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Abstract

In the last two decades, there has been an increasingly interest in studying nonlinear partial differential equations with non-standard growth conditions. This interest is justified by their applications in many domains: finance, image restoration, non-Newtonian fluids (characterized by their rapidly change of physical state from the liquid state to the solid state under the influence of different stimuli, such as electric or magnetic fields). Such fluids have many applications in several branches of engineering, including seismic protection, the automotive industry (clutches, shock absorbers, ...), military applications, etc.

In this thesis, our objective is to establish existence, regularity and uniqueness results of solutions for nonlinear, elliptical and parabolic partial differential equations of Dirichlet or Neumann type in Musielak-Orlicz-Soboev spaces and their particular cases (spaces of Orlicz-Sobolev and Lebesgu-Sobolev). Our work consists of four chapters.

In the first chapter, we make a brief reminder of different concepts and tools which we make frequent use in the other chapters.

In the second chapter, we consider a nonlinear eigenvalue problem for some elliptic equations governed by general operators including the p-Laplacian. The natural framework in which we consider such equations is that of Orlicz-Sobolev spaces. we exhibit two positive constants λ_0 and λ_1 with $\lambda_0 \leq \lambda_1$ such that λ_1 is an eigenvalue of the problem while any value $\lambda < \lambda_0$ cannot be so. By means of Harnack-type inequalities and a strong maximum principle, we prove the isolation of λ_1 on the right side. We emphasize that throughout this chapter no Δ_2-condition is needed.

In the third chapter, we prove a continuous embedding that allows us to obtain a boundary trace imbedding result for anisotropic Musielak-Orlicz spaces, which we then apply to obtain an existence result for Neumann problems with nonlinearities on the boundary associated to some anisotropic nonlinear elliptic equations in Musielak-Orlicz spaces constructed from Musielak-Orlicz functions on which and on their conjugates we do not assume the Δ_2-condition. The uniqueness is also studied.

The fourth chapter is devoted to finding the existence, the regularity and the uniqueness of solution of a parabolic problem with a Hardy potential and a singular term in Sobolev space.

Résumé

Au cours des deux dernières décennies, l'étude des équations aux dérivées partielles (EDP's) à croissance non standard a suscité un vif intérêt dans diverses directions de la recherche. Cet intérêt est justifié par leurs applications dans de nombreux domaines en : finance, restauration d'image, fluides non newtoniens (caractérisées par leur changement brutale d'état physique de l'état liquide a l'état solide sous l'influence de différents stimuli externes, comme les champs électriques ou magnétiques). De tels fluides ont de nombreuses applications dans plusieurs branches de l'ingénierie, y compris la protection antisismique, l'industrie automobile (embrayages, amortisseurs, ...), applications militaires, ... etc.

Dans cette thèse, notre objectif est d'établir des résultats d'existence, régularité et unicité des solutions pour des équations aux dérivées partielles non linéaires, elliptiques et paraboliques de type Dirichlet ou Neumann dans les espaces de Musielak-Orlicz-Soboev et leurs cas particuliers (espaces d'Orlicz-Sobolev et Lebesgu-Sobolev). Notre travail se compose de quatre chapitres.

Dans le premier chapitre, nous faisons un bref rappel de différentes notions et outils dont nous faisons un usage fréquent dans les autres chapitres.

Dans le deuxième chapitre nous considérons un problème de valeur propre non linéaire pour certaines équations elliptiques régies par des opérateurs généraux dont le p-laplacien. Le cadre naturel dans lequel nous considérons de telles équations est celui des espaces d'Orlicz-Sobolev. Nous présentons deux constantes positives λ_1 et λ_2 avec $\lambda_1 \leq \lambda_2$ tel que λ_1 est une valeur propre du problème alors que toute valeur $\lambda < \lambda_1$ ne peut pas être ainsi. Au moyen d'inégalités de type Harnack et d'un principe maximum fort, nous prouvons l'isolation de λ_1 du côté droit. Nous soulignons que tout au long du ce chapitre on n'a pas besoin de la condition $(\Delta_2\ )$.

Dans le troisième chapitre, nous prouvons une injection continue qui nous permet d'obtenir un résultat d'injection de trace de frontière pour les espaces de Musielak-Orlicz anisotropes, que nous appliquons ensuite pour obtenir un résultat d'existence et d'unicité pour un problème anisotrope de type Neumann avec des non-linéarités sur la frontière dans les espaces de Musielak-Orlicz construites à partir des fonctions de Musielak-Orlicz sur lesquelles et sur leur conjugués, nous ne supposons pas la condition $(\Delta_2\ )$.

Le quatrième chapitre est consacré à trouver l'existence, la regularité et l'unicité de solution d'un problème parabolique avec un potentiel de Hardy et un terme singulier dans les espaces de Sobolev.

Notations :

Ω	:	*open set of* $\mathbb{R}^N, N \in \mathbb{N}^*$,
$\partial\Omega$	:	*topological border of* Ω,
$x = (x_1, x_2, \cdots, x_N)$	:	*generic point of* $\mathbb{R}^N$,
$\lvert \cdot \rvert$	:	*Lebesgue measure* ,
dx	:	*surface measurement on* Ω,
$d\sigma$	:	*surface measurement on* $\partial\Omega$,
ν_i	:	i^{th} *component of the outer normal unit vector* ,
∇u	:	*gradient of* u,
$supp(f)$	:	*support of a function* f,
f^+	=	$\max(f, 0)$,
f^-	=	$\min(f, 0)$,
$\mathcal{D}(\Omega)$	:	*space of differentiable functions with compact support in* Ω,
$\mathcal{D}_+(\Omega)$	:	*space of positive functions of* $\mathcal{D}$,
$C_0(\Omega)$	:	*space of continuous functions with compact support in* Ω,
$C_\infty(\Omega)$	:	*space of indefinitely differentiable functions on* Ω,
T_k	:	*truncation function of level* k,
p'	:	*Hölder conjugate exponent of* p,
p^*	:	*Sobolev conjugate exponent of* p,
M^*	:	*complementary function of a function* M,
M^{**}	:	*second complementary function of a function* M,
M_*	:	*Sobolev conjugate function of a function* M,
$\Lambda_{N,2}$	:	*best constant in the Hardy inequality* ,

General introduction

Mathematics consists first of all of a language, which makes it possible to transcribe quantitative problems: this is modeling. Once this transcription tools are available to understand and resolve problems from real world phenomena that use the laws of physics (mechanics, thermodynamics, electromagnetism, etc.), these laws are, generally, written under the form of balance sheets which translate mathematically into Differential Equations Ordinary or by Partial Differential Equations.

Partial differential equations (PDEs) are also used in many other areas: in chemistry to model reactions, in economics to study market behavior, in finance to study derivatives and in image processing to restore degradations.

PDEs probably appeared for the first time during the birth of rational mechanics during the 17th century (Newton, Leibniz ...). Then the "catalog" of PDEs was enriched as the sciences and in particular physics developed. If only a few names have to be retained, we must cite that of Euler, then those of Navier and Stokes, for the equations of fluid mechanics, those of Fourier for the equation of heat, of Maxwell for those of electromagnetism, of Schrödinger and Heisenberg for the equations of quantum mechanics, and of course of Einstein for the PDEs of the theory of relativity.

However, the systematic study of PDEs is much more recent, and it was not until the 20th century that mathematicians began to develop the necessary arsenal. A giant leap was made by Schwartz when he gave birth to the theory of distributions (around the 1950s), and at least comparable progress is due to Hörmander for the development of pseudo-differential calculus (in the early 1970s). It is certainly good to keep in mind that the study of PDEs remains a very active area of research at the start of the 21st century. Besides, this research does not only have an impact in the applied sciences, but also plays a very important role in the current development of mathematics itself, both in geometry and in analysis.

The mathematical analysis of these partial differential equations requires an appropriate choice of functional spaces and a clear definition of the concept of solution (existence and sometimes uniqueness).

Furthermore, the work presented in this thesis concerning to prove existence, regularity and uniqueness of solutions of some partial differential equations of elliptic and parabolic with conditions at the edge of type Dirichlet or Neumann involving operators of type Leray-Lions [59] in the Musielak-Orlicz-Sobolev spaces (which is defined in Chapter 1 below) and in their particular cases (Lebesgue-Sobolev spaces and Orlicz-Sobolev spaces which which will are defined below). This work is split over the following four chapters.

The first chapter is entitled "Preliminaries (Recalls and definitions)". In this chapter, we make a brief reminder of different concepts and tools which we make frequent use in the other chapters.

The second chapter, entitled "On a nonlinear eigenvalue problem for generalized Laplacian in Orlicz-Sobolev spaces" (based on paper [91]) is devoted to study the problem

$$\begin{cases} -\mathrm{div}(\phi(|\nabla u|)\nabla u) & = \lambda\rho(x)\phi(|u|)u \quad \text{in } \Omega, \\ u & = 0 \quad\quad\quad\quad\quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω be an open bounded subset in $\mathbb{R}^N$, $N \geq 2$, having the segment property, $\phi : (0, \infty) \to (0, \infty)$ is a continuous function, so that defining the function $m(t) = \phi(|t|)t$ we suppose that m is strictly increasing and satisfies $m(t) \to 0$ as $t \to 0$ and $m(t) \to \infty$ as $t \to \infty$. The weight function $\rho \in L^\infty(\Omega)$ is such that $\rho \geq 0$ a.e. in Ω and $\rho \neq 0$ in Ω. This problem is studied in the Orlicz-Sobolev spaces $W_0^1 L_M(\Omega)$ (see Chapter 1 below) built upon the N-function (which will be defined below) $M(t) = \displaystyle\int_0^{|t|} m(s)ds$. Throughout this chapter, we do not impose the Δ_2-condition (see definition 1.2.1 below) neither on M nor on its complementary $N-$function in the sense of Young (which we define precisely later). Therefore we lose a wide range of facilitating properties of function spaces that one normally works with. Namely, if M does not satisfy the Δ_2-condition. This chapter comprises three sections. In the first section we exhibit two positive constants

$$\lambda_0 = \inf_{u \in W_0^1 L_M(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \phi(|\nabla u|)|\nabla u|^2 dx}{\displaystyle\int_\Omega \rho(x)\phi(|u|)|u|^2 dx}$$

and

$$\lambda_1 = \inf\Big\{ \int_\Omega M(|\nabla u|)dx \;\Big|\; u \in W_0^1 L_M(\Omega),\; \int_\Omega \rho(x)M(|u|)dx = 1 \Big\}.$$

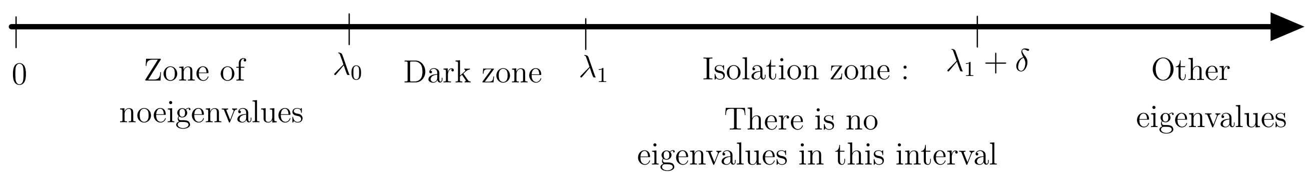
We already know from Mustonen-Tienari [68] and Gossez-Manásevich [47] that λ_1 is an eigenvalue of (1). Unlike the model case $\phi(t) = |t|^{p-2}$, $1 < p < +\infty$, we can not say that λ_1 is the first eigenvalue of (1). We prove that $\lambda_0 \leq \lambda_1$ and each $\lambda < \lambda_0$ is not an eigenvalue of (1). In the second section we prove a weak comparison principle under the condition (see [76])

$$\int_0^\delta \frac{ds}{H^{-1}(M(s))} = +\infty, \tag{2}$$

where H is the function defined for all $t \geq 0$ by

$$H(t) = tm(t) - M(t) = M^*(m(t)),$$

enable us to get a strong maximum principle. We show under (2) and by using this strong maximum principle that every eigenfunction u associated with λ_1 has a constant sign in Ω, that is, either $u > 0$ in Ω or $u < 0$ in Ω, by using this result we prove that if v an eigenfunction associated with $\lambda > \lambda_1$, then $v^+ \ngeqq 0$ and $v^- \ngeqq 0$ in Ω. That is v changes sign in Ω. Finally, we prove our main goal of this chapter Theorem 2.2.3 showing that λ_1 is isolated from the right-hand side, that is, there exists $\delta > 0$ such that in the interval $(\lambda_1, \lambda_1 + \delta)$ there are no eigenvalues. We can summarize the results of this section as follows.



In the third section we prove some important lemmas that are necessary for accomplishment of the proofs of the results obtained in the above section. First, we prove that any solution u of (1), associated with $\lambda > 0$, uniformly bounded in L^∞, that is a constant $c > 0$ not depending on u such that $\|u\|_\infty \leq c$. By using the uniform boundedness of the solutions and classical ideas we show Harnack-type inequalities and finally by these Harnack-type inequalities we prove the Hölder regularity.

The third chapter is entitled ”Imbedding results in Musielak-Orlicz-Sobolev spaces with an application to anisotropic nonlinear Neumann problems”, (based on paper [92]) comprises four sections. In the first, we consider $\vec{\phi} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^N$, the vector function $\vec{\phi} = (\phi_1, \cdots, \phi_N)$ where for every $i \in \{1, \cdots, N\}$, ϕ_i is a Musielak-Orlicz function (see Chapter 1) and we give the definition of the anisotropic Musielak-Orlicz-Sobolev space $W^1L_{\vec{\phi}}(\Omega)$, which equals to the anisotropic variable exponent Sobolev space $W^1L^{\vec{p}(\cdot)}(\Omega)$ defined in [21], if for $i \in \{1, \cdots, N\}$, $\phi_i(x,t) = t^{p_i(x)}$ and $p_i \in C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) \ : \ \inf_{x\in\overline{\Omega}} h(x) > 1\}$, also $W^1L_{\vec{\phi}}(\Omega) = W^1L^{p(\cdot)}(\Omega)$, if $\phi_1(x,t) = \cdots = \phi_N(x,t) = t^{p(x)}$ and $p \in C_+(\overline{\Omega})$, where $W^1L^{p(\cdot)}(\Omega)$ is the variable exponent Sobolev space defined in [39]. In the second section we assume the following conditions

$$\int_0^1 \frac{(\phi^{**}_{min})^{-1}(x,t)}{t^{1+\frac{1}{N}}}dt < +\infty \text{ and } \int_1^{+\infty} \frac{(\phi^{**}_{min})^{-1}(x,t)}{t^{1+\frac{1}{N}}}dt = +\infty, \ \ \forall x \in \overline{\Omega}, \tag{3}$$

there exist two positive constants $\nu < \frac{1}{N}$ and c_0, such that

$$\Big|\frac{\partial(\phi^{**}_{min})_*(x,t)}{\partial x_i}\Big| \leq c_0\Big[(\phi^{**}_{min})_*(x,t) + ((\phi^{**}_{min})_*(x,t))^{1+\nu}\Big], \tag{4}$$

for all $t \in \mathbb{R}$ and for almost every $x \in \Omega$, provided that for every $i = 1, \cdots, N$ the derivative $\frac{\partial(\phi^{**}_{min})_*(x,t)}{\partial x_i}$ exists, where $(\phi^{**}_{min})_*$ is the Sobolev conjugate of ϕ^{**}_{min} defined by

$$(\phi^{**}_{min})^{-1}_*(x,s) = \int_0^s \frac{(\phi^{**}_{min})^{-1}(x,t)}{t^{1+\frac{1}{N}}}dt, \ \ \text{for } x \in \overline{\Omega} \text{ and } s \in [0,+\infty),$$

where ϕ^{**}_{min} is the second complementary function (see (3.12) below) of ϕ_{min} and $\phi_{min}(x,s) = \min\limits_{i=1,\cdots,N} \phi_i(x,s)$. It may readily be checked that $(\phi^{**}_{min})_*$ is a Musielak-Orlicz function. Under (3) and (4), we prove the continuous imbedding $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow L_{(\phi^{**}_{min})_*}(\Omega)$, the compact imbedding $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow L_A(\Omega)$, where A is a Musielak-Orlicz function grows essentially more slowly (see (1.3) below) than $(\phi^{**}_{min})_*$, denote $A \ll (\phi^{**}_{min})_*$ and the trace imbedding $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow L_{\psi_{min}}(\partial\Omega)$, where

$$\psi_{min}(x,t) = [(\phi^{**}_{min})_*(x,t)]^{\frac{N-1}{N}}.$$

In the third section we apply the results proved in the above section to obtain the existence and uniqueness of the solution of the problem

$$\left\{\begin{array}{rll} -\sum_{i=1}^N \partial_{x_i} a_i(x,\partial_{x_i}u) + b(x)\varphi_{max}(x,|u(x)|) & = f(x,u) & \text{in } \Omega,\\ u & \geq 0 & \text{in } \Omega,\\ \sum_{i=1}^N a_i(x,\partial_{x_i}u)\nu_i & = g(x,u) & \text{on } \partial\Omega, \end{array}\right. \tag{5}$$

which is exactly the problem studied by Boureanu and Rădulescu [21] in the particular case where, for $i \in \{1, \cdots, N\}$, $\phi_i(x,t) = t^{p_i(x)}$, with $p_i \in C_+(\overline{\Omega})$. Here, $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $\varphi_{max}(x,s) = \frac{\partial\phi_{max}}{\partial s}(x,s)$, where $\phi_{max}(x,s) = \max\limits_{i=1,\cdots,N} \phi_i(x,s)$ and for every $i = 1, \cdots, N$, we denote by ν_i the i^{th} component of the outer normal unit vector and $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that there exist a locally integrable Musielak-Orlicz function (see definition 3.1.1 below) P_i with $P_i \ll \phi_i$, a positive constant c_i and a nonnegative function $d_i \in E_{\phi_i^*}$ ($E_{\phi_i^*}$ defined in (1.5) below) satisfying for all $s, t \in \mathbb{R}$ and for almost every $x \in \Omega$ the following assumptions

$$|a_i(x, s)| \leq c_i[d_i(x) + (\phi_i^*)^{-1}(x, P_i(x, s))],$$

$$\phi_i(x, |s|) \leq a_i(x, s)s \leq A_i(x, s),$$

$$(a_i(x, s) - a_i(x, t)) \cdot (s - t) > 0, \text{ for all } s \neq t,$$

the function $A_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined by

$$A_i(x, s) = \int_0^s a_i(x, t)dt.$$

We also assume that there exist a locally integrable Musielak-Orlicz function R with $R \ll \phi_{max}$ and a nonnegative function $D \in E_{\phi_{max}^*}(\Omega)$, such that for all $s, t \in \mathbb{R}$ and for almost every $x \in \Omega$

$$|\varphi_{max}(x, s)| \leq D(x) + (\phi_{max}^*)^{-1}(x, R(x, s))$$

For what concerns the data, we suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}^+$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. We define the antiderivatives $F : \Omega \times \mathbb{R} \to \mathbb{R}$ and $G : \partial\Omega \times \mathbb{R} \to \mathbb{R}$. We assume that there exist two positive constants k_1 and k_2 and two locally integrable Musielak-Orlicz functions M and H satisfy the Δ_2-condition and differentiable with respect to their second arguments with $M \ll \phi_{min}^{**}$, $H \ll \phi_{min}^{**}$ and $H \ll \psi_{min}$ such that the functions f and g satisfy for all $s \in \mathbb{R}_+$ the following assumptions

$$|f(x, s)| \leq k_1 m(x, s) \text{ for a.e. } x \in \Omega,$$

$$|g(x, s)| \leq k_1 h(x, s) \text{ for a.e. } x \in \partial\Omega,$$

where $\psi_{min}(x, s) = [(\phi_{min}^{**})_*(x, s)]^{\frac{N-1}{N}}$, $m(x, s) = \frac{\partial M}{\partial s}(x, s)$ and $h(x, s) = \frac{\partial H}{\partial s}(x, s)$. Finally, for the function b involved in (5), we assume that there exists a constant $b_0 > 0$ such that b satisfies the hypothesis

$$b \in L^\infty(\Omega), b(x) \geq b_0, \text{ for a.e. } x \in \Omega.$$

In the fourth section we prove some important lemmas that are necessary for accomplishment of the proofs of the results obtained in the above section.

The fourth chapter is entitled ”Semilinear heat equation with Hardy potential and singular terms” (based on paper [93]) and is concerned with the study of the following parabolic problem involving the Hardy potential and singular term

$$\begin{cases} u_t - \triangle u &= \mu \frac{u}{|x|^2} + \frac{f(x,t)}{u^\sigma} \quad \text{in } \Omega_T, \\ u(x,t) &> 0 \quad \text{in } \Omega \times (0,T), \\ u(x,t) &= 0 \quad \text{in } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{in } \Omega. \end{cases} \tag{6}$$

with Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 3$, containing the origin, σ and μ are positive constants and the data f and u_0 satisfy

$$f \geq 0,\ f \in L^m(\Omega_T), m \geq 1$$

and $u_0 \in L^\infty(\Omega)$ such that

$$\forall w \subset\subset \Omega\ \exists d_w > 0 : u_0 \geq d_w \text{ in } w. \tag{7}$$

We also assume that

$$\begin{cases} f \in L^{\frac{2N}{2N+(\sigma-1)(N-2)}}(\Omega_T) \quad \text{if} \quad \sigma \leq 1, \\ f \in L^1(\Omega_T) \quad \text{if} \quad \sigma > 1 \end{cases} \tag{8}$$

and under (7) and (8), we start by studying first the case $\mu < \Lambda_{N,2} := \frac{(N-2)^2}{4}$ distinguishing two cases where $\sigma \geq 1$ and $f \in L^1(\Omega_T)$ and the case where $\sigma < 1$ with $f \in L^{m_1}(\Omega_T)$, $m_1 = \frac{2N}{2N+(\sigma-1)(N-2)}$. Then we investigate the case $\mu = \Lambda_{N,2}$ and $\sigma = 1$ with data $f \in L^1(\Omega_T)$. In both cases we prove the existence of a weak solution obtained as limit of approximations that belongs to a suitable Sobolev space. The approach we use consists in approximating the singular equation with a regular problem, where the standard techniques (e.g., fixed point argument) can be applied and then passing to the limit to obtain the weak solution of the original problem. The regularity of weak solutions is analyzed according to the Lebesgue summability of f and σ. Furthermore, we prove the uniqueness of finite energy solutions when the source term f has a compact support by extending the formulation of weak solutions to a large class of test functions. Finally, in the case where $\mu > \Lambda_{N,2}$ we prove a nonexistence result. This chapter is presented as follows. The first Section contains all the main results (existence, regularity and uniqueness) and also graphic presentations allowing to better locate the obtained results. In the second section we first prove an existence result for approximate regular problems of the problem (2.1) and then we give the proof of all the main results Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.3, Theorem 4.2.4, Theorem 4.2.5 and Theorem 4.2.6. At the end, some results that are necessary for the accomplishment of the work are given in an appendix to make the chapter quite self contained.
Chapter 2 published in Nonlinear Analysis [91].

Chapter 3 Submitted to Electronic Journal of differential equations [92].

Chapter 4 Submitted to Journal of Evolution Equations [93].

Introduction générale

Les mathématiques consistent d'abord en un langage, qui permet de transcrire des problèmes de nature quantitative : C'est la modlisation. Une fois cette transcription faite, des outils sont disponibles pour comprendre et résoudre les problèmes issus des phénomènes du monde réel qui utilise les lois de la physique (mécanique, thermodynamique, électromagnétisme, etc.), ces lois sont, généralement, écrites sous la forme de bilans qui se traduisent mathématiquement par des Equations Différentielles Ordinaires ou par des Equations aux Dérivées Partielles.

Les équations aux dérivées partielles (EDPs) interviennent aussi dans beaucoup d'autres domaines : en chimie pour modéliser les réactions, en économie pour étudier le comportement des marchés, en finance pour étudier les produits dérivés et en traitement d'images pour restaurer les dégradations.

Les EDPs sont probablement apparues pour la première fois lors de la naissance de la mécanique rationnelle au cours du 17ème siècle (Newton, Leibniz...). Ensuite le "catalogue" des EDPs s'est enrichi au fur et à mesure du développement des sciences et en particulier de la physique. S'il ne faut retenir que quelques noms, on se doit citer celui d'Euler, puis ceux de Navier et Stokes, pour les équations de la mécanique des fluides, ceux de Fourier pour l'équation de la chaleur, de Maxwell pour celles de l'électromagnétisme, de Schrödinger et Heisenberg pour les équations de la mécanique quantique, et bien sûr de Einstein pour les EDPs de la théorie de la relativité.

Cependant, l'étude systématique des EDPs est bien plus récente, et c'est seulement au cours du 20ème siècle que les mathématiciens ont commencé à développer l'arsenal nécessaire. Un pas de géant a été accompli par Schwartz lorsqu'il a fait naître la théorie des distributions (autour des années 1950), et un progrès au moins comparable est dû à Hörmander pour la mise au point du calcul pseudodifférentiel (au début des années 1970). Il est certainement bon d'avoir à l'esprit que l'étude des EDPs reste un domaine de recherche très actif en ce début de 21ème siècle. D'ailleurs, ces recherches n'ont pas seulement un retentissement dans les sciences appliquées, mais jouent aussi un rôle très important dans le développement actuel des mathématiques elles-mêmes, à la fois en géométrie et en analyse.

L'analyse mathématique de ces équations aux dérivées partielles nécessite un choix approprié des espaces fonctionnels et une définition claire de la notion de solution (l'existence et parfois l'unicité).

Par ailleurs, les travaux présentés dans cette thèse concernant la preuve de l'existence, de la régularité et unicité des solutions de certaines équations aux dérivées partielles elliptiques et paraboliques avec conditions en bordure de type Dirichlet ou Neumann impliquant des opérateurs de type Leray-Lions [59] dans les espaces Musielak-Orlicz-Sobolev et dans leurs cas particuliers (les espaces de Lebesgue-Sobolev et les espaces d'Orlicz-Sobolev). Ce travail est réparti sur les quatre chapitres suivants.

Le premier chapitre est intitulé "Preliminaries (Recalls and definitions)". Dans ce chapitre, nous rappelons brièvement les différents concepts et outils que nous utilisons fréquemment dans les autres chapitres.

Le deuxième chapitre, intitulé "On a nonlinear eigenvalue problem for generalized Laplacian in Orlicz-Sobolev spaces" (basé sur le papier [91]) est consacré à étudier le problème

$$\begin{cases} -\operatorname{div}(\phi(|\nabla u|)\nabla u) &= \lambda\rho(x)\phi(|u|)u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{9}$$

où Ω un sous-ensemble borné ouvert de $\mathbb{R}^N$, $N \geq 2$, satisfait la propriété de segment, $\phi : (0,\infty) \to (0,\infty)$ est une fonction continue. Définissons la fonction $m(t) = \phi(|t|)t$ et supposons que m est strictement croissante et satisfaisant $m(t) \to 0$ si $t \to 0$ et $m(t) \to \infty$ si $t \to \infty$. La fonction de poids $\rho \in L^\infty(\Omega)$ vérifiant $\rho \geq 0$ p.p. dans Ω et $\rho \neq 0$ dans Ω. Ce problème est étudié dans l'espace d'Orlicz-Sobolev $W_0^1L_M(\Omega)$ (voir Chapitre 1) construit par la $N-$fonction (qui sera définie plus tard) $M(t) = \displaystyle\int_0^{|t|} m(s)ds$. Tout au long de ce chapitre, nous n'imposons pas la condition Δ_2 (voir la définition 1.2.1) ni sur M ni sur sa $N-$fonction complémentaire dans le sens de Young (que nous définissons précisément plus tard). Par conséquent, nous perdons un large éventail de propriétés facilitantes des espaces fonctionnels avec lesquels on travaille normalement. À savoir, si M ne satisfait pas la condition Δ_2. Ce chapitre comprend trois sections. Dans la première section, nous exposons deux constantes positives

$$\lambda_0 = \inf_{u \in W_0^1 L_M(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \phi(|\nabla u|)|\nabla u|^2 dx}{\displaystyle\int_\Omega \rho(x)\phi(|u|)|u|^2 dx}$$

et

$$\lambda_1 = \inf\Big\{ \int_\Omega M(|\nabla u|)dx \;\Big|\; u \in W_0^1 L_M(\Omega),\ \int_\Omega \rho(x)M(|u|)dx = 1 \Big\}.$$

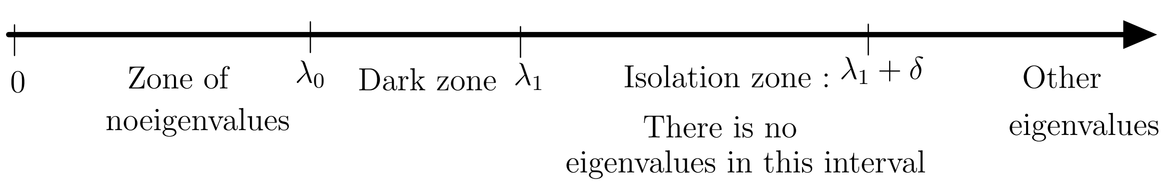
Nous savons déjà d'après [68] et Gossez-Manásevich [47] que λ_1 est une valeur propre de (9). Contrairement au cas du modèle $\phi(t) = |t|^{p-2}$, $1 < p < +\infty$, on ne peut pas dire que λ_1 est la première valeur propre de (9). Montrons que $\lambda_0 \leq \lambda_1$ et toute $\lambda < \lambda_0$ n'est pas une valeur propre de (9). Dans la deuxième section, nous prouvons un principe de comparaison faible sous la condition (see [76])

$$\int_0^\delta \frac{ds}{H^{-1}(M(s))} = +\infty, \tag{10}$$

où H est une fonction définie pour tout $t \geq 0$ par

$$H(t) = tm(t) - M(t) = M^*(m(t)),$$

qui nous permet d'obtenir un principe du maximum fort. Montrons sous la condition (10) et en utilisant ce principe du maximum fort que toute fonction propre u associée à λ_1 garde un signe constant, c-à-d soit $u > 0$ dans Ω ou $u < 0$ dans Ω en utilisant ce résultat nous montrons que si v est une fonction propre associée à $\lambda > \lambda_1$, alors $v^+ \not\equiv 0$ et $v^- \not\equiv 0$ dans Ω, c-à-d change de signe dans Ω. Enfin, nous montrons notre objectif principal de ce chapitre le Théorème 2.2.3 qui prouve que λ_1 est isolée du côté droit, c-à-d, il existe $\delta > 0$ tel que dans l'intervalle $(\lambda_1, \lambda_1 + \delta)$ il n'y a pas de valeurs propres. Nous pouvons résumer les résultats de cette section dans la figure suivante.



Dans la troisième section, nous prouvons quelques lemmes importants qui sont nécessaires pour la réalisation des preuves des résultats obtenus dans la section ci-dessus. Tout d'abord, nous montrons que toute solution u de (9), associée à $\lambda > 0$, uniformément bornée dans L^∞, c-à-d il existe une constante $c > 0$ ne dépend pas de u tel que $\|u\|_\infty \leq c$. En utilisant la bornitude uniforme des solutions et des idées classiques on montre les inégalités de type Harnack et finalement et par ces inégalités de type Harnack on prouve l'inégalité de Hölder.

Le troisième chapitre est intitulé "Imbedding results in Musielak-Orlicz-Sobolev spaces with an application to anisotropic nonlinear Neumann problems", (basé sur le papier [92]) comprend quatre sections.

Dans la première, considérons $\vec{\phi} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^N$, la fonction vectorielle $\vec{\phi} = (\phi_1, \cdots, \phi_N)$ où pour tout $i \in \{1, \cdots, N\}$, ϕ_i est une fonction de Musielak-Orlicz (voir le Chapter 1) et on donne la définition de l'espace de Musielak-Orlicz-Sobolev anisotropique $W^1 L_{\vec{\phi}}(\Omega)$, qui est équivaut à l'espace de Sobolev à exposant variables anisotropique $W^1 L^{\vec{p}(\cdot)}(\Omega)$ défini dans [21], si pour $i \in \{1, \cdots, N\}$, $\phi_i(x,t) = t^{p_i(x)}$ et $p_i \in C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) \ : \ \inf_{x \in \overline{\Omega}} h(x) > 1\}$, aussi $W^1 L_{\vec{\phi}}(\Omega) = W^1 L^{p(\cdot)}(\Omega)$, si $\phi_1(x,t) = \cdots = \phi_N(x,t) = t^{p(x)}$ et $p \in C_+(\overline{\Omega})$, où $W^1 L^{p(\cdot)}(\Omega)$ est l'espace de Sobolev à exposant variables défini dans [39]. Dans la deuxième section, supposons les conditions suivantes

$$\int_0^1 \frac{(\phi_{min}^{**})^{-1}(x,t)}{t^{1+\frac{1}{N}}} dt < +\infty \text{ and } \int_1^{+\infty} \frac{(\phi_{min}^{**})^{-1}(x,t)}{t^{1+\frac{1}{N}}} dt = +\infty, \quad \forall x \in \overline{\Omega}, \tag{11}$$

où ils existes deux constantes positifs $\nu < \frac{1}{N}$ et c_0, tel que

$$\Big| \frac{\partial (\phi_{min}^{**})_*(x,t)}{\partial x_i} \Big| \leq c_0 \Big[(\phi_{min}^{**})_*(x,t) + ((\phi_{min}^{**})_*(x,t))^{1+\nu} \Big], \tag{12}$$

pour tout $t \in \mathbb{R}$ et pour p.p. $x \in \Omega$, à condition que pour tout $i = 1, \cdots, N$ la dérivée $\frac{\partial (\phi_{min}^{**})_*(x,t)}{\partial x_i}$ existe, où $(\phi_{min}^{**})_*$ est la conjuguée de Sobolev de ϕ_{min}^{**} définie par

$$(\phi_{min}^{**})_*^{-1}(x,s) = \int_0^s \frac{(\phi_{min}^{**})^{-1}(x,t)}{t^{1+\frac{1}{N}}} dt, \quad \text{for } x \in \overline{\Omega} \text{ and } s \in [0, +\infty),$$

où ϕ_{min}^{**} est la deuxième fonction complémentaire (voir (3.12)) de ϕ_{min} et $\phi_{min}(x,s) = \min_{i=1,\cdots,N} \phi_i(x,s)$. On peut facilement vérifier que $(\phi_{min}^{**})_*$ est une fonction de Musielak-Orlicz. Sous les conditions (11) et (12), on montre l'injection continu $W^1 L_{\vec{\phi}}(\Omega) \hookrightarrow L_{(\phi_{min}^{**})_*}(\Omega)$, et l'injection compact $W^1 L_{\vec{\phi}}(\Omega) \hookrightarrow L_A(\Omega)$, où A est une fonction de Musielak-Orlicz croit essentiellement moins vite (voir (1.3)) que $(\phi_{min}^{**})_*$, dénoté $A \ll (\phi_{min}^{**})_*$ et l'injection de trace $W^1 L_{\vec{\phi}}(\Omega) \hookrightarrow L_{\psi_{min}}(\partial\Omega)$, où

$$\psi_{min}(x,t) = [(\phi_{min}^{**})_*(x,t)]^{\frac{N-1}{N}}.$$

Dans la troisième section nous appliquons les résultats prouvés dans la section ci-dessus pour obtenir l'existence et l'unicité de solution du problème

$$\begin{cases} -\sum_{i=1}^N \partial_{x_i} a_i(x,\partial_{x_i}u) + b(x)\varphi_{max}(x,|u(x)|) & = f(x,u) \quad \text{in } \Omega, \\ u & \geq 0 \quad \text{in } \Omega, \\ \sum_{i=1}^N a_i(x,\partial_{x_i}u)\nu_i & = g(x,u) \quad \text{on } \partial\Omega, \end{cases} \tag{13}$$

qui est exactement le problème étudié par Boureanu et Rădulescu [21] dans le cas particulier, pour $i \in \{1,\cdots,N\}$, $\phi_i(x,t) = t^{p_i(x)}$, avec $p_i \in C_+(\overline{\Omega})$. Ici, $\partial_{x_i} = \frac{\partial}{\partial_{x_i}}$, $\varphi_{max}(x,s) = \frac{\partial \phi_{max}}{\partial s}(x,s)$, où $\phi_{max}(x,s) = \max_{i=1,\cdots,N} \phi_i(x,s)$ et pour tout $i = 1,\cdots,N$, nous désignons par ν_i le i^{th} composante du vecteur d'unité normale externe et $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ est une fonction Carathéodory telle qu'ils existent une fonction de Musielak-Orlicz localement intégrable (voir la définition 3.1.1) P_i avec $P_i \ll \phi_i$, un constante positif c_i et une fonction négative $d_i \in E_{\phi_i^*}$ ($E_{\phi_i^*}$ défini dans (1.5)) satisfaisante pour touts $s,t \in \mathbb{R}$ et pour p.p. $x \in \Omega$ les conditions suivantes

$$|a_i(x,s)| \leq c_i[d_i(x) + (\phi_i^*)^{-1}(x,P_i(x,s))],$$

$$\phi_i(x,|s|) \leq a_i(x,s)s \leq A_i(x,s),$$

$$(a_i(x,s) - a_i(x,t))\cdot(s-t) > 0, \text{ for all } s \neq t,$$

la fonction $A_i : \Omega \times \mathbb{R} \to \mathbb{R}$ est définie par

$$A_i(x,s) = \int_0^s a_i(x,t)dt.$$

Supposons aussi qu'elle existe une fonction de Musielak-Orlic R localement intégrable avec $R \ll \phi_{max}$ et une fonction positive $D \in E_{\phi_{max}^*}(\Omega)$, tel que pour touts $s,t \in \mathbb{R}$ et pour p.p. $x \in \Omega$

$$|\varphi_{max}(x,s)| \leq D(x) + (\phi_{max}^*)^{-1}(x,R(x,s))$$

Pour ce qui concerne les données, on suppose que $f : \Omega\times\mathbb{R} \to \mathbb{R}^+$ et $g : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ sont des fonctions Carathéodory. Définissons les primitives $F : \Omega\times\mathbb{R} \to \mathbb{R}$ et $G : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ et supposons qu'ils existent deux positifs constantes k_1 et k_2 et deux fonctions de Musielak-Orlicz M and H localement intégrables satisfaisantes la conditions Δ_2- et différentiables par rapport à leurs seconds arguments avec $M \ll \phi_{min}^{**}$, $H \ll \phi_{min}^{**}$ et $H \ll \psi_{min}$ tel que f et g satisfaisantes pour tout $s \in \mathbb{R}_+$ les conditions suivantes

$$|f(x,s)| \leq k_1 m(x,s) \text{ for a.e. } x \in \Omega,$$

$$|g(x,s)| \leq k_1 h(x,s) \text{ for a.e. } x \in \partial\Omega,$$

où $\psi_{min}(x,s) = [(\phi^{**}_{min})_*(x,s)]^{\frac{N-1}{N}}$, $m(x,s) = \frac{\partial M}{\partial s}(x,s)$ et $h(x,s) = \frac{\partial H}{\partial s}(x,s)$. Finalement, pour la fonction b impliqué dans (13), on suppose qu'il existe un constante $b_0 > 0$ tel que b satisfaisante

$$b \in L^\infty(\Omega), b(x) \geq b_0, \text{ for a.e. } x \in \Omega.$$

Dans la quatrième section, montrons quelques lemmes importants qui sont nécessaires pour la réalisation des preuves des résultats obtenus dans les sections précédentes.

Le quatrième chapitre est intitulé "Semilinear heat equation with Hardy potential and singular terms" (basé sur le papier [93]) et s'intéresse à l'étude du problème parabolique suivant

$$\begin{cases} u_t - \triangle u &= \mu \frac{u}{|x|^2} + \frac{f(x,t)}{u^\sigma} & \text{in } \Omega_T, \\ u(x,t) &> 0 & \text{in } \Omega \times (0,T), \\ u(x,t) &= 0 & \text{in } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) & \text{in } \Omega. \end{cases} \tag{14}$$

où Ω est un sous-ensemble ouvert borné de $\mathbb{R}^N$, $N \geq 3$, contenant l'origine, σ et μ sont des positifs constantes et les données f et u_0 satisfaisantes

$$f \geq 0, \ f \in L^m(\Omega_T), m \geq 1$$

et $u_0 \in L^\infty(\Omega)$ tel que

$$\forall w \subset\subset \Omega \ \exists d_w > 0 : u_0 \geq d_w \text{ in } w. \tag{15}$$

Supposons que

$$\begin{cases} f \in L^{\frac{2N}{2N+(\sigma-1)(N-2)}}(\Omega_T) & \text{if} \quad \sigma \leq 1, \\ f \in L^1(\Omega_T) & \text{if} \quad \sigma > 1 \end{cases} \tag{16}$$

et sous les conditions (15) et (16), commençons par étudier d'abord le cas $\mu < \Lambda_{N,2} := \frac{(N-2)^2}{4}$ en distinguant deux cas : le cas $\sigma \geq 1$ et $f \in L^1(\Omega_T)$ et le case où $\sigma < 1$ avec $f \in L^{m_1}(\Omega_T)$, $m_1 = \frac{2N}{2N+(\sigma-1)(N-2)}$. Ensuite, nous étudions le cas $\mu = \Lambda_{N,2}$ et $\sigma = 1$ avec des données $f \in L^1(\Omega_T)$. Dans les deux cas, nous prouvons l'existence d'une solution faible obtenue comme limite d'approximations appartenant à un espace de Sobolev approprié. L'approche que nous utilisons consiste à approximer l'équation singulière avec un problème régulier, où les techniques standard (par exemple, argument de point fixe) peut être appliqué pour passé à la limite pour obtenir la solution faible du problème d'origine. La régularité des solutions faibles est analysée selon la sommabilité de Lebesgue de f et σ. De plus, nous prouvons l'unicité des solutions d'énergie finie lorsque le terme source f a un support compact en étendant la formulation de solutions faibles à une large classe de fonctions test. Finalement, dans le cas $\mu > \Lambda_{N,2}$ on montre un résultat de non-existence. Ce chapitre est présenté comme suit. La première section contient tous les résultats principaux (existence, régularité et l'unicité) et aussi des présentations graphiques permettant de mieux localiser les résultats obtenus. Dans la deuxième section, nous prouvons d'abord un résultat d'existence pour le problème approché du (14) puis nous donnons la preuve de tous les résultats principaux : Théorèm 4.2.1, Théorèm 4.2.2, Théorèm 4.2.3, Théorèm 4.2.4, Théorèm 4.2.5 et Théorèm 4.2.6. A la fin, quelques résultats nćessaires pour compléter le travail sont donnés en annexe pour rendre le chapitre assez autonome.

Chapter 2 publié in "Nonlinear Analysis" [91].

Chapter 3 Soumis au "Electronic Journal of differential equations" [92].

Chapter 4 Soumis au "Journal of Evolution Equations" [93].

Chapter 1

Preliminaries (Recalls and Definitions)

1.1 Lebesgue and Sobolev spaces

Sobolev spaces are ubiquitous in the study of elliptical and parabolic partial differential equations. It therefore makes sense to make a brief presentation before tackling these equations.
Let Ω be an open subset of $\mathbb{R}^N$. For $1 \leq p \leq \infty$ we denote by $L^p(\Omega)$ the space of Lebesgue measurable functions $u : \Omega \to \mathbb{R}^N$ such that, if $p < +\infty$

$$\|u\|_p = \Big(\int_\Omega |u(x)|^p dx\Big)^{\frac{1}{p}} < +\infty,$$

and if $p = \infty$

$$\|u\|_\infty = ess \sup_{x\in\Omega} |u(x)|.$$

For the definition, the main properties and results on Lebesgue spaces we refer to [23, 54]. For a function u in a Lebesgue space, we set by $\frac{\partial u}{\partial x_i}$ (or simply u_{x_i}) its partial derivative in the direction x_i defined in the sense of distributions, that is

$$< u_{x_i}, \phi >= -\int_\Omega u\phi_{x_i} dx,$$

so, we denote by $\nabla u = (u_{x_1}, u_{x_2}, \cdots, u_{x_N})$ the gradient of the function u.

The Sobolev space $W^{1,p}(\Omega)$, with $1 \leq p \leq \infty$, is the space of functions $u \in L^p(\Omega)$ such that $\nabla u \in (L^p(\Omega))^N$, endowed with its natural norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_p + \|\nabla u\|_p,$$

while $W_0^{1,p}(\Omega)$ is defined as the completion of $\mathcal{D}(\Omega)$ (the space of $\mathcal{C}^\infty$ functions with compact support in Ω) with respect to this norm. For $1 < p < \infty$, the dual space of $L^p(\Omega)$ is identified with $L^{p'}(\Omega)$, where $p = \frac{p}{p-1}$ is the Hölder conjugate exponent of p, and the dual space of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,p'}(\Omega)$. We know that if Ω is bounded, then any element $T \in W^{-1,p'}(\Omega)$ can be written, (see [23]), in the form $T = -\mathrm{div}(F)$ where $F \in (L^{p'}(\Omega))^N$.

1.2 Orlicz-Sobolev spaces

1.2.1 $N-$functions.

A function $M : \mathbb{R} \to \mathbb{R}$ is said to be an $N-$function if it is a continuous, real-valued, non-negative, convex function, which has superlinear growth near zero and infinity, i.e., $\lim_{t\to 0} \frac{M(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{M(t)}{t} = \infty$, and $M(t) = 0$ if and only if $t = 0$.
A function $M : \mathbb{R} \to \mathbb{R}$ is an $N-$function if and only if it can be represented as an integral

$$M(t) = \int_0^{|t|} m(s)ds,$$

where $m : [0, \infty[\tau[0, \infty[$ is increasing, right-continuous, $m(t) = 0$ if and only if $t = 0$, and $\lim_{t\to\infty} m(t) = \infty$ (see [53]). The complementary function M^* to a function M is defined by

$$M^*(s) = \sup_{t\in\mathbb{R}_+} \{st - M(t)\},$$

for $s \in \mathbb{R}_+$. Next we present some basic inequalities connected with $N-$function (see [53]).

