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Contributions to the study of some nonlinear elliptic and parabolic problems

JURY

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In loving memory of my dear Mother

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ABSTRACT

The aim of this work is specifically the study of some kinds of nonlinear elliptic and parabolic partial differential equations. More precisely, this work is organized in two parts. In the first part we investigate the existence and multiplicity of solutions for some class of elliptic equations. Firstly, we deal with a problem related to the p**-Laplacian** operator with a p-gradient term and a Dirichlet boundary condition type. Secondly, we deal with a problem involving a more general operator with a potential, and a source term that does not satisfy the well known **Ambrosetti-Rabinowitz** condition. In the second part, we study the asymptotic behavior of some parabolic equations. The first subject, concerns mainly the study of some **doubly nonlinear** parabolic problems associated with a nonlinear boundary condition. In the second subject, we deal also with parabolic equations, we show the existence of **periodic solutions** for a fairly general problem associated with an operator in **divergence** form of Leary-Lions type with **variable exponent**.

Keywords: p-Laplacian; Ambrosetti-Rabinowitz condition; variable exponent; doubly nonlinear equation; periodic solutions.

RÉSUMÉ

L'objectif de ce travail est d'apporter une certaine contribution à l'étude de quelques problèmes non linéaires de type elliptique ou parabolique. Plus précisément, ce travail est organisé en deux parties. La première partie est consacrée à l'étude de l'existence et de la multiplicité des solutions pour certaines classes d'équations elliptiques. Dans un premier temps, nous étudions un problème lié à l'opérateur **p-Laplacien** avec croissance d'ordre p en le gradient et une condition aux limites de type Dirichlet. Nous étudions ensuite un problème faisant intervenir un opérateur assez général avec un potentiel et un terme source qui ne vérifie pas la condition d'**Ambrosetti-Rabinowitz**. Dans la seconde partie, nous étudions le comportement asymptotique de quelques équations de type parabolique. Le premier sujet, concerne principalement l'étude de problèmes paraboliques **doublement non linéaires** avec une condition aux limites de type non linéaire. Restant dans le cadre des équations paraboliques, nous montrons dans le deuxième sujet, l'existence de **solutions périodiques** pour un problème assez général associé à un opérateur sous forme divergentielle de type Leary-Lions à **exposant variable**.

Mots Clés : p-Laplacien ; condition d'Ambrosetti-Rabinowitz ; exposant variable ; équation doublement non linéaire ; solutions périodiques ;

CONTENTS

RÉSUMÉ DE LA THÈSE

Cette thèse concerne l'étude de certains équations aux dérivées partielles non linéaires de type elliptique ou parabolique. Les problèmes que nous avons étudiés dans ce travail comportent des opérateurs de type divergentiel. De tels opérateurs apparaissent dans de nombreux modèles cinétiques de réactions chimiques, de dynamique des populations, de physique des plasmas ainsi que dans certains modèles d'écoulement de fluides non newtoniens.

Avant de faire une présentation des résultats que nous avons obtenus, nous signalons que le Chapitre 1 (les espaces de Lebesgue-Sobolev à exposant variables) et le Chapitre 2 (les espaces de Musielak-Sobolev) ont été consacrés aux différents rappels nécessaires à une bonne lecture du reste de la thèse.

Résumé du chapitre 3 :

Soit Ω un domaine ouvert borné dans $\mathbb{R}^N$ ($N \geq 3$) avec une frontière assez régulière. Dans ce chapitre nous nous sommes intéressés au problème elliptique suivant

$$(P) \begin{cases} -\Delta_p u = c(x)|u|^{q-1}u + \mu|\nabla u|^p + h(x) & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases}$$

où $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ est l'opérateur p-Laplacien avec $1 < p < N$. Nous étudions ce problème sous les hypothèses suivantes

$$(H) \begin{cases} c, h \text{ appartiennent à } L^k(\Omega) \text{ pour un certains } k > \frac{N}{p},\ h \text{ change de signe,} \\ c \gneqq 0 \ \text{ p.p dans } \ \Omega, q > 0, \text{ et } \mu \in \mathbb{R}^*. \end{cases}$$

Dans la littérature, il existe de nombreux résultats concernant l'existence, l'unicité et la multiplicité des solutions pour des modèles comme (P) sous diverses hypothèses sur c, h, q et μ. Nous renvoyons le lecteur à [29, 62, 61] pour quelques résultats importants liés au résultat principal de la présente étude (voir aussi l'introduction du Chapitre 3). Nos résultats complètent et étendent certains des résultats des études citées ci-dessus.

Le but principal de ce chapitre est de montrer l'existence au moins deux solutions faibles bornées pour le problème (P) sous l'hypothèse (H). Pour atteindre cet objectif, nous utiliserons une approche variationnelle qui est l'un des moyens les plus utilisés pour traiter les problèmes elliptiques. Précisément, nous appliquerons le théorème du Col (pour la première solution) et l'argument standard de semi-continuité inférieure (pour la deuxième solution). Afin de pouvoir appliquer ces derniers théorèmes, nous aurons besoin, en principe, d'une formulation variationnelle (à savoir un problème dont les solutions peuvent être obtenues en tant que points critiques d'une fonctionnelle associée). Cependant, en raison de la présence du terme p-gradient, c'est-à-dire $|\nabla u|^p$, notre problème (P) n'a pas de formulation variationnelle. Donc, pour surmonter cette difficulté, nous effectuons le changement de variable bien connu de **Kazdan-Kramer**, à savoir, $v = (e^{\frac{\mu u}{p-1}} - 1)/\mu$ où $\mu > 0$ (pour $\mu < 0$, nous changeons u par $(-u)$ dans (P), ce qui nous ramène comme au premier cas). Après un calcul simple, nous obtenons un nouveau problème (P')

équivalent à (P), donné comme suivant

$$(P')\begin{cases} -\Delta_p v = c(x)g(v) + h(x)f(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

où

$$g(s) = \frac{(p-1)^{q-p+1}}{\mu^q}(1+\mu s)^{p-1}|\ln(1+\mu s)|^{q-1}\ln(1+\mu s), \;\; \text{avec } s > \frac{-1}{\mu}, \tag{0.1}$$

et

$$f(s) = \frac{(1+\mu s)^{p-1}}{(p-1)^{p-1}}. \tag{0.2}$$

Par conséquent, nous pouvons associer à (P) une fonctionnelle d'énergie $\mathcal{I}$ définie par :

$$\mathcal{I}(v) = \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega c(x)G(v) - \int_\Omega h(x)F(v), \tag{0.3}$$

avec $G(s) = \int_0^s g(t)dt$ et $F(s) = \int_0^s f(t)dt$. Désormais, chercher les solutions faibles bornées du problème (P') revient exactement à chercher les points critiques de sa fonctionnelle d'énergie $\mathcal{I}$. Il est à noter que, selon le fameux papier d'Ambrosetti et Rabinowitz [8], l'étape majeure pour appliquer le théorème du Col est de montrer que la fonctionnelle $\mathcal{I}$ satisfait la condition de Palais-Smale au niveau $\tilde{c}$ (voir Définition 3.3) ; et ceci est lié directement à la condition d'Ambrosetti-Rabinowitz ((A-R)), à savoir

$$\text{il existe } \theta > p \text{ et } s_0 > 0 \text{ tel que } 0 < \theta G(s) \leq sg(s), \;\; \text{tandis que } |s| > s_0.$$

Malheureusement, cette condition (A-R) est quelque peu restrictive, car elle exclut de nombreuses non-linéarités g comme dans notre cas ici (voir (0.1)). Pour cette raison, plusieurs recherches ont été réalisées afin de surmonter la condition (A-R) (voir [28, 101, 78, 50, 60]) ; et c'est dans ce cadre que s'inscrit l'un des objectifs de la présente étude. Par conséquent, dans ce travail, nous utiliserons une condition assez faible que la condition (A-R). Autrement dit, le point clé pour établir la condition de Palais-Smale au niveau $\tilde{c}$ est de montrer que la non-linéarité g satisfait une condition de type **non quadraticité à l'infini** (en anglais **nonquadraticity condition at infinity**) à savoir

$$(NQ) \quad H(s) = g(s)s - pG(s) \to +\infty, \;\; \text{quant } s \to +\infty.$$

Nous signalons aussi que puisque la fonction h change de signe, alors la réalisation de la condition de Palais-Smale au niveau $\tilde{c}$ est plus délicate (voir [51, 62]).

Résumé du chapitre 4 :

Soit $\Omega \subset \mathbb{R}^N (N \geq 2)$ un domaine ouvert borné de frontière assez régulière. Soit $\phi : \Omega \times [0, +\infty) \to [0, +\infty)$ une fonction de Carathéodory telle que, pour tout $x \in \Omega$, nous avons :

$$(\phi)\begin{cases} \phi(x,0) = 0, \quad \phi(x,t).t \text{ est strictement croissante}, \\ \phi(x,t).t > 0, \; \forall t > 0 \text{ et } \phi(x,t).t \to +\infty \text{ quand } t \to +\infty. \end{cases}$$

Dans ce chapitre, nous étudions le problème elliptique quasi-linéaire suivant

$$(P)\begin{cases} -\mathrm{div}(\phi(x,|\nabla u|)\nabla u) + V(x)|u|^{q(x)-2}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

où V est un potentiel appartenant à $L^{s(x)}(\Omega)$, q et $s : \overline{\Omega} \to (1, \infty)$ sont des fonctions continues et $f : \Omega \times \mathbb{R} \to \mathbb{R}$ est une fonction de Carathéodory qui satisfait certaines conditions de croissance bien appropriées.

Concernant les hypothèses précises sur les fonctions q, s, f et V nous renvoyons le lecteur au Chapitre 4, Section 4.2.

Le problème (P) apparaît dans plusieurs branches de la physique mathématique et a été étudié de manière approfondie ces dernières années. Du point de vue de la motivation, ce problème a ses origines dans des sujets d'actualité brûlants comme le traitement d'image et les fluides électrorhéologiques. Pour plus de détails sur ces deux applications, nous renvoyons le lecteur au Chapitre 1, Section 1.2.

L'opérateur divergentiel "$\operatorname{div}(\phi(x,|\nabla u|)\nabla u)$" intervenant dans (P) généralise plusieurs opérateurs bien connus dans la littérature. On citera, à titre d'exemple les cas suivants : $\phi(x,t) = t^{p(x)-2}$, où p est une fonction continue sur $\overline{\Omega}$ avec la condition $\min_{x\in\overline{\Omega}} p(x) > 1$, alors l'opérateur "$\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$" est le $p(x)$-Laplacien. Ce dernier opérateur est une généralisation naturelle de l'opérateur p-Laplacien, à savoir, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ où $p > 1$ est un réel. On peut également citer deux opérateurs très intéressants : l'opérateur à double phase lorsque $\phi(x,t) = t^{p-2} + a(x)t^{q-2}$ (où p et q sont des réels) et l'opérateur à double phase à exposant variable dans le cas où $\phi(x,t) = t^{p(x)-2} + a(x)t^{q(x)-2}$ (où p et q sont des fonctions).

Récemment, il y a eu un grand nombre des travaux dédiés à l'étude de l'existence et la multiplicité des solutions faibles pour des problèmes de type (P). Pour quelques résultats importants liés à cette présente étude, nous renvoyons le lecteur à [2, 24, 26, 43, 53, 52, 70, 73] (voir aussi l'introduction du Chapitre 4). Après une étude approfondie de la littérature liée au sujet, les résultats qui ont été obtenus peuvent être résumés en deux axes : i) $V \equiv 0$ et f avec ou sans la condition (A-R), ii) $V \not\equiv 0$ et f avec ou sans la condition (A-R). Pour le premier axe, les problèmes étudiés ont été considérés dans un cadre fonctionnel plus général (les espaces de Musielak-Sobolev), tandis que, pour le deuxième axe ont été considérés dans un cadre fonctionnel assez particulier (les espaces de Lebesgue-Sobolev à exposant variable). Par conséquent, parmi les principales motivations de notre étude actuelle est de considérer à la fois le potentiel $V \not\equiv 0$ et la non-linéarité f sans la condition (A-R) pour le problème quasi-linéaire (P) dans le cadre fonctionnel d'espace de Musielak-Sobolev. Il est à noter que l'intérêt d'abandonner la condition (A-R) à été expliqué dans le résumé précédent.

L'objectif de ce chapitre est de montrer l'existence de solutions faibles pour le problème (P). Premièrement, nous établissons quelques résultats techniques nécessaires pour la démonstration des théorèmes principaux de ce travail. Ainsi, en se basant sur le Théorème 2.3.12, nous démontrons un résultat d'injection compacte dans le cadre des espaces de Musielak-Sobolev. Deuxièmement, par utilisation de l'argument standard de semi-continuité inférieure, nous démontrons l'existence d'une solution faible dans le cas où le potentiel V change de signe et la non-linéarité f satisfait une certaine hypothèse (notée (f_0) dans le Chapitre 4). Troisièmement, en se basant sur un théorème classique des opérateurs monotones, nous démontrons l'existence d'une solution faible unique dans le cas où le potentiel V est positif et la non-linéarité f est indépendante de la seconde variable, à savoir, $f(x,t) \equiv f(x)$. Quatrièmement, en utilisant le théorème du Col, nous démontrons l'existence d'une solution faible non triviale dans le cas où le potentiel V a un signe constant et la non-linéarité f ne vérifie pas la condition (A-R). Dans ce dernier cas, f vérifie certaines conditions au voisinage de zéro et à l'infini, et la principale condition (A-R) a été remplacée par l'hypothèse notée (f_1) dans ce chapitre (voir aussi (f_0')-(f_3)). Finalement, en utilisant le théorème de Fountain, nous démontrons l'existence d'une infinité de solutions dans le cas où le potentiel V a un signe constant et la non-linéarité f est impaire par rapport à la seconde variable t et vérifie aussi les hypothèses (f_0')-(f_3) signalées ci-dessus.

Résumé du chapitre 5 :

Soit $\Omega \subset \mathbb{R}^N (N \geq 1)$ un domaine ouvert borné de frontière assez régulière. Dans ce chapitre, nous étudions le problème parabolique doublement non linéaire suivant

$$(P) \begin{cases} \partial_t(\beta(u)) - \Delta_p u + h(x,t,u) = 0, & \text{dans } \Omega \times (0,\infty), \\ -|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(u), & \text{sur } \partial\Omega \times (0,\infty), \\ \beta(u(0)) = \beta(u_0), & \text{dans } \Omega, \end{cases}$$

où $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ $(1 < p < \infty)$, est l'opérateur p-Laplacien, $\frac{\partial}{\partial \nu}$ désigne la normale extérieure à $\partial\Omega$ en un point $x \in \partial\Omega$, β est une fonction continue croissante localement lipschitzienne sur $\mathbb{R}$ avec $\beta(0) = 0$ et $u_0 \in L^\infty(\Omega)$ est la condition initiale. Concernant les hypothèses précises sur les fonctions h et g nous renvoyons le lecteur au Chapitre 5, Section 5.2.

Les équations paraboliques non linéaires définies par (P), ou certains cas particuliers de celle-ci, sont étudiées par plusieurs mathématiciens en raison de leur intérêt mathématique et parce qu'elles décrivent de nombreux phénomènes en mécanique, physique et biologie. Pour être plus précis, nous donnons quelques exemples des modèles importants. Pour $\beta(u) = u$, $g = 0$ et $p = 2$, le problème (P) s'inscrit dans le type des équations réaction-diffusion ; tandis que pour $p \neq 2$ le problème (P) représente quelques équations de la filtration élastique non newtonienne et les phénomènes de glaciologie (voir [63, 74, 85]). Pour $\beta(u) = |u|^{\frac{1}{m}}\operatorname{sign}(u)$, avec $m > 1$ et $p = 2$, le problème (P) décrit alors le flux non stationnaire à travers un milieu poreux ; alors que pour $p \neq 2$, ce problème modélise l'écoulement polytropique non stationnaire d'un fluide dans un milieu poreux dont la contrainte tangentielle dépend de la puissance de la vitesse (voir [63]). De plus, les problèmes de type (P) représentent également quelques modèles d'évolution d'une population biologique (voir [56, 55]). La condition de non linéarité sur le bord, à savoir, $-|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(u)$, peut être physiquement interprétée comme une loi de rayonnement non linéaire prescrit à la limite du corps matériel (voir [9, 69] et les références y contenues).

Dans ce travail, nous nous préoccupons de l'existence et de l'unicité des solutions bornées et de l'existence d'un attracteur global pour le problème (P). Ici, nous nous concentrons sur une condition aux limites non linéaire de type Neumann, car la condition aux limites de type Dirichlet a été largement traitée dans la littérature (voir [19, 32, 36, 39]). Le but de cette présente étude est double : d'une part donner des conditions suffisantes pour lesquelles notre problème (P) est globalement bien posé dans un espace fonctionnel approprié ; et d'autre part déterminer des conditions suffisantes sous lesquelles le système dynamique associé à (P) admet un attracteur global compact dans $L^\infty(\Omega)$. Pour atteindre le premier objectif, nous montrons tout d'abord l'existence de solutions classiques après avoir régularisé notre problème (P). Ensuite, afin d'étudier la convergence de ces solutions, nous montrons quelques estimations a priori dans des espaces fonctionnels appropriés. Finalement, inspirés par les travaux [7, 32], nous prouvons l'unicité des solutions. Pour atteindre le deuxième objectif, nous utilisons la théorie des systèmes dynamiques (voir [96]). Précisément, nous commençons à reformuler notre problème (P) en un système dynamique en lui associant une famille d'opérateurs non linéaires $(S(t))_{t\geq 0}$. Puis, sous quelques hypothèses supplémentaires sur β, g et h, nous montrons que les solutions obtenues sont höldériennes. Ce dernier résultat nous permet de montrer que les opérateurs $(S(t))_{t\geq 0}$ sont uniforméments compacts quand t est assez large, ce qui est à son tour une étape importante pour montrer l'existence d'un attracteur global dans $L^\infty(\Omega)$.

Résumé du chapitre 6 :

Soit $\Omega \subset \mathbb{R}^N (N \geq 1)$ un domaine ouvert borné de frontière assez régulière et $T > 0$ un réel fixe. Dans ce chapitre, nous étudions le problème périodique-parabolique non linéaire suivant

$$(P)\begin{cases} \partial_t u + \mathcal{A}u = f(x,t,u,\nabla u) & \text{dans } \Omega \times (0,T), \\ u = 0 & \text{sur } \partial\Omega \times (0,T), \\ u(0) = u(T) & \text{dans } \Omega, \end{cases}$$

où $\mathcal{A}u = -\operatorname{div}(\mathbf{A}(\cdot,\cdot,u,\nabla u))$ est un opérateur de type Leray-Lions à exposant variable qui agit d'un espace fonctionnel V_0 vers son dual topologique V_0' et f est une fonction de Carathéodory, dont la croissance par rapport à $|\nabla u|$ est au plus d'ordre $p(x)$ avec $p(\cdot)$ est fonction continue sur $\overline{\Omega}$ à valeurs dans $(1,\infty)$. Pour plus de détails sur les hypothèses de $\mathbf{A}$ et f, nous revoyons le lecteur au Chapitre 6, Section 6.2.

Les problèmes non linéaires définis par (P) apparaissent dans plusieurs applications. Par exemple dans les modèles de fluides électrorhéologiques, pour lequels il apparaît un terme donné par $\int_\Omega |Du(x)|^{p(x)}dx$ (voir [88] ou Chapitre 1, Section 1.2). Une autre application importante concerne le cas où f ne dépend que de (x,t) et en prenant $\mathbf{A}(x,t,s,\xi) = |\xi|^{p(x)-2}\xi$. Alors, le problème (P) peut être vu comme une sorte d'équation de diffusion non linéaire dont le coefficient de diffusion prend la forme $|\nabla u|^{p(x)-2}$ (voir [6]). Pour plus d'applications, nous renvoyons le lecteur à [104, 25].

Dans la littérature, il existe de nombreux résultats concernant l'existence et l'unicité de solutions pour des problèmes comme (P). Lorsque $p(x) := p$ est un exposant réel, nous renvoyons le lecteur à [31, 20, 40, 54] pour quelques résultats importants. Dans le cas où $p(\cdot)$ est un exposant variable, certains cas particuliers de problèmes ont été étudiés par de nombreux auteurs [6, 49, 17, 102], au moyen de différentes méthodes telles que : opérateurs sous-différentiels, méthodes de Galerkin, théorie des semi-groupes, etc. L'objectif principal de cet présente étude est d'étendre les résultats de [40] au cas des exposants variables en utilisant la méthode des sous et sur-solutions. Précisément, nous montrons l'existence au moins d'une solution périodique pour le problème (P) en supposons l'existence d'une sous et d'une sur-solution bien ordonnées. La méthode de sous et sur-solutions, lorsqu'elle est applicable, a plus d'avantages par rapport aux autres méthodes ; par exemple : elle nous donne quelques informations sur le comportement de la solution (explosion ou extinction) et elle détermine parfois le signe de la solution (positive ou négative). Néanmoins, cette méthode est assez compliquée car elle nécessite de montrer l'existence de sous et des sur-solutions bien ordonnées, ce qui n'est généralement pas facile à obtenir. En effet, dans de nombreux cas d'application, les sous et sur-solutions sont obtenues à partir de la fonction propre associée à la première valeur propre de certains opérateurs (par exemple le p-Laplacien). Mais, quand on traite des cas avec des exposants variables, il est bien connu par exemple que le $p(x)$-Laplacien n'a pas en général une première valeur propre (voir [44]) et donc, nous devons trouver une sous et une sur solution au moyen d'autres idées (voir notre exemple d'application dans la Section 6.4).

INTRODUCTION

The study of partial differential equations (PDEs) dates from the 18th century. There is much variety of physical models introduced in the work of Euler, Navier and Stokes (for the equations of fluid mechanics), Fourier (for the heat equation), Maxwell (Maxwell's equations for electromagnetism), Schrödinger and Heisenberg (for the equations of quantum mechanics). The middle of the 19th century is enriched by the work of Riemann, Poincaré and Hilbert. From there, the PDEs receive their title of nobility because they answer many questions that scientists ask themselves. The PDEs appear also in others branches of applications: in chemistry, biology, economics, image processing, etc... The situations depending on time are reflected more particularly by equations of evolution taking into account any interactions between objects and events. As far as we are concerned, we deal with some kinds of nonlinear elliptic and parabolic partial differential equations. The primary concern of the mathematician when faced with a partial differential equation is to give it meaning in a suitable functional spaces and proving the existence, multiplicity and uniqueness of solutions. When we study some nonlinear partial differential equations, it is well known that we search for an appropriate functional space on which we resolve them. For example, the p-Laplacian equations correspond to the classical Sobolev space setting; the $p(x)$-Laplacian equations correspond to the Sobolev space with variable exponent setting, etc... Hence, the first and second chapter of this thesis are devoted to the various recalls concerning some functional spaces which will be used in the forthcoming chapters. Indeed, in the first one, we introduce Lebesgue-Sobolev spaces with variable exponent and state some of their basic properties, such as reflexivity, separability, duality and results concerning embeddings and density of smooth functions. These spaces appeared in the literature in 1931 in the paper by Orlicz [84]. More precisely, Orlicz introduced the class of measurable functions u for which $\int_0^1 |u(x)|^{p(x)}dx < \infty$ and suggested that a variety of results about integrable functions with real constant p can be generalized to certain classes of functions that are integrable with a real function p. Then, in the 1950's, Nakano [83] developed the theory of modular function spaces, that is, the class of real valued functions u on a domain Ω for which $\int_\Omega \varphi(x, |u(x)|)dx < \infty$, where $\varphi : \Omega \times [0, +\infty) \to [0, +\infty)$ is a suitable function. Moreover, in the appendix [p. 284], Nakano explicitly mentioned Lebesgue spaces with variable exponents as an example of more general spaces he considered. The major step in the investigation of Lebesgue spaces with variable exponents appeared in the early 1990s by the work of Kovacik and Rakosuik [66]. It is worth mentioning that after ten years, Fan and Zhao [44] proved the same properties in [66] by different methods. These spaces are motivated by some interesting models such as: electrorheological fluids model developed by Rajagopal and Růžička [89] and image restoration model proposed by Chen, Levine and Rao [25].

In the second chapter of this thesis, we study the Musielak-Sobolev spaces which provide the framework for a variety of different function spaces, including classical (weighted) Lebesgue, Orlicz spaces and Lebesgue spaces with variable exponents. Particularly, this chapter will be used in the study of some kind of quasilinear elliptic problems which can be found in Chapter 4 .

The third chapter of this thesis is devoted to the study of some class of elliptic problem of p-Laplacian type with a p-Gradient term. As a model case, we consider

$$(P) \begin{cases} -\Delta_p u = c(x)|u|^{q-1}u + \mu|\nabla u|^p + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 3$) with a smooth boundary, $1 < p < N$, $q > 0$, $\mu \in \mathbb{R}^*$, and c and h belong to $L^k(\Omega)$ for some $k > \frac{N}{p}$ and satisfying some suitable conditions. Notice that, the differential operator p-Laplacian defined by $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ (observe that for $p = 2$ it is precisely the Laplacian) appears in pure mathematics such as problems of curves as well as in applied mathematics problems. Indeed, it appears in various fields in experimental sciences: nonlinear reaction-diffusion problems, dynamics of populations, non-Newtonian fluids flows, flows through porous medias and petrol extraction. As an example, in the 1970s, M.C. Pélissier [85] models the flow of mountain glaciers by partial differential equations involving the p-Laplacian. She explains its presence (p-Laplacian) by the fact that the ice can be considered as a pseudo-plastic fluid and satisfies a nonlinear strain law. In the literature, there are many results concerning the existence, the uniqueness, and the multiplicity of solutions for models like problems (P). Precisely, the investigation of nonlinear elliptic partial differential equations with a gradient dependence up to the critical growth was essentially initiated by Boccardo, Murat and Puel [21, 23, 22]. Firstly, we note that the change of sign of c plays a crucial role in the study of the problem (P) regarding the uniqueness and the existence of bounded solutions. In this setting, we refer to [61] for more details. In the case where $c(x) \leq -\alpha_0$ a.e. in Ω for some $\alpha_0 > 0$, now referred to as the coercive case, Boccardo, Murat and Puel [21, 23, 22], proved the existence of bounded solutions for more general divergential form problems with quadratic growth in the gradient. For the same problem as the previous one, Barles and Murat [15] and Barles et at. [14] dealt with the uniqueness question. For weakly coercive case $c \equiv 0$, there had been many contributions (see e.g. [1, 75, 86]). However, for $c \leq 0$ a.e in Ω (i.e that may vanish only on some parts of the domain), the uniqueness of solutions was left open until the recent paper authored by Arcoya et at. [12]. In that paper, the result was proved for the situation $p = 2$, $q = 1$, and under some sufficient conditions on c and h. We refer also the reader to [11] for more general uniqueness results. The non-coercive case, that is, when $c(x) \gneqq 0$ a.e. in Ω, the problem (P) behaves very differently and becomes rather complex than for $c \leq 0$ a.e in Ω. In that situation, the question of non-uniqueness has been being an open problem given by Sirakov [95] and it has received considerable attention by many authors. Moreover, it should be pointed out that the sign of h and whether μ is a function or a constant, generate additional difficulties for solving (P). In this setting, Jeanjean and Sirakov [61] showed the existence of two bounded solutions assuming that: $\mu \in \mathbb{R}^*$, h without sign condition, and $\|\mu h\|_{L^{N/2}(\Omega)}$ is small enough. This result was extended by Coster and Jeanjean [29] for μ a bounded function with $\mu(x) \geq \mu_1 > 0$ and by assuming some regularity on c and h. Finally, in the case where c is without sign condition with $c(x) \gneqq 0$ a.e. in Ω, Jenajean and Quoirin [62] showed the existence of two bounded positive solutions by assuming $h \gneqq 0$, $\mu > 0$ and c^+ and μh are suitably small. We note that all the above quoted multiplicity results were restricted to the Laplacian operator with quadratic growth in the gradient, i.e. in the case where $p = 2$ and $q = 1$. Moreover, it is interesting to mention that when c is without sign condition, the solutions obtained are positive. In this thesis, for any $1 < p < N$, we prove the multiplicity of bounded solutions for the problem (P) under the following assumption

$$\begin{cases} c, h \text{ belongs to } L^k(\Omega) \text{ for some } k > \frac{N}{p}, h \text{ without sign condition,} \\ c \gneqq 0 \text{ a.e. in } \Omega, q > 0, \text{ and } \mu \in \mathbb{R}^*. \end{cases}$$

The fourth chapter of this thesis is devoted to study the following quasilinear elliptic problem

$$(P) \begin{cases} -\operatorname{div}(\phi(x, |\nabla u|)\nabla u) + V(x)|u|^{q(x)-2}u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$), q and $s : \bar{\Omega} \to (1, \infty)$ are continuous functions, V is a given function in a generalized Lebesgue space $L^{s(x)}(\Omega)$, $f(x, u)$ is a Carathéodory function satisfying suitable growth conditions and $\phi : \Omega \times [0, +\infty) \to [0, +\infty)$ is a Carathéodory function such that for all $x \in \Omega$, we have

$$(\phi) \begin{cases} \phi(x, 0) = 0, \quad \phi(x, t).t \text{ is strictly increasing,} \\ \phi(x, t).t > 0, \ \forall t > 0 \text{ and } \phi(x, t).t \to +\infty \text{ as } t \to +\infty. \end{cases}$$

In particular, when $\phi(x,t) = t^{p(x)-2}$, where p is a continuous function on $\overline{\Omega}$ with the condition $\min_{x\in\bar{\Omega}} p(x) > 1$, the operator introduced in (P) is exactly the $p(x)$-Laplacian, i.e. $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. This differential operator is a natural generalization of the p-Laplacian operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ (with $1 < p < \infty$) as already mentioned above. Notice that the $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian (for example, it is nonhomogeneous), so more complicated analysis has to be carefully carried out. The studies for $p(x)$-Laplacian problems have been extensively considered by many researchers in various ways (see e.g. [2, 43, 64, 70]). It should be noted that our problem (P) enables the presence of many other operators such as double-phase and variable exponent double-phase operators. Recently, this kind of problem (P) is a subject of much research. Let us give a review of some interesting results related to our work. At first, we recall that when the nonlinearity f satisfies the well-known Ambrosetti-Rabinowitz condition ((A-R) condition for short); which, for the p-Laplacian operator, then, this asserts that there exist two constants $M > 0$ and $\theta > p$, such that

$$0 < \theta F(x,t) \leq f(x,t)t, \quad \forall |t| \geq M,$$

where $F(x,t) = \int_0^t f(x,s)ds$. This last condition implies the existence of two positive constants c_1, c_2 such that

$$F(x,t) \geq c_1|t|^\theta - c_2, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{0.4}$$

This means that f is p-superlinear at infinity in the sense that

$$\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^p} = +\infty. \tag{0.5}$$

On the one hand, there are several nonlinearities which are p-superlinear but do not satisfy the (A-R) condition, on the other hand, the (A-R) condition is one of the main tools for finding solutions to elliptic problems of variational type. However, many recent types of research have been made to drop the (A-R) condition (see e.g. [24, 26, 52, 70] and references therein). Indeed, in [24], Carvalho, Goncalves and Silva, studied a more general quasilinear equation in the framework of Orlicz-Sobolev spaces; precisely, when the function ϕ considered in (P) is independent of x, i.e. $\phi(x,t) = \phi(t)$. In that paper, the authors established the existence of at least a nontrivial solution where the nonlinearity f satisfies, among other conditions, the following assumptions: there exist an N-function Γ (cf. [90]) and positive constants C, R such that

$$\Gamma\left(\frac{F(x,t)}{|t|^{\phi_0}}\right) \leq C\bar{F}(x,t), \quad \forall (x,|t|) \in \Omega \times [R,+\infty), \tag{0.6}$$

and

$$\lim_{|t|\to+\infty} \frac{f(x,t)}{|t|^{\phi^0-1}} = +\infty, \quad \lim_{|t|\to 0} \frac{f(x,t)}{|t|\phi(t)} = \lambda, \tag{0.7}$$

where $\bar{F}(x,t) := f(x,t)t - \phi^0 F(x,t)$, $\lambda \geq 0$ and ϕ_0, ϕ^0 are defined in relation (2.2) below with some specific assumptions. In the few last years, studies on double phase problems have attracted more and more interest and many results have been obtained. Especially, in [52], Ge, Lv and Lua, proved the existence of a nontrivial solution and obtained infinitely many solutions for the problem (P) with $\phi(x,t) = t^{p-2} + a(x)t^{q-2}$, where $a : \overline{\Omega} \mapsto [0,+\infty)$ is Lipschitz continuous, $1 < p < q < N$, $\frac{q}{p} < 1 + \frac{1}{N}$ and the nonlinearity f satisfies:

$$\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^q} = +\infty, \quad \lim_{|t|\to 0} \frac{f(x,t)}{|t|^{p-1}} = 0, \tag{0.8}$$

$$\tilde{F}(x,t) \leq \tilde{F}(x,s) + \mu_1, \quad \forall (x,t) \in \Omega \times (0,s) \text{ or } \forall (x,t) \in \Omega \times (s,0), \tag{0.9}$$

and

$$\tilde{H}(ts) \leq \tilde{H}(t) + \mu_2, \quad \forall t \geq 0 \text{ and } s \in [0,1], \tag{0.10}$$

where $\tilde{H}(t) := q\Phi(t) - \phi(t)t^2$ and $\tilde{F}(x,t) := f(x,t)t - qF(x,t)$ with $\Phi(t) = \int_0^t \phi(s)sds$. In [53], Ge and Chen, however, considered the same previous problem and proved the existence of infinitely many solutions; but the nonlinearity f is supposed to satisfy the assumption (0.6) above, where $\Gamma(t) = |t|^\sigma$

with $\sigma > \max\{1, \frac{N}{p}\}$, and $F(x,t) \geq 0$ for any $(x,|t|) \in \Omega \times [R, +\infty)$ is such that $\lim\limits_{|t|\to+\infty} \frac{F(x,t)}{|t|^{\phi^0}} = +\infty$ with $\phi^0 = q$ and $\phi_0 = p$. We note that all the above quoted results were restricted to the case where the potential $V \equiv 0$. Concerning the case where the potential $V \not\equiv 0$ on Ω, recently, in [91], Rădulescu and Zhang established the existence of nontrivial non-negative and non-positive solutions, and obtained infinitely many solutions for the quasilinear equation $-\text{div}A(x, \nabla u) + V(x)|u|^{\alpha(x)-2}u = f(x,u)$ in $\mathbb{R}^N$, where the divergence type operator has behaviors like $|\zeta|^{q(x)-2}$ for small $|\zeta|$ and like $|\zeta|^{p(x)-2}$ for large $|\zeta|$, where $1 < \alpha(\cdot) \leq p(\cdot) < q(\cdot) < N$. In that paper, it is supposed that the potential $V \in L^1_{loc}(\mathbb{R}^N)$ verifies $V(\cdot) \geq V_0 > 0$, $V(x) \to +\infty$ as $|x| \to +\infty$ and that the nonlinearity f satisfies some growth condition with the following assumption instead of (A-R) condition: there exist constants $M, C_1, C_2 > 0$ and a function a such that

$$C_1|t|^{q(x)}[\ln(e+|t|)]^{a(x)-1} \leq C_2\frac{f(x,t)t}{\ln(e+|t|)} \leq f(x,t)t - s(x)F(x,t),\ \forall (x,|t|) \in \mathbb{R}^N \times [M, +\infty), \tag{0.11}$$

where $\text{ess}_{x\in\mathbb{R}^N}\inf(a(x) - q(x)) > 0$, $q(\cdot) \leq s(\cdot)$ and $\text{ess}_{x\in\mathbb{R}^N}\inf(p^*(x) - s(x)) > 0$ with $p^*(x) := \frac{Np(x)}{N-p(x)}$. Related to this subject, we refer the readers to some important results concerning the study of the eigenvalue problems (see [18, 64, 65, 77] and the references therein). A main motivation of this chapter is that, to the best of our knowledge, there is little research considering both the potential $V \not\equiv 0$ and nonlinearity f without (A-R) condition for more general quasilinear equation in the framework of Musielak-Sobolev spaces. Hence, in this work, our main goal is to show the existence of weak solutions to the problem (P) when the nonlinearity f satisfies some set of growth conditions and similar condition to that in (0.6).

The fifth chapter of this thesis is devoted to study a doubly nonlinear parabolic problem of p-Laplacian type with a nonlinear boundary condition. Precisely, we consider the following problem

$$(P)\begin{cases}\partial_t(\beta(u)) - \Delta_p u + h(x,t,u) = 0, & \text{in } \Omega \times (0,\infty),\\ -|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(u), & \text{on } \partial\Omega \times (0,\infty),\\ \beta(u(0)) = \beta(u_0), & \text{in } \Omega,\end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. Here, $\frac{\partial}{\partial \nu}$ denotes the outer unit normal to $\partial\Omega$ at x, β is an increasing locally Lipschitz function on $\mathbb{R}$ with $\beta(0) = 0$ and $u_0 \in L^\infty(\Omega)$ is initial datum. Partial differential equations of the form (P), or some special cases of it, are studied by several authors because of their mathematical interest and because they describe many phenomena in mechanics, biology and physics (see e.g. [74, 85, 63, 56, 55]). In this work, we shall focus on a Neumann type nonlinear boundary condition, since the Dirichlet boundary condition have been widely treated in the literature (see [19, 32, 36, 39]). In fact, in [39], El Hachimi and El Ouardi, extend some of the results obtained in [36]. Precisely, they proved the existence and uniqueness of solutions, and the existence of a global attractor for problem (P) with Dirichlet boundary condition and initial datum in $L^2(\Omega)$ and h satisfying some set of conditions. For $p = 2$, Andreu et al. in [10], proved the existence and uniqueness of bounded solution and the existence of a global attractor where the initial datum is in $L^\infty(\Omega)$ with nonlinear boundary condition. Therefore, this work is inspired by the results of El Hachimi and El Ouardi [39] and Andreu et al. [10]. Here, assuming the initial datum in $L^\infty(\Omega)$ and the assumptions on h quite weaker than in [39], we shall extend the results in [39] concerning only the existence and the uniqueness of the solutions to the problem (P). Moreover, by adding some supplementary assumptions on the data β, h and g, and following some ideas in [10] combined with some results in [98], we prove the existence of a global attractor in $L^\infty(\Omega)$ for (P) with $1 < p < +\infty$, when the initial datum $u_0 \in L^\infty(\Omega)$. We point out that, the conditions on nonlinearities h and g used here differ from those imposed in [10]. We also note that the choice of the space $L^\infty(\Omega)$ is motivated by the fact that the solutions obtained are bounded for bounded initial data and that the compactness of the trajectories is obtained by using a result of [33].

The last chapter (Chapter 6) of this thesis is devoted to the study of some quasilinear parabolic problems with variable exponents. As a model case, we consider

$$(P)\begin{cases} \partial_t u + \mathcal{A}u = f(x,t,u,\nabla u) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u(T) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ ($N \geq 1$) with a smooth boundary $\partial\Omega$, $T > 0$. Here, $\mathcal{A}u := -\mathrm{div}(\mathbf{A}(\cdot,\cdot,u,\nabla u))$ denotes a Leray-Lions's type operator with variable exponents acting from some functional space V_0 into its topological dual V_0' (see Section 6.2) and f is a nonlinear Carathéodory function whose growth with respect to $|\nabla u|$ is at most of order $p(x)$ (see hypothesis A4)). For this type of problem, the generalized Lebesgue-Sobolev space (see Chapter 1) is the adequate functional spaces for solutions. Nonlinear problems defined by (P) arises in many applications; for instance, in electrorheological fluids (see [88]) and image restoration (see [25]). There is an extensive literature on the existence of solutions for problems like (P). Let us give a review of some results concerning the case $p(x) := p$ is a real constant. In [31], Deuel and Hess, proved the existence of at least one periodic solution for problem (P) in the case where the natural growth of f with respect to $|\nabla u|$ is of order less than p; which means, $|f(x,t,u,\nabla u)| \leq k(x,t) + c|\nabla u|^{p-\delta}$ for some $\delta > 0$, $k(x,t) \in L^{1+\delta}(\Omega \times (0,T))$, and c being a positive constant. In [54], Grenon extends the result of [31] to the case where the natural growth of f with respect to $|\nabla u|$ is at most of order p; but instead of a periodicity condition the author considered an initial one. Notice that in the two previous works, the hypothesis of existence of well-ordered sub and supersolutions is assumed. Following [31], El Hachimi and Lamrani in [40] extend the results in [54], where the authors obtained the existence of periodic solutions, under the same hypotheses as in [54]. For variable exponents, some particular cases of problems has been studied by many authors (see e.g [6, 49, 17, 102]). The main goal of this chapter is to extend the results in [40] to the variable exponents case by using the sub and supersolutions method. It is well known that this method, when it is applicable, has more advantages compared to other methods . For example, we can give some information on the behavior of the solution (blow-up or extinction) and on the sign of the solution (positive or negative). Nevertheless, this method is quite complicated because it requires well-ordered sub and supersolutions, which is not usually easy to get. Indeed, in many application cases, sub and supersolutions are obtained from eigenfunction associated to the first eigenvalue of some operators (say the p-Laplacian.) But, when dealing some with variable exponents, it is well known that the $p(x)$-Laplacian does not have in general a first eigenvalue (see [44]) and therefore, one has to find sub and supersolution by means of other ideas.

CHAPTER 1

LEBESGUE-SOBOLEV SPACES WITH VARIABLE EXPONENT

In the past few years the subject of variable exponent spaces has undergone a vast development. Nevertheless, the standard reference for basic properties has been the article [66] by Kovacik and Rakosuik from 1991. (The same properties were derived by different methods by Fan and Zhao [44] 10 years later.) In this chapter we shall introduce Lebesgue and Sobolev spaces with variable exponent and state some of their basic properties, such as reflexivity, separability, duality and first results concerning embeddings and density of smooth functions.

1.1. A brief history of function spaces with variable exponents

Lebesgue spaces with variable exponents appeared in the literature in 1931 in the paper by Orlicz [84]. In that paper, the author considered the question of Hölder's inequality in the space $\ell^{q(\cdot)}$ and then he generalized the same question to the Lebesgue space with variable exponent $L^{q(\cdot)}$ on the real line. After this, he was interested in the study of function spaces L^{Φ} that contain all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\rho(\lambda u) = \int_{\Omega} \Phi(\lambda |u(x)|) dx < \infty,$$

for some $\lambda > 0$ and Φ satisfying some natural assumptions, where Ω is an open set in $\mathbb{R}^N$ (see [82]). We point out that in [82] the case $|u(x)|^{q(x)}$ corresponding to variable exponents was not included. In the 1950's these problems were systematically studied by Nakano [83], who developed the theory of "modular function spaces". Nakano, in the appendix [p. 284], explicitly mentioned Lebesgue spaces with variable exponents as an example of more general spaces he considered. Lebesgue spaces with variable exponents on the real line reappeared independently in the Russian literature, where they were studied as spaces of interest in their own right, notably Tsenov in 1961 [97] and Sharapudinov in 1979 [93]. The question raised by Tsenov and solved by Sharapudinov is the minimization of the integral

$$\int_a^b |u(x) - v(x)|^{q(x)} dx,$$

where u is a given function and v varies over a finite dimensional subspace of $L^{q(x)}([a,b])$. In [93] Sharapudinov also introduced the Luxemburg norm for the Lebesgue space and showed some classical results such as the separability and reflexivity. In the mid-1980s Zhikov [103] started a new direction of investigation, which applied the Lebesgue spaces with variable exponents to problems in the calculus variational integrals with non-standard growth conditions. Precisely, he was concerned with minimizing the functionals

$$F(u) = \int_{\Omega} f(x, \nabla u) dx,$$

when f satisfies the non-standard growth condition

$$-c_0 + c_1|t|^p \leq f(x,t) \leq c_0 + c_2|t|^s,$$

where the c_i are positive real constants and $0 < p < s$. A particular example of such function is $f(x,t) = |t|^{q(x)}$, where $p \leq q(x) \leq s$.

