^{\frac{p(1+\nu)}{\hat{m}}},
\end{aligned}$$

where $C_f = ||f||_{L^m(Q)}$. Applying Hölder's inequality in last inequality, we have

$$\begin{aligned}
&\int_0^{t_1}\int_{A_{k,n}(t)} |f_n||w_k|^p dxdt \\
&\leq C_f \left[\int_0^{t_1}\left(\int_{A_{k,n}(t)} |w_k|^{\hat{m}} dx\right)^{\frac{1}{1+\nu}}\left(\int_{A_{k,n}(t)} dx\right)^{\frac{\nu}{1+\nu}} dt\right]^{\frac{p(1+\nu)}{\hat{m}}} \\
&\leq C_f \left(\int_0^{t_1}\int_{A_{k,n}(t)} |w_k|^{\hat{m}} dxdt\right)^{\frac{p}{\hat{m}}}\left(\int_0^{t_1}\int_{A_{k,n}(t)} dxdt\right)^{\frac{p\nu}{\hat{m}}} \\
&= C_f \left(\int_0^{t_1}\int_{A_{k,n}(t)} |w_k|^{\hat{m}} dxdt\right)^{\frac{p}{\hat{m}}}\left(\int_0^{t_1} |A_{k,n}(t)|dt\right)^{\frac{p\nu}{\hat{m}}} \\
&= C_f \left(\int_0^{t_1}\int_{A_{k,n}(t)} |w_k|^{\hat{m}} dxdt\right)^{\frac{p}{\hat{m}}} \Theta(k)^{\frac{p\nu}{\hat{m}}}.
\end{aligned} \tag{6.20}$$

Here, $\Theta(k)$ stands for the function

$$\Theta(k) = \int_0^{t_1} |A_{k,n}(t)|dt.$$

Define $\hat{\delta} = \frac{1}{1+\nu}\frac{m-1}{m}$. Since $m > 1$ and $\nu > 0$, it's not hard to check that $0 < \delta < 1$. Furthermore $\frac{1}{\hat{m}} = \hat{\delta}\left(\frac{1}{p} - \frac{1}{N}\right) + \frac{1-\hat{\delta}}{p}$.

Thus, by using the Lemma 2.9 (here $\rho = p,\ h = p,\ \sigma = \hat{m},\ v = w_k$), we have

$$\begin{aligned}
&||w_k||^p_{L^{\hat{m}}(0,t_1;L^{\hat{m}}(A_{k,n}(t)))} \leq C_{gn}\left(\int_0^{t_1} ||\nabla w_k||^{\hat{\delta}\hat{m}}_{L^p(A_{k,n}(t))}||w_k||^{(1-\hat{\delta})\hat{m}}_{L^p(A_{k,n}(t))} dt\right)^{\frac{p}{\hat{m}}} \\
&\leq C_{gn}||w_k||^{(1-\hat{\delta})p}_{L^\infty(0,t_1;L^p(A_{k,n}(t)))}\left(\int_0^{t_1} ||\nabla w_k||^{\hat{\delta}\hat{m}}_{L^p(A_{k,n}(t))} dt\right)^{\frac{p}{\hat{m}}}.
\end{aligned}$$

Applying Young's inequality we get

$$\begin{aligned}
&||w_k||^p_{L^{\hat{m}}(0,t_1;L^{\hat{m}}(A_{k,n}(t)))} \\
&\leq C_{gn}(1-\hat{\delta})||w_k||^p_{L^\infty(0,t_1;L^p(A_{k,n}(t)))} + C_{gn}\hat{\delta}\left(\int_0^{t_1} ||\nabla w_k||^{\hat{\delta}\hat{m}}_{L^p(A_{k,n}(t))} dt\right)^{\frac{p}{\hat{m}\hat{\delta}}}.
\end{aligned}$$

By (6.18) and the definition of $\hat{\delta}$, we obtain $\hat{m}\hat{\delta} = p$ and thus we get

$$||w_k||^p_{L^{\hat{m}}(0,t_1;L^{\hat{m}}(A_{k,n}(t)))} \leq C_{\hat{\delta}}||w_k||^p_{V((0,t_1)\times A_{k,n}(t))}, \tag{6.21}$$

where

$$||w_k||^p_{V((0,t_1)\times A_{k,n}(t))} = ||w_k||^p_{L^\infty(0,t_1;L^p(A_{k,n}(t)))} + ||\nabla w_k||^p_{L^p(0,t_1;L^p(A_{k,n}(t)))}$$

and $C_{\hat{\delta}} = \max\{C_{gn}(1-\hat{\delta}), C_{gn}\hat{\delta}\}$. Hence

$$\begin{aligned}
\int_0^{t_1}\int_{A_{k,n}(t)} |f_n||w_k|dxdt &\leq C_f||w_k||^p_{L^{\hat{m}}(0,t_1;L^{\hat{m}}(A_{k,n}(t)))}\Theta(k)^{\frac{p\nu}{\hat{m}}} \\
&\leq C_f C_{\hat{\delta}}\Theta(k)^{\frac{p\nu}{\hat{m}}}||w_k||_{V((0,t_1)\times A_{k,n}(t))},
\end{aligned} \tag{6.22}$$

where $C_f = ||f||_{L^m(Q)}$. On the other hand, the second term in the right hand side in (6.17) satisfies

$$\int_0^{t_1}\int_{A_{k,n}(t)} |f_n| dxdt \leq \left(\int_0^{t_1}\int_{A_{k,n}(t)} |f_n|^m dxdt\right)^{\frac{1}{m}} \left(\int_0^{t_1}\int_{A_{k,n}(t)} dxdt\right)^{\frac{m-1}{m}}$$
$$\leq ||f||_{L^m(Q)} \left(\int_0^{t_1} |A_{k,n}(t)| dt\right)^{\frac{m-1}{m}}$$
$$= C_f \Theta(k)^{\frac{p(1+\nu)}{\hat{m}}}, \tag{6.23}$$

where $C_f = ||f||_{L^m(Q)}$. Using (6.16), (6.21) and (6.22), we get

$$\begin{aligned}
&C_6 ||w_k||^p_{V((0,t_1)\times A_{k,n}(t))}\\
&= C_6 \left(||w_k||^p_{L^\infty(0,t_1;L^p(A_{k,n}(t)))} + ||w_k||^p_{L^p(0,t_1;L^p(A_{k,n}(t)))}\right)\\
&\leq \int_0^{t_1}\int_{A_{k,n}(t)} |f_n||\varphi| dxdt\\
&\leq M \int_0^{t_1}\int_{A_{k,n}(t)} |f_n||w_k| dxdt + \varphi(d)\int_0^{t_1}\int_{A_{k,n}(t)} |f_n| dxdt\\
&\leq M C_{\hat{\delta}} C_f \Theta(k)^{\frac{p\nu}{\hat{m}}} ||w_k||^p_{V((0,t_1)\times A_{k,n}(t))} + \varphi(d) C_f \Theta(k)^{\frac{p(1+\nu)}{\hat{m}}},
\end{aligned}$$

hence

$$\begin{aligned}
&C_6 ||w_k||^p_{V((0,t_1)\times A_{k,n}(t))}\\
&\leq M C_{\hat{\delta}} C_f \Theta(k)^{\frac{p\nu}{\hat{m}}} ||w_k||^p_{V((0,t_1)\times A_{k,n}(t))} + \varphi(d) C_f \Theta(k)^{\frac{p(1+\nu)}{\hat{m}}}.
\end{aligned}$$

We choose now t_1 small enough in order to get

$$M C_f C_{\hat{\delta}} t_1^{\frac{p\nu}{\hat{m}}} |\Omega|^{\frac{p\nu}{\hat{m}}} < C_6. \tag{6.24}$$

We can conclude that

$$||w_k||^p_{V((0,t_1)\times A_{k,n}(t))} \leq C_7 \Theta(k)^{\frac{p(1+\nu)}{\hat{m}}},$$

where $C_7 = \dfrac{C_f \varphi(d)}{C_6 - M C_f C_{\hat{\delta}} t_1^{\frac{p\nu}{\hat{m}}} |\Omega|^{\frac{p\nu}{\hat{m}}}}$, then, by (6.20) we get

$$||w_k||_{L^{\hat{m}}(0,t_1;L^{\hat{m}}(A_{k,n}(t)))} \leq C_{\hat{\delta}} C_7 \Theta(k)^{\frac{p(1+\nu)}{\hat{m}}}. \tag{6.25}$$

Let $h > k$. Observe that on the set $A_{h,n}(t)$ one has $G_k(H(u_n)) > h - k$. Thus,

$$\begin{aligned}
&||w_k||_{L^{\hat{m}}(0,t_1;L^{\hat{m}}(A_{k,n}(t)))}\\
&= \left|\left|\varphi\left(\frac{G_k(H(u_n))}{p}\right)\right|\right|^p_{L^{\hat{m}}(0,t_1;L^{\hat{m}}(A_{k,n}(t)))}\\
&\geq \left|\left|\frac{G_k(H(u_n))}{p}\right|\right|^p_{L^{\hat{m}}(0,t_1;L^{\hat{m}}(A_{k,n}(t)))}\\
&= \left(\frac{1}{p}\right)^p ||G_k(H(u_n))||^p_{L^{\hat{m}}(0,t_1;L^{\hat{m}}(A_{k,n}(t)))}\\
&= \left(\frac{1}{p}\right)^p \left(\int_0^{t_1}\int_{A_{k,n}(t)} (G_k(H(u_n)))^{\hat{m}} dxdt\right)^{\frac{p}{\hat{m}}}\\
&\geq \left(\frac{1}{p}\right)^p \left(\int_0^{t_1}\int_{A_{h,n}(t)} (G_k(H(u_n)))^{\hat{m}} dxdt\right)^{\frac{p}{\hat{m}}}\\
&\geq \left(\frac{1}{p}\right)^p \left(\int_0^{t_1}\int_{A_{h,n}(t)} (h-k)^{\hat{m}} dxdt\right)^{\frac{p}{\hat{m}}}\\
&\geq \left(\frac{1}{p}\right)^p (h-k)^p \left(\int_0^{t_1} |A_{h,n}(t)| dt\right)^{\frac{p}{\hat{m}}}\\
&= \left(\frac{1}{p}\right)^p (h-k)^p \Theta^{\frac{p}{\hat{m}}}(h),
\end{aligned}$$

which to get with (6.24) yield

$$\Theta(h) \leq \frac{M'}{(h-k)^{\hat{m}}}\Theta(k)^{1+\nu}, \quad \forall h > k \geq 1,$$

where $M' = (p^p C_{\hat{\delta}} C_7)^{\frac{\hat{m}}{p}}$. Note that

$$\nu > 0 \Longleftrightarrow 1 + \frac{N}{p} < m.$$

Therefore, by Lemma 2.10 with $\Theta = \varrho$, there exists a positive constant $\gamma_1 > 1$, independent of n, such that $\varrho(\gamma_1) = 0$, which means that

$$|u_n| \leq (\gamma_1(1-\theta)+1)^{\frac{1}{1-\theta}} - 1, \text{ a.e. in } \Omega \times [0, t_1].$$

