N d'ordre : 401/22

Université Sultan Moulay Slimane
Faculté des Sciences et Techniques

Centre d'Études Doctorales: Sciences et Techniques

Formation Doctorale: Mathématiques et Physique Appliquées

THÈSE

Présentée par

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Pour l'obtention du grade de

DOCTEUR

Discipline: Mathématiques

Spécialité: Mathématiques

Contributions to Nonlinear Evolution Equations and Partial Functional Differential Equations : Quantitative and Qualitative Analysis

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Soutenue le Samedi 12 Février 2022 à 09h00 devant la commission d'examen :

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Dedication

This the

- My mother
- My father
- My sisters
- My wife
- My sons
- Profe

Publications and Talks

List of Publications

❶ Khalil Ezzinbi, Khalid Hilal, **Mohamed Ziat**. μ-Pseudo compact almost automorphic weak solutions for some partial functional differential inclusions. **Applicable Analysis**. (2021), 1-23.

❷ Brahim Es-sebbar, Khalil Ezzinbi, Samir Fatajou, **Mohamed Ziat**. Compact almost automorphic weak solutions for some monotone differential inclusions: Applications to parabolic and hyperbolic equations. **Journal of Mathematical Analysis and Applications**. 486(1), (2020), 123805.

❸ El Hadi Ait Dads, Brahim Es-sebbar, Khalil Ezzinbi, **Mohamed Ziat**. Behavior of bounded solutions for some almost periodic neutral partial functional differential equations. **Mathematical Methods in the Applied Sciences**. 40(7), (2017), 2377-2397.

❹ Khalil Ezzinbi, **Mohamed Ziat**. Positive μ-pseudo almost periodic solutions for some nonlinear infinite delay integral equations arising in epidemiology using Hilbert's projective metric. **Nonautonomous Dynamical Systems**. 5(1), (2018), 89-101.

❺ Khalil Ezzinbi, **Mohamed Ziat**. Nonlocal Integro-Differential Equations Without the Assumption of Equicontinuity on the Resolvent Operator in Banach Spaces. **Differential Equations and Dynamical Systems**. (2018), 1-19.

❻ El Hadi Ait Dads, Nadia Drisi, Khalil Ezzinbi, **Mohamed Ziat**. Exponential dichotomy and (μ,ν)-pseudo almost automorphic solutions for some ordinary differential equations. **Communications in Optimizations Theory**. 2016(2016), Article ID 6 (29 May 2016) pge 1-15.

List of Submitted Articles

❶ Khalil Ezzinbi, Khalid Hilal, **Mohamed Ziat**. Compact almost automorphic weak solutions of some ϕ-strongly maximal monotone partial differential inclusions with Stepanov almost automorphic forcing terms.

List of Talks

① "μ-Pseudo Compact Almost Automorphic Solutions for some Differential Inclusions" , co-authored with Khalid Hilal and Khalil Ezzinbi. Workshop International sur la théorie des Points Fixes et Applications. Faculty of Sciences and Technologies, Béni Mellal. Morocco, 05-06 Avril 2019.

② "Pseudo Almost Periodic Solutions for Some Differential Inclusions with Stepanov-Like Pseudo Almost Periodic Forcing Terms", co-authored with Khalid Hilal and Khalil Ezzinbi. The 6th International Congress of the Moroccan Society of Applied Mathematics (SM2A). November 07-09, 2019, Béni Mellal, Morocco.

③ "Compact Almost Automorphic Weak Solutions of Some ϕ-Strongly Monotone Partial Differential Inclusions with Stepanov Almost Automorphic Forcing Terms", co-authored with Khalid Hilal and Khalil Ezzinbi. 1^{st} Conference on Artificial Intelligence and Mathematics for Challenges in Applied Sciences and Engineering (CAIM-CASES2021). December 10-12, 2021, Agadir, Morocco.

Acknowledgments

First of all, I would like to thank **Allah** for giving me the will and the courage to be able to finish this thesis.

To be supervised by Professor **Khalid Hilal** (Sultan Moulay Slimane University) and Professor **Khalil Ezzinbi** (Cadi Ayyad University) is for me a great honor. I want to express my deep gratitude to them. I warmly thank them for their scientific and human guidance, their sympathy, their availability and their precious advice. It is thanks to them that this thesis saw light.

I would like to express my deep gratitude to Professor **Lalla Saadia Chadli** (Sultan Moulay Slimane University) for having accepted to chair the thesis jury.

I am very honored that Professors **Lalla Saadia Chadli** (Sultan Moulay Slimane University) and **Abdelhai El Azzouzi** (Sidi Mohamed Ben Abdellah University) have agreed to report this thesis. I am very grateful to them for their interest in my work.

Many thanks also go to Professor **Jalila El Ghordaf** (Sultan Moulay Slimane University) who has granted me the honor of kindly studying my thesis and of being examiner.

My sincere regards also to Professor **Said Melliani** who welcomed me to the Laboratory of Applied Mathematics and Scientific Computing (**Lmacs**) at the Faculty of Sciences and Technologies of Béni Mellal and, who has always encouraged and supported me without limit.

I would also like to thank Professors **M'hamed Elomari**, **Ahmed Kajouni** and **Abderrahmane Raji** (Sultan Moulay Slimane University) for accepting to be part of the thesis jury.

I warmly thank all the Professors of the laboratory **Lmacs** for their human quality and their sympathy. I am very touched by their warm welcome to the laboratory.

I would also like to thank the Head of department of Mathematics Professor **Mohamed Oukessou**.

Many thanks go to all my colleagues, doctors and doctoral students, from the laboratory **Lmacs** for the welcome and help during the research period.

I profoundly thank as well my colleague **NADIA OURCHANE** for all our discussions about Mathematics.

I also thank the entire administrative and technical team of the Faculty of Sciences and Technologies of Béni Mellal. Many thanks to all the Cleaning Staff, too.

I am also grateful to my parents for their moral and financial support. They have never ceased to support me in difficult times. I appreciate everything that they have done to shape my life and support my education. I also thank my sisters for their moral support.

I extremely thank my wife and my children who shared with me all the moments of anguish as well as those of joy during the preparation of my thesis.

Last but not least, thanks to all those I fail to mention here, who assisted me in any way over these years.

—MOHAMED ZIAT—

Notations & Abbreviations

a.e.	almost everywhere
$\emptyset$	empty set
$\mathbb{N}$	set of positive integers
$\mathbb{N}^*$	set of non-negative integers
$\mathbb{Z}$	group of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{R}^+$	set of non-negative real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{R}^N$	N-dimensional Euclidean space
$D(A)$	domain of an operator A
$G(A)$	graph of an operator A
$\rho(A)$	resolvent set of an operator A
X	a Banach space
$\mathcal{H}$	a real Hilbert space
X^*	topological dual of a space X
$BC(\mathbb{R}, E)$	space of all E-valued bounded continuous functions on $\mathbb{R}$
$C([a, b], E)$	space of all E-valued continuous functions on $[a, b]$
$C^1([a, b], E)$	space of all functions $f \in C([a, b], E)$ such that $f' \in C([a, b], E)$
$C^2([a, b], E)$	space of all functions $f \in C([a, b], E)$ such that $f', f'' \in C([a, b], E)$
$C([0, +\infty), E)$	space of all E-valued continuous functions on $[0, +\infty)$
$C^1([0, +\infty), E)$	space of all functions $f \in C([0, +\infty), E)$ such that $f' \in C([0, +\infty), E)$
$C_0(\Omega)$	space of all continuous, complex-valued functions on Ω vanishing at infinity
$C^\infty(\mathbb{R}, \mathbb{R})$	space of all infinitely differentiable functions from $\mathbb{R}$ into $\mathbb{R}$
$C_0^\infty(\mathbb{R}, \mathbb{R})$	space of all infinitely differentiable functions from $\mathbb{R}$ into $\mathbb{R}$ vanishing at infinity
$L^1([a, b], E)$	space of all E-valued integrable functions on $[a, b]$
$L^p([a, b], E)$	space of all E-valued p-integrable functions on $[a, b]$
$L^p_{loc}(\mathbb{R}, E)$	space of all p-locally integrable fonctions $f : \mathbb{R} \longrightarrow E$
$L^p(\Omega)$	space of all p-integrable functions on Ω
$\mathcal{L}(E, F)$	space of all bounded linear operators from E to F
$\mathcal{L}(E)$	space of all bounded linear operators from E to E
$B(x, r)$	open ball of center x and radius r
$\overline{A}, \mathring{A}, \partial A$	closure, interior, and boundary of a subset A
$\overline{co}(A)$	closed convex hull of a subset A
2^X	family of all subsets of a set X

$H^1(\Omega)$	Sobolev space of order 1 on Ω
$H^2(\Omega)$	Sobolev space of order 2 on Ω
$H_0^1(\Omega)$	space of functions in $H^1(\Omega)$ that vanish at $\partial\Omega$
∇u	Gradient operator
Δu	Laplacian operator
$\partial\varphi(x)$	subdifferential of a function φ at point x
$x_n \rightharpoonup x$	the sequence (x_n) converges weakly to x
$\mathbf{1}_A$	characteristic function of a subset $A \neq \emptyset$
$\mathcal{M}$	set of all positive measures μ on the Lebesgue σ-field of $\mathbb{R}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < +\infty$, for all $a, b \in \mathbb{R}$ $(a \leq b)$
$\chi(B)$	Hausdorff measure of noncompactness of a set B
$\alpha(B)$	Kuratowski measure of noncompactness of a set B
$\oplus$	direct sum of two spaces
$M_n(\mathbb{R})$	space of real square matrices of order n
$\langle ., . \rangle$	duality pairing between E and E^*

Abstract

In this thesis, we have dealt with the quantitative and qualitative analysis for some nonlinear evolution equations and partial functional differential equations :

• A nonlinear infinite delay integral equations arising in epidemiology was considered. Without any hypotheses of monotonicity on the forcing terms, we gave sufficient conditions which guarantee the existence and uniqueness of positive μ-pseudo almost periodic solutions. This allowed us to improve and extend some results in literature. Our working tools are based on Hilbert's projective metric and the contraction mapping principle.

• Using the variation of constant formula and the spectral decomposition of the phase space, we studied the nature of bounded solutions of some neutral partial functional differential equations with finite delay when the forcing term is almost periodic in a weaker sense. More specifically, we proved under a compactness condition that all bounded solutions on $\mathbb{R}$ are almost periodic when the forcing term is only Stepanov almost periodic. Therefore, we showed that the existence of a bounded solution on $\mathbb{R}^+$ is sufficient to guarantee the existence of an almost periodic solution.

• Moreover, using tools of exponential dichotomy and the contraction mapping principle, we gave sufficient conditions for the existence and uniqueness of (μ, ν)-pseudo almost automorphic solutions for some ordinary differential equations when the forcing term is (μ, ν)-pseudo almost automorphic.

• Also, we gave sufficient conditions for the existence of compact almost automorphic weak solutions for some differential inclusions governed by a maximal monotone operator. We showed that the existence of a uniformly continuous weak solution on $\mathbb{R}^+$ having a relatively compact range over $\mathbb{R}^+$ implies the existence of a compact almost automorphic weak solution when the forcing term is compact almost automorphic. We showed also the existence of a unique bounded weak solution which is globally attractive and compact almost automorphic when the multivalued operator is strongly maximal monotone. This last result was then extended to the μ-pseudo compact almost automorphic case.

• Finally, we used the measure of noncompactness, Mönch fixed point Theorem and the theory of resolvent operators to get the existence of mild solutions for some delay nonlocal integro-differential equations in Banach spaces. Hypotheses of our results do not impose equicontinuity on the resolvent operator. As a consequence, the results obtained in this part improve, extend and complement many other important results in the literature.

Résumé

Dans cette thèse, nous nous sommes intéressés à l'étude quantitative et qualitative de certaines équations d'évolution non linéaires et certaines équations aux dérivées partielles fonctionnelles :

• Des équations intégrales non linéaires à retard infini apparaissant en épidémiologie ont été considérées. Sans aucune hypothèse de monotonie sur les termes forcing, nous avons donné des conditions suffisantes qui garantissent l'existence et l'unicité des solutions positives μ-pseudo presque périodiques. Ceci nous a permis d'améliorer et d'étendre certains résultats de la littérature. Nos outils de travail sont basés sur la métrique projective de Hilbert et le principe de contraction de Banach.

• En utilisant une formule de la variation de la constante et la décomposition spectrale de l'espace de phase, nous avons étudié la nature de solutions bornées pour une classe d'équations aux dérivées partielles fonctionnelles à retard fini de type neutre lorsque le terme forcing est presque périodique au sens faible. Plus précisément, nous avons prouvé sous une condition de compacité que toutes les solutions bornées sur $\mathbb{R}$ sont presque périodiques lorsque le terme forcing est seulement presque périodique au sens de Stepanov. Puis, nous avons montré que l'existence d'une solution bornée sur $\mathbb{R}^+$ est suffisante pour assurer l'existence d'une solution presque périodique.

• De plus, en utilisant des outils de la dichotomie exponentielle et la théorie du point fixe, nous avons donné des conditions suffisantes pour l'existence et l'unicité de solutions (μ, ν)-pseudo presque automorphes pour quelques équations différentielles ordinaires lorsque le terme forcing est (μ, ν)-pseudo presque automorphe.

• Aussi, nous avons donné des conditions suffisantes pour l'existence de solutions faibles compactes presque automorphes pour certaines inclusions différentielles générées par un opérateur maximal monotone. Nous avons montré que l'existence d'une solution faible uniformément continue sur $\mathbb{R}^+$ à image relativement compacte dans $\mathbb{R}^+$ implique l'existence d'une solution faible compacte presque automorphe lorsque le terme forcing est compact presque automorphe. Lorsque l'opérateur est fortement maximal monotone, nous avons montré l'existence et l'unicité d'une solution faible bornée qui est globalement attractive et compacte presque automorphe. Ce dernier résultat a été ensuite étendu au cas μ-pseudo compact presque automorphe.

• Enfin, nous avons utilisé la mesure de non compacité, le théorème du point fixe de Mönch et la théorie des opérateurs résolvants de Grimmer pour obtenir l'existence de solutions

intégrales pour certaines équations intégro-différentielles à retard fini avec une condition non locale dans un espace de Banach. Les hypothèses de nos résultats n'imposent pas l'équicontinuité à l'opérateur résolvant. En conséquence, les résultats obtenus dans cette partie améliorent, étendent et complètent de nombreux autres résultats importants de la littérature.

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General Introduction

0.1 The Effects of Delays

In many models, it is assumed that the futur state of the system under consideration is independent of the past and is determined solely by the present. However, there are situations where the evolution of a process depends not only on its current state, but also on past states of the system. The delays appear because of the necessary time for the system to respond to certain evolution, or because a certain threshold (limit value) must be attained before the system can be activated. Indeed, the delay can appear as a transport delay, as a communication lag, as a phase of proliferation in the cell model, as a period of pregnancy in population dynamics, as a hatching period, as a slow replacement of food supplies, etc. The nature of the delay can be multiple. Discrete, when a time T preceding t is expressed in the form $t-r$, indicating that the state of the system at time t depends on its state at time $t-r$. Continuous, or distributed, when the evolution at time t depends on everything that has taken place between time t and an earlier time.

Numerous investigations have shown that temporal delays in a system have important influence on its qualitative behavior:

• *The delay can have an effect on the oscillatory and nonoscillatory behaviour of solutions*. In fact, let us consider the following differential equation

$$x'(t) + ax(t-r) = 0, \tag{0.1.1}$$

where $a, r \in (0, +\infty)$. It has been shown in [128] that all nontrivial solutions of (0.1.1) are oscillatory if

$$r > \frac{1}{ea}$$

and (0.1.1) has a nonoscillatory solution if

$$r \leq \frac{1}{ea}.$$

Another example of oscillations in a Lotka-Volterra system has been investigated by Gopalsamy [99].

• *The delay can destabilize the stability of steady state*. An example may be found in the book of

Hale [111] where the trivial solution of

$$3x'(t) = -x(t)$$

is globally asymptotically stable, but the trivial solution of

$$x'(t) + 2x'(t-r) = -x(t)$$

is unstable for any positive delay r. The destabilizing effect of delay can be seen in general scalar neutral differential equations with a single delay $r \geq 0$

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} x(t) + \sum_{k=0}^{n} b_k \frac{d^k}{dt^k} x(t-r) = 0. \tag{0.1.2}$$

It can be demonstrated ([126], chapter 3) that the trivial solution of equation (0.1.2) loses its stability for any $r > 0$ when $|b_n| > 0$.

• *The delay can have an effect on the number of periodic solutions.* Reference [78] contains a numerical simulations to the following equation

$$x'(t) = x(t)(1 - x(t-r)). \tag{0.1.3}$$

When $r = 1.2$, (0.1.3) has a unique positive periodic solution, which is the asymptotically stable steady state $x(t) = 1$ (Figure 1). But when $r = 1.62$, (0.1.3) has two positive periodic solutions: the steady state $x_1(t) = 1$ and the bifurcating nonconstant periodic solution x_2 (Figure 2).

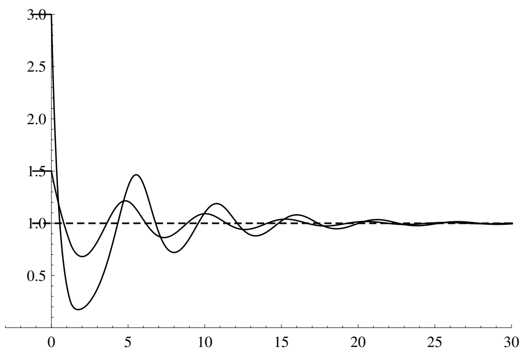

Figure 1: Periodic solution when r=1.2

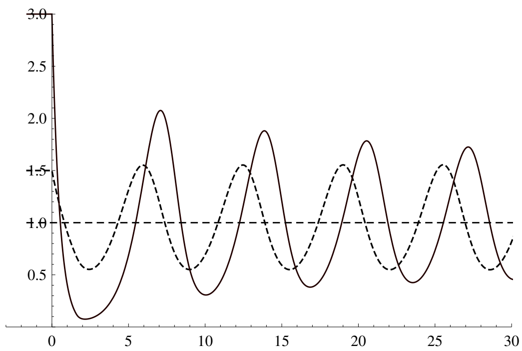

Figure 2: Periodic solutions when r=1.62

According to the previous effects, one can see that delays play an important role in real world problems. Thus, a real system should be modeled by differential equation with time delay. The abstract form of an equation with finite delay is the following [108, p. 28]

$$\begin{cases} x'(t) = F(t, x(t), x_t) \quad \text{for } \; t \geq 0 \\ x_0 = \varphi \in C([-r, 0], X), \end{cases}$$

where $F : \mathbb{R}^+ \times X \times C([-r,0],X) \longrightarrow X$ is a function satisfying some conditions, X is a Banach space, $C([-r,0],X)$ (the phase space) denotes the Banach space of continuous functions $x : [-r,0] \longrightarrow X$ with the supremum norm and x_t denotes the history function of $C([-r,0],X)$ of the state from the time $t-r$ up to the present time t, and is defined by

$$x_t(\theta) = x(t+\theta) \quad \text{for } \theta \in [-r,0].$$

An important example of delay equations is the following logistic equation

$$N'(t) = \gamma N(t)\left(1 - \frac{N(t-r)}{K}\right), \tag{0.1.4}$$

suggested by Hutchinson [120] to describe species populations struggling for a limited self-renewing food resources. Here, the delay r is the production time of food resources. The food resources at time t are determined by the population number at time $t-r$. The constant γ is related to the reproduction of species, and represents the difference between birth and death rates. Usually, γ is called the Maltus coefficient of linear growth. The constant K is the average population number, and is related to the ability of the environment to sustain the population. At the same time, Equation (0.1.4) can be used to study hatching periods, pregnancy, egg laying, etc.

The importance of delay systems has been highlighted by Volterra in [174] in the study of viscoelastic materials and in the interaction of species, by York [185] in epidemiology, by wang [175] in control theory and by Hetzer [118] in climatology. Biological motivations for modeling and theoretically studying delay systems can be found in the books of May [142] and Smith [163]. Concrete examples of delay models in mathematical biology are contained in the books of Cushing [66], MacDonald [137], and Copalsamy [101]. We also cite the book by Hale [108] which took the study of delay equations to a very advanced level.

0.2 Delay Equations Models

0.2.1 Model arising in epidemiology

As in [59], consider a fixed isolated population, about which the following assumptions were made.

1. The population has a constant size N.

2. The population is divided into two disjoint classes: susceptibles (those who can contract the disease) and infectives (those currently infectious).

The fractions of the total population in these classes are denoted by $S(t)$ and $I(t)$ respectively, so that

$$S(t) + I(t) = 1.$$

It is further assumed that the numbers in each class are large enough that $S(t)$, $I(t)$ can be treated as continuous variables.

3. The disease is not lethal and confers no immunity; that is, this is an $S-I-S$ model. An individual who moves from the susceptible to the infective class will return to the susceptible class.

4. There is no latent period; that is, there is no time delay between exposure and becoming infectious.

5. The length of time an individual is infective is a fixed positive constant τ.

6. The population is homogeneous and uniformly mixing. The contact rate, defined as the average number of effective contacts with other individuals per infective per time period, is a specified, periodic function of time. An effective contact is an interaction which results in infection of the other individual if he is susceptible. The contact rate is denoted here by $a(t)$.

From these definitions, we can infer that the total number of susceptibles who are infected in one time period at time t by one infective is $a(t)S(t)$, since $S(t)$ is the fraction of contacted individuals who are susceptible. The number of susceptibles who are infected in one time period at time t by $NI(t)$ infectives is $Na(t)S(t)I(t)$. This can be expressed in the form $Na(t)I(t)[1-I(t)]$. We shall set

$$f(t,I) = a(t)I(1-I),$$

so that $Nf(t, I(t))$ is the number of new infections per unit of time, at time t. The assumption that this number is proportional to the product $I(t)S(t)$ is the "law of mass action".

By assumptions 3 and 4, the number of individuals who recover per unit time at time t is exactly equal to the number of new infections at time $t-\tau$. Therefore, we obtain the delay differential equation

$$I'(t) = f(t, I(t)) - f(t-\tau, I(t-\tau)). \tag{0.2.1}$$

Equation (0.2.1) has an integrated form

$$I(t) = \int_{t-\tau}^{t} f(s, I(s))ds + c.$$

On the other hand, in the epidemic model, one should normally choose $c = 0$, since if there is no new infection $(f = 0)$ there should be no infectives. Then,

$$I(t) = \int_{t-\tau}^{t} f(s, I(s))ds. \tag{0.2.2}$$

The study of the asymptotic behavior of a variant of (0.2.2) will be one of our objectives in this thesis.

0.2.2 Model of Wu and Xia

Wu and Xia [180, 181] considered a ring of N mutually identical coupled lossless transmission line (LLTL) networks which are interconnected by a common resistor R. Each of which is a uniformly distributed lossless transmission line with the series inductance L_s and parallel capacitance C_s per unit length of the line. Taking an x-axis in the direction of the line, with two ends of the normalized line at $x = 0$ and $x = 1$ and denoting by $V_k(t) = v_k(1,t)$ the voltage at the boundary of each network k (See Figure 3 below), they obtained the following

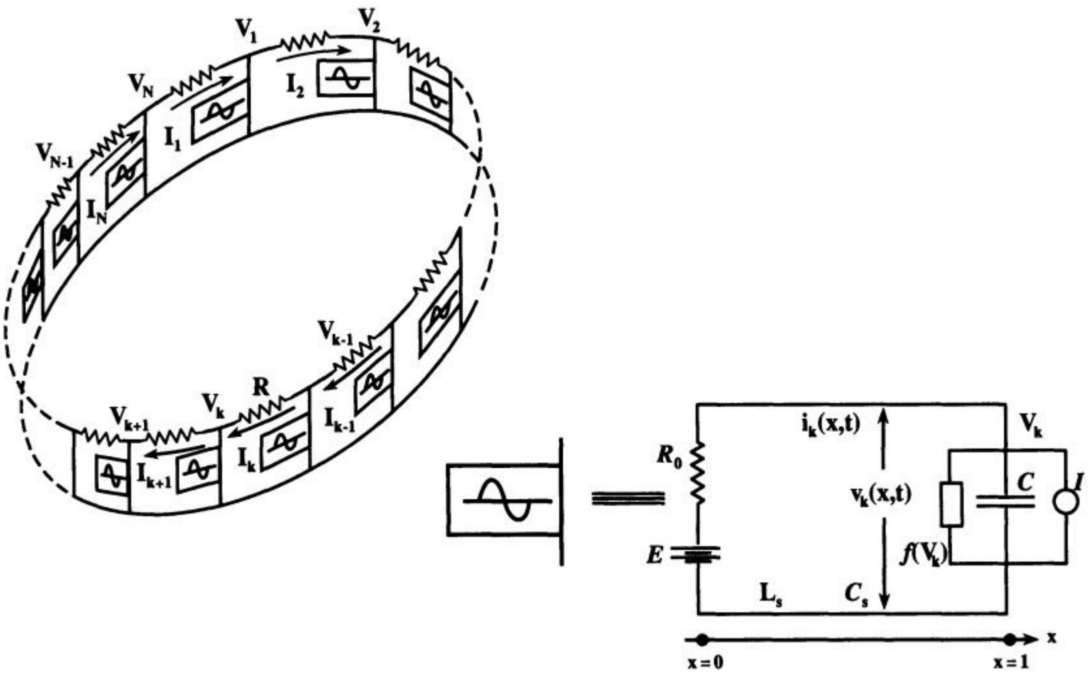

Figure 3: N mutually coupled lossless transmission line networks

equation for each $k = 1, 2, \dots, N$

$$\frac{d}{dt}\mathcal{D}V_t^k = -\frac{1}{ZC}V^k(t) - \frac{q}{ZC}V^k(t-r) - \frac{1}{C}f(\mathcal{D}V_t^k) + \frac{1}{RC}\mathcal{D}\left(V_t^{k+1} - 2V_t^k + V_t^{k-1}\right) \qquad (0.2.3)$$

where $Z = \sqrt{L_s \div C_s}$, $q = \dfrac{Z - R_0}{Z + R_0}$, $r = 2\sqrt{L_s C_s}$ is the delay due to the propagation velocity of the waves $\sigma = \dfrac{1}{\sqrt{L_s C_s}}$, $V_t^k \in C\left([-r, 0], \mathbb{R}\right)$ is defined by $V_t^k(\theta) = V_k(t+\theta)$ for each $\theta \in [-r, 0]$ and $\mathcal{D} : C([-r, 0], \mathbb{R}) \longrightarrow \mathbb{R}$ is the operator defined by

$$\mathcal{D}\varphi = \varphi(0) - q\varphi(-r) \quad \text{for } \varphi \in C\left([-r, 0], \mathbb{R}\right).$$

We now consider a continuous system of transmission lines on a circle by taking the parameter $\dfrac{1}{RC}$ in (0.2.3) as $\dfrac{K}{h^2}$, where h is the spacing between the lines and $K > 0$ is a constant independent of N which represents the diffusive interaction with neighboring oscillators. We obtain from the limit as $h \longrightarrow 0$ the following partial neutral functional differential equation on the unit circle S [109, 110]

$$\frac{\partial}{\partial t}\mathcal{D}v_t = K\frac{\partial^2}{\partial x^2}\mathcal{D}v_t + H(v_t), \tag{0.2.4}$$

where $x \in S$, K is a positive constant and H is a continuous function. The book of Wu [179] contains a detailed analysis of the results in [109, 110, 180, 181].
Equation (0.2.4) belongs to the following class of partial neutral functional differential equations

$$\frac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + F(t, u_t) \quad \text{for } t \in \mathbb{R}, \tag{0.2.5}$$

where $\mathcal{D} : C([-r, 0], X) \longrightarrow X$ is a continuous linear operator, X is a Banach space, $A : D(A) \subset X \longrightarrow X$ is a linear operator and $F : \mathbb{R} \times C([-r, 0], X) \longrightarrow X$. The operator A is usually an unbounded operator which generates a strongly continuous semigroup of bounded linear operator in X, which is equivalent by the Hille-Yosida Theorem to the fact that A is closed densely defined and satisfies the Hille-Yosida condition. Equation (0.2.5) has been extensively studied in the literatue. For instance, we refer to [5, 6, 8, 109, 110, 179, 180, 181]. Here, the operator A is not densely defined. The idea of studying partial neutral functional differential equations with operators satisfying only Hille-Yosida condition began with [4]. The authors proved results on existence, uniqueness, regularity, and continuous dependence. Many studies are devoted to the asymptotic behavior for (0.2.5) when $F(t, \varphi) = L(\varphi) + f(t)$ where $f : \mathbb{R} \longrightarrow X$ and $L : C([-r, 0], X) \longrightarrow X$ is a bounded linear operator, that is

$$\frac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t) \quad \text{for } t \in \mathbb{R}. \tag{0.2.6}$$

In [3, 7, 22], the authors obtained several results about periodic, almost periodic and pseudo almost automorphic solutions for (0.2.6) respectively. For partial functional differential equations (Equation (0.2.6) in the case $\mathcal{D}(\varphi) = \varphi(0)$), we mention the works [47, 48, 88, 192].

To the best of our knowledge, no one has considered the problem of almost periodic solutions to Equation (0.2.6) when the forcing term f is only Stepanov almost periodic. In this thesis, a *Bohr-Neugebauer* type result will be an aim for Equation (0.2.6) in the case where the operator A is not necessarily densely defined and satifies the Hille-Yosida condition.

0.2.3 A model in heat conduction in materials with memory

We consider a heat flow in a rigid body $\Omega \subset \mathbb{R}^N$ of a material with memory. Let $e(t, \xi)$ denote the density of internal energy, $w(t, \xi)$ the temperature, $q(t, \xi)$ the heat flux vector field in Ω, and $s(t, \xi)$ be the external heat supply at time t and position ξ. The balance law of energy then reads as:

$$e_t(t, \xi) + \text{div}q(t, \xi) = s(t, \xi) \quad \text{for } t \in \mathbb{R}, \xi \in \Omega. \tag{0.2.7}$$

If we consider only small variations of the absolute temperature and temperature gradient from equilibrium reference values, we may suppose that the internal energy $e : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ and the heat flux vector $q : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^N$ are described by the following constitutive equations:

$$e(t, \xi) = e_0 + c_0 w(t, \xi) \tag{0.2.8}$$

$$q(t, \xi) = -k_0 \nabla w(t, \xi) - \int_{-\infty}^{t} k(t-s) \nabla w(s, \xi) ds, \tag{0.2.9}$$

where $k : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is the heat flux memory kernel and the constants e_0, c_0 and k_0 denote the internal energy at equilibrium, the specific heat and the instantaneous conductivity, respectively (see [100, 144]). Taking for simplicity $k_0 = c_0 = 1$ and $f(t, \xi) = c_0^{-1} s(t, \xi)$, we get from (0.2.7), (0.2.8) and (0.2.9) that

$$w_t(t, \xi) = \Delta w(t, \xi) + \int_{-\infty}^{t} k(t-s) \Delta w(s, \xi) ds + f(t, \xi).$$

If we assume that the thermal history w of the body is known on $(-\infty, 0] \times \Omega$, and the temperature of the boundary $\partial\Omega$ is constant $(= 0)$ for all t, we are led to the following system:

$$\begin{cases} w_t(t, \xi) = \Delta w(t, \xi) + \int_{0}^{t} k(t-s) \Delta w(s, \xi) ds + f(t, \xi) \quad \text{for } (t, \xi) \in [0, b] \times \Omega \\ w(t, \xi) = 0 \quad \text{for } (t, \xi) \in [0, b] \times \partial\Omega. \end{cases} \tag{0.2.10}$$

Now, if we consider that the thermal history of the body Ω is known from the time $t - r$ (for some $r > 0$) up to the present time t, the temperature of $\partial\Omega$ is constant $(= 0)$ for all t, and the external heat supply depends on the this thermal history of the body, then, system (0.2.10) becomes the following integrodifferential equation with finite delay :

$$\begin{cases} w_t(t, \xi) = \Delta w(t, \xi) + \int_{0}^{t} k(t-s) \Delta w(s, \xi) ds + f(t, w(t-r, \xi)) \quad \text{for } (t, \xi) \in [0, b] \times \Omega \\ w(t, \xi) = \psi(t, \xi) \quad \text{for } (t, \xi) \in [-r, 0] \times \overline{\Omega}, \end{cases} \tag{0.2.11}$$

where ψ is a given initial function and r is a positive number. Now define

$$x(t)(\xi) = w(t, \xi)$$

$$Ax = \Delta x$$

$$\varphi(\theta)(\xi) = \psi(\theta, \xi) \quad \text{for } (\theta, \xi) \in [-r, 0] \times \Omega$$

$$(B(t)x)(\xi) = k(t) \Delta x(t)(\xi) \quad \text{for } (t, \xi) \in [0, b] \times \Omega.$$

Then, (0.2.11) takes the following abstract form :

$$\begin{cases} x'(t) = Ax(t) + \int_{0}^{t} B(t-s) x(s) ds + f(t, x_t) \quad \text{for } t \in [0, b] \\ x_0 = \varphi \in C([-r, 0], X), \end{cases} \tag{0.2.12}$$

where X is a Banach space. In some phenomena, nonlocal initial conditions are more realistic and practical than the classical ones in handling physical problems. As stressed in [52], the nonlocal condition $x_0 = \phi + g(x)$ has better effect in some physical problems than the classical condition $x_0 = \varphi$. In (0.2.12), if we consider $\varphi = \phi + g(x)$, we obtain the following nonlocal integro-differential equation with finite delay

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t, x_t) \quad \text{for } \ t \in [0, b] \\ x_0 = \phi + g(x) \in C([-r, 0], X). \end{cases} \tag{0.2.13}$$

Partial integro-differential equations play an important role in the mathematical modeling of many fields: physical phenomena, biological models, chemical kinetics and engineering sciences in which it is necessary to take into account the effect of the real world problems.

To the best of our knowledge, no work has reported on the existence of the mild solutions of the problem (0.2.13) by using the measure of noncompactness approach without assuming the equicontinuity (norm continuiuty) of the resolvent operator for the associated linear integral part. This problem will be one of our goals in this thesis.

0.3 Differential Inclusions

The general form of a first order differential inclusion is

$$x'(t) \in F(t, x(t)), \tag{0.3.1}$$

where $F(t, x)$ is a subset of $\mathbb{R}^n$ depending on t and x. The differential inclusions appear in many situations. Let us go through some examples that will motivate the appearances of (0.3.1) :

• *Motivation coming from the theory of differential equations with discontinuous right-hand side.* Consider the following Cauchy problem

$$x'(t) = \begin{cases} +1 \text{ if }, x(t) < 0 \\ -1 \text{ if } x(t) \geq 0, \end{cases} \tag{0.3.2}$$

subject to $x(0) = 0$. It is clear that (0.3.2) has no solutions in the usual sense. Indeed, if $x(t) < 0$ the solution has the form $x(t) = t + c_1$, and the form $x(t) = -t + c_2$ whenever $x(t) > 0$. As t increases, each solution reaches the point $x = 0$ and cannot leave it. The function $x(t) \equiv 0$ does not satisfy the equation, since for it $x'(t) = 0 \neq f(t, 0) = -1$. In general, the differential equation

$$x'(t) = f(t, x(t)), \tag{0.3.3}$$

with a discontinuous right-hand side f in the second argument may have no solutions. The first solution to deal with differential equation with a discontinuous right-hand side is due to Filippov [93]. He suggested to define solutions to (0.3.3), where f is discontinuous with respect to the second argument, as an absolutely continuous solutions to (0.3.1) where

$$F(t,x) = \bigcap_{\varepsilon>0} \overline{co} f\big(t, x(t) + \varepsilon B_n\big)$$

with B_n is the unit ball in $\mathbb{R}^n$ centered at zero. For more details about the theory of differential equations with discontinuous right-hand side, we refer to the books of Filippov [94] and Smirnov [162].

• *Motivation coming from differential inequalities.* Let $[a,b] \subset \mathbb{R}$ and f be a numerical function. Consider in $\mathbb{R}$ the following differential inequality

$$\begin{cases} u'(t) + u(t) = f(t) & \text{for } a < u(t) < b, \\ u'(t) + u(t) \geq f(t) & \text{for } u(t) = a, \\ u'(t) + u(t) \leq f(t) & \text{for } u(t) = b. \end{cases} \tag{0.3.4}$$

Let B be the multivalued function defined by

$$B(x) = \begin{cases} 0 \ \ \text{for } x \in]a,b[, \\ [0,+\infty) \ \ \text{for } x = a, \\ (-\infty, 0] \ \ \text{for } x = b, \\ \emptyset \ \ \text{for } x \notin [a,b]. \end{cases}$$

Define $F : \mathbb{R} \times \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ by

$$F(t,x) = f(t) - x + B(x).$$

Then, the differential inequality (0.3.4) becomes the following differential inclusion

$$x'(t) \in F(t, x(t)).$$

• *Motivation coming from Control Theory.* Let $U(t)$ be a given set depending on t. Consider the following control problem

$$x'(t) = f(t, x(t), u(t)), \ \ u(t) \in U(t). \tag{0.3.5}$$

Then, (0.3.5) can be transformed into (0.3.1) where

$$F(t,x) = f(t,x,U(t)) = \{f(t,x,u) : u \in U(t)\}.$$

For more details about the correspondence between control systems and differential inclusions, we refer to the books of Filippov [92] and Wazewski [176].

A very important class of differential inclusions are the ones governed by a maximal monotone operator, that is of the form

$$u'(t) + \mathcal{A}u(t) \ni f(t) \quad \text{for } t \in \mathbb{R}, \tag{0.3.6}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is a maximal monotone operator on a real Hilbert space $\mathcal{H}$ and $f : \mathbb{R} \longrightarrow \mathcal{H}$ is the forcing term. Differential inclusions of the form (0.3.6) arise in a variety of areas of biological, physical, economical and engineering applications. One of the differential inclusions that take the abstract form (0.3.6) is the following

$$\frac{d^2u(t)}{dt^2} + \omega^2 u(t) + \beta\left(\frac{du(t)}{dt}\right) \ni 0, \quad (0.3.7)$$

where $\beta : \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ is a maximal monotone operator. (0.3.7) represents the evolution of the abscissa $u(t)$ of a material point of unit mass on a line, subjected to two forces :
∘ An elastic restoring force $-\omega^2 u(t)$ proportional to the abscissa.
∘ A braking force $-\beta(\frac{du(t)}{dt})$ that increases with the speed of movement of the material point.
The case of a multivalued graph is not physically absurd, it can represent for example friction on a solid surface. For more examples of differential inclusions of the form (0.3.6), we refer to [117].

Several works are devoted to the study of (0.3.6). For instance, we refer to [27, 30, 51, 112, 114, 115, 178]. The Cauchy problem related to (0.3.6) was treated by Brézis in [51]. Baillon and Haraux [27] and Brézis [51] studied the existence of the periodic solutions of (0.3.6). Biroli [30] initiated the study for almost periodic solutions of (0.3.6). Following Biroli's work, Haraux [112, 114, 115] and wexler [178] have given important contributions to the question of almost periodic solutions to (0.3.6). The results in [27, 30, 51, 112, 114, 115, 178] are established under hypotheses on $\mathcal{A}$ (maximal monotone, strongly maximal monotone, subdiffernetial of a proper, convex and lower semicontinuous function), on f (periodic, almost periodic, Stepanov almost periodic) or on $\mathcal{H}$ (finite or infinite dimensional).

To the best of our knowledge, no author has considered the problem of compact almost automorphic weak solutions to (0.3.6) when the forcing term f is compact almost automorphic. This will be one of our aims in this thesis.

0.4 Periodic and Almost Periodic Oscillations

Some times it is not clear how a system can have periodicity properties; however, such properties exist. For instance, the retarted logistic equation (0.1.4) for $\gamma r > \frac{\pi}{2}$ is a good example to see this. It has been shown in [167] that (0.1.4) has a nonconstant periodic solution (See Figure 4). Although, the existence of periodic solutions in the logistic model (0.1.4) is caused by the delay r, we can have the same situation for systems without delay. The periodicity can appear also in electrical circuits: The charge q on the capacitor in LC circuit satisfies the following equation

$$q''(t) + \frac{1}{LC} q(t) = 0. \quad (0.4.1)$$

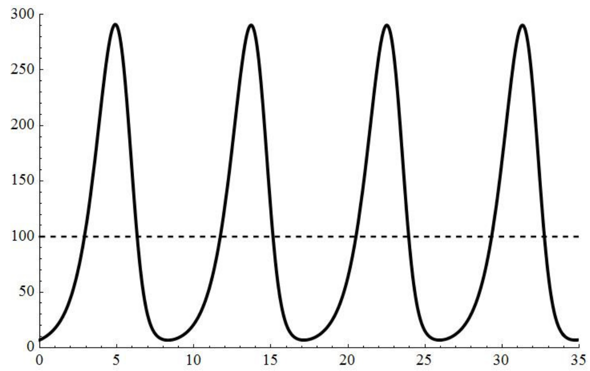

Figure 4: Periodic solution for the logistic equation (0.1.4) ($N_0 \approx 7$, K=100, $\gamma = 1$ and r=2)

Denote $\omega^2 = \dfrac{1}{LC}$. Then, Equation (0.4.1) becomes

$$q''(t) + \omega^2 q(t) = 0. \tag{0.4.2}$$

The general solution of the homogeneous second order differential equation (0.4.2) is given by

$$q(t) = q_0 \cos(\omega t + \varphi), \tag{0.4.3}$$

where q_0 and φ depend on the initial conditions. Formula (0.4.3) shows that the charge on the capacitor behaves periodically. Similar oscillations can be found also in mechanic. Figure 5 below represents a weight m on the end of a vertical spring which is fixed to a solid beam. Let k be the positive spring constant. According to Newton's law of dynamics, the

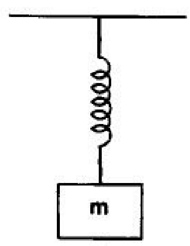

Figure 5: Simple harmonic motion

position $x(t)$ of the mass m at time t verifies the following second order ordinary differential equation

$$mx''(t) + kx(t) = 0, \tag{0.4.4}$$

which as (0.4.2) has a periodic solution. Equation (0.4.4) describes the motion of the material point without friction and free of any external forces. When other forces are involved, we are led to deal with the so-called forced oscillations. Let $f(t)$ be an external force acting on the mass m at the moment t. Then, (0.4.4) becomes

$$x''(t) + \omega^2 x(t) = \frac{1}{m} f(t), \tag{0.4.5}$$

where $\omega^2 = \dfrac{k}{m}$. Suppose that

$$f(t) = A\sin(\varpi t) \quad \text{and} \quad \varpi \neq \omega.$$

The solution of (0.4.5) with the initial conditions $x(0) = 0$ and $x'(0) = 0$ is given by

$$x(t) = \frac{A\varpi}{m(\varpi^2 - \omega^2)}\left(\frac{1}{\omega}\sin(\omega t) - \frac{1}{\varpi}\sin(\varpi t)\right). \tag{0.4.6}$$

Both terms on the right hand side of (0.4.6) represents periodic solutions. Their sum describes an oscillatory dynamics. This oscillatory dynamics is periodic if $\dfrac{\varpi}{\omega} \in \mathbb{Q}$. If $\dfrac{\varpi}{\omega} \notin \mathbb{Q}$, this oscillatory dynamics is not periodic; the type of oscillations described by the solution $x(t)$ in (0.4.6) is called an almost periodic oscillation.
From the discussion above, the occurrence of almost periodic oscillations is actually much more common than for periodic ones. Almost periodic solutions appear in evolution equations whenever there is a superposition of periodic solutions with incommensurable periods. For instance, in celestial mechanics, even if we consider that the movement of each planet is periodic, the solar system as a whole is almost periodic because there is of course no reason that the periods of the different planets are in rational relation. Figure 6 below gives an example of a numerical almost periodic function that is not periodic. The theory of

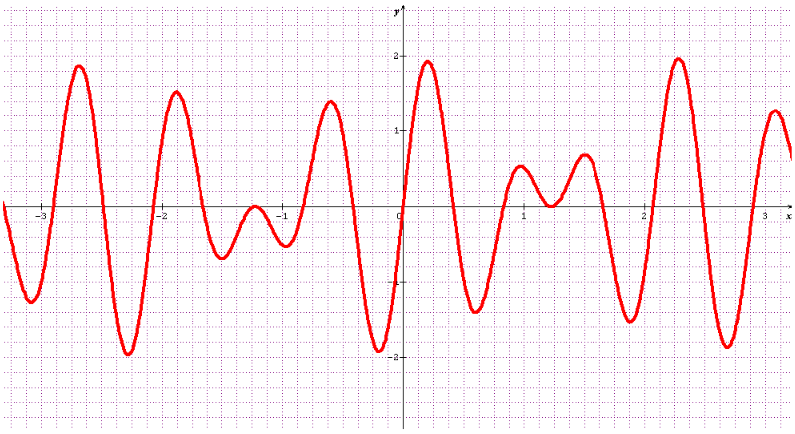

Figure 6: Graph of the almost periodic function $f(t) = \sin(\sqrt{2}\pi t) + \sin(2\pi t)$

almost periodic functions has been developed vigorously for a century. It was initiated between 1924 and 1926 by Danish mathematician Harald Bohr (1887-1951) [40]. Bohr's work was preceded by the important investigations of P. Bohr [39] and E. Esclangon [80]. This theory was developed by others, notably by Salomon Bochner (1899-1982) [32, 33] who gave two other versions of the definition of almost periodic functions equivalent to that given by Bohr, but more mathematically manageable. In the fifties of the last century, this study was taken up by Stepanov [165, 166] who defined the notion of Stepanov almost periodic function. Some fundamental properties of almost periodic functions are not verified by Stepanov almost periodic functions, for example the properties of continuity and the boundedness, which makes the search of almost periodic solutions for dynamical systems more difficult when the forcing term is Stepanov almost periodic.

The concept of almost periodicity has been generalized also into different directions: Asymptotically almost periodicity [98], pseudo almost periodicity [188, 189, 190], weighted pseudo almost periodicity [69], μ-pseudo almost periodicity [44], Stepanov pseudo almost periodicity [70] and Stepanov weighted pseudo almost periodicity [73].

The existence of almost periodic solutions for differential equations has been extensively studied in the last 50 years. Several books are devoted to cover this topics. For example, we indicate the books of Amerio and Prouse [24], Corduneannu [64], Fink [95] and Zaidman [187].

0.5 Almost Automorphic Oscillations

In his interesting book [95], Fink asked the following question : *Can an almost periodic equation have an almost automorphic solution that is not almost periodic* ? This question was discussed in [123, 155, 161]. Especially, Johnson [123] gave an affirmative answer. He gave an example of an almost periodic ordinary differential equation of the form $(*)\ u(t) = A(t)u(t) + B(t)$ with an almost automorphic solution which is not almost periodic. In fact, he constructed a nice example of a two dimensional almost periodic system whose projective flow has an almost automorphic minimal subset which is not almost periodic. He also proved that some equation in the hull of (*) has an almost automorphic solution which is not almost periodic. So, almost automorphy is a fundamental notion in the dynamical study of almost periodic differential equations. Figure 7 below gives a numerical almost automorphic function which is not almost periodic. Other examples can be found in [172]. Bochner [35, 36, 37, 38] intro-

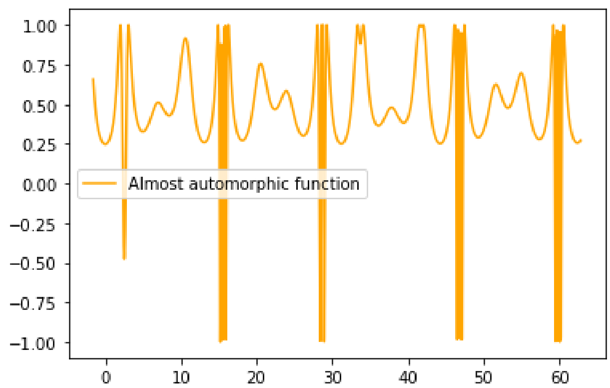

Figure 7: Graph of the almost automorphic function $f(t) = \sin\left(1 \div (2 + \sin(t) + \sin(\sqrt{2}t))\right)$

duced the concept of almost automorphy as a generalization of almost periodicity [40, 36].

This concept was then investigated in depth by Veech [172, 173] and many other authors. The almost automorphy is a weak version of almost periodicity, so many results and methods in the theory of almost periodicity are complicated in the almost automorphic framework. The name almost automorphic was given by Bochner himself because he encountered this kind of functions first in his work on differential geometry [34]. He also observed that almost automorphic functions can sometimes be used in obtaining simpler proof of certain results concerning almost periodic functions by first proving these results for almost automorphic functions.

The concept of almost automorphy has undergone many interesting generalizations, such as, Stepanov almost automorphy [150], asymptotically almost automorphy [147], pseudo almost automorphy [133, 182], weighted pseudo almost automorphy [46] and μ-pseudo almost automorphy [45].

0.6 Organization of the Thesis

This thesis is structured in 7 chapters. It is arranged as follows :

Chapter 1 : **Preliminaries for Linear and Nonlinear Functional Analysis**. In this chapter, we collect some background materials required throughout this thesis. These materials include some functional spaces, some fixed point theorems, measure of noncompactness, maximal monotone operators, C_0-semigroups and resolvent operators of Grimmer.

Chapter 2 : **Positive μ-Pseudo Almost Periodic Solutions for some Nonlinear Infinite Delay Integral Equations Arising in Epidemiology Using Hilbert's Projective Metric.** This Chapter is devoted to the following infinite delay integral equation which is a variant of (0.2.2)

$$x(t) = \alpha(t)x(t-\beta) + \int_{-\infty}^{t} a(t, t-s)f(s, x(s))ds + h(t, x(t)) \quad \text{for } t \in \mathbb{R}. \tag{0.6.1}$$

The first goal of this chapter is to improve and generalize the work [75] when the authors suppose that $f(t,x) = \sum_{i=1}^{n} f_i(t,x)g_i(t,x)$, $f_i(t,\cdot)$ is increasing in $\mathbb{R}^+$, $g_i(t,\cdot)$ is decreasing in $\mathbb{R}^+$ and $h(t,\cdot)$ is decreasing in $\mathbb{R}^+$. The main tool of [75] is a fixed point theorem in partially ordered Banach spaces.

