N° d'ordre : 344/21

Sultan Moulay Slimane University
Faculty of Sciences and Technics
Beni Mellal

Doctoral Studies Center: Sciences and Technics

Doctoral Training: Applied Mathematics and Physics

DOCTORAL THESIS

Presented by

ABDELHAMID MOUSSAOUI

Submitted in fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

Fixed Point Theorems Involving Control Functions in Metric Spaces and Fuzzy Metric Spaces

---

Defended on the 4^{th} June 2021 in front of the jury members:

Dr. Khalid HILAL	Professor	FST, USMS Beni Mellal	President/Reporter
Dr. Elhoussine AZROUL	Professor	FSDM, USMBA Fes	Reporter
Dr. Adil ABBASSI	Professor	FST, USMS Beni Mellal	Reporter
Dr. Hassan EL AMRI	Professor	ENS, UHII Casablanca	Examiner
Dr. Hanaa HACHIMI	Professor	USMS Beni Mellal	Examiner
Dr. Jalila EL GHORDAF	Professor	CRMEF Beni Mellal	Examiner
Dr. Hicham MOUSSA	Professor	FST, USMS Beni Mellal	Invited
Dr. Chakir ALLALOU	Professor	FST, USMS Beni Mellal	Invited
Dr. Said MELLIANI	Professor	FST, USMS Beni Mellal	Supervisor

Acknowledgements

First and foremost, all thanks and praises are due to **ALLAH**, the merciful, the compassionate, the almighty who has granted countless blessing so that I have been finally able to accomplish the thesis. May ALLAH's mercy, peace and blessings be upon our leader Mohammed peace be upon him.

I would like to express my sincere thanks to my esteemed supervisor **Dr. Said Melliani**, Dean of faculty of sciences and technics and Director of the Laboratory of Applied Mathematics and Scientific Computing, for his guidance, priceless advice and patience throughout the entire course of my PhD degree. His plentiful experience and immense knowledge have influenced the execution of my thesis in all the stages of my research.

My gratitude extends to the Faculty of science for the opportunity to pursue my research studies in the Mathematics department, Sultane Moulay Slimane University. Moreover, I would like to express my deep appreciation to the staff of Mathematics department for their treasured support and assistant in all the time of my research study.

I also thank my colleagues, research team and all members of the **Laboratory of Applied Mathematics and Scientific Computing** for a cherished time spent together in the research lab and in social settings. Our seminars, meetings and exchanges were of precious assistance for my research work.

Most importantly, I would like to thank my parents without whom none of my success would be possible. They are always supporting me with their endless love, faith and blessings. I also thank my sisters and brother for their spiritual support and best wishes during this challenging period.

Contents

Introduction

Fixed point theory is one of the pillars of research in functional analysis that provides diverse mathematical methods, concepts and pertinent tools for the resolution of many problems that emanate from different branches of mathematics and distinct fields of sciences and engineering. The advances made in fixed point theory, over the last 60 years or so, constitute an active and exciting area of study in modern mathematics serving essentially the study of nonlinear phenomena. In fact, the existence of solutions for several central problems can be studied throughout expressing them on an equivalent fixed point problem. Practically speaking, the operator equation $Fx = 0$ can be turned to a fixed point equation $Tx = x$, where T is a self-mapping with an appropriate domain.

Brower's fixed point theorem, Banach fixed point theorem and Tarski's fixed point theorem, are three decisive results that represent the topological, metric and the order-theoretic approaches respectively, which are in effect the three fundamental directions in fixed point theory. In this study, we are particularly interested in Metric Fixed Point Theory and its applications.

Stephan Banach, the polish mathematician, proved in his thesis in 1922 the first relevant metric fixed point theorem which is also known as Banach's Contraction Principle. This significant result offers an excellent basis for the development of metric fixed point theory. Its central subject is involved in proving the existence and uniqueness of the solutions of diverse mathematical models including variational inequality problems, nonlinear optimization problems, equilibrium problems, ordinary differential and partial differential equations, etc. Due to its vast and significant applicability in pure and applied mathematics, this principle has been generalized and extended in different approaches and several abstract spaces. In this regard, a great deal of work has been done. According to Banach's contraction, the mapping under consideration is continuous. As a result, it is natural to consider the following question: Are there any contractive conditions that do not require the mapping T to be continuous ?

Kannan [62] provided a positive answer to this question in 1968, when he proved Kannan fixed point theorem's by defining a new contraction condition. Following this direction, intensive study in this area has beguan by considering different new generalizations of contraction mappings, see e.g. Chaterjea [70], Zamfirescu [75], Reich [64], Ćirić [46] and others. Furthermore, several studies have been conducted to expand the Banach contraction by introducing various classes of auxiliary and control functions, such as the works of Geraghty [41], Hardy and Rogers [39], Berinde [80], Suzuki [74], Khan *et al.* [51] and several other researchers. In 2012, Samet *et al.*[13] introduced the notion of α-ψ-contractive type mappings by defining the concept of α-admissibility and using Bianchini-Grandolfi gauge functions, the authors inspected the existence and the uniqueness of fixed points for such mappings. In 2014, Popescu [59] suggested the concept of triangular α-orbital admissible as an improvement of the triangular α-admissible notion proposed in [13].

Recently, Khojasteh *et al.* [28] pioneered a new approach to the study of fixed point theory by coining the notion of simulation functions in order to consider a new type of nonlinear contractions known as $\mathcal{Z}$-contractions. Such notions exhibit a significant unifying power over several known results. As a result, it is possible to approach many fixed point problems from a single, common point of view. Many researchers have improved, generalized and extended the idea of simulation functions in different ways and various metric spaces, see, Roldán-López-de-Hierro *et al.* [2], Seong-Hoon Cho [69], b-simulation functions [18], ψ-simulation functions [40], etc.

As part of the ongoing phase of developing the theory, a new line of research that deals with the extension of the Banach Contraction Principle to metric spaces with a partial order has been revealed. The early findings in this direction were due to Turnici [55], Ran and Reurings [1].

The concept of fuzzy set was initiated by L.A. Zadeh [45] in 1965 as a new mathematical approach to deal with uncertainty and vagueness associated to the real-world context. It is based on the generalization of the classical concepts of crisp set and characteristic function. The theory of fuzzy sets is now well developed as an essential and practical modeling construct. One of the key issues in fuzzy topology is to obtain an appropriate and coherent concept of fuzzy metric space. This problem has been considered by many authors in a number of different ways [83, 58]. Kramosil and Michalek [36] defined fuzzy metric space by generalizing the notion of probabilistic metric space to the fuzzy setting. Furthermore, George and Veeramani [3] modified Kramosil and Michalek's definition of fuzzy metric space with the purpose to obtain a Hausdorff topology for this class of fuzzy metric space, which has a significant applications in quantum mechanics, especially in connection with both string and $\epsilon^{(\infty)}$ theory [53, 52]. Over the last years, there has been an intense interest in studying the fixed point theory in fuzzy metric spaces. In this direction, Gregori and Sapena [78] introduced the concept of fuzzy contractive mappings and obtained some fixed point results. In [19], Mihet proposed the class of ψ-contractive mappings which is larger than the fuzzy contractive mappings notion given in [78]. Later on, Wardowski presented and studied the concept of $\mathcal{H}$-contractive mappings [20].

The best proximity theory is another flourishing and influential aspect of fixed point theory which plays a fundamental role in the study of conditions that ensure the existence of optimal approximate fixed point of non-self-mapping T when the functional equation $Tx = x$ has no solution. In fact, if $T : U \longrightarrow V$ is a non-self-mapping where U and V are two nonempty subsets of metric space (X, d), it is crucial to furnish an optimal approximate solution $x \in U$ which induces the minimum error $d(x, Tx)$. Taking into account the fact that $d(x, Tx)$ is at least $d(U, V)$, a best proximity point of T is the optimal approximate solution x satisfying $d(x, Tx) = d(U, V)$. In a natural way, the best proximity theory is a noteworthy generalization of fixed point theorem. Precisely, a best proximity point turns out to be a fixed point if the mapping in question is a self-mapping. Further results of different type of contractions for the existence of a best proximity point in classical and fuzzy metric spaces can be found in [25, 76, 29, 15, 37, 22, 17, 54].

The present doctoral thesis comprises five chapters.

Chapter 1 Constitutes a brief review of basic definitions and notions related to metric spaces and fuzzy metric spaces. It also deals with fundamental results and concepts which marked the theoretical evolution of fixed point theory.

In **Chapter 2**, we introduce a new concept of α-admissible almost type $\mathcal{Z}$-contractions with respect to a simulation function ζ in the setting of complete metric spaces and we prove some results about the existence and uniqueness of fixed points for such class of mappings. Further, we establish that several existing relevant results can be derived from our main results. The presented theorems in this chapter generalize and extend some well-known theorems (see e.g. Khojasteh *et al.* [28], Samet *et al.* [67], Karapinar [24], Ćirić [46], Berinde [82], Hardy and Rogers [39], Kannan [62]).

Chapter 3 is devoted to initiate the concept $\mathcal{FZ}$-contractions involving a new class of control functions, namely $\mathcal{FZ}$-simulation functions. In the intent to unify different existing types of contractions in the framework of fuzzy metric spaces. We prove the existence and uniqueness results for such class of nonlinear contractions. Our approach generalizes the earlier known concepts of Gregori and Sapena [78], Mihet [19], Wardowski [20].

In **Chapter 4** we define a new type of nonlinear contractions by combining the ideas of $\mathcal{FZ}$-contractions defined in the previous chapter and the notion of triangular α-orbital admissible proposed by Popescu [59]. The presented concepts generalize the contractivity conditions originated by Gopal and Vetro [42], Mishra *et al.* [77], Gregori and Sapena [78] and Mihet [19]. We inspect for the existence and uniqueness of fixed points and also we provide two examples illustrating the utility of our results.

Chapter 5 deals with another flourishing and influential aspect of fixed point theory which is the best proximity theory. In this chapter, we introduce a novel approach to best proximity points theorems on a fuzzy metric backdrop, by defining new types of proximal $\mathcal{FZ}$-contractions. Notably, α-$\mathcal{FZ}$-contraction, $\alpha - \psi$-$\mathcal{FZ}$-contraction and generalized α-$\mathcal{FZ}$-contraction. We discuss the existence of the best proximity points of such classes of non-self-mappings and we deliver some corollaries and specify how one can reach more consequences from the obtained theorems.

Chapter 1

Preliminaries

1.1 Metric spaces

The concept of a metric space was first introduced by the French mathematician Maurice Fréchet [30] in 1906. This concept provided a firm foundation for many direction of research in mathematics. First, we begin with some basic definitions.

Definition 1.1.1. [30] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^+$ is said to be a metric on X if the following conditions holds for all $x, y, z \in X$:

($\mathcal{M}1$) $d(x, y) \geq 0$,

($\mathcal{M}2$) $d(x, y) = 0$ if and only if $x = y$,

($\mathcal{M}3$) $d(x, y) = d(y, x)$

($\mathcal{M}4$) $d(x, z) \leq d(y, x) + d(y, z)$

A pair (X, d) satisfying the above assumptions is a metric space.

Example 1.1.2. Let X be nonempty set and define for all $x, y \in X$

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then d is a metric on X and the pair (X, d) is called a discrete metric space.

Example 1.1.3. The real line $\mathbb{R}$ endowed with the absolute value metric

$$\begin{aligned} d\colon \mathbb{R} \times \mathbb{R} &\to \mathbb{R}^+ \\ (x, y) &\mapsto |x - y|. \end{aligned}$$

is called the usual metric space.

Example 1.1.4. Let $X = \mathbb{R}^n$ and a mapping $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ defined by

$$d(x,y) = \left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{\frac{1}{2}}$$

for all $x = (x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. Then d is a metric and the pair (X, d) is called Euclidean metric space.

Example 1.1.5. Let $\mathcal{C}([a,b])$ be the set of all continuous real valued functions defined on $[a,b]$. Define

$$d(f,g) = \sup_{t \in [a,b]} |f(t) - g(t)|$$

Then d is a metric on $\mathcal{C}([a,b])$ and the pair (X, d) is a metric space.

Definition 1.1.6. Let (X, d) be a metric space. For $x \in X$ and $r > 0$ define

$$B(x,r) = \{y \in X : d(x,y) < r\}$$

The set $B(x,r)$ is called the open ball with centre $x \in X$ and radius r.

Definition 1.1.7. A subset O of a metric space (X, d) is said to be open if given any point $x \in O$, there exists $r > 0$ such that $B(x,r) \subseteq O$.

Proposition 1.1.8. *Every open ball $B(x,r)$ is an open set.*

Definition 1.1.9. Let (X, d) be a metric space. A sequence $\{x_n\} \subseteq X$ is said to be convergent to an element $x \in X$ if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$, Thus $\lim_{n\to\infty} x_n = x$.

Definition 1.1.10. Let (X, d) be a metric space. A sequence $\{x_n\} \subseteq X$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 1.1.11. A metric space (X, d) is said to be complete if every Cauchy sequence $\{x_n\} \subseteq X$ has a limit in X.

Example 1.1.12. The real number line $\mathbb{R}$ with the absolute value metric is complete.

Example 1.1.13. The space $\mathbb{R}^n$ with the Euclidean metric is complete.

Example 1.1.14. The closed interval $[0, 1]$ with the absolute value metric is complete.

1.2 Fundamental metric contractions principles

The pioneering Banach contraction principle is a fundamental progress in the theoretical development of metric fixed point theory.

Definition 1.2.1. [8] Let (X, d) be a metric space and $T : X \longrightarrow X$ be a mapping. T is called contraction mapping if

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X, \tag{1.1}$$

Where k is a constant such that $k \in [0, 1)$.

Theorem 1.2.2. *[8]* (***Banach's Contraction Principle***). *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a contraction mapping. Then T has a unique fixed point.*

According to Banach's contraction, the mapping under consideration is continuous. As a result, it is natural to envisage the following question: Are there any contractive conditions that do not require the mapping T to be continuous?

Kannan [62] in 1968 provided a positive answer to this question by proving Kannan's fixed point theorem for the following contractive condition, known as Kannan's contraction:

Definition 1.2.3. [62] Let (X, d) be a metric space and $T : X \longrightarrow X$ be a mapping. T is called Kannan contraction mapping if there exists $k \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X \tag{1.2}$$

Theorem 1.2.4. *[62] Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a Kannan contraction mapping. Then T has a unique fixed point.*

Example 1.2.5. [62] Let $X = [0, 1]$ endowed with the ordinary euclidian metric and define

$$Tx = \begin{cases} \frac{x}{4} & \text{for } x \in [0, \frac{1}{2}) \\ \\ \frac{x}{5} & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

T is discontinuous at $x = \frac{1}{2}$, thus, Banach's condition is not satisfied. However, it satisfies Kannan's condition by taking $k = \frac{4}{9}$.

Notice that, both contractions Banach mappings and Kannan mappings are independent. Thus, the Kannan theorem cannot be considered as an extension of the Banach's contraction principle.

In 1972, Chatterjea [70] established some results by introducing a new condition similar to (1.2).

Theorem 1.2.6. *[70] Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a mapping satisfying the following condition*

$$d(Tx, Ty) \leq \varrho[d(x, Ty) + d(y, Tx)] \tag{1.3}$$

for all $x, y \in X$ where $\varrho \in [0, \frac{1}{2})$. Then T has a unique fixed point.

In 1971, Reich [64] obtained a fixed point theorem by considering new contractive condition as follows

Theorem 1.2.7. *[64] Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a mapping satisfying the following condition*

$$d(Tx, Ty) \leq \gamma d(x, y) + \delta d(x, Tx) + \beta d(y, Ty) \text{ for all } x, y \in X$$

Where γ, δ and β are nonnegative and satisfy $\gamma + \delta + \beta < 1$. Then T has a unique fixed point.

The Banach's contraction principle can be obtained from the above theorem by taking $\gamma = \delta = 0$, while $\gamma = \delta$ and $\beta = 0$ yields Kannan's result.

By combining the conditions (1.1),(1.2) and (1.3). Zamfirescu [75] proved a remarkable fixed point theorem in 1972.

Theorem 1.2.8. *[75] Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a mapping. if there exist a, b and c such that $a \in [0, 1)$, $0 \leq b$ and $c < \frac{1}{2}$ and, at least one of the following conditions is fulfilled*

$(Z1)$ $d(Tx, Ty) \leq ad(x, y)$,

$(Z2)$ $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,

$(Z3)$ $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$

Then T has a unique fixed point.

In [9] Rhoades proved that Zamfirescu's condition is equivalent to the following Ćirić contraction mapping

$$d(Tx, Ty) \leq h \max\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\}$$

for all $x, y \in X$, where $h \in [0, 1)$.

In their paper [39], Hardy and Rogers [39] obtained a new generalization of the fixed point theorem due to Reich by defining a new contractive condition covering several types of the aforementioned conditions.

Theorem 1.2.9. *[39] Let* (X, d) *be a complete metric space and* $T : X \longrightarrow X$ *a mapping such that the following condition holds*

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y)$$

for all $x, y \in X$*, where* a, b, c, e, f *are nonnegative and satisfying* $(a+b+c+e+f) < 1$. *Then* T *has a unique fixed point.*

In 1974, Ćirić [46] considered a new type of generalized contractions, defined by the following

Definition 1.2.10. [46] Let (X, d) be a metric space and $T : X \longrightarrow X$ be a mapping. T is called quasi contraction mapping if there exists $h \in [0, 1)$ such that

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{1.4}$$

for all $x, y \in X$. Rhoades [9] showed that Ćirić condition (1.4) is a generalization of the Zamfirescu result.

Boyd and Wong [12] proved a remarkable generalization of the classical Banach fixed point theorem in complete metric space by substituting the constant k in (1.1) with a control function.

Theorem 1.2.11. *[12] Let* (X, d) *be a complete metric space and* $T : X \longrightarrow X$ *a mapping such that*

$$d(Tx, Ty) \leq \psi(d(x, y)) \tag{1.5}$$

for all $x, y \in X$*, where* $\psi : [0, \infty) \longrightarrow [0, \infty)$ *is upper semicontinuous function from the right (i.e.* $r_n \searrow r \geq 0 \Rightarrow \lim_{n\to\infty} \sup \psi(r_n) \leq \psi(r)$*) and satisfies* $\psi(t) < t$ *for all* $t > 0$*. Then* T *has a unique fixed point.*

Note that if we take $\psi(t) = kt$ for all $t \in [0, \infty)$ with $k \in [0, 1)$ in (1.5), we deduce Banach's fixed point theorem [8].

In [41], Geraghty generalized the Banach's contraction principle and coined a new class of contractions mappings called Geraghty contractions, by using the following class of auxiliary functions:

Let $\mathcal{S}$ denote the class of all real functions $\beta : [0, \infty) \longrightarrow [0, 1)$ satisfying the following condition

$$\beta(t_n) \longrightarrow 1 \Rightarrow t_n \longrightarrow 0.$$

Theorem 1.2.12. *[41] Let* (X, d) *be a complete metric space and* $T : X \longrightarrow X$ *be a mapping. Assume that there exists* $\beta \in \mathcal{S}$ *such that the following condition holds*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \tag{1.6}$$

for all $x, y \in X$*. Then* T *has a unique fixed point.*

In 2004, Berinde [80, 81] presented the concept of almost contractions (known also as weak contractions) and showed that the Kannan [62] and Zamfirescu [75] operators, in addition to a large class of quasi-contractions are included in the class of weak contractions. The main results in [80] are the following theorems.

Definition 1.2.13. [80] Let (X, d) be a metric space and $T : X \longrightarrow X$ be a mapping. T is called an (δ, L)-almost contraction if there exist a constant $\delta \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \text{ for all } x, y \in X. \tag{1.7}$$

Remark 1.2.14. It's worth noting that the almost contraction (1.7) is not symmetric. However, since the distance is symmetric, the following dual condition (1.8) is implicitly included in almost contraction condition (1.7):

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(x, Ty) \text{ for all } x, y \in X. \tag{1.8}$$

Clearly, Banach contractions (1.1) are properly included in the class of almost contractions (1.7) with $\delta = k$ and $L = 0$.

The following propositions include additional examples of almost contractions.

Proposition 1.2.15. *[80] Let (X, d) be a metric space. Every Kannan's contraction* (1.2) *is an almost contraction.*

Proof. From the condition (1.2) and the triangle inequality, we have

$$\begin{aligned} d(Tx, Ty) &\leq k[d(x, Tx) + d(y, Ty)] \\ &\leq k\left([d(x, y) + d(y, Tx)] + [d(y, Tx) + d(Tx, Ty)]\right) \end{aligned}$$

Therefore

$$(1 - k)d(Tx, Ty) \leq kd(x, y) + 2kd(y, Tx)$$

Which means

$$d(Tx, Ty) \leq \frac{k}{(1-k)} d(x, y) + \frac{2k}{(1-k)} d(y, Tx)$$

Then, in light of the condition $k \in [0, \frac{1}{2})$, (1.7) holds by taking $\delta = \frac{k}{(1-k)}$ and $L = \frac{2k}{(1-k)}$. □

Proposition 1.2.16. *[80] Let (X, d) be a metric space. Every Chatterjea's contraction, i.e. every mapping $T : X \longrightarrow X$ satisfying the contractive condition* (1.3) *is an almost contraction.*

Proposition 1.2.17. *[80]Every Zamfirescu contraction, i.e., a mapping $T : X \longrightarrow X$ satisfying the conditions in Theorem 1.2.8, is an almost contraction.*

Example 1.2.18. [10] Let $X = [0, 1]$ endowed with the usual metric and define $T : X \longrightarrow X$ by

$$Tx = \begin{cases} \frac{2}{3}x & \text{for } x \in [0, \frac{1}{2}) \\ \frac{2}{3}x + \frac{1}{3} & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

Then

$\imath$) T is an almost contraction with $\delta = \frac{2}{3}$ and $L = 6$ and $Fix(T) = \{0; 1\}$.

$\jmath$) T does not fulfill any of the contraction conditions of Banach, Kannan, Chatterjea, Zamfirescu and Ćirić quasi-contraction, since T has two fixed points.

Theorem 1.2.19. *[80] Let (X, d) be a complete metric space and $T : X \longrightarrow X$ (δ, L)-almost contraction mapping then*

(1) *$Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$,*

(2) *For any $x_0 \in X$ the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ converges to some $x^* \in Fix(T)$,*

(3) *The following estimates*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1),\ n = 0, 1, 2, ...$$
$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n),\ n = 0, 1, 2, ...$$

hold, where δ is the constant (1.7).

In order to prove the uniqueness of the fixed point for such class of contraction, Breinde [80] considered an additional condition,quite similar to (1.7) as seen by the following theorem.

Theorem 1.2.20. *[80] Let (X, d) be a complete metric space and $T : X \longrightarrow X$ an almost contraction mapping for which there exist $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta d(x, y) + L_1 d(x, Tx) \text{ for all } x, y \in X.$$

Then T has a unique fixed point, i.e. $Fix(T) = \{x^*\}$.

1.3 α-ψ-contractive type mappings

In 2012, Samet, C. Vetro and P.Vetro [13] originated the idea of α-ψ-contractive type mappings and proved fixed point results for such class of mappings in the context of complete metric spaces.

Definition 1.3.1. [13] Let Ψ_1 be the family of functions $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying the following conditions:

(ψ1) ψ is nondecreasing function.

(ψ2) $\sum_{n=1}^{+\infty} \psi_n(t) < +\infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ.

Lemma 1.3.2. *[34, 35] If $\psi \in \Psi_1$, then the following hold :*

(i) $(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t \in \mathbb{R}^+$;

(ii) $\psi(t) < t$, for any $t \in \mathbb{R}^+$;

(iii) ψ is continuous at 0;

(iv) the series $\sum_{k=1}^{\infty} \psi_k(t)$ converges for all $t \in \mathbb{R}^+$.

One of the fascinating concepts, α-admissibility was presented by Samet *et al.*[13].

Definition 1.3.3. [13] Let $T : X \longrightarrow X$ be a self-mapping and $\alpha : X \times X \longrightarrow [0, \infty)$ be a function. T is said to be α-admissible if

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1 \text{ for all } x, y \in X.$$

Example 1.3.4. [13] Let $X = [0, \infty)$ and define a self mapping $T : X \longrightarrow X$ and the function $\alpha : X \times X \longrightarrow [0, \infty)$ by $Tx = \sqrt{x}$ for all $x \in X$, and

$$\alpha(x, y) = \begin{cases} e^{x-y} & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

Then, T is α-admissible.

Example 1.3.5. [13] Let $X = (0, \infty)$ and define a self mapping $T : X \longrightarrow X$ and the function $\alpha : X \times X \longrightarrow [0, \infty)$ by $Tx = \ln x$ for all $x \in X$, and

$$\alpha(x, y) = \begin{cases} 2 & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

Then, T is α-admissible.

Definition 1.3.6. [13] Let $T : X \longrightarrow X$ be a mapping on a metric space (X, d). We say that T is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \longrightarrow [0, \infty)$ and $\psi \in \Psi_1$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)), \text{ for all } x, y \in X. \tag{1.9}$$

Remark 1.3.7. Defining $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t > 0$ and some $k \in [0, 1)$ in the above definition, we derive the well known Banach contraction.

Theorem 1.3.8. *[13] Let (X, d) be a complete metric space. Suppose that $T : X \longrightarrow X$ is an α-ψ-contractive mapping and satisfies the following conditions:*

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous

Then, T has a fixed point.

Theorem 1.3.9. *[13] Let (X, d) be a complete metric space. Suppose that $T : X \longrightarrow X$ is an α-ψ-contractive mapping and satisfies the following conditions:*

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then there exists $u \in X$ such that $Tu = u$.

In order to obtain the uniqueness of the fixed point in the above theorems the authors considered the following additional condition: For all $x, y \in X$, there exist $w \in X$ such that $\alpha(x, w) \geq 1$ and $\alpha(y, w) \geq 1$.

Example 1.3.10. [13] Let $X = \mathbb{R}$ endowed with the usual metric $d(x, y) = |x - y|$ and $T : X \longrightarrow X$ the mapping defined by

$$Tx = \begin{cases} 2x - \frac{3}{2} & \text{if } x > 1, \\ \frac{x}{2} & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x < 0. \end{cases}$$

Clearly, T does not satisfy the Banach contraction principle, since

$$d(T1, T2) = 2 > d(2, 1) = 1$$

T satisfies condition (1.9) with $\psi(t) = \frac{t}{2}$ for all $t \geq 0$ and

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Further, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Indeed, for $x_0 = 1$, we get

$$\alpha(1, T1) = \alpha(1, \frac{1}{2}) = 1.$$

Now, we show that T is α-admissible. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$, which means that x, y $[0, 1]$, we have

$$Tx = \frac{x}{2}, Ty = \frac{y}{2} \in [0, 1] \text{ and } \alpha(Tx, Ty) = 1.$$

All conditions of Theorem 1.3.8 are satisfied. Thus, T has a fixed point.

1.4 Nonlinear contractions via simulation functions

In 2015, Khojasteh *et al.* [28] presented a new approach to the study of fixed point theory via the concept of simulation functions and defined a new type of nonlinear contractions, namely $\mathcal{Z}$-contractions. the authors generalized the Banach contraction principle and unified several known types of contractions in complete metric spaces. The idea of simulation functions has been extended and enriched in various directions and different metric spaces by many researchers.

Definition 1.4.1. [28] The function $\zeta : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ is said to be a simulation function, if it satisfies the following conditions:

(S_1) $\zeta(0, 0) = 0$;

(S_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(S_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then $\lim_{n\to\infty} \sup \zeta(t_n, s_n) < 0$.

The set of all simulation functions is denoted by $\mathcal{Z}$.

Example 1.4.2. [28] Let $\zeta_i : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}, i = 1, 2, 3$ be defined by

1. $\zeta_1(t, s) = \tilde{\phi}(s) - \phi(t)$ for all $t, s \in [0, \infty)$,where $\tilde{\phi}, \phi : [0, \infty) \longrightarrow [0, \infty)$ are two continuous functions such that $\tilde{\phi}(t) = \phi(t) = 0$ if and only if $t = 0$ and $\tilde{\phi}(t) < t \leq \phi(t)$ for all $t > 0$.

2. $\zeta_2(t,s) = s - \varphi(s) - t$ for all $t, s \in [0,\infty)$, where $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

3. $\zeta_3(t,s) = s - \frac{f(t,s)}{g(t,s)}t$ for all $t, s \in [0,\infty)$, where $f, g : [0,\infty) \longrightarrow (0,\infty)$ are continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t, s > 0$.

ζ_i for $i = 1, 2, 3$ are simulation functions. For other interesting examples of simulation functions, readers are refereed to [2, 56, 6].

Definition 1.4.3. [28] Let (X,d) be a metric space, $T : X \longrightarrow X$ a self-mapping and $\zeta \in \mathcal{Z}$. We say that T is a $\mathcal{Z}$-contraction with respect to ζ, if the following condition is satisfied

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0 \qquad \text{for all } x, y \in X.$$

The Banach contraction is a perfect instance of $\mathcal{Z}$-contraction with respect to the simulation function defined by $\zeta(t,s) = ks - t$, where $k \in [0,1)$.

Now, we deliver the result proved in [28] as follows.

Lemma 1.4.4. *[28] Let $T : X \longrightarrow X$ be a $\mathcal{Z}$-contraction with respect to a simulation function $\zeta \in \mathcal{Z}$ on metric space (X,d). Then T provided that has a fixed point, it is unique.*

Definition 1.4.5. [26] A self mapping T on a metric space (X,d) is said to be asymptotically regular at a point x in X, if

$$\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0$$

where T^n denotes the nth iterate of T at x.

Lemma 1.4.6. *[28] Let (X,d) be a metric space and suppose that the self mapping $T : X \longrightarrow X$ is a $\mathcal{Z}$-contraction with respect to a simulation function $\zeta \in \mathcal{Z}$. Then T is asymptotically regular at every arbitrary $x \in X$.*

Theorem 1.4.7. *[28] Let (X,d) be a complete metric space and $T : X \longrightarrow X$ be a $\mathcal{Z}$-contraction with respect to a simulation function $\zeta \in \mathcal{Z}$. Then T has unique fixed point in X and for every $x_0 \in X$ the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to the fixed point of T.*

1.5 Fuzzy metric spaces

Definition 1.5.1. [14] A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous triangular norm (or continuous t-norm) if it satisfies the following conditions

(CT1) $*$ is continuous;

(CT2) $*$ is commutative and associative;

(CT3) $a * 1 = a$ for all $a \in [0, 1]$;

(CT4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0, 1]$.

