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THÈSE

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On Some Elliptic And Parabolic Problems In Unbounded Domains

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Dédicace

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NOTATIONS

Ω	**open domain in** $I\!R^N$ **for** $N \geq 1$**.**
$\partial\Omega$	**topological boundary of** Ω **.**
$x = (x_1, \ldots, x_N)$	**point in** $I\!R^N$ **for** $N \geq 1$ **.**
$dx = dx_1 \ldots dx_N$	**Lebesgue measure in** Ω **.**
Q_T	$\Omega \times (0,T)$ **for** $T > 0$ **.**
Σ_T	$\partial\Omega \times (0,T)$ **.**
∇u	**gradient of a function** u **.**
supp f	**support of a function** f **.**
$D(\Omega)$	**space of infinitely differentiable functions with compact support in** Ω **.**
T_k	**truncation function** $k > 0$ **.**
$L^p(\Omega)$	$\{f : \Omega \mapsto I\!R \ / \ \int_\Omega \|f(x)\|^p \, dx < \infty\}$ **for** $1 \leq p < \infty$ **.**
$L^\infty(\Omega)$	$\{f : \Omega \mapsto I\!R \ / \ ess \sup_{x\in\Omega} \|f(x)\| < \infty\}$ **.**
$W^{1,p}(\Omega)$	$\{f : \Omega \mapsto I\!R \ / \ f \in L^p(\Omega)$ **and** $\nabla f \in (L^p(\Omega))^N\}$ **for** $1 \leq p \leq \infty$ **.**
$W_0^{1,p}(\Omega)$	**Closure of** $D(\Omega)$ **in** $W^{1,p}(\Omega)$ **for** $1 \leq p < \infty$ **.**
$T_0^{1,p}(\Omega)$	$\{f : \Omega \mapsto I\!R \ / \ T_k(f) \in W_0^{1,p}(\Omega), \ \forall k > 0\}$ **for** $1 \leq p < \infty$ **.**
X	**arbitrary Banach space**
X'	**dual space of the Banach space** X**.**
$C([0,T];X)$	**function space of continuous functions defined on** $[0,T]$ **with values in** X **.**
$D([0,T];X)$	**test functions with values in** X **.**
$L^p(0,T;X)$	**for** $1 \leq p < \infty, \ \{f : (0,T) \mapsto X \ / \ \int_0^T \|f(t)\|_X^p \, dt < \infty\}$ **.**
$L^\infty(0,T;X)$	$\{f : (0,T) \mapsto X \ / \ ess \sup_{t\in(0,T)} \|f(t)\|_X < \infty\}$.

INTRODUCTION GÉNÉRALE

L'étude des équations aux dérivées partielles (EDP) occupe une place importante dans les sciences, car ces équations possèdent une base solide en physique surtout à l'étude de la mécanique élastique et des fluides électrorhéologiques, elles apparaissent aussi bien en dynamique des structures que dans les théories de la gravité, des mathématiques financières (équation de Black-Scholes) ou de l'électromagnétisme (équations de Maxwell). Elles sont primordiales dans des domaines tels que la synthèse d'images, la simulation aéronautique, ou la prévision météorologique, les équations de Schrödinger et Heisenberg pour la mécanique quantique, sans oublier les célèbres équations de la théorie de la relativité. Ces équations proviennent très souvent des bilans globaux traduisant des axiomes ainsi que des paramétrisations rendant compte des lois des phénomènes observés. Elles ont également de nombreuses applications dans différents domaines de recherche (voir, par exemple, [[64]-[24]] et les références qu'ils contiennent) et soulèvent de nombreux problèmes mathématiques difficiles.

Motivations physiques et Mathématiques : L'objectif essentiel de ce travail est l'étude de certains problèmes d'équations aux dérivées partielles non linéaires (ou fortement non linéaires) de type elliptiques et paraboliques dans des domaines non-bornés en utilisant des espaces fonctionnels appropriés, faisant intervenir des opérateurs divergentiels du type de Leray-Lions. Pendant les dernières décennies, de nombreuses recherches ont été menées sur ce genre de questions, l'intérêt provient essentiellement de deux raisons : leurs difficultés mathématiques spécifiques, qui les rendent difficiles à la recherche, et, en outre, le fait que ces équations apparaissent naturellement dans plusieurs branches de la physique mathématique.

En effet, a titre d'exemples lorsque Ω est $\mathbb{R}^3$, l'équation :

$$\begin{cases} \Delta u + \dfrac{1}{1+|x|^2} e^{2u} = 0 & \text{dans } \mathbb{R}^3 \\ u > 0 & \text{sur } \mathbb{R}^3, \end{cases} \tag{0.0.1}$$

a été proposée par Eddington vers 1915 pour modéliser des amas globulaires d'étoiles, après dans le but d'améliorer le modèle d'Eddington en 1930 Matukuma a proposé le modèle suivant :

$$\begin{cases} \Delta u + \dfrac{1}{1+|x|^2} u^p = 0 & \text{dans } \mathbb{R}^3 \text{ où } p > 1, \\ u > 0 & \text{sur } \mathbb{R}^3, \end{cases} \tag{0.0.2}$$

où u représente le potentiel gravitationnel (donc $u > 0$), $\rho = \dfrac{-\Delta u}{4\pi} = \dfrac{u^p}{4\pi(1+|x|^2)}$ représente la densité et $\int \rho dx$ représente la masse totale,

Les équations (0.0.1) et (0.0.2) existent depuis longtemps, au moins au Japon. En 1938, Matukuma a trouvé une solution exacte intéressante

$$u(r, \sqrt{3}) = \left(\frac{3}{1+r^2} \right)^{\frac{1}{2}} \text{ où } p = 3.$$

Après, ces équations ont été généralisées au cadre suivant

$$\frac{1}{r^{N-1}} \left(r^{N-1} u' \right)' = - \frac{r^{\lambda-2}}{(1+r^2)^{\frac{\lambda}{2}}} u^p \quad p > 1, \quad \lambda > 0, N > 3.$$

et elles ont été étudiées par plusieurs auteurs Wang, Zhang, Alarcón, Batt, Ni et Yotsutani et al. qui ont généralisé les conjectures de Matukuma et de nombreux résultats ont été obtenus concernant l'existence et l'unicité de solutions radiales positives (voir [51], [62], [70], [71], [72], [68], [3], [15] et les références qu'ils contiennent).

Un autre exemple d'application lorsque $\Omega = \mathbb{R}^N$ l'équation :

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(\nabla u^{p-2} \nabla u) + g(u) = \nabla u^p + f & \text{in } Q_T := \Omega \times (0, T), \\ u(.,0) = u_0 & \text{in } \Omega \end{cases}$$

décrit des processus de diffusion accompagnés de termes d'absorption proportionnels à la concentration représentée par la solution u (voir [50]), on peut aussi citer entre autre l'équation non linéaire de Klein-Gordon (NLKG) qui apparaît dans la physique quantique des champs en tant que modèle pour un champ scalaire non linéaire auto-interactif (voir [67]). Dans le contexte des problèmes de transmission à longue distance ou de transmission à travers des structures particulières, avec un rapport longueur / diamètre important, comme les nanostructures on aura affaire à des domaines non-bornés.

Les applications des EDP dans des domaines non-bornés ne se restreignent pas seulement en Physique ou en Sciences de la vie ou bien en Économie mais il a des applications en Mathématiques elles-mêmes en Géométrie par exemple si on considère les équations

$$-\Delta u + Ke^{2u} = 0 \quad \text{dans} \quad \mathbb{R}^2$$

qui sont des équations de courbure gaussienne conforment en géométrie riemannienne, ces équations surviennent lorsque l'on essaie de déformer conformément à une métrique g existante sur $\mathbb{R}^2$ de sorte que la courbure scalaire (ou gaussienne) de la nouvelle métrique est la fonction prescrite K.

La théorie de résolution des équations différentielles se fond sur le choix du triplet d'évolution (V, H, V') (où V est un espace de Banach et H un espace de Hilbert tels que $V \underset{d}{\hookrightarrow} H \hookrightarrow V'$), et la majorité des méthodes utilisées se basent sur la compacité et conduisaient à des théorèmes de point fixe, tel que le théorème de Schauder, et à la notion de degré topologique. Ces méthodes sont fondées sur le principe suivant : Remplacer le problème, à l'aide de la compacité, par un problème approché en dimension finie. Parmi ces méthodes on trouve :

— La méthode des opérateurs monotones et pseudo-monotones de Browder, Minty (1963 (voir aussi [52])) et Brezis (1963 (voir aussi [74])) : Dans cette méthode, on se ramène à des sous-espaces de dimension finie par projection. Ceci permet de s'affranchir de l'hypothèse de compacité en la remplaçant par des conditions de continuité.

— La méthode de Galerkin : On procède par approximation dans une suite croissante de sous-espaces de dimension finie, ensuite on résout le problème approché (en choisissant une base spéciale), ce qu'est en général plus facile que la résolution directe en dimension infinie puis on passe à la limite pour revenir au problème initial.

Pour appliquer les techniques d'analyse non linéaire à ce genre de problèmes il semble naturel de prendre les espaces de Sobolev à exposant constant comme choix d'espace fonctionnel, et dans ce cas il existe de nombreux résultats d'existence de solutions. Cependant, avec l'émergence de problèmes non linéaires en Sciences naturelles et en Ingénierie, les espaces de Sobolev avec p constant démontrent leurs limites dans les applications. Une classe de problèmes non linéaires à croissance exponentielle variable, par exemple, constitue un nouveau domaine de recherche et reflète un nouveau type de phénomènes physiques. Comme les espaces $L^{p(x)}$ et $W^{1,p(x)}$(qui sont des extensions naturelles du cadre constant) ont été étudiés de manière approfondie par Kováčik et Rákosník ([49]), ils ont été utilisés au cours des dernières décennies pour modéliser divers phénomènes. En ([64]) Rŭžička a présenté la théorie mathématique pour l'application d'espaces de Sobolev à exposant variable dans les fluides électro-rhéologiques. Ces dernières années, les équations différentielles et les problèmes variationnels liés à des conditions de croissance de $p(x)$ ont été étudiés de manière approfondie (par exemple, Alves et Souto ([5]) ; Chabrowski et Fu ([23]) ; Mihăilescu et Rădulescu ([60]) ; Antontsev et al. ([6])). Pour p variable lorsque le domaine est borné, on trouve de nombreux résultats d'existence(voir par exemples [[9], [17], [73], [39]] pour le cadre elliptique et [[4], [14], [8], [18]] pour le cadre parabolique), mais lorsque le domaine est non-borné, les résultats d'existence sont très rares. Durant cette étude qui se compose de deux grandes parties ; une première qui vise les équations elliptiques à exposant variable et une deuxième qui traite le cadre paraboliques avec p constant et variable ; on a rencontré plusieurs difficultés dues aux exposants variables d'une part et à la non-bornitude de Ω de l'autre part.

Cette thèse est organisée comme suit :
Le premier chapitre de cette thèse est totalement consacré à définir les espaces fonctionnels appropriés (des espaces de Lebesgue-Sobolev à exposant constant et variable), dans lesquels nous allons étudier et résoudre des problèmes elliptiques et paraboliques. On commence par introduire l'ensemble des exposants variables $C_+(\overline{\Omega})$ par

$$C_+(\overline{\Omega}) = \{\text{fonction log-Hölder-continue} \quad p : \overline{\Omega} \longrightarrow I\!R \quad \text{tel que} \quad 1 < p_- \leq p_+ < N\}.$$

Cette définition nous permet d'obtenir la séparabilité et la réflexivité des espaces Banach $L^{p(x)}(\Omega)$, et d'autres résultats qui nous permettront de résoudre nos équations dans une suite d'espaces bornés. Aussi, nous allons citer quelques lemmes indispensables pour obtenir les résultats d'existence et de régularité pour les solutions.

La suite de ce travail est consacrée à l'étude de quelques problèmes de Dirichlet elliptiques et paraboliques dans le cadre des espaces de Lebesgue et Sobolev. On le divise en deux parties principales.

Plus précisément on s'intéresse dans la première partie aux problèmes stationnaires qui sont modélisés par :

$$\begin{cases} -\Delta_{p(x)}(u) + \alpha_0|u|^{p(x)-2}u = d(x)\frac{|\nabla u|^{p(x)}}{|u|^{p(x)}+1} + f - \operatorname{div} g(x) \quad \text{in } \Omega, \\ u \in W_0^{1,p(x)}(\Omega), \end{cases} \tag{0.0.3}$$

puis on généralise ces résultats en montrons l'existence et la régularité de la solution pour les problèmes fortement non linéaire du type :

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) + c(x,u) + H(x,u,\nabla u) = f(x) - \operatorname{div} g(x) \quad \text{in } \Omega, \\ u \in W_0^{1,p(.)}(\Omega), \end{cases} \tag{0.0.4}$$

Ce type de problèmes a été abordé dans le cas p constant ou $p = 2$ par plusieurs mathématiciens entre autres D. Giachetti, P. Donato (voir [34]), P. Drabek, FA. Nicolosi (voir [36]), A. Dall-Aglio, J.-P. Puel, V. De Cicco,G. Bottaro, M.E. Marina (voir [22]), P.L. Lions (voir [53], [53]), M.Chicco, M.Venturino (voir [25]), dans [34] les auteurs ont montré des résultats d'existence et de régularité pour le cas où la partie principale est un opérateur quasi-linéaire à croissance linéaire ($p = 2$), et $f(x) \in L^2(\Omega) \cap L^\infty(\Omega)$ (voir aussi [36]). Dans le cas où la partie principale est un opérateur linéaire du second ordre uniformément elliptique sous forme variationnelle à coefficients discontinus Maurizio Chicco et Mariana Venturino ont montré quelques inégalités a priori dans $L^\infty(Q)$ pour les sous-solutions (voir [25], (voir aussi [53] et [53]) pour le cadre linéaire), pour le cadre non linéaire (voir [29]) A. Dall'aglio et al. ont prouvé l'existence de solutions bornées sans supposer que $g(x, s, \xi)s \geq 0$ et avec les hypothèses :

— $f_{|\Omega \cap B_{R_0}} \in L^m(\Omega \cap B_{R_0}), m > \max(\frac{N}{p}, 1)$

— A l'extérieur de B_{R_0}, f vérifie la condition suivante $|f| \leq \dfrac{C}{|x|^r}$

où B_{R_0} est la boule de centre 0 et de rayon R_0. Dans [35], le résultat d'existence en espaces de Sobolev pondérés a été prouvé. Jeff dans [69] a prouvé l'existence de solutions variationnelles pour l'équation $Au(x) + c(x, u) = f(x)$ avec des conditions aux limites de type Dirichlet ou Neumann, où A est un opérateur différentiel partiel elliptique non linéaire sous forme de divergence et le terme $c(x, u)$ est fortement non linéaire satisfaisant la condition de signe. Dans [28] et [29] et [30], les auteur ont prouvé l'existence et une certaine régularité de la solution de l'équation

$$\begin{cases} -\Delta_p(u) + \alpha_0 |u|^{p-2}u = d(x)|\nabla u|^p + f - \operatorname{div} g(x) \quad \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega), \end{cases}$$

dans le présent travail la mise en place d'une telle approche ne peut pas être utilisée directement, en raison de la variabilité de p et le passage de la norme aux marques modulaires (intégrales) fait apparaître des constantes indésirables dans la preuve de l'estimation a priori (qui est un résultat principal pour l'existence et la régularité), pour résoudre ce problème, nous avons partitionné le domaine dans le problème approximatif et nous avons utilisé le fait que p est continu pour trouver une relation locale sur l'exposant ((voir (3.3.24)), cette relation a joué un rôle important dans la preuve),

Le cas $p(.)$ variables a été traité par exemple dans [[75], [37], [35], [44], [69], [38], [28], [30], [29]].

Lorsque $\Omega = \mathbb{R}^N$ dans [75] Q. Zhang a obtenu des conditions suffisantes d'existence de solutions radiales des équations $p(x)$-laplaciennes où $f \equiv g \equiv H \equiv 0$ en utilisant la méthode variationnelle. Dans [27] au moyen d'une approche variationnelle directe et de la théorie des espaces de Sobolev à exposant variable, des conditions suffisantes garantissant l'existence d'une infinité de solutions homocliniques radialement symétriques distinctes sont établies pour les équations (0.0.4) où $H(x,u,\nabla u) = H(x,u) \quad et \quad f \equiv g \equiv 0$, tenant en compte ces restrictions et dans le cas où $c(x,u) = v(x)|u|^{p(x)-2}u$ les auteurs dans [37] ont étudié l'existence d'une infinité de solutions pour une classe de $p(x)$-laplaciennes équations dans $\mathbb{R}^N$, lorsque la non-linéarité est sublinéaire en u à l'infini. Les auteurs de [44] ont obtenu l'existence de solutions pour le problème $p(x)$-laplacien dans le cas super-linéaire utilisant le théorème de Mountain Pass, Tandis que dans nos travaux en [10] et en [11] on a établi des résultats d'existence de régularités pour les solutions des problèmes fortement non linéaires $p(x)$-elliptique sans condition de signe sur H.

La deuxième partie de ce travail est consacrée aux problèmes d'évolution au cours du temps dans lesquels on peut prendre comme cadre de référence les équations de types :

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} a(u,\nabla u) + \mu|u|^{p(\cdot)-2}u = H(u,\nabla u) + f - div\ g & \text{in } Q_T := \Omega \times (0,T), \\ u = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ u(.,0) = u_0 & \text{in } \Omega \end{cases} \tag{0.0.5}$$

où la partie principale de l'opérateur se comporte comme l'opérateur $p(x)$-laplacien, les termes d'ordre inférieur, qui dépendent de la solution u et de son gradient ∇u, ont une croissance de puissance d'ordre $p(x) - 1$ et $p(x)$ respectivement par rapport à ces variables, et les termes sources appartiennent à des espaces de Lebesgue avec des conditions spécifiques, dans un premier temps on prend $p(x) = p$ constant puis on traite le cadre $p = p(x)$ variable. Pour le cas des domaines bornés à exposant constant, plusieurs études ont été obtenues, par exemple nous mentionnons les travaux [[42] - [79]] (et les références qui y figurent). Dans [31], l'existence de solutions non-bornées a été prouvée sous une hypothèse de signe sur le terme de premier ordre, tandis que V. Ferone et al. in [42] a prouvé l'existence de solutions non-bornées pour le problème du type (0.0.5) où $\mu = 0$. Tenant compte du fait qu'il existe différentes méthodes générales pour résoudre l'analogue (0.0.5) lorsque Ω est borné, ces arguments échouent dans la situation ci-dessus à cause de perte de compacité. En ce qui concerne les domaines non-bornés, des problèmes paraboliques de ce type ont été étudiés sous des hypothèses de régularité supplémentaires sur f et u_0 nous rappelons par exemple les résultats de [30] et [20], les auteurs de [30] ont prouvé l'existence et une certaine régularité de la solution dans le cas d'équations paraboliques où $\mu = 0$ et $g \equiv 0$, notons que dans cette thèse nous abandonnons l'hypothèse de petitesse du terme source u_0 (cité dans [30]) et f compte tenu de la présence du terme $c(x,u)$ avec quelques hypothèses spécifiques (voir la deuxième section), dans [20] le problème (0.0.5) a été étudié en supposant que $c(s) = |s|^{\sigma-1}s, \quad d \equiv 0, \quad g \equiv 0$ avec $\sigma > p-1$ et $p > 2 - 1/(N+1)$, nous étendons le résultat de base de [20] dans différentes directions en considérant en fait tout $N > p > 1$. La première préoccupation du mathématicien confronté à une équation aux dérivées partielles est de donner un sens à l'équation aux dérivées partielles dans des espaces fonctionnels appropriés et d'y démontrer l'existence et l'unicité de la solution. Lors de cette démarche, diverses difficultés liées au type de l'équation (elliptique, parabolique, hyperbolique, dégénéré, ...), à la régularité des données et aussi à la définition des espaces appropriés, dans le cas d'une équation parabolique où p est constant il est bien connu que $L^p(0,T;W_0^{1,p}(\Omega))$ peut être pris comme un espace de solutions, par contre, lorsque l'exposant est variable, alors ni $L^{p(.)}(0,T;W_0^{1,p(.)}(\Omega))$ ni $L^{p_-}(0,T;W_0^{1,p(.)}(\Omega))$, où $p_- = ess\min_{x\in\overline{\Omega}} p(x)$, constituent un espace de solutions approprié (voir [18].) Désormais, pour surmonter cette difficulté, nous définirons ci-dessous notre espace fonctionnel de solutions V comme cela a été fait par Bendahmane et al. dans [18]. Il y a beaucoup de différence entre les espaces de Lebesgue et Sobolev à exposant constant et ceux avec des exposants variables par exemple les espaces à $p(.)$ variables ne sont pas invariants en translation, lorsque p est constant, il est bien connu que $W_0^{1,p}(\Omega)$ (la fermeture de $C_0^\infty(\Omega)$ dans $W^{1,p}(\Omega)$) est identique à $H_0^{1,p}(\Omega) := \{f \in L^p(\Omega), \nabla f \in L^p(\Omega), f_{|\partial\Omega=0}\}$. Cependant, lorsque p est une fonction, il existe un phénomène intéressant de Lavrentiev [78], qui montre que les deux espaces ci-dessus ne sont pas équivalents, etc., nous renvoyons à monograph [33] pour plus de détails et plus de références. Par conséquent, les résultats mathématiques sur les équations des exposants variables sont loin d'être parfaits. Cependant, à notre connaissance, aucun résultat sur le cas de $p(x)$ - équation parabolique dans les domaines non-bornés n'a été obtenu jusqu'à présent.