Iterating this procedure successively in the sets $\Omega \times [t_1, 2t_1]$, $\Omega \times [2t_1, 3t_1]$, $\cdots$, $\Omega \times [mt_1, T]$, where $T - mt_1 \leq t_1$, (notice that the process works since in all these sets (6.24) is verified), we conclude that there is a constant C_∞, not depending on n, such that

$$\|u_n\|_\infty \leq C_\infty, \text{ a.e. in } Q = \Omega \times (0, T).$$

Let us u_n a test function in problem (6.13) and using (6.2), definition of h_n we obtain

$$\begin{aligned}
&\tfrac{1}{2}\int_\Omega u_n(x,\tau)^2 dx + \alpha \int_0^\tau \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}} dxdt\\
&+ \int_0^T \int_\Omega |u_n|^{s+1} dxdt \leq \int_0^T \int_\Omega h(u_n) f_n u_n dxdt,
\end{aligned}$$

hence

$$\alpha \int_0^\tau \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}} dxdt + \int_Q |u_n|^{s+1} dxdt \leq \int_0^T \int_\Omega h(u_n) f_n u_n dxdt.$$

By the fact that h bounded in $(0,+\infty)$, $||u_n||_{L^\infty} \leq C_\infty$ and using last inequality we obtain

$$\frac{\alpha}{(1+C_\infty)^{\theta(p-1)}} \int_Q |\nabla u_n|^p dxdt \leq ||u_n||_{L^\infty(Q)} ||h||_{L^\infty((0,+\infty))} ||f||_{L^m(Q)} |Q|^{\frac{1}{m'}}.$$

Then

$$\int_Q |\nabla u_n|^p dxdt + \int_Q |u_n|^{s+1} dxdt \leq C_8, \tag{6.26}$$

where $C_8 = \frac{C_\infty(1+C_\infty)^{\theta(p-1)}}{\alpha} ||h||_{L^\infty((0,+\infty))} ||f||_{L^m(Q)} |Q|^{\frac{1}{m'}}$ independent of n. □

Proof of Theorem 6.4. By Lemma 6.14 we have the sequence $\{u_n\}$ is bounded in $L^\infty(Q) \cap L^p(0,T;W_0^{1,p}(\Omega))$. Then, there exist a function $u \in L^\infty(Q) \cap L^p(0,T;W_0^{1,p}(\Omega))$ and a subsequence, still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \quad \text{weakly in} L^p(0,T;W_0^{1,p}(\Omega))$$
$$u_n \rightharpoonup u \quad \text{weakly* in} L^\infty(Q) \ \text{ for } \ \sigma^*(L^\infty(Q), L^1(Q)).$$

Moreover the sequence $\{\frac{\partial u_n}{\partial t}\}_n$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$, using compactness argument in [139], we obtain that

$$u_n \longrightarrow u \quad \text{strongly in } L^1(Q). \tag{6.27}$$

Hence

$$u_n \longrightarrow u \ \text{ a.e } \ \text{ in } Q \tag{6.28}$$

Now, adapting the approach of [22, Theorem 3.1], then there exists a subsequence (still denoted $\{u_n\}$) such that

$$\nabla u_n \longrightarrow \nabla u \ \text{ a.e } \ \text{ in } Q. \tag{6.29}$$

From (6.27), (6.28) and (6.3) and the continuity of $a(x,t,.,.)$, using Vitali's Theorem, we obtain

$$a(x,t,T_n(u_n),\nabla u_n) \rightharpoonup a(x,t,u,\nabla u) \ \text{ weakly } \ L^{p'}(Q). \tag{6.30}$$

We shall now prove that $|u_n|^{s-1}u_n \longrightarrow |u|^{s-1}u$ and $h_n(u_n)f_n \longrightarrow h(u)f$ strongly in $L^1(Q)$. Let E be a measurable subset of Q. By Hölder's inequality and (6.26) we have

$$\int_E |u_n|^s dxdt \leq \left(\int_E |u_n|^{s+1} dxdt \right)^{\frac{s}{s+1}} |E|^{\frac{1}{s+1}} \leq C_8^{\frac{s}{s+1}} |E|^{\frac{1}{s+1}} < \infty.$$

Hence, the sequence $\{|u_n|^s\}$ is equi-integrable and then so is $\{|u_n|^{s-1}u_n\}$. Using (6.26), (6.27) and Vilali's Theorem, we obtain $|u|^{s-1}u \in L^1(Q)$ and

$$|u_n|^{s-1}u_n \longrightarrow |u|^{s-1}u \ \text{ strongly in } \ L^1(Q). \tag{6.31}$$

Let $\psi \in L^p(0,T;W_0^{;p}(\Omega)) \cap L^\infty(\Omega)$ as a function test in (6.6), we obtain

$$-\int_Q u_n \psi_t dxdt + \int_Q a(x,t,T_n(u_n),\nabla u_n) \cdot \nabla \psi dxdt$$

$$+\int_Q |u_n|^{s-1}u_n\psi dxdt = \int_Q h_n(u_n)f_n\psi dxdt. \tag{6.32}$$

We are left to pass to the limit in the non-linear lower order term involving h. If $h(0) < \infty$ we use Lebesgue's dominated convergence theorem and we easily pass n to the limit. From now, we assume that $h(0) = \infty$. Let ψ be a non-negative function in $L^p(0,T;W_0^{;p}(\Omega)) \cap L^\infty(\Omega)$ as a test function in the weak formulation (6.32)) we have

$$-\int_Q u_n\psi_t dxdt + \int_Q a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla\psi dxdt$$

$$+\int_Q |u_n|^{s-1}u_n\psi dxdt = \int_Q h_n(u_n)f_n\psi dxdt,$$

using (6.3) and Young inequality, we obtain

$$\begin{aligned}
&\int_Q h_n(u_n)f_n\psi dxdt \leq \frac{1}{p}\int_Q |u_n|^p dxdt + \frac{1}{p'}\int_Q |\psi_t|^{p'}dxdt\\
&+\int_Q (b(x,t)+|T_n(u_n)|^{p-1}+|\nabla u_n|^{p-1})\cdot\nabla\psi dxdt\\
&+\tfrac{1}{s}\int_Q |u_n|^s dxdt + \frac{1}{s'}\int_Q |\psi|^{s'}dxdt\\
&\leq \tfrac{1}{p}||u_n||_{L^p(Q)} + \tfrac{1}{p'}||\psi_t||_{L^{p'}(Q)} + \tfrac{2^{p'-1}}{p'}||b||_{L^{p'}(Q)} + \tfrac{2^{2p'-1}}{p'}||u_n||_{L^p(Q)}\\
&+\tfrac{2^{2p'-1}}{p'}||u_n||_{L^p(0,T;W_0^{1,p}(\Omega))} + \tfrac{1}{p'}||\nabla\psi||_{L^{p'}(Q)} + \tfrac{1}{s}||u_n||_{L^s(Q)} + \tfrac{1}{s'}||\psi||_{L^{s'}(Q)}.
\end{aligned}$$

Then

$$\int_Q h_n(u_n)f_n\psi dxdt \leq C, \tag{6.33}$$

hence $\{h_n(u_n)f_n\}$ is bounded in $L^1(Q)$. We fix $\delta > 0$ and we decompose the right hand side of (6.32) in the following way

$$\int_Q h_n(u_n)f_n\psi dxdt = \int_{Q\cap\{u_n\leq\delta\}} h_n(u_n)f_n\psi dxdt$$

$$+\int_{Q\cap\{u_n>\delta\}} h_n(u_n)f_n\psi dxdt, \tag{6.34}$$

without losing generality we may assume the parameter $\delta \notin \{\beta : |\{u_n(x,t)=\beta\}| > 0\}$ which is at most countable set. The second term in (6.34) passes to the limit again by the Lebesgue Theorem as

$$h_n(u_n)f_n\psi\chi_{\{u_n>\delta\}} \leq \sup_{s\in[\delta,\infty)}[h(s)]f\psi \in L^1(Q), \tag{6.35}$$

we get

$$\lim_{n\to\infty}\int_{Q\cap\{u_n>\delta\}} h_n(u_n)f_n\psi dxdt = \int_{Q\cap\{u>\delta\}} h(u)f\psi dxdt.$$

First of all we apply the Fatou Lemma and (6.33) in order to deduce that

$$\int_Q h(u)f\psi dxdt \leq \lim_{n\to\infty}\inf\int_Q h_n(u_n)f_n\psi dxdt \leq C,$$

hence $h(u)f \in L^1(Q)$. This allows to apply once again the Lebesgue Theorem as $\delta \to 0$ obtaining

$$\lim_{\delta\to 0}\lim_{n\to\infty}\int_{Q\cap\{u_n>\delta\}} h_n(u_n)f_n\psi dxdt = \int_{Q\cap\{u>0\}} h(u)f\psi dxdt. \tag{6.36}$$

We also observe that $h(u)f \in L^1(Q)$ gives that the set $\{u = 0\}$ is contained in the set $\{f = 0\}$ up to set of zero Lebesgue measure. This means that

$$\int_{Q\cap\{u>0\}} h(u)f\psi dxdt = \int_Q h(u)f\psi dxdt, \tag{6.37}$$

and the proof is done once we have shown that the first term in the right hand side of (6.34) converges to zero a.s.,resp., $n \longrightarrow +\infty$ and $\delta \longrightarrow 0$. To this aim we define

$$V_\delta = \begin{cases} 1 & \text{if } \ \ell \le \delta, \\ \frac{2\delta-\ell}{\delta} & \text{if } \ \delta < \ell < 2\delta, \\ 0 & \text{if } \ \ell \ge 2\delta. \end{cases}$$

Take $V_\delta(u_n)\psi$ as a test function in (6.13), we get

$$\begin{aligned}&\int_Q \frac{\partial u_n}{\partial t} V_\delta(u_n)\psi dxdt + \int_Q a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla(V_\delta(u_n)\psi)dxdt \\ &+\int_Q |u_n|^{s-1}u_n V_\delta(u_n)\psi dxdt = \int_Q h_n(u_n)f_n V_\delta(u_n)\psi dxdt.\end{aligned}$$

Using integration by parties and definition of V_δ, we have

$$\int_0^T\int_Q \frac{\partial u_n}{\partial t} V_\delta(u_n)\psi dxdt = -\int_Q \Phi(u_n)\psi_t dxdt,$$

where $\Phi(\ell) = \displaystyle\int_0^\ell V_\delta(t)dt$.