Here, we study Equation (0.6.1) without any assumption of monotonicity on the functions f and h. We suppose that f and h are μ-pseudo almost periodic and we prove that (0.6.1) has a unique μ-pseudo almost periodic solution with a positive infimum.

To drop the hypotheses of monotonicity of the functions f and h, we use Hilbert's projective metric combined with the contraction mapping principle. The second goal of this chapter is to study the existence and uniqueness of μ-pseudo almost periodic solutions of the following integral equation

$$x(t) = \alpha(t)x(t-\beta) + \int_{t-\sigma(t)}^{t} f(s, x(s))ds + h(t, x(t)) \quad \text{for } t \in \mathbb{R}$$

which can be transformed into Equation (0.6.1). The results of this chapter are published in [91].

Chapter 3 : **Behavior of Bounded Solutions for Some Almost Periodic Neutral Partial Functional Differential Equations.** In this chapter, object of the paper [10], we prove a new result about the integration of Stepanov almost periodic functions. This result is then used together with a reduction principle to investigate the nature of bounded integral solutions of the following partial neutral functional differential equation $\frac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t)$ for $t \in \mathbb{R}$. More specifically, we prove under some assumptions, that if f is Stepanov almost periodic, then all bounded integral solutions on $\mathbb{R}$ are almost periodic. This is a result of Bohr Neugebauer type.

Chapter 4 : **Exponential Dichotomy and (μ, ν)-Pseudo Almost Automorphic Solutions for Some Ordinary Differential Equations.** This chapter deals with the following ordinary differential equation:

$$x'(t) = A(t)x(t) + f(t, x(t)) \quad \text{for } t \in \mathbb{R}, \tag{0.6.2}$$

where $A(t)$ is $n \times n$ matrix and the input $f : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies the following condition

$$\|f(t, x) - f(t, y)\| \leq l(t)h(\|x - y\|), \tag{0.6.3}$$

where $l \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ $(1 < p < +\infty)$ and $h : [0, +\infty) \longrightarrow [0, +\infty)$ is a nondecreasing continuous function on $[0, +\infty)$, verifying $h(r) < r$ for every $r > 0$ and $h(0) = 0$.

Our contribution is to study the existence and uniqueness of (μ, ν)-pseudo almost automorphic solutions of Equation (0.6.2) under assumption that f is (μ, ν)-pseudo almost automorphic satisfying (0.6.3). The obtained result is proved by using the exponential dichotomy and a generalized Banach contraction principle. The result presented in this chapter has been published in [9].

Chapter 5 : **Compact Almost Automorphic Weak Solutions for Some Monotone Differential Inclusions: Applications to Parabolic and Hyperbolic Equations.** We study the existence of compact almost automorphic weak solutions for the differential inclusion $u'(t) + \mathcal{A}u(t) \ni f(t)$ for $t \in \mathbb{R}$, where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is *maximal monotone* and the forcing term f is compact almost automorphic. We prove that the existence of a uniformly continuous weak solution on $\mathbb{R}^+$ having a relatively compact range over $\mathbb{R}^+$ implies the existence of a compact almost automorphic weak solution. For that goal, we use Amerio's principle [26]. We prove also the existence, uniqueness and global attractivity of a compact almost automorphic weak solution where $\mathcal{A}$ is *strongly maximal monotone*. These theoretical studies have been crowned by applications having their origins in population dynamics. We have shown the existence of compact almost automorphic solutions for parabolic and hyperbolic equations. The results presented in this chapter have been published in [83].

Chapter 6 : **μ-Pseudo Compact Almost Automorphic (Periodic) Weak Solutions For Some Partial Functional Differential Inclusions.** This chapter, object of the article [87], is a continuation of chapter 5. Its aim is to investigate the existence and uniqueness of μ-pseudo almost periodic (resp. μ-pseudo compact almost automorphic) weak solutions for the following partial functional differential inclusion :

$$x'(t) + \mathcal{A}x(t) \ni f(t, x_t) \quad \text{for } t \in \mathbb{R},$$

where $\mathcal{A} : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is a strongly maximal monotone operator on a real Hilbert space $\mathcal{H}$, $f : \mathbb{R} \times \mathcal{C} \longrightarrow \mathcal{H}$ is a Stepanov μ-pseudo almost periodic (resp. μ-pseudo compact almost automorphic) function of class r, $\mathcal{C} = C([-r, 0], \mathcal{H})$ is the Banach space of all continuous functions from $[-r, 0]$ to $\mathcal{H}$ and the history function x_t is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. Two examples are given for parabolic and subdifferential systems.

Chapter 7 : **Nonlocal Integro-Differential Equations Without the Assumption of Equicontinuity on the Resolvent Operator in Banach Spaces**. The results of this chapter are contents of the paper [90]. Here, we consider the following nonlocal integro-differential problem with delay:

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds + f(t, x_t) \quad \text{for } \ t \in [0, b] \\ x_0 = \phi + g(x) \in C([-r, 0], X), \end{cases} \tag{0.6.4}$$

where $A : D(A) \subset X \longrightarrow X$ is the infinitesimal generator of a C_0-semigroup; X is a Banach space; $B : D(B) \subset X \longrightarrow X$, $D(A) \subset D(B)$, is a closed linear operator; $f : [0, b] \times C([-r, 0], X) \longrightarrow X$ and $g : C([0, b], X) \longrightarrow C([-r, 0], X)$ are given functions; for every $t \in [0, a]$ the map x_t is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$; $\phi \in C([-r, 0], X)$ and $b, r > 0$.

The focus is on the existence of mild solutions to the problem (0.6.4), where the definition of mild solutions is given in term of the resolvent operator. The main existence theorems are based on the Mönch fixed point Theorem and they are obtained under classical assumptions : f is Carathéodory and g is completely continuous. The other relevant result contained in this chapter is a generalization of Lemma 4.2.1 in the book [124].

The novelty in our contribution is that the proof of our main results is based on the use of a measure of noncompactness without assumption of equicontinuity on the resolvent operator.

Some perspectives and comments on our results are arranged at the end of the manuscript.

Chapter 1

Preliminaries for Linear and Nonlinear Functional Analysis

Contents

The purpose of this chapter is to collect some background materials required throughout this thesis.

1.1 Some Functional Spaces

Throughout this thesis, we denote by $\mathbb{N}$ the set of positive integer numbers, by $\mathbb{Z}$ the set of integer numbers, by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the set of nonnegative real numbers. X denotes a Banach space endowed with its norm $\|\cdot\|$ (sometimes $\|\cdot\|_X$) and X^* is its dual. The space $BC(\mathbb{R}, X)$ stands for all X-valued bounded continuous functions on $\mathbb{R}$ and $C([-r, 0], X)$ stands for the space of all X-valued continuous functions on $[-r, 0]$. By $L^p_{loc}(\mathbb{R}, X)$, we denote the space of all p-locally intgrable fonctions.

1.1.1 Almost periodic functions

This subsection is devoted to the notions of almost periodicity, weakly almost periodicity and Stepanov almost periodicity. More details about these topics can be found in [24, 32, 33, 36, 40, 64, 71, 95, ?, 156, 165, 166, 186].

Definition 1.1.1 [95] A continuous function $f : \mathbb{R} \longrightarrow X$ is almost periodic (in Bohr sense) if for every $\varepsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that

$$\sup_{t\in\mathbb{R}} \|f(t+\tau) - f(t)\|_X < \varepsilon.$$

We denote by $AP(\mathbb{R}, X)$ the space of almost periodic X-valued functions.

Definition 1.1.2 [95] A continuous function $f : \mathbb{R}^+ \longrightarrow X$ is asymptotically almost periodic if there exists an almost periodic function $g : \mathbb{R} \longrightarrow X$ such that $\lim_{t\to+\infty} \|f(t) - g(t)\|_X = 0$.

Example 1.1.3 The function defined on $\mathbb{R}$ by the following

$$f(t) = \sin(\sqrt{2}t) + \sin(t)$$

is almost periodic.

Theorem 1.1.4 [95] $AP(\mathbb{R}, X)$ *is a Banach space with the supremum norm given by*

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|_X.$$

Theorem 1.1.5 [95] *Every almost periodic function is uniformly continuous on* $\mathbb{R}$*.*

Theorem 1.1.6 [24] *Every almost periodic function has a relatively compact range.*

A useful characterization of almost periodic functions was given by Bochner [36].

Theorem 1.1.7 [36] *A continuous function* $f : \mathbb{R} \longrightarrow X$ *is almost periodic if and only if for every sequence of real numbers* $(s_n)_n$ *there exist a subsequence* $(s'_n)_n \subset (s_n)_n$ *and a function* $\tilde{f}$*, such that*

$$f(t + s'_n) \longrightarrow \tilde{f}(t)$$

uniformly on $\mathbb{R}$ *as* $n \longrightarrow +\infty$*.*

Definition 1.1.8 [186] A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is almost periodic in $t \in \mathbb{R}$ uniformly with respect to $x \in X$ if for each compact set K in X, for all $\varepsilon > 0$, there exists $l > 0$, such that for any $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + l]$ with

$$\sup_{t\in\mathbb{R}} \sup_{x\in K} \|f(t+\tau, x) - f(t, x)\|_Y < \varepsilon.$$

We denote by $APU(\mathbb{R} \times X, Y)$ the set of such functions.

Definition 1.1.9 [24] A weakly continuous function $f : \mathbb{R} \longrightarrow X$ is weakly almost periodic if the scalar function $t \longmapsto \langle \varphi, f(t) \rangle$ is almost periodic for each $\varphi \in X^*$ where X^* is the dual space of X.

An almost periodic function is also weakly almost periodic. The following proposition establishes a necessary and sufficient condition for the converse to hold.

Theorem 1.1.10 [24, Theorem X, page 45] *Let $f : \mathbb{R} \longrightarrow X$ be a weakly almost periodic function. Then, f is almost periodic if and only if its range is relatively compact.*

Definition 1.1.11 [24] A sequence $(x_n)_n$ in X is scalarly convergent if the scalar sequence $(\langle \varphi, x_n \rangle)_n$ is convergent for all $\varphi \in X^*$. If $(x_n)_n$ is scalarly convergent and if in addition there exists $x \in X$ such that for all $\varphi \in X^*$, $\langle \varphi, x_n \rangle$ converges to $\langle \varphi, x \rangle$, then the sequence $(x_n)_n$ will be said to be weakly convergent, and the value x will be called the weak limit of $(x_n)_n$.

Definition 1.1.12 [24] If the space X is such that every scalarly convergent sequence is also weakly convergent, then X is said to be weakly sequentially complete, or semicomplete.

Example 1.1.13 Reflexive spaces are weakly sequentially complete.

The following Bochner type characterization for weak almost periodicity is essential.

Theorem 1.1.14 [24, Theorem IX, page 45] *Assume that X is weakly sequentially complete. Let $f : \mathbb{R} \longrightarrow X$ be a weakly continuous function. Then, f is weakly almost periodic if and only if for every sequence of real numbers $(s_n)_n$ there exist a subsequence $(s'_n)_n \subset (s_n)_n$ and a function $\tilde{f}$, such that*

$$f(t + s'_n) \longrightarrow \tilde{f}(t)$$

uniformly on $\mathbb{R}$ in the weak sense as $n \longrightarrow +\infty$.

Let us now turn to the notion of almost periodicity in the sense of Stepanov.

Definition 1.1.15 [64] A function $f \in L^p_{loc}(\mathbb{R}, X)$ is Stepanov almost periodic for some $p \geq 1$ (or S^p-almost periodic) if for every $\varepsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that

$$\sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(s + \tau) - f(s)\|_X^p ds \right)^{\frac{1}{p}} < \varepsilon.$$

We denote by $SAP^p(\mathbb{R}, X)$ the space of S^p-almost periodic X-valued functions.
Bochner pointed out that the notion of Stepanov almost periodicity can be reduced to Bohr almost periodicity using the following construction.

Definition 1.1.16 [24, 33, 156] The Bochner transform f^b of a function $f \in L^p_{loc}(\mathbb{R}, X)$ is the function $f^b : \mathbb{R} \longrightarrow L^p([0,1], X)$ defined for each $t \in \mathbb{R}$ by

$$(f^b(t))(s) = f(t+s) \quad \text{for } s \in [0,1].$$

Proposition 1.1.17 [24, 33, 156] *A function $f \in L^p_{loc}(\mathbb{R}, X)$ is S^p-almost periodic if and only if the function $f^b : \mathbb{R} \longrightarrow L^p([0,1], X)$ is almost periodic.*

If f is an S^p-almost periodic function, then from the Bochner characterization in Theorem 1.1.7, one can see that its Bochner transform $f^b : \mathbb{R} \longrightarrow L^p([0,1], X)$ is bounded on $\mathbb{R}$; that is, f satisfies

$$\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s)\|_X^p ds \right)^{\frac{1}{p}} < +\infty.$$

This leads to the following definition.

Definition 1.1.18 [24, 33, 156] Let $p \geq 1$. The space $BS^p(\mathbb{R}, X)$ consists of all functions $f \in L^p_{loc}(\mathbb{R}, X)$ such that $f^b : \mathbb{R} \longrightarrow L^p([0,1], X)$ is bounded, that is, $\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s)\|_X^p ds \right)^{\frac{1}{p}} < +\infty$. It is a normed space when equipped with the following norm

$$\|f\|_{BS^p} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s)\|_X^p ds \right)^{\frac{1}{p}}.$$

Proposition 1.1.19 $(BS^p(\mathbb{R}, X), \|\cdot\|_{BS^p})$ *is a Banach space.*

Example 1.1.20 [168] Let $H \in C_0^\infty(\mathbb{R}, \mathbb{R})$ with support in $(-\frac{1}{2}, \frac{1}{2})$ such that $H \geq 0$, $H(0) = 1$ and $\int_{\frac{1}{2}}^{\frac{1}{2}} H(t)dt = 1$. Let β_n be defined on $\mathbb{R}$ by:

$$\beta_n(t) = \sum_{k\in P_n} H(n^2(t-k))$$

with $P_n = 3^n(2\mathbb{Z} + 1)$. Let f be given by:

$$f(t) = \sum_{n\geq 1} \beta_n(t) \quad \text{for } t \in \mathbb{R}.$$

Then, $f \in C^\infty(\mathbb{R}, \mathbb{R})$ but $f \notin AP(\mathbb{R}, \mathbb{R})$ since it is not bounded in $\mathbb{R}$. However, $f \in SAP^1(\mathbb{R}, \mathbb{R})$.

Remark 1.1.21 For every $p \geq 1$, we have $AP(\mathbb{R}, X) \subsetneq SAP^p(\mathbb{R}, X)$.

Proposition 1.1.22 *A function $f \in L^p_{loc}(\mathbb{R}, X)$ is S^p-almost periodic if and only if for every sequence $(s_n)_n \subset \mathbb{R}$ there exist a subsequence $(s'_n)_n \subset (s_n)_n$ and a function $g \in L^p_{loc}(\mathbb{R}, X)$, such that*

$$\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s+s'_n) - g(s)\|_X^p ds \right)^{\frac{1}{p}} \longrightarrow 0, \tag{1.1.1}$$

as $n \longrightarrow +\infty$.

Proof. If f satisfies (1.1.1), then from

$$\begin{aligned}\sup_{t\in\mathbb{R}}\|f^b(t+s'_n)-g^b(t)\|_{L^p([0,1],X)} &= \sup_{t\in\mathbb{R}}\left(\int_0^1\|f^b(t+s'_n)(s)-g^b(t)(s)\|_X^p ds\right)^{\frac{1}{p}}\\ &= \sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|f(s+s'_n)-g(s)\|_X^p ds\right)^{\frac{1}{p}},\end{aligned}$$

it follows using Theorem 1.1.7 that f^b is almost periodic and hence f is S^p-almost periodic. Now assume that f is S^p-almost periodic, that is f^b is almost periodic. Let $(s_n)_n$ be a real sequence. Then there exist a subsequence $(s'_n)_n \subset (s_n)_n$ and a function $G : \mathbb{R} \longmapsto L^p([0,1],X)$ such that

$$\sup_{t\in\mathbb{R}}\|f^b(t+s'_n)-G(t)\|_{L^p([0,1],X)} \longrightarrow 0,$$

as $n \longrightarrow +\infty$. It follows that

$$\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|f(s+s'_n)-f(s+s'_m)\|_X^p ds\right)^{\frac{1}{p}} = \sup_{t\in\mathbb{R}}\|f^b(t+s'_n)-f^b(t+s'_m)\|_{L^p([0,1],X)} \longrightarrow 0$$

as $n, m \longrightarrow +\infty$. That is, $(f(.+s'_n))_n$ is a Cauchy sequence in the Banach space $BS^p(\mathbb{R},X)$. Therefore, there exists a function $g \in BS^p(\mathbb{R},X)$ such that

$$\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|f(s+s'_n)-g(s)\|_X^p ds\right)^{\frac{1}{p}} = \|f(.+s'_n)-g\|_{BS^p} \longrightarrow 0,$$

as $n \longrightarrow +\infty$. ■

Definition 1.1.23 [20] A function $f : \mathbb{R}\times X \longrightarrow Y$ such that $f(\cdot,x) \in L^p_{loc}(\mathbb{R},Y)$ for each $x \in X$ is S^p-almost periodic in $t \in \mathbb{R}$ uniformly with respect to $x \in X$ if for each compact set K in X, for all $\varepsilon > 0$ there exists $l > 0$ such that for every $a \in \mathbb{R}$ there exists $\tau \in [a, a+l]$ satisfying

$$\sup_{t\in\mathbb{R}}\sup_{x\in K}\left(\int_t^{t+1}\|f(s+\tau,x)-f(s,x)\|_Y^p ds\right)^{\frac{1}{p}} < \varepsilon.$$

The space of all such functions is denoted by $SAP^p(\mathbb{R}\times X, Y)$.

1.1.2 Almost automorphic functions

In this subsection we collect some notions about almost automorphy, compact almost automorphy and weakly almost automorphy. More details about these topics can be found in [35, 36, 37, 71, ?, 149, 172, 173].

Definition 1.1.24 [37, ?, 149] A continuous function $f : \mathbb{R} \longrightarrow X$ is almost automorphic if for every sequence of real numbers $(s_n)_n$ there exist a subsequence $(s'_n)_n$ and a function g such that for each $t \in \mathbb{R}$

$$\lim_{n\to+\infty} f(t+s'_n) = g(t)$$

and

$$\lim_{n\to+\infty} g(t-s_n') = f(t).$$

We denote by $AA(\mathbb{R}, X)$ the space of all almost automorphic X-valued functions.

Definition 1.1.25 [72] A continuous function $f : \mathbb{R} \longrightarrow X$ is compact almost automorphic if for each $(s_n)_n \subset \mathbb{R}$ there exist a subsequence $(s_n')_n$ and a function g such that

$$\lim_{n\to+\infty} f(t+s_n') = g(t) \quad \text{and} \quad \lim_{n\to+\infty} g(s-s_n') = f(t)$$

uniformly on any compact subset of $\mathbb{R}$.
We denote by $AA_c(\mathbb{R}, X)$ the space of all such functions.

Remark 1.1.26 **(1)** Each almost automorphic function $f : \mathbb{R} \longrightarrow X$ has a relatively compact range; hence, it is bounded.
(2) Since the convergence in Definition 1.1.24 is a pointwise convergence, the function g is only measurable and not necessarily continuous. The function g in Definition 1.1.25 is continuous.

Theorem 1.1.27 [81] *A function f is compact almost automorphic if and only if it is almost automorphic and uniformly continuous on $\mathbb{R}$.*

Example 1.1.28 [29, Example 3.1] Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that

$$f(t) = \sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right) \quad \text{for } t \in \mathbb{R}.$$

f is almost automorphic, but it is not uniformly continuous on $\mathbb{R}$. Hence, it is not almost periodic.

Example 1.1.29 [96, 173] Let θ be an irrational real number. Then, for all $n \in \mathbb{Z}$, $\cos(2\pi n\theta) \neq 0$. Let $(h_n)_n$ be the sequence defined by:

$$h_n = \operatorname{sgn}\cos(2\pi n\theta) = \begin{cases} +1 \text{ if } \cos(2\pi n\theta) > 0, \\ -1 \text{ if } \cos(2\pi n\theta) < 0. \end{cases}$$

Let f be given by

$$f(t) = h_n + (t-n)(h_{n+1} - h_n) \quad \text{for } t \in [n, n+1].$$

Then, f is compact almost automorphic, but it is not almost periodic.

Remark 1.1.30 We have that

$$AP(\mathbb{R}, X) \subsetneq AA_c(\mathbb{R}, X) \subsetneq AA(\mathbb{R}, X) \subsetneq BC(\mathbb{R}, X).$$

Theorem 1.1.31 [72] *$AA_c(\mathbb{R}, X)$ endowed with the supremum norm is a Banach space.*

Theorem 1.1.32 *Every almost automorphic function has a relatively compact range.*

Definition 1.1.33 [72] A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is compact almost automorphic in $t \in \mathbb{R}$ uniformly with respact to $x \in X$ if for every $(s_n)_n \subset \mathbb{R}$ there exist a subsequence $(s'_n)_n$ and a function g such that

$$f(t + s'_n, x) \longrightarrow g(t, x) \quad \text{as } n \longrightarrow +\infty$$

and

$$g(t - s'_n, x) \longrightarrow f(t, x) \quad \text{as } n \longrightarrow +\infty$$

uniformly on any compact set in $\mathbb{R}$ and for any $x \in X$.
The space of such functions is denoted by $AA_c(\mathbb{R} \times X, Y)$.

Theorem 1.1.34 [72] *Let $f \in AA_c(\mathbb{R} \times X, Y)$ be Lipschitzian with respect to the second argument. If $x \in AA_c(\mathbb{R}, X)$, then the composition function $t \longmapsto f(t, x(t))$ belongs to $AA_c(\mathbb{R}, Y)$.*

Definition 1.1.35 [43] A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is almost automorphic in $t \in \mathbb{R}$ uniformly with respect to $x \in X$ if the following two conditions hold:
i) for all $x \in X$, $f(\cdot, x) \in AA(\mathbb{R}, Y)$,
ii) f is uniformly continuous on each compact set K in X with respect to the second variable x, namely, for each compact set K in X, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2 \in K$, one has

$$||x_1 - x_2||_X \leq \delta \Longrightarrow \sup_{t \in \mathbb{R}} ||f(t, x_1) - f(t, x_2)||_Y \leq \varepsilon.$$

$AAU(\mathbb{R} \times X, Y)$ denotes the set of all such functions.

1.1.3 μ-Pseudo almost periodic functions and μ-pseudo almost automorphic functions

The notion of μ-pseudo almost periodicity (resp. μ-pseudo almost automorphy) was introduced by J. Blot, P. Cieutat and K. Ezzinbi in [44] (resp. in [45]) in order to correct many errors about some results developed about weighted pseudo almost periodicity [73] (resp. weighted pseudo almost automorphy [133, 182]).

We denote by $\mathfrak{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathfrak{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < +\infty$, for all $a, b \in \mathbb{R}$ $(a \leq b)$.

Example 1.1.36 [44] Let ρ be a nonnegative $\mathfrak{B}$-measurable function. Denote by μ the positive measure defined by

$$\mu(A) = \int_A \rho(t)dt \quad \text{for } A \in \mathfrak{B}, \tag{1.1.2}$$

where dt denotes the Lebesgue measure on $\mathbb{R}$. The function ρ which occurs in (1.1.2) is called the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$. In this case $\mu \in \mathcal{M}$ if and only if its Radon-Nikodym derivative ρ is locally-integrable on $\mathbb{R}$ and it satisfies

$$\int_{-\infty}^{+\infty} \rho(t)dt = +\infty.$$

Definition 1.1.37 [44] Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \longrightarrow X$ is μ-ergodic if

$$\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|f(t)\|_X d\mu(t) = 0.$$

We denote the set of all such functions by $\mathcal{E}(\mathbb{R}, X, \mu)$.

Definition 1.1.38 [44] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \longrightarrow X$ is μ-pseudo almost periodic if f is written in the form $f = g + \phi$ where $g \in AP(\mathbb{R}, X)$ and $\phi \in \mathcal{E}(\mathbb{R}, X, \mu)$.

We denote the space of all such functions by $PAP(\mathbb{R}, X, \mu)$. One can easily check that

$$AP(\mathbb{R}, X) \subset PAP(\mathbb{R}, X, \mu) \subset BC(\mathbb{R}, X).$$

From $\mu \in \mathcal{M}$, we assume the following hypothesis taken from [44]:
(**B**) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a + \tau : a \in A\}) \leq \beta\mu(A) \text{ when } A \in \mathfrak{B} \text{ satisfies } A \cap I = \emptyset.$$

The following example gives a measure satisfying (**B**).

Example 1.1.39 The Lebesgue measure λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ belongs to $\mathcal{M}$ and satisfies (**B**).

Example 1.1.40 [44] We consider the measure μ where its Radom-Nikodym derivative is

$$\rho(t) = e^t \quad \text{for } t \in \mathbb{R}.$$

Then, $\mu \in \mathcal{M}$ satisfies (**B**).

In the next theorem, we collect some properties of the space $PAP(\mathbb{R}, X, \mu)$.

Theorem 1.1.41 [44] *Let $\mu \in \mathcal{M}$ satisfy* (**B**). *Then, the following are true:*
(1) *The decomposition of a μ-pseudo almost periodic in the form $f = g + \varphi$, where $g \in AP(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$, is unique.*
(2) $(PAP(\mathbb{R}, X, \mu), \|\cdot\|_\infty)$ *is a Banach space.*
(3) *The space $PAP(\mathbb{R}, X, \mu)$ is invariant by translation, that is $f \in PAP(\mathbb{R}, X, \mu)$ implies $\tau_\beta f \in PAP(\mathbb{R}, X, \mu)$ for all $\beta \in \mathbb{R}$ where $\tau_\beta f$ is given by $\tau_\beta f(x) = f(x + \beta)$.*

Definition 1.1.42 [44] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is μ-ergodic in $t \in \mathbb{R}$ uniformly with respect to $x \in X$ if the two following conditions are true:
i) for all $x \in X$, $f(\cdot, x) \in \mathcal{E}(\mathbb{R}, Y, \mu)$,
ii) f is uniformly continuous on each compact set K in X with respect to the second variable x.
We denote by $\mathcal{E}U(\mathbb{R} \times X, Y, \mu)$ the set of all such functions.

Definition 1.1.43 [44] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is μ-pseudo almost periodic in $t \in \mathbb{R}$ uniformly with respect to $x \in X$ if f is written in the form $f = g + \phi$, where $g \in APU(\mathbb{R} \times X, Y)$ and $\phi \in \mathcal{E}U(\mathbb{R} \times X, Y, \mu)$.
We denote by $PAPU(\mathbb{R} \times X, Y, \mu)$ the set of such functions. We have

$$APU(\mathbb{R} \times X, Y) \subset PAPU(\mathbb{R} \times X, Y, \mu).$$

The following theorem give the composition of μ-pseudo almost periodic functions.

Theorem 1.1.44 [44] *Let* $\mu \in \mathcal{M}$, $f \in PAPU(\mathbb{R} \times X, Y, \mu)$ *and* $x \in PAP(\mathbb{R}, X, \mu)$. *Assume that the following hypothesis holds:*
(C) *For all bounded subset* B *of* X, f *is bounded on* $\mathbb{R} \times B$.
Then $[t \longmapsto f(t, x(t))] \in PAP(\mathbb{R}, Y, \mu)$.

Definition 1.1.45 [45] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \longrightarrow X$ is μ-pseudo almost automorphic if f is written in the form $f = g + \phi$ where $g \in AA(\mathbb{R}, X)$ and $\phi \in \mathcal{E}(\mathbb{R}, X, \mu)$.
We denote the space of all such functions by $PAA(\mathbb{R}, X, \mu)$. Then,

$$AA(\mathbb{R}, X) \subset PAA(\mathbb{R}, X, \mu) \subset BC(\mathbb{R}, X).$$

In the next theorem, we collect some properties of the space $PAA(\mathbb{R}, X, \mu)$.

Theorem 1.1.46 [45] *Let* $\mu \in \mathcal{M}$ *satisfy* **(B)**. *Then, the following are true:*
(1) *The decomposition of a* μ*-pseudo almost automorphic function in the form* $f = g + \varphi$, *where* $g \in AA(\mathbb{R}, X)$ *and* $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$, *is unique.*
(2) $(PAA(\mathbb{R}, X, \mu), \|\cdot\|_\infty)$ *is a Banach space.*
(3) *The space* $PAA(\mathbb{R}, X, \mu)$ *is invariant by translation, that is* $f \in PAA(\mathbb{R}, X, \mu)$ *implies* $\tau_\beta f \in PAA(\mathbb{R}, X, \mu)$ *for all* $\beta \in \mathbb{R}$ *where* $\tau_\beta f$ *is given by* $\tau_\beta f(x) = f(x + \beta)$.

Definition 1.1.47 [45] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is μ-pseudo almost automorphic in $t \in \mathbb{R}$ uniformly with respect to $x \in X$ if f is written in the form $f = g + \phi$, where $g \in AAU(\mathbb{R} \times X, Y)$ and $\phi \in \mathcal{E}U(\mathbb{R} \times X, Y, \mu)$.
We denote by $PAAU(\mathbb{R} \times X, Y, \mu)$ the set of such functions. We have

$$AAU(\mathbb{R} \times X, Y) \subset PAAU(\mathbb{R} \times X, Y, \mu).$$

The following theorem give the composition of μ-pseudo almost automorphic functions.

Theorem 1.1.48 [45] *Let* $\mu \in \mathcal{M}, f \in PAAU(\mathbb{R} \times X, Y, \mu)$ *and* $x \in PAA(\mathbb{R}, X, \mu)$. *Assume that hypothesis* (**C**) *holds. Then* $[t \longmapsto f(t, x(t))] \in PAA(\mathbb{R}, Y, \mu)$.

1.1.4 μ-Pseudo almost periodicity (automorphy) of class r

Definition 1.1.49 [89] Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \longrightarrow X$ is μ-ergodic of class r if

$$\lim_{\tau \to +\infty} \frac{1}{\mu([-\tau, \tau])} \int_{[-\tau,\tau]} \left(\sup_{\theta \in [t-r,t]} \|f(\theta)\|_X \right) d\mu(t) = 0.$$

We denote the space of all such functions by $\mathcal{E}(\mathbb{R}, X, \mu, r)$.

Remark 1.1.50 f is μ-ergodic of class r if $t \longmapsto f_t$ is μ-ergodic from $\mathbb{R}$ to $\mathcal{C} = C([-r, 0], X)$.

Now, we give an example of function belonging to $\mathcal{E}(\mathbb{R}, X, \mu, r)$.

Example 1.1.51 We consider the measure μ where its Radon-Nikodym derivative is

$$\rho(t) = e^t \quad \text{for } t \in \mathbb{R}.$$

Let ϕ be defined by:

$$\phi(t) = e^{-|t|} \quad \text{for } t \in \mathbb{R}.$$

For $\tau > 0$ large enough, we have that

$$\begin{aligned} \frac{1}{\mu([-\tau, \tau])} \int_{[-\tau,\tau]} \left(\sup_{s \in [-r,0]} |\phi(t+s)| \right) d\mu(t) &= \frac{1}{e^\tau - e^{-\tau}} \int_{-\tau}^{\tau} \left(\sup_{s \in [-r,0]} e^{-|t+s|} e^t \right) dt \\ &\leq \frac{1}{e^\tau - e^{-\tau}} \int_{-\tau}^{\tau} \left(\sup_{s \in [-r,0]} e^{-t} e^{-s} e^t \right) dt \\ &= \frac{2\tau e^r}{e^\tau - e^{-\tau}} \longrightarrow 0 \quad \text{as} \quad \tau \longrightarrow +\infty. \end{aligned}$$

Hence, ϕ is μ-ergodic of class r.

Definition 1.1.52 Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \longrightarrow X$ is μ-pseudo almost periodic of class r if $f = h + \phi$ where $h \in AP(\mathbb{R}, X)$ and $\phi \in \mathcal{E}(\mathbb{R}, X, \mu, r)$.
We denote by $PAP(\mathbb{R}, X, \mu, r)$ the space of all such functions. From the above definition we have that

$$AP(\mathbb{R}, X) \subsetneq PAP(\mathbb{R}, X, \mu, r) \subsetneq BC(\mathbb{R}, X).$$

Theorem 1.1.53 [89] *Let* $\mu \in \mathcal{M}$ *satisfy* (**B**). *Then, the following are true:*
(1) $\mathcal{E}(\mathbb{R}, X, \mu, r)$ *is translation invariant, therefore* $PAP(\mathbb{R}, X, \mu, r)$ *is also translation invariant.*
(2) *The space* $PAP(\mathbb{R}, X, \mu, r)$ *endowed with the uniform topology norm is a Banach space.*
(3) *The decomposition of a* μ*-pseudo almost periodic function of class* r *in the form* $f = h + \phi$, *where* $h \in AP(\mathbb{R}, X)$ *and* $\phi \in \mathcal{E}(\mathbb{R}, X, \mu, r)$, *is unique.*

Theorem 1.1.54 [89] *Let $\mu \in \mathcal{M}$ satisfy* **(B)** *and $f \in PAP(\mathbb{R}, X, \mu, r)$. Then, the function $[t \longmapsto f_t]$ belongs to $PAP(\mathbb{R}, \mathcal{C}, \mu, r)$.*

Definition 1.1.55 [20] Let $\mu \in \mathcal{M}$. A function $f \in BS^p(\mathbb{R}, X)$ is S^p-μ-ergodic if

$$\lim_{\tau \to +\infty} \frac{1}{\mu([-\tau, \tau])} \int_{[-\tau,\tau]} \left(\int_t^{t+1} \|f(s)\|_X^p ds \right)^{\frac{1}{p}} d\mu(t) = 0.$$

The collection of such functions is denoted by $\mathcal{E}^p(\mathbb{R}, X, \mu)$.

Definition 1.1.56 [20] Let $\mu \in \mathcal{M}$. A function $f \in L^p_{loc}(\mathbb{R}, X)$ is S^p-μ-pseudo almost periodic if $f = h + \phi$ where $h \in SAP^p(\mathbb{R}, X)$ and $\phi \in \mathcal{E}^p(\mathbb{R}, X, \mu)$.
We denote the space of all S^p-μ-pseudo almost periodic functions by $SPAP^p(\mathbb{R}, X, \mu)$.

Definition 1.1.57 Let $\mu \in \mathcal{M}$. A function $f \in BS^p(\mathbb{R}, X)$ is S^p-μ-ergodic of class r if

$$\lim_{\tau \to +\infty} \frac{1}{\mu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \left(\int_\theta^{\theta+1} \|f(s)\|_X^p ds \right)^{\frac{1}{p}} d\mu(t) = 0.$$

The collection of such functions is denoted by $\mathcal{E}^p(\mathbb{R}, X, \mu, r)$.

Definition 1.1.58 Let $\mu \in \mathcal{M}$. A function $f \in L^p_{loc}(\mathbb{R}, X)$ is S^p-μ-pseudo almost periodic of class r if $f = h + \phi$ where $h \in SAP^p(\mathbb{R}, X)$ and $\phi \in \mathcal{E}^p(\mathbb{R}, X, \mu, r)$.
We denote the space of all S^p-μ-pseudo almost periodic functions of class r by $SPAP^p(\mathbb{R}, X, \mu, r)$.

Remark 1.1.59 **(1)** From Proposition 1.1.17 and Definition 1.1.57 we have that
$SPAP^p(\mathbb{R}, X, \mu, r) \subset BS^p(\mathbb{R}, X)$.
(2) From Definitions 1.1.56 and 1.1.58 we have $SPAP^p(\mathbb{R}, X, \mu, r) \subset SPAP^p(\mathbb{R}, X, \mu)$.

Theorem 1.1.60 [20] *Let $\mu \in \mathcal{M}$ satisfy* **(B)**. *Then, $SPAP^p(\mathbb{R}, X, \mu)$ is translation invariant.*

Theorem 1.1.61 *Let $\mu \in \mathcal{M}$ satisfy* **(B)**. *Then, $\mathcal{E}^p(\mathbb{R}, X, \mu, r)$ is translation invariant. Therefore, $SPAP^p(\mathbb{R}, X, \mu, r)$ is translation invariant.*

Proof. Since $SAP^p(\mathbb{R}, X)$ is translation invariant, it remains to prove that if $g \in \mathcal{E}^p(\mathbb{R}, X, \mu, r)$ then $g_\alpha \in \mathcal{E}^p(\mathbb{R}, X, \mu, r)$ for all $\alpha \in \mathbb{R}$. Let $g \in \mathcal{E}^p(\mathbb{R}, X, \mu, r)$, $\alpha \in \mathbb{R}$ and consider the function G defined by:

$$G(t) = \sup_{\theta \in [t-r,t]} \left(\int_\theta^{\theta+1} \|g(s)\|_X^p ds \right)^{\frac{1}{p}} \quad \text{for } t \in \mathbb{R}.$$

Then, we have that $G \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$. On the other hand

$$\begin{aligned} G(t+\alpha) &= \sup_{\theta \in [t+\alpha-r,t+\alpha]} \left(\int_\theta^{\theta+1} \|g(s)\|_X^p ds \right)^{\frac{1}{p}} \\ &= \sup_{\theta \in [t-r,t]} \left(\int_{\theta+\alpha}^{\theta+\alpha+1} \|g(s)\|_X^p ds \right)^{\frac{1}{p}} \\ &= \sup_{\theta \in [t-r,t]} \left(\int_\theta^{\theta+1} \|g(s+\alpha)\|_X^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

Then, for $\tau > 0$ sufficiently large, we have that

$$\frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\left(\int_{\theta}^{\theta+1}\|g(s+\alpha)\|_X^p ds\right)^{\frac{1}{p}} d\mu(t) = \frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}|G(t+\alpha)|d\mu(t). \tag{1.1.3}$$

By Theorem 1.1.41, we get that

$$\lim_{\tau\to+\infty}\frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}|G(t+\alpha)|d\mu(t) = 0,$$

which implies by (1.1.3) that $g_\alpha \in \mathcal{E}^p(\mathbb{R}, X, \mu, r)$. That is $\mathcal{E}^p(\mathbb{R}, X, \mu, r)$ is invariant by translation. ■

The next theorem is about the uniqueness of the decomposition in $SPAP^p(\mathbb{R}, X, \mu)$.

Theorem 1.1.62 [20] *Let $\mu \in \mathcal{M}$ satisfy* **(B)**. *Then, the decomposition of a S^p-μ-pseudo almost periodic function in the form $f = h + \phi$, where $h \in SAP^p(\mathbb{R}, X)$ and $\phi \in \mathcal{E}^p(\mathbb{R}, X, \mu)$, is unique.*

Theorem 1.1.63 *Let $\mu \in \mathcal{M}$ satisfy* **(B)**. *Then, the decomposition of a S^p-μ-pseudo almost periodic function of class r in the form $f = h + \phi$, where $h \in SAP^p(\mathbb{R}, X)$ and $\phi \in \mathcal{E}^p(\mathbb{R}, X, \mu, r)$, is unique.*

Proof. The theorem is a consequence of Remark 1.1.59 and Theorem 1.1.62. ■

Theorem 1.1.64 *Let $\mu \in \mathcal{M}$ satisfy* **(B)**. *Then, $\mathcal{E}(\mathbb{R}, X, \mu, r) \subset \mathcal{E}^p(\mathbb{R}, X, \mu, r)$ and $PAP(\mathbb{R}, X, \mu, r) \subset SPAP^p(\mathbb{R}, X, \mu, r)$.*

Proof. Let $f \in \mathcal{E}(\mathbb{R}, X, \mu, r)$. Then,

$$\begin{aligned}
\sup_{\theta\in[t-r,t]}\left(\int_{\theta}^{\theta+1}\|f(s)\|_X^p ds\right)^{\frac{1}{p}} &= \sup_{\theta\in[t-r,t]}\left(\int_{0}^{1}\|f(s+\theta)\|_X^p ds\right)^{\frac{1}{p}} \\
&\leq \sup_{\theta\in[t-r,t]}\left(\int_{0}^{1}\sup_{s\in[0,1]}\|f(s+\theta)\|_X^p ds\right)^{\frac{1}{p}} \\
&= \sup_{\theta\in[t-r,t]}\left(\sup_{s\in[0,1]}\|f(s+\theta)\|_X^p\right)^{\frac{1}{p}} \\
&= \sup_{\theta\in[t-r,t]}\left(\sup_{s\in[0,1]}\|f(s+\theta)\|_X\right).
\end{aligned}$$

Let $s_0 \in [0,1]$ be such that $\sup_{s\in[0,1]}\|f(s+\theta)\|_X = \|f(s_0+\theta)\|_X$. Then, for $\tau > 0$ sufficiently large, we get that

$$\begin{aligned}
&\frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\left(\int_{\theta}^{\theta+1}\|f(s)\|_X^p ds\right)^{\frac{1}{p}} d\mu(t) \\
&\leq \frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}\left(\sup_{\theta\in[t-r,t]}\|f(s_0+\theta)\|_X\right) d\mu(t).
\end{aligned} \tag{1.1.4}$$

By (1) in Theorem 1.1.53, it follows that

$$\lim_{\tau\to+\infty} \frac{1}{\mu([-\tau,\tau])} \int_{[-\tau,\tau]} \left(\sup_{\theta\in[t-r,t]} \|f(s_0+\theta)\|_X \right) d\mu(t) = 0. \tag{1.1.5}$$

From (1.1.4) and (1.1.5) we deduce that $f \in \mathcal{E}^p(\mathbb{R}, X, \mu, r)$. Since $AP(\mathbb{R}, X) \subset SAP^p(\mathbb{R}, X)$, we get that $PAP(\mathbb{R}, X, \mu, r) \subset SPAP^p(\mathbb{R}, X, \mu, r)$. ■

Definition 1.1.65 Let $\mu \in \mathcal{M}$. A function $f : \mathbb{R} \times X \longrightarrow Y$ such that $f(\cdot, x) \in BS^p(\mathbb{R}, Y)$ for each $x \in X$ is S^p-μ-ergodic of class r in $t \in \mathbb{R}$ uniformly with respect to $x \in X$ if the two following hold.

(i) For all $x \in X$, $f(\cdot, x) \in \mathcal{E}^p(\mathbb{R}, Y, \mu, r)$.

(ii) f is S^p-uniformly continuous with respect to the second argument on each compact subset K in X in the following sense: for all $\varepsilon > 0$ there exists δ such that for all $x_1, x_2 \in K$ one has

$$\|x_1 - x_2\| < \delta \Longrightarrow \left(\int_t^{t+1} \|f(s, x_1) - f(s, x_2)\|_Y^p ds \right)^{\frac{1}{p}} < \varepsilon \quad \text{for } t \in \mathbb{R}.$$

We denote by $\mathcal{E}^p(\mathbb{R} \times X, Y, \mu, r)$ the set of all such functions.

Definition 1.1.66 Let $\mu \in \mathcal{M}$. A function $f : \mathbb{R} \times X \longrightarrow Y$ with $f(\cdot, x) \in L^p_{loc}(\mathbb{R}, Y)$ for each $x \in X$ is S^p-μ-pseudo almost periodic of class r in $t \in \mathbb{R}$ uniformly with respect to $x \in X$, if it can be decomposed as $f = h + \phi$, where $h \in SAP^p(\mathbb{R} \times X, Y)$ and $\phi \in \mathcal{E}^p(\mathbb{R} \times X, Y, \mu, r)$. We denote the set of such functions by $SPAP^p(\mathbb{R} \times X, Y, \mu, r)$.

Theorem 1.1.67 *Let* $\mu \in \mathcal{M}$ *satisfy* **(B)**, $f = h + \varphi \in SPAP^p(\mathbb{R} \times X, Y, \mu, r)$ *with* $h \in SAP^p(\mathbb{R} \times X, Y)$ *and* $\varphi \in \mathcal{E}^p(\mathbb{R} \times X, Y, \mu, r)$. *Assume that there exists a positive real number* L_f *such that*

$$\|f(t, x_1) - f(t, x_2)\|_Y \leq L_f \|x_1 - x_2\|_X \quad \text{for } t \in \mathbb{R} \text{ and } x_1, x_2 \in X. \tag{1.1.6}$$

If $\phi \in PAP(\mathbb{R}, X, \mu, r)$, *then the function* $[t \longmapsto f(t, \phi(t))]$ *belongs to* $SPAP^p(\mathbb{R}, Y, \mu, r)$.

Proof. Assume that $f = h + \varphi$, $\phi = \alpha + \beta$, where $h \in SAP^p(\mathbb{R} \times X, Y)$, $\varphi \in \mathcal{E}^p(\mathbb{R} \times X, Y, \mu, r)$, $\alpha \in AP(\mathbb{R}, X)$ and $\beta \in \mathcal{E}(\mathbb{R}, X, \mu, r)$. Consider the decomposition

$$f(t, \phi(t)) = h(t, \alpha(t)) + f(t, \phi(t)) - f(t, \alpha(t)) + \varphi(t, \alpha(t)).$$

We have $[t \longmapsto h(t, \alpha(t))] \in SAP^p(\mathbb{R}, Y)$ by the composition of S^p-almost periodic functions in [71]. Let us prove that $[t \longmapsto f(t, \phi(t)) - f(t, \alpha(t))] \in \mathcal{E}^p(\mathbb{R}, Y, \mu, r)$ and $[t \longmapsto \varphi(t, \alpha(t))] \in \mathcal{E}^p(\mathbb{R}, Y, \mu, r)$.

From (1.1.6), it follows that

$$\|f(t, \phi(t)) - f(t, \alpha(t))\|_Y \leq L_f \|\beta(t)\|_X \quad \text{for } t \in \mathbb{R}. \tag{1.1.7}$$

Since μ satisfies (**B**), then by Theorem 1.1.64, $\beta \in \mathcal{E}^p(\mathbb{R}, Y, \mu, r)$. Hence, (1.1.7) implies that the function $[t \longmapsto f(t, \phi(t)) - f(t, \alpha(t))]$ is S^p-μ-ergodic of class r.
To complete the proof, it is enough to prove that $[t \longmapsto \varphi(t, \alpha(t))]$ is S^p-μ-ergodic of class r.
Since φ is S^p-uniformly continuous with respect to the second argument on the compact set $K = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$, we deduce that for given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $t \in \mathbb{R}$, ξ_1 and $\xi_2 \in K$, one has

$$\|\xi_1 - \xi_2\|_X < \delta \Longrightarrow \left(\int_t^{t+1} \|\varphi(s, \xi_1) - \varphi(s, \xi_2)\|_Y^p ds\right)^{\frac{1}{p}} < \varepsilon. \tag{1.1.8}$$

Therefore, there exists $n(\varepsilon)$ and $\{x_i\}_{i=1}^{n(\varepsilon)} \subset K$, such that

$$K \subset \bigcup_{i=1}^{n(\varepsilon)} B(x_i, \delta).$$

Let

$$i(t) = \min\{i = 1, 2, \dots, k : \|\alpha(t) - x_i\|_X < \delta\}, \quad t \in \mathbb{R}.$$

Let $\theta \in [t - r, t]$ and $s \in [\theta, \theta + 1]$. Since

$$\left(\int_\theta^{\theta+1} \|\varphi(s, \alpha(s))\|_Y^p ds\right)^{\frac{1}{p}} \leq \left(\int_\theta^{\theta+1} \|\varphi(s, \alpha(s)) - \varphi(s, x_{i(s)})\|_Y^p ds\right)^{\frac{1}{p}} + \left(\int_\theta^{\theta+1} \|\varphi(s, x_{i(s)})\|_Y^p ds\right)^{\frac{1}{p}},$$

then by (1.1.8), we obtain that

$$\left(\int_\theta^{\theta+1} \|\varphi(s, \alpha(s))\|_Y^p ds\right)^{\frac{1}{p}} \leq \varepsilon + \sum_{i=1}^{n(\varepsilon)} \left(\int_\theta^{\theta+1} \|\varphi(s, x_i)\|_Y^p ds\right)^{\frac{1}{p}}.$$

Since

$$\forall i \in \{1, \dots, n(\varepsilon)\}, \quad \lim_{\tau \to +\infty} \frac{1}{\mu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \left(\int_\theta^{\theta+1} \|\varphi(s, x_i)\|_Y^{\frac{1}{p}} ds\right)^{\frac{1}{p}} d\mu(t) = 0,$$

we deduce that

$$\forall \varepsilon > 0, \quad \limsup_{\tau \to +\infty} \frac{1}{\mu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \left(\int_\theta^{\theta+1} \|\varphi(s, \alpha(s))\|_Y^p ds\right)^{\frac{1}{p}} d\mu(t) \leq \varepsilon.$$

This implies that

$$\lim_{\tau \to +\infty} \frac{1}{\mu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \left(\int_\theta^{\theta+1} \|\varphi(s, \alpha(s))\|_Y^p ds\right)^{\frac{1}{p}} d\mu(t) = 0,$$

therefore the function $[t \longmapsto \varphi(t, \alpha(t))]$ is S^p-μ-ergodic of class r. ■

Definition 1.1.68 [89] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \longrightarrow X$ is μ-pseudo compact almost automorphic of class r if $f = h + \phi$ where, $h \in AA_c(\mathbb{R}, X)$ and $\phi \in \mathcal{E}(\mathbb{R}, X, \mu, r)$.
We denote by $PAA_c(\mathbb{R}, X, \mu, r)$ the space of all such functions. Then,

$$AA_c(\mathbb{R}, X) \subsetneq PAA_c(\mathbb{R}, X, \mu, r) \subsetneq BC(\mathbb{R}, X).$$

Example 1.1.69 We consider the measure μ where its Radom-Nikodym derivative is

$$\rho(t) = e^t \quad \text{for } t \in \mathbb{R}.$$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by:

$$f(t) = h(t) + \phi(t) \quad \text{for } t \in \mathbb{R},$$

where h and ϕ are as in Examples 1.1.51 and 1.1.29. Then, f is μ-pseudo compact almost automorphic of class r.