Basé sur le fait ci-dessus et motivé par les techniques utilisées dans [[41], [66]], l'objectif principal est d'établir des résultats d'existences d'au moins une solution faible aux problèmes (0.0.3), (0.0.4), et (0.0.5) et donner une certaine régularité à ces solutions. Notre analyse est basée sur la théorie des espaces de Lebesgue-Sobolev à exposant constant et variable et la théorie des opérateurs monotones dans les espaces réflexifs de Banach. L'utilisation de fonctions test de type exponentielles permettent de se débarrasser du terme $H(x,u,\nabla u)$ et est donc un outil essentiel dans les démonstrations.

Pour obtenir le résultat d'existence, puisque Ω peut avoir une mesure infinie, nous procédons en résolvant le problème dans une séquence Ω_n d'espaces bornés telle que $\cup_{n\geq 1}\Omega_n = \Omega$, après nous passons à la limite dans les problèmes d'approximation. Dans ce but, nous ne pouvons pas utiliser les théorèmes d'injection entre les espaces $L^{p(\cdot)}(\Omega)$ ni aucun argument impliquant la mesure de Ω_n. Pour la régularité, nous utiliserons une adaptation d'une technique classique due à Stampacchia [66].

TRAVAUX EFFECTUÉS

- **TRAVAUX PUBLIÉS**

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- **TRAVAUX SOUMIS**

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Chapitre 1

PRELIMINARIES

In order to discuss the problems (0.0.3), (0.0.4), (0.0.5) we need some theories on spaces $L^{p(.)}(\Omega)$, $W^{1,p(.)}(\Omega)$ and $W_0^{1,p(.)}(\Omega)$. For this reason this chapter will be devoted to introduce some interesting definitions and basic properties of Lebesgue and Sobolev spaces with variable exponents, which are essential to prove some results of existence and regularity for solutions of the nonlinear elliptic and parabolic problems studied in this thesis.

1.1 Variable Exponent Lebesgue and Sobolev Spaces

Let Ω an open bounded set of $\mathbb{R}^N$ with $N \geq 2$. We say that a real-valued continuous function $p(.)$ is log-Hölder continuous in Ω if :

$$|p(x) - p(y)| \leq \frac{C}{|\log|x-y||} \quad \forall x, y \in \overline{\Omega} \text{ such that } |x-y| < \frac{1}{2}.$$

We denote :

$$C_+(\overline{\Omega}) = \{\text{log-Hölder continuous function } p : \overline{\Omega} \to \mathbb{R} \text{ with } 1 < p_- \leq p_+ < N\},$$

where :

$$p_- = ess \min_{x\in\overline{\Omega}} p(x) \quad p_+ = ess \sup_{x\in\overline{\Omega}} p(x).$$

We define the variable exponent Lebesgue space for $p \in C_+(\overline{\Omega})$ by :

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^{p(x)} dx < \infty\},$$

the space $L^{p(x)}(\Omega)$ under the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leq 1\big\}$$

is a uniformly convex Banach space, and therefore reflexive. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see [40, 76]).

Proposition 1.1.1 (Generalized Hölder inequality [40, 76]) *(i) For any functions $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have :*

$$|\int_\Omega uvdx| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}.$$

(ii) For all $p_1, p_2 \in C_+(\overline{\Omega})$ such that : $p_1(x) \leq p_2(x)$ a.e. in Ω, we have $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 1.1.2 ([40, 76]) *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}dx \quad \forall u \in L^{p(x)}(\Omega),$$

then, the following assertions hold

(i) $\|u\|_{L^{p(x)}(\Omega)} < 1$ (resp, $= 1, > 1$) if and only if $\rho(u) < 1$ (resp, $= 1, > 1$);

(ii) $\|u\|_{L^{p(x)}(\Omega)} > 1$ implies $\|u\|^{p_-}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq \|u\|^{p_+}_{L^{p(x)}(\Omega)}$, and $\|u\|_{L^{p(x)}(\Omega)} < 1$ implies $\|u\|^{p_+}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq \|u\|^{p_-}_{L^{p(x)}(\Omega)}$;

(iii) $\|u_n - u\|_{L^{p(x)}(\Omega)} \to 0$ if and only if $\rho(u_n - u) \to 0$, and $\|u_n - u\|_{L^{p(x)}(\Omega)} \to \infty$ if and only if $\rho(u_n - u) \to \infty$.

Now, we define the variable exponent Sobolev space by :

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm :

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)} \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$.

Proposition 1.1.3 ([40, 45]) *(i) Assuming $1 < p_- \leq p_+ < \infty$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) If $q \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \Omega$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.

(iii) Poincaré inequality : There exists a constant $C > 0$, such that

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

(vi) Sobolev-Poincaré inequality : there exists an other constant $C > 0$, such that

$$\|u\|_{L^{p*(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

Remark 1.1.1 *By (iii) of the Proposition 1.1.3, we deduce that $\|\nabla u\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms in $W_0^{1,p(x)}(\Omega)$.*

Theorem 1.1.1 ***(see [33])*** *We denote the dual of the Sobolev space $W_0^{1,p(x)}(\Omega)$ by $W^{-1,p'(x)}(\Omega)$, and for each $F \in W^{-1,p'(x)}(\Omega)$ there exists $f_0, f_1, \ldots, f_N \in L^{p'(x)}(\Omega)$ such that $F = f_0 + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}$. Moreover, for all $u \in W_0^{1,p(x)}(\Omega)$ we have*

$$\langle F, u\rangle = \int_\Omega f_0 u \, dx - \sum_{i=1}^{N} \int_\Omega f_i \frac{\partial u}{\partial x_i} \, dx,$$

where the norm on the dual space is defined by

$$\|F\|_{-1,p'(x)} = \sum_{i=0}^{N} \|f_i\|_{p'(x)}.$$

Like in [18] let us introduce the adequate space to discuss the problem (0.0.5). Let $Q_T = \Omega \times (0,T)$ with $0 < T < \infty$. Extending the definition of variable exponent $p(\cdot) : \overline{\Omega} \longmapsto]1, N[$ to $\overline{Q_T}$ by setting $p(x,t) := p(x)$ for all $(x,t) \in Q_T$, we define

$$C_+(\overline{Q_T}) = \{p(\cdot,\cdot) : \overline{Q_T} \longrightarrow I\!R \quad \text{such that} \quad p(x,t) = p(x) \in C_+(\overline{\Omega})\},$$

and we consider for $p(\cdot) \in C_+(\overline{Q_T})$ the generalized Lebesgue space

$$L^{p(x)}(Q_T) := \left\{u : Q_T \longmapsto I\!R, \quad \text{measurable,} \quad \text{such that} \quad \int_{Q_T} |u(x,t)|^{p(x)} \, dx \, dt < \infty\right\},$$

endowed with the norm

$$\|u\|_{L^{p(x)}(Q_T)} := \inf_{\lambda > 0} \left\{ \int_{Q_T} \left|\frac{u(x,t)}{\lambda}\right|^{p(x)} dx \, dt \leq 1\right\},$$

which, of course shares the same properties as $L^{p(x)}(\Omega)$. Let

$$V(Q_T)=\{u\in L^{p_-}(0,T;W^{1,p(x)}(\Omega));|\nabla u|\in L^{p(x)}(Q_T)\},$$

and

$$V_0(Q_T)=\{u\in L^{p_-}(0,T;W_0^{1,p(x)}(\Omega));|\nabla u|\in L^{p(x)}(Q_T)\},$$

endowed with the norm :

$$||u||_{V_0(Q_T)}=||\nabla u||_{L^{p(x)}(Q_T)},$$

or, the equivalent norm :

$$||u||_{V_0(Q_T)}=||\nabla u||_{L^{p_-}(0,T;W_0^{1,p(x)}(\Omega))}+||\nabla u||_{L^{p}(x)(Q_T)},$$

is a separable and reflexive Banach space. Using Poincaré's inequality and the continuous embedding $L^{p(x)}(Q_T)\hookrightarrow L^{p_-}(0,T;L^{p(x)}(\Omega))$ we can deduce the equivalence of the two norms. We state some further properties of $V_0(Q_T)$ in the following lemma.

Lemma 1.1.1 ([18])

Let $V(Q_T)$ as defined above and $V'(Q_T)$ its dual space. Then,

— *we have the following continuous dense embeddings :*

$$L^{p_+}(0,T;W_0^{1,p(x)}(\Omega))\hookrightarrow^d V_0(Q_T)\hookrightarrow^d L^{p_-}(0,T;W_0^{1,p(x)}(\Omega)).$$

In particular, since $D(Q_T)$ is dense in $L^{p_+}(0,T;W_0^{1,p(x)}(\Omega))$, it is also dense in $V_0(Q_T)$ and for the corresponding dual spaces we have :

$$L^{p_-{}'}(0,T;(W_0^{1,p(x)}(\Omega))')\hookrightarrow V_0'(Q_T)\hookrightarrow L^{p_+{}'}(0,T;(W_0^{1,p(x)}(\Omega))').$$

— *One can represent the elements of $V_0'(Q_T)$ as follows : let $G\in V_0'(Q_T)$, then there exists $F=(f_1,f_2,...,f_N)\in(L^{p'(x)}(Q_T))^N$ such that $G=-div(F)$ and*

$$<G,u>_{V'(Q_T),V_0(Q_T)}=\int_0^T\int_\Omega F.\nabla u dxdt,$$

for any $u\in V_0(Q_T)$.

1.2 Some Important Technical Propositions and Lemmas

Proposition 1.2.1 *For any $u\in L^{p(x)}(Q_T)$ and $v\in L^{p'(x)}(Q_T)$, we have*

$$\left|\int_{Q_T}u\,v\,dx\,dt\right|\le\Big(\frac{1}{p_-}+\frac{1}{p'_-}\Big)\|u\|_{L^{p(x)}(Q_T)}\,\|v\|_{L^{p'(x)}(Q_T)}\,.$$

Lemma 1.2.1 *Let* $g \in L^{r(x)}(Q_T)$ *and* $g_n \in L^{r(x)}(Q_T)$ *with* $\|g_n\|_{L^{r(x)}(Q_T)} \leq C$ *for* $1 < r(x) < \infty$. *If* $g_n(x,t) \to g(x,t)$ *a.e. on* Q_T, *then* $g_n \rightharpoonup g$ *in* $L^{r(x)}(Q_T)$.

Proof. The proof is similar to the proof of (Lemma 3.3, [16]) by taking Q_T instead Ω.

Definition 1.2.1 *For all* $k > 0$ *and* $s \in \mathbb{R}$, *the truncation function* $T_k(.)$ *is defined by*

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k. \end{cases}$$

and we define

$$T_0^{1,p(x)}(\Omega) := \{\text{measurable function} \quad u \quad \text{such that} \quad T_k(u) \in W_0^{1,p(x)}(\Omega) \quad \forall k > 0\}.$$

Proposition 1.2.2 ***(see. [?], [?])*** *Let* $u \in T_0^{1,p(x)}(\Omega)$, *there exists a unique measurable function* $v : \Omega \longmapsto \mathbb{R}^N$ *such that*

$$v.\chi_{\{|u| \leq k\}} = \nabla T_k(u) \qquad \text{for} \quad a.e. \ x \in \Omega \qquad \text{and} \qquad \text{for all} \quad k > 0.$$

We will define the gradient of u *as the function* v, *and we will denote it by* $v = \nabla u$. *Moreover, if* $u \in W_0^{1,1}(\Omega)$, *then* v *coincides with the standard distributional gradient of* u,.

Lemma 1.2.2 *Let* $\lambda \in \mathbb{R}$ *and let* u *and* v *be two functions which are finite almost everywhere, and which belong to* $\mathcal{T}_0^{1,p(x)}(\Omega)$, *then*

$$\nabla(u + \lambda v) = \nabla u + \lambda \nabla v \quad \text{a.e. in } \Omega,$$

where ∇u, ∇v *and* $\nabla(u + \lambda v)$ *are the gradients of* u, v *and* $u + \lambda v$ *introduced in the Proposition 1.2.2.*

Lemma 1.2.3 ***(see [2])*** *Let* $g \in L^{r(x)}(\Omega)$ *and* $g_n \in L^{r(x)}(\Omega)$ *with* $\|g_n\|_{r(x)} \leq C$ *for* $1 < r(x) < \infty$. *If* $g_n(x) \to g(x)$ *a.e. on* Ω, *then* $g_n \rightharpoonup g$ *in* $L^{r(x)}(\Omega)$.

Lemma 1.2.4 *Let* $u \in W_0^{1,p(x)}(\Omega)$ *then* $T_k(u) \in W_0^{1,p(x)}(\Omega)$ *for all* $k > 0$. *Moreover, we have*

$$T_k(u) \longrightarrow u \qquad \text{in} \quad W_0^{1,p(x)}(\Omega) \quad \text{as} \quad k \to \infty.$$

Proof

Let $k > 0$, for $u \in W_0^{1,p(x)}(\Omega)$ we have $T_k(u) \in W_0^{1,p(x)}(\Omega)$, and

$$\int_\Omega |T_k(u) - u|^{p(x)} dx + \int_\Omega |\nabla T_k(u) - \nabla u|^{p(x)} dx = \int_{\{|u|>k\}} |T_k(u) - u|^{p(x)} dx + \int_{\{|u|>k\}} |\nabla u|^{p(x)} dx,$$

since $T_k(u) \to u$ as $k \to \infty$ and by using the dominated convergence theorem, we have

$$\int_{\{|u|>k\}} |T_k(u) - u|^{p(x)} dx + \int_{\{|u|>k\}} |\nabla u|^{p(x)} dx \longrightarrow 0 \qquad \text{as} \quad k \to \infty,$$

then $\|T_k(u) - u\|_{1,p(x)} \longrightarrow 0$ as $k \to \infty$.

Lemma 1.2.5 *Let* $(u_n)_n$ *be a bounded sequence in* $W_0^{1,p(x)}(\Omega)$ *.*
If $u_n \rightharpoonup u$ *in* $W_0^{1,p(x)}(\Omega)$ *(weak) then* $T_k(u_n) \rightharpoonup T_k(u)$ *in* $W_0^{1,p(x)}(\Omega)$ *(weak).*

Proof

We have

$$\begin{aligned} u_n \rightharpoonup u \;\; in \;\; W_0^{1,p(x)}(\Omega)\,(weak) \;\; &\Rightarrow \;\; u_n \to u \;\; in \;\; L^{q(x)}(\Omega)\,(strong), \quad \text{for all} \quad 1 \le q(x) < p^*(x), \\ &\Rightarrow \;\; u_n \to u \qquad a.e \quad in \;\; \Omega, \\ &\Rightarrow \;\; T_k(u_n) \to T_k(u) \qquad a.e \quad in \;\; \Omega, \end{aligned}$$

and since

$$T'_k(s) = \begin{cases} 1 & \text{if} \quad |s| \le k, \\ 0 & \text{if} \quad |s| > k, \end{cases} \qquad \text{then} \quad \int_\Omega |\nabla T_k(u_n)|^{p(x)} dx \le \int_\Omega |\nabla u_n|^{p(x)} dx \; < \infty,$$

we deduce that $(T_k(u_n))_n$ is bounded in $W_0^{1,p(x)}(\Omega)$, then $T_k(u_n) \to v_k$ a.e in Ω, therefore $v_k = T_k(u)$ and we obtain

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in} \quad W_0^{1,p(x)}(\Omega)\,(weak).$$

The following Lemma due to Gronwall-Bellman would be of great importance in the estimation phase (see [55] and references therein).

Lemma 1.2.6 ([55])

Let $z(t)$ *and* $f(t)$ *be nonnegative continuous functions on* $0 \le t \le T$*, for which the inequality*

$$z(t) \le c + \int_0^t z(s) f(s) ds, \qquad t \in [0, T]$$

holds, where $c \geq 0$ is a constant. Then :

$$z(t) \leq c \quad exp(\int_0^t f(s)ds), \qquad t \in [0,T].$$

Characterization of the time mollification of a function u.

To deal with time derivative, we introduce a time mollification of a function u belonging to some Lebesgue space. Thus we define for all $\sigma \geq 0$ and all $(x,t) \in Q_T$

$$u_\sigma = \sigma \int_\infty^t \tilde{u}(x,s)e^{\sigma(s-t)}ds$$

where $\tilde{u}(x,s) = u(x,s)\chi_{(0;T)}(s)$. Note that in this section, we omit the proof of each of the above Proposition and Lemmas (since it is a slight modification of its analogous in [2]).

Proposition 1.2.3 ([2])

— *If $u \in L^{p(x)}(Q_T)$, then u_σ is measurable in Q_T, $\frac{\partial u_\sigma}{\partial t} = \sigma(u - u_\sigma)$ and $||u_\sigma||_{L^{p(x)}(Q_T)} \leq ||u||_{L^{p(x)}(Q_T)}$.*
— *If $u \in W_0^{1;p(x)}(Q_T)$, then $u_\sigma \to u$ in $W_0^{1;p(x)}(Q_T)$ as $\sigma \to \infty$.*
— *If $u_n \to u$ in $W_0^{1;p(x)}(Q_T)$, then $(u_n)_\sigma \to u_\sigma$ in $W_0^{1;p(x)}(Q_T)$.*

Lemma 1.2.7 ([2])
Assume that :

$$\frac{\partial u_n}{\partial t} = t_n + s_n \in D'(Q_T),$$

where t_n and s_n are bounded respectively in $V'(Q_T)$ and in $L^1(Q_T)$. If u_n is bounded in $V(Q_T)$, then $u_n \to u$ in $L^{p(x)}_{loc}(Q_T)$. Further $u_n \to u$ strongly in $L^1(Q_T)$.