Lemma 1.2.1 *Let M be an $N-$function, then*

(1) for every $t, s \geq 0$ and a.e. $x \in \Omega$ we have the so-called Young inequality

$$ts \leq M(t) + M^*(s).$$

(2)

$$M(t) \leq tM^{*-1}(M(t)) \leq 2M(t), \text{ for all } t \geq 0.$$

Definition 1.2.1 *An $N-$function satisfies the Δ_2-condition denoted $M \in \Delta_2$, if there exists constant $k > 0$ such that*

$$M(2t) \leq kM(t), \text{ for all } t \geq 0. \tag{1.1}$$

It is readily seen that this will be the case if and only if for every $r > 1$ there exists a positive constant $k = k(r)$ such that for all $t \geq 0$

$$M(rt) \leq kM(t), \text{ for all } t \geq 0.$$

If (1.1) holds only for $t \geq$ some t_0, then M is said to satisfy the Δ_2-condition near infinity.

For two $N-$functions P and Q, we said that Q dominates P, denote $P \prec Q$ if there exist $k > 0$ such that:

$$P(t) \leq Q(kt) \text{ for all } t \geq 0. \tag{1.2}$$

Similarly, Q dominates P near infinity if there exist $k > 0$ and $t_0 \geq 0$ such that (1.2) holds only for $t \geq t_0$. In this case there exists $K > 0$ such that:

$$P(t) \leq Q(kt) + K \text{ for all } t \geq 0.$$

We shall say that the $N-$functions P and Q are equivalent and we write $P \sim Q$ if $P \prec Q$ and $Q \prec P$. It follows from the definition that the $N-$functions P and Q are equivalent if and only if there exist positive constants k_1, k_2 and t_0 such that

$$P(k_1 t) \leq Q(t) \leq P(k_2 t) \text{ for all } t \geq t_0.$$

We say that P increases essentially more slowly than Q near infinity, denote $P \ll Q$, if for every $k > 0$; $\lim_{t\to\infty} \frac{P(kt)}{Q(t)} = 0$. This is the case if and only if $\lim_{t\to\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$. We also have, (see [54]), the equivalence $P \ll Q \Leftrightarrow Q^* \ll P^*$.

1.2.2 Orlicz spaces.

Let M an $N-$function and Ω an open subset of $\mathbb{R}^N$. The Orlicz space $L_M(\Omega)$ is defined as the space of (equivalence classes of) real-valued measurable functions u on Ω for which it exists $\lambda > 0$ ($\lambda = \lambda(u)$) such that :

$$\int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)dx < +\infty.$$

Recall that $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_M = \inf\Big\{\lambda > 0, \int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)dx \leq 1\Big\}.$$

We define the Orlicz class $K_M(\Omega)$ as the set of real-valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))dx < +\infty,$$

$K_M(\Omega)$ is also a convex subset of $L_M(\Omega)$.
The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.
The dual of $E_M(\Omega)$ can be identified with $L_{M^*}(\Omega)$ by means of the pairing $\displaystyle\int_\Omega uv\ dx$ and the dual norm of $L_{M^*}(\Omega)$ is equivalent to $\|\cdot\|_{M^*}$.

Theorem 1.2.1 *[83] Let M be an $N-$function and Ω be a bounded open subset of $\mathbb{R}^N$. Then,*

(1) $E_M(\Omega) \subset K_M(\Omega) \subset L_M(\Omega)$,

(2) $E_M(\Omega) = L_M(\Omega)$ if and only if $M \in \Delta_2$,

(3) $E_M(\Omega)$ is separable,

(4) $L_M(\Omega)$ is reflexive if and only if $M \in \Delta_2$ and $M^* \in \Delta_2$.

The Orlicz norm $\|u\|_{(M)}$ is defined by

$$\|u\|_{(M)} = \sup \int_\Omega u(x)v(x)dx$$

where the supremum is taken over all $v \in E_{M^*}(\Omega)$ such that $\|u\|_{M^*} \leq 1$, for which

$$\|u\|_M \leq \|u\|_{(M)} \leq 2\|u\|_M$$

holds for all $u \in L_M(\Omega)$ (see [53]). Now, we define the Orlicz version of Hölder's inequality

$$\int_\Omega |u(x)v(x)|dx \leq \|u\|_M \|v\|_{(M^*)}$$

for all $u \in L_M(\Omega)$ and $v \in L_{M^*}(\Omega)$.

Let E be a subset of Ω, the Luxemburg norm, associated to an $N-$function M, of the characteristic function χ_E of E is (see [53])

$$\|\chi_E\|_M = \frac{1}{M^{-1}\Big(\frac{1}{|E|}\Big)}.$$

Let $\{u_n\}$ be a sequence of $L_M(\Omega)$, we say that $\{u_n\}$ converge to $u \in L_M(\Omega)$ in the modular sense, if there exists $\lambda > 0$ such that

$$\int_\Omega M\Big(\frac{u_n - u}{\lambda}\Big)dx \to 0 \text{ as } n \to +\infty.$$

Let X and Y be two Banach spaces with bilinear bicontinuous pairing $< \cdot, \cdot >_{X,Y}$. Let $(u_n)_n$ be a sequence of X, we say $(u_n)_n$ converge to $u \in X$ with respect to the topology $\sigma(X, Y)$, denote $u_n \to u$ $\sigma(X, Y)$ in X, if $< u_n, v > \to < u, v >$ for all $v \in Y$. For example if $X = L_M(\Omega)$ and $Y = L_{M^*}(\Omega)$, then the pairing is defined by

$$< u, v > = \int_\Omega u(x)v(x)dx$$

for all $u \in X$ and $v \in Y$.

1.2.3 Orlicz-Sobolev spaces

Let M be an $N-$function and Ω an open subset of $\mathbb{R}^N$. The Orlicz-Sobolev spaces $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). The Orlicz-Sobolev space is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_M.$$

So, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N + 1$ copies of L_M. This product is denoted by ΠL_M. The space $W_0^1E_M(\Omega)$ is defined as the norm closure of $\mathcal{D}(\Omega)$ (the space of $\mathcal{C}^\infty$ functions with compact support in Ω) in $W^1E_M(\Omega)$, while $W_0^1E_M(\Omega)$ is defined as the closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$ with respect to the weak topology $\sigma(\Pi L_M, \Pi E_{M^*})$.

A sequence $\{u_n\}_n \subset W^1L_M(\Omega)$ is said to be convergent to $u \in W^1L_M(\Omega)$ in the modular sense in $W^1L_M(\Omega)$, if there exists $\lambda > 0$ such that

$$\int_\Omega M\Big(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\Big)dx \to 0, \text{ as } n \to +\infty, \text{ for all } |\alpha| \leq 1,$$

which implies convergence for $\sigma(L_M, L_{M^*})$.

We define $W^{-1}L_{M^*}(\Omega)$ and $W^{-1}E_{M^*}(\Omega)$ as the spaces of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{M^*}(\Omega)$ and $E_{M^*}(\Omega)$ respectively that is

$$W^{-1}L_{M^*}(\Omega) = \Big\{\phi \in \mathcal{D}'(\Omega) : \phi = \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} D^\alpha \phi_\alpha \text{ with } \phi_\alpha \in L_{M^*}(\Omega)\Big\}$$

and

$$W^{-1}E_{M^*}(\Omega) = \Big\{\phi \in \mathcal{D}'(\Omega) : \phi = \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} D^\alpha \phi_\alpha \text{ with } \phi_\alpha \in E_{M^*}(\Omega)\Big\}.$$

They are Banach spaces under the usual quotient norm. If, Ω has the segment property then the space $\mathcal{D}(\Omega)$ dense in $W_0^1L_M(\Omega)$ for the topology $\sigma(L_M, L_{M^*})$ (see [44]). Then, we can define the action of a distribution in $W^{-1}L_{M^*}(\Omega)$ on an element of $W_0^1L_M(\Omega)$.

1.3 Musielak-Orlicz-Sobolev spaces

Let Ω be an open subset of $\mathbb{R}^N$. A real function $\phi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$, will be called a Musielak-Orlicz function, if it satisfies the following conditions:

(i) $\phi(\cdot, t)$ is a measurable function on Ω.

(ii) $\phi(x, \cdot)$ is a convex, nondecreasing function with $\phi(x,t) = 0$ if only if $t = 0$, $\phi(x,t) > 0$ for all $t > 0$ and for almost every $x \in \Omega$,

$$\lim_{t\to 0^+} \frac{\phi(x,t)}{t} = 0 \text{ and } \lim_{t\to+\infty} \inf_{x\in\Omega} \frac{\phi(x,t)}{t} = +\infty.$$

We give here some examples on the Musielak-Orlicz functions.

$$\begin{aligned}
\phi(x,t) &= t^{p(x)} \text{ such that } \sup_{x\in\Omega} p(x) < +\infty, \\
\phi(x,t) &= t^{p(x)} \log(1+t), \\
\phi(x,t) &= t(\log(1+t))^{p(x)}, \\
\phi(x,t) &= (e^t)^{p(x)} - 1.
\end{aligned}$$

The complementary function ϕ^* of the Musilek-Orlicz function ϕ is defined by

$$\phi^*(x,s) = \sup_{t\geq 0}\{st - \phi(x,t)\}.$$

It can be checked that ϕ^* is also a Musielak-Orlicz function (see [67]). Moreover, for every t, $s \geq 0$ and a.e. $x \in \Omega$ we have the so-called Young inequality (see [67])

$$ts \leq \phi(x,t) + \phi^*(x,s).$$

For any function $h : \mathbb{R} \to \mathbb{R}$ the second complementary function $h^{**} = (h^*)^*$ (cf. (3.11)), is convex and satisfies

$$h^{**}(x) \leq h(x),$$

with equality when h is convex. Roughly speaking, h^{**} is a convex envelope of h, that is the biggest convex function smaller or equal to h.

Let ϕ and ψ be two Musielak-Orlicz functions. We say that ψ grows essentially more slowly than ϕ, denote $\psi \ll \phi$, if

$$\lim_{t\to+\infty} \sup_{x\in\Omega} \frac{\psi(x,t)}{\phi(x,ct)} = 0, \tag{1.3}$$

for every constant $c > 0$ and for almost every $x \in \Omega$. We remark that if ψ is a locally integrable, then $\psi \ll \phi$ implies that for all $c > 0$ there exists a nonnegative integrable function h, such that

$$\psi(x,t) \leq \phi(x,ct) + h(x), \text{ for all } t \in \mathbb{R} \text{ and for a.e. } x \in \Omega.$$

The Musielak-Orlicz space $L_\phi(\Omega)$ is defined by

$$L_\phi(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \text{ measurable } / \int_\Omega \phi\Big(x, \frac{u(x)}{\lambda}\Big) < +\infty \text{ for some } \lambda > 0 \Big\}.$$

Endowed with the so-called Luxemborg norm

$$\|u\|_\phi = \inf \Big\{ \lambda > 0 / \int_\Omega \phi\Big(x, \frac{u(x)}{\lambda}\Big) dx \leq 1 \Big\},$$

$(L_\phi(\Omega), \|\cdot\|_\phi)$ is a Banach space. Since $\displaystyle\lim_{t\to+\infty} \inf_{x\in\Omega} \frac{\phi(x,t)}{t} = +\infty$ and if Ω has finite measure then we have

$$L_\phi(\Omega) \hookrightarrow L^1(\Omega). \tag{1.4}$$

We will also use the space $E_\phi(\Omega)$ defined by

$$E_\phi(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \text{ measurable } / \int_\Omega \phi\Big(x, \frac{u(x)}{\lambda}\Big) < +\infty \text{ for all } \lambda > 0 \Big\}. \tag{1.5}$$

The following Hölder's inequality (see [67])

$$\int_\Omega |u(x)v(x)| dx \leq 2\|u\|_\phi \|v\|_{\phi^*}$$

holds for every $u \in L_\phi(\Omega)$ and $v \in L_{\phi^*}(\Omega)$, where ϕ and ϕ^* are two complementary Musielak-Orlicz functions. Define ϕ^{-1} for every $s \geq 0$ by

$$\phi^{-1}(x,s) = \sup\{\tau \geq 0 : \phi(x,\tau) \leq s\}.$$

Now, we give the definition of the anisotropic Musielak-Orlicz-Sobolev space.

Definition 1.3.1 *Let $\vec{\phi} : \Omega \times \mathbb{R}^+ \longrightarrow \mathbb{R}^N$, the vector function $\vec{\phi} = (\phi_1, \cdots, \phi_N)$ where for every $i \in \{1, \cdots, N\}$, ϕ_i is a Musielak-Orlicz function. We define the anisotropic Musielak-Orlicz-Sobolev space by*

$$W^1 L_{\vec{\phi}}(\Omega) = \Big\{ u \in L_{\phi_{max}}(\Omega);\ \partial_{x_i} u \in L_{\phi_i}(\Omega) \text{ for all } i = 1, \cdots, N \Big\}.$$

Since Ω has finite measure, then by the continuous imbedding (1.4), we get that $W^1 L_{\vec{\phi}}(\Omega)$ is a Banach space with respect to the following norm

$$\|u\|_{W^1 L_{\vec{\phi}}(\Omega)} = \|u\|_{\phi_{max}} + \sum_{i=1}^N \|\partial_{x_i} u\|_{\phi_i}.$$

Moreover, since Ω has finite measure we have the continuous embedding $W^1 L_{\vec{\phi}}(\Omega) \hookrightarrow W^{1,1}(\Omega)$.

Chapter 2

On a nonlinear eigenvalue problem for generalized Laplacian in Orlicz-Sobolev spaces

In this chapter, we consider a nonlinear eigenvalue problem for some elliptic equations governed by general operators including the p-Laplacian. The natural framework in which we consider such equations is that of Orlicz-Sobolev spaces. we exhibit two positive constants λ_0 and λ_1 with $\lambda_0 \leq \lambda_1$ such that λ_1 is an eigenvalue of the problem while any value $\lambda < \lambda_0$ cannot be so. By means of Harnack-type inequalities and a strong maximum principle, we prove the isolation of λ_1 on the right side. We emphasize that throughout the paper no Δ_2-condition is needed.

2.1 Introduction

Let Ω be an open bounded subset in $\mathbb{R}^N$, $N \geq 2$, having the segment property. In this paper we investigate the existence and the isolation of an eigenvalue for the following weighted Dirichlet problem

$$\begin{cases} -\mathrm{div}(\phi(|\nabla u|)\nabla u) & = \lambda\rho(x)\phi(|u|)u \quad \text{in } \Omega, \\ u & = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\phi : (0, \infty) \to (0, \infty)$ is a continuous function, so that defining the function $m(t) = \phi(|t|)t$ we suppose that m is strictly increasing and satisfies $m(t) \to 0$ as $t \to 0$ and $m(t) \to \infty$ as $t \to \infty$. The weight function $\rho \in L^\infty(\Omega)$ is such that $\rho \geq 0$ a.e. in Ω and $\rho \neq 0$ in Ω.

If $\phi(t) = |t|^{p-2}$ with $1 < p < +\infty$ the problem (2.1) is reduced to the eigenvalue problem for the

p-Laplacian

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda\rho(x)|u|^{p-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

while for $p = 2$ and $\rho = 1$ it is reduced to the classical eigenvalue problem for the Laplacian

$$\begin{cases} -\triangle u = \lambda u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.3}$$

It is known that the problem (2.3) has a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ such that $\lambda_n \to \infty$ as $n \to \infty$. Moreover, the eigenvalues of the problem (2.3) have multiplicities and the first one is simple. Anane [10] proved the existence, simplicity and isolation of the first eigenvalue $\lambda_1 > 0$ of the problem (2.2) assuming some regularity on the boundary $\partial\Omega$. The simplicity of the first eigenvalue of the problem (2.2) with $\rho = 1$ was proved later by Lindqvist [58] without any regularity on the domain Ω. For more results on the first eigenvalue of the p-Laplacian we refer for example to [72, 77].

In the general setting of Orlicz-Sobolev spaces, the following eigenvalue problem

$$\begin{cases} -\mathrm{div}(A(|\nabla u|^2)\nabla u) = \lambda\psi(u), & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.4}$$

was studied in [42] in the Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$ where $\Phi(s) = \int_0^s A(|t|^2)t dt$ and ψ is an odd increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$. In [42] the authors proved the existence of a minimum of the functional $u \to \int_\Omega \Phi(|\nabla u|)dx$ which is subject to a constraint and they proved the existence of principal eigenvalues of the problem (2.4) by using a non-smooth version of the Ljusternik theorem and by assuming the Δ_2-condition on the N-function Φ and it's complementary $\overline{\Phi}$. Mustonen and Tienari [68] studied the eigenvalue problem

$$\begin{cases} -\mathrm{div}\Big(\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\Big) = \lambda\rho(x)\frac{m(|u|)}{|u|}u, & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.5}$$

in the Orlicz-Sobolev space $W_0^1 L_M(\Omega)$, where $M(s) = \int_0^s m(t)dt$ with $m(t) = \phi(|t|)t$ and $\rho = 1$, without assuming the Δ_2-condition neither on M nor on its conjugate N-function M^*. Consequently, the functional $u \to \int_\Omega M(|\nabla u|)dx$ is not necessarily continuously differentiable and so classical variational methods can not be applied. They prove the existence of eigenvalues λ_r of problem (2.5) with $\rho = 1$ and for every $r > 0$, by proving the existence of a minimum of the real valued functional $\int_\Omega M(|\nabla u|)dx$ under the constraint $\int_\Omega M(u)dx = r$. By the implicit function theorem they proved that every solution of such minimization problem is a weak solution of the problem (2.5). This result was then extended in [47] to (2.5) with $\rho \neq 1$ and without assuming the Δ_2-condition by using a different approach based on a generalized version of Lagrange multiplier rule. The problem (2.1) was studied in [66] under the restriction that both the corresponding N-function and its complementary function satisfy the Δ_2-condition. In reflexive Orlicz-Sobolev spaces, other results related to this topic can be found in [62, 64].

In the present paper we define

$$\lambda_0 = \inf_{u \in W_0^1 L_M(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega \phi(|\nabla u|)|\nabla u|^2 dx}{\displaystyle\int_\Omega \rho(x)\phi(|u|)|u|^2 dx} \tag{2.6}$$

and

$$\lambda_1 = \inf \Big\{ \int_\Omega M(|\nabla u|)dx \;\Big|\; u \in W_0^1 L_M(\Omega),\ \int_\Omega \rho(x) M(|u|)dx = 1 \Big\}. \tag{2.7}$$

In the particular case where $\phi(t) = |t|^{p-2}$, $1 < p < +\infty$, we obtain $\lambda_0 = \lambda_1$ and so $\lambda_0 = \lambda_1$ is the first isolated and simple eigenvalue of the problem (2.2) (see [10]).

However, in the non reflexive Orlicz-Sobolev structure the situation is more complicated since we can not expect that $\lambda_0 = \lambda_1$. Precisely, we can not assert whether $\lambda_0 = \lambda_1$ or $\lambda_0 < \lambda_1$. We think that this is an open problem and we expect that the answer strongly depends on the N-function M. If $\lambda_0 < \lambda_1$, another open problem is to seek whether λ_1 is the smallest eigenvalue of problem (1). In other words to investigate the existence of eigenvalues of problem (1) in the interval $[\lambda_0, \lambda_1)$. Nonetheless, we show that $\lambda_0 \leq \lambda_1$ and that any value $\lambda < \lambda_0$ can not be an eigenvalue of the problem (3.57). Following the lines of [47], we also show that λ_1 is an eigenvalue of problem (2.1) associated to an eigenfunction u which is a weak solution of (2.1) (see Definition 2.2.1 below). It is in our purpose in this paper to prove that λ_1 is isolated from the right-hand side. To do so, we first prove some Harnack-type inequalities that enable us to show that u is Hölder continuous and then by a strong maximum principle we show that u has a constant sign. Besides, we prove that any eigenfunction associated to another eigenvalue than λ_1 necessarily changes its sign. This allows us to prove that λ_1 is isolated from the right hand side.

Let Ω be an open subset in $\mathbb{R}^N$ and let $M(t) = \int_0^{|t|} m(s)ds$, $m(t) = \phi(|t|)t$. The natural framework in which we consider the problem (2.1) is the Orlicz-Sobolev space defined by

$$W^1L_M(\Omega) = \Big\{u \in L_M(\Omega) : \partial_i u := \frac{\partial u}{\partial x_i} \in L_M(\Omega), i = 1, \cdots, N\Big\},$$

where $L_M(\Omega)$ stands for the Orlicz space defined as follows

$$L_M(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)dx < \infty \text{ for some } \lambda > 0\Big\}.$$

The spaces $L_M(\Omega)$ and $W^1L_M(\Omega)$ are Banach spaces under their respective norms

$$\|u\|_M = \inf\Big\{\lambda > 0 : \int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)dx \leq 1\Big\} \text{ and } \|u\|_{1,M} = \|u\|_M + \|\nabla u\|_M.$$

The closure in L_M of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The complementary function M^* of the N-function M is defined by

$$M^*(x, s) = \sup_{t\geq 0}\{st - M(x, t)\}.$$

Observe that by the convexity of M follows the inequality

$$\|u\|_M \leq \int_\Omega M(|u(x)|)dx + 1 \text{ for all } u \in L_M(\Omega). \tag{2.8}$$

Denote by $W_0^1L_M(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^1L_M(\Omega)$ with respect to the weak* topology $\sigma(\Pi L_M, \Pi E_{M^*})$. It is known that if Ω has the segment property, then the four spaces

$$(W_0^1L_M(\Omega), W_0^1E_M(\Omega); W^{-1}L_{M^*}(\Omega), W^{-1}E_{M^*}(\Omega))$$

form a complementary system (see [44]). If Ω is bounded in $\mathbb{R}^N$ then by the Poincaré inequality [44, lemma 5.7], $\|u\|_{1,M}$ and $\|\nabla u\|_M$ are equivalent norms in $W_0^1L_M(\Omega)$.

Let $J : D(J) \to \mathbb{R} \cup \{+\infty\}$ and $B : W_0^1L_M(\Omega) \to \mathbb{R}$ are the two functionals defined by

$$J(u) = \int_\Omega M(|\nabla u|)dx \tag{2.9}$$

and

$$B(u) = \int_\Omega \rho(x)M(|u|)dx, \tag{2.10}$$

respectively. The functional J takes values in $\mathbb{R} \cup \{+\infty\}$. Since $W_0^1L_M(\Omega) \subset E_M(\Omega)$ (see [47]), then the functional B is real valued on $W_0^1L_M(\Omega)$. Set

$$K = \{u \in W_0^1L_M(\Omega) : B(u) = 1\}.$$

In general, the functional J is not finite nor of class $\mathcal{C}^1$ (see [68] p. 158).

2.2 Main results

We will show that λ_1 given by relation (2.7) is an eigenvalue of the problem (2.1) and isolated from the right hand side, while any $\lambda < \lambda_0$ is not an eigenvalue of (2.1). In the sequel we assume that Ω is a bounded domain (unless otherwise stated) in $\mathbb{R}^N$ having the segment property.

Definition 2.2.1 *A function u is said to be a weak solution of (2.1) associated with $\lambda \in \mathbb{R}$ if*

$$\begin{cases} u \in W_0^1 L_M(\Omega), m(|\nabla u|) \in L_{M^*}(\Omega) \\ \displaystyle\int_\Omega \phi(|\nabla u|)\nabla u \cdot \nabla \psi dx = \lambda \int_\Omega \rho(x)\phi(|u|)u\psi dx, \text{ for all } \psi \in W_0^1 L_M(\Omega) \end{cases} \tag{2.11}$$

In this definition, both of the two integrals in (2.11) make sense. Indeed, for all $u \in W_0^1 L_M(\Omega)$ since $m(|\nabla u|) = \phi(|\nabla u|)|\nabla u| \in L_{M^*}(\Omega)$, the first term is well defined. From the Young inequality and the integral representation of M, we easily get $M^*(m(u)) \leq um(u) \leq M(2u)$. So that since $u \in E_M(\Omega)$ the integral on the right-hand side also makes sense.

Definition 2.2.2 *We said that λ is an eigenvalue of the problem (2.1), if there exists a function $v \neq 0$ belonging to $W_0^1 L_M(\Omega)$ such that (λ, v) satisfy (2.11). The function v will be called an eigenfunction associated with the eigenvalue λ.*

2.2.1 Existence result

We start with the next result that can be found in [68, Lemma 3.2]. For the convenience of the reader we give here a slightly different proof.

Lemma 2.2.1 *Let J and B be defined by (2.9) and (2.10). Then*
(i) B is $\sigma(\Pi L_M, \Pi E_{M^*})$ continuous,
(ii) J is $\sigma(\Pi L_M, \Pi E_{M^*})$ lower semi-continuous.

Proof 2.2.1 *(i) Let $u_n \to u$ for $\sigma(\Pi L_M, \Pi E_{M^*})$ in $W_0^1 L_M(\Omega)$. By the compact embedding $W_0^1 L_M(\Omega) \hookrightarrow E_M(\Omega)$, $u_n \to u$ in $E_M(\Omega)$ in norm. Hence $M(2(u_n - u)) \to 0$ in $L^1(\Omega)$. By the dominated convergence theorem, there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ with $u_n \to u$ a.e. in Ω and there exists $h \in L^1(\Omega)$ such that*

$$M(2(u_n - u)) \leq h(x) \text{ a.e. in } \Omega$$

for a subsequence. Therefore,

$$|u_n| \leq |u| + \frac{1}{2}M^{-1}(h),$$

so

$$M(u_n) \leq \frac{1}{2}M(2u) + \frac{1}{2}h(x)$$

and since $\rho \geq 0$ *for a.e. in* Ω, *then*

$$\rho(x)M(u_n) \leq \frac{1}{2}\rho(x)M(2u) + \frac{1}{2}\rho(x)h(x) \in L^1(\Omega).$$

Thus, the assertion (i) follows from the dominated convergence theorem.

To show (ii) we assume that $u_n \to u$ *for* $\sigma(\Pi L_M, \Pi E_{M^*})$ *in* $W_0^1L_M(\Omega)$, *that is*

$$\int_\Omega u_n v dx \to \int_\Omega uvdx \text{ and } \int_\Omega \partial_i u_n v dx \to \int_\Omega \partial_i u v dx,$$

for all $v \in E_{M^*}$. *This holds, in particular, for all* $v \in L^\infty(\Omega)$. *Hence,*

$$\partial_i u_n \to \partial_i u \text{ and } u_n \to u \text{ in } L^1(\Omega) \text{ for } \sigma(L^1, L^\infty). \tag{2.12}$$

Since the embedding $W_0^1L_M(\Omega) \hookrightarrow L^1(\Omega)$ *is compact, then* $\{u_n\}$ *is relatively compact in* $L^1(\Omega)$. *By passing to a subsequence,* $u_n \to v$ *strongly in* $L^1(\Omega)$. *In view of (2.12),* $v = u$ *and* $u_n \to u$ *strongly in* $L^1(\Omega)$. *Passing once more to a subsequence, we obtain that* $u_n \to u$ *almost everywhere on* Ω. *Since* $\zeta \mapsto M(|\zeta|)$ *is convex for* $\zeta \in \mathbb{R}^N$, *we can use [34, Theorem 2.1, Chapter 8], to obtain*

$$J(u) = \int_\Omega M(|\nabla u|)dx \leq \liminf \int_\Omega M(|\nabla u_n|)dx = \liminf J(u_n).$$

The first result of this paper is given by the following theorem.

Theorem 2.2.1 *The infimum in (2.7) is achieved at some function* $u \in K$ *which is a weak solution of (3.57) and thus* u *is an eigenfunction associated to the eigenvalue* λ_1. *Furthermore,* $\lambda_0 \leq \lambda_1$ *and each* $\lambda < \lambda_0$ *is not an eigenvalue of problem (2.1).*

Proof 2.2.2 *We split the proof of Theorem 2.2.1 into three steps.*
Step 1 *: We show that the infimum in (2.7) is achieved at some* $u \in K$. *By (2.8) we have*

$$J(u) = \int_\Omega M(|\nabla u|)dx \geq \|\nabla u\|_M - 1.$$

So, J *is coercive. Let* $\{u_n\} \subset W_0^1L_M(\Omega)$ *be a minimizing sequence, i.e.* $u_n \in K$ *and* $u_n \to \inf_{v\in K} J(v)$. *The coercivity of* J *implies that* $\{u_n\}$ *is bounded in* $W_0^1L_M(\Omega)$ *which is in the dual* *of a separable Banach space. By the Banach-Alaoglu-Bourbaki theorem, there exists* $u \in W_0^1 L_M(\Omega)$ *such that for a subsequence still indexed by* n, $u_n \to u$ *for* $\sigma(\Pi L_M, \Pi E_{M^*})$ *in* $W_0^1 L_M(\Omega)$. *As a consequence of Lemma 2.2.1 the set* K *is closed with respect to the topology* $\sigma(\Pi L_M, \Pi E_{M^*})$ *in* $W_0^1 L_M(\Omega)$. *Thus,* $u \in K$. *Since* J *is* $\sigma(\Pi L_M, \Pi E_{M^*})$ *lower semi-continuous, it follows*

$$J(u) \leq \liminf J(u_n) = \inf_{v \in K} J(v),$$

which shows that u *is a solution of* (2.7).

Step 2 : *The function* $u \in K$ *found in Step 1 is such that* $m(|\nabla u|) \in L_{M^*}(\Omega)$ *and satisfies* (2.11). *This was already proved in [47, Theorem 4.2].*

Step 3 : *Let* λ_0 *be given by (2.6). Any value* $\lambda < \lambda_0$ *cannot be an eigenvalue of problem (2.1). Indeed, suppose by contradiction that there exists a value* $\lambda \in (0, \lambda_0)$ *which is an eigenvalue of problem* (2.1). *It follows that there exists* $u_\lambda \in W_0^1 L_M(\Omega) \setminus \{0\}$ *such that*

$$\int_\Omega \phi(|\nabla u_\lambda|)\nabla u_\lambda \cdot \nabla v dx = \lambda \int_\Omega \rho(x)\phi(|u_\lambda|)u_\lambda v dx \textit{ for all } v \in W_0^1 L_M(\Omega).$$

Thus, in particular for $v = u_\lambda$ *we can write*

$$\int_\Omega \phi(|\nabla u_\lambda|)|\nabla u_\lambda|^2 dx = \lambda \int_\Omega \rho(x)\phi(|u_\lambda|)|u_\lambda|^2 dx.$$

The fact that $u_\lambda \in W_0^1 L_M(\Omega) \setminus \{0\}$ *ensures that* $\int_\Omega \rho(x)\phi(|u_\lambda|)|u_\lambda|^2 dx > 0$. *By the definition of* λ_0, *we obtain*

$$\begin{aligned}\int_\Omega \phi(|\nabla u_\lambda|)|\nabla u_\lambda|^2 dx &\geq \lambda_0 \int_\Omega \rho(x)\phi(|u_\lambda|)|u_\lambda|^2 dx \\ &> \lambda \int_\Omega \rho(x)\phi(|u_\lambda|)|u_\lambda|^2 dx \\ &= \int_\Omega \phi(|\nabla u_\lambda|)|\nabla u_\lambda|^2 dx.\end{aligned}$$

Which yields a contradiction. Therefore, we conclude that $\lambda_0 \leq \lambda_1$. *The proof of Theorem 2.2.1 is now complete.*

2.2.2 Isolation result

In this subsection we first show a maximum principle which enables us to prove that any eigenfunction associated to λ_1 has a constant sign in Ω. This property is then used to prove that λ_1 is isolated from the right-hand side.

Let w be an eigenfunction of problem (2.1) associated to the eigenvalue λ_1. Since $|w| \in K$ it follows that $|w|$ achieves also the infimum in (2.7), which implies that $|w|$ is also an eigenfunction associated to λ_1. So we can assume that w is non-negative, that is

$$w(x) \geq 0 \text{ for } x \in \Omega.$$

Since by Theorem 2.3.2 (given in Appendix), the eigenfunction w is bounded, we set

$$0 \leq \delta := \sup_{\Omega} w < +\infty.$$

For $t \in (0, \delta)$ the function $f(t) = \phi(t)t = m(t) > 0$ is continuous and strictly increasing. Let $F(s) = \int_0^s f(t)dt$. We assume that

$$\int_0^\delta \frac{ds}{H^{-1}(M(s))} = +\infty, \tag{2.13}$$

where H is the function defined for all $t \geq 0$ by

$$H(t) = tm(t) - M(t) = M^*(m(t)).$$

The assumption (2.13) is known to be a necessary condition for the strong maximum principle to hold (see [76] and the references therein). Hereafter, under (2.13) we can compare w to a suitable function given by [76, Lemma 2].

The proof of the strong maximum principle will be given after proving the following two Lemmas.

Lemma 2.2.2 *Denote by $B(y, R)$ an open ball in Ω of radius R and centered at $y \in \Omega$ and consider the annulus*

$$E_R = \Big\{x \in B(y, R) : \frac{R}{2} \leq |y - x| < R\Big\}.$$

Assume that (2.13) holds. Then there exists a function $v \in C^1$ with $0 < v < \delta$, $v' < 0$ in E_R and $w \geq v$ on ∂E_R. Moreover, v satisfies

$$-\int_\Omega \phi(|\nabla v|)\nabla v \cdot \nabla\psi dx \leq \int_\Omega f(v)\psi dx, \tag{2.14}$$

for every $\psi \in W_0^1 L_M(\Omega)$ and $\psi \leq 0$.

Proof 2.2.3 *Let $r = |y - x|$ for $x \in \overline{E}_R$. The function $v(x) = v(r)$ given by [76, Lemma 2] satisfies for every positive numbers k, l, and for $\epsilon \in (0, \delta)$*

$$[m(|v'|)]' + \frac{k}{r}m(|v'|) + lf(v) \leq 0,$$

$0 < v < \epsilon < \delta$, $v' < 0$ in E_R and $v(x) = 0$ if $|y - x| = R$. In addition, for $x \in E_R$ with $|y - x| = \frac{R}{2}$ we have $v(x) < \epsilon < \inf_{\{x:|y-x|=\frac{R}{2}\}} w(x) < \delta$. Hence, follows $w \geq v$ on ∂E_R. Moreover, by the radial symmetric expression of $div(\phi(|\nabla v|)\nabla v)$, we have

$$div(\phi(|\nabla v|)\nabla v) - f(v) \quad = -[m(|v'|)]' - \frac{(N-1)}{r} m(|v'|) - f(v) \geq 0,$$

where we recall that $v' < 0$ and use [76, Lemma 2]. Multiplying the above inequality by $\psi \in W_0^1 L_M(\Omega)$ with $\psi \leq 0$ and then integrating over Ω we obtain (2.14). The proof is achieved.

Lemma 2.2.3 (Weak comparison principle) *Assume that (2.13) holds. Let v be the $\mathcal{C}^1$-function given by Lemma 2.2.2 with $0 < v < \delta$ in Ω and $w \geq v$ on $\partial\Omega$. Then $w \geq v$ in Ω.*

Proof 2.2.4 *Let $h = w - v$ in Ω. Assume by contradiction that there exists $x_1 \in \Omega$ such that $h(x_1) < 0$. Fix $\epsilon > 0$ so small that $h(x_1) + \epsilon < 0$. By Theorem 2.3.4 (see Appendix) the function w is continuous in Ω, then so is the function h. Since $h \geq 0$ on $\partial\Omega$, the support Ω_0 of the function $h_\epsilon = \min\{h + \epsilon, 0\}$ is a compact subset in Ω. By Theorem 2.3.1 (see Appendix), the function h_ϵ belongs to $W_0^1 L_M(\Omega)$. Taking it as a test function in (2.11) and (2.14) it yields*

$$\int_{\Omega_0} \phi(|\nabla w|)\nabla w \cdot (\nabla w - \nabla v)dx = \lambda_1 \int_{\Omega_0} \rho(x)\phi(|w|)w h_\epsilon dx$$

and

$$-\int_{\Omega_0} \phi(|\nabla v|)\nabla v \cdot (\nabla w - \nabla v)dx \leq \int_{\Omega_0} \phi(|v|)v h_\epsilon dx.$$

Summing up the two formulations, we obtain

$$\int_{\Omega_0} [\phi(|\nabla w|)\nabla w - \phi(|\nabla v|)\nabla v] \cdot (\nabla w - \nabla v)dx \leq \int_{\Omega_0} (\lambda_1 \rho(x) m(w) + m(v))h_\epsilon dx. \tag{2.15}$$

The left-hand side of (2.15) is positive due to Lemma 2.3.1 (given in Appendix), while the right-hand side of (2.15) is non positive, since $h_\epsilon < 0$ in Ω_0. Therefore,

$$\int_{\Omega_0} [\phi(|\nabla w|)\nabla w - \phi(|\nabla v|)\nabla v] \cdot (\nabla w - \nabla v)dx = 0$$

implying $\nabla h_\epsilon = 0$ and so $h + \epsilon > 0$ which contradicts the fact that $h(x_1) + \epsilon < 0$.

Now we can prove our strong maximum principle.

Theorem 2.2.2 (Strong maximum principle) *Assume that (2.13) holds. Then, if w is a non-negative eigenfunction associated with λ_1, then $w > 0$ in Ω.*

Proof 2.2.5 *Let $B(y,R)$ be an open ball of Ω of radius R and centered at a fixed arbitrary $y \in \Omega$. We shall prove that $w(x) > 0$ for all $x \in B(y,R)$. Let v be the $\mathcal{C}^1$-function given by Lemma 2.2.2 with $w \geq v$ on ∂E_R where*

$$E_R = \Big\{x \in B(y,R) : \frac{R}{2} \leq |y-x| < R\Big\}.$$

Applying Lemma 2.2.3 we get $w \geq v > 0$ in E_R. For $|y-x| < \frac{R}{2}$ we consider

$$E_{\frac{R}{2}} = \Big\{x \in B(y,R) : \frac{R}{4} \leq |y-x| < \frac{R}{2}\Big\}.$$

We can us similar arguments as in the proof of Lemma 2.2.2 to obtain that there is $v \in C^1$ in $E_{\frac{R}{2}}$, with $v > 0$ in $E_{\frac{R}{2}}$ and $w \geq v$ on $\partial E_{\frac{R}{2}}$. Applying again Lemma 2.2.3 we obtain $w \geq v > 0$ in $E_{\frac{R}{2}}$. So, by the same way we can conclude that $w(x) > 0$ for any $x \in B(y,R)$.

Now we are ready to prove that the associated eigenfunction of λ_1 has necessarily a constant sign in Ω.

Proposition 2.2.1 *Assume that (2.13) holds. Then, every eigenfunction u associated to the eigenvalue λ_1 has constant sign in Ω, that is, either $u > 0$ in Ω or $u < 0$ in Ω.*

Proof 2.2.6 *Let u be an eigenfunction associated to the eigenvalue λ_1. Then u achieves the infimum in (2.7). Since $|u| \in K$ it follows that $|u|$ achieves also the infimum in (2.7), which implies that $|u|$ is also an eigenfunction associated to λ_1. Therefore, applying Theorem 2.2.2 with $|u|$ instead of w, we obtain $|u| > 0$ for all $x \in \Omega$ and since u is continuous (see Theorem 2.3.4 in Appendix), then, either $u > 0$ or $u < 0$ in Ω.*

Before proving the isolation of λ_1, we shall prove that every eigenfunction associated to another eigenvalue $\lambda > \lambda_1$ changes in force its sign in Ω. Denote by $|E|$ the Lebesgue measure of a subset E of Ω.

Proposition 2.2.2 *Assume that (2.13) holds. If $v \in W_0^1 L_M(\Omega)$ is an eigenfunction associated to an eigenvalue $\lambda > \lambda_1$. Then $v^+ \not\equiv 0$ and $v^- \not\equiv 0$ in Ω. Moreover, if we set $\Omega^+ = \{x \in \Omega : v(x) > 0\}$ and $\Omega^- = \{x \in \Omega : v(x) < 0\}$, then*

$$\min\{|\Omega^+|, |\Omega^-|\} \geq \min\left\{\frac{1}{M^*\left(\frac{dc}{\min\{a,1\}}\right)}, \frac{1}{M^*\left(\frac{dc}{\min\{b,1\}}\right)}\right\} \tag{2.16}$$

where $a = \int_\Omega v^+(x)dx$, $b = \int_\Omega v^-(x)dx$, $c = c(\lambda, |\Omega|, \|v\|_\infty, \|\rho\|_\infty)$ and d is the constant in the Poincaré norm inequality (see [44, Lemma 5.7]).

Proof 2.2.7 *By contradiction, we assume that there exists an eigenfunction v associated to $\lambda > \lambda_1$ such that $v > 0$. The case $v < 0$ being completely analogous so we omit it . Let $u > 0$ be an eigenfunction associated to λ_1. Let Ω_0 be a compact subset of Ω and define the two functions*

$$\eta_1(x) = \begin{cases} u(x) - v(x) + \sup_\Omega v & \text{if} \quad x \in \Omega_0 \\ 0 & \text{if} \quad x \notin \Omega_0 \end{cases}$$

and

$$\eta_2(x) = \begin{cases} v(x) - u(x) - \sup_\Omega v & \text{if} \quad x \in \Omega_0 \\ 0 & \text{if} \quad x \notin \Omega_0. \end{cases}$$

Pointing out that v is bounded (Theorem 2.3.2 in Appendix), the two functions η_1 and η_2 are admissible test functions in (2.11) (see Theorem 2.3.1 in Appendix). Thus, we have

$$\int_\Omega \phi(|\nabla u|)\nabla u \cdot \nabla \eta_1 dx = \lambda_1 \int_\Omega \rho(x)\phi(|u|)u\eta_1 dx$$

and

$$\int_\Omega \phi(|\nabla v|)\nabla v \cdot \nabla \eta_2 dx = \lambda \int_\Omega \rho(x)\phi(|v|)v\eta_2 dx.$$

By summing up and using Lemma 2.3.1 (in Appendix), we get

$$\begin{aligned} 0 &\leq \int_\Omega [\phi(|\nabla u|)\nabla u - \phi(|\nabla v|)\nabla v] \cdot (\nabla u - \nabla v) dx \\ &= \int_\Omega \rho(x)\Big(\lambda_1 m(u) - \lambda m(v)\Big)(u - v + \sup_\Omega v) dx. \end{aligned}$$

We claim that

$$\lambda_1 m(u) \leq \lambda m(v).$$

Indeed, suppose that $\lambda_1 m(u) > \lambda m(v)$ and let us define the two admissible test functions

$$\eta_3(x) = \begin{cases} u(x) - v(x) - \sup_\Omega u & \text{if} \quad x \in \Omega_0, \\ 0 & \text{if} \quad x \notin \Omega_0 \end{cases}$$

and

$$\eta_4(x) = \begin{cases} v(x) - u(x) + \sup_\Omega u & \text{if} \quad x \in \Omega_0, \\ 0 & \text{if} \quad x \notin \Omega_0. \end{cases}$$

As above, inserting η_3 and η_4 in (3.4) and then summing up we obtain

$$\begin{aligned} 0 &\leq \int_\Omega [\phi(|\nabla u|)\nabla u - \phi(|\nabla v|)\nabla v] \cdot (\nabla u - \nabla v) dx \\ &= \int_\Omega \rho(x)[\lambda_1 m(u) - \lambda m(v)](u - v - \sup_\Omega u) dx \leq 0, \end{aligned}$$

implying by Lemma 2.3.1 that $v = u$, *but such an equality can not occur since* $\lambda > \lambda_1$ *which proves our claim. Finally, we conclude that the function* v *can not have a constant sign in* Ω.

Next we prove the estimate (2.16). *According to the above* $v^+ > 0$ *and* $v^- > 0$. *Choosing* $v^+ \in W_0^1 L_M(\Omega)$ *as a test function in* (2.11), *we get*

$$\int_\Omega m(|\nabla v^+|)|\nabla v^+|dx = \lambda \int_\Omega \rho(x)m(v^+)v^+dx.$$

Since $M(t) \leq m(t)t \leq M(2t)$ *for* $t \geq 0$, *we obtain*

$$\int_\Omega M(|\nabla v^+|)dx \leq \lambda \|\rho(\cdot)\|_\infty \int_\Omega M(2v^+)dx.$$

We already know that by Theorem 2.3.2 (in Appendix) the function v *is bounded, then we get*

$$\int_\Omega M(|\nabla v^+|)dx \leq \lambda \|\rho(\cdot)\|_\infty M(2\|v\|_\infty)|\Omega|. \tag{2.17}$$

So, (2.8) *and* (2.17) *imply that there exists a positive constant* c, *such that*

$$\|\nabla v^+\|_M \leq c. \tag{2.18}$$

On the other hand, by the Hölder inequality [53] and the Poincaré type inequality [44, Lemma 5.7], we have

$$\int_\Omega v^+(x)dx \leq \|\chi_{\Omega^+}\|_{M^*}\|v^+\|_M \leq d\|\chi_{\Omega^+}\|_{M^*}\|\nabla v^+\|_M,$$

d *being the constant in Poincaré type inequality. Hence, using* (2.18) *to get*

$$\int_\Omega v^+(x)dx \leq cd\|\chi_{\Omega^+}\|_{M^*}. \tag{2.19}$$

We have to distinguish two cases, the case $\int_\Omega v^+(x)dx > 1$ *and* $\int_\Omega v^+(x)dx \leq 1$.