In the next theorem, we collect some properties of the space $PAA_c(\mathbb{R}, X, \mu, r)$.

Theorem 1.1.70 [89] *Let $\mu \in \mathcal{M}$ satisfy* (**B**)*. Then, the following are true:*
(1) *$\mathcal{E}(\mathbb{R}, X, \mu, r)$ is translation invariant, therefore $PAA_c(\mathbb{R}, X, \mu, r)$ is also translation invariant.*
(2) *The space $PAA_c(\mathbb{R}, X, \mu, r)$ endowed with the uniform topology norm is a Banach space.*
(3) *The decomposition of a μ-pseudo compact almost automorphic function of class r in the form $f = h + \phi$, where $h \in AA_c(\mathbb{R}, X)$ and $\phi \in \mathcal{E}(\mathbb{R}, X, \mu, r)$, is unique.*

Remark 1.1.71 If **(3)** in Theorem 1.1.70 is satisfied, then h and ϕ are called respectively the compact almost automorphic component and the μ-ergodic perturbation of class r of f.

Theorem 1.1.72 [89] *Let $\mu \in \mathcal{M}$ satisfy* (**B**) *and $f \in PAA_c(\mathbb{R}, X, \mu, r)$. Then, the function $[t \longmapsto f_t]$ belongs to $PAA_c(\mathbb{R}, \mathcal{C}, \mu, r)$.*

Now, we consider the following space:

$$\mathcal{E}(\mathbb{R} \times X, Y, \mu, r) = \left\{ f \in BC(\mathbb{R} \times X, Y) : \lim_{\tau \to +\infty} \frac{1}{\mu([-\tau, \tau])} \int_{[-\tau,\tau]} \left(\sup_{\theta \in [t-r,t]} \|f(\theta, x)\|_Y \right) d\mu(t) = 0 \right\},$$

uniformly on each compact sets of X.

Definition 1.1.73 [89] Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is μ-pseudo compact almost automorphic of class r if $f = h + \phi$ where $h \in AA_c(\mathbb{R} \times X, Y)$ and $\phi \in \mathcal{E}(\mathbb{R} \times X, Y, \mu, r)$. We denote by $PAA_c(\mathbb{R} \times X, Y, \mu, r)$ the space of all such functions.

Theorem 1.1.74 *Let $\mu \in \mathcal{M}$, $f = h + \varphi \in PAA_c(\mathbb{R} \times X, Y, \mu, r)$ with $h \in AA_c(\mathbb{R} \times X, Y)$, $\varphi \in \mathcal{E}(\mathbb{R} \times X, Y, \mu, r)$ and $\phi \in PAA_c(\mathbb{R}, X, \mu, r)$. Assume that f satisfies* (**C**) *and*
i) *There exists a positive real number L_h such that*

$$\|h(t, x_1) - h(t, x_2)\|_Y \leq L_h \|x_1 - x_2\|_X \quad \text{for } t \in \mathbb{R} \text{ and } x_1, x_2 \in X.$$

ii) *There exists a function $L_f : \mathbb{R} \longrightarrow [0, +\infty)$ such that*

$$\|f(t, x_1) - f(t, x_2)\|_Y \leq L_f(t) \|x_1 - x_2\|_X \quad \text{for } t \in \mathbb{R} \text{ and } x_1, x_2 \in X.$$

If

$$\limsup_{\tau\to+\infty} \frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}\left(\sup_{\theta\in[t-r,t]} L_f(\theta)\right)d\mu(t) < +\infty \tag{1.1.9}$$

and

$$\lim_{\tau\to+\infty} \frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}\left(\sup_{\theta\in[t-r,t]} L_f(\theta)\right)\xi(t)d\mu(t) = 0 \quad \text{for each } \xi \in \mathcal{E}(\mathbb{R},\mathbb{R},\mu), \tag{1.1.10}$$

then the composition function $[t \longmapsto f(t,\phi(t))]$ *belongs to* $PAA_c(\mathbb{R}, Y, \mu, r)$.

Proof. Similar to the proof of Theorem 9 in [122]. ■

1.1.5 (μ,ν)-Pseudo almost automorphic functions

The notion of (μ,ν)-pseudo almost automorphy was introduced by E. Ait Dads, K. Ezzinbi and M. Miraoui in [11].

From $\mu,\nu \in \mathcal{M}$, we assume the following hypothesis taken from [11].

(**D**) $\limsup_{r\to+\infty} \dfrac{\mu([-r,r])}{\nu([-r,r])} < +\infty.$

Definition 1.1.75 [11] Let $\mu,\nu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \longrightarrow X$ is (μ,ν)-ergodic if

$$\lim_{r\to+\infty} \frac{1}{\nu([-r,r])}\int_{[-r,r]} \|f(t)\|_X d\mu(t) = 0.$$

$\mathcal{E}(\mathbb{R},X,\mu,\nu)$ denotes the set of such functions.

Definition 1.1.76 [11] Let $\mu,\nu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \longrightarrow X$ is (μ,ν)-pseudo almost automorphic if it can be written in the form $f = g + \phi$, where $g \in AA(\mathbb{R},X)$ and $\phi \in \mathcal{E}(\mathbb{R},X,\mu,\nu)$.

$PAA(\mathbb{R},X,\mu,\nu)$ denotes the space of such functions.

Theorem 1.1.77 [11] *Let* $\mu,\nu \in \mathcal{M}$ *satisfy* (**B**). *Then,* $PAA(\mathbb{R},X,\mu,\nu)$ *is invariant under translations.*

Theorem 1.1.78 [11] *Let* $\mu,\nu \in \mathcal{M}$ *satisfy* (**D**). *Then, the decomposition of a* (μ,ν)*-pseudo almost automorphic in the form* $f = g + \phi$, *where* $g \in AA(\mathbb{R},X)$ *and* $\phi \in \mathcal{E}(\mathbb{R},X,\mu,\nu)$, *is unique.*

Theorem 1.1.79 [11] *Let* $\mu,\nu \in \mathcal{M}$ *satisfy* (**B**) *and* (**D**). *Then,* $(PAA(\mathbb{R},X,\mu,\nu), \|\cdot\|_\infty)$ *is a Banach space.*

Definition 1.1.80 [11] Let $\mu,\nu \in \mathcal{M}$. A continuous function $f : \mathbb{R}\times X \longrightarrow Y$ is (μ,ν)-ergodic in $t \in \mathbb{R}$ uniformly with respect to $x \in X$ if the two following conditions are true:

i) for all $x \in X$, $f(\cdot,x) \in \mathcal{E}(\mathbb{R},Y,\mu,\nu)$,

ii) f is uniformly continuous on each compact set K in X with respect to the second variable x.

$\mathcal{E}U(\mathbb{R}\times X, Y,\mu,\nu)$ denotes the set of such functions.

Definition 1.1.81 [11] Let $\mu, \nu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \times X \longrightarrow Y$ is (μ, ν)-pseudo almost automorphic in $t \in \mathbb{R}$ uniformly with respect to $x \in X$ if it is written in the form $f = g + \phi$, where $g \in AAU(\mathbb{R} \times X, Y)$ and $\phi \in \mathcal{E}U(\mathbb{R} \times X, Y, \mu, \nu)$.
$PAAU(\mathbb{R} \times X, Y, \mu, \nu)$ denotes the space of such functions.

Theorem 1.1.82 [11] *Let $\mu, \nu \in \mathcal{M}$, $f \in PAAU(\mathbb{R} \times X, Y, \mu, \nu)$ and $x \in PAA(\mathbb{R}, X, \mu, \nu)$. Assume that* (**C**) *and* (**D**) *hold. Then,* $[t \longmapsto f(t, x(t))] \in PAA(\mathbb{R}, Y, \mu, \nu)$.

1.2 Fixed Point Theory

In this section, we give some fixed point theorems that we will use in the next chapters.

Theorem 1.2.1 [102] *Let (E, d) be a complete metric space and let f be a mapping from E into E satisfying*

$$d(f(x), f(y)) \leq kd(x, y) \quad \text{for all } x \text{ and } y \in E,$$

where $0 \leq k < 1$. Then f has exactly one fixed point in E.

Theorem 1.2.2 [56] *Let (E, d) be a complete metric space and let f be a mapping from E into E satisfying*

$$d(f^n(x), f^n(y)) \leq kd(x, y) \quad \text{for all } x \text{ and } y \in E,$$

for some integer n and $0 \leq k < 1$. Then f has exactly one fixed point in E.

Theorem 1.2.3 [102] *Let (E, d) be a complete metric space. If f be a mapping from E into E satisfying*

$$d(f(x), f(y)) \leq \Phi(d(x, y)) \quad \text{for all } x \text{ and } y \in E,$$

where Φ is a positive nondecreasing continuous function on $[0, +\infty)$, verifying $\Phi(r) < r$ for every $r > 0$ and $\Phi(0) = 0$, then f has exactly one fixed point in E.

The following theorem is due to H. Mönch [146].

Theorem 1.2.4 [146] *Let D be a closed convex subset of a Banach space E and $0 \in D$. Assume that $F : D \longrightarrow E$ is a continuous map which satisfies Mönch's condition, that is,* ($M \subseteq D$ *is countable,* $M \subseteq \overline{co}(\{0\} \cup F(M)) \Longrightarrow \overline{M}$ *is compact*)*. Then F has a fixed point in D.*

We close this section by the following lemma.

Lemma 1.2.5 [114, Lemma 30, p. 220] *Let S be a contraction defined on a closed convex subset of a real Hilbert space $(\mathcal{H}, |\cdot|)$. Then*

$$|Sx - Sy| = |x - y| \Longrightarrow S\left(\frac{x+y}{2}\right) = \frac{1}{2}\big(S(x) + S(y)\big).$$

1.3 Measure of Noncompactness

The measure of noncompactness is a very useful tool in nonlinear analysis. It was initiated by Kuratowski [127] and Darbo [67] who are applied to the theories of differential and integral equations. In this section, we give some definitions, properties and examples about measure of noncompactness that we will use in this thesis. More details about this facts can be found in the monographs [1, 21, 28, 124].

Definition 1.3.1 Let E^+ be a positive cone of an ordered Banach space $(E, \leq)$. A function Ψ defined on the set of all bounded subsets of a Banach space X with values in E^+ is called a *measure of noncompactness* if $\Psi(\overline{co}(\Omega)) = \Psi(\Omega)$ for all bounded subset $\Omega \subset X$, where $\overline{co}(\Omega)$ stands for the closed convex hull of Ω.

A measure of noncompactness Ψ is said to be:

❶ *monotone* if for all bounded subsets Ω_1, Ω_2 of X, $\Omega_1 \subset \Omega_2$ implies $\Psi(\Omega_1) \leq \Psi(\Omega_2)$,

❷ *nonsingular* if $\Psi(\{a\} \cup \Omega) = \Psi(\Omega)$ for every $a \in X$ and every nonempty subset $\Omega \subset X$,

❸ *algebraically semiadditive* if $\Psi(\Omega_1 + \Omega_2) \leq \Psi(\Omega_1) + \Psi(\Omega_2)$ for all bounded subsets Ω_1, Ω_2 of X,

❹ *regular* if $\Psi(\Omega) = 0$ if and only if Ω is relatively compact in X,

❺ *semi-homogeneous:* if $\Psi(t\Omega) = |t|\Psi(\Omega)$ for every $t \in \mathbb{R}$ and all bounded subset Ω of X.

Example 1.3.2 We give some examples of measure of noncompactness.

❶ The *Hausdorff measure of noncompactness* $\chi(\cdot)$ defined on each bounded subset of the Banach space X is given by

$$\chi(\Omega) = \inf\left\{\varepsilon > 0 : \Omega \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon),\ x_i \in X \text{ for } i = 1, \ldots, n\right\}.$$

χ is monotone, nonsingular, algebraically semiadditive and regular.

❷ The *Kuratowski measure of noncompactness* $\alpha(\cdot)$ defined on each bounded subset of the Banach space X is given by

$$\alpha(\Omega) = \inf\left\{\varepsilon > 0 : \Omega \text{ can be covered by a finitely many sets of diameter } \leq \varepsilon\right\}.$$

α is monotone, nonsingular, algebraically semiadditive and regular.

❸ The modulus of fiber noncompactness χ_1 defined for bounded set $\Omega \subset C([a,b],X)$ is given by

$$\chi_1(\Omega) = \sup_{t\in[a,b]} \chi(\Omega(t))$$

where $\Omega(t) = \{x(t) : x \in \Omega\}$ for $t \in [a,b]$.

❹ The *modulus of equicontinuity* is defined by

$$\mathrm{mod}_C(\Omega) = \lim_{\delta\to 0} \sup_{x\in\Omega} \max_{|t_1-t_2|\leq\delta} \|x(t_1) - x(t_2)\|$$

where $\Omega \subset C([a,b],X)$ is bounded. Note that $\mathrm{mod}_C(\Omega) = 0$ if and only if Ω is equicontinuous.

The measure of noncompactness χ_1 and mod_C are not regular.

1.4 Maximal Monotone Operators and Differential Inclusions

Throughout this section, $\mathcal{H}$ is a real Hilbert space endowed with its norm $|\cdot|$ and its inner product $(\cdot,\cdot)_{\mathcal{H}}$ and $2^{\mathcal{H}}$ is the powerset of $\mathcal{H}$. In this section we give some tools about maximal monotone operators. More details can be found in the monograph of Brézis [51].

Definition 1.4.1 Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be a multivalued operator. Its domain is defined by

$$D(\mathcal{A}) = \{x \in \mathcal{H} : \mathcal{A}x \text{ is nonempty in } \mathcal{H}\}.$$

i) The range of $\mathcal{A}$ is defined by

$$R(\mathcal{A}) = \bigcup_{x\in\mathcal{H}} \mathcal{A}x.$$

ii) The graph of $\mathcal{A}$ is defined by

$$G(\mathcal{A}) = \{(x,y) \in \mathcal{H}^2 : y \in \mathcal{A}x\}.$$

iii) $\mathcal{A}$ is *monotone* if

$$(\mathcal{A}x - \mathcal{A}y, x - y)_{\mathcal{H}} \geq 0 \quad \text{for all } x, y \in D(\mathcal{A}),$$

which means that for each $x_1 \in D(A)$ and $x_2 \in D(A)$, one has

$$(y_1 - y_2, x_1 - x_2)_{\mathcal{H}} \geq 0 \quad \text{for all } y_1 \in \mathcal{A}x_1 \text{ and } y_2 \in \mathcal{A}x_2.$$

iv) $\mathcal{A}$ is *maximal monotone* if it is monotone and $G(\mathcal{A})$ is maximal with respect to inclusion among the graphs of all monotone operators.

v) $\mathcal{A}$ is *α-strongly maximal monotone* ($\alpha > 0$) if it is maximal monotone and

$$(\mathcal{A}x - \mathcal{A}y, x - y)_{\mathcal{H}} \geq \alpha|x-y|^2 \quad \text{for all } x, y \in D(\mathcal{A}).$$

Remark 1.4.2 Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be a multivalued operator and $\alpha > 0$. Then,

$\mathcal{A}$ is maximal monotone if and only if $\mathcal{A} + \alpha I$ is α-strongly maximal monotone.

Theorem 1.4.3 [51] *If $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is maximal monotone, then $\overline{D(\mathcal{A})}$ is convex.*

Theorem 1.4.4 [114] *Let $\mathcal{A}$ be a maximal monotone operator on $\mathcal{H}$. Then, for every $x \in D(\mathcal{A})$, the set $\mathcal{A}x$ is closed and convex.*

Remark 1.4.5 From the last theorem, the following operator is well defined

$$\mathcal{A}^0 x = \mathrm{Proj}_{\mathcal{A}x}(0) \quad \text{for } x \in D(\mathcal{A}),$$

where $\mathrm{Proj}_C(x)$ is the projection of x on a closed convex $C \subset \mathcal{H}$. The operator $\mathcal{A}^0$ plays a key role in many questions related to solving nonlinear equations.

Consider the following differential inclusion:

$$\begin{cases} u'(t) + \mathcal{A}u(t) \ni f(t) \quad \text{for } t \in [a, b], \\ u(a) = u_0 \in \mathcal{H}. \end{cases} \tag{1.4.1}$$

Definition 1.4.6 [51] Let $f \in L^1([a, b], \mathcal{H})$. A continuous function $u : [a, b] \longrightarrow \mathcal{H}$ is a *strong solution* of the differential inclusion (1.4.1) if u is absolutely continuous on each compact of $]a, b[$, $u(t) \in D(\mathcal{A})$ a.e. on $[a, b]$ and (1.4.1) is satisfied a.e. on $[a, b]$.

Definition 1.4.7 [51] A function u is a *weak solution* of (1.4.1) if there exist $f_n \in L^1([a, b], \mathcal{H})$ and $u_n \in C([a, b], \mathcal{H})$ such that u_n is a strong solution of

$$u_n'(t) + \mathcal{A}u_n(t) \ni f_n(t)$$

on $[a, b]$, $f_n \longrightarrow f$ in $L^1([a, b], \mathcal{H})$ and $u_n \longrightarrow u$ in $C([a, b], \mathcal{H})$.

Lemma 1.4.8 [51] *Let $f,\ f_n \in L^1([a, b], \mathcal{H})$. Assume that x_n is a weak solution of $x_n'(t) + \mathcal{A}x_n(t) \ni f_n(t)$ on $[a, b]$. If $x_n \longrightarrow x$ uniformly on $[a, b]$ and $f_n \longrightarrow f$ in $L^1([a, b], \mathcal{H})$, then x is a weak solution of (1.4.1) on $[a, b]$.*

Definition 1.4.9 Let $f \in L^1_{loc}(\mathbb{R}, \mathcal{H})$. A function u is a weak solution of (1.4.1) on $\mathbb{R}$ if it is a weak solution of (1.4.1) on every compact interval of $\mathbb{R}$.

The following results regarding the existence and estimations of weak solutions are needed in Chapters 5 and 6.

Theorem 1.4.10 [114, Theorem 36, p.76] *Assume that $\mathcal{A}$ is maximal monotone and $f \in L^1([a,b],\mathcal{H})$. If $u_0 \in \overline{D(\mathcal{A})}$, then there exists a unique weak solution of* (1.4.1). *Moreover, if u and v are two weak solutions of $u'(t) + \mathcal{A}u(t) \ni f(t)$ and $v'(t) + \mathcal{A}v(t) \ni g(t)$, respectively, then,*

$$|u(t) - v(t)| \leq |u(s) - v(s)| + \int_s^t |f(\sigma) - g(\sigma)| d\sigma \quad \text{for } a \leq s \leq t \leq b. \tag{1.4.2}$$

Theorem 1.4.11 [178] *Assume that $\mathcal{A}$ is α-strongly maximal monotone. Let I be an interval of $\mathbb{R}$ and $\widetilde{f}, \widehat{f} \in L^1_{loc}(I,\mathcal{H})$. If $\widetilde{u}$ and $\widehat{u}$ are weak solutions on I of $\widetilde{u}'(t)+\mathcal{A}\widetilde{u}(t) \ni \widetilde{f}(t)$ and $\widehat{u}'(t)+\mathcal{A}\widehat{u}(t) \ni \widehat{f}(t)$, respectively, then for any s and t in I, $s \leq t$, we have*

$$|\widetilde{u}(t) - \widehat{u}(t)| \leq e^{-\alpha(t-s)}|\widetilde{u}(s) - \widehat{u}(s)| + \int_s^t e^{-\alpha(t-\sigma)}|\widetilde{f}(\sigma) - \widehat{f}(\sigma)| d\sigma.$$

1.5 C_0-Semigroups and Resolvent Operators of Grimmer

1.5.1 C_0-semigroups

Definition 1.5.1 [79] A family $(T(t))_{t\geq 0}$ of bounded linear operators on a Banach space X is called a *strongly continuous semigroup* (or a C_0-*semigroup*) if the following properties hold :

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s) \quad \text{for } t, s \geq 0$ (The semigroup property),

(iii) The orbit maps $t \longmapsto T(t)x$ are continuous from $\mathbb{R}^+$ to X for every $x \in X$.

Remark 1.5.2 If $(T(t))_{t\geq 0}$ satisfies only (i) and (ii), then $(T(t))_{t\geq 0}$ is called a *semigroup*.

Definition 1.5.3 [79] Let $(T(t))_{t\geq 0}$ be a semigroup on a Banach space X and let $D(A)$ be the subspace of X defined by

$$D(A) = \left\{x \in X : \lim_{h\to 0} \frac{1}{h}(T(t)x - x) \ \text{ exists}\right\}.$$

For every $x \in D(A)$ we define

$$Ax = \lim_{h\to 0} \frac{1}{h}\big(T(t)x - x\big).$$

The operator $A : D(A) \subset X \longrightarrow X$ is called the *infinitesimal generator* of $(T(t))_{t\geq 0}$.

Example 1.5.4 [157, p. 44] (**Strongly continuous semigroup of left translations**) Let X be a Banach space of continuous functions on $[0,1]$ which are equal to zero at $x = 1$ with supremum norm. Dfine

$$\begin{cases} (T(t)f)(s) = f(t+s) & \text{for } t+s \leq 1 \\ (T(t)f)(s) = 0 & \text{for } t+s > 1. \end{cases}$$

Then, $(T(t))_{t\geq 0}$ is a strongly continuous semigroup. Its infinitesimal generator A is given by

$$\begin{cases} D(A) = \{f : f \in C^1([0,1]) \cap X, f' \in X\} \\ Af = f'. \end{cases}$$

Example 1.5.5 [79] (Multiplication Semigroups) Let Ω be a locally compact space, $q : \Omega \longrightarrow \mathbb{C}$ a continuous function with real part bounded above, that is, $\sup_{w\in\Omega} \Re q(w) < +\infty$. On the Banach space $X = C_0(\Omega)$ (with sup-norm), define the multiplication operators

$$T_q(t)f := e^{tq}.f \quad \text{for } f \in X \text{ and } t \geq 0.$$

Then, the family $(T_q(t))_{t\geq 0}$ is a strongly continuous semigroup on X, called the multiplication semigroup, and its generator is given by the multiplication operator

$$\begin{cases} D(A) = \{f \in X : q.f \in X\} \\ Af = q.f. \end{cases}$$

Proposition 1.5.6 [79] *For every strongly continuous semigroup $(T(t))_{t\geq 0}$, there exist constant $\omega \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t} \quad \text{for all } t \geq 0.$$

The following theorem, due to Hille and Yosida, is a characterization of the generators of strongly continuous semigroups.

Theorem 1.5.7 [79] (**Hille-Yosida Theorem**) *A linear operator A is the infinitesimal generator of a*

(i) *strongly continuous semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq Me^{\omega t}$ if and only if*

(a) *A is closed and $\overline{D(A)} = X$,*

(b) *The resolvent set $\rho(A)$ of A contains $(\omega, +\infty)$ and for every $\lambda > \omega$*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n} \quad \text{for all } n \in \mathbb{N}^*.$$

(ii) *contraction strongly continuous semigroup $(T(t))_{t\geq 0}$ if and only if*

(a) *A is closed and $\overline{D(A)} = X$,*

(b) *The resolvent set $\rho(A)$ of A contains $(0, +\infty)$ and for every $\lambda > 0$*

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}.$$

Definition 1.5.8 [79] A semigroup $(T(t))_{t\geq 0}$ on a Banach space X is called *equicontinuous* (*we say also norm continuous or uniformly continuous*) if $\{T(t)x : x \in B\}$ is equicontinuous at any $t > 0$ for all bounded subsets B in X. This is equivalent to say that

$$\lim_{t\to t'} \|T(t) - T(t')\| = 0 \quad \text{for all } t' > 0.$$

Example 1.5.9 [79] The multiplication semigroup $(T_q(t))_{t\geq 0}$ generated by $q : \Omega \longrightarrow \mathbb{C}$ is equicontinuous if and only if q is bounded.

Theorem 1.5.10 [79, p. 120] *The strongly continuous semigroup of left translations defined in Example 1.5.4 is not equicontinuous.*

Definition 1.5.11 [79] A semigroup $(T(t))_{t\geq 0}$ is called *compact* if $T(t)$ is compact for all $t > 0$.

Example 1.5.12 [79, Example 1.4.34, p. 123] The operator defined by

$$\begin{cases} D(A) = \{f \in C^2([0,1],\mathbb{R}) : f'(0) = 0 = f'(1)\} \\ Af = f'', \end{cases}$$

generates a compact semigroup on $X = C([0,1],\mathbb{R})$.

1.5.2 Resolvent operators of Grimmer

In the following, X is a Banach space, A and $B(t)$ are closed linear operators on X. Y represents the Banach space $D(A)$ equipped with the graph norm defined by

$$|y|_Y := |Ay| + |y| \quad \text{for } y \in Y.$$

Let Z and W be Banach spaces. We denote by $\mathcal{L}(Z,W)$ the Banach space of all bounded linear operators from Z to W endowed with the operator norm and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$. The notation $C([0,+\infty), Y)$ stands for the space of all continuous functions from $[0,+\infty)$ into Y. We consider the following integro-differential equation

$$\begin{cases} y'(t) = Ay(t) + \int_0^t B(t-s)y(s)ds \quad \text{for } t \geq 0 \\ y(0) = y_0 \in X. \end{cases} \tag{1.5.1}$$

It was not until 1982 that Grimmer [103] proved the existence and uniqueness of resolvent operators for this integro-differential equation that give the variation of parameter formula for the solutions.

Definition 1.5.13 [103] A *resolvent operator* for Equation (1.5.1) is a bounded linear operator valued function $R(t) \in \mathcal{L}(X)$ for $t \geq 0$ having the following properties:
(a) $R(0) = I$, the identity map on X and $|R(t)| \leq \lambda e^{\beta t}$ for some constants λ and β.

(b) For each $x \in X$, $R(t)x$ is strongly continuous for $t \geq 0$.

(c) $R(t) \in \mathcal{L}(Y)$ for $t \geq 0$. For $x \in Y$, $R(\cdot)x \in C^1([0,+\infty), X) \cap C([0,+\infty), Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds \ \text{ for } t \geq 0. \end{aligned}$$

For more properties about resolvent operators theory, we refer the readers to [57, 68, 103].

Example 1.5.14 [68] Let $X = \mathbb{R}$, $Ay = y$, and $B(t) = -2y$ in Equation (1.5.1). Then, we have

$$R(t)x_0 = e^t(\cos t + \sin t)x_0 \qquad \text{and} \qquad T(t)x_0 = e^{2t}x_0.$$

Remark 1.5.15 Example 1.5.14 shows also that, in general, the resolvent operator $(R(t))_{t\geq 0}$ for Equation (1.5.1) does not satisfy the semigroup law, namely,

$$R(t+s) \neq R(t)R(s) \qquad \text{for } t, s > 0.$$

Definition 1.5.16 Let $(R(t))_{t\geq 0}$ be a resolvent operator on X. We say that $(R(t))_{t\geq 0}$ is *equicontinuous* (we say also *norm continuous* or *immediately norm continuous*) if $\{R(t)x : x \in B\}$ is equicontinuous at any $t > 0$ for all bounded subsets B in X. This is equivalent to say that

$$\lim_{t\to t'} \|R(t) - R(t')\| = 0 \quad \text{for all } t' > 0.$$

Chapter 2

Positive μ-Pseudo Almost Periodic Solutions for some Nonlinear Infinite Delay Integral Equations Arising in Epidemiology Using Hilbert's Projective Metric [(1)]

Contents

2.1 Introduction

In 1976, Cooke and Kaplan [59] initiated the study of the nonlinear delay integral equation

$$x(t) = \int_{t-\sigma}^{t} f(s, x(s))ds \quad \text{for } t \in \mathbb{R}. \tag{2.1.1}$$

They considered Equation (2.1.1) as a model to explain the evolution of some infectious diseases. More details can be found in page 3.

Equation (2.1.1) has been studied also in the periodic case in [12, 49, 50, 107, 125, 130, 131, 154, 164] and in [97] in the almost periodic case.

When the delay is state-dependent, the existence of positive almost periodic solutions to the integral equation

$$x(t) = \int_{t-\sigma(t)}^{t} f(s, x(s))ds \quad \text{for } t \in \mathbb{R}$$

[(1)] This work has been done in collaboration with Professor Khalil Ezzinbi and has been published in Nonautonomous Dynamical Systems [91] was considered in [13, 86, 97, 171]. The approach used in [86, 97, 171] is the Hilbert projective metric.

Ding *et al*. studied in [76] the existence of positive almost automorphic solutions for the neutral nonlinear delay integral equation

$$x(t) = \gamma x(t-\tau) + (1-\gamma)\int_{t-\sigma}^{t} f(s, x(s))ds \quad \text{for } t \in \mathbb{R} \tag{2.1.2}$$

where $0 \leq \gamma < 1$, $f(t,x) = \sum_{i=1}^{n} f_i(t,x)g_i(t,x)$, $f_i(t,\cdot)$ is nondecreasing in $\mathbb{R}^+$ and $g_i(t,\cdot)$ is nonincreasing in $\mathbb{R}^+$. Equation (2.1.2) was also studied in the almost periodic case by E. Ait Dads and K. Ezzinbi in [14] where $f(t,\cdot)$ is nondecreasing in $\mathbb{R}^+$.

In 2011, Ding *et al* [75] considered the existence of poitive solutions for the following infinite delay integral equation

$$x(t) = \alpha(t)x(t-\beta) + \int_{-\infty}^{t} a(t, t-s)f(s, x(s))ds + h(t, x(t)) \quad \text{for } t \in \mathbb{R}, \tag{2.1.3}$$

where $f(t,x) = \sum_{i=1}^{n} f_i(t,x)g_i(t,x)$, $x \longmapsto f_i(\cdot,x)$ is increasing in $\mathbb{R}^+$ and $x \longmapsto g_i(\cdot,x)$ and $x \longmapsto h(\cdot,x)$ are decreasing in $\mathbb{R}^+$. They discussed the existence of positive pseudo almost periodic solutions by establishing a new fixed point theorem in partially ordered Banach spaces. The result of [75] cannot be applied when one of the function $x \longmapsto f(\cdot,x)$ or $x \longmapsto h(\cdot,x)$ is neither decreasing nor increasing in $\mathbb{R}^+$.

In the case of $\alpha \equiv 0$ and $h \equiv 0$, Equation (2.1.3) is reduced to the following equation

$$x(t) = \int_{-\infty}^{t} a(t, t-s)f(s, x(s))ds \quad \text{for } t \in \mathbb{R},$$

which was considered by many authors : Ait Dads *et al*. [15, 16], Cieutat and Ezzinbi [58], respectively, showed the existence of positive almost periodic (and positive almost automorphic), positive pseudo almost periodic, and positive pseudo almost automorphic solutions by using Hilbert's projective metric.

In this chapter, we study the existence and uniqueness of positive μ-pseudo almost periodic solutions for Equation (2.1.3) without any monotonicity on the functions f and h and by using the contraction mapping principle associated with Hilbert's projective metric.

Then, we apply our result to a finite delay integral equation when the delay is state dependent. Namely,

$$x(t) = \alpha(t)x(t-\beta) + \int_{t-\sigma(t)}^{t} f(s, x(s))ds + h(t, x(t)) \quad \text{for } t \in \mathbb{R}. \tag{2.1.4}$$

2.2 Hilbert's Projective Metric

In this section, we give some facts on Hilbert's projective metric. For more details about these topic, we refer the reader to [170].

Let X be real Banach space. A closed convex set K in X is called a convex cone if the following conditions are satisfied:

(a) if $x \in K$, then $\lambda x \in K$ for $\lambda \geq 0$

(b) if $x \in K$ and $-x \in K$, then $x = 0$.

A cone K induces a partial ordering $\leq$ in X by

$$0 \leq x \leq y \qquad \text{if and only if} \qquad y - x \in K.$$

A cone K is called *normal* if there exists a constant k such that

$$x \leq y \quad \text{implies that} \quad \|x\| \leq k\|y\|$$

where $\|\cdot\|$ is the norm on X. If K is now a general cone in a Banach space X and x and y are elements of $K \setminus \{0\}$, we say that x and y are *comparable* if there exist real numbers $\alpha > 0$ and $\beta > 0$ such that

$$\alpha x \leq y \leq \beta x.$$

This define an quivalence relation on $K \setminus \{0\}$ and divides $K \setminus \{0\}$ into disjoint subsets which we call *components* of K. If x and y are comparable, we define the numbers $m(y/x)$ and $M(y/x)$ by

$$m(y/x) := \sup\{\alpha > 0; \alpha x \leq y\} \tag{2.2.1}$$

$$M(y/x) := \inf\{\beta > 0; y \leq \beta x\}. \tag{2.2.2}$$

We define a metric which was introduced by Thompson [170]. If x and $y \in K \setminus \{0\}$ are comparable, define $d(x, y)$ by

$$\begin{aligned} d(x,y) &:= \max(\log M(y/x), \log M(x/y)) \\ &= \max(\log M(y/x), -\log m(y/x)). \end{aligned} \tag{2.2.3}$$

If C is a component of K, one can easily prove [170] that d gives a metric on C. Moreover, Thompson proves the following result.

Theorem 2.2.1 [170] *Let K be a normal cone in a Banach space X and let C be a component of K. Then C is a complete metric space with respect to the metric d.*

Proposition 2.2.2 [170] *Let K be a normal cone in a Banach space X with nonempty interior $\overset{\circ}{K}$. Then $\overset{\circ}{K}$ is a component of K.*

It follows from Theorem 2.2.1 and Proposition 2.2.2 that if K is a normal cone with nonempty interior $\overset{\circ}{K}$, then $\overset{\circ}{K}$ is a complete metric space with respect to the metric d.

2.3 Existence and Uniqueness of μ-Pseudo Almost Periodic Solutions of (2.1.3) and (2.1.4)

In this section, we state a result of the existence and uniqueness of μ-pseudo almost periodic solutions of Equations (2.1.3) and (2.1.4) with a positive infimum.

Equations (2.1.3) and (2.1.4) will be studied under the following hypotheses.

(H1) $\alpha \in PAP(\mathbb{R}, \mathbb{R}, \mu)$ such that $\alpha(t) \geq 0$ for $t \in \mathbb{R}$ and $\sup_{t\in\mathbb{R}} |\alpha(t)| < 1$.

(H2) f and h are nonnegative functions belonging to $PAPU(\mathbb{R} \times \mathbb{R}, \mathbb{R}, \mu)$.

(H3) There exist φ and $\phi : (0, 1) \times (0, +\infty) \longrightarrow (0, 1]$ and $k \in [0, 1)$ such that $\varphi(\lambda, x) > \lambda^k$ and $\phi(\lambda, x) > \lambda^k$ and

$$\lambda x \leq y \leq \lambda^{-1}x \Longrightarrow \begin{cases} f(t, y) \geq \varphi(\lambda, x)f(t, x) \\ h(t, y) \geq \phi(\lambda, x)h(t, x) \end{cases} \tag{2.3.1}$$

for each $x,\ y \in (0, +\infty)$, $t \in \mathbb{R}$ and $\lambda \in (0, 1)$.

(H4) $a : \mathbb{R}\times\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that the function $t \longmapsto a(t, \cdot)$ is in $PAP(\mathbb{R}, L^1(\mathbb{R}^+), \mu)$. Moreover, there exists $b \in L^1(\mathbb{R}^+)$ such that $|a^{ap}(t, s)| \leq b(s)$ for all $t \in \mathbb{R}$ and a.e. $s \in \mathbb{R}^+$, where $t \longmapsto a^{ap}(t, \cdot)$ is the almost periodic component of $t \longmapsto a(t, \cdot)$.

(H5) There exists $x_0 > 0$ such that

$$\inf_{t\in\mathbb{R}} \int_0^{+\infty} a(t, s)f(t - s, x_0)ds > 0.$$

To give the proof of the main results, we need the following three lemmas.

Lemma 2.3.1 *Let $\mu \in \mathcal{M}$ satisfy* **(B)** *and $a : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that the function $t \longmapsto a(t, \cdot)$ is in $PAP(\mathbb{R}, L^1(\mathbb{R}^+), \mu)$. We assume that there exists $b \in L^1(\mathbb{R}^+)$ such that $|a^{ap}(t, s)| \leq b(s)$ for all $t \in \mathbb{R}$ and a.e. $s \in \mathbb{R}^+$. If $f \in PAP(\mathbb{R}, \mathbb{R}, \mu)$, then the function F defined by*

$$F(t) = \int_{-\infty}^{t} a(t, t - s)f(s)ds$$

is also μ-pseudo almost periodic. Furthermore, the almost periodic component of F is given by

$$F^{ap}(t) = \int_{-\infty}^{t} a^{ap}(t, t - s)f^{ap}(s)ds. \tag{2.3.2}$$

Proof. Let $t \longmapsto a^e(t, \cdot)$ be the μ-ergodic component of $t \longmapsto a(t, \cdot)$. To prove that F is μ-pseudo almost periodic and the almost periodic component is given by (2.3.2), we must prove that $\lim_{r\to+\infty} A(r) = 0$ where

$$A(r) = \frac{1}{\mu([-r, r])} \int_{[-r,r]} \left| \int_{-\infty}^{t} \Big(a(t, t - s)f(s) - a^{ap}(t, t - s)f^{ap}(s) \Big) ds \right| d\mu(t).$$

From $\mu(\mathbb{R}) = +\infty$, it follows $\mu([-r, r]) > 0$ for r sufficiently large. Since

$$A(r) = \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left| \int_{-\infty}^{t} \Big(a^e(t, t-s) f(s) + a^{ap}(t, t-s) f^e(s) \Big) ds \right| d\mu(t),$$

we obtain that

$$\begin{aligned} A(r) &\leq \|f\|_\infty \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|a^e(t,\cdot)\|_{L^1(\mathbb{R}^+)} d\mu(t) \\ &+ \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_0^{+} |a^{ap}(t,s) f^e(t-s)| ds \right) d\mu(t). \end{aligned}$$

By the hypothesis, one has

$$\lim_{r \to +\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|a^e(t,\cdot)\|_{L^1(\mathbb{R}^+)} d\mu(t) = 0.$$

Using Fubini's Theorem, we get

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_0^{+\infty} |a^{ap}(t,s) f^e(t-s)| ds \right) d\mu(t) \leq \int_0^{+\infty} \left(\frac{1}{\mu([-r,r])} \int_{[-r,r]} |f^e(t-s)| d\mu(t) \right) b(s) ds.$$

Since μ satisfies (**B**), then from Theorem 1.1.41, we have $[t \longmapsto f^e(t-s)] \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$ for every $s \in \mathbb{R}$. Therefore,

$$\lim_{r \to +\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} |f^e(t-s)| d\mu(t) = 0,$$

$$\left| b(s) \left(\frac{1}{\mu([-r,r])} \int_{[-r,r]} |f^e(t-s)| d\mu(t) \right) \right| \leq \|f^e\|_\infty b(s) \quad \text{for } s \in [0, +\infty).$$

Using Lebesgue's dominated convergence Theorem, we deduce that $\lim_{r \to +\infty} A(r) = 0$. This ends the proof. ■

Lemma 2.3.2 *Let $\mu \in \mathcal{M}$. Assume that $f \in PAP(\mathbb{R}, \mathbb{R}, \mu)$ and $g \in PAP(\mathbb{R}, \mathbb{R}, \mu)$. Then we have $fg \in PAP(\mathbb{R}, \mathbb{R}, \mu)$.*

Proof. Since one has $fg = f^{ap} g^{ap} + f^{ap} g^e + f^e g$, it suffices to show that $f^{ap} g^e + f^e g$ is μ ergodic. Put

$$I(r) := \frac{1}{\mu([-r,r])} \int_{[-r,r]} |(f^{ap} g^e + f^e g)(t)| d\mu(t).$$

We must prove that $\lim_{r \to +\infty} I(r) = 0$. For r sufficiently large we have

$$I(r) \leq \frac{\|f^{ap}\|_\infty}{\mu([-r,r])} \int_{[-r,r]} |g^e(t)| d\mu(t) + \frac{\|g\|_\infty}{\mu([-r,r])} \int_{[-r,r]} |f^e(t)| d\mu(t). \tag{2.3.3}$$

By the hypothesis, one has

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} |g^e(t)| d\mu(t) \longrightarrow 0 \quad \text{as} \quad r \longrightarrow +\infty, \tag{2.3.4}$$

and

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} |f^e(t)| d\mu(t) \longrightarrow 0 \quad \text{as} \quad r \longrightarrow +\infty. \tag{2.3.5}$$

Then, by (2.3.3)-(2.3.5), we obtain that $\lim_{r \to +\infty} I(r) = 0$. This ends the proof. ■

Lemma 2.3.3 *Suppose that the following hypotheses hold:*

(h1) $f \in C(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$ *and there exists* $x_1 > 0$ *such that* $f(\cdot, x_1) \in BC(\mathbb{R}, \mathbb{R})$.

(h2) *There exists a map* $\xi : (0,1) \times (0,+\infty) \longrightarrow [0,1)$ *satisfying* $\xi(\lambda, x) > \lambda^k$ *and*

$$\lambda x \leq y \leq \lambda^{-1} x \Longrightarrow f(t,y) \geq \xi(\lambda, x) f(t,x)$$

for each $x,\ y > 0$, $t \in \mathbb{R}$ *and* $\lambda \in (0,1)$.

Then, we have

i) $f(t,y) \geq \left(\min\left(\frac{x}{y}, \frac{y}{x}\right)\right)^k f(t,x)$ for $x,\ y > 0$ and $t \in \mathbb{R}$.

ii) *For each* $[a,b] \subset (0,+\infty)$, f *is bounded on* $\mathbb{R} \times [a,b]$.

Proof. i) Let $x,\ y > 0$ and $t \in \mathbb{R}$. If $x = y$ then $f(t,y) = \left(\min\left(\frac{x}{y}, \frac{y}{x}\right)\right)^k f(t,x)$. Suppose that $x \neq y$ and take $\lambda = \min\left(\frac{x}{y}, \frac{y}{x}\right)$. Then, $\lambda x \leq y \leq \lambda^{-1} x$. Hypothesis **(h2)** gives us

$$f(t,y) \geq \xi(t,x) f(t,x) > \lambda^k f(t,x) = \left(\min\left(\frac{x}{y}, \frac{y}{x}\right)\right)^k f(t,x).$$

ii) Let $x \in [a,b]$. By **i)**, it follows that

$$f(t,x_1) \geq \left(\min\left(\frac{x}{x_1}, \frac{x_1}{x}\right)\right)^k f(t,x) \geq \left(\min\left(\frac{a}{x_1}, \frac{x_1}{b}\right)\right)^k f(t,x).$$

The desired result follows from Hypothesis **(h1)**. The proof is now complete. ■

Now, we give the first result of this chapter which shows the existence and uniqueness of positive μ-pseudo almost periodic solutions for Equation (2.1.3).

Theorem 2.3.4 *Let* $\mu \in \mathcal{M}$ *satisfy* (**B**). *Suppose that* **(H1)-(H5)** *hold. Then, Equation* (2.1.3) *has a unique* μ*-pseudo almost periodic solution* x_* *with a positive infimum :*

$$\inf_{t \in \mathbb{R}} x_*(t) > 0.$$

Proof. Let $X = PAP(\mathbb{R}, \mathbb{R}, \mu)$ be the space of μ-pseudo almost periodic functions endowed with the norm defined by $\|f\|_\infty = \sup_{t \in \mathbb{R}} |f(t)|$. Since $\mu \in \mathcal{M}$ satisfies (**B**), then by Theorem 1.1.41, X is a Banach space. Let K be the cone of nonnegative functions in $PAP(\mathbb{R}, \mathbb{R}, \mu)$. Then, K is a normal convex cone. Furthermore, we have

$$0 \leq x \leq y \qquad \Longrightarrow \qquad \|x\|_\infty \leq \|y\|_\infty.$$

The interior of K is given by $\overset{\circ}{K} = \{x \in PAP(\mathbb{R}, \mathbb{R}, \mu) : \inf_{t \in \mathbb{R}} x(t) > 0\}$. Define the operator A on $BC(\mathbb{R}, \mathbb{R})$ by

$$(Ax)(t) = \alpha(t) x(t-\beta) = \alpha(t) \tau_\beta x(t) \quad \text{for } t \in \mathbb{R}.$$

From **(H1)**, A is bounded linear operator from $BC(\mathbb{R},\mathbb{R})$ to $BC(\mathbb{R},\mathbb{R})$ and

$$\|A\| \leq \sup_{t\in\mathbb{R}} |\alpha(t)| < 1.$$

It follows that $(I-A)^{-1}$ exists and $B = (I-A)^{-1} = \sum_{k=0}^{+\infty} A^k$. Furthermore,

$$B = I + \sum_{n\geq 1} \left(\prod_{k=0}^{n-1} \tau_{k\beta}\alpha \right) \tau_{n\beta}. \tag{2.3.6}$$

If we set $u(t) = x(t) - \alpha(t)x(t-\beta) = \big((I-A)x\big)(t)$, then Equation (2.1.3) is equivalent to the following equation

$$u(t) = \int_{-\infty}^{t} a(t, t-s) f\big(s, (Bu)(s)\big) ds + h\big(t, (Bu)(t)\big). \tag{2.3.7}$$

Since μ satisfies (**B**), it follows from Theorem 1.1.41 that $PAP(\mathbb{R},\mathbb{R},\mu)$ is invariant by translation. In addition, since $\alpha \in PAP(\mathbb{R},\mathbb{R},\mu)$, Lemma 2.3.2 yields that

$$B(PAP(\mathbb{R},\mathbb{R},\mu)) \subset PAP(\mathbb{R},\mathbb{R},\mu). \tag{2.3.8}$$

From **(H3)** and **ii**) of Lemma 2.3.3 we deduce that f and h satisfy condition (**C**) of Theorem 1.1.44. Using **(H2)**, Theorem 1.1.44 and Lemma 2.3.3 we obtain that

$$u \in PAP(\mathbb{R},\mathbb{R},\mu) \Longrightarrow \Big[t \mapsto f\big(t, (Bu)(t)\big)\Big] \in PAP(\mathbb{R},\mathbb{R},\mu), \Big[t \mapsto h\big(t, (Bu)(t)\big)\Big] \in PAP(\mathbb{R},\mathbb{R},\mu).$$

On the other hand, in view of Hypothesis **(H4)** and Lemma 2.3.1, we get that

$$u \in PAP(\mathbb{R},\mathbb{R},\mu) \Longrightarrow \left[t \mapsto \int_{-\infty}^{t} a(t, t-s) f\big(s, (Bu)(s)\big) ds\right] \in PAP(\mathbb{R},\mathbb{R},\mu).$$

Now, we have proved that

$$u \in PAP(\mathbb{R},\mathbb{R},\mu) \Longrightarrow \left[t \mapsto \int_{-\infty}^{t} a(t, t-s) f\big(s, (Bu)(s)\big) ds + h\big(t, (Bu)(t)\big)\right] \in PAP(\mathbb{R},\mathbb{R},\mu). \tag{2.3.9}$$

We claim that $x \in \overset{\circ}{K}$ is a solution of Equation (2.1.3) if and only if $u \in \overset{\circ}{K}$ is a solution of Equation (2.3.7). It suffices to prove that $x \in \overset{\circ}{K}$ if and only if $u \in \overset{\circ}{K}$.

Let $x \in \overset{\circ}{K}$ be a solution of Equation (2.3.1). Then, there exists $\varepsilon > 0$ such that

$$\varepsilon < x(t) < \varepsilon^{-1} \quad \text{for } t \in \mathbb{R}.$$

In view of **(H3)** and Lemma 2.3.3, it follows that

$$\begin{aligned} u(t) &= x(t)-\alpha(t)x(t-\beta) \\ &= \int_{-\infty}^{t} a(t,t-s)f(s,x(s))ds + h(t,x(t)) \\ &\geq \int_{-\infty}^{t} a(t,t-s)f(s,x(s))ds \\ &\geq \int_{-\infty}^{t} a(t,t-s)\left(\min\left(\frac{x_0}{x(s)},\frac{x(s)}{x_0}\right)\right)^k f(s,x_0)ds \\ &\geq \varepsilon^k\left(\min\left(x_0,\frac{1}{x_0}\right)\right)^k \int_{-\infty}^{t} a(t,t-s)f(s,x_0)ds \\ &\geq \varepsilon^k\left(\min\left(x_0,\frac{1}{x_0}\right)\right)^k \inf_{t\in\mathbb{R}}\int_{0}^{+\infty} a(t,s)f(t-s,x_0)ds. \end{aligned}$$

Taking into account **(H5)**, we get that $\inf_{t\in\mathbb{R}} u(t) > 0$. On the other hand, since x and α are μ-pseudo almost periodic, u is also μ-pseudo almost periodic. Hence $u \in \overset{\circ}{K}$.

Now, suppose that $u \in \overset{\circ}{K}$ and let $t \in \mathbb{R}$. By (2.3.6), we have $B(u)(t) \geq u(t)$. Since $x(t) = (Bu)(t)$ we obtain that $x(t) \geq u(t)$. Then, $\inf_{t\in\mathbb{R}} x(t) \geq \inf_{t\in\mathbb{R}} u(t) > 0$. Using (2.3.8), it follows that x is μ-pseudo almost periodic and consequently $x \in \overset{\circ}{K}$.

