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Controllability and inverse problems of some systems governed by degenerate parabolic equations

Présentée par :

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Liste des publications

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Communications orales

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Abstract

In this thesis, we study the inverse source problem and controllability properties of some systems governed by degenerate parabolic equations.

Concerning the inverse problem, in Chapter 2, we address the issue of retrieving, simultaneously, n source terms in a linear coupled degenerate parabolic system. Using a global Carleman estimate with a single locally distributed observation, we derive a Lipschitz stability estimate in determining the n source terms from local measurements of only one component of the system.

For the controllability issue, in Chapter 3, we establish the null controllability of a coupled system of $n \geq 2$ degenerate parabolic equations involving singular potentials. First, under appropriate assumptions on the coupling terms, the wellposedness issue is treated using semigroup theory and some weighted inequalities of Hardy Poincaré's type. Then, employing a Carleman estimate with single internal observation, we prove the null controllability of the system with a single distributed control. In Chapter 4, we deal with the control problem of a degenerate parabolic equation with a memory term. We start by deriving the distributed null controllability of a nonhomogeneous equation with regular solutions. Then, as a consequence, using the Kakutani's fixed point theorem, we deduce the null controllability property for the initial memory problem.

Coming to the Chapter 5, we investigate the boundary controllability of some 2×2 degenerate parabolic systems. In particular, we provide necessary and sufficient conditions for the approximate and null controllability of the system when a unique control force is exerted on a part of the boundary through a Dirichlet condition. Finally, in Chapter 6, we analyze the approximate and null controllability properties of a degenerate heat equation when a pointwise control force acts on a single point inside the spatial domain. Our technique is essentially based on a spectral analysis and the moment method.

Keywords: Inverse source problem, degenerate parabolic equation, degenerate/singular system, controllability, observability, parabolic equation with memory, Carleman estimates, moment method, minimal time.

Résumé

Dans cette thèse, nous étudions le problème inverse de source et les propriétés de contrôlabilité de quelques systèmes gouvernés par des équations paraboliques dégénérées.

Concernant le problème inverse, au Chapitre 2, nous abordons la question d'identification simultanée de n termes sources dans un système parabolique linéaire dégénéré couplé. En utilisant une estimation globale de type Carleman avec une seule observation localement distribuée, nous dérivons un résultat de stabilité Lipschitzienne dans la détermination de n termes sources à partir des mesures locales sur une seule composante du système.

Pour le problème de la contrôlabilité, au Chapitre 3, nous étudions la contrôlabilité à zéro d'un système couplé de $n \geq 2$ équations paraboliques dégénérées faisant intervenir des potentiels singuliers. Tout d'abord, sous des hypothèses adéquates sur les termes de couplage, la question du caractère bien posé du problème est abordée en utilisant la théorie des semigroupes combinée par quelques inégalités de Hardy Poincaré. Ensuite, en employant une estimation de Carleman avec une seule observation interne, nous démontrons la contrôlabilité à zéro du système par un seul contrôle distribué. Le Chapitre 4 traite un problème de contrôle pour une équation parabolique dégénérée avec terme mémoire. Dans un premier temps, nous établissons la contrôlabilité à zéro d'une équation non-homogène dont les solutions sont régulières. Ensuite, en utilisant le théorème du point fixe de Kakutani, nous déduisons la propriété de contrôlabilité à zéro du problème mémoire initiale.

Dans le Chapitre 5, nous étudions la contrôlabilité frontière d'un système parabolique dégénéré. En particulier, nous fournissons des conditions nécessaires et suffisantes pour la contrôlabilité approchée et à zéro lorsqu'une seule force de contrôle est exercée sur une partie du bord au moyen d'une condition de Dirichlet. Enfin, dans le Chapitre 6, nous étudions les propriétés de contrôlabilité approchée et à zéro d'une équation de la chaleur dégénérée lorsqu'une force de contrôle ponctuelle s'exerce en un seul point à l'intérieur du domaine spatial. Notre technique est essentiellement basée sur une analyse spectrale ainsi que la méthode des moments.

Mots clés: Problème inverse de source, équation parabolique dégénérée, système dégénéré/singulier, équation parabolique avec mémoire, contrôlabilité à zéro, observabilité, estimations de Carleman, méthode des moments, temps minimal.

Contents

General introduction

Over the last three decades, the inverse problems have become the most rapidly developing fields of modern science thanks to their substantial implications on mathematical modeling of numerous problems coming from a variety of areas such as medicine, biology, ecology, industry and from image processing for the restoration of blurred images.

In mathematical physics, the objective of solving a direct problem is to find an exact (or approximate) functions that describe a physical process at all points of a given domain and all instants of an interval of time (if the phenomenon is nonstationary). Conversely, the inverse problem aims to find unknown quantities in governing model from partial information on the solution of the direct problem. The given measurements (also called *observation*) represent the data of the inverse problem, whereas the unknown is called the solution to the problem. The additional information depends strongly on the model under study as well as the unknown coefficient. Of course, these observations have to be realistic from a physical point of view; for instance, in the usual situation, we do not have the complete information on the solutions over the whole space time domain where the model evolves. However, this information is most frequently stated as the specification of the value of the solution on a part of the boundary or in a region inside the domain.

It is well known that most of the inverse problems are ill-posed or improperly posed in the sense of Hadamard; hence they are more complicated to solve than the direct problems. Mathematically speaking, this means that either the problem has many possible solutions, or has no solution in the desired class, or the solution is unstable, that is to say; an arbitrarily small perturbation in the data may lead to a sufficiently large error in the solution, which may make the derived solution meaningless. Thus, in the analysis of inverse problems, three main issues arise naturally, namely, stability, uniqueness and reconstruction of the unknown quantity.

In this work, we restrict ourselves to study stability and uniqueness problems. Concerning the stability, we are interested in deriving the so-called stability estimates, that is, to find an estimation of unknown source terms or coefficients using partial measurements. On the other hand, for the uniqueness issue, we establish whether the adopted extra observation data on the solution may uniquely identify the unknown functions. The uniqueness and stability issues for inverse problems has been the object of a vast number of publications. Quoting all these papers is beyond the scope of this work. The reader interested in a complete introduction on this topic can refer to the recent books [29, 70, 117, 126].

The fundamental tool we employ when trying to derive uniqueness and stability relies on the so-called *Carleman estimates*. These are L^2 weighted energy estimates for the solutions of PDEs. They were first introduced by T. Carleman [64] in 1939, for proving the uniqueness results for ill-posed Cauchy problems. Since then, these estimates have been rapidly developed and their area of application has gone beyond its original field: nowadays they play a peculiar role in the study of inverse problems and controllability issues for PDEs [29, 55, 106, 115]. Originally, Carleman estimates were first proposed in the theory of inverse problems, by Bukhgeim and Klibanov [46] in 1981, for proving the conditional estimate of *Hölder's* type for the classical heat equation; their arguments were based on the use of local Carleman inequalities. Then, this result has been improved in the recent paper [115] by Imanuvilov and Yamamoto (1998), obtaining *unconditional Lipschitz stability estimates*, by means of global Carleman estimates. After these fundamental contributions to the study of inverse problems, there have been great articles appearing in several dimensions of scope. The inverse problems of two coupled parabolic equations have been established in numerous papers (see, e.g., [1, 32, 73, 74, 75, 76, 145, 163]). We refer to [29, 166] for a more detailed survey concerning the applicability of Carleman estimates to the stability of inverse problems. We underline the fact that all the papers mentioned above deal with the nondegenerate setting. That is to say; the diffusion coefficients are uniformly coercive. In the last two decades, a growing interest has been devoted to the study of degenerate parabolic operators with degeneracy occurring at the boundary or in the interior of the space domain, namely

$$Py := y_t - (ay_x)_x, \quad x \in (0,1), \tag{0.0.1}$$

with $a(x_0) = 0$, being $x_0 \in [0,1]$.

For such models, inverse problems of a scalar equation were studied in [37, 55, 63, 154, 155]. The main result in these works is the development of adequate Carleman estimates, which are a crucial tool to obtain Lipschitz stability for term sources, initial data, potentials and diffusion coefficients. The case of coupled systems with two parabolic equations is considered in [38, 39]. In Chapter 2, we will extend the previous results to the context of a coupled system of $n \geq 2$ degenerate parabolic equations.

Another way to investigate evolution systems is to try to affect its evolution by employing various external forces. In other words, to determine whether a system can behave precisely according to our wishes (or in a manner arbitrarily close to it), via some quantities (called "controls") applied through actuators.

Mathematically speaking, a control problem can be written under the following abstract form

$$\frac{dy}{dt} = F(y,u), \quad t > 0, \qquad y(0) = y_0, \tag{0.0.2}$$

where y_0 is the initial datum, y is the state of the system that we are willing to control; it belongs to an appropriate state space H. On the other hand, u is the control function that lies in a set of admissible controls U. To control system (0.0.2) means to find an appropriate function $u \in U$ such that the associated solution can be steered from a given initial state to an arbitrary target in a finite time: this is *controllability*, one of the central notions in this thesis. The system (0.0.2) is said to be *exactly controllable* when any desired state of H can be achieved from any initial datum in an arbitrary finite time $T > 0$. In a particular case, if the state of this system can be steered to the equilibrium, then we speak of *null controllability*. Then, the system has *approximate controllability* property if, from any initial datum, the system can be driven to a state arbitrary close to the prescribed target. Finally, the *controllability to trajectories* ensures that every trajectory (i.e. the value at the final time of a solution of the uncontrolled equation) can be achieved from any initial state y_0. See Section 1.4 in Chapter 1 for more details.

Now, let us briefly review some existing results concerning the controllability of parabolic systems. We focus on those results that are very much connected with the topics of this work.

The controllability of scalar and non-scalar uniformly parabolic systems has attracted the interest of many researchers and important progress has been made during the last decades. To our best knowledge, the first results on the controllability of the scalar equation, date back by about half a century, concerns the one-dimensional heat equation. They have been established by H.O. Fattorini and D.L. Russell (see [86, 87]) through the so-called *moment method* (for more details, see Subsection 1.4.3.2 in Chapter 1 of this thesis). Later on, the null controllability problem for parabolic equation (with both boundary and distributed controls) has been established independently by Lebeau and Robbiano [129] and Fursikov and Imanuvilov [106] in arbitrary space dimensions via Carleman estimates. Also, controllability of degenerate parabolic equations has been investigated by many authors in the last years. Let us mention the pioneering work [56], where the authors prove that the degenerate equation

$$u_t - (x^\alpha u_x)_x = 0, \quad x \in (0,1), \tag{0.0.3}$$

with suitable boundary conditions, is null controllable by means of distributed controls or boundary controls (located at the non degenerate point) whenever $\alpha \in [0,2)$. Their approach consists in deriving suitable observability estimate for the adjoint system through new Carleman estimates. On the contrary, the null controllability fails to hold when the aforementioned condition on α is violated (i.e. when $\alpha > 2$), see [57]. After those first results, several other works appeared extending them in various situations, such as problems involving a more general diffusion coefficient and nondivergence form operators, see [5, 52, 55, 63, 104]. To our best knowledge, the question of controllability of (0.0.3) via an internal control supported on a single point inside the space domain, has not been addressed. This will be the subject of Chapter 6 of this work.

We remark the fact that all the papers cited above, consider the case where the degeneracy occurs at the boundary of the space domain. To our knowledge, [104, 105] are the first works dealing with controllability for operators with mere degeneracy at the interior of the space domain. Later on, parabolic operators that couples a degenerate diffusion coefficient with a singular potential has been considered in numerous works, obtaining substantial progress. Among these papers, we mention [95, 96, 101, 103], where the authors obtain results concerning well-posednessn, Carleman estimates and controllability.

Coming to the point of *control of non-scalar systems*, it is by now well understood to the control community that, the controllability of coupled systems with a low number of controls, is a challenging issue. We mention [15, 16, 22, 107, 79] among some very initial works on the distributed controllability problems for coupled parabolic systems. In particular, in [14, 18] the authors provide a necessary and sufficient condition (more precisely a Kalman rank condition) for the distributed null-controllability of $n \times n$ parabolic systems. It is also worth mentioning [108], where the authors studied the distributed null-controllability of the so-called *cascade system* of $n \geq 1$ coupled parabolic equations with a single control force. The extension of such results to the context of a degenerate/singular *cascade system* will be the subject of Chapter 3.

Further, concerning the boundary control problem, we must say that the boundary controllability of non-scalar systems (with less number of controls) are fascinating and challenging problems in the field of control theory. This is mainly due to the fact that the very powerful Carleman approach is often inefficient in this framework. There are only a few results on this setting and most of them concern the one-dimensional case with constant coefficients. We refer to [17, 88], where a necessary and sufficient condition is exhibited. In Chapter 6, as in the papers above, we will discuss the boundary controllability properties of a coupled system of two degenerate parabolic equations.

On the other hand, it should be noted that, in all the papers quoted so far, the considered systems have the property that the mathematical-physical description of their state at a given instant is affected only by its current state. However, in various fields of science and engineering such as in the heat conduction in materials with memory, the theory of population dynamics and nuclear dynamics reactors (see, e.g., [120, 128, 165, 169]), it is essential to take into account the effect of the past story while describing the system as a function at a given point of time.

For instance, in the analysis of space-time-dependent reactor dynamics, if the effect of the linear temperature is taken into account and the reactor model is considered as an infinite rod, then the group neutron flux $z := z(t,x)$ and the temperature $w := w(t,x)$ in the reactor can be formulated as a coupled parabolic-ODE system (see, e.g., [121, 128]):

$$\begin{cases} z_t - (a(x)z_x)_x = (\gamma_1 w + \gamma_2 - 1)\Sigma_f z, \\ \rho\gamma w_t = \gamma_3 \Sigma_g z, \end{cases} \quad t > 0, \quad x \in \mathbb{R},$$

where a is the diffusion coefficient and γ, Σ_f, Σ_g, γ_i $(i = 1,2,3)$ are physical quantities. By integrating the ODE equation over $(0,t)$ and plugging it into the first equation, we obtain a parabolic equation involving nonlinear memory term:

$$y_t - (a(x)y_x)_x + by\int_0^t y(\tau, x)\,d\tau + cy = 0, \quad t > 0, \quad x \in \mathbb{R}, \tag{0.0.4}$$

where b, c are constants depending on the initial temperature and different physical parameters.

In this thesis, more precisely in Chapter 4, we consider a degenerate linear version of (0.0.4) with a more general zero order memory, namely

$$\begin{cases} y_t - (a(x)y_x)_x = \int\limits_0^t b(t, s, x)y(s, x)\,ds + 1_\omega u & (t, x) \in (0, T) \times (0, 1), \\ y(0, x) = y_0(x), & x \in (0, 1), \end{cases}$$

under appropriate boundary conditions, with $T > 0$ and $a \geq 0$ in $[0, 1]$. We aim to establish the null controllability property for such a model. We emphasize that significant difficulties arise from both the degeneracy of the diffusion coefficient as well as the particular form of the memory term. This makes the problem under investigation exciting and completely different from the existing works (see e.g., [76, 110, 153]).

Apart from this introduction, this thesis consists of six chapters:

- **Chapter 1:** We review the main concepts and results related to the notions of inverse problem and controllability. We focus on those aspects that will be used in the sequel.
- **Chapter 2:** We deal with the analysis of an inverse source problem for a coupled system of n degenerate parabolic equations. In particular, we investigate stability estimate of Lipschitz type in recovering the n source terms in such a system from the data of only a single component of the vector solution on an arbitrary interior domain. The proof of this result is mainly based on appropriate Carleman estimates with a unique local observation.
- **Chapter 3:** We consider a coupled system of $n \geq 2$ degenerate/singular parabolic equations, with degeneracy and singularity occurring in the interior of the space domain. First, we provide conditions guaranteeing that the system under study is well-posed in appropriate weighted Hilbert spaces. Then, the null controllability result, with a single control, is proved by duality argument by means of an observability inequality for the adjoint system.
- **Chapter 4:** We establish the null controllability of a degenerate parabolic equation involving a memory term, under the action of a distributed control. Using appropriate Carleman estimates, we prove the null controllability of an equation with source term. Then, the controllability result for the memory system is obtained under a suitable assumption on the memory kernel, via a classical fixed point argument.
- **Chapter 5:** Controllability properties of some 2×2 coupled degenerate parabolic systems with a constant coupling matrix when a scalar control force is exerted on a part of the boundary of the spatial domain is studied. In particular, we provide necessary and sufficient conditions for the null and approximate controllability. Our approach is based on spectral analysis and the moment method.
- **Chapter 6:** We study the controllability properties of degenerate heat equation when a control force acts on a single point in the interior of the space domain. We discuss both approximate and null controllability. We give a necessary and sufficient condition for the approximate controllability. On the other hand, we show that a minimal time of pointwise null controllability arises. The fundamental ingredient for deriving these results is the moment method.

A final comments on the notation: by Q we shall denote the square $(0, T) \times (0, 1)$, and by C, C_T universal positive constant, which are allowed to vary from line to line.

Main contents of the thesis

In this thesis, we are interested in inverse source problem and controllability properties of some systems governed by degenerate parabolic equations.

In the following, we present a preliminary survey of the contents of each chapter, introducing the main results that we achieved with more details.

Chapter 2: Inverse problem for degenerate coupled systems

In this chapter, we analyze the simultaneous identification of n source terms $f_1, \cdots, f_n$ in the following degenerate coupled parabolic system

$$\begin{cases} \partial_t y_1 - d_1(a(x)y_{1x})_x + \displaystyle\sum_{j=1}^{2} b_{1j}y_j = f_1, & (t,x) \in Q, \\ \partial_t y_2 - d_2(a(x)y_{2x})_x + \displaystyle\sum_{j=1}^{3} b_{2j}y_j = f_2, & (t,x) \in Q, \\ \quad \vdots & \\ \partial_t y_n - d_n(a(x)y_{nx})_x + \displaystyle\sum_{j=1}^{n} b_{nj}y_j = f_n, & (t,x) \in Q, \\ y_1(0,x) = y_1^0(x), ..., y_n(0,x) = y_n^0(x), & x \in (0,1), \end{cases} \tag{0.0.5}$$

associated to appropriate boundary conditions and where $(y_k^0)_{1\le k\le n} \in L^2(0,1)^n$, the potentials $b_{kj} \in L^\infty(Q)$ $(1 \le k, j \le n)$, the diffusion coefficient a vanishes at $x = 0$ (i.e., $a(0) = 0$) and satisfies suitable assumptions.

In particular, we are interested in solving the following problem: is it possible to retrieve the source terms $f_1, ..., f_n$ from a reduced number of interior observations of the vector solution on a subregion ω of $(0,1)$?

The key ingredient relies on suitable Carleman estimates for (0.0.5) with a reduced number of locally distributed observations.

In this purpose, we consider $t_0 \in (0,T)$ and denote

$$Q_{t_0} := (t_0,T) \times (0,1), \quad \omega_{t_0} := (t_0,T) \times \omega, \quad T' := \frac{T+t_0}{2}.$$

Then, as a first step, we derive a new Carleman inequality with a locally distributed observation for a single degenerate equation, and hence, a Carleman estimate for the coupled system (0.0.5) by means of n components of the vector solution localized in $\omega'_{t_0} := (t_0,T) \times \omega'$, with $\omega' \Subset \omega$, of the form:

$$\sum_{k=1}^{n} \mathcal{J}(y_k) \le C \sum_{k=1}^{n} \Big(\iint_{Q_{t_0}} f_k^2 e^{2s\Phi}\, dx\, dt + \iint_{\omega'_{t_0}} s^3\theta^3 y_k^2 e^{2s\Phi}\, dx\, dt \Big), \tag{0.0.6}$$

for all $s \ge s_0$ (s_0 being a suitable large constant), where $\theta(t)$ is a smooth function, going to $+\infty$ at $t = 0, T$, and φ, Φ are appropriate negative functions going to $-\infty$ at $t = 0, T$. Here

$$\mathcal{J}(y) := \iint_{Q_{t_0}} \Big(\frac{1}{s\theta} y_t^2 + s\theta^{\frac{3}{2}} |\eta\psi| y^2 + s\theta a(x) y_x^2 + s^3\theta^3 \frac{x^2}{a(x)} y^2 \Big) e^{2s\varphi}\, dx\, dt.$$

Major steps in the proof of the previous inequality are based, in particular, on the following Hardy-Poincaré type inequality:

$$\int_0^1 \frac{a(x)}{x^2} y^2(x)\, dx \le C_{HP} \int_0^1 a(x) y_x^2(x)\, dx.$$

Note that, the estimate (0.0.6) could be used to prove Lipschitz stability in determining all the source terms in (0.0.5) from interior measurements of all components of the solution. However, thanks to the cascade structure of the considered system, i.e., $b_{kj} = 0$ ($1 \leq k \leq n$, $k+1 \leq j \leq n$), we can eliminate $n-1$ local terms in the right hand side of (0.0.6). More precisely, we can prove that the following holds:

$$\sum_{k=1}^{n} \mathcal{J}(y_k) \leq C\Big(\sum_{k=1}^{n} \iint_{Q_{t_0}} s^R\theta^R f_k^2 e^{2s\Phi_k}\,dx\,dt + \iint_{\omega_{t_0}} y_1^2\,dx\,dt\Big), \tag{0.0.7}$$

for some positive constant R, under the following assumption on the coupling terms

$$\operatorname{supp}(b_{k-1k}) \cap \omega_{t_0} \neq \emptyset, \quad \forall k : 2 \leq k \leq n. \tag{0.0.8}$$

Here, Φ_k ($1 \leq k \leq n$) is an appropriate weight function satisfying $\Phi_k \to -\infty$ at $t = 0, T$. Next, let $C_0 > 0$, we define

$$\mathcal{S}(C_0) := \{f \in H^1(0,T;L^2(0,1)) : |f_t(t,x)| \leq C_0|f(T',x)|, \text{ for almost all } (t,x) \in Q\}.$$

Hence, taking the above Carleman estimate into account, and following the approach introduced by Imanuvilov and Yamamoto [115], for all $f_k \in \mathcal{S}(C_0)$ ($1 \leq k \leq n$), we derive a stability estimate of the form:

$$\sum_{k=1}^{n} \|f_k\|^2_{L^2(Q)} \leq C\Big(\sum_{k=1}^{n} \|(ay_{kx})_x(T',\cdot)\|^2_{L^2(0,1)} + \|y_1\|^2_{L^2(\omega_{t_0})} + \|y_{1t}\|^2_{L^2(\omega_{t_0})}\Big), \tag{0.0.9}$$

where C is a positive constant independent of these source terms.

Furthermore, if we restrict ourselves to the case where the source terms take the form

$$f_k(t,x) = g_k(x)r_k(t,x), \quad \forall k : 1 \leq k \leq n,$$

where $r_k \in \mathrm{C}^1([0,T]\times[0,1])$ are given smooth functions and $g_k \in L^2(0,1)$ the unknown functions, then the uniqueness result follows as a direct consequence of (0.0.9).

Chapter 3: Controllability of degenerate/singular cascade systems

Let $n \geq 1$ and ω be an arbitrary nonempty open set of $(0,1)$. We consider the following coupled system of n degenerate/singular parabolic equations, where only the first equation is controlled:

$$\begin{cases} \partial_t y_1 - d_1(a(x)y_{1x})_x - \displaystyle\sum_{j=1}^{n} \frac{\lambda_{1j}}{b_{1j}} y_j + \sum_{j=1}^{n} a_{1j}y_j = v1_\omega, & (t,x) \in Q, \\ \partial_t y_2 - d_2(a(x)y_{2x})_x - \displaystyle\sum_{j=1}^{n} \frac{\lambda_{2j}}{b_{2j}} y_j + \sum_{j=1}^{n} a_{2j}y_j = 0, & (t,x) \in Q, \\ \quad\vdots & \\ \partial_t y_n - d_n(a(x)y_{nx})_x - \displaystyle\sum_{j=1}^{n} \frac{\lambda_{nj}}{b_{nj}} y_j + \sum_{j=1}^{n} a_{nj}y_j = 0, & (t,x) \in Q, \\ y_k(t,0) = y_k(t,1) = 0, \quad 1 \leq k \leq n, & t \in (0,T), \\ y_k(0,x) = y_k^0(x), \quad 1 \leq k \leq n, & x \in (0,1), \end{cases} \tag{0.0.10}$$

where $d_k > 0$, $1 \leq k \leq n$, $\lambda_{kj} > 0$, $a_{kj} \in L^\infty(Q)$ ($1 \leq k, j \leq n$), the coefficients a, b_{kj}, $1 \leq k, j \leq n$ are positives, vanishes at the same interior point $x_0 \in (0,1)$ and satisfying suitable assumptions near this point, $(y_1^0, \cdots, y_n^0) \in L^2(0,1)^n$ is the initial condition and $v \in L^2(Q)$ is the scalar control force.

The first object of this chapter is to study the well-posedness of the coupled system (0.0.10) through the semigroup theory and the following *improved* Hardy-Poincaré inequality

$$d_k \int_0^1 a y_x^2\, dx - \lambda_{kk} \int_0^1 \frac{y^2}{b_{kk}}\, dx \geq \Lambda_k \int_0^1 a y_x^2\, dx \tag{0.0.11}$$

in a suitable weighted Hilbert space, for some constants $\Lambda_k \in (0,\, d_k]$.

Our second aim is to analyze the *indirect null controllability* property of the degenerate/singular parabolic system (0.0.10), that is, to control all the equations in (0.0.10) by means of one control force, with the hope that one can act indirectly on the uncontrolled equations thanks to the coupling terms $a_{k,j}$ and b_{kj}. In particular, our main result is the following.

Theorem 0.0.1. *Suppose that the coupling terms $a_{k,j}$ and b_{kj} and the constants λ_{kj} satisfy appropriate assumptions. Then, given $(y_1^0, \cdots, y_n^0) \in L^2(0,1)^n$, there exists a control function $v \in L^2(Q)$ such that the corresponding solution to (0.0.10) satisfies*

$$y_k(T,\cdot) = 0 \text{ in } (0,1),\ \forall k : 1 \leq k \leq n. \tag{0.0.12}$$

The main idea of the proof of Theorem 0.0.1 consists in deriving an appropriate Carleman estimate (with one locally distributed observation) for any solution $Z = (z_k)_{1\leq k\leq n}$ of the following adjoint system

$$\begin{cases} \partial_t z_k + d_k(a(x) z_{kx})_x + \displaystyle\sum_{j=k-1}^{k+1} \frac{\lambda_{jk}}{b_{jk}} z_j - \sum_{j=1}^{k+1} a_{jk} z_j = 0, & (t,x) \in Q, \\ z_k(t,0) = 0, \quad z_k(t,1) = 0, \quad t \in (0,T), \\ z_k(T,x) = z_k^T(x), \quad x \in (0,1), \end{cases} \tag{0.0.13}$$

where $z_k^T \in L^2(0,1)$ and $1 \leq k \leq n$ and deduce an observability inequality, of the form

$$\|Z(0,\cdot)\|^2_{L^2(0,1)^n} \leq C_T \iint_{(0,T)\times\omega} z_1^2(t,x) dx\, dt, \tag{0.0.14}$$

for a positive constant $C_T > 0$ depending only on T and ω.

The estimate (0.0.14) is often also referred to as *indirect observability* since it provides a quantitative estimate of the total energy for the adjoint system at $t = 0$ in terms of the observed quantity of only one component, by means of the observability constant C_T.

Let us now briefly explain the main steps of the proof strategy of the observability inequality (0.0.14). We are going to proceed in three steps.

Step 1. We prove a Carleman inequality with n locally distributed observations, in a subregion $\omega' \Subset \omega$ of the following form

$$\sum_{k=1}^n \mathcal{I}(z_k) \leq C \sum_{k=1}^n \iint_{(0,T)\times\omega'} s^2\theta^2 z_k^2 e^{2s\Phi}\, dx\, dt, \tag{0.0.15}$$

where

$$\mathcal{I}(z_k) := \iint_Q \left(s\theta a(x) z_{kx}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)} z_k^2 \right) e^{2s\varphi}\, dx\, dt,$$

for all $s \geq s_0$ (s_0 being a suitable constant), where $\theta(t)$, $\Phi(t,x)$ and $\varphi(t,x)$ are suitable weight functions.

The presence of the degeneracy and singularity, in the considered problem, at the same time generates significant difficulties in the proof of (0.0.15). Thus, a convenient choice of the weight functions Φ, φ, which bring some correction to the degeneracy/singularity, combined with an appropriate application of the standard cut-off argument and the following Hardy-Poincaré inequality

$$\int_0^1 \frac{y^2}{b_{kk}}\,dx \leq C^k \int_0^1 a y_x^2\,dx \tag{0.0.16}$$

for some constants $C^k > 0$, will be the key ingredients in order overcome these difficulties.

Notice that, the estimate (0.0.15) could be used to derive the null controllability result of (0.0.10) via n control forces. However, in practice, it is worthwhile to control the n components of the system via a low number of controls and the best would be to do it by a single one.

Step 2. We try to reduce the number of the local observations in the right hand side of (0.0.15) so that we get a Carleman inequality with single observation in the control zone $(0,T)\times\omega$, of the form

$$\sum_{k=1}^{n} \mathcal{I}(z_k) \leq C \iint_{(0,T)\times\omega} z_1^2\,dx\,dt. \tag{0.0.17}$$

The main technical tool for obtaining this result will be the following inequality

$$J_{\omega_0'}(l,\Phi_k,z_k) \leq \varepsilon \sum_{j=k}^{k+1} \mathcal{I}(z_k) + C_k\Big(1+\frac{1}{\varepsilon}\Big) \sum_{j=1}^{k-1} J_{\omega_0}(l_j,\Phi_{k-1},z_j), \tag{0.0.18}$$

for some constants $l, l_j > 0$ and a sufficiently small ε with $\omega_0' \Subset \omega_0 \Subset \omega' \Subset \omega$ and

$$J_\omega(d,\phi,z) := s^d \iint_{Q_\omega} \theta^d z^2 e^{2s\phi}\,dx\,dt$$

which is valid under a technical condition on the coupling terms. More precisely, (0.0.18) holds true provided that the support of the coupling terms

$$-a_{kk-1} + \frac{\lambda_{kk-1}}{b_{kk-1}} \qquad \forall k \in \{2,\cdots,n\}$$

intersect the control domain $(0,T)$.

Step 3. Finally, employing the Carleman inequality (0.0.17) together with some energy estimates, we obtain the observability inequality (0.0.14), which yields the indirect null controllability result for the coupled degenerate/singular system (0.0.10) by means of standard duality arguments (see Proposition 1.4.1 in Chapter 1).

Chapter 4: Controllability of degenerate equation with memory

The purpose of this chapter is to study the null controllability property of the following degenerate parabolic equation with memory

$$\begin{cases} y_t - (a(x)y_x)_x = \displaystyle\int_0^t b(t,s,x)y(s,x)\,ds + 1_\omega u & (t,x)\in Q, \\ y(0,x) = y_0(x), & x\in(0,1), \end{cases} \tag{0.0.19}$$

with suitable boundary conditions. Here $v = v(x,t)$ is a locally distributed control which is acting on the system at a small set $\omega \subset (0,1)$, $y = y(x,t)$ is the state and y_0 is any given initial condition and the function a is the diffusion coefficient and we assume that it depends on the space variable x and degenerates at the boundary of the state space.

We are going to prove that, if the kernel b has an exponential decay at the end of the time horizon $[0,T]$, then, the system (0.0.19) is null controllable, i.e., for any $y_0 \in L^2(0,1)$, there exists a control function $v \in L^2(Q)$, such that the solution of (0.0.19) fulfills

$$y(T,\cdot) = 0 \qquad \text{in} \quad (0,1).$$

More precisely, the main result of this chapter reads as follows:

Theorem 0.0.2. *Let $T > 0$, $k \geq 0$ and assume that b satisfies*

$$(T-t)^{2k} e^{\frac{C}{(T-t)^2}} b \in L^\infty((0,T)\times Q), \tag{0.0.20}$$

for some constant $C > 0$. Then, for any $y_0 \in L^2(0,1)$, there exists $u \in L^2(Q)$ such that the associated solution y of (0.0.19) satisfies

$$y(T,\cdot) = 0 \qquad \text{in } (0,1).$$

A common strategy to show null controllability for a linear parabolic equation is based on proving an observability inequality for the associated adjoint system through appropriate Carleman estimates. Nevertheless, the usual Carleman inequalities do not seem to be appropriate for studying the controllability problem for integro-differential equations like (0.0.19), since the memory terms cannot be controlled by the local estimates. Therefore, to overcome this difficulty, we shall introduce a nonhomogeneous degenerate parabolic equation

$$\begin{cases} y_t - (a(x)y_x)_x = f + 1_\omega u & (t,x) \in Q, \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{0.0.21}$$

with $f \in L^2(Q)$, for which we prove the null controllability result via new Carleman estimates with a weight time function that do not blow up at $t = 0$. Finally, a fixed point argument is successfully applicable in an appropriate weighted space to deduce the null controllability result for the initial system (0.0.19), under a convenient condition on the kernel b.

Let us now briefly discuss the main steps of the proof of Theorem 0.0.2. We proceed in several steps:

Step 1. Modified Carleman estimate.

We derive a new Carleman estimate with weight time function not exploding at the initial time $t = 0$. More precisely, we will prove that the following inequality holds:

$$\begin{aligned} &\|v(0)\|^2_{L^2(0,1)} + \iint_Q (s\beta)^k v^2 e^{2s\Phi}\, dx\, dt \\ &\qquad \leq C\Big(\iint_Q (s\beta)^k g^2 e^{2s\sigma}\, dx\, dt + \iint_{Q_\omega} (s\beta)^{k+3} v^2 e^{2s\sigma}\, dxdt \Big) \end{aligned} \tag{0.0.22}$$

for all $k \geq 0$ and all $s \geq s_0$ (s_0 being a large constant). Here β is a smooth time function, going to $+\infty$ at $t = T$ and Φ, σ are appropriate negative time-space functions going to $-\infty$ at $t = T$.

The above estimate follows from a combination of the usual Carleman inequality for the degenerate parabolic equation and some energy estimates for the system (0.0.21).

Step 2. Null controllability for (0.0.21)

Following the classical approach, introduced in [106], with the modified Carleman estimate in the previous step, we can derive a null controllability result for (0.0.21). Nevertheless, as in the classical case [153], this result is not sufficient to obtain the controllability of the integro-differential equation (0.0.19).

To our purpose, we will need to prove the null controllability for (0.0.21) with more regular state. To this end, we introduce the following weighted space:

$$E_{s,k} = \big\{ y \text{ solution of } (0.0.21): \quad (s\beta)^{-k/2} e^{-s\sigma} y \in L^2(Q) \big\}.$$

Note that, if $y \in E_{s,k}$, then

$$\iint_Q (s\beta)^{-k} e^{-2s\sigma} y^2 \, dxdt < +\infty.$$

Since the weight σ decay to 0 as $t \to T^-$, the boundedness of the above integral implies

$$y(T,\cdot) = 0 \quad \text{in } (0,1).$$

More precisely, the main result of this step is the following.

Theorem 0.0.3. *Let $T > 0$ and $k \geq 0$. Assume $(s\beta)^{-k/2} e^{-s\Phi} f \in L^2(Q)$ with $s \geq s_0$. Then, for any $y_0 \in H_a^1(0,1)$, there exists $u \in L^2(Q)$ such that the associated solution y of system (0.0.21) belongs to $E_{s,k}$.*

Moreover, there exists a positive constant C such that the couple (y,u) satisfies

$$\begin{aligned}
&\iint_Q (s\beta)^{-k} e^{-2s\sigma} y^2 \, dx\, dt + \iint_{Q_\omega} (s\beta)^{-(k+3)} e^{-2s\sigma} u^2 \, dxdt \\
&\qquad \leq C\Big(\iint_Q (s\beta)^{-k} e^{-2s\Phi} f^2 \, dx\, dt + \int_0^1 y_0^2 \, dx \Big).
\end{aligned} \tag{0.0.23}$$

Here $H_a^1(0,1)$ is a suitable weighted Sobolev space to be specified later on.

Step 3. Null controllability for an intermediate system

Let $w \in E_{s,k,R} = \big\{ w \in E_{s,k} : \quad \|(s\beta)^{-k/2} e^{-s\sigma} w\|_{L^2(Q)} \leq R \big\}$, where $R > 0$ is an arbitrary constant.

As an immediate consequence of the result in the previous step, one can deduce that the following system is null controllable

$$\begin{cases}
y_t - (a(x)y_x)_x = \displaystyle\int_0^t b(t,s,x) w(s,x) \, ds + 1_\omega u & (t,x) \in Q, \\
y(0,x) = y_0(x), & x \in (0,1).
\end{cases} \tag{0.0.24}$$

More precisely, one has.

Proposition 0.0.1. *Let $T > 0$ and $k \geq 0$. Assume that the memory kernel b satisfies (0.0.20). Then, for all $w \in E_{s,kR}$ and for any $y_0 \in H_a^1(0,1)$, there exists $u \in L^2(Q)$ such that the associated solution y of system (0.0.24) belongs to $E_{s,k}$.*

Step 4. Conclusion. By Proposition 0.0.1 and the Kakutani's fixed point theorem, we can deduce the null controllability of the memory system (0.0.19) with regular initial data. Then, by means of a standard argument, we can extend this result to the case of initial conditions of $L^2(0,1)$.

Remark 1. It is worth mentioning that, from the results in Guerrero and Imanuvilov [110], it seems that the null controllability property of parabolic equations with memory may fail without any additional conditions on the kernel. On the other hand, observe that the condition (0.0.20) just restricts the function b very near T, which is due to the fact that the weight function β in the Carleman estimate (0.0.23) blows up at $t = T$.

Chapter 5: Boundary controllability for coupled degenerate systems

This chapter deals with the analysis of boundary controllability properties of the following degenerate coupled parabolic system

$$\begin{cases} \partial_t y - (x^\alpha y_x)_x = Ay, & \text{in } Q, \\ y(t,1) = Bv(t), & \text{in } (0,T), \\ \begin{cases} y(t,0) = 0, & 0 \le \alpha < 1 \\ x^\alpha y_x(t,0) = 0, & 1 \le \alpha < 2 \end{cases} & t \in (0,T), \\ y(0,x) = y_0(x), & \text{in } (0,1), \end{cases} \tag{0.0.25}$$

where $0 \le \alpha < 2$, $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ are given, $v = v(t)$ is the control function and $y = (y_1, y_2)^*$ is the state of the system.

It is known that, the boundary controllability result for a scalar degenerate parabolic equation (with a control acting at the extremity $x = 1$) can be easily derived from the corresponding distributed controllability (with a control supported on a small open set inside the domain) and vice versa. Similarly, when we exert two boundary controls, the system is null controllable at any time $T > 0$. Nevertheless, we will see that the situation is completely different in the case where we exert only one boundary control on the system. More precisely, the distributed and boundary controllability properties of coupled system (0.0.25) are not equivalent.

We recall that, the null controllability of the following controlled system

$$\begin{cases} \partial_t y - (x^\alpha y_x)_x = Ay + B1_\omega v, & \text{in } Q, \\ y(0,x) = y_0 \in L^2(0,1), & \text{in } (0,1), \end{cases} \tag{0.0.26}$$

with the same boundary conditions as in (0.0.25), holds if and only if the following algebraic Kalman's rank condition

$$\operatorname{rank}[B|AB] = 2, \tag{0.0.27}$$

is fulfilled (see, for instance, [84]).

Besides, we will prove that, unlike for system (0.0.26), the condition (0.0.27) is necessary but not sufficient for the null controllability of system (0.0.25).

To this purpose, let us first introduce the following notations:

$$\nu_\alpha = \frac{|1-\alpha|}{2-\alpha} \quad \text{and} \quad \kappa_\alpha = \frac{2-\alpha}{2} > 0,$$

where $(j_{\nu_\alpha,k})_{k\ge1}$ is the sequence of the zeros of Bessel functions of the first kind of order ν_α (for a precise definition, see Subsection 5.3.1 in Chapter 5). Hence, the main result of this chapter will be the following:

Theorem 0.0.4. *Let μ_1 and μ_2 the eigenvalues of the matrix A. Then, system (0.0.25) is null controllable at time $T > 0$ if and only if (0.0.27) holds and*

$$\kappa_\alpha^2 (j^2_{\nu_\alpha,n} - j^2_{\nu_\alpha,l}) \neq \mu_2 - \mu_1, \quad \forall n, l \in \mathbb{N}^*, \quad \textit{with} \quad n \neq l.$$

We emphasize that, as we have mentioned before, unlike the distributed controllability, Carleman estimates for the associated adjoint system to (0.0.25) do not seem to works when dealing with the boundary null controllability issue.

Furthermore, we will show that, the null boundary controllability property for system (0.0.25) is equivalent to the corresponding approximate controllability.

Chapter 6: Pointwise controllability of degenerate heat equation

In this chapter, we consider the following controlled degenerate heat equation

$$\begin{cases} y_t - (x^\alpha y_x)_x = \delta_b v(t), & (t,x) \in Q, \\ y(t,1) = 0, & t \in (0,T), \\ \begin{cases} y(t,0) = 0, & 0 \le \alpha < 1 \\ x^\alpha y_x(t,0) = 0, & 1 \le \alpha < 2 \end{cases} & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{0.0.28}$$

where $0 \le \alpha < 1$, $y_0 \in L^2(0,1)$ and $v(t)$ is the control function which acts on a single point $b \in (0,1)$. We aim to investigate controllability properties for system (0.0.28).

To this, with the same notations as above, we define the following set:

$$\mathcal{S}_{\nu_\alpha} = \left\{ \left(\frac{j_{\nu_\alpha,k}}{j_{\nu_\alpha,n}} \right)^{\frac{1}{\kappa_\alpha}}, \quad n > k \ge 1 \right\}.$$

In the first main result of this work, we provide a necessary and sufficient condition for the approximate controllability of problem (0.0.28). One has.

Theorem 0.0.5. *Equation (0.0.28) is approximately controllable at time $T > 0$ if and only if*

$$b \notin \mathcal{S}_{\nu_\alpha}. \tag{0.0.29}$$

Theorem 0.0.5 will be proved through the classical method (see Proposition 1.4.12 in Chapter 1). In particular, it relies on the validity of the following unique continuation property

$$\varphi(\cdot, b) = 0 \quad \text{on} \quad (0,T) \Rightarrow \varphi_0 = 0 \quad \text{in} \quad (0,1),$$

for all solutions of the associated adjoint problem

$$\begin{cases} \varphi_t + (x^\alpha \varphi_x)_x = 0, & (t,x) \in Q, \\ \varphi(t,1) = 0, & t \in (0,T), \\ \begin{cases} \varphi(t,0) = 0, & 0 \le \alpha < 1 \\ x^\alpha \varphi_x(t,0) = 0, & 1 \le \alpha < 2 \end{cases} & t \in (0,T), \\ \varphi(T,x) = \varphi_0(x), & x \in (0,1), \end{cases} \tag{0.0.30}$$

This property, in turn, will be established employing a spectral approach.

Let us now present the pointwise null controllability result for the system (0.0.28). This is the second main achievement in this chapter. It reads as follows.

Theorem 0.0.6. *Let $y_0 \in L^2(0,1)$ and assume that condition (0.0.29) holds. Let us define*

$$T(b,\alpha) = \limsup_{k \to +\infty} - \frac{\log(|\Phi_{\nu_\alpha,k}(b)|)}{\lambda_{\nu_\alpha,k}}.$$

Then, given $T > 0$, one has:

1. *If $T > T(b,\alpha)$, the equation (0.0.28) is null controllable at time T.*
2. *If $T < T(b,\alpha)$, the equation (0.0.28) is not null controllable at time T.*

The first point in Theorem 0.0.6 will be achieved via moment method, whereas for the second point, we proceed by duality arguments (see Proposition 1.4.1 in Chapter 1). In particular, we will show that, when $T < T(b,\alpha)$ the observability inequality

$$\|\varphi(0,\cdot)\|^2_{L^2(0,1)} \le C \int_0^T \varphi(t,b)^2 \, dt,$$

fails to hold.

From Theorem 0.0.6, we deduce that system (0.0.28) has a null controllability minimal time $T(b, \alpha)$ which strongly depends on the control position b and on the rate of the degeneracy α. At this stage, the following question automatically arise: given $T^* \in [0, +\infty]$, does there exist $\alpha \in [0, 2)$ and $b \in (0, 1)$ fulfilling (0.0.29) and such that $T(b, \alpha) = T^*$? This is an interesting and probably difficult open question that we shall discuss in the last part of this thesis.

Chapter 1

Preliminaries

This chapter is outlined as follows: In Section 1.1, we introduce some fundamental results concerning the semigroup operators. Then, we present some results on existence and uniqueness for nonhomogeneous abstract Cauchy problems. In Section 1.2, we provide a brief introduction on parabolic operators involving degenerate diffusion coefficients and singular potentials. Notably, in a first part, we focus on degenerate operators. Then, we develop a quick discussion on the operators that couple degeneracy and singularity. Next, Section 1.3 is concerned with the inverse problems for degenerate parabolic equations. In particular, we show how Carleman estimates could be used to prove Lipschitz stability estimate in recovering a source term in the problem under consideration, and then we comment on the inverse diffusion constant issue. Finally, in Section 1.4, we recall some fundamental results on controllability of linear differential systems. We begin by introducing various variants, such as, exact, approximate, null controllability and controllability to trajectories, and its reciprocal relations. Then, we present, in details, the issues of distributed and boundary controls of scalar parabolic equations.

1.1 Semigroup operators and abstract Cauchy problem

In this section, we recall some well known results on semigroup theory. We shall focus on those aspects which are useful for the next chapters. For more details and advanced results, see [19, 65, 72].

Throughout this section, H denotes a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$.

1.1.1 Semigroup operators

We begin by stating some basic definitions and properties.

Definition 1.1.1. A linear unbounded operator in H is a pair $(A, D(A))$, where $D(A)$ is a linear subspace of H and A is a linear mapping $D(A) \to H$.

Here $D(A)$ is called the domain of A.

Definition 1.1.2. 1. An operator A is said to be *dissipative* if

$$\langle Au, u \rangle \leq 0, \tag{1.1.1}$$

for all $u \in D(A)$.

2. A is said to be *maximal dissipative* if

   (i) A is dissipative;

(*ii*) for all $\lambda > 0$ and all $f \in H$, there exists $u \in D(A)$ such that $u - \lambda Au = f$; i.e., $(I - \lambda A)$ is surjective $\forall \lambda > 0$.

Definition 1.1.3. The adjoint A^* of A is the linear operator

$$\begin{aligned} A^* : D(A^*) \subset H &\to H \\ x &\mapsto A^* x, \end{aligned}$$

where

$$D(A^*) = \{u \in H : \quad \exists c > 0, \ \forall v \in D(A), \ |\langle Av, u\rangle| \leq c\|v\|\}.$$

Definition 1.1.4. Let $A : D(A) \subset H \to H$ be a linear operator such that $\overline{D(A)} = H$ and let A^* be the adjoint operator of A. Then, A is said to be *self-adjoint* if

$$D(A^*) = D(A), \quad \text{and } A = A^*.$$

Theorem 1.1.1. *[65, Corollary 2.4.8] If A is a self-adjoint dissipative operator, then A is maximal dissipative.*

Now, we are going to present some recent results on the semigroup operators. First, we begin by recalling the following definitions.

Definition 1.1.5. A family $(T(t))_{t\geq 0}$ of bounded linear operators from H into H is a strongly continuous semigroup (or simply C_0-*semigroup*) on H provided that

(*i*) $T(0) = I$;

(*ii*) $T(s + t) = T(s)T(t), \quad$ for every $t, s \geq 0$;

(*iii*) $\lim\limits_{t\to 0^+} T(t)u = u, \quad \forall u \in H$ (strong continuity).

Definition 1.1.6. A semigroup of *contractions* is a C_0-semigroup satisfying

$$\|T(t)\| \leq 1, \quad \forall t \geq 0.$$

Definition 1.1.7. The linear operator A defined by

$$D(A) := \left\{u \in H : \lim_{t\to 0^+} \frac{T(t)u - u}{t} \text{ exists}\right\}$$

and

$$Au = \lim_{t\to 0^+} \frac{T(t)u - u}{t}, \quad \text{for } u \in D(A),$$

is called the *infinitesimal generator* (or simply generator) of the semigroup T.

The next theorem due to R.S. Phillips characterizes generators of C_0-semigroups of contractions in terms of maximal dissipativity.

Theorem 1.1.2. *[19, Theorem 2.8] Let $A : D(A) \subset H \to H$ be a linear operator. Then, the following properties are equivalents:*

- *A is the infinitesimal generator of a C_0-semigroup of contractions;*
- *A is maximal dissipative;*
- *A^* is maximal dissipative.*

We end this section by the following result.

Theorem 1.1.3. *Let A be a maximal dissipative self-adjoint operator, then A generates an analytic semigroup on H.*

For a precise definition and more properties on *analytic semigroup* we refer to [83, Chapter II].

1.1.2 Abstract Cauchy problem

Consider the following nonhomogeneous Cauchy problem

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + f(t), \qquad t \in (0,T) \\ y(0) = y_0, \end{cases} \tag{CP}$$

where $f : [0,T[\to H$. Throughout this subsection, we assume that A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$.

Definition 1.1.8. Assume that $y_0 \in H$ and $f \in L^p(0,T;H)$. A function $y : [0,T[\to H$ is said to be:

i) a *classical solution* of (CP) on $[0,T]$ if $y \in C([0,T];D(A)) \cap C^1([0,T];H)$ and satisfies (CP) on $[0,T]$.

ii) a *mild solution* if $y \in C([0,T];H)$ fulfills

$$y(t) = T(t)y_0 + \int_0^t T(t-\tau)f(\tau)\,d\tau, \quad \forall t \in [0,T].$$

iii) a *strong solution* if $y \in L^p(0,T;D(A)) \cap W^{1,p}(0,T;H)$ and fulfills (CP) for a.e. $t \in [0,T]$.

Thus, the next well-posedness result holds thanks to the standard theory of semigroups.

Proposition 1.1.1. *[34, Proposition 3.3 & 3.8]*

i) For all $f \in H^1(0,T;H)$ and all $y_0 \in D(A)$, the problem (CP) admits a unique classical solution

$$y \in C([0,T];D(A)) \cap C^1([0,T];H). \tag{1.1.2}$$

ii) For all $f \in L^2(0,T;H)$ and all $y_0 \in [D(A),H]_{1/2}$, the problem (CP) admits a unique strong solution $y \in H^1(0,T;H) \cap L^2(0,T;D(A))$.

iii) If moreover, we assume that A generates an analytic semigroup. Then, for all $f \in L^2(0,T;H)$ and all $y_0 \in H$, the problem (CP) has a unique mild solution

$$y \in C([0,T];H) \cap L^2(0,T;[D(A),H]_{1/2})$$

satisfying: for all $\varepsilon \in (0,T)$,

$$y \in L^2(\varepsilon,T;D(A)) \cap H^1(\varepsilon,T;H). \tag{1.1.3}$$

Here $[D(A),H]_{1/2}$ denotes the intermediate space between H and the domain $D(A)$. For instance, $[H^2(0,1) \cap H_0^1(0,1), L^2(0,1)]_{1/2} = H_0^1(0,1)$

Remark 2. If $f \in H^1(0,T;H)$ and $y_0 \in H$, by virtue of (1.1.2) and (1.1.3), it comes that the associated unique solution to (CP) belongs to

$$y \in C([\varepsilon,T];D(A)) \cap C^1([\varepsilon,T];H). \tag{1.1.4}$$

1.2 On the degenerate and singular parabolic operators

1.2.1 Degenerate parabolic operator

Many problems that are relevant for applications are modeled by means of parabolic equations involving a diffusion coefficient that degenerates (i.e., loses the uniform ellipticity) at some points of the spatial domain. We refer the interested readers to [45, 151] for a motivating example arising in climatology, coming from the so called Energy Balance models, describing the role played by continental and oceanic areas of ice on the evolution of the climate. Other interesting contexts where the degenerate parabolic operators arise concern fluid dynamics, such as the so called "Crocco equation"[143], coming from the study of the velocity field of a laminar flow on a flat plate.

Let us consider the following linear degenerate parabolic operator

$$Py := y_t - (ay_x)_x, \quad x \in (0,1), \tag{1.2.1}$$

where the function a degenerates at the point $x = 0$ (i.e., $a(0) = 0$) and fulfills the following hypotheses:

Hypothesis 1.2.1. Weakly degenerate (WD). The function $a \in C([0,1]) \cap C^1((0,1])$ is such that $a(0) = 0$, $a > 0$ in $(0,1]$ and $\frac{1}{a} \in L^1(0,1)$.

Hypothesis 1.2.2. Strongly degenerate (SD). The function $a \in C^1([0,1])$ is such that $a(0) = 0$, $a > 0$ in $(0,1]$ and $\frac{1}{\sqrt{a}} \in L^1(0,1)$.

Example 1.2.1. A standard example of function fulfilling the Hypotheses 1.2.1 and 1.2.2 is

$$a(x) = x^\alpha, \quad \text{for } \alpha \in [0,2). \tag{1.2.2}$$

1.2.1.1 Function spaces and well-posedness

We begin by recalling the definition of some appropriate weighted Sobolev spaces. The reader is referred to [5] for more details on these spaces. First of all, we denote by $\mathcal{H}^1_a(0,1)$ the following space:

$$\mathcal{H}^1_a(0,1) := \Big\{ y \in L^2(0,1) \cap H^1_{loc}((0,1]) : \sqrt{a} y_x \in L^2(0,1) \Big\}.$$

- (WD) case:

$$H^1_a(0,1) := \Big\{ y \in \mathcal{H}^1_a(0,1) | \quad y(1) = y(0) = 0 \Big\}$$

  and

$$H^2_a(0,1) := \Big\{ y \in H^1_a(0,1) | \quad ay_x \in H^1(0,1) \Big\};$$

- (SD) case:

$$H^1_a(0,1) := \Big\{ y \in \mathcal{H}^1_a(0,1) | \quad y(1) = 0 \Big\}$$

  and

$$\begin{aligned} H^2_a(0,1) &:= \Big\{ y \in H^1_a(0,1) | \quad ay_x \in H^1(0,1) \Big\} \\ &= \Big\{ y \in L^2(0,1) | \; y \,\text{locally a.c. in}\, (0,1], ay \in H^1_0(0,1), \\ &\qquad ay_x \in H^1(0,1) \,\text{and}\, (ay_x)(0) = 0 \Big\}. \end{aligned}$$

In both cases, we also define the norms:

$$\|y\|_{H_a^1}^2 := \|y\|_{L^2(0,1)}^2 + \|\sqrt{a}y_x\|_{L^2(0,1)}^2, \qquad \|y\|_{H_a^2}^2 := \|y\|_{H_a^1}^2 + \|(ay_x)_x\|_{L^2(0,1)}^2.$$

By the definition of $H_a^1(0,1)$ it is clear that if $y \in H_a^2(0,1)$, the trace of y at $x = 1$ makes sense and this allows to consider Dirichlet condition $y(1) = 0$, in both cases of degeneracy, namely (WD) and (SD). On the contrary, the trace at $x = 0$ makes sense only in the (WD) case. Indeed, using the fact that $\dfrac{1}{a} \in L^1(0,1)$, we can prove that $H_a^1(0,1) \subset W^{1,1}(0,1)$. This property fails to hold in the (SD) case. Hence, the Dirichlet condition at $x = 0$ does not make sense anymore. However, Vancostenoble in [159, Proposition 1] proved that the functions of $H_a^2(0,1)$ satisfy the Neumann boundary condition $(ay_x)(0) = 0$.

Therefor, depending on the nature of a, we can associate to (1.2.1) the following boundary conditions:

- (WD) case:
$$y(0) = y(1) = 0; \tag{1.2.3}$$
- (SD) case:
$$(a(x)y_x)(0) = y(1) = 0. \tag{1.2.4}$$

We define the operator $(A, D(A))$ by

$$Ay := (ay_x)_x, \quad y \in D(A) = H_a^2(0,1). \tag{1.2.5}$$

For the operator $(A, D(A))$ the next result follows (see e.g. [59]).

Proposition 1.2.1. *The operator $A : D(A) \to L^2(0,1)$ is closed, self-adjoint and dissipative with dense domain.*

Now, let A be the degenerate operator defined in (1.2.5) and consider the following linear parabolic equation

$$\begin{cases} y_t - Ay = f, & (t,x) \in Q, \\ y(t,1) = 0, & t \in (0,T), \\ \begin{cases} y(t,0) = 0, & \text{(WD)}, \\ (ay_x)(t,0) = 0, & \text{(SD)}, \end{cases} & t \in (0,T), \\ y(0,x) = y_0, \end{cases} \tag{1.2.6}$$

where $y_0 \in L^2(0,1)$ and $f \in L^2(Q)$.

By invoking Proposition 1.2.1, one can show that A is the infinitesimal generator of an analytic semi-group of contractions on $L^2(0,1)$ (see Theorem 1.1.1-1.1.3).

Thus, the following well-posedness result holds (see e.g. Proposition 1.1.1 or [63, Theorem 2.1-2.2]).

Proposition 1.2.2.

- *For all $y_0 \in D(A)$ and for all $f \in H^1(0,T; L^2(0,1))$, (1.2.6) admits a unique solution*
$$y \in C\big([0,T], D(A)\big) \cap C^1(0,T; L^2(0,1)).$$
- *For all $f \in L^2(Q)$ and for all $y_0 \in L^2(0,1)$, (1.2.6) has a unique solution $y \in C(0,T; L^2(0,1)) \cap L^2(0,T; H_a^1(0,1))$ such that for every $\varepsilon \in (0,T)$ there holds*
$$y \in L^2\big(\varepsilon, T; D(A)\big) \cap H^1\big(\varepsilon, T; L^2(0,1)\big). \tag{1.2.7}$$
Moreover, if $f \in H^1(0,T; L^2(0,1))$, then for every $\varepsilon \in (0,T)$, one has
$$y \in C\big([\varepsilon, T]; D(A)\big) \cap C^1([\varepsilon, T]; L^2(0,1)).$$

We also recall the following compactness results.

Theorem 1.2.1. *[5, Section 6]*

1. *The space $H^1_a(0,1)$ is compactly embedded in $L^2(0,1)$.*
2. *The space $H^2_a(0,1)$ is compactly embedded in $H^1_a(0,1)$.*

Theorem 1.2.2. *[5, Theorem 6.4] The space $L^2\big(0,T;H^2_a(0,1)\big)\cap H^1\big(0,T;L^2(0,1)\big)$ is compactly embedded in $L^2(0,T;H^1_a(0,1))\cap C([0,T];L^2(0,1))$.*

1.2.1.2 Hardy Poincaré inequality

We emphasize that, the main ingredient in the proof of Carleman estimates stated in Sections 1.3-1.4 and Chapters 2 and 4, rely on an appropriate Hardy Poincaré's inequality. Before going further, let us first make the following assumptions on the function a.

Hypothesis 1.2.3. (WD) case. The function $a\in C([0,1])\cap C^1((0,1])$ is such that $a>0$ in $(0,1]$, $a(0)=0$ and $\exists\alpha\in[0,1)$ such that $xa'(x)\le\alpha a(x) \quad \forall x\in[0,1]$.

Hypothesis 1.2.4. (SD) case. The function $a\in C^1([0,1])$ is such that $a>0$ in $(0,1]$, $a(0)=0$ and $\exists\alpha\in[1,2)$ such that $xa'(x)\le\alpha a(x) \quad \forall x\in[0,1]$. Moreover,

$$\begin{cases} \exists\gamma\in(1,\alpha],\, x\mapsto \dfrac{a(x)}{x^\gamma} \text{ is increasing near } \; 0, \text{ when } \; \alpha>1, \\ \exists\gamma\in(0,1),\, x\mapsto \dfrac{a(x)}{x^\gamma} \text{ is increasing near } \; 0, \text{ when } \; \alpha=1. \end{cases} \tag{1.2.8}$$

The prototype $a(x)=x^\alpha$, being $\alpha\in[0,2)$, satisfies the above assumptions.

Remark 3. In the definitions above, and from now on, we have denoted with $'$ the derivative of a function depending only on one variable, while derivatives for functions of several variables will be denoted, as usual, with subscript letters, like y_x, y_t and so on.

Remark 4. Thanks to the Hypotheses 1.2.3-1.2.4, one has

$$xa'(x)\le\alpha a(x), \quad \forall x\in[0,1],$$

thus, $x\mapsto\dfrac{x^\alpha}{a(x)}$ is nondecreasing on $(0,1]$. Therefore

$$\frac{1}{a(x)}\le\frac{1}{x^\alpha a(1)}, \quad \forall x\in[0,1].$$

This implies that $\dfrac{1}{a}\in L^1(0,1)$ if $\alpha\in(0,1)$ and $\dfrac{1}{\sqrt{a}}\in L^1(0,1)$ if $\alpha\in[1,2)$. As a consequence, Hypotheses 1.2.3 and 1.2.4 yield respectively the Hypotheses 1.2.1 and 1.2.2.

Under the previous assumptions, we have.

Theorem 1.2.3. *[5, Proposition 2.1] Assume that one among the Hypothesis 1.2.3 or 1.2.4 holds. Then, there exists a positive constant C_{HP} such that*

$$\int_0^1 \frac{a}{x^2}u^2\,dx\le C_{HP}\int_0^1 au_x^2\,dx \tag{1.2.9}$$

for all $u\in H^1_a(0,1)$.

Remark 5. Note that, using the same arguments as above, with some suitable changes, we can treat the general case where a degenerates at both extremities of the interval $[0,1]$. As a typical example of a, one can consider the double power function

$$a(x) = x^{\alpha_1}(1-x)^{\alpha_2}, \quad x \in [0,1],$$

with $\alpha_1, \alpha_2 \in [0,2)$.

Finally, we would like to mention that, the case of degenerate parabolic operators in the higher space dimension have been treated by P. Cannarsa, P. Martinez and J. Vancostenoble in the recent paper [55], under some strong regularity on the involved diffusion matrix.

1.2.2 Degenerate/Singular parabolic operator

Singular potentials arise in various areas of applied science, such as quantum mechanics [43], electron capture problems [35] and in linearized combustion models [28]. This explain the increasing interest of the recent years for the parabolic operators involving such kind of potentials, namely

$$y_t - y_{xx} + \frac{\lambda}{|x|^\beta} y, \quad (t,x) \in (0,T) \times (0,1). \tag{1.2.10}$$

Since the pioneering work [24] by P. Baras and J.A. Goldstein, it is known that:

- when $\beta < 2$, global positive solutions occur, for all $\lambda \in \mathbb{R}$;
- when $\beta > 2$, instantaneous and complete blow-up appears, for all $\lambda \in \mathbb{R}$.

Thus, the case $\beta = 2$ is critical. This makes the inverse square potential $\dfrac{\lambda}{|x|^2}$ more interesting.

We point out that, in this case, the behavior of the associated Cauchy problem to (1.2.10) (with suitable boundary conditions) is mainly determined by the value of the parameter λ; there exists a critical value of λ that changes radically the well-posedness of the equation. More precisely, the following situations arise [24]:

- when $\lambda \leq \frac{1}{4}$, positive solution exists;
- when $\lambda > \frac{1}{4}$, the problem is ill-posed.

Here, the critical value $\lambda = \frac{1}{4}$ is the optimal value of the constant appearing in the following weighted Hardy's inequality [112, 144]

$$\frac{1}{4}\int_0^1 \frac{y^2}{|x|^2}\,dx \leq \int_0^1 y_x^2\,dx, \qquad \forall u \in H_0^1(0,1). \tag{1.2.11}$$

Later on, combining the result in [59] and [160] on the purely degenerate and purely singular operators, respectively, well-posedness result and new Carleman estimates where established in [95, 96, 159] for degenerate/singular operators of the form

$$y_t - (a(x)y_x)_x + \frac{\lambda}{|x|^{\beta_2}} y, \quad (t,x) \in (0,T) \times (0,1), \tag{1.2.12}$$

(for $a(x) \sim x^{\beta_1}$, $\beta_1 \in [0,2)$ and $\beta_1 + \beta_2 \leq 2$) with suitable boundary conditions and under appropriate assumptions on the involved coefficients:

$$\begin{cases} \beta_1 \in [0,2), \quad 0 < \beta_2 < 2 - \beta_1, \quad \lambda \in \mathbb{R}; \\ \beta_1 \in [0,2)\backslash\{1\}, \quad \beta_2 = 2 - \beta_1, \quad \lambda \leq \lambda^*(a,\beta). \end{cases} \tag{1.2.13}$$

Here $\lambda^*(a, \beta_1)$ denotes the optimal constant in the following inequality

$$\lambda^*(a,\beta_1)\int_0^1 \frac{y^2}{x^{2-\beta_1}}\,dx \le \int_0^1 a y_x^2\,dx. \tag{1.2.14}$$

Notice that, in the second assumption in (1.2.13), the case $\beta_1 = 1$ (and equivalently $\beta_2 = 2-\beta_1 = 1$) was removed since, in this case λ^* may be equal to 0. Hence, by virtue of (1.2.14), the fact that $\sqrt{a}y_x$ belongs to $L^2(0,1)$ does not necessarily imply that $\dfrac{y}{\sqrt{x}} \in L^2(0,1)$.

More recently, in [101, 103], G. Fragnelli and D. Mugnai complemented these results addressing the case of non smooth diffusion coefficients and potentials with mere degeneracy and singularity inside of the spatial domain, namely

$$y_t - (a(x)y_x)_x + \frac{\lambda}{b(x)}y, \quad (t,x) \in (0,T)\times(0,1), \tag{1.2.15}$$

The ways in which the functions a and b degenerate at $x_0 \in (0,1)$ may differ, and for this reason (1.2.15) has been investigated under the following assumptions:

Hypothesis 1.2.5. Double weakly degenerate case (WWD). There exists $x_0 \in (0,1)$ such that $a, b \in C^1([0,1]\setminus\{x_0\})$, $a, b > 0$ in $[0,1]\setminus\{x_0\}$, $a(x_0) = b(x_0) = 0$ and there exists $\beta_1, \beta_2 \in (0,1)$ such that $(x-x_0)a' \le \beta_1 a$ and $(x-x_0)b' \le \beta_2 b$ a.e. in $[0,1]$.

Hypothesis 1.2.6. Weakly strongly degenerate case (WSD). There exists $x_0 \in (0,1)$ such that $a \in C^1([0,1]\setminus\{x_0\})$, $b \in C^1([0,1]\setminus\{x_0\}) \cap W^{1,\infty}(0,1)$, $a, b > 0$ in $[0,1]\setminus\{x_0\}$, $a(x_0) = b(x_0) = 0$ and there exists $\beta_1 \in (0,1), \beta_2 \ge 1$ such that $(x-x_0)a' \le \beta_1 a$ and $(x-x_0)b' \le \beta_2 b$ a.e. in $[0,1]$.

Hypothesis 1.2.7. Strongly weakly degenerate case (SWD). There exists $x_0 \in (0,1)$ such that $a \in C^1([0,1]\setminus\{x_0\}) \cap W^{1,\infty}(0,1)$, $b \in C^1([0,1]\setminus\{x_0\})$, $a, b > 0$ in $[0,1]\setminus\{x_0\}$, $a(x_0) = b(x_0) = 0$ and there exists $\beta_1 \ge 1$, $\beta_2 \in (0,1)$ such that $(x-x_0)a' \le \beta_1 a$ and $(x-x_0)b' \le \beta_2 b$ a.e. in $[0,1]$.

Hypothesis 1.2.8. Double strongly degenerate case (SSD). There exists $x_0 \in (0,1)$ such that $a, b \in C^1([0,1]\setminus\{x_0\}) \cap W^{1,\infty}(0,1)$, $a, b > 0$ in $[0,1]\setminus\{x_0\}$, $a(x_0) = b(x_0) = 0$ and there exists $\beta_1, \beta_2 \ge 1$ such that $(x-x_0)a' \le \beta_1 a$ and $(x-x_0)b' \le \beta_2 b$ a.e. in $[0,1]$.

A typical example is

$$a(x) = |x-x_0|^{\beta_1} \text{ and } b(x) = |x-x_0|^{\beta_2}, \quad \text{with } \beta_1, \beta_2 \in [0,2).$$

Remark 6. We point out that, contrarily to the case of boundary degeneracy (and degeneracy/singularity), it can be shown that Dirichlet boundary conditions make sense for all situations, namely (WWD)-(SSD).

We emphasize that, the restriction $\beta_1 + \beta_2 < 2$ is related to the controllability issue. In particular, this is not necessary for the well-posedness result, at least in the case where $\lambda < 0$. Otherwise, when $\lambda > 0$, the major step in the proof of the well-posedness for the Cauchy problem associated to (1.2.15) relies heavily on a Hardy-Poincaré inequality of the form: there exists $\tilde{C} > 0$ such that

$$\int_0^1 \frac{y^2}{b}\,dx \le \tilde{C}\int_0^1 a y_x^2\,dx, \qquad \forall y \in H_{a,b}(0,1), \tag{1.2.16}$$

where $H_{a,b}(0,1)$ is an appropriate weighted Hilbert space which may coincide with the Sobolev space $H_0^1(0,1)$ in some cases. This inequality holds true if one among Hypotheses (WWD)-(SSD) is fulfilled with $\beta_1 + \beta_2 < 2$. Once again, the case $\beta_1 = \beta_2 = 1$ should be removed since, as above, an inequality of the form (1.2.16) fails to hold. Further, under the condition $\lambda \in (0, \dfrac{1}{\tilde{C}})$, the unbounded operator $Au := (ay_x)_x + \dfrac{\lambda}{b}y$ with appropriate domain $D(A)$ is dissipative in the space $H_{a,b}(0,1)$. Indeed, by invoking (1.2.16), we infer that

$$\begin{aligned} -\langle Ay, y\rangle_{L^2(0,1)} &= \int_0^1 ay_x^2\,dx - \lambda\int_0^1 \frac{y^2}{b}\,dx \\ &\geq C\|y\|^2_{H_{a,b}}, \qquad \forall y \in D(A), \end{aligned}$$

for some constant $C > 0$. On the other hand, by means of standard arguments, one may also prove that the operator $(A, D(A))$ is self-adjoint. Hence, in view of Theorems 1.1.1 and 1.1.3, as an immediate consequence of the previous results, it follows that $(A, D(A))$ generates an analytic semigroup of contractions in $L^2(0,1)$. Furthermore, a result of existence and uniqueness, analogous to one in the purely degenerate context (see Proposition 1.2.2), holds.

We refer to [101], for precise definitions and properties of the space $H_{a,b}(0,1)$ and the domain $D(A)$.

1.3 Inverse problem

We present a simple inverse problem that consists the determination of an unknown source term in a degenerate parabolic equation from some measurement data on a part of the boundary of the spatial domain. In particular, we explain how the uniqueness and stability properties can be achieved following the fundamental approach proposed by Imanuvilov and Yamamoto in [115] which is mainly based on Carleman estimates.

Consider the linear degenerate parabolic equation

$$\begin{cases} y_t - (ay_x)_x = f, & (t,x) \in Q, \\ y(t,1) = 0, & t \in (0,T) \\ \text{and} \begin{cases} y(t,0) = 0, & \text{(WD)}, \\ (ay_x)(t,0) = 0, & \text{(SD)}, \end{cases} & t \in (0,T), \\ y(0,\cdot) = y_0 \in L^2(0,1) \end{cases} \tag{1.3.1}$$

where $f \in L^2(Q)$ and the coefficient a degenerates at the extremity $x = 0$ and satisfies the Hypotheses 1.2.3 and 1.2.4.

The inverse source problem can be described in the following way: *is it possible to retrieve the source term f in (1.3.1) from the knowledge of the term $(ay_x)_x$ at some fixed time T' along with an additional boundary observation of the solution?*

To this purpose, one needs to assume that the source term f to be recovered belongs to an admissible set. Therefore, for a given constant $C_0 > 0$, we shall suppose that f lies in

$$S(C_0) := \{g \in H^1(0,T;L^2(0,1)) | \quad |g_t(t,x)| \leq C_0|g(T',x)|, \text{ for a.e. } (t,x) \in Q\}. \tag{1.3.2}$$

The answer of the previous question is the following result concerning the Lipschitz stability estimate of the parabolic equation (1.3.1) in terms of boundary measurements:

Theorem 1.3.1. *Let $T > 0$, $C_0 > 0$ and $y_0 \in L^2(0,1)$ be given. Then, there exists $C = C(T, t_0, C_0) > 0$ such that, for every $f \in S(C_0)$, the solution y of (1.3.1) fulfills*

$$\|f\|^2_{L^2(Q)} \leq C\Big(\|(ay_x)_x(T',\cdot)\|^2_{L^2(0,1)} + \|y_{t,x}(\cdot,1)\|^2_{L^2(t_0,T)}\Big). \tag{1.3.3}$$

The proof of Theorem 1.3.1 will be given later on.

Remark 7. We point out that, a Lipschitz stability estimate like (1.3.3) has been established in [55] for smooth degenerate operators in dimension two and in [37] for a one dimensional operator, in nondivergence form, involving interior degeneracy.

Remark 8. Since in Theorem 1.3.1 the source term f lies in $S(C_0)$ and in particular in $H^1(0,T;L^2(0,T))$, then, the regularity results stated in Proposition 1.2.2 guarantee that the associated solution y belongs to $C([0,T];D(A))$. Therefore, the first term in the right hand side of (1.3.3) is well defined. The second term will be discussed later on.