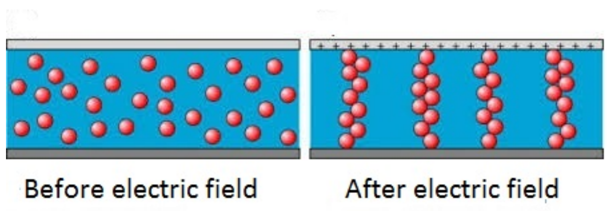
The paper by Kovacik and Rakosuik in the early 1990s [66], was considered as a major step in the investigation of Lebesgue spaces with variable exponents. This paper established several basic properties of spaces $L^{q(x)}(\Omega)$ and $W^{1,q(x)}(\Omega)$ with variable exponents in $\mathbb{R}^N$. After ten years, Fan and Zhao [44] established the same properties in [66] by different methods. At the beginning of millennium, many efforts were made to understand these spaces, for example, how to establish the connection between these spaces, conditions of coercivity and the variational integrals with nonstandard growth. Density of smooth functions in $W^{m,q(x)}(\Omega)$ and related Sobolev embedding properties are due to Edmunds and Rakosnik [**?** 37]. Pioneering regularity results for functionals with nonstandard growth are due to Acerbi and Mingione [3]. The abstract theory of Lebesgue and Sobolev spaces with variable exponents was developed in the monographs by Diening, Harjulehto, Hästö, and Ruzicka [34] and by Cruz-Uribe, Fiorenza [30].

These spaces form particular case of Orlicz spaces that we can find in [4, 82, 81, 90]. Moreover, they constitute the most appropriate functional framework for solving several nonlinear partial differential equations such as the problems involving the operator $p(x)$-Laplacian. These problems model many physical phenomena such as electrorheological fluids [92] and image restoration [25]. In the forthcoming section, we discuss these phenomena in details.

1.2. Motivation

In this section, we give two relevant examples that justify the mathematical study of models involving variable exponents.

Electrorheological fluids: Electrorheological fluids are colloidal suspensions of a certain type, consisting of dielectric particles dispersed in an insulating oil. The marvelous feature of such fluids is that they can solidify into a jelly-like state almost instantaneously when subjected to an externally applied electric field with moderate strength, with a stiffness varying proportionally to the field strength. The liquid-solid transformation is reversible. Once the applied field is removed, the original flow state is recovered. This phenomenon is known as the Winslow effect, and we can represent it as follows



Electrorheological fluids have the quality and potential for a wide field of applications. These include for example robotics, aircraft and aerospace applications. We refer the reader to [57, 89, 92] for more information.

There exist several possibilities for modeling the physics of electrorheological fluids. In [89], Rajagopal and Růžička have developed a model that takes into account the complex interaction of electromagnetic fields and the moving liquid. The constitutive equation for the motion of an electrorheological fluid is given by

$$\partial_t u + \mathrm{div} S(u) + (u \cdot \nabla)u + \nabla \pi = f, \tag{1.1}$$

where $u : \mathbb{R}^{3+1} \to \mathbb{R}^3$ is the velocity of the fluid at a point, $\nabla = (\partial_1, \partial_2, \partial_3)$ is the gradient operator, π is the pressure, f represents external forces and the stress tensor S is of the form

$$S(u)(x) = \mu(x)(1 + |Du(x)|^2)^{\frac{p-2}{2}} Du(x),$$

where $p = p(x)$ and $Du = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetric part of the gradient of u. Rajagopal and Růžička established an existence result for problem (1.1) in variable exponent spaces.

Image restoration: Image restoration is the operation of taking a corrupt/noisy image and estimating the clean, original image. Corruption may come in many forms such as motion blur, noise and camera mis-focus. Chen, Levine and Rao [25], proposed a new model for image restoration. The diffusion resulting from the model was proposed is a combination of the Gaussian smoothing and regularization based on the total variation. Precisely, given an original image f, it is assumed that it has been corrupted by some additive noise n. Then the problem is to recover the true image u

$$f = u + n.$$

The adaptive model was proposed is:

$$\min_{u \in BV(\Omega) \cup L^2(\Omega)} \int_\Omega \phi(x, \nabla u) + \frac{\lambda}{2} \int_\Omega (u - f)^2, \tag{1.2}$$

where

$$\phi(x,t) = \begin{cases} \frac{1}{q(x)} |t|^{q(x)}, & |t| \leq \beta, \\ |t| - \frac{\beta q(x) - \beta^{q(x)}}{q(x)}, & |t| > \beta, \end{cases}$$

$q(x) = 1 + \frac{1}{1 + k|\nabla G_\sigma * f(x)|}$, $G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} exp(\frac{|x|^2}{2\sigma^2})$ is the Gaussian kernel, $k > 0, \sigma > 0$ are fixed parameters and $\beta > 0$ is a user-defined threshold. For problem (1.2), Chen, Levine and Rao established the existence and uniqueness of the solution and the long-time behavior of the associated flow of the proposed model. The effectiveness of the model in image restoration is illustrated by some experimental results included in their paper.

1.3. Lebesgue spaces with variable exponents

In this part, we define Lebesgue spaces with variable exponents, $L^{q(\cdot)}$. The variable Lebesgue spaces, as their name implies, are a generalization of the classical Lebesgue L^q spaces, replacing the constant exponent q with a variable exponent function $q(\cdot)$. The resulting Banach function spaces $L^{q(\cdot)}$ have many properties similar to the L^q spaces, but they also differ in surprising and subtle ways. The spaces $L^{q(\cdot)}$ fit into the framework of Musielak spaces which we will define in the forthcoming chapter. By virtue of types of problems studied in this thesis, throughout this work, we restrict to define the Lebesgue spaces with variable exponents $L^{q(\cdot)}$ only for a continuous function $q : \overline{\Omega} \to (1, +\infty)$. However, these spaces can be defined for any measurable function $q : \Omega \to [1, +\infty]$. For more details on the basic properties of these spaces, we refer the reader to the papers [30, 34, 44, 66].

Set

$$\mathcal{C}_+(\overline{\Omega}) := \{h \in \mathcal{C}(\overline{\Omega}) : h(x) > 1 \text{ for any } x \in \overline{\Omega}\}.$$

For any $h \in \mathcal{C}_+(\overline{\Omega})$ we define

$$h^- := \min_{x \in \overline{\Omega}} h(x), \quad h^+ := \max_{x \in \overline{\Omega}} h(x).$$

Definition 1.1. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. For a measurable function $u : \Omega \to \mathbb{R}$, the mapping*

$$\rho_{q(\cdot)}(u) = \int_\Omega |u(x)|^{q(x)} dx,$$

is called modular of u with respect to $q(\cdot)$.

Remark 1.3.1. *In view of the definition in [81, p. 1], $\rho_{q(\cdot)}$ is a convex modular, that means, $\rho_{q(\cdot)}$ verifies the following properties: for any measurable functions $u, v : \Omega \to \mathbb{R}$, we have*

- $\rho_{q(\cdot)}(u) = 0 \Leftrightarrow u = 0,$
- $\rho_{q(\cdot)}(u) = \rho_{q(\cdot)}(-u),$
- $\rho_{q(\cdot)}(\alpha u + \beta v) \leq \alpha\rho_{q(\cdot)}(u) + \beta\rho_{q(\cdot)}(v), \quad \forall \alpha, \beta \geq 0,\ \alpha + \beta = 1.$

Using the modular $\rho_{q(\cdot)}$, Lebesgue spaces with variable exponent are defined as follows.

Definition 1.2. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. The set*

$$L^{q(x)}(\Omega) := \left\{ u : u : \Omega \to \mathbb{R} \text{ is measurable with } \rho_{q(\cdot)}(u) < \infty \right\},$$

is called Lebesgue space with variable exponent $q(\cdot)$ or generalized Lebesgue space.

Now we introduce the so-called Luxemburg norm on $L^{q(x)}(\Omega)$.

Definition 1.3. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. For $u \in L^{q(x)}(\Omega)$ we define its Luxemburg norm with respect to $q(\cdot)$ by*

$$\|u\|_{L^{q(x)}(\Omega)} = \|u\|_{q(x)} := \inf \left\{ \lambda > 0 : \ \rho_{q(\cdot)} \left(\frac{u}{\lambda} \right) \leq 1 \right\}.$$

Proposition 1.3.2 ([44, 66])**.** *Let $q \in \mathcal{C}_+(\overline{\Omega})$. Then, the variable exponent Lebesgue spaces $(L^{q(x)}(\Omega), \|\cdot\|_{q(x)})$ are separable Banach spaces.*

The norm $\|u\|_{q(x)}$ is in close relation with the modular $\rho_{q(\cdot)}(u)$. We have

Proposition 1.3.3 ([44])**.** *Let $u, (u_n) : \Omega \to \mathbb{R}$ be measurable functions; then,*

$$\min\{\|u\|_{q(x)}^{q_-}, \|u\|_{q(x)}^{q_+}\} \leq \rho_{q(\cdot)}(u) \leq \max\{\|u\|_{q(x)}^{q_-}, \|u\|_{q(x)}^{q_+}\}, \tag{1.3}$$

$$\|u\|_{q(x)} < 1 (= 1, > 1) \Leftrightarrow \rho_{q(\cdot)}(u) < 1 (= 1, > 1), \tag{1.4}$$

$$\|u_n - u\|_{q(x)} \to 0 \Leftrightarrow \rho_{q(\cdot)}(u_n - u) \to 0, \tag{1.5}$$

$$\|u_n\|_{q(x)} \to +\infty \Leftrightarrow \rho_{q(\cdot)}(u_n) \to +\infty. \tag{1.6}$$

Definition 1.4. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. The variable exponent Lebesgue space $L^{q'(x)}(\Omega)$ is called the conjugate space of $L^{q(x)}(\Omega)$, where $q'(x)$ is the conjugate exponent of $q(x)$, that means, $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$.*

Hölder's inequality is another important tool that can also be retrieved.

Proposition 1.3.4 ([44, 66])**.** *Let $q \in \mathcal{C}_+(\overline{\Omega})$ and $q'(x)$ its conjugate exponent. Then, for any $u \in L^{q(x)}(\Omega)$ and $v \in L^{q'(x)}(\Omega)$, the Hölder type inequality*

$$\int_\Omega |uv| dx \leq \left(\frac{1}{q^-} + \frac{1}{q'^-} \right) \|u\|_{q(x)} \|v\|_{q'(x)} \leq 2\|u\|_{q(x)} \|v\|_{q'(x)}, \tag{1.7}$$

holds true.

Among the main properties in the study of Banach spaces is the description of their dual space $(L^{q(x)}(\Omega))^*$, and closely related to this, the question of reflexivity. For this purpose we give

Proposition 1.3.5 ([44, 66]). *Let $q \in \mathcal{C}_+(\overline{\Omega})$ and $q'(x)$ its conjugate exponent. Then, the space $(L^{q(x)}(\Omega), \|\cdot\|_{q(x)})$ is reflexive, and that the mapping $I : L^{q'(x)}(\Omega) \to (L^{q(x)}(\Omega))^*$ defined by*

$$\langle I(v), w\rangle = \int_\Omega v(x)w(x)dx, \quad \forall v \in L^{q'(x)}(\Omega), \forall w \in L^{q(x)}(\Omega),$$

is a linear isomorphism and $\|I(v)\|_{(L^{q(x)}(\Omega))^*} \leq 2\|v\|_{L^{q'(x)}(\Omega)}$.

The use of different variable exponents induces different generalized Lebesgue spaces. But, since $|\Omega| < \infty$, then we can recapture classic embedding properties.

Proposition 1.3.6 ([44, 66]). *Let $q_1(x), q_2(x) \in \mathcal{C}_+(\overline{\Omega})$. Then, $q_1(x) \leq q_2(x)$ almost everywhere in Ω, if and only if, $L^{q_2(x)}(\Omega)$ embedded continuously in $L^{q_1(x)}(\Omega)$.*

Remark 1.3.7. *For a single $q \in \mathcal{C}_+(\overline{\Omega})$, obviously, we have the following chain of continuous embeddings:*

$$L^\infty(\Omega) \hookrightarrow L^{q^+(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \hookrightarrow L^{q^-(x)}(\Omega) \hookrightarrow L^1(\Omega).$$

We finish this section by recalling the following proposition.

Proposition 1.3.8 ([37]). *Let p and q be measurable functions such that $p \in L^\infty(\Omega)$ and $1 < p(x)q(x) \leq \infty$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega), u \neq 0$. Then*

$$\|u\|_{p(x)q(x)} \leq 1 \Rightarrow \|u\|^{p^+}_{p(x)q(x)} \leq \||u|^{p(x)}\|_{q(x)} \leq \|u\|^{p^-}_{p(x)q(x)},$$
$$\|u\|_{p(x)q(x)} \geq 1 \Rightarrow \|u\|^{p^-}_{p(x)q(x)} \leq \||u|^{p(x)}\|_{q(x)} \leq \|u\|^{p^+}_{p(x)q(x)}.$$

In particular when $p(x) = p$ is a constant, then

$$\||u|^p\|_{q(x)} = \|u\|^p_{pq(x)}.$$

1.4. Sobolev spaces with variable exponents

In this part, we give some basic results on the generalized Sobolev spaces $W^{1,q(x)}(\Omega)$. One motivation of studying these spaces is that solutions of partial differential equations belong naturally to Sobolev spaces (see Chapters 4, 5..).

Definition 1.5. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. We define the Sobolev space with variable exponent $q(x)$ by*

$$W^{1,q(x)}(\Omega) = \{u \in L^{q(x)}(\Omega) : \ |\nabla u| \in L^{q(x)}(\Omega)\},$$

Definition 1.6. *Let $q \in \mathcal{C}_+(\overline{\Omega})$ and $u \in W^{1,q(x)}(\Omega)$. We define the Sobolev norm by*

$$\|u\|_{W^{1,q(x)}(\Omega)} = \|u\|_{1,q(x)} := \|u\|_{q(x)} + \|\nabla u\|_{q(x)},$$

where $\|\nabla u\|_{q(x)} = \||\nabla u|\|_{q(x)}$.

Proposition 1.4.1 ([44, 66]). *The space $(W^{1,q(x)}(\Omega), \|\cdot\|_{1,q(x)})$ is separable and reflexive Banach space.*

An immediate consequence of Proposition 1.3.6 is

Proposition 1.4.2 ([44, 66]). *Let $q_1(x), q_2(x) \in \mathcal{C}_+(\overline{\Omega})$. If $q_1(x) \leq q_2(x)$ almost everywhere in Ω, then $W^{1,q_2(x)}(\Omega)$ can be embedded into $W^{1,q_1(x)}(\Omega)$.*

Now let us generalize the well-known Sobolev embedding theorem of $W^{1,q}(\Omega)$ to $W^{1,q(x)}(\Omega)$. Let us start by the following definition

Definition 1.7. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. The variable exponent defined by*

$$\frac{Nq(x)}{N-q(x)} := q^*(x), \quad \forall x \in \overline{\Omega},$$

is called Sobolev conjugate exponent.

Now we can state the Sobolev embedding result

Proposition 1.4.3 ([44]). *Let $q, r \in \mathcal{C}_+(\overline{\Omega})$. Assume that*

$$q(x) < N, \quad r(x) < q^*(x), \quad \forall x \in \overline{\Omega}.$$

Then, there is a continuous and compact embedding $W^{1,q(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$.

A natural question which may be asked is: is there a continuous embedding of $W^{1,q(x)}(\Omega)$ into $L^{q^*(x)}(\Omega)$? The following example shows that, in general, this cannot be expected.

Example 1.4.4. *Let $\Omega = \{x = (x_1, x_2) : 0 < x_1 < 1,\ 0 < x_2 < 1\} \subset \mathbb{R}^2, q(x) = 1 + x_2, u(x) = (2 + x_2)^{\frac{1}{1+x_2}}$. Then, we have $u(x) \in W^{1,q(x)}(\Omega)$ and $q^*(x) = 2(1 + x_2)/(1 - x_2)$. It is easy to check that $u \notin L^{q^*(x)}(\Omega)$.*

By assuming that the exponent $q(\cdot)$ is regular, we can get the embedding $W^{1,q(x)}(\Omega) \hookrightarrow L^{q^*(x)}(\Omega)$. To this end, we give the following definition

Definition 1.8. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. We say that $q(\cdot)$ satisfies the log-Hölder condition if,*

$$|q(x) - q(y)| \leq \frac{c}{-log|x-y|}, \quad \forall\ |x-y| < \frac{1}{2},\ x, y \in \overline{\Omega},$$

for some constant $c := c(q(\cdot)) > 0$.

Now, we can extend the Sobolev embedding theorem to variable exponents Sobolev spaces. We have

Proposition 1.4.5. *[30, 34] Let $q \in \mathcal{C}_+(\overline{\Omega})$. If $q(\cdot)$ satisfies the log-Hölder condition, then $W^{1,q(x)}(\Omega)$ can be embedded into $L^{q^*(x)}(\Omega)$.*

Since $W^{1,q(x)}(\Omega)$ is a separable Banach space, then there exists a countable dense subset. Now, we would like to identify particular families of functions that are dense. Because weak derivatives coincide with classical derivatives for smooth functions, it is natural to consider the question of when such functions are dense. We begin by defining two subspaces of $W^{1,q(x)}(\Omega)$.

Definition 1.9. *Let $q \in \mathcal{C}_+(\overline{\Omega})$. Let $W_0^{1,q(x)}(\Omega)$ be the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,q(x)}(\Omega)$, and let $H_0^{1,q(x)}(\Omega) = W^{1,q(x)}(\Omega) \cap W_0^{1,1}(\Omega)$.*

Remark 1.4.6.

1. *It is clear that if $q(x) \equiv q$ is a constant, then $H_0^{1,q}(\Omega) = W_0^{1,q}(\Omega)$. In this case, the space $\mathcal{C}^\infty(\Omega)$ is dense in $W^{1,q}(\Omega)$.*

2. *For a general function $q \in \mathcal{C}_+(\overline{\Omega})$, from the definition, we have $W_0^{1,q(x)}(\Omega) \subset H_0^{1,q(x)}(\Omega)$, and $H_0^{1,q(x)}(\Omega)$ is a closed linear subspace of $W^{1,q(x)}(\Omega)$.*

3. *In general, $H_0^{1,q(x)}(\Omega) \neq W_0^{1,q(x)}(\Omega)$. Indeed, let $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\}, 1 < \alpha_1 < 2 < \alpha_2$. Define the variable exponent q by*

$$q(x) = \begin{cases} \alpha_1, & if \quad x_1 x_2 > 0, \\ \alpha_2, & if \quad x_1 x_2 < 0, \end{cases}$$

then $H_0^{1,q(x)}(\Omega) \neq W_0^{1,q(x)}(\Omega)$. This example was given by Zhikov in [103].

4. *The identity $H_0^{1,q(x)}(\Omega) = W_0^{1,q(x)}(\Omega)$, means that, the space $\mathcal{C}_0^{\infty}(\Omega)$ is dense in $(H_0^{1,q(x)}(\Omega), \|\cdot\|_{W^{1,q(x)}(\Omega)})$.*

Using the log-Hölder condition on the variable exponent q, we get the following density result.

Proposition 1.4.7 ([30, 34, 44]). *Let $q \in \mathcal{C}_+(\overline{\Omega})$ and q satisfies the log-Hölder condition. Then, we have*

1. *$\mathcal{C}^{\infty}(\Omega)$ is dense in $W^{1,q(x)}(\Omega)$.*
2. *$H_0^{1,q(x)}(\Omega) = W_0^{1,q(x)}(\Omega)$.*

Now, we give the Poincaré inequality in generalized Sobolev space.

Proposition 1.4.8. *[30, 34] Let $q \in \mathcal{C}_+(\overline{\Omega})$ and q satisfies the log-Hölder condition. Then, we have*

$$\|u\|_{q(x)} \leq C\|\nabla u\|_{q(x)}, \quad \forall u \in W_0^{1,q(x)}(\Omega),$$

for some constant $C > 0$ depending only on the dimension N, $diam(\Omega)$, and $c(q(\cdot))$ defined in Definition 1.8. Moreover, $\|u\|_{W_0^{1,q(x)}(\Omega)} = \|\nabla u\|_{q(x)}$ is a norm in $W_0^{1,q(x)}(\Omega)$.

1.5. Some difficulties related to the Lebesgue-Sobolev spaces with variable exponents

As each functional space has their difficulties, in this part, we give some important results and techniques from classical Lebesgue-Sobolev spaces which do not hold in the Lebesgue-Sobolev spaces with variable exponents even when the exponent is very regular, i.e., q satisfies the log-Hölder condition, or $q \in \mathcal{C}^{\infty}(\overline{\Omega})$.

1. The space $L^{q(x)}(\Omega)$ is not rearrangement invariant; the translation operator $T_h : L^{q(x)}(\Omega) \to L^{q(x)}(\Omega), T_hu(x$ $u(x+h)$ is not bounded; Young's convolution inequality
$$\|u * v\|_{q(x)} \leq c\|u\|_1\|v\|_{q(x)},$$
does not hold. Moreover, the map T_h is bounded if and only if q is a constant (see Proposition 3.6. 1 in [34]).

2. The formula
$$\int_{\Omega} |u(x)|^q dx = q\int_0^{\infty} t^{q-1}|\{x \in \Omega : |u(x)| > t\}|dt$$
has no variable exponent analogue.

3. In the constant exponent case there is an obvious connection between modular and norm versions of the inequality, which does not hold in the variable exponent context. In other words, the modular Poincaré inequality
$$\rho_q(u) \leq c\rho_q(\nabla u)$$
can not, in general, hold in a modular form $\rho_{q(\cdot)}(u)$. In fact, taking the following example: let $q : (-2,2) \to [2,3]$ be a Lipschitz continuous exponent that equals 3 in $(-2,-1) \cup (1,2)$, 2 in $(-\frac{1}{2}, \frac{1}{2})$ and is linear elsewhere. Let u_λ be a Lipschitz function such that $u_\lambda(\pm 2) = 0; u_\lambda = \lambda$ in $(-1,1)$ and $|u'_\lambda| = \lambda$ in $(-2,-1) \cup (1,2)$. Then,
$$\frac{\rho_{q(x)}(u_\lambda)}{\rho_{q(x)}(u'_\lambda)} = \frac{\int_{-2}^{2} |u_\lambda|^{q(x)}dx}{\int_{-2}^{2} |u'_\lambda|^{q(x)}dx} \geq \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \lambda^2 dx}{2\int_{-2}^{-1} \lambda^3 dx} = \frac{1}{2\lambda} \to \infty,$$
as $\lambda \to 0^+$.

4. Solutions of the $p(x)$-Laplacian equation are not scalable, i.e. λu need not be a solution even if u is.

CHAPTER 2

MUSIELAK SPACES AND MUSIELAK-SOBOLEV SPACES

In the study of nonlinear partial differential equations, it is well known that more general functional space can handle differential equations with more nonlinearities. For example, the p-Laplacian equations correspond to the classical Sobolev space setting, the $p(x)$-Laplacian equations correspond to the Sobolev space with variable exponent setting, etc. In the forthcoming chapter, we deal with a more general quasilinear equation in which the Musielak-Sobolev spaces are the adequate functional spaces corresponding to their solutions. In this chapter we study the Musielak-Sobolev spaces which provide the framework for a variety of different function spaces, including classical (weighted) Lebesgue and Orlicz spaces and Lebesgue spaces with variable exponents. For more details we refer the readers to the papers [41, 58, 73, 81].

In this chapter, we present results on generalizations of variable exponent Lebesgue-Sobolev spaces (and other functional spaces) in which the role usually played by the convex function $t^{p(x)}$ is assumed by a more general convex function $\Phi(x,t)$. By virtue of the problem studied in this thesis (particularly in Chapter 4), we should note that some definitions and results given here are some restrictive.

2.1. Generalized N-functions and basic properties

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded smooth domain.

Definition 2.1. *We say that a function Φ is a generalized N-function, if for each $t \in [0,+\infty)$, $\Phi(.,t)$ is measurable and for a.e. $x \in \Omega$, $\Phi(x,.)$ is continuous, even, convex, $\Phi(x,0) = 0$, $\Phi(x,t) > 0$ for $t > 0$, and satisfies the following conditions*

$$\lim_{t\to 0^+} \frac{\Phi(x,t)}{t} = 0 \quad \textit{and} \quad \lim_{t\to +\infty} \frac{\Phi(x,t)}{t} = +\infty.$$

We denote by $N(\Omega)$ the set of generalized N-functions. Let us define

$$\Phi(x,t) = \int_0^t \phi(x,s)sds, \quad \forall t \geq 0, \tag{2.1}$$

where $\phi : \Omega \times [0,+\infty) \to [0,+\infty)$ is a Carathéodory function such that for all $x \in \Omega$, we have

$$(\phi) \begin{cases} \phi(x,0) = 0, \quad \phi(x,t).t \ \text{ is strictly increasing,} \\ \phi(x,t).t > 0, \ \forall t > 0 \ \text{ and } \ \phi(x,t).t \to +\infty \text{ as } t \to +\infty. \end{cases}$$

From Theorem 13.2 of [81], we have:

Proposition 2.1.1. $\Phi \in N(\Omega)$, *if and only if*, Φ *is of the form* (2.1).

Definition 2.2. *The function $\tilde{\Phi} : \Omega \times [0, +\infty) \to [0, +\infty)$ defined by*

$$\tilde{\Phi}(x,t) = \sup_{s>0}(ts - \Phi(x,s)), \quad \text{for } x \in \Omega \text{ and } t \geq 0,$$

is called the complementary function to Φ in the sense of Young.

Remark 2.1.2. *We observe that the function $\tilde{\Phi}$ belongs to $N(\Omega)$, and Φ is also the complementary function to $\tilde{\Phi}$.*

Proposition 2.1.3 ([81]). *Let $\Phi, \tilde{\Phi} \in N(\Omega)$ and let $\tilde{\Phi}$ be complementary to Φ in the sense of Young. Then they satisfy the Young inequality*

$$st \leq \Phi(x,t) + \tilde{\Phi}(x,s), \quad \text{for } x \in \Omega \text{ and } s,t \geq 0.$$

2.2. Musielak and Musielak-Sobolev spaces

Definition 2.3. *Let $\Phi \in N(\Omega)$, the Musielak space $L^{\Phi}(\Omega)$ is defined by*

$$L^{\Phi}(\Omega) := \left\{ u : u : \Omega \to \mathbb{R} \text{ is measurable, and } \exists \lambda > 0 \text{ such that } \int_{\Omega} \Phi\left(x, \frac{|u(x)|}{\lambda}\right) dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{\Phi}(\Omega)} = \|u\|_{\Phi} := \inf\left\{ \lambda > 0 : \int_{\Omega} \Phi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

Definition 2.4. *We say that $\Phi \in N(\Omega)$ satisfies the (Δ_2)-condition, if there exist a positive constant $C > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that*

$$\Phi(x,2t) \leq C\Phi(x,t) + h(x), \quad \text{for } x \in \Omega \text{ and } t \geq 0.$$

Now, we shall determine a sufficient condition for which a function $\Phi \in N(\Omega)$ satisfies the (Δ_2)-condition. To this end, let assuming that there exist two positive constants ϕ_0 and ϕ^0 such that

$$1 < \phi_0 \leq \frac{\phi(x,t)t^2}{\Phi(x,t)} \leq \phi^0 < N, \quad \text{for } x \in \Omega \text{ and } t > 0. \tag{2.2}$$

This relation (2.2) gives the following result:

Lemma 2.2.1. *Let $u \in L^{\Phi}(\Omega)$ and $\rho, t \geq 0$, then we have*

$$\min\left\{\rho^{\phi_0}, \rho^{\phi^0}\right\} \Phi(x,t) \leq \Phi(x,\rho t) \leq \max\left\{\rho^{\phi_0}, \rho^{\phi^0}\right\} \Phi(x,t), \tag{2.3}$$

$$\min\left\{\|u\|_{\Phi}^{\phi_0}, \|u\|_{\Phi}^{\phi^0}\right\} \leq \int_{\Omega} \Phi(x,|u(x)|)dx \leq \max\left\{\|u\|_{\Phi}^{\phi_0}, \|u\|_{\Phi}^{\phi^0}\right\}. \tag{2.4}$$

Proof. Integrating (2.2) implies (2.3). From this and the definition of the Luxemburg norm, we obtain (2.4). □

Remark 2.2.2. *The assumption $\phi^0 < \infty$ it suffices to show the Lemma 2.2.1. By virtue to what follows we need to assume (2.2) with $\phi^0 < N$.*

Remark 2.2.3. *Now, $\Phi \in N(\Omega)$ satisfies the (Δ_2)-condition is a simple consequence of Lemma 2.2.1-(2.3). More precisely, $\frac{\phi(x,t)t^2}{\Phi(x,t)} \leq \phi^0$, is a sufficient condition for that Φ verifies the (Δ_2)-condition, and it is a necessary condition if $h(x) \equiv 0$ defined in Definition 2.4. Indeed, let $\Phi(x,2t) \leq C\Phi(x,t)$ for $x \in \Omega$ and $t > 0$. Then,*

$$C\Phi(x,t) \geq \Phi(x,2t) = \int_0^{2t} \phi(x,s)sds > \int_t^{2t} \phi(x,s)sds > \phi(x,t)t^2,$$

that means,

$$\frac{\phi(x,t)t^2}{\Phi(x,t)} \leq \phi^0 := C, \quad \text{for } x \in \Omega \text{ and } t > 0.$$

For the complementary function $\tilde{\Phi}$ we have the following lemma.

Lemma 2.2.4. *Let $u \in L^{\tilde{\Phi}}(\Omega)$ and $\rho, t \geq 0$, then we have*

$$\min\left\{\rho^{(\phi_0)'}, \rho^{(\phi^0)'}\right\}\tilde{\Phi}(x,t) \leq \tilde{\Phi}(x,\rho t) \leq \max\left\{\rho^{(\phi_0)'}, \rho^{(\phi^0)'}\right\}\tilde{\Phi}(x,t), \tag{2.5}$$

$$\min\left\{\|u\|_{\tilde{\Phi}}^{(\phi_0)'}, \|u\|_{\tilde{\Phi}}^{(\phi^0)'}\right\} \leq \int_\Omega \tilde{\Phi}(x,|u(x)|)dx \leq \max\left\{\|u\|_{\tilde{\Phi}}^{(\phi_0)'}, \|u\|_{\tilde{\Phi}}^{(\phi^0)'}\right\}, \tag{2.6}$$

where $(\phi_0)' = \frac{\phi_0}{\phi_0 - 1}$ and $(\phi^0)' = \frac{\phi^0}{\phi^0-1}$.

Proof. Since $\tilde{\Phi} \in N(\Omega)$, then by Proposition 2.1.1, $\tilde{\Phi}$ can be represented as follows

$$\tilde{\Phi}(x,t) = \int_0^t \tilde{\phi}(x,s)s ds, \quad \forall t \geq 0,$$

where the function $\tilde{\phi}$ satisfies the same assumptions in (ϕ). Hence, to prove (2.5) and (2.6), we follow the same proof of Lemma 2.2.1. To this end, it suffices to show that

$$1 < (\phi^0)' \leq \frac{\tilde{\phi}(x,t)t^2}{\tilde{\Phi}(x,t)} \leq (\phi_0)', \quad \text{for } x \in \Omega \text{ and } t > 0. \tag{2.7}$$

At first, we show that

$$\tilde{\Phi}(x, \phi(x,s)s) = \phi(x,s)s^2 - \Phi(x,s) \leq \Phi(x,2s), \quad \text{for } s \geq 0. \tag{2.8}$$

Indeed, by the convexity of Φ we have

$$\Phi(x,s) + \Phi'(x,s)(t-s) \leq \Phi(x,t), \quad \text{for } s,t \geq 0, \tag{2.9}$$

and, by using $\Phi'(x,s) = \phi(x,s)s$, (2.9) becomes

$$\phi(x,s)st - \Phi(x,t) \leq \phi(x,s)s^2 - \Phi(x,s), \quad \text{for } s,t \geq 0.$$

Thus, by using Definition 2.2 we obtain

$$\begin{aligned}\tilde{\Phi}(x,\phi(x,s)s) &= \sup_{t>0}(\phi(x,s)st - \Phi(x,t)) \\ &= \phi(x,s)s^2 - \Phi(x,s) \leq \phi(x,s)s^2 \leq \int_s^{2s}\phi(x,\tau)d\tau \leq \Phi(x,2s)\end{aligned}$$

for $s \geq 0$. This shows (2.8). Now, we return to prove (2.7). Since $\tilde{\tilde{\Phi}} = \Phi$ and using (2.8) (replacing Φ with $\tilde{\Phi}$), we obtain

$$\Phi(x, \tilde{\phi}(x,s)s) = \tilde{\tilde{\Phi}}(x, \tilde{\phi}(x,s)s) = \tilde{\phi}(x,s)s^2 - \tilde{\Phi}(x,s), \tag{2.10}$$

which is equivalent to

$$\Phi(x, \tilde{\Phi}'(x,s)) = \tilde{\tilde{\Phi}}(x, \tilde{\Phi}'(x,s)) = \tilde{\Phi}'(x,s)s - \tilde{\Phi}(x,s). \tag{2.11}$$

Differentiating (2.10) (or (2.11)) and since $\tilde{\Phi}''(x,s) \neq 0$, we get

$$\Phi'(x, \tilde{\phi}(x,s)s) = s. \tag{2.12}$$

Putting $t = \tilde{\phi}(x,s)s$ in (2.2) and using (2.12), we get

$$\phi_0 \Phi(x, \tilde{\phi}(x,s)s) \leq \tilde{\phi}(x,s)s^2 \leq \phi^0 \Phi(x, \tilde{\phi}(x,s)s). \tag{2.13}$$

Substituting (2.10) in (2.13), and by simple computation we obtain (2.7). □

Remark 2.2.5. *From Lemma 2.2.4 and Remark 2.2.3, the complementary function $\tilde{\Phi}$ also satisfies (Δ_2)-condition.*

By Remarks 2.2.3 and 2.2.5 both Φ and $\tilde{\Phi}$ satisfy the (Δ_2)-condition. Hence, we have the following result (see [42, 58, 73, 81]):

Proposition 2.2.6. *The following assertions hold:*

1. $L^{\Phi}(\Omega) = \{u : u : \Omega \to \mathbb{R}$ *is measurable, and* $\int_\Omega \Phi(x, |u(x)|)\, dx < \infty\}$.
2. *For any sequence* (u_n) *in* $L^{\Phi}(\Omega)$*, we have*
   a) $\int_\Omega \Phi(x, |u_n(x)|)\, dx \to 0(\text{resp. } 1; +\infty) \Leftrightarrow \|u_n\|_\Phi \to 0(\text{resp. } 1; +\infty)$,
   b) $u_n \to u$ *in* $L^{\Phi}(\Omega) \Rightarrow \int_\Omega |\Phi(x, |u_n(x)|) - \Phi(x, |u(x)|)|\, dx \to 0$ *as* $n \to +\infty$.
3. *Let* $u \in L^{\Phi}(\Omega)$ *and* $v \in L^{\tilde{\Phi}}(\Omega)$*. Then the Hölder type inequality holds true*
$$\left| \int_\Omega u(x)v(x)dx \right| \leq 2\|u\|_\Phi \|v\|_{\tilde{\Phi}}.$$
4. $\phi(x, |u(x)|)u(x) \in L^{\tilde{\Phi}}(\Omega)$ *provided that* $u \in L^{\Phi}(\Omega)$.

Definition 2.5. *We say that* $\Phi \in N(\Omega)$ *is locally integrable if* $\Phi(., t_0) \in L^1(\Omega)$ *for every* $t_0 > 0$.

Proposition 2.2.7 ([58, 81])**.** *Let* $\Phi \in N(\Omega)$*, then the Musielak space* $(L^{\Phi}(\Omega), \|\cdot\|_\Phi)$ *is a Banach space. Moreover, if* Φ *is locally integrable, then* $L^{\Phi}(\Omega)$ *is a separable.*

We define the Musielak-Sobolev space $W^{1,\Phi}(\Omega)$ as follows.

Definition 2.6. *Let* $\Phi \in N(\Omega)$*, the Musielak-Sobolev space is defined by*
$$W^{1,\Phi}(\Omega) := \left\{ u \in L^{\Phi}(\Omega) : |\nabla u| \in L^{\Phi}(\Omega) \right\},$$
endowed with the norm
$$\|u\|_{W^{1,\Phi}(\Omega)} = \|u\|_{1,\Phi} := \|u\|_\Phi + \|\nabla u\|_\Phi,$$
where $\|\nabla u\|_\Phi = \||\nabla u|\|_\Phi$.

Remark 2.2.8. *In the particular case where* $\Phi(x, t) = \Phi(t)$ *is independent of* x*,* $W^{1,\Phi}(\Omega)$ *is actually an Orlicz-Sobolev space (see [90]) while in the case where* $\Phi(x, t) = |t|^{p(x)}$*, this space becomes the variable exponent Sobolev space* $W^{1,p(.)}(\Omega)$ *(see Chapter 1).*

2.3. Continuous and compact emdeddings

Definition 2.7. *Let* $\Phi, \Psi \in N(\Omega)$*. We say that* Φ *is weaker than* Ψ*, and denote* $\Phi \preccurlyeq \Psi$*, if there exist positive constants* K_1, K_2 *and a nonnegative function* $h \in L^1(\Omega)$ *such that*
$$\Phi(x, t) \leq K_1 \Psi(x, K_2 t) + h(x), \quad \text{for } x \in \Omega \text{ and } t \geq 0.$$

By Theorem 8.5 in [81], we have the following result

Proposition 2.3.1. *Let* $\Phi, \Psi \in N(\Omega)$ *such that* $\Phi \preccurlyeq \Psi$*. Then, we have the following continuous embeddings*
$$L^{\Psi}(\Omega) \hookrightarrow L^{\Phi}(\Omega) \text{ and } L^{\tilde{\Phi}}(\Omega) \hookrightarrow L^{\tilde{\Psi}}(\Omega).$$

Let us consider the following assertions.

(ϕ_1) $\inf_{x \in \Omega} \Phi(x, 1) = c_1 > 0$.

(ϕ_2) For every $t_0 > 0$ there exists $c = c(t_0) > 0$ such that

$$\frac{\Phi(x,t)}{t} \geq c, \quad \text{and} \quad \frac{\tilde{\Phi}(x,t)}{t} \geq c \quad \text{for } x \in \Omega \text{ and } t \geq t_0.$$

We note that, $(\phi_2) \Rightarrow (\phi_1)$. Moreover, in the case where Φ is independent of x, (ϕ_1) and (ϕ_2) hold automatically and Φ is automatically locally integrable.

Proposition 2.3.2. *If $\Phi \in N(\Omega)$ satisfies the assumption (ϕ_1), then we have the following continuous embeddings*

$$L^{\Phi}(\Omega) \hookrightarrow L^1(\Omega) \quad \textit{and} \quad W^{1,\Phi}(\Omega) \hookrightarrow W^{1,1}(\Omega).$$

Proof. Since Φ is convex and $\Phi(x,0) = 0$, then $\Phi(x,t)/t$ is strictly increasing for $t > 0$. Then, using (ϕ_1) we obtain

$$c_1 t \leq \Phi(x,1)t < \Phi(x,t), \qquad \text{for any } x \in \Omega \text{ and } t > 1,$$

which implies that $L^{\Phi}(\Omega) \hookrightarrow L^1(\Omega)$ and $W^{1,\Phi}(\Omega) \hookrightarrow W^{1,1}(\Omega)$. □

Remark 2.3.3. *The previous proposition can be showed by using Lemma 2.2.1-(2.3). Indeed, from (2.3), we have*

$$c_1 t^{\phi_0} \leq t^{\phi_0}\Phi(x,1) \leq \Phi(x,t), \qquad \textit{for any } x \in \Omega \textit{ and } t > 1,$$

which implies that, since $1 < \phi_0$, $L^{\Phi}(\Omega) \hookrightarrow L^{\phi_0}(\Omega) \hookrightarrow L^1(\Omega)$ and $W^{1,\Phi}(\Omega) \hookrightarrow W^{1,\phi_0}(\Omega) \hookrightarrow W^{1,1}(\Omega)$. Since Lemma 2.2.1 has been shown when the assumption (2.2) holds true, we omit to use it in order to show for readers that can be proving this proposition just by using the properties of Φ.

Furthermore, we have the following interesting result (see [81, p. 189]).

Proposition 2.3.4. *If $\Phi, \tilde{\Phi} \in N(\Omega)$ both are locally integrable and satisfy (ϕ_2), then the space $(L^{\Phi}(\Omega), \|\cdot\|_{\Phi})$ is reflexive, and that the mapping $J : L^{\tilde{\Phi}}(\Omega) \to (L^{\Phi}(\Omega))^*$ defined by*

$$\langle J(v), w\rangle = \int_{\Omega} v(x)w(x)dx, \quad \forall v \in L^{\tilde{\Phi}}(\Omega), \forall w \in L^{\Phi}(\Omega),$$

is a linear isomorphism and $\|J(v)\|_{(L^{\Phi}(\Omega))^*} \leq 2\|v\|_{L^{\tilde{\Phi}}(\Omega)}$.

We denote by $W_0^{1,\Phi}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{1,\Phi}(\Omega)$ and by $\mathcal{D}_0^{1,\Phi}(\Omega)$ the completion of $C_0^{\infty}(\Omega)$ in the norm $\|\nabla u\|_{\Phi}$. It is clear that $W_0^{1,\Phi}(\Omega) = \mathcal{D}_0^{1,\Phi}(\Omega)$ in the case where $\|\nabla u\|_{\Phi}$ is an equivalent norm in $W_0^{1,\Phi}(\Omega)$.

Remark 2.3.5. *By assuming Φ locally integrable and satisfies (ϕ_1), $W^{1,\Phi}(\Omega), W_0^{1,\Phi}(\Omega)$ and $\mathcal{D}_0^{1,\Phi}(\Omega)$ are clearly separable Banach spaces, and we have*

$$W_0^{1,\Phi}(\Omega) \hookrightarrow W^{1,\Phi}(\Omega) \hookrightarrow W^{1,1}(\Omega)$$
$$\mathcal{D}_0^{1,\Phi}(\Omega) \hookrightarrow \mathcal{D}_0^{1,1}(\Omega) = W_0^{1,1}(\Omega).$$

Moreover, these spaces are reflexive if $L^{\Phi}(\Omega)$ is reflexive.

In this work, we need to use some standard tools such as the Poincaré inequality and results of compactness for embeddings in Musielak-Sobolev spaces . For this reason, we shall suppose the following supplementary assumptions on Φ.

(H_1) $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with the cone property, and $\Phi \in N(\Omega)$.

(H_2) $\Phi : \overline{\Omega} \times [0,+\infty) \to [0,+\infty)$ is continuous and $\Phi(x,t) \in (0,+\infty)$ for $x \in \overline{\Omega}$ and $t \in (0,+\infty)$.

Now, let Φ satisfies (H_1) and (H_2). Then, for each $x \in \overline{\Omega}$, the function $\Phi(x,\cdot) : [0,+\infty) \to [0,+\infty)$ is a strictly increasing homeomorphism. Denote by $\Phi^{-1}(x,\cdot)$ the inverse function of $\Phi(x,\cdot)$. We also assume the following condition.

(H_3)

$$\int_0^1 \frac{\Phi^{-1}(x,t)}{t^{\frac{N+1}{N}}}dt < +\infty, \ \forall x \in \overline{\Omega}.$$

Define the function $\Phi_*^{-1} : \overline{\Omega} \times [0,+\infty) \to [0,+\infty)$ by

$$\Phi_*^{-1}(x,s) = \int_0^s \frac{\Phi^{-1}(x,\tau)}{\tau^{\frac{N+1}{N}}}d\tau, \ \text{ for } x \in \overline{\Omega} \text{ and } s \in [0,+\infty). \tag{2.14}$$

Then, by assumption (H_3), Φ_*^{-1} is well defined, and for each $x \in \overline{\Omega}$, $\Phi_*^{-1}(x,\cdot)$ is strictly increasing, $\Phi_*^{-1}(x,\cdot) \in \mathcal{C}^1((0,+\infty))$ and the function $\Phi_*^{-1}(x,\cdot)$ is concave.