We denote by T the operator associated with the right-hand side of Equation (2.3.7). Namely,

$$(Tx)(t) = \int_{-\infty}^{t} a(t,t-s)f\big(s,(Bx)(s)\big)ds + h\big(t,(Bx)(t)\big).$$

Next, we prove that the opertor T is a contraction. We consider

$$(T_1x)(t) = \int_{-\infty}^{t} a(t,t-s)f\big(s,(Bx)(s)\big)ds \qquad \text{and} \qquad (T_2x)(t) = h\big(t,(Bx)(t)\big).$$

From the discussion above (see (2.3.9)), the operator T maps $PAP(\mathbb{R},\mathbb{R},\mu)$ into itself. Let $x \in \overset{\circ}{K}$. Then, from (2.3.6) and (2.3.8), $Bx \in \overset{\circ}{K}$. Hence, there exists $\varepsilon > 0$ such that

$$\varepsilon < (Bx)(t) < \varepsilon^{-1} \quad \text{for } t \in \mathbb{R}.$$

In view of Lemma 2.3.3, we obtain

$$\begin{aligned} (Tx)(t) &\geq \int_{-\infty}^{t} a(t,t-s)f\big(s,(Bx)(s)\big)ds \\ &\geq \int_{-\infty}^{t} a(t,t-s)\left(\min\left(\frac{x_0}{(Bx)(s)},\frac{(Bx)(s)}{x_0}\right)\right)^k f(s,x_0)ds \\ &\geq \varepsilon^k\left(\min\left(x_0,\frac{1}{x_0}\right)\right)^k \int_{-\infty}^{t} a(t,t-s)f(s,x_0)ds \\ &\geq \varepsilon^k\left(\min\left(x_0,\frac{1}{x_0}\right)\right)^k \inf_{t\in\mathbb{R}}\int_{0}^{+\infty} a(t,s)f(t-s,x_0)ds. \end{aligned}$$

Therefore, by **(H5)**, we obtain that $\inf_{t\in\mathbb{R}}(Tx)(t) > 0$ and consequently $Tx \in \overset{\circ}{K}$ for all $x \in \overset{\circ}{K}$. So, T maps $\overset{\circ}{K}$ into itself.
Let $x,\ y \in \overset{\circ}{K}$ and $\lambda \in (0,1)$ such that $\lambda x \leq y \leq \lambda^{-1}x$. It follows from **(H3)** that

$$\begin{cases} f(t,y(t)) \geq \varphi(\lambda, x(t))f(t,x(t)) \\ h(t,y(t)) \geq \phi(\lambda, x(t))h(t,x(t)) \end{cases}$$

for $t \in \mathbb{R}$. We have also $\lambda y \leq x \leq \lambda^{-1}y$. Then,

$$\begin{cases} \varphi(\lambda, x(t))f(t,x(t)) \leq f(t,y(t)) \leq \big(\varphi(\lambda, x(t))\big)^{-1}f(t,x(t)) \\ \phi(\lambda, x(t))h(t,x(t)) \leq h(t,y(t)) \leq \big(\phi(\lambda, x(t))\big)^{-1}h(t,x(t)) \end{cases}$$

for $t \in \mathbb{R}$. This gives us

$$\begin{cases} \lambda^k f(t,x(t)) \leq f(t,y(t)) \leq \lambda^{-k}f(t,x(t)) \\ \lambda^k h(t,x(t)) \leq h(t,y(t)) \leq \lambda^{-k}h(t,x(t)) \end{cases} \tag{2.3.10}$$

for $t \in \mathbb{R}$. Multiplying the first inequality of (2.3.10) by $a(t,t-s)$ and integrating over $(-\infty,t)$ we obtain for $t \in \mathbb{R}$

$$\lambda^k(T_1x)(t) \leq (T_1y)(t) \leq \lambda^{-k}(T_1x)(t) \tag{2.3.11}$$

and

$$\lambda^k(T_2x)(t) \leq (T_2y)(t) \leq \lambda^{-k}(T_2x)(t). \tag{2.3.12}$$

Summing (2.3.11) and (2.3.12) we obtain that

$$\lambda^k Tx \leq Ty \leq \lambda^{-k}Tx. \tag{2.3.13}$$

For $\lambda = \left(\max\left(M(y/x), M(x/y)\right)\right)^{-1}$, we deduce that $d(x,y) = \log(\lambda^{-1})$. Using (2.2.1)-(2.2.3), inequality (2.3.13) leads to

$$d(Tx,Ty) \leq \log(\lambda^{-k}).$$

This shows that

$$d(Tx,Ty) \leq k\log(\lambda^{-1}) = kd(x,y).$$

We know from Theorem 2.2.1 and Proposition 2.2.2 that $(\overset{\circ}{K}, d)$ is a complete metric space. In view of the contraction mapping principle (Theorem 1.2.1), the operator T has a unique fixed point $x_* \in \overset{\circ}{K}$ which means that Equation (2.1.3) has a unique μ-pseudo almost periodic solution with a positive infimum. The proof is now complete. ■

Next, we give the second result of this chapter which is a corollary of Theorem 2.3.4 about Equation (2.1.4).

Corollary 2.3.5 *Let $\mu \in \mathcal{M}$ satisfy* (**B**)*. Suppose that* **(H1)-(H3)** *hold. In addition, we assume that*

i) *σ is a positive μ-pseudo almost periodic function.*

ii) *There exists $x_0 > 0$ such that*

$$\inf_{t\in\mathbb{R}} \int_{t-\sigma(t)}^{t} f(s, x_0)ds > 0. \tag{2.3.14}$$

Then, Equation (2.1.4) *has a unique μ-pseudo almost periodic solution x_* with a positive infimum.*

Proof. Equation (2.1.4) can be transformed into Equation (2.1.3) by letting

$$a(t, s) = \mathbf{1}_{[0,\sigma(t)]}(s) = \begin{cases} 1, & s \in [0, \sigma(t)] \\ 0, & s \notin [0, \sigma(t)]. \end{cases}$$

Obviously (2.3.14) of **ii)** implies **(H5)**. Now, let us prove that **(H4)** holds. By using the fact that

$$\begin{cases} \|\mathbf{1}_{[0,\sigma^{ap}(t+\tau)]} - \mathbf{1}_{[0,\sigma^{ap}(t)]}\|_{L^1(\mathbb{R}^+)} = |\sigma^{ap}(t+\tau) - \sigma^{ap}(t)| \\ \|\mathbf{1}_{[0,\sigma(t)]} - \mathbf{1}_{[0,\sigma^{ap}(t)]}\|_{L^1(\mathbb{R}^+)} = |\sigma(t) - \sigma^{ap}(t)|, \end{cases}$$

we deduce from **i)** that $t \longmapsto a(t, \cdot)$ is in $PAP(\mathbb{R}, L^1(\mathbb{R}^+), \mu)$ with $a^{ap}(t, \cdot) = \mathbf{1}_{[0,\sigma^{ap}(t)]}(\cdot)$. Furthermore, we have $|a^{ap}(t, s)| \leq \mathbf{1}_{[0,\|\sigma^{ap}\|_\infty]}(s)$ and consequently **(H4)** is satisfied. By applying Theorem 2.3.4, we obtain the desired result. ■

2.4 Application

In this section, we present an example to illustrate the main results of this chapter. We also give a remark proving the advantage of the main results.

Example 2.4.1 We consider the measure μ where its Radon-Nikodym derivative is

$$\rho(t) = \begin{cases} e^t & \text{for } t \leq 0, \\ 1 & \text{for } t > 0. \end{cases}$$

It is well known from [44, Example 3.6, p. 17] that $\mu \in \mathcal{M}$ satisfies (**B**). Let $\alpha(t) \equiv \frac{1}{5}$, $\beta = 1$ and

$$a(t, s) = \frac{3 - \sin\sqrt{2}t - \sin\pi t}{5(1 + s^2)} \quad \text{for } t \in \mathbb{R} \text{ and } s \in \mathbb{R}^+,$$

$$f(t, x) = \left(1 + \cos^2 t + \cos^2(\pi t) + \frac{\pi}{2} - \arctan(t)\right) \sqrt[4]{\sqrt{|x|}\ln\left(1 + \sqrt{|x|}\right)},$$

$$h(t, x) = \left(3 + \sin(\pi t) + \sin(\sqrt{3}t) + \frac{1}{1 + t^2}\right) \sqrt[4]{x^2 + |x|}$$

for t and $x \in \mathbb{R}$. It is clear that the Hypothesis **(H1)** is satisfied and f and h are nonnegative. On the other hand, f and h are μ-pseudo almost periodic with μ-ergodic components $f^e(t,x) = \left(\frac{\pi}{2} - \arctan(x)\right)\sqrt[4]{\sqrt{|x|}\ln\left(1+\sqrt{|x|}\right)}$ and $h^e(t,x) = \frac{\sqrt[4]{x^2+|x|}}{1+t^2}$. In fact, we have

$$\frac{1}{\mu([-r,r])}\int_0^r |\frac{\pi}{2} - \arctan(t)|dt = \frac{\frac{\pi}{2}r + \ln(\sqrt{r^2+1}) - r\arctan(r)}{1-e^{-r}+r} \longrightarrow 0 \quad \text{as } r \longrightarrow +\infty,$$

$$\frac{1}{\mu([-r,r])}\int_{-r}^0 \left|\frac{\pi}{2} - \arctan(t)\right| e^t dt \leq \frac{1}{\mu([-r,r])}\int_{-r}^0 \pi e^t dt \longrightarrow 0 \quad \text{as } r \longrightarrow +\infty.$$

For h^e we use [44, Corollary 2.15, p. 8]. Consequently, Hypothesis **(H2)** holds. Define the functions

$$\varphi(\lambda,x) = \lambda^{\frac{1}{8}}\sqrt[4]{\frac{\ln(1+\sqrt{\lambda x})}{\ln(1+\sqrt{x})}}, \qquad \phi(\lambda,x) = \sqrt[4]{\frac{\lambda^2x^2+\lambda x}{x^2+x}} \qquad \text{for } x > 0 \text{ and } \lambda \in (0,1).$$

Then, we can easily prove that (2.3.1) is satisfied. Therefore, we have

$$\varphi(\lambda,x) > \lambda^{\frac{1}{4}} > \lambda^{\frac{1}{2}}, \qquad \phi(\lambda,x) > \lambda^{\frac{1}{2}} \quad \text{for } x > 0 \text{ and } \lambda \in (0,1).$$

Hypothesis **(H3)** is now fulfilled with $k = \frac{1}{2}$. It is easy to show that Hypothesis **(H4)** holds with $b(s) = (1+s^2)^{-1}$ for $s \in \mathbb{R}^+$.
On the other hand, for $x > 0$ and $y > 0$ we have that

$$\inf_{t\in\mathbb{R}}\int_0^{+\infty} a(t,s)f(t-s,x)ds \geq \sqrt[4]{\sqrt{x}\ln\left(1+\sqrt{x}\right)}\inf_{t\in\mathbb{R}}\int_0^{+\infty}\frac{ds}{5(1+s^2)} = \frac{\pi}{10}\sqrt[4]{\sqrt{x}\ln\left(1+\sqrt{x}\right)} > 0,$$

which yields that Hypothesis **(H5)** is satisfied. All the assumptions in Theorem 2.3.4 are verified. Hence, Equation (2.1.3) with the above functions α, a, f and h has a unique μ-pseudo almost periodic solution x_* with a positive infimum.

We close this chapter by the following interesting remark.

Remark 2.4.2 (In the case $d\mu = dt$)
i) Since for each $t \in \mathbb{R}$, the function $x \longmapsto h(t,x)$ is not decreasing in $\mathbb{R}^+$, the result [75, Theorem 3.3] is not applicable to Example 2.4.1. This proves that our result is different from [75] and our consideration is significant. So, an advantage of our result is that the monotonicity is not imposed on the functions f and h.
ii) Let $p : \mathbb{R} \longrightarrow \mathbb{R}$ be a nonnegative function such that p is μ-pseudo almost periodic. Then, the function defined by the following

$$h(t,x) = \begin{cases} p(t)\sqrt[3]{|x|} & \text{for } x \in [-1,1], \\ \dfrac{p(t)}{\sqrt[3]{|x|}} & \text{for } x \in (-\infty,-1] \cup [1,+\infty), \end{cases} \tag{2.4.1}$$

is nonnegative and belongs to $PAPU(\mathbb{R} \times \mathbb{R}, \mathbb{R}, \mu)$. Define the function ϕ on $(0, 1) \times (0, +\infty)$ by $\phi(\lambda, x) = \sqrt[3]{\lambda}$. Then, for $x, y > 0$ and $\lambda \in (0, 1)$ we have

$$\phi(\lambda, x) > \lambda^{\frac{1}{2}} \quad \text{and} \quad \lambda x \leq y \leq \lambda^{-1}x \quad \Longrightarrow \quad h(t, y) \geq \phi(\lambda, x)h(t, x).$$

So, if we define the function h in Example 2.4.1 by (2.4.1), we obtain the same conclusion. Also, [75, Theorem 3.3] cannot be applied to Example 2.4.1 because the function $x \longmapsto h(t, x)$ is neither decreasing nor increasing in $\mathbb{R}^+$.

Chapter 3

Behavior of Bounded Solutions for Some Almost Periodic Neutral Partial Functional Differential Equations [1]

Contents

3.1 Introduction

The purpose of this chapter is to study the behavior of bounded integral solutions of the following class of partial neutral functional differential equations

$$\frac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{3.1.1}$$

when the forcing term $f : \mathbb{R} \longrightarrow X$ is Stepanov almost periodic and the operator A is not necessarily densely defined but satisfies the following Hille-Yosida condition:

(H0) There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$|R(\lambda, A)^n| \leq \frac{M}{(\lambda - \omega)^n} \quad \text{for } n \in \mathbb{N} \text{ and } \lambda > \omega,$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) := (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$.

[1] This work has been done in collaboration with El Hadi Ait Dads, Brahim Es-sebbar, Khalil Ezzinbi and has been published in Mathematical Methods in the Applied Sciences [10]

L is a bounded linear operator from C to X where $C := C([-r, 0], X)$ denotes the space of continuous functions from $[-r, 0]$ to X endowed with the uniform norm topology. For every $t \in \mathbb{R}$, the history function $u_t \in C$ is defined by

$$u_t(\theta) := u(t + \theta), \quad \text{for } \theta \in [-r, 0].$$

Let us consider the following differential equation in $\mathbb{R}^n$:

$$x'(t) = A(t)x(t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{3.1.2}$$

where $A(t)$ is a $n \times n$-matrix and $f : \mathbb{R} \longrightarrow \mathbb{R}^n$. One of the questions that are of interest to many researchers is when exist periodic solutions to (3.1.2) where f and B are ω periodic. In [141], Massera proved that the existence of a bounded solution of (3.1.2) on $\mathbb{R}^+$ is enough to get the existence of an ω periodic solutions. This result is known in the literature as Massera's Theorem. Bohr and Neugebauer [41] extended Massera's Theorem for Equation (3.1.2) to the almost periodic case when $A(t) = A$ a constant matrix. Moreover, they proved that every bounded solution on $\mathbb{R}$ is almost periodic. It is known that when the matrix $A(t)$ is not constant, bounded solution of (3.1.2) are not in general almost periodic [123]. We can find in [95] an example of an almost periodic scalar linear equation $x'(t) = a(t)x(t)$ in $\mathbb{R}$ where all the solutions are bounded on $\mathbb{R}$ but are not almost periodic except zero solution. In [95], Fink extended the results of Massera and Bohr-Neugebauer to infinite dimensional Banach space X when $A(t)$ is periodic. This last result, was extended to the almost automorphic case in [135, 145]. Massera's Theorem was also extended in [3] to the partial neutral functional differential equations (3.1.1) when $f : \mathbb{R} \longrightarrow X$ is almost periodic and A is a Hille-Yosida operator.

For more details about the study of the nature of bounded solutions to various classes of almost periodic equations in infinite dimensional spaces, we refer the reader to the papers [74, 119] and the books [24, 64, 132, 187].

In [24], Amerio and Prouse gave a result of integration of almost periodic functions in the sense of Stepanov (or S^p-almost periodic functions) in uniformly convex spaces when $p > 1$. The proof is based on the fact that the space $L^p([0, 1], X)$ is uniformly convex for $p > 1$ whenever X is uniformly convex. If $p = 1$, then the space $L^p([0, 1], X)$ fails to be uniformly convex.

In this chapter, we use a Bochner type characterization for Stepanov almost periodicity (see Proposition 1.1.22) which will help us to treat the integration of S^p-almost periodic functions in uniformly convex spaces when $p = 1$. Using this result, we investigate the nature of bounded integral solutions of Equation (3.1.1). More specifically, we prove under a compactness condition, that all bounded integral solutions of Equation (3.1.1) on $\mathbb{R}$ are almost periodic, when the forcing term f is only S^1-almost periodic. Moreover, we show that the existence of a bounded integral solution on $\mathbb{R}^+$ is sufficient to guarantee the existence of an almost periodic integral solution.

The principal working tools in this chapter are the variation of the constants formula and a reduction principle similar to the one developed in [7].

3.2 Variation of Constants Formula and Reduction Principle

In the sequel, we assume that the operator A satisfies the Hille-Yosida condition **(H0)**. To Equation (3.1.1), we associate the following Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t) \quad \text{for } t \geq \sigma \\ u_\sigma = \varphi \in C. \end{cases} \tag{3.2.1}$$

Definition 3.2.1 [7] A continuous function $u : [-r+\sigma, +\infty) \longrightarrow X$ is called an *integral solution* of Equation (3.2.1), if

(i) $\displaystyle\int_\sigma^t \mathcal{D}u_s ds \in D(A)$ for $t \geq \sigma$,

(ii) $\mathcal{D}u_t = \mathcal{D}\varphi + A\displaystyle\int_\sigma^t \mathcal{D}u_s ds + \int_\sigma^t [L(u_s) + f(s)]ds$ for $t \geq \sigma$,

(iii) $u_\sigma = \varphi$.

Let us introduce the part A_0 of the operator A in $\overline{D(A)}$ defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\} \\ A_0 x = Ax \quad \text{for } x \in D(A_0). \end{cases}$$

We have the following lemma.

Lemma 3.2.2 [169] *Assume that* **(H0)** *holds. Then,* A_0 *generates a strongly continuous semigroup* $(T_0(t))_{t\geq 0}$ *on* $\overline{D(A)}$.

For the existence of integral solutions, one has the following result.

Theorem 3.2.3 [5] *Assume that* **(H0)** *holds. Then, for all* $\varphi \in C$ *such that* $\mathcal{D}\varphi \in \overline{D(A)}$*, Equation* (3.2.1) *has a unique integral solution* u *on* $[-r+\sigma, +\infty)$*. Moreover,* u *is given by*

$$\begin{cases} \mathcal{D}u_t = T_0(t-\sigma)\mathcal{D}\varphi + \lim\limits_{\lambda\to+\infty}\displaystyle\int_\sigma^t T_0(t-s)B_\lambda\,[L(u_s) + f(s)]\,ds \quad \text{for } t \geq \sigma, \\ u_\sigma = \varphi, \end{cases}$$

where $B_\lambda = \lambda R(\lambda, A)$ *for* $\lambda > \omega$.

In the sequel of this chapter, for simplicity, integral solutions are called solutions. $u(., \sigma, \varphi, f)$ denotes the solution of Equation (3.2.1). The phase space C_0 of Equation (3.2.1) is given by

$$C_0 = \{\varphi \in C : \mathcal{D}\varphi \in \overline{D(A)}\}.$$

For each $t \geq 0$, we define the linear operator $U(t)$ on C_0 by

$$U(t)\varphi = u_t(., 0, \varphi, 0),$$

where $u(., 0, \varphi, 0)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt} u(t) = Au(t) + L(\nu_t) & \text{for } t \geq 0, \\ u_0 = \varphi. \end{cases}$$

We have the following result:

Proposition 3.2.4 [7] *Assume that* **(H0)** *holds. Then,* $(U(t))_{t\geq 0}$ *is a strongly continuous semigroup on* C_0*. Moreover, the operator* $\mathcal{A}_U$ *defined on* C_0 *by*

$$\begin{cases} D(\mathcal{A}_U) = \left\{\varphi \in C^1([-r, 0], X) : \mathcal{D}\varphi \in D(A), \mathcal{D}\varphi' \in \overline{D(A)} \text{ and } \mathcal{D}\varphi' = A\mathcal{D}\varphi + L(\varphi)\right\} \\ \mathcal{A}_U \varphi = \varphi' \end{cases}$$

is the infinitesimal generator of $(U(t))_{t\geq 0}$ *on* C_0*.*

In order to give the variation of constants formula associated to Equation (3.2.1), we need to extend the semigroup $(U(t))_{t\geq 0}$ to the space $C_0 \oplus \langle X_0 \rangle$, where $\langle X_0 \rangle$ is the space defined by

$$\langle X_0 \rangle = \{X_0 y : y \in X\},$$

where the function $X_0 y$ is given, for each $y \in X$ by

$$(X_0 y)(\theta) = \begin{cases} 0 & \text{if } \theta \in [-r, 0), \\ y & \text{if } \theta = 0. \end{cases}$$

The space $C_0 \oplus \langle X_0 \rangle$ equipped with the norm $|\varphi + X_0 y| = |\varphi| + |y|$, for $(\varphi, y) \in C_0 \times X$, is a Banach space. Consider the extension $\widetilde{\mathcal{A}_U}$ of the operator $\mathcal{A}_U$ on $C_0 \oplus \langle X_0 \rangle$ defined by

$$\begin{cases} D(\widetilde{\mathcal{A}_U}) = \{\varphi \in C^1([-r, 0], X) : \mathcal{D}\varphi \in D(A) \text{ and } \mathcal{D}\varphi' \in \overline{D(A)}\}, \\ \widetilde{\mathcal{A}_U}\varphi = \varphi' + X_0(A\mathcal{D}\varphi + L(\varphi) - \mathcal{D}\varphi'). \end{cases}$$

In order to compute the resolvent operator $R(\lambda, \widetilde{\mathcal{A}_U})$, we need to make the following assumption

(H1) $\mathcal{D}(e^{\lambda \cdot}c) \in D(A)$, for all $c \in D(A)$ and all complex λ, where $e^{\lambda \cdot}c \in C$ is defined by

$$(e^{\lambda \cdot}c)(\theta) = e^{\lambda\theta}c \quad \text{for } \theta \in [-r, 0].$$

Lemma 3.2.5 [7, Theorem 13] *Assume that* **(H0)** *and* **(H1)** *hold. Then, $\widetilde{\mathcal{A}_U}$ satisfies the Hille-Yosida condition on $C_0 \oplus \langle X_0 \rangle$, that is, there exist $\widetilde{M} \geq 0$ and $\widetilde{\omega} \in \mathbb{R}$ such that $(\widetilde{\omega}, +\infty) \subset \rho(\widetilde{\mathcal{A}_U})$ and*

$$\left|R(\lambda, \widetilde{\mathcal{A}_U})^n\right| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \quad \text{for } n \in \mathbb{N} \text{ and } \lambda > \widetilde{\omega}.$$

Now, we can state the variation of constants formula associated to Equation (3.2.1).

Theorem 3.2.6 [7, Theorem 16] *Assume that* **(H0)** *and* **(H1)** *hold. Then, for all $\varphi \in C_0$, the solution $u(., \sigma, \varphi, f)$ of Equation (3.2.1) is given by the following variation of constants formula*

$$u_t(., \sigma, \varphi, f) = U(t-\sigma)\varphi + \lim_{n \to +\infty} \int_\sigma^t U(t-s)\widetilde{B}_n\left(X_0 f(s)\right) ds \quad \text{for } t \geq \sigma,$$

where $\widetilde{B}_n = nR(n, \widetilde{\mathcal{A}_U})$ for $n > \widetilde{\omega}$.

The following definition is essential to describe the asymptotic behavior of the semigroup $(U(t))_{t\geq 0}$.

Definition 3.2.7 [111, Definition 3.1, p. 275] The operator $\mathcal{D}$ is said to be stable if there exist positive constants η and μ such that the solution of the homogeneous difference equation

$$\begin{cases} \mathcal{D}u_t = 0 & \text{for } t \geq 0 \\ u_0 = \phi, \end{cases}$$

where $\phi \in \{\psi \in C : \mathcal{D}\psi = 0\}$, satisfies

$$|u_t(., \phi)| \leq \mu e^{-\eta t}|\phi| \quad \text{for } t \geq 0.$$

Example 3.2.8 [180, 181] The operator $\mathcal{D}$ defined by

$$\mathcal{D}\phi = \phi(0) - q\phi(-r)$$

is stable if and only if $|q| < 1$.

The following assumptions play a crucial role to get the reduction principle.

(H2) The operator $T_0(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

(H3) The operator $\mathcal{D}$ is stable.

We have the following fundamental result on the semigroup $(U(t))_{t\geq 0}$.

Theorem 3.2.9 [7, Lemma 10] *Assume that* **(H0)**, **(H2)** *and* **(H3)** *hold. Then, the semigroup* $(U(t))_{t\geq 0}$ *is decomposed on* C_0 *as follows:*

$$U(t) = U_1(t) + U_2(t) \quad \text{for} \;\; t \geq 0,$$

where $(U_1(t))_{t\geq 0}$ *is an exponentially stable semigroup on* C_0*, which means that there are positive constants* α_0 *and* N_0 *such that*

$$|U_1(t)\phi| \leq N_0 e^{-\alpha_0 t}|\phi| \quad \text{for} \;\; t \geq 0 \text{ and } \phi \in C_0,$$

and $U_2(t)$ *is compact for every* $t > 0$.

For a bounded linear operator K on Y, we define $|K|_\alpha$ by

$$|K|_\alpha := \inf\left\{k > 0 : \;\; \alpha\left(K\left(B\right)\right) \leq k\alpha\left(B\right) \;\; \text{for any bounded set } B \text{ of } Y\right\},$$

where α is the Kuratowski measure of noncompactness defined in Chapter 1 which contains some properties of α.

The essential growth bound $\omega_{ess}\left(U\right)$ of the C_0-semigroup $\left(U\left(t\right)\right)_{t\geq 0}$ is defined by

$$\omega_{ess}\left(U\right) := \lim_{t\to+\infty} \frac{\log\left|U\left(t\right)\right|_\alpha}{t} = \inf_{t>0} \frac{\log\left|U\left(t\right)\right|_\alpha}{t}. \tag{3.2.2}$$

By Theorem 3.2.9, we have for $t > 0$

$$\left|U\left(t\right)\right|_\alpha \;\; \leq \;\; \left|U_1\left(t\right)\right|_\alpha + \left|U_2\left(t\right)\right|_\alpha \leq \left|U_1\left(t\right)\right| \leq N_0 e^{-\alpha_0 t}.$$

We deduce using (3.2.2) that $\omega_{ess}\left(U\right) \leq -\alpha_0 < 0$. Consequently, by [79, Theorem 5.3.7, p. 333], we get the following spectral decomposition.

Theorem 3.2.10 *Assume that* **(H0)**, **(H2)** *and* **(H3)** *hold. Then,* C_0 *is decomposed as follows:*

$$C_0 = S \oplus V, \tag{3.2.3}$$

where S *is* U*-invariant and there are positive constants* α *and* N *such that*

$$\left|U\left(t\right)\varphi\right| \leq Ne^{-\alpha t}|\varphi| \quad \text{for each } \varphi \in S \;\; \text{and} \;\; t \geq 0.$$

V *is a finite dimensional space and the restriction of the semigroup* $(U(t))_{t\geq 0}$ *to* V *becomes a group.*

In the sequel, $U^s(t)$ and $U^v(t)$ denote the restriction of $U(t)$, respectively, on S and V, which correspond to the above decomposition. Let $d := \dim\left(V\right)$ with a vector basis $\Phi = \{\varphi_1, \ldots, \varphi_d\}$. Then, there exist d-elements $\{\psi_1, \ldots, \psi_d\}$ in C_0^* such that

$$\begin{cases} \langle\psi_i, \varphi_j\rangle = \delta_{ij}, \\ \langle\psi_i, \varphi\rangle = 0 \qquad \text{for all } \varphi \in S \;\; \text{and} \;\; i \in \{1, \ldots, d\}, \end{cases} \tag{3.2.4}$$

where $\langle .,. \rangle$ denotes the duality pairing between C_0^* and C_0, and

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\Psi = \operatorname{col}\{\psi_1, \dots \psi_2\}$, $\langle \Psi, \Phi \rangle$ is a $(d \times d)$-matrix, where the (i,j)-component is $\langle \psi_i, \varphi_j \rangle$. Denote by Π^s and Π^v the projections, respectively, on S and V. For each $\varphi \in C_0$, we have

$$\Pi^v \varphi = \Phi \langle \Psi, \varphi \rangle .$$

In fact, for $\varphi \in C_0$, we have $\varphi = \Pi^s \varphi + \Pi^v \varphi$ with $\Pi^v \varphi = \sum_{i=1}^{d} \alpha_i \varphi_i$ and $\alpha_i \in \mathbb{R}$. By (3.2.4), we conclude that

$$\alpha_i = \langle \psi_i, \varphi \rangle .$$

Hence,

$$\Pi^v \varphi = \sum_{i=1}^{d} \langle \psi_i, \varphi \rangle \varphi_i = \Phi \langle \Psi, \varphi \rangle .$$

Because $(U^v(t))_{t \geq 0}$ is a group on V, there exists a $(d \times d)$-matrix G such that

$$U^v(t) \Phi = \Phi e^{tG} \quad \text{for } t \in \mathbb{R}.$$

For $n, n_0 \in \mathbb{N}$ such that $n \geq n_0 \geq \widetilde{\omega}$ and $i \in \{1, \dots, d\}$, we define the linear operator $x_{i,n}^*$ by

$$x_{i,n}^*(a) = \left\langle \psi_i, \widetilde{B}_n X_0 a \right\rangle \quad \text{for } a \in X.$$

Because $|\widetilde{B}_n| \leq \dfrac{n}{n - \widetilde{\omega}} \widetilde{M}$ for any $n \geq n_0$, then $x_{i,n}^*$ is a bounded linear operator from X to $\mathbb{R}$ such that

$$|x_{i,n}^*| \leq \frac{n}{n - \widetilde{\omega}} \widetilde{M} |\psi_i| \quad \text{for any } n \geq n_0.$$

Define the d-column vector $x_n^* = \operatorname{col}(x_{1,n}^*, \dots, x_{d,n}^*)$. Then, one can see that

$$\langle x_n^*, a \rangle = \left\langle \Psi, \widetilde{B}_n X_0 a \right\rangle \quad \text{for } a \in X,$$

with

$$\langle x_n^*, a \rangle_i = \left\langle \psi_i, \widetilde{B}_n X_0 a \right\rangle \quad \text{for } i = 1, \dots, d \text{ and } a \in X.$$

Consequently, we have

$$\sup_{n \geq n_0} |x_n^*| < +\infty,$$

which implies that $(x_n^*)_{n \geq n_0}$ is a bounded sequence in $\mathcal{L}(X, \mathbb{R}^d)$. We recall the following important results.

Theorem 3.2.11 [3] *Assume that* **(H0)**, **(H1)**, **(H2)** *and* **(H3)** *hold. Then, there exists* $x^* \in \mathcal{L}(X, \mathbb{R}^d)$ *such that* $(x_n^*)_{n \geq n_0}$ *converges weakly to* x^* *in the sense that*

$$\langle x_n^*, x \rangle \longrightarrow \langle x^*, x \rangle \quad \text{as } n \longrightarrow +\infty \quad \text{for all } x \in X.$$

Theorem 3.2.12 [3] *Assume that* **(H0)**, **(H1)**, **(H2)** *and* **(H3)** *hold,* f *is continuous and let* u *be a solution of Equation* (3.1.1) *on* $\mathbb{R}$. *Then, the function* z *defined by* $z(t) := \langle \Psi, u_t \rangle$ *is a solution of the ordinary differential equation*

$$\frac{d}{dt} z(t) = Gz(t) + \langle x^*, f(t) \rangle \quad \text{for } t \in \mathbb{R}. \tag{3.2.5}$$

Conversely, if z *is a solution of Equation* (3.2.5) *on* $\mathbb{R}$ *and if in addition* f *is bounded on* $\mathbb{R}$, *then the function* u *given by*

$$u(t) := \left[\Phi z(t) + \lim_{n \to +\infty} \int_{-\infty}^{t} U^s(t-s)\Pi^s \left(\widetilde{B}_n X_0 f(s) \right) ds \right](0) \quad \text{for } t \in \mathbb{R},$$

is a solution of Equation (3.1.1) *on* $\mathbb{R}$.

We obtain the same result in Theorem 3.2.12 if we weaken the boundedness assumption of the function f.

Theorem 3.2.13 *Assume that* **(H0)**, **(H1)**, **(H2)** *and* **(H3)** *hold,* f *is locally integrable and let* u *be a solution of Equation* (3.1.1) *on* $\mathbb{R}$. *Then, the function* z *defined by* $z(t) := \langle \Psi, u_t \rangle$ *is given by*

$$z(t) = e^{tG} z(0) + \int_0^t e^{(t-s)G} \langle x^*, f(s) \rangle \, ds \quad \text{for } t \in \mathbb{R}. \tag{3.2.6}$$

Conversely, if z *satisfies Equation* (3.2.6) *on* $\mathbb{R}$ *and if in addition* f *satisfies*

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} |f(s)| ds < +\infty,$$

then the function u *given by*

$$u(t) := \left[\Phi z(t) + \lim_{n \to \infty} \int_{-\infty}^{t} U^s(t-s)\Pi^s \left(\widetilde{B}_n X_0 f(s) \right) ds \right](0) \quad \text{for } t \in \mathbb{R},$$

is a solution of Equation (3.1.1) *on* $\mathbb{R}$.

Proof. The proof is similar to the one given for Theorem 3.2.12. We only have to prove that the limit

$$\lim_{n \to +\infty} \int_{-\infty}^{t} U^s(t-s)\Pi^s \left(\widetilde{B}_n X_0 f(s) \right) ds$$

exists in C_0. For $t \in \mathbb{R}$ and for n sufficiently large, we have

$$\begin{aligned}
\left| \int_{-\infty}^{t} U^s(t-s)\Pi^s \left(\widetilde{B}_n X_0 f(s) \right) ds \right| &\le 2\widetilde{M}N|\Pi^s| \int_{-\infty}^{t} e^{-\alpha(t-s)} |f(s)| ds \\
&\le 2\widetilde{M}N|\Pi^s| \sum_{k=1}^{+\infty} \left(\int_{t-k}^{t-k+1} e^{-\alpha(t-s)} |f(s)| ds \right) \\
&\le 2\widetilde{M}N|\Pi^s| \sum_{k=1}^{+\infty} \left(e^{-\alpha(k-1)} \int_{t-k}^{t-k+1} |f(s)| ds \right) \le K,
\end{aligned}$$

where $K = 2\widetilde{M}N|\Pi^s|\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}|f(s)|ds\right)\frac{1}{1-e^{-\alpha}}$. Let

$$H(n,s,t) := U^s(t-s)\Pi^s(\widetilde{B}_n X_0 f(s)) \quad \text{for } n \in \mathbb{N} \text{ and } s \le t.$$

For n and m sufficiently large and $\sigma \le t$, we have

$$\begin{aligned} \left|\int_{-\infty}^t H(n,s,t)\,ds - \int_{-\infty}^t H(m,s,t)\,ds\right| &\le \left|\int_{-\infty}^\sigma H(n,s,t)\,ds\right| + \left|\int_{-\infty}^\sigma H(m,s,t)\,ds\right| \\ &\quad + \left|\int_\sigma^t H(n,s,t)\,ds - \int_\sigma^t H(m,s,t)\,ds\right| \\ &\le 2Ke^{-c(t-\sigma)} \\ &\quad + \left|\int_\sigma^t H(n,s,t)\,ds - \int_\sigma^t H(m,s,t)\,ds\right|. \end{aligned}$$

Because $\lim_{n\to+\infty}\int_\sigma^t H(n,s,t)\,ds$ exists, it follows that

$$\limsup_{n,m\to+\infty}\left|\int_{-\infty}^t H(n,s,t)\,ds - \int_{-\infty}^t H(m,s,t)\,ds\right| \le 2Ke^{-c(t-\sigma)}.$$

By letting $\sigma \longrightarrow -\infty$, we get

$$\limsup_{n,m\to+\infty}\left|\int_{-\infty}^t H(n,s,t)\,ds - \int_{-\infty}^t H(m,s,t)\,ds\right| = 0.$$

Thus, by the completeness of the phase space C_0, we deduce that the limit

$$\lim_{n\to+\infty}\int_{-\infty}^t H(n,s,t)\,ds$$

exists in C_0. ■

3.3 A Result on Integration of Stepanov Almost Periodic Functions

In this section, we give sufficient conditions for the almost periodicity of the indefinite integral of Stepanov almost periodic functions. Let us first recall the following result.

Theorem 3.3.1 [24, Theorem II, p. 55] *Let X be a uniformly convex Banach space and $f : \mathbb{R} \longrightarrow X$ be an almost periodic function. If $F(t) := \int_0^t f(s)\,ds$ is bounded on $\mathbb{R}$, then it is almost periodic.*

The following result relaxes the assumption of almost periodicity and boundedness in Theorem 3.3.1.

Theorem 3.3.2 [24, Theorem II', p. 82] *Let X be a uniformly convex Banach space and $f : \mathbb{R} \longrightarrow X$ be an S^p-almost periodic function with $p > 1$. Let $F(t) := \int_0^t f(s)\,ds$. If $F \in BS^p(\mathbb{R}, X)$, then F is almost periodic.*

The proof of Theorem 3.3.2 relies on the fact that the space $L^p([0,1],X)$ is uniformly convex for $p > 1$, when X is uniformly convex. For $p = 1$, the space $L^p([0,1],X)$ is not uniformly convex, thus one cannot use the approach in Theorem 3.3.2 to give a similar result. To solve this problem, we make use of the Bochner type characterization in Proposition 1.1.22. We have the following result.

Theorem 3.3.3 *Let X be a uniformly convex Banach space and $f : \mathbb{R} \longrightarrow X$ be an S^1-almost periodic function. Let $F(t) := \int_0^t f(s)\,ds$. If F is bounded on $\mathbb{R}$, then it is almost periodic.*

Before proving Theorem 3.3.3, let us first study the scalar case.

Lemma 3.3.4 *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be an S^1-almost periodic function. If the integral $F(t) := \int_0^t f(s)\,ds$ is bounded on $\mathbb{R}$, then it is almost periodic.*

Proof. (of Lemma 3.3.4) We only have to prove the result for $\mathbb{R}$-valued functions. In fact for a function $f : \mathbb{R} \longrightarrow \mathbb{C}$, we have

$$\int_0^t f(s)\,ds = \int_0^t \mathrm{Re}(f(s))\,ds + i\int_0^t \mathrm{Im}(f(s))\,ds.$$

Thus, the integral $\int_0^t f(s)ds$ is bounded if and only if $\int_0^t \mathrm{Re}(f(s))\,ds$ and $\int_0^t \mathrm{Im}(f(s))\,ds$ are bounded.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an S^1-almost periodic function and $F(t) := \int_0^t f(s)\,ds$ for each $t \in \mathbb{R}$. Let $m(F) := \inf_{s\in\mathbb{R}} F(s)$, $M(F) := \sup_{s\in\mathbb{R}} F(s)$ and $(s_n)_{n\in\mathbb{N}}$ be an arbitrary sequence. We have for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$

$$F(t + s_n) = F(s_n) + \int_0^t f(s + s_n)\,ds.$$

Since F is bounded and f is S^1-almost periodic, then by the Bochner type characterization in Proposition 1.1.22, there exist a subsequence $(s'_n)_n \subset (s_n)_n$, a function $\widetilde{f} \in L^1_{loc}(\mathbb{R},\mathbb{R})$ and a constant $c \in \mathbb{R}$ such that for each $t \in \mathbb{R}$

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} |f(s + s'_n) - \widetilde{f}(s)|ds \longrightarrow 0 \tag{3.3.1}$$

and

$$F(s'_n) \longrightarrow c, \tag{3.3.2}$$

as $n \longrightarrow +\infty$. Consider the function

$$\widetilde{F}(t) := c + \int_0^t \widetilde{f}(s)\,ds \quad \text{for all } t \in \mathbb{R}.$$

We can see that for each $t \in \mathbb{R}$

$$\int_0^t f(s + s'_n)\,ds \longrightarrow \int_0^t \widetilde{f}(s)\,ds \quad \text{as } n \longrightarrow +\infty. \tag{3.3.3}$$

In fact, using (3.3.1), we obtain for $t \geq 0$

$$\begin{aligned}\left|\int_0^t f(s+s'_n)\,ds - \int_0^t \widetilde{f}(s)\,ds\right| &\leq \int_0^t |f(s+s'_n) - \widetilde{f}(s)|ds \\ &\leq \sum_{k=0}^{[t]} \int_k^{k+1} |f(s+s'_n) - \widetilde{f}(s)|ds \longrightarrow 0,\end{aligned}$$

as $n \longrightarrow +\infty$. Using the same argument for $t \leq 0$, we have

$$\begin{aligned}\left|\int_0^t f(s+s'_n)\,ds - \int_0^t \widetilde{f}(s)\,ds\right| &\leq \int_t^0 |f(s+s'_n) - \widetilde{f}(s)|ds \\ &\leq \sum_{k=[t]}^{-1} \int_k^{k+1} |f(s+s'_n) - \widetilde{f}(s)|ds \longrightarrow 0,\end{aligned}$$

as $n \longrightarrow +\infty$. From (3.3.2) and (3.3.3), we deduce that for each $t \in \mathbb{R}$

$$F(t+s'_n) \longrightarrow \widetilde{F}(t) \quad \text{as } n \longrightarrow +\infty.$$

We claim that this convergence is uniform for $t \in \mathbb{R}$. In fact, if the convergence is not uniform, then there exist $\rho > 0$ and three sequences $(p_n)_n$, $(q_n)_n$, $(t_n)_n$ with $p_n, q_n \geq n$ such that

$$|F\left(t_n + s'_{p_n}\right) - F\left(t_n + s'_{q_n}\right)| > \rho. \tag{3.3.4}$$

Let $a_n := t_n + s'_{p_n}$ and $b_n := t_n + s'_{q_n}$. Since f is S^1-almost periodic and F is bounded, there exist two subsequences, which for simplicity will be also denoted by $(a_n)_n$ and $(b_n)_n$ and two functions $\widetilde{f}_1, \widetilde{f}_2 \in L^1_{loc}(\mathbb{R},\mathbb{R})$ such that

$$\begin{cases}\sup\limits_{t\in\mathbb{R}} \displaystyle\int_t^{t+1} |f(s+a_n) - \widetilde{f}_1(s)|ds \longrightarrow 0 \\ \sup\limits_{t\in\mathbb{R}} \displaystyle\int_t^{t+1} |f(s+b_n) - \widetilde{f}_2(s)|ds \longrightarrow 0,\end{cases} \tag{3.3.5}$$

and

$$F(a_n) \longrightarrow c_1 \ \text{ and } \ F(b_n) \longrightarrow c_2, \tag{3.3.6}$$

as $n \longrightarrow +\infty$, where $c_1, c_2 \in \mathbb{R}$. We have

$$\begin{aligned}\int_t^{t+1} |f(s+a_n) - f(s+b_n)|ds &= \int_t^{t+1} |f(s+t_n+s'_{p_n}) - f(s+t_n+s'_{q_n})|ds \\ &= \int_{t+t_n}^{t+t_n+1} |f(s+s'_{p_n}) - f(s+s'_{q_n})|ds \\ &\leq \sup_{t\in\mathbb{R}} \int_t^{t+1} |f(s+s'_{p_n}) - f(s+s'_{q_n})|ds \\ &\leq \sup_{t\in\mathbb{R}} \int_t^{t+1} |f(s+s'_{p_n}) - \widetilde{f}(s)|ds + \sup_{t\in\mathbb{R}} \int_t^{t+1} |\widetilde{f}(s) - f(s+s'_{q_n})|ds.\end{aligned}$$

It follows from (3.3.1) that

$$\int_t^{t+1} |f(s+a_n) - f(s+b_n)| ds \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty. \tag{3.3.7}$$

Now, since

$$\begin{aligned}\int_t^{t+1} |\widetilde{f}_1(s) - \widetilde{f}_2(s)| ds \quad &\leq \quad \int_t^{t+1} |\widetilde{f}_1(s) - f(s+a_n)| ds + \int_t^{t+1} |f(s+a_n) - f(s+b_n)| ds \\ &+ \int_t^{t+1} |f(s+b_n) - \widetilde{f}_2(s)| ds,\end{aligned}$$

then by taking $n \longrightarrow +\infty$ using (3.3.5) and (3.3.7) we conclude that for all $t \in \mathbb{R}$

$$\int_t^{t+1} |\widetilde{f}_1(s) - \widetilde{f}_2(s)| ds = 0.$$

That is, $\widetilde{f}_1 = \widetilde{f}_2$ a.e. Using (3.3.5) and (3.3.6), it is clear that for all $t \in \mathbb{R}$

$$F(t+a_n) \longrightarrow c_1 + \int_0^t \widetilde{f}_1(s) ds := \widetilde{F}_1(t) \tag{3.3.8}$$

and

$$F(t+b_n) \longrightarrow c_2 + \int_0^t \widetilde{f}_1(s) ds := \widetilde{F}_2(t), \tag{3.3.9}$$

as $n \longrightarrow +\infty$. This implies that

$$\begin{cases} M\left(\widetilde{F}_1\right) \leq M(F) \\ M\left(\widetilde{F}_2\right) \leq M(F) \\ m\left(\widetilde{F}_1\right) \geq m(F) \\ m\left(\widetilde{F}_2\right) \geq m(F). \end{cases} \tag{3.3.10}$$

On the other hand, for all $n \in \mathbb{R}$ and $t \in \mathbb{N}$

$$\begin{cases} \widetilde{F}_1(t-a_n) = \widetilde{F}_1(-a_n) + \int_0^t \widetilde{f}_1(s-a_n) ds \\ \widetilde{F}_2(t-b_n) = \widetilde{F}_2(-b_n) + \int_0^t \widetilde{f}_1(s-b_n) ds. \end{cases} \tag{3.3.11}$$

From (3.3.8) and (3.3.9), one can see that the functions $\widetilde{F}_1$ and $\widetilde{F}_2$ are also bounded on $\mathbb{R}$. Then, there exist two subsequence $(a'_n)_n \subset (a_n)_n$ and $(b'_n)_n \subset (b_n)_n$ and two constants $d_1, d_2 \in \mathbb{R}$ such that

$$\begin{cases} \widetilde{F}_1(-a'_n) \longrightarrow d_1 \\ \widetilde{F}_2(-b'_n) \longrightarrow d_2. \end{cases} \tag{3.3.12}$$

It is clear from (3.3.5) that

$$\begin{cases} \sup_{t \in \mathbb{R}} \int_t^{t+1} |f(s) - \widetilde{f}_1(s-a_n)| ds \longrightarrow 0 \\ \sup_{t \in \mathbb{R}} \int_t^{t+1} |f(s) - \widetilde{f}_1(s-b_n)| ds \longrightarrow 0. \end{cases}$$

Using the same argument as above, we have

$$\begin{cases} \displaystyle\int_0^t \widetilde{f}_1\,(s-a'_n)\,ds \longrightarrow \int_0^t f\,(s)\,ds = F(t) \\ \displaystyle\int_0^t \widetilde{f}_1\,(s-b'_n)\,ds \longrightarrow \int_0^t f\,(s)\,ds = F(t), \end{cases} \tag{3.3.13}$$

as $n \longrightarrow +\infty$. From (3.3.11), (3.3.12) and (3.3.13), we deduce that for each $t \in \mathbb{R}$

$$\begin{cases} \widetilde{F}_1\,(t-a'_n) \longrightarrow d_1 + F\,(t) \\ \widetilde{F}_2\,(t-b'_n) \longrightarrow d_2 + F\,(t)\,, \end{cases} \tag{3.3.14}$$

as $n \longrightarrow +\infty$. In addition, we observe from (3.3.14) that

$$\begin{cases} M\,(d_1+F) \leq M\left(\widetilde{F}_1\right) \\ M\,(d_2+F) \leq M\left(\widetilde{F}_2\right) \\ m\,(d_1+F) \geq m\left(\widetilde{F}_1\right) \\ m\,(d_2+F) \geq m\left(\widetilde{F}_2\right). \end{cases} \tag{3.3.15}$$

It follows from (3.3.10) and (3.3.15) that

$$d_1 = d_2 = 0.$$

Therefore, $M\left(\widetilde{F}_1\right) = M\left(\widetilde{F}_2\right)$. However, we know from (3.3.8) and (3.3.9) that

$$\widetilde{F}_1(t) = \widetilde{F}_2(t) + c_1 - c_2 \quad \text{for } t \in \mathbb{R}.$$

We deduce that $\widetilde{F}_1 = \widetilde{F}_2$. It follows from (3.3.8) and (3.3.9) that for all $t \in \mathbb{R}$

$$|F(t+a_n) - F(t+b_n)| = |F(t+t_n+s'_{p_n}) - F(t+t_n+s'_{q_n})| \longrightarrow 0,$$

which contradicts (3.3.4). ∎

Proof. (of Theorem 3.3.3) Let $F\,(t) := \int_0^t f\,(s)\,ds$ for $t \in \mathbb{R}$. Then, for all $\phi \in X^*$ we have

$$\langle \phi, F(t)\rangle = \int_0^t \langle \phi, f(s)\rangle\,ds.$$

Since $f \in SAP^1(\mathbb{R}, X)$, then $t \longmapsto \langle \phi, f(t)\rangle \in SAP^1(\mathbb{R},\mathbb{R})$ for all $\phi \in X^*$. The boundedness of F implies the boundedness of the function $t \longmapsto \langle \phi, F(t)\rangle$. It follows by Lemma 3.3.4 that $t \longmapsto \langle \phi, F(t)\rangle$ is almost periodic for each $\phi \in X^*$. That is, F is weakly almost periodic. If $F = 0$, then F is almost periodic. Assume that $F \neq 0$. We claim that F has a relatively compact range. In fact, if the range $R_F = \{F(t) : \ t \in \mathbb{R}\}$ is not relatively compact, then there exist a constant $c > 0$ and a sequence $(t_n)_n$ such that

$$|F(t_p) - F(t_q)| > c \quad \text{for all } p \neq q. \tag{3.3.16}$$

We have for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$

$$F(t+t_n) = F(t_n) + \int_0^t f(s+t_n)ds.$$

It follows that for all $p, q \in \mathbb{N}$

$$F(t+t_p) - F(t+t_q) = F(t_p) - F(t_q) + \int_0^t (f(s+t_p) - f(s+t_q))\, ds.$$

Hence,

$$F(t_p) - F(t_q) = F(t+t_p) - F(t+t_q) - \int_0^t (f(s+t_p) - f(s+t_q))\, ds.$$

It follows from (3.3.16) that

$$|F(t+t_p) - F(t+t_q)| + \int_0^t |f(s+t_p) - f(s+t_q)|ds \geq |F(t_p) - F(t_q)| > c. \tag{3.3.17}$$

One can see that

$$\int_0^t |f(s+t_p) - f(s+t_q)|ds \leq \sum_{k=0}^{[t]} \int_k^{k+1} |f(s+t_p) - f(s+t_q)|ds.$$

From the S^1-almost periodicity of f, there exists a subsequence $(t'_n)_n \subset (t_n)_n$ such that

$$\int_k^{k+1} |f(s+t'_p) - f(s+t'_q)|ds \longrightarrow 0,$$

as $p, q \longrightarrow +\infty$. Hence for p, q large enough, we get

$$\int_0^t |f(s+t'_p) - f(s+t'_q)|ds \leq \frac{c}{2}. \tag{3.3.18}$$

Thus, using (3.3.18) and (3.3.17), we obtain

$$|F(t+t'_p) - F(t+t'_q)| > \frac{c}{2}. \tag{3.3.19}$$

Since X is uniformly convex, there exists $\delta > 0$ such that for any $u, v \in X$ with $|u| \leq 1$ and $|v| \leq 1$, if $|u-v| \geq \frac{c}{2|F|_\infty}$, then $\left|\frac{u+v}{2}\right| \leq 1-\delta$. Thus, (3.3.19) implies that

$$\left|\frac{F(t+t'_p) + F(t+t'_q)}{2}\right| \leq |F|_\infty - \delta|F|_\infty.$$

For all $\phi \in X^*$ with $|\phi| \leq 1$, we have

$$\left|\left\langle \phi, \frac{F(t+t'_p) + F(t+t'_q)}{2}\right\rangle\right| \leq \left|\frac{F(t+t'_p) + F(t+t'_q)}{2}\right| \leq |F|_\infty - \delta|F|_\infty. \tag{3.3.20}$$

Since F is weakly almost periodic and X is weakly sequentially complete (because it is reflexive), then from Theorem 1.1.14, there exist a subsequence $(t''_n)_n \subset (t'_n)_n$ and a function $\widetilde{F}$ such that for all $t \in \mathbb{R}$

$$F(t+t''_n) \longrightarrow \widetilde{F}(t) \tag{3.3.21}$$

uniformly in the weak sense. This implies that we have also

$$\widetilde{F}(t - t_n'') \longrightarrow F(t) \tag{3.3.22}$$

uniformly in the weak sense. It is clear that (3.3.21) and (3.3.22) imply that

$$|F|_\infty = |\widetilde{F}|_\infty.$$

By taking $p, q \longrightarrow +\infty$ in (3.3.20), we get for all $\phi \in X^*$ with $|\phi| \leq 1$

$$\left| \left\langle \phi, \widetilde{F}(t) \right\rangle \right| \leq |F|_\infty - \delta |F|_\infty = |\widetilde{F}|_\infty - \delta |\widetilde{F}|_\infty,$$

which yields that

$$|\widetilde{F}|_\infty = \sup_{t\in\mathbb{R}} |\widetilde{F}(t)| = \sup_{t\in\mathbb{R}} \sup_{|\phi|\leq 1,\, \phi\in X^*} \left| \left\langle \phi, \widetilde{F}(t) \right\rangle \right| \leq |\widetilde{F}|_\infty - \delta |\widetilde{F}|_\infty,$$

which is a contradiction since $|\widetilde{F}|_\infty = |F|_\infty \neq 0$. We conclude that $R_F = \{F(t) : t \in \mathbb{R}\}$ is relatively compact. We deduce from Theorem 1.1.10 that F is almost periodic. ■

3.4 Almost Periodicity of Bounded Solutions for Ordinary Differential Equations

Consider the scalar differential equation

$$x'(t) = \lambda x(t) + g(t) \quad \text{for } t \in \mathbb{R}, \tag{3.4.1}$$

where $g : \mathbb{R} \longrightarrow \mathbb{C}$ is a locally integrable function and $\lambda \in \mathbb{C}$. Since g is only locally integrable, we mean by an integral solution of Equation (3.4.1) a locally integrable function $x : \mathbb{R} \longrightarrow \mathbb{C}$ which satisfies the following integral equation

$$x(t) = x(0) + \int_0^t \lambda x(s) ds + \int_0^t g(s) ds \quad \text{for } t \in \mathbb{R}.$$

Using this convention, an integral solution of Equation (3.4.1) is locally absolutely continuous and is given by the following formula

$$x(t) = e^{t\lambda} x(0) + \int_0^t e^{(t-s)\lambda} g(s) ds \quad \text{for } t \in \mathbb{R}.$$

Moreover, an integral solution of Equation (3.4.1) satisfies (3.4.1) a.e.