Example 1.5.2. The following instances are classical examples of continuous t-norm:

a) $a * b = a.b$ Probabilistic t-norm;

b) $a * b = \min(a, b)$ Zadeh's t-norm;

c) $a * b = \max[0, a + b - 1]$ Lukasiewicz's t-norm;

Definition 1.5.3. (George and Veeramani [3]) The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space (in the sense of George and Veeramani) if X is an arbitrary set, $*$ is a continuous t-norm and M is fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions :

(GV1) $M(x, y, t) > 0$,

(GV2) $M(x, y, t) = 1$ if and only if $x = y$,

(GV3) $M(x, y, t) = M(y, x, t)$

(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(GV5) $M(x, y, .) : (0, \infty) \to (0, 1]$ is continuous .

for all $x, y, z \in X$ and $s, t > 0$.

The function $M(x, y, t)$ can be thought of as the degree of nearness of x and y with respect to the variable t.

In Definition 1.5.3, if the triangular inequality $(GV4)$ is replaced by

$$(GV4)' : M(x, y, t) * M(y, z, s) \leq M(x, z, \max\{t, s\})$$

for all $x, y, z \in X$ and $s, t > 0$, then the triple $(X, M, *)$ is said to be a non-Archimedean fuzzy metric space, as $(GV4)'$ implies $(GV4)$, every non-Archimedean fuzzy metric space is a fuzzy metric space. Furthermore, the ordered triple $(X, M, *)$ is called a strong fuzzy metric space if $(GV4)$ is replaced by

$$(GV4)'' : M(x, y, t) * M(y, z, t) \leq M(x, z, t)$$

for all $x, y, z \in X$ and $t > 0$.

Lemma 1.5.4. *[49] $M(x, y, .)$ is nondecreasing for all x, y in X.*

Example 1.5.5. [3] Let $X = \mathbb{R}$. Define $a * b = ab$ for all $a, b \in [0, 1]$ and the function $M : X \times X \times (0, \infty) \to [0, 1]$ by

$$M(x, y, t) = \left[\exp\left(\frac{|x - y|}{t}\right)\right]^{-1} \quad \text{for all } x, y \in X, t > 0.$$

Then $(X, M_d, *)$ is a fuzzy metric space.

Example 1.5.6. [3] Let (X, d) be a metric space. Define $a * b = \min(a, b)$ for all $a, b \in [0, 1]$ and

$$M(x, y, t) = \frac{kt^n}{kt^n + md(x, y)} \ , \ k, m, n \in \mathbb{R}^+$$

Then $(X, M, *)$ is a fuzzy metric space.
Letting $k = m = n = 1$, we get

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

we call this fuzzy metric induced by a metric d the standard fuzzy metric.

Example 1.5.7. ([3, 61, 79]) Let $f : X \to \mathbb{R}^+$ be a one-to-one function, $g : \mathbb{R}^+ \to [0, \infty)$ be an increasing continuous function and $\alpha, \beta > 0$. Define

$$M(x, y, t) = \left(\frac{(\min\{f(x), f(y)\})^\alpha + g(t)}{(\max\{f(x), f(y)\})^\alpha + g(t)}\right)^\beta \tag{1.10}$$

Then $(M, .)$ is a fuzzy metric on X.

In particular, by taking $\alpha = \beta = 1$ and f as the identity function, then we obtain the following examples

1. Let $X = \mathbb{R}^+$ and take g as the identity function in (1.10), we have

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}$$

   Then $(X, M, *)$ is a fuzzy metric space.

2. Let $X = (0, \infty)$ and take g as the zero function (1.10), we have

$$M(x, y, t) = \frac{\min\{x, y\}}{\max\{x, y\}}$$

   Then $(X, M, .)$ is a fuzzy metric space.

Example 1.5.8. [79] Let $\varphi : \mathbb{R}^+ \to [0,1)$ be an increasing continuous function. Define $*$ by the t-norm minimum and the function $M : X \times X \times (0,\infty) \to [0,1]$ by

$$M(x,y,t) = \begin{cases} 1 & \text{if } x = y \\ \varphi(t) & \text{if } x \neq y. \end{cases}$$

Then $(M,*)$ is a fuzzy metric on X.

In particular, by taking $\varphi(t) = k \in (0,1)$ we get the following example

$$M(x,y,t) = \begin{cases} 1 & \text{if } x = y \\ k & \text{if } x \neq y. \end{cases}$$

Here, M is called the discrete fuzzy metric because of its analogy to the standard discrete metric.

1.6 Topology induced by a fuzzy metric

Definition 1.6.1. [3] Let $(X,M,*)$ be a fuzzy metric space. We define the open ball $B_M(x,r,t)$ for $t > 0$ with centre $x \in X$ and radius r , $0 < r < 1$ as

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}$$

Definition 1.6.2. [3] A subset O of a fuzzy metric space $(X,M,*)$ is said to be open if given any point $x \in O$, there exists $0 < r < 1$, and $t > 0$ such that $B(x,r,t) \subseteq O$.

Theorem 1.6.3. *[3] Every open ball $B(x,r,t)$ is an open set.*

Definition 1.6.4. [3] Let $(X,M,*)$ be a fuzzy metric space. For$t > 0$, the closed ball with centre $x \in X$ and radius r, $0 < r < 1$ $t > 0$ is defined by

$$B[x,r,t] = \{y \in X : M(x,y,t) \geq 1 - r\}$$

Lemma 1.6.5. *Every closed ball in a fuzzy metric space is a closed set.*

Proposition 1.6.6. *[3] Let $(X,M,*)$ be a fuzzy metric space, define*

$$\tau_M = \{O \subset X : x \in O \textit{ if and only if there exist } t > 0$$

$$\textit{and } 0 < r < 1 \textit{ such that } B(x,r,t) \subset O\}$$

Then τ is a topology on X.

Remark 1.6.7. [3] Since $\{B(x,\frac{1}{n},\frac{1}{n}); n = 1,2,...\}$ is a local base at x, the above topology is first countable.

Theorem 1.6.8. *[3] Every fuzzy metric space is Hausdorff.*

Definition 1.6.9. [3] A subset A of a fuzzy metric space $(X, M, *)$ is said to be F-bounded if and only if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1-r$ for all $x, y \in A$.

Theorem 1.6.10. *[3] Every compact subset K of a fuzzy metric space X is F-bounded.*

Remark 1.6.11. Every compact set in a fuzzy metric space is closed and bounded.

Definition 1.6.12. [3] Let $(X, M, *)$ be a fuzzy metric space.

1. A sequence $\{x_n\} \subseteq X$ is said to be convergent to a point $x \in X$ if an only if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.
2. A sequence $\{x_n\} \subseteq X$ is said to be a Cauchy sequence if and only if for each $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.
3. A fuzzy metric space in which every Cauchy sequence is convergent is called a complete fuzzy metric space.
4. A fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.

1.7 Fuzzy contractive mappings and related fixed point theorems

Definition 1.7.1. [78] Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be fuzzy contractive mapping if there exists $k \in (0, 1)$ such that

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right). \tag{1.11}$$

for each $x, y \in X$ and $t > 0$.

Definition 1.7.2. [78] Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be fuzzy contractive mapping if there exists $k \in (0, 1)$ such that

$$\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \leq k\left(\frac{1}{M(x_{n,n+1}\,, t)} - 1\right).$$

for all $t > 0$, $n \in \mathbb{N}$.

Gregori and Sapena [78] extended the Banach fixed point theorem to fuzzy contractive mapping by the following theorem.

Theorem 1.7.3. *[78] Let $(X, M, *)$ be a fuzzy metric space in which fuzzy contractive sequence are Cauchy and $T : X \to X$ be a fuzzy contractive mapping. Then T has a unique fixed point.*

Let Ψ_2 be the class of all functions $\psi : (0, 1] \to (0, 1]$ such that ψ is continuous, non-decreasing and $\psi(t) > t$, for all $t \in (0, 1)$.

Definition 1.7.4. [19] Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \longrightarrow X$ is said to be fuzzy ψ-contractive mapping if

$$M(Tx, Ty, t) \geq \psi(M(x, y, t)) \text{ for all } x, y \in X, t > 0. \tag{1.12}$$

Definition 1.7.5. [20] Let $\mathcal{H}$ be the set of all functions $\eta : (0, 1] \longrightarrow [0, \infty)$ which satisfy the following two conditions:

$\mathcal{H}_1$) η transforms $(0, 1]$ onto $[0, \infty)$,

$\mathcal{H}_2$) η is strictly decreasing.

Definition 1.7.6. [20] Let $(X, M, *)$ be a fuzzy metric space and $\eta \in \mathcal{H}$. A sequence $\{x_n\}$ in X is Cauchy if and only if for each $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\eta(M(x_n, x_m, t)) < \varepsilon$ for all $n, m \geq n_0$

Proposition 1.7.7. *[20] Let $(X, M, *)$ be a fuzzy metric space and $\eta \in \mathcal{H}$. A sequence $\{x_n\}$ in X is convergent to $x \in X$ if and only if $\lim_{n\to\infty} \eta(M(x_n, x, t)) = 0$ for all $t > 0$.*

In 2013, Wardowski [20] introduced the concept of fuzzy $\mathcal{H}$-contractive mappings as generalization of notion of fuzzy contractive mappings given by Gregori and Sapena [78].

Definition 1.7.8. Let $(X, M, *)$ be a fuzzy metric space and $\eta \in \mathcal{H}$. A mapping $T : X \to X$ is said to be fuzzy $\mathcal{H}$-contractive with respect to $\eta \in \mathcal{H}$ if there exists k $(0, 1)$ such that

$$\eta(M(Tx, Ty, t)) \leq k\eta(M(x, y, t)) \text{ for all } x, y \in X, t > 0. \tag{1.13}$$

Example 1.7.9. [20] By considering $\eta(t) = \frac{1}{t} - 1$, $t \in (0, 1]$. Then this Definition reduces to the definition of fuzzy contraction due to Gregori and Sapena [78].

Theorem 1.7.10. *[20] Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ be a fuzzy $\mathcal{H}$-contractive mapping with respect to $\eta \in \mathcal{H}$ such that*

(w_1) *$\prod_{i=1}^{k} M(x, Tx, t_i) > 0$, for all $x \in X$, $k \in \mathbb{N}$ and any sequence $\{t_i\}_{i\in\mathbb{N}}) \subset (0, \infty)$,$t_i \searrow 0$;*

(w_2) *$r * s > 0 \Rightarrow \eta(r * s) \leq \eta(r) + \eta(s)$, for all $r, s \in \{M(x, Tx, t) : x \in X, t > 0\}$;*

(w_3) $\{\eta(M(x,Tx,t_i)) : i \in \mathbb{N}\}$ *is bounded for all* $x \in X$ *and any sequence* $\{t_i\}_{i\in\mathbb{N}} \subset (0,\infty)$,$t_i \searrow 0$.

Then T *has a unique fixed point* $x^* \in X$ *and for each* $x_0 \in X$ *the sequence* $\{T^n x_0\}_{n\in\mathbb{N}}$ *converges to* x^*.

Motivated by the work of Samet *et al.* [13], Gopal and Vetro [42] initiated the concept of α-ϕ-fuzzy contractive mapping and established some theorems which ensure the existence and uniqueness of a fixed point for such mappings.

Definition 1.7.11. [42] Let $(X, M, *)$ be a fuzzy metric space. We say that a mapping $T : X \to X$ is α-admissible if there exists a function $\alpha : X \times X \times (0,+\infty) \longrightarrow [0,+\infty)$ such that for all $x, y \in X$, $t > 0$

$$\alpha(x,y,t) \geq 1 \Rightarrow \alpha(Tx,Ty,t) \geq 1.$$

Definition 1.7.12. [42] Denote by Φ the class of all functions $\phi : [0,1) \to [0,1)$ which satisfy the two following conditions:

(ϕ_1) ϕ is right continuous functions,

(ϕ_2) $\phi(\ell) < \ell$ for all $\ell > 0$.

Remark 1.7.13. For every function $\phi \in \Phi$, we have $\lim_{n\to\infty} \phi^n(\ell) = 0$ where ϕ^n denotes the nth iterate of ϕ.

Definition 1.7.14. [42]Let $(X, M, *)$ be a fuzzy metric space in the sense of George and Veeramani. A mapping $T : X \to X$ is said to be an α-ϕ-fuzzy contractive mapping if there exist two functions $\alpha : X \times X \times (0,\infty) \to [0,\infty[$ and $\phi \in \Phi$ such that

$$\alpha(x,y,t)\left(\frac{1}{M(Tx,Ty,t)} - 1\right) \leq \phi\left(\frac{1}{M(x,y,t)} - 1\right) \tag{1.14}$$

for all $x, y \in X$ and for all $t > 0$.

Remark 1.7.15. Taking $\alpha(x,y,t) = 1$ for all $x, y \in X$ and $t > 0$, and $\phi(\ell) = k\ell$ for $\ell > 0$ and some $k \in (0,1)$ in Definition 1.7.14, one can deduce the definition 1.7.1 of fuzzy contractive mapping due to Gregori and Sapena. Hence, every fuzzy contractive mapping is an α-ϕ-fuzzy contractive mapping.

In order to unify diverse types of fuzzy contractive mappings, especially those provided in [78],[19] and [20], Satish Shukla *et al.* [68] developed a new type of fuzzy contractive mappings, namely, $\mathcal{Z}$-contractive mappings with the help of the folowing class of auxiliary function.

Definition 1.7.16. [68] Let $\mathcal{Z}$ be the set of all functions $\zeta : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ which satisfy the following condition:

$$\zeta(t,s) > s \text{ for all} t, s \in (0,1).$$

Example 1.7.17. The following functions $\zeta : (0,1] \times (0,1] \longrightarrow \mathbb{R}$:

(1) $\zeta(t,s) = \frac{s}{t}$,

(2) $\zeta(t,s) = \frac{1}{s+t} + t$,

(3) $\zeta(t,s) = \psi(s)$ with $\psi : (0,1] \longrightarrow (0,1]$ is a function such that $s < \psi(s)$ for all $s \in (0,1)$.

belong to the family $\mathcal{Z}$.

From the previous definition it is easy to see that $\zeta(t,t) > t$ for all $0 < t < 1$.

Definition 1.7.18. [68] Let $(X, M, *)$ be a fuzzy metric space and let $T : X \longrightarrow X$ be a self mapping. We say that T is a fuzzy $\mathcal{Z}$-contractive mapping with respect to some function $\zeta \in \mathcal{Z}$ if

$$M(Tx, Ty, t) \geq \zeta(M(Tx, Ty, t), M(x, y, t)) \text{ with } Tx \neq Ty, t > 0.$$

Example 1.7.19. Any fuzzy contractive mapping, i.e. satisfying (1.11), is $\mathcal{Z}$-contractive mapping with respect to $\zeta \in \mathcal{Z}$ defined by $\zeta(t,s) = \frac{s}{k+(1-k)s}$ for all $t, s \in (0,1)$.

Example 1.7.20. Any ψ-contractive mapping is $\mathcal{Z}$-contractive mapping with respect to $\zeta \in \mathcal{Z}$ defined by $\zeta(t,s) = \psi(s)$ for all $t, s \in (0,1)$.

Here, we must point out the fact that a fuzzy $\mathcal{Z}$-contractive mapping does not possess necessarily a fixed point, despite the completeness of fuzzy metric space. The following example confirms this fact.

Example 1.7.21. Let $X = \mathbb{N}$ be endowed with the fuzzy metric $M(n, m, t) = \min\{\frac{n}{m}, \frac{m}{n}\}$ for all $m, n \in X$ and $t > 0$. Hence, $(X, M, *)$ is a complete metric space, where $*$ is the product t-norm. Define $T : X \to X$ by $Tn = n + 1$ for all $n \in X$. Then T is a fuzzy $\mathcal{Z}$-contractive mapping with respect to the function $\zeta \in \mathcal{Z}$ defined by $\zeta(t,s) = \frac{s+t}{2}$ for $t > s$ and $\zeta(t,s) = 1$ otherwise. Note that, T is a fixed point free mapping on X.

The above case, inspired the authors to consider a fuzzy metric space having an additional property with a view to ensure the existence of fixed point for fuzzy $\mathcal{Z}$-contractive mapping.

Definition 1.7.22. [68] We say that the quadruple $(X, M, *, \zeta)$ has the property **(S)**, if for every Picard sequence $\{x_n\}$ with initial value $x \in X$, ie. $x_n = T^n x$ such that $\inf_{m>n} M(x_n, x_m, t) \leq \inf_{m>n} M(x_{n+1}, x_{m+1}, t)$ implies that

$$\lim_{n\to\infty} \inf_{m>n} \zeta(M(x_{n+1}, x_{m+1}, t), M(x_n, x_m, t)) = 1$$

Now, we deliver the main result established in [68].

Theorem 1.7.23. *[68] Let $(X, M, *)$ be a complete fuzzy metric space , $T : X \longrightarrow X$ be a fuzzy $\mathcal{Z}$-contraction and assume that the quadruple $(X, M, *, \zeta)$ has the property **(S)**. Then T has unique fixed point.*

Chapter 2

Admissible Almost Type $\mathcal{Z}$-Contractions and Fixed Point Results

The main purpose of this chapter is to introduce a new concept of α-admissible almost type $\mathcal{Z}$-contraction via simulation functions and prove some fixed point results for such a new class of nonlinear contractions in the context of complete metric spaces. Furthermore, we will show that several known fixed point results and new consequences can be easily deduced in the frameworks of standard metric spaces and metric spaces endowed with partial order by applying our main theorems. The presented results generalize, improve and unify several existing results in the literature including the Banach contraction principle [8], kannan [62], Chatterjea [70] and Ćirić [46]. This chapter comprises the results from our published paper [71].

2.1 Admissible almost type $\mathcal{Z}$-contractions

First, we introduce the following concept.

Definition 2.1.1. Let (X, d) be a metric space and $\zeta \in \mathcal{Z}$. We say that $T : X \longrightarrow X$ is an α-admissible almost $\mathcal{Z}$-contraction if there exist $\alpha : X \times X \longrightarrow [0, \infty)$ and a constant $L \geq 0$ such that

$$\zeta(\alpha(x, y)d(Tx, Ty), M(x, y) + LN(x, y)) \geq 0 \text{ for all } x, y \in X, \tag{2.1}$$

where

$$N(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\};$$
$$M(x, y) = \max\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\}.$$

Remark 2.1.2. If T is an α-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then

$$\alpha(x,y)d(Tx,Ty) < M(x,y) + LN(x,y) \text{ for all } x,y \in X.$$

2.2 Fixed point theorems for admissible almost $\mathcal{Z}$-contractions

Our first result is the following theorem.

Theorem 2.2.1. *Let (X,d) be a complete metric space and let $T : X \longrightarrow X$ be an α-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ and satisfying the following conditions:*

(i) *T is triangular α-orbital admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous.*

Then there exists $z \in X$ such that $Tz = z$.

Proof. Using condition (ii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, let $\{x_n\}$ be the iterative sequence in X defined by

$$x_{n+1} = Tx_n \text{ for all } n \in \mathbb{N}$$

If $x_{m+1} = Tx_m$ for some $m \in \mathbb{N}$, then x_m is a fixed point of T. Therefore, to continue our proof, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since T is an α-admissible mapping, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \text{ implies that } \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By induction, we obtain that

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}. \tag{2.2}$$

Applying the condition (2.1) with $x = x_n$ and $y = x_{n-1}$ and using (2.2), we get

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, x_{n-1})d(Tx_n, Tx_{n-1}), M(x_n, x_{n-1}) + LN(x_n, x_{n-1})) \\ &= \zeta(\alpha(x_n, x_{n-1})d(x_{n+1}, x_n), M(x_n, x_{n-1}) + LN(x_n, x_{n-1})) \\ &< M(x_n, x_{n-1}) + LN(x_n, x_{n-1}) - \alpha(x_n, x_{n-1})d(x_{n+1}, x_n). \end{aligned} \tag{2.3}$$

where

$$
\begin{aligned}
N(x_n, x_{n-1}) &= \min\{d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_{n-1}), d(x_{n-1}, Tx_n)\} \\
&= \min\{d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n), d(x_{n-1}, x_{n+1})\} \\
&= 0 \qquad (2.4)
\end{aligned}
$$

and

$$
\begin{aligned}
M(x_{n-1}, x_n) &= \max\{d(x_n, x_{n-1}), \frac{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})}{2}, \\
&\qquad \frac{d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)}{2}\} \\
&= \max\{d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{2}, \frac{d(x_{n-1}, x_{n+1})}{2}\} \\
&\leq \max\{d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{2}\} \\
&\leq \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} \qquad (2.5)
\end{aligned}
$$

By (2.3) and taking in account (2.2), (2.4) and (2.5), we derive that

$$
d(x_n, x_{n+1}) \leq \alpha(x_n, x_{n-1}) d(x_n, x_{n+1}) < \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}
$$

for all $n \geq 1$.

Now, if $\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ for some $n \geq 1$. Then from the above inequality we get

$$
d(x_n, x_{n+1}) \leq \alpha(x_n, x_{n-1}) d(x_n, x_{n+1}) < d(x_n, x_{n+1})
$$

which is a contradiction. Therefore

$$
\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n-1}) \text{ for all } n \geq 1. \qquad (2.6)
$$

Hence

$$
d(x_n, x_{n+1}) \leq \alpha(x_n, x_{n-1}) d(x_n, x_{n+1}) < d(x_n, x_{n-1}) \text{ for all } n \geq 1. \qquad (2.7)
$$

Consequently, we deduce that the sequence $\{d(x_n, x_{n-1})\}$ is a decreasing of positive real numbers. Thus, there exists $r \geq 0$ such that $\lim_{n \to \infty} d(x_n, x_{n-1}) = r \geq 0$. We claim that

$$
\lim_{n \to \infty} d(x_n, x_{n-1}) = 0 \qquad (2.8)
$$

On contrary assume that $r > 0$. It follows from the inequality (2.7) that

$$
\lim_{n \to \infty} \alpha(x_n, x_{n-1}) d(x_n, x_{n+1}) = r \qquad (2.9)
$$

Now, if we take the sequences $\{\delta_n = \alpha(x_n, x_{n-1})d(x_n, x_{n+1})\}$ and $\{\tau_n = d(x_n, x_{n-1})\}$ and considering (2.9), then $\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \tau_n = r$ therefore by (S_3), we get that

$$0 \le \lim_{n\to\infty} \sup \zeta(\alpha(x_n, x_{n-1})d(x_n, x_{n+1}), d(x_n, x_{n-1}) < 0 \tag{2.10}$$

a contradiction, we deduce that $r = 0$ and equation (2.8) holds.

Next, we show that $\{x_n\}$ is Cauchy sequence in X. Reasoning by the method of Reductio ad absurdum. Suppose to the contrary that $\{x_n\}$ is not a Cauchy sequence. So, there exists $\epsilon > 0$, for every $N \in \mathbb{N}$, there exist $n, m \in \mathbb{N}$ such that $N < m < n$ and $d(x_m, x_n) > \epsilon$. In account of (2.8), there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \epsilon \text{ for all } n > n_0. \tag{2.11}$$

We can find two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $m_k > n_k \ge n_0$ and

$$d(x_{m_k}, x_{n_k}) \quad > \epsilon, \text{ for all } k. \tag{2.12}$$

where m_k is the smallest index satisfying (2.12). Then

$$d(x_{m_{k-1}}, x_{n_k}) \le \epsilon \text{ for all } k, \tag{2.13}$$

Now, using (2.12), (2.13) and the triangular inequality, we obtain

$$\begin{aligned} \epsilon < d(x_{m_k}, x_{n_k}) &\le d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\le d(x_{m_k}, x_{m_k-1}) + \epsilon \end{aligned} \tag{2.14}$$

Letting $k \to \infty$ and using Equation (2.8), we derive that

$$\lim_{n\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon \tag{2.15}$$

Again, using the triangular inequality, we get

$$d(x_{m_k}, x_{n_k}) \le d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \text{ for all } k. \tag{2.16}$$

Also, we have

$$d(x_{m_k+1}, x_{n_k+1}) \le d(x_{m_k+1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}) \text{ for all } k. \tag{2.17}$$

By taking the limit as $k \to \infty$ on both sides of (2.16), (2.17) and using (2.8) we deduce that

$$\lim_{n\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \epsilon. \tag{2.18}$$

By the same reasoning as above, we get that

$$\lim_{n\to\infty} d(x_{m_k}, x_{n_k+1}) = \lim_{n\to\infty} d(x_{m_k+1}, x_{n_k}) = \epsilon. \tag{2.19}$$

As T is triangular α-orbital admissible, we have

$$\alpha(x_{m_k}, x_{n_k}) \geq 1. \tag{2.20}$$

Moreover, since T is an α-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we obtain

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_{m_k}, x_{n_k})d(Tx_{m_k}, Tx_{n_k}), M(x_{m_k}, x_{n_k}) + LN(x_{m_k}, x_{n_k})) \\ &= \zeta(\alpha(x_{m_k}, x_{n_k})d(x_{m_k+1}, x_{n_k+1}), M(x_{m_k}, x_{n_k}) + LN(x_{m_k}, x_{n_k})) \\ &< M(x_{m_k}, x_{n_k}) + LN(x_{m_k}, x_{n_k}) - \alpha(x_{m_k}, x_{n_k})d(x_{m_k+1}, x_{n_k+1}). \end{aligned} \tag{2.21}$$

Hence

$$\begin{aligned} 0 < d(x_{m_k+1}, x_{n_k+1}) &\leq \alpha(x_{m_k}, x_{n_k})d(x_{m_k+1}, x_{n_k+1}) \\ &< M(x_{m_k}, x_{n_k}) + LN(x_{m_k}, x_{n_k}) \end{aligned} \tag{2.22}$$

for all $k \geq n_1$. Where

$$\begin{aligned} M(x_{m_k}, x_{n_k}) &= \max\{d(x_{m_k}, x_{n_k}), \frac{d(x_{m_k}, Tx_{m_k}) + d(x_{n_k}, Tx_{n_k})}{2}, \\ &\qquad \frac{d(x_{m_k}, Tx_{n_k}) + d(x_{n_k}, Tx_{m_k})}{2}\} \\ &= \max\{d(x_{m_k}, x_{n_k}), \frac{d(x_{m_k}, x_{m_k+1}) + d(x_{n_k}, x_{n_k+1})}{2}, \\ &\qquad \frac{d(x_{m_k}, x_{n_k+1}) + d(x_{n_k}, x_{m_k+1})}{2}\} \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} &N(x_{m_k}, x_{n_k}) \\ &\quad = \min\{d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Tx_{n_k}), d(x_{m_k}, Tx_{n_k}), d(x_{n_k}, Tx_{m_k})\} \\ &\quad = \min\{d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{n_k+1}), d(x_{n_k}, x_{m_k+1})\} \end{aligned} \tag{2.24}$$

Taking the limit as $k \to \infty$ in (2.23)-(2.24), using (2.8), (2.15), (2.18) and (2.19) we get

$$\lim_{k\to\infty} M(x_{m_k}, x_{n_k}) = \epsilon \tag{2.25}$$

and

$$\lim_{k\to\infty} N(x_{m_k}, x_{n_k}) = 0 \tag{2.26}$$

From (2.22), (2.25) and (2.26), we derive that
$\mu_n = \alpha(x_{m_k}, x_{n_k})d(x_{m_k+1}, x_{n_k+1}) \to \epsilon$ as $\nu_n = M(x_{m_k}, x_{n_k}) + LN(x_{m_k}, x_{n_k}) \to \epsilon$, therefore by (S_3), we get

$$0 \leq \lim_{k\to\infty} \sup \zeta(\alpha(x_{m_k}, x_{n_k})d(x_{m_k+1}, x_{n_k+1}), M(x_{m_k}, x_{n_k}) + LN(x_{m_k}, x_{n_k}) < 0$$

Which is a contradiction. It follows that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Therefore, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0 \tag{2.27}$$

Furthermore, by the continuity of T, we obtain that

$$\lim_{n\to\infty} d(x_{n+1}, Tz) = \lim_{n\to\infty} d(Tx_n, Tz) = 0. \tag{2.28}$$

Taking into account (2.27), (2.28) and the uniqueness of the limit, we deduce that z is a fixed point of T and $Tz = z$.

□

In the next theorem, we substitute the continuity of T by another condition.

Theorem 2.2.2. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be an α-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ satisfying the following conditions:*

- **(i)** *T is triangular α-orbital admissible;*
- **(ii)** *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
- **(iii)** *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

Then there exists $z \in X$ such that $Tz = z$.

Proof. Following the lines of the proof of Theorem 2.2.1, we obtain that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ is a Cauchy sequence in X. Since (X, d) is complete, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. By (2.2) and the condition **(iii)**, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z) \geq 1$ for all $k \in \mathbb{N}$. Using (2.1), we obtain that

$$
\begin{aligned}
0 &\leq \zeta(\alpha(x_{n(k)}, z)d(Tx_{n(k)}, Tz), M(x_{n(k)}, z) + LN(x_{n(k)}, z)) \\
&= \zeta(\alpha(x_{n(k)}, z)d(x_{n(k)+1}, Tz), M(x_{n(k)}, z) + LN(x_{n(k)}, z)) \\
&< M(x_{n(k)}, z) + LN(x_{n(k)}, z) - \alpha(x_{n(k)}, z)d(x_{n(k)+1}, Tz).
\end{aligned}
$$

Hence

$$
\begin{aligned}
d(x_{n(k)+1}, Tz) &\leq \alpha(x_{n(k)}, z)d(x_{n(k)+1}, Tz) \\
&< M(x_{n(k)}, z) + LN(x_{n(k)}, z)
\end{aligned}
\tag{2.29}
$$

where

$$
\begin{aligned}
M(x_{n(k)}, z) &= \max\{(d(x_{n(k)}, z), \frac{d(x_{n(k)}, x_{n(k)+1}) + d(z, Tz)}{2}, \\
&\qquad \frac{d(x_{n(k)}, Tz) + d(z, x_{n(k)+1})}{2}\} \\
N(x_{n(k)}, z) &= \min\{(d(x_{n(k)}, x_{n(k)+1}), d(z, Tz), d(x_{n(k)}, Tz), d(z, x_{n(k)+1})\}
\end{aligned}
$$

Letting $k \to \infty$ in the above equalities, we get that

$$
\begin{aligned}
\lim_{k\to\infty} M(x_{n(k)}, z) &= \frac{d(z, Tz)}{2} \\
\lim_{k\to\infty} N(x_{n(k)}, z) &= 0
\end{aligned}
\tag{2.30}
$$

Suppose that $d(z, Tz) > 0$. From (2.29), we derive that

$$
d(x_{n(k)+1}, Tz) < M(x_{n(k)}, z) + LN(x_{n(k)}, z)
$$

Now, letting $k \to \infty$ in the above inequality, taking into account (2.30), we obtain that

$$
d(z, Tz) \leq \frac{d(z, Tz)}{2}
$$

a contradiction and hence $d(z, Tz) = 0$, that is $z = Tz$.