Remark 2.3.6. *Since Ω is bounded, then the assumption (H_3) places no restrictions on Φ from the point of view of embedding theory. Indeed, define $\Phi_1 : \overline{\Omega} \times [0,+\infty) \to [0,+\infty)$ by*

$$\Phi_1(x,t) = \begin{cases} \Phi(x,1)t, & \text{for } x \in \overline{\Omega} \text{ and } t \in [0,1], \\ \Phi(x,t), & \text{for } x \in \overline{\Omega} \text{ and } t > 1. \end{cases}$$

We can see that $L^{\Phi}(\Omega) = L^{\Phi_1}(\Omega)$ and $W^{1,\Phi}(\Omega) = W^{1,\Phi_1}(\Omega)$ (see [81]). Thus, in the study of embeddings of $W^{1,\Phi}(\Omega)$, we may consider Φ_1 instead of Φ. For brevity, we write Φ instead of Φ_1. In other words, we may assume that Φ satisfies the following condition which is not essential because Ω is bounded.

$$\Phi(x,t) = \Phi(x,1)t, \quad \text{for } x \in \overline{\Omega} \text{ and } t \in [0,1).$$

Now, let setting $T(x) = \lim_{s\to+\infty} \Phi_*^{-1}(x,s)$, for all $x \in \overline{\Omega}$. Then, $T(x) \in (0,+\infty]$.

Definition 2.8. *The function $\Phi_* : \overline{\Omega} \times [0,+\infty) \to [0,+\infty)$ defined by*

$$\Phi_*(x,t) = \begin{cases} s, & \text{if } x \in \overline{\Omega}, \ t \in [0,T(x)) \text{ and } \Phi_*^{-1}(x,s) = t \\ +\infty, & \text{if } x \in \overline{\Omega}, \ t \geq T(x), \end{cases}$$

is called the Sobolev conjugate function of Φ.

It is clear that $\Phi_* \in N(\Omega)$, and for each $x \in \overline{\Omega}$, $\Phi_*(x,\cdot) \in \mathcal{C}^1((0,T(x)))$. Let X be a metric space and $f : X \to (-\infty,+\infty]$ be an extended real-valued function. For $x \in X$ with $f(x) \in \mathbb{R}$, the continuity of f at x is well defined. Now, for $x \in X$ with $f(x) = +\infty$, we say that f is continuous at x if given any $M > 0$, there exists a neighborhood U of x such that $f(y) > M$ for all $y \in U$. We say that $f : X \to (-\infty,+\infty]$ is continuous on X if f is continuous at every $x \in X$. Define $Dom(f) = \{x \in X : f(x) \in \mathbb{R}\}$ and denote by $\mathcal{C}^{1-0}(X)$ the set of all locally Lipschitz continuous real-valued functions defined on X.

Remark 2.3.7. *Suppose that Φ satisfies (H_2). Then, for each $t_0 \geq 0$, $\tilde{\Phi}(x,t_0)$ and $\Phi_*(x,t_0)$ are bounded.*

Concerning the function Φ_* and the operator T, we suppose that

(H_4) $T : \overline{\Omega} \to [0,+\infty]$ is continuous on $\overline{\Omega}$ and $T \in \mathcal{C}^{1-0}(Dom(T))$;

(H_5) $\Phi_* \in \mathcal{C}^{1-0}(Dom(\Phi_*))$ and there exist positive constants $C_0, \delta_0 < \frac{1}{N}$ and $t_0 \in (0, \min_{x\in\overline{\Omega}} T(x))$ such that

$$|\nabla_x \Phi_*(x,t)| \leq C_0(\Phi_*(x,t))^{1+\delta_0},$$

for all $x \in \Omega$ and $t \in [t_0, T(x))$ provided that $\nabla_x \Phi_*(x,t)$ exists.

The following proposition gives a sufficient condition of (H_5) described in terms of Φ (see Proposition 3.1 in [41]).

Proposition 2.3.8. *Let Φ satisfy (H_1), (H_2) and (H_3), and let $\Phi \in \mathcal{C}^{1-0}(\overline{\Omega} \times [0,+\infty))$. Then,*

1. $\Phi^{-1} \in \mathcal{C}^{1-0}(\overline{\Omega} \times [0,+\infty))$, $\Phi_*^{-1} \in \mathcal{C}^{1-0}(\overline{\Omega} \times [0,+\infty))$ *and* $\Phi_* \in \mathcal{C}^{1-0}(Dom(\Phi_*))$.

2. *If Φ satisfies the (Δ_2)-condition and there exist positive constants $\epsilon_0 < \frac{1}{N}$, C and t_1 such that*

$$|\nabla_x \Phi(x,t)| \leq C_0(\Phi(x,t))^{1+\epsilon_0} \tag{2.15}$$

for all $x \in \Omega$ and $t \geq t_1$ provided that $\nabla_x \Phi(x,t)$ exists, then Φ satisfies (H_5).

Example 2.3.9. *Let us give some examples of generalized N-function Φ satisfying the assumptions (H_1)-(H_5) above.*

1. *Define $\Phi(x,t) = t^{p(x)}$, where the variable exponent $p : \overline{\Omega} \to (1, +\infty)$ is a continuous function with $1 < p^- \leq p(x) \leq p^+ < N$ (see notations in Chapter 1). It is clear that Φ satisfies the assumptions (H_1)-(H_3). In addition, if we assume that $p \in \mathcal{C}^{1-0}(\overline{\Omega})$, then the assumptions (H_4) and (H_5) hold. Indeed, firstly, $p \in \mathcal{C}^{1-0}(\overline{\Omega})$ implies $\Phi \in \mathcal{C}^{1-0}(\overline{\Omega} \times [0, +\infty))$. Secondly, by a simple computation, we get*

$$\Phi_*^{-1}(x,s) = \frac{Np(x)}{N - p(x)} s^{\frac{N-p(x)}{Np(x)}}.$$

Thus, $T(x) = +\infty$. It is clear that (H_4) is satisfied. Now, for $x \in \Omega$ and $t > 1$, we have

$$\frac{\partial \Phi(x,t)}{\partial x_j} = \frac{\partial p(x)}{\partial x_j} t^{p(x)} \ln(t), \quad \forall j \in \{1, \cdots, N\}.$$

Hence, there exist constants $t_0 > 0, \epsilon_0 > 0$ such that

$$\left| \frac{\partial \Phi(x,t)}{\partial x_j} \right| \leq C(\Phi(x,t))^{1+\epsilon_0}, \quad \forall j \in \{1, \cdots, N\}, \tag{2.16}$$

for any $x \in \Omega$ and $t \in [t_0, +\infty)$. Since Φ satisfies (Δ_2)-condition, from Proposition 2.3.8, then (H_5) is satisfied.

2. *Define $\Phi(x,t) = t^p + a(x)t^q$, where $a : \overline{\Omega} \mapsto [0, +\infty)$ is Lipschitz continuous, $1 < p < q < N$, $\frac{q}{p} < 1 + \frac{1}{N}$. It is clear that Φ satisfies (H_1)-(H_4). Now, it suffices to prove condition (2) of Proposition 2.3.8. If we put $\epsilon_0 := \frac{q}{p} - 1$ and $c_a > 0$ the Lipschitz constant of the function a, we get*

$$\left| \frac{\partial \Phi(x,t)}{\partial x_j} \right| \leq c_a t^q \leq c_a (t^p + a(x)t^q)^{\frac{q}{p}}, \quad \forall j \in \{1, \cdots, N\},$$

for any $x \in \Omega$ and $t > 0$. Hence, (2.15) holds with $C := c_a, \epsilon_0 := \frac{q}{p} - 1 < \frac{1}{N}$ and any $t_0 > 0$.

Definition 2.9. *Let $\Phi, \Psi \in N(\Omega)$. We say that Φ essentially grows more slowly than Ψ and we write $\Phi \ll \Psi$, if for any $k > 0$,*

$$\lim_{t \to +\infty} \frac{\Phi(x,kt)}{\Psi(x,t)} = 0 \text{ uniformly for } x \in \Omega.$$

Remark 2.3.10.

1. *Obviously, if $\Phi \ll \Psi$ then $\Phi \preccurlyeq \Psi$.*

2. *$\Phi \ll \Psi$, if and only if,*

$$\lim_{t \to +\infty} \frac{\Psi^{-1}(x,t)}{\Phi^{-1}(x,t)} = 0 \text{ uniformly for } x \in \Omega. \tag{2.17}$$

Proposition 2.3.11. *We have $\Phi \ll \Phi_*$.*

Proof. Since $\frac{\partial \Phi(x,t)}{\partial t}$ exists for all $x \in \Omega$ and $t \geq 0$, then integrating by parts, we get

$$\Phi_*^{-1}(x,s) = \int_0^s \frac{\Phi^{-1}(x,\tau)}{\tau^{\frac{N+1}{N}}} d\tau = -Ns^{-\frac{1}{N}} \Phi^{-1}(x,s) + \int_0^s N\tau^{-\frac{1}{N}} \frac{\partial \Phi^{-1}(x,\tau)}{\partial \tau} d\tau.$$

Then, dividing by $\Phi^{-1}(x,s)$ we obtain

$$\Phi_*^{-1}(x,s) = -Ns^{-\frac{1}{N}} + \frac{\int_0^s N\tau^{-\frac{1}{N}} \frac{\partial \Phi^{-1}(x,\tau)}{\partial \tau} d\tau}{\Phi^{-1}(x,s)}.$$

It is clear that $-Ns^{-\frac{1}{N}} \to 0$ as $s \to +\infty$. Using L'Hospital's rule (see e.g. [41, Proposition 2.2]), we get,

$$\lim_{s \to +\infty} \frac{\int_0^s N\tau^{-\frac{1}{N}} \frac{\partial \Phi^{-1}(x,\tau)}{\partial \tau} d\tau}{\Phi^{-1}(x,s)} = \lim_{s \to +\infty} \frac{Ns^{-\frac{1}{N}} \frac{\partial \Phi^{-1}(x,s)}{\partial s}}{\frac{\partial \Phi^{-1}(x,s)}{\partial s}}.$$

This shows that (2.17) holds. □

Now, we give the embedding theorems in Musielak-Sobolev spaces setting (see [41, 73]).

Theorem 2.3.12. *Assume (H_1)-(H_5) hold. Then*

1. *There is a continuous embedding $W^{1,\Phi}(\Omega) \hookrightarrow L^{\Phi_*}(\Omega)$.*
2. *Suppose that $\Psi \in N(\Omega)$, $\Psi : \overline{\Omega} \times [0,+\infty) \to [0,+\infty)$ is continuous, and $\Psi(x,t) \in (0,+\infty)$ for $x \in \Omega$ and $t \in (0,+\infty)$. If $\Psi \ll \Phi_*$. Then, the embedding $W^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L^{\Psi}(\Omega)$ is compact.*

From Remark 2.3.10, Proposition 2.3.11 and Theorem 2.3.12, we have

Theorem 2.3.13. *Assume (H_1)-(H_5) hold. Then*

1. *The embedding $W^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L^{\Phi}(\Omega)$ is compact .*
2. *The Poincaré type inequality*
$$\|u\|_\Phi \leq C\|\nabla u\|_\Phi \quad \text{for } u \in W_0^{1,\Phi}(\Omega),$$
holds.

We finish this section by giving the following lemma

Lemma 2.3.14. *Let $u \in L^{\Phi_*}(\Omega)$ and $\rho, t \geq 0$. Then, we have*

$$\min\left\{\rho^{(\phi_0)^*}, \rho^{(\phi^0)^*}\right\} \Phi_*(x,t) \leq \Phi_*(x,\rho t) \leq \max\left\{\rho^{(\phi_0)^*}, \rho^{(\phi^0)^*}\right\} \Phi_*(x,t), \tag{2.18}$$

$$\min\left\{\|u\|_{\Phi_*}^{(\phi_0)^*}, \|u\|_{\Phi_*}^{(\phi^0)^*}\right\} \leq \int_\Omega \Phi_*(x,|u(x)|)dx \leq \max\left\{\|u\|_{\Phi_*}^{(\phi_0)^*}, \|u\|_{\Phi_*}^{(\phi^0)^*}\right\}, \tag{2.19}$$

where $(\phi_0)^* = \frac{N\phi_0}{N-\phi_0}$ and $(\phi^0)^* = \frac{N\phi^0}{N-\phi^0}$.

Proof. We denote by $\theta_0(\rho) = \min\left\{\rho^{\phi_0}, \rho^{\phi^0}\right\}$ and $\theta_1(\rho) = \max\left\{\rho^{\phi_0}, \rho^{\phi^0}\right\}$. Putting $s = \Phi(x,t), \sigma = \theta_0(\rho)$ in (2.3), we get

$$\theta_1^{-1}(\sigma)\Phi^{-1}(x,s) \leq \Phi^{-1}(x,\sigma s) \leq \theta_0^{-1}(\sigma)\Phi^{-1}(x,s), \quad \text{for } \sigma, s > 0. \tag{2.20}$$

Using the definition of Φ_*^{-1} in (2.14), and by simple computation we obtain

$$\theta_1^{-1}(\sigma)\sigma^{-\frac{1}{N}}\Phi_*^{-1}(x,s) \leq \Phi_*^{-1}(x,\sigma s) \leq \theta_0^{-1}(\sigma)\sigma^{-\frac{1}{N}}\Phi_*^{-1}(x,s), \tag{2.21}$$

which implies (2.18). As the proof of Lemma 2.2.1-(2.4) we obtain (2.19). □

Part I.

Existence and multiplicity of solutions for some quasilinear elliptic problems

CHAPTER 3

MULTIPLICITY OF SOLUTIONS FOR A CLASS OF ELLIPTIC PROBLEMS OF *P*-LAPLACIAN TYPE WITH A *P*-GRADIENT TERM

We consider the following problem

$$(P)\begin{cases} -\Delta_p u = c(x)|u|^{q-1}u + \mu|\nabla u|^p + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary, $1 < p < N$, $q > 0$, $\mu \in \mathbb{R}^*$, and c and h belong to $L^k(\Omega)$ for some $k > \frac{N}{p}$. In this chapter, we assume that $c \gneqq 0$ a.e. in Ω and h without sign condition, then we prove the existence of at least two bounded solutions under the condition that $\|c\|_k$ and $\|h\|_k$ are suitably small. For this purpose, we use the Mountain Pass theorem, on an equivalent problem to (P) with variational structure. Here, the main difficulty is that the nonlinearity term considered does not satisfy Ambrosetti and Rabinowitz condition. The key idea is to replace the former condition by the **nonquadraticity condition at infinity**.

3.1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary $\partial\Omega$. In this chapter, we are concerned with the following elliptic problem

$$(P)\begin{cases} -\Delta_p u = c(x)|u|^{q-1}u + \mu|\nabla u|^p + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $1 < p < N$, $q > 0$, $\mu \in \mathbb{R}^*$, and c and h belong to $L^k(\Omega)$ for some $k > \frac{N}{p}$.

In the literature, there are many results concerning the existence, the uniqueness, and the multiplicity of solutions for models like (P) under various assumptions on c and h. At first, it is important to mention that the sign of c plays a crucial role in the problem (P) regarding uniqueness, as well as existence, of bounded solutions. In this setting, we refer to [61] for more details. In the coercive case, that is $c(x) \leq -\alpha_0$ a.e. in Ω for some $\alpha_0 > 0$, Boccardo, Murat and Puel [21, 23, 22], proved the existence of bounded solutions for more general divergence form problems with quadratic growth in the gradient by using the sub and supersolution method. Moreover, Barles and Murat [15] and Barles et at. [14] have treated the uniqueness question for similar problems. Notice that, if we allow $c(x) \leq 0$ a.e. in Ω, then Ferone and Murat [45],[46] observed that finding solutions to (P) becomes rather complex without imposing some strong regularity conditions on the data. For the particular case $c \equiv 0$, there had been many contributions [1, 75, 86]. However, for $c \leq 0$ that may vanish only on some parts of Ω, the uniqueness of solutions was left open until the recent paper authored by Arcoya et at. [12]. This last result was proved for $p = 2$, $q = 1$, and under the following condition

$$\begin{cases} c, h \text{ belong to } L^k(\Omega) \text{ for some } k > \frac{N}{2}, \mu \in L^\infty(\Omega) \text{ and } meas(\Omega \backslash Supp\ c) > 0, \\ \inf\limits_{u \in W_c, \|u\|_{H_0^1(\Omega)}} \int_\Omega \left(|\nabla u|^2 - \|\mu^+\|_{L^\infty(\Omega)} h^+(x) u^2 \right) > 0, \\ \inf\limits_{u \in W_c, \|u\|_{H_0^1(\Omega)}} \int_\Omega \left(|\nabla u|^2 - \|\mu^-\|_{L^\infty(\Omega)} h^-(x) u^2 \right) > 0. \end{cases}$$

where $W_c := \{w \in H_0^1(\Omega) : \ c(x)w(x) = 0, \text{ a.e. in } \Omega\}$. For a related uniqueness result see also Arcoya et at. [11].

The case where $c(x) \gneqq 0$ a.e. in Ω, the question of non-uniqueness has been being an open problem given by Sirakov [95] and it has received considerable attention by many authors. Moreover, it should be pointed out that the sign of h and whether μ is a function or a constant, generate additional difficulties for solving (P). In this setting, Jeanjean and Sirakov [61] showed the existence of two bounded solutions assuming that $\mu \in \mathbb{R}^*$, c and h are in $L^k(\Omega)$ for some $k > \frac{N}{2}$ and satisfying

$$\|[\mu h]^+\|_{L^{\frac{N}{2}}(\Omega)} < C_N,$$

$$\max\{\|c\|_{L^k(\Omega)}, \|[\mu h]^-\|_{L^k(\Omega)}\} < \bar{c},$$

where $\bar{c} > 0$ depends only on $N, k, meas(\Omega), |\mu|, \|[\mu h]^+\|_{L^k(\Omega)}$, and C_N is the optimal constant in Sobolev's inequality. In this last result, h is allowed to change sign. Shortly after, this result was extended by Coster and Jeanjean [29] for a bounded function μ such that $\mu(x) \geq \mu_1 > 0$ by using the degree topological method.

Finally, in the case where c is allowed to change sign and with $c(x) \gneqq 0$ a.e. in Ω, Jenajean and Quoirin [62, Theorem 1.1] showed the existence of two bounded positive solutions when $h \gneqq 0$, μ is a positive constant, and c^+ and μh are suitably small.

We would also like to mention that all the above quoted multiplicity results were restricted to the Laplacian operator with quadratic growth in the gradient, i.e. $p = 2$, and for $q = 1$. Moreover, it is interesting to mention that even when c is allowed to change sign the solutions are positive.

3.2. Main Theorem

In this work, we prove the multiplicity of bounded solutions for the problem (P) by assuming the following assumption

$$(H) \begin{cases} c, h \text{ belongs to } L^k(\Omega) \text{ for some } k > \frac{N}{p}, h \text{ is allowed to change sign,} \\ c \gneqq 0 \text{ a.e. in } \Omega, q > 0, \text{ and } \mu \in \mathbb{R}^*. \end{cases}$$

In this section, we give a brief exposition of the proof of our multiplicity result and we state the main result in this chapter. At first, without loss of generality, we solve the problem (P) by restricting it to the case where μ is a positive constant. For μ is a negative constant, we replace u by $-u$ in (P), then we conclude. Next, we observe that the problems of type (P) do not have a variational formulation due to the presence of the p-gradient term. To overcome this difficulty, we perform the Kazdan-Kramer change of variable, that is, $v = (e^{\frac{\mu u}{p-1}} - 1)/\mu$. Thus, we obtain the following equivalent problem (P')

$$(P') \begin{cases} -\Delta_p v = c(x)g(v) + h(x)f(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where
$$g(s)=\frac{(p-1)^{q-p+1}}{\mu^q}(1+\mu s)^{p-1}|\ln(1+\mu s)|^{q-1}\ln(1+\mu s),\ \text{ with } s>\frac{-1}{\mu}, \tag{3.1}$$
and
$$f(s)=\frac{(1+\mu s)^{p-1}}{(p-1)^{p-1}}. \tag{3.2}$$

Definition 3.1. *We mean by bounded weak solutions of (P'), the functions $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ satisfying*
$$\int_\Omega |\nabla v|^{p-2}\nabla v\nabla u=\int_\Omega c(x)g(v)u+\int_\Omega h(x)f(v)u,$$
for any $u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Remark 3.2.1. *Obviously, if $v>\frac{-1}{\mu}$ is a solution of (P'), then $u=\frac{p-1}{\mu}\ln(1+\mu v)$ is a solution of (P). Hence, the solutions obtained here are not necessarily positive (compare with result in [62]).*

One of the most fruitful ways to deal with (P') is the variational method, which takes into account that the weak solutions of (P') are critical points in $W_0^{1,p}(\Omega)$ of the $\mathcal{C}^1$-energy functional
$$\mathcal{I}(v)=\frac{1}{p}\int_\Omega|\nabla v|^p-\int_\Omega c(x)G(v)-\int_\Omega h(x)F(v), \tag{3.3}$$
with $G(s)=\int_0^s g(t)dt$ and $F(s)=\int_0^s f(t)dt$.

In this work, to obtain the two critical points for $\mathcal{I}$, we use the Mountain Pass Theorem to show one critical point and the standard lower semicontinuity argument to show the other. For the first one, according to the famous paper by Ambrosetti and Rabinowitz [8], the most important step is to show that $\mathcal{I}$ satisfies the Palais-Smale condition at the level $\tilde{c}$ (see Definition 3.3). The fulfillment of this condition relies on the well-known Ambrosetti-Rabinowitz condition ((A-R) for short), namely
$$\text{there exist } \theta>p \text{ and } s_0>0 \text{ such that } 0<\theta G(s)\le sg(s),\ \text{ as } |s|>s_0.$$

Unfortunately, this condition is somewhat restrictive and not being satisfied by many nonlinearities g. However, many researches have been made to drop the (A-R). We refer, for instance, to [28, 101, 78, 50, 60]. Notice that, the nonlinearity g considered here does not satisfy (A-R). Moreover, since we do not assume any sign condition on h, the fulfillment of the Palais-Smale condition turns out more delicate (see e.g. [51, 62]). To the best of our knowledge, only Jenajean and Quoirin ([62]), recently, proved the Palais-Smale condition under the assumptions c changes sign, h is positive, and without assuming (A-R). In their proof, for $p=2$ and $q=1$, the authors based one of the arguments on the positivity of h and the explicit determination of a function H;
$$H(s)=g(s)s-2G(s).$$
In our situation, as h is allowed to change sign and the analog of their function H can not be computed explicitly, due to our general consideration of p and q ($1<p<N$ and $q>0$), hence, their arguments can not be adapted.

The key point to show the Palais-Smale condition in this work is to prove that g, among other conditions, satisfies the following condition (see Lemma 3.4.3),
$$(NQ)\qquad H(s)=g(s)s-pG(s)\to+\infty,\ \text{ where } s\to+\infty.$$

The condition (NQ) is a variant of the well known **nonquadraticity condition at infinity**, which was introduced by Costa and Malgalhães [28], and is given as follows
$$(CM)\quad \text{there exist } a>0 \text{ and } \nu\ge\nu_0>0 \text{ such that } \liminf_{|s|\to\infty}\frac{H(s)}{|s|^\nu}\ge a.$$

Observe that, since $\nu > 0$, then (NQ) is weaker than (CM). Moreover, it should be noted that (NQ) was considered by Furtado and Silva in their recent paper [50]. Our result follows by using similar arguments.

Concerning the existence of the second critical point handled by the standard lower semicontinuity argument, we look for a local minimum in $W_0^{1,p}(\Omega)$ for the functional $\mathcal{I}$. Indeed, we observe that $\mathcal{I}$ takes positive values in a large sphere, due to its geometrical structure (see Proposition 3.4.1), and $\mathcal{I}(0) = 0$.

Now we state the main result of this work

Theorem 3.2.2 (Main Theorem)**.** *Assume that (H) is satisfied. If $\|c\|_k$ and $\|h\|_k$ are suitably small, then the problem (P) has at least two bounded weak solutions.*

Notation

Through this paper, we use the following notations.

1) The Lebesgue norm $(\int_\Omega |u|^p)^{\frac{1}{p}}$ in $L^p(\Omega)$ is denoted by $\|.\|_p$ for $p \in [1, +\infty[$. The norm in $L^\infty(\Omega)$ is denoted by $\|u\|_{L^\infty(\Omega)} := ess\sup_{x\in\Omega} |u(x)|$. The Hölder conjugate of p is denoted by p'.

2) The spaces $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ are equipped with Poincaré norm $\|u\| := (\int_\Omega |\nabla u|^p)^{\frac{1}{p}}$ and the dual norm $\|\cdot\|_* := \|\cdot\|_{W^{-1,p'}(\Omega)}$ respectively.

3) We denote by $B(0, R)$ the ball of radius R centered at 0 in $W_0^{1,p}(\Omega)$ and $\partial B(0, R)$ its boundary.

4) We denote by $C_i, c_i > 0$ any positive constants that are not essential in the arguments and that may vary from one line to another.

3.3. Preliminary results

In this section, we establish some preliminary results concerning the nonlinearity g. These results will be used frequently later. We start by the following lemma without proof.

Lemma 3.3.1.

1. $\frac{g(s)}{|s|^{p-2}s} \to c$ *as* $s \to 0$, *where* $c = 0$ *if* $q > p-1$ *and* $c = 1$ *if* $q = p-1$.

2. $\frac{g(s)}{|s|^{q-1}s} \to (p-1)^{q-p+1}$ *as* $s \to 0$, *for all* $q > 0$.

3. $\frac{g(s)}{s^{p-1}} \to +\infty$ *and* $\frac{G(s)}{s^p} \to +\infty$ *as* $s \to +\infty$, *for all* $q > 0$.

Lemma 3.3.2.

1. *If* $q \geq p-1$, *then we have*
$$|g(s)| \leq c_0|s|^r + c_1|s|^{p-1},$$
for all $s > -\frac{1}{\mu}$, *and for all* $r \in (p-1, p)$.

2. *If* $0 < q < p-1$, *then we have*
$$|g(s)| \leq c_1|s|^r + c_2|s|^q,$$
for all $s > -\frac{1}{\mu}$, *and for all* $r \in (p-1, p)$.

Proof. By using Lemma 3.3.1-1, there exists $\eta > 0$ such that for all $|s| < \eta$ we have

$$|g(s)| \leq c_1 |s|^{p-1}.$$

Let $\delta \in (0, 1)$. If $s \geq \eta$, then we have

$$g(s) \leq c_2(\eta, \mu, \delta) s^{p-1+\delta}. \tag{3.4}$$

Moreover, simple calculation yield

$$g'(s) = \frac{(p-1)^{q-p+1}}{\mu^{q-1}} (1 + \mu s)^{p-2} |\ln(1 + \mu s)|^{q-1} [(p-1)\ln(1 + \mu s) + q].$$

Now, if $-\frac{1}{\mu} < s \leq -\eta$, then we have $|g(s)| \leq |g(T)|$, where $T = (e^{\frac{-q}{p-1}} - 1)/\mu$. Hence,

$$|g(s)| \leq c_3(\eta, \mu, \delta) |s|^{p-1+\delta}. \tag{3.5}$$

By combining (3.4) and (3.5), the first item (1) holds. To prove the second item (2), we use Lemma 3.3.1 (2) and the same previous argument. □

3.4. Proof of Main Theorem

Recall from introduction that the proof of our Main Theorem is divided into three steps as follows. In the first step, we show the existence of the first critical point for the C^1-energy functional $\mathcal{I}$ by using the Mountain Pass Theorem due to Ambrosetti-Rabinowitz [8]. Precisely, we shall show that the energy functional $\mathcal{I}$ has a geometrical structure and then it satisfies the Palais-Smale condition at the level $\tilde{c}$. In the second step, we show the existence of the second critical point of $\mathcal{I}$ on $B(0, \rho)$ (which is a local minimum) by using the lower semicontinuity argument. Moreover, we are going to see that these critical points are not the same. Finally, we show that any solution of problem (P) is bounded.

3.4.1. First critical point: Mountain Pass Theorem

The following result shows that $\mathcal{I}$ has a geometrical structure.

Proposition 3.4.1. *Assume that (H) holds. If $\|c\|_k$ and $\|h\|_k$ are suitably small, then the functional $\mathcal{I}$ has a geometrical structure, that is, $\mathcal{I}$ satisfies the following properties*

i) there exists $\rho > 0$ such that for all v in $\partial B(0, \rho)$, $\mathcal{I}(v) \geq \beta$, where $\beta > 0$.

ii) there exists $v_0 \in W_0^{1,p}(\Omega)$ such that $\|v_0\| > \rho$ and $\mathcal{I}(v_0) \leq 0$.

Proof. i) To prove the first property we distinguish two cases on q . First case, if $0 < q < p - 1$, then by using Lemma 3.3.2 (2) and Hölder's inequality, we get

$$\int_\Omega c(x) G(v) \leq c_1 \|c\|_k \|v^{r+1}\|_{k'} + c_2 \|c\|_k \|v^{q+1}\|_{k'}.$$

We choose $r > p - 1$ with r close to $p - 1$ such that $(r + 1)k' < \frac{pN}{N-p}$, which exists due to the assumption $k > \frac{N}{p}$. Obviously, $(q + 1)k' < \frac{pN}{N-p}$. Thus, by using Sobolev's embedding we get

$$\int_\Omega c(x) G(v) \leq C_1 \|c\|_k \|v\|^{r+1} + C_2 \|c\|_k \|v\|^{q+1}.$$

Moreover, from the definition of the function f in (3.2), we have

$$|f(v)| \leq c(1 + |v|^{p-1}), \text{ for some } c > 0. \tag{3.6}$$

Using Sobolev's embedding, we get

$$\int_\Omega h(x)F(v) \leq C_3\|h\|_k + C_4\|h\|_k\|v\|^p.$$

By the definition of $\mathcal{I}$ in (3.3), we deduce that

$$\mathcal{I}(v) \geq \frac{1}{p}\|v\|^p - C_1\|c\|_k\|v\|^{r+1} - C_2\|c\|_k\|v\|^{q+1} - C_3\|h\|_k - C_4\|h\|_k\|v\|^p.$$

Now, let v in $\partial B(0,\rho)$. Then, we have

$$\mathcal{I}(v) \geq \frac{1}{p}\rho^p - \|c\|_k(C_1\rho^{r+1} + C_2\rho^{q+1}) - \|h\|_k(C_3 + C_4\rho^p).$$

We take ρ sufficiently large, and such that $\|c\|_k \leq \rho^{-r-2+p}$ and $\|h\|_k \leq \rho^{-1}$ (which are sufficiently small by hypothesis), then

$$\mathcal{I}(v) \geq \frac{1}{p}\rho^p - C\rho^{p-1} \geq \rho^{p-1}\left(\frac{1}{p}\rho - C\right) = \beta_1.$$

Second case, that is, if $q \geq p-1$, we choose again r as above such that $pk' < (r+1)k' < \frac{pN}{N-p}$. Then, by using Lemma 3.3.2 (1) and Sobolev's embedding, we get

$$\int_\Omega c(x)G(v) \leq c_1\|c\|_k\|v\|^{r+1} + c_2\|c\|_k\|v\|^p.$$

Now, as the first case, we get

$$\mathcal{I}(v) \geq \frac{1}{p}\rho^p - C'\rho^{p-1} \geq \rho^{p-1}\left(\frac{1}{p}\rho - C'\right) = \beta_2.$$

Finally, we summarize the two cases then get

$$\mathcal{I}(v) \geq \beta, \quad \text{where} \quad \beta = \min(\beta_1, \beta_2).$$

ii) To prove the second property, we show that $\mathcal{I}(tv) \to -\infty$ as $t \to +\infty$. For this, let $v \in C_0^\infty(\Omega)$ be a positive function such that $cv \gneqq 0$ a.e. on Ω. By the definition of $\mathcal{I}$ in (3.3), we have

$$\begin{aligned}\mathcal{I}(tv) &= \frac{t^p}{p}\int_\Omega |\nabla v|^p - \int_\Omega c(x)G(tv) - \int_\Omega h(x)F(tv) \\ &= t^p\left(\frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega c(x)\frac{G(tv)}{t^p v^p}v^p - \int_\Omega h(x)\frac{F(tv)}{t^p v^p}v^p\right).\end{aligned}$$

From inequality (3.6), we get

$$\int_\Omega |h(x)\frac{F(tv)}{t^p v^p}v^p| \leq c, \quad \text{as} \quad t \to +\infty.$$

Moreover, by Lemma 3.3.1 (3), we get

$$\int_\Omega c(x)\frac{G(tv)}{t^p v^p}v^p \to +\infty \quad \text{as} \quad t \to +\infty.$$

Thus, we deduce the desired result. □

Now, we recall the standard definitions of Palais-Smale sequence at the level $\tilde{c}$ and Palais-Smale condition at the level $\tilde{c}$ for $\mathcal{I}$, and we prove that the energy functional $\mathcal{I}$ defined in (3.3) has a geometrical structure.

Let us define the level at $\tilde{c}$ as follows

$$\tilde{c} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{I}(\gamma(t)),$$

where $\Gamma = \{\gamma \in \mathcal{C}([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0,\ \gamma(1) = v_0\}$ is the set of continuous paths joining 0 and v_0, where $v_0 \in W_0^{1,p}(\Omega)$ is defined in Proposition 3.4.1.

Definition 3.2. *Let (u_n) is a sequence in $W_0^{1,p}(\Omega)$. We say that (u_n) is a Palais-Smale sequence at the level $\tilde{c}$ for $\mathcal{I}$ if*

$$\mathcal{I}(u_n) \to \tilde{c}, \quad \text{and} \quad \|\mathcal{I}'(u_n)\|_{(W_0^{1,p}(\Omega))^*} \to 0.$$

Definition 3.3. *We say that $\mathcal{I}$ satisfies the Palais-Smale condition at the level $\tilde{c}$ if any Palais-Smale sequence at the level $\tilde{c}$ for $\mathcal{I}$ possesses a convergent subsequence.*

Remark 3.4.2. *We note that, since $\mathcal{I}$ has a geometrical structure, then the existence of a Palais-Smale sequence at the level $\tilde{c}$ for our energy functional $\mathcal{I}$ is ensured. This can be observed directly from the proof given in [8].*

Now, we prove that $\mathcal{I}$ satisfies the Palais-Smale condition at the level $\tilde{c}$. Precisely, we show that any Palais-Smale sequence at the level $\tilde{c}$ for $\mathcal{I}$ is bounded in $W_0^{1,p}(\Omega)$, and then, it has a strongly convergent subsequence.

They key point to prove the boundedness of the Palais-Smale sequence at the level $\tilde{c}$ in $W_0^{1,p}(\Omega)$, is to show that g verifies the nonquadraticity condition at infinity (NQ). Indeed, we have the following lemma

Lemma 3.4.3. *The function g defined in (3.1) verifies the nonquadraticity condition at infinity (NQ);*

$$(NQ) \qquad H(s) = g(s)s - pG(s) \to +\infty, \ \text{where} \ s \to +\infty.$$

Proof. To prove (NQ), we show that H is increasing and unbounded for s sufficiently large. We recall that $H(s) = g(s)s - pG(s)$. Then, by simple calculations, for s sufficiently large we get

$$H'(s) = C\mu s(1+\mu s)^{p-2}(\ln(1+\mu s))^{q-1}[(1-p)\frac{\ln(1+\mu s)}{\mu s} + q],$$

where $C = (p-1)^{q-p+1}/\mu^q$. Thus, H is increasing for s large enough. Moreover, H is unbounded. Indeed, by contradiction, if H is bounded, then there exists a positive constant M such that

$$H(s) \leq M, \quad \text{for s large enough.}$$

In addition, from the definition of H and using integration by parts on G, we get

$$H(s) = -C\frac{1}{\mu}(\ln(1+\mu s))^q(1+\mu s)^{p-1} + qC\int_0^s (1+\mu t)^{p-1}(\ln(1+\mu t))^{q-1}dt.$$

By choosing $\delta \in (p-1, p)$, we obtain

$$\frac{H(s)}{s^\delta} = -\frac{1}{\mu}\frac{(\ln(1+\mu s))^q(1+\mu s)^{p-1}}{s^\delta} + qC\frac{\int_0^s(1+\mu t)^{p-1}(\ln(1+\mu t))^{q-1}}{s^\delta} \leq \frac{M}{s^\delta}.$$

When $s \to +\infty$, we obtain $\frac{H(s)}{s^\delta} \to +\infty$ and $\frac{M}{s^\delta} \to 0$. Hence, we have a contradiction. As a conclusion, the function g verifies (NQ). □

Lemma 3.4.4. *Let (u_n) be a Palais-Smale sequence at the level $\tilde{c}$ for $\mathcal{I}$ in $W_0^{1,p}(\Omega)$. Then, (u_n) is bounded in $W_0^{1,p}(\Omega)$.*

Proof. Let (u_n) be a Palais-Smale sequence at the level $\tilde{c}$ for $\mathcal{I}$ in $W_0^{1,p}(\Omega)$. We prove by contradiction that (u_n) is bounded in $W_0^{1,p}(\Omega)$. We assume that (u_n) is unbounded in $W_0^{1,p}(\Omega)$, that is, $\|u_n\| \to +\infty$.

For all integer $n \geq 0$, we define

$$\mathcal{I}(z_n) := \max_{0\leq t\leq 1} \mathcal{I}(tu_n), \quad \text{where} \quad z_n = t_n u_n \ \text{and} \ t_n \in [0,1].$$

We are going to prove that $\mathcal{I}(z_n) \to +\infty$, and also $(\mathcal{I}(z_n))$ is bounded, which is the desired contradiction.

a) Showing that $\mathcal{I}(z_n) \to +\infty$: We set $v_n := \frac{u_n}{\|u_n\|}$, then (v_n) is bounded in $W_0^{1,p}(\Omega)$. Hence, there exists a subsequence denoted again (v_n) such that v_n converges weakly and strongly to v in $W_0^{1,p}(\Omega)$ and in $L^s(\Omega)$ for some $1 \le s < p^*$ respectively. Moreover, v_n also converges to v almost everywhere in Ω. Recall that $p^* := \frac{Np}{N-p}$, is Sobolev conjugate.

Now, we claim by contradiction that $v \equiv 0$ a.e. in Ω.

Since (u_n) is Palais-Smale type sequence, then we have

$$\mathcal{I}(u_n) \to \tilde{c} \quad \text{and} \quad \|\mathcal{I}'(u_n)\|_* \to 0. \tag{3.7}$$

Hence,

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla \varphi - \int_\Omega c(x)g(u_n)\varphi - \int_\Omega h(x)f(u_n)\varphi = \epsilon_n, \tag{3.8}$$

for all $\varphi \in W_0^{1,p}(\Omega)$ and for some $\epsilon_n \to 0$ as $n \to +\infty$. We divide both sides of (3.8) by $\|u_n\|^{p-1}$, to obtain

$$\int_\Omega c(x)\frac{g(u_n)}{\|u_n\|^{p-1}}\varphi = \frac{\epsilon_n}{\|u_n\|^{p-1}} + \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla \varphi + \int_\Omega h(x)\frac{f(u_n)}{\|u_n\|^{p-1}}\varphi. \tag{3.9}$$

On the one hand, since v_n converges weakly to v in $W_0^{1,p}(\Omega)$ and by the inequality (3.6), then for n large enough the second and the third terms of the right-hand side of (3.9) are bounded.

On the other hand, if $v \not\equiv 0$ in Ω, then $cv \not\equiv 0$ in Ω. Now, we choose $\varphi \in W_0^{1,p}(\Omega)$ such that $cv\varphi > 0$ in Ω_φ and $cv\varphi \equiv 0$ in $\Omega\backslash\Omega_\varphi$, with $|\Omega_\varphi| > 0$. Since $v_n\|u_n\| = u_n$ in Ω, then by using Lemma 3.3.1 (3), we obtain

$$\liminf c(x)\frac{g(u_n)}{\|u_n\|^{p-1}}\varphi = \liminf c(x)(v_n)^{p-1}\frac{g(v_n\|u_n\|)}{(v_n\|u_n\|)^{p-1}}\varphi = +\infty \quad \text{in} \quad \Omega_\varphi.$$

Hence, by using the Fatou's lemma in (3.9) we obtain the unbounded term in the left-hand side of (3.9). Hence, the claim, i.e. $v \equiv 0$ a.e. in Ω.

Since $\|u_n\| \to +\infty$, then there exists $M > 0$ such that $\|u_n\| > M$, for n large enough. Moreover, we have

$$\mathcal{I}(z_n) \ge \mathcal{I}\left(M\frac{u_n}{\|u_n\|}\right) = \mathcal{I}(Mv_n) = \frac{M^p}{p} - \int_\Omega c(x)G(Mv_n) - \int_\Omega h(x)F(Mv_n).$$

In what follows, we treat only the case $0 < q < p-1$. The other case follows with similar arguments. From Lemma 3.3.2 (2), we have $|G(s)| \le c_1|s|^{r+1} + c_2|s|^{q+1}$, where $p-1 < r < p$. Since $c \in L^k(\Omega)$, for some $k > \frac{N}{p}$ and v_n converges strongly to v in $L^s(\Omega)$ with $1 \le s < p^*$, then, we obtain

$$\int_\Omega c(x)G(Mv_n) \to 0 \;\text{ as } n \to +\infty,$$

due to $v \equiv 0$ a.e. in Ω. By Hölder's inequality, we get

$$\int_\Omega h(x)F(Mv_n) \le C \;\text{ as }\; n \to +\infty.$$

Hence, by choosing $M > 0$ large enough, we deduce that $\mathcal{I}(z_n) \to +\infty$, as $n \to +\infty$.

b) Showing that $\mathcal{I}(z_n)$ is bounded: To prove that $(\mathcal{I}(z_n))$ is bounded, we distinguish two cases: $t_n \le \frac{2}{\|u_n\|}$ and $t_n > \frac{2}{\|u_n\|}$.

The case $t_n \le \frac{2}{\|u_n\|}$:

Here, we only handle the proof for $q \in (0, p-1)$. The other case follows as in the proof of Proposition 3.4.1-i). By the definition of (z_n) and $\mathcal{I} \in \mathcal{C}^1(W_0^{1,p}(\Omega), \mathbb{R})$, we have $\langle \mathcal{I}'(t_nu_n), t_nu_n\rangle = 0$, which means that

$$t_n^p\|u_n\|^p = \int_\Omega c(x)g(t_nu_n)t_nu_n + \int_\Omega h(x)f(t_nu_n)t_nu_n.$$

By the definition of $\mathcal{I}$ in (3.3), we have

$$\begin{aligned} p\mathcal{I}(t_n u_n) &= t_n^p \|u_n\|^p - p\int_\Omega c(x)G(t_n u_n) - p\int_\Omega h(x)F(t_n u_n) \\ &= \int_\Omega c(x)H(t_n u_n) + \int_\Omega h(x)K(t_n u_n), \end{aligned} \tag{3.10}$$

where the function H is defined in (NQ) and $K(s) := f(s)s - pF(s)$. Moreover, from Lemma 3.3.2-2), we have

$$\begin{aligned} \int_\Omega c(x)H(t_n u_n) &\le \int_\Omega |c(x)||g(t_n u_n)t_n u_n| + p\int_\Omega |c(x)||G(t_n u_n)| \\ &\le c_1 \int_\Omega |c(x)||t_n u_n|^{r+1} + c_2 \int_\Omega |c(x)||t_n u_n|^{q+1}. \end{aligned}$$

By choosing r and q as in the proof of Proposition 3.4.1-i), we get

$$\int_\Omega c(x)H(t_n u_n) \le C_1 \|c\|_k \|t_n u_n\|^{r+1} + C_2 \|c\|_k \|t_n u_n\|^{q+1}. \tag{3.11}$$

By inequality (3.6) and Sobolev's embedding, we get

$$\begin{aligned} \int_\Omega h(x)K(t_n u_n) &\le \int_\Omega |h(x)||f(t_n u_n)t_n u_n| + p\int_\Omega |h(x)||(F(t_n u_n)t_n u_n| \\ &\le c_1\|h\|_k + c_2\|h\|_k\|t_n u_n\| + c_3\|h\|_k\|t_n u_n\|^p. \end{aligned} \tag{3.12}$$

Then, by (3.10), (3.11), and (3.12), we obtain

$$\mathcal{I}(t_n u_n) \le C,$$

for all $n \ge 0$, where C is independent of n. Thus, $(\mathcal{I}(z_n))$ is bounded, which contradicts the fact that $(\mathcal{I}(z_n))$ is unbounded (see **a)**).