Theorem 3.4.1 *If g is S^1-almost periodic, then every bounded integral solution of Equation (3.4.1) on $\mathbb{R}$ is almost periodic.*

Proof. Let x be a bounded integral solution of (3.4.1) on $\mathbb{R}$. Then, we have

$$x(t) = e^{\lambda t}\left(c + \int_0^t e^{-\lambda s} g(s)\, ds\right) \quad \text{for } t \in \mathbb{R},$$

with some constant $c \in \mathbb{C}$. We distinguish three cases.
Case 1: $\mathrm{Re}\lambda = 0$, that is $\lambda = i\theta$, for some $\theta \in \mathbb{R}$. We have

$$x(t) = e^{i\theta t} c + e^{i\theta t} \int_0^t e^{-i\theta s} g(s)\, ds \quad \text{for } t \in \mathbb{R}.$$

Since x is bounded, then the function $t \longmapsto \int_0^t e^{-i\theta s} g(s)\, ds$ is also bounded. Moreover, the function $s \longmapsto e^{-i\theta s} g(s)$ is S^1-almost periodic as a product of a periodic and an S^1-almost periodic function. We deduce from Lemma 3.3.4 that the function $t \longmapsto \int_0^t e^{-i\theta s} g(s)\, ds$ is almost periodic. This implies that $x \in AP(\mathbb{R}, \mathbb{C})$.
Case 2: $\mathrm{Re}\lambda < 0$, the solution x can be written in the following form:

$$x(t) = \int_{-\infty}^t e^{\lambda(t-s)} g(s)\, ds \quad \text{for } t \in \mathbb{R}.$$

Using the same approach as in the proof of [82, Theorem 30] one can easily prove that x is almost periodic.
Case 3: $\mathrm{Re}\lambda > 0$. The solution x can be expressed in the following form:

$$x(t) = -\int_t^{+\infty} e^{\lambda(t-s)} g(s)\, ds \quad \text{for } t \in \mathbb{R}.$$

Using the same argument as in Case 2, we can show that x is almost periodic. ■

Consider the following ordinary differential equation

$$x'(t) = Bx(t) + g(t) \quad \text{for } t \in \mathbb{R}, \tag{3.4.2}$$

where $g : \mathbb{R} \longrightarrow \mathbb{C}^n$ is a locally integrable function and $B : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ a matrix. We have the following Bohr-Neugebauer type theorem for Equation (3.4.2).

Theorem 3.4.2 *If g is S^1-almost periodic, then every bounded integral solution of Equation (3.4.2) on $\mathbb{R}$ is almost periodic.*

Proof. We use the same strategy as in [64, Proposition 5.12]. Let x be a bounded integral solution of Equation (3.4.2); that is

$$x(t) = x(0) + \int_0^t Bx(s)ds + \int_0^t g(s)ds \quad \text{for } t \in \mathbb{R}.$$

Using the linear transformation $x = Ty$ where $T : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is an invertible matrix, we get

$$y(t) = y(0) + \int_0^t Cy(s)ds + \int_0^t h(s)ds \quad \text{for } t \in \mathbb{R}, \tag{3.4.3}$$

where C is an upper-triangular matrix given by

$$\begin{aligned} C &= T^{-1}BT \\ &= \begin{pmatrix} \lambda_1 & c_{12} & c_{13} & \dots & c_{1n} \\ 0 & \lambda_2 & c_{23} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}. \end{aligned}$$

The function h is given by

$$h(t) = T^{-1}g(t) \quad \text{for } t \in \mathbb{R}.$$

We observe that Equation (3.4.3) can be written as

$$\begin{cases} y_1(t) = y_1(0) + \int_0^t \lambda_1 y_1(s)ds + \int_0^t [c_{12}y_2(s) + \cdots + c_{1n}y_n(s) + h_1(s)]\, ds \\ y_2(t) = y_2(0) + \int_0^t \lambda_2 y_2(s)ds + \int_0^t [c_{23}y_2(s) + \cdots + c_{2,n}y_n(s) + h_2(s)]\, ds \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_{n-1}(t) = y_{n-1}(0) + \int_0^t \lambda_{n-1}y_{n-1}(s)ds + \int_0^t [c_{n-1,n}y_n(s) + h_{n-1}(s)]\, ds \\ y_n(t) = y_n(0) + \int_0^t \lambda_n y_n(s)ds + \int_0^t h_n(s)ds, \end{cases}$$

where h_i, $i = 1, ..., n$ are the S^1-almost periodic components of h. The result on the scalar case (3.4.1) ensures that the bounded function y_n is almost periodic. It follows that the forcing term $c_{n-1,n}y_n + h_{n-1}$ in the last second equation is S^1-almost periodic. This process can be continued using the result in the scalar case (3.4.1) until we obtain the almost periodicity of all the components of y. Therefore, $x = Ty$ is almost periodic. ■

The next theorem shows that the existence of a bounded integral solution of Equation (3.4.2) on $\mathbb{R}$ is equivalent to the existence of a bounded integral solution on $\mathbb{R}^+$.

Theorem 3.4.3 *If Equation* (3.4.2) *has a bounded integral solution on* $\mathbb{R}^+$*, then it has a bounded integral solution on* $\mathbb{R}$*.*

Proof. Let x be a bounded integral solution of Equation (3.4.2) on $\mathbb{R}^+$. Let $(s_n)_n$ be a sequence of positive real numbers such that $s_n \longrightarrow +\infty$, as $n \longrightarrow +\infty$. Then, there exist a subsequence $(s'_n)_n \subset (s_n)_n$, a function $\tilde{g} \in L^1_{loc}(\mathbb{R}, \mathbb{C}^n)$ and a constant $c \in \mathbb{C}^n$ such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} |g(s + s'_n) - \tilde{g}(s)|ds \longrightarrow 0 \tag{3.4.4}$$

and

$$x(s'_n) \longrightarrow c \tag{3.4.5}$$

as $n \longrightarrow +\infty$. Consider the function

$$y(t) := e^{tB}c + \int_0^t e^{(t-s)B}\widetilde{g}(s)\,ds \quad \text{for all } t \in \mathbb{R}.$$

For each $t \in \mathbb{R}$, there exists $N(t) \in \mathbb{N}$ such that for all $n \geq N(t)$, $t + s'_n \geq 0$ and

$$x(t+s'_n) = e^{tB}x(s'_n) + \int_0^t e^{(t-s)B}g(s+s'_n)\,ds.$$

We have for each $t \in \mathbb{R}$

$$\int_0^t e^{(t-s)B}g(s+s'_n)\,ds \longrightarrow \int_0^t e^{(t-s)B}\widetilde{g}(s)\,ds \quad \text{as } n \longrightarrow +\infty. \tag{3.4.6}$$

In fact, using (3.4.4), we obtain for $t \geq 0$

$$\begin{aligned}
\left|\int_0^t e^{(t-s)B}g(s+s'_n)\,ds - \int_0^t e^{(t-s)B}\widetilde{g}(s)\,ds\right| &\leq \int_0^t e^{|(t-s)B|}|g(s+s'_n) - \widetilde{g}(s)|ds \\
&\leq \sum_{k=0}^{[t]} \int_k^{k+1} e^{|(t-s)B|}|g(s+s'_n) - \widetilde{g}(s)|ds \\
&\leq \sum_{k=0}^{[t]} \left(C(k,t,B)\int_k^{k+1} |g(s+s'_n) - \widetilde{g}(s)|ds\right) \longrightarrow 0
\end{aligned}$$

as $n \longrightarrow +\infty$, where $C(k,t,B) := \sup_{k \leq s \leq k+1} e^{|t-s||B|}$. Using the same argument for $t \leq 0$, we get

$$\left|\int_0^t e^{(t-s)B}g(s+s'_n)\,ds - \int_0^t e^{(t-s)B}\widetilde{g}(s)\,ds\right| \longrightarrow 0 \ \text{ as } n \longrightarrow +\infty.$$

It follows from (3.4.5) and (3.4.6) that for each fixed t in $\mathbb{R}$

$$x(t+s'_n) \longrightarrow y(t) \quad \text{as } n \longrightarrow +\infty.$$

This implies that the function y is bounded on the whole real line. Now, for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, we have

$$y(t-s'_n) = e^{tB}y(-s'_n) + \int_0^t e^{(t-s)B}\widetilde{g}(s-s'_n)\,ds. \tag{3.4.7}$$

We can extract a subsequence $(s''_n)_n \subset (s'_n)_n$ such that

$$y(-s''_n) \longrightarrow d. \tag{3.4.8}$$

Consider the function

$$z(t) := e^{tB}d + \int_0^t e^{(t-s)B}g(s)\,ds \quad \text{for all } t \in \mathbb{R}.$$

It is clear that the function z is an integral solution of Equation (3.4.2) on $\mathbb{R}$. One can see from (3.4.4) that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} |\widetilde{g}(s-s'_n) - g(s)|ds \longrightarrow 0, \tag{3.4.9}$$

as $n \longrightarrow +\infty$. Using (3.4.9), we have for each $t \in \mathbb{R}$

$$\int_0^t e^{(t-s)B}\widetilde{g}\left(s - s_n'\right) ds \longrightarrow \int_0^t e^{(t-s)B} g\left(s\right) ds \quad \text{as } n \longrightarrow +\infty. \tag{3.4.10}$$

From (3.4.7), (3.4.8) and (3.4.10), we deduce that for each fixed $t \in \mathbb{R}$

$$y\left(t - s_n''\right) \longrightarrow z\left(t\right) \quad \text{as } n \longrightarrow +\infty.$$

Therefore, z is a bounded integral solution of Equation (3.4.2) on $\mathbb{R}$. ■

The following Massera type result is a direct consequence of Theorems 3.4.2 and 3.4.3.

Corollary 3.4.4 *If Equation (3.4.2) has a bounded integral solution on $\mathbb{R}^+$, then it has an almost periodic integral solution.*

3.5 Behavior of Bounded Solutions of Equation (3.1.1)

In this section, we investigate the nature of bounded integral solutions of the neutral functional differential equation (3.1.1).

The following theorem is of Bohr-Neugebauer type.

Theorem 3.5.1 *Assume that* **(H0)-(H3)** *hold and $f : \mathbb{R} \longrightarrow X$ is S^1-almost periodic. Then, every bounded integral solution of Equation (3.1.1) on $\mathbb{R}$ is almost periodic.*

Proof. Let x be a bounded integral solution of Equation (3.1.1) on $\mathbb{R}$. Using the spectral decomposition (3.2.3), we have for each $t \in \mathbb{R}$

$$x_t = \Pi^u x_t + \Pi^s x_t. \tag{3.5.1}$$

On the one hand, we have for $t \geq \sigma$

$$\Pi^s x_t = U^s(t-\sigma)\Pi^s x_\sigma + \lim_{n\to+\infty}\int_\sigma^t U^s(t-s)\Pi^s\left(\widetilde{B}_n\left(X_0 f\left(s\right)\right)\right) ds. \tag{3.5.2}$$

Since $t \longmapsto x_t$ is bounded on $\mathbb{R}$ and $U\left(t\right)$ is exponentially stable in the subspace S, then by letting $\sigma \longrightarrow -\infty$ in (3.5.2), we obtain for all $t \in \mathbb{R}$

$$\Pi^s x_t = \lim_{n\to+\infty}\int_{-\infty}^t U^s(t-s)\Pi^s\left(\widetilde{B}_n\left(X_0 f\left(s\right)\right)\right) ds.$$

In fact, for each fixed $t \in \mathbb{R}$, we have for all $\sigma \leq t$

$$|U^s(t-\sigma)\Pi^s x_\sigma| \leq e^{-\alpha(t-\sigma)}|\Pi^s| \sup_{s\in\mathbb{R}}|x_s| \longrightarrow 0,$$

as $\sigma \longrightarrow -\infty$, and

$$\left|\lim_{n\to+\infty}\int_\sigma^t U^s(t-s)\Pi^s\left(\widetilde{B}_n\left(X_0 f\left(s\right)\right)\right) ds - \lim_{n\to+\infty}\int_{-\infty}^t U^s(t-s)\Pi^s\left(\widetilde{B}_n\left(X_0 f\left(s\right)\right)\right) ds\right|$$

$$= \left| \lim_{n \to +\infty} \int_{-\infty}^{\sigma} U^s(t-s)\Pi^s \left(\widetilde{B}_n (X_0 f(s)) \right) ds \right|$$
$$\leq \widetilde{M} N |\Pi^s| \sum_{k=1}^{+\infty} \int_{\sigma-k}^{\sigma-k+1} e^{-\alpha(t-s)} |f(s)| ds$$
$$\leq \widetilde{M} N |\Pi^s| |f|_{BS^1} \frac{e^{-\alpha(t-\sigma)}}{1-e^{-\alpha}} \longrightarrow 0,$$

as $\sigma \longrightarrow -\infty$.

Using a proof similar to the one in [82, Theorem 30], one can prove that $t \longmapsto \Pi^s x_t$ is almost periodic.

On the other hand, for each $t \in \mathbb{R}$

$$\Pi^\nu x_t = \Phi \langle \Psi, x_t \rangle = \sum_{i=1}^{d} \langle \psi_i, x_t \rangle \varphi_i. \tag{3.5.3}$$

From Theorem 3.2.13, the function $t \longmapsto \langle \Psi, x_t \rangle$ is a bounded solution of the integral equation (3.2.6) on $\mathbb{R}$. Moreover, the function $t \longmapsto \langle x^*, f(t) \rangle$ is S^1-almost periodic. It follows by Theorem 3.4.2 that $t \longmapsto \langle \Psi, x_t \rangle$ is almost periodic. We deduce from (3.5.3) that the function $t \longmapsto \Pi^\nu x_t$ is almost periodic. The almost periodicity of $t \longmapsto x_t$ follows from (3.5.1). ■

Theorem 3.5.2 *Assume that* **(H0)-(H3)** *hold and* $f : \mathbb{R} \longrightarrow X$ *is* S^1*-almost periodic. If Equation* (3.1.1) *has a bounded integral solution on* $\mathbb{R}^+$*, then it has a bounded integral solution on* $\mathbb{R}$.

Proof. Let u be a bounded integral solution of Equation (3.1.1) on $\mathbb{R}^+$. By Theorem 3.2.13, the function $t \longmapsto \langle \Psi, u_t \rangle$ is a bounded solution of the integral equation (3.2.6) on $\mathbb{R}^+$. We deduce from Theorem 3.4.3 that the integral equation (3.2.6) has a bounded solution on $\mathbb{R}$. Let y be this solution. From Theorem 3.2.13, the function x defined by

$$x(t) := \left(\Phi y(t) + \lim_{n \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s) \Pi^s \left(\widetilde{B}_n (X_0 f(s)) \right) ds \right)(0) \quad \text{for all } t \in \mathbb{R},$$

is a solution of Equation (3.1.1) on $\mathbb{R}$. It is clear from the proof of Theorem 3.2.13 that the function

$$t \longmapsto \lim_{n \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s) \Pi^s \left(\widetilde{B}_n (X_0 f(s)) \right) ds$$

is bounded on $\mathbb{R}$. Therefore the solution x is bounded on $\mathbb{R}$. ■

From Theorem 3.5.1 and Theorem 3.5.2 we have the following result.

Corollary 3.5.3 *Assume that* **(H0)-(H3)** *hold and* $f : \mathbb{R} \longrightarrow X$ *is* S^1*-almost periodic. If Equation* (3.1.1) *has a bounded integral solution on* $\mathbb{R}^+$*, then it has an almost periodic integral solution.*

3.6 Application

To apply our results, we consider the following model proposed in [180] which describes the evolution of the voltage across of a transmission line (For more details, see page 5) :

$$\begin{cases} \frac{\partial}{\partial t}\left[v(t,x)-qv(t-r,x)\right]=\frac{\partial^2}{\partial x^2}\left[v(t,x)-qv(t-r,x)\right]+av(t,x)+bv(t-r,x) \\ \qquad\qquad +\int_{t-r}^{t} h(s-t)\,v(s,x)\,ds+F(t)\,\psi(x) \quad \text{for } t\in\mathbb{R} \text{ and } x\in[0,\pi], \\ v(t,0)-qv(t-r,0)=v(t,\pi)-qv(t-r,\pi)=0 \quad \text{for } t\in\mathbb{R}, \end{cases} \tag{3.6.1}$$

where $a, b \in \mathbb{R}$, $h : [-r, 0] \longrightarrow \mathbb{R}$, $\psi : [0, \pi] \longrightarrow \mathbb{R}$ are continuous functions and $q \in (0, 1)$. The function $F : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$F(t)=\sum_{n\geq 0} F_n(t),$$

where F_n are defined for every integer $n \geq 1$ by

$$F_n(t)=\sum_{k\in P_n} H\left(n^2(t-k)\right),$$

with $P_n = 3^n(2\mathbb{Z}+1) = \{3^n(2k+1),\ k\in\mathbb{Z}\}$ and $H \in C_0^\infty(\mathbb{R},\mathbb{R})$, with support in $\left(-\frac{1}{2},\frac{1}{2}\right)$ such that

$$H \geq 0, \qquad H(0)=1 \quad \text{and} \quad \int_{\frac{-1}{2}}^{\frac{1}{2}} H(s)\,ds = 1.$$

The function F is not almost periodic, since it is not bounded. However, $F \in C^\infty(\mathbb{R},\mathbb{R}) \cap SAP^1(\mathbb{R},\mathbb{R})$ (see Example 1.1.20).

To rewrite Equation (3.6.1) in the abstract form (3.1.1), we introduce $X = C([0,\pi],\mathbb{R})$ the space of continuous functions from $[0,\pi]$ to $\mathbb{R}$ endowed with the uniform norm topology, and we define the operator $A : D(A) \subset X \longrightarrow X$ by

$$\begin{cases} D(A)=\left\{y\in C^2([0,\pi],\mathbb{R}) : y(0)=y(\pi)=0\right\}, \\ Ay=y''. \end{cases}$$

Lemma 3.6.1 [79] *The operator A satisfies the Hille-Yosida condition on the space X, namely, $(0,+\infty) \subset \rho(A)$ and*

$$|(\lambda I - A)^{-1}| \leq \frac{1}{\lambda} \quad \text{for } \lambda > 0.$$

This lemma implies that condition **(H0)** is satisfied. We can see that

$$\overline{D(A)} = \left\{y \in X : y(0)=y(\pi)=0\right\} \neq X.$$

Let A_0 be the part of the operator A in $\overline{D(A)}$. Then, A_0 is given by

$$\begin{cases} D(A_0) = \{y \in C^2([0,\pi],\mathbb{R}) : y(0) = y(\pi) = y''(0) = y''(\pi) = 0\}, \\ A_0 y = y'' \quad \text{for } y \in D(A_0). \end{cases}$$

By Lemma 3.2.2, the operator A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$. Using Example 1.5.12, $(T_0(t))_{t\geq 0}$ is compact on $\overline{D(A)}$. This implies that **(H2)** holds. Let us introduce the bounded linear operator $\mathcal{D} : C = C([-r,0],X) \longrightarrow X$ by

$$\mathcal{D}\phi = \phi(0) - q\phi(-r).$$

Since $0 < q < 1$, then $\mathcal{D}$ is stable and Hypothesis **(H3)** holds. Moreover, by definitions of the operators A and $\mathcal{D}$, we can see that **(H1)** is satisfied.

If we put

$$u(t)(x) = v(t,x) \quad \text{for } t \in \mathbb{R} \text{ and } x \in [0,\pi],$$

then Equation (3.6.1) takes the following abstract form

$$\frac{d}{dt}\mathcal{D}u(t) = A\mathcal{D}u(t) + L(u_t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{3.6.2}$$

where $L : C \longrightarrow X$ is defined by

$$L(\varphi)(x) = a\varphi(0)(x) + b\varphi(-r)(x) + \int_{-r}^{0} h(s)\varphi(s)(x)\,ds \quad \text{for } x \in [0,\pi] \text{ and } \varphi \in C$$

and $f : \mathbb{R} \longrightarrow X$ is given by

$$f(t)(x) = F(t)\psi(x) \quad \text{for } x \in [0,\pi] \text{ and } t \in \mathbb{R}.$$

It follows that L is a bounded linear operator from C to X, and $f \in SAP^1(\mathbb{R},X)$.

Lemma 3.6.2 *Under the condition* $|a| + |b| + \int_{-r}^{0} |h(\theta)|d\theta < 1 - q$, *the semigroup solution corresponding to Equation* (3.6.2) *with* $f = 0$ *is uniformly exponentially stable.*

Proof. The proof is similar to [85, Proposition 5.2]. ■

Proposition 3.6.3 *Assume that* $|a| + |b| + \int_{-r}^{0} |h(\theta)|d\theta < 1 - q$. *Then, Equation* (3.6.2) *has a unique globally attractive almost periodic integral solution. As a consequence, all other integral solutions are asymptotically almost periodic.*

Proof. From Lemma 3.6.2, we have $\omega_0(\mathcal{U}) < 0$. Hence, there exists two positive constants $\widehat{M}$ and α such that $|\mathcal{U}(t)| \leq \widehat{M}e^{-\alpha t}$ for all $t \geq 0$. We claim that all solutions of Equation (3.6.2) are bounded on $\mathbb{R}^+$. In fact, let $\phi \in C_0$ and u a solution such that $u_0 = \phi$. Then,

$$u_t = \mathcal{U}(t)\varphi + \lim_{n\to+\infty}\int_0^t \mathcal{U}(t-s)\widetilde{B}_n(X_0 f(s))\,ds \quad \text{for } t \geq 0. \tag{3.6.3}$$

It follows for $t \geq 0$ that

$$\begin{aligned}
|u_t| &\leq \widehat{M}e^{-\alpha t}|\phi| + \widehat{M}\widetilde{M}e^{-\alpha t}\int_0^t e^{\alpha s}|f(s)|ds \\
&\leq \widehat{M}e^{-\alpha t}|\phi| + \widehat{M}\widetilde{M}e^{-\alpha t}\sum_{k=0}^{[t]} e^{\alpha(k+1)}\int_k^{k+1}|f(s)|ds \\
&= \widehat{M}e^{-\alpha t}|\phi| + \widehat{M}\widetilde{M}|f|_{BS^1}e^{\alpha}\frac{e^{\alpha([t]+1-t)} - e^{-\alpha t}}{e^{\alpha}-1} \\
&\leq \widehat{M}|\phi| + \widehat{M}\widetilde{M}|f|_{BS^1}\frac{e^{2\alpha}}{e^{\alpha}-1}.
\end{aligned}$$

We deduce from Corollary 3.5.3 the existence of an almost periodic solution z. If $\widetilde{z}$ is another solution, then from the variation of constants formula (3.6.3), we have

$$|z_t - \widetilde{z}_t| \leq |U(t)||z_0 - \widetilde{z}_0| \leq \widehat{M}e^{-\alpha t}|z_0 - \widetilde{z}_0| \longrightarrow 0,$$

as $t \longrightarrow +\infty$. Thus, the almost periodic solution z is globally attractive. In addition, since $\widetilde{z}$ can be decomposed as follows

$$\widetilde{z}_t = z_t + (\widetilde{z}_t - z_t),$$

the solution $\widetilde{z}$ is asymptotically almost periodic.

The almost periodic solution z is the only solution which is bounded on $\mathbb{R}$. In fact, if $\bar{z}$ is another bounded solution on $\mathbb{R}$. Then, by using the variation of constant formula in the phase space, we have for all $\sigma, t \in \mathbb{R}$ with $\sigma \leq t$

$$z_t - \bar{z}_t = U(t-\sigma)\left(z_\sigma - \bar{z}_\sigma\right).$$

It follows that for all $\sigma, t \in \mathbb{R}$ with $\sigma \leq t$

$$|z_t - \bar{z}_t| \leq \widehat{M}e^{-\alpha(t-\sigma)}|z_\sigma - \bar{z}_\sigma|. \tag{3.6.4}$$

By letting $\sigma \longrightarrow -\infty$ in (3.6.4), we get $z_t = \bar{z}_t$ for all $t \in \mathbb{R}$. ■

Chapter 4

Exponential Dichotomy and (μ, ν)-Pseudo Almost Automorphic Solutions for Some Ordinary Differential Equations [(1)]

Contents

4.1 Introduction

In this chapter, we study the following nonlinear differential equation

$$x'(t) = A(t)x(t) + f(t, x(t)) \quad \text{for } t \in \mathbb{R}, \tag{4.1.1}$$

where $A(t)$ is a real square matrix of order n and $f : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function.

Exponential dichotomy is one of the fundamental asymptotic properties of solutions of the nonlinear system (4.1.1). It have their origins in the work of O. Perron in his classical paper [158] and has been studied with much emphasis in the last sixty years by many authors [60],[61],[129], [?],[138],[18],[19].

In the recent years, several authors have attracted much attention to the study of existence of periodic, almost periodic and pseudo almost periodic solutions to Equation (4.1.1) using exponential dichotomy approach. We just mention a few of these works. (see, e.g., [2, 18, 17, 19, 62, 177] and references therein).

(1) This work has been done in collaboration with El Hadi Ait Dads, Nadia Drisi and Khalil Ezzinbi and has been published in Communications in Optimizations Theory [9]

In his very interesting work [62], Coppel gave important results on the existence and uniqueness of almost periodic solutions to Equation (4.1.1) under the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \le L\|x - y\|$$

where L is small enough. The problem of the existence of almost periodic solutions of Equation (4.1.1) has been also studied by Weiyao in [177] under the following condition

$$\lim_{\|x\| \to +\infty} \frac{\|f(t,x)\|}{\|x\|} = 0$$

uniformly with respect to $t \in \mathbb{R}$.

An intresting result on the existence of pseudo almost periodic solutions to Equation (4.1.1) was established by Ait Dads and Arino in [18].

Recently, by means of the exponential dichotomy and Schauder's fixed point Theorem, Adesina and Ayanjinmi discussed in [2] the existence of almost periodic solutions for Equation (4.1.1). They assumed that A and f are almost periodic and f satisfies the following condition

$$\|f(t,x) - f(t,y)\| \le l(t)h(\|x - y\|), \tag{4.1.2}$$

where h is a continuous function on $[0, +\infty)$ and $h(0) = 0$. We mention that several mistakes were made in the proof of the result in [2].

In this chapter, we give the existence and uniqueness of (μ, ν)-pseudo almost automorphic solution of Equation (4.1.1) under condition (4.1.2) where $l \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ $(1 < p < +\infty)$ and $h : [0, +\infty) \longrightarrow [0, +\infty)$ is a nondecreasing continuous function on $[0, +\infty)$, verifying $h(r) < r$ for every $r > 0$ and $h(0) = 0$. Our main result is proved by using exponential dichotomy and fixed point theory.

4.2 Exponential Dichotomy

In the sequel, A denotes a continuous function from $\mathbb{R}$ to $M_n(\mathbb{R})$, where $M_n(\mathbb{R})$ is the space of real square matrices of order n. Let $X(t)$ be a fundamental matrix of the linear system

$$x'(t) = A(t)x(t) \quad \text{for } t \in \mathbb{R} \tag{4.2.1}$$

satisfying $X(0) = I$, where I is the unit matrix.

Definition 4.2.1 The linear system (4.2.1) is said to possess an exponential dichotomy on $\mathbb{R}$, if there exist constants $k \ge 1$, $\alpha > 0$ and a projection matrix P ($P^2 = P$) such that

$$\|X(t)PX^{-1}(s)\| \le ke^{-\alpha(t-s)} \quad \text{for } t \ge s,$$

$$\|X(t)(I - P)X^{-1}(s)\| \le ke^{-\alpha(s-t)} \quad \text{for } s \ge t.$$

We denote by (P, k, α) a triple of elements associated to an exponential dichotomy.

Lemma 4.2.2 [19] *Let* $A : \mathbb{R} \longrightarrow M_n(\mathbb{R})$ *be continuous function and assume that the equation* $x'(t) = A(t)x(t)$ *has an exponential dichotomy on* $\mathbb{R}$*. Then, for* $\rho \in AA(\mathbb{R}, \mathbb{R}^n)$*, the unique bounded solution of the following equation* $x'(t) = A(t)x(t) + \rho(t)$ *is almost automorphic.*

Theorem 4.2.3 *Let* $\mu, \nu \in \mathcal{M}$ *satisfy* **(B)** *and* **(D)** *and* $A : \mathbb{R} \longrightarrow M_n(\mathbb{R})$ *be continuous function. If the system* (4.2.1) *has an exponential dichotomy on* $\mathbb{R}$ *and if* $\rho : \mathbb{R} \longrightarrow \mathbb{R}^n$ *is* (μ, ν)*-pseudo almost automorphic, then the system*

$$x'(t) = A(t)x(t) + \rho(t) \quad \text{for } t \in \mathbb{R} \tag{4.2.2}$$

has a unique (μ, ν)*-pseudo almost automorphic solution.*

Proof. We known that the unique bounded solution of equation (4.2.2) is given by (see [159]):

$$x(t) = \int_{-\infty}^{+\infty} G(t, s)\rho(s)ds,$$

where

$$G(t, s) = \begin{cases} X(t)PX^{-1}(s) & \text{for } t \geq s \\ -X(t)(I - P)X^{-1}(s) & \text{for } t \leq s, \end{cases}$$

is the Green function. If $\rho(t) = g(t) + \varphi(t)$, where g is almost automorphic and φ is (μ, ν)-ergodic, then

$$x(t) = \int_{-\infty}^{+\infty} G(t, s)g(s)ds + \int_{-\infty}^{+\infty} G(t, s)\varphi(s)ds.$$

From Lemma 4.2.2, we have $t \longmapsto \int_{-\infty}^{+\infty} G(t, s)g(s)ds$ is almost automorphic. To complete the proof, we have to prove that the function

$$\Phi(t) = \int_{-\infty}^{+\infty} G(t, s)\varphi(s)ds$$

is (μ, ν)-ergodic. Since $\Phi = \Phi_1 + \Phi_2$ where

$$\Phi_1(t) = \int_{-\infty}^{t} G(t, s)\varphi(s)ds \quad \text{and} \quad \Phi_2(t) = \int_{t}^{+\infty} G(t, s)\varphi(s)ds.$$

Let $r > 0$ be sufficiently large. Then,

$$\frac{1}{\nu([-r, r])} \int_{[-r,r]} \|\Phi_1(t)\| \, d\mu(t)$$

$$
\begin{aligned}
&= \frac{1}{\nu([-r,r])}\int_{[-r,r]}\left\|\int_{-\infty}^{t} G(t,s)\varphi(s)ds\right\|d\mu(t)\\
&\leq \frac{1}{\nu([-r,r])}\int_{[-r,r]}\int_{-\infty}^{t} \|X(t)PX^{-1}(s)\varphi(s)\|dsd\mu(t)\\
&\leq \frac{1}{\nu([-r,r])}\int_{[-r,r]}\int_{-\infty}^{t} ke^{-\alpha(t-s)}\|\varphi(s)\|dsd\mu(t)\\
&\leq \frac{1}{\nu([-r,r])}\int_{[-r,r]}\int_{0}^{+\infty} ke^{-\alpha s}\|\varphi(t-s)\|dsd\mu(t)\\
&\leq k\int_{0}^{+\infty} e^{-\alpha s}\left(\frac{1}{\nu([-r,r])}\int_{[-r,r]}\|\varphi(t-s)\|d\mu(t)\right)ds.
\end{aligned}
$$

On the other hand, we have

$$
\frac{1}{\nu([-r,r])}\int_{[-r,r]}\|\Phi_2(t)\|d\mu(t)
$$

$$
\begin{aligned}
&= \frac{1}{\nu([-r,r])}\int_{[-r,r]}\left\|\int_{t}^{+\infty} G(t,s)\varphi(s)ds\right\|d\mu(t)\\
&\leq \frac{1}{\nu([-r,r])}\int_{[-r,r]}\int_{t}^{+\infty} \|X(t)(I-P)X^{-1}(s)\varphi(s)\|dsd\mu(t)\\
&\leq \frac{1}{\nu([-r,r])}\int_{[-r,r]}\int_{t}^{+\infty} ke^{-\alpha(s-t)}\|\varphi(s)\|dsd\mu(t)\\
&\leq \frac{1}{\nu([-r,r])}\int_{[-r,r]}\int_{0}^{+\infty} ke^{-\alpha s}\|\varphi(t+s)\|dsd\mu(t)\\
&\leq k\int_{0}^{+\infty} e^{-\alpha s}\left(\frac{1}{\nu([-r,r])}\int_{[-r,r]}\|\varphi(t+s)\|d\mu(t)\right)ds.
\end{aligned}
$$

Since μ and ν satisfy (**B**), then from Theorem 1.1.77, we have $[t \longmapsto \varphi(t-s)] \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$ and $[t \longmapsto \varphi(t+s)] \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$ for every $s \in \mathbb{R}$. Therefore,

$$
\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{[-r,r]}\|\varphi(t+s)\|d\mu(t) = 0,
$$

$$
\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{[-r,r]}\|\varphi(t-s)\|d\mu(t) = 0,
$$

$$
\big(\text{for all } s \in [0+\infty)\big) \quad e^{-\alpha s}\frac{1}{\nu([-r,r])}\int_{[-r,r]}\|\varphi(t+s)\|d\mu(t) \leq \|\varphi\|_\infty\frac{\mu([(-r,r])}{\nu([(-r,r])},
$$

$$
\big(\text{for all } s \in [0+\infty)\big) \quad e^{-\alpha s}\frac{1}{\nu([-r,r])}\int_{[-r,r]}\|\varphi(t-s)\|d\mu(t) \leq \|\varphi\|_\infty\frac{\mu([(-r,r])}{\nu([(-r,r])}.
$$

Using (**D**) and Lebesgue's dominated convergence Theorem, we deduce that

$$
\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{[-r,r]}\|\Phi_1(t)\|d\mu(t) = 0,
$$

and

$$
\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{[-r,r]}\|\Phi_2(t)\|d\mu(t) = 0.
$$

This completes the proof of theorem. ■

4.3 (μ,ν)-Pseudo Almost Automorphic Solutions for Equation (4.1.1)

In this section, we study the existence and uniqueness of (μ,ν)-pseudo almost automorphic solutions of Equation (4.1.1). In the following, p is a real number such that $1 < p < +\infty$ and q is its conjugate, namely
$$\frac{1}{p}+\frac{1}{q}=1.$$
The main result of this chapter is given by the following theorem.

Theorem 4.3.1 *Let* $\mu,\nu \in \mathcal{M}$ *satisfy* **(B)** *and* **(D)** *and* $A:\mathbb{R}\longrightarrow M_n(\mathbb{R})$ *be continuous function. Assume that the system* (4.2.1) *has an exponential dichotomy on* $\mathbb{R}$ *with parameters* (P,k,α) *and* $f:\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$ *is* (μ,ν)*-pseudo almost automorphic such that*

$$\|f(t,x)-f(t,y)\| \le l(t)h(\|x-y\|) \tag{4.3.1}$$

where $l\in L^p(\mathbb{R})\cap L^\infty(\mathbb{R})$ *and* $h:[0,+\infty)\longrightarrow[0,+\infty)$ *is a nondecreasing continuous function on* $[0,+\infty)$*, verifying* $h(r)<r$ *for every* $r>0$ *and* $h(0)=0$*. Then, the system* (4.1.1) *has a unique* (μ,ν)*-pseudo almost automorphic solution provided that*

$$\frac{(\alpha q)^{\frac{1}{q}}}{2k\|l\|_p} > 1. \tag{4.3.2}$$

Proof. Inequality (4.3.1) implies that

$$\begin{aligned}\|f(t,x)\| &\le \|f(t,x)-f(t,0)\|+\|f(t,0)\|\\ &\le l(t)h(\|x\|)+\|f(t,0)\|\\ &\le l(t)\|x\|+\|f(t,0)\|,\end{aligned}$$

for $t\in\mathbb{R}$ and $x\in\mathbb{R}^n$. Since the function $[t\longmapsto f(t,0)]\in PAA(\mathbb{R},\mathbb{R}^n,\mu,\nu)$, $PAA(\mathbb{R},\mathbb{R}^n,\mu,\nu)\subset BC(\mathbb{R},\mathbb{R}^n)$ and $l\in L^\infty(\mathbb{R})$, then we have

$$\|f(t,x)\|\le \sup_{t\in\mathbb{R}} l(t)\|x\|+\sup_{t\in\mathbb{R}}\|f(t,0)\| \quad \text{for } t\in\mathbb{R}.$$

Note that $\sup_{t\in\mathbb{R}} l(t) < +\infty$ and $\sup_{t\in\mathbb{R}}\|f(t,0)\| < +\infty$. Therefore, condition **(B)** holds. Let $\phi\in PAA(\mathbb{R},\mathbb{R}^n,\mu,\nu)$. By Theorem 1.1.82, $[t\longmapsto f(t,\phi(t))]\in PAA(\mathbb{R},\mathbb{R}^n,\mu,\nu)$. In view of Theorem 4.2.3, the system
$$x'(t)=A(t)x(t)+f(t,\phi(t))$$
has a unique (μ,ν)-pseudo almost automorphic solution x defined by

$$\begin{aligned}x(t) &= \int_{-\infty}^{+\infty} G(t,s)f(s,\phi(s))ds\\ &= \int_{-\infty}^{t} X(t)PX^{-1}(s)f(s,\phi(s))ds-\int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)f(s,\phi(s))ds.\end{aligned}$$

Let $c > 0$ and

$$\theta(t) = \int_{-\infty}^{t} l^p(s)ds$$

$$\|\varphi\|_c = \sup_{t\in\mathbb{R}} e^{-c\theta(t)}|\varphi(t)|.$$

Then, $\|\cdot\|_c$ is an equivalent norm to the supremum norm. In fact,

$$e^{-c\|l\|_p^p} \times \|\varphi\|_\infty \leq \|\varphi\|_c \leq \|\varphi\|_\infty. \tag{4.3.3}$$

Now, we define the mapping M on $PAA(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ by

$$M(u)(t) = \int_{-\infty}^{+\infty} G(t,s)f(s,u(s))ds \quad \text{for } t \in \mathbb{R}.$$

As in the proof of Theorem 4.2.3, M maps $PAA(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ into itself. For $u, v \in PAA(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$, we have

$$\begin{aligned}
&\|M(u)(t) - M(v)(t)\| \\
&\leq k\int_{-\infty}^{t} e^{-\alpha(t-s)}\|f(t,u(s)) - f(t,v(s))\|ds + k\int_{t}^{+\infty} e^{\alpha(t-s)}\|f(t,u(s)) - f(t,v(s))\|ds \\
&\leq k\int_{-\infty}^{t} e^{-\alpha(t-s)}l(s)h(\|u(s) - v(s)\|)ds + k\int_{t}^{+\infty} e^{\alpha(t-s)}l(s)h(\|u(s) - v(s)\|)ds \\
&\leq k\int_{-\infty}^{t} e^{-\alpha(t-s)}l(s)h(e^{c\theta(s)}\|u - v\|_c)ds + k\int_{t}^{+\infty} e^{\alpha(t-s)}l(s)h(e^{c\theta(s)}\|u - v\|_c)ds \\
&\leq k\int_{-\infty}^{t} e^{-\alpha(t-s)}l(s)h(e^{c\|l\|_p^p}\|u - v\|_c)ds + k\int_{t}^{+\infty} e^{\alpha(t-s)}l(s)h(e^{c\|l\|_p^p}\|u - v\|_c)ds \\
&= k\left(\int_{-\infty}^{t} e^{-\alpha(t-s)}l(s)ds + \int_{t}^{+\infty} e^{\alpha(t-s)}l(s)ds\right) h(e^{c\|l\|_p^p}\|u - v\|_c) \\
&\leq k\left\{\left(\int_{-\infty}^{t} e^{-\alpha q(t-s)}ds\right)^{\frac{1}{q}} \left(\int_{-\infty}^{t} l^p(s)ds\right)^{\frac{1}{p}} + \left(\int_{t}^{+\infty} e^{\alpha q(t-s)}ds\right)^{\frac{1}{q}} \left(\int_{t}^{+\infty} l^p(s)ds\right)^{\frac{1}{p}}\right\} \\
&\quad \times h(e^{c\|l\|_p^p}\|u - v\|_c) \\
&\leq \frac{2k\|l\|_p}{(\alpha q)^{\frac{1}{q}}} h\left(e^{c\|l\|_p^p}\|u - v\|_c\right).
\end{aligned}$$

Consequently,

$$\|M(u) - M(v)\|_\infty \leq \frac{2k\|l\|_p}{(\alpha q)^{\frac{1}{q}}} h\left(e^{c\|l\|_p^p}\|u - v\|_c\right).$$

Using (4.3.3) we obtain that

$$\|M(u) - M(v)\|_c \leq \frac{2k\|l\|_p}{(\alpha q)^{\frac{1}{q}}} h\left(e^{c\|l\|_p^p}\|u - v\|_c\right).$$

Now, we define the new norm N equivalent to the norm $\|\cdot\|_c$, by

$$N(\varphi) = e^{c\|l\|_p^p}\|\varphi\|_c.$$

Then,

$$N\left(M(u)-M(v)\right) \leq \frac{2k\|l\|_p}{(\alpha q)^{\frac{1}{q}}} e^{c\|l\|_p^p} h\left(N(u-v)\right).$$

If we choose the function

$$\Phi(r) := \frac{2k\|l\|_p}{(\alpha q)^{\frac{1}{q}}} e^{c\|l\|_p^p} h(r) \quad \text{for } r \geq 0,$$

we deduce that

$$N\left(M(u)-M(v)\right) \leq \Phi\left(N(u-v)\right).$$

Furthermore, Φ is a positive, continuous and nondecreasing function on $[0,+\infty)$ satisfying $\Phi(0)=0$. Using (4.3.2), $\ln\left(\frac{(\alpha q)^{\frac{1}{q}}}{2k\|l\|_p}\right)$ is nonnegative. We choose $c < \frac{1}{\|l\|_p^p} \ln\left(\frac{(\alpha q)^{\frac{1}{q}}}{2k\|l\|_p}\right)$. In this case we have $\frac{2k\|l\|_p}{(\alpha q)^{\frac{1}{q}}} e^{c\|l\|_p^p} < 1$. Consequently, $\Phi(r) < r$ for all $r > 0$. Using Theorem 1.2.3, M has exactly one fixed point $\phi \in PAA(\mathbb{R},\mathbb{R}^n,\mu,\nu)$. Consequently, $M(\phi)=\phi=x$. The proof is now complete. ■

Theorem 4.3.2 [63] *Assume that the following differential equation*

$$x'(t) = A(t)x(t) \quad \text{for } t \in \mathbb{R}$$

has an exponential dichotomy on $\mathbb{R}$ with parameters (P,k,α). Let $B:\mathbb{R} \longrightarrow M_n(\mathbb{R})$ be a bounded continuous function such that $\delta = \sup_{t\in\mathbb{R}} \|B(t)\| < \frac{\alpha}{4k^2}$. Then, the perturbed equation

$$x'(t) = [A(t)+B(t)]x(t) \quad \text{for } t \in \mathbb{R}$$

has an exponential dichotomy on $\mathbb{R}$ with parameters $\left(Q,\frac{5}{2}k^2,\alpha-2k\delta\right)$, where Q is a matrix projection and $\left(\frac{5}{2}k^2,\alpha-2k\delta\right)$ the elements of dichotomy.

The following result is a consequence of Theorems 4.3.1 and 4.3.2.