Now we state a well-known Gagliardo-Nirenberg embedding theorem in the framework of Lebesgue-Sobolev and his generalization :

Lemma 1.2.8 ***(Gagliardo-Nirenberg embedding theorem [32])***
Let Ω be a bounded open set of $\mathbb{R}^N$, T be a real positive number and v be a function in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^m(\Omega))$ with $1 < p < N$. Then $v \in L^\rho(0,T;L^\sigma(\Omega))$, where σ is a constant between p^* and m, $p \leq \rho \leq \infty$, $\frac{mN}{\sigma} + \frac{Np-m(N-p)}{\rho} = N$ and the following estimate holds :

$$\int_0^T ||v||^\rho_{L^\sigma(\Omega)}dt \leq C(N,m,p)||v||^{\rho-p}_{L^\infty(0,T;L^m(\Omega))}\int_0^T ||\nabla v||^p_{(L^p(\Omega)^N}dt.$$

Lemma 1.2.9 ([19])

Let v be a function in $W_0^{1,q(.)}(\Omega) \cap L^{\rho(.)}(\Omega)$ with $1 < q_- \leq q(x) \leq q_+ < N$ and $1 < \rho_- \leq \rho(x) \leq \rho_+ < N$. Then there exists a positive constant C, depending on N, $q(x)$ and $\rho(x)$, such that :

$$||v||_{L^{\gamma(.)}(\Omega)} \leq C||\nabla v||^{\theta}_{(L^{q(.)}(\Omega)^N}||v||^{1-\theta}_{L^{\rho(.)}(\Omega)}$$

for every θ and $\gamma(.)$ satisfying :

$$0 \leq \theta \leq 1, \qquad 1 \leq \gamma(.) \leq +\infty, \qquad \frac{1}{\gamma(.)} = \theta\left(\frac{1}{q(.)} - \frac{1}{N}\right) + \frac{1-\theta}{\rho(.)}$$

It is easy to check the following corollary :

Corollary 1.2.1

Let $v \in L^{\infty}(0,T;L^{\rho(.)}(\Omega)) \cap L^m(0,T;W_0^{1,q(.)}(\Omega))$. Then the following estimate holds

$$\int_0^T ||v||^{\sigma}_{L^{\gamma(.)}(\Omega)}dt \leq C_1||v||^{\sigma-m}_{L^{\infty}(0,T;L^{\rho(.)}(\Omega))}\int_0^T ||\nabla v||^m_{(L^{q(.)}(\Omega)^N}dt,$$

where $m \leq \sigma \leq +\infty$, C_1 is a positive constant depending on N, $q(x)$,$\rho(x)$ and σ.

Première partie

EXISTENCE AND REGULARITY RESULTS FOR STATIONARY PROBLEMS IN GENERAL DOMAINS WITH VARIABLE EXPONENT

Chapitre 2

EXISTENCE RESULTS AND BOUNDEDNESS OF SOLUTIONS FOR THE $p(x)$-LAPLACIAN EQUATIONS IN UNBOUNDED DOMAINS

Abstract

This chapter is devoted to establish the existence of solutions for the strongly nonlinear $p(x)$-elliptic problem :

$$\begin{cases} -\Delta_{p(x)}(u) + \alpha_0 |u|^{p(x)-2}u = d(x)\frac{|\nabla u|^{p(x)}}{|u|^{p(x)}+1} + f - \operatorname{div} g(x) \quad \text{in } \Omega, \\ u \in W_0^{1,p(x)}(\Omega), \end{cases}$$

where Ω is an open set of $\mathbb{R}^N$ (possibly of infinite measure), and we will give some regularity results for these solutions.

2.1 Introduction

Recently, there has been an increasing attention in the study of various mathematical problems with variable exponents. These problems are interesting in applications (see [24], [77]). For the usual problems when the exponent is constant, there are many results for existence of solutions when the domain is bounded or unbounded. For $p(.)$ variable, when the domain is bounded, on the results of existence of solutions, we refer to [9], [17], [73], when the domain is unbounded, results of existence of solutions are rare we can cite for example [75], [28].

In the case where Ω is a bounded, and for
$1 < p < N$, In [41] authors studied the problem :

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = H(x,u,\nabla u) + f - \operatorname{div} g \quad \text{in } D'(\Omega), \\ u \in W_0^{1,p}(\Omega), \end{cases}$$

the right hand side is assumed to satisfy :

$$f \in L^{N/p}(\Omega), g \in (L^{N/(p-1)}(\Omega))^N.$$

Under suitable smallness assumptions on the source data (f and g) they prove the existence of a solution u which satisfies a further regularity.

In [47] in the case of unbounded domains Guowei Dai By variational approach and the theory of the variable exponent Sobolev spaces establish the existence of infinitely many distinct homoclinic radially symmetric solutions whose $W^{1,p(x)}(\mathbb{R}^N)$-norms tend to infinity (to zero, respectively) under weaker hypotheses about nonlinearity at infinity (at zero, respectively).

The principal objective of this chapter is to prove the existence and some regularity of solutions of the following $p(x)$-Laplacian equation in open set Ω of $\mathbb{R}^N$ (possibly of infinite measure) :

$$\begin{cases} -\Delta_{p(x)}(u) + \alpha_0|u|^{p(x)-2}u = d(x)\frac{|\nabla u|^{p(x)}}{|u|^{p(x)}+1} + f - \operatorname{div} g(x) \quad \text{in } \Omega, \\ u \in W_0^{1,p(x)}(\Omega), \end{cases} \tag{2.1.1}$$

where p is log-Hölder continuous function such that $1 < p_- \leq p_+ < N$, $\Delta_{p(x)}(u) = div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplace operator, α_0 is a positive constant, d is a function in $L^\infty(\Omega)$. We assume the following hypotheses on the source terms f and g :
$f : \Omega \to \mathbb{R}$, $g : \Omega \to \mathbb{R}^N$ are a measurable function satisfying :

$$\begin{cases} f \in L^{N/p(x)}(\{x \in \Omega : 1 < |f(x)|\}), \\ f \in L^{p'(x)}(\{x \in \Omega : |f(x)| \leq 1\}) \\ g \in L^{N/(p(x)-1)}(\Omega; \mathbb{R}^N) \cap L^{p'(x)}(\Omega; \mathbb{R}^N) \end{cases} \tag{2.1.2}$$

We will proceed by solving the problem on a sequence Ω_n of bounded sets after that we pass to the limit in the approximating problems by using the a priori estimate (this a priori estimates provide the necessary compactness properties for solutions) from which the desired results are easily inferred. To this aim, we can neither use any embedding theorem between $L^{p(.)}(\Omega)$ nor any argument involving the measure of Ω_n, and under suitable assumptions on f and g we prove some regularity of a solutions u of (2.1.1). A similar result has been proved in [28] where p is constant such that $1 < p < N$ but in the present setting such an approach cannot be used exactly, because of the variability of p.

The plan of the chapter is the following : In Section 2 we will define the approximate problems, state the a priori estimates that we want to obtain. In the Section which follow we will prove strong convergence of u_n and their gradients ∇u_n, and we will conclude the proof of the main existence results. Finally, in Section 4, we prove that, if f and g have higher integrability, then every solution u of (2.1.1) is bounded. More precisely, we will assume that (2.1.2) are replaced by :

$$\begin{cases} f \in L^{q(x)}(\{x \in \Omega : 1 < |f(x)|\}) \text{ for some } q(x) > N/p(x), \\ f \in L^{p'(x)}(\{x \in \Omega : |f(x)| \leq 1\}) \\ g \in L^{r(x)}(\Omega; \mathbb{R}^N) \cap L^{p'(x)}(\Omega; \mathbb{R}^N) \text{ for some } r(x) > N/(p(x)-1) \end{cases} \tag{2.1.3}$$

2.2 Approximate problem and A priori estimate

In this section we will prove the existence result to the approximate problems. Also we will give a uniform estimate for this solutions u_n.

Approximate problem and useful lemmas

For $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ is defined by :

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases} \tag{2.2.1}$$

Let $\Omega_n = \Omega \cap B_n(0)$ where $B_n(0)$ is the Ball with center 0 and radius n, we consider the approximate problem :

$$\begin{cases} -\Delta_{p(.)}(u_n) + c(x, u_n) = H_n(x, u_n, \nabla u_n) + f_n - \operatorname{div} g_n & \text{in } \Omega_n, \\ u_n \in W_0^{1,p(.)}(\Omega_n) \cap L^\infty(\Omega_n), \end{cases} \tag{2.2.2}$$

with $c(x,u)=\alpha_0|u|^{p(x)-2}u$, $H_n(x,s,\xi)=T_n(H(x,s,\xi))$, $H(x,s,\xi)=d(x)\frac{|\xi|^{p(x)}}{|s|^{p(x)+1}}$, $f_n(x)=T_n(f(x))$ and $g_n(x)=\frac{g(x)}{1+\frac{1}{n}|g(x)|}$. Let us remark that $|H_n|\leq|H|$, $|H_n|\leq n$,$|f_n|\leq|f|$and $|g_n|\leq|g|$.

Lemma 2.2.1 ([33]) *Let p be a measurable function and $s>0$ such that $sp_{-}>1$ then $\||f|^s\|_{L^{p(x)}(\Omega)}=\|f\|^s_{L^{p(x)}(\Omega)}$ for every f in $L^{p(x)}(\Omega)$.*

Lemma 2.2.2 ([9]) *Let $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ be a Carathéodory function (measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$ for almost every x in Ω) and let us Assume that :*

$$|a(x,s,\xi)|\leq\beta(K(x)+|s|^{p(x)-1}+|\xi|^{p(x)-1}),\tag{2.2.3}$$

$$a(x,s,\xi)\xi\geq\alpha|\xi|^{p(x)},\tag{2.2.4}$$

$$[a(x,s,\xi)-a(x,s,\overline{\xi})](\xi-\overline{\xi})>0\quad \textit{for all } \xi\neq\overline{\xi} \textit{ in } \mathbb{R}^N,\tag{2.2.5}$$

hold, and let $(u_n)_n$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that $u_n\rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and

$$\int_\Omega[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u)]\nabla(u_n-u)dx\to 0,\tag{2.2.6}$$

then $u_n\to u$ in $W_0^{1,p(x)}(\Omega)$ for a subsequence.

Proof

Let $D_n=[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u)]\nabla(u_n-u)$, thanks to (2.2.5) we have D_n is a positive function, and by (2.2.6), $D_n\to 0$ in $L^1(\Omega)$ as $n\to\infty$.
Since $u_n\rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ then $u_n\to u$ a.e in Ω, and since $D_n\to 0$ a.e in Ω, there exists a subset B in Ω with measure zero such that $\forall x\in\Omega\backslash B$

$$|u(x)|<\infty,\qquad |\nabla u(x)|<\infty,\qquad K(x)<\infty,\qquad u_n\to u\qquad and\qquad D_n\to 0.$$

Taking $\xi_n=\nabla u_n$ and $\xi=\nabla u$, we have

$$\begin{aligned}D_n(x)&=[a(x,u_n,\xi_n)-a(x,u_n,\xi)](\xi_n-\xi)\\&=a(x,u_n,\xi_n)\xi_n+a(x,u_n,\xi)\xi-a(x,u_n,\xi_n)\xi-a(x,u_n,\xi)\xi_n\\&\geq\alpha|\xi_n|^{p(x)}+\alpha|\xi|^{p(x)}-\beta(K(x)+|u_n|^{p(x)-1}+|\xi_n|^{p(x)-1})|\xi|-\beta(K(x)+|u_n|^{p(x)-1}+|\xi|^{p(x)-1})|\xi_n|\\&\geq\alpha|\xi_n|^{p(x)}-C_x(1+|\xi_n|^{p(x)-1}+|\xi_n|),\end{aligned}$$

where C_x depending on x, without dependence on n. (since $u_n(x) \to u(x)$ then $(u_n)_n$ is bounded), we obtain

$$D_n(x) \geq |\xi_n|^{p(x)}\big(\alpha - \frac{C_x}{|\xi_n|^{p(x)}} - \frac{C_x}{|\xi_n|} - \frac{C_x}{|\xi_n|^{p(x)-1}}\big),$$

by the standard argument $(\xi_n)_n$ is bounded almost everywhere in Ω, (Indeed, if $|\xi_n| \to \infty$ in a measurable subset $E \in \Omega$ then

$$\lim_{n\to\infty} \int_\Omega D_n(x)\, dx \geq \lim_{n\to\infty} \int_E |\xi_n|^{p(x)}\big(\alpha - \frac{C_x}{|\xi_n|^{p(x)}} - \frac{C_x}{|\xi_n|} - \frac{C_x}{|\xi_n|^{p(x)-1}}\big)\, dx = \infty,$$

which is absurd since $D_n \to 0$ in $L^1(\Omega)$).

Let ξ^* an accumulation point of $(\xi_n)_n$, we have $|\xi^*| < \infty$ and by the continuity of $a(.,.,.)$ we obtain,

$$[a(x, u(x), \xi^*) - a(x, u(x), \xi)](\xi^* - \xi) = 0,$$

thanks to $(2.2.5)$ we have $\xi^* = \xi$, the uniqueness of the accumulation point implies that $\nabla u_n \to \nabla u$ a.e in Ω.

Since $(a(x, u_n, \nabla u_n))_n$ is bounded in $(L^{p'(x)}(\Omega))^N$ and $a(x, u_n, \nabla u_n) \to a(x, u, \nabla u)$ a.e in Ω, by the Lemma $1.2.3$, we can establish that

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \quad in \quad (L^{p'(x)}(\Omega))^N.$$

Let us taking $\bar{y}_n = a(x, u_n, \nabla u_n)\nabla u_n \ and \ \bar{y} = a(x, u, \nabla u)\nabla u$, then $\bar{y}_n \to \bar{y}$ in $L^1(\Omega)$, according to the condition $(2.2.4)$ we have

$$\alpha|\nabla u_n|^{p(x)} \leq a(x, u_n, \nabla u_n)\nabla u_n,$$

Let $z_n = \nabla u_n$, $z = \nabla u$ and $y_n = \frac{\bar{y}_n}{\alpha}$, $y = \frac{\bar{y}}{\alpha}$, in view of the *Fatou Lemma*, we get

$$\int_\Omega 2.y\, dx \leq \liminf_{n\to\infty} \int_\Omega (y_n + y - |z_n - z|^{p(x)})dx,$$

then $0 \leq -\limsup_{n\to\infty} \int_\Omega |z_n - z|^{p(x)}\, dx$, and since

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_n - z|^{p(x)}\, dx \leq \limsup_{n\to\infty} \int_\Omega |z_n - z|^{p(x)}\, dx \leq 0,$$

it follows that $\int_\Omega |\nabla u_n - \nabla u|^{p(x)}\, dx \longrightarrow 0$ as $n \to \infty$, and we get

$$\nabla u_n \longrightarrow \nabla u \quad in \quad (L^{p(x)}(\Omega))^N$$

we deduce that

$$u_n \longrightarrow u \quad in \quad W_0^{1,p(x)}(\Omega),$$

which finishes our proof.

We define the operator :

$R_n : W_0^{1,p(x)}(\Omega_n) \to W^{-1,,p'(x)}(\Omega_n)$, by :

$$\langle R_n u, v\rangle = \int_{\Omega_n} c(x,u)v - H_n(x,u,\nabla u)v dx \quad \forall v \in W_0^{1,p(x)}(\Omega_n).$$

by the Hölder inequality we have that :

for all $u, v \in W_0^{1,p(x)}(\Omega_n)$,

$$\begin{aligned}
&|\int_{\Omega_n} c(x,u)v - H_n(x,u,\nabla u)v dx| \\
&\leq (\frac{1}{p_-} + \frac{1}{p'_-})\big[\|c(x,u)\|_{L^{p'(x)}(\Omega_n)}\|v\|_{L^{p(x)}(\Omega_n)} \\
&+ \|H_n(x,u,\nabla u)\|_{L^{p'(x)}(\Omega_n)}\|v\|_{L^{p(x)}(\Omega_n)}\big] \\
&\leq (\frac{1}{p_-} + \frac{1}{p'_-})\big[\Big(\int_{\Omega_n} \big(|c(x,u)|^{p'(x)} dx + 1\Big)^{\frac{1}{p'_-}} \\
&+ \Big(\int_{\Omega_n} \big(|H_n(x,u,\nabla u)|^{p'(x)} dx + 1\Big)^{\frac{1}{p'_-}}\big]\|v\|_{W^{1,p(x)}(\Omega_n)}
\end{aligned}$$

Then :

$$\begin{aligned}
&|\int_{\Omega_n} c(x,u)v + H_n(x,u,\nabla u)v dx| \\
&\leq (\frac{1}{p_-} + \frac{1}{p'_-})\big[\Big(\int_{\Omega_n} |u|^{p(x)}\big) dx + 1\Big)^{\frac{1}{p'_-}} \\
&+ \Big(\int_{\Omega_n} n^{p'(x)} dx + 1\Big)^{\frac{1}{p'_-}}\big]\|v\|_{W^{1,p(x)}(\Omega_n)} \\
&\leq (\frac{1}{p_-} + \frac{1}{p'_-})\big[\Big(\int_{\Omega_n} |u|^{p(x)}\big) dx + 1\Big)^{\frac{1}{p'_-}} \\
&+ \big(n^{p'_+}.\,\mathrm{meas}(\Omega_n) + 1\big)^{\frac{1}{p'_-}}\big]\|v\|_{W^{1,p(x)}(\Omega_n)} \\
&\leq C_1\|v\|_{W^{1,p(x)}(\Omega_n)},
\end{aligned} \tag{2.2.7}$$

Lemma 2.2.3 *The operator $B_n = A + R_n$ is pseudo-monotone from $W_0^{1,p(x)}(\Omega_n)$ into $W^{-1,p'(x)}(\Omega_n)$. Moreover, B_n is coercive in the following sense*

$$\frac{\langle B_n v, v\rangle}{\|v\|_{W^{1,p(x)}(\Omega_n)}} \to +\infty \quad as \quad \|v\|_{W^{1,p(x)}(\Omega_n)} \to +\infty$$

$$for \quad v \in W_0^{1,p(x)}(\Omega_n).$$

where $Au = -\Delta_{p(x)}(u)$

Proof : Using Hölder's inequality we can show that the operator A is bounded, and by using (2.2.7) we conclude that B_n is bounded. For the coercivity, we have for any $u \in W_0^{1,p(x)}(\Omega_n)$,

$$\begin{aligned}
&\langle B_n u, u\rangle = \langle Au, u\rangle + \langle R_n u, u\rangle \\
&= \int_{\Omega_n} |\nabla u|^{p(x)} dx + \int_{\Omega_n} c(x,u)u - H_n(x,u,\nabla u)u dx \\
&\geq \int_{\Omega} |\nabla u|^{p(x)} dx - C_1.\|u\|_{W^{1,p(x)}(\Omega_n)} \quad \text{(using(2.2.7))} \\
&\geq \|\nabla u\|^{\delta'}_{L^{p(x)}(\Omega_n)} - C_1.\|u\|_{W^{1,p(x)}(\Omega_n)} \\
&\geq \alpha' \|u\|^{\delta'}_{W^{1,p(x)}(\Omega_n)} - C_1.\|u\|_{W^{1,p(x)}(\Omega_n)} \\
&\text{(using Poincaré's inequality)}
\end{aligned}$$

With

$$\delta' = \begin{cases} p_- & \text{if } \|\nabla u\|_{L^{p(x)}(\Omega_n)} > 1, \\ p_+ & \text{if } \|\nabla u\|_{L^{p(x)}(\Omega_n)} \leq 1, \end{cases}$$

Then, we obtain :

$$\frac{\langle B_n u, u\rangle}{\|u\|_{W^{1,p(x)}(\Omega_n)}} \to +\infty \quad \text{as } \|u\|_{W^{1,p(x)}(\Omega_n)} \to +\infty.$$

Now it remains to show that B_n is pseudo-monotone. Let $(u_k)_k$ a sequence in $W_0^{1,p(x)}(\Omega_n)$ such that :

$$\begin{aligned}
u_k &\rightharpoonup u \quad \text{in } W_0^{1,p(x)}(\Omega_n), \\
B_n u_k &\rightharpoonup \chi \quad \text{in } W^{-1,p'(x)}(\Omega_n), \\
\limsup_{k\to\infty} &\langle B_n u_k, u_k\rangle \leq \langle \chi, u\rangle.
\end{aligned} \tag{2.2.8}$$

We will prove that :

$$\chi = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \quad \text{as } k \to +\infty.$$

Firstly, since $W_0^{1,p(x)}(\Omega_n) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega_n)$, then $u_k \to u$ in $L^{p(x)}(\Omega_n)$ for a subsequence still denoted by $(u_k)_k$.