$$\begin{aligned}&\int_Q a(x,t,T_n(u_n),\nabla u_n).\nabla(V_\delta(u_n)\psi)dxdt = \int_Q a(x,t,T_n(u_n),\nabla u_n).\nabla\psi V_\delta(u_n)dxdt \\ &+\int_Q a(x,t,T_n(u_n),\nabla u_n).\nabla V_\delta(u_n)\psi dxdt = \int_Q a(x,t,T_n(u_n),\nabla u_n).\nabla\psi V_\delta(u_n)dxdt \\ &-\tfrac{1}{\delta}\int_{\{\delta<u_n<2\delta\}} a(x,t,T_n(u_n),\nabla u_n).\nabla u_n\psi dxdt \\ &\le \int_Q a(x,t,T_n(u_n),\nabla u_n).\nabla\psi V_\delta(u_n)dxdt.\end{aligned}$$

Hence

$$\begin{aligned}\int_Q h_n(u_n)f_n V_\delta\psi dxdt &= \int_{Q\cap\{u_n\le\delta\}} h_n(u_n)f_n V_\delta\psi dxdt \\ &+\int_{Q\cap\{\delta<u_n<2\delta\}} h_n(u_n)f_n V_\delta\psi dxdt.\end{aligned}$$

Then we using the above estimates, we get

$$\int_{Q\cap\{u_n\leq\delta\}} h_n(u_n)f_nV_\delta\psi dxdt \leq -\int_Q \Phi(u_n)\psi_t dxdt$$
$$+\int_Q a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla\psi V_\delta(u_n)dxdt + \int_Q |u_n|^{s-1}u_nV_\delta(u_n)\psi dxdt.$$

Using that V_δ is bounded and ϕ is continue we deduce that $\Phi(u_n)\psi_t \to \Phi(u)\psi_t$ and $|u_n|^{s-1}u_nV_\delta(u_n)\psi \to |u|^{s-1}uV_\delta\psi$ strongly in $L^1(Q)$ and $a(x,t,T_n(u_n),\nabla u_n)V_\delta(u_n) \to a(x,t,u,\nabla u)V_\delta(u)$ weakly in $L^{p'}(Q)^N$ as n tends to infinity. This implies that

$$\lim_{n\to\infty}\sup\int_{Q\cap\{u_n\leq\delta\}} h_n(u_n)f_n\psi dxdt \leq -\int_{\{u=0\}}\Phi(u)\psi_t dxdt$$
$$+\int_{\{u=0\}} a(x,t,u,\nabla u)\cdot\nabla\psi V_\delta(u)dxdt + \int_{\{u=0\}} |u|^{s-1}uV_\delta(u)\psi dxdt,$$

then

$$\lim_{n\to\infty}\sup\int_{Q\cap\{u_n\leq\delta\}} h_n(u_n)f_n\psi dxdt = 0. \tag{6.38}$$

Hence (6.37), (6.38) implies that

$$\lim_{n\to\infty}\int_Q h_n(u_n)f_n\psi dxdt = \int_Q h(u)f\psi dxdt. \tag{6.39}$$

Let $n \longrightarrow \infty$ in (6.32), by (6.27), (6.30), (6.31) and (6.39) we get

$$-\int_Q u\psi_t dxdt + \int_Q a(x,t,u,\nabla u)\cdot\nabla\psi dxdt$$
$$+\int_Q |u|^{s-1}u\psi dxdt = \int_Q h(u)f\psi dxdt.$$

Moreover, decomposing any $\psi = \psi^+ - \psi^-$, and using that (6.39) is linear in ψ, we deduce that (6.39)) holds for every $\psi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

We treated $h(s)$ unbounded as stands to 0, as regards bounded function h the proofs is easier and the only difference deals with the passage to the limit in the right hand side of (6.32). We can avoid introducing δ and we can substitute (6.35) with

$$0 \leq h_n(u_n)f_n \leq ||h||_{L^\infty(\Omega)}f.$$

Using the same argument above we have that $h_n(u_n)f_n \longrightarrow h(u)f$ strongly in $L^1(\Omega)$ as n tends to infinity. Then we can conclude as in case of an unbounded h. The proof of Theorem 6.4 is completed. □

Lemma 6.15. *Assume that hypotheses* (6.2)-(6.4) *hold true, h satisfies* ($h1$) *and the datum $f \in L^m(Q)$ with $m = \frac{N}{p} + 1$. Then for every solution u_n of* (6.13) *there exists a positive constants C_{14}, C_{15} such that*

$$||u_n||_{L^p(0,T;W_0^{1,p}(\Omega))} \leq C_{15}, \tag{6.40}$$

$$||u_n||_{L^r(Q)} \leq C_{14}, \tag{6.41}$$

with $2 \leq r < +\infty$.

Proof of Lemma 6.15. Let λ be a real positive number which will be determined lately. For $t \in (0,T]$, using $\psi(u_n) = ((1+|u_n|)^\lambda - 1)\text{sign}(u_n)\chi_{(0,t)}$, as a test function in problem (6.13), we have

$$\int_0^t \int_\Omega \frac{\partial u}{\partial t}\psi(u_n)dxd\tau + \lambda \int_0^t \int_\Omega a(x,t,T_n(u_n),\nabla u_n)\cdot \nabla u_n(1+|u_n|)^{\lambda-1}dxd\tau$$
$$+ \int_0^t \int_\Omega |u_n|^s((1+|u_n|)^\lambda - 1)dxd\tau = \int_0^t \int_\Omega h_n(u_n)f_n((1+|u_n|)^\lambda - 1)dxd\tau.$$

Using (6.2) and the fact that $((1+|u_n|)^\lambda - 1) \geq |u_n|^\lambda$ and by definition of h_n we obtain

$$\int_\Omega \Psi(u_n(x,t))dx + \alpha\lambda \int_0^t \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}}(1+|u_n|)^{\lambda-1}dxd\tau$$
$$+ \int_0^t \int_\Omega |u_n|^{s+\lambda}dxd\tau \leq \int_0^t \int_\Omega h(u_n)f_n|(1+|u_n|)^\lambda - 1|dxd\tau, \tag{6.42}$$

where

$$\Psi(\ell) = \int_0^\ell \psi(\sigma)d\sigma. \tag{6.43}$$

By definition of $\psi(\ell)$ and $\Psi(\ell)$, we get whenever $\lambda > 1$,

$$\Psi(\ell) \geq \frac{|\ell|^{\lambda+1}}{\lambda+1}, \ \ \forall \ell \in I\!R. \tag{6.44}$$

Combining (6.42), (6.44) and h bounded in $(0,+\infty)$ we have

$$\begin{aligned}
&\tfrac{1}{\lambda+1}\int_\Omega |u_n(x,t)|^{\lambda+1}dx + \alpha\lambda \int_0^t \int_\Omega |\nabla u_n|^p(1+|u_n|)^{\lambda-1-\theta(p-1)}dxd\tau \\
&+ \int_0^t \int_\Omega |u_n|^{\lambda+s}dxd\tau \leq ||h||_{L^\infty((0,+\infty))}\int_0^t \int_\Omega |f_n||(1+|u_n|)^\lambda - 1|dxd\tau \\
&\leq ||h||_{L^\infty((0,+\infty))}||f_n||_{L^m(Q)}\left(\int_0^t \int_\Omega |(1+|u_n|)^\lambda - 1|^{m'}dxd\tau\right)^{\frac{1}{m'}} \\
&\leq ||h||_{L^\infty((0,+\infty))}||f||_{L^m(Q)}\left(\int_0^t \int_\Omega |(1+|u_n|)^\lambda - 1|^{m'}dxd\tau\right)^{\frac{1}{m'}}
\end{aligned} \tag{6.45}$$

Hence

$$\begin{aligned}
&\tfrac{1}{\lambda+1}\int_\Omega |u_n(x,t)|^{\lambda+1}dx \\
&+\tfrac{\alpha\lambda p}{\lambda-1-\theta(p-1)+p}\int_0^t \int_\Omega |\nabla |u_n|^{\frac{\lambda+(1-\theta)(p-1)}{p}}|^p dxd\tau \\
&\leq ||h||_{L^\infty((0,+\infty))}||f||_{L^m(Q)}\left(\int_0^t \int_\Omega |(1+|u_n|)^\lambda - 1|^{m'}dxd\tau\right)^{\frac{1}{m'}}
\end{aligned} \tag{6.46}$$

If $\lambda \geq 1+\theta(p-1)$, we get

$$\begin{aligned}
&C_9||u_n||^{\lambda+1}_{L^\infty(0,T;L^{\lambda+1}(\Omega))} + C_{10}\int_Q |\nabla |u_n|^{\frac{\lambda+(1-\theta)(p-1)}{p}}|^p dxdt \\
&\leq C_{11}\left(\int_Q |u_n|^{\lambda m'}dxdt\right)^{\frac{1}{m'}} + C_{11},
\end{aligned} \tag{6.47}$$

where $C_9 = \frac{1}{\lambda+1}$, $C_{10} = \frac{\alpha\lambda p^p}{(\lambda+(1-\theta)(p-1))^p}$ and $C_{11} = ||h||_{L^\infty((0,+\infty))}||f||_{L^m(Q)}2^\lambda \max(1,|Q|^{\frac{1}{m'}})$. Setting $v_n = |u_n|^{\frac{\lambda+(1-\theta)(p-1)}{p}}$, then we have $|u_n| = v_n^{\frac{p}{\lambda+(1-\theta)(p-1)}}$. Using (6.47), we get

$$\begin{aligned} &C_9||v_n||^{\frac{(\lambda+1)p}{\lambda+(1-\theta)(p-1)}}_{L^\infty(0,T;L^{\frac{\lambda+(1-\theta)(p-1)}{p(\lambda+1)}}(\Omega))} + C_{10}\int_Q |\nabla v_n|^p dxdt \\ &\leq C_{11}\left(\int_Q v_n^{\frac{\lambda pm'}{\lambda+(1-\theta)(p-1)}}dxdt\right)^{\frac{1}{m'}} + C_{11}. \end{aligned} \tag{6.48}$$