The case $t_n > \frac{2}{\|u_n\|}$:
Here, we are proceeding the technique inspired by [50]. To this end, we need the following technical lemma

Lemma 3.4.5. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ the nonnegative function defined as*

$$\Phi_\epsilon(s) = \begin{cases} e^{-\epsilon/s^2}, & \text{if } \; s \neq 0, \\ 0, & \text{if } \; s = 0, \end{cases}$$

with $\epsilon > 0$. Then, we have

i) $\lim_{s\to 0} \Phi_\epsilon(s) = \lim_{s\to 0} \Phi'_\epsilon(s) = 0.$

ii) for any positive function $z \in L^k(\Omega)$ for some $k > \frac{N}{p}$ we have

$$\lim_{\epsilon\to 0} \int_\Omega \int_s^t \frac{z(x)}{\tau^{p+1}} \left(\frac{1 - \Phi_\epsilon(|\tau u_n|)}{\|u_n\|^p} \right) d\tau dx = 0, \;\; \textit{uniformly in } n \in \mathbb{N}.$$

Proof. Obviously we have i). To prove ii), we follow the same approach given in [50] for the case $p = 2$ and $z(x) = 1$, which can be immediately generalized for any positive function $z \in L^k(\Omega)$ for some $k > \frac{N}{p}$ and $p > 1$. □

Now, we resume the proof of Lemma 3.4.4. From Lemma 3.4.3, we have $H(s) \geq \sigma$, for s large enough and some $\sigma > 0$ (which will be chosen later). Moreover, if $0 < q < p-1$, then from Lemma 3.3.1 (2), we have for s sufficiently small,

$$H(s) \geq -C_1|s|^{q+1}.$$

Then, by the continuity of H, we have for all $s > -\frac{1}{\mu}$,

$$H(s) \geq \sigma\Phi_\epsilon(s) - C_2|s|^{q+1}. \tag{3.13}$$

Let $0 < s < t$, then we have

$$\begin{aligned}
\frac{\mathcal{I}(tu_n)}{t^p\|u_n\|^p} - \frac{\mathcal{I}(su_n)}{s^p\|u_n\|^p} &= -\int_\Omega c(x)\left[\frac{G(tu_n)}{t^p\|u_n\|^p} - \frac{G(su_n)}{s^p\|u_n\|^p}\right] \\
&\quad - \int_\Omega h(x)\left[\frac{F(tu_n)}{t^p\|u_n\|^p} - \frac{F(su_n)}{s^p\|u_n\|^p}\right] \\
&= \underbrace{-\int_\Omega c(x)\int_s^t \frac{d}{d\tau}\left(\frac{G(\tau u_n)}{\tau^p\|u_n\|^p}\right)d\tau dx}_{A} \\
&\quad + \underbrace{\int_\Omega -h(x)\left[\frac{F(tu_n)}{t^p\|u_n\|^p} - \frac{F(su_n)}{s^p\|u_n\|^p}\right]}_{B}.
\end{aligned} \tag{3.14}$$

Let us handle the two terms A and B respectively.

$$\begin{aligned}
A &= -\int_\Omega\int_s^t c(x)\frac{\tau^p u_n g(\tau u_n) - p\tau^{p-1}G(\tau u_n)}{\tau^{2p}\|u_n\|^p}d\tau dx \\
&= -\int_\Omega\int_s^t \frac{c(x)}{\|u_n\|^p}\frac{H(\tau u_n)}{\tau^{p+1}}d\tau dx.
\end{aligned}$$

By using (3.13), we get

$$\begin{aligned}
A &\leq \int_\Omega\int_s^t \frac{c(x)}{\|u_n\|^p}\left(C_2\frac{|u_n|^{q+1}}{\tau^{p-q}} - \sigma\frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}}\right)d\tau dx \\
&\leq \int_\Omega \frac{c(x)}{\|u_n\|^p}\left(\frac{C_2}{p-q-1}\frac{|u_n|^{q+1}}{s^{p-q-1}} - \sigma\int_s^t \frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}}d\tau\right)dx.
\end{aligned} \tag{3.15}$$

For the term B, we have

$$\begin{aligned}
B &\leq C\left(\int_\Omega |h(x)|\frac{(1+|tu_n|)^p}{t^p\|u_n\|^p} + \int_\Omega |h(x)|\frac{(1+|su_n|)^p}{s^p\|u_n\|^p}\right) \\
&\leq C\left(\int_\Omega |h(x)|\left(\frac{1}{t_n\|u_n\|} + \frac{|u_n|}{\|u_n\|}\right)^p + \int_\Omega |h(x)|\left(\frac{1}{s_n\|u_n\|} + \frac{|u_n|}{\|u_n\|}\right)^p\right).
\end{aligned} \tag{3.16}$$

By setting $s := \frac{1}{\|u_n\|}$, we obtain

$$\begin{aligned}
\frac{\mathcal{I}(tu_n)}{t^p\|u_n\|} &\leqslant \mathcal{I}(v_n) + \int_\Omega c(x)\left(\frac{C_2}{p-q-1}|v_n|^{q+1} - \sigma\int_s^t \frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}\|u_n\|^p}\right)d\tau dx \\
&+ C\left(\int_\Omega |h(x)|(\frac{1}{2}+|v_n|)^p + \int_{\Omega^+}|h(x)|(1+|v_n|)^p\right) \\
&\leqslant \mathcal{I}(v_n) + C\left[\int_\Omega c(x)|v_n|^{q+1} + \int_\Omega |h(x)| + 2\int_\Omega |h(x)||v_n|^p\right] \\
&- \sigma\int_\Omega \frac{c(x)}{p}\left(1-\frac{1}{t_n^p\|u_n\|^p}\right) + \sigma\int_\Omega \frac{c(x)}{p}\left(1-\frac{1}{t_n^p\|u_n\|^p}\right) \\
&\quad - \sigma\int_\Omega\int_s^t c(x)\frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}\|u_n\|^p}d\tau dx \\
&\leqslant \mathcal{I}(v_n) + C\left[\int_\Omega c(x)|v_n|^{q+1} + \int_\Omega |h(x)| + 2\int_\Omega |h(x)||v_n|^p\right] \\
&- \sigma\int_\Omega \frac{c(x)}{p}\left(1-\frac{1}{t_n^p\|u_n\|^p}\right) - \sigma\int_\Omega\int_s^t \frac{c(x)}{\tau^{p+1}}\left(\frac{1-\Phi_\epsilon(|\tau u_n|)}{\|u_n\|^p}\right)d\tau dx.
\end{aligned}$$

By the technical Lemma 3.4.5, we have

$$\lim_{\epsilon\to 0}\int_\Omega\int_s^t \frac{c(x)}{\tau^{p+1}}\left(\frac{1-\Phi_\epsilon(|\tau u_n|)}{\|u_n\|^p}\right)d\tau dx = 0, \quad \text{uniformly in } n \in \mathbb{N}.$$

Then,

$$\begin{aligned}
\frac{\mathcal{I}(tu_n)}{t^p\|u_n\|^p} \leqslant \frac{1}{p} - \int_\Omega c(x)G(v_n) - \int_\Omega h(x)F(v_n) + C\Bigg[\int_\Omega c(x)|v_n|^{q+1} \\
+ \int_\Omega |h(x)| + 2\int_\Omega |h(x)||v_n|^p\Bigg] - \sigma\int_\Omega \frac{c(x)}{p}\left(1-\frac{1}{2^p}\right).
\end{aligned}$$

We choose σ such that

$$\sigma > \frac{2^p(1+pC\|h\|_k)}{(2^p-1)\int_\Omega c(x)}.$$

which gives,

$$\frac{1}{p} + C\|h\|_k - \sigma\int_\Omega \frac{c(x)}{p}\left(1-\frac{1}{2^p}\right)dx < 0.$$

Since v_n converges to 0 almost everywhere in Ω, weakly in $W_0^{1,p}(\Omega)$, and strongly in $L^s(\Omega)$ for some $1 \leq s < p^*$, then, we have

$$\mathcal{I}(t_n u_n) < 0, \text{ in } \Omega \text{ for } n \text{ large enough.}$$

Hence, $(\mathcal{I}(z_n))$ is bounded. Therefore, this contradicts the fact that $(\mathcal{I}(z_n))$ is unbounded (see **a)**).

Now, If $q \geq p-1$, then from Lemma 3.3.1 (1) and the continuity of $H(s)$, we have for all $s > -\frac{1}{\mu}$,

$$H(s) \geq \sigma\Phi_\epsilon(s) - C_1|s|^{p-1}. \tag{3.17}$$

Following the computations as in (3.14), we find exactly the same terms A and B. The term B is handled as in (3.16), whereas A is handled as follows

$$\begin{aligned}
A &\leq \int_\Omega\int_s^t \frac{c(x)}{\|u_n\|^p}\left(C_1\frac{|u_n|^{p-1}}{\tau^2} - \sigma\frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}}\right), \\
&\leq \int_\Omega\int_s^t \frac{c(x)}{\|u_n\|^p}\left(C_1\frac{|u_n|^{p-1}}{s} - \sigma\frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}}\right).
\end{aligned}$$

Moreover, since $(p-1)k' < pk' < \frac{Np}{N-p}$, then by using Sobolev embedding, the rest of the proof is similar to the case $q \in (0, p-1)$. Hence, we have also the contradiction with the fact that $\mathcal{I}$ is unbounded (see **a)**). □

To finish the proof of the Palais-Smale condition for $\mathcal{I}$, we only need to show the following lemma

Lemma 3.4.6. *Any Palais-Smale sequence at the level $\tilde{c}$ of $W_0^{1,p}(\Omega)$ has a strongly convergent subsequence.*

Proof. Let (u_n) be a Palais-Smale sequence at the level $\tilde{c}$, then $\mathcal{I}'(u_n) \to 0$ in $W^{-1,p'}(\Omega)$, which means that

$$-\Delta_p u_n - c(x)g(u_n) - h(x)f(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega).$$

By Lemma 3.4.4, (u_n) is bounded in $W_0^{1,p}(\Omega)$. Hence, u_n converges weakly to u in $W_0^{1,p}(\Omega)$ and strongly in $L^s(\Omega)$ for some $1 \le s < p^*$. Therefore,

$$-\Delta_p u_n \to c(x)g(u) + h(x)f(u) \text{ in } W^{-1,p'}(\Omega). \tag{3.18}$$

We know that the operator $-\Delta_p : W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)$ is a homeomorphism ([35]). Hence, from (3.18) we get

$$u_n \to (-\Delta_p)^{-1}(c(x)g(u) + h(x)f(u)) \text{ in } W_0^{1,p}(\Omega).$$

Therefore, by the uniqueness of the limit we have

$$u_n \to u, \text{ in } W_0^{1,p}(\Omega).$$

□

3.4.2. Second critical point: lower semicontinuity argument

In this part, we use the geometrical structure of $\mathcal{I}$ (see Proposition 3.4.1) and the standard lower semicontinuity argument, we show the existence of the second critical point. We state the result as follows

Theorem 3.4.7. *Assume that $\|c\|_k$ and $\|h\|_k$ are suitably small to ensure Proposition 3.4.1. Then, the energy functional $\mathcal{I}$ possesses a critical point $v \in B(0,\rho)$ with $\mathcal{I}(v) \le 0$.*

Proof. Since $\mathcal{I}(0) = 0$, then $\inf_{v\in B(0,\rho)} \mathcal{I}(v) \le 0$. Moreover, if $h \not\equiv 0$, then we obtain that $\inf_{v\in B(0,\rho)} \mathcal{I}(v) < 0$. Indeed, we choose $v \in C_0^\infty(\Omega)$ a positive function that satisfies $cv > 0$ and $hv > 0$. From the definition of $\mathcal{I}$ in (3.3), we have for $t > 0$

$$\mathcal{I}(tv) = t^p \left(\frac{1}{p} \int_\Omega |\nabla v|^p - \int_\Omega c(x) \frac{G(tv)}{t^p v^p} v^p - \int_\Omega h(x) \frac{F(tv)}{t^p v^p} v^p \right). \tag{3.19}$$

If $q \ge p-1$, then from Lemma 3.3.1 (2), we have $G(s)/s^p \to c < +\infty$ as $s \to 0^+$. If $0 < q < p-1$, obviously, we have $G(s)/s^p \to +\infty$ as $s \to 0^+$. In addition, in both cases, we have $\frac{F(s)}{s^p} \to +\infty$ as $s \to 0^+$. Hence, by using these limits, we get from (3.19) that $\mathcal{I}(tv) < 0$ for $t > 0$ small enough.

Now, we set $m := \inf_{v\in B(0,\rho)} \mathcal{I}(v)$. Then, by Proposition 3.4.1 i), we have $\mathcal{I}(v) \ge \beta > 0$ for $\|v\| = \rho$. Moreover, there exists a sequence $(v_n) \subset B(0,\rho)$ such that $\mathcal{I}(v_n)$ converges to m. Since (v_n) is bounded in $W_0^{1,p}(\Omega)$, then there exists a subsequence denoted again (v_n) such that v_n converges to v weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^s(\Omega)$ for some $1 \le s < p^*$ respectively. Hence, we get

$$\int_\Omega h(x)F(v_n) \to \int_\Omega h(x)F(v) \text{ and } \int_\Omega c(x)G(v_n) \to \int_\Omega c(x)G(v) \quad \text{as} \quad n \to +\infty.$$

In addition, since $\|v\|^p \le \liminf_{n\to\infty} \|v_n\|^p$, then $\mathcal{I}(v) \le m = \inf_{v\in B(0,\rho)} \mathcal{I}(v)$. Hence, we conclude that v is a local minimum of $\mathcal{I}$ in $B(0,\rho)$.

□

Remark 3.4.8. *By the subsection 3.4.1, $\mathcal{I}$ has a critical point at the level $\tilde{c}$, that is, there exists w in $W_0^{1,p}(\Omega)$ such that $\mathcal{I}(w) = \tilde{c}$ and $\mathcal{I}'(w) = 0$. Since $\mathcal{I}(w) = \tilde{c} > 0 \ge \mathcal{I}(v)$, where $v \in B(0,\rho)$ is the second critical point given in Theorem 3.4.7, then w is different from v. Hence, we have two distinct weak solutions for the problem (P).*

3.4.3. Boundedness of solutions

Now, to finish the proof of our main result, it remains to show the boundedness of the solutions. Therefore, we show the following result

Proposition 3.4.9. *Any solution u of the problem (P') belongs to $L^\infty(\Omega)$.*

Proof. If $|u| \leq 1$, it is over. Otherwise, we begin by writing the problem (P') as follows

$$-\Delta_p u = a(x)(1 + |u|^{p-1}),$$

where

$$a(x) = \frac{c(x)g(u) + h(x)f(u)}{1 + |u|^{p-1}}.$$

Then, by Theorem 2.4 in [87], we can deduce the boundedness of u if we show that a belongs to $L^{\frac{p}{N(1-\epsilon)}}(\Omega)$, for some $\epsilon \in]0, 1[$. Indeed, from (3.6) and Lemma 3.3.2, we obtain

$$|a(x)| \leq C\left[|c(x)|(|u|^{r-p+1} + 1) + |h(x)|\right]. \tag{3.20}$$

Let $m > 1$ and m' it's conjugate. By using Hölder's inequality in (3.20), we obtain

$$\int_\Omega |a(x)|^{\frac{p}{N(1-\epsilon)}} \leq C\left[\|c(x)^{\frac{p}{N(1-\epsilon)}}\|_m \|u^{(r-p+1)\frac{p}{N(1-\epsilon)}}\|_{m'} + \|h^{\frac{p}{N(1-\epsilon)}}\|_m + 1\right].$$

By choosing $0 < \epsilon < 1 - \frac{(N-p)(r-p+1)}{N^2} - \frac{p}{kN}$, we have

$$\frac{p}{N(1-\epsilon)}m \leq k \quad \text{and} \quad (r-p+1)\frac{p}{N(1-\epsilon)}m' < \frac{Np}{N-p}.$$

Hence, the terms $\|c(x)^{\frac{p}{N(1-\epsilon)}}\|_m$, $\|h(x)^{\frac{p}{N(1-\epsilon)}}\|_m$, and $\|u^{(r-p+1)\frac{p}{N(1-\epsilon)}}\|_{m'}$ are finite (recall that $c, h \in L^k(\Omega)$ for some $k > \frac{N}{p}$). $\square$

CHAPTER 4

QUASILINEAR ELLIPTIC PROBLEM WITHOUT AMBROSETTI-RABINOWITZ CONDITION INVOLVING A POTENTIAL IN MUSIELAK-SOBOLEV SPACES SETTING

In this chapter, we consider the following quasilinear elliptic problem with potential

$$(P)\begin{cases}-\mathrm{div}(\phi(x,|\nabla u|)\nabla u)+V(x)|u|^{q(x)-2}u=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N\geq 2)$, V is a given function in a generalized Lebesgue space $L^{s(x)}(\Omega)$, and $f(x,u)$ is a Carathéodory function satisfying suitable growth conditions. Using variational arguments, we study the existence of weak solutions for (P) in the framework of Musielak-Sobolev spaces. The main difficulty here is that the nonlinearity $f(x,u)$ considered does not satisfy the well-known Ambrosetti-Rabinowitz condition.

4.1. Introduction

Let $\Omega\subset\mathbb{R}^N(N\geq 2)$ be a bounded smooth domain. Assume that $\phi:\Omega\times[0,+\infty)\to[0,+\infty)$ is a Carathéodory function such that for all $x\in\Omega$, we have

$$(\phi)\begin{cases}\phi(x,0)=0, \quad \phi(x,t).t \ \text{ is strictly increasing},\\ \phi(x,t).t>0,\ \forall t>0 \ \text{ and } \ \phi(x,t).t\to+\infty \text{ as } t\to+\infty.\end{cases}$$

In this chapter, we study the following quasilinear elliptic problem

$$(P)\begin{cases}-\mathrm{div}(\phi(x,|\nabla u|)\nabla u)+V(x)|u|^{q(x)-2}u=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where V is a potential belonging to $L^{s(x)}(\Omega)$, q and $s:\bar{\Omega}\to(1,\infty)$ are continuous functions and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function which satisfies some suitable growth conditions. Precise conditions concerning the functions q, s, f and V will be given hereafter.

Problem (P) appears in many branches of mathematical physics and has been studied extensively in recent years. From an application point of view, this problem has its backgrounds in such hot topics as image processing, nonlinear electrorheological fluids and elastic mechanics. We refer the readers to [25, 92] and the references therein for more background of applications. In particular, when $\phi(x,t)=$

$t^{p(x)-2}$, where p is a continuous function on $\overline{\Omega}$ with the condition $\min_{x\in\bar{\Omega}} p(x) > 1$, the operator involved in (P) is the $p(x)$-Laplacian operator, i.e. $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. This differential operator is a natural generalization of the p-Laplacian operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ where $p > 1$ is a real constant. Note that the $p(x)$-Laplacian operator possesses more complicated nonlinearities than the p-Laplacian operator (for example, it is nonhomogeneous), so more complicated analysis has to be carefully carried out.

The interest in analyzing this kind of problems is also motivated by some recent advances in the study of problems involving nonhomogeneous operators in divergence form. We refer for instance to the results in [2, 24, 26, 42, 52, 91, 70, 99]. The studies for $p(x)$-Laplacian problems have been extensively considered by many researchers in various ways (see e.g. [2, 43, 64, 70]). It should be noted that our problem (P) enables the presence of many other operators such as double-phase and variable exponent double-phase operators.

Before moving forward, we give a review of some results related to our work. We start by the case where the potential $V \equiv 0$ on Ω. Fan and Zhang in [43], proved the existence of a nontrivial solution and obtained infinitely many solutions for a Dirichlet problem involving the $p(x)$-Laplacian operator. Clément, García-Huidobro and Schmitt in [27], established the existence of a nontrivial solution for more general quasilinear equation in the framework of Orlicz-Sobolev spaces, in the case where the function ϕ considered in (P) is independent of x, i.e. $\phi(x,t) = \phi(t)$. Liu and Zhao in [73], obtained the existence of a nontrivial solution and infinitely many solutions for a quasilinear equation related to problem (P) in the framework of Musielak-Sobolev spaces (see also [42]).

In the above mentioned papers, the authors assumed, among other conditions, that the nonlinearity f satisfy to the well-known Ambrosetti-Rabinowitz condition ((A-R) condition for short); which , for the p-Laplacian operator, asserts that there exist two constants $M > 0$ and $\theta > p$, such that

$$0 < \theta F(x,t) \leq f(x,t)t, \quad \forall |t| \geq M,$$

where $F(x,t) = \int_0^t f(x,s)ds$. Clearly, this condition implies the existence of two positive constants c_1, c_2 such that

$$F(x,t) \geq c_1|t|^\theta - c_2, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{4.1}$$

This means that f is p-superlinear at infinity in the sense that

$$\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^p} = +\infty. \tag{4.2}$$

This type of condition was introduced by Ambrosetti and Rabinowitz in their famous paper [8] and has since become one of the main tools for finding solutions to elliptic problems of variational type; especially in order to prove the boundedness of Palais-Smale sequence of the energy functional associated with such a problem. Unfortunately, there are several nonlinearities which are p-superlinear but do not satisfy the (A-R) condition. For instance, if we take $f(x,t) = |t|^{p-2}t\ln(1+|t|)$, then we can check that for any $\theta > p$, $F(x,t)/|t|^\theta \to 0$ as $|t| \to +\infty$. However, many recent types of research have been made to drop the (A-R) condition (see e.g. [24, 26, 52, 70] and references therein).

In [24], the authors studied a similar problem as that in [27] and proved the existence of at least a nontrivial solution under the following assumptions on the nonlinearity f: there exist an N-function Γ (cf. [90]) and positive constants C, R such that

$$\Gamma\left(\frac{F(x,t)}{|t|^{\phi_0}}\right) \leq C\bar{F}(x,t), \quad \forall (x,|t|) \in \Omega \times [R,+\infty), \tag{4.3}$$

and

$$\lim_{|t|\to+\infty} \frac{f(x,t)}{|t|^{\phi^0-1}} = +\infty, \quad \lim_{|t|\to 0} \frac{f(x,t)}{|t|\phi(t)} = \lambda, \tag{4.4}$$

where $\bar{F}(x,t) := f(x,t)t - \phi^0 F(x,t)$, λ some nonnegative constant and ϕ_0, ϕ^0 are defined in relation (2.2) below (when $\phi(x,t) = \phi(t)$ independent of x) with specific assumptions. It should be noted that the condition (4.3) is a type of "nonquadraticity condition at infinity", which was first introduced by Costa and Magalhães in [28] for the Laplacian operator (with $\phi_0 = \phi^0 = 2$) as follows:

$$\liminf_{|t| \to +\infty} \frac{\bar{F}(x,t)}{|t|^{\sigma}} \geq a > 0,$$

holds for some $\sigma > 0$. We would also like to mention that this condition plays an important role in proving the boundedness of Palais-Smale sequences.

In [26] also, the authors considered a similar problem as that in [27] and proved the existence of a nontrivial solution under the following assumptions on the nonlinearity f: there exist $\mu_1, \mu_2 > 0$ such that

$$\lim_{|t| \to +\infty} \frac{F(x,t)}{|t|^{\phi^0}} = +\infty, \quad \lim_{|t| \to 0} \frac{f(x,t)}{|t|^{\phi^0 - 1}} = 0, \tag{4.5}$$

$$\bar{F}(x,t) \leq \bar{F}(x,s) + \mu_1, \quad \forall (x,t) \in \Omega \times (0,s) \text{ or } \forall (x,t) \in \Omega \times (s,0), \tag{4.6}$$

and

$$\bar{H}(ts) \leq \bar{H}(t) + \mu_2, \quad \forall t \geq 0 \text{ and } s \in [0,1], \tag{4.7}$$

where $\bar{H}(t) := \phi^0 \Phi(t) - \phi(t) t^2$ with $\Phi(t) = \int_0^t \phi(s) s ds$.

On the other hand, in the few last years, studies on double phase problems have attracted more and more interest and many results have been obtained. Especially, in [52] the authors proved the existence of a nontrivial solution and obtained infinitely many solutions for a double phase problem without (A-R) condition. More precisely, they considered the problem (P) (with $V \equiv 0$) with the function $\phi(x,t) = t^{p-2} + a(x) t^{q-2}$, where $a : \overline{\Omega} \mapsto [0, +\infty)$ is Lipschitz continuous, $1 < p < q < N$, $\frac{q}{p} < 1 + \frac{1}{N}$ and the nonlinearity f satisfies the assumptions (4.5) and (4.6) above with $\phi_0 = p$ and $\phi^0 = q$. In [53] however, the authors considered the same previous problem and proved the existence of infinitely many solutions; but instead of hypotheses (4.5) and (4.6) the nonlinearity f is supposed to satisfy the assumption (4.3) above where $\Gamma(t) = |t|^{\sigma}$ with $\sigma > \max\{1, \frac{N}{p}\}$, and $F(x,t) \geq 0$ for any $(x, |t|) \in \Omega \times [R, +\infty)$ is such that $\lim_{|t| \to +\infty} \frac{F(x,t)}{|t|^{\phi^0}} = +\infty$. In the same paper, the authors obtained also similar existence result under the following assumption instead of (4.3): there exist $\mu > q$ and $\theta > 0$ such that

$$\mu F(x,t) \leq t f(x,t) + \theta |t|^p, \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Recently, in [70], the authors studied the existence of a nontrivial solution and obtained infinitely many solutions for a Dirichlet problem involving $p(x)$-Laplacian operator under a new growth condition on the nonlinearity f, more precisely, they considered the following assumptions: there exist positive constants M and C such that

$$C \frac{f(x,t)t}{K(t)} \leq f(x,t)t - p(x) F(x,t), \quad \forall (x, |t|) \in \overline{\Omega} \times [M, +\infty), \tag{4.8}$$

and

$$\frac{f(x,t)t}{|t|^{p(x)} [K(t)]^{p(x)}} \to +\infty, \text{ uniformly as } |t| \to +\infty \text{ for } x \in \overline{\Omega}, \tag{4.9}$$

where $1 \leq K(\cdot) \in \mathcal{C}^1([0, +\infty), [1, +\infty))$ is increasing and $[\ln(e+t)]^2 \geq K(t) \to +\infty$ as $|t| \to +\infty$, which satisfies $tK'(t)/K(t) \leq \sigma_0 \in (0,1)$, where σ_0 is a constant. In addition, F satisfies

$$\frac{F(x,t)}{|t|^{p(x)} [\ln(e+t)]^{p(x)}} \to +\infty, \text{ uniformly as } |t| \to +\infty \text{ for } x \in \bar{\Omega}. \tag{4.10}$$

Now, we give some review results concerning the case where the potential $V \not\equiv 0$ on Ω. In [2], Abdou and Marcos, proved the existence of multiple solutions for a Dirichlet problem involving the $p(x)$-Laplacian operator with a changing sign potential V belonging to a generalized Lebesgue space $L^{s(x)}(\Omega)$

when the nonlinearity f satisfies some growth condition under (A-R) condition. In that work, the main assumptions on the variable exponents $q(\cdot), s(\cdot)$ and $p(\cdot)$ are such that: $q, s, p \in \mathcal{C}_+(\overline{\Omega})$ (see notation below) and satisfy $1 < q(x) < p(x) \leq N < s(x)$ for any $x \in \overline{\Omega}$.

Recently, in [91] the authors proved the existence of nontrivial non-negative and non-positive solutions, and obtained infinitely many solutions for the quasilinear equation $-\mathrm{div} A(x, \nabla u) + V(x)|u|^{\alpha(x)-2}u = f(x, u)$ in $\mathbb{R}^N$, where the divergence type operator has behaviors like $|\zeta|^{q(x)-2}$ for small $|\zeta|$ and like $|\zeta|^{p(x)-2}$ for large $|\zeta|$, where $1 < \alpha(\cdot) \leq p(\cdot) < q(\cdot) < N$. In that paper, it is supposed that the potential $V \in L^1_{loc}(\mathbb{R}^N)$ verifies $V(\cdot) \geq V_0 > 0$, $V(x) \to +\infty$ as $|x| \to +\infty$ and that the nonlinearity f satisfies some growth condition with the following assumption instead of (A-R) condition: there exist constants $M, C_1, C_2 > 0$ and a function a such that

$$C_1|t|^{q(x)}[\ln(e + |t|)]^{a(x)-1} \leq C_2 \frac{f(x,t)t}{\ln(e + |t|)} \leq f(x,t)t - s(x)F(x,t), \ \forall (x, |t|) \in \mathbb{R}^N \times [M, +\infty), \tag{4.11}$$

where $\mathrm{ess}_{x\in\mathbb{R}^N} \inf(a(x) - q(x)) > 0$, $q(\cdot) \leq s(\cdot)$ and $\mathrm{ess}_{x\in\mathbb{R}^N} \inf(p^*(x) - s(x)) > 0$ with $p^*(x) := \frac{Np(x)}{N-p(x)}$. Note that, the assumption (4.11) is stronger than the assumption (4.8). Related to this subject, we refer the readers to some important results concerning the study of the eigenvalue problems (see [18, 64, 65, 77] and the references therein).

A main motivation of our current study is that, to the best of our knowledge, there is little research considering both the potential $V \not\equiv 0$ and nonlinearity f without (A-R) condition for more general quasilinear equation in the framework of Musielak-Sobolev spaces. In this work, our main goal is to show the existence of weak solutions to the problem (P). Firstly, by using standard lower semicontinuity argument, we prove the existence of weak solutions under the condition that $V \in L^{s(x)}(\Omega)$ has changing sign, and the nonlinearity f satisfies the condition (f_0) below. Secondly, we establish the existence of at least a nontrivial solution and the existence of infinitely many solutions by using Mountain Pass Theorem and Fountain theorem respectively, where $V \in L^{s(x)}(\Omega)$ has constant sign and the nonlinearity f does not satisfy the (A-R) condition. For these purposes, we propose a set of growth conditions under which we are able to check the Palais-Smale condition. More precisely, we prove the boundedness of Palais-Smale sequences by using a similar condition to that in (4.3) above instead of (A-R) condition.

4.2. Energy functional and some technical results

In this section, we start by define the energy functional associated with problem (P). Then, based on Theorem 2.3.12, we establish compactness embedding results on the Musielak-Sobolev spaces setting. Finally, we give some results concerning the energy functional and some technical lemmas which will be used later.

We note that, to deal with our quasilinear elliptic problem (P), the Musielak-Sobolev spaces introduced in Chapter 2 are the adequate functional spaces corresponding to their solutions. Therefore, we need some techniques and results concerning these spaces. To this end, throughout this chapter, we shall say that the function Φ satisfies the assumption (Φ) if: ϕ satisfies the assumption (ϕ), Φ satisfies (2.2) and (H_1)-(H_5), both Φ and $\tilde{\Phi}$ are locally integrable and satisfy (ϕ_2) (for more details see Chapter 2). Hence, under the assumption (Φ), and from Chapter 2, the spaces $L^{\Phi}(\Omega)$, $W^{1,\Phi}(\Omega)$, $W_0^{1,\Phi}(\Omega)$ are separable reflexive Banach spaces. Moreover, we can apply the embedding theorems for Musielak-Sobolev spaces in Theorem 2.3.12 and Theorem 2.3.13.

Now, let us assuming the following assumption on the nonlinearity $f(x, u)$:

(f_0) There exists $\Psi \in N(\Omega)$ satisfying the assumption (2) of Theorem 2.3.12, and two positive constants ψ_0 and ψ^0 such that

$$1 < \psi_0 \leq \frac{\psi(x,t)t}{\Psi(x,t)} \leq \psi^0, \text{ for } x \in \Omega \text{ and } t > 0. \tag{4.12}$$

$$|f(x,t)| \leq C_1\psi(x, |t|) + h(x), \text{ for } (x,t) \in \Omega \times \mathbb{R}, \tag{4.13}$$

where C_1 is a positive constant, $0 \leq h \in L^{\tilde{\Psi}}(\Omega)$, and $\psi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and $\Psi(x,t) = \int_0^t \psi(x,s)ds$, for all $x \in \Omega$.

We can define the weak solution of the problem (P) as follows

Definition 4.1. *A function $u \in W_0^{1,\Phi}(\Omega)$ is said to be a weak solution of problem (P) if it holds that*

$$\int_\Omega \phi(x,|\nabla u|)\nabla u \nabla v dx + \int_\Omega V(x)|u|^{q(x)-2}uv dx = \int_\Omega f(x,u)v dx, \quad \forall v \in W_0^{1,\Phi}(\Omega).$$

We set $F(x,t) = \int_0^t f(x,s)ds$. The energy functional associated with problem (P) is given by

Definition 4.2. *The functional $\mathcal{I} : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ defined by the formula*

$$\mathcal{I}(u) = \mathcal{H}(u) + \mathcal{J}(u) - \mathcal{F}(u), \tag{4.14}$$

where,

$$\mathcal{H}(u) = \int_\Omega \Phi(x,|\nabla u|)dx, \quad \mathcal{J}(u) = \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx, \quad \text{and} \quad \mathcal{F}(u) = \int_\Omega F(x,u)dx,$$

is called the energy functional associated with problem (P).

Lemma 4.2.1. *Assuming that $s(x) > \frac{q(x)(\phi_0)^*}{(\phi_0)^*-q(x)}$ for every $x \in \overline{\Omega}$ and $\max\{\psi^0, q^+\} < (\phi_0)^*$. Then, we have the following compact embeddings*

$$W_0^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L^{s'(x)q(x)}(\Omega), \tag{4.15}$$

$$W_0^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L^{\alpha(x)}(\Omega), \tag{4.16}$$

with $\alpha(x) := \frac{s(x)q(x)}{s(x)-q(x)}$, and

$$W_0^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L^{\Psi}(\Omega). \tag{4.17}$$

Proof. Since $s(x) > \frac{q(x)(\phi_0)^*}{(\phi_0)^*-q(x)}$ for every $x \in \overline{\Omega}$, then it is clear that $s \in \mathcal{C}_+(\overline{\Omega})$ and $s(x) > q(x)$ for every $x \in \overline{\Omega}$. Furthermore, by a simple computation we have,

$$1 < s'(x)q(x) < (\phi_0)^* \quad \text{and} \quad 1 < \alpha(x) := \frac{s(x)q(x)}{s(x)-q(x)} < (\phi_0)^*, \quad \forall x \in \overline{\Omega}. \tag{4.18}$$

Thus,

$$\max_{x\in\overline{\Omega}} s'(x)q(x) := s'(x_0)q(x_0) < (\phi_0)^* \quad \text{and} \quad \max_{x\in\overline{\Omega}} \alpha(x) := \alpha(x_0) < (\phi_0)^*.$$

Using Lemma 2.3.14 and (H_5), we obtain

$$\lim_{t\to+\infty} \frac{|kt|^{s'(x)q(x)}}{\Phi_*(x,t)} \leq \frac{k^{s'(x_0)q(x_0)}}{\Phi_*(x,1)} \lim_{t\to+\infty} \frac{1}{t^{(\phi_0)^*-s'(x_0)q(x_0)}} = 0 \text{ uniformly for } x \in \Omega. \tag{4.19}$$

Using the same arguments above we show that

$$\lim_{t\to+\infty} \frac{|kt|^{\alpha(x)}}{\Phi_*(x,t)} = 0 \text{ uniformly for } x \in \Omega. \tag{4.20}$$

Hence, (4.19) and (4.2) imply that $|t|^{s'(x)q(x)} \ll \Phi_*$ and $|t|^{\alpha(x)} \ll \Phi_*$ respectively. Thus, we conclude from Theorem 2.3.12 that (4.15) and (4.16) hold. Finally, from the properties of Ψ, $\Psi(x,k)$ is bounded for any positive constant k. Using Lemma 2.2.1 and the fact that $\psi^0 < (\phi_0)^*$ we obtain for any $k > 0$

$$\lim_{t\to+\infty} \frac{\Psi(x,kt)}{\Phi_*(x,t)} \leq \frac{\Psi(x,k)}{\Phi_*(x,1)} \lim_{t\to+\infty} \frac{1}{t^{(\phi_0)^*-\psi^0}} = 0 \text{ uniformly for } x \in \Omega. \tag{4.21}$$

Hence, $\Psi \ll \Phi_*$, which implies by Theorem 2.3.12 that (4.17) holds. $\square$

Proposition 4.2.2. *Assuming that $s(x) > \frac{q(x)(\phi_0)^*}{(\phi_0)^*-q(x)}$ for every $x \in \overline{\Omega}$ and $\max\{\psi^0, q^+\} < (\phi_0)^*$. Then, the functional $\mathcal{I}$ is well defined and $\mathcal{I} \in \mathcal{C}^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ with the derivative given by*

$$\langle \mathcal{I}'(u), v\rangle = \int_\Omega \phi(x,|\nabla u|)\nabla u\nabla v dx + \int_\Omega V(x)|u|^{q(x)-2}uv dx - \int_\Omega f(x,u)v dx, \quad \forall u,v \in W_0^{1,\Phi}(\Omega).$$

Proof. Firstly, it is clear that $\mathcal{H}$ is well defined on $W_0^{1,\Phi}(\Omega)$. Furthermore, by similar arguments used in the proof of [76, Lemma 4.2], we have $\mathcal{H} \in \mathcal{C}^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ and its derivative is given by

$$\langle \mathcal{H}'(u), v\rangle = \int_\Omega \phi(x,|\nabla u|)\nabla u\nabla v dx, \quad \forall u,v \in W_0^{1,\Phi}(\Omega).$$

Secondly, the functional $\mathcal{J}$ is well defined. Indeed, by using the Hölder inequality, Proposition 1.3.8, and Lemma 4.2.1-(4.15), we have for all u in $W_0^{1,\Phi}(\Omega)$

$$\begin{aligned} |\mathcal{J}(u)| &\leq c_0\|V\|_{s(x)}\||u|^{q(x)}\|_{s'(x)} \leq c_1\|V\|_{s(x)}\max\{\|u\|^{q-}_{s'(x)q(x)}, \|u\|^{q+}_{s'(x)q(x)}\} \\ &\leq c_2\|V\|_{s(x)}\max\{\|u\|^{q-}_{1,\Phi}, \|u\|^{q+}_{1,\Phi}\}, \end{aligned} \tag{4.22}$$

where $c_i, i = 0,1,2$ are positive constants. Hence, $\mathcal{J}$ is well defined. Moreover, since $q^+ < (\phi_0)^*$ then, as in the proof of relation (4.19), the space $W_0^{1,\Phi}(\Omega)$ is compactly embedded in $L^{q(x)}(\Omega)$. From this, and using (4.16) and following the same arguments as in the proof of [64, Proposition 2], we obtain $\mathcal{J} \in \mathcal{C}^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ and

$$\langle \mathcal{J}'(u), v\rangle = \int_\Omega V(x)|u|^{q(x)-2}uv dx, \quad \forall u,v \in W_0^{1,\Phi}(\Omega).$$

Finally, using Lemma 4.2.1-(4.17) and (4.13), the functional $\mathcal{F}$ is well defined. Moreover, $\mathcal{F} \in \mathcal{C}^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ with its derivative given by

$$\langle \mathcal{F}'(u), v\rangle = \int_\Omega f(x,u)v dx, \quad \forall u,v \in W_0^{1,\Phi}(\Omega).$$

The proof of this proposition is now complete. □

Remark 4.2.3. *We observe that, in the proof of the previous proposition we need only the continuous embeddings of the space $W_0^{1,\Phi}(\Omega)$ into $L^{s'(x)q(x)}(\Omega), L^{\alpha(x)}(\Omega)$ and $L^{\Psi}(\Omega)$. The compact embeddings we will be used later.*

Remark 4.2.4. *We note that, by the previous proposition and Definition 4.1, u is a weak solution of problem (P) if and only if u is a critical point of the energy functional $\mathcal{I}$. Hence, we shall use critical point theory tools to show our main results in this chapter.*

Proposition 4.2.5.

i) The mapping $\mathcal{H}' : W_0^{1,\Phi}(\Omega) \to (W_0^{1,\Phi}(\Omega))^*$ defined by

$$\langle \mathcal{H}'(u), v\rangle = \int_\Omega \phi(x,|\nabla u|)\nabla u\nabla v dx, \quad \forall u,v \in W_0^{1,\Phi}(\Omega), \tag{4.23}$$

is bounded, coercive, strictly monotone homeomorphism, and is of type (S_+), namely,

$$u_n \rightharpoonup u \text{ in } W_0^{1,\Phi}(\Omega) \text{ and } \limsup_{n\to\infty}\langle \mathcal{H}'(u_n), u_n - u\rangle \leq 0 \text{ imply that } u_n \to u \text{ in } W_0^{1,\Phi}(\Omega),$$

where $\rightharpoonup$ and $\to$ denote the weak and strong convergence in $W_0^{1,\Phi}(\Omega)$, respectively.

ii) Assuming that $\psi^0 < (\phi_0)^*$. Then, the functional $\mathcal{F}$ is sequentially weakly continuous, namely, $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$ implies $\mathcal{F}(u_n) \to \mathcal{F}(u)$. In addition, the mapping $\mathcal{F}' : W_0^{1,\Phi}(\Omega) \to (W_0^{1,\Phi}(\Omega))^*$ defined by

$$\langle \mathcal{F}'(u), v\rangle = \int_\Omega f(x,u)v dx, \quad \forall u,v \in W_0^{1,\Phi}(\Omega), \tag{4.24}$$

is a completely continuous linear operator.

iii) Assuming that $s(x) > \frac{q(x)(\phi_0)^*}{(\phi_0)^* - q(x)}$ for every $x \in \overline{\Omega}$ and $q^+ < (\phi_0)^*$. Then, the functional $\mathcal{J}$ is sequentially weakly continuous. In addition, the mapping $\mathcal{J}' : W_0^{1,\Phi}(\Omega) \to (W_0^{1,\Phi}(\Omega))^*$ defined by

$$\langle \mathcal{J}'(u), v\rangle = \int_\Omega V(x)|u|^{q(x)-2}uvdx, \quad \forall u, v \in W_0^{1,\Phi}(\Omega), \tag{4.25}$$

is a completely continuous linear operator.

Proof. We refer the reader to [42, Theorem 2.2] for the proof of the first item and to [73, Lemma 4.1] for that of the second one. For the third item, let $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$. From Lemma 4.2.1-(4.15), $W_0^{1,\Phi}(\Omega)$ is compactly embedded in $L^{s'(x)q(x)}(\Omega)$, then $u_n \to u$ in $L^{s'(x)q(x)}(\Omega)$. This fact combined with relation (4.22) yields that $\mathcal{J}(u_n) \to \mathcal{J}(u)$. Now, it remains to show that $\langle \mathcal{J}'(u_n), u_n - u\rangle \to 0$, that is,

$$\int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n - u)dx \to 0. \tag{4.26}$$

From the assumptions, we have $1 < q(x) < (\phi_0)^*$ and $1 < \alpha(x) < (\phi_0)^*$ for every $x \in \overline{\Omega}$, where we recall that $\alpha(x) := \frac{s(x)q(x)}{s(x)-q(x)}$. Using Lemma 4.2.1-(4.16) and following the same proof of relation (4.16), the space $W_0^{1,\Phi}(\Omega)$ is compactly embedded in $L^{\alpha(x)}(\Omega)$ and in $L^{q(x)}(\Omega)$, respectively. Since (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$, then u_n converges strongly to u in $L^{\alpha(x)}(\Omega)$. Consequently, using Hölder's inequality and Proposition 1.3.8, then (4.26) holds by using the following inequality

$$\begin{aligned} \left|\int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n - u)dx\right| &\le C_0\|V\|_{s(x)}\||u_n|^{q(x)-1}\|_{q'(x)}\|u_n - u\|_{\alpha(x)} \\ &\le C_1\|V\|_{s(x)}\|u_n\|^{\tau}_{q(x)}\|u_n - u\|_{\alpha(x)}, \end{aligned} \tag{4.27}$$

where C_1 is a positive constant independent of n and $\tau \in \{q^- - 1, q^+ - 1\}$. □

We finish this section by the following lemma

Lemma 4.2.6. *Assume that $\phi^0 < q^- \le q^+ < (\phi_0)^*$, $q^+ - \frac{1}{2}\phi_0 < q^-$ and $s(x) > \frac{q(x)(\phi_0)^*}{(\phi_0)^* - q(x)}$ for every $x \in \overline{\Omega}$. Then, for any function $V \in L^{s(x)}(\Omega)$ we have*

$$\int_\Omega |V(x)||u|^{q(x)}dx \le C\|V\|_{s(x)}^{\frac{\alpha}{r^-}}\left[M_1 + M_2\left(\|u\|_{1,\Phi}^{2(q^+-\theta)} + \|u\|_{1,\Phi}^{2\frac{\theta r^+}{r^-}}\right)\right], \tag{4.28}$$

where $\alpha = r^+$ if $\|V\|_{s(x)} > 1$ and $\alpha = r^-$ if $\|V\|_{s(x)} \le 1$, and $2\frac{\theta r^+}{r^-} < 2(q^- - \theta) < 2(q^+ - \theta) < \phi_0$ for some measurable function r and positive constants M_1, M_2, C and θ.