Remark 9. • We would like to mention that t_0 is taking to be away from $t=0$ since in this case better regularity results hold considering (t_0,T) instead of $(0,T)$, see Proposition 1.2.2.

• We point out that, if one can choose the initial time t_0 closer to the final instant T, one only requires the boundary observation on a vary small interval of time.

• Note also that, in our inverse problem we require some measurements of the solution at a fixed time $T' < T$. This restriction comes from the approach followed in the proof which is mainly based on Carleman estimates. Therefore, it would be extremely interesting to know whether the stability estimate (1.3.3) remains true when $T' = T$.

Remark 10. The assumption (1.3.2) may appear as a strong restriction. Nevertheless, for a general term source, i.e., $f \in L^2(Q)$ (without any additional assumption), employing an argument of controllability for the problem under consideration, it can be shown that the estimate (1.3.3) fails to hold. We refer to [116, p. 159] where this result was first proved for the standard heat equation. See also [158, Remark 3.2] for further discussion on this issue in the context of heat equation involving inverse-square.

The second issue in our inverse problem is the *uniqueness*, namely: can we deduce that

$$(ay_{1x})_x(T',\cdot) = (ay_{2x})_x(T',\cdot),\ \text{in}\,(0,1)\ \text{ and }\ y_{1t,x}(\cdot,1) = y_{2t,x}(\cdot,1)\ \text{on}\,(t_0,T)$$

imply $f_1 \equiv f_2$ in Q.

We emphasize that, the estimate (1.3.3) yields stability of inverse source problem, however, it does not ensure the uniqueness result since the set of the admissible sources $S(C_0)$ is not a linear space. Thus, in order to overcome this difficulty, one needs to restrict the study to the situation where only the spatial part of the source term is unknown. To be more precise, we shall assume that the source term f takes the following form

$$f(t,x) = g(x)r(t,x),$$

where $g \in L^2(0,1)$ is the unknown spatial term and $r \in \mathrm{C}^1([0,T]\times[0,1])$ is a given smooth function satisfying $|r(T',x)| \geq c$ for some given constant $c > 0$. Thus, as a direct consequence of Theorem 1.3.1, we get the following result.

Theorem 1.3.2. *There exists a constants $C = C(T,t_0,C_0) > 0$ such that for all $f_1 = rg_1$ and all $f_2 = rg_2$ with $g_1, g_2 \in L^2(0,1)$ the associated solutions y_1 and y_2 of (1.3.1) fulfill*

$$\|g_1 - g_2\|^2_{L^2(0,1)} \leq C\Big(\|(a(y_{1x}-y_{2x}))_x(T',\cdot)\|^2_{L^2(0,1)} + \|y_{1t,x} - y_{2t,x}(\cdot,1)\|^2_{L^2(t_0,T)}\Big). \qquad (1.3.4)$$

We skip the proof of Theorem 1.3.2 since it is analogue to that of Theorem 2.1.2 in Chapter 2.

As an immediate consequence of the above theorem, we infer the desired uniqueness result.

As we have said before, the most powerful tool to establish the stability estimate for inverse problems of recovering coefficients or source terms of various partial differential equations takes the form of Carleman estimates. The later will be the objective of the following subsection.

1.3.1 Carleman estimates

We present suitable estimates of Carleman type satisfied by solutions of the nonhomogeneous degenerate parabolic problem (1.3.1). To this aim, proceeding as in [63], we begin by introducing the following weight functions

$$\begin{aligned} &\theta(t) := \frac{1}{(t-t_0)^4(T-t)^4}, \quad \psi(x) := \gamma\Big(\int_0^x \frac{y}{a(y)}\,dy - d\Big),\\ &\varphi(t,x) := \theta(t)\psi(x), \quad \text{and} \quad \eta(t) := T + t_0 - 2t. \end{aligned} \tag{1.3.5}$$

Here $t_0 > 0$, $\gamma > 0$ and $d > d^\star := \sup_{[0,1]} \int_0^x \frac{y}{a(y)}\,dy$. With these choices, it comes that

$$-\gamma d \le \psi(x) < 0, \quad \text{for all } x \in [0,1]. \tag{1.3.6}$$

Moreover, one can easily check that

$$\theta(t) \to +\infty, \quad \text{as} \quad t \to t_0^+, T^- \tag{1.3.7}$$

and

$$|\theta_t| \le C\theta^{5/4}, \quad |\theta_{tt}| \le C\theta^{3/2} \quad \text{in } (t_0,T). \tag{1.3.8}$$

Therefore, $\varphi(t,x) < 0$ for all $(t,x) \in Q$ and $\varphi(t,\cdot) \to -\infty$ as $t \to t_0^+, T^-$.

Remark 11. All the results stated in this thesis remain valid when the expression of the weighted time function $\theta(t) := \frac{1}{[(t-t_0)(T-t)]^4}$ is replaced by $\theta(t) := \frac{1}{[(t-t_0)(T-t)]^2}$. We refer to [2, Remark 1] for a discussion on this fact.

Moreover, if the problem is considered in $(0,T)$, these results still hold considering $\theta(t) := \frac{1}{[t(T-t)]^4}$ or $\theta(t) := \frac{1}{[t(T-t)]^2}$.

Denote by

$$I(y) := \iint_{Q_{t_0}} \Big(\frac{1}{s\theta} y_t^2 + s\theta^{\frac{3}{2}}|\eta\psi|y^2 + s\theta a(x) y_x^2 + s^3\theta^3 \frac{x^2}{a(x)} y^2\Big) e^{2s\varphi}\,dx\,dt.$$

The following Carleman inequality holds.

Theorem 1.3.3. *There exist two positive constants C and s_0, such that all solutions y of (1.3.1) fulfill, for all $s \ge s_0$,*

$$I(y) \le C\Big(\iint_{Q_{t_0}} f^2 e^{2s\varphi}\,dx\,dt + s\gamma a(1)\int_{t_0}^T \theta(t) y_x^2(t,1) e^{2s\varphi(t,1)}\,dt\Big). \tag{1.3.9}$$

The proof of the above theorem is completely similar to the one of Theorem 2.3.1 in Chapter 2. Thus we omit it.

Remark 12. Notice that, since the weight function φ is negative and θ blows up at the extremities of the time interval $[t_0,T]$, all the involved weights in the above estimate are exponentially vanishing at $t = t_0$ and $t = T$. This explains why the estimate (1.3.9) holds without any assumption on the values of the function y at $t = t_0, T$.

1.3.2 Proof of Theorem 1.3.1

In this paragraph we prove the stability estimate (1.3.3) following the arguments in [63, 115]. In proving this estimate, the Carleman inequality stated previously will play a peculiar role.

Since $y_0 \in L^2(0,1)$ and $g \in H^1(0,T;L^2(0,1))$, it follows from Proposition 1.2.2 that the solution y of (1.3.1) satisfies sufficient regularity properties to perform the following computations.

Let $z = y_t$ where y solves (1.3.1) in (t_0,T). Then, thanks to Proposition 1.2.2, y belongs to

$$C\big([t_0,T],D(A)\big) \cap C^1([t_0,T];L^2(0,1)).$$

In particular, we have that $y(t_0) \in D(A)$, hence, by (1.2.7) and using standard arguments (see for instance [63, Lemma 2.2 and Remark 2.2]), one can show that z belongs to

$$H^1\big(t_0,T;L^2(0,1)\big) \cap L^2\big(t_0,T;D(A)\big).$$

Therefore, the second term in the right hand side of the stability estimate (1.3.3) is well defined.

Now, applying the Carleman inequality (1.3.9) to the system satisfied by z, we have that

$$\begin{aligned} I(z) &\le C\Big(\iint_{Q_{t_0}} f^2 e^{2s\varphi}\,dx\,dt + s\gamma a(1)\int_{t_0}^T \theta(t) z_x^2(t,1)e^{2s\varphi(t,1)}\,dt\Big) \\ &:= J(f,z), \end{aligned} \tag{1.3.10}$$

for all $s \ge s_0$ and for a positive constant C.

We divide the remaining part of the proof into three steps.

Step 1. We claim that there exists $C = C(t_0,T) > 0$ such that

$$I(z) \le C\Big(\frac{1}{\sqrt{s}}\int_0^1 f^2(T',x)e^{2s\varphi(T',x)}\,dx + \|y_{t,x}(\cdot,1)\|^2_{L^2(t_0,T)}\Big). \tag{1.3.11}$$

By observing that

$$\sup_{t\in[0,T]} s\theta(t)e^{2s\varphi(t,1)} < +\infty,$$

one has

$$s\gamma a(1)\int_{t_0}^T \theta(t)z_x^2(t,1)e^{2s\varphi(t,1)}dt \le C\|(y_t)_x(\cdot,1)\|^2_{L^2(t_0,T)}. \tag{1.3.12}$$

On the other hand, the assumption $f \in S(C_0)$ implies that

$$\iint_{Q_{t_0}} |f_t|e^{2s\varphi}\,dx\,dt \le C_0\iint_{Q_{t_0}} |f(T',x)|e^{2s\varphi(t,x)}\,dx\,dt. \tag{1.3.13}$$

At this stage, we will need the next technical result (whose proof is postpone to the Appendix).

Lemma 1.3.1. *There exists $C > 0$, which is independent of s, such that*

$$\iint_{Q_{t_0}} f^2(T',x)e^{2s\varphi}dxdt \le \frac{C}{\sqrt{s}}\int_0^1 f^2(T',x)e^{2s\varphi(T',x)}dx.$$

Let us continue with the proof of (1.3.11). By (1.3.12) together with (1.3.13) and Lemma 1.3.1, we deduce that

$$J(f,z) \le C\big(\frac{1}{\sqrt{s}}\int_0^1 f^2(T',x)e^{2s\varphi(T',x)}\,dx + \|y_{t,x}(\cdot,1)\|^2_{L^2(t_0,T)}\Big),$$

which yields (1.3.11).
Step 2. We claim that there exists $C = C(t_0, T) > 0$ such that

$$\int_0^1 z^2(T', x)e^{2s\varphi(T',x)}\,dx \le CI(z), \tag{1.3.14}$$

where $I(z)$ is defined in (1.3.10).

From the fact that $\theta(t) \to +\infty$ as $t \to t_0^+$ and $\varphi < 0$, we get that

$$\lim_{t\to t_0} z(t, x)e^{s\varphi(t,x)} = 0, \qquad \text{for a.e. } x \in (0, 1).$$

Thus,

$$\begin{aligned}\int_0^1 z^2(T', x)e^{2s\varphi(T',x)}\,dx &= \int_0^1\int_{t_0}^{T'} \frac{\partial}{\partial t}\big(z^2e^{2s\varphi}\big)\,dxdt\\ &= \int_0^1\int_{t_0}^{T'} \Big(2s\varphi_t z^2 + 2zz_t\Big)e^{2s\varphi}\,dxdt.\end{aligned} \tag{1.3.15}$$

Next, employing the Young's inequality, we obtain

$$\int_0^1\int_{t_0}^{T'} 2zz_te^{2s\varphi}\,dxdt \le C\iint_{Q_{t_0}} \frac{z_t^2}{s\theta}e^{2s\varphi}\,dxdt + \iint_{Q_{t_0}} s\theta ze^{2s\varphi}\,dxdt. \tag{1.3.16}$$

Now, by observing that $|\varphi_t| \le C\theta^{\frac{3}{2}}|\eta\psi|$, it holds that

$$2\int_0^1\int_{t_0}^{T'} s\varphi_t z^2e^{2s\varphi}dxdt \le C\iint_{Q_{t_0}} s\theta^{3/2}|\eta\psi|z^2e^{2s\varphi}dxdt. \tag{1.3.17}$$

Plugging (1.3.16) and (1.3.17) in (1.3.15), we obtain (1.3.14). Hence, the claim of the second step is proved.
Step 3. Conclusion.

From (1.3.10), (1.3.11) and (1.3.14), we find that

$$\int_0^1 z^2(T', x)e^{2s\varphi(T',x)}\,dx \le C\Big(\frac{1}{\sqrt{s}}\int_0^1 f^2(T', x)e^{2s\varphi(T',x)}\,dx + \|y_{t,x}(\cdot, 1)\|^2_{L^2(t_0,T)}\Big). \tag{1.3.18}$$

On the other hand, in view of (1.3.1), one can write

$$\begin{aligned}f(T', \cdot) &= -(ay_x)_x(T', \cdot) + y_t(T', \cdot)\\ &= -(ay_x)_x(T', \cdot) + z(T', \cdot), \quad \text{in } (0, 1).\end{aligned}$$

Hence,

$$\int_0^1 f^2(T', x)e^{2s\varphi(T',x)}\,dx \le C\|(ay_x)_x(T', \cdot)\|^2_{L^2(0,1)} + C\int_0^1 z^2(T', x)e^{2s\varphi(T',x)}\,dx,$$

since $\sup_{x\in(0,1)} e^{2s\varphi(T',x)} < +\infty$.

Now, substituting (1.3.18) in the previous inequality, it comes that

$$\begin{aligned}\int_0^1 f^2(T', x)e^{2s\varphi(T',x)}\,dx &\le C\|(ay_x)_x(T', \cdot)\|^2_{L^2(0,1)} + C\|y_{t,x}(\cdot, 1)\|^2_{L^2(t_0,T)}\\ &\quad + \frac{C}{\sqrt{s}}\int_0^1 f^2(T', x)e^{2s\varphi(T',x)}\,dx.\end{aligned} \tag{1.3.19}$$

By the estimate (1.3.19), we conclude that, for s large enough,

$$\int_0^1 f^2(T',x)e^{2s\varphi(T',x)}\,dx \le C\|(ay_x)_x(T',\cdot)\|^2_{L^2(0,1)} + C\|y_{t,x}(\cdot,1)\|^2_{L^2(t_0,T)}.$$

Next, using once again the fact that $f \in S(C_0)$, one can write

$$\begin{aligned} |f(t,x)| &\le \int_{T'}^t |f_t(\tau,x)|d\tau + |f(T',x)| \\ &\le C|f(T',x)|, \end{aligned} \tag{1.3.20}$$

for some positive constant $C = C(C_0, T)$.

On the other hand, using the definition of φ, we have that

$$\tilde{c} \le \inf_{x\in(0,1)} e^{2s\varphi(T',x)}$$

for some constant $\tilde{c} > 0$. This together with (1.3.19) and (1.3.20) gives

$$\begin{aligned} \|f\|^2_{L^2(Q)} &\le C\int_0^1 f^2(T',x)e^{2s\varphi(T',x)}\,dx \\ &\le C\|(ay_x)_x(T',\cdot)\|^2_{L^2(0,1)} + C\|y_{t,x}(\cdot,1)\|^2_{L^2(t_0,T)} \end{aligned}$$

and hence the proof of Theorem 1.3.1 is completed.

Remark 13. Substituting the boundary observation $(y_t)_x(\cdot,1)_{|(t_0,T)}$ by a locally distributed observation $y_{t|(t_0,T)\times\omega}$, we get another stability estimate similar to the one stated in Theorem 1.3.1. This issue has recently been considered in [63] in the case where the diffusion coefficient is given by $a(x) = x^\alpha$, being $0 \le \alpha < 2$.

In Chapter 2, we will provide an extension of the results proved above for a scalar equation to the context of a general coupled system.

The construction of a source term in a partial differential equation allows one to solve other types of inverse problems, such as, the identification of potential and diffusion coefficients. In the following, we briefly discuss a recent result concerning the inverse diffusion problem for the following degenerate parabolic equation

$$\begin{cases} y_t - d(ay_x)_x = f, & (t,x) \in Q, \\ y(t,1) = 0, & t \in (0,T) \\ \text{and} \begin{cases} y(t,0) = 0, & (\text{WD}), \\ (ay_x)(t,0) = 0, & (\text{SD}), \end{cases} & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{1.3.21}$$

where $d \in I := [d_0, d_1]$, being $0 < d_0 < d_1$. More precisely, we address the following question: *is it possible to recover the constant d in (1.3.21) from partial measurements of the solution y of (1.3.21), namely the term $(ay_x)_x$ at a fixed time $t = T'$ and the spatial derivative of the solution y on a part of the boundary?*

In order to deal with this question, as pointed out by J. Trot [154], the solution of the problem under consideration needs to satisfy more refined regularity results than those stated previously. For this, it is mandatory to impose some restrictions on the sets of the initial conditions and source terms.

Then, under suitable assumptions on the data y_0 and f, the estimate stated in Theorem 1.3.1 for source terms together with a maximum principle gives the following stability estimate:

Theorem 1.3.4. *For all $d \in I$ and all $\tilde{d} \in I$, the corresponding solutions y_d and $y_{\tilde{d}}$ of problem (1.3.21) satisfy:*

$$|d-\tilde{d}|^2 \leq C\big(\|\big(a(y_{d,x}-y_{\tilde{d},x})\big)_x(T',\cdot)\|^2_{L^2(0,1)} + C\|(y_{d,t}-y_{\tilde{d},t})_x(\cdot,1)\|^2_{L^2(t_0,T)}\big). \tag{1.3.22}$$

for some positive constant $C = C(d_0, d_1, t_0, T, a, y_0, f)$.

The proof of the previous Theorem is analogous to the one of [154, Theorem 4], the only difference being that here we consider a more general diffusion coefficient a instead of x^α. However, we believe that by a simple adaptation of the proof, it can be shown that the estimate (1.3.22) holds. Being far from the purpose of this work, the details will be omitted.

As in [154], the proof of Theorem 1.3.4 relies on transforming the issue of identification of the diffusion constant in (1.3.21) into an inverse source problem for another equation like (1.3.21). Then, this later can be treated employing the stability estimate proved in Theorem 1.3.1.

Finally, it should be noted that, the stability estimates for both considered inverse problems, namely source terms and diffusion constants hold true provided that the rate of the degeneracy α (see Hypotheses 1.2.3 and 1.2.4) is less than 2. As far as we know, the question whether these estimates still valid when $\alpha > 2$ remains completely open even for the particular case $a(x) = x^\alpha$.

1.4 Controllability

In this section, we recall some fundamental results on the controllability of linear differential systems. We refer to [72, 168] for a complete and more general presentation on this topic.

Let H and U be two Hilbert spaces and consider the following linear control system:

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + Bu(t), \qquad t \in (0,T) \\ y(0) = y_0, \end{cases} \tag{S}$$

where $y_0 \in H$ is the initial datum, $y = y(t)$ is the state of the system, which takes values in the state space H, $A : D(A) \to H$ is a closed unbounded operator which generates a C_0-semigroup $(T(t))_{t\geq 0}$ on H, $u = u(t)$ denotes the control force taking values in the control space U and $B \in \mathcal{L}(U; H)$. The operator A determines the dynamic of the system whereas B describes the way the control acts on the system.

We assume that the Cauchy problem (S) is well posed in the sense of Hadamard, i.e., for all $y_0 \in H$ and $u \in L^2(0,T;U)$, it exists a unique $y \in C([0,T];H)$ fulfilling

$$\sup_{t\in[0,T]} \|y(t)\|_H \leq C(\|u\|_{L^2(0,T;U)} + \|y_0\|_H),$$

for some constant $C > 0$ depending on A, B and T.

Let us now introduce the different notions of controllability for the abstract system (S).

Definition 1.4.1. System (S) is **exactly controllable** at time T if, for any initial state $y_0 \in H$ and any desired state $y_T \in H$, there exists a control function $u \in L^2(0,T;U)$ such that the corresponding solution y fulfills

$$y(T) = y_T \quad \text{in } H.$$

Definition 1.4.2. System (S) is **null controllable** at time T if, for any initial state $y_0 \in H$, there exists a control function $u \in L^2(0,T;U)$ such that the corresponding solution y fulfills

$$y(T) = 0 \quad \text{in } H.$$

Definition 1.4.3. System (S) is **approximately controllable** at time T if, for any initial state $y_0 \in H$, any desired state $y_T \in H$ and any $\varepsilon > 0$, there exists $u \in L^2(0,T;U)$ such that the corresponding solution y satisfies

$$\|y(T) - y_T\|_H \leq \varepsilon.$$

Definition 1.4.4. System (S) is **controllable to trajectories** at time T if, for any trajectory $\tilde{y}$ (i.e., solution of (S) corresponding to $\tilde{y}_0 \in H$ and $u \equiv 0$) and any $y_0 \in H$, there exists $u \in L^2(0,T;U)$ such that the associated solution y to (S) satisfies

$$y(T) = \tilde{y}(T) \quad \text{in } H.$$

Remark 14. According to the Definition 1.4.4, the aim of the control to trajectories process consists in driving the state of the system (S) from any initial datum y_0 to an arbitrary final state of the uncontrolled system by means of a suitable control u.

Obviously, exact controllability of system (S) gives controllability to trajectories, null and approximate controllability. However, the converse does not hold for a large class of nonlinear ODEs and time-irreversible PDEs. A typical example is the heat equation, which is known to be null controllable, from both internal and boundary controls, and not exactly controllable [87, 106].

On the other hand, it is not difficult to see that, in the linear case, the controllability to trajectories and the null controllability are equivalent notions. Indeed, let $y_0, \tilde{y}_0 \in H$ and denote by $\tilde{y}$ the solution of (S) associated to $\tilde{y}_0$. We introduce the following change of variables $z := y - \tilde{y}$. Thanks to the linearity of the system, it is straightforward to check that z satisfies (S) with an initial datum $y_0 - \tilde{y}_0 \in H$. Therefore, controllability to trajectories for system (S) is reduced to null controllability property for the system satisfied by z. We emphasize that, this may no longer be true in the framework of nonlinear systems.

1.4.1 Controllability of finite dimensional systems

It is by now well known that, in the context of linear finite dimensional systems (i.e., the case where $H = \mathbb{R}^n$, $U = \mathbb{R}^m (n, m \geq 1)$ and $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)$ are two given matrices with constant real coefficients), the four concepts of controllability are equivalent. Moreover, the controllability property, for such a system, holds true if and only if the following *algebraic Kalman's rank condition*

$$\operatorname{rank}[B|AB|A^2B|\cdots|A^{n-1}B] = n \tag{1.4.1}$$

is fulfilled.

Of course the case where $m < n$ (i.e. the number of control forces is less than the number of components of the system) is the most interesting case.

Notice that, the condition (1.4.1) is independent on the initial datum y_0 and the control time T. Thus, if such a linear system is controllable in some time $T > 0$, then it is controllable for every $T > 0$. Nevertheless, the situation may be completely different in the case of PDEs.

1.4.2 Controllability and duality

In the present subsection, we characterize null and approximate controllability properties for the linear system (S) by appropriate dual formulations for its adjoint problem.

In particular, in the next result, we are going to show that the null controllability property for (S) is equivalent to an *observability inequality* for the following adjoint system (which is a backward in time problem):

$$\begin{cases} -\dfrac{dz(t)}{dt} = A^*z(t), & t \in (0,T) \\ z(T) = z_T \in H. \end{cases} \tag{D}$$

Here A^* is the adjoint of the operator A.

The following result holds.

Proposition 1.4.1. *System (S) is null controllable at time $T > 0$ if and only if there exists a positive constant C_{obs} such that solutions of the adjoint system (D) satisfy*

$$\|z(0)\|_H^2 \leq C_{obs} \int_0^T \|B^* z(t)\|_U^2 dt, \qquad \forall z_T \in H. \tag{1.4.2}$$

Here B^* is the adjoint of the control operator B.

Remark 15. We emphasize that, the constant C_{obs} in (1.4.2) depends on A, B and T.

Proof. The proof of this Theorem is inspired by [66].

($\Leftarrow$) Assume that (1.4.2) holds. Let $y_0 \in H$.

We introduce the following subspace Y of $L^2(0,T;U)$

$$Y := \{B^* z | \ z \text{ is a solution of (D) for some } z_T \in H\}.$$

For any $B^* z \in Y$, we define the following functional $F : Y \mapsto \mathbb{R}$ given by

$$F(B^* z) := -\langle z(0), y_0 \rangle_H. \tag{1.4.3}$$

By Cauchy-Schwartz inequality and (1.4.2), we have

$$\begin{aligned} \|F(B^* z)\|_H &\leq \|z(0)\|_H \|y_0\|_H \\ &\leq C_{obs} \|B^* z\|_{L^2(0,T;U)} \|y_0\|_H. \end{aligned}$$

Therefore, F is a bounded linear functional on the normed vector space H with the norm inherited from $L^2(0,T;U)$. Hence, by the Hahn-Banach Theorem, F can be extended to a bounded linear functional on $L^2(0,T;U)$ (and we keep the same notation to denote it).

Next, employing the Riesz Representation Theorem, we infer that, there exists a unique $w \in L^2(0,T;U)$ such that

$$F(v) = \langle v, w \rangle_{L^2(0,T;U)}, \qquad \forall v \in L^2(0,T;U).$$

Thus, for any $B^* z \in Y$, we have

$$F(B^* z) = \langle B^* z, w \rangle_{L^2(0,T;U)},$$

which, together with (1.4.3), gives

$$-\langle z(0), y_0 \rangle_H = \langle B^* z, w \rangle_{L^2(0,T;U)}. \tag{1.4.4}$$

We claim that

$$w = u. \tag{1.4.5}$$

Indeed, multiplying the equation (S) by z, integrating by parts over Q and using (D), we obtain

$$\begin{aligned} \langle z_T, y(T) \rangle_H - \langle z(0), y_0 \rangle_H &= \int_0^T \langle y_t(s), z(s) \rangle_H \, ds \\ &+ \int_0^T \langle z_t(s), y(s) \rangle_H \, ds \\ &= \langle B^* z, u \rangle_{L^2(0,T;U)} \end{aligned} \tag{1.4.6}$$

for all $z_T \in H$.

Combining this with (1.4.4) and (1.4.5), we conclude that

$$\langle z_T, y(T)\rangle_H = 0 \qquad \text{for any} \quad z_T \in H.$$

Consequently, $y(T) = 0$ in H and hence the null controllability result.

($\Rightarrow$) Now, we suppose that (S) is null controllable, i.e., for any $y_0 \in H$ there exists $u \in L^2(0,T;U)$ such that the corresponding solution y of (S) fulfills $y(T) = 0$ in H. We want to prove that (1.4.2) holds true.

For any $z_T \in H$, we introduce the operator $G : H \mapsto H$ given by

$$G(z_T) := z(0). \tag{1.4.7}$$

We proceed by contradiction. Assume that (1.4.2) does not hold. Then, there exists a sequence $\{z_{T,k}\}_{k\geq 1}$ such that the corresponding solution $\{z_k\}_{k\geq 1}$ to (S) with $z_T = z_{T,k}$ satisfies, for any $k \in \mathbb{N}^*$

$$k^2 \int_0^T \|B^* z_k(t)\|_U^2 dt < \|z_k(0)\|_H^2,$$

hence

$$0 \leq \int_0^T \|B^* z_k(t)\|_U^2 \, dt < \frac{1}{k^2}\|z_k(0)\|_H^2. \tag{1.4.8}$$

Let $\tilde{z}_{T,k} := \dfrac{\sqrt{k} z_{T,k}}{\|z_k(0)\|_H}$ and denote by $\tilde{z}_k$ the associated solution to (D) with $z_T = \tilde{z}_{T,k}$, hence $\tilde{z}_k = \dfrac{\sqrt{k} z_k}{\|z_k(0)\|_H}$.

Thus, by (1.4.8), it follows that

$$\begin{aligned} \int_0^T \|B^* \tilde{z}_k(t)\|_U^2 \, dt &= \frac{k}{\|z_k(0)\|_H^2} \int_0^T \|B^* z_k(t)\|_U^2 \, dt \\ &\leq \frac{1}{k}. \end{aligned} \tag{1.4.9}$$

Using (1.4.7) we also have

$$\begin{aligned} \|G(\tilde{z}_{T,k})\|_H^2 &= \|\tilde{z}_k^2(0)\|_H^2 \\ &= k. \end{aligned} \tag{1.4.10}$$

Now, proceeding as in (1.4.6) and having in mind the fact that $y(T) = 0$ in H, we find that the solution of (D) satisfies

$$-\langle z(0), y_0\rangle_H = \langle B^* z, u\rangle_{L^2(0,T;U)}, \qquad \text{for any} \quad z_T \in H.$$

Applying the above identity for $z_T = \tilde{z}_{T,k}$ and using (1.4.7), it holds that

$$-\langle G(\tilde{z}_{T,k}), y_0\rangle_H = \langle B^* \tilde{z}_k, u\rangle_{L^2(0,T;U)}. \tag{1.4.11}$$

From (1.4.9) and (1.4.11), we can easily see that $G(\tilde{z}_{T,k})$ converges (weakly) to 0 in H.

Thus, by the Principle of Uniform Boundedness, we infer that the sequence $\{G(\tilde{z}_{T,k})\}_{k\geq 1}$ is uniformly bounded in H. This clearly provides a contradiction to (1.4.9) and hence the claim follows. □

Remark 16. The usefulness of the above Proposition consists on the fact that it reduces the proof of the null controllability of (S) to the study of an observability inequality for the homogeneous system (D) which appears, at least conceptually, to be a simpler problem.

Remark 17. The observability inequality (1.4.2), when it holds, permits us to estimate the total energy of the solution of the backward system (D) at time $t = 0$ simply through a measurement of the output B^*z on $[0, T]$. However, in general, such an inequality is far from being evident and it needs a more careful analysis and major tools adapted to the considered problem; for instance, *microlocal analysis, Ingham inequalities, multiplier methods or Carleman estimates* [26, 106, 123, 132, 146, 147].

Next, we will show that the approximate controllability of (S) can be characterized by a unique continuation property for the adjoint system (D).

Proposition 1.4.2. *System (S) is approximately controllable at time $T > 0$ if and only if, for any $z_T \in H$ the solution to system (D) satisfies the following unique continuation property:*

$$B^*z = 0 \Rightarrow z = 0. \tag{1.4.12}$$

Proof. For any $y_0 \in H$ and $u \in L^2(0, T; U)$, we denote by $y(t; y_0, u)$ the solution of (S) at time $t \in [0, T]$.

We introduce the following operators $S : H \mapsto H$ and $L : L^2(0, T; U) \mapsto H$ defined by

$$S(y_0) := y(T; y_0, 0) \quad \text{and} \quad L(u) := y(T; 0, u).$$

Thanks to the linearity of the system (S), one has

$$\begin{aligned} y(T; y_0, u) &= y(T; 0, u) + y(T; y_0, 0) \\ &= L(u) + S(y_0). \end{aligned}$$

Hence, system (S) is approximately controllable if and only if

$$\begin{aligned} &\forall y_0, y_T \in H,\ \forall \varepsilon > 0,\ \exists u \in L^2(0, T; U) \quad \text{such that} \\ &\|L(u) + S(y_0) - y_T\|_H \leq \varepsilon, \end{aligned}$$

which can be written as

$$\begin{aligned} &\forall \varepsilon > 0,\ \forall w_T \in H,\ \exists u \in L^2(0, T; U) \quad \text{such that} \\ &\|L(u) - w_T\|_H \leq \varepsilon. \end{aligned}$$

Therefore,

$$\overline{L(L^2(0, T; U))} = H,$$

which is equivalent to

$$Ker(L^*) = \{0\}. \tag{1.4.13}$$

On the other hand, proceeding as in (1.4.6), one can see that the solutions of (S) and (D) satisfy

$$\begin{aligned} \langle L(u), z_T\rangle_H &= \langle y(T; 0, u), z_T\rangle_H \\ &= \langle u, B^*z\rangle_{L^2(0,T;U)} \quad \text{for any} \quad z_T \in H, \end{aligned}$$

which yields

$$L^*(z_T) = B^*z \quad \text{for any} \quad z_T \in H. \tag{1.4.14}$$

By (1.4.13) and (1.4.14), we deduce that

$$L^*(z_T) = B^*z \equiv 0 \Rightarrow z_T \equiv 0$$

and the thesis follows. □

We say that the operator A^* fulfills the *backward uniqueness* property when the solutions of system (D) satisfy

$$z(0) = 0 \Rightarrow z \equiv 0 \text{ on } [0,T]. \tag{1.4.15}$$

In other words, if the distribution of the dual system (D) vanishes at time $t = 0$, then this distribution has been identically null all long the time interval $[0,T]$.

Remark 18. Assume that solutions of the adjoint system (D) fulfill the backward uniqueness property (1.4.15). Then, the observability inequality (1.4.2) yields the unique continuation property (1.4.12). As a consequence, in this case, null controllability implies approximate controllability. Once again, this is not necessarily true in the nonlinear framework.

Once the observability inequality (1.4.2) is fulfilled for every solutions of the adjoint system (D), or, equivalently, the null controllability property of (S) holds, it is possible to obtain an estimate on the control force, with an L^2-norm, by means of the observability constant C_{obs} and the initial data y_0. More precisely, one has.

Proposition 1.4.3. *Assume that the observability inequality* (1.4.2) *holds for all solutions of* (D). *Then, system* (S) *is null controllable. Moreover, there exists some positive constant* $C_{obs} > 0$ *such that*

$$\|u\|^2_{L^2(0,T;U)} \leq C_{obs}\|y_0\|^2_H. \tag{1.4.16}$$

For a proof, see for instance [72, Theorem 2.44, p. 56] and [90].

1.4.3 Methods to study null controllability problem

In this subsection, we present some approaches that are useful to address the null controllability issue for parabolic systems. In order to understand how to use such approaches, we will apply them to some control problems that have already been solved.

1.4.3.1 Carleman estimates

The aim of this part is to explain how works the Carleman estimates to solve null controllability problem for degenerate parabolic equation

$$\begin{cases} y_t - (ay_x)_x = 1_\omega u, & (t,x) \in Q, \\ y(t,1) = 0, & t \in (0,T) \\ \text{and} \begin{cases} y(t,0) = 0, & (\text{WD}), \\ (ay_x)(t,0) = 0, & (\text{SD}), \end{cases} & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{1.4.17}$$

where 1_ω is the characteristic function of the nonempty open set $\omega \subset (0,1)$, the coefficient a degenerates at $x = 0$ and $y_0 \in L^2(0,1)$. In particular, we will show that an observability inequality for the solution of the associated adjoint problem

$$\begin{cases} -z_t - (az_x)_x = 0, & (t,x) \in Q, \\ z(t,1) = 0, & t \in (0,T) \\ \text{and} \begin{cases} z(t,0) = 0, & (\text{WD}), \\ (az_x)(t,0) = 0, & (\text{SD}), \end{cases} & t \in (0,T), \\ z(T) = z_T \in L^2(0,1), \end{cases} \tag{1.4.18}$$

is a direct consequence of suitable Carleman estimates for the involved parabolic operator. To this aim, as in the subsection 1.3.1, we begin by recalling the following weight functions:

$$\theta(t) := \frac{1}{(t(T-t))^4}, \quad \text{and} \quad \varphi(t,x) := \theta(t)\psi(x). \tag{1.4.19}$$

where ψ denotes the same space negative function defined in (1.3.5). Observe that θ satisfies the same properties in (1.3.7) (with $t_0 = 0$).

As a first step, we derive an intermediate Carleman inequality (with boundary observation) for the adjoint parabolic operator

$$Pw := -w_t - (aw_x)_x,$$

with the same boundary conditions as above. One has.

Theorem 1.4.1. *[5, Lemma 4.2] There exist two positive constants C and s_0, such that for all $s \geq s_0$,*

$$\begin{aligned}
&\iint_Q \Big(s\theta a(x)w_x^2 + s^3\theta^3\frac{x^2}{a(x)}w^2\Big)e^{2s\varphi}\,dx\,dt \\
&\leq C\Big(\iint_Q |Pw|^2 e^{2s\varphi}dxdt + s\gamma a(1)\int_0^T \theta(t)w_x^2(t,1)e^{2s\varphi(t,1)}\,dt\Big).
\end{aligned} \tag{1.4.20}$$

With the aid of Theorem 1.4.1 together with the Hardy-Poincaré inequality (1.2.9) and a standard cut-off argument, we can prove the following ω-local Carleman estimate for (1.4.18).

Theorem 1.4.2. *There exist two positive constants C and s_0, such that every solution z of (1.4.18) fulfills, for all $s \geq s_0$,*

$$\iint_Q \Big(s\theta a(x)z_x^2 + s^3\theta^3\frac{x^2}{a(x)}z^2\Big)e^{2s\varphi}\,dx\,dt \leq C\iint_{(0,T)\times\omega} z^2dxdt. \tag{1.4.21}$$

Proof. For the proof we refer to [5, Lemma 4.2]. See also Theorem 2.3.2 in Chapter 2, where this inequality is established for a coupled system in $(t_0, T) \times (0,1)$, with $t_0 > 0$, instead of Q. Nevertheless, this inequality still true in $(0,T) \times (0,1)$ with suitable changes. $\square$

Next, we shall apply the above Carleman estimates to deduce the observability inequality for the adjoint problem (1.4.18).

Proposition 1.4.4. *Let $T > 0$. Then, there exists a positive constant C_{obs} such that every solutions z of system (1.4.18) fulfills*

$$\int_0^1 z^2(0,x)dx \leq C_{obs}\iint_{(0,T)\times\omega} z^2dxdt. \tag{1.4.22}$$

Proof. Multiplying the adjoint equation (1.4.18) by z_t and integrating by parts over $(0,1)$, we obtain

$$\begin{aligned}
0 &= \int_0^1 (z_t + (az_x)_x)z_t\,dx \\
&= \int_0^1 z_t^2\,dx + [az_xz_t]_{x=0}^{x=1} - \int_0^1 az_xz_{tx}\,dx \\
&= \int_0^1 z_t^2\,dx - \int_0^1 az_xz_{tx}\,dx,
\end{aligned}$$

from which we derive

$$\frac{1}{2}\frac{d}{dt}\int_0^1 az_x^2\,dx \geq 0. \tag{1.4.23}$$

Thus, the function $t \mapsto \int_0^1 a(x) z_x^2(t,x)\,dx$ is nondecreasing for all $t \in [0,T]$. In particular,

$$\int_0^1 a z_x^2(0,x)\,dx \le \int_0^1 a z_x^2(t,x)\,dx \quad \forall t \in [0,T].$$

Integrating this inequality over $[T/4, 3T/4]$, θ being bounded therein, we get that

$$\begin{aligned}\int_0^1 a z_x^2(0,x)\,dx &\le \frac{2}{T}\int_{T/4}^{3T/4}\int_0^1 a z_x^2(t,x)\,dx\,dt \\ &\le C\int_{T/4}^{3T/4}\int_0^1 s\theta a(x) z_x^2(t,x) e^{2s\varphi(t,x)}\,dx\,dt \\ &\le C\iint_Q s\theta a(x) z_x^2 e^{2s\varphi}\,dx\,dt.\end{aligned}$$

Thus applying Theorem 1.4.2, it follows that

$$\int_0^1 a z_x^2(0,x)\,dx \le C\int_0^T\int_\omega z^2\,dxdt. \tag{1.4.24}$$

On the other hand, having in mind the fact that the function $x \mapsto \dfrac{x^2}{a(x)}$ is nondecreasing in $(0,1)$, applying the Hardy-Poincaré inequality (1.2.9), we immediately get

$$\int_0^1 z^2(0,x)dx \le C\int_0^1 a z_x^2(0,x)dx.$$

Finally, plugging the last inequality in (1.4.24), we end up with

$$\int_0^1 z^2(0,x)\,dx \le C\int_0^T\int_\omega z^2\,dxdt$$

and the conclusion follows with $C_{obs} = C$. □

As an immediate consequence of the observability inequality (1.4.22) and in view of Proposition 1.4.1, it follows that the degenerate parabolic equation (1.4.17) is null controllable. Moreover, thanks to Proposition 1.4.3 the associated control function satisfies

$$\|u\|^2_{L^2(Q)} \le C_{obs}\|y_0\|^2_{L^2(0,1)}.$$

Remark 19. Observe that, with the same kind of arguments employed above in the proof of Theorem 1.4.2, as a consequence of Theorem 1.4.1 we can deduce the boundary null controllability of the state equation (1.4.17) when the control force acts at the nondegenerate point $x = 1$. Nevertheless, the situation is different and more complex if the control acts at the *degenerate* extremity $x = 0$. We refer to [111] where this problem is treated by means of moment method and the transmutation approach, under some restrictions on the regularity of the initial datums.

We emphasize that, once we find a control function acting on a control zone $\omega \subset (0,1)$ that drives the system from an initial datum y_0 to the equilibrium state at time $t = T$, i.e., the solution of (1.4.17) fulfills $y(T) = 0$, we can stop controlling, by putting $u \equiv 0$ for $t \ge T$, and the underlying system naturally stays at rest for every $t \ge T$, that is to say,

$$y(t,\cdot) = 0, \quad \text{for all } t \ge T.$$

However, this is not the case for evolution equations involving memory terms.

1.4.3.2 Moment Method

We address the boundary null controllability result for the one-dimensional heat equation using the so called moment method developed by Fattorini and Russell [87]. The main idea relies on reducing the null controllability property to a moment problem. Then, this later will be solved by making use of a technique based on the construction of a biorthogonal family to a sequence of real exponential functions.

Of course, by using the same arguments with some minor changes, we can prove an analogous result for a degenerate parabolic equation with a diffusion coefficient of the form $a(x) = x^\alpha$, being $\alpha \in [0, 2)$. However, for the sake of simplicity we prefer to restrict our attention on the simplest nondegenerate case (i.e., $\alpha = 0$).

Consider the following controlled heat equation

$$\begin{cases} y_t - y_{xx} = 0, & (t,x) \in Q, \\ y(t,0) = 0, \quad y(t,1) = v(t), & t \in (0,T), \\ y(0,\cdot) = y_0 \in L^2(0,1). \end{cases} \tag{1.4.25}$$

Here v is the control function which is placed at the extremity $x = 1$.

Following the same arguments presented in [88], one can easily show that for a given initial datum $y_0 \in L^2(0,1)$ and a control $v \in L^2(0,T)$, system (1.4.25) has a unique solution $y \in L^2(Q) \cap C(0,T; H^{-1}(0,1))$, which depends continuously on y_0 and v, that is

$$\sup_{t\in[0,T]} \|y(t)\|_{H^{-1}(0,1)} \leq C(\|y_0\|_{L^2(0,1)} + \|v\|_{L^2(0,T)}).$$

The null controllability property for system (1.4.25) reads as follows.

Theorem 1.4.3. *Let $T > 0$. For all $y_0 \in L^2(0,1)$, it exists a control function $v \in L^2(0,T)$ such that the solution of (1.4.25) fulfills*

$$y(T,\cdot) = 0 \quad in\,(0,1).$$

Before starting the proof of the previous Theorem, we first show that the null controllability problem can be characterized in terms of an appropriate property for the following adjoint system

$$\begin{cases} -w_t - w_{xx} = 0, & (t,x) \in Q, \\ w(t,0) = w(t,1) = 0, & t \in (0,T), \\ w(T,\cdot) = w_T \in L^2(0,1), & x \in (0,1). \end{cases} \tag{1.4.26}$$

Proposition 1.4.5. *System (1.4.25) is null-controllable at time $T > 0$ if and only if, for any $y_0 \in L^2(0,1)$ there exists $v \in L^2(0,T)$ such that the following identity holds*

$$\int_0^T v(t) w_x(t,1)\, dt = \int_0^1 y_0(x) w(0,x)\, dx, \qquad \forall w_T \in L^2(0,1), \tag{1.4.27}$$

where w is the solution of system (1.4.26) associated to w_T.

Proof. Multiplying system (1.4.25) by w and the adjoint problem (1.4.26) by y, integrating by parts over Q, we obtain

$$\iint_Q y_t w\, dxdt - \iint_Q y_x w_x\, dxdt + \int_0^T w_x(t,1) v(t)\, dt = 0$$

and

$$\iint_Q w_t y\, dxdt + \iint_Q w_x y_x\, dxdt = 0.$$

Thus,

$$\iint_Q (yw)_t \, dxdt + \int_0^T w_x(t,1)v(t)\, dt = 0.$$

Consequently,

$$\int_0^T v(t)w_x(t,1)\, dt = \int_0^1 y_0(x)w(0,x)\, dx - \int_0^1 y(T,x)w_T(x)\, dx. \tag{1.4.28}$$

Notice that, if (1.4.27) is fulfilled, we get that

$$\int_0^1 y(T,x)w_T(x)\, dx = 0, \qquad \forall w_T \in L^2(0,1)$$

which yields $y(T,\cdot) = 0$.

Therefore, the solution of (1.4.25) is controllable to zero at time T and v is the null control.

Reciprocally, assume that v drives the solution of (1.4.25) to zero at time T. Thus, (1.4.27) follows immediately from (1.4.28) and the proof is concluded. □

At this point, we recall that the operator $-\partial_{xx}$ on $(0,1)$ with homogeneous Dirichlet boundary conditions has a sequence of positive eigenvalues and normalized eigenfunctions defined by

$$\lambda_k = k^2\pi^2, \qquad \varphi_k(x) = \sqrt{2}\sin(k\pi x), \quad k \geq 1,\ x \in (0,1) \tag{1.4.29}$$

and the sequence $\{\varphi_k\}_{k\geq 1}$ forms a Hilbert basis of $L^2(0,1)$.

As mentioned above, the proof of Theorem 1.4.3 is based on the construction and the estimate of a suitable biorthogonal sequence to the exponential family $\{e^{-\lambda_k t}\}_{k\geq 1}$ in $L^2(0,T)$. The main ideas are due to R.D. Russell and H.O. Fattorini (see, for instance [87] and [88, Lemma 3.1]). More precisely, we have.

Theorem 1.4.4. *Let $(\Lambda_k)_{k\geq 1}$ be a sequence of real positive numbers such that, for some $\rho > 0$, we have:*

$$\sum_{k\geq 1} \frac{1}{\Lambda_k} < +\infty \quad \text{and} \quad |\Lambda_n - \Lambda_k| \geq \rho|n-k|, \qquad \forall k,n \geq 1. \tag{1.4.30}$$

Then, there exists a biorthogonal family $\{q_k\}_{k\geq 1}$ in $L^2(0,T)$ to $\{e^{-\Lambda_k t}\}_{k\geq 1}$, i.e.,

$$\int_0^T e^{-\Lambda_k t} q_l(t)\, dt = \delta_{kl}, \quad \forall k,l \geq 1. \tag{1.4.31}$$

Here, δ_{kl} denotes the Kronecker symbol.

Moreover, the following estimation holds

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0 \quad \text{such that} \quad \|q_k\|_{L^2(0,T)} \leq C_\varepsilon e^{\varepsilon \Lambda_k}, \quad \forall k \geq 1. \tag{1.4.32}$$

Remark 20. We point out that, the convergence of the series $\displaystyle\sum_{k\geq 1} \frac{1}{\Lambda_k}$, when it is fulfilled, guarantees the existence of the biorthogonal family $\{q_k\}_{k\geq 1}$, while the gap condition

$$|\Lambda_n - \Lambda_k| \geq \rho|n-k|, \qquad \forall k,n \geq 1$$

is crucial for obtaining the estimate (1.4.32) and hence the null controllability result for problem (1.4.25) in arbitrary small times $T > 0$.

Now, we are ready to present the proof of Theorem 1.4.25. The proof strategy is divided into two main steps. The starting point is to transform the null controllability issue to a moment problem. At this step, we will provide the explicit form of the control function in terms of the biorthogonal family defined in Theorem 1.4.4. Then, in the second step, we prove that the constructed control function is well defined.

Proof of Theorem 1.4.3. **Step 1.** Using the fact that $\{\varphi_k\}_k$ forms a basis of $L^2(0,1)$ and thanks to Proposition 1.4.5, the null controllability of (1.4.25) is equivalent to:

find $v \in L^2(0,T)$ such that

$$\int_0^T v(t) w_{k,x}(t,1)\, dt = \int_0^1 y_0(x) w_k(0,x)\, dx, \qquad \forall k \geq 1,$$

where $w_k(t,\cdot) = e^{-\lambda_k(T-t)}\varphi_k$ is the solution of system (1.4.26) associated to $w_T = \varphi_k$.

Hence, v is a null control for our problem if and only if

$$\sqrt{\lambda_k} \int_0^T v(t) e^{-\lambda_k(T-t)}\, dt = e^{-\lambda_k T} \int_0^1 y_0(x)\varphi_k(x)\, dx, \qquad \forall k \geq 1,$$

which is equivalent to find $u(\cdot) := v(T-\cdot) \in L^2(0,T)$ such that

$$\int_0^T u(t) e^{-\lambda_k t}\, dt = c_k, \qquad \forall k \geq 1, \tag{1.4.33}$$

where $c_k = \dfrac{e^{-\lambda_k T}}{\sqrt{\lambda_k}} \displaystyle\int_0^1 y_0(x)\varphi_k(x)\, dx.$

Next, we are going to solve the **moment problem** (1.4.33). First, notice that the sequence $\{\lambda_k\}_k$ fulfills the conditions (1.4.30) in Theorem 1.4.4. Hence, $\{e^{-\lambda_k t}\}_{k\geq 1}$ has a biorthogonal family $\{q_k\}_{k\geq 1}$ in $L^2(0,T)$ satisfying the estimate (1.4.32). Therefore, we can formally solve (1.4.33) by setting

$$u(t) = v(T-t) = \sum_{k\geq 1} c_k q_k(t). \tag{1.4.34}$$

Indeed, plugging (1.4.34) in (1.4.33), we (formally) obtain

$$\int_0^T u(t) e^{-\lambda_l t}\, dt = \sum_{k\geq 1} c_k \int_0^T q_k(t) e^{-\lambda_l t}\, dt = c_l, \qquad \forall l \geq 1.$$

Step 2. In order to conclude, it suffices to show that $u \in L^2(0,T)$ and, equivalently, $v \in L^2(0,T)$. To this end, using the bound (1.4.32), with $\varepsilon = \dfrac{T}{2}$, it follows that

$$\begin{aligned}
\|v\|_{L^2(0,T)} = \|u\|_{L^2(0,T)} &\leq \sum_{k\geq 1} |c_k| \|q_k\|_{L^2(0,T)} \\
&\leq C_T \|y_0\|_{L^2(0,1)} \sum_{k\geq 1} \frac{e^{-\lambda_k T/2}}{\sqrt{\lambda_k}} \\
&\leq C_T \|y_0\|_{L^2(0,1)} \sum_{k\geq 1} \frac{e^{-k^2(\pi^2 T/2)}}{k} < +\infty,
\end{aligned}$$

which yields that the control function v belong to $L^2(0,T)$ and concludes the proof. □

Summarizing, in order to apply the moment method, we must, at least, check that:

- a biorthogonal family $\{q_k\}_{k\geq 1}$ exists;
- the formal series (1.4.34) that define v converges. To this end, we need an L^2 upper bound of $q_k, \quad \forall k \geq 1$.

As mentioned in [17], it is possible to consider a more general second order self-adjoint operator instead of $-\partial_{xx}$. Notably, one can consider an operator $\mathcal{A}$ of the following form

$$(\mathcal{A}y)(x) := (a(x)y'(x))' + d(x)y(x), \tag{1.4.35}$$

where $a \in C^2([0,1])$, $d \in C([0,1])$ and for a constant $c_1 > 0$, we have

$$c_1 \leq a(x), \quad x \in (0,1).$$

It is known (see for instance [78, 87]) that $-\mathcal{A}$ with homogeneous boundary conditions possesses an increasing sequence of eigenvalues $\{\lambda_k\}_{k\geq 1}$ and eigenfunctions $\{\varphi_k\}_{k\geq 1}$ so that, for $\gamma = \int_0^1 a(x)^{-\frac{1}{2}}\,dx$, we have

$$\lambda_k = \frac{\pi^2}{\gamma^2}(k+\beta)^2 + O(1), \quad |\varphi_k'(1)| = c_2\sqrt{\lambda_k}, \qquad \text{for } k \to +\infty, \tag{1.4.36}$$

for some constants $c_2, \beta > 0$.

Remark 21. Notice that, the sequence $\{\lambda_k\}_{k\geq 1}$ given above may contain negative elements. However, by observing that λ_k goes to $+\infty$ as $k \to +\infty$, one can find $k_0 \in \mathbb{N}$ such that for all $k \geq k_0, \quad \lambda_{k_0} > 0$. Thus, by choosing $M > 0$ sufficiently large and setting $\tilde{\lambda}_k = \lambda_k + M$, we construct a strictly positive increasing sequence.

By easy calculations, one can show that the sequence $\{\tilde{\lambda}_k\}_{k\geq 1}$ satisfies all the conditions in Theorem 1.4.4. Therefore, the proof of the null controllability of (1.4.25) given in Theorem 1.4.3 can be easily adapted to the operator $\mathcal{A}$ defined in (1.4.35) to obtain similar result. Indeed, in the present context, it is not difficult to prove that the associated moment problem takes the following form

$$\int_0^T u(t)e^{-\tilde{\lambda}_k t}\,dt = c_k, \qquad \forall k \geq 1,$$

where $u(t) := e^{Mt}v(T-t)$ and $c_k = \dfrac{e^{-\lambda_k T}}{|\varphi_k'(0)|}\displaystyle\int_0^1 y_0(x)\varphi_k(x)\,dx$.

Proceeding as before, one can see that a formal solution to the above moment problem takes the form

$$u(t) = \sum_{k\geq 1} c_k q_k(t),$$

where $\{q_k\}_{k\geq 1}$ is a biorthogonal family to $\{e^{-\tilde{\lambda}_k t}\}_{k\geq 1}$ in $L^2(0,T)$. Hence, the control function is given by

$$v(t) = \sum_{k\geq 1} c_k e^{-Mt} q_k(T-t).$$

Finally, using the bound of the biorthogonal family given in (1.4.32) and arguing as previously, one can show that $v \in L^2(0,T)$.

To our best knowledge, the question whether similar results hold true in the framework of a general degenerate diffusion coefficient is completely open.

Remark 22. Let us remark that, the null controllability result, via boundary and distributed controls, holds for scalar parabolic equations in arbitrary small time T. This is due to the fact that the parabolic operator possess the property of *infinite speed of propagation*. However, this is not, in general, true for various control problems. See for instance, Chapter 6 where we have proved that the pointwise null controllability for a degenerate heat equation holds if and only if the control time T is greater than a quantity $T_0 \in [0,\infty]$. See also [77] for the same result for the nondegenerate sitting.

Chapter 2

Inverse problem for degenerate coupled systems

This chapter presents an inverse source problem for a cascade system of n coupled degenerate parabolic equations. In particular, we prove stability and uniqueness results for the inverse problem of determining the source terms by observations in an arbitrary subdomain over a time interval of only one component and data of the n components at a fixed positive time over the whole spatial domain. The proof is based on the application of a Carleman estimate with a single observation acting on a subdomain.

The results obtained in this chapter are presented in the research article [12], in collaboration with Abdelkarim Hajjaj, Lahcen Maniar and Jawad Salhi.

2.1 Introduction and main results

This chapter is devoted to the question of the reconstruction of all the source terms for a degenerate parabolic system of n coupled equations, with the main particularity that we observe only one component of the system. More precisely, we consider the following parabolic linear system of n-coupled degenerate equations, with n forces:

$$\begin{cases} \partial_t y_1 - d_1(a(x)y_{1x})_x + \sum_{j=1}^{2} b_{1j}y_j = f_1, & (t,x)\in Q, \\ \partial_t y_2 - d_2(a(x)y_{2x})_x + \sum_{j=1}^{3} b_{2j}y_j = f_2, & (t,x)\in Q, \\ \quad\vdots & \\ \partial_t y_n - d_n(a(x)y_{nx})_x + \sum_{j=1}^{n} b_{nj}y_j = f_n, & (t,x)\in Q, \\ y_k(t,1)=0, \begin{cases} y_k(t,0)=0, & \text{(WD)}, \\ (ay_{kx})(t,0)=0, & \text{(SD)}, \end{cases} & \\ \quad t\in(0,T), \qquad 1\le k\le n, & \\ y_1(0,x)=y_1^0(x),\dots,y_n(0,x)=y_n^0(x), & x\in(0,1), \end{cases} \tag{2.1.1}$$

where$(y_k^0)_{1\le k\le n}\in L^2(0,1)^n$, $T>0$ fixed, the coupling terms $b_{kj}=b_{kj}(t,x)\in L^\infty(Q)$ $(1\le k,j\le n)$ and the function a is a diffusion coefficient which degenerates at 0 (i.e., $a(0)=0$) and satisfies the Hypotheses 1.2.3 and 1.2.4.

Equivalently, the previous system can be written as

$$\begin{cases} Y_t - \mathbf{D}\mathcal{A}Y + \mathbf{B}Y = F & (t,x) \in Q, \\ \mathbf{C}Y = 0, & (t,x) \in \Sigma, \\ Y(0,x) = Y^0(x), & x \in (0,1), \end{cases} \tag{2.1.2}$$

where $\Sigma := (0,T) \times \{0,1\}$, $\mathbf{D}$ is a $n \times n$ matrix, $\mathbf{B}$ is a $n \times n$ matrix, $Y = (y_k)_{1 \le k \le n}$ is the state and $F = (f_1, f_2, ..., f_n)^\star$. The operator $\mathbf{D}\mathcal{A}$ is defined by $\mathbf{D}\mathcal{A}Y = \big(d_1(ay_{1x})_x, ..., d_n(ay_{nx})_x\big)$ for $Y \in D(\mathbf{D}\mathcal{A}) \subset L^2(0,1)^n$.

The boundary condition $\mathbf{C}Y = 0$ is either $Y(0) = Y(1) = 0$ in the weakly degenerate case (WD) or $Y(1) = (aY_x)(0) = 0$ in the strongly degenerate case (SD).

We are interested in answering the following inverse problem: can we retrieve the source terms $f_1, ..., f_n$ in system (2.1.1) from incomplete data, that is to say, from a reduced number of measurements of the solution?

For this purpose, we define the zone of measurements ω to be a nonempty open subset of $(0,1)$. For $t_0 \in (0,T)$, we shall use the following notations $Q_{t_0} = (t_0,T) \times (0,1)$, $\omega_{t_0} = (t_0,T) \times \omega$ and $T' := \dfrac{T + t_0}{2}$. Let us recall that in inverse source problems, the source term has to satisfy some condition otherwise uniqueness may be false, see [158]. Let $C_0 > 0$ be given. In [115, 63], the authors make the assumption that source terms f satisfy the condition

$$|f_t(t,x)| \le C_0 |f(T',x)|, \text{ for almost all } (t,x) \in Q \tag{2.1.3}$$

Therefore they define the set $\mathcal{S}(C_0)$ of admissible source terms as

$$\mathcal{S}(C_0) := \{f \in H^1(0,T; L^2(0,1)) : f \text{ satisfies } (2.1.3)\}.$$

The main goal of the present work is to recover all the source terms f_k $(1 \le k \le n)$ using the following observation data:

$$(ay_{kx})_x(T',\cdot), \ \forall k : 1 \le k \le n, \qquad y_1|_{\omega_{t_0}} \qquad \text{and} \qquad y_{1t}|_{\omega_{t_0}}.$$

Our contributions are:

- identification of all external forces term for n-coupled degenerate parabolic system (2.1.1) from a few interior measurements,
- the reduction of the number of observations, and
- a global stability estimate (of Lipschitz type).

Apart from some of the papers mentioned in the introduction of this thesis for inverse problems and controllability of parabolic systems, to the best of our knowledge, inverse source problems for coupled systems of n equation with $n > 2$ were never considered even in the case of non degenerate parabolic coupled systems.

Motivated by this reason, the present chapter is devoted to the study of an inverse source problem for such coupled degenerate systems. More precisely, we will follow the approach introduced by Imanuvilov and Yamamoto in [115] for the treatment of uniformly parabolic problems which is based on the use of global Carleman estimates. For this purpose, we use and extend some recent Carleman estimates given in [84]. As a consequence, we prove a stability estimate of Lipschitz type in determining all the source terms from the knowledge of some measurements of only one component of the solution. To our knowledge, this work is the first one concerning Lipschitz stability results in inverse problems for degenerate coupled systems such as (2.1.1).

For fixed $T > T' > 0$, our first main result is the stability for the inverse source problem.

Theorem 2.1.1. *Let $C_0 > 0$ and assume that for some open subset $\omega_0 \subset \omega$, we have*

$$b_{k-1k} \geq b_0 > 0, \quad in \quad (t_0, T) \times \omega_0, \quad \forall k : 2 \leq k \leq n. \tag{2.1.4}$$

Then, there exists $C = C(T, t_0, C_0) > 0$ such that, for all $f_k \in \mathcal{S}(C_0)$ and $y_k^0 \in L^2(0,1)$ $(1 \leq k \leq n)$, there holds

$$\sum_{k=1}^{n} \|f_k\|^2_{L^2(Q)} \leq C\Big(\sum_{k=1}^{n} \|(ay_{kx})_x(T', \cdot)\|^2_{L^2(0,1)} + \|y_1\|^2_{L^2(\omega_{t_0})} + \|y_{1t}\|^2_{L^2(\omega_{t_0})}\Big). \tag{2.1.5}$$

A brief idea of our strategy is as follows. First, we establish a Carleman estimate with a boundary observation for a single degenerate equation. Then, using a localization argument we deduce a Carleman estimate with a distributed observation for one degenerate equation. Summing up these inequalities we obtain a Carleman estimate for the coupled system with distributed observations of each equation which could be used to show Lipschitz stability estimate in the determination of the source terms from interior measurements of all components of the system. In a second step, by using the equations we try to reduce the number of measurements obtaining a Carleman estimate with a single observation acting on a subdomain. Finally, this estimate is successfully used along with certain energy estimates to obtain the stability result for the inverse source problem of n-coupled degenerate parabolic equations by measurements of one component.

Remark 23.

- Theorem 2.1.1 provides a global Lipschitz stability estimate that extend the one obtained for a single degenerate heat equation by Cannarsa, Tort and Yamamoto [63] to the case of more general cascade coupled systems. The main difference between our work and [63] is that we consider a coupled system of degenerate parabolic equations, and the additional data are given only for one component of this system.
- Although theorem 2.1.1 provides a useful stability result, we note that it does not ensure that the inverse problem has a unique solution because the class $\mathcal{S}(C_0)$ is not a vector space.

An important case is when the unknown source terms of (2.1.1) take the form

$$f_k(t, x) = g_k(x) r_k(t, x), \quad \forall k : 1 \leq k \leq n, \tag{2.1.6}$$

where g_k are the unknown functions of $L^2(0,1)$ while $r_k \in \mathrm{C}^1([0,T] \times [0,1])$ are the given functions such that

$$\forall x \in [0,1] \quad |r_k(T', x)| > d_k, \tag{2.1.7}$$

for some given constant $d_k > 0$, $1 \leq k \leq n$. We denote by $\mathcal{E}_k$ the space

$$\mathcal{E}_k := \{g_k(x) r_k(t, x) \quad \text{for some} \quad g_k \in L^2(0,1)\}.$$

The following holds.

Lemma 2.1.1. *The space $\mathcal{E}_k$ $(1 \leq k \leq n)$ is included in $S(C_0)$ for*

$$C_0 = \sup_{1 \leq k \leq n} \Big(\frac{1}{d_k} \sup_{(t,x) \in \overline{Q}} \Big|\frac{\partial r_k}{\partial t}(t, x)\Big|\Big).$$

The proof of this Lemma is similar to the one of (2.4.16) so we omit it.

As an application of Theorem 2.1.1, we can determine $g_k(x)$ $(1 \leq k \leq n)$ in

$$\begin{cases} \partial_t y_1 - d_1(a(x)y_{1x})_x + \sum_{j=1}^{2} b_{1j}y_j = g_1 r_1, \qquad (t,x) \in Q, \\ \partial_t y_2 - d_2(a(x)y_{2x})_x + \sum_{j=1}^{3} b_{2j}y_j = g_2 r_2, \qquad (t,x) \in Q, \\ \qquad \vdots \\ \partial_t y_n - d_n(a(x)y_{nx})_x + \sum_{j=1}^{n} b_{nj}y_j = g_n r_n, \qquad (t,x) \in Q, \\ y_k(t,1) = 0, \begin{cases} y_k(t,0) = 0, & \text{(WD)}, \\ (ay_{kx})(t,0) = 0, & \text{(SD)}, \end{cases} \\ \qquad t \in (0,T), \qquad 1 \leq k \leq n, \\ y_1(0,x) = y_1^0(x), ..., y_n(0,x) = y_n^0(x), \qquad x \in (0,1). \end{cases} \tag{2.1.8}$$

Here, we prove that we can uniquely recover the spatial components g_k of the source terms from the measurement of the solution over the whole spatial domain $(0,1)$ at any fixed moment T' plus additional local observations in space and time of one component of the solution. This is our second main result:

Theorem 2.1.2. *Let $r_k \in \mathrm{C}^1([0,T] \times [0,1])$ be a function satisfying (2.1.7). Then, there exists $C = C(T, t_0, C_0) > 0$ such that, for all $\tilde{f}_k = \tilde{g}_k r_k \in \mathcal{E}_k$ and $\hat{f}_k = \hat{g}_k r_k \in \mathcal{E}_k$ the associated solutions $(\tilde{y}_1, ..., \tilde{y}_n)$ and $(\hat{y}_1, ..., \hat{y}_n)$ of (2.1.8) satisfy*

$$\begin{aligned} \sum_{k=1}^{n} \|\tilde{g}_k - \hat{g}_k\|^2_{L^2(0,1)} \leq C\Big(&\sum_{k=1}^{n} \|\big((a(\tilde{y}_k - \hat{y}_k)_x)_x\big)(T', \cdot)\|^2_{L^2(0,1)} \\ &+ \|\tilde{y}_1 - \hat{y}_1\|^2_{L^2(\omega_{t_0})} + \|\tilde{y}_{1t} - \hat{y}_{1t}\|^2_{L^2(\omega_{t_0})}\Big). \end{aligned} \tag{2.1.9}$$

In particular, Theorem 2.1.2 provides the following uniqueness result: if the solutions $(\tilde{y}_1, ..., \tilde{y}_n)$ and $(\hat{y}_1, ..., \hat{y}_n)$ of (2.1.8) associated to $\tilde{g}_k$ and $\hat{g}_k$ satisfy

$$\begin{aligned} &(a\tilde{y}_{kx})_x(T', \cdot) = (a\hat{y}_{kx})_x(T', \cdot), \forall k : 1 \leq k \leq n \quad \text{in} \quad (0,1), \\ &\tilde{y}_1 = \hat{y}_1 \quad \text{and} \quad \tilde{y}_{1t} = \hat{y}_{1t} \quad \text{in} \quad \omega_{t_0}, \end{aligned}$$

then

$$\tilde{g}_k = \hat{g}_k \quad \text{in} \quad (0,1), \forall k : 1 \leq k \leq n.$$

This chapter is organized as follows. In Section 2.2, we discuss the well-posedness of the problem (2.1.1). Then, in Section 2.3, we establish different Carleman estimates for parabolic equations and parabolic systems. Finally, in Section 2.4, we apply the Carleman estimates to prove the Lipschitz stability and uniqueness results.

2.2 Well-Posedness and regularity results

In order to study the well-posedness of system (2.1.1), we first recall the following weighted spaces (in the sequel, a.c. means absolutely continuous):

In the (WD) case:

$$H^1_a(0,1) := \Big\{ y \in L^2(0,1) : y \text{ a.c. in } [0,1],\ \sqrt{a} y_x \in L^2(0,1) \text{ and } y(1) = y(0) = 0 \Big\}$$

and

$$H_a^2(0,1) := \Big\{ y \in H_a^1(0,1) : a y_x \in H^1(0,1) \Big\}.$$

In the (SD) case:

$$H_a^1(0,1) := \Big\{ y \in L^2(0,1) : y \,\text{locally a.c. in}\, (0,1], \quad \sqrt{a} y_x \in L^2(0,1) \,\text{and}\, y(1) = 0 \Big\}$$

and

$$\begin{aligned} H_a^2(0,1) : &= \Big\{ y \in H_a^1(0,1) : a y_x \in H^1(0,1) \Big\} \\ &= \Big\{ y \in L^2(0,1) : y \,\text{locally a.c. in}\, (0,1], a y \in H_0^1(0,1), \\ &\qquad a y_x \in H^1(0,1) \,\text{and}\, (a y_x)(0) = 0 \Big\}. \end{aligned}$$

In both cases, the norms are defined as follow

$$\|y\|_{H_a^1}^2 := \|y\|_{L^2(0,1)}^2 + \|\sqrt{a} y_x\|_{L^2(0,1)}^2, \qquad \|y\|_{H_a^2}^2 := \|y\|_{H_a^1}^2 + \|(a y_x)_x\|_{L^2(0,1)}^2.$$

We recall from [5, 47] that the operator $(A, D(A))$ defined by $Ay := (a y_x)_x$, $y \in D(A) = H_a^2(0,1)$ is closed negative self-adjoint with dense domain in $L^2(0,1)$. Hence, it is infinitesimal generator of an analytic semi-group of contractions in the pivot space $L^2(0,1)$.

At this point, as the operator $\mathbf{D}\mathcal{A}$ with domain $D(\mathbf{D}\mathcal{A}) = H_a^2(0,1)^n$ is diagonal and since $\mathbf{B}$ is a bounded perturbation, the following well-posedness and regularity results hold.

Proposition 2.2.1. *(i) For all $Y^0 \in D(\mathbf{D}\mathcal{A})$ and $F \in H^1(0,T;L^2(0,1)^n)$, the problem (2.1.2) has a unique solution*

$$Y \in C\big([0,T], D(\mathbf{D}\mathcal{A})\big) \cap C^1(0,T;L^2(0,1)^n).$$

(ii) For all $F \in L^2(Q)^n$, $Y^0 \in L^2(0,1)^n$, and $\varepsilon \in (0,T)$, there exists a unique mild solution

$$Y \in H^1\big([\varepsilon,T], L^2(0,1)^n\big) \cap L^2\big(\varepsilon,T;D(\mathbf{D}\mathcal{A})\big).$$

If moreover, $F \in H^1(0,T;L^2(0,1)^n)$ and $\varepsilon \in (0,T)$, then

$$Y \in C\big([\varepsilon,T], D(\mathbf{D}\mathcal{A})\big) \cap C^1([\varepsilon,T];L^2(0,1)^n).$$

Proof. The proof of statements (i) and (ii) mainly follows from the fact that $(\mathbf{D}\mathcal{A}, D(\mathbf{D}\mathcal{A}))$ generates an analytic semi-group in the pivot space $L^2(0,1)^n$. Then it suffices to apply standard semi-groups theory: for example [34, Proposition 3.3] in the case $F \in H^1(0,T;L^2(0,1)^n)$ and [34, Proposition 3.8] in the case $F \in L^2(Q)^n$. $\square$

2.3 Global Carleman Estimates

In this section we give a new global Carleman estimate for the system (2.1.1). To this end, as in [63], we introduce the following time and space weight functions

$$\begin{aligned} &\varphi(t,x) := \theta(t)\psi(x), \quad \theta(t) := \frac{1}{(t-t_0)^4 (T-t)^4}, \\ &\psi(x) := \gamma\Big(\int_0^x \frac{y}{a(y)}\, dy - d \Big), \quad \text{and} \quad \eta(t) := T + t_0 - 2t, \end{aligned} \tag{2.3.1}$$

where $t_0 > 0$ is a fixed initial time, $T > 0$ is a final time and where the parameters γ and $d > d^\star := \sup_{[0,1]} \displaystyle\int_0^x \frac{y}{a(y)}\, dy$ are positive constants that will be chosen later.

In order to state our fundamental result, we need to show first some Carleman estimates in the case of a single parabolic degenerate equation.

2.3.1 Carleman estimate for one degenerate equation

In this subsection we shall establish a new Carleman estimate for the solution of the following parabolic equation

$$\begin{cases} y_t - d(a(x)y_x)_x = f, & (t,x) \in Q, \\ y(t,1) = 0, & t \in (0,T), \\ \begin{cases} y(t,0) = 0, & \text{for (WD)}, \\ (ay_x)(t,0) = 0, & \text{for (SD)}, \end{cases} & t \in (0,T), \\ y(0,x) = y^0(x), & x \in (0,1), \end{cases} \tag{2.3.2}$$

where $d > 0$ is a positive constant, $f \in L^2(Q)$ and $y^0 \in L^2(0,1)$.

The following Carleman estimate will be crucial for the aim of this section. Note that the Carleman estimate needed in this work is different from the one showed in [5] where, however, the Carleman inequality was derived to bound just the integrals of $s\theta a(x)y_x^2$ and $s^3\theta^3\dfrac{x^2}{a(x)}y^2$ (that were sufficient for control purposes). For inverse problems, these estimates are not sufficient to conclude. Hence, as in [115], we need to complete the above result with the estimate of the integrals of $\dfrac{1}{s\theta}y_t^2$ and $s\theta^{\frac{3}{2}}|\eta\psi|y^2$.