<u>Case 1</u> : Assume that

$$\int_\Omega v^+(x)dx > 1.$$

Thus, by (2.19) *we have*

$$\frac{1}{dc} \leq \|\chi_{\Omega^+}\|_{M^*}. \tag{2.20}$$

<u>Case 2</u> : Assume that

$$\int_\Omega v^+(x)dx \leq 1.$$

Recall that by Theorem 2.3.4 (in Appendix) the function v^+ *is continuous and as* $v^+ > 0$ *in* Ω *then* $\int_\Omega v^+(x)dx > 0$. *Therefore, by using* (2.19) *we obtain*

$$\frac{a}{dc} \leq \|\chi_{\Omega^+}\|_{M^*}, \tag{2.21}$$

where $a = \int_\Omega v^+(x)dx$. *So, by* (2.20) *and* (2.21), *we get*

$$\frac{\min\{a,1\}}{dc} \leq \|\chi_{\Omega^+}\|_{M^*},$$

where $\|\chi_{\Omega^+}\|_{M^*} = \frac{1}{M^{*-1}\left(\frac{1}{|\Omega^+|}\right)}$ *(see [53, page 79]). Hence,*

$$|\Omega^+| \geq \frac{1}{M^*\left(\frac{dc}{\min\{a,1\}}\right)}.$$

Such an estimation with v^- *can be obtained following exactly the same lines above. Then follows the inequality* (2.16).

Finally, we prove that the eigenvalue λ_1 given by the relation (2.7) is isolated from the right-hand side.

Theorem 2.2.3 *Assume that* (2.13) *holds. Then, the eigenvalue* λ_1 *is isolated from the right-hand side, that is, there exists* $\delta > 0$ *such that in the interval* $(\lambda_1, \lambda_1 + \delta)$ *there are no eigenvalues.*

Proof 2.2.8 *Assume by contradiction that there exists a non-increasing sequence* $\{\mu_n\}_n$ *of eigenvalues of* (3.57) *with* $\mu_n > \lambda_1$ *and* $\mu_n \to \lambda_1$. *Let* u_n *be an associated eigenfunction to* μ_n *and let*

$$\overline{\Omega_n^+} = \overline{\{x \in \Omega : u_n > 0\}} \text{ and } \overline{\Omega_n^-} = \overline{\{x \in \Omega : u_n < 0\}}.$$

By (2.16), *there exists* $c_n > 0$ *such that*

$$\min\{|\overline{\Omega_n^+}|, |\overline{\Omega_n^-}|\} \geq c_n. \tag{2.22}$$

Since $b_n := \int_\Omega \rho(x)M(|u_n(x)|)dx > 0$ *we define*

$$v_n(x) = \begin{cases} M^{-1}\left(\frac{M(u_n(x))}{b_n}\right) & \text{if} \quad x \in \overline{\Omega_n^+}, \\ -M^{-1}\left(\frac{M(-u_n(x))}{b_n}\right) & \text{if} \quad x \in \overline{\Omega_n^-}. \end{cases} \tag{2.23}$$

On the other hand, we have

$$|\nabla v_n| \leq \left|(M^{-1})'\left(\frac{M(|u_n|)}{b_n}\right)\right| \frac{m(|u_n|)|\nabla u_n|}{b_n} \chi_{\overline{\Omega_n^+} \cup \overline{\Omega_n^-}},$$

since u_n *is continuous, then there exists* $d_n > 0$ *such that* $\inf_{x \in \overline{\Omega_n^+} \cup \overline{\Omega_n^-}} |u_n(x)| \geq d_n$. *Let* $b = \min\{b_n\}$ *and* $d = \min\{d_n\}$. *Being* $\{u_n\}$ *uniformly bounded (Theorem 2.3.2 in Appendix), there exists a constant* $c_\infty > 0$, *not depending on* n, *such that*

$$\|u_n\|_\infty \leq c_\infty, \text{ for all } n \in \mathbb{N}. \tag{2.24}$$

Using the fact that $(M^{-1})'(\cdot)$ *is decreasing, we get*

$$|\nabla v_n| \leq \left|(M^{-1})'\left(\frac{M(d)}{\|\rho\|_\infty M(c_\infty)|\Omega|}\right)\right| \frac{m(c_\infty)}{b} |\nabla u_n| = C_0 |\nabla u_n|, \tag{2.25}$$

where $C_0 = \left|(M^{-1})'\left(\frac{M(d)}{\|\rho\|_\infty M(c_\infty)|\Omega|}\right)\right| \frac{m(c_\infty)}{b}$. *On the other hand, taking* u_n *as test function in* (2.11) *and using* (2.24) *and the inequality* $M(t) \leq m(t)t$ *for* $t > 0$, *one has*

$$\int_\Omega M(|\nabla u_n|)dx \leq \mu_n \|\rho\|_\infty m(c_\infty) c_\infty |\Omega|.$$

Since μ_n *converges to* λ_1, *there exists a constant* $C_1 > 0$, *such that*

$$\int_\Omega M(|\nabla u_n|)dx \leq C_1. \tag{2.26}$$

Therefore, by (2.25) *and* (2.26) *we obtain that* $\{v_n\}$ *is uniformly bounded in* $W_0^1 L_M(\Omega)$. *Alaoglu's theorem ensures the existence of a function* $v \in W_0^1 L_M(\Omega)$ *and a subsequence of* v_n, *still indexed by* n, *such that* $v_n \rightharpoonup v$ *for* $\sigma(\Pi L_M, \Pi E_{M^*})$. *By* (2.23), $v_n \in K$ *and since* B *is* $\sigma(\Pi L_M, \Pi E_{M^*})$ *continuous (see Lemma 2.2.1), then*

$$\int_\Omega \rho(x) M(|v(x)|)dx = B(v) = \lim_{n\to\infty} B(v_n) = 1.$$

Therefore, $v \in K$. *Since by Lemma 2.2.1 the functional* J *is* $\sigma(\Pi L_M, \Pi E_{M^*})$ *lower semi-continuous, we get*

$$J(v) = \int_\Omega M(|\nabla v|)dx \leq \liminf J(v_n) = \inf_{w\in K} J(w).$$

So that v *is an eigenfunction associated to* λ_1. *Applying Proposition 3.16, we have either* $v > 0$ *or* $v < 0$ *in* Ω. *Assume that* $v < 0$ *in* Ω *with* $v^- \not\equiv 0$. *By Egorov's Theorem,* v_n *converges uniformly to* v *except on a subset of* Ω *of null Lebesgue measure. Thus,* $v_n \leq 0$ *a.e. in* Ω *with* $v_n^- \not\equiv 0$ *outside a subset of* Ω *of null Lebesgue measure, which implies that*

$$|\overline{\Omega_n^+}| = 0,$$

which is a contradiction with the estimation (2.22).

2.3 Appendix

We prove here some important lemmas that are necessary for the accomplishment of the proofs of the above results.

Lemma 2.3.1 *Let ξ and η be vectors in $\mathbb{R}^N$. Then*

$$[\phi(|\xi|)\xi - \phi(|\eta|)\eta]\cdot(\xi-\eta) > 0, \text{ whenever } \xi \neq \eta.$$

Proof 2.3.1 *Since $\phi(t) > 0$ when $t > 0$ and $\xi\cdot\eta \leq |\xi|\cdot|\eta|$, there follows by a direct calculation*

$$[\phi(|\xi|)\xi - \phi(|\eta|)\eta]\cdot(\xi-\eta) \geq [m(|\xi|) - m(|\eta|)]\cdot(|\xi|-|\eta|)$$

and the conclusion comes from the strict monotonicity of m.

The following result can be found in [23, Lemma 9.5] in the case of Sobolev spaces.

Theorem 2.3.1 *Let A be an N-function (cf. [7]). If $u \in W^1L_A(\Omega)$ has a compact support in an open Ω having the segment property, then $u \in W_0^1L_A(\Omega)$.*

Proof 2.3.2 *Let $u \in W^1L_A(\Omega)$. We fix a compact set $\Omega' \subset \Omega$ such that supp $u \subset \Omega'$ and we denote by $\bar{u}$ the extension by zero of u to the whole of $\mathbb{R}^N$. Let J be the Friedrichs mollifier kernel defined on $\mathbb{R}^N$ by*

$$\rho(x) = ke^{-\frac{1}{1-\|x\|^2}} \text{ if } \|x\| < 1 \text{ and } 0 \text{ if } \|x\| \geq 1,$$

where $k > 0$ is such that $\int_{\mathbb{R}^N}\rho(x)dx = 1$. For $\epsilon > 0$, we define $\rho_n(x) = n^N J(nx)$. By [45], there exists $\lambda > 0$ large enough such that $A\Big(\frac{|u(x)|}{\lambda}\Big) \in L^1(\Omega)$, $A\Big(\frac{|\partial u/\partial x_i(x)|}{\lambda}\Big) \in L^1(\Omega)$, $i \in \{1,\cdots,N\}$, and

$$\int_{\mathbb{R}^N} A\Big(\frac{|\rho_n * \bar{u}(x) - \bar{u}(x)|}{\lambda}\Big)dx \to 0 \text{ as } n \to +\infty$$

and hence

$$\int_{\Omega} A\Big(\frac{|\rho_n * \bar{u}(x) - u(x)|}{\lambda}\Big)dx \to 0 \text{ as } n \to +\infty. \tag{2.27}$$

Choosing n large enough so that $0 < \frac{1}{n} < dist(\Omega', \partial\Omega)$ one has $\rho_n * \bar{u}(x) = \rho_n * u(x)$ for every $x \in \Omega'$. Hence, $\partial(\rho_n * \bar{u})/\partial x_i = \rho_n * (\partial u/\partial x_i)$ on Ω' for every $i \in \{1,\cdots,N\}$. As $\partial u/\partial x_i \in L_A(\Omega')$ we have

$$\partial(\rho_n * \bar{u})/\partial x_i \in L_A(\Omega').$$

Therefore,

$$\int_{\Omega'} A\Big(\frac{|\partial(\rho_n * \bar{u})/\partial x_i(x) - \partial u/\partial x_i(x)|}{\lambda}\Big)dx \to 0 \text{ as } n \to +\infty. \tag{2.28}$$

Observe that the functions $w_n = \rho_n * \bar{u}$ do not necessary lie in $\mathcal{C}_0^\infty(\Omega)$. Let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that $0 < \eta < 1$, $\eta(x) = 1$ for all x with $\|x\| \leq 1$, $\eta(x) = 0$ for all x with $\|x\| \geq 2$ and $|\nabla\eta| \leq 2$. Let *further* $\eta_n(x) = \eta\Big(\frac{x}{n}\Big)$ *for* $x \in \mathbb{R}^N$. *We claim that the functions* $v_n = \eta_n w_n$ *belong to* $\mathcal{C}_0^\infty(\mathbb{R}^N)$ *and satisfy*

$$\int_\Omega A\Big(\frac{|v_n(x) - u(x)|}{4\lambda}\Big)dx \to 0 \text{ as } n \to +\infty \tag{2.29}$$

and

$$\int_{\Omega'} A\Big(\frac{|\partial v_n/\partial x_i(x) - \partial u/\partial x_i(x)|}{12\lambda}\Big)dx \to 0 \text{ as } n \to +\infty \tag{2.30}$$

Indeed, by (2.27) there exist a subsequence of $\{w_n\}$ *still indexed by* n *and a function* $h_1 \in L^1(\Omega)$ *such that*

$$w_n \to u \text{ a.e. in } \Omega$$

and

$$|w_n(x)| \leq |u(x)| + \lambda A^{-1}(h_1)(x); \text{ for all } x \in \Omega \tag{2.31}$$

which together with the convexity of A *yield*

$$A\Big(\frac{|v_n(x) - u(x)|}{4\lambda}\Big) \leq \frac{1}{2}A\Big(\frac{|u(x)|}{\lambda}\Big) + \frac{1}{4}h_1(x)$$

Being the functions $A\Big(\frac{|u(x)|}{\lambda}\Big) \in L^1(\Omega)$ *and* $h_1 \in L^1(\Omega)$, *the sequence* $\Big\{A\Big(\frac{|v_n - u|}{4\lambda}\Big)\Big\}_n$ *is equi-integrable on* Ω *and since* $\{v_n\}$ *converges to* u *a.e. in* Ω, *we obtain (2.29) by applying Vitali's theorem.*

By (2.28) there exists a subsequence, relabeled again by n, *and a function* $h_2 \in L^1(\Omega')$ *such that*

$$\partial w_n/\partial x_i \to \partial u/\partial x_i \text{ a.e. in } \Omega'$$

and

$$|\partial w_n/\partial x_i(x)| \leq |\partial u/\partial x_i(x)| + h_2(x), \text{ for all } x \in \Omega'. \tag{2.32}$$

Therefore, using (2.31) and (2.32) for all $x \in \Omega'$ *we arrive at*

$$\begin{aligned} &A\Big(\tfrac{|\partial v_n/\partial x_i(x) - \partial u/\partial x_i(x)|}{12\lambda}\Big) \\ &\leq \tfrac{1}{6}\Big(A\Big(\tfrac{|u(x)|}{\lambda}\Big) + A\Big(\tfrac{|\partial u/\partial x_i(x)|}{\lambda}\Big) + h_1(x) + \tfrac{1}{2}h_2(x)\Big) \end{aligned}$$

and by Vitali's theorem we obtain (2.30).

Finally, let $K \subset \Omega'$ *be a compact set such that* $supp(u) \subset K$. *There exists a cut-off function* $\zeta \in \mathcal{C}_0^\infty(\Omega')$ *satisfying* $\zeta = 1$ *on* K. *Denoting* $u_n = \zeta v_n$, *we can deduce from (2.29) and (2.30)*

$$\int_\Omega A\Big(\frac{|u_n(x) - \zeta u(x)|}{12\lambda}\Big)dx \to 0 \text{ as } n \to +\infty$$

and

$$\int_\Omega A\Big(\frac{|\partial u_n/\partial x_i(x) - \partial(\zeta u)/\partial x_i(x)|}{12\lambda}\Big)dx \to 0 \text{ as } n \to +\infty.$$

Consequently, the sequence $\{u_n\} \subset \mathcal{C}_0^\infty(\Omega)$ *converges modularly to* $\zeta u = u$ *in* $W^1L_A(\Omega)$ *and in force for the weak topology* $\sigma(\Pi L_A, \Pi L_{\overline{A}})$ *(see [45, Lemma 6]) which in turn imply the convergence with respect to the weak* topology* $\sigma(\Pi L_A, \Pi E_{\overline{A}})$*. Thus,* $u \in W_0^1 L_A(\Omega)$*.*

Theorem 2.3.2 *For any weak solution* $u \in W_0^1 L_M(\Omega)$ *of* (2.1) *associated with* $\lambda > 0$*, there exists a constant* $c_\infty > 0$*, not depending on* u*, such that*

$$\|u\|_{L^\infty(\Omega)} \leq c_\infty.$$

Proof 2.3.3 *For* $k > 0$ *we define the set* $A_k = \{x \in \Omega : |u(x)| > k\}$ *and the two truncation functions* $T_k(s) = \max(-k, \min(s,k))$ *and* $G_k(s) = s - T_k(s)$*. By Hölder's inequality we get*

$$\begin{aligned} \int_{A_k} |G_k(u(x))|dx &\leq |A_k|^{\frac{1}{N}} \Big(\int_{A_k} |G_k(u(x))|^{\frac{N}{N-1}} dx \Big)^{\frac{N-1}{N}} \\ &\leq C(N)|A_k|^{\frac{1}{N}} \int_{A_k} |\nabla u| dx, \end{aligned}$$

where $C(N)$ *is the constant in the embedding* $W_0^{1,1}(A_k) \hookrightarrow L^{\frac{N}{N-1}}(A_k)$*. We shall estimate the integral* $\int_{A_k} |\nabla u| dx$*; to this aim we distinguish two cases : the case* $m(|\nabla u|)|\nabla u| < \lambda_1 \|\rho\|_\infty$ *and* $m(|\nabla u|)|\nabla u| \geq \lambda_1 \|\rho\|_\infty$*, where* λ_1 *is defined in (2.7).*

Case 1 : Assume that

$$m(|\nabla u|)|\nabla u| < \lambda_1 \|\rho\|_\infty. \tag{2.33}$$

Let $k_0 > 0$ *be fixed and let* $k > k_0$*. Using* (2.33) *we can write*

$$\begin{aligned} \int_{A_k} |\nabla u| dx &\leq \int_{A_k \cap \{|\nabla u| \leq 1\}} |\nabla u| dx + \int_{A_k \cap \{|\nabla u| > 1\}} |\nabla u| dx \\ &\leq |A_k| + \tfrac{1}{m(1)} \int_{A_k} m(|\nabla u|)|\nabla u| dx \\ &\leq \Big(1 + \tfrac{\lambda_1 \|\rho\|_\infty}{m(1)}\Big)|A_k|. \end{aligned}$$

Thus,

$$\int_{A_k} |G_k(u(x))| dx \leq C(N)\Big(1 + \frac{\lambda_1 \|\rho\|_\infty}{m(1)}\Big)|A_k|^{\frac{1}{N}+1}. \tag{2.34}$$

Case 2 : Assume now that

$$m(|\nabla u|)|\nabla u| \geq \lambda_1 \|\rho\|_\infty. \tag{2.35}$$

Since $u \in W_0^1 L_M(\Omega)$ *is a weak solution of problem (3.57), we have*

$$\int_\Omega \phi(|\nabla u|)\nabla u \cdot \nabla v dx = \lambda \int_\Omega \rho(x)\phi(|u|)uv dx, \tag{2.36}$$

for all $v \in W_0^1 L_M(\Omega)$. *Let* $s,\ t,\ k > 0$ *and let* $v = \frac{\lambda_1}{\lambda} \exp\left(\frac{\lambda}{\lambda_1} M(u^+)\right) T_s(G_k(T_t(u^+)))$. *So, from [46, Lemma 2] we can use* v *as a test function in* (2.36) *to obtain*

$$\begin{aligned} &\tfrac{\lambda}{\lambda_1} \int_{\{u>0\}} m(|\nabla u|)|\nabla u| m(u^+) v dx \\ &+ \tfrac{\lambda_1}{\lambda} \int_{\{k<T_t(u^+)\leq k+s\}} \phi(|\nabla u|)\nabla u \cdot \nabla T_t(u^+) \exp\left(\frac{\lambda}{\lambda_1} M(u^+)\right) dx \\ &-\lambda \int_{\{u>0\}} \rho(x)\phi(|u|) u v dx = 0. \end{aligned}$$

Since we integrate on the set $\{u > 0\}$, *by* (2.35) *we have*

$$\lambda_1 \rho(x) \leq m(|\nabla u|)|\nabla u|$$

and so we obtain

$$\frac{\lambda}{\lambda_1} \int_{\{u>0\}} \Big(m(|\nabla u|)|\nabla u| - \lambda_1 \rho(x) \Big) m(u^+) v dx \geq 0.$$

Therefore, we only have

$$\int_{\{k<T_t(u^+)\leq k+s\}} \phi(|\nabla u|)\nabla u \cdot \nabla T_t(u^+) \exp\left(\frac{\lambda}{\lambda_1} M(u^+)\right) dx = 0$$

and since $\exp\left(\frac{\lambda}{\lambda_1} M(u^+)\right) \geq 1$ *we have*

$$\int_{\{k<T_t(u^+)\leq k+s\}} \phi(|\nabla u|)\nabla u \cdot \nabla T_t(u^+) dx = 0.$$

Pointing out that

$$\int_{\{k<T_t(u^+)\leq k+s\}} \phi(|\nabla u|)\nabla u \cdot \nabla T_t(u^+) dx = \int_{\{k<u\leq k+s\}\cap\{0<u<t\}} \phi(|\nabla u|)\nabla u \cdot \nabla u dx.$$

We can apply the monotone convergence theorem as $t \to +\infty$ *obtaining*

$$\begin{aligned} \int_{\{k<u\leq k+s\}} \phi(|\nabla u|)\nabla u \cdot \nabla u dx &= \lim_{t\to\infty} \int_{\{k<u\leq k+s\}\cap\{0<u<t\}} \phi(|\nabla u|)\nabla u \cdot \nabla T_t(u^+) dx \\ &= 0. \end{aligned}$$

Applying again the monotone convergence theorem as $s \to +\infty$ *we get*

$$\int_{\{u>k\}} \phi(|\nabla u|)\nabla u \cdot \nabla u dx = \lim_{s\to\infty} \int_{\{k<u\leq k+s\}} \phi(|\nabla u|)\nabla u \cdot \nabla u dx = 0.$$

In the same way, inserting the function $v = -\frac{\lambda_1}{\lambda} \exp\left(\frac{\lambda}{\lambda_1} M(u^-)\right) T_s(G_k(T_t(u^-)))$ *that belongs to* $W_0^1 L_M(\Omega)$ *as a test function in* (2.36) *we obtain*

$$\begin{aligned} &-\tfrac{\lambda}{\lambda_1} \int_{\{u<0\}} \phi(|\nabla u|)\nabla u \cdot \nabla u m(u^-) v dx \\ &-\tfrac{\lambda_1}{\lambda} \int_{\{-k-s\leq T_t(u^-)<-k\}} \phi(|\nabla u|)\nabla u \cdot \nabla T_t(u^-) \exp\left(\frac{\lambda}{\lambda_1} M\Big(u^-\Big)\right) dx \\ &= \lambda \int_{\{u<0\}} \rho(x)\phi(|u|) u v dx. \end{aligned}$$

Then we can write

$$
\begin{aligned}
&-\tfrac{\lambda}{\lambda_1}\int_{\{u<0\}} m(|\nabla u|)|\nabla u| m(|u|) v dx \\
&-\tfrac{\lambda_1}{\lambda}\int_{\{-k-s\leq T_t(u^-)<-k\}} \phi(|\nabla u|)\nabla u\cdot\nabla T_t(u^-)\exp\Big(\frac{\lambda}{\lambda_1}M\Big(u^-\Big)\Big)dx \\
&= -\lambda\int_{\{u<0\}} \rho(x) m(|u|) v dx.
\end{aligned}
$$

Gathering the first and the last term we get

$$
\begin{aligned}
&-\tfrac{\lambda}{\lambda_1}\int_{\{u<0\}} \Big(m(|\nabla u|)|\nabla u| - \lambda_1\rho(x)\Big) m(|u|) v dx \\
&-\tfrac{\lambda_1}{\lambda}\int_{\{-k-s\leq T_t(u^-)<-k\}} \phi(|\nabla u|)\nabla u\cdot\nabla T_t(u^-)\exp\Big(\frac{\lambda}{\lambda_1}M\Big(u^-\Big)\Big)dx = 0.
\end{aligned}
$$

Here again since we have $\lambda_1\rho(x)\leq m(|\nabla u|)|\nabla u|$, *we obtain*

$$
-\frac{\lambda_1}{\lambda}\int_{\{-k-s\leq T_t(u^-)<-k\}} \phi(|\nabla u|)\nabla u\cdot\nabla T_t(u^-)\exp\Big(\frac{\lambda}{\lambda_1}M\Big(u^-\Big)\Big)dx \leq 0,
$$

that is

$$
\frac{\lambda_1}{\lambda}\int_{\{-k-s\leq T_t(u^-)<-k\}\cap\{u>-t\}} \phi(|\nabla u|)\nabla u\cdot\nabla u\exp\Big(\frac{\lambda}{\lambda_1}M\Big(u^-\Big)\Big)dx \leq 0.
$$

As $\exp\Big(\frac{\lambda}{\lambda_1}M\Big(u^-\Big)\Big)\geq 1$ *we get*

$$
\int_{\{-k-s\leq T_t(u^-)<-k\}\cap\{u>-t\}} \phi(|\nabla u|)\nabla u\cdot\nabla u dx = 0.
$$

As above using the monotone convergence theorem successively as $t\to+\infty$ *and then* $s\to+\infty$, *we arrive at*

$$
\int_{\{u<-k\}} \phi(|\nabla u|)\nabla u\cdot\nabla u dx = 0.
$$

Thus, since $m(t)=\phi(|t|)t$ *we conclude that*

$$
\int_{A_k} m(|\nabla u|)|\nabla u| dx = 0. \tag{2.37}
$$

On the other hand, by the monotonicity of the function m^{-1} *and by (2.37), we can write*

$$
\begin{aligned}
\int_{A_k}|\nabla u|dx &= \int_{A_k\cap\{m(|\nabla u|)<1\}}|\nabla u|dx + \int_{A_k\cap\{m(|\nabla u|)\geq 1\}}|\nabla u|dx \\
&\leq m^{-1}(1)|A_k| + \int_{A_k} m(|\nabla u|)|\nabla u|dx \\
&= m^{-1}(1)|A_k|.
\end{aligned}
$$

Hence,

$$
\int_{A_k}|G_k(u(x))|dx \leq C(N)m^{-1}(1)|A_k|^{\frac{1}{N}+1}. \tag{2.38}
$$

Finally, we note that the two obtained inequalities (2.34) and (2.38) are exactly the starting point of Stampacchia's L^∞-regularity proof (see [79]). In Fact, in any case we always have

$$\int_{A_k} |G_k(u(x))|dx \leq \eta |A_k|^{\frac{1}{N}+1}, \tag{2.39}$$

where $\eta := C(N)\Big(1 + m^{-1}(1) + \frac{\lambda_1 \|\rho\|_\infty}{m(1)}\Big)$. *Let* $h > k > 0$. *It is easy to see that* $A_h \subset A_k$ *and* $|G_k(u)| \geq h - k$ *on* A_h. *Thus, we have*

$$(h-k)|A_h| \leq \eta |A_k|^{\frac{1}{N}+1}.$$

The nonincreasing function ψ *defined by* $\psi(k) = |A_k|$ *satisfies*

$$\psi(h) \leq \frac{\eta}{(h-k)} \psi(k)^{\frac{1}{N}+1}.$$

Applying the first item of [79, Lemma 4.1] we obtain

$$\psi(c_\infty) = 0 \text{ where } c_\infty = C(N)\Big(1 + m^{-1}(1) + \frac{\lambda_1 \|\rho\|_\infty}{m(1)}\Big) 2^{N+1} |\Omega|^{\frac{1}{N}},$$

which yields

$$\|u\|_{L^\infty(\Omega)} \leq c_\infty = C(N)\Big(1 + m^{-1}(1) + \frac{\lambda_1 \|\rho\|_\infty}{m(1)}\Big) 2^{N+1} |\Omega|^{\frac{1}{N}}.$$

Lemma 2.3.2 *Let* Ω *be an open bounded subset in* $\mathbb{R}^N$. *Let* $B_R \subset \Omega$ *be an open ball of radius* $0 < R \leq 1$. *Suppose that* g *is a non-negative function such that* $g^\alpha \in L^\infty(B_R)$, *where* $|\alpha| \geq 1$. *Assume that*

$$\Big(\int_{B_R} g^{\alpha q k} dx\Big)^{\frac{1}{k}} \leq C \int_{B_R} g^{\alpha q} dx, \tag{2.40}$$

where $q, k > 1$ *and* C *is a positive constant. Then for any* $p > 0$ *there exists a positive constant* c *such that*

$$\sup_{B_R} g^\alpha \leq \frac{c}{R^{\frac{k}{(k-1)p}}} \Big(\int_{B_R} g^{\alpha p} dx\Big)^{\frac{1}{p}}.$$

Proof 2.3.4 *Let* $q = pk^\nu$ *where* ν *is a non-negative integer. Then using (2.40) and the fact that* $R \leq 1$ *we can have*

$$\Big(\int_{B_R} g^{\alpha p k^{\nu+1}} dx\Big)^{\frac{1}{pk^{\nu+1}}} \leq \Big(\frac{C}{R}\Big)^{\dfrac{1}{pk^\nu}} \Big(\int_{B_R} g^{\alpha p k^\nu} dx\Big)^{\frac{1}{pk^\nu}}.$$

An iteration of this inequality with respect to ν *yields*

$$\|g^\alpha\|_{L^{pk^{\nu+1}}(B_R)} \leq \Big(\frac{C}{R}\Big)^{\dfrac{1}{p}\sum_{i=0}^{\nu}\dfrac{1}{k^i}} \Big(\int_{B_R} g^{\alpha p} dx\Big)^{\frac{1}{p}}. \tag{2.41}$$

For $\beta \geq 1$, we consider ν large enough such that $pk^{\nu+1} > \beta$. Then, there exists a constant c_0 such that

$$\|g^\alpha\|_{L^\beta(B_R)} \leq c_0 \|g^\alpha\|_{L^{pk^{\nu+1}}(B_R)}.$$

Since the series in (2.41) are convergent and $g^\alpha \in L^\infty(B_R)$, Theorem 2.14 in [7] implies that

$$\sup_{B_R} g^\alpha \leq \frac{c}{R^{\frac{k}{(k-1)p}}} \Big(\int_{B_R} g^{\alpha p} dx \Big)^{\frac{1}{p}}.$$

As we need to get a Hölder estimate for weak solutions of (2.1), we use the previous lemma to prove Harnack-type inequalities. To do this, we define for a bounded weak solution $u \in W_0^1 L_M(\Omega)$ of (3.57) the two functions $v = u - \inf_{B_r} u$ and $w = \sup_{B_r} u - u$. We start by proving the following two lemmas.

Lemma 2.3.3 *Let $B_r \subset \Omega$ be an open ball of radius $0 < r \leq 1$. Then for every $p > 0$, there exists a positive constant C, depending on p, such that*

$$\sup_{B_{\frac{r}{2}}} v \leq C\bigg(\Big(r^{-N} \int_{B_r} v^p dx\Big)^{\frac{1}{p}} + r\bigg), \tag{2.42}$$

where $B_{\frac{r}{2}}$ is the ball of radius $r/2$ concentric with B_r.

Proof 2.3.5 *Since u is a weak solution of problem (2.1) then v satisfies the weak formulation*

$$\int_\Omega \phi(|\nabla v|)\nabla v \cdot \nabla \psi dx = \lambda \int_\Omega \rho(x)\phi(|v + \inf_{B_r} u|)(v + \inf_{B_r} u)\psi dx, \tag{2.43}$$

for every $\psi \in W_0^1 L_M(\Omega)$. Let Ω_0 be a compact subset of Ω such that $B_{\frac{r}{2}} \subset \Omega_0 \subset B_r$. Let $q > 1$ and let ψ be the function defined by

$$\psi(x) = \begin{cases} M(\bar{v}(x))^{q-1}\bar{v}(x) & \text{if} \quad x \in \Omega_0, \\ 0 & \text{if} \quad x \notin \Omega_0 \end{cases}$$

where $\bar{v} = v + r$. Observe that on Ω_0

$$\nabla \psi = M(\bar{v})^{q-1}\nabla \bar{v} + (q-1)M(\bar{v})^{q-2}m(\bar{v})\bar{v}\nabla \bar{v}$$

and thus by Theorem 2.3.1 we have $\psi \in W_0^1 L_M(\Omega)$. So that ψ is an admissible test function in (2.43). Taking it so it yields

$$\begin{aligned} \int_{B_r} M(\bar{v})^{q-1} m(|\nabla \bar{v}|)|\nabla \bar{v}| dx \quad &+ (q-1)\int_{B_r} M(\bar{v})^{q-2} m(\bar{v})\bar{v} m(|\nabla \bar{v}|)|\nabla \bar{v}| dx \\ &= \lambda \int_{B_r} \rho(x) M(\bar{v})^{q-1} \bar{v} m(v + \inf_{B_r} u) dx. \end{aligned}$$

Since $\bar{v}m(\bar{v}) \geq M(\bar{v})$ *and* $v + \inf_{B_r} u \leq \bar{v} + \|u\|_\infty$, *we get*

$$q \int_{B_r} M(\bar{v})^{q-1} m(|\nabla \bar{v}|)|\nabla \bar{v}| dx \leq \lambda \|\rho\|_\infty \int_{B_r} M(\bar{v})^{q-1}(\bar{v} + \|u\|_\infty) m(\bar{v} + \|u\|_\infty) dx. \tag{2.44}$$

Let

$$h(x) = \begin{cases} M(\bar{v}(x))^q & \text{if} \quad x \in \Omega_0, \\ 0 & \text{if} \quad x \notin \Omega_0. \end{cases}$$

Using the following inequality

$$am(b) \leq bm(b) + am(a), \tag{2.45}$$

with $a = |\nabla \bar{v}|$ *and* $b = \bar{v}$, *we obtain*

$$\begin{aligned} \int_{B_r} |\nabla h| dx &\leq q \int_{B_r} M(\bar{v})^{q-1} m(|\nabla \bar{v}|)|\nabla \bar{v}| dx + q \int_{B_r} M(\bar{v})^{q-1} \bar{v} m(\bar{v}) dx \\ &\leq q \int_{B_r} M(\bar{v})^{q-1} m(|\nabla \bar{v}|)|\nabla \bar{v}| dx \\ &+ q \int_{B_r} M(\bar{v})^{q-1}(\bar{v} + \|u\|_\infty) m(\bar{v} + \|u\|_\infty) dx. \end{aligned}$$

In view of (2.44), *we obtain*

$$\int_{B_r} |\nabla h| dx \leq C_2 \int_{B_r} M(\bar{v})^q dx \leq C_2 M(2\|u\|_\infty + 1)^q |\Omega|,$$

where $C_2 = \frac{(q+\lambda\|\rho\|_\infty)(1+3\|u\|_\infty)m(1+3\|u\|_\infty)}{M(r)}$. *Therefore,* $h \in W_0^{1,1}(B_r)$ *and so we can write*

$$\Big(\int_{B_r} M(\bar{v})^{\frac{qN}{N-1}} dx \Big)^{\frac{N-1}{N}} \leq C_2 C(N) \int_{B_r} M(\bar{v})^q dx,$$

where $C(N)$ *stands for the constant in the continuous embedding* $W_0^{1,1}(B_r) \hookrightarrow L^{\frac{N}{N-1}}(B_r)$. *Then, applying Lemma* 2.3.2 *with* $g = M(\bar{v})$ *and* $\alpha = 1$ *we obtain for any* $p > 0$

$$\sup_{B_r} M(\bar{v}) \leq C_3 \Big[r^{-N} \int_{B_r} M(\bar{v})^p dx \Big]^{\frac{1}{p}},$$

where $C_3 = (C_2 C(N))^{\frac{N}{p}}$. *Hence, follows*

$$\sup_{B_{\frac{r}{2}}} M(\bar{v}) \leq C_3 \Big[r^{-N} \int_{B_r} M(\bar{v})^p dx \Big]^{\frac{1}{p}}.$$

Since $\frac{t}{2} m(\frac{t}{2}) \leq M(t) \leq t m(t)$ *and* $\bar{v} = v + r = u - \inf_{B_r} u + r$ *we have* $\sup_{B_{\frac{r}{2}}} M(\bar{v}) \geq m(\frac{r}{2}) \sup_{B_{\frac{r}{2}}} \frac{\bar{v}}{2}$ *and* $M(\bar{v}) \leq \bar{v} m(1 + 2\|u\|_\infty)$, *which yields*

$$\sup_{B_{\frac{r}{2}}} \bar{v} \leq C \Big[r^{-N} \int_{B_r} \bar{v}^p dx \Big]^{\frac{1}{p}},$$

where $C = (C_2 C(N))^{\frac{N}{p}} \frac{2m(1+2\|u\|_\infty)}{m(\frac{r}{2})}$. *Hence, the inequality* (2.42) *is proved.*

Lemma 2.3.4 *Let $B_r \subset \Omega$ be an open ball of radius $0 < r \leq 1$. Then, there exist two constants $C > 0$ and $p_0 > 0$ such that*

$$\Big(r^{-N}\int_{B_r} v^{p_0}dx\Big)^{\frac{1}{p_0}} \leq C\Big(\inf_{B_{\frac{r}{2}}} v + r\Big), \tag{2.46}$$

where $B_{\frac{r}{2}}$ is the ball of radius $r/2$ concentric with B_r.

Proof 2.3.6 *Let Ω_0 be a compact subset of Ω such that $B_{\frac{r}{2}} \subset \Omega_0 \subset B_r$. Let $q > 1$ and let ψ be the function defined by*

$$\psi(x) = \begin{cases} M(\bar{v}(x))^{-q-1}\bar{v}(x) & \text{if} \quad x \in \Omega_0, \\ 0 & \text{if} \quad x \notin \Omega_0, \end{cases}$$

where $\bar{v} = v + r$. On Ω_0 we compute

$$\nabla\psi = M(\bar{v})^{-q-1}\nabla\bar{v} + (-q-1)M(\bar{v})^{-q-2}m(\bar{v})\bar{v}\nabla\bar{v}.$$

By Theorem 2.3.1 we have $\psi \in W_0^1L_M(\Omega)$. Thus, using the function ψ in (2.43) we obtain

$$\begin{aligned} &\lambda\int_{B_r}\rho(x)M(\bar{v})^{-q-1}\bar{v}m(v + \inf_{B_r} u)dx \\ &= \int_{B_r} M(\bar{u})^{-q-1}|\nabla\bar{v}|m(|\nabla\bar{v}|)dx \\ &+(-q-1)\int_{B_r} M(\bar{v})^{-q-2}m(\bar{v})\bar{v}|\nabla\bar{v}|m(|\nabla\bar{v}|)dx. \end{aligned}$$

By the fact that $\bar{v}m(\bar{v}) \geq M(\bar{v})$, we get

$$\lambda\int_{B_r}\rho(x)M(\bar{v})^{-q-1}\bar{v}m(v + \inf_{B_r} u)dx \leq -q\int_{B_r} M(\bar{v})^{-q-1}|\nabla\bar{v}|m(|\nabla\bar{v}|)dx.$$

Thus, since on B_r one has $|v + \inf_{B_r} u| \leq \bar{v} + \|u\|_\infty$ we obtain

$$q\int_{B_r} M(\bar{v})^{-q-1}|\nabla\bar{v}|m(|\nabla\bar{v}|)dx \leq \lambda\|\rho\|_\infty\int_{B_r} M(\bar{v})^{-q-1}m(\bar{v} + \|u\|_\infty)(\bar{v} + \|u\|_\infty)dx. \tag{2.47}$$

On the other hand, let h be the function defined by

$$h(x) = \begin{cases} M(\bar{v}(x))^{-q} & \text{if} \quad x \in \Omega_0, \\ 0 & \text{if} \quad x \notin \Omega_0. \end{cases}$$

Using once again (2.45) with $a = |\nabla\bar{v}|$ and $b = \bar{v}$, we obtain

$$\begin{aligned} \int_{B_r}|\nabla h|dx &\leq q\int_{B_r} M(\bar{v})^{-q-1}m(|\nabla\bar{v}|)|\nabla\bar{v}|dx + q\int_{B_r} M(\bar{v})^{-q-1}\bar{v}m(\bar{v})dx \\ &\leq q\int_{B_r} M(\bar{v})^{-q-1}m(|\nabla\bar{v}|)|\nabla\bar{v}|dx \\ &+q\int_{B_r} M(\bar{v})^{q-1}(\bar{v} + \|u\|_\infty)m(\bar{v} + \|u\|_\infty)dx, \end{aligned}$$

which together with (2.47) *yield*

$$\int_{B_r} |\nabla h| dx \le C_2 \int_{B_r} M(\bar{v})^{-q} dx \le C_2 M(r)^{-q} |\Omega|,$$

with $C_2 = \frac{(q+\lambda\|\rho\|_\infty)m(1+3\|u\|_\infty)(1+3\|u\|_\infty)}{M(r)}$. *Thus,* $h \in W_0^{1,1}(B_r)$ *and so we can write*

$$\Big(\int_{B_r} M(\bar{v})^{-\frac{qN}{N-1}} dx\Big)^{\frac{N-1}{N}} \le C_2 C(N) \int_{B_r} M(\bar{v})^{-q} dx,$$

where $C(N)$ *is the constant in the continuous embedding* $W_0^{1,1}(B_r) \hookrightarrow L^{\frac{N}{N-1}}(B_r)$. *Therefore, applying Lemma* 2.3.2 *with* $g = M(\bar{v})$ *and* $\alpha = -1$ *we get for any* $p > 0$

$$\sup_{B_r} M(\bar{v})^{-1} \le (C_2 C(N))^{\frac{N}{p}} \Big(r^{-N} \int_{B_r} M(\bar{v})^{-p} dx\Big)^{\frac{1}{p}}.$$

So that one has

$$\begin{aligned} \Big(r^{-N} \int_{B_r} M(\bar{v})^{-p} dx\Big)^{\frac{-1}{p}} &\le (C_2 C(N))^{\frac{N}{p}} \inf\nolimits_{B_r} M(\bar{v}) \\ &\le (C_2 C(N))^{\frac{N}{p}} \inf\nolimits_{B_{\frac{r}{2}}} M(\bar{v}). \end{aligned}$$

The fact that $M(\bar{v}) \ge m\Big(\frac{r}{2}\Big)\frac{\bar{v}}{2}$ *and* $M(\bar{v}) \le m(2\|u\|_\infty + 1)\bar{v}$, *yields*

$$\Big(r^{-N} \int_{B_r} \bar{v}^{-p} dx\Big)^{\frac{-1}{p}} \le C \inf_{B_{\frac{r}{2}}} \bar{v}, \tag{2.48}$$

where $C = (C_2 C(N))^{\frac{N}{p}} \frac{2m(2\|u\|_\infty+1)}{m(\frac{r}{2})}$. *Now, it only remains to show that there exist two constants* $c > 0$ *and* $p_0 > 0$, *such that*

$$\Big(r^{-N} \int_{B_r} \bar{v}^{p_0} dx\Big)^{\frac{1}{p_0}} \le c\Big(r^{-N} \int_{B_r} \bar{v}^{-p_0} dx\Big)^{\frac{-1}{p_0}}.$$

Let $B_{r_1} \subset B_r$ *and let* Ω_0 *be a compact subset of* Ω *such that* $B_{\frac{r_1}{2}} \subset \Omega_0 \subset B_{r_1}$. *Let* ψ *be the function defined by*

$$\psi(x) = \begin{cases} \bar{v}(x) & \text{if} \quad x \in \Omega_0, \\ 0 & \text{if} \quad x \notin \Omega_0. \end{cases}$$

Then, inserting ψ *as a test function in* (2.43) *we obtain*

$$\begin{aligned} \int_{B_{r_1}} m(|\nabla \bar{v}|)|\nabla \bar{v}| dx &\le \lambda\|\rho\|_\infty \int_{B_{r_1}} m(|v + \inf_{B_R} u|)\bar{v} dx \\ &\le \lambda\|\rho\|_\infty \int_{B_{r_1}} m(\bar{v} + \|u\|_\infty)(\bar{v} + \|u\|_\infty) dx. \end{aligned}$$

Since $\bar{v} \le (2\|u\|_\infty + 1)$ *and* $|B_{r_1}| = r_1^N |B_1|$ *we obtain*

$$\int_{B_{r_1}} m(|\nabla \bar{v}|)|\nabla \bar{v}| dx \le c_0 r_1^N, \tag{2.49}$$

where $c_0 = \lambda\|\rho\|_\infty m(3\|u\|_\infty + 1)(3\|u\|_\infty + 1)|B_1|$. *On the other hand, we can use* (2.45) *with* $a = |\nabla\bar{v}|$ *and* $b = \frac{\bar{v}}{r_1}$ *obtaining*

$$|\nabla\bar{v}|m\Big(\frac{\bar{v}}{r_1}\Big) \leq |\nabla\bar{v}|m(|\nabla\bar{v}|) + \frac{\bar{v}}{r_1}m\Big(\frac{\bar{v}}{r_1}\Big).$$

Pointing out that $\frac{\bar{v}}{r_1}m(\frac{\bar{v}}{r_1}) \geq M(\frac{\bar{v}}{r_1}) \geq M(\frac{\bar{v}}{r}) \geq M(1)$, *we get*

$$\frac{|\nabla\bar{v}|}{\bar{v}} \leq \frac{1}{r_1 M(1)}m(|\nabla\bar{v}|)|\nabla\bar{v}| + \frac{1}{r_1}.$$

Integrating over the ball $B_{\frac{r_1}{2}}$ *and using* (2.49) *we obtain*

$$\begin{aligned}\int_{B_{\frac{r_1}{2}}} \frac{|\nabla\bar{v}|}{\bar{v}}dx &\leq \tfrac{1}{r_1 M(1)}\int_{B_{\frac{r_1}{2}}} m(|\nabla\bar{v}|)|\nabla\bar{v}|dx + \frac{1}{r_1}|B_{\frac{r_1}{2}}| \\ &\leq \left(\tfrac{c_0}{M(1)} + \tfrac{|B_1|}{2^N}\right)r_1^{N-1}.\end{aligned}$$

The above inequality together with [84, Lemma 1.2] ensure the existence of two constants $p_0 > 0$ *and* $c > 0$ *such that*

$$\Big(\int_{B_r} \bar{v}^{p_0}dx\Big)\Big(\int_{B_r} \bar{v}^{-p_0}dx\Big) \leq cr^{2N}. \tag{2.50}$$

Finally, the estimate (2.46) *follows from* (2.48) *with* $p = p_0$ *and* (2.50).

Theorem 2.3.3 (Harnack-type inequalities) *Let* $u \in W_0^1 L_M(\Omega)$ *be a bounded weak solution of* (3.57) *and let* $B_{\frac{r}{2}}$, $0 < r \leq 1$, *be a ball with radius* $\frac{r}{2}$. *There exists a large constant* $C > 0$ *such that*

$$\sup_{B_{\frac{r}{2}}} v \leq C(\inf_{B_{\frac{r}{2}}} v + r) \tag{2.51}$$

and

$$\sup_{B_{\frac{r}{2}}} w \leq C(\inf_{B_{\frac{r}{2}}} w + r). \tag{2.52}$$

Proof 2.3.7 *Putting together* (2.42), *with the choice* $p = p_0$, *and* (2.46) *we immediately get* (2.51). *In the same way as above, one can obtain analogous inequalities to (2.46) and (2.42) for* w *obtaining the inequality* (2.52).

We are now ready to prove the following Hölder estimate for weak solutions of (3.57).

Theorem 2.3.4 (Hölder regularity) *Let* $u \in W_0^1 L_M(\Omega)$ *be a bounded weak solution of* (3.57). *Then there exist two constants* $0 < \alpha < 1$ *and* $C > 0$ *such that if* B_r *and* B_R *are two concentric balls of radii* $0 < r \leq R \leq 1$, *then*

$$osc_{B_r} u \leq C\Big(\frac{r}{R}\Big)^\alpha\Big(\sup_{B_R}|u| + C(R)\Big),$$

where $osc_{B_r} u = \sup_{B_r} u - \inf_{B_r} u$ *and* $C(R)$ *is a positive constant which depends on* R.