We have $(u_k)_k$ is a bounded sequence in $W_0^{1,p(x)}(\Omega_n)$, then $(|\nabla u_k|^{p(x)-2}\nabla u_k)_k$ is bounded in $(L^{p'(x)}(\Omega_n))^N$,

therefore, there exists a function
$\varphi \in (L^{p'(x)}(\Omega_n))^N$ such that :

$$|\nabla u_k|^{p(x)-2}\nabla u_k \rightharpoonup \varphi \quad \text{in } (L^{p'(x)}(\Omega_n))^N \text{ as } k \to \infty. \tag{2.2.9}$$

Similarly, since $(c(x,u_k) - H_n(x,u_k,\nabla u_k))_k$ is bounded in $L^{p'(x)}(\Omega_n)$, then there exists a function $\psi_n \in L^{p'(x)}(\Omega_n)$ such that :

$$c(x,u_k) - H_n(x,u_k,\nabla u_k) \rightharpoonup \psi_n \quad \text{in } L^{p'(x)}(\Omega_n) \text{ as } k \to \infty, \tag{2.2.10}$$

For all $v \in W_0^{1,p(x)}(\Omega_n)$, we have :

$$\begin{aligned}\langle \chi, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty} \int_{\Omega_n} |\nabla u_k|^{p(x)-2}\nabla u_k \nabla v dx \\ &+ \lim_{k\to\infty} \int_{\Omega_n} (c(x,u_k) - H_n(x,u_k,\nabla u_k)) v dx \\ &= \int_{\Omega_n} \varphi \nabla v dx + \int_{\Omega_n} \psi_n v dx.\end{aligned} \tag{2.2.11}$$

Using (2.2.8) and (2.2.11), we obtain :

$$\begin{aligned}&\limsup_{k\to\infty} \langle B_n(u_k), u_k\rangle \\ &= \limsup_{k\to\infty} \Big\{ \int_{\Omega_n} |\nabla u_k|^{p(x)} dx + \int_{\Omega_n} (c(x,u_k) - H_n(x,u_k,\nabla u_k)) u_k dx \Big\} \\ &\le \int_{\Omega} \varphi \nabla u dx + \int_{\Omega} \psi_n u dx,\end{aligned} \tag{2.2.12}$$

Thanks to (2.2.10), we have :

$$\int_{\Omega_n} (c(x,u_k) - H_n(x,u_k,\nabla u_k)) u_k dx \to \int_{\Omega_n} \psi_n u dx; \tag{2.2.13}$$

Therefore,

$$\limsup_{k\to\infty} \int_{\Omega_n} |\nabla u_k|^{p(x)} dx \le \int_{\Omega_n} \varphi \nabla u dx. \tag{2.2.14}$$

On the other hand, we have :

$$\int_{\Omega_n} (|\nabla u_k|^{p(x)-2}\nabla u_k - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_k - \nabla u) dx \ \ge 0, \tag{2.2.15}$$

Then

$$\int_{\Omega_n} |\nabla u_k|^{p(x)} dx \ge -\int_{\Omega_n} |\nabla u|^{p(x)} dx + \int_{\Omega_n} |\nabla u_k|^{p(x)-2}\nabla u_k \nabla u dx$$

$$+\int_{\Omega_n} |\nabla u|^{p(x)-2}\nabla u \nabla u_k dx,$$

and by (2.2.9), we get :

$$\liminf_{k\to\infty} \int_{\Omega_n} |\nabla u_k|^{p(x)} dx \geq \int_{\Omega_n} \varphi \nabla u dx,$$

this implies, thanks to (2.2.14), that :

$$\lim_{k\to\infty} \int_{\Omega_n} |\nabla u_k|^{p(x)} dx = \int_{\Omega_n} \varphi \nabla u dx. \tag{2.2.16}$$

By combining (2.2.11), (2.2.13) and (2.2.16), we deduce that :

$$\langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \quad \text{as } k \to +\infty.$$

Now, by (2.2.16) we can obtain :

$$\lim_{k\to+\infty} \int_{\Omega_n} (|\nabla u_k|^{p(x)-2}\nabla u_k - |\nabla u|^{p(x)-2}\nabla u))(\nabla u_k - \nabla u) dx = 0,$$

In view of the Lemma 2.2.2, we obtain :

$$u_k \to u, \quad W_0^{1,p(x)}(\Omega_n), \quad \nabla u_k \to \nabla u \quad \text{a.e. in } \Omega_n,$$

then

$$|\nabla u_k|^{p(x)-2}\nabla u_k \rightharpoonup |\nabla u|^{p(x)-2}\nabla u \qquad \text{in } (L^{p'(x)}(\Omega_n))^N,$$

and

$$c(x, u_k) - H_n(x, u_k, \nabla u_k) \rightharpoonup c(x, u) - H_n(x, u, \nabla u) \quad \text{in } L^{p'(x)}(\Omega_n),$$

we deduce that $\chi = B_n u$, which completes the proof.

By Lemma 2.2.3, we deduce that there exists at least one weak solution $u_n \in W_0^{1,p(x)}(\Omega_n)$ of the problem (2.2.2), (cf. [52]).

A priori estimate

Proposition 2.2.1 *Assuming that $p(.) \in C_+(\overline{\Omega})$ holds, and let u_n be any solution of* (2.2.2). *Then for every $\lambda > 0$ there exists a positive constant $C = C(N, p, \alpha_0, d, f, g, \lambda)$ such that :*

$$\|e^{\lambda|u_n|} - 1\|_{W_0^{1,p(x)}(\Omega_n)} \leq C. \tag{2.2.17}$$

Remark 2.2.1 *The previous estimate yields an estimate for the functions* $e^{\lambda|u_n|}$ *in* $L^{r(x)}_{loc}(\Omega)$ *for every* $r \in [1, +\infty)$*, every* $\lambda > 0$ *and every set* $\Omega_0 \subset\subset \Omega$*, one has*

$$\|e^{|u_n|}\|_{L^{r(x)}(\Omega_0)} \leq C(r_{\mp}, \lambda, \Omega_0)$$

Proof :

For simplicity of notation we omit the index n of the sequence. We take $\varphi(G_k(u))$ as test function in (2.2.2), where

$$G_k(s) = s - T_k(s) = \begin{cases} s-k & \text{if } s > k, \\ 0 & \text{if } |s| \leq k, \\ s+k & \text{if } s < -k. \end{cases} \tag{2.2.18}$$

$$\text{and} \quad \varphi(s) = \big(e^{\lambda|s|} - 1\big) sign(s).$$

we have :

$$\begin{aligned} &\int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + \alpha_0 \int_\Omega |u|^{p(x)-1} |\varphi(G_k(u))|, \\ &\leq d \int_\Omega |\nabla G_k(u)|^{p(x)} |\varphi(G_k(u))| + \int_\Omega |f||\varphi(G_k(u)) \\ &+ \int_\Omega |g||\nabla G_k(u)|\varphi'(G_k(u)) \\ &= I + J + K, \end{aligned} \tag{2.2.19}$$

For every s in $\mathbb{R}$ and if λ satisfies :

$$\lambda \geq 8d \tag{2.2.20}$$

we have :

$$d|\varphi(s)| \leq \frac{1}{8}\varphi'(s) \tag{2.2.21}$$

then

$$I \leq \frac{1}{8} \int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) \tag{2.2.22}$$

Before estimating J, we remark that :

$$\int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) = \int_\Omega |\nabla \Psi(G_k(u))|^{p(x)} \tag{2.2.23}$$

where

$$\Psi(s) = \int_0^{|s|} (\varphi'(t))^{\frac{1}{p(x)}} dt = \frac{p(x)}{\lambda^{\frac{1}{p'(x)}}} \big(e^{\frac{\lambda|s|}{p(x)}} - 1\big) \tag{2.2.24}$$

Moreover, we observe that there exists a positive constant $c_2 = c_2(p, \lambda)$ such that

$$|\varphi(s)| \leq c_2(\Psi(s))^{p(x)} \text{for every s such that} |s| \geq 1 \tag{2.2.25}$$

Now let us observe that p is a continuous variable exponent on $\overline{\Omega}$ then there exists a constant $\delta > 0$ such that :

$$\max_{y \in B(x,\delta)\cap\Omega} \frac{N - p(y)}{Np(y)} \leq \min_{y \in B(x,\delta)\cap\Omega} \frac{p(y)(N - p(y))}{Np(y)} \quad \text{for all } x \in \Omega. \tag{2.2.26}$$

while $\overline{\Omega}$ is compact then we can cover it with a finite number of balls B_i for $i = 1, ..., m$ from (2.2.26) we can deduce the pointwise estimate :

$$1 < p_{-,i} \leq p_{+,i} \leq \frac{p_{-,i}^2 N}{N - p_{-,i} + p_{-,i}^2} < N. \tag{2.2.27}$$

is fulfill for all $i = 1, ..., m$.

$p_{-,i}, p_{+,i}$ denote the local minimum and the local maximum of p on $\overline{B_i \cap \Omega}$ respectively

Estimation of the integral J :

Let $H \geq 1$ be a constant that we will chose later. We can estimate J by splitting it as follows :

$$\begin{aligned} J = &\sum_{i=0}^{m} \Big[\int_{B_i \cap \{|f|>H, |G_k(u)|\geq 1\}} |f||\varphi(G_k(u))| \Big] \\ &+ \int_{\{|f|>H, |G_k(u)|<1\}} |f||\varphi(G_k(u))| + \int_{\{|f|\leq H\}} |f||\varphi(G_k(u))| \\ &= J_1 + J_2 + J_3 \end{aligned}$$

By (2.2.25) J_1, can be estimated as follows

$$J_1 \leq c_2 \sum_{i=0}^{m} \Big[\int_{B_i \cap \{|f|>H, |G_k(u)|\geq 1\}} |f| \Psi(G_k(u))^{p(x)} \Big]$$

Let ϵ a positive constant to be chosen later. Using Young, Sobolev's embedding and Lemma 2.2.1 we have :

$$J_1 \leq C \int_{\{|f|>H\}} |f|^{\frac{\epsilon N}{\epsilon N + p(x) - N}} + \frac{1}{8} \sum_{i=0}^{m} \|\nabla \Psi(G_k(u))\|_{L^{p(x)}(B_i)}^{\epsilon p_{+,i}^*}$$

$$\leq C\int_{\{|f|>H\}}|f|^{\frac{\epsilon N}{\epsilon N+p(x)-N}}+\frac{1}{8}\sum_{i=0}^{m}\Big[\int_{B_i}|\nabla\Psi(G_k(u))|^{p(x)}\Big]^{\frac{\epsilon p^*_{+,i}}{p_{-,i}}}$$

where $p^*(x)=\frac{Np(x)}{N-p(x)}$ and $p^*_{+,i}=\frac{Np_{+,i}}{N-p_{+,i}}$ since (2.2.27) we can choose ϵ such that :

$$\frac{N-p_{-,i}}{Np_{-,i}}\leq\epsilon\leq\frac{p_{-,i}}{p^*_{+,i}} \tag{2.2.28}$$

Then using (2.2.28) and (2.2.23) we obtain that :

$$J_1\leq C\int_{\{|f|>H\}}|f|^{\frac{N}{p(x)}}+\frac{1}{8}\int_\Omega|\nabla G_k(u)|^{p(x)}\varphi'(G_k(u) \tag{2.2.29}$$

Remark 2.2.2 *The cases where* $\|\Psi(G_k(u))\|_{L^{p*(x)}(\Omega)}\leq 1$ *or* $\|\nabla\Psi(G_k(u))\|_{L^{p(x)}(\Omega)}\leq 1$ *are easy to see that* $J_1\leq C$ *(C depend on the data of the problem)*

On the other hand

$$J_2\leq\varphi(1)\int_{\{|f|>H\}}|f|\leq\frac{\varphi(1)}{H^{\frac{N-p_+}{p_+}}}\int_{\{|f|>H\}}|f|^{\frac{N}{p(x)}} \tag{2.2.30}$$

Finally, choosing k sufficiently large such that :

$$\alpha_0k^{p_--1}\geq 4H \tag{2.2.31}$$

We can obtain :

$$\begin{aligned}J_3&\leq\frac{\alpha_0}{4}\int_\Omega k^{p(x)-1}|\varphi(G_k(u))|\\&\leq\frac{\alpha_0}{4}\int_\Omega|u|^{p(x)-1}|\varphi(G_k(u))|\end{aligned} \tag{2.2.32}$$

Estimation of the integral K :

Thanks to Young's inequality, we have :

$$\begin{aligned}K\leq&\frac{1}{8}\int_\Omega|\nabla G_k(u)|^{p(x)}\varphi'(G_k(u))\\&+C_4\int_\Omega|g|^{p'(x)}\varphi'(G_k(u)),\\&=K_1+K_2\end{aligned} \tag{2.2.33}$$

Integral K_2 can be estimated as follows :

$$
\begin{aligned}
K_2 &\leq C_4 \int_\Omega |g|^{p'(x)} \varphi'(G_k(u)), \\
&\leq C_4 \lambda e^\lambda \int_{\{|G_k(u)|<1\}} |g|^{p'(x)} \\
&+ C_4 \sum_{i=0}^m \big[\int_{B_i \cap \{|g|>1, |G_k(u)|>1\}} |g|^{p'(x)} \varphi'(G_k(u)) \big] \\
&+ C_4 \int_{\{|g|\leq 1, |G_k(u)|>1\}} \varphi'(G_k(u)) \\
&= K_{2,1} + K_{2,2} + K_{2,3}
\end{aligned}
\tag{2.2.34}
$$

Since $\varphi'(s) \leq C_5(\Psi(s))^{p(x)}$ for every s such that $|s| \geq 1$, we have :

$$
K_{2,2} \leq C_6 \sum_{i=0}^m \big[\int_{B_i \cap \{|g|>1, |G_k(u)|>1\}} |g|^{p'(x)} \Psi(G_k(u))^{p(x)} \big]
$$

Let ϵ be a positive constant such that (2.2.28). Using Young, Sobolev's embedding and Lemma 2.2.1 we have :

$$
\begin{aligned}
K_{2,2} &\leq C_7 \int_{\{|g|>1\}} |g|^{\frac{\epsilon N p'(x)}{\epsilon N + p(x) - N}} + \frac{1}{8} \sum_{i=0}^m \|\nabla \Psi(G_k(u))\|_{L^{p(x)}(B_i)}^{\epsilon p^*_{+,i}} \\
&\leq C_7 \int_{\{|g|>1\}} |g|^{\frac{\epsilon N p'(x)}{\epsilon N + p(x) - N}} + \frac{1}{8} \sum_{i=0}^m \big[\int_{B_i} |\nabla \Psi(G_k(u))|^{p(x)} \big]^{\frac{\epsilon p^*_{+,i}}{p_{-,i}}}
\end{aligned}
$$

Then using (2.2.28) and (2.2.23) we obtain that :

$$
K_{2,2} \leq C_7 \int_{\{|g|>1\}} |g|^{\frac{N}{p(x)-1}} + \frac{1}{8} \int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u) \tag{2.2.35}
$$

The same as before in the cases where $\|\Psi(G_k(u))\|_{L^{p*(x)}(\Omega)} \leq 1$ or $\|\nabla \Psi(G_k(u))\|_{L^{p(x)}(\Omega)} \leq 1$ it is easy to check that $K_{2,2} \leq C$

Finally, using inequality

$$
\varphi'(s) \leq C_8 |\varphi(s)|, \quad \text{for every s such that} |s| \geq 1 \tag{2.2.36}
$$

and choosing $k = k(p_-, \alpha_0, \lambda)$ sufficiently large such that :

$$
\alpha_0 k^{p_- - 1} \geq 4 C_4 \tag{2.2.37}
$$

we obtain :

$$
\begin{aligned}
K_{2,3} &\leq \frac{\alpha_0}{4} \int_\Omega k^{p(x)-1} |\varphi(G_k(u))| \\
&\leq \frac{\alpha_0}{4} \int_\Omega |u|^{p(x)-1} |\varphi(G_k(u))|
\end{aligned}
\tag{2.2.38}
$$

Putting all the inequalities (2.2.19), (2.2.22), (2.2.29), (2.2.30), (2.2.32), (2.2.35), (2.2.38), (2.2.34) and (2.2.33) together, we get an estimate in $W_0^{1,p(x)}(\Omega)$ for $G_k(u)$,when k is large enough :

$$\begin{aligned}&\frac{1}{2}\int_{\Omega}|\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)) + \frac{\alpha_0}{2}\int_{\Omega}|u|^{p(x)-1}|\varphi(G_k(u))|\\&\leq C\int_{\{|f|>H\}}|f|^{\frac{N}{p(x)}} + \frac{\varphi(1)}{H^{\frac{N-p_+}{p_+}}}\int_{\{|f|>H\}}|f|^{\frac{N}{p(x)}}\\&+ C_4\lambda e^{\lambda}\int_{\Omega}|g|^{p'(x)} + C_7\int_{\Omega}|g|^{\frac{N}{p(x)-1}}\\&= C_9(N,p_-,p_+,\alpha_0,f,g,\lambda)\end{aligned} \tag{2.2.39}$$

For every λ, k satisfying (2.2.20), (2.2.31), (2.2.37) and for every $H \geq 1$. We fix now λ and k such that (2.2.39) holds.