Theorem 2.3.1. *There exist two positive constants C and s_0, such that every solution y of (2.3.2) satisfies, for all $s \geq s_0$,*

$$\begin{aligned} &\iint_{Q_{t_0}} \Big(\frac{1}{s\theta}y_t^2 + s\theta^{\frac{3}{2}}|\eta\psi|y^2 + s\theta d a(x) y_x^2 + s^3\theta^3\frac{x^2}{da(x)}y^2\Big)e^{2s\varphi}\,dx\,dt \\ &\leq C\Big(\iint_{Q_{t_0}} f^2 e^{2s\varphi}\,dx\,dt + s\gamma a(1)\int_{t_0}^T \theta(t) y_x^2(t,1)e^{2s\varphi(t,1)}\,dt\Big). \end{aligned} \tag{2.3.3}$$

Proof. The proof is based on the methods developed in [63]. Given a solution $y \in L^2(t_0,T;D(A)) \cap H^1(t_0,T;L^2(0,1))$ of (2.3.2) and a positive number $s > 0$, define $w = ye^{s\varphi}$ for a.e. $(t,x) \in Q_{t_0}$. We first prove a Carleman-type estimate for w and then we deduce the expected estimate on y. First of all, observe that w satisfies

$$P_s^+ w + P_s^- w = fe^{s\varphi},$$

where

$$\begin{aligned} P_s^+ w &= -d(aw_x)_x - s\varphi_t w - s^2 d a\varphi_x^2 w, \\ P_s^- w &= w_t + 2sda\varphi_x w_x + sd(a\varphi_x)_x w. \end{aligned} \tag{2.3.4}$$

Moreover, $w(t_0,x) = w(T,x) = 0$. This property allows us to apply the Carleman estimates established in [5] to w with Q_{t_0} in place of $(0,T)\times(0,1)$ and $d\partial_x(a\partial_x\cdot)$ instead of $\partial_x(a\partial_x\cdot)$, obtaining

$$\begin{aligned} &\|P_s^+ w\|^2_{L^2(Q_{t_0})} + \|P_s^- w\|^2_{L^2(Q_{t_0})} + \iint_{Q_{t_0}} \Big(s\theta d a(x) w_x^2 + s^3\theta^3\frac{x^2}{da(x)}w^2\Big)\,dx\,dt \\ &\leq C\Big(\iint_{Q_{t_0}} f^2 e^{2s\varphi}\,dx\,dt + s\gamma a(1)\int_{t_0}^T \theta(t) w_x^2(t,1)\,dt\Big). \end{aligned} \tag{2.3.5}$$

The operators P_s^+ and P_s^- are not exactly the ones of [5]. However, one can prove that the Carleman estimates do not change.

Using the previous estimate, we will bound the integral $\iint_{Q_{t_0}} \frac{1}{s\theta} w_t^2 \, dx \, dt$. In fact, we have

$$\frac{1}{\sqrt{s\theta}} w_t = \frac{1}{\sqrt{s\theta}} \Big(P_s^- w - 2sda\varphi_x w_x - sd(a\varphi_x)_x w \Big)$$
$$= \frac{1}{\sqrt{s\theta}} P_s^- w - 2\gamma d\sqrt{s\theta} x w_x - \gamma d \sqrt{s\theta} w.$$

Therefore, using the fact that $\frac{1}{s\theta}$ is bounded, it results

$$\begin{aligned}
&\iint_{Q_{t_0}} \frac{1}{s\theta} w_t^2 \, dx \, dt \\
&\le C\Big(\frac{1}{s\theta} \|P_s^- w\|^2_{L^2(Q_{t_0})} + \iint_{Q_{t_0}} s\theta x^2 w_x^2 \, dx \, dt + \iint_{Q_{t_0}} s\theta w^2 \, dx \, dt \Big) \\
&\le C\Big(\|P_s^- w\|^2_{L^2(Q_{t_0})} + \iint_{Q_{t_0}} s\theta \frac{x^2}{a} a w_x^2 \, dx \, dt + \iint_{Q_{t_0}} s\theta w^2 \, dx \, dt \Big).
\end{aligned} \tag{2.3.6}$$

Since the function $x \mapsto \frac{x^2}{a}$ is nondecreasing, then one has

$$\iint_{Q_{t_0}} s\theta \frac{x^2}{a} a w_x^2 \, dx \, dt \le \frac{1}{a(1)} \iint_{Q_{t_0}} s\theta a w_x^2 \, dx \, dt. \tag{2.3.7}$$

Moreover, in what follows we will also need to estimate $\iint_{Q_{t_0}} s\theta w^2 \, dx \, dt$. In particular, using Young's inequality, we have

$$\begin{aligned}
\iint_{Q_{t_0}} s\theta w^2 \, dx \, dt &= s \iint_{Q_{t_0}} \big(\theta \frac{a^{1/3}}{x^{2/3}} w^2\big)^{\frac{3}{4}} \big(\theta \frac{x^2}{a} w^2\big)^{\frac{1}{4}} \, dx \, dt \\
&\le s\frac{3}{4} \iint_{Q_{t_0}} \theta \frac{a^{1/3}}{x^{2/3}} w^2 \, dx \, dt + \frac{s}{4} \iint_{Q_{t_0}} \theta \frac{x^2}{a} w^2 \, dx \, dt.
\end{aligned}$$

Let $p(x) = x^{4/3} a^{1/3}$, then since the function $x \mapsto \frac{x^2}{a}$ is nondecreasing on $(0,1)$ one has,

$$p(x) = a\Big(\frac{x^2}{a}\Big)^{\frac{2}{3}} \le Ca(x).$$

Thanks to the Hardy-Poincaré inequality (1.2.9), we obtain

$$\int_0^1 \frac{a^{1/3}}{x^{2/3}} w^2 \, dx = \int_0^1 \frac{p(x)}{x^2} w^2 \, dx \le C \int_0^1 p(x) w_x^2 \, dx \le C \int_0^1 a(x) w_x^2 \, dx. \tag{2.3.8}$$

This gives,

$$\begin{aligned}
\iint_{Q_{t_0}} s\theta w^2 \, dx \, dt &\le C \iint_{Q_{t_0}} \Big(s\theta a w_x^2 + s^3\theta^3 \frac{x^2}{a} w^2 \Big) \, dx \, dt \\
&\le C \iint_{Q_{t_0}} \Big(s\theta d a w_x^2 + s^3\theta^3 \frac{x^2}{da} w^2 \Big) \, dx \, dt,
\end{aligned} \tag{2.3.9}$$

since $d > 0$.

From (2.3.6)-(2.3.9), we get

$$\begin{aligned}
&\iint_{Q_{t_0}} \frac{1}{s\theta} w_t^2 \, dx \, dt \\
&\le C\Big(\|P_s^- w\|^2_{L^2(Q_{t_0})} + \iint_{Q_{t_0}} s\theta d a w_x^2 \, dx \, dt + \iint_{Q_{t_0}} s^3\theta^3 \frac{x^2}{da} w^2 \, dx \, dt \Big).
\end{aligned} \tag{2.3.10}$$

In a similar way, to bound the integral $\iint_{Q_{t_0}} s\theta^{\frac{3}{2}}|\eta\psi|w^2\,dx\,dt$, we have

$$\iint_{Q_{t_0}} s\theta^{\frac{3}{2}}|\eta\psi|w^2\,dx\,dt \le C\iint_{Q_{t_0}} s\theta^{\frac{3}{2}}w^2\,dx\,dt, \tag{2.3.11}$$

since $|\eta| \le T' + t$ and $|\psi| \le \gamma d$. By using inequality (2.3.8), we infer

$$\begin{aligned}\iint_{Q_{t_0}} s\theta^{\frac{3}{2}}w^2\,dx\,dt &= s\iint_{Q_{t_0}} \big(\theta\frac{a^{1/3}}{x^{2/3}}w^2\big)^{\frac{3}{4}}\big(\theta^3\frac{x^2}{a}w^2\big)^{\frac{1}{4}}\,dx\,dt\\ &\le s\frac{3}{4}\iint_{Q_{t_0}} \theta\frac{a^{1/3}}{x^{2/3}}w^2\,dx\,dt + \frac{s}{4}\iint_{Q_{t_0}} \theta^3\frac{x^2}{a}w^2\,dx\,dt\\ &\le C\frac{3}{4}\iint_{Q_{t_0}} s\theta a w_x^2\,dx\,dt + \frac{s}{4}\iint_{Q_{t_0}} \theta^3\frac{x^2}{a}w^2\,dx\,dt.\end{aligned}$$

Therefore, since $d > 0$, we have for s large enough

$$\iint_{Q_{t_0}} s\theta^{\frac{3}{2}}|\eta\psi|w^2\,dx\,dt \le C\Big(\iint_{Q_{t_0}} s\theta d a w_x^2\,dx\,dt + \iint_{Q_{t_0}} s^3\theta^3\frac{x^2}{da}w^2\,dx\,dt\Big). \tag{2.3.12}$$

From inequalities (2.3.5), (2.3.10) and (2.3.12), one obtains

$$\begin{aligned}&\iint_{Q_{t_0}} \Big(\frac{1}{s\theta}w_t^2 + s\theta^{\frac{3}{2}}|\eta\psi|w^2 + s\theta d a(x)w_x^2 + s^3\theta^3\frac{x^2}{da(x)}w^2\Big)\,dx\,dt\\ &\le C\Big(\iint_{Q_{t_0}} f^2e^{2s\varphi}\,dx\,dt + s\gamma a(1)\int_{t_0}^{T}\theta(t)w_x^2(t,1)\,dt\Big).\end{aligned}$$

Consequently, we obtain the estimate (2.3.3) which completes the proof. □

From the boundary Carleman estimate (2.3.3), we will deduce a Carleman estimate for equation (2.3.2) on a subregion

$$\omega' := (\alpha', \beta') \Subset \omega_0 \subset \omega. \tag{2.3.13}$$

To this aim, let us set $\omega'' = (\alpha'', \beta'') \Subset \omega'$ and consider a smooth cut-off function $\xi \in C^\infty([0,1])$ such that $0 \le \xi(x) \le 1$ for $x \in (0,1)$, $\xi(x) = 1$ for $x \in [0, \alpha'']$ and $\xi(x) = 0$ for $x \in [\beta'', 1]$.

Our first intermediate Carleman estimate is thus the following.

Proposition 2.3.1. *Let $T > 0$. Then, there exist two positive constants C and s_0 such that, for every $y^0 \in L^2(0,1)$, the solution y of equation (2.3.2) satisfies, for all $s \ge s_0$*

$$\begin{aligned}&\iint_{Q_{t_0}} \Big(\frac{1}{s\theta}y_t^2 + s\theta^{\frac{3}{2}}|\eta\psi|y^2 + s\theta d a(x)y_x^2 + s^3\theta^3\frac{x^2}{da(x)}y^2\Big)\xi^2e^{2s\varphi}\,dx\,dt\\ &\le C\Big(\iint_{Q_{t_0}} \xi^2f^2e^{2s\varphi}\,dx\,dt + \iint_{\omega'_{t_0}} (f^2 + s^2\theta^2y^2)e^{2s\varphi}\,dx\,dt\Big),\end{aligned} \tag{2.3.14}$$

where $\omega'_{t_0} := (t_0, T) \times \omega'$.

In order to prove the above estimate, we need the following Caccioppoli's inequality, whose proof is inspired from to the one in [2] (for the reader's convenience, we give it in the Appendix).

Lemma 2.3.1 (Caccioppoli's inequality)**.** *Let ω' and ω'' be two nonempty open subsets of $(0,1)$ such that $\overline{\omega''} \subset \omega'$ and $\phi(t,x) = \theta(t)\varrho(x)$, where $\varrho \in C^2(\overline{\omega'}, \mathbb{R})$. Then, there exists a constant $C > 0$ such that any solution y of the equation (2.3.2) satisfies*

$$\iint_{\omega''_{t_0}} y_x^2e^{2s\phi}\,dx\,dt \le C\iint_{\omega'_{t_0}} (f^2 + s^2\theta^2y^2)e^{2s\phi}\,dx\,dt. \tag{2.3.15}$$

Proof of Proposition 2.3.1. Define $w := \xi y$ where y is the solution of (2.3.2). Then, the function w satisfies the following equation

$$\begin{cases} w_t - d(a(x)w_x)_x = \xi f - d(a(x)\xi_x y)_x - da(x)\xi_x y_x, & (t,x)\in Q,\\ w(t,1)=0 \quad \text{and} \quad \begin{cases} w(t,0)=0, & \text{(WD)}\\ (aw_x)(t,0)=0, & \text{(SD)}\end{cases} & t\in(0,T),\\ w(0,x)=\xi(x)y^0(x), & x\in(0,1).\end{cases} \tag{2.3.16}$$

Therefore, applying the Carleman estimate (2.3.3) to equation (2.3.16) and using the definition of w, we have

$$\begin{aligned} &\iint_{Q_{t_0}} \Big(\frac{1}{s\theta}w_t^2 + s\theta^{\frac{3}{2}}|\eta\psi|w^2 + s\theta da(x)w_x^2 + s^3\theta^3\frac{x^2}{da(x)}w^2\Big)e^{2s\varphi}\,dx\,dt\\ &\le C\iint_{Q_{t_0}} \Big(\xi^2 f^2 + \big(d(a(x)\xi_x y)_x + da(x)\xi_x y_x\big)^2\Big)e^{2s\varphi}\,dx\,dt. \end{aligned} \tag{2.3.17}$$

From the definition of ξ and the Caccioppoli inequality (2.3.15), we obtain

$$\begin{aligned} &\iint_{Q_{t_0}} \big(d(a(x)\xi_x y)_x + da(x)\xi_x y_x\big)^2 e^{2s\varphi}\,dx\,dt\\ &\qquad\le C\iint_{\omega''_{t_0}} (y^2+y_x^2)e^{2s\varphi}\,dx\,dt\\ &\qquad\le C\iint_{\omega'_{t_0}} (f^2+s^2\theta^2y^2)e^{2s\varphi}\,dx\,dt. \end{aligned} \tag{2.3.18}$$

Moreover, since $\xi y_x = w_x - \xi_x y$, then we get

$$\begin{aligned} &\iint_{Q_{t_0}} s\theta da(x)y_x^2\xi^2 e^{2s\varphi}\,dx\,dt\\ &\le C\Big(\iint_{Q_{t_0}} s\theta da(x)w_x^2e^{2s\varphi}\,dx\,dt + \iint_{\omega'_{t_0}} s^2\theta^2y^2e^{2s\varphi}\,dx\,dt\Big), \end{aligned} \tag{2.3.19}$$

for a positive constant C.

Finally, combining (2.3.17)-(2.3.19) we deduce the desired estimate. □

Proposition 2.3.1 gave a Carleman estimate in $(0,\alpha')$. Now, using the non degenerate Carleman estimate of [106, Lemma 1.2] which remains true when we replace $\theta(t) = \dfrac{1}{t(T-t)}$ by $\theta(t) = \dfrac{1}{(t-t_0)^4(T-t)^4}$, we are able to show a Carleman estimate to equation (2.3.2) on the interval $(\beta',1)$. For more details about this modification of the time weight function, we refer to [2, Remark 1].

Proposition 2.3.2. *Let $T>0$. Then, there exist two positive constants C and s_0 such that, for every $y^0\in L^2(0,1)$, the solution y of equation (2.3.2) satisfies, for all $s\ge s_0$*

$$\begin{aligned} &\iint_{Q_{t_0}} \Big(\frac{1}{s\theta}y_t^2 + s\theta^{\frac{3}{2}}|\eta\psi|y^2 + s\theta da(x)y_x^2 + s^3\theta^3\frac{x^2}{da(x)}y^2\Big)\zeta^2e^{2s\Phi}\,dx\,dt\\ &\le C\Big(\iint_{Q_{t_0}} \zeta^2f^2e^{2s\Phi}\,dx\,dt + \iint_{\omega'_{t_0}} (f^2+s^3\theta^3y^2)e^{2s\Phi}\,dx\,dt\Big), \end{aligned} \tag{2.3.20}$$

where $\zeta := 1-\xi$ and $\Phi(t,x) := \theta(t)\Psi(x)$, $\Psi(x) = e^{\rho\sigma} - e^{2\rho\|\sigma\|_\infty}$, with $\rho>0$ is a positive constant to be chosen later, σ is a $C^2([0,1])$ function such that $\sigma(x)>0$ in $(0,1)$, $\sigma(0)=\sigma(1)=0$ and $\sigma_x(x)\neq 0$ in $[0,1]\setminus\tilde\omega$, $\tilde\omega$ is an arbitrary open subset of ω.

Proof. The function $z := \zeta y$ is a solution of the uniformly parabolic equation

$$\begin{cases} z_t - d(a(x)z_x)_x = \zeta f - d(a(x)\zeta_x y)_x - da(x)\zeta_x y_x, & (t,x) \in Q, \\ z(t,1) = z(t,0) = 0, & t \in (0,T), \\ z(0,x) = \zeta(x)y^0(x), & x \in (0,1), \end{cases} \tag{2.3.21}$$

since z has its support in $[0,T] \times [\alpha'', 1]$.

Hence, by [106, Lemma 1.2] we have

$$\begin{aligned} &\iint_{Q_{t_0}} \left(\frac{1}{s\theta} z_t^2 + s\theta z_x^2 + s^3\theta^3 z^2\right) e^{2s\Phi}\, dx\, dt \\ &\leq C\left(\iint_{Q_{t_0}} \left(\zeta^2 f^2 + \left(d(a(x)\zeta_x y)_x + da(x)\zeta_x y_x\right)^2\right) e^{2s\Phi}\, dx\, dt + \iint_{\omega'_{t_0}} s^3\theta^3 z^2 e^{2s\Phi}\, dx\, dt \right) \\ &\leq C\left(\iint_{Q_{t_0}} \zeta^2 f^2 e^{2s\Phi}\, dx\, dt + \iint_{\omega''_{t_0}} (y^2 + y_x^2) e^{2s\Phi}\, dx\, dt + \iint_{\omega'_{t_0}} s^3\theta^3 z^2 e^{2s\Phi}\, dx\, dt \right). \end{aligned}$$

Therefore, using the Caccioppoli inequality (2.3.15) and the definitions of z and ζ we deduce

$$\begin{aligned} &\iint_{Q_{t_0}} \left(\frac{1}{s\theta} z_t^2 + s\theta z_x^2 + s^3\theta^3 z^2\right) e^{2s\Phi}\, dx\, dt \\ &\leq C\left(\iint_{Q_{t_0}} \zeta^2 f^2 e^{2s\Phi}\, dx\, dt + \iint_{\omega'_{t_0}} f^2 e^{2s\Phi}\, dx\, dt + \iint_{\omega'_{t_0}} s^3\theta^3 y^2 e^{2s\Phi}\, dx\, dt \right). \end{aligned} \tag{2.3.22}$$

From $\zeta y_x = z_x - \zeta_x y$ and $\operatorname{supp} \zeta_x \subset \omega''$, we obtain

$$\begin{aligned} \iint_{Q_{t_0}} s\theta y_x^2 \zeta^2 e^{2s\Phi}\, dx\, dt &\leq C\left(\iint_{Q_{t_0}} s\theta z_x^2 e^{2s\Phi}\, dx\, dt + \iint_{\omega''_{t_0}} s\theta y^2 e^{2s\Phi}\, dx\, dt \right) \\ &\leq C\left(\iint_{Q_{t_0}} s\theta z_x^2 e^{2s\Phi}\, dx\, dt + \iint_{\omega'_{t_0}} s^3\theta^3 y^2 e^{2s\Phi}\, dx\, dt \right), \end{aligned} \tag{2.3.23}$$

for s large enough.

Furthermore, using the fact that $s\theta^{\frac{3}{2}}|\eta\psi|z^2 \leq Cs^3\theta^3 z^2$, by (2.3.22) one has

$$\begin{aligned} &\iint_{Q_{t_0}} s\theta^{\frac{3}{2}}|\eta\psi| y^2 \zeta^2 e^{2s\Phi}\, dx\, dt = \iint_{Q_{t_0}} s\theta^{\frac{3}{2}}|\eta\psi| z^2 e^{2s\Phi}\, dx\, dt \\ &\leq C\left(\iint_{Q_{t_0}} \zeta^2 f^2 e^{2s\Phi}\, dx\, dt + \iint_{\omega'_{t_0}} f^2 e^{2s\Phi}\, dx\, dt + \iint_{\omega'_{t_0}} s^3\theta^3 y^2 e^{2s\Phi}\, dx\, dt \right). \end{aligned} \tag{2.3.24}$$

The estimates (2.3.22)-(2.3.24) lead to

$$\begin{aligned} &\iint_{Q_{t_0}} \left(\frac{1}{s\theta} y_t^2 + s\theta^{\frac{3}{2}}|\eta\psi| y^2 + s\theta y_x^2 + s^3\theta^3 y^2\right) \zeta^2 e^{2s\Phi}\, dx\, dt \\ &\leq C\left(\iint_{Q_{t_0}} \zeta^2 f^2 e^{2s\Phi}\, dx\, dt + \iint_{\omega'_{t_0}} f^2 e^{2s\Phi}\, dx\, dt + \iint_{\omega'_{t_0}} s^3\theta^3 y^2 e^{2s\Phi}\, dx\, dt \right). \end{aligned} \tag{2.3.25}$$

Taking into account the fact that $da(x) > 0$ and $\dfrac{x^2}{da(x)} > 0$ in $(\alpha', 1)$, we deduce

$$\begin{aligned} &\iint_{Q_{t_0}} \left(\frac{1}{s\theta} y_t^2 + s\theta^{\frac{3}{2}}|\eta\psi| y^2 + s\theta da(x) y_x^2 + s^3\theta^3 \frac{x^2}{da(x)} y^2\right) \zeta^2 e^{2s\Phi}\, dx\, dt \\ &\leq C\left(\iint_{Q_{t_0}} \zeta^2 f^2 e^{2s\Phi}\, dx\, dt + \iint_{\omega'_{t_0}} f^2 e^{2s\Phi}\, dx\, dt + \iint_{\omega'_{t_0}} s^3\theta^3 y^2 e^{2s\Phi}\, dx\, dt \right). \end{aligned} \tag{2.3.26}$$

This proves Proposition 2.3.2. □

2.3.2 Carleman estimate for n-coupled degenerate equations

Now, we show the main result of this section, which is the ω-Carleman estimate for the coupled system (2.1.1). For this purpose, the parameters γ, ρ and d will be chosen such that

$$d > 4^n d^\star, \quad \rho > \frac{1}{\|\sigma\|_\infty} \ln\left(\frac{4^n(d-d^\star)}{d-4^n d^\star}\right), \tag{2.3.27}$$

$$\frac{e^{2\rho\|\sigma\|_\infty}}{d-d^\star} < \gamma < \frac{4^n}{(4^n-1)d}(e^{2\rho\|\sigma\|_\infty} - e^{\rho\|\sigma\|_\infty}), \tag{2.3.28}$$

where n is the size of the system (2.1.1).

Remark 24. By (2.3.27) and proceeding as in [84, Lemma 3.1], it can be shown that the interval $\left(\frac{e^{2\rho\|\sigma\|_\infty}}{d-d^\star}, \frac{4^n}{(4^n-1)d}(e^{2\rho\|\sigma\|_\infty} - e^{\rho\|\sigma\|_\infty})\right)$ is nonempty. Indeed,

$$\begin{aligned}
&\frac{4^n}{(4^n-1)d}\left(e^{2\rho\|\sigma\|_\infty} - e^{\rho\|\sigma\|_\infty}\right) - \frac{e^{2\rho\|\sigma\|_\infty}}{d-d^\star}\\
&\qquad = \frac{4^n\,(d-d^\star)\left(e^{2\rho\|\sigma\|_\infty} - e^{\rho\|\sigma\|_\infty}\right) - (4^n-1)\,de^{2\rho\|\sigma\|_\infty}}{(4^n-1)\,d\,(d-d^\star)}\\
&\qquad = \frac{(d-4^n d^\star)\,e^{2\rho\|\sigma\|_\infty} - 4^n\,(d-d^\star)\,e^{\rho\|\sigma\|_\infty}}{(4^n-1)\,d\,(d-d^\star)}\\
&\qquad = \frac{(d-4^n d^\star)\,e^{\rho\|\sigma\|_\infty}}{(4^n-1)\,d\,(d-d^\star)}\left[e^{\rho\|\sigma\|_\infty} - \frac{4^n\,(d-d^\star)}{(d-4^n d^\star)}\right]
\end{aligned}$$

Now, having in mind the fact that $d > 4^n d^\star$, then choosing the parameter ρ in such a way $\rho > \frac{1}{\|\sigma\|_\infty} \ln\left(\frac{4^n\,(d-d^\star)}{(d-4^n d^\star)}\right)$ the thesis follows.
We can then choose γ in this interval.

From (2.3.27)-(2.3.28), we have the following result whose proof can be found in [84, Lemma 3.3].

Lemma 2.3.2. *Let the sequence Φ_k defined by*

$$\Phi_k = 4^{n-k}(\Phi - \varphi) + \varphi, \quad k = 1, ..., n. \tag{2.3.29}$$

Then, we have

- $\varphi < \Phi_k < 0, \quad k = 1, ..., n.$
- $\Phi_n = \Phi < \Phi_{n-1} < \cdots < \Phi_1.$

Using Propositions 2.3.1 and 2.3.2 and the Hardy-Poincaré inequality we deduce an intermediate important result which could be used to show a Lipschitz stability estimate for parabolic systems of determining n source terms from measurements of all components of the solution.

Theorem 2.3.2. *There exist two positive constants $C > 0$ and $s_0 > 0$ such that for all $(y_1^0, ..., y_n^0) \in (L^2(0,1))^n$, the solution $(y_1, ..., y_n)$ of (2.1.1) satisfies, for all $s \geq s_0$*

$$\sum_{k=1}^{n} \mathcal{J}(y_k) \leq C \sum_{k=1}^{n} \left(\iint_{Q_{t_0}} f_k^2 e^{2s\Phi}\,dx\,dt + \iint_{\omega'_{t_0}} s^3\theta^3 y_k^2 e^{2s\Phi}\,dx\,dt \right), \tag{2.3.30}$$

where

$$\mathcal{J}(y) := \iint_{Q_{t_0}} \left(\frac{1}{s\theta} y_t^2 + s\theta^{\frac{3}{2}}|\eta\psi|y^2 + s\theta a(x) y_x^2 + s^3\theta^3 \frac{x^2}{a(x)} y^2\right) e^{2s\varphi}\,dx\,dt. \tag{2.3.31}$$

Proof. Since y_k is the solution of the system

$$\begin{cases} \partial_t y_k - d_k(a(x)y_{kx})_x = f_k - \displaystyle\sum_{j=1}^{k+1} b_{kj}y_j, & (t,x)\in Q,\\ y_k(t,1)=0, & t\in(0,T),\\ \begin{cases} y_k(t,0)=0, \quad \text{for (WD)},\\ (ay_{kx})(t,0)=0, \quad \text{for (SD)}, \end{cases} & t\in(0,T),\\ y_k(0,x)=y_k^0(x), & x\in(0,1), \end{cases} \tag{2.3.32}$$

applying Proposition 2.3.1, for s big enough, we have

$$\begin{aligned} &\iint_{Q_{t_0}} \Big(\frac{1}{s\theta}y_{kt}^2 + s\theta^{\frac{3}{2}}|\eta\psi|y_k^2 + s\theta d_k a(x)y_{kx}^2 + s^3\theta^3\frac{x^2}{d_k a(x)}y_k^2\Big)\xi^2 e^{2s\varphi}\,dx\,dt\\ &\le C\bigg(\iint_{Q_{t_0}} \xi^2\big(f_k - \sum_{j=1}^{k+1} b_{kj}y_j\big)^2 e^{2s\varphi}\,dx\,dt\\ &\quad + \iint_{\omega'_{t_0}} \Big(\big(f_k - \sum_{j=1}^{k+1} b_{kj}y_j\big)^2 + s^2\theta^2 y_k^2\Big)e^{2s\varphi}\,dx\,dt\bigg)\\ &\le C\sum_{j=1}^{k+1}\bigg(\iint_{Q_{t_0}} \xi^2 y_j^2 e^{2s\varphi}\,dx\,dt + \iint_{\omega'_{t_0}} y_j^2 e^{2s\varphi}\,dx\,dt\bigg)\\ &\quad + C\bigg(\iint_{Q_{t_0}} f_k^2 e^{2s\varphi}\,dx\,dt + \iint_{\omega'_{t_0}} s^2\theta^2 y_k^2 e^{2s\varphi}\,dx\,dt\bigg). \end{aligned}$$

On the other hand, since $d_k>0$, we have

$$\begin{aligned} &\min(1,d_k,\frac{1}{d_k})\iint_{Q_{t_0}} \Big(\frac{1}{s\theta}y_{kt}^2 + s\theta^{\frac{3}{2}}|\eta\psi|y_k^2\\ &\quad + s\theta a(x)y_{kx}^2 + s^3\theta^3\frac{x^2}{a(x)}y_k^2\Big)\xi^2 e^{2s\varphi}\,dx\,dt\\ &\le \iint_{Q_{t_0}} \Big(\frac{1}{s\theta}y_{kt}^2 + s\theta^{\frac{3}{2}}|\eta\psi|y_k^2 + s\theta d_k a(x)y_{kx}^2 + s^3\theta^3\frac{x^2}{d_k a(x)}y_k^2\Big)\xi^2 e^{2s\varphi}\,dx\,dt. \end{aligned}$$

Hence,

$$\begin{aligned} &\iint_{Q_{t_0}} \Big(\frac{1}{s\theta}y_{kt}^2 + s\theta^{\frac{3}{2}}|\eta\psi|y_k^2 + s\theta a(x)y_{kx}^2 + s^3\theta^3\frac{x^2}{a(x)}y_k^2\Big)\xi^2 e^{2s\varphi}\,dx\,dt\\ &\le C\sum_{j=1}^{k+1}\bigg(\iint_{Q_{t_0}} \xi^2 y_j^2 e^{2s\varphi}\,dx\,dt + \iint_{\omega'_{t_0}} y_j^2 e^{2s\varphi}\,dx\,dt\bigg)\\ &\quad + C\bigg(\iint_{Q_{t_0}} f_k^2 e^{2s\varphi}\,dx\,dt + \iint_{\omega'_{t_0}} s^2\theta^2 y_k^2 e^{2s\varphi}\,dx\,dt\bigg). \end{aligned} \tag{2.3.33}$$

Moreover, since $x\mapsto \dfrac{x^2}{a}$ is nondecreasing, we have

$$\iint_{Q_{t_0}} \xi^2 y_j^2 e^{2s\varphi}\,dx\,dt \le \frac{1}{a(1)}\iint_{Q_{t_0}} \frac{a(x)}{x^2}\xi^2 y_j^2 e^{2s\varphi}\,dx\,dt.$$

Applying Hardy-Poincaé inequality to $v := \xi y_j e^{s\varphi}$, one has

$$\iint_{Q_{t_0}} \xi^2 y_j^2 e^{2s\varphi}\,dx\,dt \le C\iint_{Q_{t_0}} a(x)v_x^2\,dx\,dt,$$

and by $v_x = \xi y_{jx} e^{s\varphi} + \xi s\theta \dfrac{x}{a(x)} y_j e^{s\varphi} + \xi_x y_j e^{s\varphi}$, we obtain

$$\begin{aligned}
&\iint_{Q_{t_0}} \xi^2 y_j^2 e^{2s\varphi}\,dx\,dt \le C \iint_{Q_{t_0}} a(x)\Big(\xi y_{jx} e^{s\varphi} + \xi s\theta \frac{x}{a(x)} y_j e^{s\varphi} + \xi_x y_j e^{s\varphi}\Big)^2 dx\,dt \\
&\le C\bigg(\iint_{Q_{t_0}} \xi^2 a(x) y_{jx}^2 e^{2s\varphi}\,dx\,dt + \iint_{Q_{t_0}} \xi^2 s^2\theta^2 \frac{x^2}{a(x)} y_j^2 e^{2s\varphi}\,dx\,dt \\
&\quad + \iint_{Q_{t_0}} \xi_x^2 a(x) y_j^2 e^{2s\varphi}\,dx\,dt\bigg).
\end{aligned}$$

Now, using the fact that θ is bounded from below and since $\operatorname{supp}(\xi_x) \subset \omega'$, one has

$$\begin{aligned}
&\iint_{Q_{t_0}} \xi^2 y_j^2 e^{2s\varphi}\,dx\,dt \\
&\le C\bigg(\iint_{Q_{t_0}} \xi^2 \theta a(x) y_{jx}^2 e^{2s\varphi}\,dx\,dt + \iint_{Q_{t_0}} \xi^2 s^2\theta^2 \frac{x^2}{a(x)} y_j^2 e^{2s\varphi}\,dx\,dt\bigg) \\
&\quad + C \iint_{\omega'_{t_0}} s\theta y_j^2 e^{2s\varphi}\,dx\,dt.
\end{aligned}$$

Therefore, by taking the Carleman parameter s large enough, we obtain

$$\begin{aligned}
&\iint_{Q_{t_0}} \xi^2 y_j^2 e^{2s\varphi}\,dx\,dt \\
&\le \frac{1}{2n}\bigg(\iint_{Q_{t_0}} \Big(s\theta a(x) y_{jx}^2 + s^3\theta^3 \frac{x^2}{a(x)} y_j^2\Big)\xi^2 e^{2s\varphi}\,dx\,dt\bigg) \\
&\quad + C \iint_{\omega'_{t_0}} s^2\theta^2 y_j^2 e^{2s\varphi}\,dx\,dt.
\end{aligned} \tag{2.3.34}$$

Thus, combining (2.3.33) with (2.3.34), we deduce

$$\begin{aligned}
&\sum_{k=1}^{n} \iint_{Q_{t_0}} \Big(\frac{1}{s\theta} y_{kt}^2 + s\theta^{\frac{3}{2}} |\eta\psi| y_k^2 + s\theta a(x) y_{kx}^2 + s^3\theta^3 \frac{x^2}{a(x)} y_k^2\Big)\xi^2 e^{2s\varphi}\,dx\,dt \\
&\le \frac{1}{2n} \sum_{k=1}^{n} \sum_{j=1}^{k+1} \bigg(\iint_{Q_{t_0}} \Big(s\theta a(x) y_{jx}^2 + s^3\theta^3 \frac{x^2}{a(x)} y_j^2\Big)\xi^2 e^{2s\varphi}\,dx\,dt\bigg) \\
&\quad + C \sum_{k=1}^{n} \bigg(\iint_{Q_{t_0}} f_k^2 e^{2s\varphi}\,dx\,dt + \iint_{\omega'_{t_0}} s^2\theta^2 y_k^2 e^{2s\varphi}\,dx\,dt\bigg) \\
&\le \frac{1}{2} \sum_{k=1}^{n} \bigg(\iint_{Q_{t_0}} \Big(s\theta a(x) y_{kx}^2 + s^3\theta^3 \frac{x^2}{a(x)} y_k^2\Big)\xi^2 e^{2s\varphi}\,dx\,dt\bigg) \\
&\quad + C \sum_{k=1}^{n} \bigg(\iint_{Q_{t_0}} f_k^2 e^{2s\varphi}\,dx\,dt + \iint_{\omega'_{t_0}} s^2\theta^2 y_k^2 e^{2s\varphi}\,dx\,dt\bigg),
\end{aligned}$$

and by this, it results that

$$\begin{aligned}
&\sum_{k=1}^{n} \iint_{Q_{t_0}} \Big(\frac{1}{s\theta} y_{kt}^2 + s\theta^{\frac{3}{2}} |\eta\psi| y_k^2 + s\theta a(x) y_{kx}^2 + s^3\theta^3 \frac{x^2}{a(x)} y_k^2\Big)\xi^2 e^{2s\varphi}\,dx\,dt \\
&\le C \sum_{k=1}^{n} \Big(\iint_{Q_{t_0}} f_k^2 e^{2s\varphi}\,dx\,dt + \iint_{\omega'_{t_0}} s^2\theta^2 y_k^2 e^{2s\varphi}\,dx\,dt\Big).
\end{aligned} \tag{2.3.35}$$

Similarly, applying Proposition 2.3.2 to y_k, the solution of (2.3.32), and using the Hardy-Poincaré inequality, we obtain the estimate

$$\sum_{k=1}^{n}\iint_{Q_{t_0}}\Big(\frac{1}{s\theta}y_{kt}^2+s\theta^{\frac{3}{2}}|\eta\psi|y_k^2+s\theta a(x)y_{kx}^2+s^3\theta^3\frac{x^2}{a(x)}y_k^2\Big)\zeta^2e^{2s\Phi}\,dx\,dt$$
$$\le C\sum_{k=1}^{n}\Big(\iint_{Q_{t_0}}f_k^2e^{2s\Phi}\,dx\,dt+\iint_{\omega'_{t_0}}s^3\theta^3y_k^2e^{2s\Phi}\,dx\,dt\Big). \tag{2.3.36}$$

Since $e^{2s\varphi}\le e^{2s\Phi}$, $\dfrac{1}{2}\le\xi^2+\zeta^2\le 1$ and θ is bounded from below, then by adding (2.3.35) and (2.3.36) we obtain

$$\sum_{k=1}^{n}\iint_{Q_{t_0}}\Big(\frac{1}{s\theta}y_{kt}^2+s\theta^{\frac{3}{2}}|\eta\psi|y_k^2+s\theta a(x)y_{kx}^2+s^3\theta^3\frac{x^2}{a(x)}y_k^2\Big)e^{2s\varphi}\,dx\,dt$$
$$\le C\sum_{k=1}^{n}\Big(\iint_{Q_{t_0}}f_k^2e^{2s\Phi}\,dx\,dt+\iint_{\omega'_{t_0}}s^3\theta^3y_k^2e^{2s\Phi}\,dx\,dt\Big),$$

for s large enough. This ends the proof. □

Let us recall that our goal is to determine the source terms f_k, $k\in\{1,...,n\}$ from measurements of one component of the solution using data on a prescribed subregion ω of $(0,1)$. To this aim, the key point is given by the next lemma which play a crucial role to absorb the observation on the components y_k, $k\in\{2,...,n\}$.

Lemma 2.3.3. *Assume that Hypothesis (2.1.4) is satisfied. Let $\varepsilon>0$, $k\in\{2,...,n\}$ and two open sets $\mathcal{O}_0$ and $\mathcal{O}_1$ such that $\omega'\subset\mathcal{O}_1\subset\mathcal{O}_0\subset\omega_0$, where we recall that ω' is given in (2.3.13). Then, for all $l\in\mathbb{N}^*$, there exist positive constants $C_k,\widehat{l}$ and J such that every solution $(y_1,...,y_n)$ to (2.1.1) satisfies*

$$\mathcal{L}_{\mathcal{O}_1}(l,\Phi_k,y_k)$$
$$\le\varepsilon\sum_{j=k}^{k+1}\mathcal{J}(y_j)+C_k\Big(\sum_{j=1}^{k-2}\mathcal{L}_{\mathcal{O}_0}(\widehat{l},\Phi_{k-1},y_j)+(1+\frac{1}{\varepsilon})\mathcal{L}_{\mathcal{O}_0}(J,\Phi_{k-1},y_{k-1})\Big) \tag{2.3.37}$$
$$+\iint_{Q_{t_0}}s^{2l-2}\theta^{2l-2}(f_{k-1}^2e^{2s\Phi_{k-1}}+f_k^2e^{2s\Phi_k})\,dx\,dt,$$

with $\mathcal{L}_{\mathcal{B}}(l,\phi,y)=\displaystyle\iint_{\mathcal{B}\times(t_0,T)}s^l\theta^ly^2e^{2s\phi}\,dx\,dt$, $\widehat{l}=\max(3,2l-3)$ and $J=\max(3,2l+1,4l-5)$. In this inequality, we take $y_{k+1}\equiv 0$ when $k=n$.

Proof. The proof follows the one of [84, Lemma 3.7], but it is different due to the fact that we have to deal here with a non-homogeneous system. For this reason we only point out the difference that appear when we consider the nonhomogeneous system (2.1.1) in place of homogeneous one. Let us consider a nonnegative smooth cut-off function $\chi\in C^\infty(0,1)$, such that

$$0\le\chi(x)\le 1,\quad \chi(x)=\begin{cases}1, & x\in\mathcal{O}_1,\\ 0, & x\in(0,1)\setminus\mathcal{O}_0,\end{cases} \tag{2.3.38}$$

and $\dfrac{\chi_x}{\sqrt{\chi}}\in L^\infty(0,1)$, $\dfrac{\chi_{xx}}{\sqrt{\chi}}\in L^\infty(0,1)$.

Observe that, the $k-1th$ equation of the system (2.1.1) can be written as

$$b_{k-1k}y_k=-y_{k-1,t}+d_k(a(x)y_{k-1,x})_x-\sum_{j=1}^{k-1}b_{k-1j}y_j+f_{k-1}.$$

Multiplying the above equation by $\beta_k \chi y_k$, with $\beta_k = s^l \theta^l e^{2s\Phi_k}$, and integrating on Q_{t_0}, since $\mathcal{O}_1 \subset \omega_0$ by (2.1.4), it follows that

$$\begin{aligned} b_0 \mathcal{L}_{\mathcal{O}_1}(l, \Phi_k, y_k) &\leq \iint_{Q_{t_0}} b_{k-1k} \beta_k \chi y_k^2 \, dx\, dt \\ &= \overbrace{-\iint_{Q_{t_0}} y_{k-1,t} \beta_k \chi y_k \, dx\, dt}^{K_1} + \overbrace{\iint_{Q_{t_0}} d_{k-1} \big(a(x) y_{k-1,x}\big)_x \beta_k \chi y_k \, dx\, dt}^{K_2} \\ &\quad \overbrace{-\sum_{j=1}^{k-1} \iint_{Q_{t_0}} b_{k-1j} y_j \beta_k \chi y_k \, dx\, dt}^{K_3} + \overbrace{\iint_{Q_{t_0}} f_{k-1} \beta_k \chi y_k \, dx\, dt}^{K_4} . \end{aligned} \tag{2.3.39}$$

At this stage, we assume that the following inequality holds (its proof is similar to the one of [84, Lemma 3.7], but we repeat it in the Appendix for the reader's convenience).

Lemma 2.3.1. *For all $\varepsilon > 0$ and $l \in \mathbb{N}^*$, there exist positive constants $C_k, \widehat{l}$ and J such that*

$$\begin{aligned} K_1 + K_2 + K_3 \leq \frac{\varepsilon}{2} \sum_{j=k}^{k+1} \iint_{Q_{t_0}} \Big(s\theta a(x) y_{jx}^2 + s^3 \theta^3 \frac{x^2}{a(x)} y_j^2 \Big) e^{2s\varphi} \, dx\, dt \\ + C_k \Big(\sum_{j=1}^{k-2} \mathcal{L}_{\mathcal{O}_0}(\widehat{l}, \Phi_{k-1}, y_j) + (1 + \frac{1}{\varepsilon}) \mathcal{L}_{\mathcal{O}_0}(J, \Phi_{k-1}, y_{k-1}) \Big) \\ + \iint_{Q_{t_0}} \big| y_{k-1} \beta_k \chi f_k \big| \, dx\, dt + C \iint_{Q_{t_0}} \big| f_{k-1} s^{2l-1} \theta^{2l-1} \chi e^{2s\Phi_k} y_{k-1} \big| \, dx\, dt, \end{aligned} \tag{2.3.40}$$

where $\widehat{l} = \max(3, 2l-3)$ *and* $J = \max(3, 2l+1, 4l-5)$.

Let us continue with the proof of Lemma 2.3.3. Using Young's inequality one has

$$\begin{aligned} \iint_{Q_{t_0}} \big| y_{k-1} \beta_k \chi f_k \big| \, dx\, dt &= \iint_{Q_{t_0}} \big| y_{k-1} s^l \theta^l \chi e^{2s\Phi_k} f_k \big| \, dx\, dt \\ &\leq \iint_{Q_{t_0}} s^{2(l-1)} \theta^{2(l-1)} f_k^2 e^{2s\Phi_k} \, dx\, dt + \frac{1}{4} \iint_{Q_{t_0}} s^2 \theta^2 \chi^2 y_{k-1}^2 e^{2s\Phi_k} \, dx\, dt \end{aligned} \tag{2.3.41}$$

and

$$\begin{aligned} C \iint_{Q_{t_0}} & \big| f_{k-1} s^{2l-1} \theta^{2l-1} \chi e^{2s\Phi_{k-1}} y_{k-1} \big| \, dx\, dt \\ &\leq \iint_{Q_{t_0}} s^{2(l-1)} \theta^{2(l-1)} f_{k-1}^2 e^{2s\Phi_{k-1}} \, dx\, dt + \frac{C}{4} \iint_{Q_{t_0}} s^{2l} \theta^{2l} \chi^2 y_{k-1}^2 e^{2s\Phi_{k-1}} \, dx\, dt. \end{aligned} \tag{2.3.42}$$

Similarly,

$$\begin{aligned} K_4 &= \iint_{Q_{t_0}} f_{k-1} s^l \theta^l \chi e^{2s\Phi_k} y_k \, dx\, dt \\ &\leq \iint_{Q_{t_0}} s^{2(l-1)} \theta^{2(l-1)} f_{k-1}^2 e^{2s(2\Phi_k - \varphi)} \, dx\, dt + \frac{1}{4} \iint_{Q_{t_0}} s^2 \theta^2 \chi^2 y_k^2 e^{2s\varphi} \, dx\, dt. \end{aligned}$$

But by Lemma 2.3.2 we know that

$$e^{2s(2\Phi_k - \varphi)} \leq e^{2s\Phi_{k-1}}.$$

Therefore

$$K_4 \leq \iint_{Q_{t_0}} s^{2(l-1)}\theta^{2(l-1)} f_{k-1}^2 e^{2s\Phi_{k-1}}\,dx\,dt + \frac{1}{4}\iint_{Q_{t_0}} s^2\theta^2\chi^2 y_k^2 e^{2s\varphi}\,dx\,dt.$$

Now, at this level, using the fact that $\operatorname{supp}\chi \subset \mathcal{O}_0$ and thus $\dfrac{a(x)}{x^2}$ is bounded in $\overline{\mathcal{O}}_0$, then for s large enough,

$$K_4 \leq \iint_{Q_{t_0}} s^{2l-2}\theta^{2l-2} f_{k-1}^2 e^{2s\Phi_{k-1}}\,dx\,dt + \frac{\varepsilon}{2}\iint_{Q_{t_0}} s^3\theta^3 \frac{x^2}{a(x)} y_k^2 e^{2s\varphi}\,dx\,dt. \tag{2.3.43}$$

Putting together inequalities (2.3.39)-(2.3.43) and using the fact that $\Phi_k \leq \Phi_{k-1}$, we finally obtain

$$\begin{aligned}
&\mathcal{L}_{\mathcal{O}_1}(l,\Phi_k,y_k)\\
&\leq \varepsilon\sum_{j=k}^{k+1}\mathcal{J}(y_j) + C_k\Big(\sum_{j=1}^{k-2}\mathcal{L}_{\mathcal{O}_0}(\widehat{l},\Phi_{k-1},y_j) + (1+\frac{1}{\varepsilon})\mathcal{L}_{\mathcal{O}_0}(J,\Phi_{k-1},y_{k-1})\Big)\\
&\quad + \iint_{Q_{t_0}} s^{2l-2}\theta^{2l-2}(f_{k-1}^2 e^{2s\Phi_{k-1}} + f_k^2 e^{2s\Phi_k})\,dx\,dt.
\end{aligned}$$

□

As a consequence of (2.3.30) and Lemma 2.3.3, we deduce the following fundamental Carleman estimate for the system (2.1.1) with one observed component.

Theorem 2.3.3. *Assume Hypothesis (2.1.4). Then, there exist $R \geq 3$ (only depending on n), two positive constants C and s_0 such that every solution $(y_1,...,y_n)$ of (2.1.1) satisfies, for all $s \geq s_0$,*

$$\sum_{k=1}^{n}\mathcal{J}(y_k) \leq C\Big(\sum_{k=1}^{n}\iint_{Q_{t_0}} s^R\theta^R f_k^2 e^{2s\Phi_k}\,dx\,dt + \iint_{\omega_{t_0}} y_1^2\,dx\,dt\Big). \tag{2.3.44}$$

Proof. To prove Theorem 2.3.3 we will follow the same argument given in [108] and which is used to obtain the null controllability property for nondegenerate cascade parabolic systems with one control force. Given $\omega_0 \subset \omega$, we choose $\omega' \Subset \omega_0$ and let $Y = (y_1,...,y_n)^\star$ be the solution to (2.1.1) associated to $Y^0 \in L^2(0,1)^n$. From the definition of $\mathcal{L}_B(l,\phi,y)$ and recalling that $\Phi_n = \Phi$, by (2.3.30) we have

$$\sum_{k=1}^{n}\mathcal{J}(y_k) \leq C\sum_{k=1}^{n}\Big(\mathcal{L}_{\omega'}(3,\Phi_n,y_k) + \iint_{Q_{t_0}} f_k^2 e^{2s\Phi_n}\,dx\,dt\Big). \tag{2.3.45}$$

For $k = 2,...,n$, let us introduce the following sequence $(\widetilde{\mathcal{O}}_k)_{2\leq k\leq n}$ of open sets, such that $\omega' \Subset \widetilde{\mathcal{O}}_n \Subset \widetilde{\mathcal{O}}_{n-1} \Subset ... \Subset \widetilde{\mathcal{O}}_2 \Subset \omega_0$. We begin by applying formula (2.3.37), for $k = n$, $l = 3$, $\mathcal{O}_1 = \omega'$, $\mathcal{O}_0 = \widetilde{\mathcal{O}}_n$ and $\varepsilon = \dfrac{1}{2C}$ (with C is the positive constant appearing in (2.3.45)). Thus, from (2.3.45), we obtain

$$\begin{aligned}
\sum_{k=1}^{n}\mathcal{J}(y_k) \leq C\Bigg[&\sum_{k=1}^{n-1}\mathcal{L}_{\omega'}(3,\Phi_n,y_k) + \frac{1}{2C}\mathcal{J}(y_n)\\
&+ C_n\Big(\sum_{k=1}^{n-2}\mathcal{L}_{\widetilde{\mathcal{O}}_n}(\widehat{l},\Phi_{n-1},y_k) + (1+2C)\mathcal{L}_{\widetilde{\mathcal{O}}_n}(J,\Phi_{n-1},y_{n-1})\Big)\\
&+ \iint_{Q_{t_0}} s^{2l-2}\theta^{2l-2}(f_{n-1}^2 e^{2s\Phi_{n-1}} + f_n^2 e^{2s\Phi_n})\,dx\,dt + \sum_{k=1}^{n}\iint_{Q_{t_0}} f_k^2 e^{2s\Phi_n}\,dx\,dt\Bigg].
\end{aligned}$$

For $l_1 := \max(3, \hat{l}, J)$, using the fact that $\Phi_n \leq \Phi_{n-1}$ and $\mathcal{L}_{\omega'}(l_1, \Phi_{n-1}, y_k) \leq \mathcal{L}_{\widetilde{\mathcal{O}}_n}(l_1, \Phi_{n-1}, y_k)$, we deduce that

$$\sum_{k=1}^{n} \mathcal{J}(y_k) \leq \widetilde{C}_n \sum_{k=1}^{n-1} \mathcal{L}_{\widetilde{\mathcal{O}}_n}(l_1, \Phi_{n-1}, y_k) + C \sum_{k=1}^{n} \iint_{Q_{t_0}} f_k^2 e^{2s\Phi_n}\, dx\, dt + C \iint_{Q_{t_0}} s^{R_1}\theta^{R_1}(f_{n-1}^2 e^{2s\Phi_{n-1}} + f_n^2 e^{2s\Phi_n})\, dx\, dt, \tag{2.3.46}$$

where $\widetilde{C}_n$ is a new positive constant and $R_1 = 2l-2$. Observe that in (2.3.45) we have eliminated from the right hand side the local term involving y_n. We can go on applying (2.3.37) for $k = n-1$, $l = l_1$, $\mathcal{O}_1 = \widetilde{\mathcal{O}}_n$, $\mathcal{O}_0 = \widetilde{\mathcal{O}}_{n-1}$ and $\varepsilon = \dfrac{1}{2\widetilde{C}_n}$ and eliminate in (2.3.46) the local term $\mathcal{L}_{\widetilde{\mathcal{O}}_n}(l_1, \Phi_{n-1}, y_{n-1})$. By (a finite) iteration of this argument we obtain

$$\sum_{k=1}^{n} \mathcal{J}(y_k) \leq \widetilde{C}_2 \mathcal{L}_{\widetilde{\mathcal{O}}_2}(l_{n-1}, \Phi_1, y_1) + C \sum_{k=1}^{n} \iint_{Q_{t_0}} f_k^2 e^{2s\Phi_n}\, dx\, dt + C \sum_{k=1}^{n-1} \iint_{Q_{t_0}} s^{R_{n-1}}\theta^{R_{n-1}}(f_k^2 e^{2s\Phi_k} + f_{k+1}^2 e^{2s\Phi_{k+1}})\, dx\, dt,$$

for some positive constants l_{n-1} and R_{n-1}.

Now, since $\Phi_n = \Phi \leq \Phi_k$ and $\sup_{(t,x)\in Q} s^{l_{n-1}}\theta^{l_{n-1}} e^{2s\Phi_1} < +\infty$, choosing s large enough we readily deduce

$$\sum_{k=1}^{n} \mathcal{J}(y_k) \leq C\Big(\sum_{k=1}^{n} \iint_{Q_{t_0}} s^{R_{n-1}}\theta^{R_{n-1}} f_k^2 e^{2s\Phi_k}\, dx\, dt + \iint_{\omega_{t_0}} y_1^2\, dx\, dt\Big).$$

Finally, by setting $R = R_{n-1}$ in the previous estimate, we end the proof.

□

2.4 Stability estimate and uniqueness for the inverse source problem

In this section, we establish a stability and a uniqueness result using certain ideas from [63] and [115]. More precisely, we obtain an inequality which estimates the source terms f_k, $k \in \{1, ..., n\}$ over the entire domain $(0,1)$ with an upper bound given by some Sobolev norm of the solution Y at some fixed time $T' \in (0,T)$ and the partial knowledge of y_1 and y_{1t} over the subdomain $\omega \Subset (0,1)$. In proving these kinds of stability estimates, the global Carleman estimate obtained in Theorem 2.3.3 will play a crucial part along with certain energy estimates.

Proof of theorem 2.1.1. Let us introduce $Z := Y_t$ where $Y = (y_k)_{1\leq k\leq n}$ is the solution of (2.1.2). Then, thanks to Proposition 2.2.1, one has

$$Z = (z_k)_{1\leq k\leq n} \in L^2\big(t_0, T; D(\mathcal{A})\big) \cap H^1\big(t_0, T; L^2(0,1)^n\big)$$

and satisfies

$$\begin{cases} Z_t - \mathbf{D}\mathcal{A}Z + \mathbf{B}Z = F_t, & (t,x) \in Q_{t_0}, \\ \mathbf{C}Z = 0, & (t,x) \in \Sigma_{t_0}, \\ Z(0,x) = Y_t(0,x), & x \in (0,1), \end{cases} \tag{2.4.1}$$

where $\Sigma_{t_0} := (t_0, T) \times \{0,1\}$ and $F_t = (f_{1t}, f_{2t}, ..., f_{nt})^\star$.

Applying the Carleman estimate of Theorem 2.3.3 to problem (2.4.1), we get:

$$\sum_{k=1}^{n}\mathcal{J}(z_k)\leq C\Big(\sum_{k=1}^{n}\iint_{Q_{t_0}} s^R\theta^R f_{kt}^2 e^{2s\Phi_k}\,dx\,dt+\iint_{\omega_{t_0}} z_1^2\,dx\,dt\Big). \tag{2.4.2}$$

Let us note that, if we replace $\iint_{Q_{t_0}} s^R\theta^R f_{kt}^2 e^{2s\Phi_k}\,dx\,dt$ by $\iint_{Q_{t_0}} f_{kt}^2 e^{2s\Phi}\,dx\,dt$, the inequality (2.4.2) would be the kind of estimate that one would obtain when dealing with the more standard inverse problem that consists in retrieving the source term f in the scalar equation $y_t-(ay_x)_x=f$. Hence the next step mainly consists in adapting the reasoning of [63] to the present case, taking into account this extra term and the coupling of the equations. We shall first prove the following lemma.

Lemma 2.4.1. *There exists a constant $C=C(t_0,T)>0$ such that for every $k\in\{1,...,n\}$*

$$\int_0^1\Big(z_k(T')+\sum_{j=1}^{k+1}b_{kj}y_j(T')\Big)^2 e^{2s\varphi(T',x)}\,dx\leq C\Big(\mathcal{J}(z_k)+\sum_{j=1}^{k+1}\mathcal{J}(y_j)\Big). \tag{2.4.3}$$

Proof of Lemma 2.4.1. Since

$$\lim_{t\longrightarrow t_0}\Big(z_k(t)+\sum_{j=1}^{k+1}b_{kj}y_j(t)\Big)^2 e^{2s\varphi(t,x)}=0,\quad \text{for a.e. } x\in(0,1),$$

we can write

$$\begin{aligned}
&\int_0^1\Big(z_k(T')+\sum_{j=1}^{k+1}b_{kj}y_j(T')\Big)^2 e^{2s\varphi(T',x)}\,dx\\
&\qquad=\int_0^1\int_{t_0}^{T'}\frac{\partial}{\partial t}\Big(\Big(z_k(t)+\sum_{j=1}^{k+1}b_{kj}y_j(t)\Big)^2 e^{2s\varphi}\Big)\,dt\,dx\\
&\qquad=\overbrace{2\int_0^1\int_{t_0}^{T'}\Big(z_k+\sum_{j=1}^{k+1}b_{kj}y_j\Big)\Big(z_{kt}+\sum_{j=1}^{k+1}b_{kj}y_{jt}\Big)e^{2s\varphi}\,dx\,dt}^{I_1}\\
&\qquad\quad+\overbrace{\int_0^1\int_{t_0}^{T'}2s\varphi_t\Big(z_k+\sum_{j=1}^{k+1}b_{kj}y_j\Big)^2 e^{2s\varphi}\,dx\,dt}^{I_2}.
\end{aligned}\tag{2.4.4}$$

Using Young's inequality and taking into account the fact that $b_{kj}\in L^\infty(Q)$, we estimate

$$\begin{aligned}
I_1&=2\int_0^1\int_{t_0}^{T'}\sqrt{s\theta}\Big(z_k+\sum_{j=1}^{k+1}b_{kj}y_j\Big)\frac{1}{\sqrt{s\theta}}\Big(z_{kt}+\sum_{j=1}^{k+1}b_{kj}y_{jt}\Big)e^{2s\varphi}\,dx\,dt\\
&\leq C\iint_{Q_{t_0}}\Big(s\theta z_k^2+\sum_{j=1}^{k+1}s\theta y_j^2+\frac{z_{kt}^2}{s\theta}+\sum_{j=1}^{k+1}\frac{y_{jt}^2}{s\theta}\Big)e^{2s\varphi}\,dx\,dt.
\end{aligned}\tag{2.4.5}$$

Moreover, applying once more Young's inequality, one has

$$\begin{aligned}
\iint_{Q_{t_0}} s\theta y_j^2 e^{2s\varphi}\,dx\,dt&=\iint_{Q_{t_0}} s\theta\Big(\big(\frac{a(x)}{x^2}\big)^{\frac{1}{3}}y_j^2e^{2s\varphi}\Big)^{\frac{3}{4}}\Big(\big(\frac{x^2}{a(x)}\big)y_j^2e^{2s\varphi}\Big)^{\frac{1}{4}}\,dx\,dt\\
&\leq\frac{3}{4}\iint_{Q_{t_0}} s\theta\Big(\frac{a(x)}{x^2}\Big)^{\frac{1}{3}}y_j^2e^{2s\varphi}\,dx\,dt+\frac{1}{4}\iint_{Q_{t_0}} s\theta\frac{x^2}{a(x)}y_j^2e^{2s\varphi}\,dx\,dt.
\end{aligned}$$

Arguing as in (2.3.8), by the Hardy-Poincaré inequality applied to $y_j e^{s\varphi}$, we obtain

$$\begin{aligned}\iint_{Q_{t_0}} s\theta y_j^2 e^{2s\varphi}\,dx\,dt &\leq C\iint_{Q_{t_0}} \Big(s\theta a(x)[y_{jx}+s\theta\psi_x y_j]^2+s^3\theta^3\frac{x^2}{a(x)}y_j^2\Big)e^{2s\varphi}\,dx\,dt\\ &\leq C\iint_{Q_{t_0}} \Big(s\theta a(x)y_{jx}^2+s^3\theta^3\frac{x^2}{a(x)}y_j^2\Big)e^{2s\varphi}\,dx\,dt. \end{aligned}\tag{2.4.6}$$

Similarly, we have

$$\iint_{Q_{t_0}} s\theta z_k^2 e^{2s\varphi}\,dx\,dt \leq C\iint_{Q_{t_0}} \Big(s\theta a(x)z_{kx}^2+s^3\theta^3\frac{x^2}{a(x)}z_k^2\Big)e^{2s\varphi}\,dx\,dt. \tag{2.4.7}$$

On the other hand, since $|\varphi_t| \leq C\theta^{\frac{3}{2}}|\eta\psi|$, it follows that

$$\begin{aligned} I_2 &= \int_0^1\int_{t_0}^{T'} 2s\varphi_t\Big(z_k+\sum_{j=1}^{k+1} b_{kj}y_j\Big)^2 e^{2s\varphi}\,dx\,dt\\ &\leq C\Big(\iint_{Q_{t_0}} s\theta^{\frac{3}{2}}|\eta\psi|z_k^2 e^{2s\varphi}\,dx\,dt+\sum_{j=1}^{k+1}\iint_{Q_{t_0}} s\theta^{\frac{3}{2}}|\eta\psi|y_j^2 e^{2s\varphi}\,dx\,dt\Big). \end{aligned}\tag{2.4.8}$$

Thus, (2.4.4)-(2.4.8) yield the estimate (2.4.3). □

Going back to the proof of Theorem 2.1.1, we note that the k-th equation of the system (2.1.2) can be written as

$$z_k(T',.)-(a(x)y_{kx})_x(T',.)+\sum_{j=1}^{k+1} b_{kj}y_j(T',.) = f_k(T',.) \quad \text{in} \quad (0,1).$$

Therefore,

$$\begin{aligned} &\iint_{Q_{t_0}} s^{R+1}\theta^{R+1} f_k^2(T',x)e^{2s\Phi_k}\,dx\,dt\\ &\leq C\Bigg(\iint_{Q_{t_0}} (a(x)y_{kx})_x^2(T',x)s^{R+1}\theta^{R+1}e^{2s\Phi_k}\,dx\,dt\\ &\quad+\iint_{Q_{t_0}}\Big(z_k(T')+\sum_{j=1}^{k+1} b_{kj}y_j(T')\Big)^2 s^{R+1}\theta^{R+1}e^{2s\Phi_k}\,dx\,dt\Bigg). \end{aligned}$$

In particular, since

$$\sup_{(t,x)\in Q} s^{R+1}\theta^{R+1}(t)e^{2s\Phi_k(t,x)} < \infty,$$

the previous estimate yields

$$\begin{aligned} &\iint_{Q_{t_0}} s^{R+1}\theta^{R+1} f_k^2(T',x)e^{2s\Phi_k}\,dx\,dt\\ &\qquad\leq C\Bigg(\|(ay_{kx})_x(T',\cdot)\|^2_{L^2(0,1)}+\int_0^1\Big(z_k(T')+\sum_{j=1}^{k+1} b_{kj}y_j(T')\Big)^2 dx\Bigg). \end{aligned}$$

Furthermore,

$$\begin{aligned} &\iint_{Q_{t_0}} s^{R+1}\theta^{R+1} f_k^2(T',x)e^{2s\Phi_k}\,dx\,dt\\ &\leq C\Bigg(\|(ay_{kx})_x(T',\cdot)\|^2_{L^2(0,1)}+\int_0^1\Big(z_k(T')+\sum_{j=1}^{k+1} b_{kj}y_j(T')\Big)^2 e^{2s\varphi(T',x)}\,dx\Bigg). \end{aligned}\tag{2.4.9}$$

Finally, putting (2.3.44) and (2.4.2) into (2.4.9), we get

$$\begin{aligned}
&\sum_{k=1}^{n}\iint_{Q_{t_0}} s^{R+1}\theta^{R+1} f_k^2(T',x)e^{2s\Phi_k}\,dx\,dt \\
&\quad\leq C\Bigg(\sum_{k=1}^{n}\|(ay_{kx})_x(T',\cdot)\|^2_{L^2(0,1)} + \sum_{k=1}^{n}\Big(\mathcal{J}(z_k) + \sum_{j=1}^{k+1}\mathcal{J}(y_j)\Big)\Bigg) \\
&\quad\leq C\Bigg(\sum_{k=1}^{n}\|(ay_{kx})_x(T',\cdot)\|^2_{L^2(0,1)} + \sum_{k=1}^{n}\Big(\mathcal{J}(z_k) + \mathcal{J}(y_k)\Big)\Bigg) \\
&\quad\leq C\Bigg(\sum_{k=1}^{n}\|(ay_{kx})_x(T',\cdot)\|^2_{L^2(0,1)} + \sum_{k=1}^{n}\iint_{Q_{t_0}} s^R\theta^R(f_{kt}^2 + f_k^2)e^{2s\Phi_k}\,dx\,dt \\
&\qquad + \|y_1\|^2_{L^2(\omega_{t_0})} + \|y_{1t}\|^2_{L^2(\omega_{t_0})}\Bigg).
\end{aligned} \tag{2.4.10}$$

Next, using the assumption that $f_1, ..., f_n \in \mathcal{S}(C_0)$, one has

$$|f_{kt}(t,x)| \leq C_0|f_k(T',x)|,$$

and

$$|f_k(t,x)| \leq |f_k(T',x)| + \int_{T'}^{t}|f_{kt}(s,x)|ds \leq C|f_k(T',x)|. \tag{2.4.11}$$

Substituting this into (2.4.10), we obtain

$$\begin{aligned}
&\sum_{k=1}^{n}\iint_{Q_{t_0}} s^{R+1}\theta^{R+1} f_k^2(T',x)e^{2s\Phi_k}\,dx\,dt \\
&\leq C\Bigg(\sum_{k=1}^{n}\|(ay_{kx})_x(T',\cdot)\|^2_{L^2(0,1)} + \sum_{k=1}^{n}\iint_{Q_{t_0}} s^R\theta^R f_k^2(T',x)e^{2s\Phi_k}\,dx\,dt \\
&\qquad + \|y_1\|^2_{L^2(\omega_{t_0})} + \|y_{1t}\|^2_{L^2(\omega_{t_0})}\Bigg).
\end{aligned} \tag{2.4.12}$$

By choosing s large enough we can absorb the second term on the right-hand side and obtain

$$\begin{aligned}
&\sum_{k=1}^{n}\iint_{Q_{t_0}} s^{R+1}\theta^{R+1} f_k^2(T',x)e^{2s\Phi_k}\,dx\,dt \\
&\leq C\Bigg(\sum_{k=1}^{n}\|(ay_{kx})_x(T',\cdot)\|^2_{L^2(0,1)} + \|y_1\|^2_{L^2(\omega_{t_0})} + \|y_{1t}\|^2_{L^2(\omega_{t_0})}\Bigg).
\end{aligned} \tag{2.4.13}$$

Then we observe that, for a fixed $\varepsilon > 0$, such that

$$t_0 < T' - \varepsilon < T' < T' + \varepsilon < T,$$

we may write

$$\begin{aligned}
&\int_{T'-\varepsilon}^{T'+\varepsilon}\int_0^1 s^{R+1}\theta^{R+1} f_k^2(T',x)e^{2s\Phi_k}\,dx\,dt \leq \iint_{Q_{t_0}} s^{R+1}\theta^{R+1} f_k^2(T',x)e^{2s\Phi_k}\,dx\,dt \\
&\leq C\Bigg(\sum_{k=1}^{n}\|(ay_{kx})_x(T',\cdot)\|^2_{L^2(0,1)} + \|y_1\|^2_{L^2(\omega_{t_0})} + \|y_{1t}\|^2_{L^2(\omega_{t_0})}\Bigg).
\end{aligned}$$

Consequently,

$$2\varepsilon\kappa \sum_{k=1}^{n} \int_0^1 f_k^2(T',x)\,dx \le C\left(\sum_{k=1}^{n} \|(ay_{kx})_x(T',\cdot)\|^2_{L^2(0,1)} + \|y_1\|^2_{L^2(\omega_{t_0})} + \|y_{1t}\|^2_{L^2(\omega_{t_0})} \right), \tag{2.4.14}$$

where

$$\kappa = \min_{(t,x)\in[T'-\varepsilon,T'+\varepsilon]\times[0,1]} s^{R+1}\theta^{R+1}e^{2s\Phi_k} > 0.$$

Thus, in view of (2.4.11), we conclude that

$$\begin{aligned}
\sum_{k=1}^{n} \iint_Q f_k^2(t,x)dx\,dt &\le C \sum_{k=1}^{n} \iint_Q f_k^2(T',x)dx\,dt \\
&= CT \sum_{k=1}^{n} \int_0^1 f_k^2(T',x)dx \\
&\le C\left(\sum_{k=1}^{n} \|(ay_{kx})_x(T',\cdot)\|^2_{L^2(0,1)} + \|y_1\|^2_{L^2(\omega_{t_0})} + \|y_{1t}\|^2_{L^2(\omega_{t_0})} \right),
\end{aligned}$$

for some constant $C = C(T, t_0, C_0) > 0$. This gives (2.1.5) and completes the proof of Theorem 2.1.1. □

Proof of Theorem 2.1.2. Theorem 2.1.2 follows directly from Theorem 2.1.1: if we consider two source terms $\tilde{F} = (\tilde{f}_k)_{1\le k\le n}$ with $\tilde{f}_k = \tilde{g}_k r_k \in \mathcal{E}_k$ and $\hat{F} = (\hat{f}_k)_{1\le k\le n}$ with $\hat{f}_k = \hat{g}_k r_k \in \mathcal{E}_k$, and if we denote by $\tilde{Y} = (\tilde{y}_1, .., \tilde{y}_n)^\star$ and $\hat{Y} = (\hat{y}_1, ..., \hat{y}_n)^\star$ the associated solutions of (2.1.2), then $Z = (z_1, ..., z_n)^\star := (\tilde{y}_1 - \hat{y}_1, ..., \tilde{y}_n - \hat{y}_n)^\star$ is the solution of the problem

$$\begin{cases}
\partial_t z_1 - d_1(a(x)z_{1x})_x + \sum_{j=1}^{2} b_{1j}z_j = (\tilde{g}_1 - \hat{g}_1)r_1, & (t,x)\in Q,\\
\partial_t z_2 - d_2(a(x)z_{2x})_x + \sum_{j=1}^{3} b_{2j}z_j = (\tilde{g}_2 - \hat{g}_2)r_2, & (t,x)\in Q,\\
\qquad\vdots & \\
\partial_t z_n - d_n(a(x)z_{nx})_x + \sum_{j=1}^{n} b_{nj}z_j = (\tilde{g}_n - \hat{g}_n)r_n, & (t,x)\in Q,\\
z_k(t,1) = 0, \begin{cases} z_k(t,0) = 0, & \text{(WD)},\\ (az_{kx})(t,0) = 0, & \text{(SD)}, \end{cases} & \\
\quad t\in(0,T), \qquad 1\le k\le n, & \\
z_1(0,x) = 0, ..., z_n(0,x) = 0, & x\in(0,1).
\end{cases} \tag{2.4.15}$$

One can easily check that $\tilde{f}_k - \hat{f}_k \in \mathcal{S}(C_0)$. Indeed, for $\tilde{f}_k = \tilde{g}_k r_k \in \mathcal{E}_k$ and $\hat{f}_k = \hat{g}_k r_k \in \mathcal{E}_k$ we have, $\tilde{f}_k - \hat{f}_k \in H^1(0,1,L^2(0,1))$. Then for all $t \in [0,T]$ and for a.e. $x \in (0,1)$,

$$\begin{aligned}
\Big|\frac{\partial(\tilde{f}_k - \hat{f}_k)}{\partial t}(t,x)\Big| &= \Big|(\tilde{g}_k(x) - \hat{g}_k(x))\frac{\partial r_k}{\partial t}(t,x)\Big| \\
&\leq |(\tilde{g}_k(x) - \hat{g}_k(x))| \sup_{(t,x)\in\overline{Q}} \Big|\frac{\partial r_k}{\partial t}(t,x)\Big| \\
&\leq |(\tilde{g}_k(x) - \hat{g}_k(x))| \frac{\sup_{(t,x)\in\overline{Q}} \Big|\frac{\partial r_k}{\partial t}(t,x)\Big|}{d_k} |r_k(T',x)| \\
&\leq C_0 |(\tilde{f}_k - \hat{f}_k)(T',x)|,
\end{aligned} \tag{2.4.16}$$

where, owing to (2.1.7),

$$C_0 = \sup_{1\leq k\leq n} \left(\frac{1}{d_k} \sup_{(t,x)\in\overline{Q}} \Big|\frac{\partial r_k}{\partial t}(t,x)\Big| \right).$$

Hence, we can apply Theorem 2.1.1 to obtain (2.1.9). □

Chapter 3

Controllability of degenerate/singular cascade systems

In this chapter, we consider a class of cascade systems of n-coupled degenerate parabolic equations with singular lower order terms. We assume that both degeneracy and singularity occur in the interior of the space domain and we focus on null controllability problem. To this aim, we prove first Carleman estimates for the associated adjoint problem, then, we infer from it an indirect observability inequality. As a consequence, we deduce null controllability result when a unique distributed control is exerted on the system.

The results obtained in this chapter are presented in the research article [11], in collaboration with Abdelkarim Hajjaj, Lahcen Maniar and Jawad Salhi.