By Lemma 2.9 (here $h = p$, $\rho = \frac{p(\lambda+1)}{\lambda+(1-\theta)(p-1)}$) and using (6.48), we obtain

$$\begin{aligned} &\int_Q |v_n|^\sigma dxdt \leq C||v_n||^{\frac{p\rho}{N}}_{L^\infty(0,T;L^\rho(\Omega))}\int_Q |\nabla v_n|^p dxdt \\ &\leq C\left[\frac{C_{11}}{C_9}\left(\int_Q v_n^{\frac{\lambda pm'}{\lambda+(1-\theta)(p-1)}}dxdt\right)^{\frac{1}{m'}} + \frac{C_{11}}{C_9}\right]^{\frac{p}{N}} \\ &\times\left[\frac{C_{11}}{C_{10}}\left(\int_Q v_n^{\frac{\lambda pm'}{\lambda+(1-\theta)(p-1)}}dxdt\right)^{\frac{1}{m'}} + \frac{C_{11}}{C_{10}}\right] \\ &\leq C\left[C_{12}\left(\int_Q v_n^{\frac{\lambda pm'}{\lambda+(1-\theta)(p-1)}}dxdt\right)^{\frac{1}{m'}} + C_{12}\right]^{\frac{1}{m'}(\frac{p}{N}+1)} \\ &\leq C_{13}\left(\int_Q v_n^{\frac{\lambda pm'}{\lambda+(1-\theta)(p-1)}}dxdt\right)^{\frac{1}{m'}(\frac{p}{N}+1)} + C_{13}, \end{aligned}$$

where $C_{12} = C\max\left(\frac{C_{11}}{C_9},\frac{C_{11}}{C_{10}}\right)$ and $C_{13} = C2^{\frac{p}{N}}C_{12}^{\frac{p}{N}+1}$. By virtue of $m = \frac{N}{p}+1$, and $\sigma > \frac{\lambda pm'}{\lambda+(1-\theta)(p-1)}$, we have $\frac{1}{m'}\left(\frac{p}{N}+1\right) = 1$ and $\frac{\lambda pm'}{\sigma[\lambda+(1-\theta)(p-1)]} < 1$, applying Hölder's inequality we get

$$\int_Q |v_n|^\sigma dxdt \leq C_{13}\left(\int_Q |v_n|^\sigma dxdt\right)^{\frac{\lambda pm'}{\sigma(\lambda+(1-\theta)(p-1))}}|Q|^{1-\frac{\lambda pm'}{\sigma(\lambda+(1-\theta)(p-1))}}$$

hence applying Young's inequality with ϵ, we have

$$\int_Q |v_n|^\sigma dxdt \leq \epsilon\int_Q |v_n|^\sigma dxdt + C_\epsilon,$$

take $\epsilon = \frac{1}{2}$, we get

$$\int_Q |v_n|^\sigma dxdt \leq C_{14}, \tag{6.49}$$

where $C_{14} = 2C_\epsilon$. Then we get

$$\int_Q |u_n|^r dxdt \leq C_{14}, \tag{6.50}$$

with $r = \sigma \times \frac{\lambda+(1-\theta)(p-1)}{p} = m\frac{N(1-\theta)(p-1)+p}{N+p-pm}$. To ensure $\lambda \geq 1+\theta(p-1)$ this needs $r \geq p(\frac{N+2}{N}+\frac{\theta(p-1)}{N})$. Thus, if $r \geq p(\frac{N+2}{N}+\frac{\theta(p-1)}{N})$, is proved. If $2 \leq r \leq p(\frac{N+2}{N}+\frac{\theta(p-1)}{N})$, it is classical since Q is bounded.

By (6.46), (6.49), (6.50) and $\lambda \geq 1+\theta(p-1)$, we get

$$\int_Q |\nabla u_n|^p dxdt \leq \int_Q |\nabla u_n|^p (1+|u_n|)^{\lambda-1-\theta(p-1)} dxdt \leq C_{15}. \tag{6.51}$$

□

Lemma 6.16. *Let $s \geq 1$. Assume that hypothesis (6.2) – (6.4) hold, h satisfies $(h1)$ and $f \in L^m(Q)$ with*

$$\frac{p(N+2+\theta(p-1))}{p(N+2+\theta(p-1)) - N[\theta(p-1)+1-\gamma]} \leq m < \frac{N}{p}+1.$$

Then for every solution u_n of (6.13), there exists positive constants C_{23} and C_{24} independent of n such that

$$||u_n||_{L^p(0,T;L^p(\Omega))} \leq C_{24}, \quad ||u_n||_{L^r(Q)} \leq C_{23},$$

where r is defined in Theorem 6.8.

Lemma 6.17. *Let $s \geq 1$. Assume that hypothesis (6.2)-(6.4) hold, h satisfies $(h1)$ and the datum $f \in L^m(Q)$ with*

$$1 \leq m < \frac{p(N+2+\theta(p-1))}{p(N+2+\theta(p-1)) - N[\theta(p-1)+1-\gamma]}.$$

Then for every solution u_n of (6.13), there exists positive constants C_{23} and C_{25} independent of n such that

$$||u_n||_{L^q(0,T;L^q(\Omega))} \leq C_{25}, \quad ||u_n||_{L^r(Q)} \leq C_{23},$$

where q and r are defined in Theorem 6.9.

Proof of Lemmas 6.16, 6.17. For $t \in (0,T]$, taking $\varphi(u_n) = ((1+|u_n|)^{\delta+1}-1)\chi_{(0,t)} \times \operatorname{sign}(u_n), \delta > 0$ as test function in problem (6.13), using (6.2) and $(h1)$, we have

$$\begin{aligned} &\int_\Omega \psi(u_n(x,t))dx + \alpha(\delta+1)\int_0^t \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n)^{\theta(p-1)-\delta}} dxd\tau \\ &\quad + \int_0^t \int_\Omega |u_n|^{s+\delta+1} dxd\tau \leq C_{16} \int_0^t \int_\Omega |f_n||u_n|^{\delta+1-\gamma} dxd\tau, \end{aligned} \tag{6.52}$$

where $\psi(\ell) = \displaystyle\int_0^\ell \varphi(y)dy$. By definition of $\varphi(\ell)$ and $\psi(\ell)$, we also have if $0 < \delta < \theta(p-1)$

$$C_{17}|\ell|^{\delta+2} - C_{26} \leq \psi(\ell), \quad \forall \ell \in \mathbb{R}.$$

By using the last inequality, Hölder's inequality in (6.52), we obtain

$$\begin{aligned} &C_{17}\int_\Omega |u_n(x,t)|^{\delta+2}dx + \alpha(\delta+1)\int_0^t \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n)^{\theta(p-1)-\delta}} dxd\tau \\ &+ \int_0^t \int_\Omega |u_n|^{s+\delta+1} dxd\tau \leq C_{27}\left[1 + \left(\int_0^t \int_\Omega u_n|^{(\delta+1-\gamma)m'} dxd\tau\right)^{\frac{1}{m'}}\right], \end{aligned}$$

where C_{27} positive constant depend only $C_{16}, C_{26}, ||f||_{L^m(Q)}$ and $meas(\Omega)$. Passing to the supremum in $t \in [0,T]$, we get

$$C_{17}||u_n||^{\delta+2}_{L^\infty(0,T;L^{\delta+2}(\Omega))} + C_{18}\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)-\delta}}dxdt + \int_Q |u_n|^{s+\delta+1}dxdt \le C_{27}\left[1+\left(\int_Q |u_n|^{(\delta+1-\gamma)m'}dxdt\right)^{\frac{1}{m'}}\right] \tag{6.53}$$

where $C_{18} = \alpha(\delta+1)$. Let $1 < q < p$, applying Hölder's inequality and by last inequality, we have

$$\begin{aligned}\int_Q |\nabla u_n|^q dxdt &= \int_Q \frac{|\nabla u_n|^q (1+|u_n|)^{\frac{q(\theta(p-1)-\delta)}{p}}}{(1+|u_n|)^{\frac{q(\theta(p-1)-\delta)}{p}}}dxdt\\ &\left(\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)-\delta}}dxdt\right)^{\frac{q}{p}}\left(\int_Q (1+|u_n|)^{\frac{q(\theta(p-1)-\delta)}{p-q}}dxdt\right)^{\frac{p-q}{p}}\\ &\le C_{19}\left[1+\left(\int_Q |u_n|^{(\delta+1-\gamma)m'}dxdt\right)^{\frac{1}{m'}}\right]^{\frac{q}{p}}\left(1+\int_Q |u_n|^{\frac{q(\theta(p-1)-\delta)}{p-q}}dxdt\right)^{\frac{p-q}{p}},\end{aligned} \tag{6.54}$$

where C_{19} is a positive constant depend only $C_{16}, C_{17}, C_{18}, C_{27}$ and $meas(Q)$.

By Lemma 2.9 (where $v = u_n$, $\rho = \delta+2$, $h = q$) and using (6.53), (6.54), we obtain

$$\begin{aligned}\int_Q |u_n|^{\frac{q(N+\delta+2)}{N}}dxdt &\le ||u_n||^{\frac{q(\delta+2)}{N}}_{L^\infty(0,T;L^{\delta+2}(\Omega))}\int_Q |\nabla u_n|^q dxdt\\ &\le C_{20}\left(1+\int_Q u_n^{(\delta+1-\gamma)m'}dxdt\right)^{\frac{q}{pm'}+\frac{q}{Nm'}}\left(1+\int_Q |u_n|^{\frac{q(\theta(p-1)-\delta)}{p-q}}dxdt\right)^{\frac{p-q}{p}}.\end{aligned}$$

Let now Choosing δ and such that

$$\sigma = \frac{q(N+\delta+2)}{N} = (\delta+1-\gamma)m' = \frac{q(\theta(p-1)-\delta)}{p-q}, \tag{6.55}$$

that is

$$\delta = \frac{p(N+2) - N\theta(p-1) - Nm'(1-\gamma)}{Nm' - N - p},$$

$$\sigma = \frac{m[N(p+\gamma-1) + p(\gamma+1) - N\theta(p-1)]}{N - pm + p},$$

$$q = \frac{m[N(p+\gamma-1) + p(\gamma+1) - N\theta(p-1)]}{N + 2 - \theta(p-1)(m-1) - m(1-\gamma)}.$$

using (6.55) in last inequality, we get

$$\int_Q |u_n|^\sigma dxdt \le C_{20}\left(1+\int_Q |u_n|^\sigma dxdt\right)^{\frac{q}{Nm'}+\frac{q}{pm'}+\frac{p-q}{p}}.$$

By virtue of $m < \frac{N}{p}+1$, we have $\frac{p-q}{q}+\frac{q}{pm'}+\frac{q}{Nm'} < 1$ and applying Young's inequality, then we deduce that

$$\int_Q |u_n|^\sigma dxdt \leq C_{21}. \tag{6.56}$$

If $s > \frac{p(1+m\gamma)+N(p-1)(1-\theta)}{N-pm+p}$, then $s+\delta+1 > \sigma$, by (6.53), (6.55), (6.56), Hölder's inequality and Young's inequality, we get

$$\int_Q |u_n|^{s+\delta+1} dxdt \leq C_{22}, \tag{6.57}$$

where $s+\delta+1 = \frac{(m-1)[p(N+1-s)-N\theta(p-1)]+N(s+1-m(1-\gamma))}{N-pm+p}$. The estimates (6.56) and (6.57), implies

$$\int_Q |u_n|^r dxdt \leq C_{23}. \tag{6.58}$$

The condition $m < \frac{p(N+2+\theta(p-1))}{p(N+2+\theta(p-1))-N[\theta(p-1)+1-\gamma]}$, ensure that $\delta-\theta(p-1) < 0$, then by (6.54), (6.55) and (6.56), we can get

$$\int_Q |\nabla u_n|^q dxdt \leq C_{25}. \tag{6.59}$$

By the definitions of $\varphi(\ell)$ and $\psi(\ell)$, we can get whenever $\delta > 0$

$$\frac{|\ell|^{\delta+2}}{\delta+2} \leq \psi(\ell), \qquad \forall \ell \in \mathbb{R}.$$