Corollary 4.3.3 *Let $\mu,\nu \in \mathcal{M}$ satisfy* **(B)** *and* **(D)** *and $A,B:\mathbb{R} \longrightarrow M_n(\mathbb{R})$ be two continuous functions. Assume that the system*

$$x'(t) = A(t)x(t) \quad \text{for } t \in \mathbb{R}$$

has an exponential dichotomy on $\mathbb{R}$ with parameters (P,k,α) such that

$$\delta = \sup_{t\in\mathbb{R}} \|B(t)\| < \frac{\alpha}{4k^2}.$$

If $f:\mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is (μ,ν)-pseudo almost automorphic such that

$$\|f(t,x)-f(t,y)\| \leq l(t)h(\|x-y\|)$$

where $l \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and* $h : [0,+\infty) \longrightarrow [0,+\infty)$ *is a nondecreasing continuous function on* $[0,+\infty)$*, verifying* $h(r) < r$ *for every* $r > 0$ *and* $h(0) = 0$*. Then, the following perturbed system*

$$x'(t) = [A(t) + B(t)]x(t) + f(t, x(t)) \quad \text{for } t \in \mathbb{R}$$

has a unique (μ, ν)*-pseudo almost automorphic solution provided that*

$$\frac{\big((\alpha - 2k\delta)q\big)^{\frac{1}{q}}}{5k^2\|l\|_p} > 1.$$

Proof. Using Theorem 4.3.2, the system $x'(t) = [A(t) + B(t)]x(t)$ has an exponential dichotomy on $\mathbb{R}$ with parameters $(\frac{5}{2}k^2, \alpha - 2k\delta)$. It suffices now to apply Theorem 4.3.1. ■

Corollary 4.3.4 *Let* $\mu, \nu \in \mathcal{M}$ *satisfy* **(B)** *and* **(D)**, a_i *be n real continuous functions such that*

$$\lim_{\alpha \to +\infty} \frac{1}{\alpha}\int_0^\alpha a_i(s+t)ds = M[a_i] > 0 \quad \text{for } i \in \{1, \dots, n\}, \tag{4.3.4}$$

uniformly with respect to $t \in \mathbb{R}^+$ *and* $f : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ (μ, ν)*-pseudo almost automorphic such that*

$$\|f(t,x) - f(t,y)\| \le l(t)h(\|x - y\|) \tag{4.3.5}$$

where $l \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and* $h : [0,+\infty) \longrightarrow [0,+\infty)$ *is a nondecreasing continuous function on* $[0,+\infty)$*, verifying* $h(r) < r$ *for every* $r > 0$ *and* $h(0) = 0$*. Then, the system*

$$x'(t) = \mathit{diag}(-a_1(t), \cdots, -a_n(t))x(t) + f(t, x(t)) \quad \text{for } t \in \mathbb{R} \tag{4.3.6}$$

has a unique (μ, ν)*-pseudo almost automorphic solution provided that*

$$\frac{(\alpha q)^{\frac{1}{q}}}{2k\|l\|_p} > 1,$$

where (k, α) *is the parameters of the exponential dichotomy of the homogeneous equation associated to* (4.3.6).

Proof. Fixed $i \in \{1, \cdots, n\}$. By (4.3.4), [191, Theorem 1.6, p. 209] and [191, Theorem 1.3, p. 204], there are positive numbers k_i and α_i such that either

$$|x_i(t)x_i^{-1}(s)| \le k_i e^{-\alpha_i(t-s)} \quad (t \ge s) \tag{4.3.7}$$

or

$$|x_i(t)x_i^{-1}(s)| \le k_i e^{-\alpha_i(s-t)} \quad (t \le s). \tag{4.3.8}$$

From the proof of [191, Theorem 1.3, p. 204], $\alpha_i = \dfrac{1}{b_i}$ and $k_i = \|a_i\|b_i$ where b_i is such that $\|x_i(\cdot, \tau_1, \tau_2)\| \le b_i$ where $x_i(t, \tau_1, \tau_2) = x_i(t)\displaystyle\int_t^{+\infty} \eta(s, \tau_1, \tau_2)|x_i(s)|^{-1}ds$ for $t \in \mathbb{R}$ and

$$\eta(t, \tau_1, \tau_2) = \begin{cases} 0, & t \in (-\infty, \tau_1 - 1], \\ t - \tau_1 + 1 & t \in (\tau_1 - 1, \tau_1], \\ 1 & t \in (\tau_1, \tau_2], \\ 1 - t + \tau_2 & t \in (\tau_2, \tau_2 + 1], \\ 0 & t \in (\tau_2 + 1, +\infty). \end{cases}$$

Choosing the norm $\|A\| = \max_{i,j} |a_{i,j}|$, $\alpha = \dfrac{1}{\max_i b_i}$, $k = \max_i \|a_i\| \max_i b_i$ and using (4.3.7) and (4.3.8) we can prove that the homogeneous equation (4.3.6) with $f = 0$ has an exponential dichotomy on $\mathbb{R}$ with parameters (k, α). It suffices now to apply Theorem 4.3.1. ■

Lemma 4.3.5 [151] *If the elements of the matrix $A(t)$ of the system (4.2.1) satisfy the row dominance or the column dominance:*

$$\begin{cases} a_{ii}(t) + \sum_{j\neq i} \mid a_{ij}(t) \mid\leq -\lambda < 0, & i = 1, 2, \cdots, k, \\ a_{ii}(t) - \sum_{j\neq i} \mid a_{ij}(t) \mid\geq \lambda > 0, & i = k+1, k+2, \cdots, n \end{cases} \tag{4.3.9}$$

$$\begin{cases} a_{jj}(t) + \sum_{i\neq j} \mid a_{ij}(t) \mid\leq -\lambda < 0, & j = 1, 2, \cdots, k, \\ a_{jj}(t) - \sum_{i\neq j} \mid a_{ij}(t) \mid\geq \lambda > 0, & j = k+1, k+2, \cdots, n. \end{cases} \tag{4.3.10}$$

Then, the system (4.2.1) has an exponential dichotomy on $\mathbb{R}$ with projection $P = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$ and dichotomy constants $(1, \lambda)$.

We close this section by the following corollary.

Corollary 4.3.6 *Let $\mu, \nu \in \mathcal{M}$ satisfy* **(B)** *and* **(D)***. Assume that the coefficients of the matrix $A(t)$ of the system (4.2.1) satisfy (4.3.9) or (4.3.10) and $f : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is (μ, ν)-pseudo almost automorphic such that*

$$\|f(t, x) - f(t, y)\| \leq l(t)h(\|x - y\|)$$

where $l \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $h : [0, +\infty) \longrightarrow [0, +\infty)$ is a nondecreasing continuous function on $[0, +\infty)$, verifying $h(r) < r$ for every $r > 0$ and $h(0) = 0$. Then, the system (4.1.1) has a unique (μ, ν)-pseudo almost automorphic solution provided that

$$\frac{(\lambda q)^{\frac{1}{q}}}{2\|l\|_p} > 1.$$

Proof. Using Lemma 4.3.5, the system $x'(t) = A(t)x(t)$ has an exponential dichotomy on $\mathbb{R}$ with projection $P = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$ and dichotomy constants $(1, \lambda)$. By applying Theorem 4.3.1, we obtain the desired result. ■

4.4 Application

In this section, we give sufficient conditions ensuring the existence and uniqueness of (μ, ν)-pseudo almost automorphic solution to the following equation:

$$\begin{cases} x_1'(t) = -(t^2+8)x_1(t) + (t^2+1)x_2(t) + \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right) + \frac{|x_1(t)|}{2(|x_1(t)|+1)} e^{\frac{-t^2}{2}} \\ x_2'(t) = -(e^t+3)x_1(t) + (e^t+10)x_2(t) + \cos\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right) + \frac{|x_2(t)|}{2(|x_2(t)|+1)} e^{\frac{-t^2}{2}}, \end{cases} \tag{4.4.1}$$

for $t \in \mathbb{R}$. To represent (4.4.1) in the abstract form (4.1.1), we let

$$a_{11}(t) = -(t^2+8),\ a_{12}(t) = t^2+1,\ a_{21}(t) = -(e^t+3),\ a_{22}(t) = e^t+10 \ \text{ for } t \in \mathbb{R},$$

and

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \quad \text{for } t \in \mathbb{R},$$

$$f(t,x) = \begin{pmatrix} \sin(\frac{1}{2+\cos t+\cos\sqrt{2}t}) + \frac{|x_1|}{2(|x_1|+1)}e^{\frac{-t^2}{2}} \\ \cos(\frac{1}{2+\cos t+\cos\sqrt{2}t}) + \frac{|x_2|}{2(|x_2|+1)}e^{\frac{-t^2}{2}} \end{pmatrix} \text{ for } t \in \mathbb{R} \text{ and } x=(x_1,x_2) \in \mathbb{R}^2.$$

We consider the measure μ where its Radon-Nikodym derivative is

$$\rho_1(t) = e^{\cos(t)} \quad \text{for } t \in \mathbb{R}$$

and the measure ν where its Radon-Nikodym derivative is

$$\rho_2(t) = \begin{cases} e^t & \text{for } t \leq 0, \\ 1 & \text{for } t > 0. \end{cases}$$

From [45], $\nu \in \mathcal{M}$ satisfies (**B**). For $r > 0$ we have

$$\frac{2r}{e} \leq \mu([-r,r]) = \int_{-r}^{r} e^{\cos(t)}dt \leq 2er.$$

Then, $\mu \in \mathcal{M}$. Furthermore $\cos(\tau + a) \leq 2 + \cos(a)$ for all $\tau \in \mathbb{R}$ and $a \in A$, which implies that $\mu(\tau + A) \leq e^2\mu(A)$ and consequently μ satisfies (**B**). On the other hand,

$$\limsup_{r\to+\infty} \frac{\mu([-r,r])}{\nu([-r,r])} \leq 2e < +\infty,$$

which implies that (**D**) holds.

The elements of the matrix $A(t)$ satisfy the row dominance

$$\begin{cases} a_{11}(t) + |a_{12}(t)| \leq -\lambda = -7 < 0, \\ a_{22}(t) - |a_{21}(t)| \geq \lambda = 7. \end{cases}$$

Let us now show that the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is (μ,ν)- pseudo almost automorphic. Its almost automorphic component is the function $(t,x) \longmapsto \begin{pmatrix} \sin(\frac{1}{2+\cos t+\cos\sqrt{2}t}) \\ \cos(\frac{1}{2+\cos t+\cos\sqrt{2}t}) \end{pmatrix}$.

For $r > 0$ sufficiently large and $i \in \{1,2\}$, we have

$$\frac{1}{\nu([-r,r])}\int_{-r}^{r} \frac{|x_i|}{2(|x_i|+1)}e^{\frac{-t^2}{2}}d\mu(t) \leq \frac{|x_i|}{2(|x_i|+1)}\frac{e}{r+1-e^{-r}}.$$

Since

$$\lim_{r\to+\infty} \frac{|x_i|}{2(|x_i|+1)}\frac{e}{r+1-e^{-r}} = 0$$

then, the function $(t,x) \longmapsto \begin{pmatrix} \frac{|x_1|}{2(|x_1|+1)} e^{\frac{-t^2}{2}} \\ \frac{|x_2|}{2(|x_2|+1)} e^{\frac{-t^2}{2}} \end{pmatrix}$ is the (μ,ν)-ergodic perturbation of the function f. In order to prove that the function f satisfies (4.3.5) we take

$$l(t) = e^{\frac{-t^2}{2}} \text{ for } t \in \mathbb{R} \qquad \text{and} \qquad h(r) = \frac{r}{r+1} \text{ for } r \in \mathbb{R}^+.$$

Then, h is positive, nondecreasing and continuous on $\mathbb{R}^+$. Moreover, $h(r) < r$ for $r > 0$, $h(0) = 0$ and $l \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\|l\|_2 = \sqrt{\sqrt{\pi}}$. Furthermore, for $t \in \mathbb{R}$ and $x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2$, ($\mathbb{R}^2$ is equipped with the norm $\|(x_1,x_2)\| = |x_1| + |x_2|$), we have

$$\begin{aligned}
\|f(t,x) - f(t,y)\| &= \left\| \begin{pmatrix} \frac{|x_1|}{2(|x_1|+1)} e^{\frac{-t^2}{2}} - \frac{|y_1|}{2(|y_1|+1)} e^{\frac{-t^2}{2}} \\ \frac{|x_2|}{2(|x_2|+1)} e^{\frac{-t^2}{2}} - \frac{|y_2|}{2(|y_2|+1)} e^{\frac{-t^2}{2}} \end{pmatrix} \right\| \\
&= \frac{1}{2} l(t) \left\{ \left| \frac{|x_1|}{|x_1|+1} - \frac{|y_1|}{|y_1|+1} \right| + \left| \frac{|x_2|}{|x_2|+1} - \frac{|y_2|}{|y_2|+1} \right| \right\} \\
&= \frac{1}{2} l(t) \left\{ \frac{||x_1| - |y_1||}{(|x_1|+1)(|y_1|+1)} + \frac{||x_2| - |y_2||}{(|x_2|+1)(|y_2|+1)} \right\} \\
&\leq \frac{1}{2} l(t) \left\{ \frac{|x_1 - y_1|}{(|x_1|+1)(|y_1|+1)} + \frac{|x_2 - y_2|}{(|x_2|+1)(|y_2|+1)} \right\} \\
&\leq \frac{1}{2} l(t) \left\{ \frac{|x_1 - y_1|}{|x_1 - y_1|+1} + \frac{|x_2 - y_2|}{|x_2 - y_2|+1} \right\} \\
&= \frac{1}{2} l(t) \left\{ h(|x_1 - y_1|) + h(|x_2 - y_2|) \right\} \\
&\leq l(t) h(\|x - y\|).
\end{aligned}$$

On the other hand,

$$\frac{(\lambda q)^{\frac{1}{q}}}{2\|l\|_p} = \frac{\sqrt{7 \times 2}}{2\sqrt{\sqrt{\pi}}} > 1.$$

According to Corollary 4.3.6, Equation (4.4.1) has a unique (μ,ν)-pseudo almost automorphic solution.

Chapter 5

Compact Almost Automorphic Weak Solutions for Some Monotone Differential Inclusions : Applications to Parabolic and Hyperbolic Equations [(1)]

Contents

5.1 Introduction

Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be a multivalued operator with domain $D(\mathcal{A})$. We consider the following differential inclusions :

$$u'(t) + \mathcal{A}u(t) \ni f(t) \quad \text{for } t \in \mathbb{R} \tag{5.1.1}$$

$$u'(t) + \mathcal{A}u(t) \ni g(t, u(t)) \quad \text{for } t \in \mathbb{R} \tag{5.1.2}$$

where $f : \mathbb{R} \longrightarrow \mathcal{H}$ and $g : \mathbb{R} \times \mathcal{H} \longrightarrow \mathcal{H}$ are continuous functions. Many studies have been devoted to the existence of periodic and almost periodic solutions for the differential inclusion (5.1.1) when the operator $\mathcal{A}$ is maximal monotone on $\mathcal{H}$ and the forcing term f

(1) This work has been done in collaboration with Brahim Es-sebbar, Khalil Ezzinbi and Samir Fatajou and has been published in Journal of Mathematical Analysis and Applications [83]

is periodic or almost periodic. Brézis [51, Theorem 3.4, p. 65] proved that for any $f \in L^1([a,b],\mathcal{H})$ and $u_0 \in \overline{D(\mathcal{A})}$ there exists a unique weak solution of the following differential inclusion:

$$\begin{cases} u'(t) + \mathcal{A}u(t) \ni f(t) \quad \text{for } t \in [a,b] \\ u(a) = u_0. \end{cases}$$

Brézis [51, Theorem 3.15, p. 95] showed that if $\mathcal{A}$ is maximal monotone, then for each $f \in L^1([0,T],\mathcal{H})$ the differential inclusion

$$\begin{cases} u'(t) + \mathcal{A}u(t) \ni f(t) \\ u(0) = u(T) \end{cases}$$

has at least a weak solution. Baillon and Haraux [27] studied the following differential inclusion:

$$u'(t) + \partial\phi(u(t)) \ni f(t) \quad \text{for } t \in [0,+\infty), \tag{5.1.3}$$

where ϕ is a proper, convex and lower semicontinuous function and $f \in L^2([0,+\infty),\mathcal{H})$ is T-periodic. They proved that if a T-periodic solution of (5.1.3) exists on $\mathbb{R}$, then for each solution u of (5.1.3) on $\mathbb{R}^+$ there exists a periodic strong solution w of (5.1.3) on $\mathbb{R}$ such that

$$u(t) \rightharpoonup w(t) \quad \text{as } t \longrightarrow +\infty.$$

Haraux [112] proved that if the forcing term $f : \mathbb{R} \longrightarrow \mathcal{H}$ is S^2-almost periodic, then each weak solution of (5.1.3) on $\mathbb{R}^+$ is asymptotic to an almost periodic weak solution of (5.1.3) on $\mathbb{R}$. Haraux [114] also proved that if (5.1.1) has a uniformly continuous weak solution on $\mathbb{R}^+$ and its range over $\mathbb{R}^+$ is relatively compact, then it has an almost periodic weak solution on $\mathbb{R}$ when f is almost periodic [114, Theorem 1, p. 295]. Furthermore, Haraux [115] proved that if the forcing term $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ is S^1-almost periodic, then all bounded solutions on $\mathbb{R}$ of (5.1.1) are almost periodic.

The aim of this chapter is to study the existence of compact almost automorphic weak solutions for (5.1.1) and (5.1.2). If $\mathcal{A}$ is maximal monotone and f is compact almost automorphic, we prove that if (5.1.1) has a uniformly continuous weak solution on $\mathbb{R}^+$ having a relatively compact range over $\mathbb{R}^+$, then it has at least a compact almost automorphic weak solution on $\mathbb{R}$. Our main result is proved by using the minmax principle due of Amerio [26]. As an application, we study the following partial differential equation :

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) - \Delta u(t,x) + \beta\left(\dfrac{\partial}{\partial t}u(t,x)\right) \ni f(t,x) \quad \text{for } (t,x) \in \mathbb{R}\times\Omega \\ u(t,x) = 0 \quad \text{for } (t,x) \in \mathbb{R}\times\partial\Omega, \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ such that $\dim(\Omega) \geq 2$, β is a strongly monotone graph in $\mathbb{R}\times\mathbb{R}$ and f is a compact almost automorphic function in

$L^2(\Omega)$.

If $\mathcal{A}$ is strongly maximal monotone and f is compact almost automorphic, we prove that (5.1.1) has a unique bounded weak solution that is compact almost automorphic and globally attractive. Moreover, we use the contraction principle to prove the existence and uniqueness of compact almost automorphic weak solution for (5.1.2) where g is compact almost automorphic in t and Lipschitzian with respect to the second argument.

5.2 Compact Almost Automorphic Weak Solutions of (5.1.1) Where $\mathcal{A}$ is Maximal Monotone

In the sequel, we prove the existence of compact almost automorphic weak solutions of (5.1.1) where the operator $\mathcal{A}$ is maximal monotone.

Theorem 5.2.1 *Suppose that f is compact almost automorphic and $\mathcal{A}$ is maximal monotone. If (5.1.1) has a uniformly continuous weak solution on $\mathbb{R}^+$ having a relatively compact range over $\mathbb{R}^+$, then (5.1.1) has at least a compact almost automorphic weak solution.*

For the proof of Theorem 5.2.1, we need the following lemmas.

Lemma 5.2.2 *Let $F \in L^1_{loc}(\mathbb{R}, \mathcal{H})$ and x be a weak solution on $\mathbb{R}$ of the following differential inclusion:*

$$x'(t) + \mathcal{A}x(t) \ni F(t).$$

Assume that x is uniformly continuous on $\mathbb{R}$ and there exists a compact set K of $\mathcal{H}$ such that

$$x(t) \in K \quad \text{for all } t \in \mathbb{R}. \tag{5.2.1}$$

If there exist a sequence $(t_n)_n \subset \mathbb{R}$ and a function $G : \mathbb{R} \longrightarrow \mathcal{H}$ such that

$$F(t + t_n) \longrightarrow G(t) \quad \text{in } L^1_{loc}(\mathbb{R}, \mathcal{H}) \text{ as } n \longrightarrow +\infty,$$

then there exists a subsequence of $(t_n)_n$ denoted by $(s_n)_n$ such that

$$x(t + s_n) \longrightarrow y(t) \quad \text{as } n \longrightarrow +\infty \tag{5.2.2}$$

uniformly on any compact subset of $\mathbb{R}$, where y is a weak solution on $\mathbb{R}$ of the following differential inclusion:

$$y'(t) + \mathcal{A}y(t) \ni G(t). \tag{5.2.3}$$

Furthermore, y is uniformly continuous on $\mathbb{R}$ and $y(t) \in K$ for all $t \in \mathbb{R}$.

Proof. For each $n \in \mathbb{N}$, we define x_n and F_n on $\mathbb{R}$ by $x_n(t) = x(t + t_n)$ and $F_n(t) = F(t + t_n)$. By (5.2.1), $(x_n)_n$ satisfies $x_n(t) \in K$ for each $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Consequently $\{x_n(t) : n \in \mathbb{N}\}$

is a relatively compact set in $\mathcal{H}$ for each $t \in \mathbb{R}$. Since x is uniformly continuous on $\mathbb{R}$, then the sequence $(x_n)_n$ is uniformly equicontinuous on $\mathbb{R}$. By Arzelà-Ascoli's Theorem, $\{x_n : n \in \mathbb{N}\}$ is a relatively compact subset of $BC(\mathbb{R}, \mathcal{H})$ endowed with the topology of compact convergence. From the sequence $(t_n)_n$, we can extract a subsequence $(s_n)_n$ such that there exists $y \in BC(\mathbb{R}, \mathcal{H})$ such that $(x_n)_n$ converges to y uniformly on each compact subset of $\mathbb{R}$ and hence (5.2.2) holds. Furthermore, since x_n is a weak solution of (5.1.1) with F_n and F_n converges to G in $L^1_{loc}(\mathbb{R}, \mathcal{H})$, the use of Lemma 1.4.8 allows us to conclude that y is a weak solution of (5.2.3); moreover, $y(t) \in K$ for all $t \in \mathbb{R}$. Therefore, y is uniformly continuous on $\mathbb{R}$ because it is a limit of the sequence $(x_n)_n$, which is uniformly equicontinuous on $\mathbb{R}$. ■

Lemma 5.2.3 *Suppose that f is compact almost automorphic. If (5.1.1) has a uniformly continuous weak solution u_0 on $\mathbb{R}^+$ having a relatively compact range over $\mathbb{R}^+$, then it has a uniformly continuous weak solution u^* on $\mathbb{R}$; moreover, its range over $\mathbb{R}$ is relatively compact.*

Proof. Let $(t_n)_n \subset \mathbb{R}$ be such that

$$\lim_{n \to +\infty} t_n = +\infty.$$

If $t \in [-1, 1]$, then for sufficiently large n the sequence of functions $u_n : t \longmapsto u_0(t + t_n)$ is well defined and uniformly equicontinuous. By Arzelà-Ascoli's Theorem, there exist a function v and a subsequence $(t^1_n)_n \subset (t_n)_n$ such that

$$u_0(t + t^1_n) \longrightarrow v(t) \quad \text{as } n \longrightarrow +\infty$$

uniformly on $[-1, 1]$. Using the same argument, we deduce that for each $p \in \mathbb{N}^*$ there exists a subsequence $(t^p_n)_n \subset (t^{p-1}_n)_n \subset \cdots \subset (t_n)_n$ such that

$$u_0(t + t^p_n) \longrightarrow v(t) \quad \text{as } n \longrightarrow +\infty$$

uniformly on $[-p, p]$. Let $(t'_n)_n := (t^n_n)_n$ be Cantor's diagonal sequence. Then,

$$u_0(t + t'_n) \longrightarrow v(t) \quad \text{as } n \longrightarrow +\infty \tag{5.2.4}$$

uniformly on any compact subset of $\mathbb{R}$. Since f is compact almost automorphic, there exist a continuous function g and a subsequence $(t''_n)_n \subset (t'_n)_n$ such that

$$f(t + t''_n) \longrightarrow g(t) \quad \text{as } n \longrightarrow +\infty, \tag{5.2.5}$$

$$g(t - t''_n) \longrightarrow f(t) \quad \text{as } n \longrightarrow +\infty \tag{5.2.6}$$

uniformly on any compact subset of $\mathbb{R}$. Moreover, using (5.2.4) and (5.2.5), we find that v is a weak solution on $\mathbb{R}$ of (5.1.1) with g. Furthermore, v is uniformly continuous on $\mathbb{R}$. By applying the above argument to the returning sequence $(-t''_n)_n$, we obtain a subsequence $(t'''_n)_n \subset (t''_n)_n$ and a function u^* such that

$$v(t - t'''_n) \longrightarrow u^*(t) \quad \text{as } n \longrightarrow +\infty \tag{5.2.7}$$

uniformly on any compact subset of $\mathbb{R}$. By (5.2.6), (5.2.7) and Lemma 1.4.8, we deduce that u^* is a weak solution on $\mathbb{R}$ of (5.1.1). Furthermore, the function u^* is uniformly continuous on $\mathbb{R}$ and its range is contained in the closure of the range of u_0; hence, it is relatively compact.■

Proof. (of Theorem 5.2.1) We use Amerio's principle. Let $K = \overline{Co}(u^*(\mathbb{R}))$ be the closed convex hull of $u^*(\mathbb{R})$ in $\mathcal{H}$, where u^* is given in Lemma 5.2.3. Let Λ and Γ be the sets defined by

$$\Lambda = \Big\{u \in C(\mathbb{R}, \mathcal{H}) : u(\mathbb{R}) \subset K \text{ and } \sup_{t\in\mathbb{R}} |u(t+\sigma)-u(t)| \le \sup_{t\in\mathbb{R}} |u^*(t+\sigma)-u^*(t)| \text{ for all } \sigma \in \mathbb{R}\Big\},$$

$$\Gamma = \Big\{u \in \Lambda : \quad u \text{ is a weak solution of the differential inclusion (5.1.1) on } \mathbb{R}\Big\}.$$

We define the operator $J : \Lambda \longrightarrow \mathbb{R}^+$ by

$$J(u) = \sup_{t\in\mathbb{R}} |u(t)| \quad \text{for } u \in \Lambda.$$

We say that $\widetilde{u}$ is a *minimal weak solution* of (5.1.1) if

$$\widetilde{u} \in \Gamma \quad \text{and} \quad J(\widetilde{u}) = \inf_{u\in\Gamma} J(u).$$

We divide the proof into three steps:

Step 1. We claim that (5.1.1) has a minimal weak solution $\widehat{u}$ on $\mathbb{R}$. In fact, let

$$\delta = \inf_{u\in\Gamma} J(u).$$

Then, by Lemma 5.2.3, Γ is nonempty since $u^* \in \Gamma$. Hence, δ exists in $\mathbb{R}$. Consequently, there exists a sequence $(u_n)_n$ in Γ such that

$$\lim_{n\longrightarrow+\infty} J(u_n) = \delta. \tag{5.2.8}$$

By the definition of Γ, for each $t \in \mathbb{R}$, $\{u_n(t) : n \in \mathbb{N}\}$ is a subset of the compact K and $(u_n)_n$ is uniformly equicontinuous on $\mathbb{R}$. Using Arzelà-Ascoli's Theorem, we asssert that $\{u_n : n \in \mathbb{N}\}$ is a relatively compact subset of $BC(\mathbb{R}, \mathcal{H})$ endowed with the topology of compact convergence. Thus, there exists a subsequence of $(u_n)_n$ denoted also by $(u_n)_n$ such that

$$u_n(t) \longrightarrow \widehat{u}(t) \quad \text{as } n \longrightarrow +\infty \tag{5.2.9}$$

uniformly on any compact subset of $\mathbb{R}$. Since $u_n'(t) + \mathcal{A}u_n(t) \ni f(t)$ in the sense of weak solutions, the use of (5.2.9) together with Lemma 1.4.8 implies that $\widehat{u}$ is a weak solution on $\mathbb{R}$ of (5.1.1) and $\widehat{u} \in \Lambda$; consequently, $\widehat{u} \in \Gamma$. Hence, we obtain that

$$\delta \le J(\widehat{u}). \tag{5.2.10}$$

We note that J is lower semicontinuous with respect to the topology of compact convergence; namely, if $\lim_{n\to+\infty} x_n = x$ uniformly on compact subsets of $\mathbb{R}$, then $J(x) \leq \liminf_{n\to+\infty} J(x_n)$. By (5.2.9), we get that

$$J(\widehat{u}) \leq \liminf_{n\to+\infty} J(u_n). \tag{5.2.11}$$

By (5.2.8), (5.2.10) and (5.2.11), we deduce that

$$J(\widehat{u}) = \delta = \inf_{u\in\Gamma} J(u).$$

Step 2. We claim that the minimal weak solution $\widehat{u}$ is unique. In fact, let $u, v \in \Gamma$ be such that

$$J(u) = J(v) = \delta. \tag{5.2.12}$$

Let $(t_n)_n \subset \mathbb{R}$ be such that

$$\lim_{n\to+\infty} t_n = -\infty. \tag{5.2.13}$$

From the compact almost automorphy of f, there exist a continuous function g and a subsequence of $(t_n)_n$ denoted also by $(t_n)_n$ such that

$$f(t+t_n) \longrightarrow g(t) \quad \text{as } n \longrightarrow +\infty,$$

$$g(t-t_n) \longrightarrow f(t) \quad \text{as } n \longrightarrow +\infty$$

uniformly on any compact subset of $\mathbb{R}$. Now, let us prove that

$$u(t+t_n) \longrightarrow u_1(t) \quad \text{as } n \longrightarrow +\infty, \tag{5.2.14}$$

$$u_1(t-t_n) \longrightarrow u_2(t) \quad \text{as } n \longrightarrow +\infty, \tag{5.2.15}$$

$$v(t+t_n) \longrightarrow v_1(t) \quad \text{as } n \longrightarrow +\infty, \tag{5.2.16}$$

$$v_1(t-t_n) \longrightarrow v_2(t) \quad \text{as } n \longrightarrow +\infty \tag{5.2.17}$$

uniformly on any compact subset of $\mathbb{R}$, where u_2 and v_2 are two minimal weak solutions on $\mathbb{R}$ of (5.1.1). Since $u \in \Gamma$, it is uniformly continuous on $\mathbb{R}$ and $u(\mathbb{R}) \subset K$. Applying Lemma 5.2.2 to $x = u$, $F = f$ and the sequence $(t_n)_n$, we obtain (5.2.14) where u_1 is a weak solution on $\mathbb{R}$ of the following differential inclusion:

$$u_1'(t) + \mathcal{A}u_1(t) \ni g(t).$$

Moreover, u_1 is uniformly continuous on $\mathbb{R}$ and $u_1(\mathbb{R}) \subset K$ which implies that $u_1 \in \Lambda$. Applying again Lemma 5.2.2 to $x = u_1$, $F = g$ and the returning sequence $(-t_n)_n$, we obtain (5.2.15) where u_2 is a weak solution on $\mathbb{R}$ of (5.1.1) with $u_2 \in \Gamma$. It follows from (5.2.14) and (5.2.15) that

$$J(u_2) \leq J(u_1) \leq J(u).$$

Using (5.2.12), we obtain $J(u_2) = \delta$ and consequently u_2 is a weak minimal solution on $\mathbb{R}$ of (5.1.1). Applying the same argument to v, we obtain (5.2.16) and (5.2.17) where v_2 is a weak minimal solution on $\mathbb{R}$ of (5.1.1). Since $u'(t) + \mathcal{A}u(t) \ni f(t)$ and $v'(t) + \mathcal{A}v(t) \ni f(t)$ in the sense of weak solutions and the operator $\mathcal{A}$ is monotone, we find by using inequality (1.4.2) that the function $t \longmapsto |u(t) - v(t)|$ is nonincreasing. By (5.2.13), we obtain that

$$\lim_{n\to+\infty} |u(t+t_n) - v(t+t_n)| = \sup_{\sigma\in\mathbb{R}} |u(\sigma) - v(\sigma)|. \tag{5.2.18}$$

It follows from (5.2.14)-(5.2.17) that for each $t \in \mathbb{R}$

$$\begin{aligned} \lim_{m\to+\infty} \lim_{n\to+\infty} |u(t+t_n-t_m) - v(t+t_n-t_m)| &= \lim_{m\to+\infty} |u_1(t-t_m) - v_1(t-t_m)| \\ &= |u_2(t) - v_2(t)|. \end{aligned} \tag{5.2.19}$$

Combining (5.2.18) and (5.2.19), we obtain that for each $t \in \mathbb{R}$

$$|u_2(t) - v_2(t)| = \sup_{\sigma\in\mathbb{R}} |u(\sigma) - v(\sigma)| = c. \tag{5.2.20}$$

Consequently, we have

$$|u_2(t) - v_2(t)| = |u_2(0) - v_2(0)| \quad \text{for } t \in \mathbb{R}. \tag{5.2.21}$$

Let $S_t : \overline{D(\mathcal{A})} \longrightarrow \overline{D(\mathcal{A})}$ be the operator defined for each $x_0 \in \overline{D(\mathcal{A})}$ by

$$S_t x_0 = x(t),$$

where x is the unique weak solution on $\mathbb{R}$ of (5.1.1) with initial data $x(0) = x_0$. Taking $f = g$ in (1.4.2), we deduce that the operator S_t is contractive on the closed convex set $\overline{D(\mathcal{A})}$. It follows from (5.2.21) that

$$|S_t u_2(0) - S_t v_2(0)| = |u_2(0) - v_2(0)|.$$

Using Lemma 1.2.5, we obtain that

$$S_t\left(\frac{u_2(0) + v_2(0)}{2}\right) = \frac{1}{2}\left(\mathbb{S}_t u_2(0) + S_t v_2(0)\right) = \frac{u_2(t) + v_2(t)}{2}.$$

We conclude that $\frac{u_2 + v_2}{2}$ is also a weak solution on $\mathbb{R}$ of (5.1.1). Since $u_2(\mathbb{R}) \subset K$, $v_2(\mathbb{R}) \subset K$ and K is convex, then $\left(\frac{u_2 + v_2}{2}\right)(\mathbb{R}) \subset K$ and $\frac{u_2 + v_2}{2} \in \Gamma$. Hence,

$$\delta = \inf_{u\in\Gamma} J(u) \leq J\left(\frac{1}{2}u_2 + \frac{1}{2}v_2\right) = \sup_{t\in\mathbb{R}} \left|\frac{1}{2}u_2(t) + \frac{1}{2}v_2(t)\right|. \tag{5.2.22}$$

By the parallelogram law, we get that

$$\sup_{t\in\mathbb{R}} \left|\frac{1}{2}u_2(t) + \frac{1}{2}v_2(t)\right|^2 + \frac{1}{4}c^2 \leq \frac{1}{2}\sup_{t\in\mathbb{R}} |u_2(t)|^2 + \frac{1}{2}\sup_{t\in\mathbb{R}} |v_2(t)|^2. \tag{5.2.23}$$

By (5.2.22) and (5.2.23), we obtain that $\delta^2 + \frac{1}{4}c^2 \leq \frac{1}{2}\delta^2 + \frac{1}{2}\delta^2$. Hence, $|u_2(t) - v_2(t)| = c \leq 0$ for all $t \in \mathbb{R}$; consequently, $u_2 = v_2$, which implies by (5.2.20) that $u = v$.

Step 3. We claim that the unique minimal weak solution $\widehat{u}$ is compact almost automorphic. Let $(t'_n)_n \subset \mathbb{R}$. We have to prove that there exist a subsequence $(t_n)_n$ of $(t'_n)_n$ and a continuous function v such that

$$\widehat{u}(t + t_n) \longrightarrow v(t) \quad \text{as } n \longrightarrow +\infty, \tag{5.2.24}$$

$$v(t - t_n) \longrightarrow \widehat{u}(t) \quad \text{as } n \longrightarrow +\infty \tag{5.2.25}$$

uniformly on any compact subset of $\mathbb{R}$. From the compact almost automorphy of f, there exists a subsequence $(t_n)_n \subset (t'_n)_n$ such that

$$f(t + t_n) \longrightarrow g(t) \quad \text{as } n \longrightarrow +\infty,$$

$$g(t - t_n) \longrightarrow f(t) \quad \text{as } n \longrightarrow +\infty$$

uniformly on any compact subset of $\mathbb{R}$. Since $\widehat{u} \in \Gamma$, it is uniformly continuous on $\mathbb{R}$ and $\widehat{u}(t) \in K$ for all $t \in \mathbb{R}$. Applying Lemma 5.2.2 to $x = \widehat{u}$, $F = f$ and the sequence $(t_n)_n$, we obtain (5.2.24), where v is a weak solution on $\mathbb{R}$ of the following differential inclusion:

$$v'(t) + \mathcal{A}v(t) \ni g(t).$$

Furthermore, v is uniformly continuous on $\mathbb{R}$ and $v(t) \in K$ for all $t \in \mathbb{R}$ since $v \in \Lambda$. Using (5.2.24), we obtain that

$$J(v) \leq J(\widehat{u}). \tag{5.2.26}$$

Applying Lemma 5.2.2 to $x = v$, $F = g$ and the returning sequence $(-t_n)_n$, we find for a subsequence that

$$v(t - t_n) \longrightarrow \omega(t) \quad \text{as } n \longrightarrow +\infty \tag{5.2.27}$$

uniformly on any compact subset of $\mathbb{R}$ where $\omega \in \Gamma$. From (5.2.27) we obtain that

$$J(\omega) \leq J(v). \tag{5.2.28}$$

By (5.2.26) and (5.2.28), we get that

$$J(\omega) \leq J(\widehat{u}) = \inf_{u \in \Gamma} J(u).$$

Consequently,

$$J(\omega) = J(\widehat{u}) = \inf_{u \in \Gamma} J(u).$$

By uniqueness of the minimal weak solution of (5.1.1) (From Steps 1 and 2), we deduce that $\omega = \widehat{u}$, (5.2.25) holds and $\widehat{u}$ is compact almost automorphic. ■

5.3 Compact Almost Automorphic Weak Solutions of (5.1.1) and (5.1.2) Where $\mathcal{A}$ is Strongly Maximal Monotone

In the sequel, we prove the existence and uniqueness of compact almost automorphic weak solutions for the differential inclusions (5.1.1) and (5.1.2) where the operator $\mathcal{A}$ is strongly maximal monotone.

Theorem 5.3.1 *Assume that $\mathcal{A}$ is α-strongly maximal monotone ($\alpha > 0$) with $0 \in \mathcal{A}0$ and $f \in AA_c(\mathbb{R}, \mathcal{H})$. Then, (5.1.1) has a unique compact almost automorphic weak solution u_f that is globally attractive.*

Proof. The proof is divided in four steps:

Step 1. We claim that the differential inclusion (5.1.1) has a bounded weak solution u_f on $\mathbb{R}$. Let $n \in \mathbb{N}$ and consider the following problem:

$$\begin{cases} u'(t) + \mathcal{A}u(t) \ni f(t), \\ u(-n) = 0. \end{cases} \tag{5.3.1}$$

Then, (5.3.1) has a unique weak solution u_n on $[-n, +\infty)$. Since $\mathcal{A}0 \ni 0$, it follows by Theorem 1.4.11, with $\widetilde{u} = u_n$, $\widetilde{f} = f$, $\widehat{u} = 0$ and $\widehat{f} = 0$, that

$$|u_n(t)| \leq \int_{-n}^{t} e^{-\alpha(t-\sigma)}|f(\sigma)|d\sigma \quad \text{for } t \in [-n, +\infty).$$

The compact almost automorphy of the function f implies its boundedness (See Remark 1.1.26). Let $M_f = \sup_{t\in\mathbb{R}} |f(t)|$. Then, the last inequality gives that

$$|u_n(t)| \leq \frac{M_f}{\alpha}(1 - e^{-\alpha(t+n)}) \quad \text{for } t \in [-n, +\infty);$$

consequently,

$$|u_n(t)| \leq \frac{M_f}{\alpha} \quad \text{for } t \in [-n, +\infty). \tag{5.3.2}$$

Let $I = [a, b]$ and n and m be such that $-n \leq -m \leq a$. Using Theorem 1.4.11 for $\widetilde{u} = u_n$, $\widetilde{f} = f$, $\widehat{u} = u_m$ and $\widehat{f} = f$, we get that

$$|u_n(t) - u_m(t)| \leq e^{-\alpha(t+m)}|u_n(-m) - u_m(-m)| = e^{-\alpha(t+m)}|u_n(-m)| \quad \text{for } t \in I. \tag{5.3.3}$$

Inequalities (5.3.2) and (5.3.3) imply that

$$|u_n(t) - u_m(t)| \leq \frac{M_f}{\alpha}e^{-\alpha(a+m)} \quad \text{for } t \in I,$$

which implies that $(u_n)_n$ is a Cauchy sequence in $C(I, \mathcal{H})$ and hence it converges to u_f in $C(I, \mathcal{H})$. By Lemma 1.4.8, the function u_f is a weak solution of (5.1.1) on I. Since I is arbitrary, by (5.3.2), u_f is a bounded weak solution on $\mathbb{R}$ of (5.1.1).

Step 2. We claim that u_f is unique. Suppose that v is another bounded weak solution on $\mathbb{R}$ of (5.1.1). By Theorem 1.4.11, we have for $t, \sigma \in \mathbb{R}$, $t \geq \sigma$

$$|u_f(t) - v(t)| \leq e^{-\alpha(t-\sigma)}|u_f(\sigma) - v(\sigma)|. \tag{5.3.4}$$

Since u_f and v are bounded, then by letting $\sigma \longrightarrow -\infty$ in (5.3.4), we obtain that $u_f = v$. The global attractivity also follows from (5.3.4) by taking $\sigma = 0$ and letting $t \longrightarrow +\infty$.

Step 3. We claim that the range of u_f is relatively compact. Let $(t'_n)_n \subset \mathbb{R}$. From the almost automorphy of f, there exists a subsequence $(t_n)_n \subset (t'_n)_n$ such that for each $t \in \mathbb{R}$

$$|f(t + t_p) - f(t + t_q)| \longrightarrow 0$$

as $p, q \longrightarrow +\infty$. Let

$$\begin{cases} u_n(t) := u_f(t + t_n) & \text{for } t \in \mathbb{R}, \\ f_n(t) := f(t + t_n) & \text{for } t \in \mathbb{R}. \end{cases}$$

We claim that $(u_n(t))_n$ is a Cauchy sequence in $\mathcal{H}$ for each $t \in \mathbb{R}$. In fact, u_p and u_q are weak solutions of the following differential inclusions:

$$\begin{cases} u'_p(t) + \mathcal{A}u_p(t) \ni f_p(t) & \text{for } t \in \mathbb{R}, \\ u'_q(t) + \mathcal{A}u_q(t) \ni f_q(t) & \text{for } t \in \mathbb{R}. \end{cases} \tag{5.3.5}$$

Applying Theorem 1.4.11 to (5.3.5), we obtain that for $t \geq \sigma$

$$|u_p(t) - u_q(t)| \leq e^{-\alpha(t-\sigma)}|u_p(\sigma) - u_q(\sigma)| + \int_\sigma^t e^{-\alpha(t-s)}|f_p(s) - f_q(s)|ds.$$

Using the boundedness of u_f and letting $\sigma \longrightarrow -\infty$, we deduce that for each $t \in \mathbb{R}$

$$\begin{aligned} |u_p(t) - u_q(t)| &\leq \int_{-\infty}^t e^{-\alpha(t-s)}|f_p(s) - f_q(s)|ds \\ &= \int_{-\infty}^t e^{-\alpha(t-s)}|f(s + t_p) - f(s + t_q)|ds. \end{aligned}$$

Using Lebesgue's dominated convergence Theorem, we conclude that $(u_n(t))_n$ is a Cauchy sequence in $\mathcal{H}$ for each $t \in \mathbb{R}$. Therefore, u_f has a relatively compact range.

Step 4. We claim that the solution u_f is uniformly continuous on $\mathbb{R}$. In fact, by Theorem 1.4.11 with $\widetilde{u} = u_f(\cdot + h)$, $\widehat{u} = u_f(\cdot)$, $\widetilde{f} = f(\cdot + h)$ and $\widehat{f} = f(\cdot)$, we obtain that for $t \geq \sigma$

$$|u_f(t + h) - u_f(t)| \leq e^{-\alpha(t-\sigma)}|u_f(\sigma + h) - u_f(\sigma)| + \int_\sigma^t e^{-\alpha(t-s)}|f(s + h) - f(s)|ds.$$

Since u_f is bounded, we obtain by letting $\sigma \longrightarrow -\infty$ that for each $t \in \mathbb{R}$

$$\begin{aligned} |u_f(t + h) - u_f(t)| &\leq \int_{-\infty}^t e^{-\alpha(t-s)}|f(s + h) - f(s)|ds \\ &\leq \frac{1}{\alpha} \sup_{t \in \mathbb{R}} |f(t + h) - f(t)|, \end{aligned}$$

which implies that

$$\sup_{t\in\mathbb{R}} |u_f(t+h) - u_f(t)| \le \frac{1}{\alpha} \sup_{t\in\mathbb{R}} |f(t+h) - f(t)|.$$

By Theorem 1.1.27, we get that u_f is uniformly continuous on $\mathbb{R}$.

Step 5. We claim that u_f is compact almost automorphic. Let $(t_n)_n \subset \mathbb{R}$. Since f is compact almost automorphic, then there exist a subsequence $(t'_n)_n \subset (t_n)_n$ and a continuous function $g : \mathbb{R} \longrightarrow \mathcal{H}$ such that

$$|f(t+t'_n) - g(t)| \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty,$$

$$|g(t-t'_n) - f(t)| \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty$$

uniformly on any compact subset of $\mathbb{R}$. By Lemma 5.2.2, one can extract another subsequence $(t''_n)_n \subset (t'_n)_n \subset (t_n)_n$ such that

$$u_f(t+t''_n) \longrightarrow y(t) \quad \text{as } n \longrightarrow +\infty \tag{5.3.6}$$

uniformly on any compact subset of $\mathbb{R}$, where y is a weak solution on $\mathbb{R}$ of the following differential inclusion :

$$y'(t) + \mathcal{A}y(t) \ni g(t).$$

Since y is also uniformly continuous and its range is relatively compact, then by applying the same procedure to the function y using the returning sequence $(-t''_n)_n$, we have another subsequence $(t'''_n)_n \subset (t''_n)_n \subset (t'_n)_n \subset (t_n)_n$ such that

$$y(t-t'''_n) \longrightarrow z(t) \quad \text{as } n \longrightarrow +\infty \tag{5.3.7}$$

uniformly on any compact subset of $\mathbb{R}$, where z is a bounded weak solution of (5.1.1) on $\mathbb{R}$. From the uniqueness of the bounded weak solution (Step 2), we conclude that $z = u_f$. Thus, it follows from (5.3.6) and (5.3.7) that u_f is compact almost automorphic. ∎

Theorem 5.3.2 *Assume that $\mathcal{A}$ is α-strongly maximal monotone ($\alpha > 0$) with $0 \in \mathcal{A}0$ and $g : \mathbb{R} \times \mathcal{H} \longrightarrow \mathcal{H}$ is compact almost automorphic in t and Lipschitzian with respect to the second argument. Then, (5.1.2) has a unique compact almost automorphic weak solution provided that $\mathrm{Lip}(g) < \alpha$ where $\mathrm{Lip}(g)$ is the Lipschitz constant of g.*

Proof. Let $v : \mathbb{R} \longrightarrow \mathcal{H}$ be a compact almost automorphic function. Consider the following differential inclusion:

$$u'(t) + \mathcal{A}u(t) \ni g(t, v(t)) \quad \text{for } t \in \mathbb{R}. \tag{5.3.8}$$

By Theorem 1.1.34, the function $t \longmapsto g(t, v(t))$ is compact almost automorphic. It follows from Theorem 5.3.1 that the differential inclusion (5.3.8) has a unique compact almost automorphic weak solution u_v. Let T be defined by

$$T : AA_c(\mathbb{R}, \mathcal{H}) \longrightarrow AA_c(\mathbb{R}, \mathcal{H})$$
$$v \longmapsto u_v.$$

Then, T is well defined. Let $v, w \in AA_c(\mathbb{R}, \mathcal{H})$. Applying Theorem 1.4.11 to $\widetilde{u} = u_v$, $\widetilde{f} = g(\cdot, v(\cdot))$, $\widehat{u} = u_w$ and $\widehat{f} = g(\cdot, w(\cdot))$, we obtain that

$$|Tv(t) - Tw(t)| \leq e^{-\alpha(t-\sigma)}|u_v(\sigma) - u_w(\sigma)| + \int_\sigma^t e^{-\alpha(t-s)}|g(s, v(s)) - g(s, w(s))|ds \quad \text{for } t \geq \sigma.$$

Letting $\sigma \longrightarrow -\infty$ we obtain that for each $t \in \mathbb{R}$

$$\begin{aligned} |Tv(t) - Tw(t)| &\leq \int_{-\infty}^t e^{-\alpha(t-s)}|g(s, v(s)) - g(s, w(s))|ds \\ &\leq \frac{Lip(g)}{\alpha}|v - w|_\infty. \end{aligned}$$

This means that T is a strict contraction. We deduce that the operator T has a unique fixed point that is the unique compact almost automorphic weak solution of (5.1.2). ■

5.4 Hyperbolic and Parabolic Equations

5.4.1 A dissipative hyperbolic system

We give an existence theorem of compact almost automorphic weak solutions for the following dissipative nonlinear wave inclusion:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) - \Delta u(t,x) + \beta\left(\dfrac{\partial}{\partial t}u(t,x)\right) \ni \theta(t,x) \quad \text{for } (t,x) \in \mathbb{R} \times \Omega \\ u(t,x) = 0 \quad \text{for } (t,x) \in \mathbb{R} \times \partial\Omega. \end{cases} \tag{5.4.1}$$

We assume that

(A1) Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ such that $\dim(\Omega) \geq 2$.

(A2) β is a strongly maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta 0$ such that

$$|\beta^0 w| \leq C_1|w|^k + C_2 \quad \text{with} \quad 0 \leq k \leq \frac{N+2}{N-2}$$

where $\beta^0 w = \mathrm{Proj}_{\beta(w)}(0)$ (See Theorem 1.4.4 and Remark 1.4.5).