3.1 Introduction and Main result

In this chapter we study the null controllability by one control force for a class of systems governed by n-coupled degenerate parabolic equations in presence of singular coupling terms. More precisely, for $n \geq 1$ given, we consider the following linear parabolic system

$$\begin{cases} \partial_t y_1 - d_1(a(x)y_{1x})_x - \displaystyle\sum_{j=1}^{n} \frac{\lambda_{1j}}{b_{1j}} y_j + \sum_{j=1}^{n} a_{1j} y_j = v 1_\omega, \quad (t,x) \in Q, \\ \partial_t y_2 - d_2(a(x)y_{2x})_x - \displaystyle\sum_{j=1}^{n} \frac{\lambda_{2j}}{b_{2j}} y_j + \sum_{j=1}^{n} a_{2j} y_j = 0, \quad (t,x) \in Q, \\ \qquad \vdots \\ \partial_t y_n - d_n(a(x)y_{nx})_x - \displaystyle\sum_{j=1}^{n} \frac{\lambda_{nj}}{b_{nj}} y_j + \sum_{j=1}^{n} a_{nj} y_j = 0, \quad (t,x) \in Q, \\ y_k(t,0) = y_k(t,1) = 0, \quad 1 \leq k \leq n, \quad t \in (0,T), \\ y_k(0,x) = y_k^0(x), \quad 1 \leq k \leq n, \quad x \in (0,1), \end{cases} \tag{3.1.1}$$

where ω is a nonempty open subset of $(0,1)$, $d_k > 0$, $1 \leq k \leq n$, $T > 0$ fixed, 1_ω denotes the characteristic function of the set ω, $(y_1^0, \cdots, y_n^0) \in L^2(0,1)^n$, is the initial condition, and $v \in L^2(Q)$ is the control.

Moreover, we assume that the constants λ_{kj}, $1 \leq k,j \leq n$, satisfy suitable assumptions described below, and the functions a, b_{kj}, $1 \leq k,j \leq n$, degenerate at the same interior point $x_0 \in (0,1)$ of the spatial domain $(0,1)$ that can belong to the control set ω (for the precise assumptions we refer to section 3.2).

Equivalently, the previous system can be written as

$$\begin{cases} \partial_t Y - \mathcal{K}Y - \mathcal{B}Y + \mathcal{C}Y = e_1 v 1_\omega, & \text{in} \quad Q, \\ Y(t,0) = Y(t,1) = 0, & t \in (0,T), \\ Y(0,x) = Y^0(x), & x \in (0,1), \end{cases} \tag{3.1.2}$$

where $Y = (y_k)_{1\leq k\leq n}$, the operator $\mathcal{K}$ is given by

$$\mathcal{K} = DK, \quad D = diag(d_1, \cdots, d_n),$$

and the differential operator K is defined by

$$Ky := (a(x)y_x)_x.$$

The matrix $\mathcal{C} = (a_{kj})_{1\leq k,j\leq n}$ has its entries in $L^\infty(Q)$, $\mathcal{B}$ is the singular matrix operator given by $\mathcal{B} = (B_{kj})_{1\leq k,j\leq n}$, where $B_{kj}y := \dfrac{\lambda_{kj}}{b_{kj}}y$, and finally $e_1 = (1, 0, \cdots, 0)^\star$ is the first element of the canonical basis of $\mathbb{R}^n$.

The object of this chapter is twofold: first we analyze the well-posedness of the evolution system (3.1.1); second, we investigate the effect of the singular coupling terms on observability/controllability aspects of such kind of systems. In particular, our main controllability result will be the following.

Theorem 3.1.1. *Under the assumptions of Theorem 3.4.3, for any time $T > 0$ and any initial datum $(Y^0)^\star \in L^2(0,1)^n$, there exists a control function $v \in L^2(Q)$ such that the solution of (3.1.2) satisfies*

$$y_k(T, \cdot) = 0 \text{ in } (0,1), \ \forall k : 1 \leq k \leq n. \tag{3.1.3}$$

For the scalar case $n = 1$ (one equation and one control force), the null controllability of a degenerate/singular parabolic equation has been established by G. Fragnelli and D. Mugnai in[103]. Later on, in [113, 150] the authors considered a singular coupled system of degenerate parabolic equations (in divergence and nondivergence form) in the particular case of two equations (i.e., $n = 2$), and showed the null controllability of this system under some technical conditions on the coefficients.

In this work, as in [80, 84, 108], we want to generalize these results to the case of a general cascade system of n linear degenerate and singular parabolic equations. To this end, we will suppose that $\mathcal{B}$ and $\mathcal{C}$ have the following structure

$$\mathcal{B} = \begin{pmatrix} \frac{\lambda_{11}}{b_{11}} & \frac{\lambda_{12}}{b_{12}} & 0 & \cdots & \cdots & 0 \\ \frac{\lambda_{21}}{b_{21}} & \frac{\lambda_{22}}{b_{22}} & \frac{\lambda_{23}}{b_{23}} & 0 & \cdots & 0 \\ 0 & \frac{\lambda_{32}}{b_{32}} & \frac{\lambda_{33}}{b_{33}} & \frac{\lambda_{34}}{b_{34}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \frac{\lambda_{n-1n}}{b_{n-1n}} \\ 0 & 0 & \cdots & 0 & \frac{\lambda_{nn-1}}{b_{nn-1}} & \frac{\lambda_{nn}}{b_{nn}}, \end{pmatrix} \tag{3.1.4}$$

$$\mathcal{C} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn-1} & a_{nn} \end{pmatrix}.$$

In addition, to obtain our main null controllability result related to system (3.1.2), we assume that the singular matrix $\mathcal{B}$ is symmetric, i.e.,

$$\lambda_{kk-1} = \lambda_{k-1k} \quad \text{and} \quad b_{kk-1} = b_{k-1k}, \quad \forall k : 2 \leq k \leq n.$$

Remark 25. As we shall see later, the above assumption is used to obtain the well-posedness result using semigroup theory, but the Galerkin method would prove that system (3.1.1) is well-posed without imposing the hypothesis that matrix $\mathcal{B}$ is symmetric. However, this mentioned assumption is required to get the observability estimate.

It is well known that the main tools when dealing with null controllability properties of PDE are the so called Carleman inequalities. The main contributions are due to A. Fursikov and O. Yu. Imanuvilov, who developed the use of a Carleman inequality to the null controllability of classical (non degenerate) parabolic equations in [106].

To obtain Theorem 3.1.1, the first step relies on a Carleman estimate for the homogeneous dual problem corresponding to (3.1.1), which is proved in Theorem 3.4.3. With the Carleman estimate at hand, classical energy estimates (see Theorem 3.5.1) yield the observability estimate

$$\|Z(0,\cdot)\|^2_{L^2(0,1)^n} \leq C_T \iint_{(0,T)\times\omega} z_1^2(t,x)dx\,dt, \tag{3.1.5}$$

for the solution $Z = (z_k)_{1\leq k\leq n}$ of the dual homogeneous backward problem which, under assumption (3.1.4) (cascade system), has the form

$$\begin{cases} \partial_t z_k + d_k(a(x)z_{kx})_x + \displaystyle\sum_{j=k-1}^{k+1} \frac{\lambda_{jk}}{b_{jk}} z_j - \sum_{j=1}^{k+1} a_{jk} z_j = 0, & (t,x) \in Q, \\ z_k(t,0) = 0, \quad z_k(t,1) = 0, \quad t \in (0,T), \\ z_k(T,x) = z_k^T(x), \quad x \in (0,1), \end{cases} \tag{3.1.6}$$

where $z_k^T \in L^2(0,1)$ and $1 \leq k \leq n$.

Let us remark that only one component of the unknown is observed. One calls this property indirect observability since by observing only one component of the solution on ω, one can control all components of the state at the final time. Using the Hilbert uniqueness method, we then establish an indirect controllability result, which means that we drive back the full coupled system (3.1.1) to equilibrium at time T by only controlling the first equation of the system. We refer to [4] for a discussion of various controllability and observability concepts.

The remainder of the chapter is organized as follows. In Section 3.2 we introduce the functional analytic setting and recall some preliminary results, such as Hardy-Poincaré inequalities, that will be useful for the rest of the chapter. In Section 3.3, we study well-posedness of the problem applying the previous inequalities. In Section 3.4, we prove Carleman estimates and we use them to prove observability inequality in Section 3.5.

3.2 Basic assumptions and preliminary results

In the following we will introduce the notions of weak and strong degeneracy for the real-valued functions a and b_{kj}, $j = k-1, k \quad \forall k : 1 \leq k \leq n$, defined on the interval $[0,1]$. Accordingly, we will define suitable weighted spaces and recall some inequalities of Hardy-Poincaré type. These results will play a key role for the study of well-posedness of the system under analysis.

As in the scalar case, the situation in which a and b_{kj}, $\forall\, j = k-1, k \quad \forall k : 1 \leq k \leq n$, vanish at the point x_0 can be quite different, and for this reason we distinguish four different types of degeneracy. In particular, we consider the following cases (see, for instance [103]).

Hypothesis 3.2.1. Double weakly degenerate case (WWD). There exists $x_0 \in (0,1)$ such that $a(x_0) = b_{kk}(x_0) = 0$, $a, b_{kk} > 0$ in $[0,1] \setminus \{x_0\}$, $a, b_{kk} \in C^1([0,1] \setminus \{x_0\})$ and there exists $K, L_k \in (0,1)$ such that $(x - x_0)a' \leq Ka$ and $(x - x_0)b'_{kk} \leq L_k b_{kk}$ a.e. in $[0,1]$.

Hypothesis 3.2.2. Weakly strongly degenerate case (WSD). There exists $x_0 \in (0,1)$ such that $a(x_0) = b_{kk}(x_0) = 0$, $a, b_{kk} > 0$ in $[0,1] \setminus \{x_0\}$, $a \in C^1([0,1] \setminus \{x_0\})$, $b_{kk} \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$, there exists $K \in (0,1), L_k \in [1,2)$ such that $(x-x_0)a' \leq Ka$ and $(x-x_0)b'_{kk} \leq L_k b_{kk}$ a.e. in $[0,1]$.

Hypothesis 3.2.3. Strongly weakly degenerate case (SWD). There exists $x_0 \in (0,1)$ such that $a(x_0) = b_{kk}(x_0) = 0$, $a, b_{kk} > 0$ in $[0,1] \setminus \{x_0\}$, $a \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$, $b_{kk} \in C^1([0,1] \setminus \{x_0\})$, $\exists K \in [1,2)$, $L_k \in (0,1)$ such that $(x - x_0)a' \leq Ka$ and $(x - x_0)b'_{kk} \leq L_k b_{kk}$ a.e. in $[0,1]$.

Hypothesis 3.2.4. Double strongly degenerate case (SSD). There exists $x_0 \in (0,1)$ such that $a(x_0) = b_{kk}(x_0) = 0$, $a, b_{kk} > 0$ in $[0,1] \setminus \{x_0\}$, $a, b_{kk} \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$, there exists $K, L_k \in [1,2)$ such that $(x - x_0)a' \leq Ka$ and $(x - x_0)b'_{kk} \leq L_k b_k$ a.e. in $[0,1]$.

For the non diagonal terms we shall consider the following cases.

Hypothesis 3.2.5. The function b_{kk-1} is weakly degenerate, that is, there exists $x_0 \in (0,1)$ such that $b_{kk-1}(x_0) = 0$, $b_{kk-1} > 0$ on $[0,1] \setminus \{x_0\}$, $b_{kk-1} \in C^1([0,1] \setminus \{x_0\})$ and there exists $M_k \in (0,1)$ such that $(x - x_0)b'_{kk-1} \leq M_k b_{kk-1}$ a.e. in $[0,1]$.

Hypothesis 3.2.6. The function b_{kk-1} is strongly degenerate, that is, there exists $x_0 \in (0,1)$ such that $b_{kk-1}(x_0) = 0$, $b_{kk-1} > 0$ on $[0,1] \setminus \{x_0\}$, $b_{kk-1} \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$ and there exists $M_k \in [1,2)$ such that $(x - x_0)b'_{kk-1} \leq M_k b_{kk-1}$ a.e. in $[0,1]$.

To prove well-posedness of (3.1.1), as in [103], we start by introducing the following weighted Hilbert spaces

$$V_a^1(0,1) := \Big\{ y \in W_0^{1,1}(0,1) : \ \sqrt{a} y_x \in L^2(0,1) \Big\},$$
$$V_{a,b_{kk}}^1(0,1) := \Big\{ y \in V_a^1(0,1) : \ \frac{y}{\sqrt{b_{kk}}} \in L^2(0,1) \Big\},$$

endowed with the respective norms defined by

$$\|y\|^2_{V_a^1} := \|y\|^2_{L^2(0,1)} + \|\sqrt{a} y_x\|^2_{L^2(0,1)},$$
$$\|y\|^2_{V^1_{a,b_{kk}}} := \|y\|^2_{V_a^1} + \left\| \frac{y}{\sqrt{b_{kk}}} \right\|^2_{L^2(0,1)}.$$

For our further results, it is important to remind the following fundamental Hardy-Poincaré inequality.

Lemma 3.2.1 ([103, Proposition 2.14])**.** *If one of the Hypotheses 3.2.1, 3.2.2, 3.2.3 holds with $K + L_k \leq 2$, then there exists a constant $C^k > 0$ such that for all $y \in V^1_{a,b_{kk}}(0,1)$, we have*

$$\int_0^1 \frac{y^2}{b_{kk}}\, dx \leq C^k \int_0^1 a y_x^2 \, dx. \tag{3.2.1}$$

In our situation, due to the presence of singular coupling terms, the functional setting must contain some information on the behaviour of the functions at the singularity. Thus, we introduce the weighted Hilbert space

$$\mathcal{H}_k := \Big\{ y \in V^1_{a,b_{kk}}(0,1) : \ \frac{y}{\sqrt{b_{kk-1}}} \in L^2(0,1) \Big\},$$

endowed with the associated norm

$$\|y\|^2_{\mathcal{H}_k} := \|y\|^2_{V^1_{a,b_{kk}}} + \left\|\frac{y}{\sqrt{b_{kk-1}}}\right\|^2_{L^2(0,1)}.$$

Using the weighted space introduced above, one can prove the next result.

Lemma 3.2.2. *Assume that one among the Hypothesis 3.2.5 or 3.2.6 holds and let* $0 < K, M_k < 2$ *be such that*

$$\begin{cases} K \in [0,2) \setminus \{1\}, \quad 0 < M_k \leq 2 - K, \\ K = 1, \quad 0 < M_k < 2 - K = 1. \end{cases}$$

Then there exists a constant $C^{kk-1} > 0$ *such that for all* $y \in \mathcal{H}_k$*, we have*

$$\int_0^1 \frac{y^2}{b_{kk-1}}\,dx \leq C^{kk-1} \int_0^1 a y_x^2\,dx. \tag{3.2.2}$$

Remark 26. It is well known that when $K = M_k = 1$, an inequality of the form (3.2.2) does not hold [103, Remark 2.15]. Being such an inequality fundamental not only for the well-posedness but also to obtain the observability inequality, it is not surprising if with our techniques we cannot handle this case.

3.3 Abstract setting and well-posedness

In order to study the well-posedness of problem (3.1.1), let us make precise our assumptions on the parameters.

Hypothesis 3.3.1. Throughout this section, we assume the following hypotheses:

1. Either of the following holds:

   - One among the Hypotheses 3.2.1, 3.2.2 or 3.2.3 holds with $K + L_k \leq 2, \ \forall k : 1 \leq k \leq n$ and we assume that
   $$\lambda_{kk} \in \Big(0, \frac{d_k}{C^k}\Big), \quad \forall k : 1 \leq k \leq n. \tag{3.3.1}$$
   - Hypotheses 3.2.1, 3.2.2, 3.2.3 or 3.2.4 hold with $\lambda_{kk} < 0$.

2. We shall admit Hypothesis 3.2.5 or 3.2.6 with $0 < K, M_k < 2$, satisfying
$$\begin{cases} K \in [0,2) \setminus \{1\}, \quad 0 < M_k \leq 2 - K, \\ K = 1, \quad 0 < M_k < 2 - K = 1, \end{cases} \tag{3.3.2}$$
   and we assume that
$$\begin{cases} \lambda_{21} \in \Big(0, \dfrac{\sqrt{\Lambda_1 \Lambda_2}}{\sqrt{2}\, C^{21}}\Big), & \lambda_{nn-1} \in \Big(0, \dfrac{\sqrt{\Lambda_n \Lambda_{n-1}}}{\sqrt{2}\, C^{nn-1}}\Big), \\ \lambda_{kk-1} \in \Big(0, \dfrac{\sqrt{\Lambda_k \Lambda_{k-1}}}{2C^{kk-1}}\Big), & 3 \leq k \leq n-1, \end{cases} \tag{3.3.3}$$
   where Λ_k, $k \in \{1, \cdots, n\}$, is given in (3.3.4).

Remark 27. The upper bound for the range of the coefficients of the singular terms considered in (3.3.3) is required for the well-posedness of the problem. Thus, to look for controllability properties, we will focus our study on this range of constants.

Using the lemmas given in the previous section one can prove the next inequality, which is crucial to obtain well-posedness and observability properties.

Proposition 3.3.1. *Assume Hypothesis 3.3.1.1. Then there exists $\Lambda_k \in (0, d_k]$ such that*

$$\forall y \in V^1_{a,b_{kk}}(0,1), \qquad d_k \int_0^1 a y_x^2\, dx - \lambda_{kk} \int_0^1 \frac{y^2}{b_{kk}}\, dx \geq \Lambda_k \int_0^1 a y_x^2\, dx \tag{3.3.4}$$

Proof. If $\lambda_{kk} < 0$, the result is obvious taking $\Lambda_k = d_k$. Now, assume that $\lambda_{kk} \in \big(0, \dfrac{d_k}{C^k}\big)$. Then, by Lemma 3.2.1, we have

$$\begin{aligned} d_k \int_0^1 a y_x^2\, dx - \lambda_{kk} \int_0^1 \frac{y^2}{b_{kk}}\, dx &\geq d_k \int_0^1 a y_x^2\, dx - \lambda_{kk} C^k \int_0^1 a y_x^2\, dx \\ &= (d_k - \lambda_{kk} C^k) \int_0^1 a y_x^2\, dx \\ &\geq \Lambda_k \int_0^1 a y_x^2\, dx. \end{aligned}$$

□

Finally, we introduce the Hilbert space

$$V^2_{a,b_{kk}}(0,1) := \Big\{ y \in V^1_a : a y_x \in H^1(0,1) \quad \text{and} \quad A_k y \in L^2(0,1) \Big\},$$

where

$$A_k y := d_k (a(x) y_{kx})_x + \frac{\lambda_{kk}}{b_{kk}} y_k \quad \text{with} \quad D(A_k) = V^2_{a,b_{kk}}(0,1).$$

To study the well-posedness of the system (3.1.1) we write it as the first order evolution equation in the Hilbert space $\mathbb{H} := L^2(0,1)^n$

$$\begin{cases} \dot{Y}(t) = \mathcal{K} Y(t) + \mathcal{B} Y(t) - \mathcal{C} Y(t) + F(t) \\ Y(0) = (y_1^0, \cdots, y_n^0)^\star, \end{cases} \tag{3.3.5}$$

where

$$Y = (y_1, \cdots, y_n)^\star, \quad \text{and} \quad F(t) = e_1 v(t) 1_\omega.$$

Observe that, since $\mathcal{C}$ is a bounded perturbation, from now on we focus on the well-posedness of the following abstract inhomogeneous Cauchy problem

$$\begin{cases} \dot{Y}(t) = \mathcal{K} Y(t) + \mathcal{B} Y(t) + F(t) \\ Y(0) = (y_1^0, \cdots, y_n^0)^\star. \end{cases} \tag{3.3.6}$$

To show that $\mathbb{A} := \mathcal{K} + \mathcal{B}$ generates a C_0-semigroup on $\mathbb{H}$, we split it as

$$\mathbb{A} := \mathcal{A} + \mathcal{B}_0 \quad \text{with} \quad \mathcal{D}(\mathbb{A}) = \Big\{ Y^\star = (y_1, \cdots, y_n) \in \mathcal{H} := \prod_{k=1}^n \mathcal{H}_k : (\mathbb{A} Y)^\star \in \mathbb{H} \Big\},$$

where the operator $(\mathcal{A}, D(\mathcal{A}))$ is written in matrix form as

$$\mathcal{A} = diag(A_1, \cdots, A_n) \quad \text{with} \quad D(\mathcal{A}) = \prod_{k=1}^n D(A_k),$$

and

$$\mathcal{B}_0 = \begin{pmatrix} 0 & \dfrac{\lambda_{12}}{b_{12}} & 0 & \cdots & \cdots & 0 \\ \dfrac{\lambda_{21}}{b_{21}} & 0 & \dfrac{\lambda_{23}}{b_{23}} & 0 & \cdots & 0 \\ 0 & \dfrac{\lambda_{32}}{b_{32}} & 0 & \dfrac{\lambda_{34}}{b_{34}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \dfrac{\lambda_{n-1n}}{b_{n-1n}} \\ 0 & 0 & \cdots & 0 & \dfrac{\lambda_{nn-1}}{b_{nn-1}} & 0 \end{pmatrix}. \tag{3.3.7}$$

Let us now show that the operator $(\mathbb{A}, \mathcal{D}(\mathbb{A}))$ generates an analytic semi-group in the pivot space $\mathbb{H}$ for the equation (3.3.6). This aim relies on this fact.

Lemma 3.3.1. *Assume that hypothesis 3.3.1 is satisfied. Then, the operator $\mathbb{A}$ with domain $\mathcal{D}(\mathbb{A})$ is non-positive and self-adjoint on $\mathbb{H}$.*

Proof. Observe that $\mathcal{D}(\mathbb{A})$ is dense in $\mathbb{H}$.

(i) $\mathbb{A}$ is non-positive. By Proposition 3.3.1 and integration by parts [102, Lemma 2.1], it follows that, for any $Y^\star = (y_1, \cdots, y_n) \in \mathcal{D}(\mathbb{A})$, we have

$$\begin{aligned} -\langle \mathbb{A}Y, Y\rangle_{\mathbb{H}} &= -\langle (\mathcal{A} + \mathcal{B}_0)Y, Y\rangle_{\mathbb{H}} \\ &= \sum_{k=1}^{n} \left(d_k \int_0^1 a y_{kx}^2\, dx - \int_0^1 \frac{\lambda_{kk}}{b_{kk}} y_k^2\, dx \right) \\ &\quad - 2\sum_{k=2}^{n} \int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} y_k y_{k-1}\, dx \\ &\geq \sum_{k=1}^{n} \Lambda_k \int_0^1 a y_{kx}^2\, dx - 2\sum_{k=2}^{n} \int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} y_k y_{k-1}\, dx. \end{aligned}$$

We now apply Young's inequality and Lemma 3.2.2, to obtain

$$\begin{aligned} \int_0^1 \frac{y_k y_{k-1}}{b_{kk-1}}\, dx &\leq \delta_k \int_0^1 \frac{y_k^2}{b_{kk-1}}\, dx + \frac{1}{4\delta_k} \int_0^1 \frac{y_{k-1}^2}{b_{kk-1}}\, dx \\ &\leq \delta_k C^{kk-1} \int_0^1 a y_{kx}^2\, dx + \frac{C^{kk-1}}{4\delta_k} \int_0^1 a y_{k-1x}^2\, dx, \end{aligned}$$

where $(\delta_k)_{k=2}^n$ is a sequence of positive constants that will be chosen later on. Hence,

$$\begin{aligned} -\langle \mathbb{A}Y, Y\rangle_{\mathbb{H}} &\geq \sum_{k=1}^{n} \Lambda_k \int_0^1 a y_{kx}^2\, dx \\ &\quad - 2\sum_{k=2}^{n} \lambda_{kk-1} \left(\delta_k C^{kk-1} \int_0^1 a y_{kx}^2\, dx + \frac{C^{kk-1}}{4\delta_k} \int_0^1 a y_{k-1x}^2\, dx \right) \\ &= \left(\Lambda_1 - 2\frac{\lambda_{21} C^{21}}{4\delta_2} \right) \int_0^1 a y_{1x}^2\, dx + \left(\Lambda_n - 2\lambda_{nn-1}\delta_n C^{nn-1} \right) \int_0^1 a y_{nx}^2\, dx \\ &\quad + \sum_{k=2}^{n-1} \left(\Lambda_k - 2\left(\lambda_{kk-1}\delta_k C^{kk-1} + \lambda_{k+1k} \frac{C^{k+1k}}{4\delta_{k+1}} \right) \right) \int_0^1 a y_{kx}^2\, dx. \end{aligned}$$

Now, by hypothesis (3.3.3), one can find $(\delta_k)_{k=2}^n$ such that

$$\begin{cases} \dfrac{\lambda_{21}C^{21}}{2\Lambda_1} < \delta_2 < \dfrac{\Lambda_2}{4\lambda_{21}C^{21}}, \\ \dfrac{\lambda_{kk-1}C^{kk-1}}{\Lambda_{k-1}} < \delta_k < \dfrac{\Lambda_k}{4\lambda_{kk-1}C^{kk-1}}, \qquad 3 \le k \le n-1, \\ \dfrac{\lambda_{nn-1}C^{nn-1}}{\Lambda_{n-1}} < \delta_n < \dfrac{\Lambda_n}{2\lambda_{nn-1}C^{nn-1}}. \end{cases} \tag{3.3.8}$$

With this particular choice, we deduce that there is a constant $C > 0$ such that

$$\forall Y \in D(\mathbb{A}), \quad -\langle \mathbb{A}Y, Y\rangle_{\mathbb{H}} \ge C\|Y\|_{\mathcal{H}}^2 \ge 0, \tag{3.3.9}$$

which proves the result.
(ii) $\mathbb{A}$ is self-adjoint. Let $T : \mathbb{H} \to \mathbb{H}$ be the mapping defined in the following usual way: to each $F \in \mathbb{H}$ associate the weak solution $Y = T(F) \in \mathcal{H}$ of

$$-\langle \mathbb{A}Y, Z\rangle_{\mathbb{H}} = \langle F, Z\rangle_{\mathbb{H}}$$

for every $Z \in \mathcal{H}$. Note that T is well defined by Lax-Milgram Lemma via the part (i), which also implies that T is continuous. Now, it is easy to see that T is injective and symmetric. Thus it is self adjoint. As a consequence, $\mathbb{A} = T^{-1} : \mathcal{D}(\mathbb{A}) \to \mathbb{H}$ is self-adjoint (for example, see [71, Proposition X.2.4]). □

As a consequence of the previous Lemma, the system (3.3.6) (and thus (3.1.1)) is well-posed in the sense of semigroup theory.

Proposition 3.3.2. *Assume Hypothesis 3.3.1. Then, the operator $\mathbb{A} : \mathcal{D}(\mathbb{A}) \to \mathbb{H}$ generates an analytic contraction semigroup of angle $\pi/2$ on $\mathbb{H}$. Moreover, for all $F^\star \in L^2(Q)^n$ and $(Y^0)^\star \in \mathbb{H}$, there exists a unique weak solution of (3.3.6) that belongs to*

$$C([0,T];\mathbb{H}) \cap L^2(0,T;\mathcal{H}). \tag{3.3.10}$$

In addition, if $(Y^0)^\star \in \mathcal{D}(\mathbb{A})$ and $F^\star \in W^{1,1}(0,T,\mathbb{H})$, then

$$Y^\star = (y_1, y_2, \ldots, y_n) \in C^1(0,T;\mathbb{H}) \cap C([0,T];\mathcal{D}(\mathbb{A})). \tag{3.3.11}$$

Proof. Since $\mathbb{A}$ is a non-positive, self-adjoint operator on a Hilbert space, it is well known that $(\mathbb{A}, \mathcal{D}(\mathbb{A}))$ generates a cosine family and an analytic contractive semigroup of angle $\pi/2$ on $\mathbb{H}$ (see [19, Theorem 3.14.17]). Being $\mathbb{A}$ the generator of a strongly continuous semigroup on $\mathbb{H}$, the assertion concerning the assumption $(Y^0)^\star \in \mathbb{H}$ and the regularity of the solution Y when $(Y^0)^\star \in \mathcal{D}(\mathbb{A})$ is a consequence of the results in [23] and [65, Lemma 4.1.5 and Proposition 4.1.6]. □

3.4 Carleman estimates

The object of this section is to prove an interesting estimate of Carleman type that will be used to show the observability inequality which yields the controllability result. To this end, as in [40] or in [104, Chapter 4], we first define the following time and space weight functions. For $x \in [D, 1]$, where D is chosen in such a way that $-x_0 < D < 0$, let us introduce the function $\varphi(t,x) = \theta(t)\psi(x)$, where

$$\theta(t) := \frac{1}{[t(T-t)]^4},$$

and
$$\psi(x) := \gamma\Big[\int_{x_0}^{x} \frac{y-x_0}{\tilde{a}(y)}dy - d\Big], \tag{3.4.1}$$
for γ and d to be specified later. Here, the function $\tilde{a}$ is defined as follows:
$$\tilde{a}(x) = \begin{cases} a(x), & x \in [0,1], \\ a(-x), & x \in [-1,0]. \end{cases} \tag{3.4.2}$$

Further, we need to define the following weights function associated to nondegenerate Carleman estimates in a general interval (A,B) which are suited to our purpose. For $x \in [A,B]$, set
$$\Phi(t,x) = \theta(t)\Psi(x), \quad \Psi(x) = e^{r\zeta(x)} - e^{2\rho}, \tag{3.4.3}$$
where
$$\zeta(x) = \int_x^B \frac{dy}{\sqrt{a(y)}}, \qquad \rho = 2r\zeta(A).$$
Observe that the function $\theta(t)$ satisfy
$$\lim_{t\to 0^+}\theta(t) = \lim_{t\to T^-}\theta(t) = +\infty, \quad |\theta_t| \le c\theta^{\frac{5}{4}}, \quad |\theta_{tt}| \le c\theta^{\frac{3}{2}}$$
for some $c > 0$ depending on T.

Here the parameters γ, r and d are chosen such that
$$d > 4^n\tilde{d}^\star, \quad \rho > \ln\Big(\frac{4^n(d-\tilde{d}^\star)}{d-4^n\tilde{d}^\star}\Big), \tag{3.4.4}$$
$$\frac{e^{2\rho}-1}{d-\tilde{d}^\star} < \gamma < \frac{4^n}{(4^n-1)d}(e^{2\rho}-e^{\rho}), \tag{3.4.5}$$
where
$$\tilde{d}^\star := \sup_{[D,1]}\int_{x_0}^{x}\frac{y-x_0}{\tilde{a}(y)}\,dy,$$
and n is the size of the system (3.1.1).

For this choice of the parameters γ, r and d it is straightforward to check that the weight functions φ and Φ satisfy the following inequalities which are needed in the sequel.

Lemma 3.4.1. *One has:*

1. *For $(t,x) \in [0,T]\times[0,1]$: $\varphi(t,x) \le \Phi(t,x)$.*

2. *For $(t,x) \in [0,T]\times[0,-D]$: $\varphi(t,-x) \le \Phi(t,x)$.*

3. *For $(t,x) \in [0,T]\times[0,1]$: $\dfrac{4^n}{(4^n-1)}\Phi(t,x) < \varphi(t,x)$.*

Proof. Let us set $d^\star := \sup_{[0,1]}\int_{x_0}^{x}\frac{y-x_0}{a(y)}\,dy.$

1. From (3.4.5), we have $\gamma > \dfrac{e^{2\rho}-1}{d-\tilde{d}^\star} > \dfrac{e^{2\rho}-1}{d-d^\star}$. Thus
$$\max_{x\in[0,1]}\psi(x) = \gamma(d^\star - d) \le 1 - e^{2\rho} = \min_{x\in[0,1]}\Psi(x)$$
and the conclusion follows immediately.

2. With the aid of (3.4.5), one has
$$\max_{x\in[0,-D]}\psi(-x)\le\max_{x\in[D,1]}\psi(x)=\gamma(\tilde{d^\star}-d)\le 1-e^{2\rho}=\min_{x\in[0,1]}\Psi(x)\le\min_{x\in[0,-D]}\Psi(x).$$
Hence, $\psi(-x)\le\Psi(x)$, which completes the proof of the desired result.

3. This follows easily by (3.4.5). Indeed
$$\frac{4^n}{(4^n-1)}\max_{x\in[0,1]}\Psi(x)=\frac{4^n}{(4^n-1)}(e^\rho-e^{2\rho})<-\gamma d=\min_{x\in[0,1]}\psi(x).$$

□

We also need the following result, whose proof is similar to the one of [84, Lemma 3.3] and Lemma 2.3.2.

Lemma 3.4.2. *Let the sequence Φ_k defined by*
$$\Phi_k=4^{n-k}(\Phi-\varphi)+\varphi,\quad \forall k:1\le k\le n. \tag{3.4.6}$$
Then, we have

- $\varphi<\Phi_k<0,\quad k=1,\cdots,n.$
- $\Phi_n=\Phi<\Phi_{n-1}<\cdots<\Phi_1.$

3.4.1 Carleman estimate with boundary observation

In this subsection we show Carleman estimates with boundary observation for solutions of the following nonhomogeneous adjoint system:
$$\begin{cases} z_{kt}+d_k(a(x)z_{kx})_x+\displaystyle\sum_{j=k-1}^{k+1}\frac{\lambda_{jk}}{b_{jk}}z_j-\sum_{j=1}^{k+1}a_{jk}z_j=g_k,\\ \qquad (t,x)\in Q,\qquad 1\le k\le n,\\ z_k(t,0)=z_k(t,1)=0,\quad 1\le k\le n,\quad t\in(0,T),\\ z_k(T,x)=z_k^T(x),\quad 1\le k\le n,\quad x\in(0,1),\end{cases} \tag{3.4.7}$$
which is derived taking inspiration from the works [103] and [113]. Here $z_k^T\in L^2(0,1)$ and $g_k\in L^2(Q)$ $(1\le k\le n)$, while on the degenerate functions a, b_{kk}, b_{kk-1} we make the following assumptions.

Hypothesis 3.4.1. From now on, we assume the following hypotheses:

1. Hypothesis 3.3.1 holds. Moreover, if $K>\frac{4}{3}$, then there exists a constant $\vartheta\in(0,K]$ such that the function
$$x\mapsto\frac{a(x)}{|x-x_0|^\vartheta}\quad\begin{cases}\text{is nonincreasing on the left of } x=x_0,\\ \text{is nondecreasing on the right of } x=x_0.\end{cases} \tag{3.4.8}$$

2. Moreover, we suppose that
$$(x-x_0)b'_{kk-1}(x)\ge 0\quad\text{in}\quad[0,1]. \tag{3.4.9}$$

3. Also, if $\lambda_{kk}<0$, we require that
$$(x-x_0)b'_{kk}(x)\ge 0\quad\text{in}\quad[0,1]. \tag{3.4.10}$$

Our main result is the following.

Theorem 3.4.1. *Assume Hypothesis 3.4.1. Then, there exist two positive constant C and s_0 such that every solution $Z = (z_1, z_2, \cdots, z_n)^\star$ of (3.4.7) in*

$$\mathcal{V} := L^2(0,T; D(\mathbb{A})) \cap H^1(0,T;\mathcal{H}), \tag{3.4.11}$$

satisfies

$$\begin{aligned}&\sum_{k=1}^{n} \iint_Q \Big(s\theta a(x) z_{kx}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)} z_k^2\Big) e^{2s\varphi}\, dx\, dt \\ &\le C \sum_{k=1}^{n} \Big(\iint_Q g_k^2 e^{2s\varphi}\, dx\, dt + s\gamma \int_0^T \Big[\theta(x-x_0) a z_{kx}^2 e^{2s\varphi}\Big]_{x=0}^{x=1} dt\Big),\end{aligned} \tag{3.4.12}$$

$\forall s \ge s_0$, where γ is the constant of (3.4.1).

Throughout this chapter we will suppose that $z_0 \equiv z_{n+1} \equiv 0$.

Proof. First, observe that the system (3.4.7) can be rewritten in the following abstract form

$$\begin{cases} \partial_t Z + \mathcal{K}Z + \mathcal{B}Z - \mathcal{C}^\star Z = G, & (t,x) \in Q, \\ Z(t,0) = Z(t,1) = (0,\cdots,0)^\star, & t \in (0,T), \\ Z(T,x) = Z^T(x), & x \in (0,1), \end{cases} \tag{3.4.13}$$

where $G := (g_1, \cdots, g_n)^\star$.

Next we define, for $s > 0$, the function

$$W(t,x) = e^{2s\varphi} Z(t,x),$$

where Z is a solution of (3.4.13) in the class

$$\mathcal{V} = H^1\big(0,T;\mathcal{H}\big) \cap L^2(0,T; D(\mathbb{A})).$$

Then W solves the following system

$$\begin{cases} \partial_t(e^{-s\varphi}W) + \mathcal{K}(e^{-s\varphi}W) + \mathcal{B}(e^{-s\varphi}W) = G + \mathcal{C}^\star(e^{-s\varphi}W), & (t,x) \in Q, \\ W(t,0) = W(t,1) = (0,\cdots,0)^\star, & t \in (0,T), \\ W(T,x) = W(0,x) = (0,\cdots,0)^\star, & x \in (0,1). \end{cases}$$

Equivalently, the previous system can be written as

$$P_s^+ W + P_s^- W = G_s,$$

with

$$P_s^+ = diag(P_{s1}^+, \cdots, P_{sn}^+) + \mathcal{B}_0, \quad P_s^- = diag(P_{s1}^-, \cdots, P_{sn}^-), \quad G_s = Ge^{s\varphi} + \mathcal{C}^\star W,$$

where $\mathcal{B}_0$ is defined in (3.3.7).

The operators P_{sk}^+ and P_{sk}^- are given by

$$P_{sk}^+ w_k := d_k (a w_{kx})_x + \frac{\lambda_{kk}}{b_{kk}} w_k - s\varphi_t w_k + s^2 a \varphi_x^2 w_k,$$

and

$$P_{sk}^- w_k := w_{kt} - 2sa\varphi_x w_{kx} - s(a\varphi_x)_x w_k.$$

We have,

$$\|P_s^+W\|^2 + \|P_s^-W\|^2 + 2\langle P_s^+W, P_s^-W\rangle = \|G_s\|^2.$$

Then,

$$2\langle P_s^+W, P_s^-W\rangle \le \|G_s\|^2. \tag{3.4.14}$$

Here $\|\cdot\|$ stands for $L^2(Q)^n$ norm and $\langle\cdot,\cdot\rangle$ for the corresponding scalar product. Now, we compute the inner product $\langle P_s^+W, P_s^-W\rangle$, to obtain

$$\langle P_s^+W, P_s^-W\rangle = \sum_{k=1}^{n}\langle P_{sk}^+w_k, P_{sk}^-w_k\rangle_{L^2(Q)} + \sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\langle \frac{\lambda_{jk}}{b_{jk}}w_j, P_{sk}^-w_k\rangle_{L^2(Q)}.$$

Moreover, using the same computations of [103, Lemma.3.6], we find

$$\begin{aligned}\langle P_{sk}^+w_k, P_{sk}^-w_k\rangle_{L^2(Q)} \ge C\iint_Q \Big(s\theta a(x)w_{kx}^2 + s^3\theta^3\frac{(x-x_0)^2}{a(x)}w_k^2\Big)\,dx\,dt\\ - s\gamma\int_0^T \Big[\theta(x-x_0)aw_{kx}^2\Big]_{x=0}^{x=1}dt.\end{aligned} \tag{3.4.15}$$

Furthermore, one has

$$\begin{aligned}&\langle\frac{\lambda_{jk}}{b_{jk}}w_j, P_{sk}^-w_k\rangle_{L^2(Q)}\\ &\quad= \iint_Q \frac{\lambda_{jk}}{b_{jk}}w_j(w_{kt} - 2sa\varphi_x w_{kx} - s(a\varphi_x)_x w_k)\,dx\,dt.\end{aligned} \tag{3.4.16}$$

Using the fact that $\lambda_{kj} = \lambda_{jk}$ and $b_{kj} = b_{jk}$, an integration by parts leads to

$$\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\iint_Q \frac{\lambda_{jk}}{b_{jk}}w_jw_{kt}\,dx\,dt = \sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\int_0^1\frac{\big[w_j\,w_k\big]_{t=0}^{t=T}}{2b_{jk}}\,dx. \tag{3.4.17}$$

Similarly, one has

$$\begin{aligned}&-2s\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\iint_Q\frac{\lambda_{jk}}{b_{jk}}a\varphi_x w_jw_{kx}\,dx\,dt\\ &\qquad= -s\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\int_0^T\Big[\frac{a\varphi_x w_jw_k}{b_{jk}}\Big]_{x=0}^{x=1}dt\\ &\qquad\quad + s\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\iint_Q\frac{(a\varphi_x)_xb_{jk} - a\varphi_xb'_{jk}}{b_{jk}^2}w_jw_k\,dx\,dt.\end{aligned} \tag{3.4.18}$$

So, combining (3.4.16)-(3.4.18), we obtain

$$\begin{aligned}\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\langle\frac{\lambda_{jk}}{b_{jk}}w_j, P_{sk}^-w_k\rangle_{L^2(Q)} &= \sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\int_0^1\frac{\big[w_j\,w_k\big]_{t=0}^{t=T}}{2b_{jk}}\,dx\\ &\quad - s\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\int_0^T\Big[\frac{a\varphi_xw_jw_k}{b_{jk}}\Big]_{x=0}^{x=1}dt\\ &\quad - s\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\iint_Q\frac{a\varphi_xb'_{jk}}{b_{jk}^2}w_jw_k\,dx\,dt.\end{aligned} \tag{3.4.19}$$

As in [103, Lemma 3.6], using the definition of φ and the boundary conditions on W, the boundary terms appearing in the above identity reduce to 0.

On the other hand, applying Young's inequality and using the assumptions on b_{jk}, we obtain

$$\begin{aligned}
&- s\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\iint_Q \frac{a\varphi_x b'_{jk}}{b_{jk}^2} w_j w_k\,dx\,dt \\
&= -s\gamma\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\iint_Q \theta\frac{(x-x_0)b'_{jk}}{b_{jk}^2} w_j w_k\,dx\,dt \\
&\leq \frac{s\gamma}{2}\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\Big(\iint_Q \frac{(x-x_0)b'_{jk}}{b_{jk}^2}\theta w_j^2\,dx\,dt + \iint_Q \frac{(x-x_0)b'_{jk}}{b_{jk}^2}\theta w_k^2\,dx\,dt\Big) \\
&\leq \frac{s\gamma}{2}\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\mathcal{M}(j,k)\Big(\iint_Q \frac{\theta}{b_{jk}} w_j^2\,dx\,dt + \iint_Q \frac{\theta}{b_{jk}} w_k^2\,dx\,dt\Big),
\end{aligned} \tag{3.4.20}$$

with

$$\mathcal{M}(j,k) = \begin{cases} M_k & \text{if } \ j = k-1, \\ M_{k+1} & \text{if } \ j = k+1. \end{cases}$$

By the Hardy-Poincaré inequalities, one obtains

$$\begin{aligned}
&\frac{s\gamma}{2}\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\mathcal{M}(j,k)\Big(\iint_Q \frac{\theta}{b_{jk}} w_j^2\,dx\,dt + \iint_Q \frac{\theta}{b_{jk}} w_k^2\,dx\,dt\Big) \\
&\leq \frac{s\gamma}{2}\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\mathcal{M}(j,k)\mathbf{C}(j,k)\Big(\iint_Q \theta a w_{jx}^2\,dx\,dt + \iint_Q \theta a w_{kx}^2\,dx\,dt\Big).
\end{aligned} \tag{3.4.21}$$

Here, the constant $\mathbf{C}(j,k) > 0$ is given by

$$\mathbf{C}(j,k) = \begin{cases} C^{kk-1} & \text{if } \ j = k-1, \\ C^{k+1k} & \text{if } \ j = k+1, \end{cases}$$

where C^{kk-1} is the constant appearing in (3.2.2). By (3.4.19)-(3.4.21), we obtain

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\langle \frac{\lambda_{jk}}{b_{jk}} w_j, P_{sk}^- w_k\rangle_{L^2(Q)} \\
&\geq -\frac{s\gamma}{2}\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\mathcal{M}(j,k)\mathbf{C}(j,k)\iint_Q \theta a(w_{jx}^2 + w_{kx}^2)\,dx\,dt.
\end{aligned} \tag{3.4.22}$$

Hence, from (3.4.15) and (3.4.22), we deduce

$$\begin{aligned}\langle P_s^+W, P_s^-W\rangle \geq \sum_{k=1}^{n}\Big[C\iint_Q\Big(s\,\overbrace{\theta a(x)w_{kx}^2}^{\mathcal{A}_1}+s^3\,\overbrace{\theta^3\frac{(x-x_0)^2}{a(x)}w_k^2}^{\mathcal{A}_2}\Big)\,dx\,dt\\ -s\gamma\int_0^T\Big[\theta(x-x_0)aw_{kx}^2\Big]_{x=0}^{x=1}dt\Big]\\ -\frac{s\gamma}{2}\sum_{k=1}^{n}\sum_{\substack{j=k-1\\ j\neq k}}^{k+1}\lambda_{jk}\mathcal{M}(j,k)\mathbf{C}(j,k)\iint_Q\theta a(w_{jx}^2+w_{kx}^2)\,dx\,dt.\end{aligned}\tag{3.4.23}$$

At this stage, let us remark that one can take C as large as desired, provided that s_0 increases as well. Indeed, taken $k>0$, from

$$C\big(s\mathcal{A}_1+s^3\mathcal{A}_2\big)=kC\big(\frac{s}{k}\mathcal{A}_1+\frac{s^3}{k}\mathcal{A}_2\big),$$

we can choose $s_0'=ks_0$ and $C=kC$ large as needed. As an immediate consequence, one can prove that the distributed terms in the right hand side of (3.4.23) can be estimated as

$$\begin{aligned}\langle P_s^+W, P_s^-W\rangle \geq \sum_{k=1}^{n}\Bigg[C\iint_Q\Big(s\theta a(x)w_{kx}^2+s^3\theta^3\frac{(x-x_0)^2}{a(x)}w_k^2\Big)\,dx\,dt\\ -s\gamma\int_0^T\Big[\theta(x-x_0)aw_{kx}^2\Big]_{x=0}^{x=1}dt\Bigg],\end{aligned}\tag{3.4.24}$$

where C is a positive constant. On the other hand, we have

$$\begin{aligned}\|G_s\|^2&=\|Ge^{s\varphi}+\mathcal{C}^\star W\|^2\\ &\leq 2\sum_{k=1}^{n}\iint_Q g_k^2e^{2s\varphi}\,dx\,dt+C\sum_{k=1}^{n}\sum_{j=1}^{k+1}\iint_Q a_{jk}^2w_j^2\,dx\,dt\\ &\leq 2\sum_{k=1}^{n}\iint_Q g_k^2e^{2s\varphi}\,dx\,dt+C\sum_{k=1}^{n}\iint_Q w_k^2\,dx\,dt.\end{aligned}\tag{3.4.25}$$

Then, by the Hardy-Poincaré inequality given in [105, Proposition 2.6], we get

$$\begin{aligned}\sum_{k=1}^{n}\int_0^1 w_k^2\,dx&\leq C_0\sum_{k=1}^{n}\int_0^1\frac{a^{\frac13}(x)}{(x-x_0)^{\frac23}}w_k^2\,dx\\ &\leq C_0\sum_{k=1}^{n}\int_0^1\frac{p}{(x-x_0)^2}w_k^2\,dx\\ &\leq C_0C_{HP}\sum_{k=1}^{n}\int_0^1 pw_{kx}^2\,dx\\ &\leq C_0\max\{C_1,C_2\}C_{HP}\sum_{k=1}^{n}\int_0^1 a(x)w_{kx}^2\,dx.\end{aligned}\tag{3.4.26}$$

Here $p(x)=(a(x)|x-x_0|^4)^{1/3}$ if $K>\dfrac{4}{3}$ or $p(x)=\max\limits_{[0,1]}a^{1/3}|x-x_0|^{4/3}$ otherwise,

$$C_0:=\max\left[\left(\frac{x_0^2}{a(0)}\right)^{\frac13},\left(\frac{(1-x_0)^2}{a(1)}\right)^{\frac13}\right],C_1:=\max\left\{\left(\frac{x_0^2}{a(0)}\right)^{2/3},\left(\frac{(1-x_0)^2}{a(1)}\right)^{2/3}\right\},$$

$C_2 := \max_{[0,1]} a^{1/3} \times \max\left\{\dfrac{x_0^{4/3}}{a(0)}, \dfrac{(1-x_0)^{4/3}}{a(1)}\right\}$ and C_{HP} is the Hardy-Poincaré constant given in [105, Proposition 2.6]. Observe that the function p satisfies the assumptions of [105, Proposition 2.6] thanks to [105, Lemma 2.1]. Substituting the inequality (3.4.26) in (3.4.25), it follows that

$$\|G_s\|^2 \leq \sum_{k=1}^{n}\Big(2\iint_Q g_k^2 e^{2s\varphi}\,dx\,dt + \varepsilon\iint_Q s\theta a(x) w_{kx}^2\,dx\,dt\Big) \tag{3.4.27}$$

for $\varepsilon > 0$. Thus, by choosing ε small and s large enough, (3.4.24) and (3.4.27) imply

$$\begin{aligned}
&\sum_{k=1}^{n}\iint_Q\Big(s\theta a(x) w_{kx}^2 + s^3\theta^3\frac{(x-x_0)^2}{a(x)}w_k^2\Big)\,dx\,dt\\
&\leq C\Big(\sum_{k=1}^{n}\iint_Q g_k^2 e^{2s\varphi}\,dx\,dt + s\gamma\int_0^T\Big[\theta(x-x_0)a w_{kx}^2\Big]_{x=0}^{x=1}dt\Big),
\end{aligned}$$

where C is a positive constant. Recalling the definition of w_k, one thus obtains the asserted Carleman estimate for our original variables.

□

3.4.2 ω-Carleman estimate for the homogeneous adjoint system

In this subsection we consider the following homogeneous parabolic system

$$\begin{cases} \partial_t Z + \mathcal{K}Z + \mathcal{B}Z - \mathcal{C}^\star Z = (0,\cdots,0)^\star & (t,x)\in Q,\\ Z(T,x) = Z^T(x) \in D(\mathbb{A}^2). \end{cases} \tag{3.4.28}$$

Let us recall that $D(\mathbb{A}^2) = \Big\{Z^\star \in \mathbb{H};\quad (\mathbb{A}Z)^\star \in D(\mathbb{A})\Big\}$. Notice that $D(\mathbb{A}^2)$ is densely defined in $D(\mathbb{A})$ for the graph norm (see, for instance [42, Theorem 2.7]) and hence in $\mathbb{H}$. As in [105], define the following class of functions

$$\mathcal{W} = \Big\{Z \quad \text{is a solution of} \quad (3.4.28)\Big\}.$$

As in [42, Theorem 7.5],

$$\mathcal{W} \subset C^1([0,T]; D(\mathbb{A})) \subset \mathcal{V} \subset \mathcal{U},$$

where $\mathcal{V}$ is defined in (3.4.11) and

$$\mathcal{U} := C\big([0,T];\mathbb{H}\big) \cap L^2\big(0,T;\mathcal{H}\big).$$

To obtain Carleman estimates for the system (3.4.28), we assume that the control region ω satisfies the following assumption:

Hypothesis 3.4.2. The control set ω is such that

$$\omega = \omega_1 \cup \omega_2,$$

where ω_i $(i = 1,2)$ are intervals with $\omega_1 := (\alpha_1,\beta_1) \Subset (0,x_0)$, $\omega_2 := (\alpha_2,\beta_2) \Subset (x_0,1)$, and $x_0 \notin \bar{\omega}$.

Remark 28. In fact, it is proved in [48] that null controllability of the parabolic operator

$$Pu = u_t - (|x-x_0|^K u_x)_x, \qquad x \in (0,1),$$

which degenerates at the interior point $x_0 \in (0,1)$, under the action of a locally distributed control supported only on one side of the domain with respect to the point of degeneracy

1. fails for $K \in [1, 2)$,

2. holds true when $K \in [0, 1)$.

In particular, in order that (3.1.1) to be null controllable, the control support must lie on both sides of the degeneracy point.

Next, we introduce

$$\omega' = \omega_1' \cup \omega_2'$$

where $\omega_1' := (\alpha_1', \beta_1') \Subset \omega_1$ and $\omega_2' := (\alpha_2', \beta_2') \Subset \omega_2$.

We claim the following.

Theorem 3.4.2. *Assume Hypotheses 3.4.1 and 3.4.2. Then, there exist two positive constant C_1 and s_0 such that, every solution $(z_1, \cdots, z_n) \in \mathcal{W}$ of (3.4.28) satisfies, $\forall s \geq s_0$,*

$$\begin{aligned}
&\sum_{k=1}^{n} \iint_Q \Big(s\theta a(x) z_{kx}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)} z_k^2 \Big) e^{2s\varphi}\, dx\, dt \\
&\leq C_1 \sum_{k=1}^{n} \iint_{Q_{\omega'}} s^2\theta^2 z_k^2 e^{2s\Phi}\, dx\, dt.
\end{aligned} \tag{3.4.29}$$

Here, $Q_{\omega'} = (0, T) \times \omega'$.

For the proof of the above result, we shall use the following non degenerate non singular classical Carleman estimate in a suitable interval (A, B) (see [105] or [103, Proposition 4.8]), which will be used far away from x_0 within a localization procedure via cut-off functions.

Proposition 3.4.1. *Let z be the solution of*

$$\begin{aligned}
&z_t + (a z_x)_x + \frac{\lambda}{b(x)} z = h, \quad (t,x) \in (0,T) \times (A,B), \\
&z(t, A) = z(t, B) = 0, \quad t \in (0, T),
\end{aligned} \tag{3.4.30}$$

where $h \in L^2((0,T) \times (A,B))$, $a \in C^1([A,B])$ is a strictly positive function and $b \in C([A,B])$ is such that $b \geq b_0 > 0$ in $[A,B]$. Then, there exist two positive constants r and s_0 such that for any $s \geq s_0$,

$$\begin{aligned}
&\int_0^T \int_A^B s\theta e^{r\zeta} z_x^2 e^{2s\Phi}\, dx\, dt + \int_0^T \int_A^B s^3\theta^3 e^{3r\zeta} z^2 e^{2s\Phi}\, dx\, dt \\
&\leq C \Big(\int_0^T \int_A^B f^2 e^{2s\Phi}\, dx\, dt - \int_0^T \Big[r s\theta e^{r\zeta} z_x^2 e^{2s\Phi} \Big]_{x=A}^{x=B} dt \Big)
\end{aligned} \tag{3.4.31}$$

for some positive constant C.

Proof of Theorem 3.4.2. To prove the statement we use a technique based on cut-off functions. For this purpose, we define five points $\delta_1, \delta_2, \alpha_1'', \alpha_2'', \beta_2''$ such that

$$\alpha_1' < \alpha_1'' < \delta_1 < \beta_1' \quad \text{and} \quad \alpha_2' < \alpha_2'' < \delta_2 < \beta_2'' < \beta_2'.$$

From now on, the point D will be fixed such that $-x_0 < D < -\beta_1'$.

Now, let us consider a smooth function $\tau : [0, 1] \to [0, 1]$ such that

$$\tau(x) = \begin{cases} 1, & x \in [\delta_2, 1], \\ 0, & x \in [0, \alpha_2'']. \end{cases}$$

Define $W = \tau Z$, where Z is the solution of (3.4.28). Then, w_k $(1 \le k \le n)$, satisfies

$$\begin{cases} \partial_t w_k + d_k(a(x)w_{kx})_x - \displaystyle\sum_{j=1}^{k+1} a_{jk} w_j + \sum_{j=k-1}^{k+1} \frac{\lambda_{jk}}{b_{jk}} w_j \\ \quad = d_k(a\tau_x z_k)_x + d_k a \tau_x z_{kx}, \quad (t,x) \in (0,T) \times (\alpha_2', 1), \\ w_k(t,\alpha_2') = w_k(t,1) = 0, \quad t \in (0,T). \end{cases} \tag{3.4.32}$$

Since $x \in (\alpha_2', 1)$, observe that the system above is a nondegenerate nonsingular problem. Thus, we can apply the analogue of Proposition 3.4.1 for the component w_k in $(\alpha_2', 1)$ in place of (A, B), obtaining that there exist two positive constants C and s_0 (s_0 sufficiently large), such that w_k satisfies, for all $s \ge s_0$,

$$\begin{aligned} &\int_0^T \int_{\alpha_2'}^1 \left(s\theta e^{r\zeta} w_{kx}^2 + s^3\theta^3 e^{3r\zeta} w_k^2 \right) e^{2s\Phi} \, dx \, dt \\ &\le C \int_0^T \int_{\alpha_2'}^1 \Big(\sum_{\substack{j=1 \\ j \ne k}}^{k+1} a_{jk} w_j - \sum_{\substack{j=k-1 \\ j \ne k}}^{k+1} \frac{\lambda_{jk}}{b_{jk}} w_j + d_k(a\tau_x z_k)_x + d_k a \tau_x z_{kx} \Big)^2 e^{2s\Phi} \, dx \, dt. \end{aligned}$$

Let us remark that the boundary term in $x = 1$ is nonpositive, while the one in $x = \alpha_2'$ is 0, so that they can be neglected in the classical Carleman estimate. Moreover, taking into account the fact that $a_{jk} \in L^\infty(Q)$ and the coefficients $\dfrac{1}{b_{jk}}$ are bounded in $[\alpha_2', 1]$, we find

$$\begin{aligned} &\sum_{k=1}^n \int_0^T \int_{\alpha_2'}^1 \left(s\theta e^{r\zeta} w_{kx}^2 + s^3\theta^3 e^{3r\zeta} w_k^2 \right) e^{2s\Phi} \, dx \, dt \\ &\le C \sum_{k=1}^n \int_0^T \int_{\alpha_2'}^1 w_k^2 e^{2s\Phi} \, dx \, dt + C \sum_{k=1}^n \int_0^T \int_{\alpha_2'}^1 ((a\tau_x z_k)_x + a\tau_x z_{kx})^2 e^{2s\Phi} \, dx \, dt. \end{aligned}$$

Using the fact that τ_x and τ_{xx} are supported in $[\alpha_2'', \delta_2]$, we obtain

$$\begin{aligned} &\sum_{k=1}^n \int_0^T \int_{\alpha_2'}^1 \left(s\theta e^{r\zeta} w_{kx}^2 + s^3\theta^3 e^{3r\zeta} w_k^2 \right) e^{2s\Phi} \, dx \, dt \\ &\le C \sum_{k=1}^n \int_0^T \int_{\alpha_2'}^1 w_k^2 e^{2s\Phi} \, dx \, dt + C \sum_{k=1}^n \int_0^T \int_{\alpha_2''}^{\delta_2} (z_k^2 + z_{kx}^2) e^{2s\Phi} \, dx \, dt. \end{aligned}$$

For s large enough, we have

$$C \sum_{k=1}^n \int_0^T \int_{\alpha_2'}^1 w_k^2 e^{2s\Phi} \, dx \, dt \le \frac{1}{2} \sum_{k=1}^n \int_0^T \int_{\alpha_2'}^1 s\theta e^{r\zeta} w_k^2 e^{2s\Phi} \, dx \, dt,$$

and then

$$\begin{aligned} &\sum_{k=1}^n \int_0^T \int_{\alpha_2'}^1 \left(s\theta e^{r\zeta} w_{kx}^2 + s^3\theta^3 e^{3r\zeta} w_k^2 \right) e^{2s\Phi} \, dx \, dt \\ &\qquad \le C \sum_{k=1}^n \int_0^T \int_{\alpha_2''}^{\delta_2} (z_k^2 + z_{kx}^2) e^{2s\Phi} \, dx \, dt. \end{aligned} \tag{3.4.33}$$

At this level, we shall also use the following Caccioppoli's inequality.

Lemma 3.4.3. *[38, Proposition 5.1] Let ω' and ω'' be two non empty open subsets of $(0,1)$ such that $\overline{\omega''} \subset \omega' \subset (0,1)$ and $x_0 \notin \overline{\omega'}$. Then, there exist two positive constant $C > 0$ and $s_0 > 0$ such that any solution z of the equation (3.4.30) satisfies, $\forall s \geq s_0$,*

$$\iint_{Q_{\omega''}} z_x^2 e^{2s\Phi}\,dx\,dt \leq C \iint_{Q_{\omega'}} (f^2 + s^2\theta^2 z^2) e^{2s\Phi}\,dx\,dt,$$

with $\phi(t,x) := \theta(t)\rho(x)$, where $\rho \in C^2(\overline{\omega'},\mathbb{R})$.

In view of Lemma 3.4.3, one can estimate the right-hand side of (3.4.33) as follows

$$\begin{aligned}
&\int_0^T \int_{\alpha_2''}^{\delta_2} (z_k^2 + z_{kx}^2) e^{2s\Phi}\,dx\,dt \\
&\leq C \iint_{Q_{\omega'}} \Bigg(z_k^2 + \Big((\sum_{\substack{j=1\\ j\neq k}}^{k+1} a_{jk} z_j - \sum_{\substack{j=k-1\\ j\neq k}}^{k+1} \frac{\lambda_{jk}}{b_{jk}} z_j)^2 + s^2\theta^2 z_k^2 \Big) \Bigg) e^{2s\Phi}\,dx\,dt \\
&\leq C \sum_{j=1}^{k+1} \iint_{Q_{\omega'}} s^2\theta^2 z_j^2 e^{2s\Phi}\,dx\,dt.
\end{aligned} \tag{3.4.34}$$

Combining (3.4.33) and (3.4.34), we get

$$\begin{aligned}
\sum_{k=1}^{n} \int_0^T \int_{\alpha_2'}^{1} \Big(s\theta e^{r\zeta} w_{kx}^2 + s^3\theta^3 e^{3r\zeta} w_k^2 \Big) e^{2s\Phi}\,dx\,dt& \\
\leq C \sum_{k=1}^{n} \iint_{Q_{\omega'}} s^2\theta^2 z_k^2 e^{2s\Phi}\,dx\,dt.&
\end{aligned} \tag{3.4.35}$$

Having in mind the fact that $e^{2s\varphi} \leq e^{2s\Phi}$, one can easily check that there exists a positive constant C such that for every $(t,x) \in [0,T] \times [\alpha_2', 1]$, we have

$$a(x) e^{2s\varphi} \leq C e^{r\zeta} e^{2s\Phi} \quad \text{and} \quad \frac{(x-x_0)^2}{a(x)} e^{2s\varphi} \leq C e^{3r\zeta} e^{2s\Phi}. \tag{3.4.36}$$

By (3.4.35) and (3.4.36), using the definition of z_k, we deduce

$$\begin{aligned}
&\sum_{k=1}^{n} \int_0^T \int_{\delta_2}^{1} \Big(s\theta a(x) z_{kx}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)} z_k^2 \Big) e^{2s\varphi}\,dx\,dt \\
&= \sum_{k=1}^{n} \int_0^T \int_{\delta_2}^{1} \Big(s\theta a(x) w_{kx}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)} w_k^2 \Big) e^{2s\varphi}\,dx\,dt \\
&\leq C \sum_{k=1}^{n} \int_0^T \int_{\alpha_2'}^{1} \Big(s\theta e^{r\zeta} w_{kx}^2 + s^3\theta^3 e^{3r\zeta} w_k^2 \Big) e^{2s\Phi}\,dx\,dt \\
&\leq C \sum_{k=1}^{n} \iint_{Q_{\omega'}} s^2\theta^2 z_k^2 e^{2s\Phi}\,dx\,dt.
\end{aligned} \tag{3.4.37}$$

To complete the proof, it is sufficient to prove a similar inequality on the interval $[0, \delta_2]$. To this aim, recalling that D is chosen in such a way that $-x_0 < D < -\beta_1' < 0$, we follow a reflection procedure already introduced in [105], considering the function

$$\tilde{z}_k(t,x) = \begin{cases} z_k(t,x), & x \in [0,1], \\ -z_k(t,-x), & x \in [-1,0], \end{cases}$$

where z_k solution of the $k-$th equation of (3.4.28). Let us define

$$\tilde{a}_{jk}(t,x) = \left\{ \begin{array}{ll} a_{jk}(t,x), & x \in [0,1], \\ a_{jk}(t,-x), & x \in [-1,0], \end{array} \right. \qquad \tilde{b}_{jk}(x) = \left\{ \begin{array}{ll} b_{jk}(x), & x \in [0,\ 1], \\ b_{jk}(-x), & x \in [-1,0], \end{array} \right.$$

$$\tilde{Z} = (\tilde{z}_1, \cdots, \tilde{z}_n) \quad \text{and} \quad \tilde{\mathcal{K}}\tilde{Z} = diag\Big((\tilde{a}(x)\tilde{z}_{1x})_x, (\tilde{a}(x)\tilde{z}_{2x})_x, \cdots, (\tilde{a}(x)\tilde{z}_{nx})_x \Big).$$

Therefore, $\tilde{Z}$ solves the system

$$\left\{ \begin{array}{l} \partial_t \tilde{Z} + \tilde{\mathcal{K}}\tilde{Z} + \tilde{\mathcal{B}}\tilde{Z} - \tilde{\mathcal{C}}^{\star}\tilde{Z} = (0,\cdots,0)^{\star}, \quad (t,x) \in (0,T)\times(-1,1), \\ \tilde{Z}(t,-1) = \tilde{Z}(t,1) = (0,\cdots,0)^{\star}, \quad t \in (0,T), \end{array} \right. \tag{3.4.38}$$

where, $\tilde{\mathcal{C}}$ is the analogue of $\mathcal{C}$ with a_{jk} replaced by $\tilde{a}_{jk}$ and $\tilde{\mathcal{B}}$ is defined as $\tilde{\mathcal{C}}$. Now, consider a smooth function $\rho : [-1,1] \to [0,1]$ such that

$$\rho(x) = \begin{cases} 1, & x \in [-\alpha_1'', \delta_2], \\ 0, & x \in [-1,-\delta_1] \cup [\beta_2'', 1], \end{cases}$$

and define the function $\tilde{V} = \rho\tilde{Z}$, where $\tilde{Z}$ is the solution of (3.4.38). Then $\tilde{V}$ solves

$$\left\{ \begin{array}{l} \partial_t \tilde{V} + \tilde{\mathcal{K}}\tilde{V} + \tilde{\mathcal{B}}\tilde{V} - \tilde{\mathcal{C}}^{\star}\tilde{V} = \tilde{G}, \quad (t,x) \in (0,T)\times(-\beta_1',1), \\ \tilde{V}(t,-\beta_1') = \tilde{V}(t,1) = (0,\cdots,0)^{\star}, \quad t \in (0,T), \end{array} \right. \tag{3.4.39}$$

where $\tilde{G} := (\tilde{a}\rho_x\tilde{Z})_x + \tilde{a}\rho_x\tilde{Z}_x$.

Applying the Carleman estimate (3.4.12) to (3.4.39) (which still holds true, in $(-\beta_1',1)$ in place of $(0,1)$ and $\tilde{a}$ instead of a, since $\tilde{a} \in W^{1,1}(-\beta_1',1)$ in weakly degenerate case and $\tilde{a} \in W^{1,\infty}(-\beta_1',1)$ in strongly degenerate one), it follows that

$$\begin{aligned} &\sum_{k=1}^{n}\int_0^T\int_{-\beta_1'}^{1} \Big(s\theta\tilde{a}(x)\tilde{v}_{kx}^2 + s^3\theta^3\frac{(x-x_0)^2}{\tilde{a}(x)}\tilde{v}_k^2 \Big) e^{2s\varphi}\,dx\,dt \\ &\le C\sum_{k=1}^{n}\int_0^T\int_{-\beta_1'}^{1} \Big((\tilde{a}\rho_x\tilde{z}_k)_x + \tilde{a}\rho_x\tilde{z}_{kx} \Big)^2 e^{2s\varphi}\,dx\,dt \\ &\le C\sum_{k=1}^{n}\int_0^T\int_{-\delta_1}^{-\alpha_1''} (\tilde{z}_k^2 + \tilde{z}_{kx}^2)e^{2s\varphi}\,dx\,dt + C\sum_{k=1}^{n}\int_0^T\int_{\delta_2}^{\beta_2''} (z_k^2 + z_{kx}^2)e^{2s\varphi}\,dx\,dt. \end{aligned} \tag{3.4.40}$$

Furthermore, using the definition of $\tilde{z}_k$ and thanks to the oddness of the involved functions, one can write

$$\begin{aligned} \int_0^T\int_{-\delta_1}^{-\alpha_1''} (\tilde{z}_k^2 + \tilde{z}_{kx}^2)e^{2s\varphi}\,dx\,dt &= \int_0^T\int_{-\delta_1}^{-\alpha_1''} \big(z_k^2(-x) + z_{kx}^2(-x) \big) e^{2s\varphi(x)}\,dx\,dt \\ &= \int_0^T\int_{\alpha_1''}^{\delta_1} \big(z_k^2(x) + z_{kx}^2(x) \big) e^{2s\varphi(-x)}\,dx\,dt. \end{aligned} \tag{3.4.41}$$

At this point, by Lemma 3.4.1, for $(t,x) \in [0,T] \times [\alpha_1'', \delta_1]$, one has

$$\varphi(t,-x) \le \Phi(t,x).$$

It follows from this last inequality and (3.4.41) that

$$\int_0^T\int_{-\delta_1}^{-\alpha_1''} (\tilde{z}_k^2 + \tilde{z}_{kx}^2)e^{2s\varphi}\,dx\,dt \le \int_0^T\int_{\alpha_1''}^{\delta_1} (z_k^2 + z_{kx}^2)e^{2s\Phi}\,dx\,dt.$$

Putting the above inequality in (3.4.40) and using the fact that $\varphi \leq \Phi$, we get

$$\begin{aligned}
&\sum_{k=1}^{n}\int_0^T\int_{-\beta_1'}^{1}\Big(s\theta\tilde{a}(x)\tilde{v}_{kx}^2+s^3\theta^3\frac{(x-x_0)^2}{\tilde{a}(x)}\tilde{v}_k^2\Big)e^{2s\varphi}\,dx\,dt\\
&\leq C\sum_{k=1}^{n}\int_0^T\int_{\alpha_1''}^{\delta_1}(z_k^2+z_{kx}^2)e^{2s\Phi}\,dx\,dt+C\sum_{k=1}^{n}\int_0^T\int_{\delta_2}^{\beta_2''}(z_k^2+z_{kx}^2)e^{2s\Phi}\,dx\,dt. \qquad (3.4.42)
\end{aligned}$$

Now, proceeding as in (3.4.34), it is not difficult to see that

$$\begin{aligned}
&\int_0^T\int_{\alpha_1''}^{\delta_1}(z_k^2+z_{kx}^2)e^{2s\Phi}\,dx\,dt+\int_0^T\int_{\delta_2}^{\beta_2''}(z_k^2+z_{kx}^2)e^{2s\Phi}\,dx\,dt\\
&\leq C\iint_{Q_{\omega'}}s^2\theta^2z_k^2e^{2s\Phi}\,dx\,dt.
\end{aligned}$$

Thus, it appears that

$$\begin{aligned}
&\sum_{k=1}^{n}\int_0^T\int_{-\beta_1'}^{1}\Big(s\theta\tilde{a}(x)\tilde{v}_{kx}^2+s^3\theta^3\frac{(x-x_0)^2}{\tilde{a}(x)}\tilde{v}_k^2\Big)e^{2s\varphi}\,dx\,dt\\
&\leq C\sum_{k=1}^{n}\iint_{Q_{\omega'}}s^2\theta^2z_k^2e^{2s\Phi}\,dx\,dt.
\end{aligned}$$

Using the definitions of $\tilde{z}_k$, $\tilde{v}_k$ and ρ, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n}\int_0^T\int_0^{\delta_2}\Big(s\theta a(x)z_{kx}^2+s^3\theta^3\frac{(x-x_0)^2}{a(x)}z_k^2\Big)e^{2s\varphi}\,dx\,dt\\
&=\sum_{k=1}^{n}\int_0^T\int_0^{\delta_2}\Big(s\theta a(x)v_{kx}^2+s^3\theta^3\frac{(x-x_0)^2}{a(x)}v_k^2\Big)e^{2s\varphi}\,dx\,dt\\
&\leq\sum_{k=1}^{n}\int_0^T\int_{-\beta_1'}^{1}\Big(s\theta\tilde{a}(x)\tilde{v}_{kx}^2+s^3\theta^3\frac{(x-x_0)^2}{\tilde{a}(x)}\tilde{v}_k^2\Big)e^{2s\varphi}\,dx\,dt\\
&\leq C\sum_{k=1}^{n}\iint_{Q_{\omega'}}s^2\theta^2z_k^2e^{2s\Phi}\,dx\,dt. \qquad (3.4.43)
\end{aligned}$$

Adding (3.4.37) and (3.4.43), we finally obtain Theorem 3.4.2. □

To establish the null controllability of the parabolic system (3.1.1) with one control force, we need the following crucial Carleman estimate with one observation.