Going back to (6.52), By the above estimate , Hölder's inequality, some simplification and passing to supermum for $t \in (0,T)$, we get

$$\begin{aligned} &C_{28}|||u_n|^{\frac{\delta+p-\theta(p-1)}{p}}||_{L^\infty(0,T;L^{\frac{p(\delta+2)}{\delta+p-\theta(p-1)}}(\Omega))} + C_{29}\int_Q |\nabla |u_n|^{\frac{\delta+p-\theta(p-1)}{p}}|^p dxdt \\ &+ \int_Q |u_n|^{s+\delta+1} dxdt \leq C_{16}||f||_{L^m(Q)} \left(\int_Q |u_n|^{(\delta+1-\gamma)m'} dxdt\right)^{\frac{1}{m'}}, \end{aligned} \tag{6.60}$$

where $C_{28} = \frac{1}{\delta+2}, C_{29} = \frac{\alpha(\delta+1)p^p}{(\alpha+p-\theta(p-1))^p}$. Now applying Lemma 2.9 (where $v = |u_n|^{\frac{\delta+p-\theta(p-1)}{p}}$, $\rho = \frac{p(\delta+2)}{\delta+p-\theta(p-1)}$, $h = p$), from (6.60) and we use the same argument as before, we obtain

$$\int_Q |u_n|^\sigma dxdt \leq C_{30}; \qquad \int_Q |u_n|^r dxdt \leq C_{31}. \tag{6.61}$$

In the case $\delta \geq \theta(p-1)$ (i.e $m \geq \frac{p(N+2+\theta(p-1))}{p(N+2+\theta(p-1))-N[\theta(p-1)+1-\gamma]}$), combining (6.60), (6.61), we deduce that

$$\int_Q |\nabla u_n|^p dxdt \leq C_{24}. \tag{6.62}$$

The estimates (6.58), (6.59),(6.61) and (6.62) completed the proof of Lemma 6.16 and Lemma 6.17. □

Lemma 6.18. *Assume that hypothesis* (6.2)-(6.4) *hold, h satisfies* $(h1)$ *and* $f \in L^m(Q)$ *with*

$$1 < m < \frac{p(N+2+\theta(p-1))}{p(N+2+\theta(p-1)) - N[\theta(p-1)+1-\gamma]}.$$

Then for every solution u_n of (6.13), *there exists positive constants c_4, c_5, c_8 and c_9 independent of n such that*

(i) If $s \geq \frac{1+\theta(p-1)-m\gamma}{m-1}$, then

$$||u_n||_{L^p(0,T;L^p(\Omega))} \leq c_5,$$
$$||u_n||_{L^{(s+\gamma)m}(Q)} \leq c_4.$$

(ii) If $\frac{1+\theta(p-1)-mp\gamma}{mp-1} < s < \frac{1+\theta(p-1)-m\gamma}{m-1}$, then

$$||u_n||_{L^q(0,T;L^q(\Omega))} \leq c_9,$$
$$||u_n||_{L^r(Q)} \leq c_8,$$

where q and r are defined in Theorem 6.11.

Proof of Lemma 6.18. For $t \in (0,T)$, let $\varphi(u_n) = ((1+|u_n|)^{s(m-1)+m\gamma} - 1)sing(u_n)\chi_{(0,t)}$ as test function in (6.13),by (6.2) and the condition $(h1)$, we have

$$\begin{aligned} &\int_\Omega \psi(u_n(x,t))dxd\tau + \alpha(s(m-1)+m\gamma)\int_0^t\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)-s(m-1)-m\gamma+1}}dxd\tau \\ &+ \int_0^t\int_\Omega |u_n|^s((1+|u_n|^{s(m-1)+m\gamma})-1)dxd\tau \leq c_0\int_0^t\int_\Omega |f||u_n|^{s(m-1)+m\gamma-\gamma}dxd\tau, \end{aligned} \tag{6.63}$$

where $\psi(\ell) = \displaystyle\int_0^\ell \varphi(\sigma)d\sigma$, $\forall \ell \in \mathbb{R}$. Since $c_1|\ell|^{1+s(m-1)+m\gamma} - c_2 \leq \psi(\ell)$, $\forall \ell \in \mathbb{R}$, where c_0, c_1, c_2 are tree positive constants. By last inequality, Hölder's inequality and passing to the supremum in $t \in (0,T)$, we get

$$\begin{aligned} &c_1||u_n||^{1+s(m-1)+m\gamma}_{L^\infty(0,T;L^{1+s(m-1)+m\gamma}(\Omega))} + c_3\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)-s(m-1)-m\gamma+1}}dxdt \\ &+ \int_Q |u_n|^{(s+\gamma)m}dxdt \leq c_2\text{meas}\Omega + c_0||f||_{L^m(Q)}\left(\int_Q |u_n|^{(s+\gamma)m}dxdt\right)^{\frac{1}{m'}}, \end{aligned}$$

where $c_3 = \alpha(s(m-1)+m\gamma)$. Using Young's inequality with ϵ, we have

$$\begin{aligned} &c_1||u_n||^{1+s(m-1)+m\gamma}_{L^\infty(0,T;L^{1+s(m-1)+m\gamma}(\Omega))} + c_3\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)-s(m-1)-m\gamma+1}}dxdt \\ &+ \int_Q |u_n|^{(s+\gamma)m}dxdt \leq c_2\text{meas}\Omega + C_\epsilon + \epsilon\int_Q |u_n|^{(s+\gamma)m}dxdt. \end{aligned}$$

Taking $\epsilon = \frac{1}{2}$ in last inequality, implies that

$$c_1||u_n||^{1+s(m-1)+m\gamma}_{L^\infty(0,T;L^{1+s(m-1)+m\gamma}(\Omega))} + c_3 \int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)-s(m-1)-m\gamma+1}} dxdt$$

$$+ \int_Q |u_n|^{(s+\gamma)m} dxdt \leq c_4, \tag{6.64}$$

where c_4 is a positive constant depend of $c_0, c_2, meas\Omega, ||f||_{L^m(Q)}$.

If $s \geq \frac{1+\theta(p-1)-m\gamma}{m-1}$, then $\theta(p-1) - s(m-1) - m\gamma + 1 \leq 0$, from (6.64) we get

$$\int_Q |\nabla u_n|^p dxdt \leq \int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)-s(m-1)-m\gamma+1}} dxdt \leq c_5, \tag{6.65}$$

where $c_5 = \frac{c_4}{c_3}$. The estimates (6.64) and (6.65) completed the proof of item (i).
(ii) If $\frac{1+\theta(p-1)-mp\gamma}{mp-1} < s < \frac{1+\theta(p-1)-m\gamma}{m-1}$, then $\theta(p-1) - s(m-1) - m\gamma + 1 > 0$, let $1 < q < p$, applying Hölder's inequality, we get

$$\begin{aligned} \int_Q |\nabla u_n|^q dxdt &= \int_Q \frac{|\nabla u_n|^q (1+|u_n|)^{\frac{(\theta(p-1)-s(m-1)-m\gamma+1)q}{p}}}{(1+|u_n|)^{\frac{(\theta(p-1)-s(m-1)-m\gamma+1)q}{p}}} dxdt \\ &\leq \left(\int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)-s(m-1)-m\gamma+1}} dxdt\right)^{\frac{q}{p}} \\ &\times \left(\int_Q (1+|u_n|)^{\frac{(\theta(p-1)-s(m-1)-m\gamma+1)q}{p-q}} dxdt\right)^{\frac{p-q}{p}}, \end{aligned}$$

then by (6.64), we get

$$\int_Q |\nabla u_n|^q dxdt \leq c_6 \left(\int_Q (1+|u_n|)^{\frac{(\theta(p-1)-s(m-1)-m\gamma+1)q}{p-q}} dxdt\right)^{\frac{p-q}{p}}. \tag{6.66}$$

We now choose q in order to have

$$\frac{(\theta(p-1)-s(m-1)-m\gamma+1)q}{p-q} = (s+\gamma)m. \tag{6.67}$$

The last equality implies that

$$q = \frac{p(s+\gamma)m}{1+\theta(p-1)+s}. \tag{6.68}$$

By Lemma 2.9 (where $v = u_n$, $\rho = 1 + s(m-1) + m\gamma$, $h = q$ and $\sigma = \frac{q(N+\rho)}{N}$), from(6.64) and (6.66), we obtain

$$\int_Q |u_n|^\sigma dxdt \leq C||u_n||^{\frac{(1+s(m-1)+m\gamma)q}{N}}_{L^\infty(0,T;L^{1+s(m-1)+m\gamma}(\Omega))} \int_Q |\nabla u_n|^q dxdt \leq c_7. \tag{6.69}$$

If $s \geq \frac{p(N+1+m\gamma)-N(1+\theta(p-1))}{N-pm+p}$, then $(s+\gamma)m \geq \sigma$; if $s < \frac{p(N+1+m\gamma)-N(1+\theta(p-1))}{N-pm+p}$, then $(s+\gamma)m < \sigma$.

The estimates (6.64) and (6.69) yields

$$\int_Q |u_n|^r dxdt \leq c_8. \tag{6.70}$$

Using (6.64) and (6.67) in (6.66), we get

$$\int_Q |\nabla u_n|^q dxdt \leq c_9. \tag{6.71}$$

□

4 Proof of main results

Proof of Theorem 6.6, Theorem 6.8, Theorem 6.9 and Theorem 6.11. Because the proof of Theorem 6.6, Theorem 6.9 and Theorem 6.11 is similar to that of Theorem 6.8. Now we give the proof Theorem 6.8.