(A3) $\theta : \mathbb{R} \times \overline{\Omega} \longrightarrow \mathbb{R}$ satisfies $\dfrac{\partial \theta}{\partial t} \in S^2(\mathbb{R}, L^2(\Omega))$ where

$$S^2(\mathbb{R}, L^2(\Omega)) = \left\{ h \in L^2_{loc}(\mathbb{R}, L^2(\Omega)) : \sup_{t \in \mathbb{R}} \int_t^{t+1} |h(s)|^2_{L^2(\Omega)} ds < +\infty \right\},$$

and the function $t \longmapsto \theta(t, \cdot)$ is in $AA_c(\mathbb{R}, L^2(\Omega))$. That is for any $(t'_n)_n \subset \mathbb{R}$, there exist a subsequence $(t_n)_n$ and a continuous function $\tilde{\theta} : \mathbb{R} \times \overline{\Omega} \longrightarrow \mathbb{R}$ such that

$$\int_\Omega |\theta(t + t_n, \omega) - \tilde{\theta}(t, \omega)|^2 d\omega \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty,$$

$$\int_\Omega |\tilde{\theta}(t - t_n, \omega) - \theta(t, \omega)|^2 d\omega \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty$$

uniformly on any compact subset of $\mathbb{R}$.

Let $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$ be the Hilbert space endowed with the following norm:

$$|(\phi_1, \phi_2)|_{\mathcal{H}} = \left(\int_\Omega \left(|\nabla \phi_1(s)|^2 + |\phi_1(s)|^2 + |\phi_2(s)|^2 \right) ds \right)^{\frac{1}{2}},$$

and B be the canonical extension of β to $L^2(\Omega)$ defined in [114, p. 53] by

$$(u, v) \in G(B) \quad \text{if and only if} \quad (u(x), v(x)) \in G(\beta) \ \text{ for almost all } x \in \Omega.$$

Let

$$\begin{cases} D(\mathcal{L}) = H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \\ \mathcal{L} = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}, \end{cases}$$

$$\begin{cases} D(\mathcal{B}) = H_0^1(\Omega) \times D(B) \\ \mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}. \end{cases}$$

Lemma 5.4.1 [114, p. 93] *The operator* $D(\mathcal{A}) = D(\mathcal{L}) \cap D(\mathcal{B})$, $\mathcal{A} = \mathcal{L} + \mathcal{B}$ *is maximal monotone on* $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$.

Let $f : \mathbb{R} \longrightarrow \mathcal{H}$ be the function defined by

$$f(t)(\omega) = \begin{pmatrix} 0 \\ \theta(t, \omega) \end{pmatrix} \quad \text{for } t \in \mathbb{R} \text{ and } \omega \in \Omega.$$

Then, by assumption **(A3)**, $f \in AA_c(\mathbb{R}, \mathcal{H})$. If we take $U = \begin{pmatrix} u \\ \frac{\partial u}{\partial t} \end{pmatrix}$, then (5.4.1) takes the following abstract form:

$$U'(t) + \mathcal{A}U(t) \ni f(t) \quad \text{for } t \in \mathbb{R}. \tag{5.4.2}$$

Lemma 5.4.2 [31, Theorem 2.1] *Let* $U(t) = \begin{pmatrix} u \\ \frac{\partial u}{\partial t} \end{pmatrix}$ *be a solution which starts at* $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in D(\mathcal{A})$. *Then,*

$$\frac{\partial^2 u}{\partial t^2} \in L^\infty(\mathbb{R}^+, L^2(\Omega)) \quad \text{and} \quad \frac{\partial u}{\partial t} \in L^\infty(\mathbb{R}^+, H_0^1(\Omega)).$$

Lemma 5.4.3 [31] *Let* $U(t) = \begin{pmatrix} u \\ \frac{\partial u}{\partial t} \end{pmatrix}$ *be a solution which starts at* $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in D(\mathcal{A})$. *Then,* $U(t)$ *has a relatively compact range in the energy space* $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$.

As a consequence, we have the following result.

Theorem 5.4.4 (5.4.1) *has at least a weak solution in* $AA_c(\mathbb{R}, H_0^1(\Omega) \times L^2(\Omega))$.

Proof. Using Lemmas 5.4.2 and 5.4.3, any trajectory $(U(t))_{t\geq 0}$ which starts at $U_0 \in D(\mathcal{A})$ is uniformly continuous on $\mathbb{R}^+$ in $H_0^1(\Omega) \times L^2(\Omega)$ and its range over $\mathbb{R}^+$ is relatively compact. In view of Theorem 5.2.1, the differential inclusion (5.4.2) has at least a compact almost automorphic weak solution. ■

Remark 5.4.5 (5.4.1) has been considered in the periodic and almost periodic case in [23, 24, 25, 30, 31, 113, 114, 116].

5.4.2 A dissipative parabolic system

Consider the following system:

$$\begin{cases} \dfrac{\partial}{\partial t}w(t,x) - \Delta w(t,x) + \beta\left(w(t,x)\right) + \alpha w(t,x) \ni \gamma(w(t,x))) + h(t,x) & \text{for } (t,x) \in \mathbb{R} \times \Omega \\ w(t,x) = 0 \quad \text{for } (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases} \tag{5.4.3}$$

where $\alpha > 0$. Assume that

(B1) Ω is a smooth subset of $\mathbb{R}^N$ with a regular boundary $\partial\Omega$.

(B2) β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta 0$.

(B3) $\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitzian function such that $\gamma(0) = 0$. Let L_γ be its Lipschitz constant.

(B4) The function $t \longmapsto h(t, \cdot)$ belongs to $AA_c(\mathbb{R}, L^2(\Omega))$.

Let B be the canonical extension of β to $L^2(\Omega)$ defined in [114, p. 53] by

$$(u, v) \in G(B) \quad \text{if and only if} \quad (u(x), v(x)) \in G(\beta) \ \text{ for almost all } x \in \Omega.$$

Let A_1 be defined in $L^2(\Omega)$ by

$$\begin{cases} D(A_1) = \left\{u \in H^2(\Omega) \cap H_0^1(\Omega) : \beta\left(u\right) \in L^2(\Omega)\right\} \\ A_1 u = -\Delta u + Bu. \end{cases}$$

It is well known from [114, p. 88] that A_1 is maximal monotone. Hence, by Remark 1.4.2 the operator

$$\begin{cases} D(\mathcal{A}) = D(A_1) \\ \mathcal{A}u = A_1 u + \alpha u \end{cases}$$

is α-strongly maximal monotone. Using the fact that $0 \in \beta 0$, we get that $0 \in \mathcal{A}0$. Take $\mathcal{H} = L^2(\Omega)$.

We consider the function $f : \mathcal{H} \longrightarrow \mathcal{H}$ defined by

$$f(x)(\omega) = \gamma(x(\omega)) \quad \text{for } x \in \mathcal{H} \text{ and } \omega \in \Omega.$$

By **(B3)**, one see that f is well defined. Using **(B3)**, we obtain that f is Lipschitzian with a Lipschitz constant $L_f = L_\gamma$. Furthermore, $f \in C(\mathcal{H}, \mathcal{H})$.
Let $H : \mathbb{R} \longrightarrow \mathcal{H}$ be defined by

$$H(t)(\omega) = h(t, \omega) \quad \text{for } t \in \mathbb{R} \text{ and } \omega \in \Omega.$$

(B4) implies that $H \in AA_c(\mathbb{R}, \mathcal{H})$.
Let $g : \mathbb{R} \times \mathcal{H} \longrightarrow \mathcal{H}$ be defined by

$$g(t, x) = f(x) + H(t) \quad \text{for } t \in \mathbb{R} \text{ and } x \in \mathcal{H}.$$

We deduce that $g \in AA_c(\mathbb{R}\times\mathcal{H}, \mathcal{H})$ and g is Lipschitzian with respect to the second argument with a Lipschitz constant $L_g = L_\gamma$.
If we take $u(\cdot)(x) = w(\cdot, x)$, then, (5.4.3) takes the following abstract form:

$$u'(t) + \mathcal{A}u(t) \ni g(t, u(t)) \quad \text{for } t \in \mathbb{R},$$

in the Hilbert space $\mathcal{H}$. Now, if we suppose that $L_\gamma < \alpha$, then all the assumptions in Theorem 5.3.2 are fulfilled. Consequetly, we get the following result.

Theorem 5.4.6 *The system (5.4.3) has a unique compact almost automorphic weak solution provided that $L_\gamma < \alpha$.*

Chapter 6

μ-Pseudo Compact Almost Automorphic (Periodic) Weak Solutions for Some Partial Functional Differential Inclusions [(1)]

Contents

6.1 Introduction

Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be a multivalued operator.

The aim of this chapter is to study the existence and uniqueness of μ-pseudo almost periodic (resp. μ-pseudo compact almost automorphic) weak solutions for the following partial functional differential inclusion:

$$u'(t) + \mathcal{A}u(t) \ni f(t, u_t) \quad \text{for } t \in \mathbb{R}, \tag{6.1.1}$$

where the operator $\mathcal{A}$ is *strongly maximal monotone*, the forcing term $f : \mathbb{R} \times \mathcal{C} \longrightarrow \mathcal{H}$ is Stepanov μ-pseudo almost periodic (resp. μ-pseudo compact almost automorphic) of class r and Lipschitzian with respect to the second argument, $\mathcal{C} = C([-r, 0], \mathcal{H})$ is the Banach space of all continuous functions from $[-r, 0]$ to $\mathcal{H}$ endowed with the uniform topology and the

[(1)] This work has been done in collaboration with Khalid Hilal and Khalil Ezzinbi and has been published in Applicable Analysis [87] history function u_t is defined by:

$$u_t(\theta) = u(t+\theta) \quad \text{for } \theta \in [-r, 0].$$

For our goal, we proceed as follows. Let $F : \mathbb{R} \longrightarrow \mathcal{H}$ be a Stepanov μ-pseudo almost periodic (resp. μ-pseudo compact almost automorphic) function of class r and $G : \mathbb{R} \longrightarrow \mathcal{H}$ be its Stepanov almost periodic (resp. compact almost automorphic) component. We consider the following differential inclusions:

$$u'(t) + \mathcal{A}u(t) \ni F(t) \quad \text{for } t \in \mathbb{R}, \tag{6.1.2}$$

$$u'(t) + \mathcal{A}u(t) \ni G(t) \quad \text{for } t \in \mathbb{R}. \tag{6.1.3}$$

Firstly, we prove that (6.1.2) has a unique bounded weak solution u_F on $\mathbb{R}$ that is globally attractive. Let u_G be the unique almost periodic (resp. compact almost automorphic) weak solution of (6.1.3) which its existence is guaranteed by [178, Remark 1] (resp. by [83, Theorem 3.1]). Secondly, using some estimations of weak solutions of (6.1.2) and (6.1.3), we prove that $u_F - u_G$ is μ-ergodic of class r by substracting (6.1.3) from (6.1.2) which implies that x_F is μ-pseudo almost periodic (resp. μ-pseudo compact almost automorphic) of class r. Then, combining this result together with the contraction principle, we get the existence and uniqueness of μ-pseudo almost periodic (resp. μ-pseudo compact almost automorphic) weak solution for the partial functional differential inclusion (6.1.1) where the forcing term f is Lipschitz continuous with respect to the second argument.

6.2 Existence and Uniqueness of μ-Pseudo Almost Periodic Weak Solutions of (6.1.1)

In this section, we study the existence and uniqueness of μ-pseudo almost periodic weak solutions for the partial functional differential inclusion (6.1.1) where the forcing term is S^1-μ-pseudo almost periodic of class r.

Lemma 6.2.1 *Assume that $\mathcal{A}$ is α-strongly maximal monotone ($\alpha > 0$) with $0 \in \mathcal{A}0$ and $F : \mathbb{R} \longrightarrow \mathcal{H}$ is S^1-bounded. Then, (6.1.2) has a unique bounded weak solution u_F on $\mathbb{R}$ that is globally attractive.*

Proof. Let $n \in \mathbb{N}$ and consider the following differential inclusion:

$$\begin{cases} u'(t) + \mathcal{A}u(t) \ni F(t) \\ u(-n) = 0. \end{cases} \tag{6.2.1}$$

Let u_n be the unique weak solution of (6.2.1) on $[-n, +\infty)$. Since $0 \in \mathcal{A}0$, then by applying Theorem 1.4.11 to $\widetilde{u} = u_n$, $\widetilde{f} = F$, $\widehat{u} = 0$ and $\widehat{f} = 0$, we obtain for $t \in [-n, +\infty)$ that

$$\begin{aligned} |u_n(t)| &\leq \int_{-n}^{t} e^{-\alpha(t-\sigma)}|F(\sigma)|d\sigma \\ &\leq \int_{0}^{t+n} e^{-\alpha s}|F(t-s)|ds \\ &\leq \int_{0}^{+\infty} e^{-\alpha s}|F(t-s)|ds \\ &\leq \sum_{k=0}^{+\infty}\int_{k}^{k+1} e^{-\alpha s}|F(t-s)|ds \\ &\leq \sum_{k=0}^{+\infty} e^{-\alpha k}\int_{k}^{k+1} |F(t-s)|ds \\ &= \sum_{k=0}^{+\infty} e^{-\alpha k}\int_{t-k-1}^{t-k} |F(s)|ds. \end{aligned}$$

We conclude that

$$|u_n(t)| \leq \frac{\|F\|_{S^1}}{1-e^{-\alpha}} \quad \text{for } t \in [-n, +\infty).$$

The rest of the proof is similar to Step 1 and Step 2 of the proof of Theorem 5.2.1. ∎

Theorem 6.2.2 *Let $\mu \in \mathcal{M}$ satisfy* (**B**). *Assume that $\mathcal{A}$ is α-strongly maximal monotone* $(\alpha > 0)$ *with $0 \in \mathcal{A}0$ and $F : \mathbb{R} \longrightarrow \mathcal{H}$ is S^1-μ-pseudo almost periodic of class r. Then,* (6.1.2) *has a unique bounded weak solution which is μ-pseudo almost periodic of class r. Moreover, this solution is globally attractive.*

Proof. Let G and H be respectively the unique S^1-almost periodic component and S^1-μ-ergodic perturbation of class r of the function F. Consider the following differential inclusions:

$$u'(t) + \mathcal{A}u(t) \ni F(t) \quad \text{for } t \in \mathbb{R} \tag{6.2.2}$$

$$u'(t) + \mathcal{A}u(t) \ni G(t) \quad \text{for } t \in \mathbb{R}. \tag{6.2.3}$$

Since F is S^1-μ-pseudo almost periodic, then F is S^1-bounded (see Remark 1.1.59). Using Lemma 6.2.1, (6.2.2) has a unique bounded weak solution u_F on $\mathbb{R}$ that is globally attractive. Let u_G be the unique Bohr almost periodic weak solution of (6.2.3) which its existence is guaranteed by [178, Remark 1]. Applying Theorem 1.4.11 to $\widetilde{u} = u_F$, $\widetilde{f} = F$, $\widehat{u} = u_G$ and $\widehat{f} = G$ we get that

$$|u_F(t) - u_G(t)| \leq e^{-\alpha(t-\sigma)}|u_F(\sigma) - u_G(\sigma)| + \int_{\sigma}^{t} e^{-\alpha(t-s)}|F(s) - G(s)|ds \quad \text{for } t \geq \sigma. \tag{6.2.4}$$

Since u_F and u_G are bounded, letting σ goes to $-\infty$ in (6.2.4), we obtain that

$$|u_F(t) - u_G(t)| \leq \int_{-\infty}^{t} e^{-\alpha(t-s)}|H(s)|ds \quad \text{for } t \in \mathbb{R}. \tag{6.2.5}$$

Now, let us show that $u_F - u_G$ is μ-ergodic of class r. To do this, we will show that the function defined by

$$\psi(t) = \int_{-\infty}^{t} e^{-\alpha(t-s)}|H(s)|ds \quad \text{for } t \in \mathbb{R},$$

is μ-ergodic of class r. In fact, consider for each $k \in \mathbb{N}^*$,

$$\psi_k(t) = \int_{t-k}^{t-k+1} e^{-\alpha(t-s)}|H(s)|ds \quad \text{for } t \in \mathbb{R}.$$

We have for each $t \in \mathbb{R}$

$$\begin{aligned} |\psi_k(t)| &\leq e^{-\alpha(k-1)} \int_{t-k}^{t-k+1} |H(s)|ds \\ &= e^{-\alpha(k-1)} \int_{t}^{t+1} |H(\tau - k)|d\tau \\ &\leq e^{-\alpha(k-1)} \|H\|_{S^1}. \end{aligned} \tag{6.2.6}$$

We deduce from the well-known Weierstrass Theorem that the series $\sum_{k=1}^{+\infty} \psi_k(t)$ is uniformly convergent on $\mathbb{R}$.

For $\tau > 0$ sufficiently large we have by (6.2.6) that

$$\begin{aligned} &\frac{1}{\mu([-\tau,\tau])} \int_{[-\tau,\tau]} \left(\sup_{\theta\in[t-r,t]} |\psi_k(\theta)| \right) d\mu(t) \\ \leq\ &\frac{e^{-\alpha(k-1)}}{\mu([-\tau,\tau])} \int_{[-\tau,\tau]} \sup_{\theta\in[t-r,t]} \left(\int_{\theta}^{\theta+1} |H(\tau - k)|d\tau \right) d\mu(t). \end{aligned} \tag{6.2.7}$$

Since μ satisfies (**B**), then by Theorem 1.1.61, $\mathcal{E}^1(\mathbb{R}, X, \mu, r)$ is translation invariant. It follows that the function $[\tau \longmapsto H(\tau - k)]$ belongs to $\mathcal{E}^1(\mathbb{R}, X, \mu, r)$ for each $k \in \mathbb{N}^*$. This fact together with (6.2.7) give that $\psi_k \in \mathcal{E}(\mathbb{R}, X, \mu, r)$ for each $k \in \mathbb{N}^*$. On the other hand

$$\begin{aligned} \left| \sum_{k=1}^{N} \psi_k(t) - \psi(t) \right| &= \left| \sum_{k=1}^{N} \int_{t-k}^{t-k+1} e^{-\alpha(t-s)}|H(s)|ds - \int_{-\infty}^{t} e^{-\alpha(t-s)}|H(s)|ds \right| \\ &= \left| \sum_{k=N+1}^{\infty} \int_{t-k}^{t-k+1} e^{-\alpha(t-s)}|H(s)|ds \right| \\ &\leq \|H\|_{S^1} \sum_{k=N+1}^{+\infty} e^{-\alpha(k-1)} \longrightarrow 0 \quad \text{as } N \longrightarrow +\infty. \end{aligned}$$

Because $\sum_{k=1}^{+\infty} \psi_k(t)$ converges uniformly on $\mathbb{R}$, we deduce that

$$\psi(t) = \sum_{k=1}^{+\infty} \psi_k(t) \in \mathcal{E}(\mathbb{R}, X, \mu, r).$$

Hence, it follows by (6.2.5) that $u_F - u_G$ is μ-ergodic of class r. Consequently, if we consider the following decomposition of u_F

$$u_F = u_G + (u_F - u_G),$$

we conclude that u_F is μ-pseudo almost periodic of class r. ■

The main result of this section is the following.

Theorem 6.2.3 *Let $\mu \in \mathcal{M}$ satisfy* (**B**). *Assume that $\mathcal{A}$ is α-strongly maximal monotone* ($\alpha > 0$) *with $0 \in \mathcal{A}0$ and $f : \mathbb{R} \times \mathcal{H} \longrightarrow \mathcal{H}$ is S^1-μ-pseudo almost periodic of class r in t and Lipschitzian with respect to the second argument with $\mathrm{Lip}(f) < \alpha$ where $\mathrm{Lip}(f)$ is the Lipschitz constant of f. Then,* (6.1.1) *has a unique μ-pseudo almost periodic weak solution of class r.*

Proof. Let $v : \mathbb{R} \longrightarrow \mathcal{H}$ be a μ-pseudo almost periodic function of class r and consider the following partial functional differential inclusion:

$$u'(t) + \mathcal{A}u(t) \ni f(t, v_t) \quad \text{for } t \in \mathbb{R}. \tag{6.2.8}$$

Since μ satisfies (**B**), then by Theorem 1.1.54, the function $[t \longmapsto v_t]$ belongs to $PAP(\mathbb{R}, \mathcal{C}, \mu, r)$. By Theorem 1.1.67, the function $t \longmapsto f(t, v_t)$ is S^1-μ-pseudo almost periodic of class r. It follows from Theorem 6.2.2 that (6.2.8) has a unique μ-pseudo almost periodic weak solution u_v of class r. Hence the operator

$$T : PAP(\mathbb{R}, \mathcal{H}, \mu, r) \longrightarrow PAP(\mathbb{R}, \mathcal{H}, \mu, r)$$
$$v \longmapsto u_v$$

which assigns to each $v \in PAP(\mathbb{R}, \mathcal{H}, \mu, r)$ the unique μ-pseudo almost periodic weak solution u_v of class r of (6.2.8), is well defined. Let $v, w \in PAP(\mathbb{R}, \mathcal{H}, \mu, r)$. Applying Theorem 1.4.11 to $\tilde{u} = u_v$, $\tilde{f}(\cdot) = f(\cdot, v_.)$, $\hat{u} = u_w$ and $\hat{f}(\cdot) = f(\cdot, w_.)$, we get that

$$|Tv(t) - Tw(t)| \le e^{-\alpha(t-\sigma)}|u_v(\sigma) - u_w(\sigma)| + \int_\sigma^t e^{-\alpha(t-s)}|f(s, v_s) - f(s, w_s)|ds \quad \text{for } t \ge \sigma. \tag{6.2.9}$$

Since $u_v - u_w$ is bounded on $\mathbb{R}$, pass to the limit as $\sigma \longrightarrow -\infty$ in (6.2.9), we obtain for each $t \in \mathbb{R}$ that

$$\begin{aligned} |Tv(t) - Tw(t)| &\le \int_{-\infty}^t e^{-\alpha(t-s)}|f(s, v_s) - f(s, w_s)|ds \\ &\le \frac{\mathrm{Lip}(f)}{\alpha}\|v - w\|_\infty. \end{aligned}$$

Consequently,

$$\|Tv - Tw\|_\infty \le \frac{\mathrm{Lip}(f)}{\alpha}\|v - w\|_\infty.$$

Hence, T is a strict contraction. Since μ satisfies (**B**), Theorem 1.1.53 yields that $PAP(\mathbb{R}, \mathcal{H}, \mu, r)$ endowed with supremum norm is a Banach space. In view of the Banach fixed points Theorem, the operator T has a unique fixed point that is the unique μ-pseudo almost periodic weak solution of class r of (6.1.1). ■

6.3 Existence and Uniqueness of μ-Pseudo Compact Almost Automorphic Weak Solutions of (6.1.1)

In this section, we prove the existence and uniqueness of μ-pseudo compact almost automorphic weak solutions of class r for the partial functional differential inclusion (6.1.1).

We start by the following.

Theorem 6.3.1 *Let $\mu \in \mathcal{M}$ satisfy* (**B**). *Assume that $\mathcal{A}$ is α-strongly maximal monotone* ($\alpha > 0$) *with $0 \in \mathcal{A}0$ and $F \in PAA_c(\mathbb{R}, \mathcal{H}, \mu, r)$. Then,* (6.1.2) *has a unique bounded weak solution u_F that is μ-pseudo compact almost automorphic of class r. Moreover, u_F is globally attractive.*

Proof. Let G and H be respectively the compact almost automorphic component and the μ-ergodic perturbation of class r of the function F. Consider the following partial differential inclusions:

$$u'(t) + \mathcal{A}u(t) \ni F(t) \quad \text{for } t \in \mathbb{R}, \tag{6.3.1}$$

$$u'(t) + \mathcal{A}u(t) \ni G(t) \quad \text{for } t \in \mathbb{R}. \tag{6.3.2}$$

Since $F \in PAA_c(\mathbb{R}, \mathcal{H}, \mu, r)$ then $F \in BC(\mathbb{R}, \mathcal{H}) \subset BS^1(\mathbb{R}, \mathcal{H})$. Lemma 6.2.1 yields that (6.3.1) has a unique bounded weak solution u_F on $\mathbb{R}$ that is globally attractive. By Theorem 5.3.1, (6.3.2) has a unique compact almost automorphic weak solution u_G.
Now, let us prove that $u_F - u_G$ is μ-ergodic of class r. Applying Theorem 1.4.11 to $\widetilde{u} = u_F$, $\widetilde{f} = F$, $\widehat{u} = u_G$ and $\widehat{f} = G$, we obtain that

$$|u_F(t) - u_G(t)| \leq e^{-\alpha(t-\sigma)}|u_F(\sigma) - u_G(\sigma)| + \int_\sigma^t e^{-\alpha(t-s)}|F(s) - G(s)|ds \quad \text{for } t \geq \sigma. \tag{6.3.3}$$

Letting $\sigma \longrightarrow -\infty$ in (6.3.3), we get that

$$|u_F(t) - u_G(t)| \leq \int_{-\infty}^t e^{-\alpha(t-s)}|F(s) - G(s)|ds \quad \text{for } t \in \mathbb{R}. \tag{6.3.4}$$

Let

$$B(t) = \int_{-\infty}^t e^{-\alpha(t-s)}|F(s) - G(s)|ds \quad \text{for } t \in \mathbb{R}.$$

We claim that

$$\lim_{r \to +\infty} \frac{1}{\mu([-\tau, \tau])} \int_{[-\tau,\tau]} \left(\sup_{\theta \in [t-r,t]} |B(\theta)| \right) du(t) = 0. \tag{6.3.5}$$

In fact, let $\tau > 0$ be sufficiently large. Then,

$$\frac{1}{\mu([-\tau, \tau])} \int_{[-\tau,\tau]} \left(\sup_{\theta \in [t-r,t]} |B(\theta)| \right) d\mu(t)$$

$$
\begin{aligned}
&= \frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}\left(\sup_{\theta\in[t-r,t]}\left|\int_{-\infty}^{\theta} e^{-\alpha(\theta-s)}|F(s)-G(s)|ds\right|\right)d\mu(t)\\
&\leq \frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}\left(\sup_{\theta\in[t-r,t]}\int_{-\infty}^{\theta} e^{-\alpha(\theta-s)}|H(s)|ds\right)d\mu(t)\\
&= \frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}\left(\sup_{\theta\in[t-r,t]}\int_{0}^{+\infty} e^{-\alpha s}|H(\theta-s)|ds\right)d\mu(t)\\
&\leq \frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}\int_{0}^{+\infty}\left(e^{-\alpha s}\sup_{\theta\in[t-r,t]}|H(\theta-s)|ds\right)d\mu(t)\\
&\leq \int_{0}^{+\infty}\left(\frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}|H(\theta-s)|d\mu(t)\right)e^{-\alpha s}ds.
\end{aligned}
$$

μ satisfies (**B**), by Theorem 1.1.70, $\mathcal{E}(\mathbb{R},\mathcal{H},\mu,r)$ is translation invariant. Then, by the fact that $H\in\mathcal{E}(\mathbb{R},\mathcal{H},\mu,r)$, we have that

$$\lim_{\tau\longrightarrow+\infty}\frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}|H(\theta-s)|d\mu(t)=0 \qquad \text{for all } s\in[0,+\infty).$$

On the other hand,

$$\left|\left(\frac{1}{\mu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}|H(\theta-s)|d\mu(t)\right)e^{-\alpha s}\right|\leq\|H\|_{\infty}e^{-\alpha s} \quad \text{for all } s\in[0,+\infty).$$

Using Lebesgue's dominated convergence Theorem, we get (6.3.5). That is, B is μ-ergodic of class r. Taking into account of (6.3.4), we conclude that u_F-u_G is μ-ergodic of class r. Hence, u_F is μ-pseudo compact almost automorphic of class r. ∎

To discuss the existence and uniqueness of μ-pseudo compact almost automorphic weak solutions for the partial functional differential inclusion (6.1.1), we make the following assumption:

(**E**) Let $\mu\in\mathcal{M}$, $f=g+\varphi\in PAA_c(\mathbb{R}\times\mathcal{C},\mathcal{H},\mu,r)$ with $g\in AA_c(\mathbb{R}\times\mathcal{C},\mathcal{H})$ and $\varphi\in\mathcal{E}(\mathbb{R}\times\mathcal{C},\mathcal{H},\mu,r)$. Assume that f satisfies (**C**) and

i) There exists a positive real number L_g such that

$$|g(t,\phi_1)-g(t,\phi_2)|_{\mathcal{H}}\leq L_g|\phi_1-\phi_2|_{\mathcal{C}} \quad \text{for } t\in\mathbb{R} \text{ and } \phi_1,\phi_2\in\mathcal{C}.$$

ii) There exists a function $L_f\in L^1(\mathbb{R},\mathbb{R}^+)$ such that

$$|f(t,\phi_1)-f(t,\phi_2)|_{\mathcal{H}}\leq L_f(t)|\phi_1-\phi_2|_{\mathcal{C}} \quad \text{for } t\in\mathbb{R} \text{ and } \phi_1,\phi_2\in\mathcal{C}.$$

where L_f satisfies (1.1.9) and (1.1.10).

The first main result of this section is the following theorem.

Theorem 6.3.2 *Let $\mu\in\mathcal{M}$ satisfy* (**B**). *Assume that $\mathcal{A}$ is α-strongly maximal monotone ($\alpha>0$) with $0\in\mathcal{A}0$ and $f\in PAA_c(\mathbb{R}\times\mathcal{C},\mathcal{H},\mu,r)$ satisfies* (**E**). *Then,* (6.1.1) *has a unique μ-pseudo compact almost automorphic weak solution of class r.*

Proof. Let v be a function in $PAA_c(\mathbb{R},\mathcal{H},\mu,r)$. Consider the following partial functional differential inclusion:

$$u'(t)+\mathcal{A}u(t)\ni f(t,v_t) \quad \text{for } t\in\mathbb{R}. \tag{6.3.6}$$

By Theorem 1.1.72, the function $[t\longmapsto v_t]$ belongs to $PAA_c(\mathbb{R},\mathcal{C},\mu,r)$. Since f satisfies (**E**), Theorem 1.1.74 implies that $[t\longmapsto f(t,v_t)]\in PAA_c(\mathbb{R},\mathcal{H},\mu,r)$. In view of Theorem 6.3.1, (6.3.6) has a unique μ-pseudo compact almost automorphic weak solution u_v of class r. Hence, the following operator:

$$\begin{aligned} T: PAA_c(\mathbb{R},\mathcal{H},\mu,r) &\longrightarrow PAA_c(\mathbb{R},\mathcal{H},\mu,r)\\ v &\longmapsto u_v \end{aligned}$$

is well defined. Let us prove that T^{n_0} is a strict contraction on $PAA_c(\mathbb{R},\mathcal{H},\mu,r)$ for some n_0. In fact, let $v,w\in PAA_c(\mathbb{R},\mathcal{H},\mu,r)$. Applying Theorem 1.4.11 to $\widetilde{u}=u_v$, $\widetilde{f}(\cdot)=f(\cdot,v_\cdot)$, $\widehat{u}=u_w$ and $\widehat{f}(\cdot)=f(\cdot,w_\cdot)$, we get that

$$|Tv(t)-Tw(t)|\le e^{-\alpha(t-\sigma)}|u_v(\sigma)-u_w(\sigma)|+\int_\sigma^t e^{-\alpha(t-s)}|f(s,v_s)-f(s,w_s)|ds \quad \text{for } t\ge\sigma. \tag{6.3.7}$$

Let $\sigma\longrightarrow-\infty$ in (6.3.7). Then, we obtain that for each $t\in\mathbb{R}$

$$\begin{aligned} |Tv(t)-Tw(t)| &\le \int_{-\infty}^t e^{-\alpha(t-s)}|f(s,v_s)-f(s,w_s)|ds\\ &\le \int_{-\infty}^t e^{-\alpha(t-s)}|v_s-w_s|_{\mathcal{C}}L_f(s)ds\\ &\le \|v-w\|_\infty\int_{-\infty}^t L_f(s)ds. \end{aligned}$$

Consequently,

$$\begin{aligned} |T^2v(t)-T^2w(t)| &\le \int_{-\infty}^t e^{-\alpha(t-s)}|f(s,(Tv)_s)-f(s,(Tw)_s)|ds\\ &\le \int_{-\infty}^t e^{-\alpha(t-s)}L_f(s)|(Tv)_s-(Tw)_s|_{\mathcal{C}}ds\\ &\le \int_{-\infty}^t e^{-\alpha(t-s)}L_f(s)\sup_{\theta\in[-r,0]}|(Tv)_s(\theta)-(Tw)_s(\theta)|ds\\ &\le \int_{-\infty}^t e^{-\alpha(t-s)}L_f(s)\sup_{\theta\in[-r,0]}|(Tv)(s+\theta)-(Tw)(s+\theta)|ds\\ &\le \int_{-\infty}^t e^{-\alpha(t-s)}L_f(s)\|v-w\|_\infty\sup_{\theta\in[-r,0]}\int_{-\infty}^{s+\theta}L_f(\lambda)d\lambda ds\\ &\le \|v-w\|_\infty\int_{-\infty}^t L_f(s)\int_{-\infty}^s L_f(\lambda)d\lambda ds\\ &\le \frac{(\int_{-\infty}^t L_f(s)ds)^2}{2}\|v-w\|_\infty. \end{aligned}$$

Induction on n in the same way, gives

$$\|T^n v - T^n w\|_\infty \leq \frac{(\|L_f\|_{L^1(\mathbb{R})})^n}{n!}\|v - w\|_\infty.$$

Let n_0 be such that $\frac{(\|L_f\|_{L^1(\mathbb{R})})^{n_0}}{n_0!} < 1$. By Theorem 1.2.2, T has a unique fixed point that is the unique μ-pseudo compact almost automorphic weak solution of class r of (6.1.1). ∎

Now, we consider the following assumption :

(F) Let $\mu \in \mathcal{M}$ and $f = g + \varphi \in PAA_c(\mathbb{R} \times \mathcal{C}, \mathcal{H}, \mu, r)$ with $g \in AA_c(\mathbb{R} \times \mathcal{C}, \mathcal{H})$ and $\varphi \in \mathcal{E}(\mathbb{R} \times \mathcal{C}, \mathcal{H}, \mu, r)$ such that

i) There exists a positive constant L_g such that

$$|g(t, \phi_1) - g(t, \phi_2)|_{\mathcal{H}} \leq L_g|\phi_1 - \phi_2|_{\mathcal{C}} \quad \text{for } t \in \mathbb{R} \text{ and } \phi_1, \phi_2 \in \mathcal{C}.$$

ii) There exists a positive constant L_f such that

$$|f(t, \phi_1) - f(t, \phi_2)|_{\mathcal{H}} \leq L_f|\phi_1 - \phi_2|_{\mathcal{C}} \quad \text{for } t \in \mathbb{R} \text{ and } \phi_1, \phi_2 \in \mathcal{C}. \tag{6.3.8}$$

The second main result of this section is the following theorem.

Theorem 6.3.3 *Let $\mu \in \mathcal{M}$ satisfy* **(B)**. *Assume that $\mathcal{A}$ is α-strongly maximal monotone ($\alpha > 0$) with $0 \in \mathcal{A}0$ and $f \in PAA_c(\mathbb{R} \times \mathcal{C}, \mathcal{H}, \mu, r)$ satisfies* **(F)** *with $L_f < \alpha$. Then, (6.1.1) has a unique μ-pseudo compact almost automorphic weak solution of class r.*

Proof. Let v be in $PAA_c(\mathbb{R}, \mathcal{H}, \mu, r)$. Consider the following partial functional differential inclusion:

$$u'(t) + \mathcal{A}u(t) \ni f(t, v_t) \quad \text{for } t \in \mathbb{R}. \tag{6.3.9}$$

By Theorem 1.1.72, the function $[t \longmapsto v_t]$ belongs to $PAA_c(\mathbb{R}, \mathcal{C}, \mu, r)$. Since L_f is constant function, it satisfies (1.1.9) and (1.1.10). Using (6.3.8) we obtain that f satisfies Hypothesis **(C)** (see the proof of Theorem 6.1 in [44, p. 2441]). In view of Theorem 1.1.74, the function $[t \longmapsto f(t, v_t)]$ belongs to $PAA_c(\mathbb{R}, \mathcal{H}, \mu, r)$. It follows from Theorem 6.3.1 that (6.3.9) has a unique μ-pseudo compact almost automorphic weak solution u_v of class r. Hence, the following operator:

$$T : PAA_c(\mathbb{R}, \mathcal{H}, \mu, r) \longrightarrow PAA_c(\mathbb{R}, \mathcal{H}, \mu, r)$$
$$v \longmapsto u_v$$

is well defined. Proceeding as in the proof of Theorem 6.2.3, we get that

$$\|Tv - Tw\|_\infty \leq \frac{L_f}{\alpha}\|v - w\|_\infty$$

for each $v, w \in PAA_c(\mathbb{R}, \mathcal{H}, \mu, r)$. Hence, T is a strict contraction. Since μ satifies **(B)**, by Theorem 1.1.70, $PAA_c(\mathbb{R}, \mathcal{H}, \mu, r)$ endowed with the uniform topology norm is a Banach space. Hence, T has a unique fixed point that is the unique μ-pseudo compact almost automorphic weak solution of class r of (6.1.1). ∎

6.4 Applications

In this section, we give two examples to illustrate the main results of this chapter.

6.4.1 A dissipative parabolic system

Let Ω be a smooth subset of $\mathbb{R}^N$ with a regular boundary $\partial\Omega$, β a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ with $0\in\beta 0$ and μ a mesure satisfying Hypothesis (**B**) (see Example 1.1.40). We are concerned with the existence and uniqueness of μ-pseudo compact almost automorphic weak solution of class r for the following parabolic system:

$$\begin{cases} \dfrac{\partial}{\partial t}w(t,x)-\Delta w(t,x)+\beta\left(w(t,x)\right)+\alpha w(t,x)\ni h(t)\displaystyle\int_{-r}^{0}k(s)\zeta(w(t+s,x))ds \quad \text{on } \mathbb{R}\times\Omega \\ w(t,x)=0 \quad \text{on } \mathbb{R}\times\partial\Omega, \end{cases} \tag{6.4.1}$$

where $\alpha, r>0$, h is a μ-pseudo compact almost automorphic function of class r (see Example 1.1.69), $\zeta:\mathbb{R}\longrightarrow\mathbb{R}$ is a Lipschitzian function with Lipschitz constant L_ζ, $k:[-r,0]\longrightarrow\mathbb{R}$ is a piecewise continuous function such that $k\in L^2((-r,0),\mathbb{R})$.
We introduce the Hilbert space $\mathcal{H}=L^2(\Omega)$ and the Banach space $\mathcal{C}=C([-r,0],\mathcal{H})$. Let B be the canonical extension of β on $L^2(\Omega)$ defined in [114, p. 53] by:

$$(u,v)\in G(B) \quad \text{if and only if} \quad (u(x),v(x))\in G(\beta) \ \text{ for almost all } x\in\Omega.$$

Let A_1 be the operator defined in $L^2(\Omega)$ by:

$$\begin{cases} D(A_1)=\left\{u\in H^2(\Omega)\cap H_0^1(\Omega):\beta\left(u\right)\in L^2(\Omega)\right\} \\ A_1u=-\Delta u+Bu. \end{cases}$$

From [114, p. 88], A_1 is maximal monotone. Hence, the following operator

$$\begin{cases} D(\mathcal{A})=D(A_1) \\ \mathcal{A}u=A_1u+\alpha u \end{cases}$$

is α-strongly maximal monotone. In addition, using the fact that $0\in\beta 0$, we get that $0\in\mathcal{A}0$.
Define the function $f:\mathbb{R}\times\mathcal{C}\longrightarrow\mathcal{H}$ by:

$$f(t,\varphi)(x)=h(t)\int_{-r}^{0}k(s)\zeta(\varphi(s)(x))ds \quad \text{for } t\in\mathbb{R},\ x\in\Omega \text{ and } \varphi\in\mathcal{C}.$$

If we take $u(\cdot)(x)=w(\cdot,x)$, then (6.4.1) takes the following abstract form:

$$u'(t)+\mathcal{A}u(t)\ni f(t,u_t) \quad \text{for } t\in\mathbb{R}.$$

It is clear that f is μ-pseudo compact almost automorphic of class r. Moreover, using the fact that ζ is Lipschitzian, we get that

$$|f(t,\varphi)-f(t,\psi)|\leq\sqrt{r}L_\zeta\|h\|_\infty\|k\|_{L^2[-r,0]}|\varphi-\psi|,$$

$$|f_a(t,\varphi) - f_a(t,\psi)| \leq \sqrt{r}L_\zeta \|h_a\|_\infty \|k\|_{L^2[-r,0]} |\varphi - \psi|$$

for $t \in \mathbb{R}$ and $\varphi, \psi \in \mathcal{C}$ where h_a and f_a are the compact almost automorphic components of h and f respectively. We conclude that f satisfies (**F**). Hence, all the hypothesis in Theorem 6.3.3 are fulfilled. Consequently, we get the following result.

Theorem 6.4.1 *The system* (6.4.1) *has a unique* μ*-pseudo compact almost automorphic weak solution of class* r *provided that*

$$\sqrt{r}L_\zeta \|h\|_\infty \|k\|_{L^2[-r,0]} < \alpha.$$

6.4.2 Differential inclusion governed by a subdifferential operator

Consider the following differential inclusion:

$$u'(t) + \partial\varphi(u(t)) + \alpha u(t) \ni g(t)\xi(u(t-r)) \quad \text{for } t \in \mathbb{R}, \tag{6.4.2}$$

where $\alpha, r > 0$, φ, g and ξ are functions satisfying the following assumptions.

(A1) $\varphi : \mathbb{R}^n \longrightarrow (-\infty, +\infty]$ is proper, convex and lower semicontinuous such that $0 \in \partial\varphi(0)$.

(A2) $g : \mathbb{R} \longrightarrow \mathbb{R}$ is μ-pseudo compact almost automorphic of class r where μ is a measure satisfying (**B**).

(A3) $\xi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is Lipschitzian with a Lipschitz constant L_ξ.

Let $\mathcal{A}$ be the operator defined by:

$$\mathcal{A}x = \partial\varphi(x) + \alpha x.$$

It is clear that $D(\mathcal{A}) = D(\partial\varphi) \neq \emptyset$. In addition, on account of the fact that $0 \in \partial\varphi(0)$ we obtain that $0 \in \mathcal{A}0$.

Lemma 6.4.2 [160] *The subdifferential of a function* $\psi : \mathcal{H} \longrightarrow (-\infty, +\infty]$*, which is proper, convex, and lower semicontinuous, is maximal monotone.*

By Lemma 6.4.2, we deduce using assumption **(A1)** that the operator $\mathcal{A}$ is α-strongly maximal monotone.

Now, let us introduce the Hilbert space $\mathcal{H} = \mathbb{R}^n$. Consider the function $f : \mathbb{R} \times \mathcal{C} \longrightarrow \mathcal{H}$ defined by:

$$f(t,\phi) = g(t)\xi(\phi(-r)) \quad \text{for } t \in \mathbb{R} \text{ and } \phi \in \mathcal{C},$$

where $\mathcal{C} = C([-r,0],\mathcal{H})$. Then, (6.4.2) takes the following abstract form:

$$u'(t) + \mathcal{A}u(t) \ni f(t,u_t) \quad \text{for } t \in \mathbb{R}.$$

Assumption **(A2)** implies that f is μ-pseudo compact almost automorphic of class r and

$$f_a(t,\phi) = g_a(t)\xi(\phi(-r)) \quad \text{for } t \in \mathbb{R} \text{ and } \phi \in \mathcal{C}.$$

In addition, assumption **(A3)** leads to

$$|f(t,\phi_1) - f(t,\phi_2)|_{\mathcal{H}} \leq L_\xi \|g\|_\infty |\phi_1 - \phi_2|_{\mathcal{C}} \quad \text{for } t \in \mathbb{R} \text{ and } \phi_1, \phi_2 \in \mathcal{C}$$

and

$$|f_a(t,\phi_1) - f_a(t,\phi_2)|_{\mathcal{H}} \leq L_\xi \|g_a\|_\infty |\phi_1 - \phi_2|_{\mathcal{C}} \quad \text{for } t \in \mathbb{R} \text{ and } \phi_1, \phi_2 \in \mathcal{C},$$

where f_a and g_a are the compact almost automorphic components of f and g respectively. Hence, the function f satisfies assumption (**F**). Thanks to Theorem 6.3.3, we get the following result.

Theorem 6.4.3 *Under assumptions* **(A1)-(A3)**, *the differential inclusion* (6.4.2) *has a unique* μ-*pseudo compact almost automorphic weak solution of class* r *whenver* $L_\xi \|g\|_\infty < \alpha$.

Chapter 7

Nonlocal Integro-Differential Equations Without the Assumption of Equicontinuity on the Resolvent Operator in Banach Spaces [(1)]

Contents

7.1 Introduction

In this chapter, we study the existence of the mild solutions of the following integro-differential equation with finite delay and nonlocal initial conditions

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t, x_t) \quad \text{for } \ t \in [0, b] \\ x_0 = \phi + g(x) \in C([-r, 0], X), \end{cases} \tag{7.1.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X with domain $D(A)$, $B(t)$ is a closed linear operator on X with domain $D(B) \supset D(A)$ which is independent of t, $C([-r, 0], X)$ is the Banach space of all continuous functions from $[-r, 0]$ to X endowed with the uniform topology, the history function $x_t : [-r, 0] \longrightarrow X$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$, $\phi \in C([-r, 0], X)$, $f : [0, b] \times C([-r, 0], X) \longrightarrow X$ and $g : C([0, b], X) \longrightarrow C([-r, 0], X)$ are two continuous functions.

[(1)] This work has been done in collaboration with Khalil Ezzinbi and has been published in Differential Equations and Dynamical Systems [90]

In physics, nonlocal initial conditions are usually more precise for physical measurements and has better effect than the classical initial condition. The problem with nonlocal initial conditions have their origins in the works of Byszewski in his classical papers [52, 53, 55] in which he studied the existence of mild solutions for the nonlocal Cauchy problem:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)) & \text{for } t \in [0, b] \\ x(0) = g(x), \end{cases} \tag{7.1.2}$$

where A is the infinitesimal generator of a semigroup $(T(t))_{t\geq 0}$ of linear operators defined on a Banach space X, and the maps f and g are suitable X-valued functions.

In the recent years, many authors have attracted much attention to the study of the existence and uniqueness of mild solutions to Equation (7.1.2). We refer to [52, 53, 54, 55, 134, 152, 153, 183, 184]. In [52, 53, 55], Equation (7.1.2) was studied when f and g satisfy Lipschitz-type conditions. In [54, 134, 152, 153], the authors studied Equation (7.1.2) under conditions of compactness for $(T(t))_{t\geq 0}$. Xue [183, 184] studies Equation (7.1.2) when $(T(t))_{t\geq 0}$ is equicontinuous.

On the other hand, the resolvent operator replaces the role of the C_0-semigroup theory but does not satisfy semigroup properties. When $(R(t))_{t\geq 0}$ is a resolvent operator to Equation (7.1.1), the mild solutions are given by the following variation of constants formula

$$x(t) = R(t)[\phi(0) + g(x)(0)] + \int_0^t R(t-s)f(s, x_s)ds \quad \text{for } t \in [0, b].$$

For more details about resolvent operators we refer to [57, 104, 105, 106, 121, 143].

In [136], Lizama and Pozo discussed the existence of mild solutions to the nonlocal problem (7.1.1) without delay ($r = 0$) when the nonlocal condition g is compact and the resolvent operator $(R(t))_{t\geq 0}$ is equicontinuous. The authors used the resolvent operator theory, Sadovskii's fixed point Theorem and measure of noncompactness.

To the best of our knowledge, there is no results in the literature concerning the existence of mild solutions for Equation (7.1.1) without assumption of equicontinuity on the resolvent operator $(R(t))_{t\geq 0}$.

In this chapter, we study the existence of mild solutions for Equation (7.1.1) without assumption of equicontinuity on the resolvent operator $(R(t))_{t\geq 0}$ and without any assumption on the Banach space X. The nonlocal condition g is assumed to be compact.

The proof of the main results is based on the application of Mönch's fixed point Theorem via the measure of noncompactness with values in the cone $\mathbb{R}^2_+$, developed in [124, Example 2.1.4, p. 38]. The techniques we are going to employ here have not been used for Equation (7.1.1) before. The results obtained in this chapter generalize some results developed in [77] (when $B = 0$ and the semigroup $(T(t))_{t\geq 0}$ is not equicontinuous) and [136] (when $r = 0$ and $(R(t))_{t\geq 0}$ is equicontinuous) but the approach used here is different.

7.2 Preliminary Results

In this section, we give some lemmas which will be used in the proof of the main result. Throughout the remainder of this chapter, M_b denotes the constant $M_b = \sup_{t\in[0,b]} \|R(t)\|$.

Lemma 7.2.1 [42, p. 125] *Let* X *be a Banach space. If* $W \subset X$ *is a bounded subset, then for each* $\varepsilon > 0$, *there exists a sequence* $\{u_n\}_{n=1}^{+\infty} \subset W$ *such that*

$$\Phi(W) \leq 2\Phi(\{u_n\}_{n=1}^{+\infty}) + \varepsilon,$$

where Φ *is any measure of noncompactness.*

Definition 7.2.2 A set of function $\{f_n\}_{n=1}^{+\infty} \subset L^1([0,b],X)$ is *integrably bounded* (or *uniformly integrable*) if there exists a positive function $\nu \in L^1([0,b],\mathbb{R}^+)$ such that for all $n \geq 1$

$$\|f_n(t)\| \leq \nu(t) \quad \text{a.e. } t \in [0,b].$$

Lemma 7.2.3 [77] *If* $\{u_n\}_{n=1}^{+\infty} \subset L^1(a,b,X)$ *is uniformly integrable, then* $\chi(\{u_n(t)\}_{n\geq 1}^{+\infty})$ *is measurable and*

$$\chi\left(\left\{\int_a^t u_n(s)ds\right\}_{n=1}^{+\infty}\right) \leq 2\int_a^t \chi(\{u_n(s)\}_{n=1}^{+\infty})ds,$$

where χ *is the Hausdorff measure of noncompactness.*

Lemma 7.2.4 [139, Lemma 1.3, page 25] *Suppose that* X *is a Banach space and* f *is an integrable function from* J *to* X. *Then,*

$$\frac{1}{b-a}\int_a^b f(s)ds \in \overline{co}\Big(\big\{f(t) : t \in [a,b]\big\}\Big)$$

for all $a, b \in J$ *with* $a < b$.