As before, If we take $\varphi(T_k(u))$ as a test function in (2.2.2) we obtain :

$$\begin{aligned}&\int_{\Omega}|\nabla T_k(u)|^{p(x)}\varphi'(T_k(u)) + \alpha_0\int_{\Omega}|u|^{p(x)-1}|\varphi(T_k(u))|,\\&\leq d\int_{\Omega}|\nabla T_k(u)|^{p(x)}|\varphi(T_k(u))| + d\varphi(k)\int_{\Omega}|\nabla G_k(u)|^{p(x)}|\\&+ \int_{\Omega}|f||\varphi(T_k(u))| + \int_{\Omega}|g||\nabla T_k(u)|\varphi'(T_k(u))\\&= L_1 + L_2 + L_3 + L_4,\end{aligned} \tag{2.2.40}$$

Using (2.2.21), we have :

$$L_1 \leq \frac{1}{4}\int_{\Omega}|\nabla T_k(u)|^{p(x)}\varphi'(T_k(u) \tag{2.2.41}$$

By (2.2.39),

$$L_2 \leq C_{10}(N,p_-,p_+,\alpha_0,f,g,\lambda) \tag{2.2.42}$$

Remark 2.2.3 *if $meas(\Omega)$ is finite or if $f \in L^1(\Omega)$ it is easy to estimate the integral L_3*

In general case, let ϵ be a positive constant to be chosen later, we write

$$\begin{aligned}L_3 &\leq \varphi(k)\int_{\{|f|>1\}}|f| + \int_{\{|f|\leq 1\}}|f||\varphi(T_k(u))|\\&\leq \varphi(k)\int_{\{|f|>1\}}|f| + \epsilon\int_{\Omega}|\varphi(T_k(u))|^{p(x)}\\&+ c(\epsilon,p'_-)\int_{\{|f|\leq 1\}}|f|^{p'(x)}\end{aligned}$$

Since

$$|\varphi(T_k(u))|^{p(x)} \leq C_{11}(\lambda,p_+,p_-,k)|\varphi(T_k(u))||u|^{p(x)-1},$$

choosing ϵ such that $\epsilon C_{11} \leq \frac{\alpha_0}{2}$,we have :

$$L_3 \leq \frac{\alpha_0}{2} \int_\Omega |u|^{p(x)-1} |\varphi(T_k(u))| + C_{12}(\alpha_0, f, \lambda, p_+, p_-, k) \tag{2.2.43}$$

Finally, one has

$$L_4 \leq \frac{1}{4} \int_\Omega |\nabla T_k(u)|^{p(x)} \varphi'(T_k(u) + C_{13}(\alpha_0, \lambda, p'_-, g, p_-, k) \tag{2.2.44}$$

In conclusion, putting all the estimations ((2.2.40) - (2.2.44)) together, we get :

$$\begin{aligned} \frac{1}{2} \int_\Omega |\nabla T_k(u)|^{p(x)} \varphi'(T_k(u)) + \frac{\alpha_0}{2} \int_\Omega |u|^{p(x)-1} |\varphi(T_k(u))| \\ \leq C_{14}(N, p_-, p_+, \alpha_0, f, g, \lambda) \end{aligned} \tag{2.2.45}$$

In view of (2.2.39) and (2.2.45), we have :

$$\int_{\{|u| \leq k\}} |\nabla u|^{p(x)} e^{\lambda |u|} \leq C_{15}, \quad \int_{\{|u| > k\}} |\nabla u|^{p(x)} e^{\lambda(|u|-k)} \leq C_{15}$$

For every λ, k large enough (see (2.2.20), (2.2.31) and (2.2.37)), where C_{15} depends on λ, k and the data. Since

$$\begin{aligned} \int_\Omega |\nabla u|^{p(x)} e^{\lambda |u|} &= \int_{\{|u| \leq k\}} |\nabla u|^{p(x)} e^{\lambda |u|} \\ &+ e^{\lambda k} \int_{\{|u| > k\}} |\nabla u|^{p(x)} e^{\lambda(|u|-k)} \\ &\leq C_{16} \end{aligned}$$

If we fix the value of k (depending on λ), we obtain an estimate of $|\nabla(e^{\lambda|u|} - 1)|$ in $L^{p(x)}(\Omega)$ (depending on λ). This implies, by Sobolev's embedding, that :

$$\int_\Omega (e^{\lambda |u|} - 1)^{p*(x)} \leq C_{17} \tag{2.2.46}$$

For every λ such that (2.2.20), where C_{17} depends on λ and on the data of the problem. Note that (2.2.46) does not imply an estimate in $L^{p(x)}(\Omega)$ for $e^{\lambda|u|} - 1$, since $meas(\Omega)$ may be infinite. To obtain such an estimate, we have to combine (2.2.45) and (2.2.46), since, for every $k > 0$, one has the inequalities

$$\int_{\{|u| \leq k\}} (e^{\lambda |u|} - 1)^{p(x)} \leq C_{19} \int_\Omega |u|^{p(x)-1} |\varphi(T_k(u)|,$$

$$\int_{\{|u| > k\}} (e^{\lambda |u|} - 1)^{p(x)} \leq C_{20} \int_\Omega (e^{\lambda |u|} - 1)^{p*(x)},$$

Therefore, if $k = k(\lambda)$ is such that (2.2.45) holds, we can write

$$\begin{aligned} &\int_\Omega (e^{\lambda|u|}-1)^{p(x)} \\ &= \int_{\{|u|\le k\}} (e^{\lambda|u|}-1)^{p(x)} + \int_{\{|u|>k\}} (e^{\lambda|u|}-1)^{p(x)} \le C_{21} \end{aligned} \tag{2.2.47}$$

where C_{21} depends on λ and the data of the problem.

2.3 Main result

In this section we will prove the main result of this chapter. Let $\{u_n\}$ be any sequence of solutions of problem (2.2.2), we extend them to zero in $\Omega \setminus \Omega_n$. By (2.2.17), there exist a subsequence (still denoted by u_n) and a function $u \in W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$.

Theorem 2.3.1 *There exists at least one solution u of* (2.1.1)*; which is such that*

$$\begin{aligned} \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\psi dx + \int_\Omega c(x,u)\psi dx + \int_\Omega H(x,u,\nabla u)\psi dx \\ = \int_\Omega f\psi dx - \int_\Omega g\nabla\psi dx. \end{aligned} \tag{2.3.1}$$

for every function $\psi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. Moreover u satisfies

$$e^{\lambda|u|} - 1 \in W_0^{1,p(x)}(\Omega) \tag{2.3.2}$$

for every $\lambda \geq 0$.

The proof will be made in three steps.

Step 1 : An estimate for $\int_\Omega |\nabla G_k(u_n)|^{p(x)}$

In view of (2.2.39) we have :

$$\begin{aligned} &\int_\Omega |\nabla G_k(u_n)|^{p(x)} \\ &\le \frac{2C}{\lambda}\int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} + \frac{2\varphi(1)}{\lambda H^{\frac{N-p_+}{p_+}}}\int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} + \frac{2C_4\lambda e^\lambda}{\lambda}\int_\Omega |g|^{p'(x)} \\ &+ \frac{2C_7}{\lambda}\int_\Omega |g|^{\frac{N}{p(x)-1}} \end{aligned} \tag{2.3.3}$$

If η is an arbitrary positive number, let us choose H such that the right-hand side of (2.3.3) is smaller than η. It follows that, for every k satisfying (2.2.31), (2.2.37), every λ satisfying (2.2.20), and every $n \in \mathbb{N}$

$$\int_\Omega |\nabla G_k(u_n)|^{p(x)} \leq \eta$$

which proves :

$$\sup_n \int_\Omega |\nabla G_k(u_n)|^{p(x)} \to 0 \quad \text{as} \quad k \to \infty \tag{2.3.4}$$

Step 2 : Strong convergence of $\nabla T_k(u_n)$

In this step, we will fix $k > 0$ and prove that $\nabla T_k(u_n) \to \nabla T_k(u)$ strongly in $L^{p(x)}(\Omega_0; \mathbb{R}^N)$ as $n \to \infty$; for k fixed.

In order to prove this result we define :

$$z_n(x) = T_k(u_n) - T_k(u)$$

and we choose ψ a cut-off function such that

$$\psi \in C_0^\infty(\Omega), \quad 0 \leq \psi \leq 1, \quad \psi = 0 \quad \text{in} \quad \Omega_0$$

Let us take :

$$v = \varphi(z_n) e^{\delta |u_n|} \psi \tag{2.3.5}$$

as a test function in (2.2.2), where λ and δ are a positive constant to be chosen later, we obtain :

$$\begin{aligned}
A_n + B_n &= \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla z_n \varphi'(z_n) e^{\delta|u_n|} \psi \\
&+ \int_\Omega c(u_n) \varphi(z_n) e^{\delta|u_n|} \psi \\
&\leq d \int_\Omega |\nabla u_n|^{p(x)} |\varphi(z_n)| e^{\delta|u_n|} \psi + \int_\Omega |f| |\varphi(z_n)| e^{\delta|u_n|} \psi \\
&- \delta \int_\Omega |\nabla u_n|^{p(x)} \varphi(z_n) e^{\delta|u_n|} sign(u_n) \psi \\
&+ \int_\Omega |\nabla u_n|^{p(x)-1} |\nabla \psi| |\varphi(z_n)| e^{\delta|u_n|} \\
&+ \int_\Omega |g| |\nabla z_n| \varphi'(z_n) e^{\delta|u_n|} \psi \\
&+ \delta \int_\Omega |g| |\nabla u_n| |\varphi(z_n)| e^{\delta|u_n|} \psi + \int_\Omega |g| |\nabla \psi| |\varphi(z_n)| e^{\delta|u_n|} \\
&= C_n + D_n + E_n + F_n + G_n + H_n + L_n
\end{aligned} \tag{2.3.6}$$

Splitting Ω into $\Omega = \{|u_n| \le k\} \cup \{|u_n| > k\}$ we can write :

$$
\begin{aligned}
A_n &= \int_{\{|u_n|\le k\}} |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\nabla z_n \varphi'(z_n) e^{\delta|T_k(u_n)|}\psi \\
&+ \int_{\{|u_n|>k\}} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla z_n \varphi'(z_n) e^{\delta|u_n|}\psi \\
&= \int_{\{|u_n|\le k\}} [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)] \\
&* \nabla z_n \varphi'(z_n) e^{\delta|T_k(u_n)|}\psi \\
&+ \int_{\{|u_n|\le k\}} |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\nabla z_n \varphi'(z_n) e^{\delta|T_k(u_n)|}\psi \\
&+ \int_{\{|u_n|>k\}} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla z_n \varphi'(z_n) e^{\delta|u_n|}\psi \\
&= A_{1,n} + A_{2,n} + A_{3,n}
\end{aligned}
$$

since

$$
\begin{aligned}
&|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi\chi_{\{|u_n|\le k\}} \\
&\qquad \to |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(0)e^{\delta|T_k(u)|}\psi\chi_{\{|u|\le k\}}
\end{aligned}
$$

almost everywhere in Ω (on the set where $|u(x)| = k$ we have $|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u) = 0$)and

$$
\begin{aligned}
&||\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi\chi_{\{|u_n|\le k\}}| \\
&\qquad \le |\nabla u|^{p(x)-1}\varphi'(2k)e^{\delta k}\psi
\end{aligned}
$$

which is a fixed function in $L^{p'(x)}(\Omega)$. Therefore by Lebesgue's theorem we have

$$
\begin{aligned}
&|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi\chi_{\{|u_n|\le k\}} \\
&\qquad \to |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(0)e^{\delta|T_k(u)|}\psi\chi_{\{|u|\le k\}}
\end{aligned}
$$

strongly in $L^{p'(x)}(\Omega)$. Indeed, $\nabla z_n \rightharpoonup 0$ weakly in $L^{p(x)}(\Omega;\mathbb{R}^N)$ then $A_{2,n} \to 0$. Similarly, since $\nabla z_n\chi_{\{|u_n|>k\}} = -\nabla T_k(u)\chi_{\{|u_n|>k\}} \to 0$ strongly in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, while $|\nabla u_n|^{p(x)-2}\nabla u_n\varphi'(z_n)e^{\delta|u_n|}\psi$ is bounded in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, by (2.2.3), (2.2.17) and Remark 2.2.1 we obtain $A_{3,n} \to 0$. Therefore, we have proved that :

$$
A_n = A_{1,n} + o(1) \tag{2.3.7}
$$

For the integral B_n while $\varphi(z_n)$ has the same sign as $c(u_n)$ on the set $\{|u_n| > k\}$ we have

$$
B_n = \int_{\{|u_n|\le k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\psi
$$

$$+ \int_{\{|u_n|>k\}} c(u_n)\varphi(z_n)e^{\delta|u_n|}\psi$$
$$\geq \int_{\{|u_n|\leq k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\psi$$

the last integrand converges pointwise and it is bounded then $\int_{\{|u_n|\leq k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\psi$ goes to zero. Therefore, we obtain that :

$$B_n \geq o(1) \tag{2.3.8}$$

Let us examine C_n and D_n together. We first fix δ such that

$$\delta > d$$

Since $\varphi(z_n)sign(u_n) = |\varphi(z_n)|$ on the set $\{|u_n| > k\}$ we have

$$C_n + E_n \leq d \int_\Omega |\nabla u_n|^{p(x)}|\varphi(z_n)|e^{\delta|u_n|}\psi$$
$$- \delta \int_\Omega |\nabla u_n|^{p(x)}\varphi(z_n)e^{\delta|u_n|}sign(u_n)\psi$$
$$\leq (d+\delta) \int_{\{|u_n|\leq k\}} |\nabla T_k(u_n)|^{p(x)}|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi$$
$$+ (d-\delta) \int_{\{|u_n|>k\}} |\nabla u_n|^{p(x)}\varphi(z_n)e^{\delta|u_n|}\psi$$
$$\leq (d+\delta) \int_{\{|u_n|\leq k\}} |\nabla T_k(u_n)|^{p(x)}|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi$$
$$= (d+\delta) \int_{\{|u_n|\leq k\}} [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)$$
$$- |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)]\nabla z_n|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi$$
$$+ (d+\delta) \int_{\{|u_n|\leq k\}} |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\nabla T_k(u)|$$
$$\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi$$
$$+ (d+\delta) \int_{\{|u_n|\leq k\}} |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\nabla z_n|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi$$

The last two integrals converge to zero.If we choose λ such that :

$$\lambda \geq 2(d+\delta)$$

we have :

$$(d+\delta)|\varphi(s)| \leq \frac{\varphi'(s)}{2} \quad \text{for every s in } \mathbb{R}$$

then we can obtain :

$$C_n + E_n \leq \frac{1}{2} A_{1,n} + o(1) \tag{2.3.9}$$

Using Remark 2.2.1 we can observe that :

$$D_n \to 0 \tag{2.3.10}$$

For the term F_n we can see that $|\nabla \psi||\varphi z_n|$ converge strongly to zero in $L^{r(x)}(\Omega)$ for every $r(x) > 1$. by (2.2.17) the term $|\nabla u_n|^{p(x)-2} \nabla u_n e^{\delta |u_n|}$ is bounded in $L^{p'(x)}_{loc}(\Omega)$ then we have that :

$$F_n \to 0 \tag{2.3.11}$$

For the term G_n like before we have :

$$\begin{aligned} G_n &= \int_{\{|u| \leq k\}} |g||\nabla z_n| \varphi'(z_n) e^{\delta |u_n|} \psi \\ &+ \int_{\{|u| > k\}} |g||\nabla z_n| \varphi'(z_n) e^{\delta |u_n|} \psi \\ &= G_{1,n} + G_{2,n} \end{aligned}$$

since

$$|g| \varphi'(z_n) e^{\delta |u_n|} \psi \chi_{\{|u_n| \leq k\}} \to |g| \varphi'(0) e^{\delta |T_k u|} \psi \chi_{\{|u| \leq k\}}$$

almost everywhere in Ω and

$$|g| \varphi'(z_n) e^{\delta |u_n|} \psi \chi_{\{|u_n| \leq k\}} \leq |g| \varphi'(2k) e^{\delta k} \psi$$

Therefore by Lebesgue's theorem we have :

$$|g| \varphi'(z_n) e^{\delta |u_n|} \psi \chi_{\{|u_n| \leq k\}} \to |g| \varphi'(0) e^{\delta |T_k u|} \psi \chi_{\{|u| \leq k\}}$$

strongly in $L^{p'(x)}(\Omega)$. Indeed, $\nabla z_n \rightharpoonup 0$ weakly in $L^{p(x)}(\Omega; \mathbb{R}^N)$ then $G_{1,n} \to 0$. Similarly, since $|\nabla z_n| \chi_{\{|u_n| > k\}} = |\nabla T_k(u)| \chi_{\{|u_n| > k\}} \to 0$ strongly in $L^{p'(x)}(\Omega; \mathbb{R}^N)$, while $|g| \varphi'(z_n) e^{\delta |u_n|} \psi$ is bounded in $L^{p'(x)}(\Omega; \mathbb{R}^N)$, by (2.2.17) and remark 2.2.1 we obtain $G_{2,n} \to 0$. Therefore, we have proved that :

$$G_n \to 0 \tag{2.3.12}$$

Moreover

$$|g||\varphi(z_n)| \psi \to 0$$

almost everywhere in Ω and

$$|g||\varphi(z_n)|\psi \leq |g||\varphi(2k)|\psi$$

Therefore by Lebesgue's theorem we have :

$$|g||\varphi(z_n)|\psi \to 0$$

strongly in $L^{p'(x)}(\Omega)$. Indeed, $\nabla u_n e^{\delta|u_n|} \rightharpoonup \nabla u e^{\delta|u|}$ weakly in $L^{p(x)}(\Omega;\mathbb{R}^{\mathbb{N}})$, then :

$$H_n \to 0 \tag{2.3.13}$$

Finally, $|\nabla\psi||\varphi z_n|$ converge strongly to zero in $L^{r(x)}(\Omega)$ for every $r(x) > 1$. by (2.2.17) the term $|g|e^{\delta|u_n|}$ is bounded in $L^{p'(x)}_{loc}(\Omega)$ then we have that :

$$L_n \to 0 \tag{2.3.14}$$

Putting all inequalities (2.3.6), (2.3.7), (2.3.8), (2.3.9), (2.3.10), (2.3.11), (2.3.12), (2.3.13) and (2.3.14) we can conclude :

$$A_{1,n} \to 0 \tag{2.3.15}$$

On the other hand we have

$$\begin{aligned}&\int_{\{|u_n|>k\}}[|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)]\nabla z_n\\ &\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi = \int_{\{|u_n|>k\}}|\nabla T_k(u)|^{p(x)}\varphi'(k-T_k(u))e^{\delta k}\psi \to 0\end{aligned} \tag{2.3.16}$$

From (2.3.15) and (2.3.16) we can conclude that :

$$\begin{aligned}&\int_{\Omega_0}[|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)]\\ &(\nabla T_k(u_n) - \nabla T_k(u)) \to 0\end{aligned} \tag{2.3.17}$$

Finally, using the Lemma 2.2.2 we have :

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{strongly in } L^{p(x)}(\Omega_0;\mathbb{R}^{\mathbb{N}}) \tag{2.3.18}$$

Step 3 : End of the proof

Observing that :

$$\nabla u_n - \nabla u = \nabla T_k u_n - \nabla T_k u + \nabla G_k u_n - \nabla G_k u$$

Let Ω_0 be an open set compactly contained in Ω, using (2.3.4) and (2.3.18) we have :

$$\nabla u_n \to \nabla u \quad \text{strongly in } L^{p(x)}(\Omega_0; \mathbb{R}^{\mathbb{N}}) \tag{2.3.19}$$

To obtain (2.3.1) we have to pass to the limit in the distributional formulation of problem (2.2.2) using (2.3.19). Finally, statement (2.3.2) follows easily from Proposition 2.2.1 and (2.3.19), using Fatou's Lemma.

2.4 Boundedness of solutions

In this section we will gave some regularity on the solution of the problem (2.1.1) using an adaptation of a classical technique due to Stampacchia. To do this we need the following lemma (see [66]) :

Lemma 2.4.1 *Let ϕ be a non-negative, non-increasing function defined on the halfline $[k_0, \infty)$. Suppose that there exist positive constants A, μ,β, with $\beta > 1$, such that*

$$\phi(h) \leq \frac{A}{(h-k)^{\mu}} \phi(k)^{\beta}$$

for every $h > k \geq k_0$. Then $\phi(k) = 0$ for every $k \geq k_1$, where

$$k_1 = k_0 + A^{1/\mu} 2^{\beta/(\beta-1)} \phi(k_0)^{(\beta-1)/\mu}$$

The result that we are going to prove is the following :

Theorem 2.4.1 *Suppose that* (2.1.3) *holds. Then every solution u of* (2.1.1)*; which is specified in* (2.3) *is essentially bounded, and*

$$\|u_n\|_{L^{\infty}(\Omega)} \leq C \tag{2.4.1}$$

The proof relies on the combined use of the well-known technique by Stampacchia (see [66]) and suitable exponential test functions, as in [21].