Proof. Since we have $q^+ - \frac{1}{2}\phi_0 < q^-$, then there exists $\theta > 0$ such that $q^+ - \frac{1}{2}\phi_0 < \theta < q^-$. This fact implies that $2(q^- - \theta) < 2(q^+ - \theta) < \phi_0$ and $1 + \theta - q^+ > 0$. Let r be any measurable function satisfying,

$$\max\left\{\frac{s(x)}{1+\theta s(x)}, \frac{(\phi_0)^*}{(\phi_0)^* + \theta - q(x)}\right\} < r(x) < \min\left\{\frac{s(x)(\phi_0)^*}{(\phi_0)^* + \theta s(x)}, \frac{1}{1+\theta - q(x)}\right\}, \tag{4.29}$$

$$\theta\left(\frac{r^+}{r^-} + 1\right) < q^-, \qquad \forall x \in \Omega. \tag{4.30}$$

It is clear that $r \in L^\infty(\Omega)$ and $1 < r(x) < s(x)$. Now, by using Hölder's inequality, we get

$$\int_\Omega |V(x)||u|^{q(x)}dx \le C\|V|u|^\theta\|_{r(x)}\||u|^{q(x)-\theta}\|_{(r(x))'}. \tag{4.31}$$

Without loss of generality, we may assume that $\|V(x)|u|^{\theta}\|_{r(x)} > 1$. Using again Hölder's inequality, (1.3), and Proposition 1.3.8, we obtain

$$\begin{aligned}\|V|u|^{\theta}\|_{r(x)} &\leq \left[\int_{\Omega}|V(x)|^{r(x)}|u|^{\theta r(x)}dx\right]^{\frac{1}{r^-}}\\ &\leq C_1\||V|^{r(x)}\|^{\frac{1}{r^-}}_{\frac{s(x)}{r(x)}}\||u|^{\theta r(x)}\|^{\frac{1}{r^-}}_{(\frac{s(x)}{r(x)})'}\\ &\leq C_2\|V\|^{\frac{\alpha}{r^-}}_{s(x)}\left(1+\|u\|^{\frac{\theta r^+}{r^-}}_{\theta r(x)(\frac{s(x)}{r(x)})'}\right),\end{aligned} \tag{4.32}$$

where $\alpha = r^+$ if $\|V\|_{s(x)} > 1$ and $\alpha = r^-$ if $\|V\|_{s(x)} \leq 1$.
Using the same arguments as above, we obtain

$$\||u|^{q(x)-\theta}\|_{(r(x))'} \leq 1 + \|u\|^{q^+-\theta}_{(q(x)-\theta)(r(x))'}. \tag{4.33}$$

Since $r(x)$ is chosen such that (4.29) is fulfilled then

$$1 < \theta r(x)\left(\frac{s(x)}{r(x)}\right)' < (\phi_0)^* \quad \text{and} \quad 1 < (q(x)-\theta)(r(x))' < (\phi_0)^*, \forall x \in \Omega.$$

Since Φ_* satisfies (H_5), then by using Lemma 2.3.14, we have $|t|^{\theta r(x)\left(\frac{s(x)}{r(x)}\right)'} \preccurlyeq \Phi_*$ and $|t|^{(q(x)-\theta)(r(x))'} \preccurlyeq \Phi_*$, which imply that $L^{\Phi_*}(\Omega)$ is continuously embedded in $L^{\theta r(x)\left(\frac{s(x)}{r(x)}\right)'}(\Omega)$ and in $L^{(q(x)-\theta)(r(x))'}(\Omega)$. Therefore, from Theorem 2.3.12, $W_0^{1,\Phi}(\Omega)$ is continuously embedded in $L^{\theta r(x)\left(\frac{s(x)}{r(x)}\right)'}(\Omega)$ and in $L^{(q(x)-\theta)(r(x))'}(\Omega)$. Consequently, the relations (4.32) and (4.33) become respectively

$$\|V|u|^{\theta}\|_{r(x)} \leq C'\|V\|^{\frac{\alpha}{r^-}}_{s(x)}\left(1+\|u\|^{\frac{\theta r^+}{r^-}}_{1,\Phi}\right) \tag{4.34}$$

and

$$\||u|^{q(x)-\theta}\|_{(r(x))'} \leq C''\left(1+\|u\|^{q^+-\theta}_{1,\Phi}\right). \tag{4.35}$$

Substituting (4.34) and (4.35) into (4.31), and using Young's inequality we obtain

$$\int_{\Omega}|V(x)||u|^{q(x)}dx \leq C\|V\|^{\frac{\alpha}{r^-}}_{s(x)}\left[M_1 + M_2\left(\|u\|^{2(q^+-\theta)}_{1,\Phi} + \|u\|^{2\frac{\theta r^+}{r^-}}_{1,\Phi}\right)\right], \tag{4.36}$$

where C, M_1, and M_2 are positive constants. □

4.3. Existence of weak solution when the potential V has changing sign

In this section, we prove the existence of weak solution to the problem (P) in the case where the potential V is allowed to change sign. We have the following result

Theorem 4.3.1. *Assume that the assumptions (Φ), (f_0) and $s(x) > \frac{q(x)(\phi_0)^*}{(\phi_0)^*-q(x)}$ for every $x \in \overline{\Omega}$ hold. Furthermore, assume one of the following assumptions:*

1. $\max\{\psi^0, q^+\} < \phi_0$,
2. $\psi^0 < \phi_0 \leq \phi^0 < q^- \leq q^+ < (\phi_0)^*$, *and* $q^+ - \frac{1}{2}\phi_0 < q^-$,

then, the problem (P) has a weak solution.

To establish Theorem 4.3.1 we will prove that the functional $\mathcal{I}$ has a global minimum.

Proof of Theorem 4.3.1. Firstly, we show that $\mathcal{I}$ is coercive, namely, $\mathcal{I}(u) \to +\infty$ as $\|u\|_{1,\Phi} \to +\infty$. From (4.13), we have

$$|F(x,t)| \leq C_0\Psi(x,t) + h(x)|t|, \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Then, by applying Lemma 2.2.1, Poincaré and Hölder's inequalities, and Lemma 4.2.1-(4.15), we obtain

$$\begin{aligned}\mathcal{I}(u) &= \int_\Omega \Phi(x,|\nabla u|)dx + \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx - \int_\Omega F(x,u)dx \\ &\geq \|u\|_{1,\Phi}^{\phi_0} - c_1\|V\|_{s(x)}\|u\|_{1,\Phi}^{q_+} - c_2\|u\|_\Psi^{\psi^0} - c_3\|h\|_{\tilde{\Psi}}\|u\|_\Psi.\end{aligned}$$

From Lemma 4.2.1-(4.17), the previous inequality becomes

$$\mathcal{I}(u) \geq \|u\|_{1,\Phi}^{\phi_0} - c_1\|V\|_{s(x)}\|u\|_{1,\Phi}^{q_+} - c_2'\|u\|_{1,\Phi}^{\psi^0} - c_3'\|h\|_{\tilde{\Psi}}\|u\|_{1,\Phi}.$$

If the assumption (1) holds, then it is clear that $\mathcal{I}(u) \to +\infty$ as $\|u\|_{1,\Phi} \to +\infty$. Now, if we assume the assumption (2), then following as above arguments and using Lemma 4.2.6, we obtain

$$\begin{aligned}\mathcal{I}(u) &= \int_\Omega \Phi(x,|\nabla u|)dx + \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx - \int_\Omega F(x,u)dx \\ &\geq \|u\|_{1,\Phi}^{\phi_0} - \frac{C}{q^-}\|V\|_{s(x)}^{\frac{\alpha}{r^-}}\left[M_1 + M_2\left(\|u\|_{1,\Phi}^{2(q^+-\theta)} + \|u\|_{1,\Phi}^{2\frac{\theta r^+}{r^-}}\right)\right] - c_2'\|u\|_{1,\Phi}^{\psi^0} - c_3'\|h\|_{\tilde{\Psi}}\|u\|_{1,\Phi}.\end{aligned}$$

Since, $2\frac{\theta r^+}{r^-} < 2(q^- - \theta) < 2(q^+ - \theta) < \phi_0$ and $1 < \psi^0 < \phi_0$, then $\mathcal{I}(u) \to +\infty$ as $\|u\|_{1,\Phi} \to +\infty$. To complete the proof we show that the functional $\mathcal{I}$ is weakly lower semi-continuous, namely, $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$ implies $\mathcal{I}(u) \leq \liminf_{n\to\infty} \mathcal{I}(u_n)$. Suppose that $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$. Since the functional $\mathcal{H} \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ is strictly convex (because $\mathcal{H}'$ is strictly monotone), then we have $\mathcal{H}(u_n) > \mathcal{H}(u) + \langle \mathcal{H}'(u), u_n - u\rangle$; which implies that $\mathcal{H}$ is weakly lower semi-continuous on $W_0^{1,\Phi}(\Omega)$. Concerning the functional $\mathcal{J}$; from Proposition 4.2.5 -iii), $\mathcal{J}$ is sequentially weakly continuous, then $\mathcal{J}(u_n) \to \mathcal{J}(u)$. As the last argument, using again Proposition 4.2.5-ii), $\mathcal{F}$ is sequentially weakly continuous, which implies that $\mathcal{F}(u_n) \to \mathcal{F}(u)$. □

4.4. Existence of a unique weak solution when the potential V is positive almost everywhere on Ω

We have the following result

Theorem 4.4.1. *Assume that the assumptions* (Φ) *and* $s(x) > \frac{q(x)(\phi_0)^*}{(\phi_0)^* - q(x)}$ *for every* $x \in \overline{\Omega}$ *hold. If* $f(x,u) \equiv f(x) \in L^{(\Phi_*)'}(\Omega)$ *and* $V > 0$ *a.e. on* Ω*, then the problem* (P) *has a unique weak solution.*

Proof of Theorem 4.4.1. Define the functional $\mathcal{E} : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ by

$$\mathcal{E}(u) := \mathcal{H}(u) + \mathcal{J}(u), \tag{4.37}$$

where we recall that $\mathcal{H}$ and $\mathcal{J}$ are defined as in relation (4.14). Let us denote by $\mathcal{L} := \mathcal{E}' : W_0^{1,\Phi}(\Omega) \to (W_0^{1,\Phi}(\Omega))^*$ with

$$\langle \mathcal{L}(u), v\rangle = \langle \mathcal{H}'(u), v\rangle + \langle \mathcal{J}'(u), v\rangle. \tag{4.38}$$

Then, the operator $\mathcal{L}$ is continuous, bounded and strictly monotone. Indeed, from the proof of Proposition 4.2.2, it is clear that $\mathcal{L}$ is continuous. By Proposition 4.2.5-i) and iii), the operator $\mathcal{L}$ is bounded. The fact that $V > 0$ a.e. on Ω and using the inequalities

$$\left[(|\xi|^{q(x)-2}|\xi| - |\eta|^{q(x)-2}|\eta|)(\xi - \eta)\right] \cdot (|\xi| + |\eta|)^{2-q} \geq (q-1)|\xi - \eta|^2, \text{ if } 1 < q < 2, \tag{4.39}$$

$$(|\xi|^{q(x)-2}|\xi| - |\eta|^{q(x)-2}|\eta|)(\xi - \eta) \geq (\frac{1}{2})^q |\xi - \eta|^q, \text{ if } q \geq 2, \tag{4.40}$$

then $\mathcal{J}'$ is strictly monotone. Using again Proposition 4.2.5-i), we deduce that $\mathcal{L}$ is strictly monotone operator.

Now, since $\langle f, u \rangle := \int_\Omega f(x)u$, $\forall u \in W_0^{1,\Phi}(\Omega)$, defines a continuous linear functional on $W_0^{1,\Phi}(\Omega)$ (i.e. $f \in (W_0^{1,\Phi}(\Omega))^*$), then the problem (P) has a unique solution. □

4.5. Existence of a nontrivial weak solution when the potential V has a constant sign

In this section, we prove the existence of a nontrivial weak solution to the problem (P) in the case where the potential V has a constant sign almost everywhere on Ω, that means, $V > 0$ a.e. on Ω, or $V < 0$ a.e. on Ω. In order to obtain the third main result in this chapter, we shall add some suitable growth conditions on $f(x,u)$ and replacing the condition (f_0) assumed in Section 4.2 by the following one:

(f_0') We assume that (4.12) of (f_0) holds and that

$$|f(x,t)| \leq C_1(\psi(x,|t|) + 1), \text{ for } (x,t) \in \Omega \times \mathbb{R}, \tag{4.41}$$

where C_1 is a positive constant.

Let assuming the following conditions on $f(x,u)$:

(f_1) There exists $\Gamma \in N(\Omega)$ satisfying the assumptions of (H_2), and two positive constants γ_0 and γ^0 such that

$$1 < \frac{N}{\phi_0} < \gamma_0 \leq \frac{\gamma(x,t)t}{\Gamma(x,t)} \leq \gamma^0, \text{ for } x \in \Omega \text{ and } t > 0. \tag{4.42}$$

$$\Gamma\left(x, \frac{F(x,t)}{|t|^{\phi_0}}\right) \leq C_2 H(x,t), \text{ for } x \in \Omega \text{ and } |t| \geq M, \tag{4.43}$$

where C_2, M are positive constants, $H(x,t) = f(x,t)t - \nu F(x,t)$, for all $(x,t) \in \Omega \times \mathbb{R}$ with $\nu = \phi^0$ if $V \leq 0$ a.e. on Ω and $\nu = q^+$ if $V \geq 0$ a.e. on Ω, and $\gamma : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and $\Gamma(x,t) = \int_0^t \gamma(x,s)ds$, for all $x \in \Omega$.

(f_2) $\lim\limits_{|t|\to+\infty} \frac{F(x,t)}{|t|^{\phi^0}} = +\infty$, uniformly for $x \in \Omega$.

(f_3) $f(x,t) = o(|t|\phi(x,t))$ as $t \to 0$, uniformly for $x \in \Omega$.

We have the following existence result

Theorem 4.5.1. *Assume that the assumptions (Φ) and (f_0')-(f_3) hold. Furthermore, assume that $\phi^0 < \min\{\psi_0, q^-\}$, $\max\{\psi^0, q^+\} < (\phi_0)^*$, $q^+ - \frac{1}{2}\phi_0 < q^-$ and $s(x) > \frac{q(x)(\phi_0)^*}{(\phi_0)^* - q(x)}$ for every $x \in \overline{\Omega}$. If V has a constant sign a.e. on Ω, then the problem (P) has a nontrivial weak solution.*

In order to prove Theorem 4.5.1 we use the Mountain Pass Theorem (see [8]). Since the proof of this theorem is quite long, we will divide it into several lemmas. Firstly, we show that the functional $\mathcal{I}$ has a geometrical structure. Secondly, we show that $\mathcal{I}$ satisfies the Palais-Smale condition at level $\tilde{c}$ (see the Definition 4.4 below). To this end, we show that any Palais-Smale sequence at the level $\tilde{c}$ for $\mathcal{I}$ (see the Definition 4.3) is bounded in $W_0^{1,\Phi}(\Omega)$, and then has a strongly convergent subsequence.

Lemma 4.5.2. *Assume that the assumptions (Φ), (f_0'), (f_2) and (f_3) hold. Furthermore, assume that $\phi^0 < \min\{\psi_0, q^-\}$, $\max\{\psi^0, q^+\} < (\phi_0)^*$, $q^+ - \frac{1}{2}\phi_0 < q^-$, and $s(x) > \frac{q(x)(\phi_0)^*}{(\phi_0)^* - q(x)}$ for every $x \in \overline{\Omega}$. Then, the functional $\mathcal{I}$ has a geometrical structure, that is, $\mathcal{I}$ satisfies the following properties*

(i) *there exist $\rho > 0$ and $\beta > 0$ such that $\mathcal{I}(u) \geq \beta$ for any $u \in W_0^{1,\Phi}(\Omega)$ with $\|u\|_{1,\Phi} = \rho$.*

(ii) *there exists $u_0 \in W_0^{1,\Phi}(\Omega)$ such that $\|u_0\|_{1,\Phi} > \rho$ and $\mathcal{I}(u_0) \leq 0$.*

Proof. (i) Firstly, from (f_0') and (f_3) it follows that, for all given $\epsilon > 0$ there exists $C(\epsilon) > 0$, such that

$$|F(x,t)| \leq \epsilon\Phi(x,t) + C(\epsilon)\Psi(x,t), \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{4.44}$$

Using Lemma 2.2.1, the Poincaré inequality, and the fact that $W_0^{1,\Phi}(\Omega)$ is compactly embedded in $L^{\Psi}(\Omega)$, we obtain

$$\int_\Omega |F(x,t)|dx \leq \epsilon \max\{\|u\|_{1,\Phi}^{\phi_0}, \|u\|_{1,\Phi}^{\phi^0}\} + C'(\epsilon)\max\{\|u\|_{1,\Phi}^{\psi_0}, \|u\|_{1,\Phi}^{\psi^0}\}. \tag{4.45}$$

Using the same arguments as in the proof of relation (4.22), we obtain

$$\int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)} \leq C\|V\|_{s(x)} \max\{\|u\|_{1,\Phi}^{q_-}, \|u\|_{1,\Phi}^{q_+}\}. \tag{4.46}$$

Now, by using the definition of $\mathcal{I}$ in (4.14), Lemma 2.2.1, and the relations (4.45)-(4.46), we get

$$\begin{aligned}\mathcal{I}(u) &= \int_\Omega \Phi(x,|\nabla u|)dx + \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx - \int_\Omega F(x,u)dx \\ &\geq \min\{\|u\|_{1,\Phi}^{\phi_0}, \|u\|_{1,\Phi}^{\phi^0}\} - C\|V\|_{s(x)}\max\{\|u\|_{1,\Phi}^{q_-}, \|u\|_{1,\Phi}^{q_+}\} \\ &\quad - \epsilon\max\{\|u\|_{1,\Phi}^{\phi_0}, \|u\|_{1,\Phi}^{\phi^0}\} - C'(\epsilon)\max\{\|u\|_{1,\Phi}^{\psi_0}, \|u\|_{1,\Phi}^{\psi^0}\},\end{aligned}$$

which implies that, for all $u \in W_0^{1,\Phi}(\Omega)$ with $\|u\|_{1,\Phi} < 1$,

$$\begin{aligned}\mathcal{I}(u) &\geq \|u\|_{1,\Phi}^{\phi^0} - C\|V\|_{s(x)}\|u\|_{1,\Phi}^{q_-} - \epsilon\|u\|_{1,\Phi}^{\phi_0} - C'(\epsilon)\|u\|_{1,\Phi}^{\psi_0} \\ &\geq \frac{1}{2}\|u\|_{1,\Phi}^{\phi^0} - C\|V\|_{s(x)}\|u\|_{1,\Phi}^{q_-} - C'(\epsilon)\|u\|_{1,\Phi}^{\psi_0} \\ &= \|u\|_{1,\Phi}^{\phi^0}\left(\frac{1}{2} - C\|V\|_{s(x)}\|u\|_{1,\Phi}^{(q_-)-\phi^0} - C'(\epsilon)\|u\|_{1,\Phi}^{\psi_0-\phi^0}\right).\end{aligned} \tag{4.47}$$

Since $(q_-) - \phi^0 > 0$ and $\psi_0 - \phi^0 > 0$, then from (4.47) we can choose $\beta > 0$ and $\rho > 0$ such that $\mathcal{I}(u) \geq \beta > 0$ for any $u \in W_0^{1,\Phi}(\Omega)$ with $\|u\|_{1,\Phi} = \rho$.

(ii) From (f_2), it follows that for any $L > 0$ there exists a constant $C_L := C(L) > 0$ depending on L, such that

$$F(x,t) \geq L|t|^{\phi^0} - C_L, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{4.48}$$

Let $w \in W_0^{1,\Phi}(\Omega)$ with $w > 0$. We take $t > 1$ large enough to ensure that $\|tw\|_{1,\Phi} > 1$. Then, from (4.48) and Lemmas 2.2.1 and 4.2.6, we have

$$\begin{aligned}\mathcal{I}(tw) &= \int_\Omega \Phi(x,|t\nabla w|)dx + \int_\Omega \frac{V(x)}{q(x)}|tw|^{q(x)}dx - \int_\Omega F(x,tw)dx \\ &\leq t^{\phi^0}\|w\|_{1,\Phi}^{\phi^0} + C\|V\|_{s(x)}^{\frac{\alpha}{r^-}}\left[M_1 + M_2\left(t^{2(q^+-\theta)}\|w\|_{1,\Phi}^{2(q^+-\theta)} + t^{2\frac{\theta r^+}{r^-}}\|w\|_{1,\Phi}^{2\frac{\theta r^+}{r^-}}\right)\right] \\ &\quad - Lt^{\phi^0}\int_\Omega |w|^{\phi^0}dx + C_L|\Omega| \\ &= t^{\phi^0}\left(\|w\|_{1,\Phi}^{\phi^0} - L\int_\Omega |w|^{\phi^0}dx\right) \\ &\quad + C\|V\|_{s(x)}^{\frac{\alpha}{r^-}}\left[M_1 + M_2\left(t^{2(q^+-\theta)}\|w\|_{1,\Phi}^{2(q^+-\theta)} + t^{2\frac{\theta r^+}{r^-}}\|w\|_{1,\Phi}^{2\frac{\theta r^+}{r^-}}\right)\right] + C_L|\Omega|.\end{aligned}$$

By choosing $L > 0$ such that $\|w\|_{1,\Phi}^{\phi^0} - L\int_\Omega |w|^{\phi^0}dx < 0$ and the fact that $2\frac{\theta r^+}{r^-} < 2(q^+ - \theta) < \phi_0$, then we obtain $\mathcal{I}(tw) \to -\infty$ as $t \to +\infty$. The proof of this lemma is complete. □

Remark 4.5.3. *Note that in the proof of the geometrical structure lemma we do not need any sign condition on the potential V.*

Now, we define the level at $\tilde{c}$ as follows

$$\tilde{c} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{I}(\gamma(t)),$$

where $\Gamma = \{\gamma \in \mathcal{C}([0,1], W_0^{1,\Phi}(\Omega)) : \gamma(0) = 0,\ \gamma(1) = u_0\}$ is the set of continuous paths joining 0 and u_0, where $u_0 \in W_0^{1,\Phi}(\Omega)$ is defined in the previous lemma. Let us recall the standard definitions of Palais-Smale sequence at the level $\tilde{c}$ and Palais-Smale condition at the level $\tilde{c}$ for a functional $\mathcal{I} \in \mathcal{C}^1(E, \mathbb{R})$, where E is a Banach space.

Definition 4.3. *Let E be a Banach space with dual space E^* and (u_n) a sequence in E. We say that (u_n) is a Palais-Smale sequence at the level $\tilde{c}$ for a functional $\mathcal{I} \in \mathcal{C}^1(E, \mathbb{R})$ if*

$$\mathcal{I}(u_n) \to \tilde{c}, \quad \text{and} \quad \|\mathcal{I}'(u_n)\|_{E^*} \to 0.$$

Definition 4.4. *We say that a functional $\mathcal{I}$ satisfies the Palais-Smale condition at the level $\tilde{c}$ if any Palais-Smale sequence at the level $\tilde{c}$ for $\mathcal{I}$ possesses a convergent subsequence.*

Remark 4.5.4. *We note that, by Lemma 4.5.2 the existence of a Palais-Smale sequence at the level $\tilde{c}$ for our energy functional $\mathcal{I}$ is ensured. This can be observed directly from the proof given in [8].*

Now, in order to prove that the functional $\mathcal{I}$ satisfies the Palais-Smale condition, we shall first show that any Palais-Smale sequence for $\mathcal{I}$ is bounded. To this end, we have the following lemma:

Lemma 4.5.5. *Assume that the assumptions (Φ) and (f_0')-(f_3) hold. Furthermore, assume that $\phi^0 < \min\{\psi_0, q^-\}$, $\max\{\psi^0, q^+\} < (\phi_0)^*$, $q^+ - \frac{1}{2}\phi_0 < q^-$, and $s(x) > \frac{q(x)(\phi_0)^*}{(\phi_0)^* - q(x)}$ for every $x \in \overline{\Omega}$. If V has a constant sign a.e. on Ω, then any Palais-Samle sequence at the level $\tilde{c}$ for $\mathcal{I}$ is bounded in $W_0^{1,\Phi}(\Omega)$.*

Proof. Let (u_n) be a Palais-Smale sequence at the level $\tilde{c}$ for $\mathcal{I}$ in $W_0^{1,\Phi}(\Omega)$. We prove by contradiction that (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$. Assuming that (u_n) is unbounded in $W_0^{1,\Phi}(\Omega)$, that is, $\|u_n\|_{1,\Phi} \to +\infty$.

Let $v_n := \frac{u_n}{\|u_n\|_{1,\Phi}}$. It is clear that (v_n) is bounded in $W_0^{1,\Phi}(\Omega)$. Hence, there exists a subsequence denoted again (v_n) such that v_n converges weakly to v in $W_0^{1,\Phi}(\Omega)$. From Lemma 4.2.1-4.17, $W_0^{1,\Phi}(\Omega)$ is compactly embedded in $L^{\Psi}(\Omega)$; thus v_n converges strongly to v in $L^{\Psi}(\Omega)$, and then a.e. in Ω.

Define $\Omega_{\neq} := \{x \in \Omega : |v(x)| \neq 0\}$. We consider two possible cases: $|\Omega_{\neq}| = 0$ or $|\Omega_{\neq}| > 0$. Firstly, we assume that $|\Omega_{\neq}| = 0$, that is, $v = 0$ a.e. in Ω. From the definition of $\mathcal{I}$ in (4.14), Lemma 2.2.1, and the fact that $\|u_n\|_{1,\Phi} \to +\infty$, we get

$$\begin{aligned} \|u_n\|_{1,\Phi}^{\phi_0} &\leq \mathcal{I}(u_n) - \int_\Omega \frac{V(x)}{q(x)} |u_n|^{q(x)} dx + \int_\Omega F(x, u_n) dx \\ &\leq \mathcal{I}(u_n) + \frac{1}{q^-} \int_\Omega |V(x)||u_n|^{q(x)} dx + \int_\Omega F(x, u_n) dx, \end{aligned} \tag{4.49}$$

which implies that

$$1 \leq \frac{\mathcal{I}(u_n)}{\|u_n\|_{1,\Phi}^{\phi_0}} + \frac{1}{q^- \|u_n\|_{1,\Phi}^{\phi_0}} \int_\Omega |V(x)||u_n|^{q(x)} dx + \int_\Omega \frac{F(x, u_n)}{\|u_n\|_{1,\Phi}^{\phi_0}} dx. \tag{4.50}$$

Now, we shall show that all terms of the right-hand side of (4.50) tend to zero when n is large enough, which is the desired contradiction. Since (u_n) is a Palais-Smale sequence type, then $(\mathcal{I}(u_n))$ is bounded. Hence, the first term of the right-hand side of (4.50) tends to zero as n is large enough. For the second one, from Lemma 4.2.6 we get

$$\frac{1}{q^- \|u_n\|_{1,\Phi}^{\phi_0}} \int_\Omega |V(x)||u_n|^{q(x)} dx \leq C \|V\|_{s(x)}^{\frac{\alpha}{r^-}} \frac{M_1 + M_2 \left(\|u_n\|_{1,\Phi}^{2(q^+ - \theta)} + \|u_n\|_{1,\Phi}^{2\frac{\theta r^+}{r^-}} \right)}{q^- \|u_n\|_{1,\Phi}^{\phi_0}}. \tag{4.51}$$

Since, $2\frac{\theta r^+}{r^-} < 2(q^- - \theta) < 2(q^+ - \theta) < \phi_0$, then passing to the limit in (4.51), we obtain

$$\frac{1}{q^- \|u_n\|_{1,\Phi}^{\phi_0}} \int_\Omega |V(x)||u_n|^{q(x)} dx \to 0, \text{ as } n \to +\infty. \tag{4.52}$$

Hence, the second term tends to zero as n is large enough. For the third term, on the one hand it follows from the definition of F that

$$\int_{\{|u_n| \leq M\}} \frac{|F(x,u_n)|}{\|u_n\|_{1,\Phi}^{\phi_0}} dx \leq \frac{C(M)}{\|u_n\|_{1,\Phi}^{\phi_0}}, \tag{4.53}$$

where $C(M)$ is a positive constant depending on M defined in (4.43). On the other hand, by using Hölder's inequality, we get

$$\begin{aligned}\int_{\{|u_n|>M\}} \frac{F(x,u_n)}{\|u_n\|_{1,\Phi}^{\phi_0}} dx &= \int_{\{|u_n|>M\}} \frac{F(x,u_n)}{|u_n|^{\phi_0}} |v_n|^{\phi_0} dx \\ &\leq 2 \left\| \frac{F(x,u_n)}{|u_n|^{\phi_0}} \chi_{\{|u_n|>M\}} \right\|_\Gamma \| |v_n|^{\phi_0} \chi_{\{|u_n|>M\}} \|_{\tilde{\Gamma}}.\end{aligned}$$

Without loss of generality, we may suppose that $\left\| \frac{F(x,u_n)}{|u_n|^{\phi_0}} \chi_{\{|u_n|>M\}} \right\|_\Gamma > 1$. Then, from Lemma 2.2.1, we get

$$\left\| \frac{F(x,u_n)}{|u_n|^{\phi_0}} \chi_{\{|u_n|>M\}} \right\|_\Gamma \leq \left[\int_{\{|u_n|>M\}} \Gamma\left(x, \frac{F(x,u_n)}{|u_n|^{\phi_0}}\right) dx \right]^{\frac{1}{\gamma_0}}.$$

Hence, it follows from (4.43) that,

$$\left\| \frac{F(x,u_n)}{|u_n|^{\phi_0}} \chi_{\{|u_n|>M\}} \right\|_\Gamma \leq C \left[\int_\Omega H(x,u_n) dx \right]^{\frac{1}{\gamma_0}} + C', \tag{4.54}$$

where C and C' are positive constants independent of n.

In the case where $V \leq 0$ a.e. on Ω, then from the definition of the functional $\mathcal{I}$ we get

$$\begin{aligned}\phi^0 \mathcal{I}(u_n) - \langle \mathcal{I}'(u_n), u_n \rangle &= \int_\Omega \left[\phi^0 \Phi(x, |\nabla u_n|) - \phi(x, |\nabla u_n|) |\nabla u_n|^2 \right] dx \\ &+ \int_\Omega V(x) \left(\frac{\phi^0}{q(x)} - 1 \right) |u_n|^{q(x)} dx + \int_\Omega (f(x,u_n)u_n - \phi^0 F(x,u_n)) dx.\end{aligned} \tag{4.55}$$

From (2.2) and the fact that $\phi^0 < q^- \leq q(x)$, the first and the second terms of the right-hand side of (4.55) are nonnegative. Hence, the relation (4.55) becomes

$$\phi^0 \mathcal{I}(u_n) - \langle \mathcal{I}'(u_n), u_n \rangle \geq \int_\Omega H(x,u_n) dx. \tag{4.56}$$

It follows from (4.56) that, $\int_\Omega H(x,u_n)dx \leq C$, for n large enough.

Now, in the case where $V \geq 0$ a.e. on Ω, then from the definition of the functional $\mathcal{I}$ we get

$$\begin{aligned}q^+ \mathcal{I}(u_n) - \langle \mathcal{I}'(u_n), u_n \rangle &= \int_\Omega \left[q^+ \Phi(x, |\nabla u_n|) - \phi(x, |\nabla u_n|) |\nabla u_n|^2 \right] dx \\ &+ \int_\Omega V(x) \left(\frac{q^+}{q(x)} - 1 \right) |u_n|^{q(x)} dx + \int_\Omega H(x,u_n) dx.\end{aligned} \tag{4.57}$$

Since $q(x) \leq q^+$, then following the same arguments as for (4.56), we have also $\int_\Omega H(x,u_n)dx \leq C$, for n large enough. This fact combined with relation (4.54) yields

$$\left\| \frac{F(x,u_n)}{|u_n|^{\phi_0}} \chi_{\{|u_n|>M\}} \right\|_\Gamma \leq C, \quad \text{for } n \text{ large enough}, \tag{4.58}$$

where C is a positive constant independent of n. Now, it remains to show that $\| |v_n|^{\phi_0} \chi_{\{|u_n|>M\}} \|_{\tilde{\Gamma}} \to 0$ as $n \to +\infty$. Let $K(x,t) := \tilde{\Gamma}(x, |t|^{\phi_0})$. Since $\phi_0 > 1$ and $\tilde{\Gamma} \in N(\Omega)$, then it is clear that $K \in N(\Omega)$. Moreover, since Γ satisfies (H_2) then K verifies the assumption (2) of Theorem 2.3.12 and by Remark 2.3.7, $K(x,k)$ is bounded for each $k > 0$. Using Lemmas 2.2.1 and 2.3.14, we get

$$\lim_{t\to+\infty} \frac{K(x,kt)}{\Phi_*(x,t)} \leq \frac{K(x,k)}{\Phi_*(x,1)} \lim_{t\to+\infty} \frac{1}{t^{\phi_0(\gamma_0)'-(\phi_0)^*}},$$

where $(\gamma_0)' = \frac{\gamma_0}{\gamma_0 - 1}$ is defined as in (4.42). Since $\frac{N}{\phi_0} < \gamma_0$, then $\phi_0(\gamma_0)' < (\phi_0)^*$. From this, we get

$$\lim_{t\to+\infty} \frac{K(x,kt)}{\Phi_*(x,t)} = 0, \text{uniformly for } x \in \Omega.$$

Thus, form Theorem 2.3.12, $W_0^{1,\Phi}(\Omega)$ is compactly embedded in $L^K(\Omega)$, which implies that

$$\int_\Omega \tilde{\Gamma}(x, |v_n|^{\phi_0}) dx \to 0, \text{ as } n \to +\infty.$$

Consequently,

$$\| |v_n|^{\phi_0} \chi_{\{|u_n|>M\}} \|_{\tilde{\Gamma}} \to 0, \text{ as } n \to +\infty. \tag{4.59}$$

Hence, passing to the limit in (4.50) and using (4.52),(4.58) and (4.59), we obtain a contradiction.

Secondly, we assume that $|\Omega_{\neq}| > 0$. Then obviously, $|u_n| = |v_n| \|u_n\|_{1,\Phi} \to +\infty$ in $\Omega_{\neq}$. Hence, for some positive real M we have $\Omega_{\neq} \subset \{x \in \Omega : |u_n| \geq M\}$ for n large enough. Using Lemma 2.2.1, we get

$$\begin{aligned}
\frac{\mathcal{I}(u_n)}{\|u_n\|_{1,\Phi}^{\phi^0}} &\leq 1 + \frac{1}{q^- \|u_n\|_{1,\Phi}^{\phi^0}} \int_\Omega |V(x)||u_n|^{q(x)} dx - \int_\Omega \frac{F(x,u_n)}{\|u_n\|_{1,\Phi}^{\phi^0}} dx \\
&= 1 + \frac{1}{q^- \|u_n\|_{1,\Phi}^{\phi^0}} \int_\Omega |V(x)||u_n|^{q(x)} dx - \int_{\{|u_n|\leq M\}} \frac{|F(x,u_n)|}{\|u_n\|_{1,\Phi}^{\phi^0}} dx \\
&\quad - \int_{\{|u_n|>M\}} \frac{F(x,u_n)}{|u_n|^{\phi^0}} |v_n|^{\phi^0} dx.
\end{aligned}$$

Now, using relations (4.52), (4.53), assumption (f_2) and Fatou's Lemma, we obtain a contradiction. Hence, (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$. The proof of this lemma is complete. □

Remark 4.5.6. *The preceding lemma holds true under a slightly weaker assumption than V has a constant sign. Indeed, assume that there exists a constant ρ such that $\phi^0 \leq \rho \leq q^+$ and $V(x)\left(\frac{\rho}{q(x)} - 1\right) \geq 0$ a.e. on Ω. Then, by taking $H(x,t) = f(x,t)t - \rho F(x,t)$ and following the same arguments as in (4.55)-(4.56), we obtain the previous lemma.*

To finish the proof of the Palais-Smale condition for $\mathcal{I}$, we only need to show the following lemma:

Lemma 4.5.7. *Assume that the assumptions of Lemma 4.5.5 hold. Then, the Palais-Smale sequence at the level $\tilde{c}$ for $\mathcal{I}$ possesses a convergent subsequence.*

Proof. Let (u_n) be a Palais-Smale sequence at the level $\tilde{c}$ for $\mathcal{I}$ in $W_0^{1,\Phi}(\Omega)$. Then, $\mathcal{I}'(u_n) \to 0$ in $(W_0^{1,\Phi}(\Omega))^*$ and from Lemma 4.5.5, (u_n) is bounded in $W_0^{1,\Phi}(\Omega)$. As $W_0^{1,\Phi}(\Omega)$ is reflexive, then there exists a subsequence denoted again (u_n) such that u_n converges weakly to u in $W_0^{1,\Phi}(\Omega)$. From Proposition 4.2.5-i), the mapping $\mathcal{H}'$ is of type (S_+). Thus, to conclude the result of this lemma it suffices to show that

$$\limsup_{n\to\infty} \langle \mathcal{H}'(u_n), u_n - u \rangle \leq 0. \tag{4.60}$$

Indeed, using the definition of $\mathcal{I}'$ in Proposition 4.2.2, we have

$$\langle \mathcal{H}'(u_n), u_n - u \rangle = \langle \mathcal{I}'(u_n), u_n - u \rangle + \langle \mathcal{F}'(u_n), u_n - u \rangle - \langle \mathcal{J}'(u_n), u_n - u \rangle. \tag{4.61}$$

It is clear that,

$$\langle \mathcal{I}'(u_n), u_n - u\rangle \to 0. \tag{4.62}$$

From Proposition 4.2.5-ii), $\mathcal{F}'$ is a completely continuous linear operator. Hence,

$$\langle \mathcal{F}'(u_n), u_n - u\rangle \to 0. \tag{4.63}$$

Using again Proposition 4.2.5-iii), $\mathcal{J}'$ is a completely continuous linear operator, which implies that

$$\langle \mathcal{J}'(u_n), u_n - u\rangle \to 0. \tag{4.64}$$

Finally, it follows from (4.62), (4.63) and (4.64) that (4.60) holds. Hence, since $\mathcal{H}'$ is of type (S_+), then u_n converges strongly to u in $W_0^{1,\Phi}(\Omega)$. The proof of Theorem 4.5.1 is complete. □

4.6. Existence of infinitely many weak solutions when the potential V has a constant sign

In this section, by using Fountain theorem (see [100] for details), we prove the existence of infinitely many solutions when the potential V has a constant sign almost everywhere on Ω. Let us to state the Fountain theorem.

Let $(X, \|\cdot\|)$ be a real reflexive Banach space such that $X = \overline{\oplus_{j\in\mathbb{N}^*} X_j}$ with $dim(X_j) < +\infty$ for any $j \in \mathbb{N}^*$. For each $k \in \mathbb{N}^*$, we set $Y_k = \oplus_{j=1}^{k} X_j$ and $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$.

Proposition 4.6.1 (Fountain theorem). *Let $(X, \|\cdot\|)$ be a real reflexive Banach space and $\mathcal{I} \in \mathcal{C}^1(X, \mathbb{R})$ an even functional. If for each sufficiently large $k \in \mathbb{N}^*$, there exist $\rho_k > r_k > 0$ such that the following conditions hold:*

1. $\inf_{\{u\in Z_k, \|u\|=r_k\}} \mathcal{I}(u) \to +\infty$ *as* $k \to +\infty$,
2. $\max_{\{u\in Y_k, \|u\|=\rho_k\}} \mathcal{I}(u) \leq 0$,
3. $\mathcal{I}$ *satisfies the Palais-Smale condition for every* $c > 0$,

then $\mathcal{I}$ has a sequence of critical values tending to $+\infty$.

In order to our energy functional $\mathcal{I}$ be even, we need to assume that the nonlinearity $f(x, u)$ be odd. In other words, besides the assumptions (f_0')-(f_3), we assume

(f_4) $f(x, -t) = -f(x, t)$ for all $(x, t) \in \Omega \times \mathbb{R}$.

Now, we can state the existence result

Theorem 4.6.2. *Assume that the assumptions of Theorem 4.5.1 hold. If the function f satisfies (f_4), then the problem (P) has a sequence of weak solutions $(\pm u_n)_{n\in\mathbb{N}} \subseteq W_0^{1,\Phi}(\Omega)$ such that $\mathcal{I}(\pm u_n) \to +\infty$ as $n \to +\infty$.*

Since $W_0^{1,\Phi}(\Omega)$ is a reflexive and separable Banach space, then there exist $(e_j)_{j\in\mathbb{N}^*} \subseteq W_0^{1,\Phi}(\Omega)$ and $(e_j^*)_{j\in\mathbb{N}^*} \subseteq (W_0^{1,\Phi}(\Omega))^*$ such that

$$W_0^{1,\Phi}(\Omega) = \overline{\text{span}\{e_j : j \in \mathbb{N}^*\}}, \qquad (W_0^{1,\Phi}(\Omega))^* = \overline{\text{span}\{e_j^* : j \in \mathbb{N}^*\}}$$

and

$$\langle e_i, e_j^*\rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

For $k \in \mathbb{N}^*$ denote by

$$X_j = \text{span}\{e_j\}, \quad Y_k = \oplus_{j=1}^{k} X_j, \quad \text{and} \quad Z_k = \overline{\oplus_{j=k}^{\infty} X_j}.$$

Proof of Theorem 4.6.2. We denote by

$$\beta_k := \sup\left\{\int_\Omega \Psi(x,|u|)dx : \|u\|_{1,\Phi}=1, u\in Z_k\right\}.$$

Since $\Psi \ll \Phi_*$, then $\lim_{k\to+\infty}\beta_k = 0$ (see [73, Lemma 4.3]). Now, we verify the conditions of Fountain theorem. It follows from assumption (f_4) that $\mathcal{F}$ is even, hence the functional $\mathcal{I}$ is even. From Lemmas 4.5.5 and 4.5.7, $\mathcal{I}$ satisfies the Palais-Smale condition; hence the condition (3) of Fountain theorem holds. It remains to prove that conditions (1) and (2) in Fountain theorem hold.