Theorem 3.4.3. *Assume Hypotheses 3.4.1 and 3.4.2. Moreover, we suppose that for some open subset $\hat{\omega} \Subset w$, we have*

$$-a_{kk-1}+\frac{\lambda_{kk-1}}{b_{kk-1}}\geq b_0>0, \qquad \text{in } (0,T)\times\hat{\omega}, \qquad \forall k: 2\leq k\leq n. \qquad (3.4.44)$$

Then, there exist two positive constant C and s_0, such that every solution of (3.4.28) satisfies, $\forall s \geq s_0$,

$$\sum_{k=1}^{n}\iint_Q\Big(s\theta a(x)z_{kx}^2+s^3\theta^3\frac{(x-x_0)^2}{a(x)}z_k^2\Big)e^{2s\varphi}\,dx\,dt\leq C\iint_{Q_\omega}z_1^2\,dx\,dt. \qquad (3.4.45)$$

Remark 29. We point out that the proof of Theorem 3.4.3 is still valid if we assume (3.4.44) or

$$-a_{kk-1}+\frac{\lambda_{kk-1}}{b_{kk-1}}\leq -b_0<0, \qquad \text{in } (0,T)\times\hat{\omega}, \qquad \forall k: 2\leq k\leq n.$$

To prove the above Theorem, we will need the following lemma.

Lemma 3.4.4. *Under the assumptions of Theorem 3.4.3 and given $l \in \mathbb{N}$, $\varepsilon > 0$, $k = 2, \cdots, n$ and two open sets ω_0 and ω_0' such that $\omega' \Subset \omega_0' \Subset \omega_0 \Subset \hat{\omega} \Subset \omega$, there exist $C_k = C_k(\omega_0, \omega_0', b_0) > 0$ and $l_j \in \mathbb{N}$, $1 \leq j \leq k-1$ such that if $Z = (z_1, \cdots, z_n)^\star$ is the solution to (3.4.28), one has*

$$
\begin{aligned}
J_{\omega_0'}(l, \Phi_k, z_k) \leq \varepsilon \sum_{j=k}^{k+1} \iint_Q \Big(s\theta a(x) z_{jx}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)} z_j^2 \Big) e^{2s\varphi}\, dx\, dt \\
+ C_k \Big(1 + \frac{1}{\varepsilon} \Big) \sum_{j=1}^{k-1} J_{\omega_0}(l_j, \Phi_{k-1}, z_j),
\end{aligned}
\tag{3.4.46}
$$

with $J_\omega(d, \phi, z) := s^d \iint_{Q_\omega} \theta^d z^2 e^{2s\phi}\, dx\, dt$ and $l_j = \max(3, 2l+1)$.

Proof of Theorem 3.4.3. In order to eliminate the local terms

$$
\iint_{Q_\omega} s^2\theta^2 z_k^2 e^{2s\Phi}\, dx\, dt \quad \forall k = 2, \cdots, n,
$$

in the right hand side of (3.4.29), we proceed as in [108]. Thus, we consider a sequence of open sets $(\mathcal{O}_k)_{k=2}^{k=n}$ such that $\omega' \Subset \mathcal{O}_n \Subset \mathcal{O}_{n-1} \Subset \cdots \Subset \mathcal{O}_2 \Subset \hat{\omega}$.

At first, to absorb the term that depend on the component z_n, we apply the formula (3.4.46) for

$$
k = n, \quad l = 2, \quad \omega_0' = \omega', \quad \omega_0 = \mathcal{O}_n \quad \text{and} \quad \varepsilon = \frac{1}{2C_1},
$$

with C_1 is the constant appearing in (3.4.29). Therefore, from (3.4.29), one has

$$
\begin{aligned}
&\sum_{k=1}^{n} \iint_Q \Big(s\theta a(x) z_{kx}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)} z_k^2 \Big) e^{2s\varphi}\, dx\, dt \\
&\leq \tilde{C}_n \sum_{j=1}^{n-1} J_{\mathcal{O}_n}(l_j, \Phi_{n-1}, z_j),
\end{aligned}
\tag{3.4.47}
$$

where $\tilde{C}_n = 2\max\big(C_1, C_n(1+2C_1)\big)$. We can go on applying (3.4.46) for

$$
k = n-1, \quad l = l_{n-1}, \quad \omega_0' = \mathcal{O}_n, \quad \omega_0 = \mathcal{O}_{n-1}, \quad \text{and} \quad \varepsilon = \frac{1}{2\tilde{C}_n},
$$

and eliminate in (3.4.47) the local term $J_{\mathcal{O}_n}(l_{n-1}, \Phi_{n-1}, z_{n-1})$, obtaining

$$
\begin{aligned}
&\sum_{k=1}^{n} \iint_Q \Big(s\theta a(x) z_{kx}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)} z_k^2 \Big) e^{2s\varphi}\, dx\, dt \\
&\qquad \leq \tilde{C}_{n-1} \sum_{j=1}^{n-2} J_{\mathcal{O}_{n-1}}(l_j, \Phi_{n-2}, z_j).
\end{aligned}
$$

By (a finite) iteration of this argument, we obtain

$$
\begin{aligned}
\sum_{k=1}^{n} \iint_Q \Big(s\theta a(x) z_{kx}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)} z_k^2 \Big) e^{2s\varphi}\, dx\, dt &\leq \tilde{C}_2 J_{\mathcal{O}_2}(l_1, \Phi_1, z_1) \\
&\leq C \iint_{Q_\omega} s^{l_1}\theta^{l_1} z_1^2 e^{2s\Phi_1}\, dx\, dt,
\end{aligned}
$$

with C is a positive constant. Finally, since $\sup_{(t,x)\in Q} s^{l_1}\theta^{l_1} e^{2s\Phi_1} < +\infty$, we readily deduce (3.4.45), which concludes the proof of Theorem 3.4.3. □

Proof of Lemma 3.4.4. Let us consider a smooth cut-off function $\xi \in C^\infty(0,1)$ such that

$$\xi(x) = \begin{cases} 1, & x \in \omega_0', \\ 0, & x \in (0,1) \setminus \omega_0, \end{cases} \tag{3.4.48}$$

with $\dfrac{\xi_x}{\sqrt{\xi}}, \dfrac{\xi_{xx}}{\sqrt{\xi}} \in L^\infty(0,1)$. Multiplying the equation satisfied by z_{k-1} in (3.4.28) by $s^l\theta^l\xi e^{2s\Phi_k} z_k$ and integrating over Q, we get

$$\begin{aligned}
b_0 J_{\omega_0'}(l,\Phi_k,z_k) &\leq \iint_Q \Big(-a_{kk-1} + \frac{\lambda_{kk-1}}{b_{kk-1}} \Big) s^l\theta^l\xi e^{2s\Phi_k} z_k^2 \,dx\,dt \\
&\leq \overbrace{\Big| \iint_Q z_{k-1,t} s^l\theta^l\xi e^{2s\Phi_k} z_k \,dx\,dt \Big|}^{K_1} + \overbrace{\Big| d_{k-1} \iint_Q \big(a(x) z_{k-1,x}\big)_x s^l\theta^l\xi e^{2s\Phi_k} z_k \,dx\,dt \Big|}^{K_2} \\
&+ \overbrace{\Big| \sum_{j=1}^{k-1} \iint_Q a_{jk-1} z_j s^l\theta^l\xi e^{2s\Phi_k} z_k \,dx\,dt \Big|}^{K_3} + \overbrace{\Big| \sum_{j=k-2}^{k-1} \iint_Q \frac{\lambda_{jk-1}}{b_{jk-1}} z_j s^l\theta^l\xi e^{2s\Phi_k} z_k \,dx\,dt \Big|}^{K_4}.
\end{aligned}$$

Using the same computations as in Lemma 2.3.1 or in [84, Lemma 3.7], one can prove that

$$\begin{aligned}
K_1 + K_2 + K_3 &\leq \frac{\varepsilon}{2} \sum_{j=k}^{k+1} \iint_Q \Big(s\theta a(x) z_{jx}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)} z_j^2 \Big) e^{2s\varphi} \,dx\,dt \\
&+ C_k' \Big(1 + \frac{1}{\varepsilon} \Big) \sum_{j=1}^{k-1} \iint_{Q_{\omega_0}} s^{l_j'}\theta^{l_j'} z_j^2 e^{2s\Phi_{k-1}} \,dx\,dt,
\end{aligned} \tag{3.4.49}$$

with $l_j = \max(3, 2l+1)$ and C_k' is a positive constant depending on k. In addition, the term K_4, can be estimated using the Young's inequality, in the following way

$$\begin{aligned}
K_4 &= \Big| \sum_{j=k-2}^{k-1} \iint_Q \frac{\lambda_{jk-1}}{b_{jk-1}} z_j s^l\theta^l\xi e^{2s\Phi_k} z_k \,dx\,dt \Big| \\
&\leq \sum_{j=k-2}^{k-1} \Big| \iint_Q \Big(\frac{\lambda_{jk-1}}{b_{jk-1}} (s\theta)^{l-\frac{3}{2}} \xi \frac{\sqrt{a}}{(x-x_0)} z_j e^{s(2\Phi_k-\varphi)} \Big) \times \Big((s\theta)^{\frac{3}{2}} \frac{(x-x_0)}{\sqrt{a}} z_k e^{s\varphi} \Big) \,dx\,dt \Big| \\
&\leq \frac{1}{2\varepsilon} \sum_{j=k-2}^{k-1} \iint_Q \frac{\lambda_{jk-1}^2}{b_{jk-1}^2} (s\theta)^{2l-3} \xi^2 \frac{a}{(x-x_0)^2} z_j^2 e^{2s(2\Phi_k-\varphi)} \,dx\,dt \\
&+ \frac{\varepsilon}{2} \iint_Q s^3\theta^3 \frac{(x-x_0)^2}{a} z_k^2 e^{2s\varphi} \,dx\,dt.
\end{aligned}$$

Moreover, since $\mathrm{supp}(\xi) \subset \omega_0$ and $x_0 \notin \overline{\omega_0}$, using the fact that the functions $\dfrac{1}{b_{jk}}$ and $\dfrac{a}{(x-x_0)^2}$ are bounded in $\bar{\omega}_0$, one has

$$K_4 \leq \frac{C}{\varepsilon} \sum_{j=k-2}^{k-1} \iint_{Q_{\omega_0}} (s\theta)^{2l-3} z_j^2 e^{2s(2\Phi_k-\varphi)} \,dx\,dt + \frac{\varepsilon}{2} \iint_Q s^3\theta^3 \frac{(x-x_0)^2}{a} z_k^2 e^{2s\varphi} \,dx\,dt. \tag{3.4.50}$$

On the other hand, thanks to Lemma 3.4.2, one can easily check that

$$e^{2s(2\Phi_k-\varphi)} \leq e^{2s\Phi_{k-1}}. \tag{3.4.51}$$

Adding (3.4.49) and (3.4.50), by (3.4.51), we can deduce

$$K_1 + K_2 + K_3 + K_4 \le \varepsilon \sum_{j=k}^{k+1} \iint_Q \Big(s\theta a(x) z_{jx}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)} z_j^2 \Big) e^{2s\varphi}\, dx\, dt$$
$$+ C_k\Big(1+\frac{1}{\varepsilon}\Big) \sum_{j=1}^{k-1} \iint_{Q_{\omega_0}} s^{l_j}\theta^{l_j} z_j^2 e^{2s\Phi_{k-1}}\, dx\, dt,$$

where $l_j = \max(3, 2l+1)$. Finally,

$$J_{\omega_0'}(l, \Phi_k, z_k) \le \varepsilon \sum_{j=k}^{k+1} \iint_Q \Big(s\theta a(x) z_{jx}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)} z_j^2 \Big) e^{2s\varphi}\, dx\, dt$$
$$+ C_k\Big(1+\frac{1}{\varepsilon}\Big) \sum_{j=1}^{k-1} J_{\omega_0}(l_j, \Phi_{k-1}, z_j).$$

This concludes the proof of Lemma 3.4.4. □

3.5 Observability and null controllability

In this section, we establish an indirect observability estimate using certain ideas from [40] and [113]. For this, on the coefficients a and b_{jk} we essentially start with the assumptions made so far, with the exception of Hypothesis 3.2.4. More precisely, we shall apply the just established Carleman inequalities to prove observability inequality for the homogeneous adjoint problem (3.1.6) and deduce the null controllability for the system (3.1.1). In particular, our main observability result is the following.

Theorem 3.5.1. *Under the assumptions of Theorem 3.4.3, there exists a positive constant C_T such that for every $(z_1^T, \cdots, z_n^T) \in \mathbb{H}$, the corresponding solution Z to (3.1.6) satisfies*

$$\|Z(0,\cdot)\|_{\mathbb{H}}^2 \le C_T \iint_{Q_\omega} |z_1(t,x)|^2\, dx\, dt. \tag{3.5.1}$$

In order to prove the previous theorem, we need to prove the following observability inequality in the case of a regular final-time datum.

Theorem 3.5.2. *Under the assumptions of Theorem 3.4.3, there exists a positive constant C_T such that for every $(z_1^T, \cdots, z_n^T) \in D(\mathbb{A}^2)$, the corresponding solution Z to (3.4.28) satisfies*

$$\|Z(0,\cdot)\|_{\mathbb{H}}^2 \le C_T \iint_{Q_\omega} |z_1(t,x)|^2\, dx\, dt. \tag{3.5.2}$$

Proof. Multiplying the k-th equation of the adjoint system (3.1.6) by z_{kt} and then integrating over $(0,1)$ with respect to x, one gets that

$$0 = \sum_{k=1}^n \int_0^1 z_{kt}^2\, dx + \sum_{k=1}^n \Big[d_k a(x) z_{kt} z_{kx} \Big]_{x=0}^{x=1} - \frac{1}{2}\frac{d}{dt} \sum_{k=1}^n \int_0^1 d_k a(x) z_{kx}^2\, dx$$
$$+ \sum_{k=1}^n \sum_{j=k-1}^{k+1} \int_0^1 \frac{\lambda_{jk}}{b_{jk}} z_j z_{kt}\, dx - \sum_{k=1}^n \sum_{j=1}^{k+1} \int_0^1 a_{jk} z_j z_{kt}\, dx. \tag{3.5.3}$$

On the other hand, using the fact that the matrix $\mathcal{B}$ is symmetric, we obtain

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{j=k-1}^{k+1}\int_0^1 \frac{\lambda_{jk}}{b_{jk}} z_j z_{kt}\,dx\\&=\frac{1}{2}\sum_{k=1}^{n}\frac{d}{dt}\int_0^1 \frac{\lambda_{kk}}{b_{kk}} z_k^2\,dx+\sum_{k=1}^{n}\frac{d}{dt}\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} z_k z_{k-1}\,dx.\end{aligned}$$

Furthermore, using the fact that $a_{jk}\in L^\infty(Q)$, by Young's inequality it follows that

$$\sum_{k=1}^{n}\sum_{j=1}^{k+1}\int_0^1 a_{jk}z_j z_{kt}\,dx\le C\sum_{k=1}^{n}\int_0^1 z_k^2\,dx+\sum_{k=1}^{n}\int_0^1 z_{kt}^2\,dx$$

for some positive constant C. Thus, (3.5.3) becomes

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}\sum_{k=1}^{n}\int_0^1 d_k a(x) z_{kx}^2\,dx+\frac{1}{2}\frac{d}{dt}\sum_{k=1}^{n}\int_0^1 \frac{\lambda_{kk}}{b_{kk}} z_k^2\,dx\\&\qquad+\frac{d}{dt}\sum_{k=1}^{n}\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} z_k z_{k-1}\,dx\le C\sum_{k=1}^{n}\int_0^1 z_k^2\,dx.\end{aligned}\tag{3.5.4}$$

Then, by the same technique used in (3.4.26), we have

$$\sum_{k=1}^{n}\int_0^1 z_k^2\,dx\le C\sum_{k=1}^{n}\int_0^1 a(x) z_{kx}^2\,dx.$$

Substituting the above inequality in (3.5.4), we obtain

$$\begin{aligned}&-\frac{d}{dt}\sum_{k=1}^{n}\Big(\int_0^1 d_k a(x) z_{kx}^2\,dx-\int_0^1 \frac{\lambda_{kk}}{b_{kk}} z_k^2\,dx-2\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} z_k z_{k-1}\,dx\Big)\\&\qquad\le C\sum_{k=1}^{n}\int_0^1 a(x) z_{kx}^2\,dx\end{aligned}\tag{3.5.5}$$

for a positive constant C.

At this stage, observe that from (3.3.9), one can find $C>0$ such that

$$\begin{aligned}-\langle \mathbb{A}Z,Z\rangle_{\mathbb{H}}&=\sum_{k=1}^{n}\Big(\int_0^1 d_k a z_{kx}^2\,dx-\int_0^1 \frac{\lambda_{kk}}{b_{kk}} z_k^2\,dx-2\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} z_k z_{k-1}\,dx\Big)\\&\ge C\sum_{k=1}^{n}\int_0^1 a z_{kx}^2\,dx.\end{aligned}\tag{3.5.6}$$

Combining (3.5.5) and (3.5.6), we obtain

$$\begin{aligned}&-\frac{d}{dt}\sum_{k=1}^{n}\Big(\int_0^1 d_k a(x) z_{kx}^2\,dx-\int_0^1 \frac{\lambda_{kk}}{b_{kk}} z_k^2\,dx-2\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} z_k z_{k-1}\,dx\Big)\\&\le \hat{C}\sum_{k=1}^{n}\Big(\int_0^1 d_k a y_{kx}^2\,dx-\int_0^1 \frac{\lambda_{kk}}{b_{kk}} y_k^2\,dx-2\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} y_k y_{k-1}\,dx\Big)\end{aligned}$$

for a positive constant $\hat{C}$. Hence

$$\frac{d}{dt}\Big\{e^{\hat{C}t}\sum_{k=1}^{n}\Big(\int_0^1 d_k a(x) z_{kx}^2\,dx-\int_0^1 \frac{\lambda_{kk}}{b_{kk}} z_k^2\,dx-2\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} z_k z_{k-1}\,dx\Big)\Big\}\ge 0.$$

Consequently, the function

$$t \longmapsto e^{\hat{C}t}\sum_{k=1}^{n}\Big(\int_0^1 d_k a(x) z_{kx}^2\,dx - \int_0^1 \frac{\lambda_{kk}}{b_{kk}} z_k^2\,dx - 2\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} z_k z_{k-1}\,dx\Big)$$

is increasing for all $t \in [0,T]$. Thus,

$$\begin{aligned}\sum_{k=1}^{n}\Big(&\int_0^1 d_k a(x) z_{kx}^2(0,x)\,dx - \int_0^1 \frac{\lambda_{kk}}{b_{kk}} z_k^2(0,x)\,dx\\ &- 2\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} z_k(0,x) z_{k-1}(0,x)\,dx\Big) \le e^{\hat{C}T}\sum_{k=1}^{n}\Big(\int_0^1 d_k a(x) z_{kx}^2(t,x)\,dx\\ &- \int_0^1 \frac{\lambda_{kk}}{b_{kk}} z_k^2(t,x)\,dx - 2\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} z_k(t,x) z_{k-1}(t,x)\,dx\Big).\end{aligned}$$

Next, using Young's inequality and applying the Hardy-Poincaré inequalities (3.2.1) and (3.2.2), it results:

$$\begin{aligned}\sum_{k=1}^{n}\Big(&\int_0^1 d_k a(x) z_{kx}^2(0,x)\,dx - \int_0^1 \frac{\lambda_{kk}}{b_{kk}} z_k^2(0,x)\,dx\\ &- 2\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} z_k(0,x) z_{k-1}(0,x)\,dx\Big)\\ \le e^{\hat{C}T}&\sum_{k=1}^{n}(d_k + \lambda_{kk}C^k + \lambda_{kk-1}C^{kk-1} + \lambda_{k+1k}C^{k+1k})\int_0^1 a(x) z_{kx}^2(t,x)\,dx.\end{aligned}$$

Integrating the previous inequality over $[\frac{T}{4}, \frac{3T}{4}]$, θ being bounded therein, we find

$$\begin{aligned}\sum_{k=1}^{n}\Big(&\int_0^1 d_k a(x) z_{kx}^2(0,x)\,dx - \int_0^1 \frac{\lambda_{kk}}{b_{kk}} z_k^2(0,x)\,dx\\ &- 2\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} z_k(0,x) z_{k-1}(0,x)\,dx\Big)\\ \le \frac{2}{T}e^{\hat{C}T}&\sum_{k=1}^{n}(d_k + \lambda_{kk}C^k + \lambda_{kk-1}C^{kk-1} + \lambda_{k+1k}C^{k+1k})\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 a(x) z_{kx}^2\,dx\,dt\\ \le C_T&\sum_{k=1}^{n}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 s\theta a(x) z_{kx}^2 e^{2s\varphi}\,dx\,dt.\end{aligned}$$

Now, using the Carleman estimate (3.4.45), we readily deduce

$$\begin{aligned}\sum_{k=1}^{n}\Big(&\int_0^1 d_k a(x) z_{kx}^2(0,x)\,dx - \int_0^1 \frac{\lambda_{kk}}{b_{kk}} z_k^2(0,x)\,dx\\ &- 2\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} z_k(0,x) z_{k-1}(0,x)\,dx\Big) \le C\int_0^T\int_\omega z_1^2\,dx\,dt,\end{aligned}$$

where C is a positive constant.

Then (3.5.6) implies that there exists a constant $C > 0$ such that

$$\begin{aligned}\sum_{k=1}^{n}\int_0^1 a(x) z_{kx}^2(0,x)\,dx &\le C\sum_{k=1}^{n}\Big(\int_0^1 d_k a(x) z_{kx}^2(0,x)\,dx\\ &- \int_0^1 \frac{\lambda_{kk}}{b_{kk}} z_k^2(0,x)\,dx - 2\int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} z_k(0,x) z_{k-1}(0,x)\,dx\Big)\\ &\le C\int_0^T\int_\omega z_1^2\,dx\,dt.\end{aligned}$$

Finally, we proceed as in (3.4.26), to obtain

$$\begin{aligned}\sum_{k=1}^{n}\int_0^1 z_k^2(0,x)\,dx &\leq C_0\max\{C_1,C_2\}C_{HP}\sum_{k=1}^{n}\int_0^1 a(x)z_{kx}^2(0,x)\,dx\\ &\leq C\int_0^T\int_\omega z_1^2\,dx\,dt,\end{aligned}$$

for a positive constant C. Hence, the conclusion follows. □

By Theorem 3.5.2 and using a density argument, as in [105, Proposition 4.1], one can prove Theorem 3.5.1. As an immediate consequence, we can prove, using a standard technique (e.g., see [127, Section 7.4]), the null controllability result for the linear degenerate/singular problem (3.1.1): if (3.5.1) holds, then for every $(y_1^0,\cdots,y_n^0)\in\mathbb{H}$, there exists a control $v\in L^2(Q)$ such that the solution of the parabolic system (3.1.1) satisfies

$$y_k(T,\cdot)=0,\quad \text{in } (0,1),\quad \forall k: 1\leq k\leq n.$$

Moreover, there exists $C_T>0$ such that

$$\|v\|_{L^2(Q)}^2\leq C_T\|Y^0\|_{\mathbb{H}}^2.$$

Chapter 4

Controllability of degenerate equation with memory

In this chapter, we analyze the null controllability property for a degenerate parabolic equation involving a memory term with a locally distributed control. We first derive a null controllability result for nonhomogeneous degenerate equation via new Carleman estimates with weighted time functions that do not blow up at $t = 0$. Then, the null controllability for the initial memory system is obtained using the Kakutani's fixed point theorem.

The results obtained in this chapter are presented in the research paper [9], in collaboration with Genni Fragnelli.

4.1 Introduction

In this chapter we are concerned with the null controllability result for a degenerate parabolic equation with memory by a distributed control force. More precisely, we consider the following controlled system:

$$\begin{cases} y_t - (a(x)y_x)_x = \displaystyle\int_0^t b(t,s,x)y(s,x)\,ds + 1_\omega u & (t,x) \in Q, \\ y(t,1) = 0, & t \in (0,T), \\ \begin{cases} y(t,0) = 0, & \text{(WD)}, \\ (ay_x)(t,0) = 0, & \text{(SD)}, \end{cases} & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1). \end{cases} \tag{4.1.1}$$

Here, $Q = (0,T) \times (0,1)$ and $\omega \Subset (0,1)$ is a non-empty open set, 1_ω is the corresponding characteristic function, $u = u(t,x)$ is the control function, $y = y(t,x)$ is the state and $b = b(t,s,x) \in L^\infty((0,T) \times Q)$ is a memory kernel. Moreover, the diffusion coefficient a vanishes at the boundary $x = 0$ (i.e., $a(0) = 0$) and satisfies the Hypotheses 1.2.3 and 1.2.4.

As mentioned in the introduction of this thesis, the null controllability of parabolic equations without memory (i.e. $b \equiv 0$) is by now well understood, for both uniformly and degenerate diffusion coefficient, by means of distributed and boundary controls (see [2, 5, 55, 104, 105, 106] and the references therein).

On the other hand, in the presence of memory terms, much less is known on the controllability of the underlying system.

When $a = b = 1$, S. Guerrero and O. Imanuvilov prove in [110] that (4.1.1) fails to be null controllable with a boundary control. Indeed, there exists a set of initial states that cannot be driven to 0 in any positive final time. Then, similar result is proved by X. Zhou and H. Gao in [170] whenever b is a non-trivial constant; in this paper it is also proved that the approximate controllability holds. Later on, these results are extended in [171] to the context of one dimensional degenerate parabolic equation. In particular, the authors assume that $a(x) = x^\alpha$, being $x \in (0,1)$, $0 \le \alpha < 1$ and prove that the null controllability of (4.1.1) fails whereas the approximate property holds in a suitable state space with a boundary control acting at the extremity $x = 0$ or $x = 1$.

Thus, it is important to see which kind of conditions on b we have to require so that the null controllability of (4.1.1) holds. In [125, 149] R. Lavanya, K. Balachandran and B.R. Nagaraj obtained the null controllability of a nonlinear and non degenerate version of (4.1.1) assuming that the memory kernel is sufficiently smooth and vanishes at the neighborhood of initial and final times. In particular,

$$b(t,s,x) \equiv b(t,s) \quad \text{and} \quad \operatorname{supp} b(\cdot,s) \Subset (t_0,t_1), \quad 0 < t_0 < t < t_1 < T, \quad \forall s \in (0,T). \tag{4.1.2}$$

The proof relies on Carleman estimates and a fixed point method. This assumption has been relaxed by Q. Tao and H. Gao in [153], where the authors showed that null controllability holds provided b fulfills

$$e^{\frac{C}{(T-t)}} b \in L^\infty((0,T)\times Q) \tag{4.1.3}$$

for some positive constant C.

For related results on this subject, we refer to[139] for wave equation, [25] for viscoelasticity equation, [142] for thermoelastic system and [167] in the case of heat equation with hyperbolic memory kernel (see also the bibliography therein).

The purpose of this work is to give a suitable condition on the memory kernel b in such a way that the degenerate parabolic equation with memory (4.1.1) is null controllable, that is there exists a control $u \in L^2(Q)$ such that the associated solution of (4.1.1), corresponding to the initial data $y_0 \in L^2(0,1)$, satisfies

$$y(T,\cdot) = 0 \qquad \text{in } (0,1).$$

We include here a brief description of the proof strategy: in a first step, we focus on the following nonhomogeneous degenerate parabolic system

$$\begin{cases} y_t - (a(x)y_x)_x = f + 1_\omega u & (t,x) \in Q, \\ y(t,1) = 0, & t \in (0,T), \\ \begin{cases} y(t,0) = 0, & \text{(WD)}, \\ (ay_x)(t,0) = 0, & \text{(SD)}, \end{cases} & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{4.1.4}$$

for a given function $f \in L^2(Q)$.

In particular, we establish suitable Carleman estimates for the associated adjoint problem using some classical weight time functions that blow up to $+\infty$ as $t \to 0^-, T^+$. Then, using a weight time function not exploding in the neighborhood of $t = 0$, we derive a new modified Carleman estimate that would allow us to show null controllability of the underlying parabolic equation. As a consequence, we deduce null controllability result for some problems similar to the degenerate parabolic equation with memory. Finally, this controllability result combined with an appropriate application of Kakutani's fixed point Theorem allows us to obtain the null controllability result for the original system (4.1.1) under a suitable condition on the kernel b.

Remark 30. We believe that the null controllability of system (4.1.1) can be obtained also following the same ideas in [149, 125]. More precisely, by means of classical duality arguments, the null controllability property can be reduced to an observability inequality for the adjoint parabolic problem

$$\begin{cases} -v_t - (a(x)v_x)_x = \int_t^T b(s,t,x)v(s,x)\,ds & (t,x) \in Q, \\ v(t,1) = 0, & t \in (0,T), \\ \begin{cases} v(t,0) = 0, & \text{(WD)}, \\ (av_x)(t,0) = 0, & \text{(SD)}, \end{cases} & t \in (0,T), \\ v(T,x) = v_T(x), & x \in (0,1), \end{cases} \tag{4.1.5}$$

where $v_T \in L^2(Q)$.

Such an inequality is proved by R. Lavanya and K. Balachandran in the aforementioned references through the use of a new Carleman estimate for (4.1.5) under a strict restriction on the memory kernel. Indeed, in order to treat the integral term in (4.1.5), the coefficient b need to be sufficiently smooth and to satisfy condition (4.1.2). One could expects the same condition for system (4.1.1).

However, in this work, we follow the methodology used in [153] for the treatment of nondegenerate equation which permits us to show that system (4.1.1) is null controllable provided the coefficient b satisfies only some exponential decay at the final time $t = T$ (see (4.5.2)).

The outline of this chapter is as follows: Section 4.2 is devoted to the well-posedness of systems (4.1.1) and (4.1.4) in suitable weighted spaces. In Section 4.3, we develop a new Carleman estimate for the adjoint problem to the nonhomogeneous parabolic equation (4.1.4) and, in Section 4.4, we apply such an estimate to deduce null controllability for (4.1.4). In Section 4.5, using the Kakutani's fixed point Theorem, we prove the null controllability result for the degenerate parabolic equation with memory (4.1.1) under suitable condition on the memory kernel. Finally, in Section 4.6, we discuss various extensions of our result.

4.2 Well-posedness results

The goal of this section is to study the well-posedness results for (4.1.1) and (4.1.4). First, we recall the following weighted Sobolev spaces:

In the (WD) case:

$$H_a^1(0,1) := \Big\{ y \in L^2(0,1) : y \text{ a.c. in } [0,1],\ \sqrt{a}y_x \in L^2(0,1) \text{ and } y(1) = y(0) = 0 \Big\}$$

and

$$H_a^2(0,1) := \Big\{ y \in H_a^1(0,1) : ay_x \in H^1(0,1) \Big\}.$$

In the (SD) case:

$$H_a^1(0,1) := \Big\{ y \in L^2(0,1) : y \text{ locally a.c. in } (0,1],\quad \sqrt{a}y_x \in L^2(0,1) \text{ and } y(1) = 0 \Big\}$$

and

$$\begin{aligned} H_a^2(0,1) : &= \Big\{ y \in H_a^1(0,1) : ay_x \in H^1(0,1) \Big\} \\ &= \Big\{ y \in L^2(0,1) : y \text{ locally a.c. in } (0,1], ay \in H_0^1(0,1), \\ &\qquad ay_x \in H^1(0,1) \text{ and } (ay_x)(0) = 0 \Big\}. \end{aligned}$$

In both cases, the norms are defined as follow

$$\|y\|^2_{H^1_a} := \|y\|^2_{L^2(0,1)} + \|\sqrt{a}y_x\|^2_{L^2(0,1)}, \qquad \|y\|^2_{H^2_a} := \|y\|^2_{H^1_a} + \|(ay_x)_x\|^2_{L^2(0,1)}.$$

We recall the following well-posedness result for system (4.1.4) (see, for instance, [5, 47]).

Proposition 4.2.1. *Assume that $y_0 \in L^2(0,1)$, $f \in L^2(Q)$ and $u \in L^2(Q)$. Then, system (4.1.4) admits a unique weak solution*

$$y \in W_T := L^2(0,T;H^1_a(0,1)) \cap C([0,T];L^2(0,1)) \tag{4.2.1}$$

such that

$$\|y\|_{L^2(0,T;H^1_a(0,1))} + \|y\|_{C([0,T];L^2(0,1))} \le C\Big(\|y_0\|_{L^2(0,1)} + \|f\|_{L^2(Q)} + \|1_\omega u\|_{L^2(Q)}\Big), \tag{4.2.2}$$

for some positive constant C. Moreover, if $y_0 \in H^1_a(0,1)$, then

$$y \in Z_T := L^2(0,T;H^2_a(0,1)) \cap H^1(0,T;L^2(0,1))$$

and

$$\|y\|_{L^2(0,T;H^2_a(0,1))} + \|y\|_{H^1(0,T;L^2(0,1))} \le C\Big(\|y_0\|_{H^1_a(0,1)} + \|f\|_{L^2(Q)} + \|1_\omega u\|_{L^2(Q)}\Big), \tag{4.2.3}$$

for some positive constant C.

Existence and uniqueness of solution for system (4.1.1) are established in the following result:

Proposition 4.2.2. *Assume that $y_0 \in L^2(0,1)$ and $u \in L^2(Q)$. Then, system (4.1.1) admits a unique solution $y \in W_T$.*

We emphasis that, in order to prove null controllability result for (4.1.1) (see Theorem 4.5.2), we only need existence and uniqueness in the case $y_0 \in L^2(0,1)$.

Proof. The proof of this Proposition is a consequence of [109, Theorem 1.1].

First of all, we transform (4.1.1) into the following Cauchy problem

$$\begin{cases} y'(t) + Ay(t) = \displaystyle\int_0^t k(t,s,y(s))\,ds + f(t), \quad t \in (0,T), \\ y(0) = y_0, \end{cases} \tag{4.2.4}$$

where

$$Ay := -(ay_x)_x, \quad y \in D(A) := H^2_a(0,1)$$

and

$$f(t) := 1_\omega u(t), \quad k(t,s,y(s)) := b(t,s,\cdot)y(s), \quad \text{for a.e. } (t,s) \in (0,T)^2.$$

Next, we are going to check that (4.2.4) satisfies the assumptions in the aforementioned Theorem. To this aim, let $H^{-1}_a(0,1)$ be the dual space of $H^1_a(0,1)$ with respect to the pivot space $L^2(0,1)$, endowed with the natural norm

$$\|z\|_{H^{-1}_a} := \sup_{\|y\|_{H^1_a}=1} \langle z, y\rangle_{H^{-1}_a, H^1_a}.$$

Observe that

$$\langle Ay, z\rangle_{H^{-1}_a,H^1_a} = \int_0^1 ay_x z_x\,dx, \quad \forall z \in H^1_a(0,1),$$

$$\langle k(t,s,y), z\rangle_{H^{-1}_a,H^1_a} = \int_0^1 b(t,s,x)yz\,dx, \quad \text{for a.e. } (t,s) \in (0,T)^2, \quad \forall z \in H^1_a(0,1),$$

for any $y \in H^1_a(0,1)$.

Hence, one can check easily that the operators A and k satisfy the following properties:

(a) there exists a positive constant C such that $\|Ay\|_{H_a^{-1}} \leq C\|y\|_{H_a^1}, \quad \forall y \in H_a^1(0,1)$;

(b) there exists a positive constant C such that

$$\|Ay_1 - Ay_2\|_{H_a^{-1}} \leq C\|y_1 - y_2\|_{H_a^1}, \quad \text{for any } y_1, y_2 \in H_a^1(0,1);$$

(c) $\exists\, \gamma > 0$ and $\lambda > 0$ such that

$$\langle Ay_1 - Ay_2, y_1 - y_2 \rangle_{H_a^{-1}, H_a^1} + \lambda\|y_1 - y_2\|^2_{L^2(0,1)} \geq \gamma\|y_1 - y_2\|^2_{H_a^1},$$

for any $y_1, y_2 \in H_a^1(0,1)$;

(d) there exists a function $\beta : (0,T)^2 \mapsto \mathbb{R}^+$ such that

$$\|(k(t,s,y_1) - k(t,s,y_2))\|_{H_a^{-1}} \leq \beta(t,s)\|y_1 - y_2\|_{H_a^1}, \quad \text{for a.e. } (t,s) \in (0,T)^2,$$

for any $y_1, y_2 \in H_a^1(0,1)$.

Besides β is explicitly given by

$$\beta(t,s) := \|b(t,s,\cdot)\|_{L^\infty(0,1)}, \quad \text{for a.e. } (t,s) \in (0,T)^2.$$

Then, taking into account the fact that $b \in L^\infty((0,T) \times Q)$, $f \in L^2(Q)$ and in view of [109, Remark 1.2, 1.3], we infer that all the assumptions of [109, Theorem 1.1] are fulfilled. Consequently, the problem (4.2.4) has a unique solution

$$y \in L^2(0,T; H_a^1(0,1)) \cap L^\infty(0,T; L^2(0,1))$$

with $\quad y_t \in L^2(0,T; H_a^{-1}(0,1))$.

Moreover, by [134, Theorem 3.1, Chapter 1] we also have

$$y \in C([0,T]; L^2(0,1)).$$

Thus (4.2.1) is proved.

□

4.3 Carleman estimates

The goal of this section is to establish appropriate Carleman estimates for the following adjoint parabolic system

$$\begin{cases} -v_t - (a(x)v_x)_x = g & (t,x) \in Q, \\ v(t,1) = 0, & t \in (0,T), \\ \begin{cases} v(t,0) = 0, & \text{(WD)}, \\ (av_x)(t,0) = 0, & \text{(SD)}, \end{cases} & t \in (0,T), \\ v(T,x) = v_T(x), & x \in (0,1), \end{cases} \tag{4.3.1}$$

where $v_T \in L^2(0,1)$ and $g \in L^2(Q)$.

To our purpose, as in [5] (see also Chapter 1 in this thesis), we introduce the following weight functions

$$\begin{aligned} &\psi(x) := \gamma\Big(\int_0^x \frac{y}{a(y)}\,dy - d\Big), \quad \theta(t) := \frac{1}{\big[t(T-t)\big]^4}, \\ &\varphi(t,x) := \theta(t)\psi(x), \end{aligned} \tag{4.3.2}$$

Now, let $\tilde{\omega}$ be an arbitrary open subset of ω and $\rho \in C^2([0,1])$ be such that

$$\rho > 0, \text{ in } (0,1), \quad \rho(0) = \rho(1) = 0 \quad \text{and} \quad \rho_x \neq 0, \text{ in } [0,1]\backslash\tilde{\omega}$$

and define

$$\Psi(x) := e^{\lambda\rho(x)} - e^{2\lambda\|\rho\|_\infty}, \quad \Phi(t,x) := \theta(t)\Psi(x). \tag{4.3.3}$$

As in [2], the parameters λ, d and γ are positive constant satisfy

$$d > d^\star := \int_0^1 \frac{y}{a(y)}\, dy, \qquad \gamma > \frac{e^{2\lambda\|\rho\|_\infty}}{(d-d^*)} \tag{4.3.4}$$

and to be specified later on. It clearly follows from (4.3.4) that

$$-\gamma d \le \psi(x) < 0, \quad \text{for all } x \in [0,1], \tag{4.3.5}$$

$$\psi(x) \le \Psi(x), \text{ for all } x \in [0,1], \ \varphi(t,x) \le \Phi(t,x), \text{ for all } (t,x) \in Q \tag{4.3.6}$$

Moreover, we readily have from the definition of the function θ that

$$|\theta'(t)| \le C\theta^{\frac{3}{2}}(t), \ \forall\, t \in [0,T], \text{ and } \quad \theta(t) \to +\infty, \quad \text{as} \quad t \to 0^-, T^+. \tag{4.3.7}$$

We also remind the following Carleman estimate:

Theorem 4.3.1. *[12, Theorem 3.3] Let $T > 0$. There exist two positive constants C and s_0, such that the solution $v \in Z_T$ of (4.3.1) satisfies*

$$\begin{aligned}\iint_Q \Big(s\theta a(x)v_x^2 + s^3\theta^3\frac{x^2}{a(x)}v^2\Big)e^{2s\varphi}\, dx\, dt \\ \le C\Big(\iint_Q g^2 e^{2s\Phi}\, dx\, dt + \iint_{Q_\omega} s^3\theta^3 v^2 e^{2s\Phi}\, dxdt\Big)\end{aligned} \tag{4.3.8}$$

for all $s \ge s_0$. Here $Q_\omega = (0,T)\times\omega$.

Theorem 4.3.1 could be used to prove null controllability for (4.1.1) under the following hypothesis on the memory kernel b:

$$e^{\frac{C^*}{(T-t)^4}} b \in L^\infty((0,T)\times Q) \tag{4.3.9}$$

for some constant $C^* > 0$. However, we emphasize that, our objective is to provide null controllability for the memory equation (4.1.1) for more general memory kernel b. In this purpose, as a first step, we are going to extend the Carleman inequality proved in the previous Theorem in the following way.

Theorem 4.3.2. *Let $k \ge 0$. Then, there exist two positive constants C and s_0, such that the solution $v \in Z_T$ of (4.3.1) satisfies,*

$$\begin{aligned}\iint_Q \Big((s\theta)^{1+k}a(x)v_x^2 + (s\theta)^{(3+k)}\frac{x^2}{a(x)}v^2\Big)e^{2s\varphi}\, dx\, dt \\ \le C\Big(\iint_Q (s\theta)^k g^2 e^{2s\Phi}\, dx\, dt + \iint_{Q_\omega} (s\theta)^{k+3} v^2 e^{2s\Phi}\, dxdt\Big)\end{aligned} \tag{4.3.10}$$

for all $s \ge s_0$.

Proof. Let $\omega_2 = (x_1, x_2)$ and ω_1 be two arbitrary subintervals of ω such that $\omega_2 \Subset \omega_1$ and consider a smooth cut-off function $\chi \in C^\infty([0,1])$ such that

$$0 \le \chi \le 1, \qquad \chi(x) := \begin{cases} 1, & \text{for } x \in [0, x_1], \\ 0, & \text{for } x \in [x_2, 1]. \end{cases}$$

Then, thanks to [84, Proposition 3.4], the solution of (4.3.1) satisfies

$$\begin{aligned}&\iint_Q \Big((s\theta)^{1+k}\chi^2 a(x)v_x^2 + (s\theta)^{(3+k)}\chi^2\frac{x^2}{a(x)}v^2\Big)e^{2s\varphi}\,dx\,dt\\ &\quad\le C\Big(\iint_Q (s\theta)^k\chi^2 g^2 e^{2s\varphi}\,dx\,dt + \iint_{Q_{\omega_1}}(s\theta)^k\Big(g^2+(s\theta)^2v^2\Big)e^{2s\varphi}\,dxdt\Big).\end{aligned} \tag{4.3.11}$$

On the other hand, let $\zeta := 1-\chi$, it follows from [84, Proposition 3.5] that

$$\begin{aligned}&\iint_Q \Big((s\theta)^{1+k}\zeta^2 a(x)v_x^2 + (s\theta)^{(3+k)}\zeta^2\frac{x^2}{a(x)}v^2\Big)e^{2s\Phi}\,dx\,dt\\ &\quad\le C\Big(\iint_Q (s\theta)^k\zeta^2 g^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_{\omega_1}}(s\theta)^{k+3}v^2 e^{2s\Phi}\,dxdt\Big).\end{aligned} \tag{4.3.12}$$

Therefore, using (4.3.6), (4.3.11), (4.3.12) and the fact that $\frac{1}{2}\le \chi^2+\zeta^2\le 1$ there holds

$$\begin{aligned}&\iint_Q \Big((s\theta)^{1+k}a(x)v_x^2 + (s\theta)^{(3+k)}\frac{x^2}{a(x)}v^2\Big)e^{2s\varphi}\,dx\,dt\\ &\quad\le C\Big(\iint_Q (s\theta)^k g^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_{\omega}}(s\theta)^{k+3}v^2 e^{2s\Phi}\,dxdt\Big)\end{aligned}$$

which concludes Theorem 4.3.2. □

Next, by (4.3.10), we are going to derive a new modified Carleman inequality, that is an estimate with a weight time function exploding only at the final time $t=T$. This choice is done recalling the technique developed by A.V. Fursikov and O.Y. Imanuvilov in [106] in the context of uniformly parabolic equations. In our setting, this new weight allows us to derive a null controllability result for system (4.1.1) imposing a restriction on the kernel b only at the final time $t=T$ (see (4.5.2)). To this end, let us introduce the following weight functions:

$$\beta(t) := \begin{cases}\theta(\frac{T}{2}) = (\frac{2}{T})^8, & \text{for } t\in[0,\frac{T}{2}],\\ \theta(t), & \text{for } t\in[\frac{T}{2},T],\end{cases}\quad \tilde{\varphi}(t,x)=\beta(t)\psi(x),\quad \sigma(t,x):=\beta(t)\Psi(x)$$

and

$$\begin{aligned}\widehat{\varphi}(t) &:= \max_{x\in[0,1]}\varphi(t,x) = \gamma(d^*-d)\beta(t),\\ \varphi^*(t) &:= \min_{x\in[0,1]}\varphi(t,x) = -\gamma d\beta(t).\end{aligned} \tag{4.3.13}$$

In view of (4.3.6), we can see that the weight functions $\tilde{\varphi}$ and σ satisfy the following inequality which is needed in what follows

$$\tilde{\varphi}(t,x)\le\tilde{\Phi}(t,x),\quad \forall (t,x)\in Q. \tag{4.3.14}$$

Now, we are ready to state the following modified Carleman estimate, which reveals to be a major tool to obtain the null controllability result given in Theorem 4.5.2.

Lemma 4.3.1. *Let $k\ge 0$. Then, there exists two positive constants C and s_0 such that every solution $v\in Z_T$ of system (4.3.1) satisfies*

$$\begin{aligned}&s^k e^{2s\widehat{\varphi}(0)}\|v(0)\|^2_{L^2(0,1)} + \iint_Q (s\beta)^k v^2 e^{2s\tilde{\varphi}}\,dx\,dt\\ &\quad\le Ce^{2s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]}\Big(\iint_Q (s\beta)^k g^2 e^{2s\tilde{\Phi}}\,dx\,dt + \iint_{Q_\omega}(s\beta)^{k+3}v^2e^{2s\tilde{\Phi}}\,dxdt\Big)\end{aligned} \tag{4.3.15}$$

for all $s\ge s_0$.

Proof. Let $\xi \in C^\infty([0,T])$ be a cut-off function such that

$$0 \le \xi \le 1, \qquad \xi(t) := \begin{cases} 1, & \text{for } t \in \left[0, \dfrac{T}{2}\right], \\ 0, & \text{for } t \in \left[\dfrac{5T}{8}, T\right] \end{cases} \tag{4.3.16}$$

and define $w = \tilde{\xi} v$, where $\tilde{\xi} = \beta^{\frac{k}{2}} \xi e^{s\widehat{\varphi}(0)}$ and v solves (4.3.1).

Hence w satisfies

$$\begin{cases} -w_t - (a(x)w_x)_x = -\tilde{\xi}' v + \tilde{\xi} g, & (t,x) \in Q, \\ w(t,1) = 0, & t \in (0,T), \\ \begin{cases} w(t,0) = 0, & (WD), \\ (aw_x)(t,0) = 0, & (SD), \end{cases} & t \in (0,T), \\ w(T,x) = 0, & x \in (0,1). \end{cases} \tag{4.3.17}$$

Then, by (4.2.2) applied to the above system, one can see that

$$\|w(0)\|^2_{L^2(0,1)} + \|w\|^2_{L^2(Q)} \le C \iint_Q (-\tilde{\xi}' v + \tilde{\xi} g)^2 \, dx \, dt \tag{4.3.18}$$

for some constants $C > 0$.

We estimate from below the two terms on the left hand side of (4.3.18) in the following way:

$$\|w(0)\|^2_{L^2(0,1)} = \|\beta^{\frac{k}{2}}(0)\xi(0) e^{s\widehat{\varphi}(0)} v(0)\|^2_{L^2(0,1)} = \left(\frac{2}{T}\right)^{8k} \|e^{s\widehat{\varphi}(0)} v(0)\|^2_{L^2(0,1)} \tag{4.3.19}$$

and

$$\|w\|^2_{L^2(Q)} = \int_0^{\frac{5T}{8}} \int_0^1 \beta^k \xi^2 e^{2s\widehat{\varphi}(0)} v^2 \, dx \, dt \ge \int_0^{\frac{T}{2}} \int_0^1 \beta^k v^2 e^{2s\tilde{\varphi}} \, dx \, dt \tag{4.3.20}$$

since $\tilde{\varphi} \le \widehat{\varphi}(0)$ in Q.

Concerning the right hand side of (4.3.18), we have

$$\begin{aligned} &\iint_Q (-\tilde{\xi}' v + \tilde{\xi} g)^2 \, dx \, dt \\ &= \iint_Q \left[\left(-\frac{k}{2}\beta' \beta^{\frac{k}{2}-1} \xi - \beta^{\frac{k}{2}} \xi'\right) e^{s\widehat{\varphi}(0)} v + \beta^{\frac{k}{2}} \xi e^{s\widehat{\varphi}(0)} g\right]^2 dx \, dt \\ &\le C\bigg(\iint_Q (\beta')^2 \beta^{k-2} \xi^2 e^{2s\widehat{\varphi}(0)} v^2 \, dx \, dt \\ &\quad + \iint_Q \beta^k (\xi')^2 e^{2s\widehat{\varphi}(0)} v^2 \, dx \, dt + \iint_Q \beta^k \xi^2 e^{2s\widehat{\varphi}(0)} g^2 \, dx \, dt \bigg). \end{aligned} \tag{4.3.21}$$

Observing that $\beta' = 0$ in $[0, T/2]$, $\beta = \theta$ in $[T/2, T]$ and using (4.3.7), the fact that $\operatorname{supp} \xi \subset [0, 5T/8]$ and $\operatorname{supp} \xi' \subset [T/2, 5T/8]$, it follows that

$$e^{2s\widehat{\varphi}(0)} \iint_Q (\beta')^2 \beta^{k-2} \xi^2 v^2 \, dx \, dt \le C e^{2s\widehat{\varphi}(0)} \int_{\frac{T}{2}}^{\frac{5T}{8}} \int_0^1 \beta^{k+1} v^2 \, dx \, dt \tag{4.3.22}$$

and

$$\begin{aligned} e^{2s\widehat{\varphi}(0)} \iint_Q \beta^k (\xi')^2 v^2 \, dx \, dt &\le C e^{2s\widehat{\varphi}(0)} \int_{\frac{T}{2}}^{\frac{5T}{8}} \int_0^1 \beta^k v^2 \, dx \, dt \\ &\le C e^{2s\widehat{\varphi}(0)} \int_{\frac{T}{2}}^{\frac{5T}{8}} \int_0^1 \beta^{k+1} v^2 \, dx \, dt. \end{aligned} \tag{4.3.23}$$

Hence, by the estimates (4.3.18)-(4.3.23), we find that

$$s^k\|e^{s\widehat{\varphi}(0)}v(0)\|^2_{L^2(0,1)} + \int_0^{\frac{T}{2}}\int_0^1 (s\beta)^k v^2 e^{2s\tilde{\varphi}}\,dx\,dt \\ \leq C\Big(s^k\int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 s^k\beta^{k+1}v^2e^{2s\widehat{\varphi}(0)}\,dx\,dt + \int_0^{\frac{5T}{8}}\int_0^1 (s\beta)^k e^{2s\widehat{\varphi}(0)}g^2\,dx\,dt\Big). \tag{4.3.24}$$

Now, let us deal with the first term in the right-hand side of (4.3.24).

First, using the fact that $\beta = \theta$ and $\tilde{\varphi} = \varphi$ in $[T/2, T]$, one has

$$\int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 s^k\beta^{k+1}v^2e^{2s\tilde{\varphi}}\,dx\,dt = \int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 s^k\theta^{k+1}v^2e^{2s\varphi}\,dx\,dt. \tag{4.3.25}$$

Then, applying Young's inequality as in [5], we see that

$$\int_0^1 v^2e^{2s\varphi}\,dx = \int_0^1 \left(\left(\frac{a(x)}{x^2}\right)^{\frac{1}{3}}v^2e^{2s\varphi}\right)^{\frac{3}{4}}\left(\frac{x^2}{a(x)}v^2e^{2s\varphi}\right)^{\frac{1}{4}}dx \\ \leq \frac{3}{4}\int_0^1\left(\frac{a(x)}{x^2}\right)^{\frac{1}{3}}v^2e^{2s\varphi}\,dx + \frac{1}{4}\int_0^1\frac{x^2}{a(x)}v^2e^{2s\varphi}\,dx. \tag{4.3.26}$$

Let $p(x) = x^{4/3}a^{1/3}$, then since the function $x \mapsto \dfrac{x^2}{a}$ is nondecreasing on $(0,1)$ one has,

$$p(x) = a\Big(\frac{x^2}{a}\Big)^{\frac{2}{3}} \leq Ca(x).$$

Then, applying the Hardy-Poincaré inequality (1.2.9) to $ve^{s\varphi}$, we get

$$\int_0^1 \frac{a^{1/3}}{x^{2/3}}(ve^{s\varphi})^2\,dx = \int_0^1\frac{p(x)}{x^2}(ve^{s\varphi})^2\,dx \\ \leq C\int_0^1 p(x)(ve^{s\varphi})_x^2\,dx \leq C\int_0^1 a(x)(ve^{s\varphi})_x^2\,dx. \tag{4.3.27}$$

Using the definition of φ (see (4.3.2)), it follows that

$$\int_0^1 \frac{a^{1/3}}{x^{2/3}}(ve^{s\varphi})^2\,dx \leq C\int_0^1 a(x)\big(v_x + s\varphi_x v\big)^2e^{2s\varphi}\,dx \\ \leq C\int_0^1\Big(a(x)v_x^2 + s^2\theta^2\frac{x^2}{a(x)}v^2\Big)e^{2s\varphi}\,dx. \tag{4.3.28}$$

By (4.3.26) and (4.3.28), we obtain

$$\int_0^1 v^2e^{2s\varphi}\,dx \leq C\int_0^1\Big(a(x)v_x^2 + s^2\theta^2\frac{x^2}{a(x)}v^2\Big)e^{2s\varphi}\,dx. \tag{4.3.29}$$

Hence, from (4.3.25) and (4.3.29), we get that

$$\int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 s^k\beta^{k+1}v^2e^{2s\tilde{\varphi}}\,dx\,dt \\ \leq C\int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 s^k\theta^{k+1}\Big(a(x)v_x^2 + s^2\theta^2\frac{x^2}{a(x)}v^2\Big)e^{2s\varphi}\,dxdt.$$

Thus, applying Carleman inequality (4.3.10), one has

$$
\begin{aligned}
s^k \int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 \beta^{k+1} v^2 e^{2s\tilde{\varphi}}\,dx\,dt &\le C \int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 \Big(s^k\theta^{k+1}a(x)v_x^2 + s^{k+2}\theta^{k+3}\frac{x^2}{a(x)}v^2\Big)e^{2s\varphi}\,dxdt \\
&\le C \iint_Q \Big((s\theta)^{k+1}a(x)v_x^2 + (s\theta)^{k+3}\frac{x^2}{a(x)}v^2\Big)e^{2s\varphi}\,dxdt \\
&\le C\Big(\iint_Q (s\theta)^k g^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_\omega}(s\theta)^{k+3}v^2e^{2s\Phi}\,dxdt\Big). \qquad (4.3.30)
\end{aligned}
$$

Now observe that

$$
\varphi^*\big(\frac{5T}{8}\big) \le \tilde{\varphi}, \quad \text{in } \Big[0, \frac{5T}{8}\Big] \times \big[0, 1\big]. \qquad (4.3.31)
$$

Therefore

$$
\begin{aligned}
& s^k \int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 \beta^{k+1} v^2 e^{2s\widehat{\varphi}(0)}\,dx\,dt = s^k \int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 \beta^{k+1} v^2 e^{2s[\widehat{\varphi}(0)-\tilde{\varphi}]}e^{2s\tilde{\varphi}}\,dx\,dt \\
& \le e^{2s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]} s^k \int_{\frac{T}{2}}^{\frac{5T}{8}}\int_0^1 \beta^{k+1} v^2 e^{2s\tilde{\varphi}}\,dx\,dt \\
& \le Ce^{2s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]}\Big(\iint_Q (s\theta)^k g^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_\omega}(s\theta)^{k+3}v^2e^{2s\Phi}\,dxdt\Big) \qquad (4.3.32)
\end{aligned}
$$

for s large enough.

Moreover, in view of (4.3.14) and (4.3.31), the second term in the right hand side of (4.3.24) reads as

$$
\begin{aligned}
\int_0^{\frac{5T}{8}}\int_0^1 (s\beta)^k e^{2s\widehat{\varphi}(0)} g^2\,dx\,dt &= \int_0^{\frac{5T}{8}}\int_0^1 (s\beta)^k e^{2s[\widehat{\varphi}(0)-\tilde{\varphi}]}e^{2s\tilde{\varphi}} g^2\,dx\,dt \\
&\le e^{2s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]}\int_0^{\frac{5T}{8}}\int_0^1 (s\beta)^k e^{2s\tilde{\Phi}} g^2\,dx\,dt.
\end{aligned}
$$

Combining this last inequality with (4.3.24) and (4.3.32), it follows that

$$
\begin{aligned}
& s^k\|e^{s\widehat{\varphi}(0)}v(0)\|^2_{L^2(0,1)} + \int_0^{\frac{T}{2}}\int_0^1 (s\beta)^k v^2 e^{2s\tilde{\varphi}}\,dx\,dt \\
&\quad \le Ce^{2s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]}\Big(\iint_Q (s\theta)^k g^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_\omega}(s\theta)^{k+3}v^2e^{2s\Phi}\,dxdt \\
&\qquad + \int_0^{\frac{5T}{8}}\int_0^1 (s\beta)^k g^2 e^{2s\tilde{\Phi}}\,dx\,dt\Big). \qquad (4.3.33)
\end{aligned}
$$

On the other hand, proceeding as in (4.3.30), we also obtain

$$
\begin{aligned}
\int_{\frac{T}{2}}^{T}\int_0^1 (s\beta)^k v^2 e^{2s\tilde{\varphi}}\,dx\,dt &= \int_{\frac{T}{2}}^{T}\int_0^1 (s\beta)^k v^2 e^{2s\varphi}\,dx\,dt \\
&\le C\int_{\frac{T}{2}}^{T}\int_0^1 \Big((s\theta)^k a(x)v_x^2 + (s\theta)^{k+2}\frac{x^2}{a(x)}v^2\Big)e^{2s\varphi}\,dxdt \\
&\le C\Big(\iint_Q (s\theta)^k g^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_\omega}(s\theta)^{k+3}v^2e^{2s\Phi}\,dxdt\Big). \qquad (4.3.34)
\end{aligned}
$$

Note that, since the function $s \to s^k e^{cs}$, with $k \ge 0$ and $c < 0$, is nonincreasing for larger values of s, then, from the fact that $\beta \le \theta$ in $(0, T)$ we get that,

$$
(s\theta)^k e^{2s\Phi} = (s\theta)^k e^{2s\Psi(x)\theta(t)} \le (s\beta)^k e^{2s\Psi(x)\beta(t)} = (s\beta)^k e^{2s\tilde{\Phi}}, \quad \text{in } Q,
$$

for s large enough, where we recall that Ψ is the weight function given in (4.3.3).

Finally, combining this fact with the estimates (4.3.33) and (4.3.34), we deduce that

$$s^k\|e^{s\widehat{\varphi}(0)}v(0)\|^2_{L^2(0,1)}+\iint_Q(s\beta)^k v^2 e^{2s\tilde{\varphi}}\,dx\,dt$$
$$\leq Ce^{2s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]}\Big(\iint_Q(s\beta)^k g^2 e^{2s\tilde{\Phi}}\,dx\,dt+\iint_{Q_\omega}(s\beta)^{k+3}v^2e^{2s\tilde{\Phi}}\,dxdt\Big).$$

This ends the proof of Lemma 4.3.1. □

4.4 Null controllability for a nonhomogeneous system

In this section we will apply the Carleman estimates established in Section 4.3 to deduce the null controllability result for the nonhomogeneous problem (4.1.4). To this aim, following the arguments presented in [106, 153], we introduce, for all $k \geq 0$ and $s \geq s_0$, the following weighted space

$$E_{s,k}=\left\{y\in Z_T:\quad (s\beta)^{-k/2}e^{-s\tilde{\Phi}}y\in L^2(Q)\right\}$$

endowed with the associated norm

$$\|y\|^2_{E_{s,k}}:=\iint_Q(s\beta)^{-k}e^{-2s\tilde{\Phi}}y^2\,dxdt.$$

The parameter s_0 is defined as in Lemma 4.3.1.

Observe that, if we consider y in $E_{s,k}$, then y is continuous in time and satisfies

$$\iint_Q(s\beta)^{-k}e^{-2s\tilde{\Phi}}y^2\,dxdt<+\infty,$$

thus, from the definition of $\tilde{\Phi}$, in particular the fact that $\tilde{\Phi}<0$, we have that

$$y(T,\cdot)=0\quad\text{in }(0,1).$$

Then, we are going to prove the following:

Theorem 4.4.1. *Let $T>0$ and $k\geq 0$. Assume $(s\beta)^{-k/2}e^{-s\tilde{\varphi}}f\in L^2(Q)$ with $s\geq s_0$. Then, for any $y_0\in H^1_a(0,1)$, there exists $u\in L^2(Q)$ such that the associated solution y of system (4.1.4) belongs to $E_{s,k}$.*

Moreover, there exists a positive constant C such that the couple (y,u) satisfies

$$\iint_Q(s\beta)^{-k}e^{-2s\tilde{\Phi}}y^2\,dx\,dt+\iint_{Q_\omega}(s\beta)^{-(k+3)}e^{-2s\tilde{\Phi}}u^2\,dxdt$$
$$\leq Ce^{2s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]}\Big(\iint_Q(s\beta)^{-k}e^{-2s\tilde{\varphi}}f^2\,dx\,dt+s^{-k}\int_0^1e^{-2s\widehat{\varphi}(0)}y_0^2\,dx\Big).\tag{4.4.1}$$

Proof. The proof of this Theorem is inspired by [106, 153]. First of all, consider the following functional:

$$J(y,u)=\iint_Q(s\beta)^{-k}e^{-2s\tilde{\Phi}}y^2\,dx\,dt+\iint_{Q_\omega}(s\beta)^{-(k+3)}e^{-2s\tilde{\Phi}}u^2\,dxdt$$

where (y,u) satisfies system (4.1.4) with $u\in L^2(Q)$ and

$$y(T,\cdot)=0\qquad\text{in }(0,1).\tag{4.4.2}$$

By classical arguments (see for instance [135, 133]), one can show that J attains its minimizer at a unique point say, $(\tilde{y}, \tilde{u})$.

We are going to prove the existence of a dual variable $\tilde{z}$ such that

$$\begin{cases} \tilde{y} = (s\beta)^k e^{2s\tilde{\Phi}} \mathcal{L}^* \tilde{z} & \text{in } Q \\ \tilde{u} = -1_\omega (s\beta)^{k+3} e^{2s\tilde{\Phi}} \tilde{z} & \text{in } Q \end{cases}$$

where $\mathcal{L}^* \tilde{z} = -\tilde{z}_t - (a(x)\tilde{z}_x)_x$ and $\tilde{z}$ satisfies the boundary conditions

$$\tilde{z}(\cdot, 1) = 0 \quad \text{and} \begin{cases} \tilde{z}(\cdot, 0) = 0, & \text{(WD)} \\ (a\tilde{z}_x)(\cdot, 0) = 0, & \text{(SD)} \end{cases} \quad \text{on } (0, T). \tag{4.4.3}$$

Let us define the following linear space

$$X_a = \Big\{ w \in C^\infty(\overline{Q}) : \quad w \quad \text{satisfies (4.4.3)} \Big\}.$$

In addition, we set

$$\kappa(z, w) = \iint_Q (s\beta)^k e^{2s\tilde{\Phi}} \mathcal{L}^* z \mathcal{L}^* w \, dx \, dt + \iint_{Q_\omega} (s\beta)^{k+3} e^{2s\tilde{\Phi}} zw \, dxdt, \qquad \forall\, z, w \in X_a \tag{4.4.4}$$

and

$$\ell(w) = \iint_Q f w \, dx \, dt + \int_0^1 y_0 w(0) dx, \qquad \forall\, w \in X_a, \tag{4.4.5}$$

where f, y_0 are the functions in (4.1.4).

Observe that Carleman estimate (4.3.15) holds for all $w \in X_a$. In particular, we have

$$s^k \int_0^1 e^{2s\widehat{\varphi}(0)} w(0)^2 \, dx + \iint_Q (s\beta)^k e^{2s\tilde{\varphi}} w^2 \, dx \, dt \leq C e^{2s[\widehat{\varphi}(0) - \varphi^*(\frac{5T}{8})]} \kappa(w, w), \qquad \forall\, w \in X_a.$$

Now, let us denote by $\widetilde{X}_a$ the completion of X_a with the norm $\|w\|_{\widetilde{X}_a} = (\kappa(w, w))^{1/2}$. Thus, $\widetilde{X}_a$ is a Hilbert space with this norm.

Clearly, κ is a strictly positive, symmetric and continuous bilinear form in $\widetilde{X}_a$.

Moreover, in view of the above inequality, one can see that the linear form ℓ is continuous in $\widetilde{X}_a$. Indeed, employing the Cauchy-Schwarz inequality, for all $w \in \widetilde{X}_a$, we have

$$\begin{aligned}
|\ell(w)| &= \iint_Q f w \, dx \, dt + \int_0^1 y_0 w(0) dt \\
&\leq \Bigg(\Big(\iint_Q (s\beta)^{-k} e^{-2s\tilde{\varphi}} f^2 \, dx \, dt \Big)^{1/2} \Big(\iint_Q (s\beta)^k e^{2s\tilde{\varphi}} w^2 \, dx \, dt \Big)^{1/2} \\
&\qquad + \Big(s^{-k} \int_0^1 e^{-2s\widehat{\varphi}(0)} y_0^2 \, dx \Big)^{1/2} \Big(s^k \int_0^1 e^{2s\widehat{\varphi}(0)} w(0)^2 \, dx \Big)^{1/2} \Bigg) \\
&\leq \Bigg(\Bigg[\Big(\iint_Q (s\beta)^{-k} e^{-2s\tilde{\varphi}} f^2 \, dx \, dt \Big)^{1/2} + \Big(s^{-k} \int_0^1 e^{-2s\widehat{\varphi}(0)} y_0^2 \, dx \Big)^{1/2} \Bigg] \times \\
&\qquad \Bigg[\Big(\iint_Q (s\beta)^k e^{2s\tilde{\varphi}} w^2 \, dx \, dt \Big)^{1/2} + \Big(s^k \int_0^1 e^{2s\widehat{\varphi}(0)} w(0)^2 \, dx \Big)^{1/2} \Bigg] \Bigg) \\
&\leq C e^{s[\widehat{\varphi}(0) - \varphi^*(\frac{5T}{8})]} \Bigg[\Big(\iint_Q (s\beta)^{-k} e^{-2s\tilde{\varphi}} f^2 \, dx \, dt \Big)^{1/2} \\
&\qquad + \Big(s^{-k} \int_0^1 e^{-2s\widehat{\varphi}(0)} y_0^2 \, dx \Big)^{1/2} \Bigg] \|w\|_{\widetilde{X}_a}.
\end{aligned} \tag{4.4.6}$$

Hence, by Lax-Milgram Theorem, we infer that there exists a unique $\tilde{z} \in \widetilde{X}_a$ such that

$$\kappa(\tilde{z}, w) = \ell(w), \qquad \forall\, w \in \widetilde{X}_a. \tag{4.4.7}$$

This fact, together with (4.4.6), gives that

$$\begin{aligned}\kappa(\tilde{z}, \tilde{z}) = \ell(\tilde{z}) \leq C e^{s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]}\Big[\Big(\iint_Q (s\beta)^{-k} e^{-2s\tilde{\varphi}} f^2\, dx\, dt\Big)^{1/2} \\ + \Big(s^{-k}\int_0^1 e^{-2s\widehat{\varphi}(0)} y_0^2\, dx\Big)^{1/2}\Big] \|\tilde{z}\|_{\widetilde{X}_a}.\end{aligned}$$

This implies

$$\begin{aligned}\|\tilde{z}\|_{\widetilde{X}_a} \leq C e^{s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]}\Big[\Big(\iint_Q (s\beta)^{-k} e^{-2s\tilde{\varphi}} f^2\, dx\, dt\Big)^{1/2} \\ + \Big(s^{-k}\int_0^1 e^{-2s\widehat{\varphi}(0)} y_0^2\, dx\Big)^{1/2}\Big].\end{aligned} \tag{4.4.8}$$

Setting

$$\begin{cases} \tilde{y} = (s\beta)^k e^{2s\tilde{\Phi}} \mathcal{L}^* \tilde{z} \\ \tilde{u} = -1_\omega (s\beta)^{k+3} e^{2s\tilde{\Phi}} \tilde{z} \end{cases} \tag{4.4.9}$$

and using the definition of the bilinear form $\kappa(\cdot,\cdot)$, we can write

$$\begin{aligned}\|\tilde{z}\|^2_{\widetilde{X}_a} &= \iint_Q (s\beta)^{-k} e^{-2s\tilde{\Phi}} \big((s\beta)^k e^{2s\tilde{\Phi}} \mathcal{L}^* \tilde{z}\big)^2\, dx\, dt \\ &\quad + \iint_{Q_\omega} (s\beta)^{-(k+3)} e^{-2s\tilde{\Phi}} \big(1_\omega (s\beta)^{k+3} e^{2s\tilde{\Phi}} \tilde{z}\big)^2\, dxdt \\ &= \iint_Q (s\beta)^{-k} e^{-2s\tilde{\Phi}} \tilde{y}^2\, dx\, dt + \iint_{Q_\omega} (s\beta)^{-(k+3)} e^{-2s\tilde{\Phi}} \tilde{u}^2\, dxdt\end{aligned}$$

and, in view of (4.4.8), we can deduce

$$\begin{aligned}&\iint_Q (s\beta)^{-k} e^{-2s\tilde{\Phi}} \tilde{y}^2\, dx\, dt + \iint_{Q_\omega} (s\beta)^{-(k+3)} e^{-2s\tilde{\Phi}} \tilde{u}^2\, dxdt \\ &\quad \leq C e^{2s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]}\Big(\iint_Q (s\beta)^{-k} e^{-2s\tilde{\varphi}} f^2\, dx\, dt + s^{-k}\int_0^1 e^{-2s\widehat{\varphi}(0)} y_0^2\, dx\Big).\end{aligned} \tag{4.4.10}$$

Hence $\tilde{y} \in E_{s,k}$ and satisfies the inequality (4.4.1).

In order to complete the proof, it remains to show that $(\tilde{y}, \tilde{u})$, satisfies the parabolic problem (4.1.4) and the identity (4.4.2). First of all, by (4.4.10) it is immediate that $\tilde{y}, \tilde{u} \in L^2(Q)$.

Moreover, denote by $\hat{y}$ the weak solution of system (4.1.4) associated to the control function $u = \tilde{u}$. Then, $\hat{y}$ also solves this system in the sense of transposition, that is, $\hat{y}$ is the unique function in $L^2(Q)$ satisfying

$$\iint_Q \hat{y} h\, dx\, dt = \int_0^1 y_0 w(0) dx + \iint_Q 1_\omega \tilde{u} w\, dx\, dt + \iint_Q f w\, dx\, dt, \qquad \forall\, h \in L^2(Q), \tag{4.4.11}$$

where w is the solution of

$$\begin{cases} -w_t - (a(x) w_x)_x = h & (t,x) \in Q, \\ w(t,1) = 0, & t \in (0,T), \\ \begin{cases} w(t,0) = 0, & \text{(WD)}, \\ (a w_x)(t,0) = 0, & \text{(SD)}, \end{cases} & t \in (0,T), \\ w(T,x) = 0, & x \in (0,1). \end{cases}$$

On the other hand, substituting the expressions of $\tilde{y}$ and $\tilde{u}$, given in (4.4.9), in (4.4.7), we obtain

$$\iint_Q \tilde{y}h\,dx\,dt - \iint_{Q_\omega} 1_\omega \tilde{u}w\,dx\,dt = \iint_Q fw\,dx\,dt + \int_0^1 y_0 w(0)dx, \qquad \forall\, h \in L^2(Q). \tag{4.4.12}$$

Hence, (4.4.11) and (4.4.12) imply that $\tilde{y} = \hat{y}$ solves (4.1.4).

This completes the proof of Theorem 4.4.1. □

We underline that Theorem 4.4.1 provides null controllability property for more regular solution of (4.1.4). Such a result turns out to be fundamental for the proof of Theorem 4.5.1.

4.5 Null controllability of memory system

In this section, we analyze the null controllability result for the degenerate parabolic equation (4.1.1). First, for $k \geq 0$, we set

$$E_{s,k,R} = \left\{ w \in E_{s,k} : \quad \|(s\beta)^{-k/2} e^{-s\tilde{\Phi}} w\|_{L^2(Q)} \leq R \right\},$$

where R is an arbitrary positive constant. Clearly, $E_{s,k,R}$ is a bounded, closed and convex subset of $L^2(Q)$.