By Lemma 6.16 we have the sequence $\{u_n\}$ is bounded in $L^r(Q) \cap L^p(0,T;W_0^{1,p}(\Omega))$. Then there exist a function $u \in L^r(Q) \cap L^p(0,T;W_0^{1,p}(\Omega))$ and sub-sequence, still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \ \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{6.72}$$

$$u_n \rightharpoonup u \ \text{ weakly in } L^r(Q). \tag{6.73}$$

From (6.33), (6.72) and (6.73) we have the sequence $\{\frac{\partial u_n}{\partial t}\} = \text{div } a(x,t,T_n(u_n),\nabla u_n) + (h_n(u_n)f_n - |u_n|^{s-1}u_n)$ is bounded in the space $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$, using the compactness argument in [139], we obtain that

$$u_n \longrightarrow u \text{ strongly in } L^1(Q). \tag{6.74}$$

Hence

$$u_n \longrightarrow u \ \ a.e. \ in \ Q. \tag{6.75}$$

Now, we adapting the approach of [22, Theorem 3.1] there exists a subsequence (still denoted by $\{u_n\}$) such that

$$\nabla u_n \longrightarrow \nabla u \ \ a.e. \ in \ Q. \tag{6.76}$$

From (6.75), (6.76) and (6.3), using Vitali's Theorem, we obtain

$$a(x,t,T_n(u_n),\nabla u_n) \longrightarrow a(x,t,u,\nabla u) \text{ weakly in } L^{p'}(Q). \tag{6.77}$$

We shall now prove that $|u_n|^{s-1}u_n \longrightarrow |u|^{s-1}u$ and $h_n(u_n)f_n \longrightarrow h(u)f$ strongly in $L^1(Q)$. Indeed, let ϕ_i be a sequence of increasing, positive uniformly bounded $C^\infty(Q)$ functions, such that

$$\phi_i(s) \longrightarrow \begin{cases} 1 & \text{if } \ s \geq \delta, \\ 0 & \text{if } \ |s| < \delta. \\ -1 & \text{if } \ s \leq -\delta. \end{cases}$$

choosing $\phi_i(u_n)$ as a test function in (6.13), we get

$$\begin{aligned} \int_Q |u_n|^{s-1}u_n\phi_i(u_n)dxdt &\leq \int_Q h(u_n)f_n\phi_i(u_n)dxdt \\ &\leq ||h||_{L^\infty((0,+\infty))} \int_Q f\phi_i(u_n)dxdt. \end{aligned}$$

The limit on i implies

$$\int_{\{(x,t)\in Q:|u_n(x,t)|>\delta\}} |u_n|^s dxdt \leq ||h||_{L^\infty((0,+\infty))} \int_{\{(x,t)\in Q:|u_n(x,t)|>\delta\}} f dxdt. \tag{6.78}$$

We are going to use this inequality to show that if E is any measurable subset of Q, then

$$\lim_{|E|\longrightarrow 0} \int_E |u_n|^s dxdt = 0$$

uniformly with respect to n. Using (6.78), for any $\delta > 0$ we have

$$\begin{aligned}\int_E |u_n|^s dxdt &\leq \delta^s|E| + \int_{E\cap\{(x,t)\in Q:|u_n(x,t)|>\delta\}} |u_n|^s dxdt \\ &\leq \delta^s|E| + ||h||_{L^\infty((0,+\infty))} \int_{\{(x,t)\in Q:|u_n(x,t)|>\delta\}} f dxdt.\end{aligned}$$

The fact $f \in L^1(Q)$ allows us to say that for any given $\epsilon > 0$, there exist δ_ϵ such that

$$||h||_{L^\infty((0,+\infty))} \int_{\{(x,t)\in Q:|u_n(x,t)|>\delta_\epsilon\}} f dxdt \leq \epsilon.$$

In this way

$$\int_E |u_n|^s dxdt \leq \delta_\epsilon^s|E| + \epsilon||h||_{L^\infty((0,+\infty))},$$

and so

$$\lim_{|E|\longrightarrow 0} \int_E |u_n|^s dxdt \leq \epsilon||h||_{L^\infty((0,+\infty))} \quad \forall\epsilon > 0,$$

we thus proved that $\lim_{|E|\longrightarrow 0} \int_E |u_n|^s dxdt = 0$ uniformly with respect to n. Vitali's Theorem and (6.75) implies that

$$|u_n|^{s-1}u_n \longrightarrow |u|^{s-1}u \text{ strongly in } L^1(Q). \tag{6.79}$$

Using the same argument ones of the Theorem 6.4, we get

$$h_n(u_n)f_n \longrightarrow h(u)f \quad \text{strongly in} \quad L^1(Q). \tag{6.80}$$

Let now $\phi \in C^\infty(\overline{Q})$, which is zero in neighborhood of $\Gamma \cup (\Omega \times \{T\})$. Inserting ϕ as test function in (6.13), we get

$$\begin{aligned}-\int_Q u_n\frac{\partial\phi}{\partial t}dxdt &+ \int_Q a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla\phi dxdt \\ &+ \int_Q u_n^{s-1}u_n\phi dxdt = \int_Q h_n(u_n)f_n\phi dxdt.\end{aligned}$$

let $n \longrightarrow +\infty$ in last inequality, by (6.74), (6.77), (6.79) and (6.80), we get

$$\begin{aligned}-\int_Q u\frac{\partial\phi}{\partial t}dxdt &+ \int_Q a(x,t,u,\nabla u)\cdot\nabla\phi dxdt \\ &+ \int_Q u^{s-1}u\phi dxdt = \int_Q h(u)f\phi dxdt.\end{aligned}$$

□

Conclusion and Perspectives

In this thesis, we have proved the existence and regularity of solutions to certain singular parabolic problems with strong nonlinearities. More precisely, In the first step, we have approximated the singular problems considered by another-ones non-singular, and based on the classical results that exist in the parabolic PDEs and the application of the fixed point theorem we have proved the existence of a weak solution to the approximate problems. In the second step, we have proved some prior estimates for the weak solutions to the approximate problems, also we have shown an important property of these solutions that is the strict positivity in the interior of the parabolic cylinder, which gives meaning to a weak formulation of problems, also this property used in the proofs of convergences of the singular terms. In the thirty steps, we have used the estimates obtained in the second step and also we used the classical results of compacity, which permit passing to the limit in the approximate problems, and then we obtain the solution of the problems considered. In the last step, we have localized our attention to the study of the regularity of the solution and its gradient, which depends on the parameters $(\gamma, \theta, \mu, s, m...)$ and the summability of the data f. To achieve this regularity we have used the Gagliardo-Nirenberg inequality.

For the perspective, we are now working on creating a new mathematical model that takes into account the different aspects. More precisely, we are interested in studying the singular parabolic problems with convection and reaction terms. the simple models are the following:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u = -\operatorname{div}(|u|^{p-1} u E(x,t)) + \frac{f}{u^\gamma} & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Gamma, \\ u_0(x, t=0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{6.81}$$

and

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u = |\nabla u|^q + \frac{f}{u^\gamma} & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Gamma, \\ u_0(x, t=0) = u_0(x) & \text{in} \quad \Omega. \end{cases} \tag{6.82}$$

Another perspective is the study of the existence and regularity of solutions to certain singular parabolic problems in fractional concepts.

Bibliography

[1] B. Abdellaoui, I. Peral: Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian a critical potential. Ann. Math. **182**, 247–270 (2003)

[2] B. Abdelaoui, A. Attar, Quasilinear elliptic prblem with Hardy potential and singular term, Comm. On Pure and App. Analys., 12(3), 1363-1380 (2013).

[3] R.A. Adams. Sobolev spaces. Academic press, New York, 1975.

[4] A. Adimurth, L. Boccardo, G.R. Cirmi, L. Orsina, The regularizing effect of lower terms in elliptic problems involving Hardy potential, Adv. Nolinear Stud., 17(2), 311-317 (2017).

[5] J.A. Aguilar Crespo, I. Peral Alonso, Global behaviour of the Cauchy problem for some critical nonlinear parabolic equations, SIAM J. Math. Anal. **31**, 1270–1294 (2000).

[6] A. Alvino, V. Ferone, G. Trombetti, A priori estimates for a class of non-uniformly elliptic equations. Atti Sem. Mat. Fis. Univ. Modena Reggio Emilia **46**, 381–391 (1998).

[7] A. Alvino, L. Boccardo, V. Ferone and L. Orsina, Existence results for nonlinear elliptic equations with degenerate coercivity, Ann. Mat. Pura Appl. IV. Ser. **182**, 53-79 (2003).

[8] G. Anello and F. Faraci, Two solutions for an elliptic problem with two singular terms, Calc. Var. Partial Differential Equations, **56**, 56-91 (2017).

[9] G. Anello and F. Faraci, On a singular semilinear elliptic problem with an asymptotically linear nonlinearity, Proc. Roy. Soc. Edinburgh Sect. A, **146**, 59-77 (2016).

[10] D. Arcoya and L. Moreno-Mérida, Multiplicity of solutions for a Dirichlet problem with a strongly singular nonlinearity, Nonlinear Anal. **95**, 281-291 (2014).

[11] D.G. Aronson, J. Serrin, Local behaviour of solutions of quasilinear parabolic equations, Arch. Ration. Mech. Anal. **25**, 81 (1967).

[12] R. Aris, Mathematical Modelling Techniques, Res. Notes Math., vol. **24**, Pitman (Advanced Publishing Program), Boston, MA, 1979.

[13] P. Baras, J.A. Goldestien, The heat equations with singular potential, Tran. of the Ami. Math. Soci. **284**(1), 121-139 (1984).

[14] P. Benilan, L. Boccardo, T. Gallou¨et, R. Gariepy, M. Pierre and J. L. Vasquez, An L^1 theory of existence and uniqueness of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa, **22**, 240-273 (1995).

[15] A. Benkirane, B. El Haji, M. El Moumni, On the existence of solution for degenerate parabolic equations with singular terms, Pure and Applied Mathematics Quarterly, 14(3–4), 591–606 (2018).

[16] L. Boccardo and T. Gallouët. Nonlinear elliptic and parabolic equations involving measure data, J. Functional Anal., **87**, 149-169 (1989).

[17] L. Boccardo, F. Murat, J.P Puel, Existence resultsfor some quasilinear parabolic equations, Nonlinear Analysis **13**, 373-392 (1989).

[18] L. Boccardo, T. Gallouët, Nonlinear elliptic equations with right hand side measures, Commun. in partial diff. equations, **17** (3-4), 641-655 (1992).

[19] L. Boccardo, T. Gallouët, Strongly nonlinear elliptic equations having natural growth and L1 data., Nonlinear Anal. T.M.A., **19**(6), 573-578 (1992).

[20] L. Boccardo, A. Dall'Aglio, T. Gallou¨et and L. Orsina, Nonlinear parabolic equations with measure data, J. Funct. Anal., **147**, 237-258 (1997).

[21] L. Boccardo, A. Dall'Aglio, L. Orsina, Existence and regularity results for some elliptic equations with degenerate coercivity, Dedicated to Prof. C. Vinti (Italian) (Perugia, 1996). Atti Sem. Mat. Fis. Univ. Modena **46**(suppl.), 51-81 (1998).

[22] L. Boccardo, A. Dall'Aglio, T. Gallouët, et al.: Existence and regularity results for some nonlinear parabolic equations. Adv Math Sci Appl. **9**(2), 1017–1031 (1999).

[23] L. Boccardo, T. Gallët and J. L. Vazquez, Solutions of nonlinear parabolic equations without growth restrictions on the data, Electron. J. Differential Equations, **60**, 1-20 (2001).