(1) By (f_0'), it follows that

$$|F(x,t)| \le C(\Psi(x,t)+|t|), \quad \forall (x,t)\in\Omega\times\mathbb{R}. \tag{4.65}$$

Let $u\in Z_k$ with $\|u\|_{1,\Phi}>1$. From the definition of $\mathcal{I}$ in (4.14), Poincaré's inequality and Lemmas 2.2.1 and 4.2.6, we obtain

$$\begin{aligned}\mathcal{I}(u) &= \int_\Omega \Phi(x,|\nabla u|)dx + \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx - \int_\Omega F(x,u)dx\\ &\ge \|u\|_{1,\Phi}^{\phi_0} - C\|V\|_{s(x)}^{\frac{\alpha}{r^-}}\left[M_1+M_2\left(\|u\|_{1,\Phi}^{2(q^+-\theta)}+\|u\|_{1,\Phi}^{2\frac{\theta r^+}{r^-}}\right)\right] - C_1\int_\Omega \Psi(x,|u|)dx - C_2\|u\|_{1,\Phi},\end{aligned} \tag{4.66}$$

where we recall that $\alpha = r^+$ if $\|V\|_{s(x)}>1$ and $\alpha=r^-$ if $\|V\|_{s(x)}\le 1$. Furthermore, from Lemma 2.2.1, we have

$$\int_\Omega \Psi(x,|u|)dx = \int_\Omega \Psi\left(x,\|u\|_{1,\Phi}\frac{|u|}{\|u\|_{1,\Phi}}\right)dx \le \|u\|_{1,\Phi}^{\psi^0}\int_\Omega \Psi\left(x,\frac{|u|}{\|u\|_{1,\Phi}}\right)dx.$$

Using the definition of β_k, the relation (4.66) becomes

$$\mathcal{I}(u) \ge \|u\|_{1,\Phi}^{\phi_0} - C\|V\|_{s(x)}^{\frac{\alpha}{r^-}}\left[M_1+M_2\left(\|u\|_{1,\Phi}^{2(q^+-\theta)}+\|u\|_{1,\Phi}^{2\frac{\theta r^+}{r^-}}\right)\right] - C_1\|u\|_{1,\Phi}^{\psi^0}\beta_k - C_2\|u\|_{1,\Phi}.$$

Now, let $u_k\in Z_k$ with $\|u\|_{1,\Phi}=r_k=(2C_1\beta_k)^{\frac{1}{\phi_0-\psi^0}}$. Since $\phi_0<\psi^0$ and $\lim_{k\to+\infty}\beta_k=0$, then $r_k\to+\infty$ as $k\to+\infty$. Thus, we have

$$\begin{aligned}\mathcal{I}(u) &\ge (2C_1\beta_k)^{\frac{\phi_0}{\phi_0-\psi^0}} - C\|V\|_{s(x)}^{\frac{\alpha}{r^-}}\left[M_1+M_2\left({r_k}^{2(q^+-\theta)}+{r_k}^{2\frac{\theta r^+}{r^-}}\right)\right]\\ &\qquad - C_1(2C_1\beta_k)^{\frac{\psi^0}{\phi_0-\psi^0}}\beta_k - C_2 r_k,\end{aligned}$$

$$\mathcal{I}(u) \ge \frac{1}{2}{r_k}^{\phi_0} - C\|V\|_{s(x)}^{\frac{\alpha}{r^-}}\left[M_1+M_2\left({r_k}^{2(q^+-\theta)}+{r_k}^{2\frac{\theta r^+}{r^-}}\right)\right] - C_2 r_k.$$

Since $2\frac{\theta r^+}{r^-}<2(q^--\theta)<2(q^+-\theta)<\phi_0$ and $1<\phi_0$, then

$$\inf_{\{u\in Z_k, \|u\|=r_k\}} \mathcal{I}(u)\to+\infty \text{ as } k\to+\infty.$$

(2) Let $w\in Y_k$ with $w>0$, $\|w\|_{1,\Phi}=1$ and $t>1$. Then, from relation (4.48) and Lemmas 2.2.1 and 4.2.6, we obtain

$$\begin{aligned}\mathcal{I}(tw) &\le t^{\phi^0}\left(\|w\|_{1,\Phi}^{\phi^0} - L\int_\Omega |w|^{\phi^0}dx\right)\\ &\quad + C\|V\|_{s(x)}^{\frac{\alpha}{r^-}}\left[M_1+M_2\left(t^{2(q^+-\theta)}\|w\|_{1,\Phi}^{2(q^+-\theta)}+t^{2\frac{\theta r^+}{r^-}}\|w\|_{1,\Phi}^{2\frac{\theta r^+}{r^-}}\right)\right]+C_L|\Omega|.\end{aligned}$$

It is clear that we can choose $L>0$ so that $\|w\|_{1,\Phi}^{\phi^0}-L\int_\Omega|w|^{\phi^0}dx<0$. With this fact and since $2\frac{\theta r^+}{r^-}<2(q^+-\theta)<\phi_0$ then we have $\mathcal{I}(tw)\to-\infty$ as $t\to+\infty$. Thus, there exists $\tilde{t}>r_k>1$ such that $\mathcal{I}(\tilde{t}w)<0$. By setting $\rho_k=\tilde{t}$, then we obtain

$$\max_{\{u\in Y_k,\|u\|=\rho_k\}}\mathcal{I}(u)\le 0.$$

The proof of this theorem is complete. □

4.7. Application

In this section, we give an example of a nonlinearity f satisfying the assumptions (f_0')-(f_3), and for which our main Theorems 4.5.1 and 4.6.2 hold.

Let us fix $\Phi(x,t) = \frac{1}{p(x)}|t|^{p(x)}$ with $p \in \mathcal{C}^{1-0}(\overline{\Omega})$. Then, the operator $\operatorname{div}(\phi(x,|\nabla u|)\nabla u)$ involved in (P) is the $p(x)$-Laplacian operator, i.e. $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. In this case, we have $\phi_0 = p^-$ and $\phi^0 = p^+$ with the assumption $1 < p^- \le p(x) \le p^+ < N$.

In the case where $V \geq 0$ a.e. on Ω: We take $F(x,t) = |t|^{q^+}\ln(1+|t|)$, with $q^+ + 1 < \frac{Np^-}{N-p^-}$. The derivative with respect to t of $F(x,t)$ is given by

$$F'(x,t) := f(x,t) = q^+|t|^{q^+-2}t\ln(1+|t|) + \frac{t|t|^{q^+-1}}{1+|t|},$$

and we have

$$H(x,t) := f(x,t)t - q^+F(x,t) = \frac{|t|^{q^+}}{1+|t|}.$$

It is clear that f satisfies the assumptions (f_0'), (f_2)-(f_4). Moreover, since $\frac{F(x,t)}{|t|^\theta} \to 0$ for all $\theta > q^+$ then from (4.1), f does not satisfy the (A-R) condition. Now, it remains to show that the assumption (f_1) holds. To this end, let us consider the function $\Gamma(x,t) = |t|^\beta$, where $1 < \frac{N}{p^-} < \beta < \frac{q^+}{q^+-p^-}$. Then, $\Gamma\left(x, \frac{F(x,t)}{|t|^{p^-}}\right) = |t|^{\beta(q^+-p^-)}\ln^\beta(1+|t|)$. Since $\beta(q^+-p^-) < q^+$, then $\frac{|t|^{\beta(q^+-p^-)+1}\ln^\beta(1+|t|)}{|t|^{q^++1}} \to 0$ as $|t| \to +\infty$. Hence, the assumption (f_1) holds.

In the case where $V \leq 0$ a.e. on Ω: We can take $F(x,t) = |t|^{p^+}\ln(1+|t|)$. By the same arguments above, the choice of $\Gamma(x,t) = |t|^\beta$, where $1 < \frac{N}{p^-} < \beta < \frac{p^+}{p^+-p^-}$, ensures easily that f verifies the assumptions (f_0')-(f_4).

Consequently, in the both cases, the main Theorems 4.5.1 and 4.6.2 hold.

Remark 4.7.1.

1. *In the case where $V \leq 0$ a.e. on Ω, we can not take the same function F considered in the first case, i.e. $F(x,t) = |t|^{q^+}\ln(1+|t|)$. Indeed, in this case, the nonlinearity f satisfies the (A-R) condition.*

2. *As in the first remark, we can not consider the function $F(x,t) = |t|^{p^+}\ln(1+|t|)$ when $V \geq 0$ a.e. on Ω. Indeed, in this case we have*

$$f(x,t)t - q^+F(x,t) = (p^+ - q^+)|t|^{p^+}\ln(1+|t|) + \frac{|t|^{p^++1}}{1+|t|} < 0, \textit{ for } |t| \textit{ large enough.}$$

Hence, the nonlinearity f do not satisfy the assumption (f_1).

Part II.

Asymptotic behavior for some nonlinear parabolic problems

CHAPTER 5

GLOBAL ATTRACTOR FOR A DOUBLY NONLINEAR PARABOLIC PROBLEM WITH A NONLINEAR BOUNDARY CONDITION

The purpose of this chapter is to prove the existence and uniqueness of bounded weak solutions for a doubly nonlinear parabolic problem of p-Laplacian type with a nonlinear boundary condition. We formulate our problem as a dynamical system; then by using Hölder continuity of solutions and assuming appropriate hypotheses, we prove also the existence of a global attractor in $L^\infty(\Omega)$.

5.1. Introduction

In this chapter, we consider the following doubly nonlinear parabolic problem

$$(P)\begin{cases} \partial_t(\beta(u)) - \Delta_p u + h(x,t,u) = 0, & \text{in } \Omega\times(0,\infty), \\ -|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(u), & \text{on } \partial\Omega\times(0,\infty), \\ \beta(u(0)) = \beta(u_0), & \text{in } \Omega, \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$. Here, Δ_p denotes the p-Laplacian operator defined by $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ $(1 < p < \infty)$, $\frac{\partial}{\partial \nu}$ denotes the outer unit normal to $\partial\Omega$ at x. Precise conditions concerning β, h, u_0 and g will be given hereafter.

Partial differential equations of the form (P), or some special cases of it, are studied by several authors because of their mathematical interest and because they describe many phenomena in mechanics, biology and physics. To be more specific we give some important models. For $\beta(u) = u$, $g = 0$ and $p = 2$, the problem reduces the reaction-diffusion equation, while for $p \neq 2$ the problem represents the equation of non-Newtonian elastic filtration, glaciology phenomena (see [74, 85]). For $\beta(u) = |u|^{\frac{1}{m}}\text{sign}(u)$, with $m > 1$ and $p = 2$, (P) is the so-called porous medium equation and describes the non-stationary flow through a porous medium. For $p \neq 2$, this problem models the non-stationary polytropic flow of a fluid in a porous medium whose tangential stress has a power dependence of the velocity. We refer the reader to the review paper [63]. Furthermore, problem (P) includes also mathematical models from the evolution of a biological population (see [56, 55]).

Here, we shall focus on a Neumann type nonlinear boundary condition, since the Dirichlet boundary condition have been widely treated in the literature (see [19, 32, 36**?**]). This nonlinear condition occur in many applications in physics and can be for example physically interpreted as a nonlinear radiation law prescribed on the boundary of the material body (see [9, 69] and references therein).

In this work, we are concerned with existence and uniqueness of bounded solutions and existence of a global attractor for the problem (P). Our aim is to give sufficient conditions under which our problem (P) is globally well-posed in a suitable functional space setting and state conditions under which the dynamical system associated to (P) has a compact global attractor in $L^\infty(\Omega)$, by using the general setting of attractors (see R. Temam [96]).

Our work is inspired by the results of El Hachimi and El Ouardi [39] and Andreu et al. [10]. In fact, in [39], the authors extend some of the results obtained in [36] for problem (P) with Dirichlet boundary condition and initial datum in $L^2(\Omega)$. Here, assuming that the initial datum in $L^\infty(\Omega)$ and the assumptions on h are quite weaker than in [39], we shall extend the results in [39] concerning only the existence and the uniqueness of the solutions to the problem (P). Moreover, following the line ideas in [10] combined with some results in [98], we prove the existence of a global attractor in $L^\infty(\Omega)$ for (P) with $1 < p < +\infty$, when the initial datum $u_0 \in L^\infty(\Omega)$. We point out that, the conditions on nonlinearities h and g used here differ from those imposed in [10]. We also note that the choice of the space $L^\infty(\Omega)$, as indicated in that paper, is motivated by the fact that the solutions obtained are bounded for bounded initial data and that the compactness of the trajectories is straightforward by the results of [33]; the Hölder continuity of solutions obtained for a more general problem in [98] being an essential key to prove the uniform compactness of trajectories of the dynamical system associated with our problem (P).

The paper is organized as follows. Section 5.2, is devoted to the existence and uniqueness of bounded weak solutions of problem (P); while Section 5.4 deals with the existence of the global attractor in $L^\infty(\Omega)$ to the dynamic system associated to the problem (P).

5.2. Hypotheses and First Main Theorem

In this section, we start by introducing our hypotheses on the data and making precise the meaning of solutions of problem (P). Then, we state the existence result.

(H_1) The initial datum $u_0 \in L^\infty(\Omega)$ and $\beta : \mathbb{R} \to \mathbb{R}$ is an increasing locally Lipschitz function with $\beta(0) = 0$.

(H_2) $h : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$i)$ there exist $r_0 > 0$ and $c_0 > 0$ such that

$$h(x,t,s)\operatorname{sign}(s) \geq -c_0, \quad \text{for all} \quad |s| > r_0, \tag{5.1}$$

with sign is the function defined by

$$\operatorname{sign}(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -1 & \text{if } s < 0. \end{cases}$$

$ii)$ there exists an increasing function $a : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|h(x,t,s)| \leq a(|s|), \ \text{ for almost everywhere in } \Omega \times \mathbb{R}^+. \tag{5.2}$$

$iii)$ $\frac{\partial h}{\partial t}(x,t,s)$ exists and for all $\Lambda > 0$, there exists $C_\Lambda > 0$ such that

$$\left|\frac{\partial h}{\partial t}(x,t,s)\right| \leq C_\Lambda, \ \text{ for } \ |s| \leq \Lambda. \tag{5.3}$$

$iv)$ there exists $K > 0$ such that the map

$$s \mapsto h(x,t,s) + K\beta(s) \quad \text{is increasing.} \tag{5.4}$$

(H_3) g is an increasing Lipschitz continuous function and such that

$$g(s)\text{sign}(s) \geq 0 \quad \text{for all} \quad |s| > r_0, \tag{5.5}$$

where r_0 is defined in (5.1).

Let us denote by $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$, for any positive number T fixed.

Definition 5.1. *By a bounded weak solution of problem (P) we mean a function u such that $u \in L^\infty(Q) \cap L^p(0,T;W^{1,p}(\Omega)) \cap L^\infty(\tau,T;W^{1,p}(\Omega))$, for all $\tau > 0$, satisfying the identity*

$$\int_0^T \langle \partial_t(\beta(u)), \varphi\rangle + \int_Q |\nabla u|^{p-2}\nabla u \nabla\varphi + \int_\Sigma g(u)\varphi + \int_Q h(x,t,u)\varphi = 0, \tag{5.6}$$

for all $\varphi \in L^\infty(Q) \cap L^p(0,T;W^{1,p}(\Omega))$.
Moreover, if $\varphi \in L^p(0,T;W^{1,p}(\Omega)) \cap W^{1,1}(0,T;L^1(\Omega))$ with $\varphi(\cdot,T) = 0$ then

$$\int_0^T \langle \partial_t(\beta(u)), \varphi\rangle = -\int_0^T \int_\Omega (\beta(u) - \beta(u_0))\partial_t\varphi.$$

Remark 5.2.1.

1. *Obviously, since $u \in L^\infty(Q)$ and β is increasing , then we have $\beta(u) \in L^\infty(Q)$.*
2. *By hypotheses (H_1) to (H_3), we have*
$$\partial_t(\beta(u)) \in L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega)).$$
3. *Thanks to previous points (1) and (2), we have $\beta(u) \in C([0,T];H^{-s}(\Omega))$ for s large enough. Hence, the third condition of problem (P) makes sense.*

Definition 5.2. *Let β be a continuous increasing function with $\beta(0) = 0$. We define for $t \in \mathbb{R}$*

$$\Psi(t) = \int_0^t \beta(\tau)d\tau.$$

Then the Legendre transform Ψ^* of Ψ is defined by

$$\Psi^*(\tau) = \sup_{s\in\mathbb{R}}\{\tau s - \Psi(s)\}. \tag{5.7}$$

In particular, we have

$$\Psi^*(\beta(\tau)) = \tau\beta(\tau) - \Psi(\tau). \tag{5.8}$$

Remark 5.2.2. *From the equality (5.8) and Remark 5.2.1-(1), if u is bounded then $\Psi^*(\beta(u))$ is also bounded.*

The following lemmas will be useful hereafter (see [7, Lemma 1.5])

Lemma 5.2.3. *Let $u \in L^p(0,T;W^{1,p}(\Omega))$ such that $\partial_t(\beta(u)) \in L^{p'}(0,T;W^{-1,p'}(\Omega))$. Then, we have*

$$\langle \partial_t(\beta(u)), u\rangle = \frac{d}{dt}\int_\Omega \Psi^*(\beta(u)),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between $W^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$.

The following lemmas are central for the estimates we drive in what follows (see [96, Lemma 1.1 and Lemma 5.1 of Chapter 3])

Lemma 5.2.4 (The Uniform Gronwall Lemma)**.** *Let g, h, y be three positive locally integrable functions on $]t_0, +\infty[$ such that y' is locally integrable on $]t_0, +\infty[$, and which satisfy*

$$y' \leq gy + h \quad \text{for } \ t \geq t_0,$$

$$\int_t^{t+r} g(s)ds \leq a_1, \quad \int_t^{t+r} h(s)ds \leq a_2, \quad \int_t^{t+r} y(s)ds \leq a_3, \quad \text{for } \ t \geq t_0,$$

where r, a_1, a_2, a_3, are positive constants. Then,

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2\right) \exp(a_1), \quad \text{for all } \ t \geq t_0.$$

Lemma 5.2.5 (Ghidaglia Lemma)**.** *Let y be a positive absolutely continuous function on $(0, +\infty)$ which satisfies*

$$y' + \gamma y^p \leq \delta,$$

with $p > 1, \gamma > 0, \delta \geq 0$. Then, for all $t \geq 0$,

$$y(t) \leq \left(\frac{\delta}{\gamma}\right)^{1/p} + (\gamma(p-1)t)^{-1/(p-1)}.$$

We finish this recall by the usual Gronwall's lemma

Lemma 5.2.6 (Usual Gronwall Lemma)**.** *Let y be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies for a.e. t the differential inequality*

$$y' \leq gy + h,$$

where g and h are nonnegative integrable functions on $[0, T]$. Then, for all $t \in [0, T]$

$$y(t) \leq \exp\left(\int_0^t g(s)ds\right)\left[y(0) + \int_0^t h(s)ds\right].$$

We can now state our first main result of this chapter.

Theorem 5.2.7 (First Main Theorem)**.** *Under hypotheses (H_1) to (H_3), there exists a unique bounded weak solution of problem (P) such that $\beta(u) \in \mathcal{C}([0, T]; L^1(\Omega))$.*

5.3. Proof of the First Main Theorem

The proof of our first main theorem will be divided into three steps. Firstly, we obtain the existence of classical solutions for a regularized problem associated with problem (P) which can be solved in a classical sense by well-known results of [68]. Secondly, in order to study the convergence of these solutions, we show some a priori estimates in suitable functional spaces. Finally, inspired by the papers [7, 32], we prove the uniqueness of solutions.

5.3.1. Existence of bounded weak solutions

Classical solutions

Let $\epsilon > 0$, we consider the following regularized problem

$$(P_\epsilon) \begin{cases} \partial_t(\beta_\epsilon(u_\epsilon)) - \Delta_p^\epsilon u_\epsilon + h_\epsilon(x, t, u_\epsilon) = 0, & \text{in } Q, \\ -(|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} \frac{\partial u_\epsilon}{\partial \nu} = g_\epsilon(u_\epsilon), & \text{on } \Sigma, \\ \beta_\epsilon(u_\epsilon(0)) = \beta_\epsilon(u_{0,\epsilon}), & \text{in } \Omega, \end{cases}$$

where $\Delta_p^\epsilon u_\epsilon = \operatorname{div}((|\nabla u_\epsilon|^2+\epsilon)^{\frac{p-2}{2}}\nabla u_\epsilon)$ and

$$(\mathscr{B})\begin{cases}\beta_\epsilon \text{ is of class } \mathcal{C}^1(\mathbb{R}) \text{ such that } 0<\epsilon\le\beta'_\epsilon,\\ \beta_\epsilon(0)=0,\ \beta_\epsilon\to\beta \text{ in } \mathcal{C}_{loc}(\mathbb{R}) \text{ and } |\beta_\epsilon|\le|\beta|.\end{cases}$$

$$(\mathscr{H})\begin{cases}h_\epsilon \text{ is of class } \mathcal{C}^\infty(Q\times\mathbb{R}),\\ h_\epsilon(x,t,s)\to h(x,t,s) \text{ in } L^1(Q) \text{ for any fixed } s \text{ and in } \mathcal{C}_{loc}(\mathbb{R}) \text{ for a.e. } (x,t) \text{ in } Q,\\ h_\epsilon \text{ satisfies uniformly } (H_2).\end{cases}$$

$$(\mathscr{G})\begin{cases}g_\epsilon \text{ is of class } \mathcal{C}^\infty(\mathbb{R}),\ g_\epsilon\to g \text{ in } \mathcal{C}_{loc}(\mathbb{R}),\\ g_\epsilon \text{ satisfies uniformly } (H_3).\end{cases}$$

Finally, we regularize the initial condition by the same way as in the proof of [47, Proposition 3, p. 761]:

$$(\mathscr{U})\begin{cases}u_{0,\epsilon}\in\mathcal{C}^3(\overline{\Omega}) \text{ is such that } u_{0,\epsilon}\to u_0 \text{ in } L^1(\Omega),\ \|u_{0,\epsilon}\|_{L^\infty(\Omega)}\le\|u_0\|_{L^\infty(\Omega)}+1,\\ \text{and satisfies the compatibility condition :}\\ -(|\nabla u_{0,\epsilon}|^2+\epsilon)^{\frac{p-2}{2}}\frac{\partial u_{0,\epsilon}}{\partial\nu}=g_\epsilon(u_{0,\epsilon}).\end{cases}$$

We point out that since β_ϵ is increasing, $|\beta_\epsilon|\le|\beta|$ and $(u_{0,\epsilon})$ is bounded, then $(\beta_\epsilon(u_{0,\epsilon}))$ is also bounded.

Now, by using the classical results of Ladyzenskaya et al. [68, Chapter V], there exists a unique classical solution u_ϵ of problem (P_ϵ), for any fixed $T>0$.

A priori estimates

In this part, we shall show some essential estimate concerning the solutions of problem (P_ϵ). The sign condition on the nonlinearities f and g plays a crucial role for proving the boundedness of solutions of problem (P_ϵ). We have the following

Lemma 5.3.1. *For all $\epsilon>0$, we have*

$$\|u_\epsilon\|_{L^\infty(Q)}\le C, \tag{5.9}$$

where, $C:=C(T,\|u_0\|_{L^\infty(\Omega)})>0$ is independent on ϵ.

Proof. By using the sign conditions (5.1) and (5.5), we obtain for all $k>0$

$$h_\epsilon(x,t,u_\epsilon)[(\beta_\epsilon(u_\epsilon)-\beta_\epsilon(r_0))^+]^{k+1}\ge -c_0[(\beta_\epsilon(u_\epsilon)-\beta_\epsilon(r_0))^+]^{k+1},$$

and

$$g_\epsilon(u_\epsilon)[(\beta_\epsilon(u_\epsilon)-\beta_\epsilon(r_0))^+]^{k+1}\ge 0.$$

By multiplying the first equation of (P_ϵ) by $[(\beta_\epsilon(u_\epsilon)-\beta_\epsilon(r_0))^+]^{k+1}$ and using the above inequalities we get

$$\frac{1}{k+2}\frac{d}{dt}\int_\Omega[(\beta_\epsilon(u_\epsilon)-\beta_\epsilon(r_0))^+]^{k+2}\le c_0\int_\Omega[(\beta_\epsilon(u_\epsilon)-\beta_\epsilon(r_0))^+]^{k+1}. \tag{5.10}$$

We set $y_{\epsilon,k}(t):=\|(\beta_\epsilon(u_\epsilon)-\beta_\epsilon(r_0))^+\|_{L^{k+2}(\Omega)}$. By using Hölder inequality, we get

$$\int_\Omega[(\beta_\epsilon(u_\epsilon)-\beta_\epsilon(r_0))^+]^{k+1}\le c_1(y_{\epsilon,k}(t))^{k+1},$$

which yields from (5.10) that,

$$\frac{d}{dt}(y_{\epsilon,k}(t))\le c_1 \quad \text{for all } t>0, \tag{5.11}$$

which in turn gives, after integrating between 0 and t,

$$y_{\epsilon,k}(t)\le c(T)+y_{\epsilon,k}(0) \quad \text{for all } 0<t\le T, \tag{5.12}$$

Let $k \to +\infty$. Since $(u_{0,\epsilon})$ is bounded, then we get

$$\|(\beta_\epsilon(u_\epsilon(t)) - \beta_\epsilon(r_0))^+\|_{L^\infty(\Omega)} \le C(T, \|u_0\|_{L^\infty(\Omega)}), \quad \text{for all } 0 < t \le T. \tag{5.13}$$

Now, let $v_\epsilon := -u_\epsilon$. Clearly, v_ϵ is a solution of the following problem

$$(\tilde{P}_\epsilon) \begin{cases} \partial_t(\tilde{\beta}_\epsilon(v_\epsilon)) - \Delta_p^\epsilon v_\epsilon + \tilde{h}_\epsilon(x,t,v_\epsilon) = 0, & \text{in } Q, \\ -(|\nabla v_\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} \frac{\partial v_\epsilon}{\partial \nu} = \tilde{g}_\epsilon(v_\epsilon), & \text{on } \Sigma, \\ \tilde{\beta}_\epsilon(v_\epsilon(0)) = \tilde{\beta}_\epsilon(v_{0,\epsilon}), & \text{in } \Omega, \end{cases}$$

where $\tilde{\beta}_\epsilon(s) = -\beta_\epsilon(-s)$, $\tilde{h}_\epsilon(x,t,s) = -h_\epsilon(x,t,-s)$ and $\tilde{g}_\epsilon(s) = -g_\epsilon(-s)$. Moreover, $\tilde{\beta}_\epsilon$ satisfies the same properties $(\mathscr{B})$ of β_ϵ while $\tilde{h}_\epsilon$ and $\tilde{g}_\epsilon$ satisfy also the sign condition (5.1) and (5.5) respectively (with respect to $\tilde{\beta}_\epsilon$). Therefore, we follow the same previous argument to obtain the following estimate

$$\|(\tilde{\beta}_\epsilon(v_\epsilon(t)) - \tilde{\beta}_\epsilon(r_0))^+\|_{L^\infty(\Omega)} \le C'(T, \|u_0\|_{L^\infty(\Omega)}), \text{ for all } 0 < t \le T;$$

which is equivalent to

$$\|(-\beta_\epsilon(u_\epsilon(t)) + \beta_\epsilon(-r_0))^+\|_{L^\infty(\Omega)} \le C'(T, \|u_0\|_{L^\infty(\Omega)}), \text{ for all } 0 < t \le T. \tag{5.14}$$

Consequently, from (5.13) and (5.14) we conclude that

$$\|u_\epsilon(t)\|_{L^\infty(\Omega)} \le C(T, \|u_0\|_{L^\infty(\Omega)}), \text{ for all } 0 < t \le T.$$

□

Remark 5.3.2. *Note that since β_ϵ is increasing and $|\beta_\epsilon| \le |\beta|$, then using previous lemma, $(\beta_\epsilon(u_\epsilon))$ is also bounded.*

Lemma 5.3.3. *For all $\epsilon > 0$, we have*

$$\|u_\epsilon\|_{L^p(0,T;W^{1,p}(\Omega))} \le c, \tag{5.15}$$

where, $c > 0$ is independent on ϵ.

Proof. Mutiplying the first equation of (P_ϵ) by u_ϵ and integrating over Ω, we get

$$-\int_\Omega (|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} |\nabla u_\epsilon|^2 = \frac{d}{dt}\int_\Omega \Psi_\epsilon^*(\beta_\epsilon(u_\epsilon)) + \int_{\partial\Omega} g_\epsilon(u_\epsilon)u_\epsilon + \int_\Omega h_\epsilon(x,t,u_\epsilon)u_\epsilon,$$

where Ψ_ϵ^* is the Legendre transform of Ψ_ϵ associated to β_ϵ defined in Definition 5.2. Since g_ϵ is an increasing function and (u_ϵ) is bounded, then we get

$$|g_\epsilon(u_\epsilon)| \le \max(g_\epsilon(c), |g_\epsilon(-c)|) := \eta_\epsilon. \tag{5.16}$$

Due to the fact that $g_\epsilon \to g$ in $\mathcal{C}_{loc}(\mathbb{R})$, (η_ϵ) is a bounded sequence as $\epsilon \to 0$. Then, we integrate over $[0, T]$ and by using the boundedness of (u_ϵ), Remark 5.2.2 and condition (5.2), we obtain

$$\int_0^T \int_\Omega |\nabla u_\epsilon|^p \le \int_0^T \int_\Omega (|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} |\nabla u_\epsilon|^2 \le c,$$

which shows 5.15. □

Remark 5.3.4. *By using the estimate (5.15) and Young's inequality, we can easily see that $((|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} \nabla u_\epsilon)$ is bounded in $L^{p'}(0, T; W^{-1,p'}(\Omega))$.*

Lemma 5.3.5. *For all $\epsilon > 0$ and all $\tau > 0$, we have*

$$\|u_\epsilon(t)\|_{W^{1,p}(\Omega)} \leq c(\tau), \quad \text{for all } \ t \geq \tau, \tag{5.17}$$

$$\int_\tau^T \int_\Omega \beta'_\epsilon(u_\epsilon)(\partial_t(u_\epsilon))^2 \leq c'(\tau, T), \tag{5.18}$$

$$\int_\tau^T \int_\Omega (\partial_t \beta_\epsilon(u_\epsilon))^2 \leq c''(\tau, T). \tag{5.19}$$

Proof. By multiplying the first equation of (P_ϵ) by $\partial_t(u_\epsilon)$ and integrating over Ω, we get

$$\begin{aligned}\int_\Omega \beta'_\epsilon(u_\epsilon)(\partial_t(u_\epsilon))^2 + \frac{d}{dt}\left(\frac{1}{p}\int_\Omega (|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p}{2}}\right) + \int_{\partial\Omega} g_\epsilon(u_\epsilon)\partial_t(u_\epsilon) \\ + \int_\Omega h_\epsilon(x, t, u_\epsilon)\partial_t(u_\epsilon) = 0.\end{aligned} \tag{5.20}$$

We set

$$G_\epsilon(\xi) = \int_0^\xi g_\epsilon(s)ds \ \text{ and } \ H_\epsilon(x, t, \xi) = \int_0^\xi h_\epsilon(x, t, s)ds.$$

In order to simplify the proof, we can assume $g_\epsilon(0) = 0$. By using the boundedness of (u_ϵ), properties $(\mathscr{G})$ and $(\mathscr{H})$ on g_ϵ and h_ϵ respectively, there exist two positive constants A_1 and A_2 such that

$$|\int_{\partial\Omega} G_\epsilon(u_\epsilon)| \leq A_1 \ \text{ and } \ |\int_\Omega H_\epsilon(x, t, u_\epsilon)| \leq A_2. \tag{5.21}$$

Hence

$$\int_{\partial\Omega} G_\epsilon(u_\epsilon) + A_1 \geq 0 \ \text{ and } \ \int_\Omega H_\epsilon(x, t, u_\epsilon) + A_2 \geq 0.$$

Therefore, by using (5.3), relation (5.20) becomes

$$\begin{aligned}\int_\Omega \beta'_\epsilon(u_\epsilon)(\partial_t(u_\epsilon))^2 + \frac{d}{dt}\Big(\frac{1}{p}\int_\Omega (|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p}{2}} + \int_{\partial\Omega} G_\epsilon(u_\epsilon) + A_1 + \int_\Omega H_\epsilon(x, t, u_\epsilon) + A_2\Big) \\ \leq |\int_\Omega \int_0^{u_\epsilon} \frac{\partial h}{\partial t}(x, t, s)ds| \leq C_\Lambda,\end{aligned} \tag{5.22}$$

which implies

$$\frac{d}{dt}\left(\frac{1}{p}\int_\Omega (|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p}{2}} + \int_{\partial\Omega} G_\epsilon(u_\epsilon) + A_1 + \int_\Omega H_\epsilon(x, t, u_\epsilon) + A_2\right) \leq C_\Lambda, \tag{5.23}$$

where $\Lambda = C(T, \|u_0\|_{L^\infty(\Omega)})$ is defined in (5.9).

Let us fix $r > 0$. As in the proof of (5.15), we get

$$\int_t^{t+r} \int_\Omega \frac{1}{p}(|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p}{2}} \leq c_1(r).$$

By using again the boundedness of (u_ϵ) and properties $(\mathscr{G})$ and $(\mathscr{H})$ on g_ϵ and h_ϵ respectively, we get

$$\int_t^{t+r}\left(\int_{\partial\Omega} G_\epsilon(u_\epsilon) + \int_\Omega H_\epsilon(x, t, u_\epsilon)\right) \leq c_2(r).$$

Hence, by the uniform Gronwall Lemma 5.2.4, (5.17) holds true.

By using the first estimate (5.17), the boundedness of (u_ϵ) and the properties on g_ϵ and h_ϵ, we deduce easily the second one. Indeed, by integrating (5.22) over (τ, T) we obtain

$$\begin{aligned}\int_\tau^T \int_\Omega \beta'_\epsilon(u_\epsilon)(\partial_t(u_\epsilon))^2 \leq \frac{1}{p}\int_\Omega ((|\nabla u_\epsilon(\tau)|^2 + \epsilon)^{\frac{p}{2}} - (|\nabla u_\epsilon(T)|^2 + \epsilon)^{\frac{p}{2}}) \\ + \int_{\partial\Omega} (G_\epsilon(u_\epsilon(\tau)) - G_\epsilon(u_\epsilon(T))) \\ + \int_\Omega (H_\epsilon(x, t, u_\epsilon(\tau)) - H_\epsilon(x, t, u_\epsilon(T))) + C_\Lambda(T - \tau).\end{aligned} \tag{5.24}$$

Hence, (5.18) holds.

Finally, to show (5.19) we use the estimate (5.18). Indeed, since β is locally Lipschitz, we can choose β_ϵ such that $\beta'_\epsilon \leq L$, for some constant $L > 0$. Then, we get

$$\int_\tau^T \int_\Omega (\partial_t \beta_\epsilon(u_\epsilon))^2 \leq L \int_\tau^T \int_\Omega \beta'_\epsilon(u_\epsilon)(\partial_t(u_\epsilon))^2 \leq c''(\tau, T).$$

This shows (5.19). □

Remark 5.3.6. *The absence of an estimate on $(\partial_t u_\epsilon)$ exclude the application of Aubin's lemma to (u_ϵ) (see [94, Corollary 4]); but we can apply that lemma to $(\beta_\epsilon(u_\epsilon))$ (see below).*

Passing to the limit

In order to pass to the limit, we use the previous lemmas. Indeed, by estimates (5.9), (5.15) and (5.17), there exists a subsequence of (u_ϵ) (called again (u_ϵ)) and a function u such that

$$u_\epsilon \to u \text{ weakly}^* \text{ in } L^\infty(Q),$$

$$u_\epsilon \to u \text{ weakly in } L^p(0, T; W^{1,p}(\Omega)),$$

$$u_\epsilon \to u \text{ weakly}^* \text{ in } L^\infty(\tau, T; W^{1,p}(\Omega)), \text{ for all } \tau > 0.$$

Now, we pass to the limit on the nonlinear term g_ϵ at the boundary . For this, we have

$$g_\epsilon(u_\epsilon) \to g(u) \text{ strongly in } L^p(0, T; L^p(\partial\Omega)). \tag{5.25}$$

Indeed, from estimate (5.15), $u_\epsilon \to u$ weakly in $L^p(0, T; W^{1,p}(\Omega))$. By using Theorem 3.4.5 of Morrey [80, p. 76], we get that $u_\epsilon \to u$ strongly in $L^p(0, T; L^p(\partial\Omega))$. Moreover, since g_ϵ is a Lipschitz function, then we get

$$\begin{aligned}\int_0^T \int_{\partial\Omega} |g_\epsilon(u_\epsilon) - g(u)|^p &\leq c_1(p) \int_0^T \int_{\partial\Omega} (|g_\epsilon(u_\epsilon) - g_\epsilon(u)|^p + |g_\epsilon(u) - g(u)|^p) \\ &\leq c(p) \int_0^T \int_{\partial\Omega} (|u_\epsilon - u|^p + |g_\epsilon(u) - g(u)|^p).\end{aligned}$$

Using the fact that g_ϵ converges uniformly to g in any compact subset of $\mathbb{R}$, (5.25) holds true.

By using the growth condition (5.2) on h_ϵ and Vitali's theorem, $h_\epsilon(x, t, u_\epsilon)$ converges strongly to $h(x, t, u)$ in $L^1(Q)$.

On the other hand, there exists $\xi \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ such that

$$\Delta_p^\epsilon u_\epsilon \to \xi \text{ weakly in } L^{p'}(0, T; W^{-1,p'}(\Omega)).$$

Indeed, let $v \in L^p(0, T; W^{1,p}(\Omega))$. We have

$$\left| \int_Q (\Delta_p^\epsilon u_\epsilon) v \right| \leq \int_Q (|\nabla u_\epsilon|^2 + \epsilon)^{\frac{p-2}{2}} |\nabla u_\epsilon||\nabla v| + \int_\Sigma |g_\epsilon(u_\epsilon)||v| + \int_Q |h_\epsilon(x, t, u_\epsilon)||v|.$$

The first, second and third terms on the right-hand side of this inequality are bounded due to Remark 5.3.4, (5.16) and (5.2). Furthermore, by using the Minty argument, we get $\xi = \Delta_p u$ (see e.g. [72] and [19]).

By using (5.17) and applying the same argument as in the proof of the estimate (5.19), we obtain that $(\beta_\epsilon(u_\epsilon))$ is bounded in $L^\infty(\tau, T; W^{1,p}(\Omega))$, for all $\tau > 0$. Moreover, from (5.19) we get that $(\partial_t \beta_\epsilon(u_\epsilon))$ is bounded in $L^2(\tau, T; L^2(\Omega))$, for all $\tau > 0$. Hence, by using Aubin's lemma (see [94, Corollary 4]), we deduce that $(\beta_\epsilon(u_\epsilon))$ is relatively compact in $\mathcal{C}(]0, T]; L^1(\Omega))$. Therefore, $\beta_\epsilon(u_\epsilon) \to \zeta$ strongly in $\mathcal{C}(]0, T]; L^1(\Omega))$. Thus, by using the same arguments of [19, p. 1048], we obtain $\zeta = \beta(u)$.

In order to prove the continuity at $t = 0$, we shall use Lemma 5.3.11 below, which gives the uniform Lipschitz continuity of solutions in $L^1(\Omega)$. In a first step, we deal with initial data $u_0 \in \mathcal{C}^1(\overline{\Omega})$; then, we take a sequence $(u_{0,\epsilon})$ bounded in $W^{1,p}(\Omega)$ and satisfying to conditions $(\mathscr{U})$. We have the following result which is a consequence of Lemma 5.3.5

Corollary 5.3.7. *For all $\epsilon > 0$ we have*

$$\|u_\epsilon(t)\|_{W^{1,p}(\Omega)} \leq c, \quad \textit{for all} \;\; t > 0. \tag{5.26}$$

$$\int_0^T \int_\Omega \beta'_\epsilon(u_\epsilon)(\partial_t(u_\epsilon))^2 \leq c'(T). \tag{5.27}$$

$$\int_0^T \int_\Omega (\partial_t \beta_\epsilon(u_\epsilon))^2 \leq c''(T). \tag{5.28}$$

Now, we continue our proof of continuity at $t = 0$. We proceed as in [10]. From the previous corollary we deduce that $\beta_\epsilon(u_\epsilon) \to \beta(u)$ strongly in $\mathcal{C}([0,T]; L^1(\Omega))$ for initial data $u_0 \in \mathcal{C}^1(\overline{\Omega})$.

Let us now assume that $u_0 \in L^\infty(\Omega)$ and take a smooth sequence $(u_{0,\epsilon})$ satisfying to $(\mathscr{U})$. According to the first case, the corresponding solution $\beta(u_\epsilon)$ are continuous at $t = 0$. Since $(u_{0,\epsilon})$ is bounded and converges to u_0 in $L^1(\Omega)$, then by the dominate convergence theorem $\beta(u_{0,\epsilon}) \to \beta(u_0)$ in $L^1(\Omega)$. Moreover, we have

$$\begin{aligned}\|\beta(u(t)) - \beta(u(0))\|_{L^1(\Omega)} \leq \|\beta(u(t)) - \beta(u_\epsilon(t))\|_{L^1(\Omega)} + \|\beta(u_\epsilon(t)) - \beta(u_{0,\epsilon})\|_{L^1(\Omega)} \\ + \|\beta(u_{0,\epsilon}) - \beta(u_0)\|_{L^1(\Omega)}.\end{aligned}$$

By using Lemma 5.3.11, we get

$$\begin{aligned}\|\beta(u(t)) - \beta(u(0))\|_{L^1(\Omega)} \leq e^{Kt}\|\beta(u_0) - \beta(u_{0,\epsilon})\|_{L^1(\Omega)} + \|\beta(u_\epsilon(t)) - \beta(u_{0,\epsilon})\|_{L^1(\Omega)} \\ + \|\beta(u_{0,\epsilon}) - \beta(u_0)\|_{L^1(\Omega)}.\end{aligned}$$

Hence, all terms of the right-hand side tend to 0 as $\epsilon \to 0$. Consequently, $\beta(u) \in C([0,T]; L^1(\Omega))$.

Finally, by passing to the limit in (P_ϵ) when $\epsilon \to 0$, we obtain that u is a bounded weak solution of problem (P) in the sense of Definition 5.1.

Remark 5.3.8. *If we assume that h does not depend on t and verifies the condition (2.1') (see Section 5.4 page 75), then we can show that $\partial_t \beta(u) \in L^2(0, +\infty, L^2(\Omega))$. This can be done directly from Lemma 5.5.2 (below) by integrating 5.24. We point out that, in the paper [32], the authors obtained the same estimate on $\partial_t \beta(u)$ under the condition $u_0 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ in the case of Dirichlet boundary condition and for a general operator than p-Laplacian. However, the growth conditions considered in that paper are different from ours.*

5.3.2. Uniqueness of bounded weak solutions

The proof of the uniqueness of solutions is inspired by [7, Theorem 2.2] and [32, Theorem 3]. This result is formulated as a comparison principle and is important for the study of global attractors. Here, the main difference between the equation studied in these papers and the one we consider here lies in the fact that the boundary condition is nonlinear. Therefore, we can not use their arguments directly since the usual Gronwall Lemma 5.2.6 is failed.