□

To ensure the uniqueness of a fixed point of a α-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we shall consider the following condition:

(**C**) For all $x, y \in \text{Fix(T)}$, we have $\alpha(x, y) \geq 1$.
where Fix(T) denotes the set of fixed points of T.

Theorem 2.2.3. *Adding condition* (**C**) *to the hypotheses of Theorem* 2.2.1 (*resp. Theorem* 2.2.2*, we obtain the uniqueness of the fixed point of* T*.*

Proof. We argue by contradiction, suppose that there exist $z, z^* \in X$ such that $z = Tz$ and $z^* = Tz^*$ with $z \neq z^*$. From assumption (**C**), we have

$$\alpha(z, z^*) \geq 1. \tag{2.31}$$

Therefore, it follows from equation (2.1) and (S_2), that

$$\begin{aligned} 0 &\leq \zeta(\alpha(z, z^*)d(Tz, Tz^*), M(z, z^*) + LN(z, z^*)) \\ &= \zeta(\alpha(z, z^*)d(z, z^*), M(z, z^*) + LN(z, z^*)) \\ &< M(z, z^*) + LN(z, z^*) - \alpha(z, z^*)d(z, z^*). \end{aligned} \tag{2.32}$$

Where

$$\begin{aligned} M(z, z^*) &= \max\{d(z, z^*), \frac{d(z, Tz) + d(z^*, Tz^*)}{2}, \frac{d(z, Tz^*) + d(z^*, Tz)}{2}\} \\ &= \max\{d(z, z^*), \frac{d(z, z) + d(z^*, z^*)}{2}, \frac{d(z, z^*) + d(z^*, z)}{2}\} \\ &= d(z, z^*) \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} N(z, z^*) &= \min\{d(z, Tz), d(z^*, Tz^*), d(z, Tz^*), d(z^*, Tz)\} \\ &= \min\{d(z, z), d(z^*, z^*), d(z, z^*), d(z^*, z)\} \\ &= 0 \end{aligned} \tag{2.34}$$

From (2.32), together with (2.33) and (2.34) we deduce that

$$0 < d(z, z^*) - \alpha(z, z^*)d(z, z^*)$$

Using (2.31) it follows that

$$d(z, z^*) \leq \alpha(z, z^*)d(z, z^*) < d(z, z^*) \tag{2.35}$$

Which is a contradiction. Hence $z = z^*$.

□

2.3 Consequences and Corollaries

2.3.1 Fixed point results in metric spaces

In this section, we will show that several known fixed point results can be easily derived from our obtained results.

Corollary 2.3.1. *[24] Let (X, d) be a complete metric space. Suppose that $T : X \longrightarrow X$ is a generalized α-ψ-contractive mapping and satisfies the following conditions:*

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) either, T is continuous, or

(iii)′ if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then there exists $u \in X$ such that $Tu = u$.

Proof. Taking $L = 0$ and a simulation function $\zeta_{EB} : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ defined by $\zeta_{EB}(t, s) = \psi(s) - t$ for all $s, t \in [0, \infty)$ where $\psi \in \Psi_1$, in Theorem 2.2.3, we obtain that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)), \text{ for all } x, y \in X.$$

The mapping T is an α-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta_{EB} \in \mathcal{Z}$ and the conclusion follows. □

Corollary 2.3.2. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a given mapping. Suppose that there exists a function $\psi \in \Psi_1$ such that*

$$d(Tx, Ty) \leq \psi(M(x, y)), \text{ for all } x, y \in X.$$

Then T has a unique fixed point.

Proof. It suffices to choose the mapping $\alpha : X \times X \longrightarrow [0, \infty)$ such that $\alpha(x, y) = 1$, for all $x, y \in X$ and $L = 0$ with $\zeta = \zeta_{EB}$ in Theorem 2.2.3. □

Corollary 2.3.3. (*V.Berinde [82]*) *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a given mapping. Suppose that there exists a function $\psi \in \Psi_1$ such that*

$$d(Tx, Ty) \leq \psi(d(x, y)), \text{ for all } x, y \in X.$$

Then T has a unique fixed point.

Corollary 2.3.4. (*Ćirić [46]*) *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a given mapping. Suppose that there exists a constant $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leq k \max\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\}$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. It is enough to choose the mapping $\alpha : X \times X \longrightarrow [0, \infty)$ such that $\alpha(x, y) = 1$, for all $x, y \in X$ and $L = 0$ with $\zeta_C(t, s) = ks - t$ in Theorem 4. $\square$

Corollary 2.3.5. (*Hardy and Rogers [39]*) *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a given mapping. Suppose that there exists a constants $\beta, \gamma, \eta \geq 0$ with $\beta + 2\gamma + 2\eta \in (0, 1)$ such that*

$$d(Tx, Ty) \leq \beta d(x, y) + \gamma[d(x, Tx) + d(y, Ty)] + \eta[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$. Then T has a unique fixed point.

Corollary 2.3.6. (*Banach Contraction Principle [8]*) *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a given mapping. Suppose that there exists a constant $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y) \textit{ for all } x, y \in X$$

Then T has a unique fixed point.

Corollary 2.3.7. (*Kannan [62]*) *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a given mapping. Suppose that there exists a constant $k \in (0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \textit{ for all } x, y \in X$$

Then T has a unique fixed point.

Corollary 2.3.8. (*Chatterjea [70]*) *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a given mapping. Suppose that there exists a constant $k \in (0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)] \textit{ for all } x, y \in X$$

Then T has a unique fixed point.

2.3.2 Fixed point results in metric spaces endowed with a partial order

The trend of studying the existence of fixed points in metric space endowed with a partial order was initiated by Turnici [55]. In 2003, Ran and Reurings [1] proved the Banach contraction principle in the framework of partially ordered sets and proposed some applications of the obtained result to matrix equations. In this regard, several generalizations and extension have been appeared (see, e.g. [65],[33],[47]and [4]).

In this direction, we will illustrate that various existing results and new consequences in a metric space equipped with a partial order can be derived easily by using the same approach and technics discussed in our obtained results. First, we shall collect some basic concepts.

Let $(X, \preceq)$ be partially ordered set and $T : X \longrightarrow X$ be a given self-mapping on $(X, \preceq)$. T is called nondecreasing with respect to $\preceq$ if

$$x, y \in X, x \preceq y \Rightarrow Tx \preceq Ty.$$

In similar way, we say that a sequence $\{x_n\} \subset X$ is nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. if we further assume that d is a metric on X. We say that tripled $(X, \preceq, d)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ for which $x_n \to x$ as $n \longrightarrow \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.

Corollary 2.3.9. *Let $(X, \preceq, d)$ be a partially ordered complete metric space and $T : X \longrightarrow X$ a self-mapping on X. Assume that the following conditions are satisfied:*

- (i) *there exists a simulation function ζ T and a constant $L \geq 0$ such that*

  $$\zeta(d(Tx, Ty), M(x, y) + LN(x, y)) \geq 0 \textit{ for all } x, y \in X \textit{ with } x \preceq y,$$

  where $M(x, y)$ and $N(x, y)$ are defined as in Theorem 2.2.1.

- (ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
- (iii) *either, T is continuous or $(X, \preceq, d)$ is regular.*

Then T has a fixed point. Further, if for all $x, y \in X$ there exists $w \in X$ such that $x \preceq w$ and $y \preceq w$, then T has a unique fixed point.

Proof. Define the function $\alpha : X \times X \longrightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y \text{ or } x \succeq y, \\ 0 & \text{otherwise} \end{cases}$$

Hence, T is an α-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, that is,

$$\zeta(\alpha(x, y)d(Tx, Ty), M(x, y) + LN(x, y)) \geq 0 \text{ for all } x, y \in X,$$

Considering the condition (i), we have $\alpha(x_0, Tx_0) \geq 1$. In account of the monotone property of T, we have

$$\alpha(x_0, Tx_0) \geq 1 \Leftrightarrow x_0 \preceq Tx_0 \Rightarrow Tx_0 \leq T^2x_0 \Leftrightarrow \alpha(Tx_0, T^2x_0) \geq 1.$$

Now, if we assume that T is continuous, thus, T satisfies all conditions of Theorem 2.2.1. Therefore, T has a fixed point. Next, if $(X, \preceq, d)$ is regular, let $\{x_n\}$ be a sequence in X for which $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \longrightarrow \infty$. Since $(X, \preceq, d)$ is regular, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k. It follows from the definition of the function α that $\alpha(x_{n(k)}, x) \geq 1$. Regarding Theorem 2.2.2, we obtain the existence of a fixed point. The uniqueness part of the proof follows directly from the suggested hypothesis. □

Corollary 2.3.10. *[24] Let $(X, \preceq, d)$ be a partially ordered complete metric space and $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Assume that the following conditions are satisfied:*

(i) there exists a function $\psi \in \Psi_1$ such that

$$d(Tx, Ty) \leq \psi(M(x, y)) \text{ for all } x, y \in X \text{ with } x \preceq y,$$

(ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(iii) T is continuous or $(X, \preceq, d)$ is regular.

Then T has a fixed point. Further, if for all $x, y \in X$ there exists $w \in X$ such that $x \preceq w$ and $y \preceq w$, then T has a unique fixed point.

Proof. It follows from 2.3.9 taking $L = 0$ and the simulation function $\zeta : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ defined by $\zeta(t, s) = \psi(s) - t$ for all $s, t \in [0, \infty)$ where $\psi \in \Psi_1$. □

Chapter 3

New Types of Nonlinear Contractions in Fuzzy Metric Spaces

The main focus of this chapter is to unify diverse existing types of fuzzy contractive mappings by developing a new approach that is based on a new type of control functions, namely $\mathcal{FZ}$-simulation functions. We define a new class of nonlinear contractions called $\mathcal{FZ}$-contractions and we provide some illustrative examples of the two new notions. The presented examples show that such a class unifies and generalizes properly several existing concepts in the current literature including the classes of fuzzy contractive mappings of Gregori and Sapena [78], Mihet [19] and Wardowski [20]. We establish some fixed point theorems for such type of contraction mappings in the framework of fuzzy metric spaces. The chapter contains various instances that demonstrate the generality of our findings and the importance of the defined concepts.

3.1 A new class of control functions

The following class of control functions has been introduced in our published paper [72], where we used the term class $\mathcal{FZ}$ instead of the present $\mathcal{FZ}$-simulation functions.

Definition 3.1.1. The function $\xi : (0,1]\times(0,1] \longrightarrow \mathbb{R}$ is said to be a $\mathcal{FZ}$-simulation function, if the following properties hold :

(ξ_1) $\xi(1,1) = 0$,

(ξ_2) $\xi(t,s) < \frac{1}{s} - \frac{1}{t}$ for all $t, s \in (0,1)$,

(ξ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0,1]$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n < 1$ then $\lim_{n\to\infty} \sup \xi(t_n, s_n) < 0$.

The collection of all $\mathcal{FZ}$-simulation functions is denoted by $\mathcal{FZ}$.

Definition 3.1.2. Let $(X, M, *)$ be a fuzzy metric space, $T : X \longrightarrow X$ a mapping and $\xi \in \mathcal{FZ}$. Then T is called an $\mathcal{FZ}$-contraction with respect to ξ if the following condition is satisfied

$$\xi(M(Tx, Ty, t), M(x, y, t)) \geq 0 \text{ for all } x, y \in X, t > 0. \tag{3.1}$$

Example 3.1.3. The class of fuzzy contractive mappings introduced by Gregori and Sapena [78] is a perfect example of $\mathcal{FZ}$-contraction. It can be expressed easily from the above definition by taking the $\mathcal{FZ}$-simulation function as

$$\xi(t, s) = k\left(\frac{1}{s} - 1\right) - \frac{1}{t} + 1 \text{ for all } s, t \in (0, 1].$$

where $k \in (0, 1)$.

Example 3.1.4. The corresponding $\mathcal{FZ}$-simulation function for the fuzzy ψ-contractive mapping is defined by

$$\xi(t, s) = \frac{1}{\psi(s)} - \frac{1}{t} \text{ for all } s, t \in (0, 1] \text{ and } \psi \in \Psi$$

Example 3.1.5. The corresponding $\mathcal{FZ}$-simulation function for the fuzzy $\mathcal{H}$-contractive mapping is defined by

$$\xi(t, s) = \frac{1}{\eta^{-1}(k.\eta(s))} - \frac{1}{t} \text{ for all } s, t \in (0, 1] \text{ and } \eta \in \mathcal{H}$$

Example 3.1.6. Let $\xi_i : (0, 1] \times (0, 1] \longrightarrow \mathbb{R}, i = 1, 2, 3$ be the functions defined by

1. $\xi_1(t, s) = \phi(\frac{1}{s} - 1) - \frac{1}{t} + 1$ for all $s, t \in]0, 1]$, where $\phi : [0, \infty) \to [0, \infty)$ is a right continuous function with $\phi(r) < r$, for all $r > 0$.

2. $\xi_2(t, s) = (\frac{1}{s} - 1) - \int_0^{\frac{1}{t}-1} f(s)ds$ for all $s, t \in]0, 1]$, where $f : [0, \infty) \to [0, \infty)$ such that $\int_0^{\epsilon} f(s)ds > \epsilon$, for all $\epsilon > 0$.

3. $\xi_3(t, s) = (\frac{1}{s} - 1) - \tilde{\phi}(\frac{1}{s} - 1) - (\frac{1}{t} - 1)$, for all $s, t \in]0, 1]$, where $\tilde{\phi} : [0, \infty) \to [0, \infty)$ is a right continuous function with $\tilde{\phi}(r) < r$, for all $r > 0$.

ξ_i for $i = 1, 2, 3$ are $\mathcal{FZ}$-simulation functions.

3.2 A fixed point theorem for $\mathcal{FZ}$-contraction type mappings

Now, we state and prove the fixed point result for this new type of contraction.

Theorem 3.2.1. *Let $(X, M, *)$ be a complete strong fuzzy metric space and $T : X \longrightarrow X$ be a $\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{FZ}$. Then T has a unique fixed point.*

Proof. Let $x_0 \in X$ be any arbitrary point. We construct the Picard sequence $\{x_n\} \in X$ defined by

$$x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}.$$

If there exists some n_0 such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ implies that x_{n_0} is a fixed point of T and it completes the proof. For this reason, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. First, we prove that $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$. Arguing by contradiction, suppose to the contrary that there exists some t_0 such that

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t_0) < 1$$

On account of $(GV2)$, we have that $M(x_n, x_{n+1}, t_0) < 1$ for all $n \in \mathbb{N}$. Setting $x = x_{n-1}$ and $y = x_n$. Applying (3.1) and (ξ_2), we obtain

$$\begin{aligned} 0 &\leq \xi(M(Tx_{n-1}, Tx_n, t_0), M(x_{n-1}, x_n, t_0)) \\ &= \xi(M(x_n, x_{n+1}, t_0), M(x_{n-1}, x_n, t_0)) \\ &< \frac{1}{M(x_{n-1}, x_n, t_0)} - \frac{1}{M(x_n, x_{n+1}, t_0)}. \end{aligned}$$

Hence

$$M(x_{n-1}, x_n, t_0) < M(x_n, x_{n+1}, t_0).$$

Which means that $\{M(x_{n-1}, x_n, t_0), n \in \mathbb{N}\}$ is nondecreasing sequence of positive real numbers. Therefore, there exists $L \leq 1$ such that

$$\lim_{n\to\infty} M(x_{n-1}, x_n, t_0) = L$$

We shall show that $L = 1$, suppose on contrary that $L < 1$. Define $t_n = M(x_n, x_{n+1}, t_0)$ and $s_n = M(x_{n-1}, x_n, t_0)$ and taking into account ξ_3, we derive that

$$0 \leq \lim_{n\to\infty} \sup \xi(M(x_n, x_{n+1}, t_0), M(x_{n-1}, x_n, t_0)) < 0$$

which leads to contradiction. Therefore $L = 1$, that is

$$\lim_{n\to\infty} M(x_{n-1}, x_n, t_0) = 1 \tag{3.2}$$

Next, we show that the sequence $\{x_n\}$ is Cauchy. Reasoning by contradiction, assume that $\{x_n\}$ is not a Cauchy sequence. Consequently, there exists $\epsilon \in (0,1)$, $t_0 > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that n_k is the smallest index exceeding m_k for which $n_k > m_k \geq k$ for all $k \in \mathbb{N}$ and

$$M(x_{m_k}, x_{n_k}, t_0) \leq 1 - \epsilon, \tag{3.3}$$

and

$$M(x_{m_k}, x_{n_k - 1}, t_0) > 1 - \epsilon \tag{3.4}$$

On account of (3.3) and (3.4), the triangular inequality yields

$$\begin{aligned} 1 - \epsilon \geq M(x_{m_k}, x_{n_k}, t_0) &\geq M(x_{m_k}, x_{n_k-1}, t_0) * M(x_{n_k-1}, x_{n_k}, t_0) \\ &\geq (1 - \epsilon) * M(x_{n_k-1}, x_{n_k}, t_0) \end{aligned}$$

Taking limit as $k \to \infty$ and using (3.2), we deduce that

$$\lim_{k \to \infty} M(x_{m_k}, x_{n_k}, t_0) = 1 - \epsilon \tag{3.5}$$

Applying the same reasoning as above, we have

$$\begin{aligned} 1 - \epsilon &\geq M(x_{m_k}, x_{n_k}, t_0) \\ &\geq M(x_{m_k}, x_{m_k-1}, t_0) * M(x_{m_k-1}, x_{n_k-1}, t_0) * M(x_{n_k-1}, x_{n_k}, t_0) \end{aligned}$$

and

$$\begin{aligned} M(x_{m_k-1}, x_{n_k-1}, t_0) &\geq M(x_{m_k}, x_{n_k}, t_0) \\ &\geq M(x_{m_k-1}, x_{m_k}, t_0) * M(x_{m_k}, x_{n_k}, t_0) * M(x_{n_k}, x_{n_k-1}, t_0) \end{aligned}$$

Letting $k \to \infty$ in the last inequalities, we obtain

$$\lim_{k \to \infty} M(x_{m_k-1}, x_{n_k-1}, t_0) = 1 - \epsilon$$

Taking the sequences $\tau_k = M(x_{m_k}, x_{n_k}, t_0)$ and $\delta_k = M(x_{m_k-1}, x_{n_k-1}, t_0)$ for all $k \in \mathbb{N}$. Applying (ξ_3), we derive that

$$0 \leq \lim_{n \to \infty} \sup \xi(M(x_{m_k}, x_{n_k}, t_0, M(x_{m_k-1}, x_{n_k-1}, t_0)) < 0$$

Which is a contradiction. Then, $\{x_n\}$ is a Cauchy sequence in X. The completeness of $(X, M, *)$ ensure the existence of $u \in X$ such that $\lim_{n\to\infty} M(x_n, u, t) = 1$ for all $t > 0$, we shall show that the point u is a fixed point of T. Reasoning by contradiction, suppose that $Tu \neq u$, that is $M(u, Tu, t) < 1$. Applying (3.1) and ξ_2, we get

$$
\begin{aligned}
0 &\leq \lim_{n\to\infty} \sup \xi(M(Tx_n, Tu, t), M(x_n, u, t)) \\
&\leq \lim_{n\to\infty} \sup \xi(M(x_{n+1}, u, t), M(x_n, u, t)) \\
&\leq \lim_{n\to\infty} \sup(\frac{1}{M(x_n, u, t)} - \frac{1}{M(x_{n+1}, u, t)}).
\end{aligned}
$$

Consequently, we have $1 \leq M(u, Tu, t)$, that is $M(u, Tu, t) = 1$. Which is a contradiction, Thus we conclude that u is a fixed point of T.

Finally, we shall show the uniqueness the fixed point of T. We argue by contradiction, suppose that there are two distinct fixed points $u, v \in X$ of the mapping T, then $M(u, v, t) < 1$ for all $t > 0$. Applying (3.1) and ξ_2, we have

$$
\begin{aligned}
0 &\leq \xi(M(Tu, Tv, t), M(u, v, t)) \\
&= \xi(M(u, v, t), M(u, v, t)) \\
&< (\frac{1}{M(u, v, t)} - \frac{1}{M(u, v, t)}) = 0.
\end{aligned}
$$

A contradiction. Therefore, the fixed point of T is unique. This completes the proof. □

Example 3.2.2. Let $X = (0, \infty)$ and $M : X \times X \times (0, \infty) \longrightarrow [0, 1]$ be the fuzzy metric defined by

$$
M(x, y, t) = \frac{\min(x, y)}{\max(x, y)} \text{ for all } t \in (0, \infty), x, y > 0
$$

and denote $a * b = ab$ for all $a, b \in [0, 1]$. $(X, M, *)$ is a complete strong fuzzy metric space [3]. The mapping $T : X \longrightarrow X$ defined by $T(x) = \sqrt{x}$ is a $\mathcal{FZ}$-contraction with respect to the $\mathcal{FZ}$-simulation function

$$
\xi(t, s) = \frac{1}{\sqrt{s}} - \frac{1}{t} \quad \forall t, s \in (0, 1]
$$

Note that , all the condition of the previous Theorem are satisfied and T has a unique fixed point $x = 1 \in X$.

3.3 Consequences

As a result of Theorem 3.2.1, we derive several corollaries in this section, which can be interpreted as generalizations of different results in the literature.

Corollary 3.3.1. *Let $(X, M, *)$ be a complete strong fuzzy metric space $(X, M, *)$ and $T : X \longrightarrow X$ is mapping such that*

$$
\frac{1}{M(Tx, Ty, t)} - 1 \leq k \left(\frac{1}{M(x, y, t)} - 1 \right). \tag{3.6}
$$

for each $x, y \in X$ and $t > 0$, where $k \in (0, 1)$. Then T has unique fixed point.

Proof. Taking into account Example 3.1.3, the desired result follows from Theorem 3.2.1. □

Corollary 3.3.2. *Let $(X, M, *)$ be a complete strong fuzzy metric space $(X, M, *)$, $\psi \in \Psi_2$ and $T : X \longrightarrow X$ is mapping such that*

$$M(Tx, Ty, t) \geq \psi(M(x, y, t)) \text{ for all } x, y \in X, t > 0. \tag{3.7}$$

for each $x, y \in X$. Then T has unique fixed point.

Proof. Taking into account Example 3.1.4, the desired result follows from Theorem 3.2.1. □

Corollary 3.3.3. *Let $(X, M, *)$ be a complete strong fuzzy metric space $(X, M, *)$, $T : X \longrightarrow X$ and $\eta \in \mathcal{H}$ such that*

$$\xi(t, s) = \frac{1}{\eta^{-1}(k.\eta(s))} - \frac{1}{t} \text{ for all } s, t \in (0, 1] \text{ and } \eta \in \mathcal{H}$$

for each $x, y \in X$ and $t > 0$. Then T has unique fixed point.

Proof. In view of Example 3.1.5, the desired result follows from Theorem 3.2.1. □

3.4 Fuzzy φ-fixed point results via extended $\mathcal{FZ}$-simulation functions

Jleli *et al.* [31] recently initiated the idea of φ-fixed points and studied φ-fixed point in metric space, which strengthens the well-known Banach contraction theorem. Afterwards, Sezen *et al.* [54] defined the notion of fuzzy φ-fixed points and proved some existence and uniqueness results of fuzzy φ-fixed point in the setting of fuzzy metric spaces.

Inspired by the works of Melliani and Moussaoui [72] and Sezen *et al.* [54], Hayel *et al.* [32] introduced a novel type of fuzzy contractive mappings named as ($\mathcal{FZ}$,F,φ)-contractive mappings, which unifies different known types of contractions such as Gregori and Sapena [78], Mihet [19], Wardowski [20], Melliani and Moussaoui [72].

In these sections, we introduce the notion of extended $\mathcal{FZ}$-simulation functions, which is the first attempt to enlarge and refine the definition of $\mathcal{FZ}$-simulation functions. We provide some properties for such type of control functions and we prove that the class of extended $\mathcal{FZ}$-simulation functions (denoted also by $\mathcal{FZ}_e$) includes properly that of $\mathcal{FZ}$-simulation functions. With the help of this set of functions, we define a new type of fuzzy contractions mappings termed $(\mathcal{FZ}_e^{\varphi}, F)$-contraction by gathering the ideas of fuzzy φ-fixed point and $\mathcal{FZ}_e$-contraction. Further, we prove some φ-fixed point results and corollaries in complete fuzzy metric spaces. The obtained results, improve, extend and generalize those given in Gregori and Sapena [78], Mihet [19], Wardowski [20], Hayel *et al.* [32], Sezen *et al.* [54].

Firstly, we recollect some relevant definitions and results in this regard. Let $(X, M, *)$ be a fuzzy metric space, $T : X \to X$ be an operator and $\varphi : X \to (0, 1]$ be function. The set of all fixed points of T is denoted by

$$\mathrm{Fix}(T) := \{x \in X : Tx = x\},$$

$\mathcal{O}_\varphi$ will stand for the set of all ones of the function φ, i.e.

$$\mathcal{O}_\varphi := \{x \in X : \varphi(x) = 1\}.$$

Definition 3.4.1. [54] Let X be a non-empty set, $\varphi : X \to (0, 1]$ a given function. An element u is said to be a fuzzy φ-fixed point of the operator $T : X \to X$ if and only if u is a fixed point of T and $\varphi(u) = 1$, that is $u \in \mathrm{Fix}(T) \cap \mathcal{O}_\varphi$.

Lemma 3.4.2. *[38] Let $(X, M, *)$ be a fuzzy metric space and $\{x_n\}$ be sequence in X such that $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t >$. Suppose that $\{x_n\}$ is not a Cauchy sequence, then there exists $\epsilon \in (0, 1)$, $t_0 > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ such that*

$$M(x_{n_k}, x_{m_k}, t_0) \leq 1 - \epsilon,$$

$$and\ M(x_{n_k}, x_{m_k-1}, \frac{t_0}{2}) = 1 - \epsilon.$$

Definition 3.4.3. [32] Let $\mathcal{F}$ be the set of functions $F : (0,1]^3 \to (0,1]$ satisfying the following axioms :

(F_1) $F(u,v,w) \le \min\{u,v\}$, for all $u,v,w \in (0,1]$;

(F_2) $F(u,1,1) = u$, for all $u \in (0,1]$

(F_3) F is continuous.

Example 3.4.4. [32, 54] The following functions $F : (0,1]^3 \to (0,1]$ belong to $\mathcal{F}$:

(1) $F(u,v,w) = u.v.w$, for all $u,v,w \in (0,1]$;

(2) $F(u,v,w) = \min\{u,v\}.w$ for all $u \in (0,1]$;

We point out that the Definition 3.1.1 of $\mathcal{FZ}$-simulation functions was slightly modified in [32]. The authors considered a large family of function ξ as follows:

Definition 3.4.5. Let $\mathcal{Z_M}$ be the set of all functions $\xi : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ satisfying the following:

$(\xi_1)'$ $\xi(t,s) < \frac{1}{s} - \frac{1}{t}$ for all $t,s \in (0,1)$,

$(\xi_2)'$ if $\{t_n\}, \{s_n\}$ are sequences in $(0,1]$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n < 1$ and $s_n < t_n$ then $\lim_{n\to\infty} \sup \xi(t_n, s_n) < 0$.

Remark 3.4.6. [32] Note that $\mathcal{FZ} \subset \mathcal{Z_M}$.

Definition 3.4.7. [32] Let $(X, M, *)$ be a fuzzy metric space, $\varphi : X \to (0,1]$ a given function and $F \in \mathcal{F}$. A mapping $T : X \longrightarrow X$ is said to be an ($\mathcal{FZ}$,F,φ)-contractive mapping with respect to $\xi \in \mathcal{Z_M}$ if the following condition is satisfied

$$\xi(F(M(Tx,Ty,t), \varphi(Tx), \varphi(Ty)), F(M(x,y,t), \varphi(x), \varphi(y))) \ge 0 \tag{3.8}$$

for all $x, y \in X$ and $t > 0$.

Example 3.4.8. [32] Let $X = [0,1]$ and $M : X \times X \times (0,\infty) \longrightarrow [0,1]$ be the fuzzy metric defined by

$$M(x,y,t) = \frac{t}{t + d(x,y)}$$

where $d(x,y) = |x - y|$ for all $x, y \in X$ and $*$ is the product t-norm $a * b = a.b$ for all $a, b \in [0,1]$. Define a mapping $T : X \longrightarrow X$ by

$$Tx = \frac{x}{x+1} \text{ for all } x \in X,$$

and two functions $\varphi : X \longrightarrow (0,1]$ and $F : (0,1]^3 \longrightarrow (0,1]$ by

$$\varphi(x) = 1 \text{ and } F(a,b,c) = a.b.c, \text{ for all } x \in X \text{ and } a,b,c \in (0,1]$$

Obviously, we can see that φ is continuous and $F \in \mathcal{F}$, we have

$$\frac{1}{F(M(Tx,Ty,t),\varphi(Tx),\varphi(Ty))} - 1$$
$$= \frac{|Tx - Ty|}{t}$$
$$= \frac{1}{(x+1)(y+1)} \frac{|x-y|}{t}$$

and

$$\frac{1}{F(M(Tx,Ty,t),\varphi(Tx),\varphi(Ty))} - 1$$
$$= \frac{|x-y|}{t}$$

In particular, for each $k \in [\frac{1}{2}, 1)$ we obtain

$$k\left(\frac{1}{F(M(x,y,t),\varphi(x),\varphi(y))} - 1\right) - \left(\frac{1}{F(M(Tx,Ty,t),\varphi(Tx),\varphi(Ty))} - 1\right)$$
$$= \left(k - \frac{1}{(x+1)(y+1)}\right)\frac{|x-y|}{t} \geq 0.$$

Therefore, T is an ($\mathcal{FZ}$,F,φ)-contractive mapping with respect to $\xi(t,s) = k\left(\frac{1}{s} - 1\right) - \frac{1}{t} + 1$.