Proof 2.3.8 *From (2.51) and (2.52) we obtain*

$$\sup_{B_{\frac{r}{2}}} u - \inf_{B_r} u = \sup_{B_{\frac{r}{2}}} v \le C(\inf_{B_{\frac{r}{2}}} v + r) = C(\inf_{B_{\frac{r}{2}}} u - \inf_{B_r} u + r)$$

and

$$\sup_{B_r} u - \sup_{B_{\frac{r}{2}}} u = \sup_{B_{\frac{r}{2}}} w \le C(\inf_{B_{\frac{r}{2}}} w + r) \le C(\sup_{B_{\frac{r}{2}}} w + r) = C(\sup_{B_r} u - \sup_{B_{\frac{r}{2}}} u + r).$$

Hence, summing up both the two first terms in the left-hand side and the two last terms in the right-hand side of the above inequalities, we obtain

$$\sup_{B_r} u - \inf_{B_r} u \le C\Big(\sup_{B_r} u - \inf_{B_r} u + \inf_{B_{\frac{r}{2}}} u - \sup_{B_{\frac{r}{2}}} u + 2r\Big),$$

that is to say, what one still writes

$$osc_{B_{\frac{r}{2}}} u \le \Big(\frac{C-1}{C}\Big) osc_{B_r} u + 2r. \tag{2.53}$$

Let us fix some real number $R_1 \le R$ and define $\sigma(r) = osc_{B_r} u$. Let $n \in \mathbb{N}$ be an integer. Iterating the inequality (2.53) by substituting $r = R_1$, $r = \frac{R_1}{2}$, $\cdots$, $r = \frac{R_1}{2^n}$, we obtain

$$\begin{aligned}\sigma\Big(\tfrac{R_1}{2^n}\Big) &\le \gamma^n \sigma(R_1) + R_1 \textstyle\sum_{i=0}^{n-1} \frac{\gamma^{n-1-i}}{2^{i-1}} \\ &\le \gamma^n \sigma(R) + \tfrac{R_1}{1-\gamma},\end{aligned}$$

where $\gamma = \frac{C-1}{C}$. For any $r \le R_1$, there exists an integer n satisfying

$$2^{-n-1} R_1 \le r < 2^{-n} R_1.$$

Since σ is an increasing function, we get

$$\sigma(r) \le \gamma^n \sigma(R) + \frac{R_1}{1-\gamma}.$$

Being $\gamma < 1$, we can write

$$\gamma^n \le \gamma^{-1} \gamma^{-\frac{\log(\frac{r}{R_1})}{\log 2}} = \gamma^{-1} \Big(\tfrac{r}{R_1}\Big)^{-\frac{\log\gamma}{\log 2}}.$$

Therefore,

$$\sigma(r) \le \gamma^{-1} \Big(\frac{r}{R_1}\Big)^{-\frac{\log\gamma}{\log 2}} \sigma(R) + \frac{R_1}{1-\gamma}.$$

This inequality holds for arbitrary R_1 such that $r \le R_1 \le R$. Let now $\alpha \in (0,1)$ and $R_1 = R^{1-\alpha} r^{\alpha}$, so that we have from the preceding

$$\sigma(r) \le \gamma^{-1} \Big(\frac{r}{R}\Big)^{-(1-\alpha)\frac{\log\gamma}{\log 2}} \sigma(R) + \frac{R}{1-\gamma}\Big(\frac{r}{R}\Big)^{\alpha}.$$

Thus, the desired result follows by choosing α such that $\alpha = -(1-\alpha)\frac{\log\gamma}{\log 2}$, that is $\alpha = \frac{-\log\gamma}{\log 2 - \log\gamma}$.

Chapter 3

Imbedding results in Musielak-Orlicz-sobolev spaces with an application to anisotropic nonlinear Neumann problems

In this chapter, we prove a continuous embedding that allows us to obtain a boundary trace imbedding result for anisotropic Musielak-Orlicz spaces, which we then apply to obtain an existence result for Neumann problems with nonlinearities on the boundary associated to some anisotropic nonlinear elliptic equations in Musielak-Orlicz spaces constructed from Musielak-Orlicz functions on which and on their conjugates we do not assume the Δ_2-condition. The uniqueness is also studied.

3.1 Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $(N \geq 2)$. We denote by $\vec{\phi} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^N$ the vector function $\vec{\phi} = (\phi_1, \cdots, \phi_N)$ where for every $i \in \{1, \cdots, N\}$, ϕ_i is a Musielak-Orlicz function differentiable with respect to its second argument whose complementary Musielak-Orlicz function is denoted by ϕ_i^* (see preliminaries). We consider the follwing problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)\varphi_{max}(x, |u(x)|) & = f(x, u) \quad \text{in } \Omega, \\ u & \geq 0 \quad \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, \partial_{x_i} u)\nu_i & = g(x, u) \quad \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\partial_{x_i} = \frac{\partial}{\partial x_i}$ and for every $i = 1, \cdots, N$, we denote by ν_i the i^{th} component of the outer normal unit vector and $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that there exist a locally integrable

Musielak-Orlicz function (see definition 3.1.1 below) P_i with $P_i \ll \phi_i$, a positive constant c_i and a nonnegative function $d_i \in E_{\phi_i^*}(\Omega)$ satisfying for all s, $t \in \mathbb{R}$ and for almost every $x \in \Omega$ the following assumptions

$$|a_i(x,s)| \leq c_i[d_i(x) + (\phi_i^*)^{-1}(x, P_i(x,s))], \tag{3.2}$$

$$\phi_i(x,|s|) \leq a_i(x,s)s \leq A_i(x,s), \tag{3.3}$$

$$\big(a_i(x,s) - a_i(x,t)\big) \cdot \big(s - t\big) > 0, \text{ for all } s \neq t, \tag{3.4}$$

the function $A_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined by

$$A_i(x,s) = \int_0^s a_i(x,t)dt.$$

Here and in what follows, we define ϕ_{min} and ϕ_{max} by

$$\phi_{min}(x,s) = \min_{i=1,\cdots,N} \phi_i(x,s) \text{ and } \phi_{max}(x,s) = \max_{i=1,\cdots,N} \phi_i(x,s).$$

Let $\varphi_{max}(x,y) = \frac{\partial \phi_{max}}{\partial y}(x,y)$. We also assume that there exist a locally integrable Musielak-Orlicz function R with $R \ll \phi_{max}$ and a nonnegative function $D \in E_{\phi^*_{max}}(\Omega)$, such that for all s, $t \in \mathbb{R}$ and for almost every $x \in \Omega$

$$|\varphi_{max}(x,s)| \leq D(x) + (\phi^*_{max})^{-1}(x, R(x,s)), \tag{3.5}$$

where ϕ^*_{max} stands for the complementary function of ϕ_{max} defined below in (3.11).

For what concerns the data, we suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}^+$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. We define the antiderivatives $F : \Omega \times \mathbb{R} \to \mathbb{R}$ and $G : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ of f and g respectively by

$$F(x,s) = \int_0^s f(x,t)dt \text{ and } G(x,s) = \int_0^s g(x,t)dt.$$

We say that a Musielak-Orlicz function ϕ satisfies the Δ_2-condition, if there exists a positive constant $k > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$\phi(x, 2t) \leq k\phi(x,t) + h(x).$$

We remark that the condition (Δ_2) is equivalent to the following condition: for all $\alpha > 1$ there exists a positive constant $k > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$\phi(x, \alpha t) \leq k\phi(x,t) + h(x).$$

We assume now that there exist two positive constants k_1 and k_2 and two locally integrable Musielak-Orlicz functions M and H satisfy the Δ_2-condition and differentiable with respect to their second arguments with $M \ll \phi^{**}_{min}$, $H \ll \phi^{**}_{min}$ and $H \ll \psi_{min}$, such that the functions f and g satisfy for all $s \in \mathbb{R}_+$ the following assumptions

$$|f(x,s)| \leq k_1 m(x,s), \text{ for a.e. } x \in \Omega, \tag{3.6}$$

$$|g(x,s)| \leq k_2 h(x,s), \text{ for a.e. } x \in \partial\Omega, \tag{3.7}$$

where

$$\psi_{min}(x,t) = [(\phi^{**}_{min})_*(x,t)]^{\frac{N-1}{N}}, \ m(x,s) = \frac{\partial M(x,s)}{\partial s} \text{ and } h(x,s) = \frac{\partial H(x,s)}{\partial s}. \tag{3.8}$$

Finally, for the function b involved in (3.1), we assume that there exists a constant $b_0 > 0$ such that b satisfies the hypothesis

$$b \in L^\infty(\Omega) \text{ and } b(x) \geq b_0, \text{ for a.e. } x \in \Omega. \tag{3.9}$$

We remark that (3.4) and the relation $a_i(x,\zeta) = \nabla_\zeta A_i(x,\zeta)$ imply in particular that for any $i = 1, \cdots, N$, the function $\zeta \to A_i(\cdot,\zeta)$ is convex.

Let us put ourselves in the particular case of $\vec{\phi} = (\phi_i)_{i\in\{1,\cdots,N\}}$ where for $i \in \{1,\cdots,N\}$, $\phi_i(x,t) = |t|^{p_i(x)}$ with $p_i \in C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : \inf_{x\in\Omega} h(x) > 1\}$. Defining $p_{max}(x) = \max_{i\in\{1,\cdots,N\}} p_i(x)$ and $p_{min}(x) = \min_{i\in\{1,\cdots,N\}} p_i(x)$, one has $\phi_{max}(x,t) = |t|^{p_M(x)}$ and then $\varphi_{max}(x,t) = p_M(x)|t|^{p_M(x)-2}t$, where p_M is p_{max} or p_{min} according to whether $|t| \geq 1$ or $|t| \leq 1$ and then the space $W^1L_{\vec{\phi}}(\Omega)$ is nothing but the anisotropic space with variable exponent $W^{1,\vec{p}(\cdot)}(\Omega)$, where $\vec{p}(\cdot) = (p_1(\cdot),\cdots,p_N(\cdot))$ (see [37] for more details on this space). Therefore, the problem (3.1) can be rewritten as follows

$$\begin{cases} -\sum_{i=1}^N \partial_{x_i} a_i(x,\partial_{x_i}u) + b_1(x)|u|^{p_M(x)-2}u &= f(x,u) \quad \text{in } \Omega, \\ u &\geq 0 \quad \text{in } \Omega, \\ \sum_{i=1}^N a_i(x,\partial_{x_i}u)\nu_i &= g(x,u) \quad \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

where $b_1(x) = p_M(x)b(x)$. Boureanu and Rădulescu [21] have proved the existence and uniqueness of the weak solution of (3.10). They prove an imbedding and a trace results which they use together with a classical minimization existence result for functional reflexive framework (see [81, Theorem 1.2]). Problem (3.10) with Dirichlet boundary condition and $b_1(x) = 0$ was treated in [51]. The authors proved that if $f(\cdot,u) = f(\cdot) \in L^\infty(\Omega)$ then (3.10) admits a unique solution by using [81, Theorem 1.2].

The problem (3.10) with for all $i = 1,\cdots,N$

$$a_i(x,s) = a(x,s) = s^{p(x)-1},$$

with $p \in C^1(\overline{\Omega})$ and $b_1 = g = 0$ was treated in [43], where the authors proved the three nontrivial smooth solutions, two of which have constant sign (one positive, the other negative).

Let us mention some related results in the framework of Orlicz-Sobolev spaces. Le and Schmitt [56] proved an existence result for the following boundary value problem

$$\begin{cases} -\mathrm{div}(A(|\nabla u|^2)\nabla u) + F(x,u) = 0, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in $W_0^1L_\phi(\Omega)$ where $\phi(s) = A(|s|^2)s$ and F is a Carathéodory function satisfying some growth conditions. This result extends the one obtained in [42] with $F(x,u) = -\lambda\psi(u)$, where ψ is an odd increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$. In [42, 56] the authors assume that the N-function ϕ^* complementary to the N-function ϕ satisfies the Δ_2 condition, which is used to prove that the functional $u \to \int_\Omega \Phi(|\nabla u|)dx$ is coercive and of class $\mathcal{C}^1$, where Φ is the antiderivative of ϕ vanishing at origin.

Here, we are interested in proving the existence and uniqueness of the weak solution for problem (3.1) without any additional condition on the Musielak-Orlicz function ϕ_i or its complementary ϕ_i^* for $i = 1, \cdots, N$. Thus, the Musielak-Orlicz spaces $L_{\phi_i}(\Omega)$ are neither reflexive nor separable and hence classical existence results can not be applied.

The approach we use consists in proving first a continuous imbedding and a trace result which we then apply to solve problem (3.1). The results we prove here extend to the anisotropic Musielak-Orlicz-Sobolev spaces the continuous imbedding result obtained in [38] under extra conditions and the trace result proved in [60]. The imbedding result we obtain extends to Musielak spaces a part of the one obtained in [63] in the anisotropic case and that of Fan [39] in the isotropic case (see Remark 3.3.1). In the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ where $1 < p_+ = \sup_{x\in\Omega} p(x) < N$, other imbedding results can be found for instance in [32, 33, 52] while the case $1 \leq p_- \leq p_+ \leq N$ was investigated in [49].

To the best of our knowledge, the trace result we obtain here is new and does not exist in the literature. The main difficulty we found when we deal with problem (3.1) is the coercivity of the energy functional. We overcome this by using both our continuous imbedding and trace results. Then we prove the boundedness of a minimization sequence and by a compactness argument, we are led to obtain a minimizer which is a weak solution of problem (3.1).

Definition 3.1.1 *Let Ω be an open subset of $\mathbb{R}^N$, $(N \geq 2)$. We say that a Musielak-Orlicz function*

ϕ is locally-integrable, if for every compact subset K of Ω and every constant $c > 0$, we have

$$\int_K \phi(x,c)dx < \infty.$$

The chapter is organized as follows : Section 3.2 contains some definitions. In Section 3.3, we give and prove our main results, which we then apply in Section 3.4 to solve problem (3.1). In the last section we give the appendix which contains some important lemmas that are necessary for the accomplishment of the proofs of the results.

3.2 Preliminaries

3.2.1 Anisotropic Musielak-Orlicz-Sobolev spaces

Let Ω be an open subset of $\mathbb{R}^N$. A real function $\phi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$, will be called a Musielak-Orlicz function, if it satisfies the following conditions:

(i) $\phi(\cdot, t)$ is a measurable function on Ω.

(ii) $\phi(x, \cdot)$ is an N-function, that is a convex, nondecreasing function with $\phi(x,t) = 0$ if only if $t = 0$, $\phi(x,t) > 0$ for all $t > 0$ and for almost every $x \in \Omega$,

$$\lim_{t\to 0^+} \frac{\phi(x,t)}{t} = 0 \text{ and } \lim_{t\to+\infty} \inf_{x\in\Omega} \frac{\phi(x,t)}{t} = +\infty.$$

We will extend these Musielak-Orlicz functions into even functions on all $\Omega \times \mathbb{R}$. The complementary function ϕ^* of the Musilek-Orlicz function ϕ is defined by

$$\phi^*(x,s) = \sup_{t\geq 0}\{st - \phi(x,t)\}. \tag{3.11}$$

It can be checked that ϕ^* is also a Musielak-Orlicz function (see [67]). Moreover, for every t, $s \geq 0$ and a.e. $x \in \Omega$ we have the so-called Young inequality (see [67])

$$ts \leq \phi(x,t) + \phi^*(x,s).$$

For any function $h : \mathbb{R} \to \mathbb{R}$ the second complementary function $h^{**} = (h^*)^*$ (cf. (3.11)), is convex and satisfies

$$h^{**}(x) \leq h(x), \tag{3.12}$$

with equality when h is convex. Roughly speaking, h^{**} is a convex envelope of h, that is the biggest convex function smaller or equal to h.

Let ϕ and ψ be two Musielak-Orlicz functions. We say that ψ grows essentially more slowly than ϕ, denote $\psi \ll \phi$, if

$$\lim_{t\to+\infty} \sup_{x\in\Omega} \frac{\psi(x,t)}{\phi(x,ct)} = 0,$$

for every constant $c > 0$ and for almost every $x \in \Omega$. We remark that if ψ is a locally integrable, then $\psi \ll \phi$ implies that for all $c > 0$ there exists a nonnegative integrable function h, such that

$$\psi(x,t) \leq \phi(x,ct) + h(x), \text{ for all } t \in \mathbb{R} \text{ and for a.e. } x \in \Omega.$$

The Musielak-Orlicz space $L_\phi(\Omega)$ is defined by

$$L_\phi(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable } / \int_\Omega \phi\Big(x, \frac{u(x)}{\lambda}\Big) < +\infty \text{ for some } \lambda > 0\Big\}.$$

Endowed with the so-called Luxemborg norm

$$\|u\|_\phi = \inf\Big\{\lambda > 0 / \int_\Omega \phi\Big(x, \frac{u(x)}{\lambda}\Big)dx \leq 1\Big\},$$

$(L_\phi(\Omega), \|\cdot\|_\phi)$ is a Banach space. Observe that since $\lim_{t\to+\infty} \inf_{x\in\Omega} \frac{\phi(x,t)}{t} = +\infty$ and if Ω has finite measure then we have the following continuous imbedding

$$L_\phi(\Omega) \hookrightarrow L^1(\Omega). \tag{3.13}$$

We will also use the space $E_\phi(\Omega)$ defined by

$$E_\phi(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable } / \int_\Omega \phi\Big(x, \frac{u(x)}{\lambda}\Big) < +\infty \text{ for all } \lambda > 0\Big\}.$$

Observe that for every $u \in L_\phi(\Omega)$ the following inequality holds

$$\|u\|_\phi \leq \int_\Omega \phi(x, u(x))dx + 1. \tag{3.14}$$

For two complementary Musielak-Orlicz functions ϕ and ϕ^*, the following Hölder's inequality (see [67])

$$\int_\Omega |u(x)v(x)|dx \leq 2\|u\|_\phi \|v\|_{\phi^*} \tag{3.15}$$

holds for every $u \in L_\phi(\Omega)$ and $v \in L_{\phi^*}(\Omega)$. Define ϕ^{*-1} for every $s \geq 0$ by

$$\phi^{*-1}(x,s) = \sup\{\tau \geq 0 : \phi^*(x,\tau) \leq s\}.$$

Then, for almost every $x \in \Omega$ and for every $s \in \mathbb{R}$ we have

$$\phi^*(x, \phi^{*-1}(x, s)) \leq s, \tag{3.16}$$

$$s \leq \phi^{*-1}(x, s)\phi^{-1}(x, s) \leq 2s, \tag{3.17}$$

$$\phi(x, s) \leq s\frac{\partial \phi(x, s)}{\partial s} \leq \phi(x, 2s). \tag{3.18}$$

Definition 3.2.1 *Let $\vec{\phi} : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^N$, the vector function $\vec{\phi} = (\phi_1, \cdots, \phi_N)$ where for every $i \in \{1, \cdots, N\}$, ϕ_i is a Musielak-Orlicz function. We define the anisotropic Musielak-Orlicz-Sobolev space by*

$$W^1L_{\vec{\phi}}(\Omega) = \Big\{u \in L_{\phi_{max}}(\Omega);\ \partial_{x_i} u \in L_{\phi_i}(\Omega) \text{ for all } i = 1, \cdots, N\Big\}.$$

By the continuous imbedding (3.13), we get that $W^1L_{\vec{\phi}}(\Omega)$ is a Banach space with respect to the following norm

$$\|u\|_{W^1L_{\vec{\phi}}(\Omega)} = \|u\|_{\phi_{max}} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{\phi_i}.$$

Moreover, we have the continuous embedding $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow W^{1,1}(\Omega)$.

3.3 Main results

In this section we prove an imbedding theorem and a trace result. Let us assume the following conditions

$$\int_0^1 \frac{(\phi^{**}_{min})^{-1}(x, t)}{t^{1+\frac{1}{N}}} dt < +\infty \text{ and } \int_1^{+\infty} \frac{(\phi^{**}_{min})^{-1}(x, t)}{t^{1+\frac{1}{N}}} dt = +\infty, \quad \forall x \in \overline{\Omega}. \tag{3.19}$$

Thus, we define the Sobolev conjugate $(\phi^{**}_{min})_*$ of ϕ^{**}_{min} by

$$(\phi^{**}_{min})^{-1}_*(x, s) = \int_0^s \frac{(\phi^{**}_{min})^{-1}(x, t)}{t^{1+\frac{1}{N}}} dt, \quad \text{for } x \in \overline{\Omega} \text{ and } s \in [0, +\infty). \tag{3.20}$$

It may readily be checked that $(\phi^{**}_{min})_*$ is a Musielak-Orlicz function. We assume that there exist two positive constants $\nu < \frac{1}{N}$ and c_0, such that

$$\Big|\frac{\partial(\phi^{**}_{min})_*(x, t)}{\partial x_i}\Big| \leq c_0\Big[(\phi^{**}_{min})_*(x, t) + ((\phi^{**}_{min})_*(x, t))^{1+\nu}\Big], \tag{3.21}$$

for all $t \in \mathbb{R}$ and for almost every $x \in \Omega$, provided that for every $i = 1, \cdots, N$ the derivative $\frac{\partial(\phi^{**}_{min})_*(x,t)}{\partial x_i}$ exists.

3.3.1 An imbedding theorem

Theorem 3.3.1 *Let Ω be an open bounded subset of $\mathbb{R}^N$, $(N \geq 2)$, with the cone property. Assume that (3.19) and (3.21) are fulfilled, $(\phi^{**}_{min})_*(\cdot,t)$ is Lipschitz continuous on $\overline{\Omega}$ and ϕ_{max} is locally integrable. Then, there is a continuous embedding*

$$W^1L_{\vec{\phi}}(\Omega) \hookrightarrow L_{(\phi^{**}_{min})_*}(\Omega).$$

Some remarks about Theorem 3.3.1 are in order. We discuss how Theorem 3.3.1 include some previous results known in the literature when reducing to particular Musielak-Orlicz functions.

Remark 3.3.1 *1. Let $M(x,t) = t^{p(x)}$ and $m(x,t) = \frac{\partial M(x,t)}{\partial t} = p(x)t^{p(x)-1}$, where $p(\cdot)$ is Lipschitz continuous on $\overline{\Omega}$, with $1 < p_- = \inf_{x\in\Omega} p(x) \leq p(x) \leq p_+ = \sup_{x\in\Omega} p(x) < N$. Since $M(\cdot,t)$ and $m(\cdot,t)$ are continuous on $\overline{\Omega}$, then we can use Lemma 3.5.8 (given in Appendix) to define the following Musielak-Orlicz function*

$$\phi(x,t) = \begin{cases} \frac{t_1^{p(x)}}{t_1^{\alpha}} t^{\alpha} & \text{if } t \leq t_1, \\ t^{p(x)} & \text{if } t \geq t_1, \end{cases}$$

where $t_1 > 1$ and $\alpha > 1$ are two constants mentioned in the proof of Lemma 3.5.8. Let us now consider the particular case where, for all $i = 1, \cdots, N$,

$$\phi_i(x,t) = \phi(x,t) = \begin{cases} \frac{t_1^{p(x)}}{t_1^{\alpha}} t^{\alpha} & \text{if } t \leq t_1, \\ t^{p(x)} & \text{if } t \geq t_1. \end{cases} \tag{3.22}$$

It is worth pointing out that since Ω is of finite Lebesgue measure, it can be seen easily that $W^1L_{\vec{\phi}}(\Omega) = W^1L_{\phi}(\Omega) = W^{1,p(\cdot)}(\Omega)$. Thus, $\phi^{**}_{min}(x,t) = \phi_{min}(x,t) = \phi(x,t)$ and

$$(\phi^{**}_{min})_*(x,t) = (\phi_{min})_*(x,t) = \phi_*(x,t) = \begin{cases} \left(\frac{(N-\alpha)t}{N\alpha t_1}\right)^{\frac{N\alpha}{N-\alpha}} t_1^{\frac{Np(x)}{N-\alpha}} & \text{if } t \leq t_1, \\ \left(\frac{1}{p_*(x)} t\right)^{p_*(x)} & \text{if } t \geq t_1, \end{cases}$$

provided that $\alpha < N$. Now we shall prove that $(\phi^{**}_{min})_*$ satisfies (3.21) and our imbedding result include some previous result known in the literature. For every $t \in \mathbb{R}$ and for almost every $x \in \Omega$ we have

$$\frac{\partial(\phi^{**}_{min})_*(x,t)}{\partial x_i} = \begin{cases} \frac{N}{N-\alpha}\frac{\partial p(x)}{\partial x_i} \log(t_1)(\phi^{**}_{min})_*(x,t) & \text{if } t \leq t_1, \\ \frac{\partial p_*(x)}{\partial x_i} \log\left(\frac{t}{ep_*(x)}\right)(\phi^{**}_{min})_*(x,t) & \text{if } t \geq t_1. \end{cases}$$

- *If* $t \leq t_1$ *then*

$$\Big|\frac{\partial(\phi^{**}_{min})_*(x,t)}{\partial x_i}\Big| = \frac{N}{N-\alpha}\Big|\frac{\partial p(x)}{\partial x_i}\Big|\log(t_1)(\phi^{**}_{min})_*(x,t).$$

Since $p(\cdot)$ *is Lipschitz continuous on* $\overline{\Omega}$ *there exists a constant* $C_1 > 0$ *satisfying* $\Big|\frac{\partial p(x)}{\partial x_i}\Big| \leq C_1$ *thus we get*

$$\Big|\frac{\partial(\phi^{**}_{min})_*(x,t)}{\partial x_i}\Big| \leq C_1\frac{N}{N-\alpha}\log(t_1)(\phi^{**}_{min})_*(x,t). \tag{3.23}$$

- *If* $t \geq t_1$

$$\Big|\frac{\partial(\phi^{**}_{min})_*(x,t)}{\partial x_i}\Big| = \Big|\frac{\partial p_*(x,t)}{\partial x_i}\Big|\Big|\log\Big(\frac{t}{ep_*(x)}\Big)\Big|(\phi^{**}_{min})_*(x,t).$$

Since $p(\cdot)$ *is Lipschitz continuous on* $\overline{\Omega}$, *then it can be seen easily that* $p_*(\cdot)$ *is also Lipschitz continuous on* $\overline{\Omega}$. *Then, there exists a constant* $C_2 > 0$ *satisfying* $\Big|\frac{\partial p_*(x)}{\partial x_i}\Big| \leq C_2$, *thus we get*

$$\Big|\frac{\partial(\phi^{**}_{min})_*(x,t)}{\partial x_i}\Big| \leq C_2\Big|\log\Big(\frac{t}{ep_*(x)}\Big)\Big|(\phi^{**}_{min})_*(x,t).$$

Let $\epsilon > 0$. *If* $\epsilon < \frac{1}{N}$ *then for all* $t > 0$ *we can easily check that*

$$\log(t) \leq \frac{1}{\epsilon^2 Ne}t^{\epsilon}. \tag{3.24}$$

Now, since the Musielak-Orlicz function $(\phi^{**}_{min})_*$ *has a superlinear growth, we can choose* $A > 0$ *for which there exists* $t_0 > \max\{t_1, e\}$ *(not depending on* x*) such that* $At \leq (\phi^{**}_{min})_*(x,t)$ *whenever* $t \geq t_0$. *Therefore,*

- *If* $t \geq t_0$ *then by (3.24) we obtain*

$$\begin{aligned}\Big|\tfrac{\partial(\phi^{**}_{min})_*(x,t)}{\partial x_i}\Big| &\leq C_2\Big(\log\Big(\tfrac{t}{e}\Big) + \log(\tfrac{N^2}{N-p_+})\Big)(\phi^{**}_{min})_*(x,t)\\ &\leq \tfrac{C_2}{\epsilon^2 Ne^{1+\epsilon}}t^{\epsilon}(\phi^{**}_{min})_*(x,t) + C_2\log(\tfrac{N^2}{N-p_+})(\phi^{**}_{min})_*(x,t)\\ &\leq \tfrac{C_2}{\epsilon^2 Ne^{1+\epsilon}A^{\epsilon}}((\phi^{**}_{min})_*(x,t))^{1+\epsilon} + C_2\log(\tfrac{N^2}{N-p_+})(\phi^{**}_{min})_*(x,t).\end{aligned} \tag{3.25}$$

- *If* $t_1 < t \leq t_0$, *then*

$$\Big|\tfrac{\partial(\phi^{**}_{min})_*(x,t)}{\partial x_i}\Big| \leq C_2\Big(\log(t_0) + \log\Big(\tfrac{eN^2}{N-p_+}\Big)\Big)(\phi^{**}_{min})_*(x,t). \tag{3.26}$$

Therefore, from (3.23), (3.25) and (3.26), we get that for every $t \geq 0$ *and for almost every* $x \in \Omega$, *there is a constant* c_0 *such that*

$$\Big|\frac{\partial(\phi^{**}_{min})_*(x,t)}{\partial x_i}\Big| \leq c_0\Big((\phi^{**}_{min})_*(x,t) + ((\phi^{**}_{min})_*(x,t))^{1+\epsilon}\Big).$$

Before we show that our imbedding result includes some previous known results in the literature, we remark that the proof of Theorem 3.3.1 relies to the application of Lemma 3.5.4 *in Appendix for the function* $g(x,t) = ((\phi^{**}_{min})_*(x,t))^{\alpha}$, $\alpha \in (0,1)$, *where we have used the fact that* Ω *is bounded to ensure that* $\max_{x\in\overline{\Omega}} g(x,t) < \infty$ *for some* $t > 0$. *In the case of the variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ *built upon the the Musielak-Orlicz function given in (3.22), we do not need* Ω *to be bounded, since*

$$\phi_*(x,t) \le \max\{t_1^{\frac{N\alpha}{N-\alpha}}, t^{\frac{N^2}{N-p_+}}\} < \infty, \text{ for some } t > 0.$$

Therefore, the embedding result in Theorem 3.3.1 can be seen as an extension to the Musielak-Orlicz framework of the one obtained in [39, Theorem 1.1].

2. *Let us consider the particular case where, for* $i \in \{1,\cdots,N\}$,

$$\phi_i(x,t) = \begin{cases} \frac{t_1^{p_i(x)}}{t_1^{\alpha}} t^{\alpha} & \text{if } t \le t_1, \\ t^{p_i(x)} & \text{if } t \ge t_1 \end{cases}$$

where $t_1 > 1$, $1 < \alpha < N$ *and* $\vec{\phi} = (\phi_i)_{i\in\{1,\cdots,N\}}$ *with* $p_i \in C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \inf_{x\in\overline{\Omega}} h(x) > 1\}$, $1 < p_i(x) < N$, $N \ge 3$. *Define* $p_i^- = \inf_{x\in\Omega} p_i(x)$, $p_M(x) = \max_{i\in\{1,\cdots,N\}} p_i(x)$, $p_m(x) = \min_{i\in\{1,\cdots,N\}} p_i(x)$ *and* $\phi_{min}(x,t) = \min_{i\in\{1,\cdots,N\}} \phi_i(x,t)$. *Then,*

$$\phi^{**}_{min}(x,t) = \phi_{min}(x,t) = \begin{cases} \frac{t_1^{p_m(x)}}{t_1^{\alpha}} t^{\alpha} & \text{if } t \le t_1, \\ t^{p_m(x)} & \text{if } t \ge t_1, \end{cases}$$

whose Sobolev conjugate function is given by

$$(\phi^{**}_{min})_*(x,t) = \begin{cases} \left(\frac{(N-\alpha)t}{N\alpha t_1}\right)^{\frac{N\alpha}{N-\alpha}} t_1^{\frac{Np_m(x)}{N-\alpha}} & \text{if } t \le t_1, \\ \left(\frac{1}{(p_m)_*(x)} t\right)^{(p_m)_*(x)} & \text{if } t \ge t_1. \end{cases}$$

Let us define p^*_- *by the formulae* $p^*_- = \frac{N}{\sum_{i=1}^N \frac{1}{p_i^-} - 1}$. *Since* $p_i^- > p_m^-$, *then we obtain that*

$$p^*_- > \frac{Np_m^-}{N - p_m^-} = (p_m^-)_*. \tag{3.27}$$

Since Ω *is of finite Lebesgue measure, it can be seen easily that* $W^1L_{\vec{\phi}}(\Omega) = W^{1,\vec{p}(\cdot)}(\Omega)$. *So, by Theorem 3.3.1 we have* $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{(p_m)_*(\cdot)}(\Omega)$ *and since* $(p_m)_*(x) \ge (p_m^-)_*$ *for each* $x \in \overline{\Omega}$, *we deduce that* $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{(p_m^-)_*}(\Omega)$. *Therefore, by (3.27) the result we obtain can be found in [63, Theorem 1].*

3. *Let us now consider the case where*

$$\phi_i(x,t)=\begin{cases}\frac{t_1^{p_i(x)}\log(t_1+1)}{t_1^{\alpha}}t^{\alpha} & \text{if } t\le t_1,\\ t^{p_i(x)}\log(t+1) & \text{if } t\ge t_1,\end{cases}$$

where $t_1>1$, $1<\alpha<N$ *and for each* $i\in\{1,\cdots,N\}$ *the function* $p_i(\cdot)$ *is Lipschitz continuous on* $\overline{\Omega}$ *with* $1<\inf_{x\in\overline{\Omega}}p_i(x)\le p_i(x)\le\sup_{x\in\overline{\Omega}}p_i(x)<N-1$. *Define* $p_M(x)=\max_{i\in\{1,\cdots,N\}}p_i(x)$, $p_m(x)=\min_{i\in\{1,\cdots,N\}}p_i(x)$ *and* $\phi_{min}(x,t)=\min_{i\in\{1,\cdots,N\}}\phi_i(x,t)$. *Then,*

$$\phi_{min}(x,t)=\phi_{min}^{**}(x,t)=\begin{cases}\frac{t_1^{p_m(x)}\log(t_1+1)}{t_1^{\alpha}}t^{\alpha} & \text{if } t\le t_1,\\ t^{p_m(x)}\log(t+1) & \text{if } t\ge t_1.\end{cases}$$

Let $A(x,t)=t^{p_m(x)}\log(t+1)$, *by [60, Example 2] there exist* $\sigma<\frac{1}{N}$, $C_0>0$ *and* $t_0>0$ *such that*

$$\Big|\frac{A_*(x,t)}{\partial x_i}\Big|\le C_0(A_*(x,t))^{1+\sigma},$$

for $x\in\Omega$ *and* $t\ge t_0$. *Choosing this* $t_0>0$ *in Lemma 3.5.8 given in Appendix, we can take* $t_1>t_0+1$, *then we obtain*

$$\Big|\frac{A_*(x,t)}{\partial x_i}\Big|\le C_0(A_*(x,t))^{1+\sigma},\ \text{for all } t\ge t_1. \tag{3.28}$$

On the other hand, for $t\le t_1$ *we have*

$$(\phi_{min}^{**})_*(x,t)=\Big(\frac{(N-\alpha)t}{N\alpha t_1}\Big)^{\frac{N\alpha}{N-\alpha}}\Big(\frac{t_1^{p_m(x)}}{\log(t_1+1)}\Big)^{\frac{N}{N-\alpha}}.$$

Thus

$$\Big|\frac{\partial(\phi_{min}^{**})_*(x,t)}{\partial x_i}\Big|=\frac{N\log(t_1)}{N-\alpha}\Big|\frac{\partial p_m(x)}{\partial x_i}\Big|(\phi_{min}^{**})_*(x,t).$$

Since $p_m(\cdot)$ *is Lipschitz continuous on* $\overline{\Omega}$ *there exists a constant* $C_3>0$ *satisfying* $\Big|\frac{\partial p_m(x)}{\partial x_i}\Big|\le C_3$. *So we have*

$$\Big|\frac{\partial(\phi_{min}^{**})_*(x,t)}{\partial x_i}\Big|\le\frac{C_3N\log(t_1)}{N-\alpha}(\phi_{min}^{**})_*(x,t). \tag{3.29}$$

Therefore, by (3.28) and (3.29) the function $(\phi_{min}^{**})_*$ *satisfies the assertions of Theorem 3.3.1 and then we get the continuous embedding*

$$W^1L_{\vec{\phi}}(\Omega)\hookrightarrow L_{(\phi_{min}^{**})_*}(\Omega).$$

Proof 3.3.1 ***of Theorem 3.3.1.*** *Let $u \in W^1L_{\vec{\phi}}(\Omega)$. Assume first that the function u is bounded and $u \neq 0$. Defining $f(s) = \displaystyle\int_\Omega (\phi^{**}_{min})_*\Big(x, \frac{|u(x)|}{s}\Big)dx$, for $s > 0$, one has $\lim\limits_{s\to 0^+} f(s) = +\infty$ and $\lim\limits_{s\to\infty} f(s) = 0$. Since f is continuous on $(0, +\infty)$, there exists $\lambda > 0$ such that $f(\lambda) = 1$. Then by the definition of the Luxemburg norm, we get*

$$\|u\|_{(\phi^{**}_{min})_*} \leq \lambda. \tag{3.30}$$

On the other hand,

$$f(\|u\|_{(\phi^{**}_{min})_*}) = \int_\Omega (\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\|u\|_{(\phi^{**}_{min})_*}}\Big)dx \leq 1 = f(\lambda)$$

and since f is decreasing we get

$$\lambda \leq \|u\|_{(\phi^{**}_{min})_*}. \tag{3.31}$$

So that by (3.30) and (3.31), we get $\lambda = \|u\|_{(\phi^{**}_{min})_*}$ and

$$\int_\Omega (\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)dx = 1. \tag{3.32}$$

From (3.20) we can easily check that $(\phi^{**}_{min})_*$ satisfies the following differential equation

$$(\phi^{**}_{min})^{-1}(x, (\phi^{**}_{min})_*(x,t))\frac{\partial(\phi^{**}_{min})_*}{\partial t}(x,t) = ((\phi^{**}_{min})_*(x,t))^{\frac{N+1}{N}}.$$

Hence, by (3.17) we obtain the following inequality

$$\frac{\partial(\phi^{**}_{min})_*}{\partial t}(x,t) \leq ((\phi^{**}_{min})_*(x,t))^{\frac{1}{N}}(\phi^{**}_{min})^{*-1}(x, (\phi^{**}_{min})_*(x,t)), \text{ for a.e. } x \in \Omega. \tag{3.33}$$

Let h be the function defined by

$$h(x) = \Big[(\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big]^{\frac{N-1}{N}}. \tag{3.34}$$

Since $(\phi^{**}_{min})_*(\cdot, t)$ is Lipschitz continuous on $\overline{\Omega}$ and $(\phi^{**}_{min})_*(x, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}^+$, the function h is Lipschitz continuous on $\overline{\Omega}$. Hence, we can compute using Lemma 3.5.6 (given in Appendix) for $f = h$, obtaining for a.e. $x \in \Omega$,

$$\frac{\partial h(x)}{\partial x_i} = \frac{N-1}{N}\Big((\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)^{-\frac{1}{N}}\Big[\frac{\partial(\phi^{**}_{min})_*}{\partial t}\Big(x, \frac{u(x)}{\lambda}\Big)\frac{\partial_{x_i}u(x)}{\lambda} + \frac{\partial(\phi^{**}_{min})_*}{\partial_{x_i}}\Big(x, \frac{u(x)}{\lambda}\Big)\Big].$$

Therefore,

$$\sum_{i=1}^N \Big|\frac{\partial h(x)}{\partial x_i}\Big| \leq I_1 + I_2, \text{ for a.e. } x \in \Omega, \tag{3.35}$$

where we have set

$$I_1 = \frac{N-1}{N\lambda}\Big((\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)^{\frac{-1}{N}} \frac{\partial(\phi^{**}_{min})_*}{\partial t}\Big(x, \frac{u(x)}{\lambda}\Big) \sum_{i=1}^{N} |\partial_{x_i} u(x)|$$

and

$$I_2 = \frac{N-1}{N}\Big((\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)^{\frac{-1}{N}} \sum_{i=1}^{N} \Big|\frac{\partial(\phi^{**}_{min})_*}{\partial x_i}\Big(x, \frac{u(x)}{\lambda}\Big)\Big|.$$

We shall now estimate the two integrals $\int_\Omega I_1(x)dx$ *and* $\int_\Omega I_2(x)dx$*. By* (3.33)*, we can write*

$$I_1(x) \leq \frac{N-1}{N\lambda}(\phi^{**}_{min})^{*-1}\Big(x, (\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big) \sum_{i=1}^{N} |\partial_{x_i} u(x)|. \tag{3.36}$$

By (3.16)*, we have*

$$\int_\Omega (\phi^{**}_{min})^*\Big(x, (\phi^{**}_{min})^{*-1}\Big(x, (\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)\Big)dx \leq \int_\Omega (\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)dx \leq 1.$$

Hence

$$\Big\|(\phi^{**}_{min})^{*-1}\Big(x, (\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)\Big\|_{(\phi^{**}_{min})^*} \leq 1. \tag{3.37}$$

It follows from (3.15)*,* (3.36) *and* (3.37) *that*

$$\begin{aligned} \int_\Omega I_1(x)dx &\leq \frac{2(N-1)}{N\lambda}\Big\|(\phi^{**}_{min})^{*-1}\Big(x, (\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)\Big\|_{(\phi^{**}_{min})^*} \sum_{i=1}^{N} \Big\|\partial_{x_i} u(x)\Big\|_{\phi^{**}_{min}} \\ &\leq \frac{2(N-1)}{N\lambda} \sum_{i=1}^{N} \Big\|\partial_{x_i} u(x)\Big\|_{\phi^{**}_{min}} \\ &\leq \frac{2}{\lambda} \sum_{i=1}^{N} \Big\|\partial_{x_i} u(x)\Big\|_{\phi^{**}_{min}}. \end{aligned} \tag{3.38}$$

Recall that by the definition of ϕ_{min} *and* (3.12)*, we get* $\|\partial_{x_i} u(x)\|_{\phi^{**}_{min}} \leq \|\partial_{x_i} u(x)\|_{\phi_i}$*, so that* (3.38) *implies*

$$\int_\Omega I_1(x)dx \leq \frac{2}{\lambda} \sum_{i=1}^{N} \Big\|\partial_{x_i} u(x)\Big\|_{\phi_i}. \tag{3.39}$$

By (3.21)*, we can write*

$$I_2(x) \leq c_1\Big[\Big((\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)^{1-\frac{1}{N}} + \Big((\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big)^{1-\frac{1}{N}+\nu}\Big],$$

with $c_1 = c_0(N-1)$*. Since* $(\phi^{**}_{min})_*(\cdot, t)$ *is continuous on* $\overline{\Omega}$ *and* $\nu < \frac{1}{N}$*, we can apply Lemma* 3.5.4 *(given in Appendix) with the functions* $g(x,t) = \frac{((\phi^{**}_{min})_*(x,t))^{1-\frac{1}{N}+\nu}}{t}$ *and* $f(x,t) = \frac{(\phi^{**}_{min})_*(x,t)}{t}$ *with* $\epsilon = \frac{1}{8c_1c_*}$*, obtaining for* $t = \frac{|u(x)|}{\lambda}$

$$\Big[(\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)\Big]^{1-\frac{1}{N}+\nu} \leq \frac{1}{8c_1c_*}(\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big) + K_0\frac{|u(x)|}{\lambda}. \tag{3.40}$$

Using again Lemma 3.5.4 (given in Appendix) with the functions $g(x,t) = \frac{((\phi^{**}_{min})_*(x,t))^{1-\frac{1}{N}}}{t}$ *and* $f(x,t) = \frac{(\phi^{**}_{min})_*(x,t)}{t}$ *with* $\epsilon = \frac{1}{8c_1c_*}$, *we get by substituting* t *by* $\frac{|u(x)|}{\lambda}$

$$\left[(\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)\right]^{1-\frac{1}{N}} \leq \frac{1}{8c_1c_*}(\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big) + K_0\frac{|u(x)|}{\lambda}, \tag{3.41}$$

where c_* *is the constant in the embedding* $W^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$, *that is*

$$\|w\|_{L^{\frac{N}{N-1}}(\Omega)} \leq c_*\|w\|_{W^{1.1}(\Omega)}, \text{ for all } w \in W^{1,1}(\Omega). \tag{3.42}$$

By (3.40) and (3.41), we obtain

$$\int_\Omega I_2(x)dx \leq \frac{1}{4c_*} + \frac{2K_0c_1}{\lambda}\|u\|_{L^1(\Omega)}. \tag{3.43}$$

Puting together (3.39) and (3.43) in (3.35), we obtain

$$\begin{aligned}\sum_{i=1}^N \|\partial_{x_i}h\|_{L^1(\Omega)} &\leq \tfrac{1}{4c_*} + \tfrac{2}{\lambda}\sum_{i=1}^N \|\partial_{x_i}u(x)\|_{\phi_i} + \frac{2K_0c_1}{\lambda}\|u\|_{L^1(\Omega)} \\ &\leq \tfrac{1}{4c_*} + \tfrac{2}{\lambda}\sum_{i=1}^N \|\partial_{x_i}u(x)\|_{\phi_i} + \frac{2K_0c_1c_2}{\lambda}\|u\|_{\phi_{max}},\end{aligned}$$

where c_2 *is the constant in the continuous embedding (3.13). Then it follows*

$$\sum_{i=1}^N \|\partial_{x_i}h\|_{L^1(\Omega)} \leq \frac{1}{4c_*} + \frac{c_3}{\lambda}\|u\|_{W^1L_{\vec{\phi}}(\Omega)}, \tag{3.44}$$

with $c_3 = \max\{2, 2K_0c_1c_2\}$. *Now, using again Lemma 3.5.4 (given in Appendix) for the functions* $g(x,t) = \frac{\left[(\phi^{**}_{min})_*(x,t)\right]^{1-\frac{1}{N}}}{t}$ *and* $f(x,t) = \frac{(\phi^{**}_{min})_*(x,t)}{t}$ *with* $\epsilon = \frac{1}{4c_*}$, *we obtain for* $t = \frac{|u(x)|}{\lambda}$

$$h(x) \leq \frac{1}{4c_*}(\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big) + K_0\frac{|u(x)|}{\lambda},$$

From (3.13), we obtain

$$\|h\|_{L^1(\Omega)} \leq \frac{1}{4c_*} + \frac{K_0c_2}{\lambda}\|u\|_{L_{\phi_{max}}(\Omega)}. \tag{3.45}$$

Thus, by (3.44) and (3.45) we get

$$\|h\|_{W^{1,1}(\Omega)} \leq \frac{1}{2c_*} + \frac{c_4}{\lambda}\|u\|_{W^1L_{\vec{\phi}}(\Omega)},$$

where $c_4 = c_3 + K_0c_2$, *which shows that* $h \in W^{1,1}(\Omega)$ *and which together with (3.42) yield*

$$\|h\|_{L^{\frac{N}{N-1}}(\Omega)} \leq \frac{1}{2} + \frac{c_4c_*}{\lambda}\|u\|_{W^1L_{\vec{\phi}}(\Omega)}.$$

Having in mind (3.32), we get $\int_\Omega [h(x)]^{\frac{N}{N-1}}dx = \int_\Omega (\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\lambda}\Big)dx = 1$*. So that one has*

$$\|u\|_{(\phi^{**}_{min})_*} = \lambda \leq 2c_4c_*\|u\|_{W^1L_{\vec{\Phi}}(\Omega)}. \tag{3.46}$$

We now extend the estimate (3.46) to an arbitrary $u \in W^1L_{\vec{\phi}}(\Omega)$*. Let* T_n*,* $n > 0$*, be the truncation function at levels* $\pm n$ *defined on* $\mathbb{R}$ *by* $T_n(s) = \min\{n, \max\{s, -n\}\}$*. Since* ϕ_{max} *is locally integrable, by [7, Lemma 8.34.] one has* $T_n(u) \in W^1L_{\vec{\phi}}(\Omega)$*. So that in view of (3.46)*

$$\|T_n(u)\|_{(\phi^{**}_{min})_*} \leq 2c_4c_*\|T_n(u)\|_{W^1L_{\vec{\phi}(\Omega)}} \leq 2c_4c_*\|u\|_{W^1L_{\vec{\phi}(\Omega)}}. \tag{3.47}$$

Let $k_n = \|T_n(u)\|_{(\phi^{**}_{min})_*}$*. Thanks to (3.47), the sequence* $\{k_n\}_{n=1}^{\infty}$ *is nondecreasing and converges. If we denote* $k = \lim_{n\to\infty} k_n$*, by Fatou's lemma we have*

$$\int_\Omega (\phi^{**}_{min})_*\Big(x, \frac{|u(x)|}{k}\Big)dx \leq \liminf \int_\Omega (\phi^{**}_{min})_*\Big(x, \frac{|T_n(u)|}{k_n}\Big)dx \leq 1.$$

This implies that $u \in L_{(\phi^{**}_{min})_*}(\Omega)$ *and*

$$\|u\|_{(\phi^{**}_{min})_*} \leq k = \lim_{n\to\infty} \|T_n(u)\|_{(\phi^{**}_{min})_*} \leq 2c_4c_*\|u\|_{W^1L_{\vec{\phi}(\Omega)}}.$$

The theorem is then completely proved.