Now, we state the comparison principle result as follows

Lemma 5.3.9. *Let $\overline{u}$ and $\underline{u}$ be two solutions of problem (P) corresponding to different initial data $\overline{u}_0$ and $\underline{u}_0$ respectively, such that $\underline{u}_0 \leq \overline{u}_0$. Then, we have $\underline{u} \leq \overline{u}$ in Q.*

Remark 5.3.10. *The arguments used in both papers [7] and [32] for proving the comparison principle result are based, among other conditions, on the hypothesis that $\partial_t(\beta(\underline{u}))$ and $\partial_t(\beta(\overline{u}))$ belong to $L^1(Q)$. In this work, we show that if u is a solution of problem (P) then $\partial_t(\beta(u))$ belongs to $L^2(\tau, T; L^2(\Omega))$ for all $\tau > 0$. Therefore, since $\beta(u) \in \mathcal{C}([0,T]; L^1(\Omega))$ then $\partial_t(\beta(u))$ belongs to $L^1(Q)$.*

Proof. For small $\delta > 0$, let us set

$$\psi_\delta(z) := \min\left(1, \max\left(\frac{z}{\delta}, 0\right)\right), \quad \text{for all} \;\; z \in \mathbb{R}.$$

By taking $\varphi := \psi_\delta(\underline{u} - \overline{u})$ as a test function in the problem (P) corresponding to $\overline{u}$ and $\underline{u}$, then we obtain

$$\int_0^t \int_\Omega \partial_t(\beta(\underline{u}) - \beta(\overline{u}))\psi_\delta(\underline{u} - \overline{u}) + \int_0^t \int_\Omega (|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla \overline{u}|^{p-2}\nabla \overline{u})\nabla(\underline{u} - \overline{u})\psi'_\delta(\underline{u} - \overline{u})$$
$$+ \int_0^t \int_{\partial\Omega} (g(\underline{u}) - g(\overline{u}))\psi_\delta(\underline{u} - \overline{u}) + \int_0^t \int_\Omega (h(x,t,\underline{u}) - h(x,t,\overline{u}))\psi_\delta(\underline{u} - \overline{u}) = 0.$$

By using the monotonicity of the operator $-\Delta_p$, the second term is nonnegative. Now, let tends $\delta \to 0$. By using the fact that $\partial_t(\beta(\underline{u})), \partial_t(\beta(\overline{u})) \in L^1(Q)$, hypothesis (5.4) and the fact that g is increasing, then we obtain

$$\int_0^t \int_\Omega \partial_t(\beta(\underline{u}) - \beta(\overline{u}))\psi_\delta(\underline{u} - \overline{u}) \to \int_0^t \int_\Omega \partial_t(\beta(\underline{u}) - \beta(\overline{u}))\chi_{\{\underline{u}-\overline{u}>0\}}$$
$$= \int_\Omega (\beta(\underline{u}(t)) - \beta(\overline{u}(t)))^+,$$

$$\int_0^t \int_\Omega (h(x,t,\underline{u}) - h(x,t,\overline{u}))\psi_\delta(\underline{u} - \overline{u}) \to \int_0^t \int_\Omega (h(x,t,\underline{u}) - h(x,t,\overline{u}))\chi_{\{\underline{u}-\overline{u}>0\}}$$
$$\geq -K \int_0^t \int_\Omega (\beta(\underline{u}) - \beta(\overline{u}))^+,$$

and

$$\int_0^t \int_{\partial\Omega} (g(\underline{u}) - g(\overline{u}))\psi_\delta(\underline{u} - \overline{u}) \to \int_0^t \int_{\partial\Omega} (g(\underline{u}) - g(\overline{u}))\chi_{\{\underline{u}-\overline{u}>0\}} \geq 0, \tag{5.29}$$

where χ and $v^+ := \max(v, 0)$ denote the characteristic function and positive part of v respectively. Hence, we get

$$\int_\Omega (\beta(\underline{u}(t)) - \beta(\overline{u}(t)))^+ \leq K \int_0^t \int_\Omega (\beta(\underline{u}) - \beta(\overline{u}))^+. \tag{5.30}$$

Thus, from usual Gronwall Lemma 5.2.6 we deduce that $\beta(\underline{u}) \leq \beta(\overline{u})$. Since β is increasing, then we have in particular $\beta(\underline{u}) = \beta(\overline{u})$ in the set $\{\underline{u} - \overline{u} > 0\}$. Using this last conclusion and the following well-known inequalities

$$(|a|^{p-2}a - |b|^{p-2}b).(a-b) \geq c(p) \begin{cases} |a-b|^p, & \text{if } p \geq 2, \\ \frac{|a-b|^2}{(|a|+|b|)^{2-p}}, & \text{if } 1 < p < 2, \end{cases}$$

for any real vectors a and b, where $c(p) = 2^{2-p}$ when $p \geq 2$ and $c(p) = p - 1$ when $1 < p < 2$, then we get

$$\nabla(\underline{u} - \overline{u}) = 0 \quad \text{in the set} \quad \{0 < \underline{u} - \overline{u} < \delta\},$$

hence, $\max(0, \min(\underline{u} - \overline{u}, \delta)) =$ const; which implies $\underline{u} \leq \overline{u}$, since it is true on Σ. □

We end this section with the following lemma which affirms the uniform Lipschitz continuity of solutions in $L^1(\Omega)$. This result, will be useful in the next section.

Lemma 5.3.11. *Let $\overline{u}$ and $\underline{u}$ be two solutions of problem (P) corresponding to different initial data $\overline{u}_0$ and $\underline{u}_0$ respectively. Then, the following L^1-Lipschitz continuity holds:*

$$\|\beta(\overline{u}(t)) - \beta(\underline{u}(t))\|_{L^1(\Omega)} \leq e^{Kt}\|\beta(\overline{u}_0) - \beta(\underline{u}_0)\|_{L^1(\Omega)}, \tag{5.31}$$

Proof. by using (5.29) and following the same arguments as in [32], the lemma holds. □

5.4. Existence of global attractor in $L^\infty(\Omega)$

In this section, by using the general setting of attractors, we shall show that the problem (P) has a global attractor in $L^\infty(\Omega)$.

5.4.1. Preliminary results

In this part, we recall some basic backgrounds related to the theory of dynamical systems. Precisely, we recall some concepts in the sense of R. Temam [96]. Let E be a Banach space.

Definition 5.3 (Semigroup operator). *A family of (nonlinear) operators $(S(t))_{t\geq 0}$ acting on E into itself is called a continuous semigroup if satisfying the following properties:*

1. *$S(t)$ is continuous from E into itself for all $t \geq 0$.*
2. *$S(t+s) = S(t) \circ S(s)$ for all $t, s \geq 0$.*
3. *$S(0) = I$ (identity in E).*

Definition 5.4 (Invariant set). *A set $\mathcal{A} \subset E$ is an invariant set for the semigroup $(S(t))_{t\geq 0}$ if*

$$S(t)\mathscr{A} = \mathscr{A} \quad \text{for all} \quad t \geq 0.$$

Definition 5.5 (Global attractor). *The set $\mathscr{A} \subset E$ is called a global attractor of the dynamical system $((S(t))_{t\geq 0}, E)$ if the following conditions are satisfied:*

1. *The set $\mathscr{A}$ is a nonempty compact subset of E.*
2. *The set $\mathscr{A}$ is invariant.*
3. *The set $\mathscr{A}$ attracts each bounded subset $\mathscr{B}$ of E, that is the following holds:*

$$dist(S(t)\mathscr{B}, \mathscr{A}) \to 0 \ \text{ as } \ t \to +\infty,$$

where $dist(A, B) = \sup_{a\in A} \inf_{b\in B} \|a-b\|_E$.

In order to establish the existence of attractors, a useful concept is the related concept of absorbing sets.

Definition 5.6 (Absorbing set). *Let $\mathscr{B}$ be a subset of E and $\mathscr{U}$ an open set containing $\mathscr{B}$. We say that $\mathscr{B}$ is absorbing in $\mathscr{U}$ if the orbit of any bounded set of $\mathscr{U}$ enters into $\mathscr{B}$ after a certain time (which may depend on the set):*

$$\forall \mathscr{B}_0 \subset \mathscr{U}, \quad \mathscr{B}_0 \ \text{bounded}, \ \exists\, t_1(\mathscr{B}_0) \quad \text{such that} \ S(t)\mathscr{B}_0 \subset \mathscr{B}, \quad \forall t \geq t_1(\mathscr{B}_0).$$

Definition 5.7 (Uniformly compact operator). *We say that a family of operators $(S(t))_{t\geq 0}$ are uniformly compact fort t large if, for every bounded set $\mathscr{B}$ there exists t_0 which may depend on $\mathscr{B}$ such that*

$$\bigcup_{t\geq t_0} S(t)\mathscr{B}$$

is relatively compact in E.

5.4.2. Hypotheses and Second Main Theorem

In this part, based on the first main result and by adding some supplementary assumptions on our data β, h and g, we shall show the existence of a global attractor in $L^\infty(\Omega)$. More precisely, let us define the family of nonlinear maps $(S(t))_{t\geq 0}$ by

$$\begin{array}{rcccl} S(t) & : & L^\infty(\Omega) & \mapsto & L^\infty(\Omega) \\ & & u_0 & \mapsto & \beta(u(t)), \end{array}$$

where u is the unique bounded weak solution of problem (P) corresponding to initial datum u_0. By Theorem 5.2.7, this map is well defined. It is worth pointing out that, from Lemma 5.3.11, the nonlinear map $(S(t))_{t\geq 0}$ is continuous in L^1-norm, but is not continuous in $L^\infty(\Omega)$-norm. Therefore, as our dynamical system follows the approach of R. Temam [96], then even if we prove the uniform compactness of $(S(t))_{t\geq 0}$ and the existence of an absorbing set, the conclusion of the existence of the global attractor does not follow directly. To overcome this difficulty, we shall use Lemma 5.3.11 in spirit to [10].

Now, in order to prove the existence of a global attractor, we need to show that the solutions of problem (P) are Hölder continuous. This result will be used in the proof of the compactness of the trajectories, that is, to obtain that the nonlinear operator $(S(t))_{t\geq 0}$ is uniformly compact for t large enough; which in turn is an important step to obtain the existence of a global attractor. The basic idea to show the Hölder continuity of solutions, is to apply Theorem 1.1 of [98]. Henceforth, the crucial fact is to adapt the hypotheses of this theorem to our problem (P). For this end, we need some additional appropriate hypotheses on the function β and on the boundary datum g.

Before to state the main result of this section, we perform the following formal computations in order to justify our considered assumptions.

We set $\beta(u) = v$, and replace this change of variable in the first and second equations of (P). We get

$$\partial_t(v) - div(|(\beta^{-1}(v))'|^{p-2}(\beta^{-1}(v))'|\nabla v|^{p-2}\nabla v) + h(x,t,\beta^{-1}(v)) = 0. \tag{5.32}$$

$$-|(\beta^{-1}(v))'|^{p-2}(\beta^{-1}(v))'|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} = g(\beta^{-1}(v)). \tag{5.33}$$

Identifying the equation (5.32) with equation (1) of [98] leads to consider the following supplementary hypotheses on β

$(H_1)'$ β is strictly increasing and β^{-1} belongs to $\mathcal{C}^1(\mathbb{R})$.

$(H_2)'$ $i)$ $(\beta^{-1})'$ is degenerate near the origin in the sense that there exists an interval $[-\delta_0,\delta_0]$ around the origin such that for all $s \in [-\delta_0,\delta_0]$

$$\alpha_1|s|^{\gamma_1} \leq (\beta^{-1}(s))' \leq \alpha_2|s|^{\gamma_2}, \tag{5.34}$$

for some constants $\alpha_1,\alpha_2 > 0$ and $\gamma_1,\gamma_2 \geq 0$.

$ii)$ $(\beta^{-1})'$ is bounded from above and from below; that is, for all $s \in \mathbb{R}\backslash[-\delta_0,\delta_0]$ we have

$$\Lambda_1 \leq (\beta^{-1}(s))' \leq \Lambda_2, \tag{5.35}$$

for some positive constants $\Lambda_1 \leq \Lambda_2$.

Concerning the nonlinear term g, according to the remark d) of [98], we assume the following supplementary hypothesis

$(H_3)'$ $g \in \mathcal{C}^1(\mathbb{R})$.

In order that our nonlinear operator $(S(t))_{t\geq 0}$ satisfies the properties of semigroup, that is, $S(t+s) = S(t)\circ S(s)$, we need to assume

$(H_4)'$ $h(x,t,u) := h(x,u)$.

Finally, to show the existence of absorbing sets in the space $L^\infty(\Omega)$ for the dynamical system $(S(t))_{t\geq 0}$, we need to replace the condition (5.1) of (H_2) by the following one : there exist $c_1, c_2 > 0$ such that

$$h(x,t,s)\text{sign}(s) \geq c_1|\beta(s)|^{q-1} - c_2, \quad \text{for all} \quad |s| > r_0, \tag{2.1'}$$

with $q > \sup(2,p)$.

Note that this condition is stronger than (5.1). Consequently, Theorem 5.2.7 holds under (2.1').

Remark 5.4.1.

- *Observe that the assumption* $(H_2)'$*-i) is equivalent to* $\lim_{|s|\to 0} \beta'(s) = +\infty$, *that is,* β' *blows-up at* $s = 0$. *As a simple example for this type of functions, we have*

$$\beta(s) = |s|^{\frac{1}{m}} sign(s), \quad with \;\; m > 1. \tag{5.36}$$

Consequently, the class of doubly nonlinear porous media-type equations includes in our study.

- *We point out that, if* $\gamma_1 = \gamma_2 = 0$ *in* (5.34), *in other words, if the assumption* (5.35) *is satisfied for all* $s \in \mathbb{R}$, *then functions like* (5.36) *are preclude.*

Remark 5.4.2. *We note that since the solutions of our problem* (P) *are bounded, then we can apply Theorem 1.1 of [98]. Hence, this result is given as follows*

Theorem 5.4.3 ([98])**.** *Assume that Theorem 5.2.7 holds. Then, under hypotheses* $(H_1)'$ *to* $(H_3)'$, *the solutions of problem* (P) *are Hölder continuous in* $\overline{\Omega} \times [\epsilon, T]$, *for all* $\epsilon > 0$.

The next result is the second main theorem of this chapter which assures the existence of the global attractor for the semigroup $(S(t))_{t\geq 0}$ related to the problem (P).

Theorem 5.4.4 (Second Main Theorem)**.** *Assume that Theorem 5.2.7 holds and that hypotheses (2.1'),* $(H_1)'$ *to* $(H_4)'$ *are satisfied. Then, the semigroup* $(S(t))_{t\geq 0}$ *related to the problem* (P) *possesses a global attractor* $\mathscr{A}$ *in* $L^\infty(\Omega)$.

5.5. Proof of the Second Main Theorem

In this section, we give a proof of Theorem 5.4.4. To this end, firstly, we show that the operators $(S(t))_{t\geq 0}$ are uniformly compact for t large. Secondly, we show the existence of absorbing sets in $L^\infty(\Omega)$. Finally, we construct a set $\mathscr{A}$, then we prove that this set is a global attractor.

Lemma 5.5.1. *Let* $\mathscr{B}$ *be a bounded set of* $L^\infty(\Omega)$ *and* $t_0 > 0$. *Then,*

$$\bigcup_{t\geq t_0} S(t)\mathscr{B},$$

is relatively compact in $L^\infty(\Omega)$.

Proof. At first, the set $\bigcup_{t\geq 0} S(t)\mathscr{B}$ is bounded in $L^\infty(\Omega)$ by Lemma 5.3.1. In other words, the approximate solutions are uniformly bounded. Moreover, from Theorem 5.4.3, they are Hölder continuous for any $t \geq t_0 > 0$. Consequently, by the Ascoli-Arzelà theorem, the lemma holds. □

Lemma 5.5.2. *Under hypotheses* (H_1) *to* (H_3), *with (2.1') instead of (5.1), there exists a positive constant* ρ *such that for any* $u_0 \in L^\infty(\Omega)$, *we have*

$$\|u(t)\|_{L^\infty(\Omega)} \leq \rho, \quad for\ all \;\; t > 0.$$

Proof. Firstly, let $k > 0$. Multiply the first equation of (P_ϵ) by $[(\beta_\epsilon(u_\epsilon) - \beta_\epsilon(r_0))^+]^{k+1}$, see Lemma 5.3.1, then we get

$$\begin{aligned} \frac{1}{k+2}\frac{d}{dt}\int_\Omega [(\beta_\epsilon(u_\epsilon) - \beta_\epsilon(r_0))^+]^{k+2} + c_1 \int_\Omega [(\beta_\epsilon(u_\epsilon) - \beta_\epsilon(r_0))^+]^{q+k} \\ \leq c_2 \int_\Omega [(\beta_\epsilon(u_\epsilon) - \beta_\epsilon(r_0))^+]^{k+1}. \end{aligned} \tag{5.37}$$

Set $y_{\epsilon,k}(t) := \|(\beta_\epsilon(u_\epsilon) - \beta_\epsilon(r_0))^+\|_{L^{k+2}(\Omega)}$. Using Hölder inequality, gives

$$(y_{\epsilon,k}(t))^{q+k} \leq c_4 \int_\Omega [(\beta_\epsilon(u_\epsilon) - \beta_\epsilon(r_0))^+]^{q+k},$$

and
$$\int_{\Omega}[(\beta_\epsilon(u_\epsilon)-\beta_\epsilon(r_0))^+]^{k+1}\leq c_5(y_{\epsilon,k}(t))^{k+1}.$$
Consequently, from (5.37) we get the following inequality
$$\frac{d}{dt}(y_{\epsilon,k}(t))+\delta(y_{\epsilon,k}(t))^{q-1}\leq\lambda, \tag{5.38}$$
where δ and λ are two positive constants depending only on q. By applying Ghidaglia's Lemma 5.2.5 on (5.38), we get
$$y_{\epsilon,k}(t)\leq c_6(q)+\frac{1}{[c_7(\|u_{0,\epsilon}\|_{L^k(\Omega)})+c_8(q)t]^{\frac{1}{q-2}}},\quad \text{for all}\ \ t>0, \tag{5.39}$$
with $c_7(\|u_{0,\epsilon}\|_{L^k(\Omega)})>0$.
Let $k\to+\infty$ in (5.39). Since $\|u_{0,\epsilon}\|_{L^\infty(\Omega)}\leq\|u_0\|_{L^\infty(\Omega)}+1$, then we get
$$\|(\beta_\epsilon(u_\epsilon(t))-\beta_\epsilon(r_0))^+\|_{L^\infty(\Omega)}\leq c_9,\quad \text{for all}\ \ t>0. \tag{5.40}$$
Hence,
$$\|u_\epsilon(t)\|_{L^\infty(\Omega)}\leq c_{10},\quad \text{for all}\ \ t>0, \tag{5.41}$$
where $c_{10}>0$ depends only on $\|u_0\|_{L^\infty(\Omega)}$.

Secondly, let $v_\epsilon:=-u_\epsilon$. Then v_ϵ is a solution of problem $(\tilde{P}_\epsilon)$ (see Lemma 5.3.1). By proceeding as in the proof of Lemma 5.3.1, we obtain
$$\|(-\beta_\epsilon(u_\epsilon(t))+\beta_\epsilon(-r_0))^+\|_{L^\infty(\Omega)}\leq c_{11},\quad \text{for all}\ \ t>0. \tag{5.42}$$
Hence,
$$\|u_\epsilon(t)\|_{L^\infty(\Omega)}\leq c_{12},\quad \text{for all}\ \ t>0, \tag{5.43}$$
where, $c_{12}>0$ depends only on $\|u_0\|_{L^\infty(\Omega)}$.
Therefore, the ball $B(0,\rho)$ centered at 0 and with radius $\rho:=\max(c_{10},c_{12},r_0)$ is an absorbing set in $L^\infty(\Omega)$. □

Remark 5.5.3. *There exists also absorbing sets in $W^{1,p}(\Omega)$ for the dynamical system $(S(t))_{t\geq 0}$. Indeed, arguing as for (5.17) of Lemma 5.3.5, we get*
$$\|u_\epsilon(t)\|_{W^{1,p}(\Omega)}\leq c(\tau),\quad \textit{for all}\ \ t\geq\tau>0.$$
Furthermore, by using the weak convergences of u_ϵ to u in $L^p(0,T;W^{1,p}(\Omega))$ and the lower semi-continuity of the norm, we get
$$\|u(t)\|_{W^{1,p}(\Omega)}\leq c(\tau)=\rho,\quad \textit{for all}\ \ t\geq\tau>0.$$
Hence, the ball $B(0,\rho)$ centered at 0 and with radius ρ is an absorbing set in $W^{1,p}(\Omega)$.

Proof of Theorem 5.4.4. We set
$$\mathscr{A}:=\omega(\mathscr{K})=\{u\in L^\infty(\Omega):\exists t_n\to+\infty \text{ and } \exists u_n\in\mathscr{K} \text{ such that } S(t_n)u_n\to u \\ \text{in } L^\infty(\Omega)\},$$
where $\mathscr{K}:=\overline{S(\tau)\mathscr{B}}^{L^\infty(\Omega)}$ for some $\tau>0$. At first, from Lemma 5.5.1 and Lemma 5.5.2, the set $\mathscr{K}$ is well defined and is a compact absorbing subset of $L^\infty(\Omega)$. Now, we shall show that $\mathscr{A}$ satisfies the all conditions of Definition 5.5. By construction of $\mathscr{A}$, the first one is satisfied. Let us show the second one. Let $v=S(t)u$ with $u\in\mathscr{A}$. Since $u\in\mathscr{A}$, there exist $u_n\in\mathscr{K}$ and $t_n\to+\infty$ such that $S(t_n)u_n\to u$ in $L^1(\Omega)$. Then, Lemma 5.3.11 implies that $S(t+t_n)u_n=S(t)(S(t_n)u_n)\to S(t)u=v$ in $L^1(\Omega)$. By using the proprieties of the semigroup and the fact that $\mathscr{K}$ is an absorbing set, then we get $S(t+t_n)u_n\in\mathscr{K}$ for n large enough. Consequently, by construction of $\mathscr{K}$, we also have $S(t+t_n)u_n\to v$ in $L^\infty(\Omega)$,

hence $v \in \mathscr{A}$. By the same argument, the inverse implication follows because for $v \in \mathscr{A}$, there exist $t_n \to +\infty$ and $v_n \in \mathscr{K}$ such that $S(t_n)v_n \to v$ in $L^1(\Omega)$, we have $S(t_n - t)v_n \in \mathscr{K}$ and therefore $S(t)u = v$; so that $v \in S(t)\mathscr{A}$. A simple argument by contradiction gives condition 3) of Definition 5.5. Indeed, let B be a bounded subset in $L^\infty(\Omega)$. There exists $\delta > 0$ and $t_n \to +\infty$, $v_n \in B$ such that

$$\operatorname{dist}(S(t_n)v_n, \mathscr{A}) \geq \frac{\delta}{2} > 0, \tag{5.44}$$

for each n. Now, as $\mathscr{K}$ is a compact absorbent subset, there exists $t(B) := t_0 > 0$ such that $S(t_n)v_n$ belongs to $\mathscr{K}$ for $t_n > t_0$ and $S(t_n)v_n \to v$ in $L^\infty(\Omega)$. Whence, $v \in w(\mathscr{K}) = \mathscr{A}$ and this contradicts (5.44). $\square$

CHAPTER 6

EXISTENCE OF PERIODIC SOLUTIONS FOR SOME QUASILINEAR PARABOLIC PROBLEMS WITH VARIABLE EXPONENTS

In this chapter, we prove the existence of at least one periodic solution for some nonlinear parabolic boundary value problems associated with Leray-Lions's operators with variable exponents under the hypothesis of existence of well-ordered sub and supersolutions.

6.1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, and fixed $T > 0$. Our aim here, is to prove existence of periodic solutions for the following nonlinear parabolic problem

$$(P) \begin{cases} \partial_t u + \mathcal{A}u = f(x,t,u,\nabla u) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u(T) & \text{in } \Omega, \end{cases}$$

where $\mathcal{A}u = -\text{div}(\mathbf{A}(\cdot,\cdot,u,\nabla u))$ is a Leray-Lions's type operator with variable exponents acting from some functional space V_0 (see below) into its topological dual V_0' and where f is a nonlinear Carathéodory function, whose growth with respect to $|\nabla u|$ is at most of order $p(x)$ in the sense defined below (hypothesis A4).)

The suitable functional spaces to deal with in this type of problems are generalized Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, respectively (for more details see Chapter 1). As we have seen in Chapter 1, there are many differences between Lebesgue and Sobolev spaces with constant exponents and those with variable exponents. Recalling, for instance, that $p(x)$ need to satisfy the log-Höder condition (see Definition 1.8) in order that the Poincaré's inequality and the density of smooth functions in $W^{1,p(x)}(\Omega)$ hold true. Many difficulties arise when we study problems like (P) in the case of variable exponents. One typical difficulty when dealing with these type of problems is to define adequate functional spaces for solutions. In the case $p(x) = p$ is a constant, it is well known that $L^p(0,T;W_0^{1,p}(\Omega))$ can be taken as a space of solutions. However, when $p(x)$ is nonconstant, then nor $L^{p(x)}(0,T;W_0^{1,p(x)}(\Omega))$ neither $L^{p_-}(0,T;W_0^{1,p(x)}(\Omega))$, where $p_- = \min_\Omega p(x)$, constitute a suitable space of solutions (see [17].) Henceforth, in order to overcome this difficulty, we shall define below our functional space of solutions V_0 as it was done by Bendahmane in [17].

Nonlinear problems defined by (P) arises in many applications. For instance, in electrorheological fluids, whose essential part of the energy is given by $\int_\Omega |Du(x)|^{p(x)}dx$ (see [88] and Motivation 1.2). This type of fluids has the ability to change it's mechanical properties (for example becoming a solid gel) when an electric field is applied. Another important application is that when f depends only on (x,t) and by taking $\mathbf{A}(x,t,s,\xi) = |\xi|^{p(x)-2}\xi$; then the problem (P) can be seen as a sort of nonlinear diffusion equation whose coefficient of diffusion takes the form $|\nabla u|^{p(x)-2}$ (see [6]). For more applications, we refer the reader to [104, 25].

There is by now an extensive literature on the existence of solutions like problem (P). Let us start by recalling some known results concerning the case $p(x) := p$ is a real constant. In [31], by applying a penalty method to an appropriately associated auxiliary parabolic variational inequality, J. Deuel and P. Hess proved the existence of at least one periodic solution for problem (P) in the case where the natural growth of f with respect to $|\nabla u|$ is of order less than p; that means, $|f(x,t,u,\nabla u)| \leq k(x,t) + c|\nabla u|^{p-\delta}$ for some $\delta > 0$, $k(x,t) \in L^{1+\delta}(\Omega \times (0,T))$, and c being a positive constant. In [54], N. Grenon extends the result of [31] to the case where the natural growth of f with respect to $|\nabla u|$ is at most of order p; but instead of a periodicity condition the author considered an initial one. The proof therein is based on some regularization techniques used in [20, 79].

Let us point out that in the two previous works, the hypothesis of existence of well-ordered sub and supersolutions is assumed. Following [31], the results in [54] were extended by A. El Hachimi, A. A. Lamrani in [40], where the authors obtained the existence of periodic solutions, under the same hypotheses as in [54]. For variable exponents, some particular cases of problems has been studied by many authors [6, 49, 17, 102], by means of different methods such as: subdifferential operators, Galerkin scheme, semigroup theory, etc.

The main goal of this paper is to extend the results in [40] to the variable exponents case by using the sub and supersolutions method. It is well known that this method, when it is applicable, has more advantages compared to other methods . For example: we can give some information on the behavior of the solution (blow-up or extinction) and the sign of the solution (positive or negative). Nevertheless, this method is quite complicated because it requires well-ordered sub and supersolutions, which is not usually easy to get. Indeed, in many application cases, sub and supersolutions are obtained from eigenfunction associated to the first eigenvalue of some operators (say the p-Laplacian.) But, when dealing some with variable exponents, it is well known that the $p(x)$-Laplacian does not have in general a first eigenvalue (see [44]) and therefore, we have to find sub and supersolution by means of other ideas (see our application example in section 5).

6.2. Hypotheses and main result

Let $p : \overline{\Omega} \mapsto [1,+\infty)$ be a continuous, real-valued function. Denote by $p_- = \min_{x\in\overline{\Omega}} p(x)$ and $p_+ = \max_{x\in\overline{\Omega}} p(x)$. Throughout this chapter, we shall assume that the variable exponents $p(x)$ satisfies the log-Hölder condition (see Definition 1.8) and that $1 < p_- \leq p_+ < \infty$. For more detail concerning the Lebesgue and Sobolev spaces with variable exponent we refer the readers to Chapter 1.

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, $Q = \Omega \times (0,T)$ where $T > 0$ is fixed and $\Sigma = \partial\Omega \times (0,T)$.

We set
$$V_0 = \{f \in L^{p_-}(0,T; W_0^{1,p(x)}(\Omega));\ |\nabla f| \in L^{p(x)}(Q)\},$$
endowed with the norm
$$\|f\|_{V_0} = \|\nabla f\|_{L^{p(x)}(Q)},$$
or, the equivalent norm
$$\|f\|_{V_0} = \|f\|_{L^{p_-}(0,T;W_0^{1,p(x)}(\Omega))} + \|\nabla f\|_{L^{p(x)}(Q)}.$$

Remark 6.2.1. *The equivalence of the two norms above is obtained by using Poincaré's inequality and the continuous embedding $L^{p(x)}(Q) \hookrightarrow L^{p_-}(0,T;L^{p(x)}(\Omega))$.*

We set
$$V = \{f \in L^{p_-}(0,T; W^{1,p(x)}(\Omega));\ |\nabla f| \in L^{p(x)}(Q)\}.$$

Following lemma gives some properties of V_0

Lemma 6.2.2 ([17]). *We denote by V_0' the dual space of V_0. Then*

- *we have the following continuous dense embeddings:*

$$L^{p_+}(0,T;W_0^{1,p(x)}(\Omega)) \overset{d}{\hookrightarrow} V_0 \overset{d}{\hookrightarrow} L^{p_-}(0,T;W_0^{1,p(x)}(\Omega)).$$

  In particular, since $\mathscr{D}(Q)$ is dense in $L^{p_+}(0,T;W_0^{1,p(x)}(\Omega))$, it is also dense in V_0 and for the corresponding dual spaces we have

$$L^{(p_-)'}(0,T;(W_0^{1,p(x)}(\Omega))') \hookrightarrow V_0' \hookrightarrow L^{(p_+)'}(0,T;(W_0^{1,p(x)}(\Omega))').$$

- *one can represent the elements of V_0' as follows: let $G \in V_0'$, then there exists $F = (f_1, f_2, \cdots, f_N) \in (L^{p'(x)}(Q))^N$ such that $G =$ -div(F) and*

$$\langle G, u\rangle_{V',V_0} = \int_0^T \int_\Omega F\cdot\nabla u \, dxdt,$$

  for any $u \in V_0$.

Now, let us give the hypotheses concerning the functions $\mathbf{A}$ and f introduced in problem (P).

A1) $\mathbf{A}$ is a Carathéodory function defined on $Q\times\mathbb{R}\times\mathbb{R}^N$, with values in $\mathbb{R}^N$ such that there exist $\lambda > 0$, and $l \in L^{p'(x)}(Q)$, $l \geq 0$, so that for all $s\in\mathbb{R}$ and for all $\xi\in\mathbb{R}^N$: (say growth condition of $\mathbf{A}$)

$$|\mathbf{A}(x,t,s,\xi)| \leq \lambda(l(x,t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}), \quad \text{a.e in } Q.$$

A2) For all $s\in\mathbb{R}$ and for all $\xi,\xi'\in\mathbb{R}^N$, with $\xi\neq\xi'$: (say monotonicity condition of $\mathbf{A}$)

$$(\mathbf{A}(x,t,s,\xi) - \mathbf{A}(x,t,s,\xi'))\cdot(\xi-\xi') > 0, \quad \text{a.e in } Q.$$

A3) There exists $\alpha > 0$, so that for all $s\in\mathbb{R}$ and for all $\xi\in\mathbb{R}^N$: (say coercivity condition of $\mathbf{A}$)

$$\mathbf{A}(x,t,s,\xi)\cdot\xi \geq \alpha|\xi|^{p(x)}, \quad \text{a.e in } Q.$$

A4) f is a Carathéodory function on $Q\times\mathbb{R}\times\mathbb{R}^N$, and there exist a function $b:\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ increasing, and $h\in L^1(Q), h\geq 0$, such that: (say natural growth condition on f respect to $|\xi|$ of order $p(x)$)

$$|f(x,t,s,\xi)| \leq b(|s|)(h(x,t) + |\xi|^{p(x)}), \quad \text{for } (x,t,s,\xi)\in Q\times\mathbb{R}\times\mathbb{R}^N.$$

Remark 6.2.3. *If $u \in V_0\cap L^\infty(Q)$. Then, Under the assumptions A1), A2) and A3) we have $\mathcal{A}u \in V_0'$. Moreover, under the assumption A4) we have $f(x,t,u,\nabla u)\in L^1(Q)$.*

Definition 6.1. *A periodic solution for problem (P) is a measurable function $u: Q\mapsto\mathbb{R}$ satisfying the following conditions*

$$u\in V_0\cap L^\infty(Q),\ \ \partial_t u \in V_0' + L^1(Q), \tag{6.1}$$

$$\langle\partial_t u, \phi\rangle_{V_0'+L^1(Q),V_0\cap L^\infty(Q)} + \int_Q \mathbf{A}(x,t,u,\nabla u)\cdot\nabla\phi = \int_Q f(x,t,u,\nabla u)\phi \ \ \textit{for all}\ \ \phi\in V_0\cap L^\infty(Q), \tag{6.2}$$

$$u(x,0) = u(x,T) \ \ \textit{for all}\ \ x\in\Omega. \tag{6.3}$$

Remark 6.2.4. *Thanks to the previous remark and (6.2), we have $\partial_t u \in V_0' + L^1(Q)$. Moreover, the periodicity condition (6.3) makes sense according to the following lemma.*

Lemma 6.2.5 ([17]). *We set $\mathcal{W} := \{u\in V_0;\ \partial_t u\in V_0' + L^1(Q)\}$. Then, we have the following embedding*

$$\mathcal{W}\cap L^\infty(Q) \hookrightarrow C([0,T];L^2(\Omega)).$$

Definition 6.2. *A subsolution (in the distributional sense) of problem (P) is a function $\varphi \in V \cap L^\infty(Q)$ such that $\partial_t \varphi \in V_0' + L^1(Q)$ and*

$$\begin{cases} \partial_t \varphi + \mathcal{A}\varphi \leq f(x,t,\varphi,\nabla\varphi) & \text{in } Q, \\ \varphi \leq 0 & \text{on } \Sigma, \\ \varphi(0) \leq \varphi(T) & \text{in } \Omega, \end{cases}$$

A supersolution of problem (P) is obtained by reversing the inequalities.

We can now state the main result of this chapter

Theorem 6.2.6. *Suppose that $\mathbf{A}$ verifies the hypotheses $A1)$, $A2)$, $A3)$, and that f satisfies $A4)$. Moreover, assume the existence of a subsolution φ, and a supersolution ψ, such that $\varphi \leq \psi$ a.e in Q. Then, there exists at least one periodic solution u of problem (P), such that $\varphi \leq u \leq \psi$ a.e in Q.*

6.3. Proof of Theorem 6.2.6

Before we start the proof of the main theorem, we need the following technical lemmas which will be used later.

Lemma 6.3.1 ([71]). *Let $\pi : \mathbb{R} \to \mathbb{R}$ be a C^1 piecewise function with $\pi(0) = 0$ and $\pi' = 0$ outside a compact set. Let $\Pi(s) = \int_0^s \pi(\sigma)d\sigma$. If $u \in V_0 \cap L^\infty(Q)$ with $\partial_t u \in V_0' + L^1(Q)$, then*

$$\int_0^T \langle \partial_t u, \pi(u)\rangle = \langle \partial_t u, \pi(u)\rangle_{V_0'+L^1(Q), V_0\cap L^\infty(Q)} = \int_\Omega \Pi(u(T))dx - \int_\Omega \Pi(u(0))dx.$$

Lemma 6.3.2 ([5]). *Assume that $A1)$, $A2)$ and $A3)$ are satisfied and let (u_n) be a sequence in V_0 which converges weakly to u in V_0, and*

$$\limsup_{n\to\infty} \int_Q (\mathbf{A}(x,t,u_n,\nabla u_n) - \mathbf{A}(x,t,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u) \leq 0.$$

Then,

$$u_n \to u \ \text{ strongly in } \ V_0.$$

6.3.1. Truncation of problem (P)

Definition 6.3. *Let φ be a subsolution and ψ a supersolution of problem (P), such that $\varphi \leq \psi$ a.e in Q. For $u \in V$, the truncation function $T(u)$ is defined by*

$$T(u) = u - (u-\psi)^+ + (\varphi - u)^+.$$

We shall denote by

$$\mathbf{A}(u,\nabla u)(x,t) = \mathbf{A}(x,t,u(x,t),\nabla u(x,t))$$

and

$$F(u,\nabla u)(x,t) = f(x,t,u(x,t),\nabla u(x,t)),$$

the Nemyskii operators associated respectively to the functions $\mathbf{A}$ and f.

For almost everywhere (x,t) in Q, we define

$$\mathbf{A}^\star(u,\nabla u)(x,t) = \mathbf{A}(Tu,\nabla u)(x,t)$$

and

$$F^\star(u,\nabla u)(x,t) = F(Tu,\nabla Tu)(x,t).$$

Remark 6.3.3. *Note that, $F^\star$ is not a Carathéodory function since it is not continuous with respect to ∇u. This constraint will be overcome thanks to the following lemma.*

Lemma 6.3.4. *The operator $F^\star : u \to F^\star(u, \nabla u)$ is defined and continuous from V into $L^1(Q)$. Moreover, there exists a constant $C > 0$ such that*

$$|F^\star(u,\nabla u)(x,t)| \leq C(h^\star(x,t) + |\nabla u|^{p(x)}), \quad \text{a.e in } Q, \tag{6.4}$$

where $h^\star$ is a nonnegative function in $L^1(Q)$.

Proof. The proof of this lemma is similar to the case when $p(x)$ is a constant (see [54].) □

Denote by $\mathcal{A}^\star u = -\text{div}(\mathbf{A}^\star(u, \nabla u))$. Then $\mathcal{A}^\star$ is a Leray-Lions's type operator from V_0 into its dual V_0', that means, $\mathbf{A}^\star$ satisfies the assumptions $A1$), $A2$) and $A3$) above.

6.3.2. Penalization and regularization of problem (P)

Let $k > 0$ be a constant such that

$$-k \leq \varphi - 1 \leq \psi + 1 \leq k, \quad \text{a.e in } Q.$$

We set

$$K = \{v \in V_0, \ \ -k \leq v \leq k \ \ \text{a.e in } Q\}.$$

Then, K is a closed convex set of V_0.

Definition 6.4. *Let $\eta > 0$. Then, the penalization operator related to K is defined by $\frac{1}{\eta}\beta(u)$, where*

$$\beta(u) = [(u-k)^+]^{(p_-)-1} - [(u+k)^-]^{(p_-)-1}, \text{ for } u \in V.$$

Obviously, we have

$$\beta(u)u \geq 0 \ \text{ a.e in } \ Q, \ \text{ and } \ K \equiv \{v \in V_0, \beta(v) = 0 \text{ a.e in } Q\}.$$

Let $\epsilon > 0$. For $u \in V$, and for almost everywhere (x,t) in Q we set

$$F_\epsilon^\star(u,\nabla u)(x,t) = \frac{|F^\star(u,\nabla u)(x,t)|}{1+\epsilon|F^\star(u,\nabla u)(x,t)|}.$$

It is clear that $F_\epsilon^\star(u, \nabla u) \in L^\infty(Q)$ for all $u \in V$. Moreover, the mapping $u \to F_\epsilon^\star(u,\nabla u)$ is continuous from V into $L^1(Q)$, and from (6.4) we can easily verify that

$$|F_\epsilon^\star(u,\nabla u)(x,t)| \leq C(h^\star(x,t) + |\nabla u|^{p(x)}), \quad \text{a.e in } Q. \tag{6.5}$$

where C is a constant which is independent of ϵ.

We will now consider the following penalized-regularized problem

$$(P_{\eta,\epsilon}^\star)\begin{cases} u_{\eta,\epsilon} \in V_0, \ \partial_t u_{\eta,\epsilon} \in V_0', & \\ \partial_t u_{\eta,\epsilon} + \mathcal{A}^\star u_{\eta,\epsilon} - F_\epsilon^\star(u_{\eta,\epsilon}, \nabla u_{\eta,\epsilon}) + \frac{1}{\eta}\beta(u_{\eta,\epsilon}) = 0 & \text{in } Q, \\ u_{\eta,\epsilon} = 0 & \text{in } \Sigma, \\ u_{\eta,\epsilon}(0) = u_{\eta,\epsilon}(T) & \text{in } \Omega. \end{cases}$$

By application of theorem 1.2, p.319 in [72] we can ensure the existence of a solution of problem $(P_{\eta,\epsilon}^\star)$. Indeed, we have:

Proposition 6.3.5. *Let $D = \{u \in V_0,$ such that $\partial_t u \in V_0'$ and $u(0) = u(T)\}$. Then, the operator $\mathcal{N}u = \mathcal{A}^\star u - F_\epsilon^\star(u, \nabla u) + \frac{1}{\eta}\beta(u)$ defined from D into V_0' is bounded, coercive and pseudo-monotone. Moreover, there exists at least one solution $(u_{\eta,\epsilon})$ of problem $(P_{\eta,\epsilon}^\star)$.*

Proof. **The boundedness of $\mathcal{N}$:** From the assumption $A1)$ and the definition of $\mathcal{N}$, we have

$$|\langle \mathcal{N}u, v\rangle| \leq \lambda \left(\int_Q l(x,t)|\nabla v| + \int_Q |Tu|^{p(x)-1}|\nabla v| + \int_Q |\nabla u|^{p(x)-1}|\nabla v| \right) + \int_Q |F_\epsilon^\star(u,\nabla u)||v| + \frac{1}{\eta}\int_Q |\beta(u)||v|. \tag{6.6}$$

We treat each integral in the right-side member of (6.6).

By remark 6.2.3 (recall that φ, ψ are in the $L^\infty(Q)$), we have

$$\int_Q l(x,t)|\nabla v| + \int_Q |Tu|^{p(x)-1}|\nabla v| \leq C\|v\|_{V_0}.$$

By using Hölder's inequality, we get

$$\int_Q |\nabla u|^{p(x)-1}|\nabla v| \leq c_2 \||\nabla u|^{p(x)-1}\|_{L^{p'(x)}(Q)}\|v\|_{V_0}.$$

Then, if $\||\nabla u|^{p(x)-1}\|_{L^{p'(x)}(Q)} \leq 1$, it's over. Otherwise, from inequality (1.3), we have

$$\||\nabla u|^{p(x)-1}\|^{(p')_-}_{L^{p'(x)}(Q)} = \||\nabla u|^{p(x)-1}\|^{p'_+}_{L^{p'(x)}(Q)} \leq \int_Q \left(|\nabla u|^{p(x)-1}\right)^{p'(x)},$$

and

$$\int_Q \left(|\nabla u|^{p(x)-1}\right)^{p'(x)} \leq \max\{\|\nabla u\|^{p_-}_{L^{p(x)}(Q)}, \|\nabla u\|^{p_+}_{L^{p(x)}(Q)}\} = \max\{\|u\|^{p_-}_{V_0}, \|u\|^{p_+}_{V_0}\}.$$

Hence,

$$\int_Q |\nabla u|^{p(x)-1}|\nabla v| \leq c \max\{\|u\|^{\frac{p_-}{p'_+}}_{V_0}, \|u\|^{(p_+)-1}_{V_0}\}\|v\|_{V_0}.$$

Since $|F_\epsilon^\star| \leq \frac{1}{\epsilon}$ and $V_0 \hookrightarrow L^1(Q)$, then we get

$$\int_Q |F_\epsilon^\star(u,\nabla u)||v| \leq (c_1/\epsilon)\|v\|_{V_0}.$$

Moreover, we have

$$\frac{1}{\eta}\int_Q |\beta(u)||v| \leq \frac{1}{\eta}\left(\int_Q |(u-k)^+|^{(p_-)-1}|v| + \int_Q |(u+k)^-|^{(p_-)-1}|v|\right). \tag{6.7}$$

Now, since $u \in V_0 \hookrightarrow L^{p_-}(Q)$, we get $((u-k)^+)^{(p_-)-1} \in L^{(p_-)'}(Q)$. Then, by using Hölder's inequality in (6.7), we obtain

$$\int_Q |(u-k)^+|^{(p_-)-1}|v| \leq c_3\|(u-k)^+\|^{(p_-)-1}_{V_0}\|v\|_{V_0} \leq c_3\|u\|^{(p_-)-1}_{V_0}\|v\|_{V_0}.$$

Similarly, we obtain

$$\int_Q |(u+k)^-|^{(p_-)-1}|v| \leq c_4\|u\|^{(p_-)-1}_{V_0}\|v\|_{V_0}.$$

Whence,

$$\|\mathcal{N}u\|_{V_0'} \leq \gamma\left(1 + \|u\|^{(p_-)-1}_{V_0} + \max\{\|u\|^{\frac{p_-}{p'_+}}_{V_0}, \|u\|^{(p_+)-1}_{V_0}\}\right).$$

where γ is a positive constant.