Let $w \in E_{s,k,R}$ and consider the following system:

$$\begin{cases} y_t - (a(x)y_x)_x = \int\limits_0^t b(t,s,x)w(s,x)\,ds + 1_\omega u & (t,x) \in Q, \\ y(t,1) = 0, & t \in (0,T), \\ \begin{cases} y(t,0) = 0, & \text{(WD)}, \\ (ay_x)(t,0) = 0, & \text{(SD)}, \end{cases} & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1). \end{cases} \tag{4.5.1}$$

Hence, the next null controllability result holds.

Proposition 4.5.1. *Let T and R strictly positive and $k \geq 0$. Assume that the memory kernel satisfies,*

$$(T-t)^{2k} e^{\left(\frac{4}{T}\right)^4 \frac{s\gamma d}{(T-t)^4}} b \in L^\infty((0,T) \times Q), \tag{4.5.2}$$

where γ and d are the constants of (4.3.2) and s is the same of Lemma 4.3.1. Then, for all $w \in E_{s,kR}$ and for any $y_0 \in H^1_a(0,1)$, there exists $u \in L^2(Q)$ such that the associated solution y of system (4.5.1) belongs to $E_{s,k}$.

Notice that, condition (4.5.2) may appear as a quite strong restriction on the admissible kernel function b. Notwithstanding, it is instead a natural one, since the only thing that we are asking is its boundedness with respect to the Carleman weight. In other words, b should decay exponentially to 0 as t goes to T^-. Recall that this assumption is less restrictive, for larger values of the parameter $k > 0$, than (4.3.9).

Proof. Let $w \in E_{s,k,R}$ and let $y \in Z_T$ the solution of (4.5.1). Using the fact that $-\gamma d\beta \leq \tilde{\varphi}$ in

Q (see (4.3.5)), we get that

$$\begin{aligned}
&\iint_Q (s\beta)^{-k} e^{-2s\tilde{\varphi}} \Big(\int_0^t b(t,s,x) w(s,x)\, ds \Big)^2 \, dx\, dt \\
&\quad \le C_T \iint_Q \int_0^t (s\beta)^{-k} e^{-2s\tilde{\varphi}} b^2(t,s,x) w^2(s,x)\, ds\, dx\, dt \\
&\quad \le C_T \iint_Q \int_0^t (s\beta)^{-k} e^{2s\gamma d\beta} b^2(t,s,x) w^2(s,x)\, ds\, dx\, dt \\
&\quad \le C_T \iint_Q \int_0^t (T-t)^{4k} e^{\frac{2s\gamma d}{(T/4)^4 (T-t)^4}} b^2(t,s,x) w^2(s,x)\, ds\, dx\, dt.
\end{aligned}$$

Hence, by virtue of condition (4.5.2), we have

$$\begin{aligned}
&\iint_Q (s\beta)^{-k} e^{-2s\tilde{\varphi}} \Big(\int_0^t b(t,s,x) w(s,x)\, ds \Big)^2 \, dx\, dt \\
&\qquad\qquad \le C_T \iint_Q w^2 \, dx\, dt; \qquad (4.5.3)
\end{aligned}$$

therefore, using Hölder's inequality, the fact that $\sup_{(t,x)\in\overline{Q}} (s\beta(t))^k e^{2s\tilde{\Phi}}(t,x) < +\infty$ and $w \in E_{s,k,R}$, we conclude that

$$\begin{aligned}
&\iint_Q (s\beta)^{-k} e^{-2s\tilde{\varphi}} \Big(\int_0^t b(t,s,x) w(s,x)\, ds \Big)^2 \, dx\, dt \\
&\quad \le C_T s^{-k} \Big(\sup_{(t,x)\in\overline{Q}} (s\beta(t))^k e^{2s\tilde{\Phi}}(t,x) \Big) \iint_Q (s\beta)^{-k} e^{-2s\tilde{\Phi}} w^2 \, dx\, dt \\
&\quad \le C_T s^{-k} R^2 < +\infty.
\end{aligned}$$

This implies that $(s\beta)^{-k/2} e^{-s\tilde{\varphi}} \Big(\int_0^t b(t,s,x) w(s,x)\, ds \Big) \in L^2(Q)$. Hence, in view of Theorem 4.4.1, we deduce that there exists $u \in L^2(Q)$ such that the associated solution y of (4.5.1) belongs to $E_{s,k}$. Hence, the conclusion follows. □

As a consequence of Proposition 4.5.1 and Kakutani's fixed point Theorem, we obtain the following result.

Theorem 4.5.1. *Let $T > 0$, $k \ge 0$ and assume that (4.5.2) holds with $s \ge s_0$ such that*

$$C s^{-k} e^{-s\gamma \left(\frac{2}{T}\right)^8 d^*} \le \frac{1}{2},$$

where γ, d^* and C are the constants that appear in (4.3.4) and (4.4.1), respectively.

Then, for any $y_0 \in H^1_a(0,1)$, there exists $u \in L^2(Q)$ such that the associated solution $y \in Z_T$ of (4.1.1) satisfies

$$y(T,\cdot) = 0 \qquad \text{in } (0,1).$$

Proof of Theorem 4.5.1. For the moment take $R > 0$ sufficiently large. Define, as in [153], the multivalued mapping $\Lambda : E_{s,k,R} \subset E_{s,k} \to 2^{E_{s,k}}$ in the following way: for every $w \in E_{s,k,R}$, $\Lambda(w)$ is the set of $y \in E_{s,k}$ such that for some $u \in L^2(Q)$ satisfying

$$\iint_{Q_\omega} (s\beta)^{-(k+3)} e^{-2s\tilde{\Phi}} u^2 \, dxdt \leq Ce^{2s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]}\Big(R^2 + \int_0^1 e^{-2s\widehat{\varphi}(0)} y_0^2 \, dx\Big), \tag{4.5.4}$$

the associated solution y of (4.5.1) satisfies

$$y \in E_{s,k} \quad \text{and} \quad y(T,\cdot) = 0 \qquad \text{in } (0,1). \tag{4.5.5}$$

Thus, our task is reduced to prove that Λ admit at least one fixed point in $E_{s,k,R}$. To this aim, it suffices to check that Λ satisfies the assumptions of Kakutani's fixed point Theorem (see, e.g., [89, Theorem 2.3] or [119]). Next, we are going to check that all the conditions to apply such a theorem in $L^2(Q)$-topology are satisfied.

Clearly, $\Lambda(w)$ is a closed set of $L^2(Q)$. Moreover, thanks to Proposition 4.5.1, $\Lambda(w)$ is non empty. The fact that the identity in (4.5.5) is stable by convex combinations yields the convexity of $\Lambda(w)$.

Now, let us prove that $\Lambda(E_{s,k,R}) \subset E_{s,k,R}$ for a sufficiently large R. Using the inequality (4.4.1), condition (4.5.2) and proceeding as in (4.5.3), we have

$$\begin{aligned}
&\iint_Q (s\beta)^{-k} e^{-2s\tilde{\Phi}} y^2 \, dx\, dt + \iint_{Q_\omega} (s\beta)^{-(k+3)} e^{-2s\tilde{\Phi}} u^2 \, dxdt \\
&\leq Ce^{2s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]}\Big(\iint_Q (s\beta)^{-k} e^{-2s\tilde{\varphi}} \Big(\int_0^t b(t,s,x) w(s,x)\, ds\Big)^2 dx\, dt \\
&\qquad + s^{-k} \int_0^1 e^{-2s\widehat{\varphi}(0)} y_0^2 \, dx\Big) \\
&\leq Ce^{2s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]}\Big(s^{-k} \iint_Q w^2 \, dx\, dt + s^{-k} \int_0^1 e^{-2s\widehat{\varphi}(0)} y_0^2 \, dx\Big).
\end{aligned}$$

Therefore, applying Hölder's inequality, we obtain

$$\begin{aligned}
&\iint_Q (s\beta)^{-k} e^{-2s\tilde{\Phi}} y^2 \, dx\, dt + \iint_{Q_\omega} (s\beta)^{-(k+3)} e^{-2s\tilde{\Phi}} u^2 \, dxdt \\
&\leq Cs^{-k} e^{2s[\widehat{\varphi}(0)-\varphi^*(\frac{5T}{8})]}\Big(\Big(\sup_{(t,x)\in\overline{Q}} (s\beta(t))^k e^{2s\tilde{\Phi}(t,x)}\Big)\Big(\iint_Q (s\beta)^{-k} e^{-2s\tilde{\Phi}} w^2 \, dx\, dt\Big) \\
&\qquad + \int_0^1 e^{-2s\widehat{\varphi}(0)} y_0^2 \, dx\Big).
\end{aligned}$$

In particular, since $\tilde{\varphi} \leq \widehat{\varphi}(0)$, $\tilde{\Phi} \leq \widehat{\Phi}(0)$ in Q (see (4.3.13)), $\sup_{(t,x)\in\overline{Q}} (s\beta(t))^k e^{\frac{s}{2}\tilde{\Phi}(t,x)} < +\infty$ and $w \in E_{s,k,R}$, the last inequality becomes

$$\begin{aligned}
&\iint_Q (s\beta)^{-k} e^{-2s\tilde{\Phi}} y^2 \, dx\, dt + \iint_{Q_\omega} (s\beta)^{-(k+3)} e^{-2s\tilde{\Phi}} u^2 \, dxdt \\
&\leq Cs^{-k}\Big(e^{s[2\widehat{\varphi}(0)-2\varphi^*(\frac{5T}{8})+\frac{3}{2}\widehat{\Phi}(0)]} R^2 + e^{-2s\varphi^*(\frac{5T}{8})} \int_0^1 y_0^2 \, dx\Big).
\end{aligned} \tag{4.5.6}$$

On the other hand, by taking the parameter ρ in (4.3.3) so that $\rho > \dfrac{ln3}{\|\sigma\|_\infty}$, one can show that the interval $\Big(\dfrac{e^{2\rho\|\sigma\|_\infty}}{d-d^\star}, \dfrac{3(e^{2\rho\|\sigma\|_\infty} - e^{\rho\|\sigma\|_\infty})}{2(d-d^\star)}\Big)$ is nonempty, and thus, we can choose the constant

γ (see (4.3.2)) in such a way

$$\frac{e^{2\rho\|\sigma\|_\infty}-1}{d-d^*}<\gamma<\frac{3\big(e^{2\rho\|\sigma\|_\infty}-e^{\rho\|\sigma\|_\infty}\big)}{2(d-d^\star)}.$$

Thus, as a straightforward consequence, one has

$$\frac{3}{2}\hat{\hat{\Phi}}(t)\leq\widehat{\varphi}(t)\qquad\text{for every}\quad t\in(0,T).\tag{4.5.7}$$

Using (4.5.7), the definitions of $\widehat{\varphi}$ and φ^*, and choosing $d\geq 10d^*$, we find

$$\begin{aligned}2\widehat{\varphi}(0)-2\varphi^*(\frac{5T}{8})+\frac{3}{2}\widehat{\Phi}(0)&\leq 3\widehat{\varphi}(0)-2\varphi^*(\frac{5T}{8})\\&=3\gamma(d^*-d)\beta(0)+2\gamma d\beta\Big(\frac{5T}{8}\Big)\\&=\gamma\Big(\frac{2}{T}\Big)^8\Big[3(d^*-d))+2d\Big(\frac{16}{15}\Big)^4\Big]\\&=\gamma\Big(\frac{2}{T}\Big)^8\Big[3d^*-d\Big(3-2\Big(\frac{16}{15}\Big)^4\Big)\Big]\\&<-\gamma\Big(\frac{2}{T}\Big)^8d^*<0.\end{aligned}$$

By assumption s is such that

$$Cs^{-k}e^{s[2\widehat{\varphi}(0)-2\varphi^*(\frac{5T}{8})+\frac{3}{2}\widehat{\Phi}(0)]}\leq\frac{1}{2},$$

thus, we immediately obtain, from this last inequality and (4.5.6),

$$\begin{aligned}\iint_Q(s\beta)^{-k}e^{-2s\tilde{\Phi}}y^2\,dx\,dt+\iint_{Q_\omega}(s\beta)^{-(k+3)}e^{-2s\tilde{\Phi}}u^2\,dxdt&\\\leq\Big(\frac{1}{2}R^2+Ce^{-2s\varphi^*(\frac{5T}{8})}s^{-k}\int_0^1y_0^2\,dx\Big).&\end{aligned}\tag{4.5.8}$$

Hence, for R sufficiently large, we have

$$\iint_Q(s\beta)^{-k}e^{-2s\tilde{\Phi}}y^2\,dx\,dt\leq R^2.$$

As a consequence, $\Lambda(E_{s,k,R})\subset E_{s,k,R}$.

Furthermore, let $\{w_n\}$ be a sequence of $E_{s,k,R}$. Thanks to Proposition 4.2.1, the associated solutions $\{y_n\}$ are bounded in Z_T. Then, in view of Aubin-Lions Theorem, this implies that $\Lambda(E_{s,k,R})$ is relatively compact in $L^2(Q)$.

Let us finally check that Λ is upper-semicontinuous under the $L^2(Q)$-topology. To this aim, let $\{w_n\}$ be a sequence satisfying $w_n\to w$ in $E_{s,k,R}$ and $y_n\in\Lambda(w_n)$ such that $y_n\to y$ in $L^2(Q)$. Our objective is to prove that $y\in\Lambda(w)$. At first, observe that for any $w_n\in E_{s,k,R}$, we can find at least one control $u_n\in L^2(Q)$ such that the associated solution y_n belongs to $L^2(Q)$. By virtue of Proposition 4.2.1 and (4.5.8), we deduce that there is a subsequence satisfying

$$\begin{aligned}u_n&\to u&&\text{weakly in }L^2(Q)\\y_n&\to\tilde{y}&&\text{weakly in }Z_T\quad\text{and}&&(4.5.9)\\&&&\text{strongly in }C(0,T;L^2(0,1)).&&(4.5.10)\end{aligned}$$

This yields $y=\tilde{y}$ in $L^2(Q)$.

Since (y_n, u_n) satisfies the system

$$\begin{cases} y_{n,t} - (a(x)y_{n,x})_x = \int_0^t b(t,s,x)w_n(s,x)\,ds + 1_\omega u_n, & (t,x) \in Q, \\ y_n(t,1) = 0, & t \in (0,T), \\ \begin{cases} y_n(t,0) = 0, & \text{(WD)}, \\ (ay_{n,x})(t,0) = 0, & \text{(SD)}, \end{cases} & t \in (0,T), \\ y_n(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{4.5.11}$$

hence passing to weak limit, it follows that the couple (y, u) satisfies (4.5.1). This provides that $y \in \Lambda(w)$ and, therefore, Λ is upper semicontinuous.

Consequently, using the Kakutani's fixed point Theorem in the $L^2(Q)$-topology for the mapping Λ, we infer that there is at least one $y \in E_{s,k,R}$ such that $y \in \Lambda(y)$. Thus, by the definition of Λ, there exists at least one couple (y, u) satisfying all the conditions in Theorem 4.5.1. The uniqueness of y follows by Proposition 4.2.2. Hence, the proof of Theorem 4.5.1 is complete. □

Remark 31.
- Let us recall that, without any hypothesis on the kernel b, the null controllability of (4.1.1) fails (see [110, 170]). Hence, the decaying condition (4.5.2) could be necessary.
- A condition similar to (4.5.2) already appears in the work of Q. Tao and H. Gao in [153] for uniformly parabolic equations (see (4.1.3)). Hence, the null controllability result stated in Theorem 4.5.1 for the degenerate equation with memory can be seen as an extension to the one obtained in [153].
- The difference on the powers of the exponential terms in (4.5.2) and (4.1.3) is mainly due to the different weighted time functions considered in these two contexts.
- Owing to Remark 11, we can actually decrease the exponent 4 in the assumption (4.5.2) to the exponent 2. In particular, in place of (4.5.2) we can assume

$$(T-t)^{2k} e^{\frac{\hat{C}}{(T-t)^2}} b \in L^\infty((0,T) \times Q),$$

where $\hat{C} = \left(\frac{4}{T}\right)^2 s\gamma d$.

Clearly, Theorem 4.5.1 holds also in a general domain $(t^*, T) \times (0,1)$ with suitable changes. Thanks to this fact, the following null controllability result holds for memory system (4.1.1).

Theorem 4.5.2. *Let $T > 0$, $k \geq 0$ and assume that (4.5.2) holds with s as in Theorem 4.5.1. Then, for any $y_0 \in L^2(0,1)$, there exists $u \in L^2(Q)$ such that the associated solution $y \in W_T$ of (4.1.1) satisfies*

$$y(T,\cdot) = 0 \qquad \text{in } (0,1).$$

Proof. Consider the following homogeneous parabolic problem:

$$\begin{cases} w_t - (a(x)w_x)_x = \int_0^t b(t,s,x)w(s,x)\,ds & (t,x) \in (0,\frac{T}{2}) \times (0,1), \\ w(t,1) = 0, & t \in (0,\frac{T}{2}), \\ \begin{cases} w(t,0) = 0, & \text{(WD)}, \\ (aw_x)(t,0) = 0, & \text{(SD)}, \end{cases} & t \in (0,\frac{T}{2}), \\ w(0,x) = y_0(x), & x \in (0,1), \end{cases}$$

where y_0 is the initial condition in (4.1.1).

By Proposition 4.2.2, the solution of this system belongs to

$$W_T^* := L^2\left(0, \frac{T}{2}; H_a^1(0,1)\right) \cap C\left(\left[0, \frac{T}{2}\right]; L^2(0,1)\right).$$

Then, there exists $t^* \in (0, \frac{T}{2})$ such that $w(t^*, \cdot) := w^*(\cdot) \in H_a^1(0,1)$.

Now, we consider the following controlled parabolic system:

$$\begin{cases} z_t - (a(x)z_x)_x = \int\limits_0^t b(t,s,x)z(s,x)\,ds + 1_\omega h & (t,x) \in (t^*,T) \times (0,1), \\ z(t,1) = 0, & t \in (t^*,T), \\ \begin{cases} z(t,0) = 0, & \text{(WD)}, \\ (az_x)(t,0) = 0, & \text{(SD)}, \end{cases} & t \in (t^*,T), \\ z(t^*,x) = w^*(x), & x \in (0,1). \end{cases}$$

Hence, thanks to Theorem 4.5.1, there exists $h \in L^2((t^*,T) \times (0,1))$ such that the associated solution $z \in Z_T^* := L^2(t^*,T; H_a^2(0,1)) \cap H^1(t^*,T; L^2(0,1))$ satisfies

$$z(T,\cdot) = 0 \qquad \text{in } (0,1).$$

Finally, setting

$$y := \begin{cases} w, & \text{in} & [0,t^*], \\ z, & \text{in} & [t^*,T] \end{cases} \qquad \text{and} \quad u := \begin{cases} 0, & \text{in} & [0,t^*], \\ h, & \text{in} & [t^*,T], \end{cases}$$

one can prove that $y \in W_T$ solves the system (4.1.1) associated to u and is such that

$$y(T,\cdot) = 0 \qquad \text{in } (0,1).$$

Hence, the thesis follows. □

Remark 32. In the present context, by Theorem 4.5.2, one can deduce immediately the null controllability result for (4.1.1) when the control acts at the *nondegenerate* point $x = 1$. Indeed, it is sufficient to use a standard technique and a localization argument as in [5, Remark 4.6.2]. Of course, the situation is completely different in the case when the control acts at the *degenerate* point $x = 0$. We refer to [55] for a discussion of this issue.

Remark 33. Observe that, as in the context of parabolic equation without memory (i.e., $b = 0$), the null controllability for (4.1.1) proved in Theorem 4.5.2 yields the exact controllability to trajectories, that is, for any trajectory $\overline{y}$ (i.e. solution of (4.1.1) corresponding to $u \equiv 0$ and $y_0 \in L^2(0,1)$) and any $y_0 \in L^2(0,1)$, there exists $u \in L^2(Q)$ such that the associated solution to (4.1.1) satisfies

$$\overline{y}(T,x) = y(T,x), \qquad x \in (0,1).$$

Indeed, let us consider a trajectory $\overline{y}$ and introduce the following change of variables $z = y - \overline{y}$, where y is a solution of (4.1.1). Hence, z satisfies the following controlled system:

$$\begin{cases} z_t - (a(x)z_x)_x = \int\limits_0^t b(t,s,x)z(s,x)\,ds + 1_\omega u & (t,x) \in Q, \\ z(t,1) = 0, & t \in (0,T), \\ \begin{cases} z(t,0) = 0, & \text{(WD)}, \\ (az_x)(t,0) = 0, & \text{(SD)}, \end{cases} & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,1), \end{cases}$$

where $z_0 = y_0 - \overline{y}_0$.

According to Theorem 4.5.2 there exists $u \in L^2(Q)$ such that

$$z(T, x) = 0, \qquad x \in (0, 1).$$

Consequently,

$$\overline{y}(T, x) = y(T, x), \qquad x \in (0, 1).$$

4.6 Extensions

In this section we discuss some extensions of the above null controllability results.

4.6.1 Null controllability in the case $a(1) = 0$

In this subsection we address the null controllability result for the following degenerate parabolic equation with memory

$$\begin{cases} y_t - (a(x)y_x)_x = \int_0^t b(t,s,x)y(s,x)\,ds + 1_\omega u & (t,x) \in Q, \\ y(t,0) = 0, & t \in (0,T), \\ \begin{cases} y(t,1) = 0, & \text{(WD)}, \\ (ay_x)(t,1) = 0, & \text{(SD)}, \end{cases} & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{4.6.1}$$

where $y_0 \in L^2(0,1)$ and a degenerates at the extremity $x = 1$, i.e., $a(1) = 0$. In order to present our main result we need to introduce the functional spaces where our problem will be well posed. As before, we distinguish the two following cases:

- Weakly degenerate case (WD)

$$\begin{cases} a \in C([0,1]) \cap C^1([0,1)),\ a(1) = 0,\ a > 0 \quad \text{in} \quad [0,1), \\ \exists\, \tilde{\alpha} \in [0,1), \quad \text{such that} \quad (x-1)a'(x) \le \tilde{\alpha} a(x), \quad \forall\, x \in [0,1], \end{cases} \tag{4.6.2}$$

- Strongly degenerate (SD)

$$\begin{cases} a \in C^1([0,1]),\ a(1) = 0,\ a > 0 \quad \text{in} \quad [0,1), \\ \exists\, \tilde{\alpha} \in [1,2), \quad \text{such that} \quad (x-1)a'(x) \le \tilde{\alpha} a(x), \quad \forall\, x \in [0,1], \\ \begin{cases} \exists\, \tilde{\beta} \in (1, \tilde{\alpha}],\ x \mapsto \dfrac{a(x)}{(1-x)^{\tilde{\beta}}} \quad \text{is nonincreasing near} \quad 0, \quad \text{if} \quad \tilde{\alpha} > 1, \\ \exists\, \tilde{\beta} \in (0,1),\ x \mapsto \dfrac{a(x)}{(1-x)^{\tilde{\beta}}} \quad \text{is nonincreasing near} \quad 0, \quad \text{if} \quad \tilde{\alpha} = 1. \end{cases} \end{cases} \tag{4.6.3}$$

Clearly, the prototype is $a(x) = (1-x)^{\tilde{\alpha}}, \quad \tilde{\alpha} \in (0,2)$.

Let us introduce the weighted spaces H^1_a and H^2_a as follows:

Case (WD).

$$H^1_a(0,1) := \Big\{ y \in L^2(0,1) : y \text{ a.c. in } [0,1], \quad \sqrt{a} y_x \in L^2(0,1) \text{ and } y(0) = y(1) = 0 \Big\}$$

and

$$H^2_a(0,1) := \Big\{ y \in H^1_a(0,1) : a y_x \in H^1(0,1) \Big\}.$$

Case (SD).

$$H^1_a(0,1) := \Big\{y \in L^2(0,1) : y \text{ locally a.c. in } [0,1), \quad \sqrt{a}y_x \in L^2(0,1) \text{ and } y(0) = 0\Big\}$$

and

$$\begin{aligned} H^2_a(0,1) : &= \Big\{y \in H^1_a(0,1) : ay_x \in H^1(0,1)\Big\} \\ &= \Big\{y \in L^2(0,1) : y \text{ locally a.c. in } [0,1), \quad ay \in H^1_0(0,1), \\ &\qquad ay_x \in H^1(0,1) \text{ and } (ay_x)(1) = 0\Big\}. \end{aligned}$$

Using the above spaces, one can prove that the well-posedness results given in Propositions 4.2.1 and 4.2.2 still hold. On the contrary, setting $\varphi := \theta\tilde{\psi}$, where θ is defined as in (4.3.2) and

$$\tilde{\psi} := \tilde{\gamma}\Big(\int_x^1 \frac{1-y}{a(y)}\,dy - \tilde{d}\Big), \tag{4.6.4}$$

with $\tilde{\gamma}$ and $\tilde{d} > \displaystyle\int_0^1 \frac{1-y}{a(y)}\,dy$ positive constants, the next null controllability result holds.

Theorem 4.6.1. *Let $T > 0$, $k \geq 0$ and assume that*

$$(T-t)^{2k} e^{\left(\frac{4}{T}\right)^4 \frac{s\tilde{\gamma}\tilde{d}}{(T-t)^4}} b \in L^\infty((0,T)\times Q), \tag{4.6.5}$$

with s as in Theorem 4.5.1. Then, for any $y_0 \in L^2(0,1)$, there exists $u \in L^2(Q)$ such that the associated solution $y \in W_T$ of (4.6.1) satisfies

$$y(T,\cdot) = 0 \qquad \textit{in } (0,1).$$

Proof. The proof of this theorem follows the same strategy of Theorem 4.5.2; of course using symmetric arguments. The main difference is that here, in place of (1.2.9) and (4.3.8), we use the following Hardy Poincaré inequality:

there is a positive constant C such that, for every $y \in H^1_a(0,1)$, the following inequality holds

$$\int_0^1 \frac{a(x)}{(1-x)^2} y^2(x)\,dx \leq C\int_0^1 a(x)|y_x(x)|^2\,dx,$$

and the following Carleman estimate:

there exist two positive constants C and s_0, such that the solution $v \in Z_T$ of (4.3.1) satisfies

$$\begin{aligned} &\iint_Q \Big(s\theta a(x)v_x^2 + s^3\theta^3\frac{(1-x)^2}{a(x)}v^2\Big)e^{2s\varphi}\,dx\,dt \\ &\qquad \leq C\Big(\iint_Q g^2 e^{2s\Phi}\,dx\,dt + \iint_{Q_\omega} s^3\theta^3 v^2 e^{2s\Phi}\,dxdt\Big) \end{aligned}$$

for all $s \geq s_0$.

As the procedure is completely similar, we omit the details of the proof. □

4.6.2 Null controllability in the case $a(0) = a(1) = 0$

In this subsection we will extend the null controllability result proved above to the degenerate parabolic equation with memory

$$\begin{cases} y_t - (a(x)y_x)_x = \displaystyle\int_0^t b(t,s,x)y(s,x)\,ds + 1_\omega u & (t,x) \in Q, \\ \begin{cases} y(t,0) = 0 = y(t,1), & \text{(WWD)}, \\ (ay_x)(t,0) = 0 = y(t,1), & \text{(SWD)}, \\ y(t,0) = 0 = (ay_x)(t,1), & \text{(WSD)}, \\ (ay_x)(t,0) = 0 = (ay_x)(t,1), & \text{(SSD)}, \end{cases} & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{4.6.6}$$

where $y_0 \in L^2(0,1)$ and a vanishes at both extremities of the interval $(0,1)$ and satisfies, as in [140], one of the four following cases:

- weakly-weakly degenerate case (WWD):

$$\begin{cases} a \in C([0,1]) \cap C^1((0,1)),\ a(0) = a(1) = 0,\ a > 0 \quad \text{in} \quad (0,1), \\ \exists\,\alpha \in [0,1), \quad \text{such that} \quad xa'(x) \le \alpha a(x), \quad \forall\, x \in [0,1], \\ \exists\,\tilde\alpha \in [0,1), \quad \text{such that} \quad (x-1)a'(x) \le \tilde\alpha a(x), \quad \forall\, x \in [0,1], \end{cases}$$

- strongly-weakly degenerate case (SWD):

$$\begin{cases} a \in C([0,1]) \cap C^1([0,1)),\ a(0) = a(1) = 0,\ a > 0 \quad \text{in} \quad (0,1), \\ \exists\,\alpha \in [1,2), \quad \text{such that} \quad xa'(x) \le \alpha a(x), \quad \forall\, x \in [0,1], \\ \begin{cases} \exists\,\beta \in (1,\alpha],\ x \mapsto \dfrac{a(x)}{x^\beta} \quad \text{is nondecreasing near} \quad 0, \quad \text{if} \quad \alpha > 1, \\ \exists\,\beta \in (0,1),\ x \mapsto \dfrac{a(x)}{x^\beta} \quad \text{is nondecreasing near} \quad 0, \quad \text{if} \quad \alpha = 1, \end{cases} \\ \exists\,\tilde\alpha \in [0,1), \quad \text{such that} \quad (x-1)a'(x) \le \tilde\alpha a(x), \quad \forall\, x \in [0,1], \end{cases}$$

- weakly-strongly degenerate case (WSD):

$$\begin{cases} a \in C([0,1]) \cap C^1((0,1]),\ a(0) = a(1) = 0,\ a > 0 \quad \text{in} \quad (0,1), \\ \exists\,\alpha \in [0,1), \quad \text{such that} \quad xa'(x) \le \alpha a(x), \quad \forall\, x \in [0,1], \\ \exists\,\tilde\alpha \in [1,2), \quad \text{such that} \quad (x-1)a'(x) \le \tilde\alpha a(x), \quad \forall\, x \in [0,1], \\ \begin{cases} \exists\,\tilde\beta \in (1,\tilde\alpha],\ x \mapsto \dfrac{a(x)}{(1-x)^{\tilde\beta}} \quad \text{is nonincreasing near} \quad 0, \quad \text{if} \quad \tilde\alpha > 1, \\ \exists\,\tilde\beta \in (0,1),\ x \mapsto \dfrac{a(x)}{(1-x)^{\tilde\beta}} \quad \text{is nonincreasing near} \quad 0, \quad \text{if} \quad \tilde\alpha = 1. \end{cases} \end{cases}$$

- strongly-strongly degenerate case (SSD):

$$\begin{cases} a \in C^1([0,1]),\ a(0) = a(1) = 0,\ a > 0 \quad \text{in} \quad (0,1), \\ \exists\,\alpha \in [1,2), \quad \text{such that} \quad xa'(x) \le \alpha a(x), \quad \forall\, x \in [0,1], \\ \begin{cases} \exists\,\beta \in (1,\alpha],\ x \mapsto \dfrac{a(x)}{x^\beta} \quad \text{is nondecreasing near} \quad 0, \quad \text{if} \quad \alpha > 1, \\ \exists\,\beta \in (0,1),\ x \mapsto \dfrac{a(x)}{x^\beta} \quad \text{is nondecreasing near} \quad 0, \quad \text{if} \quad \alpha = 1, \end{cases} \\ \exists\,\tilde\alpha \in [1,2), \quad \text{such that} \quad (x-1)a'(x) \le \tilde\alpha a(x), \quad \forall\, x \in [0,1], \\ \begin{cases} \exists\,\tilde\beta \in (1,\tilde\alpha],\ x \mapsto \dfrac{a(x)}{(1-x)^{\tilde\beta}} \quad \text{is nonincreasing near} \quad 0, \quad \text{if} \quad \tilde\alpha > 1, \\ \exists\,\tilde\beta \in (0,1),\ x \mapsto \dfrac{a(x)}{(1-x)^{\tilde\beta}} \quad \text{is nonincreasing near} \quad 0, \quad \text{if} \quad \tilde\alpha = 1. \end{cases} \end{cases}$$

A typical example is $a(x) = x^\alpha(1-x)^{\tilde\alpha}$, with $\alpha, \tilde\alpha \in [0,2)$.

As previously, in order to study the well-posedness of problem (4.6.6), we shall define four different classes of weighted spaces.

Case (WWD).

$$H^1_a(0,1) := \Big\{ y \in L^2(0,1) : y \text{ a.c. in } [0,1], \quad \sqrt{a} y_x \in L^2(0,1) \text{ and } y(0) = y(1) = 0 \Big\}$$

and

$$H^2_a(0,1) := \Big\{ y \in H^1_a(0,1) : a y_x \in H^1(0,1) \Big\}.$$

Case (SWD).

$$H^1_a(0,1) := \Big\{ y \in L^2(0,1) : y \text{ locally a.c. in } (0,1], \quad \sqrt{a} y_x \in L^2(0,1) \text{ and } y(1) = 0 \Big\}$$

and

$$\begin{aligned} H^2_a(0,1) : &= \Big\{ y \in H^1_a(0,1) : a y_x \in H^1(0,1) \Big\} \\ &= \Big\{ y \in L^2(0,1) : y \text{ locally a.c. in } (0,1], \quad a y \in H^1_0(0,1), \\ &\qquad a y_x \in H^1(0,1) \text{ and } (a y_x)(0) = 0 \Big\}. \end{aligned}$$

Case (WSD).

$$H^1_a(0,1) := \Big\{ y \in L^2(0,1) : y \text{ locally a.c. in } [0,1), \quad \sqrt{a} y_x \in L^2(0,1) \text{ and } y(0) = 0 \Big\}$$

and

$$\begin{aligned} H^2_a(0,1) : &= \Big\{ y \in H^1_a(0,1) : a y_x \in H^1(0,1) \Big\} \\ &= \Big\{ y \in L^2(0,1) : y \text{ locally a.c. in } [0,1), \quad a y \in H^1_0(0,1), \\ &\qquad a y_x \in H^1(0,1) \text{ and } (a y_x)(1) = 0 \Big\}. \end{aligned}$$

Case (SSD).

$$H^1_a(0,1) := \Big\{ y \in L^2(0,1) : y \text{ locally a.c. in } (0,1), \quad \sqrt{a} y_x \in L^2(0,1) \Big\}$$

and

$$\begin{aligned} H^2_a(0,1) : &= \Big\{ y \in H^1_a(0,1) : a y_x \in H^1(0,1) \Big\} \\ &= \Big\{ y \in L^2(0,1) : y \text{ locally a.c. in } (0,1), \quad a y \in H^1_0(0,1), \\ &\qquad a y_x \in H^1(0,1) \text{ and } (a y_x)(0) = (a y_x)(1) = 0 \Big\}. \end{aligned}$$

Again, the well-posedness results proved in Propositions 4.2.1 and 4.2.2 still hold and, as a consequence of Theorems 4.5.2 and 4.6.1, one can deduce the following null controllability result for (4.6.6).

Theorem 4.6.2. *Let $T > 0$, $k \geq 0$ and assume*

$$(T-t)^{2k} e^{\left(\frac{4}{T}\right)^4 \frac{s\bar\gamma\bar d}{(T-t)^4}} b \in L^\infty((0,T)\times Q), \tag{4.6.7}$$

with s as in Theorem 4.5.1, where $\bar\gamma = \max\{\gamma, \tilde\gamma\}, \bar d = \max\{d, \tilde d\}$. Then, for any $y_0 \in L^2(0,1)$, there exists $u \in L^2(Q)$ such that the associated solution $y \in W_T$ of (4.6.6) satisfies

$$y(T,\cdot) = 0 \qquad in\ (0,1).$$

Here $\gamma, \tilde\gamma, d$ and $\tilde d$ are the constants given in (4.3.2) and in (4.6.4).

Proof. Consider the following parabolic system

$$\begin{cases} w_t - (a(x)w_x)_x = \int\limits_0^t b(t,s,x)w(s,x)\,ds + 1_\omega u_1 & (t,x) \in (0,T) \times (0,\beta'), \\ w(t,\beta') = 0, & t \in (0,T), \\ \begin{cases} w(t,0) = 0, & \text{(WD)}, \\ (aw_x)(t,0) = 0, & \text{(SD)}, \end{cases} & t \in (0,T), \\ w(0,x) = y_0(x), & x \in (0,\beta'), \end{cases} \tag{4.6.8}$$

where $\omega \Subset (\lambda', \beta') \Subset (0,1)$ and y_0 is the initial condition in (4.6.6).

Thus, by Theorem 4.5.2, we know that there exists a control $u_1 \in L^2((0,T) \times (0,\beta'))$ such that the associated solution $w \in W_T$ of (4.6.8) satisfies

$$w(T,\cdot) = 0, \quad \text{in}\,(0,\beta').$$

Now, define $\tilde{w}$ the trivial extension of w in $[0,1]$. Hence

$$\tilde{w}(T,\cdot) = 0, \quad \text{in}\,(0,1).$$

In a similar way, we consider the following parabolic system

$$\begin{cases} z_t - (a(x)z_x)_x = \int\limits_0^t b(t,s,x)z(s,x)\,ds + 1_\omega u_2 & (t,x) \in (0,T) \times (\lambda',1), \\ z(t,\lambda') = 0, & t \in (0,T), \\ \begin{cases} z(t,1) = 0, & \text{(WD)}, \\ (az_x)(t,1) = 0, & \text{(SD)}, \end{cases} & t \in (0,T), \\ z(0,x) = y_0(x), & x \in (\lambda',1). \end{cases} \tag{4.6.9}$$

Then, thanks to Theorem 4.6.1, there exists a control $u_2 \in L^2((0,T) \times (\lambda',1))$ such that the associated solution $z \in W_T$ solution of (4.6.9) satisfies

$$z(T,\cdot) = 0, \quad \text{in}\,(\lambda',1).$$

Now, define $\tilde{z}$ the trivial extension of z in $[0,1]$. Hence

$$\tilde{z}(T,\cdot) = 0, \quad \text{in}\,(0,1).$$

Next, consider

$$\tilde{u}_1(t,x) = \begin{cases} u_1(t,x), & (t,x) \in (0,T) \times (0,\beta'), \\ 0, & (t,x) \in (0,T) \times (\beta',1), \end{cases}$$

and

$$\tilde{u}_2(t,x) = \begin{cases} 0, & (t,x) \in (0,T) \times (0,\lambda'), \\ u_2(t,x), & (t,x) \in (0,T) \times (\lambda',1). \end{cases}$$

Let $\chi \in C^\infty([0,1])$ be a smooth cut-off function such that

$$0 \le \chi(x) \le 1, \quad \chi(x) = \begin{cases} 1, & x \in (0,\lambda''), \\ 0, & x \in (\beta'',1), \end{cases} \tag{4.6.10}$$

where $(\lambda'', \beta'') \Subset \omega$ and set $y = \chi\tilde{w} + (1-\chi)\tilde{z}$.

Then, one can easily verifies that

$$y_t = \chi\tilde{w}_t + (1-\chi)\tilde{z}_t,$$

and

$$\begin{aligned}(ay_x)_x =&\chi(a\tilde{w}_x)_x + (1-\chi)(a\tilde{z}_x)_x + ((a\tilde{w})_x\chi_x + a\tilde{w}\chi_{xx} + a\tilde{w}_x\chi_x)\\ &- ((a\tilde{z})_x\chi_x + a\tilde{z}\chi_{xx} + a\tilde{z}_x\chi_x).\end{aligned}$$

Therefore, we find that

$$\begin{aligned}y_t - (ay_x)_x - \int_0^t b(t,s,x)y(s,x)\,ds &= \chi\Big(\tilde{w}_t - (a\tilde{w}_x)_x - \int_0^t b(t,s,x)\tilde{w}(s,x)\,ds\Big)\\ &+ (1-\chi)\Big(\tilde{z}_t - (a\tilde{z}_x)_x - \int_0^t b(t,s,x)\tilde{z}(s,x)\,ds\Big)\\ &- \Big((a\tilde{w})_x\chi_x + a\tilde{w}\chi_{xx} + a\tilde{w}_x\chi_x\Big) + \Big((a\tilde{z})_x\chi_x + a\tilde{z}\chi_{xx} + a\tilde{z}_x\chi_x\Big)\\ &= 1_\omega\chi u_1 + 1_\omega(1-\chi)u_2 - \Big((a\tilde{w})_x\chi_x + a\tilde{w}\chi_{xx} + a\tilde{w}_x\chi_x\Big)\\ &\quad + \Big((a\tilde{z})_x\chi_x + a\tilde{z}\chi_{xx} + a\tilde{z}_x\chi_x\Big).\end{aligned}$$

Observe that the supports of χ_x and χ_{xx} are contained in $(\lambda'', \beta'') \Subset \omega$. Then, we can write

$$y_t - (ay_x)_x = \int_0^t b(t,s,x)y(s,x)\,ds + 1_\omega u$$

where $u \in L^2(Q)$ satisfies

$$\begin{aligned}1_\omega u = &1_\omega\chi\tilde{u}_1 + 1_\omega(1-\chi)\tilde{u}_2 - \Big((a\tilde{w})_x\chi_x + a\tilde{w}\chi_{xx} + a\tilde{w}_x\chi_x\Big)\\ &+ \Big((a\tilde{z})_x\chi_x + a\tilde{z}\chi_{xx} + a\tilde{z}_x\chi_x\Big).\end{aligned}$$

Moreover, using the definitions of $\tilde{w}$, $\tilde{z}$ and χ, it follows that

$$\begin{aligned}&y(t,0) = \Big(\chi\tilde{w} + (1-\chi)\tilde{z}\Big)(t,0) = 0, && t \in (0,T),\\ &y(t,1) = \Big(\chi\tilde{w} + (1-\chi)\tilde{z}\Big)(t,1) = 0, && t \in (0,T),\\ &(ay)_x(t,0) = \Big(\chi_x a\tilde{w} + \chi(a\tilde{w}_x) - \chi_x a\tilde{z} + (1-\chi)(a\tilde{z}_x)\Big)(t,0) = 0, && t \in (0,T),\\ &(ay)_x(t,1) = \Big(\chi_x a\tilde{w} + \chi(a\tilde{w}_x) - \chi_x a\tilde{z} + (1-\chi)(a\tilde{z}_x)\Big)(t,1) = 0, && t \in (0,T),\end{aligned}$$

from which we get the boundary conditions given in (4.6.6).

In addition, we have

$$\begin{aligned}y(0,x) &= \chi(x)\tilde{w}(0,x) + (1-\chi(x))\tilde{z}(0,x)\\ &= \chi(x)y_0(x) + (1-\chi(x))y_0(x) = y_0(x), \qquad x \in (0,1).\end{aligned}$$

In conclusion, y solves the memory system (4.6.6), and satisfies

$$y(T,\cdot) = \chi\tilde{w}(T,\cdot) + (1-\chi)\tilde{z}(T,\cdot) = 0 \qquad \text{in } (0,1).$$

Hence the claim follows. □

Chapter 5

Boundary controllability for coupled degenerate systems

The objective of this chapter is to study the controllability properties for one dimensional linear system of two coupled degenerate parabolic equations with one control force acting at the boundary of the space domain. In particular, we give necessary and sufficient conditions for the approximate and null controllability results. Our proofs are based on the moment method together with some properties of Bessel functions and their zeros.

The results obtained in this chapter are presented in the preprint [10].

5.1 Introduction

In this chapter, we deal with controllability issues for a class of coupled systems of one-dimensional degenerate parabolic equations, by a boundary control located at the end point of the interval $(0, 1)$. More precisely, we consider the following control system:

$$\begin{cases} \partial_t y - (x^\alpha y_x)_x = Ay, & \text{in } Q, \\ y(t,1) = Bv(t), & \text{in } (0, T), \\ \begin{cases} y(t,0) = 0, & 0 \le \alpha < 1 \\ x^\alpha y_x(t,0) = 0, & 1 \le \alpha < 2 \end{cases} & t \in (0,T), \\ y(0,x) = y_0(x), & \text{in } (0,1), \end{cases} \tag{5.1.1}$$

where $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ are given. Here $v = v(t)$ is the control function which only acts at one boundary point for all times and $y = (y_1, y_2)^*$ is the state variable. Further, $\alpha \ge 0$ represents the order of degeneracy of the diffusion coefficient that may vanish at $x = 0$.

We will see that, for every $v \in L^2(0,T)$ and $y_0 \in H_\alpha^{-1}(0,1)^2$, system (5.1.1) admits a unique weak solution defined by transposition that satisfies

$$y \in L^2(Q)^2 \cap C^0\big([0,T], H_\alpha^{-1}(0,1)^2\big).$$

Observe that the previous regularity permits to pose the boundary controllability of the degenerate system (5.1.1) in the space $H_\alpha^{-1}(0,1)^2$, defined later in section 5.2.

With the previous notations, we recall the following definitions:

Definition 5.1.1. System (5.1.1) is approximately controllable in $H_\alpha^{-1}(0,1)^2$ at time $T > 0$ if for every $y_0, y_d \in H_\alpha^{-1}(0,1)^2$ and any $\varepsilon > 0$, there exists a control function $v \in L^2(0,T)$ such that the solution y to system (5.1.1) satisfies

$$\|y(T,\cdot) - y_d\|_{H_\alpha^{-1}(0,1)^2} \le \varepsilon.$$

On the other hand, it will be said that system (5.1.1) is null controllable at time $T > 0$ if for every $y_0 \in H_\alpha^{-1}(0,1)^2$, there exists a control $v \in L^2(0,T)$ such that the solution y to system (5.1.1) satisfies

$$y(T,\cdot) = 0, \quad \text{in} \quad H_\alpha^{-1}(0,1)^2.$$

To our knowledge, the first distributed null-controllability for degenerate coupled parabolic systems has been treated in [61]. In particular, the authors consider a *cascade system* with the same diffusion coefficient, and recover distributed controllability results similar to those obtained in [108]. For more general systems of degenerate equations we refer to [2, 3, 84].

The main goal of this chapter is to provide an answer to the null and approximate controllability issues for the degenerate system (5.1.1) where the control is exerted at the boundary point $x = 1$. With this purpose, let us remind the result from [88], which asserts that a necessary condition for the controllability of this kind of systems is given by the following Kalman's rank condition:

$$rank[B|AB] = 2.$$

Moreover, as explained in [88], by taking $P = [B|AB]$, the change of variables

$$\tilde{y} = P^{-1}y,$$

leads to the following reformulation of (5.1.1):

$$\begin{cases} \partial_t \tilde{y} - (x^\alpha \tilde{y}_x)_x = \tilde{A}\tilde{y}, & \text{in } (0,T)\times(0,1), \\ \tilde{y}(t,1) = \tilde{B}v, & \text{in } (0,\,T), \\ \begin{cases} \tilde{y}(t,0) = 0, & 0 \le \alpha < 1 \\ x^\alpha \tilde{y}_x(t,0) = 0, & 1 \le \alpha < 2 \end{cases} & t \in (0,T), \\ \tilde{y}(0,x) = P^{-1}y_0(x), & \text{in } (0,1), \end{cases} \tag{5.1.2}$$

where

$$\tilde{A} = P^{-1}AP = \begin{pmatrix} 0 & a_1 \\ 1 & a_2 \end{pmatrix} \quad \text{and} \quad \tilde{B} = P^{-1}B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore, the controllability properties of system (5.1.1) can be obtained directly from the controllability results of system (5.1.2) passing through this mentioned change of variables. For simplicity, it will be assumed in the rest of this chapter that A and B are given by

$$A = \begin{pmatrix} 0 & a_1 \\ 1 & a_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5.1.3}$$

We would like to emphasize that imposing a control that acts at the nondegenerate point does not imply a simple adaptation of the previous distributed controllability results. For example, at a first glance, one may think that our boundary controllability results can be obtained directly by standard extension and localization arguments from the corresponding distributed controllability results as in the case of scalar parabolic equations. But this is not the case and the situation is quite different for non-scalar parabolic systems. Indeed, as pointed out in [88, 17], the boundary controllability is not equivalent and is more complex than distributed controllability. To be precise, while the Kalman's rank condition is a necessary and sufficient condition for the null controllability at any time in the distributed case, it was proved in [88] that it is necessary, but not sufficient, for the boundary controllability for coupled parabolic systems.

In this setting, our main result gives necessary and sufficient conditions for the null controllability of the system (5.1.1) (see Theorem 5.5.2). To this aim, we will follow the strategy initiated by Fattorini and Russell [87, 86], which is based on the moment method and results on biorthogonal sequences. In particular, we use techniques similar to those in [15, 31, 88], but adapted to our nonstandard degenerate situation.

Moreover, the proof of the null controllability property will rely on the following known result which relates the existence and bounds of biorthogonal families to complex exponentials to some gap conditions (see [31]).

Theorem 5.1.1. *Let $\{\Lambda_n\}_{n\geq 1}$ be a sequence of complex numbers fulfilling the following assumptions:*

1. $\Lambda_n \neq \Lambda_m, \quad \forall n, m \geq 1, \quad$ *with* $\quad n \neq m,$
2. $\mathcal{R}(\Lambda_n) > 0 \quad$ *for every* $\quad n \geq 1,$
3. *for some* $\delta > 0$
$$\mathcal{I}(\Lambda_n) \leq \delta\sqrt{\mathcal{R}(\Lambda_n)}, \quad \forall n \geq 1,$$
4. $\{\Lambda_n\}_{n\geq 1}$ *is non decreasing in modulus,*
$$|\Lambda_n| \leq |\Lambda_{n+1}| \quad \forall n \geq 1,$$
5. $\{\Lambda_n\}_{n\geq 1}$ *satisfies the following gap condition: for some* $\rho, q > 0,$
$$\left\{ \begin{array}{l} |\Lambda_n - \Lambda_m| \geq \rho|n^2 - m^2| \quad \forall n, m : |n - m| \geq q, \\ \inf\limits_{n\neq m,\ |n-m|<q} |\Lambda_n - \Lambda_m| > 0, \end{array} \right. \tag{5.1.4}$$
6. *for some* $p, s > 0,$
$$|p\sqrt{r} - \mathcal{N}(r)| \leq s, \quad \forall r > 0, \tag{5.1.5}$$
where $\mathcal{N}$ is the counting function associated with the sequence $\{\Lambda_n\}_{n\geq 1}$ that is the function defined by
$$\mathcal{N}(r) = Card\{n : \quad |\Lambda_n| \leq r\}, \qquad \forall r > 0. \tag{5.1.6}$$

Then, there exists $T_0 > 0$, such that for any $T \in (0, T_0)$, we can find a family $\{q_n\}_{n\geq 1} \subset L^2(-T/2, T/2)$ biorthogonal to $\{e^{-\Lambda_n t}\}_{n\geq 1}$ i.e., a family $\{q_n\}_{n\geq 1}$ in $L^2(-T/2, T/2)$ such that
$$\int_{-T/2}^{T/2} q_n(t)e^{-\Lambda_m t}\, dt = \delta_{nm}.$$
Moreover, there exists a positive constant $C > 0$ independent of T for which
$$\|q_n\|_{L^2(-T/2,T/2)} \leq Ce^{C\sqrt{\mathcal{R}(\Lambda_n)}+\frac{C}{T}}, \qquad \forall n \geq 1. \tag{5.1.7}$$
Here for $z \in \mathbb{C}$, $\mathcal{R}(z)$ and $\mathcal{I}(z)$ denote the real and imaginary parts of z.

The rest of the chapter is organized as follows. In Section 5.2, we prove the well-posedness of the problem (5.1.1) in appropriate weighted spaces using the transposition method and recall some characterizations of the controllability. In section 5.3, we discuss the spectral analysis related to scalar degenerate operators and present a description of the spectrum associated with system (5.1.1) which will be useful for developing the moment method. Section 5.4 is devoted to studying the boundary approximate controllability problem for the system (5.1.1). Finally, in section 5.5, we prove the boundary null controllability result.

5.2 Preliminary results

5.2.1 Function spaces and well-posedness

Let us start introducing the functional setting associated with degenerate operators from [5]. First of all, we denote by $\mathcal{H}^1_\alpha(0,1)$ the following weighted Sobolev space:

$$\mathcal{H}^1_\alpha(0,1) := \Big\{ y \in L^2(0,1) \cap H^1_{loc}((0,1]) : x^{\alpha/2} y_x \in L^2(0,1) \Big\}.$$

As pointed out in the first chapter of this thesis, since equation (5.1.1) is degenerate, different boundary conditions have to be imposed at $x = 0$ depending on the value of α. Indeed, for any $u \in \mathcal{H}^1_\alpha(0,1)$, the trace of u at $x = 1$ obviously makes sense which allows to consider homogeneous Dirichlet condition at $x = 1$. On the other hand, the trace of u at $x = 0$ only makes sense when $0 \leq \alpha < 1$. This leads us to consider, as in [56], the following weighted Hilbert spaces:

1. In the weakly degenerate case (WD), $0 \leq \alpha < 1$:

$$H^1_\alpha(0,1) := \Big\{ y \in L^2(0,1) : y \,\text{absolutely continuous in}\, [0,1], \\ x^{\alpha/2} y_x \in L^2(0,1) \,\text{and}\, y(1) = y(0) = 0 \Big\}$$

and

$$H^2_\alpha(0,1) := \Big\{ y \in H^1_\alpha(0,1) : x^\alpha y_x \in H^1(0,1) \Big\}.$$

2. In the strongly degenerate case (SD), $1 \leq \alpha < 2$:

$$H^1_\alpha(0,1) := \Big\{ y \in L^2(0,1) : y \,\text{locally absolutely continuous in}\, (0,1], \\ x^{\alpha/2} y_x \in L^2(0,1) \,\text{and}\, y(1) = 0 \Big\}$$

and

$$\begin{aligned} H^2_\alpha(0,1) :&= \Big\{ y \in H^1_\alpha(0,1) : x^\alpha y_x \in H^1(0,1) \Big\} \\ &= \Big\{ y \in L^2(0,1) : y \,\text{locally absolutely continuous in}\, (0,1], x^\alpha y \in H^1_0(0,1), \\ &\qquad x^\alpha y_x \in H^1(0,1) \,\text{and}\, (x^\alpha y_x)(0) = 0 \Big\}. \end{aligned}$$

In both cases, the norms are defined as follows

$$\|y\|^2_{H^1_\alpha} := \|y\|^2_{L^2(0,1)} + \|x^{\alpha/2} y_x\|^2_{L^2(0,1)}, \qquad \|y\|^2_{H^2_\alpha} := \|y\|^2_{H^1_\alpha} + \|(x^\alpha y_x)_x\|^2_{L^2(0,1)}.$$

Let $H^{-1}_\alpha(0,1)$ be the dual space of $H^1_\alpha(0,1)$ with respect to the pivot space $L^2(0,1)$, endowed with the natural norm

$$\|z\|_{H^{-1}_\alpha} := \sup_{\|y\|_{H^1_\alpha}=1} \langle z, y \rangle_{H^{-1}_\alpha, H^1_\alpha}.$$

In what follows, for simplicity, we will always denote by $\langle \cdot, \cdot \rangle$ the standard scalar product of either $L^2(0,1)$ or $L^2(0,1)^2$, by $\langle \cdot, \cdot \rangle_{X',X}$ the duality pairing between the Hilbert space X and its dual X'. On the other hand, we will use $\| \cdot \|_{H^1_\alpha}$ (resp. $\| \cdot \|_{H^{-1}_\alpha}$) for denoting the norm of $H^1_\alpha(0,1))^2$ (resp. $H^{-1}_\alpha(0,1)^2$).

Now, we are ready to investigate the well-posedness of the system (5.1.1). To this aim, let us consider the nonhomogeneous adjoint problem:

$$\begin{cases} -\partial_t\varphi - (x^\alpha \varphi_x)_x = A^*\varphi + g, & \text{in } (0,T)\times(0,1), \\ \varphi(t,1) = 0, & \text{on } (0,T), \\ \begin{cases} \varphi(t,0) = 0, & 0 \le \alpha < 1 \\ x^\alpha \varphi_x(t,0) = 0, & 1 \le \alpha < 2 \end{cases} & t \in (0,T), \\ \varphi(T,x) = \varphi_0, & \text{in } (0,1), \end{cases} \tag{5.2.1}$$

where φ_0 and g are functions in appropriate spaces.

Let us start with a result concerning the well-posedness of system (5.2.1) which is by now classical (see, for instance [56, Theorem 2.1]). One has

Proposition 5.2.1. *Assume that $\varphi_0 \in H^1_\alpha(0,1)^2$ and $g \in L^2(Q)^2$. Then, system (5.2.1) admits a unique strong solution*

$$\varphi \in L^2(0,T;H^2_\alpha(0,1)^2) \cap C^0([0,T];H^1_\alpha(0,1)^2)$$

such that

$$\|\varphi\|_{L^2(0,T;H^2_\alpha(0,1)^2)} + \|\varphi\|_{C^0([0,T];H^1_\alpha(0,1)^2)} \le C\Big(\|\varphi_0\|_{H^1_\alpha} + \|g\|_{L^2(Q)^2}\Big), \tag{5.2.2}$$

for some positive constant C.

In view of proposition 5.2.1, the following definition makes sense:

Definition 5.2.1. Let $y_0 \in H^{-1}_\alpha(0,1)^2$ and $v \in L^2(0,T)$ be given. It will be said that $y \in L^2(Q)^2$ is a solution by transposition to (5.1.1) if, for each $g \in L^2(Q)^2$, the following identity holds

$$\iint_Q y\cdot g\,dx\,dt = \langle y_0, \varphi(0,\cdot)\rangle_{H^{-1}_\alpha, H^1_\alpha} - \int_0^T B^*(x^\alpha\varphi_x)(t,1)\,v(t)\,dt, \tag{5.2.3}$$

where φ is associated to g through the following backward system

$$\begin{cases} -\partial_t\varphi - (x^\alpha \varphi_x)_x = A^*\varphi + g, & \text{on } (0,T)\times(0,1), \\ \varphi(t,1) = 0, & \text{in } (0,T), \\ \begin{cases} \varphi(t,0) = 0, & 0 \le \alpha < 1 \\ x^\alpha \varphi_x(t,0) = 0, & 1 \le \alpha < 2 \end{cases} & t \in (0,T), \\ \varphi(T,x) = 0, & \text{in } (0,1). \end{cases} \tag{5.2.4}$$

With this definition we can state the result of existence and uniqueness of solution to system (5.1.1).

Proposition 5.2.2. *Assume that $y_0 \in H^{-1}_\alpha(0,1)^2$ and $v \in L^2(0,T)$. Then, system (5.1.1) admits a unique solution by transposition y that satisfies*

$$\begin{cases} y \in L^2(Q)^2 \cap C^0\big([0,T], H^{-1}_\alpha(0,1)^2\big), \quad \partial_t y \in L^2\big(0,T;(H^2_\alpha(0,1)^2)'\big), \\ \partial_t y - (x^\alpha y_x)_x = Ay \quad \text{in} \quad L^2\big(0,T;(H^2_\alpha(0,1)^2)'\big), \\ y(0,\cdot) = y_0 \quad \text{in} \quad H^{-1}_\alpha(0,1)^2. \end{cases} \tag{5.2.5}$$

and

$$\|y\|_{L^2(Q)^2} + \|y\|_{C^0(H^{-1}_\alpha)} + \|y_t\|_{L^2((H^2_\alpha(0,1)^2)')} \le C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1}_\alpha}\big), \tag{5.2.6}$$

for a constant $C = C(T) > 0$.

Proof. The proof of this Proposition is based on some ideas from [88].
Let $y_0 \in H^{-1}_\alpha(0,1)^2$, $v \in L^2(0,T)$ and consider the following functional

$$\mathcal{T} : L^2(Q)^2 \to \mathbb{R}$$

given by

$$\mathcal{T}(g) = \langle y_0, \varphi(0,\cdot)\rangle_{H^{-1}_\alpha, H^1_\alpha} - \int_0^T B^*(x^\alpha \varphi_x)(t,1)\, v(t)\, dt,$$

where $\varphi \in C^0\big([0,T]; H^1_\alpha(0,1)^2\big) \cap L^2\big(0,T; H^2_\alpha(0,1)^2\big)$ is the solution of the adjoint system (5.2.4) associated to $g \in L^2(Q)^2$. From the estimate (5.2.2), it follows that

$$\big|\mathcal{T}(g)\big| \le C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1}_\alpha}\big)\|g\|_{L^2(Q)^2},$$

for all $g \in L^2(Q)^2$. Hence, $\mathcal{T}$ is bounded. As a consequence, by Riesz Representation Theorem, there exists a unique $y \in L^2(Q)^2$ satisfying (5.2.3). Moreover,

$$\|y\|_{L^2(Q)^2} = \|\mathcal{T}\| \le C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1}_\alpha}\big),$$

and y satisfies the equality $\partial_t y - (x^\alpha y_x)_x = Ay \quad \text{in} \quad \mathcal{D}'(Q)^2$.

Next, we are going to prove that the solution y of system (5.1.1) is more regular. To be precise, we show that $(x^\alpha y_x)_x \in L^2\big(0,T; (H^2_\alpha(0,1)^2)'\big)$ and

$$\|(x^\alpha y_x)_x\|_{L^2((H^2_\alpha(0,1)^2)')} \le C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H^{-1}_\alpha}\big). \tag{5.2.7}$$

For doing that, let us take two sequences $\{y_0^m\}_{m\ge1} \subset H^1_\alpha(0,1)^2$ and $\{v^m\}_{m\ge1} \subset H^1_0(0,T)$ such that

$$y_0^m \to y_0 \quad \text{in} \quad H^{-1}_\alpha(0,1)^2 \quad \text{and} \quad v^m \to v \quad \text{in} \quad L^2(0,T).$$

Now, the strategy consists in transforming our original system (5.1.1) into a problem with homogeneous boundary condition and a source term. To this end, let us introduce the change of variables

$$y^m(t,x) = \tilde{y}^m(t,x) + x^{2-\alpha} v^m(t) B,$$

where y^m is the solution of (5.1.1) associated to y_0^m and v^m. Then, formally, the new function $\tilde{y}^m$ satisfies the problem

$$\begin{cases} \partial_t \tilde{y}^m - (x^\alpha \tilde{y}^m_x)_x = A\tilde{y}^m + \tilde{f}^m(t,x), & \text{on } (0,T)\times(0,1), \\ \tilde{y}^m(t,1) = 0, & \text{in} \quad (0,T), \\ \begin{cases} \tilde{y}^m(t,0) = 0, & 0 \le \alpha < 1 \\ x^\alpha \tilde{y}^m_x(t,0) = 0, & 1 \le \alpha < 2 \end{cases} & t \in (0,T), \\ \tilde{y}^m(0,x) = y_0^m(x), & \text{in} \quad (0,1), \end{cases} \tag{5.2.8}$$

where $\tilde{f}^m(t,x) = [(2-\alpha)v^m(t) - x^{2-\alpha} v_t^m(t)]B + x^{2-\alpha} v^m(t) AB$. Moreover, an easy computation shows that the function $x \mapsto x^{2-\alpha}$ belongs to $\mathcal{H}^1_\alpha(0,1)$ which of course implies that $\tilde{f}^m \in L^2(Q)^2$. In view of the previous regularity assumptions we can apply Proposition 5.2.1, to deduce that system (5.2.8) has a unique solution

$$\tilde{y}^m \in L^2\big(0,T; H^2_\alpha(0,1)^2\big) \cap C^0\big(0,T; H^1_\alpha(0,1)^2\big).$$

Therefore, the problem (5.1.1) for v^m and y_0^m has a unique solution $y^m \in L^2\big(0,T; \mathcal{H}^1_\alpha(0,1)^2\big)$ which satisfies

$$\iint_Q y^m \cdot g \; dtdx = \langle y_0^m, \varphi(0,\cdot)\rangle_{H^{-1}_\alpha, H^1_\alpha} - \int_0^T B^*(x^\alpha \varphi_x)(t,1)\, v^m(t)\, dt, \quad \forall m \ge 1,$$

for all $g \in L^2(Q)^2$, where φ is the solution of the system (5.2.4) associated to g. Using this last identity and (5.2.3), we obtain

$$\begin{cases} \|y^m\|_{L^2(Q)^2} \leq C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H_\alpha^{-1}}\big) \\ y^m \to y \quad \text{in } L^2(Q)^2 \quad \text{and} \quad (x^\alpha y_x^m)_x \to (x^\alpha y_x)_x \quad \text{in } \mathcal{D}'(Q)^2. \end{cases} \tag{5.2.9}$$

On the other hand, integrations by parts lead to

$$\int_0^T \langle (x^\alpha y_x^m)_x, \psi\rangle \, dt = \iint_Q y^m \cdot (x^\alpha \psi_x)_x \, dtdx - \int_0^T B^*(x^\alpha \psi_x)(t,1)\, v^m(t)\, dt, \quad \forall m \geq 1$$

for every $\psi \in L^2\big(0,T; H_\alpha^2(0,1)^2\big)$. From this equality we infer that the sequence $(x^\alpha y_x^m)_x$ is bounded in $L^2\big(0,T;(H_\alpha^2(0,1)^2)'\big)$. Combining with (5.2.9), we deduce that $(x^\alpha y_x)_x$ belongs to $L^2\big(0,T;(H_\alpha^2(0,1)^2)'\big)$ and satisfies estimate (5.2.7). With the previous property in mind and the identity $\partial_t y - (x^\alpha y_x)_x = Ay$, we also see that $y_t \in L^2\big(0,T;(H_\alpha^2(0,1)^2)'\big)$ and

$$\|y_t\|_{L^2((H_\alpha^2(0,1)^2)')} \leq C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H_\alpha^{-1}}\big)$$

for a positive constant C. Therefore $y \in C([0,T];X^2)$, where X is the interpolation space $X = [L^2(0,1),(H_\alpha^2(0,1))']_{1/2} = H_\alpha^{-1}(0,1)$ (see [69, Theorem 11.4]). In conclusion, we get

$$\|y\|_{C(H_\alpha^{-1}(0,1)^2)} \leq C\big(\|v\|_{L^2(0,T)} + \|y_0\|_{H_\alpha^{-1}}\big).$$

Finally, one can easily check that $y(0,\cdot) = y_0$ in $H_\alpha^{-1}(0,1)^2$. This ends the proof. □

5.2.2 Duality

As it is well known, the controllability of system (5.1.1) can be characterized in terms of appropriate properties of the solutions of the corresponding homogeneous adjoint problem (see for instance [17, Theorem. 2.1], or [72, Theorem. 2.44]). Thus, we introduce the homogeneous backward adjoint problem associated with system (5.1.1)

$$\begin{cases} -\partial_t \varphi - (x^\alpha \varphi_x)_x = A^*\varphi, & \text{in } (0,T)\times(0,1), \\ \varphi(t,1) = 0, & \text{on } (0,\, T), \\ \begin{cases} \varphi(t,0) = 0, & 0 \leq \alpha < 1 \\ x^\alpha \varphi_x(t,0) = 0, & 1 \leq \alpha < 2 \end{cases} & t \in (0,T), \\ \varphi(T,x) = \varphi_0, & \text{in } (0,1), \end{cases} \tag{5.2.10}$$

where $\varphi_0 \in L^2(0,1)^2$.

In order to provide these characterizations, we use the following result which gives a relation between the solutions of systems (5.1.1) and (5.2.10) (For a proof, see for instance [88] or [156]).

Proposition 5.2.3. *Let B the matrix given by $B = \begin{pmatrix}1 & 0\end{pmatrix}^*$. Let us consider $y_0 \in H_\alpha^{-1}(0,1)^2$, $v \in L^2(0,T)$ and $\varphi_0 \in H_\alpha^1(0,1)^2$. Then, the solution y of system (5.1.1) associated to y_0 and v, and the solution φ of the adjoint system (5.2.10) associated to φ_0 satisfy*

$$\int_0^T v(t) B^*(x^\alpha \varphi_x)(t,1)\, dt = -\langle y(T), \varphi_0\rangle_{H_\alpha^{-1},H_\alpha^1} + \langle y_0, \varphi(0,\cdot)\rangle_{H_\alpha^{-1},H_\alpha^1}. \tag{5.2.11}$$

We have the following equivalent formulation of the approximate controllability:

Proposition 5.2.4. *System (5.1.1) is approximately controllable at time T if and only if for all initial condition $\varphi_0 \in H_\alpha^1(0,1)^2$ the solution to system (5.2.10) satisfies the unique continuation property*

$$B^*(x^\alpha \varphi_x)(\cdot,1) = 0 \quad \textit{on} \quad (0,T) \Rightarrow \varphi_0 = 0 \quad \textit{in} \quad (0,1) \quad (\textit{i.e.,} \quad \varphi = 0 \quad \textit{in} \quad Q).$$

5.3 Spectral analysis

5.3.1 Spectral properties of scalar degenerate operators

The knowledge of the eigenvalues and associated eigenfunctions of the degenerate diffusion operator $y \mapsto -(x^\alpha y')'$, will be essential for our purposes. It is worth mentioning that, an explicit expression of the eigenvalues is given in [111] for the weakly degenerate case $\alpha \in (0,1)$, and in [141] for the strongly degenerate case $\alpha \in [1,2)$, and depends on the Bessel functions of first kind (see [130, 162]). For this reason, we will start by giving a brief account of some results concerning the Bessel functions that will be useful in the rest of this thesis.

For a real number ν, we denote by J_ν the Bessel function of the first kind of order ν defined by the following Taylor series expansion around $x = 0$:

$$J_\nu(x) = \sum_{m\geq 0} \frac{(-1)^m}{m!\,\Gamma(1+\nu+m)} \left(\frac{x}{2}\right)^{2m+\nu},$$

where $\Gamma(.)$ is the Gamma function.

We recall that $\forall \nu \in \mathbb{R}$, the Bessel function J_ν satisfies the following differential equation

$$x^2 y_{xx} + x y_x + (x^2 - \nu^2) y = 0 \qquad x \in (0, +\infty).$$

Besides, the function J_ν has an infinite number of real zeros which are simple with the possible exception of $x = 0$ (see [81]). We denote by $(j_{\nu,n})_{n\geq 1}$ the strictly increasing sequence of the positive zeros of J_ν:

$$j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n} < \cdots$$

and we recall that

$$j_{\nu,n} \to +\infty \quad \text{as} \quad n \to +\infty$$

and the following bounds on the zeros $j_{\nu,n}$, which are provided in [138]:

- $\forall \nu \in \left(0, \frac{1}{2}\right]$, $\forall n \geq 1$,

$$\left(n + \frac{\nu}{2} - \frac{1}{4}\right)\pi \leq j_{\nu,n} \leq \left(n + \frac{\nu}{4} - \frac{1}{8}\right)\pi. \tag{5.3.1}$$

- $\forall \nu \geq \frac{1}{2}$, $\forall n \geq 1$,

$$\left(n + \frac{\nu}{4} - \frac{1}{8}\right)\pi \leq j_{\nu,n} \leq \left(n + \frac{\nu}{2} - \frac{1}{4}\right)\pi. \tag{5.3.2}$$

Moreover, we have the following results (see [124, Proposition 7.8]):

Lemma 5.3.1. *Let $j_{\nu,n}, n \geq 1$ be the positive zeros of the Bessel function J_ν. Then, the following holds:*

- *The difference sequence $(j_{\nu,n+1} - j_{\nu,n})_n$ converges to π as $n \longrightarrow +\infty$.*
- *The sequence $(j_{\nu,n+1} - j_{\nu,n})_n$ is strictly decreasing if $|\nu| > \frac{1}{2}$, strictly increasing if $|\nu| < \frac{1}{2}$, and constant if $\nu = \frac{1}{2}$.*

We also have that the Bessel functions enjoy the following integral formula (see [162]):

$$\int_0^1 x J_\nu(j_{\nu,n} x) J_\nu(j_{\nu,m} x)\, dx = \frac{\delta_{nm}}{2} [J'_\nu(j_{\nu,n})]^2, \quad n, m \in \mathbb{N}^*,$$

where, δ_{nm} is the Kronecker symbol.

Now, we give the explicit expression of the spectrum of the operator $y \mapsto -(x^\alpha y')'$, i.e., the nontrivial solutions (λ, Φ) of

$$\begin{cases} -(x^\alpha \Phi'(x))' = \lambda \Phi(x), \quad x \in (0,1), \\ \Phi(1) = 0, \\ \begin{cases} \Phi(0) = 0, \quad \text{in the (WD) case} \\ (x^\alpha \Phi_x)(0) = 0, \quad \text{in the (WD) case.} \end{cases} \end{cases} \tag{5.3.3}$$

Let $\kappa_\alpha = \dfrac{2-\alpha}{2} > 0$. From now on, we set

$$\begin{cases} \nu_\alpha = \dfrac{1-\alpha}{2-\alpha} \in \Big(0, \dfrac{1}{2}\Big], \quad \text{in the (WD) case} \\ \nu_\alpha = \dfrac{\alpha-1}{2-\alpha} \geq 0, \quad \text{in the (SD) case.} \end{cases}$$

Then, one has (see [111, 141]):

Proposition 5.3.1. *The admissible eigenvalues λ for problem (5.3.3) are given by*

$$\lambda_{\nu_\alpha,n} = \kappa_\alpha^2 j_{\nu_\alpha,n}^2, \qquad \forall n \geq 1. \tag{5.3.4}$$

and the associated normalized (in $L^2(0,1)$) eigenfunctions takes the form

$$\Phi_{\nu_\alpha,n}(x) = \frac{\sqrt{2\kappa_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}(j_{\nu_\alpha,n} x^{\kappa_\alpha}), \qquad x \in (0,1), \quad n \geq 1. \tag{5.3.5}$$

Moreover, the family $(\Phi_{\nu_\alpha,n})_{n\geq 1}$ forms an orthonormal basis of $L^2(0,1)$.

In both cases of degeneracy, the spectrum of the associated degenerate operator satisfies the following properties.