Proof : Since (2.2.39) we can obtain an estimate for $\int_{\Omega} |u|^{p(x)-1} \varphi(G_k(u))$ then for some constant $k_0 = k(\lambda)$ sufficiently large we have

$$meas(A_{k_0}) < 1 \tag{2.4.2}$$

where

$$A_k = \{x \in \Omega : |u| > k\}$$

as before we can take the test function $\varphi(G_k(u))$ then we have :

$$\begin{aligned} &\int_{A_k} |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + \alpha_0 \int_{A_k} |u|^{p(x)-1} |\varphi(G_k(u))|, \\ &\leq d \int_{A_k} |\nabla G_k(u)|^{p(x)} |\varphi(G_k(u))| + \int_{A_k \cap \{|f|>1\}} |f||\varphi(G_k(u))| \\ &+ \int_{A_k \cap \{|f|\leq 1\}} |\varphi(G_k(u))| + \int_{A_k} |g||\nabla G_k(u)|\varphi'(G_k(u)) \end{aligned} \tag{2.4.3}$$

As in the proof of Proposition 2.2.1 one has :

$$\begin{aligned} &\int_{A_k} |g||\nabla G_k(u)|\varphi'(G_k(u)) \\ &\leq \frac{1}{4} \int_{A_k} |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + C_{22} \int_{A_k} |g|^{p'(x)} \varphi'(G_k(u)) \end{aligned}$$

and if $\lambda \geq 4d$ and $k \geq k_0(\lambda)$(large enough) where

$$\alpha_0 k_0^{p_- - 1} \geq 4 \tag{2.4.4}$$

then

$$\begin{aligned} &\frac{1}{2} \int_{A_k} |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + \frac{3\alpha_0}{4} \int_{A_k} |u|^{p(x)-1} |\varphi(G_k(u))|, \\ &\leq \int_{(A_k \setminus A_{k+1}) \cap \{|f|>1\}} |f||\varphi(G_k(u))| \\ &+ \int_{A_{k+1} \cap \{|f|>1\}} |f||\varphi(G_k(u))| + C_{22}\varphi'(1) \int_{A_k \setminus A_{k+1}} |g|^{p'(x)} \\ &+ C_{22} \int_{A_{k+1}} |g|^{p'(x)} \varphi'(G_k(u)) \end{aligned} \tag{2.4.5}$$

using Hölder inequality we have :

$$\begin{aligned} &\int_{(A_k \setminus A_{k+1}) \cap \{|f|>1\}} |f||\varphi(G_k(u))| \\ &\leq \varphi(1)\left(\frac{1}{q_-} + \frac{1}{q'_-}\right) \|f\|_{L^{q(x)}(\{|f|>1\})} (meas(A_k))^{\frac{1}{q'_+}} \end{aligned}$$

by Hölder's inequality and interpolation we obtain :

$$\begin{aligned} &\int_{A_{k+1} \cap \{|f|>1\}} |f||\varphi(G_k(u))| \\ &\leq \|f\|_{L^{q_-}(A_{k+1} \cap \{|f|>1\})} \|\varphi(G_k(u))\|_{L^{p*_-/p_-}(A_{k+1})}^{\frac{N}{p_- q_-}} \|\varphi(G_k(u))\|_{L^1(A_{k+1})}^{1-\frac{N}{p_- q_-}} \end{aligned}$$

while (2.2.25), (2.4.2) and using Young's and sobolev's inequalities we can deduce that :

$$\begin{aligned}
&\int_{A_{k+1}\cap\{|f|>1\}} |f||\varphi(G_k(u))| \\
&\leq \frac{1}{8}\|\nabla\psi(G_k(u))\|^{p_-}_{L^{p(x)}(A_k)} \\
&+ C_{23}\|f\|^{\frac{p_-q_-}{p_-q_--N}}_{L^{q_-}(\{|f|>1\})}\|\varphi(G_k(u))\|_{L^1(A_k)} \\
&\leq \frac{1}{8}\int_{A_k} |\nabla\psi(G_k(u))|^{p(x)} + 1dx \\
&+ C_{23}\|f\|^{\frac{p_-q_-}{p_-q_--N}}_{L^{q_-}(\{|f|>1\})}\|\varphi(G_k(u))\|_{L^1(A_k)} \\
&\leq \frac{1}{8}\int_{A_k} |\nabla(G_k(u))|^{p(x)}\varphi'(G_k(u))dx + \frac{1}{8}meas(A_k) \\
&+ C_{23}\|f\|^{\frac{p_-q_-}{p_-q_--N}}_{L^{q_-}(\{|f|>1\})}\|\varphi(G_k(u))\|_{L^1(A_k)}
\end{aligned}$$

Therfore, choosing k_0 such that :

$$C_{23}\|f\|^{\frac{p_-q_-}{p_-q_--N}}_{L^{q_-}(\{|f|>1\})} \leq \frac{\alpha_0 k_0^{p_--1}}{4} \tag{2.4.6}$$

the second integral in the right-hand side of (2.4.5) can be absorbed by the left-hand side.

In view of Hölder's inequality and (2.4.2) and (2.1.3) we have :

$$\begin{aligned}
&C_{22}\varphi'(1)\int_{A_k\setminus A_{k+1}} |g|^{p'(x)} \\
&\leq C_{23}(\int_{A_k} |g|^{p'(x)})^{\eta}(meas(A_k))^{1-\frac{p'_+}{r_-}} \\
&\leq C_{24}(\|g\|_{L^{r(x)}(A_k)})^{\delta"}(meas(A_k))^{1-\frac{p'_+}{r_-}}
\end{aligned}$$

where

$$\eta = \begin{cases} \frac{p'_-}{r_+} & \text{if } \int_{A_k} |g|^{p'(x)} \leq 1, \\ \frac{p'_+}{r_-} & \text{if } \int_{A_k} |g|^{p'(x)} > 1. \end{cases}$$

$$\delta" = \begin{cases} \frac{\eta}{r_-} & \text{if } \|g\|_{L^{r(x)}(A_k)} \geq 1, \\ \frac{\eta}{r_+} & \text{if } \|g\|_{L^{r(x)}(A_k)} < 1. \end{cases}$$

Finally, with similar calculations, using (2.2.36) we have :

$$C_{22}\int_{A_{k+1}} |g|^{p'(x)}\varphi'(G_k(u))$$

$$\leq C_{24}\int_{A_{k+1}\cap\{|g|>1\}}|g|^{p'(x)}|\varphi(G_k(u))|$$
$$+\, C_{24}\int_{A_{k+1}\cap\{|g|\leq 1\}}|\varphi(G_k(u))|$$

If we choose k_0 such that :

$$\alpha_0 k_0^{p_- -1} > 4C_{24} \tag{2.4.7}$$

then

$$C_{22}\int_{A_{k+1}}|g|^{p'(x)}\varphi'(G_k(u))$$
$$\leq C_{24}\int_{A_{k+1}\cap\{|g|>1\}}|g|^{p'_+}|\varphi(G_k(u))|$$
$$+\,\frac{\alpha_0}{4}\int_{A_k}|u|^{p(x)-1}|\varphi(G_k(u))|$$

by Hölder's inequality and interpolation we obtain :

$$C_{22}\int_{A_{k+1}}|g|^{p'(x)}\varphi'(G_k(u))$$
$$\leq C_{24}\|g\|^{p'_+}_{L^{r_-}(\Omega,\mathbb{R}^N)}\|\varphi(G_k(u))\|^{\frac{p'_+N}{p_-r_-}}_{L^{p*_-/p_-}(A_k)}\|\varphi(G_k(u))\|^{1-\frac{p'_+N}{p_-r_-}}_{L^1(A_k)}$$
$$+\,\frac{\alpha_0}{4}\int_{A_k}|u|^{p(x)-1}|\varphi(G_k(u))|$$

as before while (2.2.25), (2.4.2) and using Young's and sobolev's inequalities we can deduce that :

$$C_{22}\int_{A_{k+1}}|g|^{p'(x)}\varphi'(G_k(u))$$
$$\leq \frac{1}{8}\int_{A_k}|\nabla(G_k(u))|^{p(x)}\varphi'(G_k(u))dx+\frac{1}{8}meas(A_k)$$
$$+\, C_{25}\|g\|^{\frac{p'_+p_-r_-}{p_-r_- -p'_+N}}_{L^{r_-}(\Omega,\mathbb{R}^N)}\|\varphi(G_k(u))\|_{L^1(A_k)}$$
$$+\,\frac{\alpha_0}{4}\int_{A_k}|u|^{p(x)-1}|\varphi(G_k(u))|$$

Therefore, by taking k_0 satistying (2.4.2), (2.4.4), (2.4.6), (2.4.7) and the further condition :

$$C_{25}\|g\|^{\frac{p'_+p_-r_-}{p_-r_- -p'_+N}}_{L^{r_-}(\Omega,\mathbb{R}^N)}\leq\frac{\alpha_0 k_0^{p_- -1}}{4} \tag{2.4.8}$$

one obtains, for every $k\geq k_0$:

$$\frac{1}{4}\int_{A_k}|\nabla G_k(u)|^{p(x)}\varphi'(G_k(u))$$

$$\leq \varphi(1)(\frac{1}{q_-} + \frac{1}{q'_-})\|f\|_{L^{q(x)}(\{|f|>1\})}(meas(A_k))^{\frac{1}{q'_+}} + \frac{1}{4}meas(A_k)$$

$$+ C_{24}(\|g\|_{L^{r(x)}(A_k)})^{\delta"}(meas(A_k))^{1-\frac{p'_+}{r_-}}$$

$$\leq C_{26}(meas(A_k))^m$$

where $m = \min(\frac{1}{q'_+}, 1 - \frac{p'_+}{r_-})$ in view of (2.2.23) and Sobolev's inequality we can obtain :

$$(\int_{A_k} |\psi(G_k(u))|^{p*(x)})^{\beta} \leq \|\psi(G_k(u))\|_{L^{p*(x)}(A_k)}$$

$$\leq C_{27}(meas(A_k))^{\frac{m}{\alpha}}$$

Where :

$$\alpha = \begin{cases} p_+ & \text{if } \|\nabla\psi(G_k(u))\|_{L^{p(x)}(A_k)} \leq 1, \\ p_- & \text{if } \|\nabla\psi(G_k(u))\|_{L^{p(x)}(A_k)} > 1. \end{cases}$$

$$\beta = \begin{cases} \frac{N-p_-}{Np_-} & \text{if } \|\psi(G_k(u))\|_{L^{p*(x)}(A_k)} \leq 1, \\ \frac{N-p_+}{Np_+} & \text{if } \|\psi(G_k(u))\|_{L^{p*(x)}(A_k)} > 1. \end{cases}$$

We now take $h - k > 1$ and recall that there exists $C_{28}(\lambda, p_+, p_-)$such that $|\psi(s)| \geq C_{28}|s|$ for every $s \in \mathbb{R}$ so that

$$[C_{28}(h-k)]^{p*_-}meas(A_h) \leq \int_{A_h} |\psi(G_k(u))|^{p*(x)}$$

$$\leq \int_{A_k} |\psi(G_k(u))|^{p*(x)}$$

$$\leq C_{29}(meas(A_k))^{\frac{m}{\alpha\beta}}$$

Then it follows for every h and k (such that $h > k \geq k_0$) that

$$meas(A_h) \leq \frac{C_{30}}{[h-k]^{p*_-}}(meas(A_k))^{\frac{m}{\alpha\beta}}$$

Since by (2.1.3), $\frac{m}{\alpha\beta} > 1$ Lemma 2.4.1 applied to the function $\phi(h) = meas(A_h)$ gives :

$$\|u_n\|_{L^\infty(\Omega)} \leq C$$

Chapitre 3

EXISTENCE RESULTS AND BOUNDEDNESS OF SOLUTIONS FOR SOME STRONGLY NONLINEAR $p(x)$-ELLIPTIC EQUATIONS IN GENERAL DOMAINS

Abstract

In this chapter, we prove an existence result and some regularity for the solution of the strongly nonlinear $p(x)$-elliptic problem of the form :

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) + c(x,u) + H(x,u,\nabla u) = f(x) - \operatorname{div} g(x) \quad \text{in } \Omega, \\ u \in W_0^{1,p(.)}(\Omega), \end{cases}$$

where we suppose that $c(x,s)$ and $H(x,s,\xi)$ are a nonlinear terms satisfying a natural growth condition but no sign condition on H. The hypothesis on the source terms load to the existence of solutions. The domain Ω is allowed to have infinite measure.

3.1 Introduction

Let Ω an open set of $\mathbb{R}^N$ with $N \geq 2$, possibly of infinite measure. In this chapter we are interested in establishing an existence result for the following elliptic problem :

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) + c(x,u) + H(x,u,\nabla u) = f(x) - \operatorname{div} g(x) \quad \text{in } \Omega, \\ u \in W_0^{1,p(.)}(\Omega), \end{cases} \tag{3.1.1}$$

As an example of (3.1.1) we can refer to the problem :

$$\begin{cases} -\Delta_{p(x)}(u) + \gamma |u|^{p(x)-2}u = h(x)|\nabla u|^{p(x)} + f(x) - \operatorname{div} g(x) \quad \text{in } (\Omega), \\ u \in W_0^{1,p}(\Omega), \end{cases}$$

where $-\Delta_{p(x)}(u) = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplace operator, γ is a positive constant and $h(x)$ is a function in $L^\infty(\Omega)$.

In the case where Ω is a bounded, and for $2 - \frac{1}{N} < p_- \leq p_+ < N$ Azroul, Hjiaj, Touzani in [9] studied the problem :

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) + g(x,u,\nabla u) = f - \operatorname{div} \phi(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where the right hand side is satisfying :

$$f \in L^1(\Omega) \quad \text{and} \quad \phi \in C^0(\mathbb{R}^N).$$

They prove the existence of at least one entropy solution $u \in W_0^{1,q(x)}(\Omega)$ for all continuous functions $q(.)$ such that : $1 < q(x) < \bar{q}(x) = \frac{N(p(x)-1)}{N-1}$.

In [41] Vincenzo Ferone and Franois Murat studied the problem :

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = H(x,u,\nabla u) + f - \operatorname{div} g \quad \text{in } D'(\Omega), \\ u \in W_0^{1,p}(\Omega), \end{cases}$$

where Ω is a bounded and the right hand side is assumed to satisfy :

$$f \in L^{N/p}(\Omega), g \in (L^{N/(p-1)}(\Omega))^N.$$

Under suitable smallness assumptions on the source data they prove the existence of a solution u which satisfies a further regularity. The authors in [21] proved an L^∞ estimate for the bounded solutions of

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + a_0(x,u,\nabla u) = H(x,u,\nabla u) + f - \operatorname{div} g \quad \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega), \end{cases}$$

when f belongs to $L^q(\Omega)$ and g belongs to $(L^r(\Omega))^N$ with $r = p'q$ and $max(1, N/p) < q \leq \infty$.

With respect to the regularity results both local Lipschitz-continuity and local boundedness of weak solutions, obtained from the pioneering contributions in [58] [77] [61] [26] [65] without imposing boundary conditions. The authors M. Eleuteri, P. Marcellini, E. Mascolo have proved in [39] that any vector-valued minimizer of an energy integral over an open set Ω, with variable exponent $p(x)$ in the Sobolev class $W^{1,r}_{loc}(\Omega)$ for some $r > N$, is locally Lipschitz continuous with suitable assumptions. We emphasize here that energies with variable exponents are also studied in the framework of the $p; q$-growth(according to Marcellini's terminology). In the case of $p(x)$ is a continuous function, the key idea is to think that in small domains (say small balls), p is the minimum of $p(x)$ and q is the maximum values of $p(x)$ with q arbitrarily close to p(we refer to Marcellini [[56], [58], [57], [59]] and to Mingione [61] for a survey on this subject). Therefore, since proving local regularity results is something that can be done by reducing the problem to an arbitrarily small open subset of Ω, and then concluding with a standard covering argument. In this chapter we study a general differential equation which non always can be written, in an equivalent form, as a minimization problem for an appropriate energy integral and we obtained a global regularity of weak solutions, however by imposing the zero Dirichlet boundary condition.

The case of unbounded domains was treated for example in [[75], [37], [35], [69], [38], [28], [30], [29]]. In [75] Q. Zhang obtained sufficient conditions of the existence of radial solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$. In [37] authors studied the existence of infinitely many solutions for a class of $p(x)$-Laplacian equations in $\mathbb{R}^N$, when the non-linearity is sublinear in u at infinity. In [35] the existence result in weighted Sobolev spaces was proved. Jeff In [69] proved the existence of variational solutions for the equation $Au(x) + c(x,u) = f(x)$ with boundary conditions of either Dirichlet or Neumann type, where A is a nonlinear elliptic partial differential operator in divergence form and the term $c(x,u)$ is strongly nonlinear satisfying the sign condition. In [28] and [29] and [30] authors consider p as a constant exponent and they proved the existence and some regularity on solution in the case of nonlinear elliptic and parabolic equations, in the present work such an approach cannot be used directly, because of the variability of p and the transition from the norm to the modular makes appears undesirable constants in the proof of the a priori estimate (which is a principal result for the existence and the regularity), to solve this problem we have partitioned the domain in the approximate problem and we used the fact that p is continuous to find a local relation on the exponent ((see (3.3.24)), this relation played an important role in the proof), based on the above fact and motivated by techniques used in [[41], [66]], the main purpose of this part is devoted to investigate the existence of at least one weak solution for problem (3.1.1) and to give some regularity for this solution. Our analysis is built on the variable exponent Lebesgue-Sobolev space theory and theory of operators of monotone type in reflexive Banach spaces. The use of test functions of exponential type allows to get rid of the term $H(x, u, \nabla u)$ and therefore is an essential tool in the proof.

To obtain the existence result, since Ω can have infinite measure, we proceed by solving the problem on a sequence Ω_n of bounded sets, after that we pass to the limit in the approximating problems. To this aim, we can neither use any embedding theorem between $L^{p(\cdot)}(\Omega)$ nor any argument involving the measure of Ω_n. For regularity we will use an adaptation of a classical technique due to Stampacchia [66].

This chapter is organized as follows :
In the second section we introduce some assumptions on the data of the problem. The third section contains the existence of solutions for the approximate problem, some important lemmas useful to prove our main results and some essential a priori estimates. The fourth section is devoted to show the existence of solutions for the problem (3.1.1) by passing to the limit in the weak formulation of the approximate problem. In order to do this we need to use a local version of the technique used in [41]. Finally in the last section we prove that, if f and g have higher integrability, then every solution u of (3.1.1) is bounded.

3.2 Essential assumption

Let Ω be an open set of $\mathbb{R}^N$ ($N \geq 2$), We consider the problem :(3.1.1), and $p \in C_+(\bar{\Omega})$.

assumptions on $a(x, s, \xi)$:

We consider a Leray-Lions operator from $W_0^{1,p(x)}(\Omega)$ into its dual $W^{-1,p'(x)}(\Omega)$, defined by the formula :

$$Au = -\operatorname{div}\ a(x, u, \nabla u),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) which satisfies the following conditions :

$$|a(x,s,\xi)| \leq \beta\left(K(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}\right), \tag{3.2.1}$$

$$a(x,s,\xi)\xi \geq \alpha|\xi|^{p(x)}, \tag{3.2.2}$$

$$\left[a(x,s,\xi) - a(x,s,\overline{\xi})\right](\xi - \overline{\xi}) > 0 \quad \text{for all } \xi \neq \overline{\xi} \text{ in } \mathbb{R}^N, \tag{3.2.3}$$

for a.e. $x \in \Omega$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $K(x)$ is a positive function lying in $L^{p'(x)}(\Omega) \cap L^{r(x)}_{loc}(\Omega)$ for some $r(x) > p'(x)$ and $\alpha, \beta > 0$.

assumptions on $c(x,s)$:

$c : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function ; there exists $\alpha_0, \beta_0 > 0$ such that, for every (x,s) in $\Omega \times \mathbb{R}$,

$$|c(x,s)| \leq \beta_0\left(K_0(x) + |s|^{p(x)-1}\right), \tag{3.2.4}$$

$$c(x,s)s \geq \alpha_0|s|^{p(x)}, \tag{3.2.5}$$

where $K_0(x)$ is a positive function in $L^{p'(x)}(\Omega)$.

assumptions on $H(x,s,\xi)$:

The nonlinear term $H(x,s,\xi)$ is a Carathéodory function which satisfies :

$$|H(x,s,\xi)| \leq d|\xi|^{p(x)}, \tag{3.2.6}$$

where $d > 0$, for a.e. $x \in \Omega$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

assumptions on $f(x)$ and $g(x)$:

$f : \Omega \to \mathbb{R}$, $g : \Omega \to \mathbb{R}^N$ measurable functions satisfying :

$$\begin{gathered} f \in L^{N/p(x)}\left(\{x \in \Omega : 1 < |f(x)|\}\right), f \in L^{p'(x)}\left(\{x \in \Omega : |f(x)| \leq 1\}\right) \\ g \in L^{N/(p(x)-1)}(\Omega;\mathbb{R}^N) \cap L^{p'(x)}(\Omega;\mathbb{R}^N) \end{gathered} \tag{3.2.7}$$

To obtain some regularity on the solution u we will assume that (3.2.7) are replaced by :

$$\begin{gathered} f \in L^{q(x)}\left(\{x \in \Omega : 1 < |f(x)|\}\right) \text{ for some } q(x) > N/p(x), \\ f \in L^{p'(x)}\left(\{x \in \Omega : |f(x)| \leq 1\}\right) \\ g \in L^{r(x)}(\Omega;\mathbb{R}^N) \cap L^{p'(x)}(\Omega;\mathbb{R}^N) \text{ for some } r(x) > N/(p(x)-1) \end{gathered} \tag{3.2.8}$$

Notations : The symbol $\rightharpoonup$ will denote the weak convergence, and the constants C_i, $i = 1, 2, \dots$ used in each step of proof are independent.