Coercivity of $\mathcal{N}$: From the definition of $\mathcal{N}$, we have

$$\langle \mathcal{N}u, u\rangle = \langle \mathcal{A}^\star u, u\rangle - \langle F^\star_\epsilon(u,\nabla u), u\rangle + \langle \frac{1}{\eta}\beta(u), u\rangle.$$

Furthermore, we have

$$\langle \mathcal{A}^\star u, u\rangle = \int_Q \mathbf{A}^\star(u,\nabla u)\cdot\nabla u \geq \alpha \int_Q |\nabla u|^{p(x)} \geq \alpha \min\{\|u\|_{V_0}^{p_-}, \|u\|_{V_0}^{p_+}\},$$

$$\langle F^\star_\epsilon(u,\nabla u), u\rangle = \int_Q \frac{F^\star(u,\nabla u)}{1+\epsilon|F^\star(u,\nabla u)|}u \leq \int_Q \frac{|F^\star(u,\nabla u)|}{1+\epsilon|F^\star(u,\nabla u)|}|u| \leq \frac{1}{\epsilon}\|u\|_{L^1(Q)} \leq \frac{c}{\epsilon}\|u\|_{V_0},$$

and

$$\langle \frac{1}{\eta}\beta(u), u\rangle = \frac{1}{\eta}\int_Q \beta(u)u \geq 0.$$

Hence,

$$\frac{\langle \mathcal{N}u, u\rangle}{\|u\|_{V_0}} \geq \alpha \min\{\|u\|_{V_0}^{(p_-)-1}, \|u\|_{V_0}^{(p_+)-1}\} - \frac{c}{\epsilon}.$$

Whence,

$$\frac{\langle \mathcal{N}u, u\rangle}{\|u\|_{V_0}} \to +\infty, \text{ when } \|u\|_{V_0} \to +\infty.$$

Pseudo-monotonicity of $\mathcal{N}$: Let $(u_n) \in D$ and $u \in D$, such that u_n converges weakly to u in V_0 and $\partial_t u_n$ converges weakly to $\partial_t u$ in V_0'. By Lemma 6.2.2, u_n converges weakly to u in $L^{p_-}(0,T;W_0^{1,p(x)}(\Omega))$ and $\partial_t u_n$ converges weakly to $\partial_t u$ in $L^{(p_+)'}(0,T;W^{-1,p'(x)}(\Omega))$.

Moreover, we assume that

$$\lim_{n\to\infty}\sup\langle \mathcal{N}u_n, u_n - u\rangle \leq 0, \tag{6.8}$$

and we shall prove that

$$\lim_{n\to\infty}\inf\langle \mathcal{N}u_n, u_n - v\rangle \geq \langle \mathcal{N}u, u - v\rangle \text{ for all } v \in V_0. \tag{6.9}$$

Choose $s > \frac{N}{2}+1$ such that $W^{-1,p'(x)}(\Omega) \hookrightarrow H^{-s}(\Omega)$; then $\partial_t u_n$ converges weakly to $\partial_t u$ in $L^{(p_+)'}(0,T;H^{-s}(\Omega)$
We set $B_0 = W_0^{1,p(x)}(\Omega)$, $B = L^{p(x)}(\Omega)$ and $B_1 = H^{-s}(\Omega)$. Then, we have the following embeddings

$$B_0 \overset{c}{\hookrightarrow} B \hookrightarrow B_1, \tag{6.10}$$

where $B_0 \overset{c}{\hookrightarrow} B$ means that B_0 is compactly embedded in B. By theorem of Aubin-Lions's, p. 57-58 in [72], we deduce that u_n converges strongly to u in $L^{p_-}(0,T;L^{p(x)}(\Omega))$, which embedded into $L^{p_-}(Q)$.

Furthermore, we have

$$\frac{1}{\eta}\int_Q |\beta(u_n)||u_n - u| \leq \frac{1}{\eta}\left[\int_Q |(u_n-k)^+|^{(p_-)-1}|u_n-u| + \int_Q |(u_n+k)^-|^{(p_-)-1}|u_n-u|\right]. \tag{6.11}$$

By using Hölder's inequality and the embedding of V_0 into $L^{p_-}(Q)$ in (6.11), we get

$$\frac{1}{\eta}\int_Q |\beta(u_n)||u_n - u| \leq \frac{c}{\eta}\left(\|u_n\|_{V_0}^{(p_-)-1}\|u_n - u\|_{L^{p_-}(Q)}\right).$$

Since (u_n) is bounded in V_0 and u_n converges strongly to u in $L^{p_-}(Q)$, we obtain

$$\frac{1}{\eta}\int_Q |\beta(u_n)||u_n - u| \to 0 \text{ when } n \to +\infty. \tag{6.12}$$

On the other hand, since $|F_\epsilon^\star| \leq \frac{1}{\epsilon}$ and $L^{p_-}(Q)$ is embedded in $L^1(Q)$, we get

$$\int_Q |F_\epsilon^\star(u_n, \nabla u_n)||u_n - u| \leq \frac{c}{\epsilon}\|u_n - u\|_{L^{p_-}(Q)} \to 0 \text{ when } n \to \infty. \tag{6.13}$$

We develop each term of $\langle \mathcal{N} u_n, u_n - u\rangle$ and use (6.8), (6.12) and (6.13), to obtain

$$\lim_{n\to\infty} \sup \langle \mathcal{N} u_n, u_n - u\rangle = \lim_{n\to\infty} \sup \int_Q \mathbf{A}(Tu_n, \nabla u_n) \cdot \nabla(u_n - u) \leq 0. \tag{6.14}$$

Applying Vitali's theorem and weak convergence of $\mathbf{A}(Tu_n, \nabla u)$ in $(L^{p'(x)}(Q))^N$, we obtain

$$\lim_{n\to\infty} \int_Q \mathbf{A}(Tu_n, \nabla u) \cdot \nabla(u_n - u) = 0. \tag{6.15}$$

By using (6.14) and (6.15), then we obtain

$$\lim_{n\to\infty} \int_Q [\mathbf{A}(Tu_n, \nabla u_n) - \mathbf{A}(Tu_n, \nabla u)] \cdot (\nabla u_n - \nabla u) \leq 0.$$

Now, thanks to Lemma 6.3.2, we get

$$u_n \to u \ \text{ strongly in } V_0 \ \text{ that means } \nabla u_n \to \nabla u \ \text{ strongly in } (L^{p(x)}(Q))^N.$$

Hence, the inequality (6.9) holds.

Finally, by using theorem 1.2, p. 319 in [72], we deduce the existence of at least one solution $(u_{\eta,\epsilon})$ of problem $(P^\star_{\eta,\epsilon})$. □

6.3.3. A priori estimates

In this section, we are going to obtain some estimations on the sequence solutions $(u_{\eta,\epsilon})$ of problem $(P^\star_{\eta,\epsilon})$ independently of η and ϵ.

Estimates on $(u_{\eta,\epsilon})_\eta$ in V_0 and $L^\infty(Q)$

Let us fix ϵ, and denote by $(u_\eta) \equiv: (u_{\eta,\epsilon})$. Then, we have the following lemma

Lemma 6.3.6. *The sequences $(\frac{1}{\eta}\beta(u_\eta))_\eta$ and $(u_\eta)_\eta$ are bounded in $L^{(p_-)'}(Q)$ and V_0 respectively.*

Proof. From the definition of $\frac{1}{\eta}\beta(u_\eta)$, we deduce that

$$\left\|\frac{1}{\eta}\beta(u_\eta)\right\|_{L^{(p_-)'}(Q)} \leq \left\|\frac{(u_\eta - k)^+}{\eta^{\frac{1}{(p_-)-1}}}\right\|^{(p_-)-1}_{L^{p_-}(Q)} + \left\|\frac{(u_\eta + k)^-}{\eta^{\frac{1}{(p_-)-1}}}\right\|^{(p_-)-1}_{L^{p_-}(Q)}.$$

Then, we only need to show that:

$$\left(\frac{(u_\eta - k)^+}{\eta^{\frac{1}{(p_-)-1}}}\right)_\eta \text{ and } \left(\frac{(u_\eta + k)^-}{\eta^{\frac{1}{(p_-)-1}}}\right)_\eta \text{ are bounded in } L^{p_-}(Q).$$

Since $(u_\eta - k)^+ \in V_0$, then by multiplying $(P^\star_{\eta,\epsilon})$ by $(u_\eta - k)^+$, we get

$$\langle \partial_t u_\eta, (u_\eta - k)^+\rangle + \int_Q \mathbf{A}^\star(u_\eta, \nabla u_\eta) \cdot \nabla(u_\eta - k)^+ - \int_Q F_\epsilon^\star(u_\eta, \nabla u_\eta)(u_\eta - k)^+ + \frac{1}{\eta}\int_Q ((u_\eta - k)^+)^{p_-} = 0.$$

Since $u_\eta(0) = u_\eta(T)$, then from Lemma 6.3.1, we deduce that $\langle \partial_t u_\eta, (u_\eta - k)^+ \rangle = 0$. Hence

$$\frac{1}{\eta}\int_Q ((u_\eta - k)^+)^{p_-} = \int_Q F_\epsilon^\star(u_\eta, \nabla u_\eta)(u_\eta - k)^+ - \int_Q \mathbf{A}^\star(u_\eta, \nabla u_\eta) \cdot \nabla (u_\eta - k)^+. \tag{6.16}$$

Under the assumption $A3)$, the second integral in the right-hand side of equality (6.16) is nonnegative. On the other hand, we have $|F_\epsilon^\star| \le 1/\epsilon$. Let us divide both sides of equality (6.16) by $\eta^{1/(p_{-1})-1}$. By using Hölder's inequality, we obtian

$$\int_Q \frac{((u_\eta - k)^+)^{p_-}}{\eta^{(p_-)'}} \le C\left[\int_Q \frac{((u_\eta - k)^+)^{p_-}}{\eta^{(p_-)'}}\right]^{1/p_-},$$

where C is independent on η. Hence, $\left(\frac{(u_\eta - k)^+}{\eta^{\frac{1}{(p_-)-1}}}\right)_\eta$ is bounded in $L^{p_-}(Q)$.

Using $(-(u_\eta + k)^-)$ as a test function, we prove in the same way that $\left(\frac{(u_\eta + k)^-}{\eta^{\frac{1}{(p_-)-1}}}\right)_\eta$ is bounded in $L^{p_-}(Q)$.

Now we prove that $(u_\eta)_\eta$ is bounded in V_0. Multiplying $(P_{\eta,\epsilon}^\star)$ by u_η and using the assumption $A3)$, Lemma 6.3.1, and the fact that $\beta(u_\eta)u_\eta \ge 0$, we obtain

$$\alpha \int_Q |\nabla u_\eta|^{p(x)} \le \int_Q \mathbf{A}^\star(u_\eta, \nabla u_\eta) \cdot \nabla u_\eta \le \int_Q F_\epsilon^\star(u_\eta, \nabla u_\eta) u_\eta \le \frac{c}{\epsilon}\|u_\eta\|_{V_0}.$$

By using the inequality (1.3), then we get

$$\min\{\|u_\eta\|_{V_0}^{(p_-)-1}, \|u_\eta\|_{V_0}^{(p_+)-1}\} \le c(\epsilon, \alpha).$$

Since $(p_-) - 1$ and $(p_+) - 1$ are strictly greater than 0, we deduce that $(u_\eta)_\eta$ is bounded in V_0. □

Lemma 6.3.7. *The sequence $(\partial_t u_\eta)_\eta$ is bounded in V_0'.*

Proof. Let $v \in V_0$, from the first equation of problem $(P_{\eta,\epsilon}^\star)$, we get

$$\langle \partial_t u_\eta, v\rangle = -\langle \mathcal{A}^\star u_\eta, v\rangle + \langle F_\epsilon^\star(u_\eta, \nabla u_\eta), v\rangle - \langle \frac{1}{\eta}\beta(u_\eta), v\rangle.$$

Thus,

$$|\langle \partial_t u_\eta, v\rangle| \le \int_Q |\mathbf{A}^\star(u_\eta, \nabla u_\eta)||\nabla v| + \int_Q |F_\epsilon^\star(u_\eta, \nabla u_\eta)||v| + \int_Q \frac{1}{\eta}|\beta(u_\eta)||v|. \tag{6.17}$$

We treat each integral in the right-hand side of (6.17). We claim first that $\mathbf{A}^\star(u_\eta, \nabla u_\eta)$ is bounded in $(L^{p'(x)}(Q))^N$. Indeed, if $\|\mathbf{A}^\star(u_\eta, \nabla u_\eta)\|_{L^{p'(x)}(Q)} \le 1$, the claim is obvious. Otherwise, we have

$$\|\mathbf{A}^\star(u_\eta, \nabla u_\eta)\|_{L^{p'(x)}(Q)}^{(p')_-} \le \int_Q |\mathbf{A}^\star(u_\eta, \nabla u_\eta)|^{p'(x)}.$$

Since $p'(x) \le (p')_+$, and $(a+b)^p \le 2^{p-1}(a^p + b^p)$ for all $a, b \ge 0$ and $p > 1$, then according to the assumption $A1)$, we get

$$\int_Q |\mathbf{A}^\star(u_\eta, \nabla u_\eta)|^{p'(x)} \le c(\lambda, (p')_+)\left[\int_Q l(x,t)^{p'(x)} + \int_Q |Tu_\eta|^{p(x)} + \int_Q |\nabla u_\eta|^{p(x)}\right]. \tag{6.18}$$

By using the inequality (1.3) for the third integral in the right-hand side of (6.18), and the fact that (u_η) is bounded in V_0 (by Lemma 6.3.6), we can deduce the boundedness of $\mathbf{A}^\star(u_\eta, \nabla u_\eta)$ in $(L^{p'(x)}(Q))^N$.

Since $|F_\epsilon^\star| \le \frac{1}{\epsilon}$, $\frac{1}{\eta}\beta(u_\eta)$ is bounded in $L^{(p_-)'}(Q)$ (by Lemma 6.3.6) and $v \in V_0 \hookrightarrow L^{p_-}(Q) \hookrightarrow L^1(Q)$, then we use Hölder's inequality in (6.17), to obtain the desired result. □

As in (6.10), by Aubin-Lions's theorem, we can extract a subsequence, still denoted by (u_η) which is relatively compact in $L^{p_-}(0,T;L^{p(x)}(\Omega)) \hookrightarrow L^{p_-}(Q)$. Furthermore, there exists $u_\epsilon \in V_0$ such that: for all $\epsilon > 0$ fixed, we have as $\eta \to 0$

$$u_\eta \to u_\epsilon \quad \text{strongly in } L^{p_-}(Q) \text{ and a.e in } Q, \tag{6.19}$$

$$u_\eta \rightharpoonup u_\epsilon \quad \text{weakly in } V_0, \tag{6.20}$$

$$\partial_t u_\eta \rightharpoonup \partial_t u_\epsilon \quad \text{weakly in } V_0'. \tag{6.21}$$

Now, as $F_\epsilon^\star(u_\eta, \nabla u_\eta)$ and $\frac{1}{\eta}\beta(u_\eta)$ are bounded in $L^{(p_-)'}(Q)$, independently of η, then there exist β_ϵ and F_ϵ in $L^{(p_-)'}(Q)$, such that

$$\frac{1}{\eta}\beta(u_\eta) \rightharpoonup \beta_\epsilon \quad \text{in} \quad L^{(p_-)'}(Q), \tag{6.22}$$

and

$$F_\epsilon^\star(u_\eta, \nabla u_\eta) \rightharpoonup F_\epsilon \quad \text{in} \quad L^{(p_-)'}(Q). \tag{6.23}$$

In addition, as $\mathbf{A}^\star(u_\eta, \nabla u_\eta)$ is bounded in $(L^{p'(x)}(Q))^N$, then there exists χ_ϵ in $(L^{p'(x)}(Q))^N$ such that

$$\mathbf{A}^\star(u_\eta, \nabla u_\eta) \rightharpoonup \chi_\epsilon \quad \text{in} \quad (L^{p'(x)}(Q))^N \hookrightarrow (L^{(p_-)'}(Q))^N. \tag{6.24}$$

The estimations in V_0 and $L^\infty(Q)$ obtained above do not allow us to take directly the limit in the problem $(P_{\eta,\epsilon}^\star)$, due to the fact that the term $F_\epsilon^\star(u_\eta, \nabla u_\eta)$ is bounded only in $L^1(Q)$. To overcome this difficulty we need the strong convergence of (u_η) in V_0. To this end, we shall prove the following lemma

Lemma 6.3.8. *(u_η) converges strongly to (u_ϵ) in V_0, when η tends to zero.*

Proof. The proof is almost the same as in the case when the exponents $p(x) = p$ is a constant (see [54]). Thus, we give here only a sketch. Since $\mathbf{A}^\star$ satisfies the hypothesis $A1), A2), A3)$, and the sequence (u_η) converges weakly to u_ϵ in V_0, then we shall apply Lemma 6.3.2. For this, it suffices to show that

$$\limsup_{\eta \to 0} \int_Q \left(\mathbf{A}^\star(u_\eta, \nabla u_\eta) - \mathbf{A}^\star(u_\eta, \nabla u_\epsilon)\right) \cdot (\nabla u_\eta - \nabla u_\epsilon) = 0. \tag{6.25}$$

We consider $\mu > 0$ and we subtract $(P_{\eta,\epsilon}^\star)$ from $(P_{\mu,\epsilon}^\star)$, we get

$$\partial_t u_\eta - \partial_t u_\mu + \mathcal{A}^\star u_\eta - \mathcal{A}^\star u_\mu - F_\epsilon^\star(u_\eta, \nabla u_\eta) + F_\epsilon^\star(u_\mu, \nabla u_\mu) + \frac{1}{\eta}\beta(u_\eta) - \frac{1}{\mu}\beta(u_\mu) = 0.$$

We multiply this equation by $u_\eta - u_\mu$, and use Lemma 6.3.1, to obtain

$$\int_Q \left(\mathbf{A}^\star(u_\eta, \nabla u_\eta) - \mathbf{A}^\star(u_\mu, \nabla u_\mu)\right) \cdot (\nabla u_\eta - \nabla u_\mu) - \int_Q \left(F_\epsilon^\star(u_\eta, \nabla u_\eta) - F_\epsilon^\star(u_\mu, \nabla u_\mu)\right)(u_\eta - u_\mu) \tag{6.26}$$
$$+ \int_Q \left(\frac{1}{\eta}\beta(u_\eta) - \frac{1}{\mu}\beta(u_\eta)\right)(u_\eta - u_\mu) = 0.$$

Firstly, we take the lim sup when η tends to 0 and secondly the lim sup when μ tends to 0 in (6.26). By using (6.19), (6.20), (6.21), (6.22) and (6.23), we obtain

$$\limsup_{\eta \to 0} \int_Q \mathbf{A}^\star(u_\eta, \nabla u_\eta) \cdot \nabla u_\eta - \int_Q \chi_\epsilon \nabla u_\epsilon - \int_Q \chi_\epsilon \nabla u_\epsilon + \limsup_{\mu \to 0} \int_Q \mathbf{A}^\star(u_\mu, \nabla u_\mu) \cdot \nabla u_\mu = 0.$$

So, for $\mu = \eta$, we get

$$\limsup_{\eta \to 0} \int_Q \mathbf{A}^\star(u_\eta, \nabla u_\eta) \cdot \nabla u_\eta = \int_Q \chi_\epsilon \nabla u_\epsilon. \tag{6.27}$$

On the other hand, since u_η converges to u_ϵ a.e in Q, by using assumption $A1$), we get

$$\mathbf{A}^\star(u_\eta, \nabla u_\epsilon) \to \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon), \quad \text{strongly in } (L^{p'(x)}(Q))^N.$$

Moreover, from (6.19) and (6.20), we obtain

$$\int_Q \mathbf{A}^\star(u_\eta, \nabla u_\epsilon) \cdot \nabla(u_\eta - u_\epsilon) \to 0, \quad \text{when} \quad \eta \to 0. \tag{6.28}$$

Finally, we use (6.27) and (6.28) to obtain (6.25). □

Now, since the mapping $u \to F_\epsilon^\star(u, \nabla u)$ is continuous from V into $L^1(Q)$, the previous lemma allows to pass to the limit in the term $F_\epsilon^\star(u_\eta, \nabla u_\eta)$ which converges to $F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)$ in $L^1(Q)$. Moreover, we can also deduce the strong convergence of $\mathcal{A}^\star u_\eta$ to $\mathcal{A}^\star u_\epsilon$ in V_0'.

Furthermore, since $\frac{1}{\eta}\beta(u_\eta)$ is bounded in $L^{(p_-)'}(Q)$, and u_η converges strongly to u_ϵ in V_0, then $\beta(u_\epsilon) = 0$ a.e in Q, which implies that u_ϵ is in K. Thus, u_ϵ is in $L^\infty(Q)$, which is a fundamental difference with u_η (the role of the penalty operator $\frac{1}{\eta}\beta(u_\eta)$.)

Finally, we pass to the limit in $(P_{\eta,\epsilon}^\star)$, when η tends to zero, to obtain the following problem

$$(P_\epsilon^\star)\begin{cases} u_\epsilon \in V_0 \cap L^\infty(Q),\ \partial_t u_\epsilon \in V_0' + L^1(Q), & \\ \partial_t u_\epsilon + \mathcal{A}^\star u_\epsilon - F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon) + \beta_\epsilon = 0 & \text{in } Q, \\ u_\epsilon(0) = u_\epsilon(T) & \text{in } \Omega, \end{cases}$$

and one can easily deduce that

$$\forall v \in K,\ \langle \partial_t u_\epsilon, v - u_\epsilon \rangle + \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \cdot \nabla(v - u_\epsilon) - \int_Q F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)(v - u_\epsilon) \geq 0. \tag{6.29}$$

Estimates on $(u_\epsilon)_\epsilon$ in V_0

At this stage, we got a nonlinear problem $(P_\epsilon^\star)$ which only depends on the parameter ϵ. So, in order to pass to the limit when ϵ tends to zero, we need some a priori estimates in V_0.

Lemma 6.3.9. *The sequence (u_ϵ) is bounded in V_0.*

Proof. We prove this result by using the test function $z_s(u_\epsilon) = \exp(su_\epsilon^2)u_\epsilon$, where s is such that

$$\alpha z_s'(u_\epsilon) - C|z_s(u_\epsilon)| \geq \frac{\alpha}{2}, \tag{6.30}$$

where α is defined in $A3$) and C in (6.5). As u_ϵ is in $V_0 \cap L^\infty(Q)$, then $z_s(u_\epsilon)$ is in $V_0 \cap L^\infty(Q)$.

By multiplying $(P_\epsilon^\star)$ by $z_s(u_\epsilon)$, we obtain

$$\langle \partial_t u_\epsilon, z_s(u_\epsilon) \rangle + \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \cdot \nabla z_s(u_\epsilon) + \int_Q \beta_\epsilon z_s(u_\epsilon) = \int_Q F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon) z_s(u_\epsilon). \tag{6.31}$$

From the periodicity condition of u_ϵ, the first term in the left hand-side of (6.31) equals zero. We use (6.19), (6.22) and the sign condition of β, we get $\int_Q \beta_\epsilon z_s(u_\epsilon) \geq 0$. Moreover, by using (6.5), the coercivity assumption $A3$), and the fact that u_ϵ is in K, we obtain

$$\alpha \int_Q z_s'(u_\epsilon)|\nabla u_\epsilon|^{p(x)} \leq C\left(1 + \int_Q |z_s(u_\epsilon)||\nabla u_\epsilon|^{p(x)}\right).$$

Now, by using the (6.30) and the inequality (1.3), we get

$$\min\{\|u_\epsilon\|_{V_0}^{p_-}, \|u_\epsilon\|_{V_0}^{p_+}\} \leq \frac{2C}{\alpha},$$

where C is independent of ϵ. Hence, (u_ϵ) is bounded in V_0. □

Lemma 6.3.10. *The sequence* $(\partial_t u_\epsilon)$ *is bounded in* $V_0' + L^1(Q)$.

To prove this lemma, it suffices to show from the problem $(P_\epsilon^\star)$ that β_ϵ is bounded in $L^1(Q)$. In other words, we need the following estimate, whose proof is similar to that in [54], p. 296

$$\|\frac{1}{\eta}\beta(u_{\eta,\epsilon})\|_{L^1(Q)} \le C_1 + \int_Q C(h^\star(x,t) + |\nabla u_{\eta,\epsilon}|^{p(x)}), \tag{6.32}$$

where C_1 is independent of η and ϵ, and where C is defined in (6.5).

Proof. Let $v \in V_0 \cap L^\infty(Q)$, then from the equation of problem $(P_\epsilon^\star)$, we have

$$|\langle \partial_t u_\epsilon, v\rangle| \le \int_Q |\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon)||\nabla v| + \int_Q |F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)||v| + \int_Q |\beta_\epsilon||v|. \tag{6.33}$$

In a similar way as in the proof of Lemma 6.3.7, and since (u_ϵ) is bounded in V_0, we obtain

$$\int_Q |\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon)||\nabla v| \le C\|v\|_{V_0}.$$

We use (6.5), inequality (1.3) and the boundedness of (u_ϵ) in V_0, to obtain

$$\int_Q |F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)||v| \le C'\|v\|_{V_0}.$$

Now, by using (6.32) and since $v \in L^\infty(Q)$, we obtain

$$\int_Q |\beta_\epsilon||v| \le C''\|v\|_{L^\infty(Q)}.$$

Finally, we have

$$\|\partial_t u_\epsilon\|_{V_0'+L^1(Q)} \le C, \quad \text{where } C \text{ is independent of } \epsilon.$$

□

Passage to the limit in ϵ: We fix $s > \frac{N}{2} + 1$, so that $H_0^s(\Omega) \hookrightarrow L^\infty(\Omega)$, and then $L^1(\Omega) \hookrightarrow H^{-s}(\Omega)$. We have also, $H_0^s(\Omega) \hookrightarrow W^{1,p(x)}(\Omega)$, and consequently, $W^{-1,p'(x)}(\Omega) \hookrightarrow H^{-s}(\Omega)$. From Lemma 6.2.2 we have $V_0' \hookrightarrow L^{(p_+)'}(0,T;(W_0^{1,p(x)}(\Omega))')$. Thus, from the previous Lemma $(\partial_t u_\epsilon)$ is bounded in $L^1(0,T;H^{-s}(\Omega))$. Moreover, from the compactness theorem of [94](p. 85, Corollary 4) and (6.10), the sequence (u_ϵ) is relatively compact in $L^{p_-}(Q)$. So, we can extract a subsequence still denoted by (u_ϵ), such that, when ϵ tends to zero we have

$$u_\epsilon \to u \quad \text{strongly in } L^{p_-}(Q), \quad \text{and a.e in } Q, \tag{6.34}$$
$$u_\epsilon \to u \quad \text{weak}^* \text{ in } L^\infty(Q), \tag{6.35}$$
$$\partial_t u_\epsilon \rightharpoonup \partial_t u \quad \text{weakly in } V_0' + L^1(Q). \tag{6.36}$$

Now, as $\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon)$ is bounded in $(L^{p'(x)}(Q))^N$, there exists χ in $(L^{p'(x)}(Q))^N$ such that

$$\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \rightharpoonup \chi \quad \text{in} \quad (L^{p'(x)}(Q))^N \hookrightarrow (L^{(p_-)'}(Q))^N. \tag{6.37}$$

In addition, by using (6.34), it is clear that u is in K .

Lemma 6.3.11. *The sequence* (u_ϵ) *converges strongly to some* u *in* V_0.

Proof. The idea of proof is to apply the Lemma 6.3.2, since u_ϵ converges weakly to u in V_0 and $\mathbf{A}^\star$ satisfies $A1)$, $A2)$ and $A3)$.

We consider $\epsilon' > 0$ and we subtract $(P^\star_\epsilon)$ from $(P^\star_{\epsilon'})$, we obtain

$$\partial_t(u_\epsilon - u_{\epsilon'}) + \mathcal{A}^\star u_\epsilon - \mathcal{A}^\star u_{\epsilon'} - F^\star_\epsilon(u_\epsilon, \nabla u_\epsilon) + F^\star_{\epsilon'}(u_{\epsilon'}, \nabla u_{\epsilon'}) + \beta_\epsilon - \beta_{\epsilon'} = 0. \tag{6.38}$$

Now, we multiply (6.38) by the same type of test function $z_s(u_\epsilon - u_{\epsilon'})$ used in the proof of Lemma 6.3.9, we get

$$\begin{aligned}\langle \partial_t(u_\epsilon - u_{\epsilon'}), z_s(u_\epsilon - u_{\epsilon'})\rangle + \int_Q (\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) - \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'})) \cdot \nabla(u_\epsilon - u_{\epsilon'}) z'_s(u_\epsilon - u_{\epsilon'}) \\ + \int_Q (F^\star_{\epsilon'}(u_{\epsilon'}, \nabla u_{\epsilon'}) - F^\star_\epsilon(u_\epsilon, \nabla u_\epsilon)) z_s(u_\epsilon - u_{\epsilon'}) + \int_Q (\beta_\epsilon - \beta_{\epsilon'}) z_s(u_\epsilon - u_{\epsilon'}) = 0.\end{aligned} \tag{6.39}$$

Thanks to the periodicity condition of u_ϵ, the first term of (6.39) equals zero. By (6.19), (6.22) and the sign condition of β, the last term of (6.39) is nonnegative. By (6.5), the equation (6.39), then implies that

$$\begin{aligned}\int_Q (\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) - \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'})) \cdot \nabla(u_\epsilon - u_{\epsilon'}) z'_s(u_\epsilon - u_{\epsilon'}) \leq C \int_Q (h^\star(x,t) + |\nabla u_\epsilon|^{p(x)}) |z_s(u_\epsilon - u_{\epsilon'})| \\ + C \int_Q (h^\star(x,t) + |\nabla u_{\epsilon'}|^{p(x)}) |z_s(u_\epsilon - u_{\epsilon'})|.\end{aligned} \tag{6.40}$$

Using the coercivity condition $A3)$, we get

$$\begin{aligned}\int_Q (\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) - \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'})) \cdot \nabla(u_\epsilon - u_{\epsilon'}) z'_s(u_\epsilon - u_{\epsilon'}) \leq 2C \int_Q h^\star(x,t) |z_s(u_\epsilon - u_{\epsilon'})| \\ + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \cdot \nabla u_\epsilon |z_s(u_\epsilon - u_{\epsilon'})| + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'}) \cdot \nabla u_{\epsilon'} |z_s(u_\epsilon - u_{\epsilon'})| \\ \leq 2C \int_Q h^\star(x,t) |z_s(u_\epsilon - u_{\epsilon'})| + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \cdot \nabla(u_\epsilon - u_{\epsilon'}) |z_s(u_\epsilon - u_{\epsilon'})| \\ + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \cdot \nabla u_{\epsilon'} |z_s(u_\epsilon - u_{\epsilon'})| - \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'}) \cdot \nabla(u_\epsilon - u_{\epsilon'}) |z_s(u_\epsilon - u_{\epsilon'})| \\ + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'}) \cdot \nabla u_\epsilon |z_s(u_\epsilon - u_{\epsilon'})|.\end{aligned} \tag{6.41}$$

By condition (6.30), we deduce that

$$\begin{aligned}\frac{1}{2} \int_Q (\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) - \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'})) \cdot \nabla(u_\epsilon - u_{\epsilon'}) \leq 2C \int_Q h^\star |z_s(u_\epsilon - u_{\epsilon'})| \\ + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \cdot \nabla u_{\epsilon'} |z_s(u_\epsilon - u_{\epsilon'})| + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'}) \cdot \nabla u_\epsilon |z_s(u_\epsilon - u_{\epsilon'})|.\end{aligned} \tag{6.42}$$

Following the same steps of Lemma 6.3.8, we obtain the desired result, namely

$$\limsup_{\epsilon \to 0} \int_Q (\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) - \mathbf{A}^\star(u_\epsilon, \nabla u)) \cdot (\nabla u_\epsilon - \nabla u) \leq 0.$$

□

Now, we prove that u is between φ and ψ almost everywhere in Q, where φ and ψ are respectively sub and supersolution of problem (P) with $\varphi \leq \psi$ a.e in Q.

Lemma 6.3.12. *We have $\varphi \leq u \leq \psi$ a.e in Q.*

Proof. We shall prove that $\varphi \leq u$ a.e in Q. One can verify easily that: $v = u_\epsilon + (\varphi - u_\epsilon)^+$ is in K. Then, we can take it as a function test in (6.29). Hence, we obtain

$$\langle \partial_t u_\epsilon, (\varphi - u_\epsilon)^+ \rangle + \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \cdot \nabla(\varphi - u_\epsilon)^+ - \int_Q F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)(\varphi - u_\epsilon)^+ \geq 0. \tag{6.43}$$

Since φ is a subsolution, and $(\varphi - u_\epsilon)^+$ in $V_0 \cap L^\infty(Q)$, we obtain

$$\langle \partial_t \varphi, (\varphi - u_\epsilon)^+ \rangle + \int_Q \mathbf{A}^\star(\varphi, \nabla \varphi) \cdot \nabla(\varphi - u_\epsilon)^+ - \int_Q F(\varphi, \nabla \varphi)(\varphi - u_\epsilon)^+ \leq 0. \tag{6.44}$$

By subtracting (6.43) from (6.44), and by using Lemma 6.3.1, we get

$$\int_Q (\mathbf{A}(\varphi, \nabla \varphi) - \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon)) \cdot \nabla(\varphi - u_\epsilon)^+ + \int_Q (F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon) - F(\varphi, \nabla \varphi))(\varphi - u_\epsilon)^+ \leq 0. \tag{6.45}$$

Thanks to Lemma 6.3.11, we pass to the limit when ϵ tends to zero in (6.45) and get

$$\int_Q (\mathbf{A}(\varphi, \nabla \varphi) - \mathbf{A}^\star(u, \nabla u)) \cdot \nabla(\varphi - u)^+ + \int_Q (F^\star(u, \nabla u) - F(\varphi, \nabla \varphi))(\varphi - u)^+ \leq 0. \tag{6.46}$$

Furthermore, from the definition of $\mathbf{A}^\star$ and $F^\star$, we have

$$(F^\star(u, \nabla u) - F(\varphi, \nabla \varphi))(\varphi - u)^+ = 0 \ \text{ a.e in } Q \ \text{ and } \ \mathbf{A}^\star(u, \nabla u) \cdot \nabla(\varphi - u)^+ = \mathbf{A}(\varphi, \nabla u) \cdot \nabla(\varphi - u)^+.$$

Therefore, we obtain

$$\int_Q (\mathbf{A}(\varphi, \nabla \varphi) - \mathbf{A}(\varphi, \nabla u)) \cdot \nabla(\varphi - u)^+ \leq 0,$$

that is

$$\int_{\{\varphi \geq u\}} (\mathbf{A}(\varphi, \nabla \varphi) - \mathbf{A}(\varphi, \nabla u)) \cdot \nabla(\varphi - u) \leq 0.$$

According to $A2$), this implies that $\nabla(\varphi - u) = 0$ a.e in $\{(x,t) \in Q, \varphi \geq u\}$. Then, $\varphi - u = 0$ a.e in $\{(x,t) \in Q, \varphi \geq u\}$ which means that $\varphi \leq u$ a.e in Q. By a similar proof, we can obtain $u \leq \psi$ a.e in Q. □

To complete the proof of theorem 6.2.6 we need the following lemma

Lemma 6.3.13. *β_ϵ tends to zero in $L^1(Q)$.*

Proof. By taking $(u_\epsilon - k + 1)^+ \in V_0 \cap L^\infty(Q)$ as a test function in $(P_\epsilon^\star)$, and using the periodicity condition of u_ϵ and assumption $A3$), we obtain

$$\int_Q \beta_\epsilon (u_\epsilon - k + 1)^+ \leq F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)(u_\epsilon - k + 1)^+. \tag{6.47}$$

On the other hand, we have

$$\int_Q |\beta_\epsilon| = \int_{\{|u_\epsilon| < k\}} |\beta_\epsilon| + \int_{\{u_\epsilon = k\}} |\beta_\epsilon| + \int_{\{u_\epsilon = -k\}} |\beta_\epsilon|. \tag{6.48}$$

The definition of β gives $\beta(u_{\epsilon,\eta}) = 0$ if $|u_{\epsilon,\eta}| \leq k$, hence, $\frac{1}{\eta}\beta(u_{\epsilon,\eta})$ tends to 0, when η tends to 0 that is

$$\frac{1}{\eta}\beta(u_{\epsilon,\eta})\chi_{\{|u_\epsilon| < k\}} \to 0 \ \text{ a.e in } \ Q. \tag{6.49}$$

From (6.49), the boundedness of $\frac{1}{\eta}\beta(u_{\epsilon,\eta})\chi_{\{|u_\epsilon| < k\}}$ in $L^{(p_-)'}(Q)$ (see Lemma 6.3.6) and by using Lemma 4.2 of [13], we get

$$\frac{1}{\eta}\beta(u_{\epsilon,\eta})\chi_{\{|u_\epsilon| < k\}} \to 0 \ \text{ when } \eta \to 0 \ \text{ in } \ L^{(p_-)'}(Q) \ \text{ weakly}. \tag{6.50}$$

By (6.22), we deduce that

$$\beta_\epsilon = 0 \ \text{ a.e in } \{|u_\epsilon| < k\}. \tag{6.51}$$

Since $|u_\epsilon| < k$ a.e in Q, then from (6.51), we obtain

$$\int_Q \beta_\epsilon (u_\epsilon - k + 1)^+ = \int_{\{u_\epsilon = k\}} |\beta_\epsilon|.$$

So, (6.47) becomes

$$\int_{\{u_\epsilon = k\}} |\beta_\epsilon| \leq F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)(u_\epsilon - k + 1)^+. \tag{6.52}$$

Since $-k+1 \leq \varphi \leq u \leq \psi \leq k-1$, $(u_\epsilon - k + 1)^+$ tends to 0 almost everywhere in Q and in $L^\infty(Q)$ weak*, and $F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)$ converges strongly to $F(u, \nabla u)$ in $L^1(Q)$. Then, we can deduce from (6.52) that

$$\lim_{\epsilon \to 0} \int_{\{u_\epsilon = k\}} |\beta_\epsilon| = 0.$$

In the same way, we show that $\lim\limits_{\epsilon \to 0} \int_{\{u_\epsilon = -k\}} |\beta_\epsilon| = 0$. Whence, the desired result. □

Conclusion.

Now, we can pass to the limit in each term of problem $(P_\epsilon^\star)$. In other words, we have

$$\begin{gathered}
\mathcal{A}^\star(u_\epsilon, \nabla u_\epsilon) \to \mathcal{A}^\star(u, \nabla u) \ \text{ in } \ V_0' \ \text{ strongly,} \\
F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon) \to F^\star(u, \nabla u) \ \text{ in } \ L^1(Q) \ \text{ strongly,} \\
\beta_\epsilon \to 0 \ \text{ in } \ L^1(Q) \ \text{ strongly,} \\
\partial_t u_\epsilon \to \partial_t u \ \text{ in } \ V_0' + L^1(Q) \ \text{ strongly.}
\end{gathered}$$

Therefore, u satisfies

$$\partial_t u + \mathcal{A}^\star(u, \nabla u) - F^\star(u, \nabla u) = 0.$$

From Lemma 6.3.12 we have $\varphi \leq u \leq \psi$. Then, we get $\mathcal{A}^\star(u, \nabla u) = \mathcal{A}(u, \nabla u)$ and $F^\star(u, \nabla u) = F(u, \nabla u)$.

Concerning the periodicity condition, since (u_ϵ) is bounded in $V_0 \cap L^\infty(Q)$ and $(\partial_t u_\epsilon)$ is bounded in $V_0' + L^1(Q)$, then $(\partial_t u_\epsilon)$ is bounded in $L^1(0, T; H^{-s}(\Omega))$. So, (u_ϵ) is relatively compact in $L^{p_-}(Q)$. Hence, $u_\epsilon(0) \to u(0)$ in $L^{p_-}(Q)$ and $u_\epsilon(T) \to u(T)$ in $L^{p_-}(Q)$. As $u_\epsilon(0) = u_\epsilon(T)$, then we deduce that $u(0) = u(T)$. Finally, u is a periodic solution of problem (P).

6.4. Application

In this section we construct a subsolution and a supersolution for the following nonlinear parabolic problem associated with $p(x)$-Laplacian (concerning their physical interpretation see our introduction or [6] for more details):

$$(P) \begin{cases} \partial_t u - \Delta_{p(x)} u = f(x,t) & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u(T) & \text{in } \Omega, \end{cases}$$

where $\Omega \equiv B(0, R) = \{x \in \mathbb{R}^N |\ |x| < R\}$ is the unit ball, with $R > 0$ large enough. Moreover, assmue that $p(x) \in C^1(\mathbb{R}^N)$ is radial, that means $p(x) = p(|x|) = p(r)$, with $|x| = r < R$, and satisfies the assumptions of our Section 2.

Let $M = \|f\|_{L^\infty(Q)} < \infty$. We set

$$\psi(r) = \int_r^R \left[\frac{M}{N}t\right]^{\frac{1}{p(t)-1}} dt, \quad \text{and} \quad \varphi(r) = -\psi(r).$$

It is clear that $\varphi(r) \leq 0 \leq \psi(r)$. Moreover, ψ and φ are supersolution and subsolution, respectively of problem (P). Indeed, we have

$$-\Delta_{p(r)}\psi(r) = -\frac{1}{r^{N-1}}(r^{N-1}|\psi'(r)|^{p(r)-2}\psi'(r))'.$$

Since

$$\psi'(r) = -\left(\frac{M}{N}r\right)^{\frac{1}{p(r)-1}},$$

then

$$|\psi'(r)|^{p(r)-2}\psi'(r) = -\frac{M}{N}r.$$

Now, since $\psi(r)$ is independent of t, then we obtain

$$\partial_t\psi(r) - \Delta_{p(r)}\psi(r) = -\Delta_{p(r)}\psi(r) = M = \|f\|_{L^\infty(Q)} \geq f(x,t).$$

Moreover, if $r \in \partial\Omega$ (ie. $r = R$), then $\psi(r) = 0$. Hence, ψ is a supersolution of problem (P) in the sense of definition 6.2.

We repeat the same previous calculations, to obtain

$$\partial_t\varphi(r) - \Delta_{p(r)}\varphi(r) = -\Delta_{p(r)}\varphi(r) = -M = -\|f\|_{L^\infty(Q)} \leq f(x,t),$$

as far as $\varphi(r) = 0$ if $r \in \partial\Omega$. Hence, φ is a subsolution of problem (P) in the sense of definition 6.2.

Hence, applying our main result, theorem 6.2.6, we deduce the existence of at least one periodic solution $u(x,t)$ of problem (P) such that $\varphi \leq u \leq \psi$ a.e in Q.

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Abstract

The aim of this work is specifically the study of some kinds of nonlinear elliptic and parabolic partial differential equations. More precisely, this work is organized in two parts. In the first part we investigate the existence and multiplicity of solutions for some class of elliptic equations. Firstly, we deal with a problem related to the p-Laplacian operator with a p-gradient term and a Dirichlet boundary condition type. Secondly, we deal with a problem involving a more general operator with a potential, and a source term that does not satisfy the well known Ambrosetti-Rabinowitz condition. In the second part, we study the asymptotic behavior of some parabolic equations. The first subject, concerns mainly the study of some doubly nonlinear parabolic problems associated with a nonlinear boundary condition. In the second subject, we deal also with parabolic equations, we show the existence of periodic solutions for a fairly general problem associated with an operator in divergence form of Leary-Lions type with variable exponent.

Keywords: p-Laplacian; Ambrosetti-Rabinowitz condition; variable exponent; doubly nonlinear equation; periodic solutions.

Résumé

L'objectif de ce travail est d'apporter une certaine contribution à l'étude de quelques problèmes non linéaires de type elliptique ou parabolique. Plus précisément, ce travail est organisé en deux parties. La première partie est consacrée à l'étude de l'existence et de la multiplicité des solutions pour certaines classes d'équations elliptiques. Dans un premier temps, nous étudions un problème lié à l'opérateur de type p-Laplacien avec croissance d'ordre p dans le gradient et une condition aux limites de type Dirichlet. Nous étudions ensuite un problème faisant intervenir un opérateur assez général avec un potentiel et un terme source qui ne vérifie pas la condition d'Ambrosetti-Rabinowitz. Dans la seconde partie, nous étudions le comportement asymptotique de quelques équations de type parabolique. Le premier sujet, concerne principalement l'étude de problèmes paraboliques doublement non linéaires avec une condition aux limites de type non linéaire. Restant dans le cadre des équations paraboliques, nous montrons dans le deuxième sujet, l'existence de solutions périodiques pour un problème assez général associé à un opérateur sous forme divergentielle de type Leary-Lions à exposant variable.

Mots-clés: ***p***-**Laplacien; condition d'Ambrosetti-Rabinowitz; exposant variable; doublement non linéaire; solutions périodiques;**

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