Corollary 3.3.1 *Let* Ω *be an open bounded subset of* $\mathbb{R}^N$*,* $(N \geq 2)$*, with the cone property. Assume that (3.19), (3.21) are fulfilled,* $(\phi^{**}_{min})_*(\cdot, t)$ *is Lipschitz continuous on* $\overline{\Omega}$ *and* ϕ_{max} *is locally integrable. Let* A *be a Musielak-Orlicz function where the function* $A(\cdot, t)$ *is continuous on* $\overline{\Omega}$ *and* $A \ll (\phi^{**}_{min})_*$*. Then, the following embedding*

$$W^1L_{\vec{\phi}}(\Omega) \hookrightarrow L_A(\Omega).$$

is compact.

Proof 3.3.2 *Let* $\{u_n\}$ *is a bounded sequence in* $W^1L_{\vec{\phi}}(\Omega)$*. By Theorem 3.3.1,* $\{u_n\}$ *is bounded in* $L_{(\phi^{**}_{min})_*}(\Omega)$*. Since the embedding* $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ *is continuous and the imbedding* $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ *is compact, we deduce that there exists a subsequence of* $\{u_n\}$ *still denoted by* $\{u_n\}$ *which converges in measure in* Ω*. Since* $A \ll (\phi^{**}_{min})_*$*, by Lemma 3.5.5 (given in Appendix) the sequence* $\{u_n\}$ *converges in norm in* $L_A(\Omega)$*.*

3.3.2 A trace result

We prove here a trace result which is a useful tool to prove the coercivity of some energy functionals. Recall that $\psi_{min}(x,t) = [(\phi_{min}^{**})_*(x,t)]^{\frac{N-1}{N}}$ is a Musielak-Orlicz function. Indeed, we have

$$\frac{\partial}{\partial t}(\psi_{min})^{-1}(x,t) = \frac{\partial}{\partial t}(\phi_{min}^{**})_*^{-1}\Big(x, t^{\frac{N}{N-1}}\Big).$$

By (3.20), we get

$$\frac{\partial}{\partial t}(\psi_{min})^{-1}(x,t) = \frac{N}{N-1} t^{\frac{1}{N-1}} \frac{(\phi_{min}^{**})^{-1}\Big(x, t^{\frac{N}{N-1}}\Big)}{t^{\frac{N}{N-1}+\frac{1}{N-1}}} = \frac{N}{N-1}\frac{(\phi_{min}^{**})^{-1}\Big(x, t^{\frac{N}{N-1}}\Big)}{t^{\frac{N}{N-1}}}.$$

Being the inverse of a Musielak-Orlicz function, it is clear that $(\phi_{min}^{**})^{-1}$ satisfies

$$\lim_{\tau\to+\infty}\frac{(\phi_{min}^{**})^{-1}(x,\tau)}{\tau} = 0 \text{ and } \lim_{\tau\to 0^+}\frac{(\phi_{min}^{**})^{-1}(x,\tau)}{\tau} = +\infty.$$

Moreover, $(\phi_{min}^{**})^{-1}(x,\cdot)$ is concave so that if $0<\tau<\sigma$, then

$$\frac{(\phi_{min}^{**})^{-1}(x,\tau)}{(\phi_{min}^{**})^{-1}(x,\sigma)} \geq \frac{\tau}{\sigma}.$$

Hence, if $0 < s_1 < s_2$,

$$\frac{\frac{\partial}{\partial t}(\psi_{min})^{-1}(x,s_1)}{\frac{\partial}{\partial t}(\psi_{min})^{-1}(x,s_2)} = \frac{(\phi_{min}^{**})^{-1}\Big(x, s_1^{\frac{N}{N-1}}\Big)}{(\phi_{min}^{**})^{-1}\Big(x, s_2^{\frac{N}{N-1}}\Big)}\frac{s_2^{\frac{N}{N-1}}}{s_1^{\frac{N}{N-1}}} \geq \frac{s_1^{\frac{N}{N-1}}}{s_2^{\frac{N}{N-1}}}\frac{s_2^{\frac{N}{N-1}}}{s_1^{\frac{N}{N-1}}} = 1.$$

It follows that $\frac{\partial}{\partial t}(\psi_{min})^{-1}(x,t)$ is positive and decreases monotonically from $+\infty$ to 0 as t increases from 0 to $+\infty$ and thus ψ_{min} is a Musielak-Orlicz function.

Theorem 3.3.2 *Let Ω be an open bounded subset of $\mathbb{R}^N$, $(N \geq 2)$, with the cone property. Assume that (3.19), (3.21) are fulfilled, $(\phi_{min}^{**})_*(\cdot,t)$ is Lipschitz continuous on $\overline{\Omega}$ and ϕ_{max} is locally integrable. Let ψ_{min} the Musielak-Orlicz function defined in (3.8). Then, the following boundary trace embedding $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow L_{\psi_{min}}(\partial\Omega)$ is continuous.*

Remark 3.3.2 *In the case where for all $i = 1,\cdots,N$,*

$$\phi_i(x,t) = \phi(x,t) = \begin{cases} \frac{t_1^{p(x)}}{t_1^{\alpha}} t^{\alpha} & \text{if } t \leq t_1, \\ t^{p(x)} & \text{if } t \geq t_1. \end{cases}$$

with $p \in L^\infty(\Omega)$, $1 \le \inf_{x\in\Omega} p(x) \le \sup_{x\in\Omega} p(x) < N$, $|\nabla p| \in L^{\gamma(\cdot)}(\Omega)$, *where* $\gamma \in L^\infty(\Omega)$ *and* $\inf_{x\in\Omega}\gamma(x) > N$. *It is worth pointing out that since* Ω *is of finite Lebesgue measure, it can be seen easily that* $W^1L_{\vec{\phi}}(\Omega) = W^1L_\phi(\Omega) = W^{1,p(\cdot)}(\Omega)$. *Then* $\phi^{**}_{min}(x,s) = \phi_{min}(x,t) = \phi(x,t)$ *and so*

$$(\phi^{**}_{min})_*(x,t) = (\phi_{min})_*(x,t) = \phi_*(x,t) = \begin{cases} \left(\frac{(N-\alpha)t}{N\alpha t_1}\right)^{\frac{N\alpha}{N-\alpha}} t_1^{\frac{Np(x)}{N-\alpha}} & \text{if } t \le t_1, \\ \left(\frac{1}{p_*(x)}t\right)^{p_*(x)} & \text{if } t \ge t_1. \end{cases}$$

As above we can prove that $(\phi^{**}_{min})_*$ *satisfies the conditions of Theorem 3.3.2 and then our trace result is an extension to Musielak-Orlicz framework of the one proved by Fan in [36].*

Proof 3.3.3 ***of Theorem 3.3.2.*** *Let* $u \in W^1L_{\vec{\phi}}(\Omega)$. *As the embedding* $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow L_{(\phi^{**}_{min})_*}(\Omega)$ *is continuous, the function* u *belongs to* $L_{(\phi^{**}_{min})_*}(\Omega)$ *and then* u *belongs to* $L_{\psi_{min}}(\Omega)$. *Clearly* $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ *and by the Gagliardo trace theorem (see [41]) we have the embedding* $W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$. *Hence, we conclude that for all* $u \in W^1L_{\vec{\phi}}(\Omega)$ *there holds* $u|_{\partial\Omega} \in L^1(\partial\Omega)$. *Therefore, for every* $u \in W^1L_{\vec{\phi}}(\Omega)$ *the trace* $u|_{\partial\Omega}$ *is well defined. Assume first that* u *is bounded and* $u \ne 0$. *Since* $(\phi^{**}_{min})_*(\cdot,t)$ *is continuous on* $\partial\Omega$, *the function* u *belongs to* $L_{\psi_{min}}(\partial\Omega)$. *Let*

$$k = \|u\|_{L_{\psi_{min}}(\partial\Omega)} = \inf\Big\{\lambda > 0;\ \int_{\partial\Omega} \psi_{min}\Big(x, \frac{u(x)}{\lambda}\Big)dx \le 1\Big\}.$$

We have to distinguish the two cases : $k \ge \|u\|_{L_{(\phi^{**}_{min})_*}(\Omega)}$ *and* $k < \|u\|_{L_{(\phi^{**}_{min})_*}(\Omega)}$. *Suppose first that*

Case 1 : Assume that

$$k \ge \|u\|_{L_{(\phi^{**}_{min})_*}(\Omega)}. \tag{3.48}$$

Going back to (3.34), we can repeat exactly the same lines with $l(x) = \psi_{min}\Big(x, \frac{u(x)}{k}\Big)$ *instead of the function* h, *obtaining*

$$\|l\|_{W^{1,1}(\Omega)} \le \Big[\frac{1}{4c} + \frac{c_3}{k}\|u\|_{W^1L_{\vec{\phi}}(\Omega)} + \|l\|_{L^1(\Omega)}\Big], \tag{3.49}$$

where c *is the constant in the imbedding* $W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$, *that is*

$$\|w\|_{L^1(\partial\Omega)} \le c\|w\|_{W^{1,1}(\Omega)}, \text{ for all } w \in W^{1,1}(\Omega). \tag{3.50}$$

Since $(\phi^{**}_{min})_*(\cdot,t)$ *is continuous on* $\overline{\Omega}$, *using Lemma 3.5.4 (given in Appendix) with the functions* $f(x,t) = \frac{(\phi^{**}_{min})_*(x,t)}{t}$, $g(x,t) = \frac{l(x)}{t}$ *and* $\epsilon = \frac{1}{4c}$, *we obtain for* $t = \frac{|u(x)|}{k}$

$$l(x) \le \frac{1}{4c}(\phi^{**}_{min})_*\Big(x, \frac{u(x)}{k}\Big) + K_0\frac{|u(x)|}{k}. \tag{3.51}$$

By (3.48), we have

$$\int_\Omega (\phi^{**}_{min})_*\Big(x, \frac{u(x)}{k}\Big)dx \le \int_\Omega (\phi^{**}_{min})_*\Big(x, \frac{u(x)}{\|u\|_{(\phi^{**}_{min})_*}}\Big)dx \le 1.$$

Integrating (3.51) *over* Ω, *we obtain*

$$\|l\|_{L^1(\Omega)} \leq \frac{1}{4c} + \frac{K_0 c_2}{k}\|u(x)\|_{L_{\phi_{max}}(\Omega)} \leq \frac{1}{4c} + \frac{K_0 c_2}{k}\|u\|_{W^1 L_{\vec{\phi}}(\Omega)}, \tag{3.52}$$

where c_2 *is the constant of the imbedding* (3.13). *Thus, by virtue of* (3.49) *and* (3.52) *we get*

$$\|l\|_{W^{1,1}(\Omega)} \leq \frac{1}{2c} + \frac{C_4}{k}\|u\|_{W^1 L_{\vec{\phi}}(\Omega)},$$

where $C_4 = c_2 K_0 + c_3$. *This implies that* $l \in W^{1,1}(\Omega)$ *and by* (3.50) *we arrive at*

$$\|l\|_{L^1(\partial\Omega)} \leq \frac{1}{2} + \frac{cC_4}{k}\|u\|_{W^1 L_{\vec{\phi}}(\Omega)}.$$

As $\|l\|_{L^1(\partial\Omega)} = \int_{\partial\Omega} |l(x)|dx = \int_{\partial\Omega} \psi_{min}\Big(x, \frac{u(x)}{k}\Big)dx = 1$, *we get*

$$\|u\|_{L_{\psi_{min}}(\partial\Omega)} = k \leq 2cC_4\|u\|_{W^1 L_{\vec{\phi}}(\Omega)}.$$

Case 2 : *Assume that*

$$k < \|u\|_{(\phi^{**}_{min})_*}.$$

By Theorem 3.3.1, there is a constant $c > 0$ *such that*

$$\|u\|_{L_{\psi_{min}}(\partial\Omega)} = k < \|u\|_{(\phi^{**}_{min})_*} \leq c\|u\|_{W^1 L_{\vec{\phi}}(\Omega)}.$$

Finally, in both cases there exists a constant $c > 0$ *such that*

$$\|u\|_{L_{\psi_{min}}(\partial\Omega)} \leq c\|u\|_{W^1 L_{\vec{\phi}}(\Omega)}.$$

For an arbitrary $u \in W^1 L_{\vec{\phi}}(\Omega)$, *we proceed as in the proof of Theorem 3.3.1 by truncating the function* u.

3.4 Application to some anisotropic elliptic equations

In this section, we apply the above results to get the existence and the uniqueness results of the weak solution for the problem (3.1).

3.4.1 Properties of the energy functional

Definition 3.4.1 *Let* Ω *be an open bounded subset of* $\mathbb{R}^N$, $(N \geq 2)$. *By a weak solution of problem* (3.1), *we mean a function* $u \in W^1 L_{\vec{\phi}}(\Omega)$ *satisfying for all* $v \in C^\infty(\overline{\Omega})$ *the identity*

$$\int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i} v dx + \int_\Omega b(x)\varphi_{max}(x, u)v dx - \int_\Omega f(x, u)v dx - \int_{\partial\Omega} g(x, u)v ds = 0. \tag{3.53}$$

We note that all the terms in (3.53) make sense. Indeed, for the first term in the right hand side in (3.53), we can write by using (3.18)

$$\int_\Omega \phi_i^*(x, \phi_i^{*-1}(x, P_i(x, \partial_{x_i} u(x))))dx \leq \int_\Omega P_i(x, \partial_{x_i} u(x))dx \leq \int_\Omega p_i(x, \partial_{x_i} u(x))\partial_{x_i} u(x)dx,$$

where P_i is the Musielak-Orlicz function given in (3.2) and $p_i(x,s) = \frac{\partial P_i(x,s)}{\partial s}$. Since P_i is locally integrable and $P_i \ll \phi_i$, we can use Lemma 3.5.7 (given in Appendix) obtaining $p_i(\cdot, \partial_{x_i} u(\cdot)) \in L_{P_i^*}(\Omega)$. So that by Hölder's inequality (3.15), we get

$$\int_\Omega \phi_i^*(x, \phi_i^{*-1}(x, P_i(x, \partial_{x_i} u(x))))dx \leq 2\|p_i(\cdot, \partial_{x_i} u(\cdot))\|_{P_i^*}\|\partial_{x_i} u\|_{P_i} < \infty.$$

Thus, $\phi_i^{*-1}(\cdot, P_i(\cdot, \partial_{x_i} u(\cdot))) \in L_{\phi_i^*}(\Omega)$. Since $v \in C^\infty(\overline{\Omega})$ and ϕ_{max} is locally integrable, then $v \in W^1 L_{\vec{\phi}}(\Omega)$. So we can use the growth condition (3.2) and again the Hölder inequality (3.15), to write

$$\int_\Omega a_i(x, \partial_{x_i} u)\partial_{x_i} v dx \leq 2c_i\|d_i(\cdot)\|_{\phi_i^*}\|\partial_{x_i} v\|_{\phi_i} + 2c_i\|\phi_i^{*-1}(\cdot, P_i(\cdot, \partial_{x_i} u(\cdot)))\|_{\phi_i^*}\|\partial_{x_i} v\|_{\phi_i} < \infty. \tag{3.54}$$

For the second term, the inequality (3.18) enables us to write

$$\int_\Omega \phi_{max}^*(x, \phi_{max}^{*-1}(x, R(x, u(x))))dx \leq \int_\Omega R(x, u(x))dx \leq \int_\Omega r(x, u(x))u(x)dx,$$

where R is the Musielak-Orlicz function given in (3.5) and $r(x,s) = \frac{\partial R(x,s)}{\partial s}$. Since R is locally integrable and $R \ll \phi_{max}$, Lemma 3.5.7 (given in Appendix) gives

$$\int_\Omega \phi_{max}^*(x, \phi_{max}^{*-1}(x, R(x, \partial_i u)))dx \leq 2\|r(\cdot, u(\cdot))\|_{R^*}\|u\|_R < \infty,$$

which shows that $\varphi_{max}(\cdot, u(\cdot)) \in L_{\phi_{max}^*}(\Omega)$. Thus,

$$\int_\Omega b(x)\varphi_{max}(x, u)v dx \leq 2\|b\|_\infty\|\varphi_{max}(\cdot, u(\cdot))\|_{\phi_{max}^*}\|v\|_{\phi_{max}} < \infty. \tag{3.55}$$

We now turn to the third term in the right hand side in (3.53). By using (3.6) and the Hölder inequality (3.15), one has

$$\int_\Omega f(x, u)v dx \leq k_1\|m(\cdot, u(\cdot))\|_{L_{M^*}(\Omega)}\|v\|_{L_M(\Omega)}. \tag{3.56}$$

Since M is locally integrable and $M \ll \phi_{min}^{**}$, then $M \ll \phi_{max}$ and Lemma 3.5.7 (given in Appendix) ensures that $\int_\Omega f(x,u)v dx < \infty$. For the last term in the right hand side in (3.53), using (3.7) to have

$$\int_{\partial\Omega} g(x, u)v ds \leq k_2 \int_{\partial\Omega} h(x, u)v ds.$$

Since the primitive H of h is a locally integrable function satisfies $H \ll \phi^{**}_{min}$, thus we can us a similar way as in Lemma 3.5.7 (given in Appendix) to get that $h(x,u) \in L_{H^*}(\partial\Omega)$ and since $\partial\Omega$ is a bounded set, then the imbedding (3.13) gives that $h(x,u) \in L^1(\partial\Omega)$. On the other hand, since $v \in C^\infty(\overline{\Omega})$, then $v \in L^\infty(\partial\Omega)$. Therefor,

$$\int_{\partial\Omega} g(x,u)vds < \infty.$$

Define the functional $I : W^1L_{\vec{\phi}}(\Omega) \to \mathbb{R}$ by

$$I(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_i u)dx + \int_\Omega b(x)\phi_{max}(x,u)dx - \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)ds. \tag{3.57}$$

Some basic properties of I are established in the following lemma.

Lemma 3.4.1 *Let Ω be an open bounded subset of $\mathbb{R}^N$, $(N \geq 2)$. Then*

(i) The functional I is well defined on $W^1L_{\vec{\phi}}(\Omega)$.

(ii) The functional I has a Gâteaux derivative $I'(u)$ for every $u \in W^1L_{\vec{\phi}}(\Omega)$. Moreover, for every $v \in W^1L_{\vec{\phi}}(\Omega)$

$$\begin{aligned} \langle I'(u), v\rangle &= \int_\Omega \sum_{i=1}^N a_i(x,\partial_i u)\partial_i v dx + \int_\Omega b(x)\varphi_{max}(x,u)vdx \\ &\quad - \int_\Omega f(x,u)vdx - \int_{\partial\Omega} g(x,u)vds. \end{aligned}$$

So that, the critical points of I are weak solutions to problem (3.1).

Proof 3.4.1 *(i) For almost every $x \in \Omega$ and for every $\zeta \in \mathbb{R}$, we can write*

$$A_i(x,\zeta) = \int_0^1 \frac{d}{dt}A_i(x,t\zeta)dt = \int_0^1 a_i(x,t\zeta)\zeta dt.$$

Then, by (3.2) we get

$$A_i(x,\zeta) \leq c_i d_i(x)\zeta + \int_0^1 \phi_i^{*-1}(x, P_i(x,t\zeta))\zeta dt \leq c_i d_i(x)\zeta + \phi_i^{*-1}(x, P_i(x,\zeta))\zeta.$$

In a similar way as in (3.54), we arrive at

$$\Big|\int_\Omega A_i(x,\partial_i u(x))dx\Big| < \infty.$$

Hence, the first term in the right hand side in (3.57) is well defined. For the second term, using (3.18), the Hölder inequality (3.15) and (3.55), we obtain

$$\Big|\int_\Omega b(x)\phi_{max}(x,u(x))dx\Big| \leq 2\|b\|_\infty \|\varphi_{max}(\cdot,u(\cdot))\|_{\phi^*_{max}}\|u\|_{\phi_{max}} < \infty,$$

while for the third term, by (3.6) and (3.18) we can write

$$\int_\Omega \big|F(x,u(x))\big|dx \le k_1 \int_\Omega m(x,u)udx.$$

Then, using the Hölder inequality (3.15) together with (3.56) we get

$$\int_\Omega \big|F(x,u(x))\big|dx \le 2k_1\|m(\cdot,u(\cdot))\|_{M^*}\|u\|_M < \infty.$$

For the last term in the right hand side in (3.57), we can use (3.7) and (3.18) to have

$$\int_{\partial\Omega} G(x,u)ds \le k_2 \int_{\partial\Omega} H(x,u)ds \le \int_{\partial\Omega} h(x,u)uds.$$

Since H is a locally integrable function satisfies $H \ll \phi^{**}_{min}$, then $H \ll \phi_{max}$ and we can us a similar method as in Lemma 3.5.7 (given in Appendix) to get that $h(x,u) \in L_{H^*}(\partial\Omega)$. Thus, the Hölder inequality (3.15) implies that

$$\int_{\partial\Omega} G(x,u)ds < \infty.$$

(ii) For every $i = 1,\cdots,N$ define the functional $\Lambda_i : W^1L_{\vec{\phi}}(\Omega) \to \mathbb{R}$ by

$$\Lambda_i(u) = \int_\Omega A_i(x,\partial_i u(x))dx.$$

Denote by B, L_1, $L_2 : W^1L_{\vec{\phi}}(\Omega) \to \mathbb{R}$ the functionals $B(u) = \displaystyle\int_\Omega b(x)\phi_{max}(x,u(x))dx$ $L_1(u) = \displaystyle\int_\Omega F(x,u(x))dx$ and $L_2(u) = \displaystyle\int_{\partial\Omega} G(x,u(x))ds$. We observe that for $u \in W^1L_{\vec{\phi}}(\Omega)$, $v \in C^\infty(\overline{\Omega})$ and $r > 0$

$$\frac{1}{r}[\Lambda_i(u+rv) - \Lambda_i(u)] = \int_\Omega \frac{1}{r}\Big[A_i\Big(x,\frac{\partial}{\partial x_i}u(x) + r\frac{\partial}{\partial x_i}v(x)\Big) - A_i\Big(x,\frac{\partial}{\partial x_i}u(x)\Big)\Big]dx$$

and

$$\frac{1}{r}\Big[A_i\Big(x,\frac{\partial}{\partial x_i}u(x) + r\frac{\partial}{\partial x_i}v(x)\Big) - A_i\Big(x,\frac{\partial}{\partial x_i}u(x)\Big)\Big] \longrightarrow a_i\Big(x,\frac{\partial}{\partial x_i}u(x)\Big)\frac{\partial}{\partial x_i}v(x),$$

as $r \to 0$ for almost every $x \in \Omega$. On the other hand, by the mean value theorem, there exists $\nu \in [0,1]$ such that

$$\begin{aligned}&\frac{1}{r}\Big|A_i\Big(x,\frac{\partial}{\partial x_i}u(x) + r\frac{\partial}{\partial x_i}v(x)\Big) - A_i\Big(x,\frac{\partial}{\partial x_i}u(x)\Big)\Big|\\ &= \Big|a_i\Big(x,\frac{\partial}{\partial x_i}u(x) + \nu r\frac{\partial}{\partial x_i}v(x)\Big)\Big|\Big|\frac{\partial}{\partial x_i}v(x)\Big|.\end{aligned}$$

Hence, by using the last equality and (3.2) we get

$$\begin{aligned}&\frac{1}{r}\Big|A_i\Big(x,\frac{\partial}{\partial x_i}u(x) + r\frac{\partial}{\partial x_i}v(x)\Big) - A_i\Big(x,\frac{\partial}{\partial x_i}u(x)\Big)\Big| \le\\ &c_i\Big[d_i(x) + \phi_i^{*-1}\Big(x,\phi_i\Big(x,\frac{\partial}{\partial x_i}u(x) + \nu r\frac{\partial}{\partial x_i}v(x)\Big)\Big)\Big]\Big|\frac{\partial}{\partial x_i}v(x)\Big|.\end{aligned}$$

Next, by the Hölder inequality (3.15) we get

$$c_i\Big[d_i(x)+\phi_i^{*-1}\Big(x,\phi_i\Big(x,\frac{\partial}{\partial x_i}u(x)+\nu r\frac{\partial}{\partial x_i}v(x)\Big)\Big)\Big]\Big|\frac{\partial}{\partial x_i}v(x)\Big|\in L^1(\Omega).$$

The dominated convergence can be applied to yield

$$\lim_{r\to 0}\frac{1}{r}[\Lambda_i(u+rv)-\Lambda_i(u)]=\int_\Omega a_i\Big(x,\frac{\partial}{\partial x_i}u(x)\Big)\frac{\partial}{\partial x_i}v(x)dx:=\langle\Lambda_i'(u),v\rangle,$$

for every $i=1,\cdots,N$. *By a similar calculus as in above, we can show that* $\langle B'(u),v\rangle=\int_\Omega b(x)\varphi_{max}(x,u)vdx$, $\langle L_1'(u),v\rangle=\int_\Omega f(x,u)vdx$ *and* $\langle L_2'(u),v\rangle=\int_\Omega g(x,u)vdx$.

3.4.2 An existence result

Our main existence result is the following.

Theorem 3.4.1 *Let* Ω *be an open bounded subset of* $\mathbb{R}^N$, $(N\geq 2)$, *with the cone property. Assume that (3.2), (3.3), (3.4), (3.6), (3.7), (3.9), (3.19) and (3.21) are fulfilled and suppose that* ϕ_{max} *and* ϕ_{min}^* *are locally integrable and* $(\phi_{min}^{**})_*(\cdot,t)$ *is Lipschitz continuous on* $\overline{\Omega}$. *Then, problem (3.1) admits at least a weak solution in* $W^1L_{\vec{\phi}}(\Omega)$.

Proof 3.4.2 *We divide the proof into three steps.*

Step 1 *: Weak* lower semicontinuity property of* I. *Define the functional* $J:W^1L_{\vec{\phi}}(\Omega)\to\mathbb{R}$ *by*

$$J(u)=\int_\Omega\sum_{i=1}^N A_i(x,\partial_i u)dx+\int_\Omega b(x)\phi_{max}(x,u)dx,$$

so that

$$I(u)=J(u)-L_1(u)-L_2(u).$$

First, we claim that J *is sequentially weakly lower semicontinuous. Indeed, since* $u\mapsto\phi_{max}(x,u)$ *is continuous, it is enough to show that the functional*

$$u\mapsto K(u)=\int_\Omega\sum_{i=1}^N A_i(x,\partial_i u)dx,$$

is sequentially weakly lower semicontinuous. To do this, let* $u_n\overset{*}{\rightharpoonup}u$ *in* $W^1L_{\vec{\phi}}(\Omega)$ *in the sense*

$$\int_\Omega u_n\varphi dx\to\int_\Omega u\varphi dx \text{ for all } \varphi\in E_{\phi_{max}^*} \text{ and } \int_\Omega\partial_i u_n\varphi dx\to\int_\Omega\partial_i u\varphi dx, \tag{3.58}$$

for all $\varphi \in E_{\phi_i^*}$. *By the definition of* ϕ_{min} *and* ϕ_{max}, (3.58) *holds true for every* $\varphi \in E_{\phi_{min}^*}(\Omega)$. *Being* ϕ_{min}^* *locally integrable, one has* $L^\infty(\Omega) \subset E_{\phi_{min}^*}(\Omega)$. *Therefore, for every* $i \in \{1, \cdots, N\}$

$$\partial_i u_n \rightharpoonup \partial_i u \text{ and } u_n \rightharpoonup u \text{ in } L^1(\Omega), \tag{3.59}$$

for the weak topology $\sigma(L^1, L^\infty)$. *As the embedding* $W^1L_{\vec{\phi}}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ *is continuous, the compact embedding* $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ *implies that the sequence* $\{u_n\}$ *is relatively compact in* $L^1(\Omega)$. *Therefore, there exist a subsequence still indexed by* n *and a function* $v \in L^1(\Omega)$, *such that* $u_n \to v$ *strongly in* $L^1(\Omega)$. *In view of* (3.59), *we have* $v = u$ *almost everywhere on* Ω *and* $u_n \to u$ *in* $L^1(\Omega)$. *Passing once more to a subsequence, we can have* $u_n \to u$ *almost everywhere on* Ω. *Since* $\zeta \to A_i(x, \zeta)$ *is convex, by* (3.3) *we can use [34, Theorem 2.1, Chapter 8] obtaining*

$$K(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_i u)dx \leq \liminf \int_\Omega \sum_{i=1}^N A_i(x, \partial_i u_n)dx = \liminf K(u_n).$$

We now prove that $L_1(u) = \int_\Omega F(x, u)dx$ *and* $L_2(u) = \int_{\partial\Omega} G(x, u)ds$ *are continuous. Since* $M \ll \phi_{min}^{**}$, *it follows that* $M \ll (\phi_{min}^{**})_*$, *then by Corollary 3.3.1 we get* $u_n \to u$ *in* $L_M(\Omega)$. *Thus, there exists* n_0 *such that for every* $n \geq n_0$, $\|u_n - u\|_M < \frac{1}{2}$. *By* (3.6), *we get*

$$\int_\Omega |F(x, u_n(x))|dx \leq k_1 \int_\Omega M(x, u_n(x))dx.$$

Let $\theta_n = \|u_n - u\|_M$. *By the convexity of* M, *we can write*

$$\begin{aligned} M(x, u_n(x)) &= M\Big(x, \theta_n\Big(\tfrac{u_n(x)-u(x)}{\theta_n}\Big) + (1-\theta_n)\tfrac{u(x)}{1-\theta_n}\Big) \\ &\leq \theta_n M\Big(x, \tfrac{u_n(x)-u(x)}{\theta_n}\Big) + (1-\theta_n) M\Big(x, \tfrac{u(x)}{1-\theta_n}\Big). \end{aligned}$$

Hence,

$$\int_\Omega M(x, u_n(x))dx \leq \theta_n + (1-\theta_n) \int_\Omega M\Big(x, \frac{u(x)}{1-\theta_n}\Big)dx. \tag{3.60}$$

Moreover,

$$M\Big(x, \frac{u(x)}{1-\theta_n}\Big) \leq M(x, 2u(x)).$$

Since M *is locally integrable and* $M \ll \phi_{max}$, *there exists a nonnegative function* $h \in L^1(\Omega)$, *such that*

$$\int_\Omega M(x, 2|u(x)|)dx \leq \int_\Omega \phi_{max}\Big(x, \frac{|u(x)|}{\|u\|_{\phi_{max}}}\Big)dx + \int_\Omega h(x)dx < \infty.$$

Thus, the Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty} \int_\Omega M\Big(x, \frac{u(x)}{1-\theta_n}\Big)dx = \int_\Omega M(x, u(x))dx$$

and therefore, by (3.60), we have

$$\limsup_{n\to\infty}\int_\Omega M(x,u_n(x))dx \le \int_\Omega M(x,u(x))dx.$$

In addition, by Fatou's Lemma we get

$$\int_\Omega M(x,u(x))dx \le \liminf_{n\to\infty}\int_\Omega M(x,u_n(x))dx.$$

Therefore, we have proved that

$$\lim_{n\to+\infty}\int_\Omega M(x,u_n(x))dx = \int_\Omega M(x,u(x))dx.$$

Thus, by [50, Theorem 13.47], we get that $M(x,u_n(x))\to M(x,u(x))$ strongly in $L^1(\Omega)$ which implies that $M(x,u_n(x))$ is equi-integrable, then so is $F(x,u_n(x))$ and since $F(x,u_n)\to F(x,u)$ almost everywhere on Ω, then by Vitali's theorem we get $L_1(u_n)\to L_1(u)$. Similarly, we can show that $L_2(u_n)\to L_2(u)$. Thus, L_1 and L_2 are continuous and since J is weakly lower semicontinuous, we conclude that I is weakly* lower semicontinuous.*

Step 2 *: Coercivity of the functional I. By (3.3), (3.9) and (3.14), we can write*

$$\begin{aligned}
I(u) &\ge \int_\Omega\sum_{i=1}^N \phi_i(x,\partial_i u)dx + b_0\int_\Omega \phi_{max}(x,u)dx - \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)ds\\
&\ge \sum_{i=1}^N \|\partial_i u\|_{\phi_i} + b_0\|u\|_{\phi_{max}} - N - b_0 - \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)ds\\
&\ge \min\{1,b_0\}\|u\|_{W^1L_{\vec\phi}(\Omega)} - \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)ds - N - b_0.
\end{aligned}$$

By (3.6) and (3.7), we get

$$I(u) \ge \min\{1,b_0\}\|u\|_{W^1L_{\vec\phi}(\Omega)} - k_1\int_\Omega M(x,u)dx - k_2\int_{\partial\Omega} H(x,u)ds - N - b_0.$$

As $M \ll (\phi^{**}_{min})_*$ and $H \ll \psi_{min}$, by Theorem 3.3.1 and Theorem 3.3.2 there exist two positive constant $C_1>0$ and $C_2>0$ such that $\|u\|_{L_M(\Omega)} \le C_1\|u\|_{W^1L_{\vec\phi}(\Omega)}$ and $\|u\|_{L_H(\partial\Omega)} \le C_2\|u\|_{W^1L_{\vec\phi}(\Omega)}$. Since M and H satisfy the Δ_2-condition, there exist two positive constants $r_1>0$ and $r_2>0$ and two nonnegative functions $h_1\in L^1(\Omega)$ and $h_2\in L^1(\partial\Omega)$ such that

$$\begin{aligned}
I(u) &\ge \min\{1,b_0\}\|u\|_{W^1L_{\vec\phi}(\Omega)} - k_1r_1\int_\Omega M\Big(x,\frac{|u(x)|}{C_1\|u\|_{W^1L_{\vec\phi}(\Omega)}}\Big)dx\\
&\quad -k_2r_2\int_{\partial\Omega} H\Big(x,\frac{|u(x)|}{C_2\|u\|_{W^1L_{\vec\phi}(\Omega)}}\Big)ds - \int_\Omega h_1(x)dx - \int_{\partial\Omega} h_2(x)ds - N - b_0\\
&\ge \min\{1,b_0\}\|u\|_{W^1L_{\vec\phi}(\Omega)} - k_1r_1\int_\Omega M\Big(x,\frac{|u(x)|}{\|u\|_{L_M(\Omega)}}\Big)dx\\
&\quad -k_2r_2\int_{\partial\Omega} H\Big(x,\frac{|u(x)|}{\|u\|_{L_H(\partial\Omega)}}\Big)ds - \int_\Omega h_1(x)dx - \int_{\partial\Omega} h_2(x)ds - N - b_0.\\
&\ge \min\{1,b_0\}\|u\|_{W^1L_{\vec\phi}(\Omega)} - \int_\Omega h_1(x)dx - \int_{\partial\Omega} h_2(x)ds - k_1r_1 - k_2r_2 - N - b_0,
\end{aligned}$$

which implies

$$I(u) \to \infty \text{ as } \|u\|_{W^1L_{\vec{\phi}}(\Omega)} \to \infty.$$

Step 3 *: Existence of a weak solution. Since I is coercive, for an arbitrary $\lambda > 0$ there exists $R > 0$ such that*

$$\|u\|_{W^1L_{\vec{\phi}}(\Omega)} > R \Rightarrow I(u) > \lambda.$$

Let $E_\lambda = \{u \in W^1L_{\vec{\phi}}(\Omega) : I(u) \leq \lambda\}$ and denote by $B_R(0)$ the closed ball in $W^1L_{\vec{\phi}}(\Omega)$ of radius R centered at origin. We claim that $\alpha = \inf_{v \in W^1L_{\vec{\phi}}(\Omega)} I(v) > -\infty$. If not, for all $n > 0$ there is a sequence $u_n \in E_\lambda$ such that $I(u_n) < -n$. As $E_\lambda \subset B_R(0)$, by the Banach-Alaoglu-Bourbaki theorem there exists $u \in B_R(0)$ such that, passing to a subsequence if necessary, we can assume that $u_n \rightharpoonup u$ weak in $W^1L_{\vec{\phi}}(\Omega)$. So that the weak* lower semicontinuity of I implies $I(u) = -\infty$ which contradicts the fact that I is well defined on $W^1L_{\vec{\phi}}(\Omega)$. Therefore, for every $n > 0$ there exists a sequence $u_n \in E_\lambda$ such that $I(u_n) \leq \alpha + \frac{1}{n}$. Thus, there exists $u \in B_R(0)$ such that, for a subsequence still indexed by n, $u_n \rightharpoonup u$ weak* in $W^1L_{\vec{\phi}}(\Omega)$. Since I is weakly* lower semicontinuous, we get*

$$I(u) = J(u) - L_1(u) - L_2(u) \leq \liminf_{n\to\infty} \Big(J(u_n) - L_1(u_n) - L_2(u_n)\Big) = \liminf_{n\to\infty} I(u_n) \leq \alpha.$$

Note that u belongs also to E_λ, which yields $I(u) = \alpha \leq \lambda$. This shows that $I(u) = \min\{I(v) : v \in W^1L_{\vec{\phi}}(\Omega)\}$. Moreover, inserting $v = -u^-$ as test function in (3.53) and then using (3.18), we obtain $u \geq 0$. The theorem is completely proved.

3.4.3 Uniqueness result

In order to prove the uniqueness of the weak solution we have found, we need to assume the following monotony assumptions

$$\big(f(x,s) - f(x,t)\big)\big(s-t\big) < 0 \text{ for a.e. } x \in \Omega \text{ and for all } s,t \in \mathbb{R} \text{ with } s \neq t \tag{3.61}$$

$$\big(g(x,s) - g(x,t)\big)\big(s-t\big) < 0 \text{ for a.e. } x \in \Omega \text{ and for all } s,t \in \mathbb{R} \text{ with } s \neq t \tag{3.62}$$

$$\big(\varphi_{max}(x,s) - \varphi_{max}(x,t)\big)\big(s-t\big) > 0 \text{ for a.e. } x \in \Omega \text{ and for all } s,t \in \mathbb{R} \text{ with } s \neq t. \tag{3.63}$$

Theorem 3.4.2 *If in addition to the hypothesis (3.4) the conditions (3.61), (3.62) and (3.63) are fulfilled, then the weak solution u to problem (3.1) is unique.*

Proof 3.4.3 *Suppose that there exists another solution w. We choose $v = u - w$ as test function in (3.53) obtaining*

$$\int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i}(u-w)dx + \int_\Omega b(x)\varphi_{max}(x,u)(u-w)dx$$
$$- \int_\Omega f(x,u)(u-w)dx - \int_{\partial\Omega} g(x,u)(u-w)ds = 0.$$

We replace u by w in (3.53) and we take $v = w - u$. We obtain

$$\int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} w)\partial_{x_i}(w-u)dx + \int_\Omega b(x)\varphi_{max}(x,w)(w-u)dx$$
$$- \int_\Omega f(x,w)(w-u)dx - \int_{\partial\Omega} g(x,w)(w-u)ds = 0.$$

By combining the previous two equalities, we obtain

$$\int_\Omega \sum_{i=1}^N \Big[a_i(x, \partial_{x_i} u) - a_i(x, \partial_{x_i} w)\Big](\partial_{x_i} u - \partial_{x_i} w)dx$$
$$+ \int_\Omega b(x)\Big[\varphi_{max}(x,u) - \varphi_{max}(x,w)\Big](u-w)dx$$
$$- \int_\Omega \Big[f(x,u) - f(x,w)\Big](u-w)dx - \int_{\partial\Omega} \Big[g(x,u) - g(x,w)\Big](u-w)ds = 0.$$

In view of (3.4), (3.61), (3.62) and (3.63), we obtain $u = w$ a.e. in Ω.

3.5 Appendix

We recall here some important lemmas that are necessary for the accomplishment of the proofs of the above results.

Lemma 3.5.1 *[67, Theorem 7.10.] Let Ω be an open bounded subset of $\mathbb{R}^N$, and let ϕ be a locally integrable Musielak-Orlicz function. Then E_ϕ is a separable space.*

Lemma 3.5.2 *Let Ω be an open bounded subset of $\mathbb{R}^N$, and let ϕ be a locally integrable Musielak-Orlicz function. For every $\eta \in L_{\phi^*}(\Omega)$, the linear functional F_η defined for every $\zeta \in E_\phi(\Omega)$ by*

$$F_\eta(\zeta) = \int_\Omega \zeta(x)\eta(x)dx \tag{3.64}$$

belongs to the dual space of $E_\phi(\Omega)$, denoted $E_\phi(\Omega)^*$, and its norm $\|F_\eta\|$ satisfies

$$\|F_\eta\| \leq 2\|\eta\|_{\phi^*}, \tag{3.65}$$

where $\|F_\eta\| = \sup\{|F_\eta(u)|, \|u\|_{L_M(\Omega)} \leq 1\}$.