Theorem 3.4.9. *[32] Let $(X, M, *)$ be a complete fuzzy metric space, $\varphi : X \to (0,1]$ a continuous function and $F \in \mathcal{F}$. If $T : X \longrightarrow X$ an ($\mathcal{FZ}$,F,φ)-contractive mapping with respect to $\xi \in \mathcal{Z}_{\mathcal{M}}$. Then $Fix(T) \subseteq \mathcal{O}_\varphi$ and T has a unique fixed point.*

Example 3.4.10. [32] Let $X = (0,\infty)$ and M be a fuzzy set on $X \times X \times (0,\infty)$ defined by $M(x,y,t) = \frac{\min(x,y)}{\max(x,y)}$ for all $t \in (0,\infty), x, y > 0$ and $*$ the product t-norm. Then $(X, M, *)$ is a complete non-Archimedean fuzzy metric space. Define the mapping $T : X \longrightarrow X$ by

$$Tx = \begin{cases} \sqrt{x} & \text{if } x \in (0,1), \\ 1 & \text{if } x \in [1,\infty) \end{cases}$$

and two functions $\varphi : X \longrightarrow (0,1]$ and $F : (0,1]^3 \longrightarrow (0,1]$ by

$$\varphi(x) = \begin{cases} x & \text{if } x \in (0,1), \\ 1 & \text{if } x \in [1,\infty). \end{cases}$$

By choosing

$$\xi(t,s) = \begin{cases} 1 & \text{if } (t,s) = (1,1), \\ \frac{1}{\sqrt{s}} - \frac{1}{t} & \text{otherwise .} \end{cases}$$

T is an ($\mathcal{FZ}$,F,φ)-contractive mapping with respect to ξ.

3.5 Extended $\mathcal{FZ}$-simulation functions

Following this direction, we introduce the notion of extended $\mathcal{FZ}$-simulation functions in order to enlarge, refine and extend the concept of $\mathcal{FZ}$-simulation functions.

Definition 3.5.1. The function $e : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ is said to be an extended $\mathcal{FZ}$-simulation function, if the following properties hold :

($\mathcal{E}1$) $e(t,s) < \frac{1}{s} - \frac{1}{t}$ for all $t, s \in (0,1)$,

($\mathcal{E}2$) if $\{t_n\}, \{s_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = a < 1$ and $s_n < a$ then $\lim_{n\to\infty} \sup e(t_n, s_n) < 0$.

($\mathcal{E}3$) for some sequence $\{t_n\}$ in $(0,1)$ we have

$$\lim_{n\to\infty} t_n = a \in (0,1], e(t_n, a) \geq 0 \Longrightarrow a = 1.$$

We denote the collection of all extended $\mathcal{FZ}$-simulation functions by $\mathcal{FZ}_e$.

Proposition 3.5.2. *Every $\mathcal{FZ}$-simulation function is an $\mathcal{FZ}_e$-simulation function.*

Proof. Let $\xi : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ be an $\mathcal{FZ}$-simulation function. It is easy to show that ξ satisfies ($\mathcal{E}1$) and ($\mathcal{E}2$), we shall prove ($\mathcal{E}3$). Reasoning by contradiction, Let $\{t_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = a \leq 1$ and $\xi(t_n, a) \geq 0$. Assume that $a < 1$, and applying ($\xi3$) with $s_n = a \in (0,1)$, we get

$$\lim_{n\to\infty} \sup \xi(t_n, a) = \lim_{n\to\infty} \sup \xi(t_n, s_n) < 0$$

which yields to a contradiction, hence $a = 1$. □

The converse inclusion is not true, we confirm this by the following example.

Example 3.5.3. Let $e : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ be the function defined by

$$e(t,s) = \begin{cases} 1, & \text{if } t = s = 1 \\ \frac{1}{\psi(s)} - \frac{1}{t}, & \text{otherwise.} \end{cases}$$

Clearly, e is not $\mathcal{FZ}$-simulation function, since $e(1,1) \neq 0$ and ($\xi1$) is not satisfied. Now, we show that e is an extended $\mathcal{FZ}$-simulation function. for all $t, s \in (0,1)$ we have, $e(t,s) = \frac{1}{\psi(s)} - \frac{1}{t} < \frac{1}{s} - \frac{1}{t}$ which proves ($\mathcal{E}1$). if $\{t_n\}, \{s_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = a < 1$ and $s_n < a$, using the fact $\psi(u) > u$, for all $u \in (0,1)$, we have

$$\lim_{n\to\infty} \sup e(t_n, s_n) = \frac{1}{\psi(a)} - \frac{1}{a} < 0$$

Therefore, e satisfies $(\mathcal{E}2)$.
Let $\{t_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = a \in (0,1], e(t_n, a) \geq 0$, we shall prove that $a = 1$. Suppose that $a < 1$, we have

$$e(t_n, a) = \frac{1}{\psi(a)} - \frac{1}{t_n} \geq 0$$

Taking limit as $n \to \infty$, we get that

$$\frac{1}{\psi(a)} - \frac{1}{a} \geq 0$$

Hence, $\psi(a) \leq a$ which contradicts the fact that $\psi(u) > u$, for all $u \in (0,1)$. Therefore $a = 1$ and e is an extended $\mathcal{FZ}$-simulation function.

3.6 Fixed point results for $(\mathcal{FZ}_e^{\varphi}, F)$-contraction

First, we introduce the following concept of $(\mathcal{FZ}_e^{\varphi}, F)$-contraction.

Definition 3.6.1. Let $(X, M, *)$ be a fuzzy metric space, $\varphi : X \to (0,1]$ a given function and $F \in \mathcal{F}$. A mapping T is said to be an $(\mathcal{FZ}_e^{\varphi}, F)$-contraction. if there exists $e \in \mathcal{FZ}_e$ such that

$$e(F(M(Tx,Ty,t), \varphi(Tx), \varphi(Ty)), \mathcal{N}_F^{\varphi}(x,y,t)) \geq 0 \tag{3.9}$$

for all $x, y \in X$ and all $t > 0$. Where

$$\begin{aligned} \mathcal{N}_F^{\varphi}(x,y,t) = \min\{&F(M(x,y,t), \varphi(x), \varphi(y)), F(M(x,Tx,t), \varphi(x), \quad \varphi(Tx)), \\ &F(M(y,Ty,t), \varphi(y), \varphi(Ty))\}. \end{aligned} \tag{3.10}$$

Our first main result is the following one.

Theorem 3.6.2. *Let $(X, M, *)$ be a complete fuzzy metric space, $\varphi : X \to (0,1]$ be a given function and $F \in \mathcal{F}$. Suppose that the following conditions hold :*

(i) *$T : X \longrightarrow X$ is an $(\mathcal{FZ}_e^{\varphi}, F)$-contraction with respect to $e \in \mathcal{FZ}_e$*

(ii) *φ is continuous.*

Then $Fix(T) \subseteq \mathcal{O}_{\varphi}$. Moreover, the operator T has a unique fixed point.

Proof. First, we show that $Fix(T) \subseteq \mathcal{O}_{\varphi}$. Assume that $u \in X$ is a fixed point of T. Applying (3.9) with $x = y = u$, we obtain

$$\begin{aligned} 0 &\leq e(F(M(Tu,Tu,t), \varphi(Tu), \varphi(Tu)), \mathcal{N}_F^{\varphi}(u,u,t)) \\ &= e(F(1, \varphi(u), \varphi(u)), \mathcal{N}_F^{\varphi}(u,u,t)) \end{aligned} \tag{3.11}$$

Where

$$\begin{aligned}\mathcal{N}_F^{\varphi}(u,u,t)) &= \min\{F(M(u,u,t),\varphi(u),\varphi(u)),F(M(u,Tu,t),\varphi(u),\varphi(Tu)),\\ &\qquad F(M(u,Tu,t),\varphi(u),\varphi(Tu))\}\\ &= \min\{F(1,\varphi(u),\varphi(u)),F(1,\varphi(u),\varphi(u)),F(1,\varphi(u),\varphi(u))\}\\ &= F(1,\varphi(u),\varphi(u))\end{aligned}$$

We claim that $F(1,\varphi(u),\varphi(u))=1$. Suppose, on the contrary, that $F(1,\varphi(u),\varphi(u))<1$. Regarding $(\mathcal{E}1)$, the inequality (3.11) yields that

$$\begin{aligned}0 &\le e(F(M(Tu,Tu,t),\varphi(Tu),\varphi(Tu)),\mathcal{N}_F^{\varphi}(u,u,t))\\ &= e(F(1,\varphi(u),\varphi(u)),\mathcal{N}_F^{\varphi}(u,u,t))\\ &< \frac{1}{\mathcal{N}_F^{\varphi}(u,u,t))}-\frac{1}{F(1,\varphi(u),\varphi(u))}\\ &= \frac{1}{F(1,\varphi(u),\varphi(u))}-\frac{1}{F(1,\varphi(u),\varphi(u))}\\ &= 0\end{aligned}$$

Which is a contradiction. Then

$$F(1,\varphi(u),\varphi(u))=1$$

From (F_1), we deduce that

$$F(1,\varphi(u),\varphi(u))=1\le\min\{1,\varphi(u)\}\le\varphi(u)$$

Which means that $\varphi(u)=1$, and hence, $u\in\mathcal{O}_{\varphi}$, and so

$$\mathrm{Fix}(T)\subseteq\mathcal{O}_{\varphi}.$$

Next, let $x_0\in X$ be an arbitrary point and $\{x_n\}$ be the Picard sequence defined by

$$x_n=T^nx_0,n\in\mathbb{N}$$

If there exists some $m\in\mathbb{N}$ such that $x_m=x_{m+1}$, then x_m is fixed point of T and it completes the proof. For this reason, assume that $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}$, which means that $M(x_n,x_{n+1},t)<1$ for all $t>0$.

If there exists some $k_0\in\mathbb{N}$ such that $F(M(x_{k_0},x_{k_0+1},t),\varphi(x_{k_0}),\varphi(x_{k_0+1}))=1$, then we could deduce from condition (F_1) that

$$F(M(x_{k_0},x_{k_0+1},t),\varphi(x_{k_0}),\varphi(x_{k_0+1}))\le\min\{M(x_{k_0},x_{k_0+1},t),\varphi(x_{k_0})\}$$

$$\le M(x_{k_0},x_{k_0+1},t)<1$$

Which is a contradiction. As consequence,

$$F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})) < 1 \text{ for all } n \in \mathbb{N}$$

Since T is an $(\mathcal{FZ}_e^{\varphi}, F)$-contraction with respect to $e \in \mathcal{FZ}_e$, we have

$$0 \leq e(F(M(Tx_n, Tx_{n+1}, t), \varphi(Tx_n), \varphi(Tx_{n+1})), \mathcal{N}_F^{\varphi}(x_n, x_{n+1}, t)) \tag{3.12}$$

Now, we define $\vartheta_n = F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})) < 1$, $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathcal{N}_F^{\varphi}(x_n, x_{n+1}, t)) &= \min\{F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})), \\
&\quad F(M(x_n, Tx_n, t), \varphi(x_n), \varphi(Tx_n)), \\
&\qquad F(M(x_{n+1}, Tx_{n+1}, t), \varphi(x_{n+1}), \varphi(Tx_{n+1}))\} \\
&= \min\{F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1}), \\
&\quad F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1}), \\
&\qquad F(M(x_{n+1}, x_{n+2}, t), \varphi(x_{n+1}), \varphi(x_{n+2})) \\
&= \min\{\vartheta_n, \vartheta_n, \vartheta_{n+1}\} \\
&= \min\{\vartheta_n, \vartheta_{n+1}\} < 1.
\end{aligned}$$

Regarding $(\mathcal{E}1)$, the inequality (3.12) yields that

$$\begin{aligned}
0 &\leq e(F(M(Tx_n, Tx_{n+1}, t), \varphi(Tx_n), \varphi(Tx_{n+1})), \mathcal{N}_F^{\varphi}(x_n, x_{n+1}, t)) \\
&= e(F(M(x_{n+1}, x_{n+2}, t), \varphi(x_{n+1}), \varphi(x_{n+2})), \min\{\vartheta_n, \vartheta_{n+1}\}) \\
&= e(\vartheta_{n+1}, \min\{\vartheta_n, \vartheta_{n+1}\}) \\
&< \frac{1}{\min\{\vartheta_n, \vartheta_{n+1}\})} - \frac{1}{\vartheta_{n+1}}
\end{aligned}$$

It follows that

$$\min\{\vartheta_n, \vartheta_{n+1}\} < \vartheta_{n+1}$$

Therefore $\vartheta_n < \vartheta_{n+1}$. Then, it follows that the sequence $\{\vartheta_n\}$ is a nondecreasing of positive real numbers in $(0, 1]$. Consequently, there exists $l(t) \leq 1$ such that $\lim_{n\to\infty} F_n = l(t) \geq 1$ for all $t > 0$. We shall prove that $l(t) = 1$ using the method of Reductio ad Absurdum. On the contrary, we assume that $l(t) < 1$ for some $t_0 > 0$. Denote $\tau_n = \vartheta_{n+1}$ and $\delta_n = \min\{\vartheta_n, \vartheta_{n+1}\}$, we have

$$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \delta_n = l(t)$$

Since $\{\delta_n\}$ is strictly nondecreasing we have $\delta_n < l(t)$. Regarding $(\mathcal{E}2)$, we deduce that

$$\lim_{n\to\infty} \sup e(\tau_n, \delta_n) < 0,$$

which is in contradiction with

$$e(F_{n+1}, \min\{\vartheta_n, \vartheta_{n+1}\}) = e(\tau_n, \delta_n) \geq 0, \text{for all } n \in \mathbb{N}$$

Accordingly, we obtain that

$$\lim_{n\to\infty} \vartheta_n = F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})) = 1 \tag{3.13}$$

Due to (F_1), we have

$$\begin{aligned} F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})) &\leq \min\{M(x_n, x_{n+1}, t), \varphi(x_{n+1}))\} \\ &\leq \varphi(x_{n+1}) \end{aligned}$$

and

$$\begin{aligned} F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})) &\leq \min\{M(x_n, x_{n+1}, t), \varphi(x_{n+1}))\} \\ &\leq M(x_n, x_{n+1}, t) \end{aligned}$$

Taking $n \to \infty$ and keeping (3.13) in mind, we obtain

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1, \text{ for all } t > 0. \tag{3.14}$$

$$\text{and } \lim_{n\to\infty} \varphi(x_n) = 1. \tag{3.15}$$

Next, we show that $\{x_n\}$ is Cauchy sequence in X. Arguing by contradiction, we assume that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $\epsilon \in (0, 1)$, $t_0 > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ such that

$$M(x_{n_k}, x_{m_k}, t_0) < 1 - \epsilon. \tag{3.16}$$

Taking in account Lemma 1.5.4, we have

$$M(x_{n_k}, x_{m_k}, \frac{t_0}{2}) < 1 - \epsilon. \tag{3.17}$$

By choosing m_k as the smallest index satisfying (3.17), we get

$$M(x_{n_k}, x_{m_k-1}, \frac{t_0}{2}) \geq 1 - \epsilon. \tag{3.18}$$

On account of (3.16) and (3.18), the triangular inequality yields

$$\begin{aligned} 1 - \epsilon &> M(x_{n_k}, x_{m_k}, t_0) \\ &\geq M(x_{n_k}, x_{m_k-1}, \frac{t_0}{2}) * M(x_{m_k-1}, x_{m_k}, \frac{t_0}{2}) \\ &\geq (1 - \epsilon) * M(x_{m_k-1}, x_{m_k}, \frac{t_0}{2}) \end{aligned}$$

Taking the limit of both sides as $k \to \infty$, using (3.14) and (T_3), we derive that

$$\lim_{n\to\infty} M(x_{n_k}, x_{m_k}, t_0) = 1 - \epsilon \tag{3.19}$$

Since T is an $(\mathcal{FZ}_e^{\varphi}, F)$-contraction with respect to $e \in \mathcal{FZ}_e$, we have

$$\begin{aligned}
0 &\leq e(F(M(Tx_{n_k-1}, Tx_{m_k-1}, t_0), \varphi(Tx_{n_k-1}), \varphi(Tx_{m_k-1})), \mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0)) \\
&= e(F(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k})), \mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0)) \\
&< \frac{1}{\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0)} - \frac{1}{F(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k}))}.
\end{aligned}$$

Which implies that

$$\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0) \quad < F(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k})) \tag{3.20}$$

Where

$$\begin{aligned}
\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0) &= \min\{F(M(x_{n_k-1}, x_{m_k-1}, t_0), \varphi(x_{n_k-1}), \varphi(x_{m_k-1})), \\
&\qquad F(M(x_{n_k-1}, Tx_{n_k-1}, t_0), \varphi(x_{n_k-1}), \varphi(Tx_{n_k-1})), \\
&\qquad\quad F(M(x_{m_k-1}, Tx_{m_k-1}, t_0), \varphi(x_{m_k-1}), \varphi(Tx_{m_k-1}))\} \\
&= \min\{F(M(x_{n_k-1}, x_{m_k-1}, t_0), \varphi(x_{n_k-1}), \varphi(x_{m_k-1}), \\
&\qquad F(M(x_{n_k-1}, x_{n_k}, t_0), \varphi(x_{n_k-1}), \varphi(x_{n_k}), \\
&\qquad\quad F(M(x_{m_k-1}, x_{m_k}, t_0), \varphi(x_{m_k-1}), \varphi(x_{m_k}))
\end{aligned}$$

Now, if $\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0) = F(M(x_{m_k-1}, x_{m_k}, t_0), \varphi(x_{m_k-1}), \varphi(x_{m_k}))$, hence (3.20) becomes

$$F(M(x_{m_k-1}, x_{m_k}, t_0), \varphi(x_{m_k-1}), \varphi(x_{m_k})) < F(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k}))$$

Passing to the limit as $k \to \infty$ in the above inequality, using (3.14), (3.15), (F_2) and taking into account the continuity of F yields

$$\begin{aligned}
F(1,1,1) = 1 &\leq F(\lim_{k\to\infty} M(x_{n_k}, x_{m_k}, t_0), 1, 1) \\
&= \lim_{k\to\infty} M(x_{n_k}, x_{m_k}, t_0)
\end{aligned}$$

and then $\lim_{k\to\infty} M(x_{n_k}, x_{m_k}, t_0) = 1$, which contradicts the inequality (3.16). In similar way, if we consider

$$\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0)) = F(M(x_{n_k-1}, x_{n_k}, t_0), \varphi(x_{n_k-1}), \varphi(x_{n_k}))$$

Then, again we obtain a contradiction. Therefore, we must have

$$\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0) = F(M(x_{n_k-1}, x_{m_k-1}, t_0), \varphi(x_{n_k-1}), \varphi(x_{m_k-1})).$$

Then, (3.20) gives rise to

$$\begin{aligned}\lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0) &\leq F(\lim_{k\to\infty} M(x_{n_k}, x_{m_k}, t_0), 1, 1)\\ &= \lim_{k\to\infty} M(x_{n_k}, x_{m_k}, t_0)\\ &= 1-\epsilon \end{aligned} \tag{3.21}$$

By the triangular inequality, we have

$$M(x_{n_k-1}, x_{m_k-1}, t_0) \quad \geq M(x_{n_k-1}, x_{n_k}, \tfrac{t_0}{2}) * M(x_{n_k}, x_{m_k-1}, \tfrac{t_0}{2})$$

Letting $k \to \infty$ in the last inequality and using (3.14) and (3.14), we get

$$\lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0) \quad \geq 1 * (1-\epsilon) = 1-\epsilon. \tag{3.22}$$

From (3.21) and (3.22), we derive that

$$\lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0) = 1-\epsilon. \tag{3.23}$$

On the other hand, by (3.14), (3.18) and regarding (F_2), we have

$$\begin{aligned}\lim_{k\to\infty} F(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k})) &= F(1-\epsilon, 1, 1)\\ &= 1-\epsilon\end{aligned}$$

In particular, it follows from (3.20),(F_1) and (3.16), that

$$\begin{aligned}\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0) &< F(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k})),\\ &\leq \min\{M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k})\}\\ &\leq M(x_{n_k}, x_{m_k}, t_0)\\ &< 1-\epsilon\end{aligned}$$

Take the sequences $\alpha_k = F(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k}))$ and $\beta_k = \mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0))$ for all $k \in \mathbb{N}$. From the above observations, we conclude that $\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = 1-\epsilon$ and $\beta_k < 1-\epsilon$. Thus, we can Apply the axiom $(\mathcal{E}2)$ to these sequences, as consequence

$$0 \leq \lim_{k\to\infty} \sup e(\alpha_k, \beta_k) < 0.$$

Which is a contradiction. Thus, we deduce that $\{x_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is a complete fuzzy metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} M(x_n, u, t) = 1. \tag{3.24}$$

Due to continuity of φ, (8) and (17), we derive that

$$\varphi(u) = 1 \tag{3.25}$$

Therefore, $u \in \mathcal{O}_\varphi$. Next, we shall show that u is a fuzzy φ-fixed point of T. Arguing by contradiction. Suppose that $M(u, Tu, t) < 1$ for some $t > 0$. Define the sequences

$$\mu = F(M(u,Tu,t), 1, \varphi(Tu)) \ , \ \acute{\alpha}_n = F(M(x_{n+1},Tu,t), \varphi(x_{n+1}), \varphi(Tu))$$
$$\text{and } \acute{\beta}_n = \mathcal{N}_F^\varphi(x_n, u, t)) \text{ for all } n \in \mathbb{N}$$

Using (F_1), we obtain

$$\begin{aligned} \mu = F(M(u,Tu,t), 1, \varphi(Tu)) &\leq \min\{M(u,Tu,t), 1\} \\ &= M(u,Tu,t) \\ &< 1 \end{aligned} \tag{3.26}$$

Taking the limit as $n \to \infty$ and using the continuity of F

$$\begin{aligned} \lim_{n\to\infty} \acute{\alpha}_n &= F(M(x_{n+1},Tu,t), \varphi(x_{n+1}), \varphi(Tu)) \\ &= F(M(u,Tu,t), 1, \varphi(Tu)) \\ &= \mu \end{aligned}$$

On the other hand,

$$\begin{aligned} \acute{\beta}_n = \mathcal{N}_F^\varphi(x_n, u, t)) &= \min\{F(M(x_n,u,t), \varphi(x_n), \varphi(u), \\ &\qquad F(M(x_n,Tx_n,t), \varphi(x_n), \varphi(Tx_n)), \\ &\qquad\quad F(M(u,Tu,t), \varphi(u), \varphi(Tu))\} \\ &= \min\{F(M(x_n,u,t), \varphi(x_n), 1), \\ &\qquad F(M(x_n,x_{n+1},t), \varphi(x_n), \varphi(x_{n+1}), \\ &\qquad\quad F(M(u,Tu,t), \varphi(u), \varphi(Tu))\}. \end{aligned}$$

As F is continuous, we have

$$\lim_{n\to\infty} F(M(x_n,u,t), \varphi(x_n), 1) = F(1,1,1) = 1$$
$$\lim_{n\to\infty} F(M(x_n,x_{n+1},t), \varphi(x_n), \varphi(x_{n+1})) = F(1,1,1) = 1$$

Particulary, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\acute{\beta}_n = F((M(u,Tu,t), 1, \varphi(Tu)) = \mu$$

and $\{\acute{\alpha}_n\}_{n\geq n_0} \subset (0,1]$ is a sequence converging to $\mu < 1$, such that for all $n \geq n_0$,

$$\begin{aligned} e(\acute{\alpha}_n, \mu) &= e(\acute{\alpha}_n, \acute{\beta}_n) \\ &= e(F(M(x_{n+1},Tu,t), \varphi(x_{n+1}), \varphi(Tu)), \mathcal{N}_F^\varphi(x_n,u,t))\}) \\ &= e(F(M(Tx_n,Tu,t), \varphi(Tx_n), \varphi(Tu)), \mathcal{N}_F^\varphi(x_n,u,t))\}) \\ &\geq 0. \end{aligned}$$

Regarding $(\mathcal{E}3)$, the last inequality yields that $\mu = 1$. which contradicts (3.26). As a consequence $M(u, Tu, t) = 1$, which means that u is a fuzzy φ-fixed point of T.

As a final step, we shall show the uniqueness of fuzzy φ-fixed point of T. We argue by contradiction, suppose that there are two distinct φ-fixed points $u, v \in X$ of the mapping T. Then $M(u, v, t) < 1$ for all $t > 0$. Since we have $Fix(T) \subseteq \mathcal{O}_\varphi$, it follows that $\varphi(u) = \varphi(v) = 1$. Now, using 3.9, we have

$$0 \leq e(F(M(Tu, Tv, t), \varphi(Tu), \varphi(Tv)), \mathcal{N}_F^\varphi(u, v, t)) \tag{3.27}$$

Where

$$\begin{aligned}
\mathcal{N}_F^\varphi(u, v, t)) &= \min\{F(M(u, v, t), \varphi(u), \varphi(v)), F(M(u, Tu, t), \varphi(u), \varphi(Tu)), \\
&\qquad F(M(v, Tv, t), \varphi(v), \varphi(Tv))\} \\
&= \min\{F(M(u, v, t), 1, 1), F(1, 1, 1), F(1, 1, 1)\} \\
&= F(M(u, v, t), 1, 1) \\
&= M(u, v, t)
\end{aligned}$$

Regarding $(\mathcal{E}1)$, the inequality (3.27) yields that

$$\begin{aligned}
0 &\leq e(F(M(Tu, Tv, t), \varphi(Tu), \varphi(Tv)), \mathcal{N}_F^\varphi(u, v, t)) \\
&= e(F(M(u, v, t), 1, 1), M(u, v, t)) \\
&= e(M(u, v, t), M(u, v, t)) \\
&< \frac{1}{M(u, v, t)} - \frac{1}{M(u, v, t)} = 0
\end{aligned}$$

A contradiction, thus $u = v$. Therefore, the fuzzy φ-fixed point of T is unique. This completes the proof. □

Corollary 3.6.3. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \longrightarrow X$. Suppose that there exists some $e \in \mathcal{FZ}_e$ such that for all $x, y \in X, t > 0$*

$$e(M(Tx, Ty, t), \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\}) \geq 0$$

Then T has a unique fixed point.

Proof. The result follows by defining $F(a, b, c) = a.b.c$ for all $a, b, c \in (0, 1]$ and $\varphi(x) = 1$ for all $x \in X$ in Theorem 3.6.2. □

Corollary 3.6.4. *Let $(X, M, *)$ be a complete fuzzy metric space, and let $T : X \longrightarrow X$ be a given mapping. Suppose that there exists some $\psi \in \Psi$ such that for all $x, y \in X, t > 0$*

$$M(Tx, Ty, t) \geq \psi(\min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\})$$

Then T has a unique fixed point.

Proof. Define $e : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ by $e(t,s) = \frac{1}{\psi(s)} - \frac{1}{t}$ for all $s,t \in (0,1]$, $F(a,b,c) = a.b.c$ for all $a,b,c \in (0,1]$ and $\varphi(x) = 1$ for all $x \in X$. by proposition 1, one can see that e is an extended simulation function. The result follows from Theorem 3.6.2. □

Corollary 3.6.5. *Let $(X, M, *)$ be a complete fuzzy metric space, and let $T : X \longrightarrow X$ be a given mapping and $\eta \in \mathcal{H}$ such that*

$$\eta(F((M(Tx,Ty,t),\varphi(Tx),\varphi(Ty))) \leq k\eta(\mathcal{N}_F^{\varphi}(x,y,t))$$

$x, y \in X, t > 0$ where $k \in (0,1)$. Then T has a unique φ-fixed point.

Proof. The result follows by defining $e : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ by $e(t,s) = \frac{1}{\eta^1(k\eta(s))} - \frac{1}{t}$ for all $s,t \in (0,1]$ in Theorem 3.6.2. □

Corollary 3.6.6. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \longrightarrow X$ be a given mapping. Assume that*

$$\frac{1}{F(M(Tx,Ty,t),\varphi(Tx),\varphi(Ty))} - 1 \leq \phi\left(\frac{1}{\mathcal{N}_F^{\varphi}(x,y,t)} - 1\right)$$

for all $x, y \in X$ and $t > 0$, where $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(t) < t$, for all $t > 0$ and $\phi(0) = 0$. Then T has a unique fuzzy φ-fixed point.

3.7 Best Proximity point results

In this section, some best proximity results in fuzzy metric spaces are deduced from our main results .

Let U and V be two nonempty subsets of a fuzzy metric space $(X, M, *)$ and $T : A \longrightarrow B$ a non-self-mapping.
The following notations will be used in the sequel.

$$U_0 = \{u \in U : M(u,v,t) = M(U,V,t) \text{ for some } v \in V\};$$

$$V_0 = \{v \in V : M(u,v,t) = M(U,V,t) \text{ for some } u \in U\}.$$

where

$$M(U,V,t) = \sup\{M(u,v,t) : u \in U, v \in V\};$$

The set of all best proximity points of non-self-mapping $T : U \longrightarrow V$ will be denoted by

$$B_{est}(T) = \{u \in: M(u,Tu,t) = M(U,V,t)\}.$$

Definition 3.7.1. [54] Let X be a non-empty set, $\varphi : X \to (0, 1]$ a given function and $T : U \longrightarrow V$ a non-self mapping. An element $u^* \in U$ is said to be a fuzzy φ-best proximity point of T if and only if u^* is a best proximity point of T and $\varphi(u^*) = 1$.