3.3 Approximate problem and A priori estimate

In this we prove the existence result to the approximate problem. Also we will give a uniform estimate for this solutions u_n.

Approximate problem and useful lemmas

Let $\Omega_n = \Omega \cap B_n(0)$ where $B_n(0)$ is the Ball with center 0 and radius n, we consider the approximate problem :

$$\begin{cases} -\operatorname{div} a(x, u_n, \nabla u_n) + c(x, u_n) + H_n(x, u_n, \nabla u_n) = f_n(x) - \operatorname{div} g_n(x) \quad \text{in } \Omega_n, \\ u_n \in W_0^{1,p(.)}(\Omega_n) \cap L^\infty(\Omega_n), \end{cases} \tag{3.3.1}$$

with $H_n(x, s, \xi) = T_n(H(x, s, \xi))$ $f_n(x) = T_n(f(x))$ and $g_n(x) = \frac{g(x)}{1+\frac{1}{n}|g(x)|}$ where $T_k(.)$ is the truncation function defined by :

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases} \tag{3.3.2}$$

. Let us remark that $|H_n| \leq |H|$, $|H_n| \leq n$,$|f_n| \leq |f|$and $|g_n| \leq |g|$.

Lemma 3.3.1 ([9]) *Let us Assume* (3.2.1)-(3.2.3) *hold, and let* $(u_n)_n$ *be a sequence in* $W_0^{1,p(x)}(\Omega)$ *such that* $u_n \rightharpoonup u$ *in* $W_0^{1,p(x)}(\Omega)$ *and*

$$\int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)] \nabla(u_n - u)dx \to 0, \tag{3.3.3}$$

then $u_n \to u$ *in* $W_0^{1,p(x)}(\Omega)$ *for a subsequence.*

We define the operator $R_n : W_0^{1,p(x)}(\Omega_n) \to W^{-1,,p'(x)}(\Omega_n)$, by :

$$\langle R_n u, v\rangle = \int_{\Omega_n} c(x, u)v + H_n(x, u, \nabla u)v dx \quad \forall v \in W_0^{1,p(x)}(\Omega_n).$$

by the Hölder inequality, and the growth condition (3.2.4) we have that :

for all $u, v \in W_0^{1,p(x)}(\Omega_n)$,

$$
\begin{aligned}
&|\int_{\Omega_n} c(x,u)v + H_n(x,u,\nabla u)v dx| \\
&\leq (\frac{1}{p_-} + \frac{1}{p'_-}) \left[\|c(x,u)\|_{L^{p'(x)}(\Omega_n)} \|v\|_{L^{p(x)}(\Omega_n)} + \|H_n(x,u,\nabla u)\|_{L^{p'(x)}(\Omega_n)} \|v\|_{L^{p(x)}(\Omega_n)} \right] \\
&\leq (\frac{1}{p_-} + \frac{1}{p'_-}) \left[\Big(\int_{\Omega_n} \big(|c(x,u)|^{p'(x)} dx + 1\Big)^{\frac{1}{p'_-}} + \Big(\int_{\Omega_n} \big(|H_n(x,u,\nabla u)|^{p'(x)} dx + 1\Big)^{\frac{1}{p'_-}} \right] \|v\|_{W^{1,p(x)}(\Omega_n)} \\
&\leq (\frac{1}{p_-} + \frac{1}{p'_-}) \left[\Big(\int_{\Omega_n} \big((\beta_0(K_0(x) + |u|^{p(x)-1})\big)^{p'(x)} dx + 1\Big)^{\frac{1}{p'_-}} + \Big(\int_{\Omega_n} n^{p'(x)} dx + 1\Big)^{\frac{1}{p'_-}} \right] \|v\|_{W^{1,p(x)}(\Omega_n)},
\end{aligned}
$$

Then :

$$
\begin{aligned}
&|\int_{\Omega_n} c(x,u)v + H_n(x,u,\nabla u)v dx| \\
&\leq (\frac{1}{p_-} + \frac{1}{p'_-}) \left[\Big(\int_{\Omega_n} C_0(K_0(x)^{p'(x)} + |u|^{p(x)}) dx + 1\Big)^{\frac{1}{p'_-}} + \Big(\int_{\Omega_n} n^{p'(x)} dx + 1\Big)^{\frac{1}{p'_-}} \right] \|v\|_{W^{1,p(x)}(\Omega_n)} \\
&\leq (\frac{1}{p_-} + \frac{1}{p'_-}) \left[\Big(\int_{\Omega_n} C_0(K_0(x)^{p'(x)} + |u|^{p(x)}) dx + 1\Big)^{\frac{1}{p'_-}} + \Big(n^{p'_+}. \operatorname{meas}(\Omega_n) + 1\Big)^{\frac{1}{p'_-}} \right] \|v\|_{W^{1,p(x)}(\Omega_n)} \\
&\leq C_1 \|v\|_{W^{1,p(x)}(\Omega_n)},
\end{aligned} \tag{3.3.4}
$$

Lemma 3.3.2 *The operator $B_n = A + R_n$ is pseudo-monotone from $W_0^{1,p(x)}(\Omega_n)$ into $W^{-1,p'(x)}(\Omega_n)$. Moreover, B_n is coercive in the following sense*

$$
\frac{\langle B_n v, v\rangle}{\|v\|_{W^{1,p(x)}(\Omega_n)}} \to +\infty \quad \textit{as} \quad \|v\|_{W^{1,p(x)}(\Omega_n)} \to +\infty \quad \textit{for} \quad v \in W_0^{1,p(x)}(\Omega_n).
$$

Proof : Using Hölder's inequality and the growth condition (3.2.1), we can show that the operator A is bounded, and by using (3.3.4) we conclude that B_n is bounded. For the coercivity, we have for any $u \in W_0^{1,p(x)}(\Omega_n)$,

$$
\begin{aligned}
\langle B_n u, u\rangle &= \langle Au, u\rangle + \langle R_n u, u\rangle \\
&= \int_{\Omega_n} a(x,u,\nabla u)\nabla u dx + \int_{\Omega_n} c(x,u)u + H_n(x,u,\nabla u)u dx \\
&\geq \alpha \int_{\Omega} |\nabla u|^{p(x)} dx - C_1.\|u\|_{W^{1,p(x)}(\Omega_n)} \quad \text{(using(3.3.4)and(3.2.2))} \\
&\geq \alpha \|\nabla u\|^{\delta'}_{L^{p(x)}(\Omega_n)} - C_1.\|u\|_{W^{1,p(x)}(\Omega_n)} \\
&\geq \alpha' \|u\|^{\delta'}_{W^{1,p(x)}(\Omega_n)} - C_1.\|u\|_{W^{1,p(x)}(\Omega_n)}, \quad \text{(using the Poincaré inequality)}
\end{aligned}
$$

With

$$\delta' = \begin{cases} p_- & \text{if } \|\nabla u\|_{L^{p(x)}(\Omega_n)} > 1, \\ p_+ & \text{if } \|\nabla u\|_{L^{p(x)}(\Omega_n)} \leq 1, \end{cases}$$

Then, we obtain :

$$\frac{\langle B_n u, u\rangle}{\|u\|_{W^{1,p(x)}(\Omega_n)}} \to +\infty \quad \text{as } \|u\|_{W^{1,p(x)}(\Omega_n)} \to +\infty.$$

It remains now to show that B_n is pseudo-monotone. Let $(u_k)_k$ a sequence in $W_0^{1,p(x)}(\Omega_n)$ such that :

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{in } W_0^{1,p(x)}(\Omega_n), \\ B_n u_k &\rightharpoonup \chi \quad \text{in } W^{-1,p'(x)}(\Omega_n), \\ \limsup_{k\to\infty} &\langle B_n u_k, u_k\rangle \leq \langle \chi, u\rangle. \end{aligned} \tag{3.3.5}$$

We will prove that :

$$\chi = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \quad \text{as } k \to +\infty.$$

Firstly, since $W_0^{1,p(x)}(\Omega_n) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega_n)$, then $u_k \to u$ in $L^{p(x)}(\Omega_n)$ for a subsequence still denoted by $(u_k)_k$. We have $(u_k)_k$ is a bounded sequence in $W_0^{1,p(x)}(\Omega_n)$, then by the growth condition $(a(x,u_k,\nabla u_k))_k$ is bounded in $(L^{p'(x)}(\Omega_n))^N$, therefore, there exists a function $\varphi \in (L^{p'(x)}(\Omega_n))^N$ such that :

$$a(x,u_k,\nabla u_k) \rightharpoonup \varphi \quad \text{in } (L^{p'(x)}(\Omega_n))^N \text{ as } k \to \infty. \tag{3.3.6}$$

Similarly, since $(c(x,u_k) + H_n(x,u_k,\nabla u_k))_k$ is bounded in $L^{p'(x)}(\Omega_n)$, then there exists a function $\psi_n \in L^{p'(x)}(\Omega_n)$ such that :

$$c(x,u_k) + H_n(x,u_k,\nabla u_k) \rightharpoonup \psi_n \quad \text{in } L^{p'(x)}(\Omega_n) \text{ as } k \to \infty, \tag{3.3.7}$$

For all $v \in W_0^{1,p(x)}(\Omega_n)$, we have :

$$\begin{aligned} \langle \chi, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty} \int_{\Omega_n} a(x,u_k,\nabla u_k)\nabla v dx + \lim_{k\to\infty} \int_{\Omega_n} (c(x,u_k) + H_n(x,u_k,\nabla u_k)) v dx \\ &= \int_{\Omega_n} \varphi \nabla v dx + \int_{\Omega_n} \psi_n v dx. \end{aligned} \tag{3.3.8}$$

Using (3.3.5) and (3.3.8), we obtain :

$$\begin{aligned} &\limsup_{k\to\infty} \langle B_n(u_k), u_k\rangle \\ &= \limsup_{k\to\infty} \Big\{ \int_{\Omega_n} a(x,u_k,\nabla u_k)\nabla u_k dx + \int_{\Omega_n} (c(x,u_k) + H_n(x,u_k,\nabla u_k)) u_k dx \Big\} \\ &\leq \int_{\Omega} \varphi \nabla u dx + \int_{\Omega} \psi_n u dx, \end{aligned} \tag{3.3.9}$$

Thanks to (3.3.7), we have :

$$\int_{\Omega_n} \left(c(x, u_k) + H_n(x, u_k, \nabla u_k)\right) u_k dx \to \int_{\Omega_n} \psi_n u dx; \tag{3.3.10}$$

Therefore,

$$\limsup_{k\to\infty} \int_{\Omega_n} a(x, u_k, \nabla u_k)\nabla u_k dx \leq \int_{\Omega_n} \varphi \nabla u dx. \tag{3.3.11}$$

On the other hand, using (3.2.3), we have :

$$\int_{\Omega_n} \left(a(x, u_k, \nabla u_k) - a(x, u_k, \nabla u)\right)(\nabla u_k - \nabla u)\, dx\, . \geq 0, \tag{3.3.12}$$

Then

$$\begin{aligned}&\int_{\Omega_n} a(x, u_k, \nabla u_k)\nabla u_k dx \\ &\geq -\int_{\Omega_n} a(x, u_k, \nabla u)\nabla u dx + \int_{\Omega_n} a(x, u_k, \nabla u_k)\nabla u dx + \int_{\Omega_n} a(x, u_k, \nabla u)\nabla u_k dx,\end{aligned}$$

and by (3.3.6), we get :

$$\liminf_{k\to\infty} \int_{\Omega_n} a(x, u_k, \nabla u_k)\nabla u_k dx \geq \int_{\Omega_n} \varphi \nabla u dx,$$

this implies, thanks to (3.3.11), that :

$$\lim_{k\to\infty} \int_{\Omega_n} a(x, u_k, \nabla u_k)\nabla u_k dx = \int_{\Omega_n} \varphi \nabla u dx. \tag{3.3.13}$$

By combining (3.3.8), (3.3.10) and (3.3.13), we deduce that :

$$\langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \quad \text{as } k \to +\infty.$$

Now, by (3.3.13) we can obtain :

$$\lim_{k\to+\infty} \int_{\Omega_n} \left(a(x, u_k, \nabla u_k) - a(x, u_k, \nabla u)\right)(\nabla u_k - \nabla u)\, dx = 0,$$

In view of the Lemma 3.3.1, we obtain :

$$u_k \to u, \quad W_0^{1,p(x)}(\Omega_n), \quad \nabla u_k \to \nabla u \quad \text{a.e. in } \Omega_n,$$

then

$$a(x, u_k, \nabla u_k) \rightharpoonup a(x, u, \nabla u) \qquad \text{in } (L^{p'(x)}(\Omega_n))^N,$$

and

$$c(x, u_k) + H_n(x, u_k, \nabla u_k) \rightharpoonup c(x, u) + H_n(x, u, \nabla u) \quad \text{in } L^{p'(x)}(\Omega_n),$$

we deduce that $\chi = B_n u$, which completes the proof. By Lemma 3.3.2, we deduce that there exists at least one weak solution
$u_n \in W_0^{1,p(x)}(\Omega_n)$ of the problem (3.3.1), (cf. [52]).

A priori estimate

Proposition 3.3.1 *Assuming that* (3.2.1)-(3.2.7) *hold,* $p(.) \in C_+(\overline{\Omega})$ *and let* u_n *be any solution of* (3.3.1). *Then for every* $\lambda > 0$ *there exists a positive constant* $C = C(N, p, \alpha, \alpha_0, d, f, g, \lambda)$ *such that*

$$\|e^{\lambda|u_n|} - 1\|_{W_0^{1,p(x)}(\Omega_n)} \leq C. \tag{3.3.14}$$

Remark 3.3.1 *The previous estimate yields an estimate for the functions* $e^{\lambda|u_n|}$ *in* $L^{r(x)}_{loc}(\Omega)$ *for every* $r \in [1, +\infty)$*, every* $\lambda > 0$ *and every set* $\Omega_0 \subset\subset \Omega$*, one has*

$$\|e^{|u_n|}\|_{L^{r(x)}(\Omega_0)} \leq C(r_{\mp}, \lambda, \Omega_0)$$

Proof : For simplicity of notation we will always omit the index n of the sequence. We take $\varphi(G_k(u))$ as test function in (3.3.1), where

$$G_k(s) = s - T_k(s) = \begin{cases} s-k & \text{if } s > k, \\ 0 & \text{if } |s| \leq k, \\ s+k & \text{if } s < -k. \end{cases} \quad \text{and} \quad \varphi(s) = \big(e^{\lambda|s|} - 1\big) sign(s). \tag{3.3.15}$$

In view of hypotheses (3.2.2),(3.2.5) and (3.2.6) we have :

$$\begin{aligned} &\alpha \int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + \alpha_0 \int_\Omega |u|^{p(x)-1} |\varphi(G_k(u))|, \\ &\leq d \int_\Omega |\nabla G_k(u)|^{p(x)} |\varphi(G_k(u))| + \int_\Omega |f||\varphi(G_k(u))| + \int_\Omega |g||\nabla G_k(u)| \varphi'(G_k(u)) \\ &= I + J + K, \end{aligned} \tag{3.3.16}$$

Estimation of the integral I :

For every s in $\mathbb{R}$ and if λ satisfies :

$$\lambda \geq \frac{8d}{\alpha} \tag{3.3.17}$$

we have :

$$d|\varphi(s)| \leq \frac{\alpha}{8} \varphi'(s) \tag{3.3.18}$$

then

$$I \leq \frac{\alpha}{8} \int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) \tag{3.3.19}$$

Before estimating J, we remark that :

$$\int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) = \int_\Omega |\nabla \Psi(G_k(u))|^{p(x)} \tag{3.3.20}$$

where

$$\Psi(s)=\int_0^{|s|}(\varphi'(t))^{\frac{1}{p(x)}}dt=\frac{p(x)}{\lambda^{\frac{1}{p'(x)}}}\left(e^{\frac{\lambda|s|}{p(x)}}-1\right) \tag{3.3.21}$$

Moreover, we observe that there exists a positive constant $C_2=C_2(p,\lambda)$ such that

$$|\varphi(s)|\le C_2(\Psi(s))^{p(x)}\text{for every s such that}|s|\ge 1 \tag{3.3.22}$$

Now let us observe that p is a continuous variable exponent on $\overline{\Omega}$ then there exists a constant $\delta>0$ such that :

$$\max_{y\in B(x,\delta)\cap\Omega}\frac{N-p(y)}{Np(y)}\le\min_{y\in B(x,\delta)\cap\Omega}\frac{p(y)(N-p(y))}{Np(y)}\quad\text{for all } x\in\Omega. \tag{3.3.23}$$

while $\overline{\Omega}$ is compact then we can cover it with a finite number of balls B_i for $i=1,...,m$ from (3.3.23) we can deduce the point-wise estimate :

$$1<p_{-,i}\le p_{+,i}\le\frac{p_{-,i}^2N}{N-p_{-,i}+p_{-,i}^2}<N. \tag{3.3.24}$$

is satisfy for all $i=1,...,m$.