Lemma 3.5.3 *Let Ω be an open bounded subset of $\mathbb{R}^N$ and let ϕ be a locally integrable Musielak-Orlicz function. Then, the dual space of $E_\phi(\Omega)$ can be identified to the Musielak-Orlicz space $L_{\phi^*}(\Omega)$.*

Proof 3.5.1 *According to Lemma 3.5.2 any element $\eta \in L_{\phi^*}(\Omega)$ defines a bounded linear functional F_η on $L_\phi(\Omega)$ and also on $E_\phi(\Omega)$ which is given by (3.64). It remains to show that every bounded linear functional on $E_\phi(\Omega)$ is of the form F_η for some $\eta \in L_{\phi^*}(\Omega)$. Given $F \in E_\phi(\Omega)^*$, we define the complex measure λ by setting*

$$\lambda(E) = F(\chi_E),$$

where E is a measurable subset of Ω having finite measure and χ_E stands for the characteristic function of E. Due to the fact that ϕ is locally integrable, the measurable function $\phi\Big(\cdot, \phi^{-1}\Big(x_0, \frac{1}{2|E|}\Big)\chi_E(\cdot)\Big)$ belongs to $L^1(\Omega)$ for any $x_0 \in \Omega$. Hence, there is an $\epsilon > 0$ such that for any measurable subset Ω' of Ω, one has

$$|\Omega'| < \epsilon \Rightarrow \int_{\Omega'} \phi\Big(x, \phi^{-1}\Big(x_0, \frac{1}{2|E|}\Big)\chi_E(x)\Big)dx \leq \frac{1}{2}.$$

As $\phi(\cdot, s)$ is measurable on E, Luzin's theorem implies that for $\epsilon > 0$ there exists a closed set $K_\epsilon \subset E$, with $|E \setminus K_\epsilon| < \epsilon$, such that the restriction of $\phi(\cdot, s)$ to K_ϵ is continuous. Let k be the point where the maximum of $\phi(\cdot, s)$ is reached in the set K_ϵ.

$$\int_E \phi\Big(x, \phi^{-1}\Big(k, \frac{1}{2|E|}\Big)\Big)dx = \int_{K_\epsilon} \phi\Big(x, \phi^{-1}\Big(k, \frac{1}{2|E|}\Big)\Big)dx + \int_{E\setminus K_\epsilon} \phi\Big(x, \phi^{-1}\Big(k, \frac{1}{2|E|}\Big)\Big).$$

For the first term in the right hand side of the equality, we can write

$$\int_{K_\epsilon} \phi\Big(x, \phi^{-1}\Big(k, \frac{1}{2|E|}\Big)\Big)dx \leq \int_E \phi\Big(k, \phi^{-1}\Big(k, \frac{1}{2|E|}\Big)\Big)dx \leq \frac{1}{2}.$$

Since $|E \setminus K_\epsilon| < \epsilon$, the second term can be estimated as

$$\int_{E\setminus K_\epsilon} \phi\Big(x, \phi^{-1}\Big(k, \frac{1}{2|E|}\Big)\Big) \leq \frac{1}{2}.$$

Thus, we get

$$\int_\Omega \phi\Big(x, \phi^{-1}\Big(k, \frac{1}{2|E|}\Big)\chi_E(x)\Big)dx \leq 1.$$

Therefore, we obtain

$$|\lambda(E)| \leq \|F\|\|\chi_E\|_\phi \leq \frac{\|F\|}{\phi^{-1}\Big(k, \frac{1}{2|E|}\Big)}.$$

As the right-hand side tends to zero when $|E|$ converges to zero, the measure λ is absolutely continuous with respect to the Lebesgue measure and so by Radon-Nikodym's Theorem (see for instance [7, Theorem 1.52]), it can be expressed in the form

$$\lambda(E) = \int_E \eta(x)dx,$$

for some nonnegative function $\eta \in L^1(\Omega)$ unique up to sets of Lebesgue measure zero. Thus,

$$F(\zeta) = \int_\Omega \zeta(x)\eta(x)dx$$

holds for every measurable simple function ζ. Note first that since Ω is bounded and ϕ is locally integrable, any measurable simple function lies in $E_\phi(\Omega)$ and the set of measurable simple functions is dense in $(E_\phi(\Omega), \|\cdot\|_\phi)$. Indeed, for nonnegative $\zeta \in E_\phi(\Omega)$, there exists a sequence of increasing measurable simple functions ζ_j converging almost everywhere to ζ and $|\zeta_j(x)| \leq |\zeta(x)|$ on Ω. By the theorem of dominated convergence one has $\zeta_j \to \zeta$ in $E_\phi(\Omega)$. For an arbitrary $\zeta \in E_\phi(\Omega)$, we obtain the same result splitting ζ into positive and negative parts.

Let $\zeta \in E_\phi(\Omega)$ and let ζ_j be a sequence of measurable simple functions converging to ζ in $E_\phi(\Omega)$. By Fatou's Lemma and the inequality (3.65) we can write

$$\begin{aligned} \Big|\int_\Omega \zeta(x)\eta(x)dx\Big| &\leq \liminf_{j\to+\infty}\int_\Omega |\zeta_j(x)\eta(x)|dx = \liminf_{j\to+\infty} F(|\zeta_j| sgn\eta) \\ &\leq 2\|\eta\|_{\phi^*}\liminf_{j\to+\infty}\|\zeta_j\|_\phi \leq 2\|\eta\|_{\phi^*}\|\zeta\|_\phi, \end{aligned}$$

which implies that $\eta \in L_{\phi^*}(\Omega)$. Thus, the linear functional $F_\eta(\zeta) = \int_\Omega \zeta(x)\eta(x)dx$ and F defined both on $E_\phi(\Omega)$ have the same values on the set of measurable simple functions, so they agree on $E_\phi(\Omega)$ by a density argument.

Lemma 3.5.4 *Let Ω be an open bounded subset of $\mathbb{R}^N$. Let $f, g : \Omega \times (0, +\infty) \to (0, +\infty)$ be continuous nondecreasing functions with respect to there second argument and $g(\cdot, t)$ is continuous on $\overline{\Omega}$ with $\lim_{t\to\infty} \frac{f(x,t)}{g(x,t)} = +\infty$, then for all $\epsilon > 0$, there exists a positive constant K_0 such that*

$$g(x,t) \leq \epsilon f(x,t) + K_0, \text{ for all } t > 0.$$

Proof 3.5.2 *Let $\epsilon > 0$ be arbitrary. There exists $t_0 > 0$ such that $t \geq t_0$ implies $g(x,t) \leq \epsilon f(x,t)$. Then, for all $t \geq 0$,*

$$g(x,t) \leq \epsilon f(x,t) + K(x),$$

where $K(x) = \sup_{t\in(0,t_0)} g(x,t)$. Being $g(\cdot,t)$ continuous on $\overline{\Omega}$, one has $g(x,t) \leq \epsilon f(x,t) + K_0$ with $K_0 = \max_{x\in\overline{\Omega}} K(x)$.

Lemma 3.5.5 *Let Ω be an open bounded subset of $\mathbb{R}^N$. Let A, B be two Musielak-Orlicz functions such that $B \ll A$, with $B(\cdot,t)$ is continuous on $\overline{\Omega}$. If a sequence $\{u_n\}$ is bounded in $L_A(\Omega)$ and converges in measure in Ω, then it converges in norm in $L_B(\Omega)$.*

Proof 3.5.3 *Fix $\epsilon > 0$. Defining $v_{j,k}(x) = \frac{u_j(x)-u_k(x)}{\epsilon}$, we shall show that $\{u_j\}$ is a Cauchy sequence in the Banach space $L_B(\Omega)$. Clearly $\{v_{j,k}\}$ is bounded in $L_A(\Omega)$, say $\|v_{j,k}\|_A \leq K$ for all j and k. Since $B \ll A$ there exists a positive number t_0 such that for $t \geq t_0$ one has*

$$B(x,t) \leq \frac{1}{4} A\Big(x, \frac{t}{K}\Big).$$

On the other hand, since $B(\cdot,t)$ is continuous on $\overline{\Omega}$. Let x_0 be the point where the maximum of $B(\cdot,t)$ is reached in $\overline{\Omega}$. Let $\delta = \frac{1}{4B(x_0,t_0)}$ and set

$$\Omega_{j,k} = \Big\{x \in \Omega : |v_{j,k}| \geq B^{-1}\Big(x_0, \frac{1}{2|\Omega|}\Big)\Big\}.$$

Since $\{u_j\}$ converges in measure, there exists an integer N_0 such that $|\Omega_{j,k}| \leq \delta$ whenever j, $k \geq N_0$. Defining

$$\Omega'_{j,k} = \{x \in \Omega_{j,k} : |v_{j,k}| \geq t_0\} \text{ and } \Omega''_{j,k} = \Omega_{j,k} \setminus \Omega'_{j,k},$$

one has

$$\begin{aligned} \int_\Omega B(x,|v_{j,k}(x)|)dx &= \int_{\Omega\setminus\Omega_{j,k}} B(x,|v_{j,k}(x)|)dx + \int_{\Omega'_{j,k}} B(x,|v_{j,k}(x)|)dx \\ &+ \int_{\Omega''_{j,k}} B(x,|v_{j,k}(x)|)dx. \end{aligned} \tag{3.66}$$

For the first term in the right hand side of (3.66), we can write

$$\begin{aligned} \int_{\Omega\setminus\Omega_{j,k}} B(x,|v_{j,k}(x)|)dx &\leq \int_{\Omega\setminus\Omega_{j,k}} B\Big(x, B^{-1}\Big(x_0, \frac{1}{2|\Omega|}\Big)\Big)dx \\ &\leq \int_\Omega B\Big(x_0, B^{-1}\Big(x_0, \frac{1}{2|\Omega|}\Big)\Big)dx \\ &\leq \tfrac{1}{2}. \end{aligned}$$

Since $B \ll A$, the second term in the right hand side of (3.66) can be estimated as follows

$$\int_{\Omega'_{j,k}} B(x,|v_{j,k}(x)|)dx \leq \frac{1}{4}\int_\Omega A\Big(x, \frac{|v_{j,k}|}{K}\Big)dx \leq \frac{1}{4},$$

while for the third term in the right hand side of (3.66), we get

$$\int_{\Omega''_{j,k}} B(x,|v_{j,k}(x)|)dx \leq \int_{\Omega_{j,k}} B(x,t_0)dx \leq \delta B(x_0,t_0) = \frac{1}{4}.$$

Finally, puting all the above estimates in (3.66), we arrive at

$$\int_\Omega B(x,|v_{j,k}(x)|)dx \leq 1, \text{ for every } j,k \geq N_0,$$

which yields $\|u_j - u_k\|_B \leq \epsilon$. Thus, $\{u_j\}$ converges in the Banach space $L_B(\Omega)$.

Lemma 3.5.6 *Let $u \in W^{1,1}_{loc}(\Omega)$ and let $F : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ be a Lipschitz continuous function. If $f(x) = F(x, u(x))$ then $f \in W^{1,1}_{loc}(\Omega)$. Moreover, for every $j = 1, \cdots, N$, the weak derivative $\partial_{x_j} f$ of f is such that*

$$\partial_{x_j} f(x) = \frac{\partial F(x, u(x))}{\partial x_j} + \frac{\partial F(x, u(x))}{\partial t} \partial_{x_j} u(x), \text{ for a.e. } x \in \Omega.$$

Proof 3.5.4 *Let $\varphi \in D(\Omega)$ and let $\{e_j\}_{j=1}^N$ be the standard basis in $\mathbb{R}^N$. We can write*

$$\begin{aligned}
&- \int_\Omega F(x, u(x)) \partial_{x_j} \varphi(x) dx \\
&= - \lim_{h \to 0} \int_\Omega F(x, u(x)) \frac{\varphi(x) - \varphi(x - he_j)}{h} dx \\
&= \lim_{h \to 0} \int_\Omega \frac{F(x + he_j, u(x + he_j)) - F(x, u(x))}{h} \varphi(x) dx \\
&= \lim_{h \to 0} \int_\Omega \frac{F(x + he_j, u(x + he_j)) - F(x, u(x + he_j))}{h} \varphi(x) dx \\
&\quad + \lim_{h \to 0} \int_\Omega \frac{F(x, u(x + he_j)) - F(x, u(x))}{h} \varphi(x) dx \\
&= \lim_{h \to 0} \int_\Omega Q_1(x, h) \varphi(x) dx \\
&\quad + \lim_{h \to 0} \int_\Omega Q_2(x, h) \frac{u(x + he_j) - u(x)}{h} \varphi(x) dx,
\end{aligned}$$

where

$$Q_1(x, h) = \begin{cases} \dfrac{F(x + he_j, u(x + he_j)) - F(x, u(x + he_j))}{h} & \text{if } h \neq 0, \\ \dfrac{\partial F(x, u(x))}{\partial x_j} & \text{if } h = 0 \end{cases}$$

and

$$Q_2(x, h) = \begin{cases} \dfrac{F(x, u(x + he_j)) - F(x, u(x))}{u(x + he_j) - u(x)} & \text{if } u(x + he_j) \neq u(x), \\ \dfrac{\partial F(x, u(x))}{\partial t} & \text{otherwise.} \end{cases}$$

Since $F(\cdot, \cdot)$ is Lipschitz continuous, there exist two constants k_1 and $k_2 > 0$ independent of h, such that $\|Q_1(\cdot, h)\|_\infty \leq k_1$ and $\|Q_2(\cdot, h)\|_\infty \leq k_2$. Thus, for some sequence of values of h tending to zero, $Q_1(\cdot, h)$ converges to $\frac{\partial F(x,u(x))}{\partial x_j}$ and $Q_2(\cdot, h)$ converges to $\frac{\partial F(x,u(x))}{\partial t}$ both in $L^\infty(\Omega)$ for the weak-star topology $\sigma^*(L^\infty(\Omega), L^1(\Omega))$. On the other hand, since $u \in W^{1,1}(supp(\varphi))$ we have

$$\lim_{h \to 0} \int_{supp(\varphi)} \frac{u(x + he_j) - u(x)}{h} \varphi(x) dx = \int_{supp(\varphi)} \partial_j u(x) \varphi(x) dx.$$

It follows that

$$- \int_\Omega F(x, u(x)) \partial_{x_j} \varphi(x) dx = \int_\Omega \frac{\partial F(x, u(x))}{\partial x_j} \varphi(x) dx + \int_\Omega \frac{\partial F(x, u(x))}{\partial t} \partial_{x_j} u(x) \varphi(x) dx,$$

which completes the proof of the lemma.

Lemma 3.5.7 *Let Ω be an open subset of $\mathbb{R}^N$. Let A and ϕ be two Musielak-Orlicz functions with ϕ is locally integrable, differentiable with respect to its second argument and $\phi \ll A$. Then, $\varphi(\cdot, s) \in L_{\phi^*}(\Omega)$ for every $s \in L_A(\Omega)$, where $\varphi(x,s) = \frac{\partial \phi(x,s)}{\partial s}$.*

Proof 3.5.5 *Let $s \in L_A(\Omega)$. By (3.18), we can write*

$$\begin{aligned}\int_\Omega \phi^*(x, \varphi(x,s))dx &= \int_\Omega \int_0^{\varphi(x,s)} \varphi^{-1}(x,\tau)d\tau dx \leq \int_\Omega |s|\varphi(x,|s|)dx \\ &\leq \int_\Omega \phi(x, 2|s|)dx.\end{aligned}$$

It's obvious that if $s = 0$ then $\varphi(\cdot, s) \in L_{\phi^*}(\Omega)$. Assume that $s \neq 0$. Since ϕ is locally integrable and $\phi \ll A$, there exists a nonnegative function $h \in L^1(\Omega)$ such that $\phi(x, 2|s|) \leq A\Big(x, \frac{|s|}{\|s\|_A}\Big) + h(x)$ for a.e. $x \in \Omega$. Thus,

$$\int_\Omega \phi(x,2|s|)dx \leq \int_\Omega A\Big(x, \frac{|s|}{\|s\|_A}\Big)dx + \int_\Omega h(x)dx < \infty.$$

Hence, $\varphi(\cdot, s) \in L_{\phi^*}(\Omega)$.

Let $\phi : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ be a real function such that the partial function $\phi(x,\cdot)$ is convex. The function ϕ is called the principal part of the Musielak-Orlicz function M if $M(x,t) = \phi(x,t)$ for large values of the argument t.

Lemma 3.5.8 *Let $t_0 > 0$ be arbitrary and let $\phi : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}^+$ be a real function where the partial function $\phi(x,\cdot)$ is convex. Define the function $\varphi(x,t) = \frac{\partial \phi(x,t)}{\partial t}$. If $\phi(\cdot,t)$ and $\varphi(\cdot,t)$ are continuous on $\overline{\Omega}$ and $\lim\limits_{t\to+\infty} \inf\limits_{x\in\Omega} \varphi(x,t) = +\infty$. Then $\phi(x,t)$ is the principal part of a Musielak-Orlicz function $M(x,t)$.*

Proof 3.5.6 *Since $\lim\limits_{t\to+\infty} \inf\limits_{x\in\Omega} \varphi(x,t) = +\infty$, then there exists $t_1 > t_0 + 1$ (not depending on x) such that $\sup\limits_{x\in\overline{\Omega}} \varphi(x, t_0+1) + \sup\limits_{x\in\overline{\Omega}} \phi(x,t_0) \leq \varphi(x,t_1)$. Thus, we have*

$$\begin{aligned}\inf_{x\in\overline{\Omega}} \phi(x,t_1) \leq \phi(x,t_1) &= \int_{t_0}^{t_0+1} \varphi(x,\tau)d\tau + \int_{t_0+1}^{t_1} \varphi(x,\tau)d\tau + \phi(x,t_0) \\ &\leq \sup_{x\in\overline{\Omega}} \varphi(x,t_0+1) + \sup_{x\in\overline{\Omega}} \phi(x,t_0) + (t_1 - t_0 - 1)\varphi(x,t_1) \\ &\leq (t_1 - t_0)\varphi(x,t_1) \\ &< t_1 \varphi(x,t_1) \\ &\leq t_1 \sup_{x\in\overline{\Omega}} \varphi(x,t_1),\end{aligned}$$

from which it follows that $\alpha = \frac{t_1 \sup_{x\in\overline{\Omega}} \varphi(x,t_1)}{\inf_{x\in\overline{\Omega}} \phi(x,t_1)} > 1$.

We define the function $M(x,t)$ *by*

$$M(x,t) = \begin{cases} \frac{\phi(x,t_1)}{t_1^\alpha} t^\alpha & \text{if } t \leq t_1, \\ \phi(x,t) & \text{if } t \geq t_1. \end{cases}$$

The function $M(x,t)$ *is a Musielak-Olicz function inasmuch as its derivative,*

$$\frac{\partial M(x,t)}{\partial t} = \begin{cases} \frac{\alpha\phi(x,t_1)}{t_1^\alpha} t^{\alpha-1} & \text{if } t \leq t_1, \\ \varphi(x,t) & \text{if } t \geq t_1, \end{cases}$$

is a function which is positive for $t > 0$, *right-continuous for* $t \geq 0$ *non-decreasing, and such that it satisfies* $\lim_{t\to+\infty} \dfrac{\partial M(x,t)}{\partial t} = +\infty$.

Chapter 4

Semilinear heat equation with Hardy potential and singular terms

In this chapter, we analyze the question of existence and nonexistence of positive solutions for the following parabolic problem

$$\begin{cases} \partial_t u - \Delta u &= \mu \dfrac{u}{|x|^2} + \dfrac{f}{u^\sigma} \quad \text{in } \Omega_T := \Omega \times (0,T), \\ u &> 0 \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{in } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathrm{R}^N$, $N \geq 3$, is a bounded open, $\sigma \geq 0$ and $\mu \geq 0$ are real constants and $f \in L^m(\Omega_T)$, $m \geq 1$, and u_0 are nonnegative functions. The study we lead shows that the existence of solutions depends on σ and the summability of the datum f as well as on the interplay between μ and the best constant in the Hardy inequality. Regularity results of positive solutions, when they exist, are also obtained. Furthermore, we prove uniqueness of finite energy solutions.

4.1 Introduction

Let Ω be a bounded open subset of R^N, $N \geq 3$, containing the origin. Set $\Omega_T := \Omega \times (0,T)$ where $T > 0$ is a real constant. In this paper we investigate the existence and regularity as well as the uniqueness of solutions to the following parabolic problem

$$\begin{cases} \partial_t u - \Delta u & = \mu \frac{u}{|x|^2} + \frac{f}{u^\sigma} \quad \text{in } \Omega_T, \\ u & > 0 \quad \text{in } \Omega_T, \\ u & = 0 \quad \text{in } \Gamma := \partial\Omega \times (0,T), \\ u(x,0) & = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{4.1}$$

where $\sigma \geq 0$ and $\mu \geq 0$. The source terms f and u_0 satisfy

$$f \geq 0,\ f \in L^m(\Omega_T), m \geq 1 \tag{4.2}$$

and $u_0 \in L^\infty(\Omega)$ such that

$$\forall w \subset\subset \Omega\ \exists d_w > 0 : u_0 \geq d_w \text{ in } w. \tag{4.3}$$

It is clear that problem (4.1) is strongly related to the following classical Hardy inequality which asserts that

$$\Lambda_{N,2} \int_\Omega \frac{|u|^2}{|x|^2} dx \leq \int_\Omega |\nabla u|^2 dx, \tag{4.4}$$

for all $u \in C_0^\infty(\Omega)$, where $\Lambda_{N,2} = (\frac{N-2}{2})^2$ is optimal and not achieved (see for instance [24, 85] and [13] when $\Omega = \mathrm{R}^N$). The presence of a term with negative exponent generally induces a difficulty in defining the notion of solution for the problem (4.1).

In the literature, singular problems like (4.1) are considered and intensively studied in various situations depending on σ or μ. If $\sigma = 0$ and $\mu > 0$ the problem (4.1) is reduced to the following heat equation involving the Hardy potential

$$\begin{cases} \partial_t u - \Delta u & = \mu \frac{u}{|x|^2} + f \quad \text{in } \Omega_T, \\ u & = 0 \quad \text{in } \partial\Omega \times (0,T), \\ u(x,0) & = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{4.5}$$

and is studied first by Baras and Goldstein in their pioneering work [17]. When the data $0 \leq f \in L^1(\Omega_T)$ and u_0 is a positive L^1-function or a positive Radon measure on Ω are not both identically zero (otherwise the result is false since $u \equiv 0$ is a solution), Baras and Goldstein [17] have proved that if $0 \leq \mu \leq \Lambda_{N,2}$ then there exists a positive global solution for the problem (4.5), while if $\mu > \Lambda_{N,2}$ there is no solution.

Problem (4.5) with $-\mathrm{div} a(x,t,\nabla u)$ instead of $-\Delta$ was studied in [74], where the author proved that all the solutions have the same asymptotic behaviour, that is they all tend to the solution of the original problem which satisfies a zero initial condition. In [75] the authors studied the influence of

the presence of the Hardy potential and the summability of the datum f on the regularity of the solutions of problem (4.5) with the nonlinear operator $-\mathrm{div}a(x,t,u,\nabla u)$ in the principal part.
The singular Hardy potential appears in the context of combustions theory (see [85] and references therein) and quantum mechanics (see [17] and [85] and references therein). There is a wide literature about problems involving the Hardy potential where the existence and regularity of solutions as well as nonexistence of solutions are analyzed, for instance, we refer to [2, 3, 4, 5, 6, 8, 12, 19, 20, 48, 61, 65, 89].

Problems involving singularities (like (4.1) with $\mu = 0$) describe naturally several physical phenomena. Stationary cases include the semilinear equation $-\Delta u = f(x)u^{-\sigma}$, $x \in \Omega \subset \mathrm{R}^N$, that can be obtained as a generalization to the higher dimension from a one dimensional ODE ($N = 1$) by some transformations of boundary layer equations for the class of non-Newtonian fluids called pseudoplastic (see [69, 35]). As far as we know, semilinear equations with singularities arise in various contexts of chemical heterogeneous catalysts [11], non-Newtonian fluids as well as heat conduction in electrically conducting materials (the term u^σ describes the resistivity of the material), see for instance, [69, 40]. In view of this physical interpretation various generalisations of this later equation considered in the framework of partial differential equations ($N \geq 2$) has been the subject of study in many papers. For the mathematical analysis account, the seminal papers [26, 82] constitute the starting point of a wide literature about singular semilinear elliptic equations. Far from being complete we quote the list [9, 19, 22, 29, 30, 55, 57, 70, 87, 88, 94].
It is worth recalling that due to the meaning of the unknowns (concentrations, populations,...), only the positive solutions are relevant in most cases.
As far as the parabolic setting is concerned for problems as in (4.1) with $\mu = 0$, the literature is not rich enough. For problems like (4.1) with p-Laplacian operator, existence results of nonnegative solutions are obtained in [28] for data with higher summability while in [71] the authors proved the existence of nonnegative distributional solutions for non regular data (L^1 and measure) and the uniqueness is proved for energy solutions. Other related problems with singular terms can be found in [16, 14, 15].
In the case where $\sigma \neq 0$ and $\mu = 0$, problem (4.1) with a quite more general diffusion operator including the Laplacian one is studied in [27]. The authors considered nonnegative data having suitable Lebesgue-type summabilities and assumed the strict positivity on the initial data inside the parabolic cylinder. They have shown, via Harnack's inequality, that this strict positivity is inherited by the constructed solution to the problem, thus giving a meaning to the notion of solution

considered. Some regularity results are obtained according to the regularity of f and the values of σ.

Our main goal in this paper is to study the problem (4.1) in the presence of the two singular terms, that is $\mu > 0$ and $\sigma \geq 0$ extending to the evolution case some results obtained for the elliptic problem (with the Δ_p operator instead of Laplacian one) studied in [1]. Abdellaoui and Attar [1] investigated the interplay between the summability of f and σ providing the largest class of the datum f for which the problem admits a solution in the sense of distributions. Uniqueness and regularity results on the distributional solutions are also established. In the same spirit, the parabolic case with $\mu = 0$ was investigated in [27]. Our work fits in the context of recent work on equations involving the Hardy potential in the case of nonexistence of solutions. We start by studying first the case $\mu < \Lambda_{N,2} := \frac{(N-2)^2}{4}$ distinguishing two cases where $\sigma \geq 1$ and $f \in L^1(\Omega_T)$ and the case where $\sigma < 1$ with $f \in L^{m_1}(\Omega_T)$, $m_1 = \frac{2N}{2N+(\sigma-1)(N-2)}$. Then we investigate the case $\mu = \Lambda_{N,2}$ and $\sigma = 1$ with data $f \in L^1(\Omega_T)$. In both cases we prove the existence of a weak solution obtained as limit of approximations that belongs to a suitable Sobolev space. The approach we use consists in approximating the singular equation with a regular problem, where the standard techniques (e.g., fixed point argument) can be applied and then passing to the limit to obtain the weak solution of the original problem. The regularity of weak solutions is analyzed according to the Lebesgue summability of f and σ. Furthermore, we prove the uniqueness of finite energy solutions when the source term f has a compact support by extending the formulation of weak solutions to a large class of test functions. Finally, in the case where $\mu > \Lambda_{N,2}$ we prove a nonexistence result.

The chapter is presented as follows. The Section (4.2) contains all the main results (existence, regularity and uniqueness) and also graphic presentations allowing to better locate the obtained results. In section (4.3) we first prove an existence result for approximate regular problems of the problem (4.1) and then we give the proof of all the main results Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.3, Theorem 4.2.4, Theorem 4.2.5 and Theorem 4.2.6. At the end, some results that are necessary for the accomplishment of the work are given in an appendix to make the paper quite self contained.

4.2 Main results

We begin by stating the definition of weak solution and finite energy solution of the problem (4.1) and then we state and comment the main results.

Definition 4.2.1 *1)- By a weak solution of the problem (4.1) we mean a function $u \in L^1(0,T;W^{1,1}_{loc}(\Omega))$ such that $\frac{f}{u^\sigma} \in L^1(0,T;L^1_{loc}(\Omega))$ and*

$$-\int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u\partial_t\phi dxdt + \int_{\Omega_T} \nabla u\cdot\nabla\phi dxdt = \int_{\Omega_T}\Big(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\Big)\phi dxdt, \tag{4.6}$$

for every $\phi \in C_0^\infty(\Omega\times[0,T))$.

2)- We call a finite energy solution of the problem (4.1) a weak solution u that satisfies $u \in L^2(0,T;H^1_0(\Omega))$ with $\partial_t u \in L^2(0,T;H^{-1}(\Omega)) + L^1(0,T;L^1_{loc}(\Omega))$.

In Definition 4.2.1 above all the integrals make sense. Generated by the singular terms, the only difficulty is raised in the right-hand side. Indeed, by Hardy's inequality the integral $\displaystyle\int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt$ is finite while we make use of a comparison result with a solution of a problem in [27, Proposition 2.2], where the hypothesis (4.3) is used, for the integral $\displaystyle\int_{\Omega_T}\Big|\frac{f\phi}{u^\sigma}\Big|dxdt$ to be finite. This gives the reason behind the hypothesis $\frac{f}{u^\sigma} \in L^1(0,T;L^1_{loc}(\Omega))$.

Throughout this paper, we will make use of the two real auxiliary truncation functions T_k and G_k defined for fixed $k>0$ respectively as $T_k(s) = \max(-k,\min(s,k))$ and $G_k(s) = (|s|-k)^+\mathrm{sign}(s)$. Throughout the paper we define

$$m_1 := \frac{2N}{2N-(1-\sigma)(N-2)}.$$

Observe that $m_1 \geq 1$ if and only if $\sigma \leq 1$. We will prove the existence of solution for the problem (4.1) under the assumption that the datum f satisfies

$$\begin{cases} f \in L^{m_1}(\Omega_T) & \text{if} \quad 0\leq\sigma\leq 1,\\ f\in L^1(\Omega_T) & \text{if} \quad \sigma\geq 1,\end{cases} \tag{4.7}$$

4.2.1 The case $\mu < \Lambda_{N,2}$: existence of weak solutions

The first existence result is the following.

Theorem 4.2.1 *Let Ω be a bounded open subset of R^N, $N\geq 3$, containing the origin. Assume that u_0 and f are nonnegative functions satisfying (4.3) and (4.7) respectively. If $\mu < \Lambda_{N,2}$ then the problem (4.1) has a positive weak solution u such that*

1. *if $0\leq\sigma\leq 1$ then u is a finite energy solution,*

2. *if* $\sigma > 1$ *then* $u \in L^2(0,T;H^1_{loc}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ *with* $G_k(u) \in L^2(0,T;H^1_0(\Omega))$. *Moreover, if* $\frac{4\sigma}{(\sigma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$ *then* $u^{\frac{\sigma+1}{2}} \in L^2(0,T;H^1_0(\Omega))$,

3. *if* $\sigma > 1$ *and supp*$(f) \subset\subset \Omega$ *then* u *is a finite energy solution.*

Remark 4.2.1 *Let us notice that in absence of the Hardy potential (i.e.* $\mu = 0$*), the result corresponding to the case* $\sigma \leq 1$ *is already obtained in [27, Theorem 1.3 (i)], when* $p = 2$ *and the source term* f *belongs to* $L^{m_2}(\Omega_T)$, $m_2 := \frac{2(N+2)}{2(N+2)-N(1-\sigma)}$. *Note that since* $m_1 < m_2$, *the result we prove here is a refinement of that in [27, Theorem 1.3 (i)]. While in the case* $\sigma > 1$ *we obtain the same result to that in [27, Theorem 1.3 (ii)]. Note that if* $\sigma = 1$ *the above results coincide.*

Observe that $1 \leq m_1 \leq \frac{2N}{N+2}$ *for any* $0 \leq \sigma \leq 1$. *We point out that in the case where* $\sigma = 0$, *which yields* $m_1 = \frac{2N}{N+2}$, *we find the result already established in [75, Theorem 1.2] for data* $f \in L^r(0,T;L^q(\Omega))$ *with* $r = q \geq \frac{2N}{N+2}$. *It is worth recalling here that* $\frac{2N}{N+2}$ *is the Hölder conjugate exponent of the Sobolev exponent* $\frac{2N}{N-2}$ *and by duality argument, data belonging to the Lebesgue space of exponent* $\frac{2N}{N+2}$ *are in force in the dual space* $L^2(0,T;H^{-1}(\Omega))$.

4.2.2 The case $\mu = \Lambda_{N,2}$: existence of infinite energy solutions

In the following result we deal with the case where $\mu = \Lambda_{N,2}$. The weak solutions found do not generally belong to the energy space.

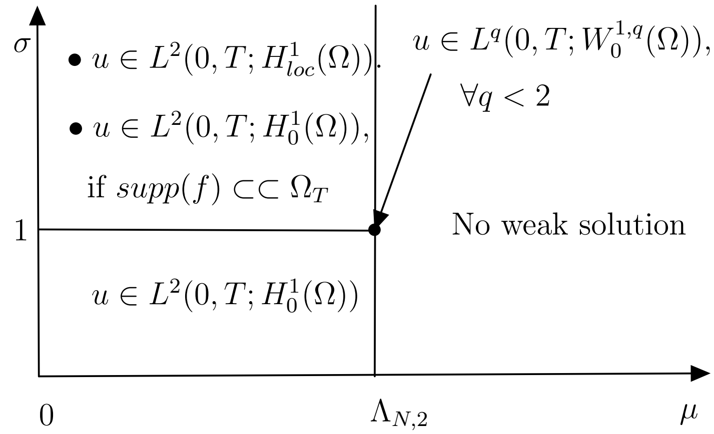
Theorem 4.2.2 *Let* Ω *be a bounded open subset of* $\mathbb{R}^N$, $N \geq 3$, *containing the origin. Suppose that (4.3) is filled and assume that* $\sigma = 1$ *and* $f \in L^1(\Omega_T)$. *If* $\mu = \Lambda_{N,2}$ *then the problem (4.1) has a weak solution* u *such that* $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, *for every* $q < 2$.

4.2.3 The case $\mu > \Lambda_{N,2}$: nonexistence of weak solutions

If we assume $\mu > \Lambda_{N,2}$ then the problem (4.1) has no weak solution. This is stated in the following theorem.

Theorem 4.2.3 *Let* Ω *be a bounded open subset of* $\mathbb{R}^N$, $N \geq 3$, *containing the origin. Assume that (4.3) and (4.7) hold. If* $\mu > \Lambda_{N,2}$ *then the problem (4.1) has no positive weak solution.*

The following figure summarizes the different existence results according to the interactions between the singularities.



4.2.4 Regularity of weak solutions

In the following theorem we give some regularity results for the weak solution u of the problem (4.1) obtained in Theorem 4.2.1.

Theorem 4.2.4 *Let Ω be a bounded open subset of R^N, $N \geq 3$, containing the origin. Assume that (4.2) and (4.3) hold and suppose that $\sigma > 0$ and $\mu < \Lambda_{N,2}$. Then*

(i) if $\sigma \geq 1$ and $m \geq 1$ one has

(a) if $m > \frac{N}{2} + 1$ then $u \in L^\infty(\Omega_T)$,

(b) if $1 \leq m < \frac{N}{2} + 1$, then $u^{\frac{\gamma+1}{2}} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ where $\gamma = \frac{Nm(1+\sigma)-N+2m-2}{N-2m+2}$ provided that $\frac{4\gamma}{(\gamma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$.

(ii) If $0 \leq \sigma \leq 1$ one has

(c) if $m > \frac{N}{2} + 1$ then $u \in L^\infty(\Omega_T)$,

(d) if $m_1 \leq m < \frac{N}{2} + 1$ then $u^{\frac{\gamma+1}{2}} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ where $\gamma = \frac{Nm(1+\sigma)-N+2m-2}{N-2m+2}$ provided that $\frac{4\gamma}{(\gamma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$.

Remark 4.2.2 *1. Observe that since $0 \leq \sigma \leq 1$ and $N \geq 3$ one has $1 \leq m_1 := \frac{2N}{2N-(1-\sigma)(N-2)} < \frac{N}{2} + 1$.*

2. If $\sigma \geq 1$ and $1 \leq m < \frac{N}{2} + 1$ then $\gamma \geq m\sigma \geq 1$.

3. If $0 \leq \sigma \leq 1$ and $m_1 \leq m < \frac{N}{2} + 1$ then $\gamma \geq m\sigma \geq 0$.

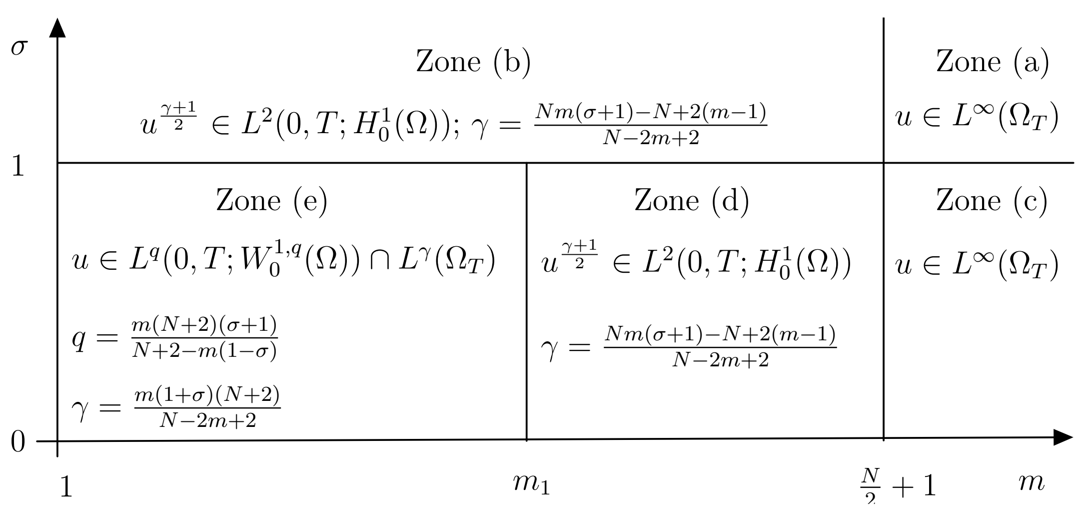
4. *Notice that* $0 \leq \frac{4\gamma}{(\gamma+1)^2} \leq 1$ *and since* $\mu < \Lambda_{N,2}$ *the assumption* $\frac{4\gamma}{(\gamma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$ *is necessary in order to get the results stated in Theorem 4.2.4.*

In the case where $0 \leq \sigma \leq 1$, the regularity results obtained in the previous Theorem 4.2.4 concerns the weak solutions corresponding to data $f \in L^m(\Omega_T)$, with $m \geq m_1$. When we decrease the summability of the data that is $f \in L^m(\Omega_T)$, with $1 < m < m_1$, we obtain solutions lying in a bigger space than the energy one. Actually, we have the following result.

Theorem 4.2.5 *Let* Ω *be a bounded open subset of* R^N, $N \geq 3$, *containing the origin. Assume that (4.3) holds and* $f \in L^m(\Omega_T)$, *with* $1 < m < m_1$ *and suppose that* $0 \leq \sigma < 1$ *and* $\mu < \Lambda_{N,2}$. *Then the problem (1.1) has a positive weak solution* $u \in L^q(0,T; W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$, *with* $q = \frac{m(N+2)(1+\sigma)}{N+2-m(1-\sigma)}$ *and* $\gamma = \frac{m(1+\sigma)(N+2)}{N-2m+2}$.

Remark 4.2.3 *We point out that for the particular case* $\sigma = 0$ *we obtain that the solution* u *belongs to* $L^q(0,T; W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$ *with* $q = \frac{m(N+2)}{N+2-m}$ *and* $\gamma = \frac{m(N+2)}{N-2m+2}$. *These are exactly the same exponents as those obtained in nonsingular case in [18, Theorem 1.9] when* $f \in L^{m_3}(\Omega_T)$, $m_3 := \frac{2(N+2)}{2(N+2)-N}$. *Observe that since for* $\sigma = 0$ *we have* $m_1 = \frac{2N}{N+2} < m_3$, *the result we prove is a refinement of the one in [18, Theorem 1.9]. This is not surprising since the effect of Hardy's potential vanishes for* $\mu < \Lambda_{N,2}$ *as it is shown in the the proof of Theorem 4.2.5. Remark that we cannot consider the case where* $\sigma = 0$ *and* $m = 1$, *since the test functions we use in order to obtain the regularity stated in Theorem 4.2.5 cannot be chosen.*

The following figure summarizes the previous regularity results considering the singularity in function of the summability of the source term f.



4.2.5 Uniqueness of finite energy solutions

As far as the uniqueness is concerned, we give the following result for the finite energy solutions in the case of data with compact support.

Theorem 4.2.6 *Let Ω be a bounded open subset of R^N, $N \geq 3$, containing the origin. Suppose that (4.3) is fulfilled, $\mu < \Lambda_{N,2}$ and $\sigma \geq 0$. If $f \in L^m(\Omega_T)$, with $m \geq 1$ and supp$(f) \subset\subset \Omega_T$ then the energy solution $u \in L^2(0,T;H_0^1(\Omega))$ of the problem (4.1) is unique.*

4.3 Proofs of the Results

4.3.1 Approximate problems

Let us consider the following sequence of approximate schemes

$$\begin{cases} \partial_t u_n - \Delta u_n &= \mu \frac{T_n(u_n)}{|x|^2+\frac{1}{n}} + \frac{f_n}{(|u_n|+\frac{1}{n})^\sigma} \quad \text{in } \Omega \times (0,T), \\ u_n(x,t) &> 0 \quad \text{in } \Omega \times (0,T), \\ u_n(x,t) &= 0 \quad \text{in } \partial\Omega \times (0,T), \\ u_n(x,0) &= u_0(x) \quad \text{in } \Omega, \end{cases} \tag{4.8}$$

where $f_n = T_n(f) = \min(f,n)$. The case $\sigma = 0$ leads to the variational framework since $m_1 = \frac{2N}{N+2}$ is the Hölder conjugate exponent of the Sobolev exponent $2^* := \frac{2N}{N-2}$ and then by the Sobolev embedding and a duality argument we obtain $f \in L^{m_1}(\Omega_T) \hookrightarrow L^2(0,T;H^{-1}(\Omega))$ and the existence of u_n can be found in [59]. If $0 < \sigma \leq 1$, the proof of the existence of a solution u_n to the approximate problem (4.8), which is based on the fixed point theorem of Schauder, is now classical. For the convenience of the reader we give it here.

Lemma 4.3.1 *Assume that $0 < \sigma \leq 1$ and $\mu \leq \Lambda_{N,2}$. For each integer $n \in \mathrm{N}$ the approximate problem (4.8) has a nonnegative solution $u_n \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ such that $\partial_t u_n \in L^2(0,T;H^{-1}(\Omega))$.*

Proof 4.3.1 *Let $v \in L^2(\Omega_T)$ and let $n \in \mathrm{N}$ be fixed. We consider $w \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega)) \cap L^\infty(\Omega_T)$ the unique solution (depending* *on v and n) of the following problem*

$$\begin{cases} \partial_t w - \Delta w &= \mu \frac{T_n(w)}{|x|^2+\frac{1}{n}} + \frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}} \quad \text{in } \Omega_T \\ w(x,t) &= 0 \quad \text{in } \partial\Omega \times (0,T) \\ w(x,0) &= u_0(x) \quad \text{in } \Omega. \end{cases} \tag{4.9}$$

The existence of w can be found in [59]. Let us consider the map S defined by $S(v) = w$. Taking w as test function in (4.9) we get

$$\|\nabla w\|^2_{L^2(\Omega_T)} \leq \mu \int_{\Omega_T} \frac{T_n(w)w}{|x|^2+\frac{1}{n}} dxdt + \int_{\Omega_T} \frac{f_n w}{\left(|v|+\frac{1}{n}\right)^{\sigma}} dxdt + \|u_0\|^2_{L^2(\Omega)}.$$

Thus, by the Hölder inequality we arrive at

$$\|\nabla w\|^2_{L^2(\Omega_T)} \leq |\Omega_T|^{\frac{1}{2}} \left(\mu n^2 + n^{\sigma+1}\right) \Big(\int_{\Omega_T} w^2 dxdt \Big)^{\frac{1}{2}} + \|u_0\|^2_{L^2(\Omega)}.$$

The Poincaré inequality yields

$$\|w\|^2_{L^2(\Omega_T)} \leq C_1 \|w\|_{L^2(\Omega_T)} + C_2,$$

where $C_1 = C_P^2 |\Omega_T|^{\frac{1}{2}} (\mu n^2 + n^{\sigma+1})$, $C_2 = C_P^2 \|u_0\|^2_{L^2(\Omega)}$ and C_p is the constant in the Poincaré inequality. Therefore by the Young inequality we obtain

$$\|w\|_{L^2(\Omega_T)} \leq C := \sqrt{C_1^2 + 2C_2}.$$

Defining the ball $B = \Big\{ v \in L^2(\Omega_T) : \|v\|_{L^2(\Omega_T)} \leq C \Big\}$ of $L^2(\Omega_T)$ we have proved that the map $S : L^2(\Omega_T) \to L^2(\Omega_T)$ is well defined. In order to apply Schauder's Point fixed Theorem over S to guarantee the existence of solution of (4.8), we need to check that the map S is continuous and compact.

First, we prove the continuity of S. In order to do this, let $\{v_k\}_k \subset L^2(\Omega_T)$ be a sequence such that

$$\lim_{k \to +\infty} \|v_k - v\|_{L^2(\Omega_T)} = 0.$$

Denote by $w_k := S(v_k)$ and $w := S(v)$. Then w_k is the solution of

$$\begin{cases} \partial_t w_k - \Delta w_k &= \mu \frac{T_n(w_k)}{|x|^2+\frac{1}{n}} + \frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}} \quad \text{in } \Omega_T \\ w_k(x,t) &= 0 \quad \text{in } \partial\Omega \times (0,T) \\ w_k(x,0) &= u_0(x) \quad \text{in } \Omega, \end{cases} \tag{4.10}$$

we will prove that

$$\lim_{k\to+\infty}\|w_k-w\|_{L^2((\Omega_T)}=0.$$

We point out that up to a subsequence, we can assume that $v_k\to v$ *a.e. in* Ω_T. *So that one has* $\frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}}$ *converges to* $\frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}}$ *a.e. in* Ω_T. *Furthermore, since*

$$\frac{|f_n|}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}}\le n^{\sigma+1},$$

by the dominated convergence theorem we obtain

$$\frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}}\to\frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}}\ \text{in } L^2(\Omega_T). \tag{4.11}$$

Thus, testing by w_k-w *in the difference equations solved by* w_k *and* w *and using the fact that* $w_k(x,0)=w(x,0)=u_0$ *and the Hölder inequality, to have*

$$\frac{1}{2}\int_\Omega((w_k(x,T)-w(x,T)))^2dx+\int_{\Omega_T}|\nabla(w_k-w)|^2dxdt-\mu\int_{\Omega_T}\frac{(w_k-w)^2}{|x|^2+\frac{1}{n}}dxdt$$
$$\le\left(\int_{\Omega_T}\left|\frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}}-\frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}}\right|^2dxdt\right)^{\frac{1}{2}}\|w_k-w\|_{L^2(\Omega_T)}.$$

If $\mu<\Lambda_{N,2}$ *then by the Poincaré inequality we obtain*

$$\left(1-\frac{\mu}{\Lambda_{N,2}}\right)\|w_k-w\|_{L^2(\Omega_T)}\le C_p^2\left(\int_{\Omega_T}\left|\frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}}-\frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}}\right|^2dxdt\right)^{\frac{1}{2}},$$

where C_p *is the Poincaré constant. While if* $\mu=\Lambda_{N,2}$ *then by [85, Theorem 2.1] there exists a constant* $C(\Omega)>0$ *such that*

$$C(\Omega)\|w_k-w\|_{L^2(\Omega_T)}\le\left(\int_{\Omega_T}\left|\frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}}-\frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}}\right|^2dxdt\right)^{\frac{1}{2}}.$$

Having in mind (4.11) we conclude that the sequence $\{w_k\}_k$ *converges to* w *in* $L^2(\Omega_T)$ *and so* S *is continuous.*

We turn now to prove that S *is compact. Let* $\{v_k\}_{k\in\mathbb{N}}$ *be a sequence such that* $\|v_k\|_{L^2(\Omega_T)}\le c$. *Testing with* $w_k=S(v_k)$ *in (4.10) and using (4.4) and the Hölder inequality we arrive at*

$$\|w_k\|^2_{L^2(0,T;H_0^1(\Omega))}\le|\Omega_T|^{\frac{1}{2}}\left(\mu n^2+n^{\sigma+1}\right)\left(\int_{\Omega_T}w_k^2dxdt\right)^{\frac{1}{2}}+\|u_0\|^2_{L^2(\Omega)}.$$

By the Poincaré and Young inequalities we obtain

$$\|w_k\|_{L^2(0,T;H_0^1(\Omega))}\le C, \tag{4.12}$$

where C is a positive constant independent of k. Since the ball B is invariant under S, one has $w_k \in B$. By (4.12) the sequence $\{w_k\}_k$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$ and then so is the sequence $\{\partial_t w_k\}_k$ in $L^1(0,T;H^{-1}(\Omega))$. Therefore, by [78, Corollary 4] there exists a subsequence of $\{w_k\}_{k\in\mathbb{N}}$, still indexed by k, that strongly converges to some limit function $\bar{w} \in L^2(\Omega_T)$. Because of the continuity of S we get $\bar{w} = S(v)$ and so S is compact. Given these conditions on S, Schauder's Fixed point Theorem provides the existence of $u_n \in L^2(0,T;H_0^1(\Omega))$ such that u_n solves (4.8).