Definition 3.7.2. Let U and V be two nonempty closed subsets of a fuzzy metric space $(X, M, *)$. We say that the operator $T : U \to V$ is an $(\mathcal{FZ}_e^{\varphi}, F)$-fuzzy proximal contraction with respect to $e \in \mathcal{FZ}_e$, if there exist a function $\varphi : X \to (0, 1]$, $F \in \mathcal{F}$, such that

$$\begin{cases} M(u, Tx, t) = M(U, V, t) \\ M(v, Ty, t) = M(U, V, t) \end{cases} \Rightarrow e(F(M(u, v, t), \varphi(u), \varphi(v)), \mathcal{N}_F^{\varphi}(x, y, t)) \geq 0 \tag{3.28}$$

for all $u, v, x, y \in U$ and $t > 0$. Where

$$\mathcal{N}_F^{\varphi}(x, y, t) = \min\{F(M(x, y, t), \varphi(x), \varphi(y)), F(M(x, u, t), \varphi(x), \quad \varphi(u)), F(M(y, v, t), \varphi(y), \varphi(v))\}.$$

Theorem 3.7.3. *Let U and V be two nonempty subsets of a complete fuzzy metric space $(X, M, *)$ such that U_0 is nonempty and $\varphi : X \to (0, 1]$, $F \in \mathcal{F}$. Suppose that $T : U \to V$ is an $(\mathcal{FZ}_e^{\varphi}, F)$-fuzzy proximal contraction with respect to $e \in \mathcal{FZ}_e$. Suppose also*

- **(i)** *U_0 is closed with respect to the topology induced by M*
- **(ii)** *$T(U_0) \subseteq V_0$;*
- **(ii)** *φ is continuous*

Then T has a unique fuzzy φ-best proximity point, that is, there exists $x^* \in U$ such that $B_{est}(T) \cap \mathcal{O}_{\varphi} = \{x^*\}$.

Proof. First, we show that $B_{est}(T) \subseteq \mathcal{O}_{\varphi}$. Assume that $\rho \in U$ is a best proximity point of T, which means that $M(\rho, T\rho, t) = M(U, V, t)$. Applying (3.28) with $\sigma = u = v = x = y$, we have

$$0 \leq e(F(1, \varphi(\sigma), \varphi(\sigma)), \mathcal{N}_F^{\varphi}(\sigma, \sigma, t))$$

Where

$$\begin{aligned} \mathcal{N}_F^{\varphi}(\sigma, \sigma, t)) &= \min\{F(M(\sigma, \sigma, t), \varphi(\sigma), \varphi(\sigma)), F(M(\sigma, \sigma, t), \varphi(\sigma), \varphi(\sigma)), \\ &\qquad F(M(\sigma, \sigma, t), \varphi(\sigma), \varphi(\sigma))\} \\ &= F(1, \varphi(\sigma), \varphi(\sigma)) \end{aligned}$$

We shall indicate that $F(1, \varphi(\sigma), \varphi(\sigma)) = 1$. Reasoning by contradiction, Suppose that $F(1, \varphi(\sigma), \varphi(\sigma)) < 1$ and using $(\mathcal{E}1)$, we derive

$$\begin{aligned}
0 &\leq e(F(M(\rho, \rho, t), \varphi(\rho), \varphi(\rho)), \mathcal{N}_F^{\varphi}(\sigma, \sigma, t)) \\
&= e(F(1, \varphi(\sigma), \varphi(\sigma)), \mathcal{N}_F^{\varphi}(\sigma, \sigma, t)) \\
&< \frac{1}{\mathcal{N}_F^{\varphi}(\sigma, \sigma, t)} - \frac{1}{F(1, , \varphi(\sigma), \varphi(\sigma))} \\
&= \frac{1}{F(1, , \varphi(\sigma), \varphi(\sigma))} - \frac{1}{F(1, , \varphi(\sigma), \varphi(\sigma))} \\
&= 0
\end{aligned}$$

Which is a contradiction. Therefore,

$$F(1, \varphi(\sigma), \varphi(\sigma)) = 1$$

By (F_1), we obtain

$$F(1, \varphi(\sigma), \varphi(\sigma)) = 1 \leq \min\{1, \varphi(\sigma)\} \leq \varphi(\sigma)$$

which yields $\varphi(\sigma) = 1$, and then $B_{est}(T) \subseteq \mathcal{O}_{\varphi}$.

Next, let $x_0 \in X$ be an element in U_0. Taking into account that $Tx_0 \in T(U_0) \subseteq V_0$ we can find $x_1 \in U_0$ such that $M(x_1, Tx_0, t) = M(U, V, t)$. Since $Tx_1 \in T(U_0) \subseteq V_0$, so that there exists $x_2 \in U_0$ such that $M(x_2, Tx_1, t) = M(U, V, t)$. Recursively, a sequence $\{x_n\} \subset U_0$ can be constructed as follows

$$M(x_{n+1}, Tx_n, t) = M(U, V, t) \text{ for all } n \in \mathbb{N}. \tag{3.29}$$

If $x_k = x_{k+1}$ for some $k \in \mathbb{N}$, then

$$M(x_k, Tx_k, t) = M(x_{k+1}, Tx_k, t) = M(U, V, t)$$

Therefore, x_k is the required best proximity point and the proof is completed. Due to this reason, for the rest of the proof, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, that is $M(x_n, x_{n+1}, t) < 1$ for all $t > 0$.

Now, if there exists some $n_0 \in \mathbb{N}$ such that $F(M(x_{n_0}, x_{n_0+1}, t), \varphi(x_{n_0}), \varphi(x_{n_0+1})) = 1$, the condition (F_1) yields that

$$\begin{aligned}
1 = F(M(x_{n_0}, x_{n_0+1}, t), \varphi(x_{n_0}), \varphi(x_{n_0+1})) &\leq \min\{M(x_{n_0}, x_{n_0+1}, t), \varphi(x_{n_0})\} \\
&\leq M(x_{n_0}, x_{n_0+1}, t) < 1
\end{aligned}$$

A contradiction. Accordingly, we deduce that

$$F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})) < 1 \text{ for all } n \in \mathbb{N}$$

Next, we denote $\gamma_n = F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})) < 1$, $n \in \mathbb{N}$. We have

$$0 \leq e(F(M(x_{n+1}, x_{n+2}, t), \varphi(x_{n+1}), \varphi(x_{n+2})), \mathcal{N}_F^{\varphi}(x_n, x_{n+1}, t)) \tag{3.30}$$

Where

$$\begin{aligned}
\mathcal{N}_F^{\varphi}(x_n, x_{n+1}, t)) &= \min\{F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})), \\
&\quad F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})), \\
&\qquad F(M(x_{n+1}, x_{n+2}, t), \varphi(x_{n+1}), \varphi(x_{n+2}))\} \\
&= \min\{\gamma_n, \gamma_n, \gamma_{n+1}\} \\
&= \min\{\gamma_n, \gamma_{n+1}\} < 1.
\end{aligned}$$

Using the property $(\mathcal{E}1)$, we deduce that

$$\begin{aligned}
0 &\leq e(F(M(x_{n+1}, x_{n+2}, t), \varphi(x_{n+1}), \varphi(x_{n+2})), \mathcal{N}_F^{\varphi}(x_n, x_{n+1}, t)) \\
&= e(\gamma_{n+1}, \min\{\gamma_n, \gamma_{n+1}\}) \\
&< \frac{1}{\min\{\gamma_n, \gamma_{n+1}\})} - \frac{1}{\gamma_{n+1}}
\end{aligned}$$

which yields $\gamma_n < \gamma_{n+1}$. Therefore, we deduce that $\{\gamma_n\}$ is an increasing sequence of non-negative real numbers in $(0, 1]$. Thus, there exists $h(t) \leq 1$ such that $\lim_{n\to\infty} \gamma_n = h(t) \geq 1$ for all $t > 0$. In particular, as $\{\gamma_n\}$ is strictly increasing, then $h(t) < \gamma_n$. We shall prove that $h(t) = 1$ for all $t > 0$. Suppose, on contrary, that $h(t) < 1$ for some $t > 0$. If we choose the sequences $\varpi_n = \gamma_{n+1}$ and $\theta_n = \min\{\gamma_n, \gamma_{n+1}\}$, we have $\lim_{n\to\infty} \varpi_n = \lim_{n\to\infty} \theta_n = h(t)$ and $\theta_n < h(t)$, by the condition $(\mathcal{E}2)$, we derive that

$$\lim_{n\to\infty} \sup e(\varpi_n, \theta_n) < 0,$$

which contradicts Equation (3.30). Accordingly, we deduce that

$$\lim_{n\to\infty} \gamma_n = F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})) = 1 \text{ for all } t > 0. \tag{3.31}$$

Moreover, using (F_1) we get

$$\begin{aligned}
F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})) &\leq \min\{M(x_n, x_{n+1}, t), \varphi(x_{n+1}))\} \\
&\leq \varphi(x_{n+1})
\end{aligned}$$

and

$$\begin{aligned}
F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})) &\leq \min\{M(x_n, x_{n+1}, t), \varphi(x_{n+1}))\} \\
&\leq M(x_n, x_{n+1}, t)
\end{aligned}$$

Which implies

$$\lim_{n\to\infty} \varphi(x_n) = 1 \text{ and } \lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1, \forall t > 0. \tag{3.32}$$

As a next step, we shall prove that the sequence $\{x_n\}$. Reasoning by contradiction, assume that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $\epsilon \in (0, 1)$,

$t_0 > 0$ and subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, so that for $m_k > n_k \geq k$ we have

$$M(x_{n_k}, x_{m_k}, t_0) < 1 - \epsilon. \tag{3.33}$$

By Lemma 1, we have

$$M(x_{n_k}, x_{m_k}, \frac{t_0}{2}) < 1 - \epsilon. \tag{3.34}$$

If we choose m_k as the least natural number satisfying (3.31), we have

$$M(x_{n_k}, x_{m_k-1}, \frac{t_0}{2}) \geq 1 - \epsilon. \tag{3.35}$$

Taking into account (3.33) and (3.35), we deduce that

$$\begin{aligned} 1 - \epsilon &\geq M(x_{n_k}, x_{m_k}, t_0) \\ &\geq M(x_{n_k}, x_{m_k-1}, \frac{t_0}{2}) * M(x_{m_k-1}, x_{m_k}, \frac{t_0}{2}) \\ &> (1 - \epsilon) * M(x_{m_k-1}, x_{m_k}, \frac{t_0}{2}) \end{aligned}$$

Letting $k \to \infty$ and using (3.30), we get

$$\lim_{n\to\infty} M(x_{n_k}, x_{m_k}, t_0) = 1 - \epsilon \tag{3.36}$$

Let us denote

$$\begin{aligned} r_k &= F(M(x_{n_k}, x_{m_k}, t), \varphi(x_{n_k}), \varphi(x_{m_k})) \text{ and} \\ s_k &= \mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t)) \text{ for all } k \in \mathbb{N} \end{aligned}$$

Since T is an $(\mathcal{FZ}_e^{\varphi}, F)$-fuzzy proximal contraction and

$$M(x_{n_k}, Tx_{n_k-1}, t)) = M(x_{m_k}, Tx_{m_k-1}, t)) = M(U, V, t)$$

for all $k \in \mathbb{N}$. So, by (3.28) we have

$$\begin{aligned} 0 &\leq e(F(M(x_{n_k}, x_{m_k}, t), \varphi(x_{n_k}), \varphi(x_{m_k})), \mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t)) \\ &< \frac{1}{\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t))} - \frac{1}{F(M(x_{n_k}, x_{m_k}, t), \varphi(x_{n_k}), \varphi(x_{m_k}))} \end{aligned}$$

Hence

$$\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t)) < F(M(x_{n_k}, x_{m_k}, t), \varphi(x_{n_k}), \varphi(x_{m_k})) \tag{3.37}$$

Where

$$\mathcal{N}_F^{\varphi}(x_{n_k}, x_{m_k}, t)) = \min\{F(M(x_{n_k-1}, x_{m_k-1}, t), \varphi(x_{n_k-1}), \varphi(x_{m_k-1})),$$
$$F(M(x_{n_k-1}, x_{n_k}, t), \varphi(x_{n_k-1}), \varphi(x_{n_k})),$$
$$F(M(x_{m_k-1}, x_{m_k}, t), \varphi(x_{m_k-1}), \varphi(x_{m_k}))\}$$

By following a similar reasoning to that in the proof of Theorem 3.6.2, one can show that

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0) = 1 - \epsilon \text{ and}$$
$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} F(M(x_{n_k}, x_{m_k}, t), \varphi(x_{n_k}), \varphi(x_{m_k})) = 1 - \epsilon$$

Particularly, it follows from (3.37), (F_1) and (3.33), that

$$\begin{aligned} s_k &< F(M(x_{n_k}, x_{m_k}, t), \varphi(x_{n_k}), \varphi(x_{m_k})) \\ &\leq \min\{M(x_{n_k}, x_{m_k}, t), \varphi(x_{n_k})\} \\ &\leq M(x_{n_k}, x_{m_k}, t) \\ &< 1 - \epsilon \end{aligned}$$

On account of the above observations, we deduce that $\lim_{k\to\infty} r_k = \lim_{k\to\infty} s_k = 1 - \epsilon$ and $s_k < 1 - \epsilon$. Regarding the axiom $\mathcal{E}2$, we obtain

$$0 \leq \lim_{k\to\infty} \sup e(r_k, s_k) < 0.$$

Which is a contradiction. This contradiction proves that $\{x_n\}$ is a Cauchy sequence. Since U_0 is closed subset of the complete fuzzy metric space $(X, M, *)$, there exists $x^* \in U_0$ such that

$$\lim_{n\to\infty} M(x_n, x^*, t) = 1. \tag{3.38}$$

By the continuity of φ, (3.32) and (3.38), we have

$$\varphi(x^*) = 1. \tag{3.39}$$

As $T(U_0) \subseteq V_0$ and $x^* \in U_0$, there exists $\omega \in U_0$ such that

$$\lim_{n\to\infty} M(w, Tx^*, t) = M(U, V, t). \tag{3.40}$$

Now, we shall prove that $x^* = w$, reasoning by contradiction. Suppose that $M(x^*, w, t) < 1$ for some $t > 0$. Define

$$a = F(M(x^*, w, t), 1, \varphi(w)) \ , \ \acute{r}_n = F(M(x_{n+1}, w, t), \varphi(x_{n+1}), \varphi(w))$$

$$\text{and } \acute{s}_n = \mathcal{N}_F^{\varphi}(x_n, x^*, t)) \text{ for all } n \in \mathbb{N}$$

Using (F_1), we obtain

$$\begin{aligned} a = F(M(x^*, w, t), 1, \varphi(w)) &\leq \min\{M(x^*, w, t), 1\} \\ &= M(x^*, w, t) < 1 \end{aligned} \tag{3.41}$$

Taking the limit as $n \to \infty$ and using the continuity of F

$$\begin{aligned} \lim_{n\to\infty} \acute{r}_n &= \lim_{n\to\infty} F(M(x_{n+1}, w, t), \varphi(x_{n+1}), \varphi(w)) \\ &= F(M(x^*, w, t), 1, \varphi(w)) \\ &= a \end{aligned}$$

On the other hand,

$$\begin{aligned} \acute{s}_n = \mathcal{N}_F^{\varphi}(x_n, x^*, t)) &= \min\{F(M(x_n, x^*, t), \varphi(x_n), \varphi(x^*), \\ &\qquad F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})), \\ &\qquad\quad F(M(x^*, w, t), \varphi(x^*), \varphi(w))\} \\ &= \min\{F(M(x_n, x^*, t), \varphi(x_n), 1), \\ &\qquad F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})), \\ &\qquad\quad F(M(x^*, w, t), \varphi(x^*), \varphi(w))\} \end{aligned}$$

Due to the continuity of F, we have

$$\lim_{n\to\infty} F(M(x_n, x^*, t), \varphi(x_n), 1) = F(1, 1, 1) = 1$$
$$\lim_{n\to\infty} F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})) = F(1, 1, 1) = 1$$

As consequence, there exists $n_0 \in \mathbb{N}$ such that

$$\acute{s}_n = F(M(x^*, w, t), 1, \varphi(w)) = a, \quad n \geq n_0.$$

In particular, $\{\acute{r}_n\}_{n\geq n_0} \subset (0, 1]$ is a sequence converging to $a < 1$ and such that for all $n \geq n_0$,

$$e(\acute{r}_n, a) = e(\acute{r}_n, \acute{s}_n) = e(F(M(x_{n+1}, w, t), \varphi(x_{n+1}), \varphi(w)), \mathcal{N}_F^{\varphi}(x_n, x^*, t))) \geq 0$$

by means of (3.28). The previous inequality with the axiom $(\mathcal{E}3)$ ensure that $a = 1$. This contradicts (3.41). Hence

$$M(x^*, w, t) = 1,$$

In other words, $x^* = w$ and by considering (3.40), we derive that

$$M(x^*, Tx^*, t) = M(U, V, t)$$

By (3.39), we conclude that x^* is a fuzzy φ-best proximity point of T.

Finally, we shall show the uniqueness of the fuzzy φ-best proximity point of T, that is, $B_{est}(T) \cap \mathcal{O}_\varphi$ is singleton. We argue by contradiction, suppose that $x^*, w^* \in X$ are two distinct fuzzy φ-best proximity fixed points of the mapping T. Then $M(x^*, w^*, t) < 1$ for all $t > 0$. Hence

$$M(x^*, Tx^*, t) = M(A, B, t) \text{ and } M(w^*, Tw^*, t) = M(A, B, t)$$

Since T is an $(\mathcal{FZ}_e^\varphi, F)$-fuzzy proximal contraction with respect to $e \in \mathcal{FZ}_e$

$$0 \leq e(F(M(x^*, w^*, t), \varphi(x^*), \varphi(w^*)), \mathcal{N}_F^\varphi(x^*, w^*, t)) \tag{3.42}$$

Where

$$\begin{aligned}\mathcal{N}_F^\varphi(x^*, w^*, t)) &= \min\{F(M(x^*, w^*, t), \varphi(x^*), \varphi(w^*)), F(M(x^*, x^*, t), \varphi(x^*), \varphi(x^*)), \\ &\qquad F(M(w^*, w^*, t), \varphi(w^*), \varphi(w^*))\} \\ &= \min\{F(M(x^*, w^*, t), 1), F(1, 1, 1), F(1, 1, 1)\} \\ &= F(M(x^*, w^*, t), 1, 1)\end{aligned}$$

Then, using the property $(\mathcal{E}1)$, we have

$$\begin{aligned}0 &\leq e(F(M(x^*, w^*, t), \varphi(x^*), \varphi(w^*)), \mathcal{N}_F^\varphi(x^*, w^*, t)) \\ &< \frac{1}{F(M(x^*, w^*, t), \varphi(x^*), \varphi(w^*))} - \frac{1}{\mathcal{N}_F^\varphi(x^*, w^*, t)} \\ &= \frac{1}{F(M(x^*, w^*, t), 1, 1)} - \frac{1}{F(M(x^*, w^*, t), 1, 1)} \\ &= 0\end{aligned}$$

Which leads to a contradiction. Hence $M(x^*, w^*, t) < 1$, which implies $x^* = w^*$. This completes the proof. □

Corollary 3.7.4. *Let U and V be two nonempty subsets of a complete fuzzy metric space $(X, M, *)$ such that U_0 is nonempty, $\varphi : X \to (0, 1]$ and $F \in \mathcal{F}$.*

$$\begin{cases} M(u, Tx, t) = M(U, V, t) \\ M(v, Ty, t) = M(U, V, t) \end{cases} \Rightarrow F(M(u, v, t), \varphi(u), \varphi(v)) \geq \psi(\mathcal{N}_F^\varphi(x, y, t)) \tag{3.43}$$

(i) *U_0 is closed with respect to the topology induced by M*

(ii) *$T(U_0) \subseteq V_0$;*

(ii) *φ is continuous*

Then T has a unique fuzzy φ-best proximity point, that is, there exists $x^* \in U$ such that $B_{est}(T) \cap \mathcal{O}_\varphi = \{x^*\}$.

$$\eta(F((M(Tx, Ty, t), \varphi(Tx), \varphi(Ty))) \leq k\eta(\mathcal{N}_F^\varphi(x, y, t))$$

$x, y \in X, t > 0$ *where* $k \in (0, 1)$. *Then T has a unique φ-fixed point.*

Remark 3.7.5. It is clear that several consequences can be expressed as a corollaries of our main results by defining the function e in proper.

Chapter 4

Fixed Point Results for α-η-$\mathcal{FZ}$-Contractions Type Mappings

The main purpose of this chapter is to generalize the concept of $\mathcal{FZ}$-contraction mappings by introducing two classes of mappings, generalized α-η-$\mathcal{FZ}$-contractions and modified α-η-$\mathcal{FZ}$-contractions, based on the notion of α-admissible function with respect to η. We prove some fixed point results for such mappings in the framework of fuzzy metric spaces. We provide various deduced results and two examples to clarify the utility of our results. The presented concepts in this chapter enrich, extend and generalize different types of contraction mappings in the current literature, essentially the contractive conditions initiated by Gopal and Calogero vetro [42], Mishra *et al.* [77], Gregori and Sapena [78], Mihet [19] and Melliani *et al.* [72].

4.1 Introduction and Preliminaries

Gopal and Vetro [42] recently developed the idea of α-ϕ-fuzzy contractive mapping, which was inspired by the work of Samet *et al.* [13]. In 2016, Mishra, Vetro and Kumam [77] proved certain fixed point theorems which provide a generalization of several interesting results in the literature. Then they came up with the concept of a modified α-ϕ-fuzzy contractive mapping by using the following notion of α-admissibility.

Definition 4.1.1. [77] Let $(X, M, *)$ be a fuzzy metric space, and let $\alpha, \eta : X \times X \times (0, +\infty) \longrightarrow [0, +\infty)$ be two functions. A mapping $T : X \to X$ is said to be an α-admissible with respect to η if for all $x, y \in X$

$$\alpha(x, y, t) \geq \eta(x, y, t) \Rightarrow \alpha(Tx, Ty, t) \geq \eta(Tx, Ty, t) \text{ for all } t > 0.$$

Remark 4.1.2. Setting $\eta(x,y,t) = 1$ for all $x, y \in X$ in the previous definition, we deduce the Definition 1.7.11. Additionally, If $\alpha(x,y,t) = 1$ for all $x, y \in X$ and $t > 0$, then T is said to be η-subadmissible.

Definition 4.1.3. [77] Let $(X, M, *)$ be a fuzzy metric space in the sense of George and Veeramani. A mapping $T : X \to X$ is said to be a modified α-ϕ-fuzzy contractive mapping if there exist three functions $\alpha, \eta : X \times X \times (0,\infty) \to [0,\infty[$ and $\phi \in \Phi$ such that, for all $x, y \in X$ and for all $t > 0$, we have

$$\alpha(x,y,t) \geq \eta(x,y,t) \Rightarrow \left(\frac{1}{M(Tx,Ty,t)} - 1\right) \leq \phi\left(\frac{1}{N(x,y,t)} - 1\right)$$

where $N(x,y,t) = \min\{M(x,y,t), M(x,Tx,t), M(y,Ty,t)\}$.

Remark 4.1.4. Note that If we take $\eta(x,y,t) = 1$ and $N(x,y,t) = M(x,y,t)$, then this definition reduces to definition of an α-ϕ-fuzzy contractive mapping given in [42].

In the line with [60], we use the concept of α-orbital admissible mappings and triangular α-orbital admissible mappings in the following form.

Definition 4.1.5. Let $T : X \longrightarrow X$ be a mapping and $\alpha : X \times X \longrightarrow [0,\infty)$ be a function. Then T is said to be α-orbital admissible with respect η if

$$\alpha(x,Tx,t) \geq \eta(x,Tx,t) \Longrightarrow \alpha(Tx,T^2x) \geq \eta(Tx,T^2x,t).$$

Moreover, T is called a triangular α-orbital admissible with respect to η if it satisfies the following conditions

(**A1**) T is α-orbital admissible with respect to η ;

(**A2**) $\alpha(x,y,t) \geq \eta(x,y,t)$ and $\alpha(y,Ty,t) \geq \eta(y,Ty,t) \Rightarrow \alpha(x,Ty,t) \geq \eta(x,Ty,t)$.

Remark 4.1.6. Note that if we take $\eta(x,y,t) = 1$, then Definition 4.1.5 reduces to the definition of triangular α-orbital admissible mapping (resp, α-orbital admissible mapping).

4.2 Generalized α-η-$\mathcal{FZ}$-contractions

In this direction we introduce the following concepts.

Definition 4.2.1. Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be an α-η-$\mathcal{FZ}$-contraction, if there exist two functions $\alpha, \eta : X \times X \times (0,\infty) \to [0,\infty)$ such that, for all $x, y \in X$ and for all $t > 0$, we have

$$\alpha(x,y,t) \geq \eta(x,y,t) \Rightarrow \xi(M(Tx,Ty,t), M(x,y,t)) \geq 0. \tag{4.1}$$

Remark 4.2.2. Setting $\alpha(x,y,t) = \eta(x,y,t) = 1$ for all $x,y \in X$, $t > 0$ in (4.1), then this definition reduces to the Definition (3.1.2) of $\mathcal{FZ}$-contraction mappings.

Definition 4.2.3. Let $(X, M, *)$ be a fuzzy metric. A mapping $T : X \to X$ is said to be a generalized α-η-$\mathcal{FZ}$-contraction if there exist two functions $\alpha, \eta : X \times X \times (0,\infty) \to [0,\infty)$ such that, for all $x, y \in X$ and for all $t > 0$, we have

$$\alpha(x,y,t) \geq \eta(x,y,t) \Rightarrow \xi(M(Tx,Ty,t), N(x,y,t)) \geq 0 \tag{4.2}$$

where $N(x,y,t) = \min\{M(x,y,t), M(x,Tx,t), M(y,Ty,t)\}$.

Remark 4.2.4.

- If $\alpha(x,y,t) = \eta(x,y,t) = 1$ for all $x,y \in X$, $t > 0$ and $N(x,y,t) = M(x,y,t)$, then this definition reduces to the Definition (3.1.2) of $\mathcal{FZ}$-contraction, thus it will imply the definition of the fuzzy contractive mapping considered by Gregori and Sapena [78].
- If $\phi \in \Phi$ and we define $\xi(t,s) = \phi\left(\frac{1}{s} - 1\right) - \frac{1}{t} + 1$ for all $s,t \in (0,1]$, then this definition reduces to Definition (4.1.3) of modified α-ϕ-fuzzy contractive mapping given in by Mishra, Vetro and Kumam [77].

Our first main result is the following theorem.

Theorem 4.2.5. *Let $(X, M, *)$ be a complete fuzzy metric space in the sense of George and Veeramani and let $T : X \longrightarrow X$ be a generalized α-η-$\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{FZ}$ satisfying the following conditions:*

(i) *T is triangular α-orbital admissible with respect to η;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq \eta(x_0, Tx_0, t)$;*

(iii) *T is continuous.*

Then there exists $u \in X$ such that $Tu = u$.

Proof. Using condition (ii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq \eta(x_0, Tx_0, t)$. Define a sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n \text{ for all } n \in \mathbb{N}$$

If $x_{m+1} = Tx_m$ for some $m \in \mathbb{N}$, then T has a fixed point. Therefore, to continue our proof, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since T is an α-admissible mapping with respect to η, we have

$$\alpha(x_0, x_1, t) = \alpha(x_0, Tx_0, t) \geq \eta(x_0, Tx_0, t) = \eta(x_0, x_1, t) \text{ implies that}$$
$$\alpha(x_1, x_2, t) = \alpha(Tx_0, Tx_1, t) \geq \eta(Tx_0, Tx_1, t) = \eta(x_1, x_2, t).$$

Recursively, we get that

$$\alpha(x_n, x_{n+1}, t) \geq \eta(x_n, x_{n+1}, t), \text{ for all } n \in \mathbb{N}. \tag{4.3}$$

Regarding that T is a generalized α-η-$\mathcal{FZ}$-contraction, if we consider in (4.2) $x = x_n$ and $y = x_{n-1}$, we get

$$\begin{aligned} 0 &\leq \xi(M(Tx_n, Tx_{n-1}, t), N(x_n, x_{n-1}, t)) \\ &= \xi(M(x_{n+1}, x_n, t), N(x_n, x_{n-1}, t)) \\ &< \frac{1}{N(x_n, x_{n-1}, t)} - \frac{1}{M(x_{n+1}, x_n, t)}. \end{aligned}$$

Hence

$$\frac{1}{M(x_{n+1}, x_n, t)} < \frac{1}{N(x_n, x_{n-1}, t)}$$

which is equivalent to

$$N(x_n, x_{n-1}, t) < M(x_{n+1}, x_n, t) \tag{4.4}$$

where

$$\begin{aligned} N(x_n, x_{n-1}, t) &= \min\{M(x_n, x_{n-1}, t), M(x_n, Tx_n, t), M(x_{n-1}, Tx_{n-1}, t)\} \\ &= \min\{M(x_n, x_{n-1}, t), M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t)\} \\ &= \min\{M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t)\} \end{aligned} \tag{4.5}$$

Now if $\min\{M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t)\} = M(x_n, x_{n+1}, t)$ then

$$M(x_n, x_{n+1}, t) < M(x_{n+1}, x_n, t)$$

Which is a contradiction. It follows that

$$\min\{M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t)\} = M(x_{n-1}, x_n, t)$$

By (4.4), we obtain that

$$M(x_{n-1}, x_n, t) < M(x_n, x_{n+1}, t) \text{ for all } n \in \mathbb{N}$$

Hence, we deduce that the sequence $\{M(x_n, x_{n+1}, t)\}$ is a nondecreasing of positive real numbers in $[0, 1]$. Thus, there exists $l(t) \leq 1$ such that $\lim_{n\to\infty} M(x_n, x_{n-1}, t) = l(t) \geq 1$ for all $t > 0$. We claim that

$$\lim_{n\to\infty} M(x_n, x_{n-1}, t) = 1 \tag{4.6}$$

On contrary assume that $l(t_0) < 1$ for some $t_0 > 0$. Now, if we take the sequences $\{\tau_n = M(x_n, x_{n+1}, t_0)$ and $\{\delta_n = M(x_{n-1}, x_n, t_0)\}$ and considering $(\xi 3)$, we obtain the contradiction

$$0 \leq \lim_{n\to\infty} \sup \xi(M(x_n, x_{n+1}, t_0), M(x_{n-1}, x_n, t_0)) < 0$$

Which yields $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$.