Lemma 5.3.2. *Let $(\lambda_{\nu_\alpha,k})_{k\geq 1}$ be the sequence of eigenvalues of the spectral problem (5.3.3). Then, the following properties hold:*

1. *For all $n, m \in \mathbb{N}^\star$, there is a constant $\rho_\alpha > 0$ such that the sequence of eigenvalues $(\lambda_{\nu_\alpha,n})_{n\geq 1}$ satisfy the gap condition:*

$$|\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,m}| \geq \rho |n^2 - m^2|, \qquad \forall n, m \geq 1. \tag{5.3.6}$$

2. *The series $\displaystyle\sum_{n\geq 1} \frac{1}{\lambda_{\nu_\alpha,n}}$ is convergent.*

Remark 34. Note that the gap condition (5.3.6) is stronger than the following separation property

$$|\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,m}| \geq \rho |n - m|, \qquad \forall n, m \geq 1. \tag{5.3.7}$$

Proof. 1. Let $n, m \in \mathbb{N}^\star$ with $n \geq m$. We have

$$\begin{aligned} &\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,m} \\ &= \kappa_\alpha^2 (j_{\nu_\alpha,n}^2 - j_{\nu_\alpha,m}^2) \\ &= \kappa_\alpha^2 (j_{\nu_\alpha,n} - j_{\nu_\alpha,m})(j_{\nu_\alpha,n} + j_{\nu_\alpha,m}) \\ &= \kappa_\alpha^2 \Big((j_{\nu_\alpha,n} - j_{\nu_\alpha,n-1}) + \ldots + (j_{\nu_\alpha,m+1} - j_{\nu_\alpha,m}) \Big)(j_{\nu_\alpha,n} + j_{\nu_\alpha,m}). \end{aligned} \tag{5.3.8}$$

Now, if $\alpha \in [0,1)$. In this situation $\nu_\alpha \in \big(0, \frac{1}{2}\big]$, then thanks to Lemma 5.3.1 we immediately have that

$$j_{\nu_\alpha,n} - j_{\nu_\alpha,n-1} \geq j_{\nu_\alpha,2} - j_{\nu_\alpha,1}, \quad \forall n \geq 2.$$

Therefore,

$$\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,m} \geq \kappa_\alpha^2 (n-m)(j_{\nu_\alpha,2} - j_{\nu_\alpha,1})(j_{\nu_\alpha,n} + j_{\nu_\alpha,m}).$$

Using (5.3.1), the last inequality becomes:

$$\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,m} \geq \frac{7}{8}\pi^2\kappa_\alpha^2 (n-m)\big(n+m+\nu_\alpha - \frac{1}{2}\big).$$

Taking into account the fact that

$$\big(n+m+\nu_\alpha - \frac{1}{2}\big) > \frac{n+m}{2},$$

we deduce that there exists $\rho_\alpha = \frac{7}{16}\pi^2\kappa_\alpha^2$ such that

$$\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,m} \geq \rho_\alpha (n^2 - m^2).$$

If $\alpha \in [1,2)$. We discuss two different sub-cases: $\alpha \in [1, \frac{4}{3}]$ (i.e. $\nu_\alpha \leq \frac{1}{2}$) and $\alpha \in [\frac{4}{3}, 2)$ (i.e. $\nu_\alpha \geq \frac{1}{2}$).

When $\nu_\alpha \leq \frac{1}{2}$ the proof is similar to the one treated above. When $\nu_\alpha \geq \frac{1}{2}$, by Lemma 5.3.1, the sequence $(j_{\nu_\alpha,n} - j_{\nu_\alpha,n-1})_{n\geq 2}$ is decreasing and converge to π, and thus

$$j_{\nu_\alpha,n} - j_{\nu_\alpha,n-1} \geq \pi, \quad \forall n \geq 2.$$

Hence, (5.3.8) becomes

$$\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,m} \geq \kappa_\alpha^2 \pi (n-m)(j_{\nu_\alpha,n} + j_{\nu_\alpha,m}). \tag{5.3.9}$$

Owing to (5.3.2),

$$\begin{aligned} j_{\nu_\alpha,n} + j_{\nu_\alpha,m} &\geq \big(n + \frac{\nu_\alpha}{4} - \frac{1}{8}\big)\pi + \big(m + \frac{\nu_\alpha}{4} - \frac{1}{8}\big)\pi \\ &\geq \big(n+m+\frac{\nu_\alpha}{2} - \frac{1}{4}\big)\pi \geq n+m. \end{aligned} \tag{5.3.10}$$

Combining (5.3.9) and (5.3.10), the thesis follows with $\rho_\alpha = \pi\kappa_\alpha^2$.

Thus, in every case there holds

$$\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,m} \geq \rho_\alpha (n^2 - m^2).$$

After reversing the roles of n and m, one has

$$\lambda_{\nu_\alpha,m} - \lambda_{\nu_\alpha,n} \geq \rho_\alpha (m^2 - n^2).$$

Consequently,

$$|\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,m}| \geq \rho_\alpha |n^2 - m^2|, \qquad \forall n, m \geq 1.$$

2. This point follows easily form (5.3.1) and (5.3.2). Indeed, when $\nu_\alpha \leq \frac{1}{2}$, from (5.3.1) we have

$$\big(n - \frac{1}{4}\big)\pi \leq j_{\nu_\alpha,n} \leq n\pi, \quad \forall n \geq 1.$$

Therefore

$$\sum_{n\geq 1}\frac{1}{\lambda_{\nu_\alpha,n}}\leq\frac{1}{\kappa_\alpha^2\pi^2}\sum_{n\geq 1}\frac{1}{\left(n-\frac{1}{4}\right)^2}\leq\frac{4}{\kappa_\alpha^2\pi^2}\sum_{n\geq 1}\frac{1}{n^2}<+\infty.$$

Similarly, if $\nu_\alpha\geq\dfrac{1}{2}$, then from (5.3.2) we obtain

$$n\pi\leq j_{\nu_\alpha,n}\leq\left(n+\frac{\nu_\alpha}{2}\right)\pi,\quad\forall n\geq 1.$$

Thus

$$\sum_{n\geq 1}\frac{1}{\lambda_{\nu_\alpha,n}}\leq\frac{1}{\kappa_\alpha^2\pi^2}\sum_{n\geq 1}\frac{1}{n^2}<+\infty.$$

This proves the second point and finishes the proof.

□

5.3.2 Spectral properties of vectorial degenerate operators

Let us consider the degenerate vectorial operators

$$L:=\begin{pmatrix}-\partial_x(x^\alpha\partial_x\cdot) & 0\\ 0 & -\partial_x(x^\alpha\partial_x\cdot)\end{pmatrix}-A: D(L)\subset L^2(0,1)^2\to L^2(0,1)^2$$

and also its adjoint

$$L^*:=\begin{pmatrix}-\partial_x(x^\alpha\partial_x\cdot) & 0\\ 0 & -\partial_x(x^\alpha\partial_x\cdot)\end{pmatrix}-A^*$$

with domains $D(L)=D(L^*)=H^2_\alpha(0,1)^2$.

This section will be devoted to giving some spectral properties of the operators L and L^* which will be useful for developing the moment method.

We have the following result:

Proposition 5.3.2. *1. The spectra of L and L^* are given by*

$$\sigma(L)=\sigma(L^*)=\left\{\lambda^{(1)}_{\nu_\alpha,n},\lambda^{(2)}_{\nu_\alpha,n}\right\}_{n\geq 1}=\left\{\kappa_\alpha^2 j^2_{\nu_\alpha,n}-\mu_1,\ \kappa_\alpha^2 j^2_{\nu_\alpha,n}-\mu_2\right\}_{n\geq 1}\tag{5.3.11}$$

where μ_1 and μ_2 are the eigenvalues of the matrix A defined by :

- *Case 1:* $a_2^2+4a_1>0$,

$$\mu_1=\frac{1}{2}\left(a_2-\sqrt{a_2^2+4a_1}\right)\quad\text{and}\quad\mu_2=\frac{1}{2}\left(a_2+\sqrt{a_2^2+4a_1}\right),\tag{5.3.12}$$

- *Case 2:* $a_2^2+4a_1<0$,

$$\mu_1=\frac{1}{2}\left(a_2+i\sqrt{-(a_2^2+4a_1)}\right)\quad\text{and}\quad\mu_2=\frac{1}{2}\left(a_2-i\sqrt{-(a_2^2+4a_1)}\right),\tag{5.3.13}$$

2. For each $n\geq 1$, the corresponding eigenfunctions of L (resp., L^*) associated to $\lambda^{(1)}_{\nu_\alpha,n}$ and $\lambda^{(2)}_{\nu_\alpha,n}$ are respectively given by

$$\psi^{(1)}_n=U_1\Phi_{\nu_\alpha,n},\qquad\psi^{(2)}_n=U_2\Phi_{\nu_\alpha,n},\tag{5.3.14}$$

with

$$U_1=\frac{a_1}{\mu_2^2+a_1}\begin{pmatrix}-\mu_2\\ 1\end{pmatrix}\quad\text{and}\quad U_2=\begin{pmatrix}-\mu_1\\ 1\end{pmatrix}.$$

(resp.,

$$\Psi^{(1)}_n=V_1\Phi_{\nu_\alpha,n},\qquad\Psi^{(2)}_n=V_2\Phi_{\nu_\alpha,n},\tag{5.3.15}$$

with

$$V_1=\begin{pmatrix}-\dfrac{\mu_2}{a_1}\\ 1\end{pmatrix}\quad\text{and}\quad V_2=\frac{a_1}{\mu_1^2+a_1}\begin{pmatrix}-\dfrac{\mu_1}{a_1}\\ 1\end{pmatrix}).$$

Proof. The proof of this Theorem is inspired by [107]. We will prove the result for the operator L. The same reasoning provides the proof for its adjoint L^*.

We look for a complex λ and a function $\psi \in H^2(0,1;\mathbb{C}^2) \cap H^1_a(0,1;\mathbb{C}^2)$ such that $\psi \not\equiv 0$ and $L(\psi) = \lambda\psi$. Using the fact that the function $\Phi_{\nu_\alpha,n}$ is the eigenfunction of the degenerate elliptic operator $-\partial_x(x^\alpha \partial_x \cdot)$ associated to the eigenvalues $\lambda_{\nu_\alpha,n} = \kappa_\alpha^2 j^2_{\nu_\alpha,n}$, we can find ψ as

$$\psi(x) = \sum_{n\geq 1} a_n \Phi_{\nu_\alpha,n}(x), \quad \forall x \in (0,1),$$

where $\{a_n\}_{n\geq 1} \subset \mathbb{C}^2$ and, for some $k \geq 1$, $a_k \neq 0$. From the identity $L(\psi) = \lambda\psi$ we deduce

$$\sum_{n\geq 1} \left(\kappa_\alpha^2 j^2_{\nu_\alpha,n} I - A - \lambda I\right) a_n \Phi_{\nu_\alpha,n}(x) = 0, \quad \forall x \in (0,1).$$

From this identity, it is clear that the eigenvalues of the operator L correspond to the eigenvalues of the matrices

$$\kappa_\alpha^2 j^2_{\nu_\alpha,n} I - A, \qquad \forall n \geq 1$$

and the associated eigenfunctions of L are given choosing $a_n = z_k \delta_{kn}$, for any $n \geq 1$, where $z_k \in \mathbb{C}^2$ is an associated eigenvector of $j^2_{\nu_\alpha,n} I - A$, that is to say, $\psi_n(\cdot) = z_n \Phi_{\nu_\alpha,n}(\cdot)$.

Taking into account the expression of the characteristic polynomial of $\kappa_\alpha^2 j^2_{\nu_\alpha,n} I - A$:

$$P(z) = z^2 - z(2\lambda_{\nu_\alpha,n} - a_2) + \lambda_{\nu_\alpha,n}(\lambda_{\nu_\alpha,n} - a_2) - a_1, \qquad n \geq 1,$$

a direct computation provides the formulas (5.3.11) and (5.3.14) as eigenvalues and associated eigenfunctions of the operator L. This ends the proof. □

Let us now check that the sequence of eigenvalues of L and L^* fulfills the conditions in Theorem 5.1.1. One has

Proposition 5.3.3. *Assume that condition (5.4.2) holds. Then, one can construct a family from the spectrum* $\{\lambda^{(1)}_{\nu_\alpha,n}, \lambda^{(2)}_{\nu_\alpha,n}\}_{n\geq 1}$ *defined by*

$$\begin{aligned} \{\Lambda_{\nu_\alpha,n}\}_{n\geq 1} &= \{\lambda_{\nu_\alpha,k} + \mu_2 - \mu_1 : k \geq 1\} \cup \{\lambda_{\nu_\alpha,k} : k \geq 1\} \\ &= \{\lambda^{(1)}_{\nu_\alpha,n} + \mu_2, \lambda^{(2)}_{\nu_\alpha,n} + \mu_2\}_{n\geq 1}, \end{aligned} \tag{5.3.16}$$

which satisfies the hypotheses in Theorem 5.1.1.

Proof. We distinguish between three cases depending on the spectrum of matrix A.

Case 1: A has two real eigenvalues μ_1 and μ_2, chosen such that $\mu_1 < \mu_2$.
Let us consider the sequence $\{\Lambda_{\nu_\alpha,n}\}_{n\geq 1}$ defined in (5.3.16) by

$$\{\Lambda_{\nu_\alpha,n}\}_{n\geq 1} = \{\lambda^{(1)}_{\nu_\alpha,n} + \mu_2, \lambda^{(2)}_{\nu_\alpha,n} + \mu_2\}_{n\geq 1}.$$

The hypothesis 1) holds true if and only if the condition (5.4.2) is satisfied. In addition, the hypotheses 2) and 3) are obviously satisfied by definition.

Let us now show the hypothesis 4). To this aim, it suffices to prove that the indexes can be fixed in such a way that

$$\Lambda_{\nu_\alpha,n} \leq \Lambda_{\nu_\alpha,n+1}, \quad \forall n \geq 1.$$

Since $\mu_2 - \mu_1 > 0$, we observe that $\Lambda_{\nu_\alpha,1} = \lambda_{\nu_\alpha,1} > 0$, and thus $\Lambda_{\nu_\alpha,n} > 0, \quad \forall n \geq 1$.

We start by studying the case where $\nu_\alpha \in \left(0, \frac{1}{2}\right]$, that is $\alpha \in [0,1) \cup [1, \frac{4}{3}]$. Denoting by $[x]$ the integer part of x, we see that, whenever

$$n \geq n_0 := \left[\frac{\mu_2 - \mu_1}{2\kappa_\alpha^2 \pi (j_{\nu_\alpha,2} - j_{\nu_\alpha,1})} + \frac{3}{4} - \frac{\nu_\alpha}{2}\right] + 1 \tag{5.3.17}$$

one has

$$\lambda^{(1)}_{\nu_\alpha,n-1} < \lambda^{(2)}_{\nu_\alpha,n} < \lambda^{(1)}_{\nu_\alpha,n} < \lambda^{(2)}_{\nu_\alpha,n+1}.$$

Indeed, using the expressions of $\lambda^{(1)}_{\nu_\alpha,n}$ and $\lambda^{(2)}_{\nu_\alpha,n}$ and the bound given in (5.3.1), we get

$$\begin{aligned}\lambda^{(2)}_{\nu_\alpha,n} - \lambda^{(1)}_{\nu_\alpha,n-1} &= \kappa_\alpha^2 (j^2_{\nu_\alpha,n} - j^2_{\nu_\alpha,n-1}) + \mu_1 - \mu_2 \\ &= \kappa_\alpha^2 (j_{\nu_\alpha,n} - j_{\nu_\alpha,n-1})(j_{\nu_\alpha,n} + j_{\nu_\alpha,n-1}) + \mu_1 - \mu_2 \\ &\geq \kappa_\alpha^2 \pi \big(2n + \nu_\alpha - \frac{3}{2}\big)(j_{\nu_\alpha,n} - j_{\nu_\alpha,n-1}) + \mu_1 - \mu_2.\end{aligned}$$

By Lemma 5.3.1, we infer that

$$\lambda^{(2)}_{\nu_\alpha,n} - \lambda^{(1)}_{\nu_\alpha,n-1} \geq \kappa_\alpha^2 \pi \big(2n + \nu_\alpha - \frac{3}{2}\big)(j_{\nu_\alpha,2} - j_{\nu_\alpha,1}) + \mu_1 - \mu_2. \tag{5.3.18}$$

Now, in order to obtain $\lambda^{(2)}_{\nu_\alpha,n} - \lambda^{(1)}_{\nu_\alpha,n-1} > 0$, it suffices to take $n \geq n_0$ where the integer $n_0 \geq 1$ is defined in (5.3.17) and the claim follows.

Let us now treat the case $\nu_\alpha \geq \frac{1}{2}$ (i.e., $\alpha \in [\frac{4}{3}, 2)$). Proceeding in a similar manner, by applying (5.3.2) instead of (5.3.1), we obtain

$$\begin{aligned}\lambda^{(2)}_{\nu_\alpha,n} - \lambda^{(1)}_{\nu_\alpha,n-1} &= \kappa_\alpha^2 (j^2_{\nu_\alpha,n} - j^2_{\nu_\alpha,n-1}) + \mu_1 - \mu_2 \\ &\geq \kappa_\alpha^2 \pi^2 \big(2n - \frac{5}{4} + \frac{\nu_\alpha}{2}\big) + \mu_1 - \mu_2 \\ &\geq \kappa_\alpha^2 \pi^2 (2n - 1) + \mu_1 - \mu_2.\end{aligned} \tag{5.3.19}$$

At this point, we see that, whenever

$$n \geq n_0 := \Big[\frac{\mu_2 - \mu_1}{2\kappa_\alpha^2 \pi^2} + \frac{1}{2}\Big] + 1, \tag{5.3.20}$$

one has

$$\lambda^{(1)}_{\nu_\alpha,n-1} < \lambda^{(2)}_{\nu_\alpha,n} < \lambda^{(1)}_{\nu_\alpha,n} < \lambda^{(2)}_{\nu_\alpha,n+1}.$$

This shows hypothesis 4).

Let us move to prove hypothesis 5). For this purpose, we are going to use the following notations:

$$\Lambda_{\nu_\alpha,2n-1} = \lambda^{(2)}_{\nu_\alpha,n} + \mu_2, \qquad \Lambda_{\nu_\alpha,2n} = \lambda^{(1)}_{\nu_\alpha,n} + \mu_2, \quad \forall n \geq 1.$$

At first, let us check that the sequence $\big\{\Lambda_{\nu_\alpha,n}\big\}_{n\geq 1}$ satisfies

$$\inf_{n,k\geq 1: n\neq k} |\Lambda_{\nu_\alpha,n} - \Lambda_{\nu_\alpha,k}| > 0. \tag{5.3.21}$$

Using (5.3.18) and (5.3.19), then for $i = 1, 2$, we get

$$\lambda^{(2)}_{\nu_\alpha,n} - \lambda^{(1)}_{\nu_\alpha,n-1},\ \lambda^{(i)}_{\nu_\alpha,n} - \lambda^{(i)}_{\nu_\alpha,n-1} \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Thus, there exists $k_0 \in \mathbb{N}$ and $C > 0$ such that

$$|\Lambda_{\nu_\alpha,n} - \Lambda_{\nu_\alpha,n-1}| > C, \quad \forall n \geq k_0.$$

Combining the above estimate with the hypothesis 1), the property (5.3.21) holds. As a consequence, the second property of (5.1.4) is satisfied for any $q \geq 1$.

Now, we proceed to prove the second property of (5.1.4). By (5.3.6), we have

$$|\Lambda_{\nu_\alpha,2n} - \Lambda_{\nu_\alpha,2k}| = |\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,k}| \geq \rho_\alpha |n^2 - k^2| = \frac{\rho_\alpha}{4}|(2n)^2 - (2k)^2|,$$

and

$$|\Lambda_{\nu_\alpha,2n-1} - \Lambda_{\nu_\alpha,2k-1}| = |\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,k}| \geq \rho_\alpha |n^2 - k^2| \geq \frac{\rho_\alpha}{4}|(2n-1)^2 - (2k-1)^2|.$$

Moreover, denoting $\tilde{n} = 2n$ and $\tilde{k} = 2k-1$ and using (5.3.6), we see that

$$\begin{aligned} |\Lambda_{\nu_\alpha,\tilde{n}} - \Lambda_{\nu_\alpha,\tilde{k}}| &= \left|\lambda^{(1)}_{\nu_\alpha,n} - \lambda^{(2)}_{\nu_\alpha,k}\right| \\ &= \left|\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,k} + (\mu_2 - \mu_1)\right| \\ &\geq \rho_\alpha |n^2 - k^2| - (\mu_2 - \mu_1) \\ &= \frac{\rho_\alpha}{4}|\tilde{n}^2 - (\tilde{k}+1)^2| - (\mu_2 - \mu_1) \\ &= \frac{\rho_\alpha}{4}|\tilde{n}^2 - \tilde{k}^2 - 2\tilde{k} - 1| - (\mu_2 - \mu_1). \end{aligned} \tag{5.3.22}$$

Observe that, if $\tilde{n} < \tilde{k}$, we have,

$$|\Lambda_{\nu_\alpha,\tilde{n}} - \Lambda_{\nu_\alpha,\tilde{k}}| \geq \frac{\rho_\alpha}{4}(\tilde{k}^2 - \tilde{n}^2)\Big(1 - \frac{4(\mu_2-\mu_1)}{\rho_\alpha(\tilde{k}^2 - \tilde{n}^2)}\Big)$$

Taking into account the fact that $\tilde{k} + \tilde{n} \geq 2$ and choosing $q_1 \geq \dfrac{4(\mu_2 - \mu_1)}{\rho_\alpha}$, then $\forall \tilde{k}, \tilde{n} \geq 1$ with $|\tilde{k} - \tilde{n}| \geq q_1$, we obtain

$$\begin{aligned} |\Lambda_{\nu_\alpha,\tilde{n}} - \Lambda_{\nu_\alpha,\tilde{k}}| &\geq \frac{\rho_\alpha}{4}(\tilde{k}^2 - \tilde{n}^2)\Big(1 - \frac{2(\mu_2-\mu_1)}{\rho_\alpha q_1}\Big) \\ &\geq \frac{\rho_\alpha}{8}(\tilde{k}^2 - \tilde{n}^2) \end{aligned} \tag{5.3.23}$$

On the other hand, if $\tilde{n} > \tilde{k}$, one has

$$|\Lambda_{\nu_\alpha,\tilde{n}} - \Lambda_{\nu_\alpha,\tilde{k}}| \geq \frac{\rho_\alpha}{4}(\tilde{n}^2 - \tilde{k}^2)\Big(1 - \Big(\frac{4(\mu_2-\mu_1)}{\rho_\alpha} + 2\tilde{k} + 1\Big)\frac{1}{(\tilde{n}^2 - \tilde{k}^2)}\Big).$$

Therefore, taking $q_2 \geq \dfrac{4(\mu_2 - \mu_1)}{\rho_\alpha} + 4$ so that $\forall \tilde{k}, \tilde{n} \geq 1$ with $|\tilde{k} - \tilde{n}| \geq q_2$ and having in mind the fact that $2\tilde{k} + 1 \leq 4\tilde{k}$, it follows that

$$\begin{aligned} |\Lambda_{\nu_\alpha,\tilde{n}} - \Lambda_{\nu_\alpha,\tilde{k}}| &\geq \frac{\rho_\alpha}{4}(\tilde{n}^2 - \tilde{k}^2)\Big(1 - \Big(\frac{4(\mu_2-\mu_1)}{\rho_\alpha} + 2\tilde{k} + 1\Big)\frac{1}{2\tilde{k}q_2}\Big) \\ &\geq \frac{\rho_\alpha}{4}(\tilde{n}^2 - \tilde{k}^2)\Big(1 - \frac{1}{q_2}\Big(\frac{2(\mu_2-\mu_1)}{\rho_\alpha} + 2\Big)\Big) \\ &\geq \frac{\rho_\alpha}{8}(\tilde{n}^2 - \tilde{k}^2) \end{aligned} \tag{5.3.24}$$

Thus, choosing $q = max\{q_1, q_2\}$, the gap condition in (5.1.4) follows immediately.

Let us now show the hypothesis 6). From the definition of Λ_{ν_α} in (5.3.16), for any $r > 0$, we can see that:

$$\begin{aligned} \mathcal{N}(r) &= Card\{k : \ \lambda_{\nu_\alpha,k} \leq r\} + Card\{k : \ \lambda_{\nu_\alpha,k} + \mu_2 - \mu_1 \leq r\} \\ &= \mathcal{N}_1(r) + \mathcal{N}_2(r) = n_1 + n_2. \end{aligned}$$

We now proceed to prove suitable estimates for n_1 and n_2.
As before, we distinguish two cases depending on the value of ν_α. First, we consider the case where $\nu_\alpha \leq \dfrac{1}{2}$. We look for n_1 satisfying

$$\lambda_{\nu_\alpha,n_1} \leq r.$$

By (5.3.1), it follows that

$$\kappa_\alpha^2\big(n_1+\frac{\nu_\alpha}{2}-\frac{1}{4}\big)^2\pi^2 \le r.$$

So that

$$n_1 \le \frac{\sqrt{r}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{2}+\frac{1}{4}. \tag{5.3.25}$$

On the other hand, using (5.3.2), one has

$$\lambda_{\nu_\alpha,n_1+1} > r$$

which implies that

$$n_1 > \frac{\sqrt{r}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{4}-\frac{7}{8}. \tag{5.3.26}$$

Summarizing, n_1 is a nonnegative integer such that

$$\frac{\sqrt{r}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{4}-\frac{7}{8} < n_1 \le \frac{\sqrt{r}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{2}+\frac{1}{4}. \tag{5.3.27}$$

The case $\nu_\alpha \ge \frac{1}{2}$ can be treated in a similar way, but, instead of working with bounds (5.3.1), we will use (5.3.2) to obtain

$$\frac{\sqrt{r}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{2}-\frac{3}{4} < n_1 \le \frac{\sqrt{r}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{4}+\frac{1}{8}. \tag{5.3.28}$$

Next we are going to estimate n_2. Let us start by the case $\nu_\alpha \le \frac{1}{2}$. Using arguments similar to the ones used above, we can see that

$$\lambda_{\nu_\alpha,n_2}+\mu_2-\mu_1 \le r$$

and

$$\lambda_{\nu_\alpha,n_2+1}+\mu_2-\mu_1 > r$$

imply

$$\frac{\sqrt{r+\mu_1-\mu_2}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{4}-\frac{7}{8} < n_2 \le \frac{\sqrt{r+\mu_1-\mu_2}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{2}+\frac{1}{4}.$$

Then, using the fact that $\sqrt{a}-\sqrt{b}\le\sqrt{a-b}$ and $\sqrt{a-b}\le\sqrt{a}$ provided $a \ge b > 0$, we deduce

$$\frac{\sqrt{r}}{\kappa_\alpha \pi}-\frac{\sqrt{\mu_2-\mu_1}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{4}-\frac{7}{8} < n_2 \le \frac{\sqrt{r}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{2}+\frac{1}{4}. \tag{5.3.29}$$

Similarly, in the case where $\nu_\alpha \ge \frac{1}{2}$, we get

$$\frac{\sqrt{r}}{\kappa_\alpha \pi}-\frac{\sqrt{\mu_2-\mu_1}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{2}-\frac{3}{4} < n_2 \le \frac{\sqrt{r}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{4}+\frac{1}{8}. \tag{5.3.30}$$

Next, combining (5.3.27) and (5.3.29), it follows that for $\nu_\alpha \le \frac{1}{2}$,

$$\frac{2\sqrt{r}}{\kappa_\alpha \pi}-\frac{\sqrt{\mu_2-\mu_1}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{2}-\frac{7}{4} < \mathcal{N}(r) \le \frac{2\sqrt{r}}{\kappa_\alpha \pi}-\nu_\alpha+\frac{1}{2}. \tag{5.3.31}$$

The bounds (5.3.28) together with (5.3.30) gives for $\nu_\alpha \ge \frac{1}{2}$,

$$\frac{2\sqrt{r}}{\kappa_\alpha \pi}-\frac{\sqrt{\mu_2-\mu_1}}{\kappa_\alpha \pi}-\nu_\alpha-\frac{3}{2} < \mathcal{N}(r) \le \frac{2\sqrt{r}}{\kappa_\alpha \pi}-\frac{\nu_\alpha}{2}+\frac{1}{4} \tag{5.3.32}$$

Finally, choosing $p = \dfrac{2}{\kappa_\alpha \pi}$ and for $\nu_\alpha \leq \dfrac{1}{2}$ (resp. $\nu_\alpha \geq \dfrac{1}{2}$)

$$s = \max\{\frac{\sqrt{\mu_2 - \mu_1}}{\kappa_\alpha \pi} + \frac{\nu_\alpha}{2} + \frac{7}{4},\ -\nu_\alpha + \frac{1}{2}\} = \frac{\sqrt{\mu_2 - \mu_1}}{\kappa_\alpha \pi} + \frac{\nu_\alpha}{2} + \frac{7}{4}$$

(resp. $s = \max\{\dfrac{\sqrt{\mu_2 - \mu_1}}{\kappa_\alpha \pi} + \nu_\alpha + \dfrac{3}{2},\ -\dfrac{\nu_\alpha}{2} + \dfrac{1}{4}\} = \dfrac{\sqrt{\mu_2 - \mu_1}}{\kappa_\alpha \pi} + \nu_\alpha + \dfrac{3}{2}$), we get

$$|p\sqrt{r} - \mathcal{N}(r)| \leq s,$$

which proves the claim.

Case 2: A has two complex eigenvalues μ_1 and μ_2.

In this case $a_2^2 + 4a_1 < 0$,

$$\mu_1 = \frac{a_2}{2} + i\hat{\beta}, \quad \text{and} \quad \mu_2 = \frac{a_2}{2} - i\hat{\beta},$$

where $\hat{\beta} := \dfrac{1}{2}\sqrt{-(a_2^2 + 4a_1)}$.

Now, we consider the complex sequence $\{\Lambda_{\nu_\alpha,n}\}_{n\geq 1}$, with

$$\Lambda_{\nu_\alpha,2n} = \lambda^{(1)}_{\nu_\alpha,n} + \mu_2 = \lambda_{\nu_\alpha,n} - 2i\hat{\beta}, \qquad \Lambda_{\nu_\alpha,2n-1} = \lambda^{(2)}_{\nu_\alpha,n} + \mu_2 = \lambda_{\nu_\alpha,n}, \qquad \forall n \geq 1.$$

Let us check if the hypotheses in Theorem 5.1.1 hold true for $\left\{\Lambda_{\nu_\alpha,n}\right\}_{n\geq 1}$.
First, it is clear that the sequence $\left\{\Lambda_{\nu_\alpha,n}\right\}_{n\geq 1}$ always satisfies the hypothesis 1). Furthermore, the hypothesis 2) follows directly from the fact that

$$\mathcal{R}(\Lambda_{\nu_\alpha,2n}) = \mathcal{R}(\Lambda_{\nu_\alpha,2n-1}) = \lambda_{\nu_\alpha,n} > 0.$$

The hypothesis 3) is clearly fulfilled. Indeed, one has

$$\mathcal{I}(\Lambda_{\nu_\alpha,2n}) = 2\hat{\beta} \leq \delta\sqrt{\mathcal{R}(\Lambda_{\nu_\alpha,2n})}$$

and

$$\mathcal{I}(\Lambda_{\nu_\alpha,2n-1}) = 0 \leq \delta\sqrt{\mathcal{R}(\Lambda_{\nu_\alpha,2n-1})},$$

for some suitable $\delta > 0$.

Let us now show hypothesis 4). To this end, it suffices to prove that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ $|\Lambda_{\nu_\alpha,2n}| \leq |\Lambda_{\nu_\alpha,2n+1}|$. Using (5.3.6), we have

$$\begin{aligned}
|\Lambda_{\nu_\alpha,2n+1}|^2 - |\Lambda_{\nu_\alpha,2n}|^2 &= \lambda^2_{\nu_\alpha,n+1} - \lambda^2_{\nu_\alpha,n} - 4\hat{\beta}^2 \\
&\geq (\lambda_{\nu_\alpha,n+1} - \lambda_{\nu_\alpha,n})^2 - 4\hat{\beta}^2 \\
&\geq \rho_\alpha^2 |(n+1)^2 - n^2|^2 - 4\hat{\beta}^2 \\
&= \rho_\alpha^2 (2n+1)^2 - 4\hat{\beta}^2.
\end{aligned}$$

Now, in order to obtain $|\Lambda_{\nu_\alpha,n+1}|^2 - |\Lambda_{\nu_\alpha,n}|^2 \geq 0$, it suffices to take $n \geq n_0$ where the integer $n_0 \geq 1$ is given by

$$n_0 = \left[\frac{\hat{\beta}}{\rho_\alpha} - \frac{1}{2}\right] + 1.$$

This completes the proof of 4).

Let us now check if the hypothesis 5) holds true. First, observe that the second property is actually satisfied using (5.3.6) and hypothesis 1).

Concerning the first property, arguing as done in the real case, by (5.3.6), there exists $\rho_\alpha > 0$ such that

$$|\Lambda_{\nu_\alpha,2n} - \Lambda_{\nu_\alpha,2k}| \geq \frac{\rho_\alpha}{4}|(2n)^2 - (2k)^2|,$$

and

$$|\Lambda_{\nu_\alpha,2n-1} - \Lambda_{\nu_\alpha,2k-1}| \geq \frac{\rho_\alpha}{4}|(2n-1)^2 - (2k-1)^2|.$$

Moreover, denoting $\tilde{n} = 2n$ and $\tilde{k} = 2k - 1$, and proceeding as in (5.3.23) and (5.3.24), one can see that there exists $q \geq 4$ such that $\forall \tilde{k}, \tilde{n} \geq 1$ with $|\tilde{k} - \tilde{n}| \geq q$

$$\begin{aligned}
|\Lambda_{\nu_\alpha,\tilde{n}} - \Lambda_{\nu_\alpha,\tilde{k}}|^2 &= |\Lambda_{\nu_\alpha,2n} - \Lambda_{\nu_\alpha,2k-1}|^2 \\
&= \left|\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,k}\right|^2 + 4\hat{\beta}^2 \\
&\geq \left|\lambda_{\nu_\alpha,n} - \lambda_{\nu_\alpha,k}\right|^2 \geq \left(\frac{\rho_\alpha}{4}|\tilde{n}^2 - \tilde{k}^2 - 2\tilde{k} - 1|\right)^2 \\
&\geq \left(\frac{\rho_\alpha}{8}(\tilde{k}^2 - \tilde{n}^2)\right)^2
\end{aligned}$$

which provides the desired result.

Finally, it remains to prove hypothesis 6). For any $r > 0$, we define

$$\begin{aligned}
\mathcal{N}(r) &= Card\{k : \ \lambda_{\nu_\alpha,k} \leq r\} + Card\{k : \ (\lambda^2_{\nu_\alpha,k} + 4\hat{\beta}^2)^{1/2} \leq r\} \\
&= \mathcal{N}_1(r) + \tilde{\mathcal{N}_2}(r) = n_1 + \tilde{n}_2.
\end{aligned}$$

where the estimates of n_1 are given in (5.3.27) and (5.3.28), and $\tilde{n}_2$ can be estimated in a similar way as n_2. Indeed, we can see that

$$\lambda_{\nu_\alpha,\tilde{n}_2} \leq (\lambda^2_{\nu_\alpha,\tilde{n}_2} + 4\hat{\beta}^2)^{1/2} \leq r$$

and then $\tilde{n}_2$ satisfies for $\nu_\alpha \leq \dfrac{1}{2}$ (resp. $\nu_\alpha \geq \dfrac{1}{2}$) the estimate (5.3.25) (resp. the upper bound in (5.3.28)). On the other hand, since

$$\lambda_{\nu_\alpha,\tilde{n}_2+1} + 2\hat{\beta} \geq (\lambda^2_{\nu_\alpha,\tilde{n}_2+1} + 4\hat{\beta}^2)^{1/2} > r$$

then one can gets the same estimates in (5.3.29) and (5.3.30) with $2\hat{\beta}$ in place of $\mu_2 - \mu_1$. Again, as in the real case, one can show that there exists some suitable parameters p and s for which the inequality (5.1.5) holds.

Case 3: A has a double eigenvalue.

In this case $a_2^2 + 4a_1 = 0$. We denote by $\mu = \dfrac{a_2}{2} \in \mathbb{R}$ the eigenvalue of A. Thus, the sequence $\left\{\Lambda_{\nu_\alpha,n}\right\}_{n\geq 1}$ is then reduced to $\{\lambda_{\nu_\alpha,n}\}_{n\geq 1}$. In view of (5.3.6), and reasoning as in the first case, we automatically conclude that $\left\{\Lambda_{\nu_\alpha,n}\right\}_{n\geq 1}$ fulfills all the hypotheses in Theorem 5.1.1. This complete the proof of Proposition (5.3.3). □

We will finish this section giving a result on the set of eigenfunctions of the operators L and L^*. It reads as follows:

Proposition 5.3.4. *Let us consider the sequences*

$$\mathcal{B} = \left\{\psi_n^{(1)}, \psi_n^{(2)}, \quad n \geq 1\right\} \quad \textit{and} \quad \mathcal{B}^* = \left\{\Psi_n^{(1)}, \Psi_n^{(2)}, \quad n \geq 1\right\}. \tag{5.3.33}$$

Then,

1. *$\mathcal{B}$ and $\mathcal{B}^*$ are biorthogonal families in $L^2(0,1)^2$.*
2. *$\mathcal{B}$ and $\mathcal{B}^*$ are complete sequences in $L^2(0,1)^2$.*

3. *The sequences $\mathcal{B}$ and $\mathcal{B}^*$ are biorthogonal Riesz bases of $L^2(0,1)^2$.*

4. *The sequence $\mathcal{B}^*$ is a basis of $H^1_\alpha(0,1)^2$ and $\mathcal{B}$ is its biorthogonal basis in $H^{-1}_\alpha(0,1)^2$.*

Proof. From the expressions of $\psi_n^{(i)}$ and $\Psi_n^{(i)}$, we can write

$$\psi_n^{(i)} = U_i \Phi_{\nu_\alpha,n} \quad \text{and} \quad \Psi_n^{(i)} = V_i \Phi_{\nu_\alpha,n}, \quad i = 1,2, \quad n \geq 1,$$

where $U_i, V_i \in \mathbb{R}^2$ and $\Phi_{\nu_\alpha,n}$ is given (5.3.5).

1. It is not difficult to check that $\{U_i\}_{i=1,2}$ and $\{V_i\}_{i=1,2}$ are biorthogonal families of $\mathbb{R}^2$. Moreover, since $(\Phi_{\nu_\alpha,n})_{n\geq 1}$ is an orthonormal basis for $L^2(0,1)$, we readily deduce

$$\langle \psi_n^{(i)}, \Psi_k^{(j)} \rangle = (U_i)^{tr} V_j \langle \Phi_{\nu_\alpha,n}, \Phi_{\nu_\alpha,k} \rangle = \delta_{ij}\delta_{nk}, \quad \forall n,k \geq 1, \quad i,j = 1,2.$$

This proves the claim.

2. We will use [114, Lemma 1.44]. For this purpose, let us consider $f = (f_1, f_2)^{tr} \in L^2(0,1)^2$ such that

$$\langle f, \psi_n^{(i)} \rangle = 0, \quad \forall n \geq 1, \quad i = 1,2.$$

If we denote $f_{i,n}$ $(i = 1,2)$ the corresponding Fourier coefficients of the function $f_i \in L^2(0,1)$ with respect to the basis $(\Phi_{\nu_\alpha,n})_{n\geq 1}$, then the previous equality can be written as

$$(f_{1,n}, f_{2,n})[U_1|U_2] = 0_{\mathbb{R}^2}, \quad \forall n \geq 1.$$

Using the fact that $det[U_1|U_2] \neq 0$, we deduce $f_{1,n} = f_{2,n} = 0$, for all $n \geq 1$. This implies that $f_1 = f_2 = 0$ (since $(\Phi_{\nu_\alpha,n})_{n\geq 1}$ is an orthonormal basis in $L^2(0,1)$) and, therefore, $f = 0$ which proves the completeness of $\mathcal{B}$. A similar argument can be used for $\mathcal{B}^*$ and the conclusion follows immediately.

3. By [114, Theorem 7.13], we know that $\{\psi_n^{(1)}, \psi_n^{(2)}\}_{n\geq 1}$ is a Riesz basis for $L^2(0,1)^2$ if and only if $\{\psi_n^{(1)}, \psi_n^{(2)}\}_{n\geq 1}$ is a complete Bessel sequence and possesses a biorthogonal system that is also a complete Bessel sequence. Using the previous properties 1) and 2), we only have to prove that the sequence $\{\psi_n^{(1)}, \psi_n^{(2)}\}_{n\geq 1}$ and $\{\Psi_n^{(1)}, \Psi_n^{(2)}\}_{n\geq 1}$ are Bessel sequences. This amounts to prove that the series

$$S_1(f) = \sum_{n\geq 1} \left[\langle f, \psi_n^{(1)} \rangle^2 + \langle f, \psi_n^{(2)} \rangle^2\right] \quad \text{and} \quad S_2(f) = \sum_{n\geq 1} \left[\langle f, \Psi_n^{(1)} \rangle^2 + \langle f, \Psi_n^{(2)} \rangle^2\right]$$

converge for any $f = (f_1, f_2)^{tr} \in L^2(0,1)^2$.
From the definition of the functions $\psi_n^{(i)}$ and $\Psi_n^{(i)}$, it is easy to see that there exists some constant $C > 0$ such that

$$S_1(f) \leq C \sum_{n\geq 1} \left(|f_{1,n}|^2 + |f_{2,n}|^2\right) \quad \text{and} \quad S_2(f) \leq C \sum_{n\geq 1} \left(|f_{1,n}|^2 + |f_{2,n}|^2\right).$$

Recall that $f_{i,n}$ is the Fourier coefficient of the function $f_i \in L^2(0,1)$ $(i = 1,2)$ with respect to $\Phi_{\nu_\alpha,n}$. Accordingly, the series $S_1(f)$ and $S_2(f)$ converge since $(\Phi_{\nu_\alpha,n})_{n\geq 1}$ is an orthonormal basis for $L^2(0,1)$. We obtain thus the proof of desired result.

4. For showing item 4) we make use of [114, Theorem 5.12]. First, we have that

$$H^1_\alpha(0,1) \subset L^2(0,1) \subset \left(H^1_\alpha(0,1)\right)' = H^{-1}_\alpha(0,1).$$

Therefore, $\mathcal{B}^* \subset H^1_\alpha(0,1)^2$ and is complete in this space since it is in $L^2(0,1)^2$. On the other hand, by the definition of the duality pairing, we have

$$\langle \psi_n^{(i)}, \Psi_k^{(j)} \rangle_{H_\alpha^{-1},H_\alpha^1} = \langle \psi_n^{(i)}, \Psi_k^{(j)} \rangle = \delta_{ij}\delta_{nk}, \quad \forall n,k \geq 1, \quad i,j = 1,2.$$

Thus, $\mathcal{B} \subset H_\alpha^{-1}(0,1)^2$ and is biorthogonal to $\mathcal{B}^*$, which also yields that $\mathcal{B}^*$ is minimal in $H^1_\alpha(0,1)^2$ thanks to [114, Lemma 5.4]. To conclude the proof, it remains to prove that for any $f = (f_1, f_2) \in H^1_\alpha(0,1)^2$, the series

$$S(f) = \sum_{n\geq 1} \left[\langle \psi_n^{(1)}, f \rangle_{H_\alpha^{-1},H_\alpha^1} \Psi_n^{(1)} + \langle \psi_n^{(2)}, f \rangle_{H_\alpha^{-1},H_\alpha^1} \Psi_n^{(2)} \right]$$

converges in $H^1_\alpha(0,1)^2$.

Using again the definitions of $\psi_n^{(i)}$ and $\Psi_n^{(i)}$, one can prove that

$$\langle \psi_n^{(1)}, f \rangle_{H_\alpha^{-1},H_\alpha^1} \Psi_n^{(1)} = \frac{a_1}{\mu_2^2 + a_1} \begin{pmatrix} \dfrac{\mu_2^2}{a_1} f_{1,n} - \dfrac{\mu_2}{a_1} f_{2,n} \\ -\mu_2 f_{1,n} + f_{2,n} \end{pmatrix} \Phi_{\nu_\alpha,n} \tag{5.3.34}$$

and

$$\langle \psi_n^{(2)}, f \rangle_{H_\alpha^{-1},H_\alpha^1} \Psi_n^{(2)} = \frac{a_1}{\mu_1^2 + a_1} \begin{pmatrix} \dfrac{\mu_1^2}{a_1} f_{1,n} - \dfrac{\mu_1}{a_1} f_{2,n} \\ -\mu_1 f_{1,n} + f_{2,n} \end{pmatrix} \Phi_{\nu_\alpha,n} \tag{5.3.35}$$

where $f_{i,n}$ is the Fourier coefficient of the function $f_i \in H^1_\alpha(0,1)$, $i = 1,2$.

But, we know that the series $\sum_{n\geq 1} f_{i,n}\Phi_{\nu_\alpha,n}$, $i = 1,2$ converges in $H^1_\alpha(0,1)$ since $(\Phi_{\nu_\alpha,n})_{n\geq 1}$ is an orthogonal basis for $H^1_\alpha(0,1)$ and $f_1, f_2 \in H^1_\alpha(0,1)$. This implies that, the series $\sum_{n\geq 1} \langle \psi_n^{(1)}, f \rangle_{H_\alpha^{-1},H_\alpha^1} \Psi_n^{(1)}$ and $\sum_{n\geq 1} \langle \psi_n^{(2)}, f \rangle_{H_\alpha^{-1},H_\alpha^1} \Psi_n^{(2)}$ converge in $H^1_\alpha(0,1)^2$ and assure the convergence of $S(f)$ in $H^1_\alpha(0,1)^2$. This concludes the proof of the result.

□

5.4 Boundary approximate controllability

We will devote this section to proving the approximate controllability at time $T > 0$ of system (5.1.1). In fact, our first main result is the following one.

Theorem 5.4.1. *Let $\alpha \in [0,2)$ and consider μ_1 and μ_2 the eigenvalues of the matrix A. Then, system (5.1.1) is approximately controllable at time $T > 0$ if and only if conditions*

$$rank[B|AB] = 2 \tag{5.4.1}$$

and

$$\kappa_\alpha^2 (j^2_{\nu_\alpha,n} - j^2_{\nu_\alpha,l}) \neq \mu_2 - \mu_1, \quad \forall n,l \in \mathbb{N}^*, \quad \textit{with} \quad n \neq l. \tag{5.4.2}$$

The previous theorem is a direct consequence of the following result.

Theorem 5.4.2. *Let $\alpha \in [0,2)$ and consider μ_1 and μ_2 the eigenvalues of the matrix A. Then, system (5.1.2) is approximately controllable at time $T > 0$ if and only if (5.4.2) holds.*

For the proof of Theorem 5.4.2, we are going to apply the following known result provided in [17] and [88].

Theorem 5.4.3. *Let $T > 0$. Suppose that $\{\Lambda_n\}_{n\geq 1}$ is a sequence of complex numbers such that, for some $\delta, \rho > 0$, one has*

$$\begin{cases} \Re(\Lambda_n) \geq \delta|\Lambda_n|, \qquad |\Lambda_n - \Lambda_m| \geq \rho|n-m|, \quad \forall n, m \geq 1, \\ \displaystyle\sum_{n\geq 1} \frac{1}{|\Lambda_n|} < +\infty. \end{cases} \tag{5.4.3}$$

Then, there exists a family $\{q_n\}_{n\geq 1} \subset L^2(0,T)$ biorthogonal to $\{e^{-\Lambda_n t}\}_{n\geq 1}$ i.e., a family $\{q_n\}_{n\geq 1}$ in $L^2(0,T)$ such that

$$\int_0^T q_n(t) e^{-\Lambda_m t}\, dt = \delta_{nm}, \quad \forall n, m \geq 1.$$

Moreover, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ for which

$$\|q_n\|_{L^2(0,T)} \leq C_\varepsilon e^{\varepsilon \Re(\Lambda_n)}, \qquad \forall n \geq 1.$$

Using the previous result and similar techniques as in Proposition 5.3.3, we obtain the following result.

Proposition 5.4.1. *The family defined in (5.3.16) satisfies all the hypothesis of Theorem 5.4.3 (i.e., (5.4.3)).*

Proof of Theorem 5.4.2. As said in section 5.2, in order to prove this theorem we will follow a duality approach leading us to study a unique continuation property for the adjoint system.

Necessary condition: By contradiction, let us assume that condition (5.4.2) does not hold, i.e., that there is $n_0, l_0 \in \mathbb{N}^*$ with $n_0 \neq l_0$ such that

$$\lambda^{(1)}_{\nu_\alpha, n_0} = \lambda^{(2)}_{\nu_\alpha, l_0} := \lambda.$$

Let us see that the approximate controllability property does not hold. Owing to Proposition 5.2.4, it suffices to show that the unique continuation property for the adjoint system (5.2.10) is no longer valid. Indeed, let us take $\varphi_0 = a\Psi^{(1)}_{n_0} + b\Psi^{(2)}_{l_0} \in H^1_\alpha(0,1)^2$, with $a, b \in \mathbb{R}$ to be determined. In this case, it is not difficult to see that the corresponding solution to the adjoint problem (5.2.10) is given by

$$\varphi(t,x) = (a\Psi^{(1)}_{n_0}(x) + b\Psi^{(2)}_{l_0}(x))e^{-\lambda(T-t)}, \quad \forall (t,x) \in Q.$$

On the other hand, direct computations show that

$$(x^\alpha(\Phi_{\nu_\alpha,n})_x)(1) = \frac{\sqrt{2}\kappa_\alpha^{\frac{3}{2}} j_{\nu_\alpha,n}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} J'_{\nu_\alpha}(j_{\nu_\alpha,n}). \tag{5.4.4}$$

Coming back to the definition of $\Psi^{(i)}_n$, and taking into account the previous property, one obtains

$$\begin{aligned}
&B^*(x^\alpha \varphi_x)(t,1) \\
&= B^*\big(a(x^\alpha \Psi^{(1)}_{n_0,x})(1) + b(x^\alpha \Psi^{(2)}_{l_0,x})(1)\big)e^{-\lambda(T-t)} \\
&= \sqrt{2}\kappa_\alpha^{\frac{3}{2}}\Big(aB^*V_1 j_{\nu_\alpha,n_0}\frac{J'_{\nu_\alpha}(j_{\nu_\alpha,n_0})}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n_0})|} + bB^*V_2 j_{\nu_\alpha,l_0}\frac{J'_{\nu_\alpha}(j_{\nu_\alpha,l_0})}{|J'_{\nu_\alpha}(j_{\nu_\alpha,l_0})|}\Big)e^{-\lambda(T-t)} \\
&= -\sqrt{2}\kappa_\alpha^{\frac{3}{2}}\Big(a\frac{\mu_2}{a_1} j_{\nu_\alpha,n_0}\frac{J'_{\nu_\alpha}(j_{\nu_\alpha,n_0})}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n_0})|} + b\frac{\mu_1}{\mu_1^2 + a_1} j_{\nu_\alpha,l_0}\frac{J'_{\nu_\alpha}(j_{\nu_\alpha,l_0})}{|J'_{\nu_\alpha}(j_{\nu_\alpha,l_0})|}\Big)e^{-\lambda(T-t)}
\end{aligned}$$

Choosing

$$a = \frac{\mu_1}{\mu_1^2 + a_1} j_{\nu_\alpha, l_0} \frac{J'_{\nu_\alpha}(j_{\nu_\alpha,l_0})}{|J'_{\nu_\alpha}(j_{\nu_\alpha,l_0})|},$$
$$b = -\frac{\mu_2}{a_1} j_{\nu_\alpha,n_0} \frac{J'_{\nu_\alpha}(j_{\nu_\alpha,n_0})}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n_0})|}$$

we have that $B^*(x^\alpha \varphi_x)(t,1) = 0$ but $\varphi_0 \neq 0$, which proves that the unique continuation property for the adjoint system (5.2.10) fails to be true. This ends the proof of the necessary part.

Sufficient condition: Let us now assume that the condition (5.4.2) holds and prove the unique continuation property for system (5.2.10).

Let us consider $\varphi_0 \in H^1_\alpha(0,1)^2$ and suppose that the corresponding solution φ of the adjoint problem (5.2.10) satisfies

$$B^*(x^\alpha \varphi_x)(t,1) = 0, \quad \forall t \in (0,T). \tag{5.4.5}$$

Observe that, thanks to Proposition 5.3.4, φ_0 can be written as

$$\varphi_0 = \sum_{n\geq 1} (b_n \Psi_n^{(1)} + c_n \Psi_n^{(2)}),$$

where

$$b_n = \langle \psi_n^{(1)}, \varphi_0 \rangle_{H_\alpha^{-1}, H_\alpha^1} \quad \text{and} \quad c_n = \langle \psi_n^{(2)}, \varphi_0 \rangle_{H_\alpha^{-1}, H_\alpha^1}.$$

Hence, the corresponding solution φ of system (5.2.10) associated to φ_0 is given by

$$\varphi(t,x) = \sum_{n\geq 1} \left(b_n \Psi_n^{(1)} e^{-\lambda^{(1)}_{\nu_\alpha,n}(T-t)} + c_n \Psi_n^{(2)} e^{-\lambda^{(2)}_{\nu_\alpha,n}(T-t)} \right).$$

Therefore, using (5.4.4) we get that

$$\begin{aligned}
0 &= B^*(x^\alpha \varphi_x)(T-t,1) \\
&= \sum_{n\geq 1} B^* \big(b_n (x^\alpha \Psi^{(1)}_{n,x})(1) e^{-\lambda^{(1)}_{\nu_\alpha,n} t} + c_n (x^\alpha \Psi^{(2)}_{n,x})(1) e^{-\lambda^{(2)}_{\nu_\alpha,n} t} \big) \\
&= \sqrt{2} \kappa_\alpha^{\frac{3}{2}} \sum_{n\geq 1} \frac{J'_{\nu_\alpha}(j_{\nu_\alpha,n})}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} j_{\nu_\alpha,n} \big(b_n B^* V_1 e^{-\lambda^{(1)}_{\nu_\alpha,n} t} + c_n B^* V_2 e^{-\lambda^{(2)}_{\nu_\alpha,n} t} \big) \\
&= -\sqrt{2} \kappa_\alpha^{\frac{3}{2}} \sum_{n\geq 1} \frac{J'_{\nu_\alpha}(j_{\nu_\alpha,n})}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} \frac{j_{\nu_\alpha,n}}{a_1} \big(b_n \mu_2 e^{-\lambda^{(1)}_{\nu_\alpha,n} t} + c_n \mu_1 \frac{a_1}{\mu_1^2 + a_1} e^{-\lambda^{(2)}_{\nu_\alpha,n} t} \big) \\
&= -\sqrt{2} \kappa_\alpha^{\frac{3}{2}} \sum_{n\geq 1} \frac{J'_{\nu_\alpha}(j_{\nu_\alpha,n})}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} \frac{j_{\nu_\alpha,n}}{a_1} e^{\mu_2 t} \big(b_n \mu_2 e^{-(\lambda^{(1)}_{\nu_\alpha,n} + \mu_2) t} + c_n \mu_1 \frac{a_1}{\mu_1^2 + a_1} e^{-(\lambda^{(2)}_{\nu_\alpha,n} + \mu_2) t} \big).
\end{aligned}$$

From Proposition 5.4.1, we can apply Theorem 5.4.3 in order to deduce the existence of a biorthogonal family $\{q_n^{(1)}, q_n^{(2)}\}_{n\geq 1}$ to $\{e^{-(\lambda^{(1)}_{\nu_\alpha,n} + \mu_2)t}, e^{-(\lambda^{(2)}_{\nu_\alpha,n} + \mu_2)t}\}_{n\geq 1}$ in $L^2(0,T)$. Then, the previous identity, in particular, yields

$$\begin{cases}
\displaystyle\int_0^T B^*(x^\alpha \varphi_x)(s,1)\, e^{-\mu_2 s}\, q_n^{(1)}(s)\, ds = -\sqrt{2} \kappa_\alpha^{\frac{3}{2}} \frac{J'_{\nu_\alpha}(j_{\nu_\alpha,n})}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} \frac{j_{\nu_\alpha,n}}{a_1} \mu_2 b_n = 0, \quad \forall n \geq 1 \\
\displaystyle\int_0^T B^*(x^\alpha \varphi_x)(s,1)\, e^{-\mu_2 s} q_n^{(2)}(s)\, ds = -\sqrt{2} \kappa_\alpha^{\frac{3}{2}} \frac{J'_{\nu_\alpha}(j_{\nu_\alpha,n})}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} \frac{j_{\nu_\alpha,n}}{a_1} \mu_1 \frac{a_1}{\mu_1^2 + a_1} c_n = 0, \ \forall n \geq 1.
\end{cases}$$

and $b_n = c_n = 0$ for any $n \geq 1$. In conclusion, $\varphi_0 = 0$. This proves the continuation property for the solutions to the adjoint problem (5.2.10) and, thanks to Proposition 5.2.4, the approximate controllability of system (5.1.2) at any positive time T. $\square$

Remark 35. Notice that, condition (5.4.2) is equivalent to the simplicity of the spectrum of L and L^*.

5.5 Boundary null controllability

In this section, we will address the problem of the null controllability at time $T>0$ of system (5.1.1). In particular, our main aim is to characterize the boundary controllability properties of (5.1.1) (a degenerate system of two equations) when we apply just one control on a part of the boundary. In this sense, one has:

Theorem 5.5.1. *Let $\alpha \in [0,2)$ and consider by μ_1 and μ_2 the eigenvalues of A. Then system (5.1.1) is null controllable at any time $T>0$ if and only if conditions (5.4.1) and (5.4.2) hold.*

Moreover, there exists some positive constant C independent of T such that the control force satisfies

$$\|v\|_{L^2(0,T)} \leq Ce^{CT+\frac{C}{T}}\|y_0\|_{H_\alpha^{-1}}, \tag{5.5.1}$$

The above result is a consequence of the following theorem.

Theorem 5.5.2. *Let $\alpha \in [0,2)$ and consider by μ_1 and μ_2 the eigenvalues of A. Then system (5.1.2) is null controllable at any time $T>0$ if and only if conditions (5.4.1) and (5.4.2) hold. Moreover, the control force satisfies the estimate (5.5.1).*

It is worth mentioning that the Theorem 5.4.3 can also be applied to get the null controllability result for the system (5.1.1). However, it does not permit to deduce the estimate (5.5.1) on the null-control. For this reason, to obtain the null controllability result together with such an estimate, we will use Theorem 5.1.1.

Proof of Theorem 5.5.2. The main technical tool for proving this result is the moment method.

To prove Theorem 5.5.2, we transform the controllability problem into a moment problem. Using Proposition 5.2.3, we deduce that the control $v \in L^2(0,T)$ drives the solution of (5.1.1) to zero at time T if and only if $v \in L^2(0,T)$ satisfies

$$\int_0^T B^*(x^\alpha \varphi_x)(t,1)\,v(t)\,dt = \langle y_0, \varphi(0,\cdot)\rangle_{H_\alpha^{-1},H_\alpha^1}, \qquad \forall \varphi_0 \in H_\alpha^1(0,1)^2 \tag{5.5.2}$$

where $\varphi \in C^0\big([0,T];H_\alpha^1(0,1)^2\big)\cap L^2\big(0,T;H_\alpha^2(0,1)^2\big)$ is the solution of the adjoint system (5.2.10) associated to φ_0.

Observe that, using Proposition 5.3.4, the corresponding solution φ of system (5.2.10) associated to φ_0 is given by

$$\varphi(t,x) = \sum_{k\geq 1}\Big(\langle \psi_k^{(1)},\varphi_0\rangle_{H_\alpha^{-1},H_\alpha^1}\Psi_k^{(1)}e^{-\lambda_{\nu_\alpha,k}^{(1)}(T-t)} + \langle \psi_k^{(2)},\varphi_0\rangle_{H_\alpha^{-1},H_\alpha^1}\Psi_k^{(2)}e^{-\lambda_{\nu_\alpha,k}^{(2)}(T-t)}\Big).$$

Since $\mathcal{B}^*$ is a basis for $H_\alpha^1(0,1)^2$, we find that $\varphi(t,x) = \Psi_n^{(i)}(x)e^{-\lambda_{\nu_\alpha,n}^{(i)}(T-t)}$ is the solution of system (5.2.10) associated with $\varphi_0 = \Psi_n^{(i)}$. Therefore, we can deduce that the identity (5.5.2) is equivalent to

$$\int_0^T B^*(x^\alpha \Psi_{n,x}^{(i)})(1)v(t)e^{-\lambda_{\nu_\alpha,n}^{(i)}(T-t)}dt = e^{-\lambda_{\nu_\alpha,n}^{(i)}T}\langle y_0, \Psi_n^{(i)}\rangle_{H_\alpha^{-1},H_\alpha^1}, \quad \forall n\geq 1, \quad i=1,2.$$

Taking into account the expressions of $\Psi_n^{(i)}$ (see (5.3.15)), we infer that $v \in L^2(0,T)$ is a null control for system (5.1.1) associated to y_0 if and only if

$$\begin{aligned}\frac{\sqrt{2}\kappa_\alpha^{\frac{3}{2}} j_{\nu_\alpha,n}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} J'_{\nu_\alpha}(j_{\nu_\alpha,n})B^*V_i \int_0^T v(t)e^{-\lambda_{\nu_\alpha,n}^{(i)}(T-t)}dt \\ = e^{-\lambda_{\nu_\alpha,n}^{(i)}T}\langle y_0, \Psi_n^{(i)}\rangle_{H_\alpha^{-1},H_\alpha^1}, \quad \forall n\geq 1, \quad i=1,2\end{aligned}$$

and equivalently,

$$\int_0^T v(t)e^{-\lambda^{(i)}_{\nu_\alpha,n}(T-t)}dt = C^{(i)}_{\nu_\alpha,n}, \quad \forall n \geq 1, \quad i = 1,2, \tag{5.5.3}$$

where $C^{(i)}_{\nu_\alpha,n}$ is given by

$$C^{(i)}_{\nu_\alpha,n} = \frac{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|e^{-\lambda^{(i)}_{\nu_\alpha,n}T}}{\sqrt{2}\kappa_\alpha^{\frac{3}{2}} j_{\nu_\alpha,n}J'_{\nu_\alpha}(j_{\nu_\alpha,n})B^*V_i}\langle y_0, \Psi^{(i)}_n\rangle_{H^{-1}_\alpha,H^1_\alpha}, \quad \forall n \geq 1, \quad i = 1,2. \tag{5.5.4}$$

At this stage, the strategy to solve the moment problem (5.5.3) is to use the concept of biorthogonal family. In fact, Proposition 5.3.3 and Theorem 5.1.1 guarantee the existence of $T_0 > 0$, such that for any $T \in (0,T_0)$, there exists a biorthogonal family $\{q^{(1)}_n, q^{(2)}_n\}_{n\geq 1}$ to $\{e^{-(\lambda^{(1)}_{\nu_\alpha,n}+\mu_2)t}, e^{-(\lambda^{(2)}_{\nu_\alpha,n}+\mu_2)t}\}_{n\geq 1}$ in $L^2(-T/2,T/2)$ which also satisfies

$$\|q^{(i)}_n\|_{L^2(-T/2,T/2)} \leq Ce^{\sqrt{\mathcal{R}(\lambda^{(i)}_{\nu_\alpha,n}+\mu_2)+\frac{C}{T}}}, \qquad \forall n \geq 1, \quad i = 1,2. \tag{5.5.5}$$

for some positive constant C independent of T.
Performing the change of variable $s = T/2 - t$ in (5.5.3), the controllability problem reduces then to the following moment problem: Given $y_0 \in H^{-1}_\alpha(0,1)^2$ find $v \in L^2(0,T)$ such that $u(s) = v(T/2-s)e^{\mu_2 s} \in L^2(-T/2,T/2)$ satisfies

$$\int_{-T/2}^{T/2} u(s)e^{-(\lambda^{(i)}_{\nu_\alpha,n}+\mu_2)s}ds = \widehat{C}^{(i)}_{\nu_\alpha,n}, \quad \forall n \geq 1, \quad i = 1,2, \tag{5.5.6}$$

with $\widehat{C}^{(i)}_{\nu_\alpha,n} = e^{\lambda^{(i)}_{\nu_\alpha,n}T/2}C^{(i)}_{\nu_\alpha,n}$

Then, a formal solution to the moment problem (5.5.6) is given by

$$u(s) = \sum_{n\geq 1}(\widehat{C}^{(1)}_{\nu_\alpha,n}q^{(1)}_n(s) + \widehat{C}^{(2)}_{\nu_\alpha,n}q^{(2)}_n(s)).$$

Thus,

$$v(s) = \sum_{n\geq 1}\left(\widehat{C}^{(1)}_{\nu_\alpha,n}q^{(1)}_n(T/2-s) + \widehat{C}^{(2)}_{\nu_\alpha,n}q^{(2)}_n(T/2-s)\right)e^{-\mu_2(T/2-s)}. \tag{5.5.7}$$

The only remaining point is to prove that $v \in L^2(0,T)$ (in fact (5.5.1)). This comes directly from the estimate (5.5.5) and the fact that

$$\|\Psi^{(i)}_n\|_{H^1_\alpha} = \|V_i\Phi_{\nu_\alpha,n}\|_{H^1_\alpha} \leq C\sqrt{\lambda_{\nu_\alpha,n}} = C\kappa_\alpha j_{\nu_\alpha,n}, \quad \forall n \geq 1, \quad i = 1,2,$$

for some positive constant C.

Indeed, the previous inequality implies

$$\begin{aligned} |\langle y_0, \Psi^{(i)}_n\rangle_{H^{-1}_\alpha,H^1_\alpha}| &\leq \|y_0\|_{H^{-1}_\alpha}\|\Psi^{(i)}_n\|_{H^1_\alpha} \\ &\leq C\kappa_\alpha j_{\nu_\alpha,n}\|y_0\|_{H^{-1}_\alpha}, \quad \forall n \geq 1, \quad i = 1,2. \end{aligned}$$

Moreover, from (5.5.4), by using the expressions of κ_α, we obtain

$$|\widehat{C}^{(i)}_{\nu_\alpha,n}| \leq \frac{C}{\sqrt{2-\alpha}}e^{-\lambda^{(i)}_{\nu_\alpha,n}T/2}\|y_0\|_{H^{-1}_\alpha}, \quad \forall n \geq 1, \quad i = 1,2. \tag{5.5.8}$$

Now, taking into account the definition of $\lambda_{\nu_\alpha,n}$, we get for a new constants C not depending on n and T

$$|e^{-\lambda^{(i)}_{\nu_\alpha,n}T/2}| \leq e^{CT}e^{-\lambda_{\nu_\alpha,n}T/2}$$

and

$$\sqrt{\mathcal{R}(\lambda^{(i)}_{\nu_\alpha,n}+\mu_2)} \leq C\sqrt{\lambda_{\nu_\alpha,n}}, \quad \forall n \geq 1. \tag{5.5.9}$$

Coming back to the expression (5.5.7) of the null control v and using the previous estimates, we get

$$\|v\|_{L^2(0,T)} \leq \frac{Ce^{CT}}{\sqrt{2-\alpha}}\|y_0\|_{H_\alpha^{-1}} \sum_{n\geq 1} e^{-\lambda_{\nu_\alpha,n}T/2} e^{C\sqrt{\lambda_{\nu_\alpha,n}}+\frac{C}{T}}. \tag{5.5.10}$$

Using Young's inequality,

$$C\sqrt{\lambda_{\nu_\alpha,n}} \leq \frac{\lambda_{\nu_\alpha,n}T}{4} + \frac{C^2}{T}$$

we see that

$$\|v\|_{L^2(0,T)} \leq \frac{Ce^{CT+\frac{C}{T}}}{\sqrt{2-\alpha}}\|y_0\|_{H_\alpha^{-1}} \sum_{n\geq 1} e^{-\lambda_{\nu_\alpha,n}T/4}.$$

On the other hand, by (5.3.1) and (5.3.2), it can be easily checked that there exist a constant $C>0$ such that

$$C\kappa_\alpha^2 n^2 \leq \lambda_{\nu_\alpha,n} = \kappa_\alpha^2 j^2_{\nu_\alpha,n}, \quad \forall n \geq 1.$$

Finally,

$$\begin{aligned}
\|v\|_{L^2(0,T)} &\leq \frac{C}{\sqrt{2-\alpha}} e^{CT+\frac{C}{T}} \|y_0\|_{H_\alpha^{-1}} \sum_{n\geq 1} e^{-C\kappa_\alpha^2 n^2 T} \\
&\leq \frac{C}{\sqrt{2-\alpha}} e^{CT+\frac{C}{T}} \|y_0\|_{H_\alpha^{-1}} \int_0^\infty e^{-C\kappa_\alpha^2 T s^2}\, ds \\
&= \frac{C}{(2-\alpha)^{\frac{3}{2}}} e^{CT+\frac{C}{T}} \|y_0\|_{H_\alpha^{-1}} \sqrt{\frac{\pi}{T}} \\
&\leq Ce^{CT+\frac{C}{T}} \|y_0\|_{H_\alpha^{-1}},
\end{aligned}$$

where C independent of T. This inequality shows that $v \in L^2(0,T)$ and yields the desired estimate on the null control in the case where $T < T_0$. On the other hand, when $T \geq T_0$, it suffices to set the null control function to 0 for the time interval $(T_0/2, T)$. Indeed, v is given by

$$v(t) = \begin{cases} v_0(t), & t \in [0, T_0/2] \\ 0, & t \in [T_0/2, T] \end{cases}$$

and consequently, the following estimate follows

$$\|v\|_{L^2(0,T)} \leq Ce^{\frac{2C}{T_0}} \|y_0\|_{H_\alpha^{-1}}. \tag{5.5.11}$$

This completes the proof of Theorem 5.5.2.

$\square$

Chapter 6

Pointwise controllability of degenerate heat equation

This chapter is devoted to the controllability of the degenerate heat equation controlled by an internal force acting at a single point inside the space domain. We give a necessary and sufficient condition for the approximate controllability. On the other hand, we provide a minimal time for null controllability. Our approach is mainly based on the moment method developed by Fattorini and Russell [87].

The results obtained in this chapter are presented in the research article [13], in collaboration with Jawad Salhi.

6.1 Introduction

The aim of this chapter is to address the pointwise controllability of a parabolic equation in one space dimension, which degenerates at the boundary of the space domain. To be more precise, for $0 \leq \alpha < 2$, we consider the following problem:

$$\begin{cases} y_t - (x^\alpha y_x)_x = \delta_b v(t), & (t,x) \in Q, \\ y(t,1) = 0, & t \in (0,T), \\ \begin{cases} y(t,0) = 0, & 0 \leq \alpha < 1 \\ x^\alpha y_x(t,0) = 0, & 1 \leq \alpha < 2 \end{cases} & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{6.1.1}$$

where $y_0 \in L^2(0,1)$, $T > 0$ and δ_b denotes the Dirac mass supported at a given point $b \in (0,1)$, on which one acts via a control function $v(t)$. This is the so-called pointwise control.

The proof of null controllability of the system (6.1.1) will rely on the celebrated moment method initially developed in [86, 87]. Let us recall quickly this method in the classical non-degenerated situation. We consider the 1-D Laplace operator ∂_{xx} with domain $D(\partial_{xx}) := H^2(0,\pi) \cap H_0^1(0,\pi)$ and state space $H := L^2(0,\pi)$. In other words, given the operator ∂_{xx}, let us consider the following controlled heat equation on $(0,T) \times (0,\pi)$, with Dirichlet boundary conditions:

$$\begin{cases} y_t - y_{xx} = f(x)v(t), & (t,x) \in (0,T) \times (0,\pi), \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,\pi), \end{cases} \tag{6.1.2}$$

where $y_0 \in L^2(0,\pi)$, $v \in L^2(0,T)$ is the control and $f \in H^{-1}(0,\pi)$ is an imposed profile for this control.

It is well-known that $-\partial_{xx} : D(\partial_{xx}) \to L^2(0,\pi)$ admits a sequence of eigenvalues and normalized eigenfunctions given by

$$\lambda_k = k^2, \qquad e_k(x) := \sqrt{\frac{2}{\pi}}\sin(kx), \qquad k \geq 1, \quad x \in (0,\pi), \tag{6.1.3}$$

and the sequence $\{e_k\}_{k\geq 1}$ is a Hilbert basis of $L^2(0,\pi)$.

The starting point of this moment method is to decompose the initial datum y_0 and the control profile f in the basis of the eigenfunctions $\{e_k\}_{k\geq 1}$ associated to the operator $-\partial_{xx}$, i.e.

$$f(x) = \sum_{k\geq 1} f_k e_k(x), \qquad y_0(x) = \sum_{k\geq 1} y_k^0 e_k(x),$$

with $\{f_k\}_{k\geq 1}, \{y_k^0\}_{k\geq 1} \in l^2(\mathbb{N}^\star) := \{\{\beta_j\} \in \mathbb{R}^{\mathbb{N}} : \sum_{j\in\mathbb{N}^\star} |\beta_j|^2 < +\infty\}$.