Moreover, inserting $-u_n^-$ as test function in (4.8) yields $u_n \geq 0$ and from Lemma 4.4.5 (in Appendix) we conclude that $\{u_n\}_n$ is an increasing sequence in n.

4.3.2 Proof of Theorem 4.2.1

The main argument is to get a priori estimates on $\{u_n\}_n$ and then to pass to the limit as $n \to +\infty$. We divide the proof in four cases, the case where $\sigma = 1$, the case $\sigma < 1$, the case $\sigma > 1$ and the case $\sigma > 1$ with $supp(f) \subset\subset \Omega_T$.

Case 1 : $\sigma = 1$.

Taking $u_n\chi(0,\tau)(t)$ as test function in (4.8), with $0 \leq \tau \leq T$, we get

$$\frac{1}{2}\int_\Omega (u_n(x,\tau))^2 dx + \int_0^\tau \int_\Omega |\nabla u_n|^2 dxdt \leq \mu \int_0^\tau \int_\Omega \frac{u_n^2}{|x|^2 + \frac{1}{n}} dxdt$$
$$+ \int_{\Omega_T} f dxdt + \|u_0\|^2_{L^2(\Omega)}.$$

Then, by using (4.4) we obtain

$$\frac{1}{2}\int_\Omega (u_n(x,\tau))^2 dx + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right) \int_0^\tau \int_\Omega |\nabla u_n|^2 dxdt \leq \|f\|_{L^1(\Omega_T)} + \|u_0\|^2_{L^2(\Omega)}.$$

Passing to the supremum in $\tau \in [0,T]$, we obtain

$$\frac{1}{2}\sup_{0\leq\tau\leq T}\int_\Omega (u_n(x,\tau))^2 dx + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right) \int_{\Omega_T} |\nabla u_n|^2 dxdt \leq \|f\|_{L^1(\Omega_T)} + \|u_0\|^2_{L^2(\Omega)}.$$

This shows that the sequence $\{u_n\}_n$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Then, there exist a subsequence of $\{u_n\}_n$ still indexed by n and a function $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^2(0,T;H_0^1(\Omega))$. Moreover, the boundedness of $\{\partial_t u_n\}_n$ in $L^2(0,T;H^{-1}(\Omega))$ implies that the sequence $\{u_n\}_n$ is relatively compact in $L^1(\Omega_T)$ (see [78, Corollary 4]) and hence for a subsequence, indexed again by n, we have $u_n \to u$ a.e. in Ω_T.

Let $\phi \in C_0^\infty(\Omega \times [0,T))$. Using ϕ as test function in (4.8) we obtain

$$\begin{aligned} &-\int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u_n \partial_t \phi dtdx + \int_{\Omega_T} \nabla u_n \cdot \nabla \phi dxdt \\ &= \mu \int_{\Omega_T} \frac{T_n(u_n)\phi}{|x|^2 + \frac{1}{n}} dxdt + \int_{\Omega_T} \frac{f_n \phi}{|u_n| + \frac{1}{n}} dxdt. \end{aligned} \tag{4.13}$$

Notice that since $u_n \rightharpoonup u$ weakly in $L^2(0,T; H_0^1(\Omega))$, we immediately have

$$\lim_{n\to+\infty} \int_{\Omega_T} \nabla u_n \cdot \nabla \phi dxdt = \int_{\Omega_T} \nabla u \cdot \nabla \phi dxdt$$

and

$$\lim_{n\to+\infty} \int_{\Omega_T} u_n \partial_t \phi dtdx = \int_{\Omega_T} u \partial_t \phi dtdx.$$

As regards the first integral in the right-hand side of (4.13), we know that the sequence $\{u_n\}$ is increasing to its limit u so we have

$$\Big| \frac{T_n(u_n)\phi}{|x|^2 + \frac{1}{n}} \Big| \le \frac{|u\phi|}{|x|^2}.$$

Applying Hölder's and Hardy's inequalities we obtain

$$\int_{\Omega_T} \frac{|u\phi|}{|x|^2} dxdt \le \|\phi\|_\infty (\Lambda_{N,2})^{-\frac{1}{2}} \Big(\int_{\Omega_T} |\nabla u|^2 dxdt \Big)^{\frac{1}{2}} \Big(\int_{\Omega_T} \frac{dxdt}{|x|^2} \Big)^{\frac{1}{2}}.$$

As $N \ge 3$ and Ω bounded, a straightforward calculation yields the existence of a positive constant C_1 such that

$$\int_\Omega \frac{dx}{|x|^2} \le C_1. \tag{4.14}$$

Therefore, the function $\frac{|u\phi|}{|x|^2}$ lies in $L^1(\Omega_T)$ and since $\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}} \to \frac{u\phi}{|x|^2}$ a.e. in Ω_T the Lebesgue dominated convergence theorem gives

$$\lim_{n\to+\infty} \int_{\Omega_T} \frac{T_n(u_n)\phi}{|x|^2} dxdt = \int_{\Omega_T} \frac{u\phi}{|x|^2} dxdt.$$

On the other hand, the support $supp(\phi)$ of the function ϕ is a compact subset of Ω_T and so by Lemma 4.4.4 (in Appendix) there exists a constant $C_{supp(\phi)} > 0$ such that $u_n \ge C_{supp(\phi)}$ in $supp(\phi)$. Then,

$$\Big| \frac{f_n \phi}{u_n + \frac{1}{n}} \Big| \le \frac{\|\phi\|_\infty}{C_{supp(\phi)}} |f| \in L^1(\Omega_T).$$

So that by the Lebesgue dominated convergence theorem we can get

$$\lim_{n\to+\infty} \int_{\Omega_T} \frac{f_n \phi}{u_n + \frac{1}{n}} dxdt = \int_{\Omega_T} \frac{f\phi}{u} dxdt.$$

Now passing to the limit as n tends to ∞ in (4.13) we obtain

$$-\int_\Omega u_0 \phi(x,0)dx - \int_{\Omega_T} u \partial_t \phi dtdx + \int_{\Omega_T} \nabla u \cdot \nabla \phi dxdt = \mu \int_{\Omega_T} \frac{u\phi}{|x|^2} dxdt + \int_{\Omega_T} \frac{f\phi}{u} dxdt$$

for all $\phi \in C_0^\infty(\Omega_T)$, namely u is a finite energy solution to (4.1).

Case 2 : $\sigma < 1$.

For $\tau \in (0,T)$ let us use as a test function in (4.8) the function $u_n\chi_{(0,\tau)}(t)$ which belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. By Hölder's inequality and (4.4) we arrive at

$$\begin{aligned}&\frac{1}{2}\int_\Omega |u_n(x,\tau)|^2dx + \Big(1-\frac{\mu}{\Lambda_{N,2}}\Big)\int_0^\tau\int_\Omega |\nabla u_n|^2dxdt\\ &\leq \|f\|_{L^{m_1}(\Omega_T)}\Big(\int_0^\tau\int_\Omega |u_n|^{(1-\sigma)m_1'}dxdt\Big)^{\frac{1}{m_1'}} + \frac{1}{2}\|u_0\|_{L^2(\Omega)},\end{aligned}$$

where $m_1 := \frac{2N}{2N-(1-\sigma)(N-2)}$ and $m_1' := \frac{m_1}{m_1-1}$. Setting $2^* := \frac{2N}{N-2}$ one has $(1-\sigma)m_1' = 2^*$. By Sobolev's inequality there exists a positive constant C such that

$$\begin{aligned}&\frac{1}{2}\int_\Omega |u_n(x,\tau)|^2dx + \Big(1-\frac{\mu}{\Lambda_{N,2}}\Big)\|u_n\|^2_{L^2(0,\tau;H_0^1(\Omega))}\\ &\leq C\|f\|_{L^{m_1}(\Omega_T)}\|u_n\|^{1-\sigma}_{L^2(0,\tau;H_0^1(\Omega))} + \frac{1}{2}\|u_0\|_{L^2(\Omega)}.\end{aligned}$$

For every real numbers a, $b \geq 0$ and for every $\epsilon > 0$, by Young's inequality we have

$$ab \leq \epsilon a^p + C_\epsilon b^q, \tag{4.15}$$

where $p \geq 1$ and $q \geq 1$ are such that $1 = \frac{1}{p} + \frac{1}{q}$. Since $\frac{2^*}{m_1'} = 1-\sigma < 2$ we apply (4.15) with $a = \|u_n\|^{\frac{2^*}{m_1'}}_{L^2(0,\tau;H_0^1(\Omega))}$, $b = C\|f\|_{L^{m_1}(\Omega_T)}$, $p = \frac{2m_1'}{2^*}$ and $q = \frac{2m_1'}{2m_1'-2^*}$, to get

$$\begin{aligned}&\frac{1}{2}\int_\Omega |u_n(x,\tau)|^2dx + \Big(1-\frac{\mu}{\Lambda_{N,2}}-\epsilon\Big)\|u_n\|^2_{L^2(0,\tau;H_0^1(\Omega))}\\ &\leq C_\epsilon(C\|f\|_{L^{m_1}(\Omega_T)})^{\frac{2m_1'}{2m_1'-2^*}} + \frac{1}{2}\|u_0\|_{L^2(\Omega)}.\end{aligned}$$

Choosing ϵ such that $1-\frac{\mu}{\Lambda_{N,2}}-\epsilon > 0$ and passing to the supremum in $\tau \in [0,T]$ we obtain

$$\frac{1}{2}\sup_{0\leq\tau\leq T}\int_\Omega (u_n(x,\tau))^2dx + \Big(1-\frac{\mu}{\Lambda_{N,2}}-\epsilon\Big)\int_{\Omega_T}|\nabla u_n|^2dxdt \leq C_3,$$

with $C_3 = C_\epsilon(C\|f\|_{L^{m_1}(\Omega_T)})^{\frac{2m_1'}{2m_1'-2^*}} + \frac{1}{2}\|u_0\|_{L^2(\Omega)}$. Therefore, the sequence $\{u_n\}_n$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$. Thus there exist a subsequence of $\{u_n\}_n$, still labelled by n, and a function $u \in L^2(0,T;H_0^1(\Omega))$ such that

$$u_n \rightharpoonup u \text{ weakly in } L^2(0,T;H_0^1(\Omega)).$$

Now we shall prove that u is a weak solution of (4.1). For this, let us insert as test function in (4.8) an arbitrary function $\phi \in C_0^\infty(\Omega\times[0,T))$.

$$\begin{aligned}&-\int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u_n\partial_t\phi dtdx + \int_{\Omega_T}\nabla u\cdot\nabla\phi dxdt\\ &= \mu\int_{\Omega_T}\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}dxdt + \int_{\Omega_T}\frac{f_n\phi}{(u_n+\frac{1}{n})^\sigma}dxdt.\end{aligned}$$

As in the first case, we can pass to the limit in the above equality to conclude that u is a finite energy solution of (4.1).

Case 3 : $\sigma > 1$.

In order to prove that $\{u_n\}_n$ is uniformly bounded in $L^2(0,T;H^1_{loc}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, we will prove that the sequence $G_k(u_n)$ is uniformly bounded in $L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and $T_k(u_n)$ is uniformly bounded in $L^2(0,T;H^1_{loc}(\Omega)) \cap L^\infty(0,T;L^{\sigma+1}(\Omega))$. Let us first prove that $G_k(u_n)$ is uniformly bounded in $L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Inserting $G_k(u_n)\chi_{(0,\tau)}$, with $0 \le \tau \le T$, as a test function in (4.8) we obtain

$$\begin{aligned}&\int_0^\tau \int_\Omega \partial_t u_n G_k(u_n)dxdt + \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt \\ &= \mu \int_{\Omega_\tau} \frac{T_n(u_n)G_k(u_n)}{|x|^2+\frac{1}{n}}dxdt + \int_{\Omega_\tau} \frac{f_n G_k(u_n)}{(u_n+\frac{1}{n})^\sigma}dxdt \\ &\le \mu \int_{\Omega_\tau} \frac{u_n G_k(u_n)}{|x|^2}dxdt + \int_{\Omega_T} \frac{f_n G_k(u_n)}{(u_n+\frac{1}{n})^\sigma}dxdt.\end{aligned} \tag{4.16}$$

Observe that the function $G_k(u_n)$ is different from zero only on the set $B_{n,k} := \Big\{(x,t) \in \Omega_\tau : u_n(x,t) > k\Big\}$, and so we have

$$\begin{aligned}\int_0^\tau \int_\Omega \partial_t u_n G_k(u_n)dxdt &= \frac{1}{2}\int_{B_{n,k}} \partial_t (u_n-k)^2 dxdt = \frac{1}{2}\int_{\Omega_\tau} \partial_t (G_k(u_n(x,\tau)))^2 dxdt \\ &= \frac{1}{2}\int_\Omega (G_k(u_n(x,\tau)))^2 dx - \frac{1}{2}\int_\Omega (G_k(u_n(x,0)))^2 dx.\end{aligned}$$

Since $\displaystyle\int_\Omega (G_k(u_n(x,0)))^2 dx \le \int_\Omega (u_0(x))^2 dx$ and $u_n + \frac{1}{n} \ge k$ on $B_{n,k}$ inequality (4.16) becomes

$$\begin{aligned}\frac{1}{2}\int_\Omega (G_k(u_n(x,\tau)))^2 dx &+ \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt \\ &\le \mu \int_{\Omega_\tau} \frac{u_n G_k(u_n)}{|x|^2}dxdt + C_4,\end{aligned}$$

with $C_4 = \|u_0\|^2_{L^2(\Omega)} + \frac{1}{k^{\sigma-1}}\|f\|_{L^1(\Omega_T)}$. Moreover, since $u_n G_k(u_n) = (G_k(u_n))^2 + kG_k(u_n)$ on the set $B_{n,k}$ we get

$$\begin{aligned}&\frac{1}{2}\int_\Omega (G_k(u_n(x,\tau)))^2 dx + \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt - \mu \int_{\Omega_\tau} \frac{(G_k(u_n))^2}{|x|^2}dxdt \\ &\le \mu k \int_{\Omega_\tau} \frac{G_k(u_n)}{|x|^2}dxdt + C_4.\end{aligned}$$

Taking into account that $\mu < \Lambda_{N,2}$ by (4.4) we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega (G_k(u_n(x,\tau)))^2 dx + \Big(1 - \frac{\mu}{\Lambda_{N,2}}\Big)\int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt \\ &\le \mu k \int_{\Omega_\tau} \frac{G_k(u_n(x,t))}{|x|^2}dxdt + C_4.\end{aligned} \tag{4.17}$$

We shall now estimate the term $\mu k\displaystyle\int_{\Omega_\tau}\frac{G_k(u_n(x,t))}{|x|^2}dxdt$. Let us fix α such that $1<\alpha<2$ and set $\beta=\frac{\alpha}{\alpha-1}$. By Young's inequality we can write

$$\mu k\int_{\Omega_\tau}\frac{G_k(u_n)}{|x|^2}dxdt\leq\frac{\mu}{\alpha}\int_{\Omega_\tau}\frac{(G_k(u_n))^\alpha}{|x|^2}dxdt+\frac{\mu}{\beta}\int_{\Omega_\tau}\frac{k^\beta}{|x|^2}dxdt.$$

Having in mind (4.14) we have

$$\mu k\int_{\Omega_\tau}\frac{G_k(u_n)}{|x|^2}dxdt\leq\mu\int_{\Omega_\tau}\frac{(G_k(u_n))^\alpha}{|x|^2}dxdt+C_5,$$

where $C_5=\frac{C_1\mu k^\beta}{\beta}$. Then the Hölder inequality yields

$$\begin{aligned}\mu k\int_{\Omega_\tau}\frac{G_k(u_n)}{|x|^2}dxdt\ &\leq\mu\Big(\int_{\Omega_\tau}\frac{(G_k(u_n))^2}{|x|^2}dxdt\Big)^{\frac{\alpha}{2}}\Big(\int_{\Omega_\tau}\frac{dxdt}{|x|^2}\Big)^{\frac{2-\alpha}{2}}+C_5\\ &\leq C_6\Big(\int_{\Omega_\tau}\frac{(G_k(u_n))^2}{|x|^2}dxdt\Big)^{\frac{\alpha}{2}}+C_5,\end{aligned}$$

where $C_6=\mu C_1^{\frac{2-\alpha}{2}}$ and by (4.4) we obtain

$$\mu k\int_{\Omega_\tau}\frac{G_k(u_n(x,t))}{|x|^2}dxdt\leq C_7\Big(\int_{\Omega_\tau}|\nabla G_k(u_n)|^2dxdt\Big)^{\frac{\alpha}{2}}+C_5,$$

where $C_7=\frac{C_6}{\Lambda_{N,2}}$. For arbitrary $\epsilon>0$, applying the Young inequality (4.15) with $a=\displaystyle\int_{\Omega_\tau}|\nabla G_k(u_n)|^2dxdt$, $b=C_7$ and $p=\frac{2}{\alpha}$, we get

$$\mu k\int_{\Omega_\tau}\frac{G_k(u_n(x,t))}{|x|^2}dxdt\leq\epsilon\int_{\Omega_\tau}|\nabla G_k(u_n)|^2dxdt+C_8,\tag{4.18}$$

where $C_8=C_5+C_\epsilon C_7^{\frac{2-\alpha}{2}}$. Choosing ϵ such that $1-\frac{\mu}{\Lambda_{N,2}}-\epsilon>0$ and gathering (4.17) and (4.18), we deduce that

$$\frac{1}{2}\int_\Omega(G_k(u_n(x,\tau)))^2dx+\Big(1-\frac{\mu}{\Lambda_{N,2}}-\epsilon\Big)\int_{\Omega_\tau}|\nabla G_k(u_n)|^2dxdt\leq C_9,\tag{4.19}$$

where $C_9=C_8+C_4$. Passing to the supremum in $\tau\in[0,T]$, we conclude that the sequence $\{G_k(u_n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$.

We now turn to prove that the sequence $\{T_k(u_n)\}_n$ is uniformly bounded in $L^2(0,T;H^1_{loc}(\Omega))\cap L^\infty(0,T;L^{\sigma+1}(\Omega))$. Using $T_k^\sigma(u_n)\chi_{(0,\tau)}$, $0\leq\tau\leq T$, as a test function in (4.8) we obtain

$$\begin{aligned}&\frac{1}{\sigma+1}\int_\Omega(T_k(u_n(x,\tau)))^{\sigma+1}dx+\int_{\Omega_\tau}(T_k(u_n))^{\sigma-1}|\nabla T_k(u_n)|^2dxdt\\ &\leq k^{\sigma-1}\mu\int_{\Omega_T}\frac{u_n^2}{|x|^2}dxdt+\int_{\Omega_T}fdxdt+\frac{1}{\sigma+1}\|u_0\|_{L^{\sigma+1}(\Omega)},\end{aligned}\tag{4.20}$$

where we have dropped $\sigma>1$ in the second integral on the left-hand side and written $T_k^\sigma(u_n)=T_k^{\sigma-1}(u_n)T_k(u_n)$ in the first integral on the right-hand side of the inequality. As

$u_n = T_k(u_n) + G_k(u_n)$, the the first term on the right-hand side of the above inequality can be estimated as

$$\begin{aligned}\int_{\Omega_T} \frac{u_n^2}{|x|^2} dxdt &= \int_{\Omega_T} \frac{(T_k(u_n))^2}{|x|^2} dxdt + \int_{\Omega_T} \frac{(G_k(u_n))^2}{|x|^2} dxdt \\ &+2 \int_{\Omega_T} \frac{T_k(u_n) G_k(u_n)}{|x|^2} dxdt \\ &\leq k^2 \int_{\Omega_T} \frac{dxdt}{|x|^2} + \int_{\Omega_T} \frac{(G_k(u_n))^2}{|x|^2} dxdt + 2k \int_{\Omega_T} \frac{G_k(u_n)}{|x|^2} dxdt.\end{aligned}$$

So that by (4.4), (4.14), (4.18) and (4.19) there exists a real constant $C_{10} > 0$ such that

$$\int_{\Omega_T} \frac{u_n^2}{|x|^2} dxdt \leq C_{10}.$$

Then, it follows that the inequality (4.20) reads as

$$\tfrac{1}{\sigma+1} \int_{\Omega} (T_k(u_n(x,\tau)))^{\sigma+1} dx + \int_{\Omega_\tau} (T_k(u_n))^{\sigma-1} |\nabla T_k(u_n)|^2 dxdt \leq C_{11}, \tag{4.21}$$

with $C_{11} = k^{\sigma-1} \mu C_{10} + \|f\|_{L^1(\Omega_T)} + \frac{1}{\sigma+1} \|u_0\|_{L^{\sigma+1}(\Omega)}$. On the other hand, let $\Omega' \subset\subset \Omega$. By Lemma 4.4.4 (in Appendix) there exists $C_{\Omega'} > 0$ such that

$$T_k(u_n(x,t)) \geq C_0 := \min\{k, C_{\Omega'}\}, \tag{4.22}$$

for all $(x,t) \in \Omega' \times [0,T]$. Thus, by (4.21) and (4.22) we get

$$\tfrac{1}{\sigma+1} \int_{\Omega} (T_k(u_n(x,\tau)))^{\sigma+1} dx + C_0^{\sigma-1} \int_0^\tau \int_{\Omega'} |\nabla T_k(u_n)|^2 dxdt \quad \leq C_{11}$$

Passing to the supremum in $\tau \in [0,T]$, we get that the sequence $\{T_k(u_n)\}_{n\in\mathrm{N}}$ is uniformly bounded in $L^2(0,T; H^1_{loc}(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$. Therefore, we conclude that the sequence $\{u_n\}_{n\in\mathrm{N}}$ is uniformly bounded in $L^2(0,T; H^1_{loc}(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$. As a consequence, there exist a subsequence of $\{u_n\}_{n\in\mathrm{N}}$, relabeled again by n, and a function $u \in L^2(0,T; H^1_{loc}(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^2(0,T; H^1_{loc}(\Omega))$. Inserting an arbitrary $\phi \in C_0^\infty(\Omega \times [0,T))$ as a test function in (4.8), it follows

$$\begin{aligned}&-\int_{\Omega} u_0(x)\phi(x,0) dx - \int_{\Omega_T} u_n \partial_t \phi dtdx + \int_{\Omega_T} \nabla u_n \cdot \nabla \phi dxdt \\ &= \mu \int_{\Omega_T} \frac{T_n(u_n)\phi}{|x|^2 + \frac{1}{n}} dxdt + \int_{\Omega_T} \frac{f_n \phi}{(u_n + \frac{1}{n})^\sigma} dxdt.\end{aligned}$$

Then the passage to the limit as in the first case shows that u is a weak solution of (4.1).

Now assume that $\sigma > 1$ is such that $\frac{4\sigma}{(\sigma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$. For $0 \leq \tau \leq T$ let us use $u_n^\sigma \chi_{(0,\tau)}(t)$ as a test function in (4.8). By the Hardy inequality (4.4) we arrive at

$$\frac{1}{\sigma+1} \int_{\Omega} (u_n(x,\tau))^{\sigma+1} dx + \Big(\frac{4\sigma}{(\sigma+1)^2} - \frac{\mu}{\Lambda_{N,2}}\Big) \int_0^\tau \int_{\Omega} |\nabla u_n^{\frac{\sigma+1}{2}}|^2 dxdt \leq C,$$

where $C = \|f\|_{L^1(\Omega_T)} + \frac{1}{\sigma+1}\|u_0\|_{L^{\sigma+1}(\Omega)}$. Hence following closely the same computations as above, we get $u^{\frac{\sigma+1}{2}} \in L^2(0,T;H_0^1(\Omega))$.

Case 4 : Suppose that $\sigma > 1$ and $supp(f) \subset\subset \Omega_T$.

Testing by $u_n\chi(0,\tau)(t)$ in (4.8) and using (4.4) we get

$$\begin{aligned}&\frac{1}{2}\int_\Omega (u_n(x,\tau))^2dx + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right)\int_0^\tau\int_\Omega |\nabla u_n|^2 dxdt \\ &\leq \int_{\Omega_T} \frac{f}{u_n^{\sigma-1}}dxdt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.\end{aligned}$$

Applying Lemma 4.4.4 (in Appendix) there exists $C > 0$ such that $u_n \geq C$ in $supp(f)$. Whence, passing to the supremum in $\tau \in [0,T]$ we obtain

$$\begin{aligned}&\frac{1}{2}\sup_{0\leq\tau\leq T}\int_\Omega (u_n(x,\tau))^2dx + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right)\int_\Omega |\nabla u_n|^2 dxdt \\ &\leq \frac{1}{C^{\sigma-1}}\int_{supp(f)} f dxdt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.\end{aligned}$$

Thus, the sequence $\{u_n\}_n$ is bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Therefore, there exist a function $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and a subsequence of $\{u_n\}_n$, still indexed by n, such that $u_n \rightharpoonup u$ in $L^2(0,T;H_0^1(\Omega))$ and then u is a finite energy solution of the problem (4.1).

4.3.3 Proof of Theorem 4.2.2

Let $0 \leq \tau \leq T$. Taking $u_n\chi_{(0,\tau)}(t)$ as a test function in (4.8), we get

$$\begin{aligned}&\frac{1}{2}\int_\Omega (u_n(x,\tau))^2dx + \int_0^\tau\int_\Omega |\nabla u_n|^2 dxdt - \Lambda_{N,2}\int_0^\tau\int_\Omega \frac{u_n^2}{|x|^2}dxdt \\ &\leq \|f\|_{L^1(\Omega_T)} + \frac{1}{2}\|u_0\|_{L^2(\Omega)}.\end{aligned}$$

Passing to the supremum in $\tau \in [0,T]$ and using Theorem 4.4.1 (in Appendix) we conclude that the sequence $\{u_n\}_n$ is uniformly bounded in $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, for all $q < 2$. As a consequence, there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a function $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^q(0,T;W_0^{1,q}(\Omega))$. Arguing in a similar way as in the case 1, we conclude that u is a weak solution of the problem (4.1).

4.3.4 Proof of Theorem 4.2.3

Suppose that $\mu > \Lambda_{N,2}$. Arguing by contradiction, assum that (4.1) admits a positive weak solution u. Thus u is also a weak solution to the problem

$$\begin{cases} \partial_t u - \Delta u - \Lambda_{N,2}\frac{u}{|x|^2} &= (\mu - \Lambda_{N,2})\frac{u}{|x|^2} + \frac{f}{u^\sigma} \quad \text{in } \Omega_T, \\ u &> 0 \quad \text{in } \Omega_T, \\ u &= 0 \quad \text{in } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{in } \partial\Omega \times (0,T). \end{cases}$$

By virtue of Lemma 4.4.2 (in Appendix) we have

$$\Big((\mu - \Lambda_{N,2})\frac{u}{|x|^2} + \frac{f}{u^\sigma}\Big)|x|^{-\alpha_1} \in L^1(B_{r_1}(0) \times (t_1,t_2)),$$

for any small enough parabolic cylinder $B_{r_1}(0) \times (t_1,t_2) \subset\subset \Omega_T$ where α_1 is defined in (4.31). As in our equation $\lambda = \Lambda_{N,2}$ we have $\alpha_1 = \frac{N-2}{2}$. Since $u > 0$ and $f \geq 0$ we have in particular

$$(\mu - \Lambda_{N,2})\frac{u}{|x|^2}|x|^{-\frac{N-2}{2}} \in L^1(B_{r_1}(0) \times (t_1,t_2)). \tag{4.23}$$

On the other hand, since

$$\partial_t u - \Delta u - \Lambda_{N,2}\frac{u}{|x|^2} = (\mu - \Lambda_{N,2})\frac{u}{|x|^2} + \frac{f}{u^\sigma} \geq 0$$

by Lemma 4.4.1 (in Appendix) there exists a constant $C > 0$ such that

$$u \geq C|x|^{-\frac{N-2}{2}}. \tag{4.24}$$

Gathering (4.23) and (4.24) we obtain

$$|x|^{-N} \in L^1(B_{r_1}(0) \times (t_1,t_2))$$

which is a contradiction. Therefore, if $\mu > \Lambda_{N,2}$ the problem (4.1) has no positive weak solution.

4.3.5 Proof of Theorem 4.2.4

The proofs of (i) and (ii) are similar. We only give the proof of (i).

• **Proof of** (a) – We shall establish an a priori L^∞-estimate for the solution u_n of (4.8). To do so, we use standard ideas that can be found in several nonsingular cases as for instance in

[25, 31, 80, 86, 90, 95]. Despite being classic, we give the proof for the convenience of the reader. Let $k \geq k_0 := \max(1, \|u_0\|_\infty)$. We choose $G_k(u_n)\chi_{(0,\tau)}$, $0 \leq \tau \leq T$, as a test function in (4.8), we get

$$\int_0^\tau \int_\Omega \partial_t u_n G_k(u_n) dxdt + \int_{A_{k,n}} |\nabla G_k(u_n)|^2 dxdt \\ \leq \mu \int_{A_{k,n}} \frac{u_n G_k(u_n)}{|x|^2} dxdt + \int_{A_{k,n}} \frac{f G_k(u_n)}{(u_n + \frac{1}{n})^\sigma} dxdt,$$

where we have set $A_{k,n} = \{(x,t) \in \Omega_\tau : u_n(x,t) > k\}$. Observe that since $G_k(u_n)$ is different from zero only on the set $A_{k,n}$ and according to the choice of k, one has

$$\int_0^\tau \int_\Omega \partial_t u_n G_k(u_n) dxdt = \frac{1}{2} \int_\Omega G_k(u_n(x,\tau))^2 dx.$$

In addition, on the set $A_{k,n}$ we have $u_n + \frac{1}{n} > k_0$. Thus, using first Hölder's inequality and then Hardy's inequality, we arrive at

$$\frac{1}{2} \int_\Omega G_k(u_n(x,\tau))^2 dx + \int_{A_{k,n}} |\nabla G_k(u_n)|^2 dxdt \\ \leq \frac{\mu}{\Lambda_{N,2}} \int_{A_{k,n}} |\nabla G_k(u_n)|^2 dxdt + \frac{1}{k_0^\sigma} \int_{A_{k,n}} f G_k(u_n) dxdt.$$

Then passing to the supremum in $\tau \in (0,T)$ we obtain

$$\frac{1}{2} \|G_k(u_n)\|^2_{L^\infty(0,T;L^2(\Omega))} + \Big(1 - \frac{\mu}{\Lambda_{N,2}}\Big) \|G_k(u_n)\|^2_{L^2(0,T;H_0^1(\Omega))} \\ \leq \frac{1}{k_0^\sigma} \int_{\Omega_T} f G_k(u_n) dxdt. \tag{4.25}$$

On the other hand, since $G_k(u_n) \in L^\infty(\Omega_T) \cap L^2(0,T;H_0^1(\Omega))$ then
$G_k(u_n) \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Therefore, by [31, Proposition 3.1] there exists a positive constant c such that

$$\int_{\Omega_T} G_k(u_n)^{\frac{2N+4}{N}} dxdt \leq c^{\frac{2N+4}{N}} \Big(\int_{\Omega_T} |\nabla G_k(u_n)|^2 dxdt \Big) \Big(\|G_k(u_n)\|^2_{L^\infty(0,T;L^2(\Omega))} \Big)^{\frac{2}{N}}.$$

Setting $\Gamma_{N,2} := 1 - \frac{\mu}{\Lambda_{N,2}}$ and $C_1 := \frac{c^{\frac{2N+4}{N}} 2^{\frac{2}{N}}}{\Gamma_{N,2} k_0^{\sigma(1+\frac{2}{N})}}$, we obtain using (4.25)

$$\int_{\Omega_T} G_k(u_n)^{\frac{2N+4}{N}} dxdt \leq C_1 \Big(\int_{\Omega_T} f G_k(u_n) dxdt \Big)^{1+\frac{2}{N}}.$$

Observe that both integrals are on the subset $A_{k,n}$. Using Hölder's inequality in the right-hand side term with exponents $\frac{2N+4}{N}$ and $\frac{2N+4}{N+4}$, we get

$$\int_{A_{k,n}} G_k(u_n)^{\frac{2N+4}{N}} dxdt \leq C_1 \Big(\int_{A_{k,n}} f^{\frac{2N+4}{N+4}} dxdt \Big)^{\frac{N+4}{2N}} \Big(\int_{A_{k,n}} G_k(u_n)^{\frac{2N+4}{N}} dxdt \Big)^{\frac{1}{2}},$$

from which it follows

$$\int_{A_{k,n}} G_k(u_n)^{\frac{2N+4}{N}} dxdt \leq C_1^2 \Big(\int_{A_{k,n}} f^{\frac{2N+4}{N+4}} dxdt \Big)^{\frac{N+4}{N}}.$$

Since $f \in L^m(\Omega_T)$ with $m > \frac{N}{2} + 1 > \frac{2N+4}{N+4}$, we use again Hölder's inequality obtaining

$$\int_{A_{k,n}} G_k(u_n)^{\frac{2N+4}{N}} dxdt \leq C_1^2 \|f\|_{L^m(\Omega_T)}^{\frac{2N+4}{N}} |A_{k,n}|^{\frac{N+4}{N} - \frac{2N+4}{mN}}.$$

Now let $h > k$. It's easy to see that $A_{h,n} \subset A_{k,n}$ and $G_k(u_n) \geq h - k$ on $A_{h,n}$, so that one has

$$|A_{h,n}|(h-k)^{\frac{2N+4}{N}} \leq C_1^2 \|f\|_{L^m(\Omega_T)}^{\frac{2N+4}{N}} |A_{k,n}|^{\frac{N+4}{N} - \frac{2N+4}{mN}}.$$

Setting $\psi(k) = |A_{k,n}|$, we get

$$\psi(h) \leq \frac{C_2}{(h-k)^\alpha} \psi(k)^\beta,$$

where $C_2 = C_1^2 \|f\|_{L^m(\Omega_T)}^{\frac{2N+4}{N}}$, $\alpha = \frac{2N+4}{N}$ and $\beta = \frac{N+4}{N} - \frac{2N+4}{mN}$. Since $m > \frac{N}{2} + 1$ we have $\beta > 1$ and then we can apply the first item of [80, Lemma 4.1] to conclude that there exists a constant C_∞, such that $\psi(C_\infty) = 0$, that is

$$\|u_n\|_\infty \leq C_\infty.$$

• **Proof of** (b) – Using $u_n^\gamma \chi_{(0,\tau)}$, $0 < \tau < T$, as a test function in (4.8) and applying the Hölder's inequality and (4.4) we arrive at

$$\begin{aligned} &\frac{1}{\gamma+1} \int_\Omega (u_n(x,\tau))^{\gamma+1} dx + \Big(\gamma \Big(\frac{2}{\gamma+1} \Big)^2 - \frac{\mu}{\Lambda_{N,2}} \Big) \int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt \\ &\leq \|f\|_{L^m(\Omega_T)} \Big(\int_{\Omega_T} u_n^{(\gamma-\sigma)m'} dxdt \Big)^{\frac{1}{m'}} + \|u_0\|_{L^{\gamma+1}(\Omega)}^{\gamma+1}. \end{aligned} \tag{4.26}$$

Note that $\sigma \leq \gamma = \frac{Nm(\sigma+1)-N+2m-2}{N-2m+2}$. Since we have supposed that $\gamma \Big(\frac{2}{\gamma+1} \Big)^2 - \frac{\mu}{\Lambda_{N,2}} > 0$, we discuss the two cases $\sigma = \gamma$ and $\sigma < \gamma$. Thus, if $\sigma = \gamma$ we immediately have

$$\begin{aligned} &\frac{1}{\gamma+1} \int_\Omega (u_n(x,\tau))^{\gamma+1} dx + \Big(\gamma \Big(\frac{2}{\gamma+1} \Big)^2 - \frac{\mu}{\Lambda_{N,2}} \Big) \int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt \\ &\leq |\Omega_T|^{\frac{1}{m'}} \|f\|_{L^m(\Omega_T)} + \|u_0\|_{L^{\gamma+1}(\Omega)}^{\gamma+1}. \end{aligned}$$

While if $\sigma < \gamma$, we compute $(\gamma - \sigma)m' = (\gamma+1)\frac{N+2}{N} < (\gamma+1)\frac{N}{N-2}$. Therefore, applying Hölder's inequality and then the Sobolev inequality in the first integral on the right-hand side of (4.26) we obtain

$$\begin{aligned} &\frac{1}{\gamma+1} \int_\Omega (u_n(x,\tau))^{\gamma+1} dx + \Big(\gamma \Big(\frac{2}{\gamma+1} \Big)^2 - \frac{\mu}{\Lambda_{N,2}} \Big) \|u_n^{\frac{\gamma+1}{2}}\|_{L^2(0,T;H_0^1(\Omega))}^2 \\ &\leq C \|f\|_{L^m(\Omega_T)} \|u_n^{\frac{\gamma+1}{2}}\|_{L^2(0,T;H_0^1(\Omega))}^{\frac{\gamma-\sigma}{\gamma+1}} + \|u_0\|_{L^{\gamma+1}(\Omega)}^{\gamma+1}, \end{aligned}$$

where $C > 0$ is a constant not depending on n. Now we use the Young inequality (4.15) with $a = \|u_n^{\frac{\gamma+1}{2}}\|_{L^2(0,T;H_0^1(\Omega))}^{\frac{\gamma-\sigma}{\gamma+1}}$, $b = C\|f\|_{L^m(\Omega_T)}$, $p = \frac{\gamma+1}{\gamma-\sigma}$ and $q = \frac{\gamma+1}{\sigma+1}$ and then we pass to the supremum over $\tau \in [0, T]$, we obtain

$$\begin{aligned}&\frac{1}{\gamma+1}\|u_n^{\frac{\gamma+1}{2}}\|^2_{L^\infty(0,T;L^2(\Omega))} + \Big(\gamma\Big(\frac{2}{\gamma+1}\Big)^2 - \frac{\mu}{\Lambda_{N,2}} - \epsilon\Big)\|u_n^{\frac{\gamma+1}{2}}\|^2_{L^2(0,T;H_0^1(\Omega))}\\ &\le C_\epsilon(C\|f\|_{L^m})^{\frac{\gamma+1}{\sigma+1}} + \|u_0\|^{\gamma+1}_{L^{\gamma+1}(\Omega)}.\end{aligned}$$

Finally we choose ϵ such that $\gamma\Big(\frac{2}{\gamma+1}\Big)^2 - \frac{\mu}{\Lambda_{N,2}} - \epsilon > 0$. Consequently, in both cases the sequence $\{u_n^{\frac{\gamma+1}{2}}\}_n$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Whence, there exist a subsequence of $\{u_n^{\frac{\gamma+1}{2}}\}_n$, still indexed by n, and a function $v \in L^2(0,T;H_0^1(\Omega))$ such that $u_n^{\frac{\gamma+1}{2}} \rightharpoonup v$ weakly in $L^2(0,T;H_0^1(\Omega))$. Now according to the proof of the second item of Theorem 4.2.1, we know that $u_n \rightharpoonup u$ weakly in $L^2(0,T;H^1_{loc}(\Omega))$ so that identifying almost everywhere the limits one has $v = u^{\frac{\gamma+1}{2}} \in L^2(0,T;H_0^1(\Omega))$.