Next, we show that $\{x_n\}$ is Cauchy sequence in X. Reasoning by contradiction, assume that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $\epsilon \in (0,1)$, $t_0 > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ such that

$$M(x_{n_k}, x_{m_k}, t_0) \leq 1 - \epsilon. \tag{4.7}$$

Taking in account Lemma 1.5.4 we derive that

$$M(x_{n_k}, x_{m_k}, \frac{t_0}{2}) \leq 1 - \epsilon. \tag{4.8}$$

By choosing m_k as the smallest index satisfying (4.8), we have

$$M(x_{n_k}, x_{m_k-1}, \frac{t_0}{2}) > 1 - \epsilon. \tag{4.9}$$

On account of (4.7),(4.9) and the triangular inequality, we obtain

$$\begin{aligned} 1 - \epsilon &\geq M(x_{n_k}, x_{m_k}, t_0) \\ &\geq M(x_{n_k}, x_{m_k-1}, \frac{t_0}{2}) * M(x_{m_k-1}, x_{m_k}, \frac{t_0}{2}) \\ &> (1-\epsilon) * M(x_{m_k-1}, x_{m_k}, \frac{t_0}{2}) \end{aligned}$$

Taking limit as $k \to \infty$ and using (4.6), we derive that

$$\lim_{n\to\infty} M(x_{n_k}, x_{m_k}, t_0) = 1 - \epsilon \tag{4.10}$$

Furthermore, since T is triangular α-orbital admissible with respect to η, we have

$$\alpha(x_{n_k-1}, x_{m_k-1}, t_0) \geq \eta(x_{n_k-1}, x_{m_k-1}, t_0). \tag{4.11}$$

Using the fact that T is a generalized α-η-$\mathcal{FZ}$-contraction mapping with respect to $\xi \in \mathcal{FZ}$, we obtain that

$$\begin{aligned} 0 &\leq \xi(M(Tx_{n_k-1}, Tx_{m_k-1}, t_0), N(x_{n_k-1}, x_{m_k-1}, t_0)) \\ &= \xi(M(x_{n_k}, x_{m_k}, t_0), N(x_{n_k-1}, x_{m_k-1}, t_0)) \\ &< \frac{1}{N(x_{n_k-1}, x_{m_k-1}, t_0)} - \frac{1}{M(x_{n_k}, x_{m_k}, t_0)}. \end{aligned}$$

Which implies that

$$N(x_{n_k-1}, x_{m_k-1}, t_0) \quad < M(x_{n_k}, x_{m_k}, t_0) \tag{4.12}$$

Where

$$\begin{aligned} N(x_{n_k-1}, x_{m_k-1}, t_0) &= \min\{M(x_{n_k-1}, x_{m_k-1}, t_0), M(x_{n_k-1}, Tx_{n_k-1}, t_0), \\ &\qquad M(x_{m_k-1}, Tx_{m_k-1}, t_0)\} \\ &= \min\{M(x_{n_k-1}, x_{m_k-1}, t_0), M(x_{n_k-1}, x_{n_k}, t_0), \\ &\qquad M(x_{m_k-1}, x_{m_k}, t_0)\}. \end{aligned} \tag{4.13}$$

Taking the limit as $k \to \infty$ in (4.12) and using (4.6), we get

$$\begin{aligned} \lim_{k\to\infty} N(x_{n_k-1}, x_{m_k-1}, t_0) &= \min\{\lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0), 1, 1\} \\ &\leq \lim_{k\to\infty} M(x_{n_k}, x_{m_k}, t_0) \\ &= 1 - \epsilon \end{aligned} \tag{4.14}$$

By the triangular inequality, we have

$$M(x_{n_k-1}, x_{m_k-1}, t_0) \quad \geq M(x_{n_k-1}, x_{n_k}, \tfrac{t_0}{2}) * M(x_{n_k}, x_{m_k-1}, \tfrac{t_0}{2}) \tag{4.15}$$

Again, taking the limit as $k \to \infty$ in the last inequality and using (4.6) and (4.9), we get

$$\lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0) \quad \geq 1 - \epsilon. \tag{4.16}$$

From (4.14) and (4.15), we derive that

$$\lim_{k\to\infty} N(x_{n_k-1}, x_{m_k-1}, t_0) = 1 - \epsilon. \tag{4.17}$$

Taking the sequences $s_k = N(x_{n_k-1}, x_{m_k-1}, t_0)$ and $t_k = M(x_{m_k}, x_{n_k}, t_0)$, then $\lim_{k\to\infty} s_k = \lim_{k\to\infty} t_k = 1 - \epsilon$. Applying ($\xi 3$), we get

$$0 \leq \lim_{k\to\infty} \sup \xi(M(x_{m_k}, x_{n_k}, t_0), N(x_{n_k-1}, x_{m_k-1}, t_0)) < 0$$

which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is a complete fuzzy metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} M(x_n, u, t) = 1. \tag{4.18}$$

As T is continuous, we derive that

$$\lim_{n\to\infty} M(x_{n+1}, Tu, t) = \lim_{n\to\infty} M(Tx_n, Tu, t) = 1 \tag{4.19}$$

and by the uniqueness of the limit, we conclude that u is fixed point of T, that is $Tu = u$.

□

Theorem 4.2.6. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \longrightarrow X$ be a generalized α-η-$\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{FZ}$ satisfying the following conditions:*

- **(i)** *T is triangular α-orbital admissible with respect to η;*
- **(ii)** *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq \eta(x_0, Tx_0, t)$;*
- **(iii)** *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$, $t > 0$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x, t) \geq \eta(x_{n(k)}, x, t)$ for all $k \in \mathbb{N}$ and $t > 0$.*

Then there exists $u \in X$ such that $Tu = u$.

Proof. Following the lines of the proof of Theorem 4.2.5, we obtain that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ is a Cauchy sequence in X. Since $(X, M, *)$ is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. By the condition (**iii**), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u, t) \geq \eta(x_{n(k)}, u, t)$ for all $k \in \mathbb{N}$,$t > 0$. Applying (4.2), we obtain that

$$\begin{aligned} 0 &\leq \xi(M(Tx_{n(k)}, Tu, t), N(x_{n(k)}, u, t)) \\ &= \xi(M(x_{n(k)+1}, Tu, t), N(x_{n(k)}, u, t)) \end{aligned}$$

Where

$$\begin{aligned} N(x_{n(k)}, u, t) &= \min\{M(x_{n(k)}, u, t), M(x_{n(k)}, Tx_{n(k)}, t), M(u, Tu, t)\} \\ &= \min\{M(x_{n(k)}, u, t), M(x_{n(k)}, x_{n(k)+1}, t), M(u, Tu, t)\} \end{aligned} \tag{4.20}$$

As $k \to \infty$, we get

$$\begin{aligned} \lim_{k\to\infty} N(x_{n(k)}, u, t) &= \min\{1, 1, M(u, Tu, t)\} \\ &= M(u, Tu, t) \end{aligned}$$

Suppose that $Tu \neq u$. Then $M(u, Tu, t) < 1$. Now by choosing the sequences $\{\tau_k = M(x_{n(k)+1}, Tu, t)\}$ and $\{\sigma_k = N(x_{n(k)}, u, t)\}$. On account of the above observations, we have $\lim_{k\to\infty} \tau_k = \lim_{k\to\infty} \sigma_k < 1$.
Applying the property $(\xi 3)$, it follows that

$$0 \leq \lim_{k\to\infty} \sup \xi(M(x_{n(k)+1}, Tu, t), N(x_{n(k)}, u, t)) < 0 \tag{4.21}$$

which is a contradiction. Thus we have $M(u, Tu, t) = 1$, which is equivalent to $Tu = u$.

□

To ensure the uniqueness of a fixed point of a generalized α-η-$\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{Z}$, we will consider the following condition:

(**U**) For all $x, y \in \text{Fix(T)}$, we have $\alpha(x, y, t) \geq \eta(x, y, t)$.
where Fix(T) denotes the set of fixed points of T.

Theorem 4.2.7. *Adding condition* (**U**) *to the hypotheses of Theorem 4.2.5* (*resp. Theorem* 4.2.6), *we obtain the uniqueness of the fixed point of* T.

Proof. We argue by contradiction, suppose that $u, v \in X$ are two distinct fixed points of the mapping. Then $M(u, v, t) < 1$ for all $t > 0$. From assumption (**U**), we have

$$\alpha(u, v, t) \geq \eta(u, v, t). \tag{4.22}$$

Therefore, it follows from equation (4.2) and ($\xi 2$), that

$$\begin{aligned}
0 &\leq \xi(M(Tu, Tv, t), N(u, v, t)) \\
&= \xi(M(Tu, Tv, t), \min\{M(u, v, t), M(u, Tu, t), M(v, Tv, t)\}) \\
&= \xi(M(u, v, t), \min\{M(u, v, t), 1, 1\}) \\
&= \xi(M(u, v, t), M(u, v, t)) \\
&< \frac{1}{M(u, v, t)} - \frac{1}{M(u, v, t)} = 0.
\end{aligned} \tag{4.23}$$

Which is a contradiction. Therefore, the fixed point of T is unique. This completes the proof. □

4.3 Examples and Consequences

Example 4.3.1. Let $X = [0, \infty)$ endowed with the fuzzy metric $M : X \times X \times (0, \infty) \longrightarrow [0, 1)$ defined by $M(x, y, t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$, $t > 0$ and $*$ the t-norm given by $a * b = a.b$ for all $a, b \in [0, 1]$. We define $T : X \longrightarrow X$ by

$$Tx = \begin{cases} \frac{x^2}{6} & \text{if } x, y \in [0, 1] \\ \\ \frac{3}{2} & \text{otherwise} \end{cases}$$

and $\alpha, \eta : X \times X \times (0, \infty) \longrightarrow [0, \infty)$ by

$$\alpha(x, y, t) = \begin{cases} 2 + xy & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$$\eta(x, y, t) = \begin{cases} 1 + xy & \text{if } x, y \in [0, 1] \\ 6 & \text{otherwise} \end{cases}$$

Let $(x, y) \in X \times X$, from the definition of α and η, we have that $\alpha(x, y, t) \geq \eta(x, y, t)$ for all $t > 0$ iff $x, y \in [0, 1]$. Suppose that $\alpha(x, Tx, t) \geq \eta(x, Tx, t)$ therefore $x, Tx \in [0, 1]$, and hence $Tx, TTx \in [0, 1]$, which implies that $\alpha(Tx, T^2x, t) \geq \eta(Tx, T^2x, t)$. Thus T is α-orbital admissible with respect to η. Suppose that $\alpha(x, y, t) \geq \eta(x, y, t)$ and $\alpha(y, Ty, t) \geq \eta(y, Ty, t)$ for all $t > 0$, then $x, y, Ty \in [0, 1]$ which implies that $\alpha(x, Ty, t) \geq \eta(x, Ty, t)$. Hence T is triangular α-orbital admissible with respect to η.

Clearly, for any $x_0 \in [0, 1]$ we have $\alpha(x_0, Tx_0, t) \geq \eta(x_0, Tx_0, t)$ for all $t > 0$. Next, let $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}, t) \geq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$. Hence $\{x_n\} \subseteq [0, 1]$ and then $x \in [0, 1]$, which implies that $\alpha(x_n, x, t) \geq \eta(x_n, x, t)$.

Now, we show that T is a generalized α-η-$\mathcal{FZ}$-contraction mapping, i.e. we have to prove that (4.2) is satisfied. We define $\xi : (0, 1] \times (0, 1] \longrightarrow \mathbb{R}$ by $\xi(t, s) = \frac{1}{3}(\frac{1}{s} - 1) - \frac{1}{t} + 1$. let $(x, y) \in X \times X$ such that $\alpha(x, y, t) \geq \eta(x, y, t)$ for all $t > 0$. From the definition of α and η we have $x, y \in [0, 1]$, then $Tx = \frac{x^2}{6}$ and $Ty = \frac{y^2}{6}$. Since

$$\begin{aligned}
\frac{1}{M(Tx, Ty, t)} - 1 &= \frac{t + |Tx - Ty|}{t} - 1 \\
&= \frac{|x^2 - y^2|}{6t} \\
&\leq \frac{|x - y|}{3t} \\
&\leq \frac{1}{3t} \max\{|x - y|, |x - Tx|, |y - Ty|\} \\
&= \frac{1}{3}(\frac{1}{N(x, y, t)} - 1)
\end{aligned}$$

We define $\xi : (0, 1] \times (0, 1] \longrightarrow \mathbb{R}$ by $\xi(t, s) = \frac{1}{3}(\frac{1}{s} - 1) - \frac{1}{t} + 1$. It follows that $\xi(M(Tx, Ty, t), N(x, y, t)) \geq 0$.

Therefore, T satisfies all the hypothesis of Theorem 4.2.6 and $x = 0$, $x = \frac{3}{2}$ are fixed points of T.
Note that, T is not a fuzzy contractive mapping (see Definition 1.7.1) in the sense of Gregori and Sapena [78]. Indeed, by choosing $x = 1$ and $y = \frac{3}{2}$, there is no $k \in (0, 1)$ satisfying

$$\begin{aligned}
\frac{1}{M(Tx, Ty, t)} - 1 = \frac{|Tx - Ty|}{t} &= \frac{8}{6t} \\
&\leq \frac{k}{2t} = k(\frac{1}{M(x, y, t)} - 1)
\end{aligned}$$

Now, we derive several corollaries which can be expressed easily from our main result.

Corollary 4.3.2. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \longrightarrow X$ be an α-admissible. Assume that there exists a $\mathcal{FZ}$-simulation mapping ξ such that for all $x, y \in X$ and $t > 0$,*

$$\alpha(x, y, t) \geq 1 \Rightarrow \xi(M(Tx, Ty, t), N(x, y, t)) \geq 0 \tag{4.24}$$

where $N(x, y, t) = \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\}$. And suppose that the following assertions hold:

- **(i)** *T is triangular α-orbital admissible;*
- **(ii)** *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$;*
- **(iii)** *either, T is continuous, or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$, $t > 0$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x, t) \geq 1$ for all $k \in \mathbb{N}$ and $t > 0$.*

Then T has a fixed point. Furthermore, if for all $x, y \in$ Fix(T) and for all $t > 0$, we have $\alpha(x, y, t) \geq 1$, then T has a unique fixed point.

Proof. The result follows by defining $\eta(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ in Theorem 4.2.7. □

Corollary 4.3.3. *Let $(X, M, *)$ be a complete fuzzy metric space, $T : X \longrightarrow X$ and $\alpha, \eta : X \times X \times (0, \infty) \longrightarrow [0, \infty)$ be a mappings. Assume that there exists a function $\psi \in \Psi_2$ such that, for all $x, y \in X$ and $t > 0$,*

$$\alpha(x, y, t) \geq \eta(x, y, t) \Rightarrow M(Tx, Ty, t) \geq \psi(N(x, y, t))$$

where $N(x, y, t) = \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\}$. Furthermore we suppose that :

- **(i)** *T is triangular α-orbital admissible with respect to η;*
- **(ii)** *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq \eta(x_0, Tx_0, t)$;*
- **(iii)** *either T is continuous mapping or, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$, $t > 0$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x, t) \geq \eta(x_{n(k)}, x, t)$ for all $k \in \mathbb{N}$ and $t > 0$.*
- **(v)** *for all $x, y \in$ Fix(T) and $t > 0$, we have $\alpha(x, y, t) \geq \eta(x, y, t)$.*

Then T has a unique fixed point.

Proof. Define $\xi : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ by

$$\xi(t,s) = \frac{1}{\psi(s)} - \frac{1}{t} \text{ for all } s,t \in (0,1].$$

Since $\xi \in \mathcal{FZ}$ the desired results follow from Theorem 4.2.7. □

Corollary 4.3.4. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \longrightarrow X$ be a modified α-ϕ-fuzzy contractive mapping satisfying the following conditions:*

(i) *T is triangular α-orbital admissible with respect to η;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq \eta(x_0, Tx_0, t)$;*

(iii) *either T is continuous, or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$, $t > 0$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x, t) \geq \eta(x_{n(k)}, x, t)$ for all $k \in \mathbb{N}$ and $t > 0$.*

Then T has a fixed point. Furthermore, if for all $x, y \in$ Fix(T) and for all $t > 0$, we have $\alpha(x, y, t) \geq \eta(x, y, t)$, then T has a unique fixed point.

Proof. Follows from Theorem 4.2.7 by taking $\xi(t,s) = \phi(\frac{1}{s} - 1) - \frac{1}{t} + 1$ for all $s,t \in (0,1]$. □

Corollary 4.3.5. *Let $(X, M, *)$ be a complete fuzzy metric space. Let $T : X \longrightarrow X$ and $\alpha : X \times X \times (0,\infty) \longrightarrow [0,\infty)$ be a mappings. Assume that there exists $\phi \in \Phi$ such that, for all $x, y \in X$ and $t > 0$,*

$$\alpha(x,y,t) \geq 1 \Rightarrow \left(\frac{1}{M(Tx,Ty,t)} - 1 \right) \leq \phi \left(\frac{1}{N(x,y,t)} - 1 \right)$$

where $N(x,y,t) = \min\{M(x,y,t), M(x,Tx,t), M(y,Ty,t)\}$. Suppose that the following conditions are satisfied:

(i) *T is triangular α-orbital admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$;*

(iii) *T is continuous mapping or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$, $t > 0$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x, t) \geq 1$ for all $k \in \mathbb{N}$ and $t > 0$.*

Then T has a fixed point. Furthermore, if for all $x, y \in$ Fix(T) and for all $t > 0$, we have $\alpha(x, y, t) \geq 1$, then T has a unique fixed point.

Proof. Follows from Theorem 4 by taking $\eta(x,y,t) = 1$ for all $x, y \in X$ and $t > 0$, and $\xi(t,s) = \phi(\frac{1}{s} - 1) - \frac{1}{t} + 1$ for all $s,t \in (0,1]$. □

Corollary 4.3.6. *Let $(X, M, *)$ be a complete fuzzy metric space, $T : X \longrightarrow X$ and $\alpha, \eta : X \times X \times (0, \infty) \longrightarrow [0, \infty)$ be a mappings. Assume that there exists a function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ with $\varphi(\lambda) > 0$ for all $\lambda > 0$ and $\varphi(0) = 0$, such that for all $x, y \in X$ and $t > 0$,*

$$\alpha(x, y, t) \geq \eta(x, y, t) \Rightarrow \left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \left(\frac{1}{N(x, y, t)} - 1\right) - \varphi\left(\frac{1}{N(x, y, t)} - 1\right)$$

where $N(x, y, t) = \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\}$. Furthermore we suppose that :

- **(i)** *T is triangular α-orbital admissible with respect to η;*
- **(ii)** *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq \eta(x_0, Tx_0, t)$;*
- **(iii)** *either T is continuous mapping or, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$, $t > 0$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x, t) \geq \eta(x_{n(k)}, x, t)$ for all $k \in \mathbb{N}$ and $t > 0$.*
- **(v)** *for all $x, y \in Fix(T)$ and $t > 0$, we have $\alpha(x, y, t) \geq \eta(x, y, t)$.*

Then T has a unique fixed point.

Proof. Follows from Theorem 4.2.7 by taking $\xi(t, s) = \left(\frac{1}{s} - 1\right) - \varphi\left(\frac{1}{s} - 1\right) - \frac{1}{t} + 1$ for all $s, t \in (0, 1]$. □

Corollary 4.3.7. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \longrightarrow X$ a mapping satisfying*

$$\xi(M(Tx, Ty, t), N(x, y, t)) \geq 0 \tag{4.25}$$

with respect to a $\mathcal{FZ}$-simulation function $\xi \in \mathcal{FZ}$. Then T has a unique fixed point.

4.4 Modified α-η-$\mathcal{FZ}$-contractions

Definition 4.4.1. Let $(X, M, *)$ be a fuzzy metric space in the sense of George and Veeramani. A mapping $T : X \to X$ is said to be a modified $\mathcal{FZ}$-contraction if there exist two functions $\alpha, \eta : X \times X \times (0, \infty) \to [0, \infty)$ such that, for all $x, y \in X$ and for all $t > 0$, we have

$$\alpha(x, Tx, t)\alpha(y, Ty, t) \geq \eta(x, Tx, t)\eta(y, Ty, t) \Rightarrow \xi(M(Tx, Ty, t), \ \mathbb{N}(x, y, t)) \geq 0 \tag{4.26}$$

where $\mathbb{N}(x, y, t) = \min\{M(x, y, t), \max\{M(x, Tx, t), M(y, Ty, t)\}\}$.

Remark 4.4.2.

- If $\psi \in \Psi_2$ and we define $\xi : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ by $\xi(t,s) = \frac{1}{\psi(s)} - \frac{1}{t}$ for all $s,t \in (0,1]$ as $\mathcal{FZ}$-simulation function, then this definition reduces to definition of modified α-η-ψ-fuzzy contractive mapping [73], i.e.

$$\alpha(x,Tx,t)\alpha(y,Ty,t) \geq \eta(x,Tx,t)\eta(y,Ty,t) \Rightarrow M(Tx,Ty,t) \geq \psi(\mathbb{N}(x,y,t))$$

Theorem 4.4.3. *Let $(X, M, *)$ be a complete fuzzy metric space in the sense of George and Veeramani and let $T : X \longrightarrow X$ be a modified $\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{FZ}$ satisfying the following conditions:*

(i) *T is triangular α-orbital admissible with respect to η;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq \eta(x_0, Tx_0, t)$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$, $t > 0$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x, t) \geq \eta(x_{n(k)}, x, t)$ and $\alpha(x, Tx, t) \geq \eta(x, Tx, t)$ for all $k \in \mathbb{N}$ and $t > 0$.*

Then T has a fixed point.

Proof. Following the same reasoning in Theorem 4.2.5, we get that

$$\alpha(x_n, x_{n+1}, t) \geq \eta(x_n, x_{n+1}, t), \text{ for all } n \in \mathbb{N}. \tag{4.27}$$

it follows that

$$\alpha(x_n, x_{n+1}, t)\alpha(x_{n-1}, x_n, t) \geq \eta(x_n, x_{n+1}, t)\eta(x_{n-1}, x_n, t), \tag{4.28}$$

for all $t > 0$ and $n \in \mathbb{N}$. Regarding that T is a modified $\mathcal{FZ}$-contractive mapping, if we consider in (4.26) $x = x_n$ and $y = x_{n-1}$, we get

$$\begin{aligned} 0 &\leq \xi(M(Tx_n, Tx_{n-1}, t), \mathbb{N}(x_n, x_{n-1}, t)) \\ &= \xi(M(x_{n+1}, x_n, t), \mathbb{N}(x_n, x_{n-1}, t)) \\ &< \frac{1}{\mathbb{N}(x_n, x_{n-1}, t)} - \frac{1}{M(x_{n+1}, x_n, t)}. \end{aligned}$$

Hence

$$\frac{1}{M(x_{n+1}, x_n, t)} < \frac{1}{\mathbb{N}(x_n, x_{n-1}, t)}$$

which is equivalent to

$$\mathbb{N}(x_n, x_{n-1}, t) < M(x_{n+1}, x_n, t) \tag{4.29}$$

where

$$\begin{aligned}\mathbb{N}(x_n, x_{n-1}, t) &= \min\{M(x_n, x_{n-1}, t), \max\{M(x_n, Tx_n, t), M(x_{n-1}, Tx_{n-1}, t)\}\} \\ &= \min\{M(x_n, x_{n-1}, t), \max\{M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t)\}\} \\ &= M(x_{n-1}, x_n, t) \qquad (4.30)\end{aligned}$$

By (4.28), we get

$$M(x_{n-1}, x_n, t) < M(x_n, x_{n+1}, t) \text{ for all } n \in \mathbb{N}$$

Hence, we deduce that the sequence $\{M(x_n, x_{n+1}, t)\}$ is a nondecreasing of positive real numbers in $[0, 1]$. Thus, there exists $s(t) \leq 1$ such that $\lim_{n\to\infty} M(x_n, x_{n-1}, t) = s(t) \geq 1$ for all $t > 0$. We claim that

$$\lim_{n\to\infty} M(x_n, x_{n-1}, t) = 1 \qquad (4.31)$$

On contrary assume that $s(t_0) < 1$ for some $t_0 > 0$. Now, if we take the sequences $\{\tau_n = M(x_n, x_{n+1}, t_0)$ and $\{\delta_n = M(x_{n-1}, x_n, t_0)\}$ and considering ($\xi 3$), we obtain the contradiction

$$0 \leq \lim_{n\to\infty} \sup \xi(M(x_n, x_{n+1}, t_0), M(x_{n-1}, x_n, t_0)) < 0$$

Which yields $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$. Next, we show that $\{x_n\}$ is Cauchy sequence in X. Reasoning by contradiction, assume that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $\epsilon \in (0, 1)$, $t_0 > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ such that

$$M(x_{n_k}, x_{m_k}, t_0) \leq 1 - \epsilon. \qquad (4.32)$$

Taking in account Lemma 1.5.4, we derive that

$$M(x_{n_k}, x_{m_k}, \frac{t_0}{2}) \leq 1 - \epsilon. \qquad (4.33)$$

By choosing m_k as the smallest index satisfying (4.32), we have

$$M(x_{n_k}, x_{m_k-1}, \frac{t_0}{2}) > 1 - \epsilon. \qquad (4.34)$$

On account of (4.31),(4.33) and the triangular inequality, we have

$$\begin{aligned}1 - \epsilon &\geq M(x_{n_k}, x_{m_k}, t_0) \\ &\geq M(x_{n_k}, x_{m_k-1}, \frac{t_0}{2}) * M(x_{m_k-1}, x_{m_k}, \frac{t_0}{2}) \\ &> (1 - \epsilon) * M(x_{m_k-1}, x_{m_k}, \frac{t_0}{2})\end{aligned}$$

Taking limit as $k \to \infty$ and using (4.30), we derive that

$$\lim_{n\to\infty} M(x_{n_k}, x_{m_k}, t_0) = 1 - \epsilon \tag{4.35}$$

Since we have

$$\alpha(x_{n_k-1}, Tx_{n_k-1}, t_0)\alpha(x_{m_k-1}, Tx_{m_k-1}, t_0) \geq \eta(x_{n_k-1}, Tx_{n_k-1}, t_0)\eta(x_{m_k-1}, Tx_{m_k-1}, t_0).$$

and using the fact that T is an admissible modified $\mathcal{FZ}$-contraction mapping with respect to $\xi \in \mathcal{FZ}$, we obtain that

$$\begin{aligned} 0 &\leq \xi(M(Tx_{n_k-1}, Tx_{m_k-1}, t_0), \mathbb{N}(x_{n_k-1}, x_{m_k-1}, t_0)) \\ &= \xi(M(x_{n_k}, x_{m_k}, t_0), \mathbb{N}(x_{n_k-1}, x_{m_k-1}, t_0)) \\ &< \frac{1}{\mathbb{N}(x_{n_k-1}, x_{m_k-1}, t_0)} - \frac{1}{M(x_{n_k}, x_{m_k}, t_0)}. \end{aligned}$$

Which implies that

$$\mathbb{N}(x_{n_k-1}, x_{m_k-1}, t_0) \quad < M(x_{n_k}, x_{m_k}, t_0), \tag{4.36}$$

where

$$\begin{aligned} \mathbb{N}(x_{n_k-1}, x_{m_k-1}, t_0) &= \min\{M(x_{n_k-1}, x_{m_k-1}, t_0), \\ &\qquad \max\{M(x_{n_k-1}, Tx_{n_k-1}, t_0), M(x_{m_k-1}, Tx_{m_k-1}, t_0)\}\} \\ &= \min\{M(x_{n_k-1}, x_{m_k-1}, t_0), \\ &\qquad \max\{M(x_{n_k-1}, x_{n_k}, t_0), M(x_{m_k-1}, x_{m_k}, t_0)\}\}. \end{aligned} \tag{4.37}$$

By (4.30), we have

$$\begin{aligned} \lim_{k\to\infty} \max\{M(x_{n_k-1}, x_{n_k}, t_0), M(x_{m_k-1}, x_{m_k}, t_0)\} &= \max\{1, 1\} \\ &= 1. \end{aligned} \tag{4.38}$$

Taking the limit as $k \to \infty$ in (4.36) and using (4.38), we get

$$\begin{aligned} \lim_{k\to\infty} \mathbb{N}(x_{n_k-1}, x_{m_k-1}, t_0) &= \min\{\lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0), 1\} \\ &\leq \lim_{k\to\infty} M(x_{n_k}, x_{m_k}, t_0) \\ &= 1 - \epsilon. \end{aligned} \tag{4.39}$$

By the triangular inequality, we have

$$M(x_{n_k-1}, x_{m_k-1}, t_0) \quad \geq M(x_{n_k-1}, x_{n_k}, \tfrac{t_0}{2}) * M(x_{n_k}, x_{m_k-1}, \tfrac{t_0}{2}) \tag{4.40}$$

Again, taking the limit as $k \to \infty$ in the last inequality and using (4.30) and (4.33), we get

$$\lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0) \quad \geq 1 - \epsilon. \tag{4.41}$$

From (4.40) and (4.38), we derive that

$$\lim_{k\to\infty} \mathbb{N}(x_{n_k-1}, x_{m_k-1}, t_0) = 1 - \epsilon. \tag{4.42}$$

Taking the sequences $s_k = \mathbb{N}(x_{n_k-1}, x_{m_k-1}, t_0)$ and $t_k = M(x_{m_k}, x_{n_k}, t_0)$, then $\lim_{k\to\infty} s_k = \lim_{k\to\infty} t_k = 1 - \epsilon$. Applying ($\xi 3$), we get

$$0 \leq \lim_{k\to\infty} \sup \xi(M(x_{m_k}, x_{n_k}, t_0), \mathbb{N}(x_{n_k-1}, x_{m_k-1}, t_0)) < 0.$$

Which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is a complete fuzzy metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} M(x_n, u, t) = 1.$$

From the condition (iii), $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$, $t > 0$ and $x_n \to u \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u, t) \geq \eta(x_{n(k)}, u, t)$ and $\alpha(u, Tu, t) \geq \eta(u, Tu, t)$ for all $k \in \mathbb{N}$ and $t > 0$. Hence

$$\alpha(x_{n(k)}, Tx_{n(k)}, t)\alpha(u, Tu, t) \geq \eta(x_{n(k)}, Tx_{n(k)}, t)\eta(u, Tu, t)$$

$$\begin{aligned} 0 &\leq \xi(M(Tx_{n(k)}, Tu, t), \mathbb{N}(x_{n(k)}, u, t)) \\ &= \xi(M(x_{n(k)+1}, Tu, t), \mathbb{N}(x_{n(k)}, u, t)) \\ &< \frac{1}{\mathbb{N}(x_{n(k)}, u, t)} - \frac{1}{M(x_{n(k)+1}, Tu, t)} \end{aligned}$$

It follows that

$$\mathbb{N}(x_{n(k)}, u, t) < M(x_{n(k)+1}, Tu, t) \tag{4.43}$$

where

$$\begin{aligned} \mathbb{N}(x_{n(k)}, u, t) &= \min\{M(x_{n(k)}, u, t), \max\{M(x_{n(k)}, Tx_{n(k)}, t), M(u, Tu, t)\}\} \\ &= \min\{M(x_{n(k)}, u, t), \max\{M(x_{n(k)}, x_{n(k)+1}, t), M(u, Tu, t)\}\} \end{aligned}$$

Letting $k \to \infty$ in (4.42), we get

$$\lim_{k\to\infty} \mathbb{N}(x_{n(k)}, u, t) = 1 \leq M(u, Tu, t)$$

Therefore, $M(u, Tu, t) = 1$, which is equivalent to $Tu = u$. This completes the proof. □

4.5 Corollaries and Examples

By defining $\alpha(x, y, t) = 1$ for all $x, y \in X$ and all $t > 0$, we obtain the following result

Corollary 4.5.1. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \longrightarrow X$ be η-subadmissible a mapping. satisfying the following conditions:*

$$1 \geq \eta(x, Tx, t)\eta(y, Ty, t) \Rightarrow \xi(M(Tx, Ty, t), \mathbb{N}(x, y, t)) \geq 0$$

where $\mathbb{N}(x, y, t) = \min\{M(x, y, t), \max\{M(x, Tx, t), M(y, Ty, t)\}\}$.