For an abstract operator $S : L^1([0,b],X) \longrightarrow C([0,b],X)$, we consider the following properties taken from [124].

(S1) There exists $N > 0$ such that

$$\|Sf(t) - Sg(t)\| \leq N\int_0^t \|f(s) - g(s)\|ds$$

for every $f, g \in L^1([0,b],X)$ and $t \in [0,b]$.

(S2) For any compact $K \subset X$ and sequence $\{f_n\}_{n=1}^{+\infty} \subset L^1([0,b],X)$ such that $\{f_n(t)\}_{n=1}^{+\infty} \subset K$ for a.e. $t \in [0,b]$, the weak convergence $f_n \rightharpoonup f_0$ implies the strong convergence $Sf_n \longrightarrow Sf_0$.

Consider the operator $\Phi : L^1([0,b],X) \longrightarrow C([0,b],X)$ defined by

$$(\Phi f)(t) = \int_0^t R(t-s)f(s)ds \quad \text{for } t \in [0,b].$$

The next lemma plays a key role in this chapter and it is a generalization of [124, Lemma 4.2.1, p. 111]. Note that the proof of [124, Lemma 4.2.1, p. 111] is based on the use of the semigroup law property which is not true, in general, for a resolvent operator as mentionned in Remark 1.5.15.

Lemma 7.2.5 *The operator Φ satisfies the conditions* **(S1)** *and* **(S2)**.

Proof. Let $f, g \in L^1([0,b],X)$ and $t \in [0,b]$. We have

$$\begin{aligned} \|\Phi f(t) - \Phi g(t)\| &= \left\| \int_0^t R(t-s)(f(s)-g(s))ds \right\| \\ &\leq M_b \int_0^t \|f(s)-g(s)\| ds, \end{aligned} \tag{7.2.1}$$

which implies that **(S1)** holds. Now let $K \subset X$ be a compact set and $\{f_n\}_{n=1}^{+\infty} \subset L^1([0,b],X)$ a sequence such that $\{f_n(t)\}_{n=1}^{+\infty} \subset K$ for a.e. $t \in [0,b]$. Then, there exists $F > 0$ such that $\|f_n(t)\| \leq F$ for $n \geq 1$ and a.e. $t \in [0,b]$. By (7.2.1), Φ is a bounded linear operator from the space $L^1([0,b],X)$ into $C([0,b],X)$. Consequently, we have

$$f_n \rightharpoonup f_0 \Longrightarrow \Phi f_n \rightharpoonup \Phi f_0. \tag{7.2.2}$$

To end the proof, we claim that the sequence $\{\Phi f_n\}_{n=1}^{+\infty}$ is relatively compact in $C([0,b],X)$. By Arzelà-Ascoli's Theorem, it suffices to prove that the sequence $\{\Phi f_n\}_{n=1}^{+\infty} \subset C([0,b],X)$ is equicontinuous on $[0,b]$ and $\{\Phi f_n(t)\}_{n=1}^{+\infty}$ is relatively compact in X for each $t \in [0,b]$. For the first assertion, we start by the equicontinuity at 0. For $n \geq 1$ and $t \in [0,b]$ we have

$$\|\Phi f_n(t) - \Phi f_n(0)\| \leq M_b F t,$$

which gives that $\|\Phi f_n(t) - \Phi f_n(0)\| \longrightarrow 0$ as $t \longrightarrow 0$ uniformly with respect to $n \geq 1$. Now, let $0 < t_1 < t_2 \leq b$ and $n \geq 1$. We have

$$\begin{aligned} \|\Phi f_n(t_2) - \Phi f_n(t_1)\| &\leq \int_{t_1}^{t_2} \|R(t_2-s)f_n(s)\| ds \\ &\quad + \int_0^{t_1} \|R(t_2-s)f_n(s) - R(t_1-s)f_n(s)\| ds \\ &\leq (t_2-t_1)M_b F \\ &\quad + \int_0^{t_1} \sup_{x\in K} \left\| R(t_2-t_1+u)x - R(u)x \right\| du \end{aligned}$$

The strong continuity of the resolvent operator $(R(t))_{t\geq 0}$ ensures that

$$\sup_{x\in K} \left\| R(t_2-t_1+u)x - R(u)x \right\| \longrightarrow 0 \text{ as } t_2 \longrightarrow t_1.$$

Since

$$\sup_{x\in K}\left\|R(t_2-t_1+u)x-R(u)x\right\|\leq 2M_bF,$$

Lebesgue's dominated convegence Theorem leads to

$$\int_0^{t_1}\sup_{x\in K}\left\|R(t_2-t_1+u)x-R(u)x\right\|du\longrightarrow 0 \text{ as } t_2\longrightarrow t_1.$$

We deduce that

$$\lim_{t_1\to t_2}\|\Phi f_n(t_2)-\Phi f_n(t_1)\|=0$$

uniformly with respect to $n\geq 1$, which means that $\{\Phi f_n\}_{n=1}^{+\infty}$ is equicontinuous on $[0,b]$. Now, we prove the second assertion. Let $t\in[0,b]$, $(\sigma_n)\in[0,t]$ and $(x_n)\in K$. Since $[0,t]$ and K are compact, there exist two subsequences (σ_{n_k}) and (x_{n_k}) of (σ_n) and (x_n) respectively and $\sigma_0\in[0,t]$ and $x_0\in K$ such that $\sigma_{n_k}\longrightarrow\sigma_0$ and $x_{n_k}\longrightarrow x_0$ as $k\longrightarrow+\infty$. Now we have

$$\begin{aligned}\|R(\sigma_{n_k})x_{n_k}-R(\sigma_0)x_0\| &\leq \|R(\sigma_{n_k})x_{n_k}-R(\sigma_{n_k})x_0\|+\|R(\sigma_{n_k})x_0-R(\sigma_0)x_0\|\\ &\leq M_b\|x_{n_k}-x_0\|+\|R(\sigma_{n_k})x_0-R(\sigma_0)x_0\|.\end{aligned}$$

Using the strong continuity of the resolvent operator $(R(t))_{t\geq0}$, we get that $\|R(\sigma_{n_k})x_0-R(\sigma_0)x_0\|\longrightarrow 0$ as $k\longrightarrow+\infty$. Consequently

$$\lim_{k\to\infty}\|R(\sigma_{n_k})x_{n_k}-R(\sigma_0)x_0\|=0.$$

Due to the Bolzano-Weierstraβ property, we conclude that the set $\{R(\sigma)x:\sigma\in[0,t],\ x\in K\}$ is compact in X. Hence, the set

$$\overline{co}\Big(\{R(\sigma)x:\sigma\in[0,t],\ x\in K\}\Big)$$

is also compact in X. According to Lemma 7.2.4, we have

$$\int_0^t R(t-s)f_n(s)ds\in t\overline{co}\Big(\{R(\sigma)x:\sigma\in[0,t],\ x\in K\}\Big).$$

Afterwards

$$\{\Phi f_n(t)\}_{n=1}^{+\infty}\subset t\overline{co}\Big(\{R(\sigma)x:\sigma\in[0,t],\ x\in K\}\Big),$$

which implies that the set $\{\Phi f_n(t)\}_{n=1}^{+\infty}$ is relatively compact in X for each $t\in[0,b]$. Thus, in view of Arzelà-Ascoli's Theorem, $\{\Phi f_n\}_{n=1}^{+\infty}$ is relatively compact in $C([0,b],X)$ and so the convergence in (7.2.2) is strong. i.e., $\Phi f_n\longrightarrow\Phi f_0$. ■

Lemma 7.2.6 [124, Theorem 4.2.2, p. 112]. *Let* $\{f_n\}_{n=1}^{+\infty}\subset L^1([0,b],X)$ *be integrably bounded. Assume that*

$$\chi(\{f_n(t)\}_{n=1}^{+\infty})\leq q(t)$$

for a.e. $t\in[0,b]$ *where* $q\in L^1([0,b],\mathbb{R}^+)$. *If* S *satisfies conditions* **(S1)** *and* **(S2)** *then*

$$\chi(\{(Sf_n)(t)\}_{n=1}^{+\infty})\leq 2N\int_0^t q(s)ds \text{ for } t\in[0,b],$$

where $N>0$ *is the constant in condition* **(S1)**.

Definition 7.2.7 [124] The sequence $\{f_n\}_{n=1}^{+\infty} \subset L^1([0,b],X)$ is *semicompact* if it is integrably bounded and the set $\{f_n(t)\}_{n=1}^{+\infty}$ is relatively compact for a.e. $t \in [0,b]$.

Lemma 7.2.8 [124, Theorem 5.1.1, p. 122]. *Let* $S : L^1([0,b],X) \longrightarrow C([0,b],X)$ *be an operator satisfying the conditions* **(S1)** *and* **(S2)**. *Then for every semicompact sequence* $\{f_n\}_{n=1}^{+\infty} \subset L^1([0,b],X)$, *the sequence* $\{Sf_n\}_{n=1}^{+\infty}$ *is relatively compact in* $C([0,b],X)$.

7.3 Main Results

In this section, we give some existence results of the nonlocal integro-differential equation (7.1.1). Before stating and proving theses results, we give the following definition of mild solutions of (7.1.1).

Definition 7.3.1 A continuous function $x : [-r,b] \longrightarrow X$ is said to be a mild solution of the nonlocal Equation (7.1.1) if $x(0) = \phi(0) + g(x)(0)$ and

$$x(t) = R(t)[\phi(0) + g(x)(0)] + \int_0^t R(t-s)f(s,x_s)ds \quad \text{for} \quad t \in [0,b].$$

Equation (7.1.1) will be studied under the following hypotheses.

(H1) A is the infinitesimal generator of a strongly continuous semigroup on X.

(H2) For all $t \geq 0$, $B(t)$ is closed linear operator from $D(A)$ to X and $B(t) \in \mathcal{L}(Y,X)$. For any $y \in Y$, the map $t \longmapsto B(t)y$ is bounded, differentiable and the derivative $t \longmapsto B'(t)y$ is bounded uniformly continuous on $\mathbb{R}^+$.

(H3) i) $f : [0,b] \times C([-r,0],X) \longrightarrow X$ satisfies the Carathéodory-type condition, i.e., $f(\cdot,\varphi) : [0,b] \longrightarrow X$ is measurable for all $\varphi \in C([-r,0],X)$ and $f(t,\cdot) : C([-r,0],X) \longrightarrow X$ is continuous for a.e. $t \in [0,b]$.

ii) There exist a function $m \in L^1([0,b],\mathbb{R}^+)$ and a nondecreasing continuous function $\Omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$\|f(t,\varphi)\| \leq m(t)\Omega(\|\varphi\|)$$

for a.e. $t \in [0,b]$ and all $\varphi \in C([-r,0],X)$.

iii) There exists a function $\eta \in L^1([0,b],\mathbb{R}^+)$ such

$$\chi(f(t,D)) \leq \eta(t) \sup_{-r\leq\theta\leq 0} \chi(D(\theta))$$

for a.e. $t \in [0,b]$ and any bounded subset $D \subset C([-r,0],X)$.

(H4) $g : C([0, b], X) \longrightarrow C([-r, 0], X)$ is continuous and compact. Moreover

$$\|g(\varphi)\| \leq c\|\varphi\| + d, \tag{7.3.1}$$

for all $\varphi \in C([0, b], X)$.

The class of all functions satisfying **(H3)** iii) is nonempty. The next example gives a function verifying a such hypothesis. A second function will be given in the next section.

Example 7.3.2 Let X be a Banach space and f be defined by

$$f(t, \varphi) = e^{-t} \int_{-r}^{0} F(\theta, \varphi(\theta))d\theta$$

for $t \in [0, b]$ and $\varphi \in C([-r, 0], X)$, where $F : [-r, 0] \times X \longrightarrow X$ satisfies the following condition.

(C) i) There exist $\alpha \in L^1([-r, 0], \mathbb{R}^+)$ and $\beta \in L^1([-r, 0], \mathbb{R}^+)$ such that

$$\|F(\theta, x)\| \leq \alpha(\theta)\|x\| + \beta(\theta)$$

for $\theta \in [-r, 0]$ and $x \in X$.

ii) There exists $\xi > 0$ such that for any bounded subset $D \subset X$

$$\chi(F(\theta, D)) \leq \xi\chi(D) \quad \text{for a.e. } \theta \in [-r, 0].$$

Then, f satisfies the Hypothesis **(H3)** iii). In fact, let D be a bounded subset of $C([-r, 0], X)$. From **(C)** ii) we have

$$\chi(F(\theta, D(\theta))) \leq \xi\chi(D(\theta)) \quad \text{for a.e. } \theta \in [0, b],$$

where $F(\theta, D(\theta)) = \{F(\theta, \varphi(\theta)) \in \mathbb{X},\ \varphi \in D\}$. Using **(C)** i), we obtain $\|f(t, \varphi)\| \leq \|\alpha\|_{L^1}\|\varphi\| + \|\beta\|_{L^1}$ for all $t \in [0, b]$ and $\varphi \in C([-r, 0], X)$. Therefore, for each $t \in [0, b]$, $f(t, D)$ is bounded. Let $\varepsilon > 0$ be fixed. By means of Lemma 7.2.1, there exists a sequence $\{\varphi_n\}_{n=1}^{+\infty} \subset D$ such that

$$\begin{aligned} \chi(f(t, D)) &\leq 2\chi(f(t, \{\varphi_n\}_{n=1}^{+\infty})) + \varepsilon \\ &= 2\chi\left(e^{-t}\int_{-r}^{0} F(\theta, \{\varphi_n(\theta)\}_{n=1}^{+\infty})d\theta\right) + \varepsilon. \end{aligned}$$

On the other hand, using again **(C)** i), it follows that

$$\|F(\theta, \varphi_n(\theta))\| \leq \rho\alpha(\theta) + \beta(\theta) \quad \text{for } n \geq 1 \text{ and } \theta \in [-r, 0],$$

where $\rho = \sup_{\psi \in D} \|\psi\|$. Thus, $F(\theta, \{\varphi_n(\theta)\}_{n=1}^{+\infty})$ is uniformly integrable. From Lemma 7.2.3, we get

$$\begin{aligned} \chi(f(t,D)) &\leq 4e^{-t}\int_{-r}^{0} \chi(F(\theta, \{\varphi_n(\theta)\}_{n=1}^{+\infty}))d\theta + \varepsilon \\ &\leq 4e^{-t}\int_{-r}^{0} \chi(F(\theta, D(\theta)))d\theta + \varepsilon \\ &\leq 4\xi e^{-t}\int_{-r}^{0} \chi(D(\theta))d\theta + \varepsilon \\ &\leq 4\xi r e^{-t} \sup_{-r\leq\theta\leq 0} \chi(D(\theta))) + \varepsilon \end{aligned}$$

for $t \in [0, b]$. Since $\varepsilon > 0$ is arbitrary, it follows that

$$\chi(f(t,D)) \leq 4\xi r e^{-t} \sup_{-r\leq\theta\leq 0} \chi(D(\theta))).$$

Hence, the function f satisfies the Hypothesis **(H3)** iii) with $\eta(t) = 4\xi r e^{-t}$ for $t \in [0, b]$.

Now, we give a measure of noncompactness in the space $C([a, b], X)$ with values in the cone $\mathbb{R}^2_+$. It permits us to dropping the equicontinuity on the resolvent operator.

Example 7.3.3 [124, Example 2.1.4, p. 38]. We consider the measure of noncompactness

$$\nu(\Omega) = \max_{D\in\Delta(\Omega)} \Big(\gamma(D), \mathrm{mod}_C(D)\Big) \tag{7.3.2}$$

in $C([a, b], X)$ where $\Delta(\Omega)$ is the collection of all denumerable subsets of Ω, mod_C is given in Example 1.3.2 and γ is the measure of noncompactness defined by

$$\gamma(D) = \sup_{t\in[a,b]} e^{-Lt}\chi(D(t)) \tag{7.3.3}$$

where L is a constant and χ is the Hausdorff measure of noncompactness. The range for the measure of noncompactness ν is in the cone $\mathbb{R}^2_+$, max is taken in the sense of the ordering induced by this cone. ν is monotone, nonsingular and regular. Furthermore, there exists $\widetilde{D} \in \Delta(\Omega)$ such that the maximum on the right-hand side of (7.3.2) is achieved on $\widetilde{D}$.

The following theorem gives an existence result of the resolvent operator for Equation (1.5.1).

Theorem 7.3.4 [103] *Assume that* **(H1)** *and* **(H2)** *hold. Then, there exists a unique resolvent operator of Equation* (1.5.1).

The first result of this chapter is the following.

Theorem 7.3.5 *Assume that* **(H1)-(H4)** *hold. Then, for each* $\phi \in C([-r, 0], X)$, *the nonlocal problem* (7.1.1) *has at least one mild solution on* $[-r, b]$ *provided that there exists a constant* $R_0 > 0$ *satisfying*

$$M_b(\|\phi\| + cR_0 + d + \Omega(R_0)\|m\|_{L^1}) \leq R_0. \tag{7.3.4}$$

Proof. For each $x \in C([-r,b],X)$ the restriction of x on $[0,b]$, $x|_{[0,b]} \in C([0,b],X)$. For simplicity, we write $g(x|_{[0,b]})$ as $g(x)$. Our goal is to show that the solution operator $\mathbf{K} : C([-r,b],X) \longrightarrow C([-r,b],X)$ defined by the following

$$(\mathbf{K}x)(t) = \begin{cases} \underbrace{\phi(t) + g(x)(t)}_{=(K_1x)(t)} & \text{for } t \in [-r,0], \\ \underbrace{R(t)(\phi(0) + g(x)(0)) + \int_0^t R(t-s)f(s,x_s)ds}_{=(K_2x)(t)} & \text{for } t \in [0,b] \end{cases}$$

has a fixed point in the closed ball $\overline{B}_{R_0} = \{x \in C([-r,b],X) : \|x\| \leq R_0\}$ where R_0 is the constant appearing in the inequality (7.3.4). In order to apply Theorem 1.2.4, we devide the proof into three steps.

Step 1. We claim that the operator solution $\mathbf{K}$ maps $\overline{B}_{R_0}$ into itself. For every $x \in \overline{B}_{R_0}$, we have for $t \in [-r,0]$

$$\begin{aligned} \|(\mathbf{K}x)(t)\| &\leq \|\phi(t)\| + \|g(x)(t)\| \\ &\leq \|\phi\| + \|g(x)\| \\ &\leq \|\phi\| + c\|x\| + d \\ &\leq M_b(\|\phi\| + cR_0 + d). \end{aligned} \tag{7.3.5}$$

For $t \in [0,b]$

$$\begin{aligned} \|(\mathbf{K}x)(t)\| &\leq \|R(t)(\phi(0) + g(x)(0))\| + \left\| \int_0^t R(t-s)f(s,x_s) \right\| \\ &\leq M_b(\|\phi\| + \|g(x)\|) + M_b \int_0^b \Omega(\|x_s\|)m(s)ds \\ &\leq M_b(\|\phi\| + c\|x\| + d) + M_b\Omega(\|x\|)\|m\|_{L^1} \\ &\leq M_b(\|\phi\| + cR_0 + d + \Omega(R_0)\|m\|_{L^1}). \end{aligned} \tag{7.3.6}$$

It follows from the inequalities (7.3.4)-(7.3.6) that $\mathbf{K}\overline{B}_{R_0} \subseteq \overline{B}_{R_0}$.

Step 2. We claim that the operator $\mathbf{K}$ is continuous on $\overline{B}_{R_0}$. Let $(x_n)_{n\geq 1}$ be a sequence in $\overline{B}_{R_0}$ such that

$$\lim_{n\to+\infty} \|x_n - x\| = 0. \tag{7.3.7}$$

We have for each $s \in [0,b]$

$$\begin{aligned} \|x_{n,s} - x_s\| &= \sup_{\theta\in[-r,0]} \|x_{n,s}(\theta) - x_s(\theta)\| \\ &= \sup_{\theta\in[-r,0]} \|x_n(\theta+s) - x(\theta+s)\| \\ &\leq \sup_{\theta'\in[-r,b]} \|x_n(\theta') - x(\theta')\| \\ &= \|x_n - x\|. \end{aligned}$$

Using (7.3.7), we get that

$$\lim_{n\to+\infty} \|x_{n,s} - x_s\| = 0 \ \text{ for } s \in [0, b].$$

Hypothesis **(H3)** i) implies that

$$\lim_{n\to+\infty} \|f(s, x_{n,s}) - f(s, x_s)\| = 0 \ \text{ for } s \in [0, b].$$

Due to **(H3)** ii) we get that

$$\|f(s, x_{n,s}) - f(s, x_s)\| \leq 2\Omega(R_0)m(s) \ \text{ for } s \in [0, b].$$

In view of Lebesgue's dominated convergence Theorem, we have that

$$\lim_{n\to+\infty} \int_0^t \|f(s, x_{n,s}) - f(s, x_s)\| ds = 0 \ \text{ for } s \in [0, b]. \tag{7.3.8}$$

For $t \in [0, b]$ we have

$$\|(\mathbf{K}x_n)(t) - (\mathbf{K}x)(t)\|_X$$
$$\leq M_b\|g(x_n) - g(x)\|_{C([-r,0],X)} + M_b \int_0^b \|f(s, x_{n,s}) - f(s, x_s)\|_X ds.$$

Taking into account **(H4)** and (7.3.8) we get that

$$\lim_{n\to+\infty} \|\mathbf{K}x_n - \mathbf{K}x\|_{C([0,b],X)} = 0.$$

On the other hand

$$\lim_{n\to+\infty} \|\mathbf{K}x_n - \mathbf{K}x\|_{C([-r,0],X)} = \lim_{n\to+\infty} \|g(x_n) - g(x)\|_{C([-r,0],X)} = 0.$$

The proof of step 2 is now complete.

Step 3. We claim that the solution operator **K** satisfies Mönch's condition.

Suppose $A \subset B_{R_0}$ is countable and $A \subset \overline{co}(\{0\} \cup \mathbf{K}(A))$. We shall show that A is relatively compact by using the measure of noncompactness ν defined in Example 7.3.3 where the constant $L > 0$ is chosen such that

$$2M_b \sup_{t\in[0,b]} \int_0^t e^{-L(t-s)}\eta(s)ds < 1. \tag{7.3.9}$$

Since

$$\nu(\mathbf{K}(A)) = \max_{D\in\Delta(\mathbf{K}(A))} \Big(\gamma(D), \text{mod}_C(D)\Big),$$

there exists a denumerable set $\{y_n\}_{n=1}^{+\infty}$ such that $\{y_n\}_{n=1}^{+\infty} \subset \mathbf{K}(A)$ and $\{y_n\}_{n=1}^{+\infty}$ achieves the maximum $\nu(\mathbf{K}(A))$. i.e.

$$\nu(\mathbf{K}(A)) = \Big(\gamma(\{y_n\}_{n=1}^{+\infty}), \text{mod}_C(\{y_n\}_{n=1}^{+\infty})\Big).$$

So, there exists a set $\{x_n\}_{n=1}^{+\infty} \subset A$ such that

$$y_n(t) = (\mathbf{K}x_n)(t) \quad \text{for } n \geq 1 \text{ and } t \in [-r, b]. \tag{7.3.10}$$

According to **(H3)** iii) and (7.3.3) we have for $s \in [0, b]$

$$\begin{aligned} \chi\left(\{f(s, x_{n,s})\}_{n=1}^{+\infty}\right) &\leq \eta(s) \sup_{-r\leq\theta\leq 0} \chi\left(\{x_{n,s}(\theta)\}_{n=1}^{+\infty}\right) \\ &\leq \eta(s) \sup_{-r\leq\theta\leq 0} \chi\left(\{x_n(s+\theta)\}_{n=1}^{+\infty}\right) \\ &\leq \eta(s) \sup_{-r\leq\tau\leq s} \chi\left(\{x_n(\tau)\}_{n=1}^{+\infty}\right) \\ &\leq \eta(s) \sup_{-r\leq\tau\leq s} e^{L\tau}\gamma\left(\{x_n\}_{n=1}^{+\infty}\right) \\ &\leq \eta(s)e^{Ls}\gamma\left(\{x_n\}_{n=1}^{+\infty}\right). \end{aligned} \tag{7.3.11}$$

On the other hand, using **(H3)** ii) and the fact that $\{x_n\}_{n=1}^{+\infty} \subset A \subset B_{R_0}$, we get

$$(\text{for all } n \geq 1) \quad \|f(s, x_{n,s})\| \leq \Omega(R_0)m(s) \quad \text{for } s \in [0, b]. \tag{7.3.12}$$

Thanks to **(H3)** ii), **(H3)** iii), (7.3.11) and (7.3.12), all the assumptions of Lemma 7.2.6 hold. Consequently we find that

$$\chi\left(\{(\Phi f(\bullet, x_{n,\bullet})(t)\}_{n=1}^{+\infty}\right) \leq 2M_b\gamma\left(\{x_n\}_{n=1}^{+\infty}\right) \int_0^t e^{Ls}\eta(s)ds \tag{7.3.13}$$

for $t \in [0, b]$. Since **(H4)** holds, by the strong continuity of the resolvent operator $(R(t))_{t\geq 0}$ and Arzelà-Ascoli's Theorem, we have that the sets

$$\{(K_1x_n)(t) : n \geq 1\} = \{\phi(t) + g(x_n)(t) : n \geq 1\} \quad \text{for } t \in [-r, 0],$$

$$\{\phi(0) + g(x_n)(0) : n \geq 1\}$$

are relatively compact in X. Consequently

$$\chi\left(\{(K_1x_n)(t)\}_{n=1}^{+\infty}\right) = 0 \quad \text{for } t \in [-r, 0], \tag{7.3.14}$$

and

$$\chi\left(\left\{R(t)\left(\phi(0) + g(x_n)(0)\right)\right\}_{n=1}^{+\infty}\right) = 0 \quad \text{for } t \in [0, b]. \tag{7.3.15}$$

Taking into account (7.3.10), (7.3.13)-(7.3.15) we get that

$$\begin{aligned}
\gamma\Big(\{y_n\}_{n=1}^{+\infty}\Big) &= \sup_{t\in[-r,b]} e^{-Lt}\chi\Big(\{y_n(t)\}_{n=1}^{+\infty}\Big)\\
&= \sup_{t\in[0,b]} e^{-Lt}\chi\Big(\{(K_2x_n)(t)\}_{n=1}^{+\infty}\Big)\\
&\le \sup_{t\in[0,b]} e^{-Lt}\chi\Big(\{R(t)(\phi(0)+g(\{x_n\}_{n=1}^{+\infty})(0))+(\Phi f(\cdot,x_{n,\cdot})(t))\}_{n=1}^{+\infty}\Big)\\
&\le \sup_{t\in[0,b]} e^{-Lt}2M_b\gamma\Big(\{x_n\}_{n=1}^{+\infty}\Big)\int_0^t e^{Ls}\eta(s)ds\\
&\le \gamma\Big(\{x_n\}_{n=1}^{+\infty}\Big)2M_b\sup_{t\in[0,b]}\int_0^t e^{-L(t-s)}\eta(s)ds.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\gamma\Big(\{x_n\}_{n=1}^{+\infty}\Big)\le\gamma(A) &\le \gamma\Big(\overline{co}(\{0\}\cup\mathbf{K}(A))\Big)\le\gamma\Big(\{y_n\}_{n=1}^{+\infty}\Big)\\
&\le \gamma\Big(\{x_n\}_{n=1}^{+\infty}\Big)2M_b\sup_{t\in[0,b]}\int_0^t e^{-L(t-s)}\eta(s)ds.
\end{aligned}$$

The inequality (7.3.9) gives

$$\gamma\Big(\{x_n\}_{n=1}^{+\infty}\Big)=0. \tag{7.3.16}$$

Combining (7.3.11) and (7.3.16) we get the following

$$\chi\Big(\{f(s,x_{n,s})\}_{n=1}^{+\infty}\Big)=0 \quad \text{for all } s\in[0,b].$$

This gives that the sequence $\{f(\bullet,x_{n,\bullet})\}_{n=1}^{+\infty}\subset L^1([0,b],X)$, is semicompact. In view of Lemma 7.2.8, the sequence $\Phi(\{f(\bullet,x_{n,\bullet})\}_{n=1}^{+\infty})$ is relatively compact in $C([0,b],X)$.

From **(H4)**, the set $\{K_1x_n\}_{n=1}^{+\infty}$ is relatively compact in $C([-r,0],X)$. Using the strong continuity of the resolvent operator $(R(t))_{t\ge0}$, we known that the set $\{R(\cdot)(\phi(0)+g(x_n)(0)) : n\ge1\}$ is relatively compact in $C([0,b],X)$ (see (7.3.15)). Finally we get that $\{K_2x_n\}_{n=1}^{+\infty}$ is relatively compact in $C([0,b],X)$ because $(K_2x_n)(t)=R(t)(\phi(0)+g(x_n)(0))+\Phi f(\bullet,x_{n,\bullet})(t)$ for $t\in[0,b]$.

Now let us prove that $\{y_n\}_{n=1}^{+\infty}=\{\mathbf{K}x_n\}_{n=1}^{+\infty}$ is relatively compact in $C([-r,b],X)$. To do this, we use Arzelà Ascoli's Theorem. That is, we need to prove that for each $t\in[-r,b]$, the set $\big\{(\mathbf{K}x_n)(t),n\ge1\big\}$ is relatively compact in X and the sequence $\{\mathbf{K}x_n\}_{n=1}^{+\infty}$ is equicontinuous on $[-r,b]$. For $t\in[-r,b]$, we have

$$\big\{(\mathbf{K}x_n)(t),n\ge1\big\}=\begin{cases}\big\{(K_1x_n)(t),n\ge1\big\} & \text{for } t\in[-r,0],\\ \big\{(K_2x_n)(t),n\ge1\big\} & \text{for } t\in[0,b].\end{cases}$$

So, $\big\{(\mathbf{K}x_n)(t),n\ge1\big\}$ is relatively compact in X because $\big\{(K_1x_n)(t),n\ge1\big\}$ is relatively compact in X for each $t\in[-r,0]$ and $\big\{(K_2x_n)(t),n\ge1\big\}$ is relatively compact in X for each $t\in[0,b]$. For the equicontinuity, letting $t\in[-r,b]$ and $\varepsilon>0$, we must prove that

$$(\exists\delta>0)(\forall n\ge1)(\forall t'\in[-r,b]):\ |t-t'|<\delta\Longrightarrow\|(\mathbf{K}x_n)(t)-(\mathbf{K}x_n)(t')\|<\varepsilon. \tag{7.3.17}$$

• If $t,\ t' \in [-r, 0]$ then

$$\|(\mathbf{K}x_n)(t) - (\mathbf{K}x_n)(t')\| = \|(K_1x_n)(t) - (K_1x_n)(t')\|,$$

so, (7.3.17) holds because $\{K_1x_n\}_{n=1}^{+\infty}$ is equicontinuous on $[-r, 0]$.
• If $t,\ t' \in [0, b]$ then

$$\|(\mathbf{K}x_n)(t) - (\mathbf{K}x_n)(t')\| = \|(K_2x_n)(t) - (K_2x_n)(t')\|,$$

so, (7.3.17) holds because $\{K_2x_n\}_{n=1}^{+\infty}$ is equicontinuous on $[0, b]$.
• If $t \in [-r, 0]$ and $t' \in [0, b]$ then, by the fact that $(K_1x_n)(0) = (K_2x_n)(0) = (\mathbf{K}x_n)(0)$, we have

$$\begin{aligned} &\|(\mathbf{K}x_n)(t) - (\mathbf{K}x_n)(t')\| \\ \leq\ & \|(K_1x_n)(t) - (K_1x_n)(0)\| + \|(K_2x_n)(t') - (K_2x_n)(0)\|. \end{aligned} \tag{7.3.18}$$

We can find $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$(\forall n \geq 1) : t \in]-\delta_1, 0[\Longrightarrow \|(K_1x_n)(t) - (K_1x_n)(0)\| < \frac{\varepsilon}{2}, \tag{7.3.19}$$

$$(\forall n \geq 1) : t' \in]0, \delta_2[\Longrightarrow \|(K_2x_n)(t) - (K_2x_n)(0)\| < \frac{\varepsilon}{2}. \tag{7.3.20}$$

If we choose $\delta = \min\{\delta_1, \delta_2\}$ then (7.3.17) follows by using (7.3.18)-(7.3.20). Hence, the sequence $\{\mathbf{K}x_n\}_{n=1}^{+\infty}$ is equicontinuous on $[-r, b]$. Therefore $\gamma(\{y_n\}_{n=1}^{+\infty}) = 0$ and $\mathrm{mod}_C(\{y_n\}_{n=1}^{+\infty}) = 0$.
From the monotonocity, the nonsingularity and the regularity of the measure of noncompactness ν and the condition $A \subset \overline{\mathrm{co}}(\{0\} \cup \mathbf{K}(A))$ we have

$$\nu(A) \leq \nu\Big(\overline{\mathrm{co}}(\{0\} \cup \mathbf{K}(A))\Big) \leq \nu(\mathbf{K}(A)) = \Big(\gamma(\{y_n\}_{n=1}^{+\infty}), \mathrm{mod}_C(\{y_n\}_{n=1}^{+\infty})\Big) = (0, 0).$$

This means that A is relatively compact. In view of the Theorem 1.2.4, the operator $\mathbf{K}$ has at least one fixed point $x \in \overline{B}_{R_0}$, which is a mild solution of the nonlocal problem (7.1.1). ■

Now, we give another existence theorem without condition (7.3.1).

Theorem 7.3.6 *Assume that* **(H1)-(H4)** *hold except for condition* (7.3.1). *Then, for each* $\phi \in C([-r, 0], X)$*, the nonlocal problem* (7.1.1) *has at least one mild solution on* $[-r, b]$ *provided that*

$$\limsup_{k \to +\infty} \frac{M_b}{k}\Big(\|\phi\| + g_k + \Omega(k)\|m\|_{L^1}\Big) < 1, \tag{7.3.21}$$

where $g_k = \sup\{\|g(\varphi)\| : \|\varphi\| \leq k\}$.

Proof. We should only find a closed convex subset $W \subset C([-r,b],X)$ such that $\mathbf{K}$ maps W into itself and complete the proof similary to Theorem 7.3.5. From the inequality (7.3.21), there exists a constant $k > 0$ such that

$$\|\phi\| + g_k < k$$

and

$$M_b\Big(\|\phi\| + g_k + \Omega(k)\|m\|_{L^1}\Big) < k.$$

Let $W = \{x \in C([-r,b],X) : \|x\|_{[-r,b]} \leq k\}$. For every $x \in W$, we have

$$\begin{aligned} \|(\mathbf{K}x)(t)\| &\leq \|\phi(t)\| + \|g(x)(t)\| \\ &\leq \|\phi\| + \|g(x)\| \\ &\leq \|\phi\| + g_k \\ &\leq k \end{aligned}$$

for $t \in [-r, 0]$, and

$$\begin{aligned} \|(\mathbf{K}x)(t)\| &\leq \|R(t)(\phi(0) + g(x)(0))\| + \left\| \int_0^t R(t-s)f(s,x_s) \right\| \\ &\leq M_b(\|\phi\| + \|g(x)\|) + M_b \int_0^b \Omega(\|x_s\|)m(s)ds \\ &\leq M_b(\|\phi\| + g_k + \Omega(k)\|m\|_{L^1}) \\ &\leq k \end{aligned}$$

for $t \in [0, b]$. So, $\mathbf{K}$ maps W into itself. ■

7.4 Application

In this section, we apply the results obtained in the previous section to the following partial integro-differential equation:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,y) = \dfrac{\partial}{\partial y}u(t,y) + \displaystyle\int_0^t \zeta(t-s)\frac{\partial}{\partial y}u(s,y)ds + \kappa(t)h(u(t-1,y)) \quad \text{for } t, y \in [0,1], \\ u(t,1) = 0 \quad \text{for } t \in [0,1], \\ u(\theta,y) = \cos\big(\frac{\pi}{2}y\big) + \lambda\displaystyle\int_{-1}^{\theta}(1-y)\sin(u(0,-\eta))d\eta \quad \text{for } \theta \in [-1,0] \text{ and } y \in [0,1], \end{cases} \tag{7.4.1}$$

where λ is a positive constant and ζ, κ and h are functions satisfying the following assumptions:

(A1) $\zeta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is bounded and C^1 function such that ζ' is bounded and uniformly continuous on $\mathbb{R}^+$.

(A2) $\kappa : [0,1] \longrightarrow \mathbb{R}$ is integrable.

(A3) $h : \mathbb{R} \longrightarrow \mathbb{R}$ is Lipschitzian with a Lipschitz constant L_h and $h(0) = 0$.

To rewrite Equation (7.4.1) in the abstract form, we introduce the Banach space $X = \{f \in C([0,1],\mathbb{R}) : f(1) = 0\}$ of continuous functions from $[0,1]$ to $\mathbb{R}$ vanishing at 1, equipped with the supremum norm.
We define the operator $A : D(A) \subset X \longrightarrow X$ by

$$\begin{cases} D(A) = \{f \in C^1([0,1],\mathbb{R}) : f'(1) = f(1) = 0\} \\ Af = f'. \end{cases}$$

It is well known from Example 1.5.4 that A is the generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on X given by

$$\begin{cases} (T(t)f)(s) = f(t+s) & \text{for } t+s \leq 1 \\ (T(t)f)(s) = 0 & \text{for } t+s > 1, \end{cases}$$

which implies that **(H1)** is satisfied. Furthermore, by Theorem 1.5.10, $(T(t))_{t\geq 0}$ is not equicontinuous.
Let $B : D(A) \subset X \longrightarrow X$ be the operator defined by

$$B(t)f = \zeta(t)Af \quad \text{for } t \in [0,1] \text{ and } f \in D(A).$$

Then, **(H2)** follows from **(A1)**. Define the functions $f : [0,1] \times C([-1,0],X) \longrightarrow X$, $g : C([0,1],X) \longrightarrow C([-1,0],X)$ and ϕ by

$$f(t,\varphi)(y) = \kappa(t)h(\varphi(-1)(y)) \quad \text{for } t \in [0,1] \text{ and } y \in [0,1],$$

$$g(\psi)(\theta)(y) = \lambda \int_{-1}^{\theta} (1-y)\sin(\psi(0)(-\eta))d\eta \quad \text{for } \theta \in [-1,0] \text{ and } y \in [0,1], \tag{7.4.2}$$

$$\phi(\theta,y) = \cos\left(\frac{\pi}{2}y\right) \quad \text{for } \theta \in [-1,0] \text{ and } y \in [0,1].$$

If we take $x(\cdot)(y) = u(\cdot,y)$, then Equation (7.4.1) takes the following abstract form

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t,x_t) \quad \text{for } t \in [0,1] \\ x_0 = \phi + g(x) \in C([-1,0],X). \end{cases} \tag{7.4.3}$$

In view of Theorem 7.3.4, Equation (7.4.3) has a unique resolvent operator $(R(t))_{t\geq 0}$.

Theorem 7.4.1 [84] *Let A be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ and let $(B(t))_{t\geq 0}$ satisfy* **(H2)**. *Then, the resolvent operator $(R(t))_{t\geq 0}$ for Equation (1.5.1) is equicontinuous for $t > 0$ if and only if $(T(t))_{t\geq 0}$ is equicontinuous for $t > 0$.*

According to Theorem 7.4.1, the resolvent operator $(R(t))_{t\geq 0}$ is not equicontinuous.

The function f is well defined. In fact, let $t \in [0,1]$ and $\varphi \in C([-1,0],X)$. If $y_n \longrightarrow y$ in $[0,1]$, then for each $t \in [0,1]$ we have that $\varphi(-1)(y_n) \longrightarrow \varphi(-1)(y)$. From **(A3)**, it follows that $h(\varphi(-1)(y_n)) \longrightarrow h(\varphi(-1)(y))$. Consequently,

$$f(t,\varphi)(y_n) \longrightarrow f(t,\varphi)(y).$$

Using **(A3)** ($h(0) = 0$) we get $f(t,\varphi)(1) = \kappa(t)h(\varphi(-1)(1)) = \kappa(t)h(0) = 0$. We conclude that $f(t,\varphi) \in X$ which means that f is well defined.

Thanks to **(A2)** and **(A3)**, we can also prove that the function f satisfies **(H3)** i).

Assumption **(A3)** implies, for each $t \in [0,1]$ and $\varphi \in C([-1,0],X)$, that

$$\begin{aligned}
\|f(t,\varphi)\| = \sup_{y\in[0,1]} |f(t,\varphi)(y)| &\leq |\kappa(t)| \sup_{y\in[0,1]} |h(\varphi(-1)(y))| \\
&= |\kappa(t)| \sup_{y\in[0,1]} |h(\varphi(-1)(y)) - h(0)| \\
&\leq |\kappa(t)|L_h \sup_{y\in[0,1]} |\varphi(-1)(y)| \\
&= |\kappa(t)|L_h\|\varphi(-1)\| \\
&\leq |\kappa(t)|L_h\|\varphi\|.
\end{aligned}$$

This means that **(H3)** ii) holds with $\Omega(z) = L_h z$ for $z \in \mathbb{R}^+$ and $m(t) = |\kappa(t)|$ for $t \in [0,1]$. On the other hand, due to **(A3)**, we get

$$\|f(t,\varphi) - f(t,\psi)\| \leq L_h|\kappa(t)|\|\varphi(-1) - \psi(-1)\|. \tag{7.4.4}$$

Now, let us prove the following lemma.

Lemma 7.4.2 *The function* f *satisfies Hypothesis* **(H3)** iii) *with* $\eta(\cdot) = L_h|\kappa(\cdot)|$.

Proof. Let D be a bounded subset of $C([-1,0],X)$ and $t \in [0,1]$. Then $D(-1)$ is a bounded subset of X. Let us pose $\lambda = \chi(D(-1))$ and fix some $\varepsilon > 0$. Then there exists $\{x_1, x_2, \ldots, x_n\} \subset X$ such that

$$D(-1) \subset \bigcup_{i=1}^{n} B(x_i, \lambda + \varepsilon). \tag{7.4.5}$$

We can find $\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \subset C([-1,0],X)$ such that $\varphi_i(-1) = x_i$ for each $i \in \{1,2,\ldots,n\}$. Put $z_i = f(t,\varphi_i)$ for $i \in \{1,2,\ldots,n\}$. Let $z \in f(t,D)$. Then there exists $\varphi \in D$ such that $z = f(t,\varphi)$. Since $\varphi(-1) \in D(-1)$, by (7.4.5) there exists $i_0 \in \{1,\ldots,n\}$ such that

$$\|\varphi(-1) - x_{i_0}\| < \lambda + \varepsilon. \tag{7.4.6}$$

By virtue of (7.4.4) and (7.4.6), it follows that

$$\begin{aligned}
\|z - z_{i_0}\| &= \|f(t,\varphi) - f(t,\varphi_{i_0})\| \\
&\leq L_h|\kappa(t)|\|\varphi(-1) - \varphi_{i_0}(-1)\| \\
&= L_h|\kappa(t)|\|\varphi(-1) - x_{i_0}(-1)\| \\
&< L_h|\kappa(t)|[\lambda + \varepsilon].
\end{aligned}$$

We have proved that for each $z \in f(t, D)$ there exists $i_0 \in \{1, \ldots, n\}$ such that $z \in B\big(z_{i_0}, L_h|\kappa(t)|[\lambda + \varepsilon]\big)$. Hence we conclude that

$$f(t, D) \subset \bigcup_{i=1}^{n} B\big(z_i, L_h|\kappa(t)|[\lambda + \varepsilon]\big).$$

On the basis of the definition of Hausdorff's measure of noncompactness we get

$$\chi(f(t, D)) < L_h|\kappa(t)|[\chi(D(-1)) + \varepsilon].$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\chi(f(t, D)) \leq L_h|\kappa(t)|\chi(D(-1)).$$

Afterwards

$$\chi(f(t, D)) \leq L_h|\kappa(t)| \sup_{\theta \in [-1,0]} \chi(D(\theta)).$$

This means that **(H3)** iii) is satisfied with $\eta(\cdot) = L_h|\kappa(\cdot)|$ and the proof is now complete. ■

Lemma 7.4.3 *The function* g *defined by* (7.4.2) *satisfies Hypothesis* **(H4)**.

Proof. For the sake of convenience, we divide the proof into several steps.

Step1. We show that the function g is well defined. Let $u \in C([0, 1], X)$, $\theta, \theta' \in [-1, 0]$ and $y, y' \in [0, 1]$.

We have

$$\begin{aligned} |g(u)(\theta)(y) - g(u)(\theta)(y')| &\leq \lambda\left(\int_{-1}^{\theta} |\sin(u(0)(-\eta))|d\eta\right)|y - y'| \leq \lambda(\theta + 1)|y - y'| \\ &\leq \lambda|y - y'|. \end{aligned} \tag{7.4.7}$$

Clearly, $g(u)(\theta)(1) = 0$ which together with the inequality (7.4.7) lead to $g(u)(\theta) \in X$. On the other hand

$$\begin{aligned} |g(u)(\theta)(y) - g(u)(\theta')(y)| &\leq \lambda(1 - y)\int_{\theta'}^{\theta} |\sin(u(0)(-\eta))|d\eta \\ &\leq \lambda|\theta - \theta'|. \end{aligned}$$

Hence

$$\|g(u)(\theta) - g(u)(\theta')\| \leq \lambda|\theta - \theta'|. \tag{7.4.8}$$

We conclude that $g(u) \in C([-1, 0], X)$.

Step2. We prove that the function g is continuous. Let $u, u' \in C([0, 1], X)$.

For each $\theta \in [-1, 0]$ and $y \in [0, 1]$ we have

$$\begin{aligned} |g(u)(\theta)(y) - g(u')(\theta)(y)| &\leq \lambda(1 - y)\int_{-1}^{\theta} |\sin(u(0)(-\eta)) - \sin(u'(0)(-\eta))|d\eta \\ &\leq \lambda(1 - y)\int_{-1}^{\theta} |u(0)(-\eta) - u'(0)(-\eta)|d\eta \\ &\leq \lambda(1 - y)(\theta + 1)\|u - u'\| \\ &\leq \lambda\|u - u'\|, \end{aligned} \tag{7.4.9}$$

which yields that

$$\|g(u) - g(u')\| \leq \lambda \|u - u'\|. \tag{7.4.10}$$

We conclude that the function g is continuous on $C([0,1], X)$.

Step3. We prove that the function g satisfies inequality (7.3.1).

Observing that $g(0) = 0$, it follows from inequality (7.4.10) that $\|g(u)\| \leq \lambda \|u\|$ for each $u \in C([0,1], X)$. So (7.3.1) holds with $c = \lambda$ and $d = 0$.

Step4. We show that the function g is compact.

Let $B \subset C([0,1], X)$ be a bounded subset of diameter $\delta(B)$. We have to prove that $g(B)$ is relatively compact in $C([-1,0], X)$. To do this, we must prove that:

i) The family $g(B)$ is equicontinuous on $[-1,0]$.

ii) The family $g(B)(\theta)$ is relatively compact in X for each $\theta \in [-1,0]$.

Assertion i) follows from (7.4.8). Next, we show that ii) holds. Firstly, the equicontinuity of the family $g(B)(\theta)$ on $[0,1]$ follows from (7.4.7). Secondly, using (7.4.9) we deduce that $\delta(g(B)(\theta)(y)) \leq \lambda \delta(B)$ which means that the family $g(B)(\theta)(y)$ is bounded. The proof is now complete. ■

The inequality (7.3.4) of Theorem 7.3.5 is equivalent to the existence of $R_0 > 0$ such that

$$R_0 \Big(1 - M_1 (\lambda + L_h \|\kappa\|_{L^1}) \Big) \geq M_1 \|\phi\| = M_1,$$

where $M_1 = \sup_{t \in [0,1]} \|R(t)\|$. This is equivalent to say that

$$M_1 (\lambda + L_h \|\kappa\|_{L^1}) < 1. \tag{7.4.11}$$

At this point, if we suppose that (7.4.11) holds, then all the assumptions in Theorem 7.3.5 are fulfilled. Thus, we have the following result.

Theorem 7.4.4 *Under the assumptions* **(A1)-(A3)**, *the nonlocal problem* (7.4.1) *has at least one mild solution on* $[-1,1]$ *provided that* (7.4.11) *holds.*

Conclusion and Perspectives

In this thesis, we have used functional analysis methods such as semigroups, operator theory and fixed point theory. These techniques have allowed to obtain new results for the study of special solutions such as bounded, almost periodic, almost automorphic, μ-pseudo almost periodic and μ-pseudo almost automorphic solutions which reflect an important dynamic for the models studied in the thesis. Especially, in Chapters 5, we studied the existence and uniqueness of compact almost automorphic weak solutions of the differential inclusion (5.1.1) in the case where the multivalued operator $\mathcal{A}$ is maximal monotone or strongly maximal monotone. Our results open doors to tackle the following problems:

• Suppose that $\mathcal{A}$ is monotone (resp. strongly monotone), non maximal and closed. What is the hypothesis that we must add so that the differential inclusion (5.1.1) has a compact almost automorphic weak solutions ?

• Study the existence of compact almost automorphic weak solutions of the following second order differential inclusion:

$$u''(t) + \mathcal{A}u(t) \ni f(t) \quad \text{for } t \in \mathbb{R}$$

in the case where $\mathcal{A}$ is maximal monotone (resp. strongly maximal monotone).

• Study the existence of compact almost automorphic weak solutions of the following differential inclusion:

$$u'(t) + \mathcal{A}(t)u(t) \ni f(t) \quad \text{for } t \in \mathbb{R}$$

in the case where, for each $t \in \mathbb{R}$, the multivalued operator $\mathcal{A}(t)$ is maximal monotone (resp. strongly maximal monotone).

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