4.3.6 Proof of Theorem 4.2.5

The ideas we use are standard and we follow the lines of [27, Theorem 4.1, (i)-(b)]. Let us choose $u_n^{2\delta-1}\chi_{(0,\tau)}$, $0 < \tau < T$, as test function in (3.1) where δ is a positive real constant verifying $\frac{1}{2} < \delta < 1$ that will be chosen after few lines. We get

$$\begin{aligned}&\frac{1}{2\delta}\int_\Omega (u_n(x,\tau))^{2\delta}dx + \frac{(2\delta-1)}{\delta^2}\int_{\Omega_\tau}|\nabla u_n^\delta|^2 dxdt\\ &\le \mu\int_{\Omega_\tau}\frac{u_n^{2\delta}}{|x|^2}dxdt + \int_{\Omega_\tau} f u_n^{(2\delta-1-\sigma)}dxdt + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)}.\end{aligned}$$

Passing to the supremum in $\tau \in (0,T)$ and applying Hardy's inequality (4.4) and then Hölder's inequality, we obtain

$$\begin{aligned}&\frac{1}{2\delta}\|u_n^\delta\|^2_{L^\infty(0,T;L^2(\Omega))} + \Big(\frac{2\delta-1}{\delta^2} - \frac{\mu}{\Lambda_{N,2}}\Big)\int_{\Omega_T}|\nabla u_n^\delta|^2 dxdt\\ &\le \int_{\Omega_T} f u_n^{(2\delta-1-\sigma)}dxdt + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)}\\ &\le \|f\|_{L^m(\Omega_T)}\Big(\int_{\Omega_T} u_n^{(2\delta-1-\sigma)m'}dxdt\Big)^{\frac{1}{m'}} + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.27}$$

We point out that the function $\delta \mapsto \frac{2\delta-1}{\delta^2}$ is non increasing and reaches its minimum on $(\frac{1}{2}, 1)$ in 1, that is $\frac{2\delta-1}{\delta^2} > 1$ and as we have assumed $\mu < \Lambda_{N,2}$ we get $\Lambda_\delta := \frac{2\delta-1}{\delta^2} - \frac{\mu}{\Lambda_{N,2}} > 0$. Since $u_n \in L^\infty(\Omega_T) \cap L^2(0,T;H_0^1(\Omega))$ then $u_n \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Thus, by [31, Proposition 3.1] there exists a positive constant c such that

$$\int_{\Omega_T}(u_n^\delta)^{\frac{2N+4}{N}}dxdt \le c^{\frac{2N+4}{N}}\Big(\int_{\Omega_T}|\nabla u_n^\delta|^2 dxdt\Big)\Big(\|u_n^\delta\|^2_{L^\infty(0,T;L^2(\Omega))}\Big)^{\frac{2}{N}}.$$

Then, using (4.27) we obtain

$$\begin{aligned}\int_{\Omega_T} (u_n^\delta)^{\frac{2N+4}{N}} dxdt &\le \frac{(2\delta)^{\frac{2}{N}} c^{\frac{2N+4}{N}}}{\Lambda_\delta}\Big(\|f\|_{L^m(\Omega_T)} \Big(\int_{\Omega_T} u_n^{(2\delta-1-\sigma)m'} dxdt \Big)^{\frac{1}{m'}} \\ &\quad + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)} \Big)^{1+\frac{2}{N}} \\ &\le \frac{(4\delta)^{\frac{2}{N}} c^{\frac{2N+4}{N}}}{\Lambda_\delta}\Big(\|f\|^{1+\frac{2}{N}}_{L^m(\Omega_T)} \Big(\int_{\Omega_T} u_n^{(2\delta-1-\sigma)m'} dxdt \Big)^{\frac{N+2}{Nm'}} \\ &\quad + \frac{1}{(2\delta)^{1+\frac{2}{N}}}\|u_0^\delta\|^{\frac{2N+4}{N}}_{L^2(\Omega)} \Big).\end{aligned}$$

Now we choose δ to be such that $\delta\frac{2N+4}{N} = (2\delta-1-\sigma)m'$, that is $\delta = \frac{mN(1+\sigma)}{2N-4(m-1)}$. Observe that since $1 < m < m_1 < \frac{N}{2}+1$ one has $N-2(m-1) > 0$ and $\delta > \frac{1+\sigma}{2} \ge \frac{1}{2}$. To check the upper bound on δ, we notice that $\delta < 1$ is equivalent to $m < \frac{2N+4}{N(1+\sigma)+4}$. Such an inequality is always satisfied since for $\sigma < 1$ we have $m < m_1 < \frac{2N+4}{N(1+\sigma)+4}$. Therefore, with this choice of δ we obtain

$$\begin{aligned}\|u_n\|^{(2\delta-1-\sigma)m'}_{L^{(2\delta-1-\sigma)m'}(\Omega_T)} &\le \frac{(4\delta)^{\frac{2}{N}} c^{\frac{2N+4}{N}}}{\Lambda_\delta}\|f_n\|^{\frac{2}{N}+1}_{L^m(\Omega_T)}\|u_n\|^{\frac{(N+2)(2\delta-1-\sigma)}{N}}_{L^{(2\delta-1-\sigma)m'}(\Omega_T)} \\ &\quad + \frac{(4\delta)^{\frac{2}{N}} c^{\frac{2N+4}{N}}}{\Lambda_\delta}\frac{1}{(2\delta)^{1+\frac{2}{N}}}\|u_0^\delta\|^{\frac{2N+4}{N}}_{L^2(\Omega)}.\end{aligned}$$

Since $m < \frac{N}{2}+1$ we have

$$(2\delta-1-\sigma)m' > \frac{(N+2)(2\delta-1-\sigma)}{N}$$

and so by virtue of Young's inequality the sequence $\{u_n\}_n$ is uniformly bounded in $L^\gamma(\Omega_T)$ with

$$\gamma = (2\delta-1-\sigma)m' = \frac{m(N+2)(1+\sigma)}{N-2m+2} > 1.$$

Now we shall obtain an estimation on ∇u_n. Notice that from (4.27) we get

$$\Lambda_\delta\delta^2\int_{\Omega_T}\frac{|\nabla u_n|^2}{u_n^{2(1-\delta)}}dxdt \le \|f_n\|^{\frac{2}{N}+1}_{L^m(\Omega_T)}\|u_n\|^{(2\delta-1-\sigma)}_{L^\gamma(\Omega_T)} + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)}$$

and since $\{u_n\}_n$ is uniformly bounded in $L^\gamma(\Omega_T)$, we deduce the existence of a positive constant C, not depending on n, such that

$$\int_{\Omega_T}\frac{|\nabla u_n|^2}{u_n^{2(1-\delta)}}dxdt \le C.$$

Let now $q \ge 1$ be such that $q < 2$. An application of Hölder's inequality with exponents $\frac{2}{q}$ and $\frac{2}{2-q}$ yields

$$\begin{aligned}\int_{\Omega_T}|\nabla u_n|^q dxdt &= \int_{\Omega_T}\frac{|\nabla u_n|^q}{u_n^{q(1-\delta)}}u_n^{q(1-\delta)}dxdt \\ &\le \Big(\int_{\Omega_T}\frac{|\nabla u_n|^2}{u_n^{2(1-\delta)}}dxdt\Big)^{\frac{q}{2}}\Big(\int_{\Omega_T}u_n^{\frac{(1-\delta)2q}{2-q}}dxdt\Big)^{\frac{2-q}{2}} \\ &\le C^{\frac{q}{2}}\Big(\int_{\Omega_T}u_n^{\frac{(1-\delta)2q}{2-q}}dxdt\Big)^{\frac{2-q}{2}}.\end{aligned}$$

Now we impose the condition $\gamma = \frac{(1-\delta)2q}{2-q}$ that gives $q = \frac{m(N+2)(\sigma+1)}{N+2-m(1-\sigma)}$. Observe that $q \geq m(\sigma+1) > 1$ and since $\sigma < 1$ we have $m \leq m_1 < \frac{2N+4}{N(1+\sigma)+4}$ which implies $q < 2$. Thus, the sequence $\{u_n\}_n$ is uniformly bounded in $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$. Therefore, there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a function $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$ such that $u_n \rightharpoonup u$ weakly in $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$ and $u_n \to u$ a.e. in Ω_T. Using $\phi \in C_0^\infty(\Omega \times [0,T))$ as test function in (4.8) we obtain

$$\begin{aligned} &-\int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u_n \partial_t \phi dt dx + \int_{\Omega_T} \nabla u_n \cdot \nabla \phi dx dt \\ &= \mu \int_{\Omega_T} \frac{T_n(u_n)\phi}{|x|^2 + \frac{1}{n}} dx dt + \int_{\Omega_T} \frac{f_n \phi}{|u_n| + \frac{1}{n}} dx dt. \end{aligned} \tag{4.28}$$

Notice that since $u_n \rightharpoonup u$ weakly in $L^q(0,T;W_0^{1,q}(\Omega))$, we immediately have

$$\lim_{n\to+\infty} \int_{\Omega_T} \nabla u_n \cdot \nabla \phi dx dt = \int_{\Omega_T} \nabla u \cdot \nabla \phi dx dt$$

and

$$\lim_{n\to+\infty} \int_{\Omega_T} u_n \partial_t \phi dt dx = \int_{\Omega_T} u \partial_t \phi dt dx.$$

As regards the first integral in the right-hand side of (4.28), we know that the sequence $\{u_n\}$ is increasing to its limit u so we have

$$\Big|\frac{T_n(u_n)\phi}{|x|^2 + \frac{1}{n}}\Big| \leq \frac{|u\phi|}{|x|^2}.$$

Applying Hölder's and Hardy's inequalities with exponents 2δ and $\frac{2\delta}{2\delta-1}$ we obtain

$$\int_{\Omega_T} \frac{|u\phi|}{|x|^2} dx dt \leq \|\phi\|_\infty (\Lambda_{N,2})^{\frac{-1}{2\delta}} \Big(\int_{\Omega_T} |\nabla u^\delta|^2 dx dt\Big)^{\frac{1}{2\delta}} \Big(\int_{\Omega_T} \frac{dx dt}{|x|^2}\Big)^{\frac{2\delta-1}{2\delta}}.$$

From (4.14) and (4.27) we deduce that the sequence $\{u_n^\delta\}$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$ and thus there exist a subsequence of $\{u_n^\delta\}$, still indexed by n, and a function $v \in L^2(0,T;H_0^1(\Omega))$ such that $u_n^\delta \rightharpoonup v$ weakly in $L^2(0,T;H_0^1(\Omega))$ and $u_n^\delta \to v$ a.e. in Ω_T. But we also have $u_n^\delta \rightharpoonup v$ weakly in $L^q(0,T;W_0^{1,q}(\Omega))$ and hence follows $v = u^\delta \in L^2(0,T;H_0^1(\Omega))$. Which shows that the function $\frac{|u\phi|}{|x|^2}$ lies in $L^1(\Omega_T)$. Furthermore, since $\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}} \to \frac{u\phi}{|x|^2}$ a.e. in Ω_T, the Lebesgue dominated convergence theorem gives

$$\lim_{n\to+\infty} \int_{\Omega_T} \frac{T_n(u_n)\phi}{|x|^2} dx dt = \int_{\Omega_T} \frac{u\phi}{|x|^2} dx dt.$$

On the other hand, the support $supp(\phi)$ of the function ϕ is a compact subset of Ω_T and so by Lemma A.5 (in Appendix) there exists a constant $C_{supp(\phi)} > 0$ such that $u_n \geq C_{supp(\phi)}$ in $supp(\phi)$. Then,

$$\Big|\frac{f_n \phi}{u_n + \frac{1}{n}}\Big| \leq \frac{\|\phi\|_\infty}{C_{supp(\phi)}} |f| \in L^1(\Omega_T).$$

So that by the Lebesgue dominated convergence theorem we get

$$\lim_{n\to+\infty}\int_{\Omega_T}\frac{f_n\phi}{u_n+\frac{1}{n}}dxdt=\int_{\Omega_T}\frac{f\phi}{u}dxdt.$$

We point out that we also have $u\geq C_{supp(\phi)}$ in $supp(\phi)$. Now passing to the limit as n tends to ∞ in (4.28) we obtain

$$\begin{aligned}&-\int_{\Omega}u_0\phi(x,0)dx-\int_{\Omega_T}u\partial_t\phi dtdx+\int_{\Omega_T}\nabla u\cdot\nabla\phi dxdt\\&=\mu\int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt+\int_{\Omega_T}\frac{f\phi}{u}dxdt\end{aligned}$$

for all $\phi\in C_0^\infty(\Omega_T)$, namely u is a finite energy solution to (1.1).

4.3.7 Proof of Theorem 4.2.6

Let $u,v\in L^2(0,T;H_0^1(\Omega))$ be two energy solutions of the problem (4.1) corresponding to the same data u_0 satisfying (4.3) and $f\in L^m(\Omega_T)$, $m\geq 1$. Since the datum f is compactly supported in Ω_T, then $\partial_t u\in L^2(0,T;H^{-1}(\Omega))+L^1(\Omega_T)$. Let $k>0$ and $r>k$. The function $T_k((u-v)_+)\in L^2(0,T;H_0^1(\Omega))\cap L^\infty(\Omega_T)$ is an admissible test function in the formulation of solution (4.38) in Lemma 4.4.6 (in Appendix). Taking it so in the difference of formulations (4.38) solved by u and v, we obtain

$$\begin{aligned}&\int_{\Omega_T}\partial_t(u-v)_+T_k((u-v)_+)dxdt+\int_{\Omega_T}|\nabla T_k((u-v)_+)|^2dxdt\\&\leq\int_{\{(u-v)_+\leq k\}}\frac{(T_k((u-v)_+))^2}{|x|^2}dxdt+k\mu\int_{\{(u-v)_+>k\}}\frac{(u-v)_+}{|x|^2}dxdt\\&+\int_{\Omega_T}f\Big(\frac{1}{u^\sigma}-\frac{1}{v^\sigma}\Big)T_k((u-v)_+)dxdt\end{aligned}$$

Setting $\Theta_k(s)=\int_0^s T_k(\nu)d\nu$ and dropping the negative term, we get

$$\begin{aligned}&\int_{\Omega}\Theta_k((u-v)_+(x,T))dx+\int_{\Omega_T}|\nabla T_k((u-v)_+)|^2dxdt\\&\leq\int_{\{(u-v)_+\leq k\}}\frac{(T_k((u-v)_+))^2}{|x|^2}dxdt+k\mu\int_{\{(u-v)_+>k\}}\frac{(u-v)_+}{|x|^2}dxdt\\&+\int_{\Omega}\Theta_k((u-v)_+(x,0))dx.\end{aligned}$$

Using $\displaystyle\int_\Omega \Theta_k((u-v)_+(x,T))dx \geq 0$, the fact that $u(x,0)=v(x,0)=u_0(x)$, Hardy's inequality (4.4) and Hölder's inequality, we arrive at

$$\begin{aligned}&\int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt\\ &\leq \frac{\mu}{\Lambda_{N,2}}\int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt\\ &+k\mu\Big(\int_{\{(u-v)_+>k\}}\frac{((u-v)_+)^2}{|x|^2}dxdt\Big)^{\frac12}\Big(\int_{\{(u-v)_+>k\}}\frac{dxdt}{|x|^2}\Big)^{\frac12}.\end{aligned}$$

Having in mind (4.14) and using again (4.4) we reach that

$$\begin{aligned}\int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt &\leq \frac{\mu}{\Lambda_{N,2}}\int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt\\ &+\frac{k\mu T^{\frac12}C_1^{\frac12}}{\Lambda_{N,2}^{\frac12}}\Big(\int_{\{(u-v)_+>k\}}|\nabla (u-v)_+|^2 dxdt\Big)^{\frac12}.\end{aligned}\tag{4.29}$$

On the other hand, taking $T_r(G_k((u-v)_+))\in L^2(0,T;H_0^1(\Omega))\cap L^\infty(\Omega_T)$ as a test function in the problems solved by u and v and subtracting the two equations we obtain

$$\begin{aligned}&\int_{\Omega_T}\partial_t(u-v)_+T_r(G_k((u-v)_+))dxdt+\int_{\{k<(u-v)_+<k+r\}}|\nabla (u-v)_+|^2 dxdt\\ &\leq \mu\int_{\{(u-v)_+>k\}}\frac{(u-v)_+^2}{|x|^2}dxdt+\int_{\Omega_T} f\Big(\frac{1}{u^\sigma}-\frac{1}{v^\sigma}\Big)T_r(G_k((u-v)_+))dxdt.\end{aligned}$$

Setting $\Theta_{k,r}(s)=\displaystyle\int_0^s T_r(G_k(\nu))d\nu$ and dropping the negative term, the above inequality becomes

$$\begin{aligned}&\int_\Omega \Theta_{k,r}((u-v)_+(x,T))dx+\int_{\{k<(u-v)_+<k+r\}}|\nabla (u-v)_+|^2 dxdt\\ &\leq \mu\int_{\{(u-v)_+>k\}}\frac{(u-v)_+^2}{|x|^2}dxdt+\int_\Omega \Theta_{k,r}((u-v)_+(x,0))dx.\end{aligned}$$

Note that $\displaystyle\int_\Omega \Theta_{k,r}((u-v)_+(x,T))dx \geq 0$ and $\displaystyle\int_\Omega \Theta_{k,r}((u-v)_+(x,0))dx = 0$. Whence, by (4.4) we obtain

$$\int_{\{k<(u-v)_+<k+r\}}|\nabla (u-v)_+|^2 dxdt \leq \frac{\mu}{\Lambda_{N,2}}\int_{\{(u-v)_+>k\}}|\nabla (u-v)_+|^2 dxdt.$$

Then, passing to the limit as r tends to $+\infty$ we get

$$\int_{\{k<(u-v)_+\}}|\nabla (u-v)_+|^2 dxdt \leq \frac{\mu}{\Lambda_{N,2}}\int_{\{k<(u-v)_+\}}|\nabla (u-v)_+|^2 dxdt.\tag{4.30}$$

Therefore, gathering (4.29) and (4.30) we obtain

$$\begin{aligned}\int_{\Omega_T}|\nabla (u-v)_+|^2 dxdt &\leq \frac{\mu}{\Lambda_{N,2}}\int_{\Omega_T}|\nabla (u-v)_+|^2 dxdt\\ &+\frac{k\mu C_1}{\Lambda_{N,2}}\Big(\int_{\{(u-v)_+>k\}}|\nabla (u-v)_+|^2 dxdt\Big)^{\frac12}.\end{aligned}$$

Passing now to the limit as k tends to 0 we obtain

$$\int_{\Omega_T} |\nabla(u-v)_+|^2 dxdt \leq \frac{\mu}{\Lambda_{N,2}} \int_{\Omega_T} |\nabla(u-v)_+|^2 dxdt,$$

which, recalling that $u - v \in \mathcal{C}([0,T]; L^1(\Omega))$ (see [73, Theorem 1.1]) implies $(u-v)_+(\cdot,\tau) = 0$ for any $\tau \in [0,T]$ and for almost every $x \in \Omega$. By the u/v symmetry we conclude that $u = v$ a.e. in Ω_T.

4.4 Appendix

We give here some important lemmas that are necessary for the accomplishment of the proofs of the above results.

Theorem 4.4.1 *[85, Theorem 2.2] Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$. Then for every $1 \leq q < 2$ there exists a positive constant $C = C(\Omega, q)$ such that for all $u \in H_0^1(\Omega)$ we have*

$$C\Big(\int_\Omega |\nabla u|^q dx\Big)^{\frac{2}{q}} \leq \int_\Omega |\nabla u|^2 dx - \Lambda_{N,2} \int_\Omega \frac{u^2}{|x|^2} dx.$$

Let

$$\alpha_1 := \frac{N-2}{2} - \sqrt{\Big(\frac{N-2}{2}\Big)^2 - \lambda} \tag{4.31}$$

be the smallest root of $\alpha^2 - (N-2)\alpha + \gamma = 0$. It is well known that this root yields the radial solution $|x|^{-\alpha_1}$ to the homogeneous problem

$$-\Delta v - \lambda \frac{v}{|x|^2} = 0.$$

The following lemma provides a local comparison result with this radial solution.

Lemma 4.4.1 *[5, Lemma 2.2] Assume that u is a non-negative function defined in Ω such that $u \neq 0$, $u \in L^1_{loc}(\Omega_T)$. If u satisfies*

$$\partial_t u - \Delta u - \lambda \frac{u}{|x|^2} \geq 0, \ \text{in } \mathcal{D}'(\Omega_T)$$

with $\Omega_T := \Omega \times (0,T)$, $\lambda \leq \Lambda_{N,2}$ and $B_r(0) \subset\subset \Omega$, then there exists a constant $C = C(N, r, t_1, t_2)$ such that for each cylinder $B_{r_1}(0) \times (t_1, t_2) \subset \Omega \times (0,T)$, $0 < r_1 < r$,

$$u \geq C|x|^{-\alpha_1} \ \text{in } B_{r_1}(0) \times (t_1, t_2),$$

where α_1 is the constant defined in (4.31).

Lemma 4.4.2 *Let $0 < \lambda \leq \Lambda_{N,2}$ and $g \in L^1(0,T; L^1_{loc}(\Omega))$, $g \geq 0$. If u is a weak solution of the problem*

$$\begin{cases} \partial_t u - \Delta u &= \lambda \dfrac{u}{|x|^2} + g \quad \text{in } \Omega_T := \Omega \times (0,T), \\ u &= 0 \quad \text{in } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{in } \Omega, \end{cases} \tag{4.32}$$

where $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, then g satisfies

$$\int_{t_1}^{t_2} \int_{B_{r_1}(0)} |x|^{-\alpha_1} g dx dt < +\infty,$$

for any ball $B_{r_1}(0) \subset\subset \Omega$, where α_1 is defined in (4.31).

Proof 4.4.1 *We use similar arguments as in [5, Remark 2.4]. Let $B_r(0) \subset\subset \Omega$ and $\phi \in L^2(0,T; H^1_0(\Omega)) \cap L^\infty(\Omega_T)$ be a weak solution of the problem*

$$\begin{cases} \partial_t \phi - \Delta \phi - \lambda \frac{\phi}{|x|^2} &= 1 \quad in\ \Omega_T, \\ \phi &= 0 \quad in\ \partial\Omega \times (0,T), \\ \phi(x,0) &= 1 \quad in\ \Omega. \end{cases} \tag{4.33}$$

Multiplying (4.32) by $T_n(\phi)$ and integrating over $B_r(0) \times (0,T)$ we obtain

$$\int_0^T \int_{B_r(0)} \partial_t u T_n(\phi) dx dt - \int_0^T \int_{B_r(0)} \Delta u T_n(\phi) dx dt - \lambda \int_0^T \int_{B_r(0)} \frac{u}{|x|^2} T_n(\phi) dx dt$$
$$= \int_0^T \int_{B_r(0)} g T_n(\phi) dx dt.$$

Since u is a weak solution of (4.32) the above integrals make sense for each integer n. By classical integration by parts formula, one has

$$\int_{B_r(0)} u(x,T) T_n(\phi(x,T)) dx - \int_{B_r(0)} u(x,0) dx - \int_0^T \int_{B_r(0)} u \partial_t (T_n(\phi)) dx dt$$
$$- \int_0^T \int_{B_r(0)} u \Delta(T_n(\phi)) dx dt - \lambda \int_0^T \int_{B_r(0)} \frac{u}{|x|^2} T_n(\phi) dx dt = \int_0^T \int_{B_r(0)} g T_n(\phi) dx dt. \tag{4.34}$$

Since $T_n(\phi) \to \phi$ in $L^1(\Omega_T)$ and a.e. in Ω_T and $\phi \in L^\infty(\Omega_T)$, we can apply the Lebesgue dominated convergence theorem in the (4.34) to get

$$\int_{B_r(0)} u(x,T)\phi(x,T) dx - \int_{B_r(0)} u_0(x) dx - \int_0^T \int_{B_r(0)} u \partial_t \phi dx dt - \int_0^T \int_{B_r(0)} u \Delta \phi dx dt$$
$$-\lambda \int_0^T \int_{B_r(0)} \frac{u}{|x|^2} \phi dx dt = \int_0^T \int_{B_r(0)} g \phi dx dt.$$

As ϕ is a solution of (4.33), we get

$$\int_{B_r(0)} u(x,T)\phi(x,T)dx - \int_{B_r(0)} u_0 dx - 2\int_0^T\int_{B_r(0)} u\partial_t\phi dxdt + \int_0^T\int_{B_r(0)} udxdt$$
$$= \int_0^T\int_{B_r(0)} g\phi dxdt.$$

Applying again the integration by parts formula we obtain

$$-\int_{B_r(0)} u(x,T)\phi(x,T)dx + \int_{B_r(0)} u_0(x)dx + 2\int_0^T\int_{B_r(0)} \partial_t u\phi dxdt + \int_0^T\int_{B_r(0)} udxdt$$
$$= \int_0^T\int_{B_r(0)} g\phi dxdt.$$

By Lemma 4.4.1, for every cylinder $B_{r_1}(0)\times(t_1,t_2)\subset B_r(0)\times(0,T)$, $0<r_1<r$ there exists a constant $C>0$ such that

$$\begin{aligned}\int_{t_1}^{t_2}\int_{B_{r_1}(0)} |x|^{-\alpha_1} gdxdt \;&\le \int_{B_r(0)} u(x,T)\phi(x,T)dx + \int_{B_r(0)} u_0dx + 2\int_0^T\int_{B_r(0)} |\partial_t u\phi|dxdt\\ &+\int_0^T\int_{B_r(0)} udxdt.\end{aligned}$$

Since $u\in L^1(0,T;L^1_{loc}(\Omega))$, $u_0\in L^\infty(\Omega)$, $\phi\in L^\infty(\Omega_T)$ and $\partial_t u\in L^2(0,T;H^{-1}_{loc}(\Omega))+L^1(0,T;L^1_{loc}(\Omega))$ conclude that

$$\int_{t_1}^{t_2}\int_{B_{r_1}(0)} |x|^{-\alpha_1} gdxdt < +\infty.$$

We will now compare the solution u_n of (4.8) with the solution w_n of the problem

$$\begin{cases}\partial_t w_n - \Delta w_n &= \frac{f_n}{(w_n+\frac{1}{n})^\sigma} \quad \text{in } \Omega\times(0,T),\\ \quad w_n(x,t) &= 0 \quad \text{in } \partial\Omega\times(0,T),\\ \quad w_n(x,0) &= u_0(x) \quad \text{in } \Omega,\end{cases} \tag{4.35}$$

where $f = min(f,n)$ and u_0 satisfies (4.3). Recall that 4.35 has a weak solution w_n (see [27, Lemma 2.1]).

Lemma 4.4.3 *Let u_n be a solution of (4.8) and w_n be a solution of (4.35). Then, $w_n\le u_n$ a.e. in Ω_T.*

Proof 4.4.2 *Consider the problems solved by w_n and u_n, subtracting the two equations, we get*

$$\begin{aligned}\partial_t(w_n-u_n)-\Delta(w_n-u_n) \;&= -\mu\frac{T_n(u_n)}{|x|^2+\frac{1}{n}} + f_n\Big(\frac{1}{(w_n+\frac{1}{n})^\sigma} - \frac{1}{(u_n+\frac{1}{n})^\sigma}\Big)\\ &\le f_n\Big(\frac{1}{(w_n+\frac{1}{n})^\sigma} - \frac{1}{(u_n+\frac{1}{n})^\sigma}\Big).\end{aligned} \tag{4.36}$$

Using $(w_n - u_n)_+\chi_{(0,\tau)}$, $0 \le \tau \le T$, *as test function in (4.36) it follows that*

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (w_n - u_n)_+^2(x,\tau)dx + \int_{\Omega_\tau} |\nabla(w_n - u_n)_+|^2 dxdt \\
&\le \int_{\Omega_\tau} f_n\Big(\frac{(u_n+\frac{1}{n})^\sigma - (w_n+\frac{1}{n})^\sigma}{(u_n+\frac{1}{n})^\sigma(w_n+\frac{1}{n})^\sigma}\Big)(w_n - u_n)_+ dxdt \\
&\le 0,
\end{aligned}$$

where we have used $w_n(x,0) = u_n(x,0) = u_0(x)$. *Hence we get* $\int_{\Omega_T} |\nabla(w_n - u_n)_+|^2(x,\tau)dx = 0$. *Recalling that* $w_n - u_n \in \mathcal{C}([0,T]; L^1(\Omega))$ *(see [73, Theorem 1.1]) implies* $(w_n - u_n)_+(\cdot,\tau) = 0$ *for every* $0 \le \tau \le T$ *and for almost every* $x \in \Omega$. *Thus,* $w_n \le u_n$ *a.e. in* Ω_T.

Lemma 4.4.4 *Let* u_n *be a solution of (4.8). Then for every* $\Omega' \subset\subset \Omega$ *there exists* $C_{\Omega'} > 0$ *(not depending on n), such that* $u_n \ge C_{\Omega'}$ *in* $\Omega' \times [0,T]$.

Proof 4.4.3 *The proof follows by combining [27, Proposition 2.2] and Lemma 4.4.3.*

Lemma 4.4.5 *Assume that* $\mu \le \Lambda_{N,2}$ *and let* u_n *be a solution of* (4.8). *Then the sequence* $\{u_n\}_{n\in\mathbb{N}}$ *is increasing with respect to* $n \in \mathbb{N}$.

Proof 4.4.4 *Subtracting the two equations corresponding to the problems solved by* u_n *and* u_{n+1}, *we obtain*

$$\begin{aligned}
\partial_t(u_n - u_{n+1}) - \Delta(u_n - u_{n+1}) &\le \mu\tfrac{T_{n+1}(u_n) - T_{n+1}(u_{n+1})}{|x|^2 + \frac{1}{n+1}} \\
&+ f_{n+1}\Big(\frac{1}{\big(u_n + \frac{1}{n+1}\big)^\sigma} - \frac{1}{\big(u_{n+1} + \frac{1}{n+1}\big)^\sigma}\Big).
\end{aligned} \tag{4.37}$$

Inserting $(u_n - u_{n+1})_+$ *as a test function in (4.37) and using the fact that* T_{n+1} *is a* 1*-Lipschitzian function, we get*

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_T} \partial_t(u_n - u_{n+1})_+^2 dxdt + \int_{\Omega_T} |\nabla(u_n - u_{n+1})_+|^2 dxdt \\
&\le \int_{\Omega_T} f_{n+1}(u_n - u_{n+1})_+\Big(\frac{1}{\big(u_n + \frac{1}{n+1}\big)^\sigma} - \frac{1}{\big(u_{n+1} + \frac{1}{n+1}\big)^\sigma}\Big)dxdt \\
&+\mu\int_{\Omega_T} \frac{(u_n - u_{n+1})_+^2}{|x|^2} dxdt.
\end{aligned}$$

Dropping the non-negative parabolic term and using the fact that

$$(u_n - u_{n+1})_+\Big(\frac{1}{\big(u_n + \frac{1}{n+1}\big)^\sigma} - \frac{1}{\big(u_{n+1} + \frac{1}{n+1}\big)^\sigma}\Big) \le 0,$$

we obtain

$$\int_{\Omega_T} |\nabla(u_n - u_{n+1})_+|^2 dxdt \le \mu \int_{\Omega_T} \frac{(u_n - u_{n+1})_+^2}{|x|^2} dxdt.$$

Thus, if $\mu < \Lambda_{N,2}$ *the Hardy inequality (4.4) yields*

$$\int_{\Omega_T} |\nabla(u_n - u_{n+1})_+|^2 dxdt = 0,$$

while if $\mu = \Lambda_{N,2}$ *we can apply the Theorem 4.4.1 to obtain*

$$\int_{\Omega_T} |\nabla(u_n - u_{n+1})_+|^q dxdt = 0,$$

for all $q < 2$*. Therefore, in both cases we get* $(u_n - u_{n+1})_+ = 0$ *a.e. in* Ω_T*, that is* $u_n \le u_{n+1}$ *a.e. in* Ω_T*.*

Lemma 4.4.6 *Let* $u \in L^2(0,T;H_0^1(\Omega))$ *be a finite energy solution of (4.1) with a datum* $f \in L^1(\Omega_T)$ *such that* $supp(f) \subset\subset \Omega_T$*. Then* u *satisfies* $\frac{u\phi}{|x|^2} \in L^1(\Omega_T)$*,* $\frac{f\phi}{u^\sigma} \in L^1(\Omega_T)$ *and*

$$\int_{\Omega_T} \partial_t u\phi dxdt + \int_{\Omega_T} \nabla u \cdot \nabla\phi dxdt = \int_{\Omega_T} \Big(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\Big)\phi dxdt, \tag{4.38}$$

for every $\phi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$*.*

Proof 4.4.5 *Let* $\phi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ *be a nonnegative function. A direct application of Hardy's inequality yields* $\mu\frac{u\phi}{|x|^2} \in L^1(\Omega_T)$*, while since* f *is compactly supported in* Ω_T*, by Lemma 4.4.4 there exists a constant* $C_{supp(f)} > 0$ *such that* $u \ge C_{supp(f)}$ *in* $supp(f)$ *so that one has*

$$\int_{\Omega_T} \frac{|f\phi|}{u^\sigma} dxdt \le C^\sigma_{supp(f)} \|\phi\|_\infty \|f\|_{L^1(\Omega_T)} < \infty.$$

We argue as in [71, Lemma 4.2] considering a sequence of function $\phi_n \in C_0^\infty(\Omega_T)$*, with* $\phi_n \ge 0$ *and* $\phi_n \to \phi$ *in* $L^2(0,T;H_0^1(\Omega))$*, with* $\|\phi_n\|_\infty \le \|\phi\|_\infty$*. Inserting* ϕ_n *as a test function in 4.6 and integrating by parts, we obtain*

$$\int_{\Omega_T} \partial_t u\phi_n dxdt + \int_{\Omega_T} \nabla u \cdot \nabla\phi_n dxdt = \int_{\Omega_T} \Big(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\Big)\phi_n dxdt. \tag{4.39}$$

Since $\phi_n \to \phi$ *in* $L^2(\Omega_T)$ *then, for a subsequence still indexed by* n*, we may assume that* $\phi_n \to \phi$ *a.e. in* Ω_T*. As* f *is compactly supported in* Ω_T *we have*

$$\Big(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\Big)\phi_n \le \|\phi\|_\infty\Big(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\Big) \in L^1(\Omega_T).$$

Thus, by the Lebesgue dominated convergence theorem we obtain

$$\lim_{n\to\infty}\int_{\Omega_T}\Big(\mu\frac{u}{|x|^2}+\frac{f}{u^\sigma}\Big)\phi_n dxdt=\int_{\Omega_T}\Big(\mu\frac{u}{|x|^2}+\frac{f}{u^\sigma}\Big)\phi dxdt.$$

Since $\partial_t u\in L^2(0,T;H^{-1}(\Omega))+L^1(\Omega_T)$ *we use the convergence* $\phi_n\to\phi$ *in* $L^2(0,T;H_0^1(\Omega))$ *and again the Lebesgue dominated convergence theorem in (4.39) obtaining*

$$\int_{\Omega_T}\partial_t u\phi dxdt+\int_{\Omega_T}\nabla u\cdot\nabla\phi dxdt=\int_{\Omega_T}\Big(\mu\frac{u}{|x|^2}+\frac{f}{u^\sigma}\Big)\phi dxdt.$$

Bibliography

[1] B. Abdellaoui and A. Attar. Quasilinear elliptic problem with Hardy potential and singular term. *Commun. Pure Appl. Anal.*, 12(3):1363–1380, (2013).

[2] B. Abdellaoui, S.E. Miri, I. Peral, and M. Touaoula. Some remarks on quasilinear parabolic problems with singular potential and a reaction term. *NoDEA Nonlinear Differential Equations Appl.*, 21(4):453–490, (2014).

[3] B. Abdellaoui and I. Peral. Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian with a critical potential. *Ann. Mat. Pura Appl. (4)*, 182(3):247–270, (2003).

[4] B. Abdellaoui, I. Peral, and A. Primo. Elliptic problems with a Hardy potential and critical growth in the gradient: non-resonance and blow-up results. *J. Differential Equations*, 239(2):386–416, (2007).

[5] B. Abdellaoui, I. Peral, and A. Primo. Influence of the Hardy potential in a semilinear heat equation. *Proc. Roy. Soc. Edinburgh Sect. A*, 139(5):897–926, (2009).

[6] B. Abdellaoui, I. Peral, and A. Primo. Optimal results for parabolic problems arising in some physical models with critical growth in the gradient respect to a Hardy potential. *Adv. Math.*, 225(6):2967–3021, (2010).

[7] R.A. Adams and J.F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, (2003).

[8] A. Adimurthi, L. Boccardo, G.R. Cirmi, and L. Orsina. The regularizing effect of lower order terms in elliptic problems involving Hardy potential. *Adv. Nonlinear Stud.*, 17(2):311–317, (2017).

[9] C.O. Alves, J.V. Goncalves, and L.A. Maia. Singular nonlinear elliptic equations in $\mathbf{R}^N$. *Abstr. Appl. Anal.*, 3(3-4):411–423, (1998).

[10] A. Anane. Simplicité et isolation de la première valeur propre du p-laplacien avec poids. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(16):725–728, 1987.

[11] R. Aris. *The mathematical theory of diffusion and reaction in permeable catalysts*, volume I and II. Oxford University Press, New York, (1975).

[12] A. Attar, S. Merchán, and I. Peral. A remark on the existence properties of a semilinear heat equation involving a Hardy-Leray potential. *J. Evol. Equ.*, 15(1):239–250, (2015).

[13] J.P. Azorero and I. Peral. Hardy inequalities and some critical elliptic and parabolic problems. *J. Differential Equations*, 144(2):441–476, (1998).

[14] M. Badra, K. Bal, and J. Giacomoni. Existence results to a quasilinear and singular parabolic equation. *Discrete Contin. Dyn. Syst.*, (Dynamical systems, differential equations and applications. 8th AIMS Conference. Suppl. Vol. I):117–125, (2011).

[15] M. Badra, K. Bal, and J. Giacomoni. Some results about a quasilinear singular parabolic equation. *Differ. Equ. Appl.*, 3(4):609–627, (2011).

[16] M. Badra, K. Bal, and J. Giacomoni. A singular parabolic equation: existence, stabilization. *J. Differential Equations*, 252(9):5042–5075, (2012).

[17] P. Baras and A. G., Jerome. The heat equation with a singular potential. *Trans. Amer. Math. Soc.*, 284(1):121–139, (1984).

[18] L. Boccardo, A. Dall'Aglio, T. Gallouët, and L. Orsina. Nonlinear parabolic equations with measure data. *J. Funct. Anal.*, 147(1):237–258, (1997).

[19] L. Boccardo and L. Orsina. Semilinear elliptic equations with singular nonlinearities. *Calc. Var. Partial Differential Equations*, 37(3-4):363–380, (2010).

[20] L. Boccardo, L. Orsina, and I. Peral. A remark on existence and optimal summability of solutions of elliptic problems involving Hardy potential. *Discrete Contin. Dyn. Syst.*, 16(3):513–523, (2006).

[21] M.-M. Boureanu and V. Rădulescu. Anisotropic Neumann problems in Sobolev spaces with variable exponent. *Nonlinear Anal.*, 75(12):4471–4482, 2012.

[22] B. Brandolini, F. Chiacchio, and C. Trombetti. Symmetrization for singular semilinear elliptic equations. *Ann. Mat. Pura Appl. (4)*, 193(2):389–404, (2014).

[23] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, (2011).

[24] H. Brezis and L. Vázquez. Blow-up solutions of some nonlinear elliptic problems. *Rev. Mat. Univ. Complut. Madrid*, 10(2):443–469, (1997).

[25] G.R. Cirmi and M.M. Porzio. L^∞-solutions for some nonlinear degenerate elliptic and parabolic equations. *Ann. Mat. Pura Appl.*, 169(4):67–86, (1995).

[26] M.G. Crandall, P.H. Rabinowitz, and L. Tartar. On a Dirichlet problem with a singular nonlinearity. *Comm. Partial Differential Equations*, 2(2):193–222, (1977).

[27] I. de Bonis and L.M. De Cave. Degenerate parabolic equations with singular lower order terms. *Differential Integral Equations*, 27(9-10):949–976, (2014).

[28] I. de Bonis and D. Giachetti. Nonnegative solutions for a class of singular parabolic problems involving p-Laplacian. *Asymptot. Anal.*, 91(2):147–183, (2015).

[29] L.M. De Cave. Nonlinear elliptic equations with singular nonlinearities. *Asymptot. Anal.*, 84(3-4):181–195, (2013).

[30] J.I. Diaz, J.M. Morel, and L. Oswald. An elliptic equation with singular nonlinearity. *Comm. Partial Differential Equations*, 12(12):1333–1344, (1987).

[31] E. DiBenedetto. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, (1993).

[32] L. Diening, P. Harjulehto, P. Hästö, and M. Ružička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.

[33] D.E. Edmunds and J. Rákosník. Sobolev embeddings with variable exponent. *Studia Math.*, 143(3):267–293, 2000.

[34] I. Ekeland and R. Temam. *Convex analysis and variational problems.* North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1976. Translated from the French, Studies in Mathematics and its Applications, Vol. 1.

[35] H.V. Emden and A.W. Shaker. Multigrid methods for a semilinear pde in the theory of pseudoplastic fluids. *Calhoun*, pages 231–242, (1993).

[36] X. Fan. Boundary trace embedding theorems for variable exponent Sobolev spaces. *J. Math. Anal. Appl.*, 339(2):1395–1412, 2008.

[37] X. Fan. Anisotropic variable exponent Sobolev spaces and $\vec{p}(\cdot)-$Laplacian equations. *Complex Var. Elliptic Equ.*, 56(7-9):623–642, 2011.

[38] X. Fan. An imbedding theorem for Musielak-Sobolev spaces. *Nonlinear Anal.*, 75(4):1959–1971, 2012.

[39] X. Fan, J.S. Shen, and D. Zhao. Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$. *J. Math. Anal. Appl.*, 262(2):749–760, 2001.

[40] W. Fulks and J.S. Maybee. A singular non-linear equation. *Osaka Math. J.*, 12:1–19, (1960).

[41] E. Gagliardo. Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili. *Rend. Sem. Mat. Univ. Padova*, 27:284–305, 1957.

[42] M. García-Huidobro, V. K. Le, R. Manásevich, and K. Schmitt. On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting. *NoDEA Nonlinear Differential Equations Appl.*, 6(2):207–225, 1999.

[43] L. Gasiński and N.S. Papageorgiou. Anisotropic nonlinear Neumann problems. *Calc. Var. Partial Differential Equations*, 42(3-4):323–354, 2011.

[44] J. P. Gossez. Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients. *Trans. Amer. Math. Soc.*, 190:163–205, 1974.

[45] J. P. Gossez. Some approximation properties in Orlicz-Sobolev spaces. *Studia Math.*, 74(1):17–24, 1982.

[46] J. P. Gossez. A strongly nonlinear elliptic problem in Orlicz-Sobolev spaces. In *Nonlinear functional analysis and its applications, Part 1 (Berkeley, Calif., 1983)*, volume 45 of *Proc. Sympos. Pure Math.*, pages 455–462. Amer. Math. Soc., Providence, RI, 1986.

[47] J. P. Gossez and R. Manásevich. On a nonlinear eigenvalue problem in Orlicz-Sobolev spaces. *Proc. Roy. Soc. Edinburgh Sect. A*, 132(4):891–909, 2002.

[48] J. Han, Y. Wang, and P. Niu. Existence of solutions to the parabolic equation with a singular potential of the Sobolev-Hardy type. *Acta Math. Sci. Ser. B (Engl. Ed.)*, 32(5):1901–1918, (2012).

[49] P. Harjulehto and P. Hästö. Sobolev inequalities with variable exponent attaining the values 1 and n. *Publ. Mat.*, 52(2):347–363, 2008.

[50] E. Hewitt and K. Stromberg. *Real and abstract analysis. A modern treatment of the theory of functions of a real variable.* Springer-Verlag, New York, 1965.

[51] B. Kone, S. Ouaro, and S. Traore. Weak solutions for anisotropic nonlinear elliptic equations with variable exponents. *Electron. J. Differential Equations*, pages No. 144, 11, 2009.

[52] O. Kováčik and J. Rákosník. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.*, 41(116)(4):592–618, 1991.

[53] M. A. Krasnosel'skii and J. B. Rutic'kii. *Convex functions and Orlicz spaces.* Translated from the first Russian edition by Leo F. Boron. P. Noordhoff Ltd., Groningen, 1961.

[54] A. Kufner, O. John, and S. Fučík. *Function spaces.* Noordhoff International Publishing, Leyden; Academia, Prague, 1977. Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis.

[55] A.C. Lazer and P.J. McKenna. On a singular nonlinear elliptic boundary-value problem. *Proc. Amer. Math. Soc.*, 111(3):721–730, (1991).

[56] V. Le and K. Schmitt. Quasilinear elliptic equations and inequalities with rapidly growing coefficients. *J. London Math. Soc. (2)*, 62(3):852–872, 2000.

[57] A.R. Leggat and S.E. Miri. Anisotropic problem with singular nonlinearity. *Complex Var. Elliptic Equ.*, 61(4):496–509, (2016).

[58] P. Lindqvist. On the equation div $(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. *Proc. Amer. Math. Soc.*, 109(1):157–164, 1990.

[59] J.L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod; Gauthier-Villars, Paris, (1969).

[60] D. Liu, B. Wang, and P. Zhao. On the trace regularity results of Musielak-Orlicz-Sobolev spaces in a bounded domain. *Commun. Pure Appl. Anal.*, 15(5):1643–1659, 2016.

[61] S. Merchán and I. Peral. Remarks on the solvability of an elliptic equation with a supercritical term involving the Hardy-Leray potential. *J. Math. Anal. Appl.*, 394(1):347–359, (2012).

[62] M. Mihailescu, V. Radulescu, and D. Repovs. On a non-homogeneous eigenvalue problem involving a potential: an orlicz-sobolev space setting. *J. Math. Pures Appl. (9)*, 93(2):132–148, 2010.

[63] M. Mihăilescu, P. Pucci, and V. Rădulescu. Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent. *J. Math. Anal. Appl.*, 340(1):687–698, 2008.

[64] M. Mihăilescu and V. Rădulescu. Eigenvalue problems associated with nonhomogeneous differential operators in Orlicz-Sobolev spaces. *Anal. Appl. (Singap.)*, 6(1):83–98, 2008.

[65] S.E. Miri. Quasilinear elliptic problem with Hardy potential and a reaction-absorbtion term. *Differ. Equ. Appl.*, 5(1):111–125, (2013).

[66] M. Montenegro and S. Lorca. The eigenvalue problem for quasilinear elliptic operators with general growth. *Appl. Math. Lett.*, 25(7):1045–1049, 2012.

[67] J. Musielak. *Orlicz spaces and modular spaces*, volume 1034 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[68] V. Mustonen and M. Tienari. An eigenvalue problem for generalized Laplacian in Orlicz-Sobolev spaces. *Proc. Roy. Soc. Edinburgh Sect. A*, 129(1):153–163, 1999.

[69] A. Nachman and A. Callegari. A nonlinear singular boundary value problem in the theory of pseudoplastic fluids. *SIAM J. Appl. Math.*, 38(2):275–281, (1980).

[70] F. Oliva and F. Petitta. On singular elliptic equations with measure sources. *ESAIM Control Optim. Calc. Var.*, 22(1):289–308, (2016).

[71] F. Oliva and F. Petitta. A nonlinear parabolic problem with singular terms and nonregular data. *Nonlinear Anal.*, 194:111472, 13, (2020).

[72] M. Ôtani and T. Teshima. On the first eigenvalue of some quasilinear elliptic equations. *Proc. Japan Acad. Ser. A Math. Sci.*, 64(1):8–10, 1988.

[73] A. Porretta. Existence results for nonlinear parabolic equations via strong convergence of truncations. *Ann. Mat. Pura Appl. (4)*, 177:143–172, (1999).

[74] M.M. Porzio. Quasilinear parabolic and elliptic equations with singular potentials. In *2017 MATRIX annals*, volume 2 of *MATRIX Book Ser.*, pages 223–242. Springer, Cham, (2019).

[75] M.M. Porzio and A. Primo. Summability and existence results for quasilinear parabolic equations with Hardy potential term. *NoDEA Nonlinear Differential Equations Appl.*, 20(1):65–100, (2013).

[76] P. Pucci and J. Serrin. A note on the strong maximum principle for elliptic differential inequalities. *J. Math. Pures Appl. (9)*, 79(1):57–71, 2000.

[77] S. Sakaguchi. Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 14(3):403–421 (1988), 1987.

[78] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, (1987).

[79] G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, 15(fasc. 1):189–258, 1965.

[80] G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, 15(fasc., fasc. 1):189–258, (1965).

[81] M. Struwe. *Variational methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, second edition, 1996. Applications to nonlinear partial differential equations and Hamiltonian systems.

[82] C.A. Stuart. Existence and approximation of solutions of non-linear elliptic equations. *Math. Z.*, 147(1):53–63, (1976).

[83] M. Tienari. A degree theory for a class of mappings of monotone type in Orlicz-Sobolev spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (97):68, 1994.

[84] N. S. Trudinger. On Harnack type inequalities and their application to quasilinear elliptic equations. *Comm. Pure Appl. Math.*, 20:721–747, 1967.

[85] J.L. Vazquez and E. Zuazua. The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential. *J. Funct. Anal.*, 173(1):103–153, (2000).

[86] V. Vespri. L^∞-estimates for nonlinear parabolic equations with natural growth conditions. *Rend. Sem. Mat. Univ. Padova*, 90:1–8, (1993).

[87] M. Yao and J. Zhao. Positive solution of a singular non-linear elliptic boundary value problem. *Appl. Math. Comput.*, 148(3):773–782, (2004).

[88] S. Yijing and Z. Duanzhi. The role of the power 3 for elliptic equations with negative exponents. *Calc. Var. Partial Differential Equations*, 49(3-4):909–922, (2014).

[89] A. Youssfi, E. Azroul, and H. Hjiaj. On nonlinear elliptic equations with Hardy potential and L^1-data. *Monatsh. Math.*, 173(1):107–129, (2014).

[90] A. Youssfi, A. Benkirane, and Y. El Hadfi. On bounded solutions for nonlinear parabolic equations with degenerate coercivity. *Mediterr. J. Math.*, 13(5):3029–3040, (2016).

[91] A. Youssfi and M M.O. Khatri. On a nonlinear eigenvalue problem for generalized laplacian in orlicz-sobolev spaces. *J. Nonlinear Anal.*, 190, 2020.

[92] A. Youssfi and M.M.O. Khatri. Imbedding results in musielak-orlicz-sobolev spaces with an application to anisotropic nonlinear neumann problems. *Submitted to Electronic Journal of differential equations.*

[93] A. Youssfi and M.M.O. Khatri. Semilinear heat equation with hardy potential and singular terms. *Submitted to Journal of Evolution Equations.*

[94] Z. Zhang and J. Cheng. Existence and optimal estimates of solutions for singular nonlinear Dirichlet problems. *Nonlinear Anal.*, 57(3):473–484, (2004).

[95] S. Zhou. A priori L^∞-estimate and existence of solutions for some nonlinear parabolic equations. *Nonlinear Anal.*, 42(5, Ser. A: Theory Methods):887–904, (2000).