Then, it is classical that find $v \in L^2(0,T)$ such that $y(T,\cdot) = 0$ is equivalent to find $v \in L^2(0,T)$ such that

$$f_k \int_0^T e^{-\lambda_k t} v(t)\, dt = -e^{-\lambda_k T} y_k^0, \quad \forall k \geq 1. \tag{6.1.4}$$

Finding $v \in L^2(0,T)$ such that (6.1.4) holds, is the so-called moment problem. A necessary condition for the existence of a solution to this problem for any $y_0 \in L^2(0,\pi)$ is:

$$f_k \neq 0, \qquad \forall k \geq 1.$$

In [86, 16], the authors solved the previous moment problem by proving the existence of a biorthogonal family $\{q_k\}_{k\geq 1}$ to $\{e^{-\lambda_k t}\}_{k\geq 1}$ in $L^2(0,T)$ which, in particular, satisfies the next additional property: for every $\varepsilon > 0$ there exists a constant $C(\varepsilon,T) > 0$ such that

$$\|q_k\|_{L^2(0,T)} \leq C(\varepsilon,T)e^{\varepsilon\lambda_k}, \qquad \forall k \geq 1. \tag{6.1.5}$$

Then, the control is obtained as a linear combination of $\{q_k\}_{k\geq 1}$, that is,

$$v = -\sum_{k\geq 1} \frac{e^{-\lambda_k T}}{f_k} y_k^0 q_k,$$

and the previous bounds (6.1.5) are used to prove that this series converges in $L^2(0,T)$ for any positive time $T > T_0$, where

$$T_0 := \limsup_{k\to\infty} I_k(f) \in [0,+\infty], \quad \text{with} \quad I_k(f) := -\frac{\log(|f_k|)}{k^2}.$$

More precisely, it is proved in [16] that:

1. Equation (6.1.2) is null controllable at any time $T > T_0$.
2. Equation (6.1.2) is not null controllable at any time $T < T_0$.

In the same framework, it is worth to mention also the recent work [137], where the author proved the first results related to the cost of the null controllability in the case of a minimal time on control for the one dimensional heat equation (6.1.2).

Let us also underline the reference [77], where a study of (6.1.2) is performed in the particular case $f(x) := \delta_b \in H^{-1}(0,\pi)$, with $b \in (0,\pi)$. That is to say: Given $T > 0$ and $y_0 \in H^{-1}(0,\pi)$, can we find a control $v \in L^2(0,T)$ such that the solution $y \in C([0,T]; H^{-1}(0,\pi))$ of

$$\begin{cases} y_t - y_{xx} = \delta_b v(t), & (t,x) \in (0,T) \times (0,\pi), \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,\pi), \end{cases} \tag{6.1.6}$$

satisfies

$$y(T,\cdot) = 0 \qquad \text{in} \quad (0,\pi). \tag{6.1.7}$$

Using again the existence of a biorthogonal family in $L^2(0,T)$ to the exponentials $\{e^{-\lambda_k t}\}_{k\geq 1}$ and the bounds (6.1.5), S. Dolecki exhibited a minimal time $T_0 \in [0,+\infty]$, depending on b, such that system (6.1.6) is not null controllable at time T if $T < T_0$ and is null controllable at time T when $T > T_0$. This minimal time of controllability is given by

$$T_0(b) = \limsup_{k\to\infty} -\frac{\log(|\sin(kb)|)}{k^2}. \tag{6.1.8}$$

To our knowledge, this was the first result on null controllability of parabolic problems where a minimal time of control appears.

In this chapter we will focus on the pointwise controllability of degenerate parabolic equations. The particularity is that the control is exerted only on some point in the interior of the spatial domain. With respect with both boundary and distributed parabolic control problems, we will see new phenomena such as conditions on the time control and geometric conditions on the location of the control.

Remark 36. In what follows, we will keep the same notations of Chapter 5.

This chapter is outlined as follows. In Section 6.2, we analyze the well-posedness for the equation (6.1.1). Section 6.3 is devoted to studying the pointwise approximate controllability problem for equation (6.1.1). Finally, in Section 6.4, we prove the pointwise null controllability result.

6.2 Well-posedness results

Let us define the following symmetric continuous bilinear form a on $H^1_\alpha(0,1)$ by

$$a : \left\{ \begin{array}{l} H^1_\alpha \times H^1_\alpha \to \mathbb{R} \\ (y,z) \mapsto \displaystyle\int_0^1 \sqrt{x^\alpha} y_x \sqrt{x^\alpha} z_x \, dx. \end{array} \right. \tag{6.2.1}$$

We immediately see that a is $H^1_\alpha(0,1)$-$L^2(0,1)$ coercive, i.e.

$$\exists \gamma > 0, \exists \lambda \in \mathbb{R}, \forall y \in H^1_\alpha(0,1), \quad a(y,y) + \lambda \|y\|^2_{L^2(0,1)} \geq \gamma \|y\|^2_{H^1_\alpha(0,1)}. \tag{6.2.2}$$

Then, the equation (6.1.1) is well-posed. To be precise, one has (see [135, Theorem III.1.2]):

Theorem 6.2.1. *For any $y_0 \in L^2(0,1)$ and $v \in L^2(0,T)$, equation (6.1.1) possesses a unique solution y satisfying $y \in L^2(0,T;H^1_\alpha(0,1)) \cap C^0([0,T];L^2(0,1))$ and*

$$\begin{aligned} &\|y\|_{L^2(0,T;H^1_\alpha(0,1))} + \|y\|_{C^0([0,T];L^2(0,1))} \\ &\leq C\Big(\|y_0\|_{L^2(0,1)} + \|\delta_b\|_{H^{-1}_\alpha}\|v\|_{L^2(0,T)}\Big), \end{aligned}$$

for some positive constant C.

6.3 Approximate controllability

In this section, we deal with the notion of approximate controllability (which is weaker than null controllability), that can be stated as follows: For every $\varepsilon > 0$ and $y_0, y_T \in L^2(0,1)$, find a control $v \in L^2(0,T)$ such that the solution y of (6.1.1) satisfies

$$\|y(T) - y_T\|_{L^2(0,1)} \leq \varepsilon. \tag{6.3.1}$$

It is nowadays well-known (see for instance [72, Theorem 2.43]) that the approximate controllability at time $T > 0$ of (6.1.1) is equivalent to the unique continuation property for the adjoint parabolic equation: equation (6.1.1) is approximately controllable at time $T > 0$ if and only if its adjoint equation

$$\begin{cases} \varphi_t + (x^\alpha \varphi_x)_x = 0, & (t,x) \in Q, \\ \varphi(t,1) = 0, & t \in (0,T), \\ \begin{cases} \varphi(t,0) = 0, & 0 \le \alpha < 1 \\ x^\alpha \varphi_x(t,0) = 0, & 1 \le \alpha < 2 \end{cases} & t \in (0,T), \\ \varphi(T,x) = \varphi_0(x), & x \in (0,1), \end{cases} \tag{6.3.2}$$

satisfies the following unique continuation property

$$\forall \varphi_0 \in L^2(0,1), \quad \Big(\varphi(\cdot,b) = 0 \quad \text{on} \quad (0,T)\Big) \Rightarrow \varphi_0 = 0 \quad \text{in} \quad (0,1). \tag{6.3.3}$$

Let us present our pointwise approximate controllability results, that is, our first main result related to system (6.1.1). To this end, we introduce the following set

$$\mathcal{S}_{\nu_\alpha} = \left\{ \left(\frac{j_{\nu_\alpha,k}}{j_{\nu_\alpha,n}}\right)^{\frac{1}{\kappa_\alpha}}, \quad n > k \ge 1 \right\}, \tag{6.3.4}$$

where $(j_{\nu_\alpha,k})_{k\ge1}$ is the sequence of the zeros of Bessel functions defined in Section 5.3.

One has:

Theorem 6.3.1. *Equation (6.1.1) is approximately controllable at time $T > 0$ if and only if*

$$b \notin \mathcal{S}_{\nu_\alpha}. \tag{6.3.5}$$

Remark 37. We point out that, condition (6.3.5) is equivalent to the so-called *Fattorini-Hautus test* [85]:

$$\Phi_{\nu_\alpha,n}(b) \neq 0, \quad \text{for any} \quad n \ge 1.$$

Here $\Phi_{\nu_\alpha,n}$ denotes the eigenfunctions defined in Section 5.3.

Remark 38. The approximate controllability result stated in Theorem 6.3.1 does not depend on the final time T: approximate controllability of equation (6.1.1) at some time $T > 0$ is equivalent to the approximate controllability of equation (6.1.1) at any time $T > 0$. On the other hand, condition (6.3.5) characterizes the approximate controllability property of equation (6.1.1). Thus, (6.3.5) is a necessary condition for the null controllability of this system at some time $T > 0$.

Proof. **Necessary condition:** By contradiction, let us assume that condition (6.3.5) does not hold, i.e., that there is $n_0 > k_0 \ge 1$ for which $j_{\nu_\alpha,n_0} b^{\kappa_\alpha} = j_{\nu_\alpha,k_0}$. Let us see that the unique continuation property for the adjoint equation (6.3.2) is no longer valid. Indeed, let us take $\varphi_0 = \Phi_{\nu_\alpha,n_0} \in L^2(0,1)$. Thus, it is not difficult to see that the corresponding solution to the adjoint problem (6.3.2) is given by

$$\varphi(t,x) = e^{-\lambda_{\nu_\alpha,n_0}(T-t)} \Phi_{\nu_\alpha,n_0}(x), \qquad (t,x) \in Q.$$

Therefore, $\varphi(t,b) = e^{-\lambda_{\nu_\alpha,n_0}(T-t)} \Phi_{\nu_\alpha,n_0}(b) = 0$ on $(0,T)$ but $\varphi_0 \neq 0$. So, system (6.1.1) is not approximately controllable at time $T > 0$. This proves the necessary part of Theorem 6.3.1.

Sufficient condition: Let us now assume that condition (6.3.5) holds. The task now is to prove that the unique continuation property for the solutions of the adjoint problem (6.3.2) holds. To this end, let us fix $\varphi_0 \in L^2(0,1)$ and assume that the corresponding solution φ of (6.3.2) satisfies

$$\varphi(t,b) = 0, \qquad \forall t \in (0,T).$$

Using the fact that $(\Phi_{\nu_\alpha,k})_{k\geq 1}$ is a basis for $L^2(0,1)$, we can write

$$\varphi_0(x) = \sum_{k\geq 1} a_k \Phi_{\nu_\alpha,k}(x),$$

where the previous series converges in $L^2(0,1)$. In this series the coefficients are given by $a_k = \langle \varphi_0, \Phi_{\nu_\alpha,k}\rangle$ for any $k \geq 1$. Then, the solution φ of system (6.3.2) is given by

$$\varphi(t,x) = \sum_{k\geq 1} a_k e^{-\lambda_{\nu_\alpha,k}(T-t)} \Phi_{\nu_\alpha,k}(x).$$

On the other hand, since the sequence $(\lambda_{\nu_\alpha,k})_{k\geq 1}$ satisfies the hypotheses of Lemma 5.3.2, we know that there exists a biorthogonal family $\{q_{\nu_\alpha,k}\}_{k\geq 1}$ to the exponentials made upon the $\lambda_{\nu_\alpha,k}$, see Theorem 6.4.2.

Hence, by [114, Lemma 5.4] we infer that the family $\{e^{-\lambda_{\nu_\alpha,k}t}\}_{k\geq 1}$ is minimal, which implies that is ω-independent, see [114, Theorem 5.8 and Definition 5.7].

Recall that we have assumed $\varphi(\cdot,b) = 0$ on the interval $(0,T)$. Then, the expression of $\varphi(\cdot,b)$ together with the property of the exponentials imply $a_k = 0$ for any $k \geq 1$. This proves the continuation property for the solutions to the adjoint problem (6.3.2) and, thus the approximate controllability of (6.1.1) at any positive time T. □

6.4 Null controllability

We recall that the main problem that we will address is the null pointwise controllability for equation (6.1.1), employing a control located at an interior point $b \in (0,1)$. In other words, given $y_0 \in L^2(0,1)$ we wish to find a control function $v \in L^2(0,T)$ that drives the solution y of (6.1.1) at rest in a finite time $T > 0$. This is our second main result. It reads as follows:

Theorem 6.4.1. *Let $y_0 \in L^2(0,1)$ and assume that condition (6.3.5) holds. Let us define*

$$T(b,\alpha) = \limsup_{k\to+\infty} -\frac{\log(|\Phi_{\nu_\alpha,k}(b)|)}{\lambda_{\nu_\alpha,k}}. \tag{6.4.1}$$

Then, given $T > 0$, one has:

1. *If $T > T(b,\alpha)$, the equation (6.1.1) is null controllable at time T.*

2. *If $T < T(b,\alpha)$, the equation (6.1.1) is not null controllable at time T.*

Remark 39. The minimal time $T(b,\alpha)$ depends on the control position b but also on the rate of the degeneracy α.

Proof. **Positive pointwise controllability result.** Let us assume that $T > T(b,\alpha)$. Our objective is to prove that equation (6.1.1) is exactly controllable to zero at time T. Following the ideas of [54, 87], we may reduce the controllability issue to a moment problem. First, we treat the problem with formal computations. We will present a rigorous justification in a second moment.

Step 1: Reduction to a moment problem. Let us start expanding the initial condition $y_0 \in L^2(0,1)$ with respect to the basis of the eigenfunctions $(\Phi_{\nu_\alpha,k})_{k\geq 1}$. Indeed, we know that there exist a sequence $(\mu^0_{\alpha,k})_{k\geq 1} \in l^2(\mathbb{N}^\star)$ such that, for all $x \in (0,1)$,

$$y_0(x) = \sum_{k\geq 1} \mu^0_{\alpha,k} \Phi_{\nu_\alpha,k}(x).$$

Next, we expand also the solution y of (6.1.1) as

$$y(t,x)=\sum_{k\geq 1}\beta_{\alpha,k}(t)\Phi_{\nu_\alpha,k}(x),\quad x\in(0,1),\, t\geq 0\quad\text{with}\quad \sum_{k\geq 1}\beta^2_{\alpha,k}(t)<+\infty.$$

Therefore, the null controllability condition $y(T,x)=0$ becomes

$$\beta_{\alpha,k}(T)=0,\quad \forall k\geq 1. \tag{6.4.2}$$

On the other hand, we observe that $\varphi_k(t,x):=e^{-\lambda_{\nu_\alpha,k}(T-t)}\Phi_{\nu_\alpha,k}(x)$ is solution of the adjoint problem:

$$\begin{cases} (\varphi_k)_t+\big(x^\alpha(\varphi_k)_x\big)_x=0, & (t,x)\in Q,\\ \varphi_k(t,0)=\varphi_k(t,1)=0, & t\in(0,T),\\ \begin{cases}\varphi_k(t,0)=0, & 0\leq\alpha<1\\ x^\alpha(\varphi_k)_x(t,0)=0, & 1\leq\alpha<2\end{cases} & t\in(0,T),\\ \varphi_k(T,x)=\Phi_{\nu_\alpha,k}, & x\in(0,1).\end{cases} \tag{6.4.3}$$

Combining (6.1.1) and (6.4.3) we obtain

$$\begin{aligned}
&\Phi_{\nu_\alpha,k}(b)\int_0^T v(t)e^{-\lambda_{\nu_\alpha,k}(T-t)}\,dt\\
&\quad=\int_0^T\!\!\int_0^1\Big[\varphi_k\big(y_t-(x^\alpha y_x)_x\big)+y\big((\varphi_k)_t+(x^\alpha(\varphi_k)_x)_x\big)\Big]dx\,dt\\
&\quad=\int_0^1\varphi_k y|_0^T\,dx-\int_0^T\varphi_k x^\alpha y_x|_0^1\,dt+\int_0^T yx^\alpha(\varphi_k)_x|_0^1\,dt\\
&\quad=\int_0^1 y(T,x)\varphi_k(T,x)\,dx-\int_0^1 y(0,x)\varphi_k(0,x)\,dx\\
&\quad=\int_0^1 y(T,x)\Phi_{\nu_\alpha,k}(x)\,dx-\int_0^1 y(0,x)\Phi_{\nu_\alpha,k}(x)e^{-\lambda_{\nu_\alpha,k}T}\,dx\\
&\quad=\beta_{\alpha,k}(T)-\mu^0_{\alpha,k}e^{-\lambda_{\nu_\alpha,k}T}.
\end{aligned}$$

Therefore, there exists a control function $v\in L^2(0,T)$ such that the solution satisfies $y(T,x)=0$ for any $x\in(0,1)$ if, and only if, there exists $v\in L^2(0,T)$ such that:

$$\Phi_{\nu_\alpha,k}(b)\int_0^T v(t)e^{-\lambda_{\nu_\alpha,k}(T-t)}\,dt=-\mu^0_{\alpha,k}e^{-\lambda_{\nu_\alpha,k}T},\quad\forall k\geq 1.$$

A necessary condition for the existence of a solution for any $y_0\in L^2(0,1)$ is:

$$\Phi_{\nu_\alpha,k}(b)\neq 0,\quad\forall k\geq 1.$$

This latter condition is fulfilled by (6.3.5) (see Remark 37).

We are thus led to find a function $v\in L^2(0,T)$ that satisfies the following problem

$$\int_0^T v(t)e^{-\lambda_{\nu_\alpha,k}(T-t)}\,dt=-\frac{e^{-\lambda_{\nu_\alpha,k}T}\mu^0_{\alpha,k}}{\Phi_{\nu_\alpha,k}(b)},\quad\forall k\geq 1. \tag{6.4.4}$$

After a change of variable in the integral, we arrive to the reduction of the null controllability issue to the problem ($h(t)=v(T-t)$)

$$\begin{cases}\text{Find}\quad h\in L^2(0,T)\quad\text{such that}\\ \displaystyle\int_0^T h(t)e^{-\lambda_{\nu_\alpha,k}t}\,dt=-\frac{e^{-\lambda_{\nu_\alpha,k}T}\mu^0_{\alpha,k}}{\Phi_{\nu_\alpha,k}(b)},\quad\forall k\geq 1.\end{cases} \tag{6.4.5}$$

This is a moment problem in $L^2(0,T)$ with respect to the family $\{e^{-\lambda_{\nu_\alpha,k}t}\}_{k\geq 1}$.

Step 2: Formal solution of the moment problem. We present here the formal computations that show that the moment problem (6.4.4) has a solution h. For defining the function h satisfying (6.4.5), in what follows we firstly need to introduce a sequence $\{q_{\nu_\alpha,k}\}_{k\geq 1}$ in $L^2(0,T)$ which is biorthogonal to $\{e^{-\lambda_{\nu_\alpha,k}t}\}_{k\geq 1}$. The existence of such a sequence is a consequence of the convergence of the series $\sum_{n\geq 1}\dfrac{1}{\lambda_{\nu_\alpha,n}}$ by the celebrated Muntz theorem, and it is guaranteed by the following result.

Theorem 6.4.2 ([88, Lemma 3.1])**.** *Let $(\lambda_{\nu_\alpha,k})_{k\geq 1}$ be defined by (5.3.4). Then there exists a biorthogonal family $\{q_{\nu_\alpha,k}\}_{k\geq 1}$ in $L^2(0,T)$ to $\{e^{-\lambda_{\nu_\alpha,k}t}\}_{k\geq 1}$, i.e.,*

$$\int_0^T e^{-\lambda_{\nu_\alpha,k}t} q_{\nu_\alpha,l}(t)\,dt = \delta_{kl}, \quad \forall k,l\geq 1. \tag{6.4.6}$$

Here, δ_{kl} denotes the Kronecker symbol.

Moreover, the following estimation holds

$$\forall \varepsilon>0, \exists C_{\varepsilon,T}>0 \quad \textit{such that} \quad \|q_{\nu_\alpha,k}\|_{L^2(0,T)} \leq C_{\varepsilon,T} e^{\varepsilon\lambda_{\nu_\alpha,k}}, \quad \forall k\geq 1. \tag{6.4.7}$$

Hence, we may formally solve the moment problem above by defining

$$h(t)=\sum_{k\geq 1} h_k(t), \quad \text{with} \quad h_k(t) = -\frac{e^{-\lambda_{\nu_\alpha,k}T}\mu^0_{\alpha,k}}{\Phi_{\nu_\alpha,k}(b)} q_{\nu_\alpha,k}(t). \tag{6.4.8}$$

Indeed, if this series makes sense (and if the following computation can be justified) we have

$$\begin{aligned}\int_0^T h(t)e^{-\lambda_{\nu_\alpha,k}t}\,dt &= \sum_{l\geq 1} -\frac{e^{-\lambda_{\nu_\alpha,l}T}\mu^0_{\alpha,l}}{\Phi_{\nu_\alpha,l}(b)} \int_0^T e^{-\lambda_{\nu_\alpha,k}t} q_{\nu_\alpha,l}(t)\,dt \\ &= -\frac{e^{-\lambda_{\nu_\alpha,k}T}\mu^0_{\alpha,k}}{\Phi_{\nu_\alpha,k}(b)},\end{aligned}$$

and the claim will be proved.

Step 3: L^2 regularity of the control and controllability result. We consider h given by (6.4.8). We have to check that h belongs to $L^2(0,T)$ if $T>T(b,\alpha)$. Indeed, from the definition of the minimal time $T(b,\alpha)$ (see (6.4.1)) and for any fixed $\varepsilon>0$, we can infer that there exists a positive constant $C_{\nu_\alpha,\varepsilon}$ such that

$$\frac{1}{|\Phi_{\nu_\alpha,k}(b)|} \leq C_{\nu_\alpha,\varepsilon} e^{\lambda_{\nu_\alpha,k}(T(b,\alpha)+\varepsilon)}, \qquad \forall k\geq 1.$$

Hence, we can use the bound (6.4.7) and get a new positive constant $C_{\nu_\alpha,\varepsilon,T}$ for which

$$\begin{aligned}\|h_k\|_{L^2(0,T)} &\leq \|y_0\|_{L^2(0,1)} \frac{e^{-\lambda_{\nu_\alpha,k}T}}{|\Phi_{\nu_\alpha,k}(b)|}\|q_{\nu_\alpha,k}\|_{L^2(0,T)} \\ &\leq C_{\nu_\alpha,\varepsilon}\|y_0\|_{L^2(0,1)} e^{-\lambda_{\nu_\alpha,k}(T-T(b,\alpha)-\varepsilon)}\|q_{\nu_\alpha,k}\|_{L^2(0,T)} \\ &\leq C_{\nu_\alpha,\varepsilon,T}\|y_0\|_{L^2(0,1)} e^{-\lambda_{\nu_\alpha,k}(T-T(b,\alpha)-2\varepsilon)}.\end{aligned}$$

By the estimate above with $\varepsilon = \dfrac{T-T(b,\alpha)}{4}$, we deduce that

$$\|h_k\|_{L^2(0,T)} \leq C_{\nu_\alpha,T}\|y_0\|_{L^2(0,1)} e^{-\lambda_{\nu_\alpha,k}(T-T(b,\alpha))/2}.$$

Next, observe that

$$\sum_{k\geq 1} \|h_k\|_{L^2(0,T)} \leq C_{\nu_\alpha,T}\|y_0\|_{L^2(0,1)} \sum_{k\geq 1} e^{-\lambda_{\nu_\alpha,k}(T-T(b,\alpha))/2} < +\infty,$$

where the last series is convergent due to the presence of the exponential with negative sign. Indeed, using Lemma 5.3.6, we have

$$\begin{aligned}
&\sum_{k\geq 1} e^{-\lambda_{\nu_\alpha,k}(T-T(b,\alpha))/2} \\
&\qquad = \frac{2}{T-T(b,\alpha)} \sum_{k\geq 1} \left(\lambda_{\nu_\alpha,k} \frac{T-T(b,\alpha)}{2} e^{-\lambda_{\nu_\alpha,k}(T-T(b,\alpha))/2}\right) \frac{1}{\lambda_{\nu_\alpha,k}} \\
&\qquad \leq \frac{2}{T-T(b,\alpha)} \left(\sup_{x>0} xe^{-x}\right) \sum_{k\geq 1} \frac{1}{\lambda_{\nu_\alpha,k}} < +\infty.
\end{aligned}$$

This immediately ensures the absolute convergence of the series which defines the control h. This allows to conclude that

$$h = -\sum_{k\geq 1} \frac{e^{-\lambda_{\nu_\alpha,k}T}\mu^0_{\alpha,k}}{\Phi_{\nu_\alpha,k}(b)} q_{\nu_\alpha,k} \in L^2(0,T),$$

and therefore, that the degenerate heat equation (6.1.1) is null controllable at time T when $T > T(b,\alpha)$.

Negative pointwise controllability result. In order to finish the proof of Theorem 6.4.1, let us prove that if $0 < T < T(b,\alpha)$, then equation (6.1.1) is not null controllable at time T. We argue by contradiction. Assume that equation (6.1.1) is null controllable at time $T < T(b,\alpha)$. By duality, this last fact is equivalent to the existence of a positive constant C such that every solution φ of the adjoint problem (6.3.2) satisfies, the following observability estimate:

$$\|\varphi(0,\cdot)\|^2_{L^2(0,1)} \leq C \int_0^T \varphi(t,b)^2\,dt. \tag{6.4.9}$$

Let us work with the particular solutions associated with initial data $\varphi_0 = \Phi_{\nu_\alpha,k}$. With this choice, the solution φ_k of (6.3.2) is given by

$$\varphi_k(t,x) = e^{-\lambda_{\nu_\alpha,k}(T-t)}\Phi_{\nu_\alpha,k}(x), \qquad \forall k \geq 1.$$

Thus, the observability inequality (6.4.9) becomes

$$\begin{aligned}
e^{-2\lambda_{\nu_\alpha,k}T} &\leq C\Phi_{\nu_\alpha,k}(b)^2 \int_0^T e^{-2\lambda_{\nu_\alpha,k}(T-t)}\,dt \\
&\leq C\frac{1}{2\lambda_{\nu_\alpha,k}}\left(1 - e^{-2\lambda_{\nu_\alpha,k}T}\right)\Phi_{\nu_\alpha,k}(b)^2, \\
&\leq C\frac{1}{2\lambda_{\nu_\alpha,k}}\Phi_{\nu_\alpha,k}(b)^2, \\
&\leq \frac{C}{2\lambda_{\nu_\alpha,1}}\Phi_{\nu_\alpha,k}(b)^2, \qquad \forall k \geq 1,
\end{aligned} \tag{6.4.10}$$

that is to say, for a new constant $C > 0$ not depending on k, one has,

$$1 \leq Ce^{2\lambda_{\nu_\alpha,k}T}\Phi_{\nu_\alpha,k}(b)^2. \tag{6.4.11}$$

From the definition of $T(b,\alpha)$, we obtain the existence of an increasing unbounded subsequence $\{k_n\}_{n\geq 1}$ such that

$$T(b,\alpha) = \lim_{n\to+\infty} -\frac{\log(|\Phi_{\nu_\alpha,k_n}(b)|)}{\lambda_{\nu_\alpha,k_n}}.$$

If we assume that $T(b,\alpha) < +\infty$, then, for every $\varepsilon > 0$, there exits a positive integer n_ε such that

$$T(b,\alpha) - \varepsilon \leq -\frac{\log(\left|\Phi_{\nu_\alpha,k_n}(b)\right|)}{\lambda_{\nu_\alpha,k_n}}, \qquad \forall n \geq n_\varepsilon.$$

This last inequality together with (6.4.11) provide the new inequality

$$1 \leq Ce^{-2\lambda_{\nu_\alpha,k_n}(T(b,\alpha)-T-\varepsilon)}, \qquad \forall n \geq n_\varepsilon.$$

The previous inequality gives a contradiction if we take $\varepsilon \in (0, \dfrac{T(b,\alpha)-T}{2})$. This ends the proof. $\square$

Let us end this section by proving that the minimal time $T(b,\alpha)$ is well-defined and satisfies $T(b,\alpha) \in [0,+\infty]$. One has

Theorem 6.4.3. *Let us assume the hypotheses of Theorem 6.4.1 and let $T(b,\alpha)$ be the number given by (6.4.1). Then,*

$$T(b,\alpha) \in [0,+\infty].$$

Proof. Owing to Remark 37, condition (6.3.5) implies that

$$\Phi_{\nu_\alpha,k}(b) \neq 0, \quad \forall k \geq 1.$$

Hence,

$$\begin{aligned} 0 < |\Phi_{\nu_\alpha,k}(b)| &= |\langle \delta_b, \Phi_{\nu_\alpha,k}\rangle_{H_\alpha^{-1},H_\alpha^1}| \\ &\leq \|\delta_b\|_{H_\alpha^{-1}} \|\Phi_{\nu_\alpha,k}\|_{H_\alpha^1} \\ &= \|\delta_b\|_{H_\alpha^{-1}} \sqrt{\lambda_{\nu_\alpha,k}}. \end{aligned}$$

Therefore, we deduce that there exists a constant $\sigma > 0$ such that

$$0 < |\Phi_{\nu_\alpha,k}(b)| \leq \sigma\lambda_{\nu_\alpha,k}, \qquad \forall k \geq 1.$$

Thus,

$$\log\Big(\frac{1}{|\Phi_{\nu_\alpha,k}(b)|}\Big) \geq \log\Big(\frac{1}{\sigma\lambda_{\nu_\alpha,k}}\Big), \qquad \forall k \geq 1,$$

that is to say,

$$-\frac{\log(|\Phi_{\nu_\alpha,k}(b)|)}{\lambda_{\nu_\alpha,k}} \geq -\frac{\log(\sigma\lambda_{\nu_\alpha,k})}{\lambda_{\nu_\alpha,k}}, \qquad \forall k \geq 1.$$

Using the fact that $\lambda_{\nu_\alpha,k} \to +\infty$ as $k \to +\infty$, it follows that $T(b,\alpha) \in [0,+\infty]$. This ends the proof of the result. $\square$

Appendix

A Proof of Lemma 1.3.1

Denote by $\nu(t) = \theta(t)^{-1}$. After straightforward computations, we get that

$$\dot{\theta}(T') = 0 \quad \text{and} \quad \ddot{\theta}(t) \geq \frac{8}{\nu(t)^5} = 8\theta(t)^{\frac{5}{4}} \geq 8\theta(T')^{\frac{5}{4}}, \quad \forall t \in (0,T). \tag{A.1}$$

Employing the standard Taylor's formula to the function θ in (t_0, T'), there exists $\tilde{T} \in (t_0, T')$ such that

$$\begin{aligned}\theta(t) &= \theta(T') + \dot{\theta}(T')(t-T') + \frac{\ddot{\theta}(\tilde{T})}{2}(t-T')^2 \\ &\geq \theta(T') + 4\theta(T')^{\frac{5}{4}}(t-T')^2,\end{aligned}$$

where we have used (A.1). Now, recalling that $\varphi = \theta\psi$, from the last inequality, we get

$$\varphi(t,x) \leq \varphi(T',x) - c_0(t-T')^2,$$

where $c_0 = -4 \max\limits_{x\in(0,1)} \psi(x)\theta(T')^{\frac{5}{4}} = 4\gamma(d-d^*)\theta(T')^{\frac{5}{4}} > 0$. Therefore,

$$\begin{aligned}\int_{t_0}^{T} e^{2s\varphi(t,x)}\,dt &\leq e^{2s\varphi(T',x)} \int_{t_0}^{T} e^{-2sc_0(t-T')^2}\,dt \\ &= \frac{e^{2s\varphi(T',x)}}{\sqrt{2c_0 s}} \int_{-\infty}^{+\infty} e^{-r^2}\,dr \\ &\leq \frac{C}{\sqrt{s}} e^{2s\varphi(T',x)}\end{aligned}$$

for some positive constant C independent of s.

Consequently,

$$\iint_{Q_{t_0}} f^2(T',x)e^{2s\varphi(t,x)}\,dx\,dt \leq \frac{C}{\sqrt{s}} \int_0^1 f^2(T',x)e^{2s\varphi(T',x)}\,dx.$$

B Proof of Lemma 2.3.1

Using the equation satisfied by y_k

$$y_{kt} - d_k(a(x)y_{kx})_x + \sum_{j=1}^{k} b_{kj}y_j = f_k$$

and integrating by parts, we obtain

$$\begin{aligned}
K_1 &= \iint_{Q_{t_0}} y_{k-1,t}\beta_k\chi y_k \,dx\,dt \\
&= -\iint_{Q_{t_0}} y_{k-1}\beta_{k,t}\chi y_k \,dx\,dt - \iint_{Q_{t_0}} y_{k-1}\beta_k\chi y_{k,t}\,dx\,dt \\
&= -\iint_{Q_{t_0}} y_{k-1}\beta_{k,t}\chi y_k \,dx\,dt + \iint_{Q_{t_0}} y_{k-1}\beta_k\chi \left(a(x)y_{k,x}\right)_x \,dx\,dt \\
&\quad -\sum_{j=1}^{k}\iint_{Q_{t_0}} y_{k-1}\beta_k\chi b_{k,j}y_j\,dx\,dt - \iint_{Q_{t_0}} y_{k-1}\beta_k\chi b_{k,k+1}y_{k+1}\,dx\,dt \\
&\quad + \iint_{Q_{t_0}} y_{k-1}\beta_k\chi f_k\,dx\,dt = \sum_{i=1}^{5} K_1^{(i)}.
\end{aligned}$$

Recalling that $\beta_k = s^l\theta^l e^{2s\Phi_k}$, we have that there exists $C>0$ such that $|\beta_{k,t}| \leqslant Cs^{l+1}e^{2s\Phi_k}\theta^{l+1}$. Thus, using Young's inequality, we obtain

$$\begin{aligned}
\left|K_1^{(1)}\right| &= \left|\iint_{Q_{t_0}} y_{k-1}\beta_{k,t}\chi y_k\,dx\,dt\right| \\
&\leqslant Cs^{l+1}\iint_{Q_{t_0}} e^{2s\Phi_k}\theta^{l+1}\chi|y_{k-1}y_k|\,dx\,dt \\
&\leqslant C\iint_{Q_{t_0}}\left(s^{3/2}e^{s\varphi}\theta^{3/2}|y_k|\sqrt{\chi\cdot\frac{2\varepsilon}{C}\cdot\frac{x^2}{a(x)}}\right) \\
&\times\left(s^{l-1/2}e^{s(2\Phi_k-\varphi)}\theta^{l-1/2}|y_{k-1}|\sqrt{\chi\cdot\frac{C}{2\varepsilon}\cdot\frac{a(x)}{x^2}}\right)\,dx\,dt \\
&\leqslant \varepsilon s^3\iint_{Q_{t_0}} e^{2s\varphi}\theta^3\chi\frac{x^2}{a(x)}y_k^2\,dx\,dt + \frac{C}{\varepsilon}s^{2l-1}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{2l-1}\chi\frac{a(x)}{x^2}y_{k-1}^2\,dx\,dt.
\end{aligned}$$

Using the fact that $\operatorname{supp}\chi\subset\mathcal{O}_0$ and hence $\dfrac{a(x)}{x^2}$ is bounded on $\overline{\mathcal{O}_0}$, it follows that

$$\left|K_1^{(1)}\right| \leqslant \varepsilon s^3\iint_{Q_{t_0}} e^{2s\varphi}\theta^3\chi\frac{x^2}{a(x)}y_k^2\,dx\,dt + \frac{C}{\varepsilon}s^{2l-1}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{2l-1}\chi y_{k-1}^2\,dx\,dt. \tag{B.1}$$

Integrating by parts in $K_1^{(2)}$, we obtain

$$\begin{aligned}
K_1^{(2)} = &-\underbrace{\iint_{Q_{t_0}} \beta_k\chi y_{k-1,x}a(x)y_{k,x}\,dx\,dt}_{J_1} - \underbrace{\iint_{Q_{t_0}} y_{k-1}\chi\beta_{k,x}a(x)y_{k,x}\,dx\,dt}_{J_2} \\
&-\underbrace{\iint_{Q_{t_0}} y_{k-1}\beta_k\chi_x a(x)y_{k,x}\,dx\,dt}_{J_3}.
\end{aligned}$$

Once again, using Young's inequality, we get

$$\begin{aligned}
J_1 &\leqslant C\iint_{Q_{t_0}} s^l e^{2s\Phi_k}\theta^l\chi^{1/2}y_{k-1,x}a(x)y_{k,x}\,dx\,dt \\
&\leqslant \varepsilon s\iint_{Q_{t_0}} e^{2s\varphi}\theta a(x)y_{k,x}^2\,dx\,dt + \frac{C}{\varepsilon}s^{2l-1}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{2l-1}\chi a(x)y_{k-1,x}^2\,dx\,dt
\end{aligned}$$

and observing that $|\beta_{k,x}| \leqslant Cs^{l+1}\theta^{l+1}e^{2s\Phi_k}$, and $\chi(x)a(x) \leqslant C\sqrt{\chi a(x)}$ on $\overline{\mathcal{O}_0}$, one has

$$\begin{aligned}J_2 &\leqslant C \iint_{Q_{t_0}} s^{l+1}e^{2s\Phi_k}\theta^{l+1}\sqrt{\chi a(x)}y_{k-1}y_{k,x}\,dx\,dt\\ &\leqslant \varepsilon s \iint_{Q_{t_0}} e^{2s\varphi}\theta a(x)y_{k,x}^2\,dx\,dt + \frac{C}{\varepsilon}s^{2l+1}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{2l+1}\chi y_{k-1}^2\,dx\,dt.\end{aligned}$$

For the term J_3, having in mind facts that $a(x) \leqslant C\sqrt{a(x)}$ on $\overline{\mathcal{O}_0}$ and $\dfrac{\chi_x}{\sqrt{\chi}} \in L^\infty(0,1)$, we obtain

$$\begin{aligned}J_3 &\leqslant C \iint_{Q_{t_0}} s^{l}e^{2s\Phi_k}\theta^{l}\sqrt{\chi a(x)}y_{k-1}y_{k,x}\,dx\,dt\\ &\leqslant \varepsilon s \iint_{Q_{t_0}} e^{2s\varphi}\theta a(x)y_{k,x}^2\,dx\,dt + \frac{C}{\varepsilon}s^{2l-1}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{2l-1}\chi y_{k-1}^2\,dx\,dt.\end{aligned}$$

Hence

$$\begin{aligned}\left|K_1^{(2)}\right| \leqslant& 3\varepsilon s \iint_{Q_{t_0}} e^{2s\varphi}\theta a(x)y_{k,x}^2\,dx\,dt + \frac{C}{\varepsilon}\left(s^{2l+1}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{2l+1}\chi y_{k-1}^2\,dx\,dt\right.\\ &\left.+s^{2l-1}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{2l-1}\chi a(x)y_{k-1,x}^2\,dx\,dt\right).\end{aligned} \tag{B.2}$$

For the term $K_1^{(3)}$, we have

$$\left|K_1^{(3)}\right| \leqslant \sum_{j=1}^{k-1} C\iint_{Q_{t_0}} s^l e^{2s\Phi_k}\theta^l\chi y_j y_{k-1}\,dx\,dt + C\iint_{Q_{t_0}} s^l e^{2s\Phi_k}\theta^l \chi y_k y_{k-1}\,dx\,dt$$

and

$$\begin{aligned}&\sum_{j=1}^{k-1} C\iint_{Q_{t_0}} s^l e^{2s\Phi_k}\theta^l\chi y_j y_{k-1}\,dx\,dt\\ &\leqslant \frac{1}{2}s^{2l-3}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{2l-3}\chi y_{k-1}^2\,dx\,dt + C\sum_{j=1}^{k-1} s^3\iint_{Q_{t_0}} e^{2s\varphi}\theta^3\chi y_j^2\,dx\,dt.\end{aligned}$$

Similarly,

$$\begin{aligned}\iint_{Q_{t_0}} s^l e^{2s\Phi_k}\theta^l\chi y_k y_{k-1}\,dx\,dt \leqslant& \varepsilon s^3\iint_{Q_{t_0}} e^{2s\varphi}\theta^3\chi\frac{x^2}{a(x)}y_k^2\,dx\,dt\\ &+\frac{1}{4\varepsilon}s^{2l-3}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{2l-3}\chi\frac{a(x)}{x^2}y_{k-1}^2\,dx\,dt.\end{aligned}$$

Therefore

$$\begin{aligned}K_1^{(3)} \leqslant& \varepsilon s^3\iint_{Q_{t_0}} e^{2s\varphi}\theta^3\chi\frac{x^2}{a(x)}y_k^2\,dx\,dt\\ &+C\left(1+\frac{1}{\varepsilon}\right)s^{\kappa_1}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{\kappa_1}\chi y_{k-1}^2\,dx\,dt\\ &+C\sum_{j=1}^{k-2} s^3\iint_{Q_{t_0}} e^{2s\varphi}\theta^3\chi y_j^2\,dx\,dt,\end{aligned} \tag{B.3}$$

where $\kappa_1 = \max(3, 2l-3)$.

For the last term $K_1^{(4)}$, we have

$$\left|K_1^{(4)}\right| = \left|-\iint_{Q_{t_0}} y_{k-1} s^l e^{2s\Phi_k}\theta^l \chi b_{k,k+1} y_{k+1}\,dx\,dt\right| \leqslant C\iint_{Q_{t_0}} s^l e^{2s\Phi_k}\theta^l \chi y_{k-1}y_{k+1}\,dx\,dt$$
$$\leqslant \varepsilon s^3 \iint_{Q_{t_0}} e^{2s\varphi}\theta^3 \frac{x^2}{a(x)}\chi y_{k+1}^2\,dx\,dt + \frac{C}{\varepsilon}s^{2l-3}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{2l-3}\frac{a(x)}{x^2}\chi y_{k-1}^2\,dx\,dt.$$

Thus, as in $K_1^{(1)}$, we obtain

$$\left|K_1^{(4)}\right| \leqslant \varepsilon s^3 \iint_{Q_{t_0}} e^{2s\varphi}\theta^3 \frac{x^2}{a(x)}\chi y_{k+1}^2\,dx\,dt + \frac{C}{\varepsilon}s^{2l-3}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{2l-3}\chi y_{k-1}^2\,dx\,dt \tag{B.4}$$

Combining (B.1)-(B.4), we end up with

$$\begin{aligned}|K_1| \leqslant& \varepsilon\Big(\mathcal{J}(y_k)+\mathcal{J}(y_{k+1})\Big) + C\sum_{j=1}^{k-2} s^3 \iint_{Q_{t_0}} e^{2s\varphi}\theta^3\chi y_j^2\,dx\,dt\\ &+ C\left(1+\frac{1}{\varepsilon}\right)s^{\kappa_2}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{\kappa_2}\chi y_{k-1}^2\,dx\,dt\\ &+\frac{C}{\varepsilon}s^{2l-1}\iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)}\theta^{2l-1}\chi a(x) y_{k-1,x}^2\,dx\,dt\end{aligned} \tag{B.5}$$

where $\kappa_2 = \max(3, 2l+1)$.

Going back to the term K_2, one has

$$\begin{aligned}K_2 &= \iint_{Q_{t_0}} d_{k-1}\left(a(x)y_{k-1,x}\right)_x \beta_k\chi y_k\,dx\,dt\\ &= \underbrace{-\iint_{Q_{t_0}} d_{k-1}a(x)y_{k-1,x}\left(\beta_k\chi\right)_x y_k\,dx\,dt}_{K_2^{(1)}} \underbrace{-\iint_{Q_{t_0}} d_{k-1}a(x)y_{k-1,x}\beta_k\chi y_{k,x}\,dx\,dt}_{K_2^{(2)}}.\end{aligned}$$

Taking into account the fact that $|(\beta_k\chi)_x| \leqslant Cs^{l+1}\theta^{l+1}e^{2s\Phi_k}\chi^{1/2}$ and that the functions a and $\frac{a}{x^2}$ are bounded on $\overline{\mathcal{O}_0}$, we find that

$$\begin{aligned}\left|K_2^{(1)}\right| \leqslant& C\iint_{Q_{t_0}} s^{l+1}\theta^{l+1}e^{2s\Phi_k}\chi^{1/2}y_{k-1,x}y_k\,dx\,dt\\ \leqslant& \varepsilon s^3\iint_{Q_{t_0}} e^{2s\varphi}\theta^3\frac{x^2}{a(x)}y_k^2\,dx\,dt + \frac{C}{\varepsilon}s^{2l-1}\iint_{Q_{t_0}}\theta^{2l-1}e^{2s(2\Phi_k-\varphi)}a(x)\chi y_{k-1,x}^2\,dx\,dt.\end{aligned}$$

Analogously,

$$\begin{aligned}\left|K_2^{(2)}\right| \leqslant& \varepsilon s\iint_{Q_{t_0}}\theta a(x)\chi e^{2s\varphi}y_{k,x}^2\,dx\,dt\\ &+\frac{C}{\varepsilon}s^{2l-1}\iint_{Q_{t_0}} a(x)\theta^{2l-1}e^{2s(2\Phi_k-\varphi)}\chi y_{k-1,x}^2\,dx\,dt.\end{aligned}$$

Therefore,

$$|K_2| \leqslant \varepsilon\mathcal{J}(y_k) + 2\frac{C}{\varepsilon}s^{2l-1}\iint_{Q_{t_0}} a(x)\theta^{2l-1}e^{2s(2\Phi_k-\varphi)}\chi y_{k-1,x}^2\,dx\,dt. \tag{B.6}$$

For the term K_3, one has

$$\begin{aligned}|K_3| &\leqslant \sum_{j=1}^{k-1} C \iint_{Q_{t_0}} s^l \theta^l e^{2s\Phi_k} \chi y_k y_j \,dx\,dt \\ &\leqslant \varepsilon s^3 \iint_{Q_{t_0}} \theta^3 e^{2s\varphi} \chi \frac{x^2}{a(x)} y_k^2 \,dx\,dt \\ &\quad + \frac{C}{\varepsilon} \sum_{j=1}^{k-1} s^{2l-3} \iint_{Q_{t_0}} \theta^{2l-3} e^{2s(2\Phi_k-\varphi)} \frac{a(x)}{x^2} \chi y_j^2 \,dx\,dt.\end{aligned}$$

This yields that

$$|K_3| \leqslant \varepsilon s^3 \iint_{Q_{t_0}} \theta^3 e^{2s\varphi} \chi \frac{x^2}{a(x)} y_k^2 \,dx\,dt + \frac{C}{\varepsilon} \sum_{j=1}^{k-1} s^{2l-3} \iint_{Q_{t_0}} \theta^{2l-3} e^{2s(2\Phi_k-\varphi)} \chi y_j^2 \,dx\,dt. \tag{B.7}$$

Combining (B.5), (B.6) with (B.7), we arrive at

$$\begin{aligned}K_1 + K_2 + K_3 \leqslant& \varepsilon(\mathcal{J}(y_k) + \mathcal{J}(y_{k+1})) + K_1^{(5)} + C \sum_{j=1}^{k-2} s^{\widehat{l}} \iint_{Q_{t_0}} e^{2s\tilde{\varphi}} \theta^{\widehat{l}} \chi y_j^2 \,dx\,dt \\ &+ C\left(1 + \frac{1}{\varepsilon}\right) s^{\kappa_2} \iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)} \theta^{\kappa_2} \chi y_{k-1}^2 \,dx\,dt \\ &+ \frac{C}{\varepsilon} s^{2l-1} \iint_{Q_{t_0}} e^{2s(2\Phi_k-\varphi)} \theta^{2l-1} \chi a(x) y_{k-1,x}^2 \,dx\,dt,\end{aligned} \tag{B.8}$$

with $\widehat{l} = \max(3, 2l-3)$, $\tilde{\varphi} = \max(\varphi, 2\Phi_k - \varphi)$ and κ_2 defined in (B.5).

Next, we proceed to estimate the last term in the above inequality.

Set $\tilde{\Phi}_k = 2\Phi_k - \varphi$, multiply the equation satisfied by y_{k-1} by $s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k}\, y_{k-1}$ and integrate over Q, it comes that

$$\begin{aligned}&\iint_{Q_{t_0}} s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k} a y_{k-1,x}^2 \,dx\,dt \\ &= \frac{1}{d_{k-1}} \iint_{Q_{t_0}} s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k} d_{k-1} a y_{k-1,x}^2 \,dx\,dt \\ &= \frac{1}{2d_{k-1}} \iint_{Q_{t_0}} s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k} a (y_{k-1}^2)_t \,dx\,dt \\ &\quad - \frac{1}{d_{k-1}} \iint_{Q_{t_0}} b_{k-1,k} s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k} y_{k-1} y_k \,dx\,dt \\ &\quad - \sum_{j=1}^{k-1} \frac{1}{d_{k-1}} \iint_{Q_{t_0}} b_{k-1,j} s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k} y_j y_{k-1} \,dx\,dt \\ &\quad - \frac{1}{d_{k-1}} \iint_{Q_{t_0}} \left(s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k}\right)_x y_{k-1} d_{k-1} a y_{k-1,x} \,dx\,dt \\ &\quad - \frac{1}{d_{k-1}} \iint_{Q_{t_0}} f_{k-1} s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k} y_{k-1} \,dx\,dt \\ &= \sum_{j=1}^{5} H_j.\end{aligned}$$

Observe that

$$\left|\partial_t \left(s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k}\right)\right| \leqslant C s^{2l+1}\theta^{2l+1}\chi e^{2s\tilde{\Phi}_k}$$

for some positive constant C. Hence

$$|H_1| \leqslant C \iint_{\mathcal{O}_1\times(0,T)} s^{2l+1}\theta^{2l+1}e^{2s\tilde{\Phi}_k}y_{k-1}^2\,dx\,dt. \tag{B.9}$$

Again, by Young's inequality one has

$$\begin{aligned}
|H_2| &= \left|\frac{1}{d_{k-1}}\iint_{Q_{t_0}} b_{k-1,k}s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k}y_{k-1}y_k\,dx\,dt\right| \\
&\leqslant C\iint_{Q_{t_0}} s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k}y_{k-1}y_k\,dx\,dt \\
&\leqslant C\iint_{Q_{t_0}}\left(\sqrt{\frac{2\varepsilon}{C}}s^{\frac{3}{2}}\theta^{\frac{3}{2}}\sqrt{\chi}e^{s\varphi}\sqrt{\frac{x^2}{a}}y_k\right) \\
&\times\left(\sqrt{\frac{C}{2\varepsilon}}s^{2l-\frac{5}{2}}\theta^{2l-\frac{5}{2}}\sqrt{\chi}e^{s\left(2\tilde{\Phi}_k-\varphi\right)}\sqrt{\frac{a}{x^2}}y_{k-1}\right)\,dx\,dt \\
&\leqslant \varepsilon\iint_{Q_{t_0}} s^3\theta^3e^{2s\varphi}\chi\frac{x^2}{a}y_k^2\,dx\,dt + \frac{C}{\varepsilon}\iint_{Q_{t_0}} s^{4l-5}\theta^{4l-5}\chi e^{2s\left(2\tilde{\Phi}_k-\varphi\right)}\frac{a}{x^2}y_{k-1}^2\,dx\,dt.
\end{aligned}$$

Hence

$$|H_2| \leqslant \varepsilon\iint_{Q_{t_0}} s^3\theta^3e^{2s\varphi}\chi\frac{x^2}{a}y_k^2\,dx\,dt + \frac{C}{\varepsilon}\iint_{Q_{t_0}} s^{4l-5}\theta^{4l-5}\chi e^{2s\left(2\tilde{\Phi}_k-\varphi\right)}y_{k-1}^2\,dx\,dt. \tag{B.10}$$

Proceeding as above we can see that

$$\begin{aligned}
|H_3| &\leqslant \sum_{j=1}^{k-2} C\iint_{Q_{t_0}} s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k}y_jy_{k-1}\,dx\,dt \\
&+ C\iint_{Q_{t_0}} s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k}y_{k-1}^2\,dx\,dt \\
&\leqslant \sum_{j=1}^{k-2} C\iint_{Q_{t_0}}\left(\sqrt{\frac{2\varepsilon}{C}}s^{l-\frac{1}{2}}\theta^{l-\frac{1}{2}}\sqrt{\chi}e^{s\varphi}y_j\right)\left(\sqrt{\frac{C}{2\varepsilon}}s^{l-\frac{1}{2}}\theta^{l-\frac{1}{2}}\sqrt{\chi}e^{s(2\tilde{\Phi}_k-\varphi)}y_{k-1}\,dx\,dt\right) \\
&+ C\iint_{Q_{t_0}} s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k}y_{k-1}^2\,dx\,dt \\
&\leqslant \sum_{j=1}^{k-2}\varepsilon\iint_{Q_{t_0}} s^{2l-1}\theta^{2l-1}\chi e^{2s\varphi}y_j^2\,dx\,dt + \frac{C}{\varepsilon}\iint_{Q_{t_0}} s^{2l-1}\theta^{2l-1}\chi e^{2s\left(2\tilde{\Phi}_k-\varphi\right)}y_{k-1}^2\,dx\,dt \\
&+ C\iint_{Q_{t_0}} s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k}y_{k-1}^2\,dx\,dt.
\end{aligned} \tag{B.11}$$

Now, observe that $\left|\left(s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k}\right)_x\right| \leqslant Cs^{2l}\theta^{2l}e^{2s\tilde{\Phi}_k}$ on $\mathcal{O}_0\times(0,T)$, hence

$$\begin{aligned}
|H_4| &\leqslant \iint_{Q_{t_0}}\left|\left(s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k}\right)_x y_{k-1}ay_{k-1,x}\right|\,dx\,dt \\
&\leqslant C\iint_{\mathcal{O}_1\times(0,T)} s^{2l}\theta^{2l}e^{2s\tilde{\Phi}_k}y_{k-1}\sqrt{a}y_{k-1,x}\,dx\,dt \\
&\leqslant \frac{1}{2}\iint_{\mathcal{O}_1\times(0,T)} s^{2l-1}\theta^{2l-1}e^{2s\tilde{\Phi}_k}ay_{k-1,x}^2\,dx\,dt \\
&\quad + C\iint_{\mathcal{O}_1\times(0,T)} s^{2l+1}\theta^{2l+1}e^{2s\tilde{\Phi}_k}y_{k-1}^2\,dx\,dt.
\end{aligned} \tag{B.12}$$

Since $\tilde{\Phi}_k \le 2\tilde{\Phi}_k - \varphi$, it comes from (B.9)-(B.12) that

$$\begin{aligned}
&\iint_{Q_{t_0}} s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k} a y_{k-1,x}^2\, dx\, dt \\
&\leqslant C\left(1+\frac{1}{\varepsilon}\right)\iint_{Q_{\omega'}} s^J\theta^J e^{2s\left(2\tilde{\Phi}_k-\varphi\right)} y_{k-1}^2\, dx\, dt + \varepsilon\iint_{Q_{t_0}} s^3\theta^3 e^{2s\varphi}\chi\frac{x^2}{a} y_k^2\, dx\, dt \\
&+\sum_{j=1}^{k-2}\varepsilon\iint_{Q_{t_0}} s^3\theta^3\chi e^{2s\varphi} y_j^2\, dx\, dt + \frac{1}{d_{k-1}}\iint_{Q_{t_0}}\left|f_{k-1}s^{2l-1}\theta^{2l-1}\chi e^{2s\tilde{\Phi}_k} y_{k-1}\right| dx\, dt, \qquad \text{(B.13)}
\end{aligned}$$

where $J = \kappa_2 = \max(3, 2l+1, 4l-5)$.

Observing that $\Phi_{k-1} - \left(2\tilde{\Phi}_k - \varphi\right) = -2\varphi > 0$, it follows that

$$2\tilde{\Phi}_k - \varphi \le \Phi_{k-1} \quad \text{and} \quad \tilde{\varphi} \le \Phi_{k-1}.$$

Finally, from (B.8) and (B.13), we deduce that

$$\begin{aligned}
K_1 + K_2 + K_3 \leqslant &\varepsilon\Big(\mathcal{J}(y_k) + \mathcal{J}(y_{k+1})\Big) + C\sum_{j=1}^{k-2} s^{\hat{l}}\iint_{\mathcal{O}_0\times(0,T)} e^{2s\Phi_{k-1}}\theta^{\hat{l}} y_j^2\, dx\, dt \\
&+ C\left(1+\frac{1}{\varepsilon}\right) s^J\iint_{\mathcal{O}_0\times(0,T)} e^{2s\Phi_{k-1}}\theta^J y_{k-1}^2\, dx\, dt \\
&+\iint_{Q_{t_0}}\left|y_{k-1}s^l\theta^l e^{2s\Phi_k}\chi f_k\right| dx\, dt \\
&+ C\iint_{Q_{t_0}}\left|f_{k-1}s^{2l-1}\theta^{2l-1}\chi e^{2s\Phi_{k-1}} y_{k-1}\right| dx\, dt, \qquad \text{(B.14)}
\end{aligned}$$

where we have used the fact that $\tilde{\Phi}_k \le \Phi_{k-1}$. This completes the proof of (2.3.40).

C Caccioppoli's inequality

In this section, we will prove the Caccioppoli's inequality (2.3.15) for an inhomogeneous degenerate parabolic equation. Note that, this inequality is different from the one shown in [5] for the homogeneous case.

Proof of Lemma 2.3.1. Define a smooth cut-off function $\tau \in C^\infty(0,1)$ such that $0 \le \tau \le 1$ in $(0,1)$, $\mathrm{supp}(\tau) \subset \omega'$ and $\tau \equiv 1$ on ω''. Since y solves (2.3.2), we have

$$\begin{aligned}
0 &= \int_{t_0}^T \frac{d}{dt}\left(\int_0^1 \tau^2 y^2 e^{2s\phi}\, dx\right) dt \\
&= 2s\iint_{Q_{t_0}} \tau^2\phi_t y^2 e^{2s\phi}\, dx\, dt + 2\iint_{Q_{t_0}}\Big(d\big(a(x)y_x\big)_x + f\Big)\tau^2 y e^{2s\phi}\, dx\, dt.
\end{aligned}$$

Then, integrating by parts and using the fact that $d > 0$, we find

$$\begin{aligned}
\iint_{Q_{t_0}} a\tau^2 y_x^2 e^{2s\phi}\, dx\, dt = s&\iint_{Q_{t_0}}\frac{\tau^2}{d}\phi_t y^2 e^{2s\phi}\, dx\, dt + \frac{1}{2}\iint_{Q_{t_0}}\Big((e^{2s\phi}\tau^2)_x a\Big)_x y^2\, dx\, dt \\
&+\iint_{Q_{t_0}}\frac{\tau^2}{d} f y e^{2s\phi}\, dx\, dt.
\end{aligned}$$

Since τ is supported in ω', $\tau \equiv 1$ in ω'', $|\dot{\theta}| \leq C\theta^2$, $a \in C^1(\overline{\omega'})$, $\min\limits_{x\in\omega''} a(x) > 0$ and $\varrho \in C^2(\overline{\omega'})$ then, using Young's inequality one obtains

$$\begin{aligned}
\min_{x\in\omega''}\{a(x)\} \iint_{\omega''_{t_0}} y_x^2 e^{2s\phi}\,dx\,dt &\leq \iint_{Q_{t_0}} a\tau^2 y_x^2 e^{2s\phi}\,dx\,dt \\
&\leq C\Bigg(\iint_{\omega'_{t_0}} \Big(s|\dot{\theta}| + s^2\theta^2\Big) y^2 e^{2s\phi}\,dx\,dt \\
&\quad + \iint_{\omega'_{t_0}} f^2 e^{2s\phi}\,dx\,dt + \iint_{\omega'_{t_0}} y^2 e^{2s\phi}\,dx\,dt\Bigg) \\
&\leq C \iint_{\omega'_{t_0}} \Big(f^2 + s^2\theta^2 y^2\Big) e^{2s\phi}\,dx\,dt
\end{aligned}$$

and the proof is complete. $\square$

Conclusion and perspectives

In this thesis, we have studied the following problems:

In Chapter 2, we addressed the question of Lipschitz stability regarding the inverse problem of retrieving, simultaneously, n source terms in a coupled system of $n \geq 2$ degenerate parabolic equations by means of measurements of one component of the solution. Such a result is derived employing an appropriate Carleman estimate with one locally distributed observation.

In Chapter 3, we treated the distributed controllability for a coupled system of degenerate parabolic equations with singular potentials. The main particularity is the fact that the coupling is also done in the singular terms. By means of a Carleman inequality with only one observation for the problem under analysis, we obtained the null controllability employing one single distributed control supported in a suitable open subset of the domain.

In Chapter 4, we provided a sufficient condition on the null controllability of an integro-differential degenerate parabolic equation. In particular, we have shown that, by assuming an exponential decay in time on the kernel memory at the end of the time horizon $[0, T]$, the null controllability holds. Besides this, from the result in [110], it is clear that nontrivial constant kernels cannot be handled when dealing with controllability problems for memory systems of type (4.1.1). Instead, some additional assumption has to be imposed. In this work, we considered kernels depending on space and time variables. Nevertheless, we do not know whether the proposed decaying condition is the best possible or if, instead, sharper results can be proved.

In Chapter 5, we provided necessary and sufficient conditions for the approximate and null controllability properties of a linear coupled system of two degenerate parabolic equations when a control force acts on a part of the boundary. As a consequence and unlike the scalar case, we infer that the distributed and boundary null controllability properties of coupled degenerate parabolic systems are in general not equivalent. Indeed, *Kalman rank condition* is a necessary condition for the controllability of both systems but is not a sufficient condition for the boundary controllability problem.

In Chapter 6, we have established the approximate and null controllability properties of the one-dimensional degenerate heat equation with a pointwise control. In this setting, the position of the control force and the rate of degeneracy α can have an important influence on the controllability properties of the control system. Notably, we showed that a minimal time of pointwise null controllability, $T_0 \in [0, +\infty]$, arises in such a way that the underlying equation is null controllable at time T if $T > T_0$ and is not when $T < T_0$.

In the following, we present some perspectives that are somehow linked with the topics we have addressed in this thesis.

1) Controllability of a coupled system with different diffusion coefficients

In this work, we have addressed the controllability issue of coupled parabolic systems involving the same diffusion coefficients. The same problem would be of interest in the case of different diffusion coefficients.

- **Distributed control**

Let us consider the following distributed controlled system:

$$\begin{cases} y_{1t} - (k_1(x) y_{1x})_x + a_{11} y_1 + a_{12} y_2 = 1_\omega u, & \text{in } Q, \\ y_{2t} - (k_2(x) y_{2x})_x + a_{22} y_2 + a_{21} y_1 = 0, & \text{in } Q, \\ y_i(t,1) = 0, & i = 1,2, \quad t \in (0,T), \\ \begin{cases} y_i(t,0) = 0, & (WD) \\ (k_i y_{ix})(t,0) = 0, & (SD) \end{cases} & i = 1,2, \quad t \in (0,T), \\ y_i(0,x) = y_{0i}(x), & i = 1,2, \quad \text{in } (0,1), \end{cases} \tag{.1}$$

where ω is an open subset of $(0,1)$, the coefficients $a_{ij} \in L^\infty(Q)$, $i,j = 1,2$, $u \in L^2(Q)$ is the control force, k_1, k_2 are two diffusion coefficients vanishing at the extremity $x = 0$ and $(y_{01}, y_{02}) \in L^2(0,1)^2$.

The first results on null controllability of the coupled parabolic system (.1) have been established in [61]. The authors concern mainly the case where $k_1 = k_2$ and with particular coupling terms:

$$a_{12} = 0 \quad \text{in } Q \quad \text{and} \quad a_{21} = 1_{\mathcal{O}} \tag{.2}$$

for some non empty open set $\mathcal{O} \Subset (0,1)$ satisfying $\mathcal{O} \cap \omega \neq \emptyset$. Then, the previous results have been extended by Hajjaj et al. in [3] to the case where the system (.1) has a *cascade structure,* i.e., $a_{12} = 0$, with different diffusion coefficients. To this aim, the authors apply the Carleman estimates developed for a single equation in [5], with suitable weight functions, for the two degenerate equations of the associated adjoint system to (.1) for proving the null controllability result.

Later on, the non-cascade version of the system (.1) has been considered in [2]. In particular, the authors consider the special diffusion coefficients $k_i(x) := x^{\alpha_i}$, being $\alpha_i \in [0,1)$, or the so-called *weakly-weakly degenerate systems*, under Dirichlet boundary conditions and prove that the null controllability holds. Their approach is based on the use of new global Carleman estimates with an appropriate weight function. Nevertheless, the proposed weight function is no longer suitable for the strongly degenerate setting and, consequently, the extension of such a controllability result to the *weakly-strongly* or *strongly-strongly* degenerate parabolic systems are completely open. We refer to [2, Section 7] for further discussions on this issue.

- **Boundary control**

Now, let us consider the following boundary controlled system:

$$\begin{cases} y_{1t} - d_1 (x^{\alpha_1} y_{1x})_x + b_{11} y_1 + b_{12} y_2 = 0, & \text{in } Q, \\ y_{2t} - d_2 (x^{\alpha_2} y_{2x})_x + b_{22} y_2 + b_{21} y_1 = 0, & \text{in } Q, \\ y_1(t,1) = v(t), \quad y_2(t,1) = 0, & t \in (0,\,T), \\ \begin{cases} y_i(t,0) = 0, & 0 \le \alpha < 1 \\ x^{\alpha_i} y_{ix}(t,0) = 0, & 1 \le \alpha < 2 \end{cases} & i = 1,2, \quad t \in (0,T), \\ y_i(0,x) = y_{0i}(x), & i = 1,2, \quad \text{in } (0,1), \end{cases} \tag{.3}$$

where $d_1, d_2 > 0$, $0 \le \alpha_1, \alpha_2 < 2$, the coefficients $b_{ij} \in \mathbb{R}$, $i,j = 1,2$, $(y_{01}, y_{02}) \in L^2(0,1)^2$ is the initial condition and $v \in L^2(0,T)$ is a scalar control force.

The boundary controllability of coupled degenerate parabolic systems like (.3) is completely open when $\alpha_1 \neq \alpha_2$.

We will present here a simple example which shows that when the diffusion coefficients are not similar, the situation can be much more complex and unnatural difficulties arise when we try to control a coupled system of two degenerate parabolic equations form the boundary.

We will be concerned with the non-degenerate cascade version of (.3). More precisely, we consider the situation where $\alpha_1 = \alpha_2 = 0$, $b_{21} = 1$ and $b_{11} = b_{12} = b_{22} = 0$, namely

$$\begin{cases} y_{1t} - d_1 y_{1xx} = 0, & \text{in } Q, \\ y_{2t} - d_2 y_{2xx} + y_1 = 0, & \text{in } Q, \\ y_1(t,1) = v(t), \quad y_2(t,1) = 0 & t \in (0,T), \\ y_1(t,0) = y_1(t,0) = 0, & t \in (0,T), \\ y_1(0,x) = y_{01}(x), \; y_2(0,x) = y_{02}(x) & \text{in } (0,1). \end{cases} \tag{.4}$$

In [88] it is shown that the system (.4) is approximately controllable at any positive time $T > 0$ if and only if the square root of d_1/d_2 is an irrational number, i.e.,

$$\sqrt{d_1/d_2} \neq \mathbb{Q}.$$

On the other hand, the boundary null controllability property holds if the control time T is greater than a minimal time $T_0 \in [0, +\infty]$ which depends on the diffusion constants d_1 and d_2. Otherwise, the null controllability fails.

Moreover, using the Diophantine approximation theory, in [16], the authors proved that, it is possible to select two positive numbers d_1 and d_2 for which the system under consideration is approximately controllable at any time $T > 0$ and never null controllable (i.e., $T_0 = +\infty$).

To the best of our knowledge, the previous result has never been extended to the context of a coupled degenerate system of the form (.3).

Hence, as the first step in this direction, we suggest a coupled system of degenerate-nondegenerate parabolic equations. This and other related questions are being considered and will be addressed elsewhere.

2) Boundary control of a degenerate/singular parabolic system

Let us consider the following degenerate/singular parabolic equation:

$$\begin{cases} y_t - (x^\alpha y_x)_x - \dfrac{\lambda}{x^{2-\alpha}} y = 0, & \text{in } Q, \\ y(t,0) = v(t), \quad y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & \text{in } (0,1), \end{cases} \tag{.5}$$

where $y_0 \in L^2(0,1)$ is the initial data, $0 \leq \alpha < 1$ and $\lambda \leq (1-\alpha)^2/4$ are two real parameters.

In [36], Biccari et al. analyzed the null controllability of the degenerate/singular parabolic equation (.5) when a scalar control force acts at the degenerate point $x = 0$. Through the classical moment method, the authors show that this equation is null-controllable. They also provide suitable estimates for the control cost.

Following [88], since we have explicit knowledge of the spectrum of the operator $-(x^\alpha y_x)_x - \dfrac{\lambda}{x^{2-\alpha}} y$, we believe that the moment method could also be used for analyzing boundary controllability for a coupled system of two degenerate/singular parabolic equations.

3) Control of a degenerate/singular system involving first-order terms

As in [108], we believe that the results found in Chapter 3 could be extended to more general cascade systems by introducing first-order coupling terms. However, the employment of a weight function for the Carleman inequality for the degenerate/singular part and a classical weight for the classical parabolic equation is not of use in this situation. We think that the weight function introduced in [60] (see also [92]) could be used for analyzing controllability properties of a degenerate/singular system involving coupling terms of first-order.

4) Bilinear control

The focus of this thesis was the controllability of degenerate systems via boundary, pointwise and interior locally distributed controls that enter the model as an additive term describing the effect of some external forces on the process at hand. However this is not always realistic to act on the system in such a way. In the spirit of the works [94, 122, 157], it would be interesting to study the problem of bilinear or multiplicative controllability for this class of systems.

5) On the minimal time of pointwise null controllability

In Chapter 6, we have shown that system (6.1.1) is null controllable if and only if the control time T is greater than a minimal time $T(b,\alpha) \in [0,+\infty]$. However, we do not know, if for a given $T^* \in [0,+\infty]$, one can find some $\alpha \in (0,2)$ and $b \in (0,1)$ satisfying (6.3.5) and so that $T(b,\alpha) = T^*$. We point out that, this problem was completely solved for the peculiar case $\alpha = 0$ (i.e. the nondegenerate heat equation) (see [16, 77]). In particular, it has been proved that:
for any $T^* \in [0,+\infty]$*, there exists* $b \in (0,1)$ *satisfying* $b \notin \mathcal{S}_{\nu_0} := \mathbb{Q} \cap [0,1]$ *such that*

$$T(b,0) = T^*.$$

In fact, the authors proved that the minimal time $T(b,0)$ strongly depends on the Diophantine approximation properties of the irrational number b. The situation is completely different in the case where $0 < \alpha < 2$, since in this framework the set

$$\mathcal{S}_{\nu_\alpha} = \left\{ \left(\frac{j_{\nu_\alpha,k}}{j_{\nu_\alpha,n}}\right)^{\frac{1}{\kappa_\alpha}}, \quad n > k \geq 1 \right\}$$

may contain booth rational and irrational numbers and then the approach used in the previous papers is not suitable anymore. Therefore, the extension of such a result to the degenerate setting requires new ideas and more investigation on the theory of zeros of Bessel functions.

6) Memory-type null controllability

We recall that, in the context of the parabolic equation without memory, once there exists a control function acting on a control region $\omega \subset (0,1)$ that drives the system from an initial state y_0 to the equilibrium at time $t = T$, i.e., $y(T,\cdot) = 0$, we can stop controlling, by setting $u \equiv 0$ for $t \geq T$, and the underlying system naturally stays at rest for all $t \geq T$, i.e.,

$$y(t,\cdot) = 0, \qquad \forall\, t \geq T.$$

Unfortunately, this is not the case for the parabolic equation with memory like (4.1.1). Indeed, due to the effect of the accumulated memory at time $t = T$, i.e., $\int_0^T b(T,s,\cdot)y(s,\cdot)\,ds$, the null state of this system at T cannot be kept for $t \geq T$ in the absence of control function.

Hence, it could be of interest to consider a more general concept of null controllability for a system of type (4.1.1). In particular, we look for a control function that drives both the state and the memory term to 0 at time $t = T$.

This problem has been addressed by S. Ivanov and L. Pandolfi in [118] for the parabolic equation with memory and through a distributed control:

$$y_t - y_{xx} = \int_0^t b(t-s)y_{xx}(s)ds + 1_\omega u, \qquad (t,x) \in Q.$$

In [118], it is proved that this system cannot be controlled to rest for large classes of memory kernels and controls. In fact, the presence of the memory terms makes the controllability of this system to be impossible if the control is located in a fixed subset ω.

On the other hand, to obtain controllability result as explained in [66] and [67], the support of the control function needs to move to cover the domain where the equation evolves in the control time horizon. We refer to [66] where this problem is discussed in the context of the heat equation. The extension to the degenerate problem is the subject of future work.

7) Other inverse problems

In this thesis, we have addressed an inverse problem that consists of the identification of source terms. However, other types of inverse problems could be considered. For instance, in the context of the uniformly parabolic equation, the inverse problem concerning the identification of the initial conditions has been established in [148], whereas [74] provide a simultaneous reconstruction of one coupling terms and initial conditions from a single local observation of the solution of a coupled system of two parabolic equations. In the context of degenerate systems, only a few results are known. For instance, J. Tort [154] established the inverse problem of retrieving the diffusive constant in a degenerate parabolic equation.

Therefore, the extension of all the previous results to the context of degenerate or degenerate/singular scalar and coupled parabolic systems become, in our opinion, is a very interesting issue.

Finally, it would certainly be interesting to establish numerical reconstruction problems like the one considered in [131] in the context of the uniformly parabolic equation. See also [68], where the authors introduce a non-iterative method for recovering the space-dependent source and the initial data simultaneously in a parabolic equation from two over-specified measurements. The first attempts in the degenerate setting can be found in [8, 20, 21].

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