(i) *there exists $x_0 \in X$ such that $1 \geq \eta(x_0, Tx_0, t)$;*

(ii) *if $\{x_n\}$ is a sequence in X such that $1 \geq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$, $t > 0$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $1 \geq \eta(x_{n(k)}, x, t)$ and $1 \geq \eta(x, Tx, t)$ for all $k \in \mathbb{N}$ and $t > 0$.*

Then T has a fixed point.

Example 4.5.2. Let $X = [0, \infty)$ endowed with the fuzzy metric $M : X \times X \times (0, \infty) \longrightarrow [0, 1)$ defined by $M(x, y, t) = e^{-\frac{|x-y|}{t}}$ for all $x, y \in X$, $t > 0$ and $*$ the t-norm given by $a * b = a.b$ for all $a, b \in [0, 1]$. We define $T : X \longrightarrow X$ by

$$Tx = \begin{cases} \frac{x^2}{4} & \text{if } x, y \in [0, 1] \\ 3x^2 + 1 & \text{otherwise} \end{cases}$$

and $\alpha, \eta : X \times X \times (0, \infty) \longrightarrow [0, \infty)$ by $\alpha \equiv 1$,

$$\eta(x, y, t) = \begin{cases} \frac{1}{4} \text{ if } x, y \in [0, 1] \\ 9 \qquad \text{otherwise} \end{cases}$$

Define $\xi : (0, 1] \times (0, 1] \longrightarrow \mathbb{R}$ by $\xi(t, s) = \frac{1}{\sqrt{s}} - \frac{1}{t}$. Let $(x, y) \in X \times X$, from the definition of α and η, we have that $1 \geq \eta(x, y, t)$ for all $t > 0$ iff $x, y \in [0, 1]$. Suppose that $1 \geq \eta(x, y, t)$, then $x, y, \in [0, 1]$, since we have for all $x \in [0, 1]$ $Tx \in [0, 1]$ it follows that $1 \geq \eta(Tx, Ty, t)$. Hence T is η-subadmissible.

Clearly, for any $x_0 \in [0, 1]$ we have $1 \geq \eta(x_0, Tx_0, t)$ for all $t > 0$. Next, let $\{x_n\}$ be a sequence such that $1 \geq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$. Hence $\{x_n\} \subseteq [0, 1]$ and then $x \in [0, 1]$, which implies that $1 \geq \eta(x_n, x, t)$.

Now, we have to prove that (4.26) is satisfied. Let $(x, y) \in X \times X$ such that $1 \geq \eta(x, Tx, t)\eta(y, Ty, t)$ for all $t > 0$. From the definition of α and η we have

$x, y \in [0,1]$. On the other hand, since

$$\begin{aligned}\frac{1}{M(Tx,Ty,t)} &= \frac{1}{e^{-\frac{|Tx-Ty|}{t}}}\\ &= \frac{1}{e^{-\frac{|x^2-y^2|}{4t}}}\\ &\leq \frac{1}{e^{-\frac{|x-y|}{2t}}}\\ &\leq \frac{1}{\sqrt{M(x,y,t}}\end{aligned}$$

It follows that $\xi(M(Tx,Ty,t), N(x,y,t)) \geq 0$. As all conditions of corollary 4.5.1 are fulfilled, then T has a fixed point, here $x = 0$ is a fixed point to T.

By defining $\eta(x,y,t) = 1$ and $\xi : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ by $\xi(t,s) = \frac{1}{\psi(s)} - \frac{1}{t}$ for all $s,t \in (0,1]$ in Theorem 4.4.3, we have the following corollary.

Corollary 4.5.3. *Let $(X, M, *)$ be a complete fuzzy metric space, $T : X \longrightarrow X$ and $\alpha, \eta : X \times X \times (0,\infty) \longrightarrow [0,\infty)$ be a mappings. Assume that there exists a function $\psi \in \Psi$ such that, for all $x, y \in X$ and $t > 0$,*

$$\alpha(x,Tx,t)\alpha(y,Ty,t) \geq 1 \Rightarrow M(Tx,Ty,t) \geq \psi(N(x,y,t)) \tag{4.44}$$

where $N(x,y,t) = \min\{M(x,y,t), \max\{M(x,Tx,t), M(y,Ty,t)\}\}$. Furthermore we suppose that :

- **(i)** *T is triangular α-orbital admissible;*
- **(ii)** *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0,t) \geq \eta(x_0,Tx_0,t)$;*
- **(iii)** *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1},t) \geq \eta(x_n,x_{n+1},t)$ for all $n \in \mathbb{N}$, $t > 0$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x,t) \geq \eta(x_{n(k)},x,t)$ for all $k \in \mathbb{N}$ and $t > 0$.*

Then T has a fixed point.

Chapter 5

Best Proximity Problems For Proximal $\mathcal{FZ}$-Contractions

The best proximity theory is another expanding and prominent aspect of fixed point theory which plays a fundamental role in the investigation of conditions that guarantee the existence of an optimal approximate fixed point when the functional equation $Tx = x$ has no solution. Indeed, a non-self mapping $T : U \longrightarrow V$ does not possess necessarily a fixed point, where U and V are two nonempty subsets of a classical metric space (X, d). It is of fundamental importance to provide an optimal approximate solution $x \in U$ that produces the least amount of error $d(x, Tx)$. Taking into account the fact that $d(x, Tx)$ is at least $d(U, V)$, a best proximity point of T is the optimal approximate solution x satisfying $d(x, Tx) = d(U, V)$. Best proximity theory is a fascinating generalization of fixed point theorems. In fact, if the mapping in question is a self-mapping, the best proximity point turned out to be a fixed point in a natural way.

In the present chapter, following this line of research interest, we present a simulation functions approach to best proximity point problems in fuzzy metric spaces. We initiate new concepts of α-ψ-$\mathcal{FZ}$-contraction, α-$\mathcal{FZ}$-contraction and generalized α-$\mathcal{FZ}$-contraction. We discuss the existence of the best proximity point of such classes of non-self-mappings involving control functions in the structure of complete fuzzy metric spaces. The presented results generalize and extend various existing results in the literature.

5.1 Best proximity point results for α-ψ-$\mathcal{FZ}$-contraction type mappings

Let U and V be two nonempty subsets of a fuzzy metric space $(X, M, *)$. We will employ the following notations:

$$U_0(t) = \{u \in U : M(u, v, t) = M(U, V, t) \text{ for some } v \in V\},$$

$$V_0(t) = \{v \in V : M(u, v, t) = M(U, V, t) \text{ for some } u \in U\},$$

where

$$M(U, V, t) = \sup\{M(u, v, t) : u \in U, v \in V\}.$$

First, we introduce the following concepts.

Definition 5.1.1. Let U and V be two non-empty subsets of fuzzy metric space $(X, M, *)$ and $\alpha : X \times X \times (0, \infty) \to [0, \infty)$. We say that $T : U \to V$ is an α-proximal admissible if

$$\begin{cases} \alpha(x, y, t) \geq 1, \\ M(u, Tx, t) \quad = M(v, Ty, t) = M(U, V, t) \end{cases} \Rightarrow \alpha(u, v, t) \geq 1. \tag{5.1}$$

for all $u, v, x, y \in X$ and $t > 0$.

Definition 5.1.2. Let U and V be two non-empty subsets of fuzzy metric space $(X, M, *)$ and $\alpha : X \times X \times (0, \infty) \to [0, \infty)$. We say that $T : U \to V$ is an α-$\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{FZ}$ if T is an α-proximal admissible, such that

$$\begin{cases} \alpha(x, y, t) \geq 1 \\ M(u, Tx, t) = M(U, V, t) \\ M(v, Ty, t) = M(U, V, t) \end{cases} \Rightarrow \xi(M(u, v, t), M(x, y, t)) \geq 0 \tag{5.2}$$

for all $u, v, x, y \in U$ and $t > 0$.

Definition 5.1.3. Let U and V be two nonempty subsets of fuzzy metric space $(X, M, *)$, $\alpha : X \times X \times (0, \infty) \to [0, \infty)$ and $\psi \in \Psi_2$. We say that $T : U \to V$ is an α-ψ-$\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{FZ}$ if T is an α-proximal admissible such that

$$\begin{cases} \alpha(x, y, t) \geq 1 \\ M(u, Tx, t) = M(U, V, t) \\ M(v, Ty, t) = M(U, V, t) \end{cases} \Rightarrow \xi(M(u, v, t), \psi(M(x, y, t))) \geq 0 \tag{5.3}$$

for all $u, v, x, y \in U$ and $t > 0$.

Remark 5.1.4. Note that Definition 5.1.3 cannot be reduced to Definition 5.1.2, since $\psi(t) = t$ does not belong to Ψ_2.

Definition 5.1.5. Let U and V be two non-empty subsets of fuzzy metric space $(X, M, *)$ and $\alpha : X \times X \times (0, \infty) \to [0, \infty)$. We say that $T : U \to V$ is a generalized α-$\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{FZ}$ if T is an α-proximal admissible, such that

$$\begin{cases} \alpha(x, y, t) \geq 1 \\ M(u, Tx, t) = M(U, V, t) \\ M(v, Ty, t) = M(U, V, t) \end{cases} \Rightarrow \xi(M(u, v, t)), \mathcal{R}(x, y, t)) \geq 0 \tag{5.4}$$

for all $u, v, x, y \in U$ and $t > 0$, where

$$\mathcal{R}(x, y, t) = \min\{M(x, y, t), \frac{M(x, u, t)M(y, v, t)}{M(x, y, t)}\}.$$

Our first main result is the following theorem.

Theorem 5.1.6. *Let U and V be non-empty subsets of a complete fuzzy metric space $(X, M, *)$, $\alpha : X \times X \times (0, \infty) \to [0, \infty)$, $\psi \in \Psi$ and $\xi \in \mathcal{FZ}$ is non-increasing with respect to its second argument. Assume that $T : U \longrightarrow V$ is an α-ψ-$\mathcal{FZ}$-contraction with respect to ξ and*

- **(i)** *T is triangular weak-α-admissible,*
- **(ii)** *U is closed with respect to the topology induced by M,*
- **(iii)** $T(U_0) \subseteq V_0$,
- **(iv)** *there exists $x_0, x_1 \in U$ such that $M(x_0, Tx_0, t) = M(U, V, t)$ and $\alpha(x_0, x_1, t) \geq 1$ for all $t > 0$,*
- **(v)** *T is continuous.*

Then there exists $z \in U$ such that $M(z, Tz, t) = M(U, V, t)$ for all $t > 0$, that is, T has a best proximity point $z \in U$.

Proof. Due to the condition(iv), there exists $x_0, x_1 \in U$ such that $\alpha(x_0, x_1, t) \geq 1$ and

$$M(x_1, Tx_0, t) = M(U, V, t)$$

Regarding (iii), we deduce that $Tx_1 \in V_0$ hence there exists $x_2 \in U$ such that

$$M(x_2, Tx_1, t) = M(U, V, t).$$

Since $\alpha(x_0, x_1, t) \geq 1$ and T is an α-proximal admissible, it follows that $\alpha(x_1, x_2, t) \geq 1$. Recursively, a sequence $\{x_n\} \subset U_0$ can be defined as follows

$$\alpha(x_n, x_{n+1}, t) \geq 1 \text{ for all } n \in \mathbb{N}, \tag{5.5}$$

$$\text{and } M(x_{n+1}, Tx_n, t) = M(U, V, t) \text{ for all } n \in \mathbb{N}. \tag{5.6}$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, we obtain

$$M(x_{n_0}, Tx_{n_0}, t) = M(x_{n_0+1}, Tx_{n_0}, t) = M(U, V, t),$$

which means that x_{n_0} is a best proximity point of T. Therefore, to continue our proof, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Making use of (5.5) and (5.6), we obtain that

$$M(x_n, Tx_{n-1}, t) = M(x_{n+1}, Tx_n, t) = M(U, V, t), \text{ for all } n \in \mathbb{N}.$$

Regarding that T is an α-ψ-$\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{FZ}$, together with (5.5), (5.6) and (ξ_2), we obtain that

$$\begin{aligned}0 &\leq \xi(M(x_n, x_{n+1}, t), \psi(M(x_{n-1}, x_n, t))) \\ &< \frac{1}{\psi(M(x_{n-1}, x_n, t))} - \frac{1}{M(x_n, x_{n+1}, t)}.\end{aligned}$$

Consequently, we have

$$M(x_{n-1}, x_n, t) < \psi(M(x_{n-1}, x_n, t)) < M(x_n, x_{n+1}, t)$$

Which means that $\{M(x_n, x_{n+1}, t)\}$ is a nondecreasing sequence of positive real numbers in $(0, 1]$. Then, there exists $l(t) \leq 1$ such that $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = l(t) \geq 1$ for all $t > 0$. We shall prove that $l(t) = 1$. Reasoning by contradiction, suppose that $l(t_0) < 1$ for some $t_0 > 0$. Now, if we take the sequences $\{\tau_n = M(x_n, x_{n+1}, t_0)$ and $\{\delta_n = M(x_{n-1}, x_n, t_0)\}$ and considering (ψ_1), (ξ_3) and that ξ is non-increasing with respect to its second argument, we get

$$\begin{aligned}0 &\leq \lim_{n\to\infty} \sup \xi(M(x_n, x_{n+1}, t_0), \psi(M(x_{n-1}, x_n, t_0))) \\ &\leq \lim_{n\to\infty} \sup \xi(M(x_n, x_{n+1}, t_0), M(x_{n-1}, x_n, t_0)) \\ &< 0,\end{aligned}$$

A contradiction. Which yields

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1 \text{ for all } t > 0. \tag{5.7}$$

Next, we show that the sequence $\{x_n\}$ is Cauchy. Reasoning by contradiction, suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $\epsilon \in (0, 1)$, $t_0 > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k \geq k$ for all $k \in \mathbb{N}$ such that

$$M(x_{m_k}, x_{n_k}, t_0) \leq 1 - \epsilon. \tag{5.8}$$

Taking in account Lemma 1.5.4 we derive that

$$M(x_{m_k}, x_{n_k}, \frac{t_0}{2}) \leq 1 - \epsilon. \tag{5.9}$$

By choosing m_k as the smallest index satisfying (5.9), we have

$$M(x_{m_k}, x_{n_k-1}, \frac{t_0}{2}) > 1 - \epsilon. \tag{5.10}$$

On account of (5.8),(5.10) and the triangular inequality, we have

$$\begin{aligned}1 - \epsilon &\geq M(x_{m_k}, x_{n_k}, t_0) \\ &\geq M(x_{m_k}, x_{n_k-1}, \frac{t_0}{2}) * M(x_{n_k-1}, x_{n_k}, \frac{t_0}{2}) \\ &> (1 - \epsilon) * M(x_{n_k-1}, x_{n_k}, \frac{t_0}{2})\end{aligned}$$

Taking limit as $k \to \infty$ in both sides of the above inequality and using (5.8), we derive that

$$\lim_{n\to\infty} M(x_{m_k}, x_{n_k}, t_0) = 1 - \epsilon \tag{5.11}$$

On other hand, we have

$$M(x_{m_k-1}, x_{n_k-1}, t_0) \geq M(x_{m_k-1}, x_{m_k}, \frac{t_0}{3}) * M(x_{m_k}, x_{n_k}, \frac{t_0}{3}) * M(x_{n_k}, x_{n_k-1}, \frac{t_0}{3})$$

and

$$M(x_{m_k}, x_{n_k}, t_0) \geq M(x_{m_k}, x_{m_k-1}, \frac{t_0}{3}) * M(x_{m_k-1}, x_{n_k-1}, \frac{t_0}{3}) * M(x_{n_k-1}, x_{n_k}, \frac{t_0}{3})$$

imply that

$$\lim_{n\to\infty} M(x_{m_k-1}, x_{n_k-1}, t_0) = 1 - \epsilon. \tag{5.12}$$

Furthermore, since T is triangular weak-α-admissible and taking into account (5.5), we deduce that

$$\alpha(x_n, x_m, t) \geq 1 \text{ for all } n, m \in \mathbb{N} \text{ with } n > m. \tag{5.13}$$

So that,

$$\alpha(x_{m_k}, x_{n_k}, t_0) \geq 1 \text{ and} \tag{5.14}$$

$$M(x_{m_k}, Tx_{m_k-1}, t_0) = M(x_{n_k}, Tx_{n_k-1}, t_0) = M(U, V, t_0) \text{ for all } k \in \mathbb{N}. \tag{5.15}$$

Regarding the fact that T is an α-ψ-$\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{FZ}$, making use of (5.13) and (5.14), we have

$$0 \leq \xi(M(x_{m_k}, x_{n_k}, t_0), \psi(M(x_{m_k-1}, x_{n_k-1}, t_0))) \text{ for all } k \in \mathbb{N}.$$

From (5.11) and (5.12) we see that the sequences $\{\mu_k = M(x_{m_k}, x_{n_k}, t_0)$ and $\{\nu_k = M(x_{m_k-1}, x_{n_k-1}, t_0)\}$ have the same limit $1 - \epsilon < 1$, taking into account that ξ is non-increasing with respect to its second argument, by the property ξ_3 we conclude that

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \sup \xi(\mu_k, \psi(\nu_k)) \\ &\leq \lim_{n\to\infty} \sup \xi(\mu_k, \nu_k) \\ &< 0. \end{aligned}$$

Which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence in U. Since U is closed subset of a complete fuzzy metric space $(X, M, *)$, there exists $z \in U$ such that

$$\lim_{n\to\infty} M(x_n, z, t) = 1. \tag{5.16}$$

As T is continuous, we conclude that Tx_n converges to Tz, thus

$$\lim_{n\to\infty} M(Tx_n, Tz, t) = 1. \tag{5.17}$$

Due to the continuity of the fuzzy metric M, we have $M(x_{n+1}, Tx_n, t) \to M(z, Tz, t)$. From (5.6), we deduce

$$M(U, V, t) = \lim_{n\to\infty} M(x_{n+1}, Tx_n, t) = M(z, Tz, t).$$

Which means that, $z \in U$ is a best proximity point of non-self mapping T. □

In the next theorem, we replace the continuity hypothesis of T in Theorem 5.1.6 by the following condition :

(H) : if $\{x_n\}$ is a sequence in U such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$, $t > 0$ and $x_n \to x \in U$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x, t) \geq 1$ for all $k \in \mathbb{N}$ and $t > 0$.

Theorem 5.1.7. *Let U and V be non-empty subsets of a complete fuzzy metric space $(X, M, *)$ and $\alpha : X \times X \times (0, \infty) \to [0, \infty)$, $\psi \in \Psi_2$ and $\xi \in \mathcal{FZ}$ is non-increasing with respect to its second argument. Assume that $T : U \longrightarrow V$ is an α-ψ-$\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{FZ}$ and*

- **(i)** *T is triangular weak-α-admissible,*
- **(ii)** *U is closed with respect to the topology induced by M,*
- **(iii)** *$T(U_0) \subseteq V_0$,*
- **(iv)** *there exists $x_0, x_1 \in U$ such that $M(x_1, Tx_0, t) = M(U, V, t)$ and $\alpha(x_0, x_1, t) \geq 1$ for all $t > 0$,*
- **(v)** **(H)** *holds.*

Then there exists $z \in U$ such that $M(z, Tz, t) = M(U, V, t)$ for all $t > 0$, that is, T has a best proximity point $z \in U$.

Proof. Following the lines of the proof of Theorem 5.1.6, we deduce that there exists a Cauchy sequence $\{x_n\}$ in U_0 which converges to $z \in U_0$. Since $T(U_0) \subseteq V_0$, we have $Tz \in V_0$ and then

$$M(a_1, Tz, t) = M(U, V, t) \text{ for some } a_1 \in U_0.$$

By the condition **(H)**, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n_k}, z, t) \geq 1 \text{ for all } k \in \mathbb{N}t > 0. \tag{5.18}$$

Regarding that T is an α-proximal admissible and

$$M(a_1, Tz, t) = M(x_{n_k+1}, Tx_{n_k}, t) = M(U, V, t) \tag{5.19}$$

we obtain that $\alpha(x_{n_k+1}, a_1, t) \geq 1$. Hence

$$0 \leq \xi(M(a_1, x_{n_k+1}, t), \psi(M(z, x_{n_k}, t))) \text{ for all } k \in \mathbb{N}.$$

Applying the property (ξ_2), it follows that

$$M(z, x_{n_k}, t) < \psi(M(z, x_{n_k}, t)) < M(a_1, x_{n_k+1}, t)$$

which yields $\lim_{k\to\infty} M(a_1, x_{n_k+1}, t) = 1$. Then $a_1 = z$, from (5.19) we derive that $M(z, Tz, t) = M(U, V, t)$. This completes the proof. □

5.2 Best proximity theorems for generalized α-$\mathcal{FZ}$-contractions

Theorem 5.2.1. *Let U and V be non-empty subsets of a complete fuzzy metric space $(X, M, *)$ and $\alpha : X \times X \times (0, \infty) \to [0, \infty)$. Assume that $T : U \longrightarrow V$ is a generalized α-$\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{FZ}$ and*

- **(i)** *T is triangular weak-α-admissible,*
- **(ii)** *U is closed with respect to the topology induced by M,*
- **(iii)** *$T(U_0) \subseteq V_0$,*
- **(iv)** *there exists $x_0, x_1 \in U$ such that $M(x_1, Tx_0, t) = M(U, V, t)$ and $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$,*
- **(v)** *T is continuous.*

Then there exists $z \in U$ such that $M(z, Tz, t) = M(U, V, t)$ for all $t > 0$, that is, T has a best proximity point $z \in U$.

Proof. Using the condition (iv), there exists $x_0, x_1 \in U$ such that $\alpha(x_0, x_1, t) \geq 1$ and $M(x_1, Tx_0, t) = M(U, V, t)$. Regarding (iii), we have $Tx_1 \in V_0$ which yields that there exists $x_2 \in U$ such that

$$M(x_2, Tx_1, t) = M(U, V, t).$$

Since $\alpha(x_0, x_1, t) \geq 1$ and T is an α-proximal admissible, it follows that $\alpha(x_1, x_2, t) \geq 1$. We recursively construct the sequence $\{x_n\} \subset U_0$ as follows

$$\alpha(x_n, x_{n+1}, t) \geq 1, \tag{5.20}$$

$$\text{and } M(x_{n+1}, Tx_n, t) = M(U, V, t) \text{ for all } n \in \mathbb{N}. \tag{5.21}$$

Supposing that there exists some $m_0 \in \mathbb{N}$ such that $x_{m_0+1} = x_{m_0}$. Hence,

$$M(x_{m_0}, Tx_{m_0}, t) = M(x_{m_0+1}, Tx_{m_0}, t) = M(U, V, t),$$

Which means that x_{m_0} is a best proximity point of T and the proof is finished. Therefore, to continue our proof, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Making use of (5.20) and (5.21), we obtain

$$M(x_n, Tx_{n-1}, t) = M(x_{n+1}, Tx_n, t) = M(U, V, t) \text{ for all } n \in \mathbb{N} \tag{5.22}$$

Regarding that T is a generalized α-$\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{FZ}$

$$0 \leq \xi(M(x_n, x_{n+1}, t), \mathcal{R}(x_{n-1}, x_n, t)) \tag{5.23}$$

where

$$\begin{aligned}
\mathcal{R}(x_{n-1}, x_n, t) &= \min\{M(x_{n-1}, x_n, t), \frac{M(x_{n-1}, x_n, t)M(x_n, x_{n+1}, t)}{M(x_{n-1}, x_n, t)}\} \\
&= \min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}
\end{aligned}$$

Now, if

$$\min\{M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t)\} = M(x_n, x_{n+1}, t)$$

Applying (ξ_2), we get that

$$\begin{aligned}
0 &\leq \xi(M(x_n, x_{n+1}, t), \mathcal{R}(x_{n-1}, x_n, t)) \\
&< \frac{1}{\mathcal{R}(x_{n-1}, x_n, t)} - \frac{1}{M(x_n, x_{n+1}, t)},
\end{aligned} \tag{5.24}$$

Thus

$$\mathcal{R}(x_{n-1}, x_n, t) = M(x_n, x_{n+1}, t) < M(x_n, x_{n+1}, t) \tag{5.25}$$

Which is a contradiction. Consequently,

$$\mathcal{R}(x_{n-1}, x_n, t) = \min\{M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t)\} = M(x_{n-1}, x_n, t)$$

By (5.24), we obtain that

$$M(x_{n-1}, x_n, t) < M(x_n, x_{n+1}, t) \text{ for all } n \in \mathbb{N}$$

Hence, we deduce that $\{M(x_n, x_{n+1}, t)\}$ is a nondecreasing sequence of positive real numbers in $(0, 1]$. Thus, there exists $s(t) \leq 1$ such that $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = s(t) \geq 1$ for all $t > 0$. We shall prove that $s(t) = 1$. Reasoning by contradiction, suppose that $s(t_0) < 1$ for some $t_0 > 0$. Now, if we take the sequences $\mathcal{T}_n = M(x_n, x_{n+1}, t_0)$ and $\mathcal{S}_n = M(x_{n-1}, x_n, t_0)$ and considering (ξ_3), we obtain

$$0 \leq \lim_{n\to\infty} \sup \xi(\mathcal{T}_n, \mathcal{S}_n) < 0,$$

A contradiction. Therefore,

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1 \text{ for all } t > 0. \tag{5.26}$$

Next, we show that the sequence $\{x_n\}$ is Cauchy. Reasoning by contradiction, assume that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $\epsilon \in (0,1)$, $t_0 > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k \geq k$ for all $k \in \mathbb{N}$ such that

$$M(x_{m_k}, x_{n_k}, t_0) \leq 1 - \epsilon. \tag{5.27}$$

Taking in account Lemma 1.5.4, we derive that

$$M(x_{m_k}, x_{n_k}, \frac{t_0}{2}) \leq 1 - \epsilon. \tag{5.28}$$

By choosing m_k as the smallest index satisfying (5.28), we have

$$M(x_{m_k}, x_{n_k-1}, \frac{t_0}{2}) > 1 - \epsilon. \tag{5.29}$$

Making use of (5.27),(5.29) and the triangular inequality, we get

$$\begin{aligned} 1 - \epsilon &\geq M(x_{m_k}, x_{n_k}, t_0) \\ &\geq M(x_{m_k}, x_{n_k-1}, \frac{t_0}{2}) * M(x_{n_k-1}, x_{n_k}, \frac{t_0}{2}) \\ &> (1-\epsilon) * M(x_{n_k-1}, x_{n_k}, \frac{t_0}{2}) \end{aligned}$$

Passing to the limit $k \to \infty$ in both sides of the above inequality and using (5.26), we derive that

$$\lim_{n\to\infty} M(x_{m_k}, x_{n_k}, t_0) = 1 - \epsilon \tag{5.30}$$

On other hand,

$$M(x_{m_k-1}, x_{n_k-1}, t_0) \geq M(x_{m_k-1}, x_{m_k}, \frac{t_0}{3}) * M(x_{m_k}, x_{n_k}, \frac{t_0}{3}) * M(x_{n_k}, x_{n_k-1}, \frac{t_0}{3})$$

and

$$M(x_{m_k}, x_{n_k}, t_0) \geq M(x_{m_k}, x_{m_k-1}, \frac{t_0}{3}) * M(x_{m_k-1}, x_{n_k-1}, \frac{t_0}{3}) * M(x_{n_k-1}, x_{n_k}, \frac{t_0}{3})$$

imply that

$$\lim_{n\to\infty} M(x_{m_k-1}, x_{n_k-1}, t_0) = 1 - \epsilon. \tag{5.31}$$

Furthermore, since T is triangular weak-α-admissible, we deduce that

$$\alpha(x_n, x_m, t) \geq 1 \text{ for all } n, m \in \mathbb{N} \text{ with } n > m. \tag{5.32}$$

Thus

$$\alpha(x_{m_k}, x_{n_k}, t_0) \geq 1 \text{ and} \tag{5.33}$$

$$M(x_{m_k}, Tx_{m_k-1}, t_0) = M(x_{n_k}, Tx_{n_k-1}, t_0) = M(U, V, t_0) \text{ for all } k \in \mathbb{N}. \tag{5.34}$$

Regarding the fact that T a generalized α-$\mathcal{FZ}$-contraction with respect to $\xi \in \mathcal{FZ}$ and using (5.33)-(5.34), we obtain that

$$0 \leq \xi(M(x_{m_k}, x_{n_k}, t_0), \mathcal{R}(x_{m_k-1}, x_{n_k-1}, t_0)) \text{ for all } k \in \mathbb{N}. \tag{5.35}$$

Where

$$\mathcal{R}(x_{m_k-1}, x_{n_k-1}, t_0) = \min\{M(x_{m_k-1}, x_{n_k-1}, t_0), \frac{M(x_{m_k-1}, x_{m_k}, t_0)M(x_{n_k-1}, x_{n_k}, t_0)}{M(x_{m_k-1}, x_{n_k-1}, t_0)}\}$$

Letting $k \to \infty$ in the above equality and using (5.26), we deduce that

$$\lim_{k\to\infty} \mathcal{R}(x_{m_k-1}, x_{n_k-1}, t_0) = \min\{\frac{1}{1-\epsilon}, 1-\epsilon\} = 1 - \epsilon$$

Taking the sequences $\mu_k = M(x_{m_k}, x_{n_k}, t_0)$ and $\nu_k = \mathcal{R}(x_{m_k-1}, x_{n_k-1}, t_0)$ for all $k \in \mathbb{N}$. Applying ξ_3, we derive that

$$0 \leq \lim_{n\to\infty} \sup \xi(\mu_k, \nu_k) < 0, \tag{5.36}$$

Which is a contradiction. Then, $\{x_n\}$ is a Cauchy sequence in U. Since U is closed subset of a complete fuzzy metric space $(X, M, *)$, there exists $z \in U$ such that

$$\lim_{n\to\infty} M(x_n, z, t) = 1. \tag{5.37}$$

As T is continuous, we obtain that Tx_n converges toTz, thus

$$\lim_{n\to\infty} M(Tx_n, Tz, t) = 1. \tag{5.38}$$

Due to the continuity of the fuzzy metric M, we have $M(x_{n+1}, Tx_n, t) \to M(z, Tz, t)$. in view of (5.22), we get

$$M(U, V, t) = \lim_{n\to\infty} M(x_{n+1}, Tx_n, t) = M(z, Tz, t). \tag{5.39}$$

Thus, $z \in U$ is a best proximity point of non-self mapping T.