[24] L. Boccardo, H. Brezis, Some remarks on a class of elliptic equations with degenerate coercivity. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. **8**, 521-530 (2003).

[25] Boccardo L., Orsina L., Semilinear elliptic equations with nonlinearities, Calc. Var. and PDEs. **37**(3-4), 363-380 (2010).

[26] Boccardo L., A Contribution to the theory of Quasilinear elliptic equations and application to the minimization of integral functional, Milan J. Math. Vol. **79**, 193-206 (2011).

[27] L. Boccardo, L. Orsina, M.M. Porzio, Existence results for quasilinear elliptic and parabolic problems with quadratic gradient terms and sources. Adv. Calc. Var., **4**, 397-419 (2011).

[28] L. Boccardo, G. Croce, L. Orsina, Nonlinear degenerate elliptic problems with W01,1 solutions. Manuscr. Math. **137**, 419–439 (2012).

[29] L. Boccardo, M.M. Porzio, Some degenerate parabolic prblemes: existence and decay properties. Discrete Contin. Dyn. Syst. Ser. S **7**(4), 383–428 (2014).

[30] L. Boccardo, L.Moreno-Mérida, L. Orsina, A class of quasilinear Dirichlet problems with unbounded coefficients and singular quadratic lower order terms. Milan J. Math., **83** 157-176 (2015).

[31] B. Bougherara, J. Giacomoni and J. Hernández, Existence and regularity of weak solutions for singular elliptic problems, Electron. J. Differ. Equ. Conf., **22**, 19-30 (2015).

[32] H. Brezis and W. Strauss, Semilinear elliptic equations in L1, T. Math. Soc. Japan, **25**, 565-590 (1973).

[33] H. Brezis, F.E. Browder, Strongly nonlinear parabolic boundary value problems, Proc. natn. Acad. Sci. U.S.A., **76**, 38-40 (1979).

[34] H. Brezis, F.E. Browder, Strongly nonlinear parabolic variational inequalities, Proc. natn. Acad. Sci. U.S.A., **77**, 713-715 (1980).

[35] H. Brezis; *Analyse fonctionnelle. Théorie et applications*, Masson, Paris, 1983.

[36] H. Brezis, Functional Analysis, Sobolev spaces and Partial Differential Equations, Universitext. Springer, New York, 2011.

[37] M.G. Crandall,, P.H. Rabinowitz and L. Tartar: On a Dirichlet problem with a singular nonlinearity. Commun. Partial Differ. Equ., **2**, 193–222 (1977).

[38] A. Canino, Minimax methods for singular elliptic equations with an application to a jumping problem, J. Differential Equations, **221**, 210-223 (2006).

[39] A. Canino and M. Degiovanni, A variational approach to a class of singular semilinear elliptic equations, J. Convex Anal., **11**, 147-162 (2004).

[40] J. Carmona, P.J. Matinez-Aparicio, A. Suarez, Existence and non-existence of positive solutions for nonlinear elliptic singular equations with natural growth, Non. Analy; Theory, Methods and Applications, **89**, 157-169 (2013).

[41] J. Carmona, P.J. Matinez-Aparicio, Homogenization of singular quasilinear elliptic problems with natural growth in a domain with many small holer, Disc. Conti. Dyn. Sys., **37**(1), 15 (2017).

[42] J. Carmona, T. Leonori, S. Lopez-Martinez, P.J. Martinez-Aparicio, Quasilinear elliptic problems with singular and homogeneous lower order terms, Nonlinear Analysis, **179**, 105-130 (2019).

[43] J. Carmona, S. Lopez-Martinez, P.J. Martinez-Aparicio, Singular elliptic problems with changin sign datum: existence and homogenization, Revisita Matematica complutense, **33**(1), 39-62 (2020).

[44] Charkaoui A., Alaa N.E., Weak periodic solution for semilinear parabolic problems with singular nonlinearities and L^1 data, Mediterr. J. Math. **17**, 108(2020).

[45] M. del Pino: A global estimate for the gradient in a singular elliptic boundary value problem. Proc. R. Soc. Edinb. Sect. A, **122**, 341–352 (1992).

[46] H.T. Banks, Modeling and Control in the Biomedical Sciences, in: Lecture Notes in Biomathematics, vol. **6**, Springer-Verlag, Berlin, (1975).

[47] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vazquez, An L^1-theory of existence and uniqueness of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **22**, 241-273 (1995).

[48] X. Cabré, Y. Martel, Existence versus explosion instantanée pour des èquations de la chaleur linèaires avec potentiel singulier, C. R. Acad. Sci. Paris **329**, 937-978 (1999).

[49] A. Canino, B. Sciunzi and A. Trombetta, Existence and uniqueness for $p-$Laplace equations involving singular nonlinearities, NoDEA Nonlinear Differential Equations Appl., **23** (2016).

[50] P. Cianci2017, G.R. Cirmi, S. D'Asero, S Leonardi, Morrey estimate for solutions of singular quadratic nonlinear equations, Ann. Mat. Pura. App. (1923-), **196**(5), 1739-1758 (2017).

[51] G. R. Cirmi and M. M. Porzio, L^1-Solution for some nonlinear degenerate elliptic and parabolic equations, Ann. Mat. Pura Appl., (4), **169**, 67-86 (1995).

[52] G.R. Cirmi, S. D'Asero, S. Leonardi, Gradient estimate for solutions of nonlinear singular elliptic equations below the duality exponent, Mathematical Methods in the applied Sciences, **41**(1), 261-269 (2018).

[53] G. Croce, The regularizing effects of some lower order terms in an elliptic equation with degenerate coercivity. Rend. Mat. Appl., **27**, 299–314 (2007).

[54] G. Croce, An elliptic problem with two singularities, Asymptotic Analysis, **78**, 1–10 (2012).

[55] Croce, G.: An elliptic problem with degenerate coercivity and a singular quadratic gradient lower order term. Discrete Contin. Dyn. Syst. **5**, 507–530 (2015).

[56] A. Dall'Aglio and L. Orsina, Existence results for some nonlinear parabolic equations with nonregular data, Differential Integral Equations, **5**, 1335-1354 (1992).

[57] A. Dall'Aglio, D. Giachetti, J.P. Puel: Nonlinear parabolic equations with natural growth in general domains. Boll. Unione Mat. Ital.-B **3**, 653–684 (2005).

[58] A. Dall'Aglio, D. Giachetti, S. Segura de León: Nonlinear parabolic problems with a very general quadratic gradient term. Differ. Integral Equ. **20**(4), 361–396 (2007).

[59] A. Dall'Aglio, L. Orsina, Nolinear parabolic equations with natural growth conditions and L^1 data, Nonlinear Analysis, Methods and Applications, **27**(1), 59-73 (1996).

[60] A. Dall'Aglio, L. Orsina, F. Petitta, Existence of solutions for degenerate parabolic equations with singular terms, Nonlinear Analysis, **131**, 273-288 (2016).

[61] R. Dautray, J.L. Lions, Mathematical analysis and numerical methods for science and technology, Vol. **5**, Springer-Verlag (1992).

[62] L.M. De Cave, Nonlinear elliptic equations with singular nonlinearities, Asymptotic Analysis, **84**, 181–195 (2013).

[63] L.M. De Cave, F. Oliva, Elliptic equations with general singular lower order term and measure data, Nonlinear Analysis, **128**, 391-411 (2015).

[64] L.M. De Cave, R. Durastanti, F. Oliva, Existence and uniqueness results for possibly singular nonlinear elliptic equations with measure data, Nonlinear Diff. Equ. App., **25**:18 (2018).

[65] F. Della Pietra, G. di Blasio, Comparison, existence and regularity results for a class of non-uniformly elliptic equations. Differ. Equ. Appl. **2**, 79–103 (2010).

[66] F. Della Pietra, G. di Blasio, Existence and comparison results for non-uniformly parabolic problems. Mediterr. J. Math. **7**(3), 323–340 (2010).

[67] J.I. Diaz, Nonlinear Partial Differential Equations and Free Boundaries, Vol. I, in: Research Notes in Mathematics, vol. **106**, Pitman, Boston, MA, (1985), Advanced Publishing Program.

[68] I. De Bonis and L.M. De Cave, Degenerate parabolic equations with singular lower order terms, Differential and Integral Equations **27**(9/10), 949-976 (2014).

[69] I. De Bonis, D. Giachetti, Nonnegative solutions for a class of singular parabolic problems involving p-Laplacian, Asymptotic Analysis, **91**, 147-183 (2015).

[70] V. De Cicco, D. Giachetti, F. Oliva, F. Petitta, The Direchlet problem for singular elliptic equations with general nonlinearities, Calc. Vari. Part. Diff. Equ., **58**(4), 1-40 (2019).

[71] R. Durastanti and F. Oliva, The Dirichlet problem for possibly singular elliptic equations with degenerate coercivity. arXiv:2105.13453v1 (2021).

[72] R. Durastanti, Asymptotic behavior and existence of solutions for singular elliptic equations, Ann. Mat. Pura. Appl., **199**, 925-954 (2020).

[73] R. Durastanti, F. Oliva, Comparaison principal of elliptic equations with mixed singularities, Potential Analysis, 1-18 (2021).

[74] E. DiBenedetto, *Degenerate parabolic equations,* Springer-Verlag, New York, (1993).

[75] Y. El Hadfi, A. Benkirane, A. Youssfi ,Existence and regularity results for parabolic equations with degenerate coercivity, Complex Variables and Elliptic Equations, **63**(5), 715-729 (2018).

[76] Y. El Hadfi, M. El Ouardy, A. Ifzarne, A. Sbai: On nonlinear parabolic equations with singular lower order term. J Elliptic Parabol Equ, **8**, 49-75 (2021).

[77] M. El Ouardy, Y. El Hadfi and A. Ifzarne, Existence and regularity results for a singular parabolic equations with degenerate coercivity, Disc. Conti. Dyn. Sys. Series S, **15**(1), 117-141 (2022). doi:10.3934/dcdss.2021012.

[78] M. El Ouardy, Y. El Hadfi, Some nonlinear parabolic problems with singular natural growth term, **77**, 95 (2022). doi:10.1007/s00025-022-01631-6.

[79] M. El Ouardy, Y.El Hadfi and A. Sbai, Existence of positive solutions to nonlinear singular parabolic equations with Hardy potential, J. Pseudo-Differ. Oper. Appl. **13**, 28 (2022). doi:10.1007/s11868-022-00457-8.

[80] Fulks, W., and Maybee, J. S., A singular non-linear equation, Osaka J. Math., **12**(1), 1-19 (1960).

[81] L.M. De Cave, F. Oliva, On the regularizing effect of some absorption and singular lower order terms in classical Direchlet problems with L^1 data, Journal of Elliptic and Parabolic Equations,**2**, 73-85 (2016).