□

5.3 Best proximity results for α-$\mathcal{FZ}$-contractions

Theorem 5.3.1. *Let U and V be non-empty subsets of a complete fuzzy metric space $(X, M, *)$, $\alpha : X \times X \times (0, \infty) \to [0, \infty)$ and $\xi \in \mathcal{FZ}$. Assume that $T : U \longrightarrow V$ is an α-$\mathcal{FZ}$-contraction with respect to ξ and*

- **(i)** *T is triangular weak-α-admissible,*
- **(ii)** *U is closed with respect to the topology induced by M,*
- **(iii)** *$T(U_0) \subseteq V_0$,*
- **(iv)** *there exists $x_0, x_1 \in U$ such that $M(x_0, Tx_0, t) = M(U, V, t)$ and $\alpha(x_0, x_1, t) \geq 1$ for all $t > 0$,*
- **(v)** *T is continuous or* (**H**) *holds.*

Then there exists $z \in U$ such that $M(z, Tz, t) = M(U, V, t)$ for all $t > 0$, that is, T has a best proximity point $z \in U$.

Proof. following the same arguments as those given in the proof of Theorem 5.2.1, we know that there exists a Cauchy sequence $\{x_n\}$ in U which converges to $z \in U$. Further

$$\lim_{n\to\infty} M(x_n, z, t) = 1 \text{ for all } n \in \mathbb{N}t > 0. \tag{5.40}$$

If T is continuous,then

$$\lim_{n\to\infty} M(Tx_n, Tz, t) = 1 \text{ for all } n \in \mathbb{N}t > 0. \tag{5.41}$$

Taking into account (5.6), (5.40) and (5.41) we obtain that

$$M(U, V, t) = \lim_{n\to\infty} M(x_{n+1}, Tx_n, t) = M(z, Tz, t).$$

Which means that, $z \in U$ is a best proximity point of non-self mapping T.
Now, suppose that (**H**) holds. Since $T(U_0) \subseteq V_0$, we have $Tz \in V_0$ and then

$$M(a_1, Tz, t) = M(U, V, t) \text{ for some } a_1 \in U_0.$$

By the condition (**H**), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n_k}, z, t) \geq 1 \text{ for all } k \in \mathbb{N}t > 0. \tag{5.42}$$

Regarding that T is an α-proximal admissible and

$$M(a_1, Tz, t) = M(x_{n_k+1}, Tx_{n_k}, t) = M(U, V, t) \tag{5.43}$$

we obtain that $\alpha(x_{n_k+1}, a_1, t) \geq 1$. Hence

$$0 \leq \xi(M(a_1, x_{n_k+1}, t), M(z, x_{n_k}, t)) \text{ for all } k \in \mathbb{N}.$$

Applying the property (ξ_2), it follows that

$$M(z, x_{n_k}, t) < M(a_1, x_{n_k+1}, t)$$

which yields $\lim_{k\to\infty} M(a_1, x_{n_k+1}, t) = 1$. Then $a_1 = z$, from (5.19) we derive that $M(z, Tz, t) = M(U, V, t)$. This completes the proof. □

Remark 5.3.2. Note that Theorem 5.3.1 cannot be deduced by combining Theorem 5.1.6 and Theorem 5.2.1, since the function $\psi(t) = t$ does not belong to Ψ_2. Moreover, in Theorem 5.1.6 and 5.1.7, we have an additional condition that ξ is non-increasing in its second argument.

5.4 Consequences

In this section, we shall illustrate that several consequences of the existence results can be easily concluded from our main results.

Corollary 5.4.1. *Let U and V be nonempty subsets of a complete fuzzy metric space $(X, M, *)$, $\alpha : X \times X \times (0, \infty) \to [0, \infty)$, $\psi \in \Psi_2$. Assume that $T : U \longrightarrow V$ is an α-admissible proximal mapping such that*

$$\begin{cases} \alpha(x, y, t) \geq 1 \\ M(u, Tx, t) = M(U, V, t) \\ M(v, Ty, t) = M(U, V, t) \end{cases} \Rightarrow M(u, v, t) \geq \psi(M(x, y, t))$$

for all $u, v, x, y \in U$ and $t > 0$. Suppose also

- **(i)** *T is triangular weak-α-admissible,*
- **(ii)** *U is closed with respect to the topology induced by M,*
- **(iii)** *$T(U_0) \subseteq V_0$,*
- **(iv)** *there exists $x_0, x_1 \in U$ such that $M(x_0, Tx_0, t) = M(U, V, t)$ and $\alpha(x_0, x_1, t) \geq 1$ for all $t > 0$,*
- **(v)** *T is continuous or* **(H)** *holds.*

Then there exists $z \in U$ such that $M(z, Tz, t) = M(U, V, t)$ for all $t > 0$, that is, T has a best proximity point $z \in U$.

Proof. Define $\xi : (0, 1] \times (0, 1] \longrightarrow \mathbb{R}$ by

$$\xi(t, s) = \frac{1}{\psi(s)} - \frac{1}{t} \text{ for all } s, t \in (0, 1].$$

Since $\xi \in \mathcal{FZ}$ the desired results follow from Theorem 5.3.1. □

Corollary 5.4.2. *Let U and V be nonempty subsets of a complete fuzzy metric space $(X, M, *)$, $\alpha : X \times X \times (0, \infty) \to [0, \infty)$, $\eta \in \mathcal{H}$. Assume that $T : U \longrightarrow V$ is an α-admissible proximal mapping such that*

$$\begin{cases} \alpha(x, y, t) \geq 1 \\ M(u, Tx, t) = M(U, V, t) \\ M(v, Ty, t) = M(U, V, t) \end{cases} \Rightarrow \eta(M(u, v, t)) \leq k\eta(M(x, y, t)),$$

for all $u, v, x, y \in U$ and $t > 0$. Suppose also

(i) *T is triangular weak-α-admissible,*

(ii) *U is closed with respect to the topology induced by M,*

(iii) *$T(U_0) \subseteq V_0$,*

(iv) *there exists $x_0, x_1 \in U$ such that $M(x_0, Tx_0, t) = M(U, V, t)$ and $\alpha(x_0, x_1, t) \geq 1$ for all $t > 0$,*

(v) *T is continuous or **(H)** holds.*

Then there exists $z \in U$ such that $M(z, Tz, t) = M(U, V, t)$ for all $t > 0$, that is, T has a best proximity point $z \in U$.

Proof. It follows from Theorem 5.3.1 using the $\mathcal{FZ}$-simulation function $\xi(t, s) = \frac{1}{\eta^{-1}(k.\eta(s))} - \frac{1}{t}$ for all $s, t \in (0, 1]$.

□

Corollary 5.4.3. *Let U and V be nonempty subsets of a complete fuzzy metric space $(X, M, *)$, $\alpha : X \times X \times (0, \infty) \to [0, \infty)$. Assume that $T : U \longrightarrow V$ is an α-admissible proximal mapping such that*

$$\begin{cases} \alpha(x, y, t) \geq 1 \\ M(u, Tx, t) = M(U, V, t) \\ M(v, Ty, t) = M(U, V, t) \end{cases} \Rightarrow \left(\frac{1}{M(u, v, t)} - 1\right) \leq \phi\left(\frac{1}{M(x, y, t)} - 1\right)$$

for all $u, v, x, y \in U$ and $t > 0$, where $\phi : [0, \infty) \longrightarrow [0, \infty)$ with $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$. Suppose also

(i) *T is triangular weak-α-admissible,*

(ii) *U is closed with respect to the topology induced by M,*

(iii) *$T(U_0) \subseteq V_0$,*

(iv) *there exists $x_0, x_1 \in U$ such that $M(x_0, Tx_0, t) = M(U, V, t)$ and $\alpha(x_0, x_1, t) \geq 1$ for all $t > 0$,*

(v) *T is continuous or **(H)** holds.*

Then there exists $z \in U$ such that $M(z, Tz, t) = M(U, V, t)$ for all $t > 0$, that is, T has a best proximity point $z \in U$.

Proof. Follows from Theorem 5.3.1 by taking $\xi(t, s) = \phi(\frac{1}{s} - 1) - \frac{1}{t} + 1$ for all $s, t \in (0, 1]$. □

5.5 Application to fixed point theory

In this section, as an application of the obtained theorems, we will deduce some new fixed point results in the context of fuzzy metric spaces.
First, note that if

$$\begin{cases} U = V = X \\ M(u, Tx, t) = M(U, V, t) \\ M(v, Ty, t) = M(U, V, t) \end{cases}$$

It follows that $Tx = u$ and $Ty = v$. Consequently, taking $U = V = X$ in Theorem 5.3.1, we get the following fixed point result

Corollary 5.5.1. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \longrightarrow X$ be a mapping satisfying*

$$\alpha(x, Tx, t) \geq 1 \Rightarrow \xi(M(Tx, Ty, t), M(x, y, t)) \geq 0$$

and assume that

(i) *T is triangular weak-α-admissible,*

(ii) *there exists $x_0, x_1 \in U$ such that $\alpha(x_0, x_1, t) \geq 1$ for all $t > 0$,*

(iii) *T is continuous, or* (**H**) *holds.*

Then T has a fixed point.

Corollary 5.5.2. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \longrightarrow X$ be a mapping satisfying*

$$\alpha(x, Tx, t) \geq 1 \Rightarrow \xi(M(Tx, Ty, t), \mathcal{R}(x, y, t)) \geq 0$$

where

$$\mathcal{R}(x, y, t) = \min\{M(x, y, t), \frac{M(x, u, t)M(y, v, t)}{M(x, y, t)}\}.$$

and assume that

(i) *T is triangular weak-α-admissible,*

(ii) *there exists $x_0, x_1 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$,*

(iii) *T is continuous.*

Then T has a fixed point point.

Proof. The result follows by taking $U = V = X$ in Theorem 5.2.1. $\square$

Remark 5.5.3. We must point to the fact that, by defining the control function ξ and the admissible mapping $\alpha(x, y)$ in a proper way, it is possible to particularize and derive a number of varied consequences. We skip making such a number of corollaries since they seem clear.

Conclusion, Research Scope and Perspectives

In this study, we introduced new types of nonlinear contractions and established new fixed point results in different contexts. First, we defined the class of admissible almost type $\mathcal{Z}$-contractions in complete metric space by combining the ideas of admissible functions and simulation functions and proved some fixed point results for such type of mappings. Furthermore, we showed that the defined concept generalize several existing and known concepts of contractions in the literature, including the famous Banach contraction mappings. Additionally, we discussed several results which can be obtained as consequences of our main theorems in the framework of metric spaces and metric spaces endowed with a partial order.

In the framework of fuzzy metric spaces, we proposed a novel approach to the study of fixed point theory by introducing a new type of control functions. On the basis of the same class, we defined a new type of fuzzy contraction mappings, $\mathcal{FZ}$-contractions, with the purpose to unify, enrich and cover different existing types of contractions in the fuzzy metric spaces backdrop. Moreover, we proved the existence and uniqueness of fixed point for the newly defined contractions. Some illustrative examples are presented to clarify the unifying power of our concepts. Indeed, motivated by the work of Melliani and Moussaoui [72], Hayel *et al.*[32] presented new kind of fuzzy contractive mappings named as ($\mathcal{FZ}$,F,φ)-contractive mappings. Following this direction, we introduced the notion of extended $\mathcal{FZ}$-simulation functions, which is intended to enlarge and refine the definition of $\mathcal{FZ}$-simulation functions. Then, we conceived the idea of $(\mathcal{FZ}_e^{\varphi}, F)$-contraction and established some φ-fixed point results in complete fuzzy metric spaces. The obtained results, improve, extend and generalize those given by Gregori and Sapena [78], Mihet [19], Wardowski [20], Hayel *et al.* [32], Sezen *et al.* [54]. In the same lines, we established the notions of generalized α-η-$\mathcal{FZ}$-contractions and modified α-η-$\mathcal{FZ}$-contractions, based on the notion of α-admissible function with respect to η. We proved some fixed point results for such mappings in the framework of fuzzy metric spaces. The presented concepts enrich, extend and generalize different types of contraction mappings in the current literature, in particular, the contractive conditions initiated by Gopal and Calogero vetro [42], Mishra et al. [77], Gregori and Sapena [78], Mihet [19] and Melliani *et al.* [72].

Finally, we presented a simulation function approach to best proximity point problems in fuzzy metric spaces. We initiated some new types of proximal fuzzy contraction, α-ψ-$\mathcal{FZ}$-contraction, α-$\mathcal{FZ}$-contraction and generalized α-$\mathcal{FZ}$-contraction, we discussed the existence results of best proximity point of such classes of non-self mappings involving control functions in the structure of complete fuzzy metric spaces. The obtained results, enrich, generalize and extend various notions to the case of non-self mappings.

The study discussed in this dissertation offers a number of directions for future works. On the one hand, the newly introduced concepts can be fertile groundwork of further generalizations and extensions to different settings. For instance, our obtained results in the structure of fuzzy metric spaces, can be further studied in the frame of partially ordered fuzzy metric spaces, complex-valued fuzzy metric spaces, fuzzy b-metric spaces and generalized fuzzy metric spaces. Moreover, further extensions can be provided to L-fuzzy mappings and fuzzy neutrosophic soft mappings. Other attractive areas of research are the study of the existence and uniqueness of common fixed point and coincidence points of two operators. Thus, It would be important to see how the given results concerning $\mathcal{FZ}$-contractions can be used to obtain new findings in this regard.

Bibliography

[1] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," Proceedings of the American Mathematical Society, vol. 132, no. 5, pp. 1435–1443, 2004.

[2] A.F. Roldán-López-de-Hierro, E. Karapınar, C. Roldán-López-de-Hierro, J. Martínez-Moreno, Coincidence point theorems on metric spaces via simulation functions, J. Comput. Appl. Math. 275 (2015) 345-355.

[3] A. George and P.Veeramani,On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64(3) (1994),395–399.

[4] A. Petruşel and I. A. Rus, "Fixed point theorems in ordered L-spaces," Proceedings of the American Mathematical Society, vol. 134, no. 2, pp. 411–418, 2006.

[5] A.Moussaoui, N.Hussain, S.Melliani, Global optimal solutions for proximal fuzzy contractions involving control functions, Journal of Mathematics, in press, 2021.

[6] Argoubi H., Samet S., Vetro C., Nonlinear contractions involving simulation functions in a metric space with a partial order,J. Nonlinear Sci. Appl., 2015, 8, 1082-1094.

[7] Aydi, H., Felhi, A.: Fixed points in modular spaces via α-admissible mappings and simulation functions.J. Nonlinear Sci. Appl. 9(6), 3686–3701 (2016).

[8] Banach, S. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. Fund. Math. 3, 133–181 (1922).

[9] B.D. Rhoades, A comparaison of various definitions of contractive mappings. Trans. Amer. Math. Soc 226 (1977),257-290.

[10] Berinde, V. and Păcurar, M., Iterative approximation of fixed points of single-valued almost contractions, in Fixed Point Theory and Graph Theory, 29–97, Elsevier/Academic Press, Amsterdam, 2016.

[11] Border. KC, Fixed Point Theorems with Applications to Economics and Game Theory. Cambridge University Press, New York (1985).

[12] Boyd, D.W., Wong, J.S.W.: On nonlinear contractions. Proc. Amer. Math. Soc. 20, 458–464 (1969).

[13] B. Samet, C.Vetro, and P.Vetro, "Fixed point theorems for α-ψ contractive type mappings," Nonlinear Analysis. Theory, Methods and Applications A, vol. 75, no. 4, pp. 2154-2165, 2012.

[14] B. Schweizer and Sklar, Statistical metric spaces, Pacific J.Math 10(1)(1960), 313-334.

[15] Calogero Vetro, Peyman Salimi, Best Proximity Point Results in Non-Archimedean Fuzzy Metric Spaces, Fuzzy Information and Engineering, 5:4, 417-429,(2013).

[16] Chanda, A., Damjanovic, B., Dey, L.K.: Fixed point results on θ-metric spaces via simulation functions. Filomat 31(11), 3365–3375 (2017).

[17] De la Sen M, Abbas M, Saleem N. On optimal fuzzy best proximity coincidence points of fuzzy order preserving proximal $\psi(\sigma, \alpha)$-lower-bounding asymptotically contractive mappings in non-Archimedean fuzzy metric spaces. Springerplus. 2016;5(1):1478.

[18] Demma, M., Saadati,R., Vetro, p.: Fixed point results on b- metric space via Picard sequences and b-simulations. Iran.J. Math.Sci. Inform. 11(1),123-136(2016).

[19] D. Mihet, Fuzzy ψ-contractive mappings in non-archimedean fuzzy metric spaces, Fuzzy Sets and Systems 159(6) (2008),739–744.

[20] D. Wardowki, Fuzzy contractive mappings and fixed points in fuzzy metric spaces, Fuzzy Sets Syst.222(2013) 108-114.

[21] Efe Ok. EA, Real Analysis with Economic Applications. Princeton University Press, Princeton (2007).

[22] E. Karapinar and F. Khojasteh, An approach to best proximity points via simulation functions, J. Fixed Point Theory Appl. 19 (2017).

[23] E. Karapinar, Fixed points results via simulation functions, 393 Filomat 30(8) (2016), 2343-2350.

[24] E. Karapinar , B.Samet: Generalized α-ψ-contractive type mappings and related fixed point theorems with applications. Abstr. Appl. Anal.2012, Article ID 793486 (2012).

[25] Fan. K, Extensions of two fixed point theorems of F.E. Browder. Math. Z. 112(3), 234-240 (1969)

[26] F. E. Browder,W. V. Petrysyn, The solution by iteration of nonlinear functional equation in Banach spaces, Bull. Amer. Math. Soc. 72 (1966), 571-576.

[27] Felhi, A., Aydi, H., Zhang, D.: Fixed points for α-admissible contractive mappings via simulation functions.J. Nonlinear Sci. Appl. 9(10), 5544–5560 (2016)

[28] F. Khojasteh, S .Shukla and S. Radenović, A new approach to the study of fixed point theory for simulation functions, Filomat 29(6) (2015), 1189–1194.

[29] F. Tchier, C. Vetro, F. Vetro, Best approximation and variational inequality problems involving a simulation function, Fixed Point Theory Appl.2016 (26) (2016).

[30] Fréchet, M. Sur quelques points du calcul fonctionnel. Rend. Circ.Mat. Palermo, 22, 1–74(1906).

[31] Jleli M, Samet B. A generalized metric space and related fixed point theorems. Fixed Point Theory Appl 2015;61

[32] Hayel N.S, I.A. Khan, M. Imdad and W.M. Alfaqih, New fuzzy φ-fixed point results employing a new class of fuzzy contractive mappings

[33] I. Altun and H. Simsek, "Some fixed point theorems on ordered metric spaces and application," Fixed Point Theory and Applications, vol. 2010, Article ID 621469, 17 pages, 2010.

[34] I. A. Rus, A. Petruşel, G. Petruşel, Fixed Point Theory, Cluj University Press, (2008). 2.2, 2

[35] I. A. Rus, M. A. Şerban, Some generalizations of a Cauchy lemma and applications, Topics in Mathematics, Computer Science and Philosophy, Presa Univ. Clujean, Cluj-Napoca, (2008), 173-181.2.1, 2.3

[36] I. Kramosil and J. Michalek, Fuzzy metrics and statistical metric spaces, Kybernetika 11(5) (1975), 336–344.

[37] Iram Iqbal, Nawab Hussain, Marwan Amin Kutbi: Existence of the solution to variational inequality, optimization problem, and elliptic boundary value problem through revisited best proximity point results. J. Comput. Appl. Math. 375: 112804 (2020)

[38] G. Babu, K. Sarma and Y.G. Aemro, Generalization of fixed point results for (α, η, β)-contractive mappings in fuzzy metric spaces, Bangmod Int J.Math & Comp Sci 3(1-2)(2017),35-52.

[39] G.E. Hary, T.D. Rogers, A generalization of a fixed point theorem of Reich. Canad.Math. Bull. 16, 201–206 (1973)

[40] Gh. Heidary Joonaghany, A. Farajzadeh, M. Azhini, F. Khojasteh, A new common fixed point theorem for Suzuki type contractions via generalized ψ-simulation functions, Sahand Communications in Mathematical Analysis, 16(1) (2019), 129–148

[41] Geraghty, M. (1973). On contractive mappings. Proc. Am. Math. Soc., 40, 604–608.

[42] Gopal, D, Vetro, C: Some new fixed point theorems in fuzzy metric spaces. Iran. J. Fuzzy Syst. 11(3), 95-107 (2014)

[43] Khojasteh, F., Karapınar, E., Radenović S.: θ-metric space: A generalization. Math. Probl. Eng. (2013)(Article ID 504609)

[44] Kumar, M., Sharma, R.: A new approach to the study of fixed point theorems for simulation functions in G-metric spaces. Bol. Soc. Paran. Mat. 37(2), 113-119 (2019)

[45] L. A. Zadeh, Fuzzy sets. Inform. and Control, 8:338-353 (1965).

[46] L. B. Ćirić, "Fixed points for generalized multi-valued contractions," Matematiĉki Vesnik, vol. 9, no. 24,pp. 265–272, 1972.

[47] L. Ćirić, N. Cakicć, M. Rajović, and J. S. Ume, "Monotone generalized nonlinear contractions in partially ordered metric spaces," Fixed Point Theory and Applications, vol. 2008, Article ID 131294, 11 pages, 2008.

[48] Mathews, GS, Partial metric topology. Reseach Report 212, Dept. of Computer Science University of Warwick (1992)

[49] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems 27(3) (1988), 385–389.

[50] Mohamed A. Khamsi, William A. Kirk, An Introduction to Metric Spaces and Fixed Point Theory, Pure and Applied Mathematics (New York). Wiley-Interscience, New York,2001.

[51] M.S. Khan, M.Swalech and S.Sessa, fixed point theorem by altering distances betwen the points, Bull. Aust. Math. Soc. 30.1-9(1984)

[52] MS. El Naschie, A review of E-infinity theory and the mass spectrum of high energy particle physics, Chaos. Soliton. Fract. 19 (2004) 209–236.

[53] MS. El Naschie, On the uncertainty of Cantorian geometry and two-slit experiment, Chaos. Soliton. Fract. 9 (1998) 517–529.

[54] M.S. Sezen and D. Türkoğlu, Some fixed point theorems of (F, φ)-fuzzy contractions in fuzzy metric spaces, Journal of Inequalities & Special Functions 8(4) (2017).

[55] M. Turinici, Abstract comparison principles and multivariable Gronwall-Bellman inequalities, J. Math. Anal. Appl. 117(1986), 100–127.

[56] Nastasi A., Vetro P., Fixed point results on metric and partial metrric spaces via simulations functions, J. Nonlinear Sci. Appl.,2015, 8, 1059–1069

[57] N. Mlaiki, N.Y. $\ddot{O}zg\ddot{u}r$, N. Taş, New Fixed-Point Theorems on an S-metric Space via Simulation Functions, Mathematics 2019, 7, 583

[58] O. Kaleva and S. Seikkala, On fuzzy metric spaces, Fuzzy Sets and Systems, 12 (1984), 215-229.

[59] O. Popescu, Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl., 2014 (2014), 12 pages

[60] P. Chuadchawna, A. Kaewcharoen, S. Plubtieng Fixed point theorems for generalized α-η-ψ-Geraghty contraction type mappings in α-η-complete metric spaces, J. Nonlinear Sci. Appl. 9 (2016), 471-485

[61] P. Veeramani, Best approximation in fuzzy metrics paces,Journal of Fuzzy Mathematics, 9 (2001)75–80.

[62] R. Kannan, "Some results on fixed points," Bulletin of the Calcutta Mathematical Society, vol. 10, pp. 71–76, 1968.

[63] Romaguera, S, Valero, O: A quantitative computational model for complete partial metric spaces via formal balls.Math. Struct. Comput. Sci. 19(3), 541-563 (2009)

[64] Reich, S. Fixed point of contractive functions. Bollettino dell Unione Matematica Italiana 1972, 4, 26–42.

[65] R. P. Agarwal, M. A. El-Gebeily, and D. O'Regan, "Generalized contractions in partially ordered metric spaces," Applicable Analysis, vol. 87, no. 1, pp. 109–116, 2008.

[66] Said Melliani, Abdelhamid Moussaoui and L.S. Chadli, Admissible Almost Type $\mathcal{Z}$-Contractions and Fixed Point Results, Int. J. Math & Math. Sci, Volume 2020, 9104909:1-9104909:7 (2020)

[67] Samet, B., Vetro, C., Vetro, P., Fixed point theorem for α-ψ-contractive type mappings. Nonlinear Anal. 75 (2012), 2154-2165.

[68] Satish Shukla, Dhananjay Gopal, Wutiphol Sintunavarat, A new class of fuzzy contractive mappings and fixed point theorems, Fuzzy Sets and Systems, Volume 350,2018, Pages 85-94,

[69] Seong-Hoon Cho, Fixed Point Theorems for $\mathcal{L}$-Contractions in Generalized Metric Spaces, Abstract and Applied Analysis, Volume 2018, Article ID 1327691, 6 pages.

[70] S. K. Chatterjea, "Fixed-point theorems," Comptes Rendus de lÁcadémie Bulgare des Sciences, vol. 25, pp. 727–730, 1972.

[71] S. Melliani, Abdelhamid Moussaoui and L.S. Chadli, Admissible Almost Type $\mathcal{Z}$-Contractions and Fixed Point Results, Int. J. Math & Math. Sci, Volume 2020, 9104909:1-9104909:7 (2020)

[72] S. Melliani and A. Moussaoui, Fixed point theorem using a new class of fuzzy contractive mappings, Journal of Universal Mathematics 1(2) (2018), 148–154.

[73] S. Phiangsungnoen, Y. J. Choc and P. Kumam. Fixed Point Results for Modied Various Contractions in Fuzzy Metric Spaces via α-admissibl. Filomat, 30(7)(2016), 1869-1881.

[74] Suzuki T, A generalized Banach contraction principle that characterizes metric completeness. Proc. Am. Math. Soc. 2008, 136: 1861–1869.

[75] T. Zamfirescu, Fixed point theorems in metric spaces. Arch. Math. (Basel) 23, 292–298 (1972)

[76] T. Suzuki, M. Kikkawa, C. Vetro The existence of best proximity points in metric spaces with the property UC Nonlinear Anal., 71 (2009), pp. 2918-2926

[77] U. Mishra, C.Vetro and P.Kumam, On modified α-ϕ-fuzzy contractive mappings and an application to integral equations, Journal of Inequalities and Applications (2016) 2016:67.

[78] V. Gregori and A. Sapena, On fixed-point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 125(2) (2002),245–252.

[79] V. Gregori,S. Morillas, A. Sapena, Examples of fuzzy metrics and applications,Fuzzy Sets and Systems 170(2011)95-111.

[80] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Analysis Forum, 9(2004), No. 1, 43-53.

[81] V. Berinde, General constructive fixed point theorems for Ćirić type almost contraction in metric spaces, Carpath. J. Math. 24(2008), 1019

[82] V. Berinde, Itertive Approximation of Fixed Points, Editura Efemeride, Baia Mare, Romania,2002

[83] Z. Deng, Fuzzy pseudometric spaces, J. Math. Anal. Appl., 86 (1982), 74-95.1