[82] F.Farroni, L. Greco, G. Moscariello, G. Zecca, Nonlinear evolution problemswith singular coefficients in the lower order terms, Nonlinear Diff. Equa. Appl., **28**, 38 (2021).

[83] V. Ferone, M.R. Posteraro, J.M. Rakotoson: Nolinear parabolic problems with critical growth and unbounded data. Indiana Univ. Math. J. **50**, 1201–1215 (2001).

[84] J. Garcia Azorero, I. Peral, Hardy inequalities and some critical ellitic and parabolic problems, J. Diff. Equa., **144**, 441-476 (1998).

[85] J.A. Goldstein and I. Kombe, Instantaneous blow up, Contemp. Math., **327**, 141-149 (2003).

[86] J.A. Goldstein and I. Kombe, Nonlinear degenerate parabolic equations with singular lower order term, Adv. Diff. Equations, **8**(10), 1153-1192 (2003).

[87] J.A. Goldstein and Q.S. Zhang, Linear parabolic equations with strong singular potentials, Trans. Amer. Soc., **355**, 197-211 (2003).

[88] J.A. Goldstein and Q.S. Zhang, On a degenerate heat equation with a singular potential, J. of Functional Analysis, **186**(2), 342-359 (2001).

[89] D. Giachetti, P.J. Martinez-Aparicio, F. Murat, A semilinear elliptic equations with a mild singularity at $u = 0$: existence and homogenization, Journal de Mathématiques Pures et Appliquées, **107**(1), 41-77 (2017).

[90] M. Ghergu, V. Rădulescu, Multi-parameter bifurcation and asymptotics for the singular Lane–Emden–Fowler equation with a convection term, Proc. Roy. Soc. Edinburgh Sect. A, **135** (1), 61–83 (2005).

[91] N. Grenon and A. Mercaldo, Existence and regularity results for solutions to nonlinear parabolic equations, Adv. Differential Equations, **10**, 1007-1034 (2005).

[92] N. Grenon, A. Mercaldo: Existence and regularity results for solutions to nonlinear parabolic equations. Adv. Diff. Equa., **10**(9), 1007–1034 (2015).

[93] E. Hewitt and K. Stromberg; *Real and abstract analysis*, Springer-Verlag, Berlin Heidelberg New York, (1965).

[94] W.L. Perry, A monotone iterative technique for solution of pth order (p ¡ 0) reaction diffusion problems in permeable catalysis, J. Comput. Chem., **5**(4), 353–357 (1984).

[95] O. Kavian; *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, Mathématiques et Applications, vol. **13**, Springer Verlag, (1993).

[96] I. Kombe, The linear heat equations with highly oscillating potential, Proc. Amer. Math. Soc. **132**, 2683-2691 (2004).

[97] A. Kufner, O. John, B. Opic; *Function spaces Academia*,Praha, (1977).

[98] Ladyženskaja O., Solonnikov V.A., Ural'ceva N.N., Linear and Quasilinear Equations of Parabolic Type, in: Translations of the American Mathematical Society, American Mathematical Society, Providence, (1968).

[99] A.V. Lair and A.W. Shaker: Classical and weak solutions of a singular semilinear elliptic problem. J. Math. Anal. Appl. **211**, 371–385 (1997).

[100] R.Landes, V. Mustonev, A strongly nonlinear parabolic initial boundary value problem, Ark. Mat., **25**, 29-40 (1987).

[101] R. Landes, V. Mostenov, On parabolic initial-boundary value problems with critical growth for the gradient, Ann. Inst. H. Poincaré, Anal. Nonlinéaire, **11**, 135-158 (1994).

[102] M. Lattore, F. Oliva, F. petitta, S.S. De León, The Dirichlet problem for the 1-Laplacian with a general L^1 data, Nonlinearity, **34**(3), 1791 (2021).

[103] A.C. Lazer, P.J. McKenna: On a singular nonlinear elliptic boundary-value problem. Proc. Am. Math. Soc., **111**, 721–730 (1991).

[104] C. Leone, A. Porretta, Entropy solutions for nonlinear elliptic equations in L^1, Nonlinear Anal. T.M.A., **32**, 325-334 (1998).

[105] F. Leoni, B. Pellacci: Local estimates and global existence for strongly nonlinear parabolic equations with locally integrable data. J. Evol. Equ., **6**, 113–144 (2006).

[106] F.Q. Li, Regularity of solutions to nonlinear parabolic equations with a lower-order term, Potential Anal. **16**, 393-400 (2002).

[107] F.Q. Li, Existence and regularity results forsome parabolic equations with degenerate coercivity, Ann. Acad. Sci. Fen. Mat., **37**, 605-633 (2012).

[108] J.L. Lions. Quelques méthodes de résolution des problème aux limites non lineaires, Dundo, Paris (1969).

[109] J. Leray, J. L. Lions, Quelques résultats de Visik sur les problémes non linéaires par les méthodes de MintyBrowder, Bull. Soc. Math. France, **93**, 97-107 (1965).

[110] C.D. Luning, W.L. Perry, Positive solutions of negative exponent generalized Emden–Fowler boundary value problems, SIAM J. Math. Anal., **12**(6), 874-879 (1981).

[111] M. Magliocca: Existence results for a Cauchy-Direchlet parabolic problem with a repulsive gradient term. Nonlinear Anal. **166**, 102–143 (2018).

[112] M. Magliocca, F. Oliva, On some parabolic equations involving superlinear singular gradient terms. J. Evol. Equ., **21**, 2547-2590 (2021).

[113] P.J. Martinez-Aparicio, F. Petitta, Parabolic equations with nonlinear singularities, Nonlinear Analysis, **74**, 114-131 (2011).

[114] L. Moreno-Mérida, A quasilinear Dirichlet problem with quadratic growth respect to the gradient and L1 data, Nonlinear Anal., **15**, 450-459 (2014) .

[115] A. Mokrane, Existence of bounded solutions for some nonlinear parabolic equations, Proc. R. Soc. Edinb., **107**, 313-326 (1987).

[116] A. marah, H. Redwane, K. zaki, Existence and regularity results for non linear parabolic equations with quadratic growth with respect to the gradient, Ren. Circ. Mat. di Palermo, Ser(2020) doi:10.1007/s12215-020-00530-5.

[117] A. Nachman, A. Callegari, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, SIAM J. Appl. Math., **38**(2), 275–281 (1980).

[118] F. Oliva, F. Petitta, On singular elliptic equations with measure source, ESAIM: Control, Optimisation and Calcul Variations, **22**(1), 289-308 (2016).

[119] F. Oliva, F. Petitta, Finite and infinite energy solutions for singular elliptic problems: Existence and uniqueness, J. Diff. Equa., **264**(1), 311-340 (2018).

[120] F. Oliva, Existence and uniqueness of solution to some singular equations with natural growth, Ann. Math. Pur. Appl., **200**, 287-314 (2020).

[121] F. Oliva, Regularizing effect of absorption terms in singular problems, J. Math. Analy. Appl., **472**(1), 1136-1166 (2019).

[122] F. Oliva, F. Petitta, A nonlinear parabolic problem with singular terms and nonregular data. Nonlinear Analysis, **194**, 111472 (2020).

[123] L. Orsina, M.M. Porzio, $L^\infty(Q)$-estimate an existence of solutions for some nonlinear parabolic equations, Boll.Un. Mat. Ital., **6**, 631-647 (1992).

[124] F. Petitta, A. Porretta, On the notion of renormalized solution to nonlinear parabolic equations with general measure data. J. Elliptic Parabol. Equ., **1**, 201–214 (2015).

[125] A. Porretta, Some remarks on the regularity of solutions for a class of elliptic equations with measure data, Houston Journal of Math., **26**(1), 183-213 (2000).

[126] A. Porretta, Existence for elliptic equations in L1 having lower order terms with natural growth, Portugaliae.Math., **57**, 179-190 (2000).

[127] M.M. Porzio, Existence of solutions for some noncoercive parabolic equations. Discrete Contin. Dyn. Syst. **5**(3), 553–568 (1999).

[128] M.M. Porzio: Local regularity results for some parabolic equations. Houst. J. Math. **25**(4), 769–792 (1999).

[129] M.M.Porzio, M.A. Pozio, Parabolic equations with non-linear degenerate and space-time dependent operators. J. Evol. Equ. **8**, 31-70 (2008).

[130] M.M. Porzio, A. Primo, Summability and existence results for quasilinear parabolic equations with Hardy potential terms, Nonlinear Diff. Equ. Appl., **20**, 65-100 (2013).

[131] M.M. Porzio, On uniform and decay estimate for unbounded solutions of partial differential equations, J. Diff. Equa. **259**(11), 6970-7011 (2015).

[132] M.M. Porzio, Quasilinear parabolic equations with singular potentials. In: de. Gier, J., Praeger, C. Tao, T.(eds) (2017) MATRIX Annals. MATRIX Book Series, vol **2**. Springer, Cham.

[133] H.B. Keller, D.S. Cohen, Some positone problems suggested by nonlinear heat generation, J. Math. Mech., **16**, 1361–1376 (1967).

[134] M. Ri, S. Huang, Q. Tian, Z.P., Ma: Existence of $W_0^1(\Omega)$ solution to nonlinear elliptic equation with singular natural growth term. AIMS Math. **5**(6), 5791–5800 (2020).

[135] A. Sbai and Y. El Hadfi, Degenerate elliptic problem with a singular nonlinearity, Complex Variables and Elliptic Equations (2021). doi: 10.1080/17476933.2021.2014458.

[136] A. Sbai and Y. El Hadfi, Regularizing effect of absorption terms in singular and degenerate elliptic problems. arXiv preprint arXiv:2008.03597 (2020).

[137] L. Schwartz, Théorie des distributions *à* valeurs vectorielles I, Ann. Inst. Fourier, Grenoble, **7**, 1-141 (1957).

[138] R. Souilah: Existence and regularity results for some elliptic equations with degenerate coercivity and singular quadratic lower-order terms. Medeterr. J.Math. **16**, 87 (2019)

[139] J. Simon,*Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl., **146**, 65-96 (1987).

[140] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre* à *coefficients discontinus.* Ann. Ist. Fourier (Grenoble), **15**, 189-258 (1965).

[141] C.A. Stuart: Existence and approximation of solutions of nonlinear elliptic equations. Math. Z., **147**, 53–63 (1976).

[142] J. Tyagi, An existence of positive solution to singular elliptic equations, Boll. Union. Mat. Ital., **7**, 45-53 (2014).

[143] A. Youssfi, A. Benkirane, Y. El Hadfi, On bounded solutions for nonlinear parabolic equations with degenerate coercivity, Mediterranean Journal of Mathematics, **13**(5), 3029-3040 (2016).

[144] Y. Wang, M. Wang, Solution to nonlinear elliptic equations with a gradient, Acta Mathematica Scientia, **35B**(5), 1023-1036 (2015).