$p_{-,i},p_{+,i}$ denote the local minimum and the local maximum of p on $\overline{B_i\cap\Omega}$ respectively

Estimation of the integral J :

Let $H\ge 1$ be a constant that we will chose later. We can estimate J by splitting it as follows :

$$\begin{aligned}J=&\sum_{i=0}^{m}\left[\int_{B_i\cap\{|f|>H,|G_k(u)|\ge1\}}|f||\varphi(G_k(u))|\right]+\int_{\{|f|>H,|G_k(u)|<1\}}|f||\varphi(G_k(u))|\\&+\int_{\{|f|\le H\}}|f||\varphi(G_k(u))|\\&=J_1+J_2+J_3\end{aligned}$$

By (3.3.22) J_1, can be estimated as follows :

$$J_1\le C_2\sum_{i=0}^{m}\left[\int_{B_i\cap\{|f|>H,|G_k(u)|\ge1\}}|f|\Psi(G_k(u))^{p(x)}\right]$$

Let ϵ a positive constant to be chosen later. Using Young, Sobolev's embedding and Lemma 2.2.1 we have :

$$J_1\le C\int_{\{|f|>H\}}|f|^{\frac{\epsilon N}{\epsilon N+p(x)-N}}+\frac{\alpha}{8}\sum_{i=0}^{m}\|\nabla\Psi(G_k(u))\|_{L^{p(x)}(B_i)}^{\epsilon p^*_{+,i}}$$

$$\leq C \int_{\{|f|>H\}} |f|^{\frac{\epsilon N}{\epsilon N + p(x) - N}} + \frac{\alpha}{8} \sum_{i=0}^{m} \left[\int_{B_i} |\nabla \Psi(G_k(u))|^{p(x)} \right]^{\frac{\epsilon p^*_{+,i}}{p_{-,i}}}$$

where $p^*(x) = \frac{Np(x)}{N-p(x)}$ and $p^*_{+,i} = \frac{Np_{+,i}}{N-p_{+,i}}$ since (3.3.24) we can choose ϵ such that :

$$\frac{N - p_{-,i}}{Np_{-,i}} \leq \epsilon \leq \frac{p_{-,i}}{p^*_{+,i}} \tag{3.3.25}$$

Then using (3.3.25) and (3.3.20) we obtain that :

$$J_1 \leq C \int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} + \frac{\alpha}{8} \int_{\Omega} |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) \tag{3.3.26}$$

Remark 3.3.2 *The cases where $\|\Psi(G_k(u))\|_{L^{p*(x)}(\Omega)} \leq 1$ or $\|\nabla \Psi(G_k(u))\|_{L^{p(x)}(\Omega)} \leq 1$ are easy to see that $J_1 \leq C$ (C depend on the data of the problem)*

On the other hand

$$J_2 \leq \varphi(1) \int_{\{|f|>H\}} |f| \leq \frac{\varphi(1)}{H^{\frac{N-p_+}{p_+}}} \int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} \tag{3.3.27}$$

Finally, choosing k sufficiently large such that :

$$\alpha_0 k^{p_- - 1} \geq 4H \tag{3.3.28}$$

We can obtain :

$$J_3 \leq \frac{\alpha_0}{4} \int_{\Omega} k^{p(x)-1} |\varphi(G_k(u))| \leq \frac{\alpha_0}{4} \int_{\Omega} |u|^{p(x)-1} |\varphi(G_k(u))| \tag{3.3.29}$$

Estimation of the integral K :

Thanks to Young's inequality, we have :

$$\begin{aligned} K \leq & \frac{\alpha}{8} \int_{\Omega} |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + C_4 \int_{\Omega} |g|^{p'(x)} \varphi'(G_k(u)), \\ & = K_1 + K_2 \end{aligned} \tag{3.3.30}$$

The integral K_2 can be estimated as follows :

$$\begin{aligned} K_2 \leq\, & C_4 \int_{\Omega} |g|^{p'(x)} \varphi'(G_k(u)), \\ & \leq C_4 \lambda e^{\lambda} \int_{\{|G_k(u)|<1\}} |g|^{p'(x)} + C_4 \sum_{i=0}^{m} \Big[\int_{B_i \cap \{|g|>1, |G_k(u)|>1\}} |g|^{p'(x)} \varphi'(G_k(u)) \Big] \\ & + C_4 \int_{\{|g|\leq 1, |G_k(u)|>1\}} \varphi'(G_k(u)) \\ & = K_{2,1} + K_{2,2} + K_{2,3} \end{aligned} \tag{3.3.31}$$

Since $\varphi'(s) \leq C_5(\Psi(s))^{p(x)}$ for every s such that $|s| \geq 1$, we have :

$$K_{2,2} \leq C_6 \sum_{i=0}^{m} \Big[\int_{B_i \cap \{|g|>1, |G_k(u)|>1\}} |g|^{p'(x)} \Psi(G_k(u))^{p(x)} \Big]$$

Let ϵ be a positive constant such that (3.3.25). Using Young, Sobolev's embedding and Lemma 2.2.1 we have :

$$\begin{aligned} K_{2,2} &\leq C_7 \int_{\{|g|>1\}} |g|^{\frac{\epsilon N p'(x)}{\epsilon N + p(x) - N}} + \frac{\alpha}{8} \sum_{i=0}^{m} \|\nabla \Psi(G_k(u))\|_{L^{p(x)}(B_i)}^{\epsilon p^*_{+,i}} \\ &\leq C_7 \int_{\{|g|>1\}} |g|^{\frac{\epsilon N p'(x)}{\epsilon N + p(x) - N}} + \frac{\alpha}{8} \sum_{i=0}^{m} \Big[\int_{B_i} |\nabla \Psi(G_k(u))|^{p(x)} \Big]^{\frac{\epsilon p^*_{+,i}}{p_{-,i}}} \end{aligned}$$

Then using (3.3.25) and (3.3.20) we obtain that :

$$K_{2,2} \leq C_7 \int_{\{|g|>1\}} |g|^{\frac{N}{p(x)-1}} + \frac{\alpha}{8} \int_{\Omega} |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) \tag{3.3.32}$$

The same as before in the cases where $\|\Psi(G_k(u))\|_{L^{p*(x)}(\Omega)} \leq 1$ or $\|\nabla \Psi(G_k(u))\|_{L^{p(x)}(\Omega)} \leq 1$ it is easy to check that $K_{2,2} \leq C$

Finally, using inequality

$$\varphi'(s) \leq C_8 |\varphi(s)|, \quad \text{for every s such that} |s| \geq 1 \tag{3.3.33}$$

and choosing $k = k(p_-, \alpha, \alpha_0, \lambda)$ sufficiently large such that :

$$\alpha_0 k^{p_- - 1} \geq 4 C_4 \tag{3.3.34}$$

we obtain :

$$K_{2,3} \leq \frac{\alpha_0}{4} \int_{\Omega} k^{p(x)-1} |\varphi(G_k(u))| \leq \frac{\alpha_0}{4} \int_{\Omega} |u|^{p(x)-1} |\varphi(G_k(u))| \tag{3.3.35}$$

Putting all the inequalities (3.3.16), (3.3.19), (3.3.26), (3.3.27), (3.3.29), (3.3.32), (3.3.35), (3.3.31) and (3.3.30) together, we get an estimate in $W_0^{1,p(x)}(\Omega)$ for $G_k(u)$,when k is large enough :

$$\begin{aligned} &\frac{\alpha}{2} \int_{\Omega} |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + \frac{\alpha_0}{2} \int_{\Omega} |u|^{p(x)-1} |\varphi(G_k(u))| \\ &\leq C \int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} + \frac{\varphi(1)}{H^{\frac{N-p_+}{p_+}}} \int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} + C_4 \lambda e^{\lambda} \int_{\Omega} |g|^{p'(x)} + C_7 \int_{\Omega} |g|^{\frac{N}{p(x)-1}} \\ &= C_9(N, p_-, p_+, \alpha, \alpha_0, f, g, \lambda) \end{aligned} \tag{3.3.36}$$

For every λ, k satisfying (3.3.17), (3.3.28), (3.3.34) and for every $H \geq 1$.

We fix now λ and k such that (3.3.36) holds.

As before, In view of hypotheses (3.2.2),(3.2.5) and (3.2.6) and if we take $\varphi(T_k(u))$ as a test function in (3.3.1) we obtain :

$$\begin{aligned} &\alpha \int_\Omega |\nabla T_k(u)|^{p(x)}\varphi'(T_k(u)) + \alpha_0 \int_\Omega |u|^{p(x)-1}|\varphi(T_k(u))|, \\ &\leq d\int_\Omega |\nabla T_k(u)|^{p(x)}|\varphi(T_k(u))| + d\varphi(k)\int_\Omega |\nabla G_k(u)|^{p(x)}| + \int_\Omega |f||\varphi(T_k(u))| \\ &+ \int_\Omega |g||\nabla T_k(u)|\varphi'(T_k(u)) \\ &= L_1 + L_2 + L_3 + L_4, \end{aligned} \tag{3.3.37}$$

Using (3.3.18), we have :

$$L_1 \leq \frac{\alpha}{4}\int_\Omega |\nabla T_k(u)|^{p(x)}\varphi'(T_k(u)) \tag{3.3.38}$$

By (3.3.36),

$$L_2 \leq C_{10}(N, p_-, p_+, \alpha, \alpha_0, f, g, \lambda) \tag{3.3.39}$$

Remark 3.3.3 *if $meas(\Omega)$ is finite or if $f \in L^1(\Omega)$ it is easy to estimate the integral L_3*

In general case, let ϵ be a positive constant to be chosen later, we write :

$$\begin{aligned} L_3 &\leq \varphi(k)\int_{\{|f|>1\}} |f| + \int_{\{|f|\leq 1\}} |f||\varphi(T_k(u))| \\ &\leq \varphi(k)\int_{\{|f|>1\}} |f| + \epsilon\int_\Omega |\varphi(T_k(u))|^{p(x)} + c(\epsilon, p'_-)\int_{\{|f|\leq 1\}} |f|^{p'(x)} \end{aligned}$$

Since

$$|\varphi(T_k(u))|^{p(x)} \leq C_{11}(\lambda, p_+, p_-, k)|\varphi(T_k(u))||u|^{p(x)-1},$$

choosing ϵ such that $\epsilon C_{11} \leq \frac{\alpha_0}{2}$,we have :

$$L_3 \leq \frac{\alpha_0}{2}\int_\Omega |u|^{p(x)-1}|\varphi(T_k(u))| + C_{12}(\alpha_0, f, \lambda, p_+, p_-, k) \tag{3.3.40}$$

Finally, one has

$$L_4 \leq \frac{\alpha}{4}\int_\Omega |\nabla T_k(u)|^{p(x)}\varphi'(T_k(u) + C_{13}(\alpha, \lambda, p'_-, g, p_-, k) \tag{3.3.41}$$

In conclusion, putting all the estimations (3.3.37) - (3.3.41) together, we get :

$$\frac{\alpha}{2}\int_\Omega |\nabla T_k(u)|^{p(x)}\varphi'(T_k(u)) + \frac{\alpha_0}{2}\int_\Omega |u|^{p(x)-1}|\varphi(T_k(u))| \leq C_{14}(N, p_-, p_+, \alpha, \alpha_0, f, g, \lambda). \tag{3.3.42}$$

In view of (3.3.36) and (3.3.42), we have :

$$\int_{\{|u|\le k\}} |\nabla u|^{p(x)} e^{\lambda |u|} \le C_{15}, \quad \int_{\{|u|>k\}} |\nabla u|^{p(x)} e^{\lambda(|u|-k)} \le C_{15}$$

For every λ, k large enough (see (3.3.17), (3.3.28) and (3.3.34)), where C_{15} depends on λ, k and the data. Since

$$\int_{\Omega} |\nabla u|^{p(x)} e^{\lambda |u|} = \int_{\{|u|\le k\}} |\nabla u|^{p(x)} e^{\lambda |u|} + e^{\lambda k} \int_{\{|u|>k\}} |\nabla u|^{p(x)} e^{\lambda(|u|-k)} \le C_{16}$$

If we fix the value of k (depending on λ), we obtain an estimate of $|\nabla(e^{\lambda|u|}-1)|$ in $L^{p(x)}(\Omega)$ (depending on λ). This implies, by Sobolev ?s embedding, that :

$$\int_{\Omega} (e^{\lambda |u|} - 1)^{p*(x)} \le C_{17} \tag{3.3.43}$$

For every λ such that (3.3.17), where C_{17} depends on λ and on the data of the problem. Note that (3.3.43) does not imply an estimate in $L^{p(x)}(\Omega)$ for $e^{\lambda|u|}-1$, since $meas(\Omega)$ may be infinite. To obtain such an estimate, we have to combine (3.3.42) and (3.3.43), since, for every $k>0$, one has the inequalities

$$\int_{\{|u|\le k\}} (e^{\lambda |u|} - 1)^{p(x)} \le C_{19} \int_{\Omega} |u|^{p(x)-1} |\varphi(T_k(u)|,$$

$$\int_{\{|u|>k\}} (e^{\lambda |u|} - 1)^{p(x)} \le C_{20} \int_{\Omega} (e^{\lambda |u|} - 1)^{p*(x)},$$

Therefore, if $k=k(\lambda)$ is such that (3.3.42) holds, we can write :

$$\int_{\Omega} (e^{\lambda |u|} - 1)^{p(x)} = \int_{\{|u|\le k\}} (e^{\lambda |u|} - 1)^{p(x)} + \int_{\{|u|>k\}} (e^{\lambda |u|} - 1)^{p(x)} \le C_{21} \tag{3.3.44}$$

where C_{21} depends on λ and the data of the problem.

3.4 Main result

In this section we will prove the main result of this chapter. Let $\{u_n\}$ be any sequence of solutions of problem (3.3.1), we extend them to zero in $\Omega \setminus \Omega_n$. By (3.3.14), there exist a subsequence (still denoted by u_n) and a function $u \in W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$.

Theorem 3.4.1 *Suppose that* (3.2.1) - (3.2.7) *hold. Then there exists at least one solution* u *of* (3.1.1)*; which is such that :*

$$\int_{\Omega} a(x,u,\nabla u)\nabla\psi dx + \int_{\Omega} c(x,u)\psi dx + \int_{\Omega} H(x,u,\nabla u)\psi dx = \int_{\Omega} f\psi dx - \int_{\Omega} g\nabla\psi dx. \tag{3.4.1}$$

for every function $\psi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. Moreover u satisfies

$$e^{\lambda|u|} - 1 \in W_0^{1,p(x)}(\Omega) \tag{3.4.2}$$

for every $\lambda \geq 0$.

The proof will be made in three steps.

Step 1 : An estimate for $\int_\Omega |\nabla G_k(u_n)|^{p(x)}$

In view of (3.3.36) we have :

$$\begin{aligned}
&\int_\Omega |\nabla G_k(u_n)|^{p(x)} \\
&\leq \frac{2C}{\alpha\lambda}\int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} + \frac{2\varphi(1)}{\alpha\lambda H^{\frac{N-p_+}{p_+}}}\int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} + \frac{2C_4\lambda e^\lambda}{\alpha\lambda}\int_\Omega |g|^{p'(x)} \\
&+ \frac{2C_7}{\alpha\lambda}\int_\Omega |g|^{\frac{N}{p(x)-1}}
\end{aligned} \tag{3.4.3}$$

If η is an arbitrary positive number, let us choose H such that the right-hand side of (3.4.3) is smaller than η. It follows that, for every k satisfying (3.3.28), (3.3.34), every λ satisfying (3.3.17), and every $n \in \mathbb{N}$

$$\int_\Omega |\nabla G_k(u_n)|^{p(x)} \leq \eta$$

which proves :

$$\sup_n \int_\Omega |\nabla G_k(u_n)|^{p(x)} \to 0 \quad \text{as} \quad k \to \infty \tag{3.4.4}$$

Step 2 : Strong convergence of $\nabla T_k(u_n) in (L^{p(x)}(\Omega_0))^N$

In this step, we will fix $k > 0$ and prove that $\nabla T_k(u_n) \to \nabla T_k(u)$ strongly in $L^{p(x)}(\Omega_0;\mathbb{R}^N)$ as $n \to \infty$; for k fixed.

In order to prove this result we define

$$z_n(x) = T_k(u_n) - T_k(u)$$

and we choose ψ a cut-off function such that

$$\psi \in C_0^\infty(\Omega), \quad 0 \leq \psi \leq 1, \quad \psi = 1 \quad \text{in} \quad \Omega_0$$

Let us take

$$v = \varphi(z_n)e^{\delta|u_n|}\psi \tag{3.4.5}$$

as a test function in (3.3.1), where λ and δ a positive constant to be chosen later, we obtain

$$\begin{aligned}
&A_n + B_n = \int_\Omega a(u_n, \nabla u_n)\nabla z_n \varphi'(z_n)e^{\delta|u_n|}\psi + \int_\Omega c(u_n)\varphi(z_n)e^{\delta|u_n|}\psi \\
&\leq d\int_\Omega |\nabla u_n|^{p(x)}|\varphi(z_n)|e^{\delta|u_n|}\psi + \int_\Omega |f||\varphi(z_n)|e^{\delta|u_n|}\psi \\
&- \delta\int_\Omega a(u_n, \nabla u_n)\nabla u_n \varphi(z_n)e^{\delta|u_n|}sign(u_n)\psi + \int_\Omega |a(u_n, \nabla u_n)||\nabla\psi||\varphi(z_n)|e^{\delta|u_n|} \\
&+ \int_\Omega |g||\nabla z_n|\varphi'(z_n)e^{\delta|u_n|}\psi + \delta\int_\Omega |g||\nabla u_n||\varphi(z_n)|e^{\delta|u_n|}\psi + \int_\Omega |g||\nabla\psi||\varphi(z_n)|e^{\delta|u_n|} \\
&= C_n + D_n + E_n + F_n + G_n + H_n + L_n
\end{aligned} \tag{3.4.6}$$

Splitting Ω into $\Omega = \{|u_n| \leq k\} \cup \{|u_n| > k\}$ we can write :

$$\begin{aligned}
A_n =& \int_{\{|u_n|\leq k\}} a(T_k(u_n), \nabla T_k(u_n))\nabla z_n \varphi'(z_n)e^{\delta|T_k(u_n)|}\psi \\
&+ \int_{\{|u_n|>k\}} a(u_n, \nabla u_n)\nabla z_n \varphi'(z_n)e^{\delta|u_n|}\psi \\
&= \int_{\{|u_n|\leq k\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))]\nabla z_n \varphi'(z_n)e^{\delta|T_k(u_n)|}\psi \\
&+ \int_{\{|u_n|\leq k\}} a(T_k(u_n), \nabla T_k(u))\nabla z_n \varphi'(z_n)e^{\delta|T_k(u_n)|}\psi \\
&+ \int_{\{|u_n|>k\}} a(u_n, \nabla u_n)\nabla z_n \varphi'(z_n)e^{\delta|u_n|}\psi \\
&= A_{1,n} + A_{2,n} + A_{3,n}
\end{aligned}$$

since

$$a(T_k(u_n), \nabla T_k(u))\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi\chi_{\{|u_n|\leq k\}} \to a(T_k(u), \nabla T_k(u))\varphi'(0)e^{\delta|T_k(u)|}\psi\chi_{\{|u|\leq k\}}$$

almost everywhere in Ω (on the set where $|u(x)| = k$ we have $a(T_k(u_n), \nabla T_k(u)) = a(T_k(u_n), 0) = 0 = a(T_k(u), \nabla T_k(u))$)and

$$\begin{aligned}
&|a(T_k(u_n), \nabla T_k(u))\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi\chi_{\{|u_n|\leq k\}}| \\
&\qquad\qquad \leq \beta(K(x) + |k|^{p(x)-1} + |\nabla u|^{p(x)-1})\varphi'(2k)e^{\delta k}\psi
\end{aligned}$$

which is a fixed function in $L^{p'(x)}(\Omega)$. Therefore by Lebesgue ?s theorem we have

$$a(T_k(u_n), \nabla T_k(u))\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi\chi_{\{|u_n|\leq k\}} \to a(T_k(u), \nabla T_k(u))\varphi'(0)e^{\delta|T_k(u)|}\psi\chi_{\{|u|\leq k\}}$$

strongly in $L^{p'(x)}(\Omega)$. Indeed, $\nabla z_n \rightharpoonup 0$ weakly in $L^{p(x)}(\Omega;\mathbb{R}^N)$ then $A_{2,n} \to 0$. Similarly, since $\nabla z_n \chi_{\{|u_n|>k\}} = -\nabla T_k(u)\chi_{\{|u_n|>k\}} \to 0$ strongly in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, while $a(u_n,\nabla u_n)\varphi'(z_n)e^{\delta|u_n|}\psi$ is bounded in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, by (3.2.1), (3.3.14) and Remark 3.3.1 we obtain $A_{3,n} \to 0$. Therefore, we have proved that :

$$A_n = A_{1,n} + o(1) \tag{3.4.7}$$

For the integral B_n while $\varphi(z_n)$ has the same sign as $c(u_n)$ on the set $\{|u_n| > k\}$ we have

$$\begin{aligned} B_n &= \int_{\{|u_n|\le k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\psi + \int_{\{|u_n|>k\}} c(u_n)\varphi(z_n)e^{\delta|u_n|}\psi \\ &\ge \int_{\{|u_n|\le k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\psi \end{aligned}$$

the last integrand converges pointwise and by (3.2.4) it is bounded then
$\int_{\{|u_n|\le k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\psi$ goes to zero. Therefore, we obtain that :

$$B_n \ge o(1) \tag{3.4.8}$$

Let us examine C_n and E_n together.We first fix δ such that

$$\delta > \frac{d}{\alpha}$$

By (3.2.2) and since $\varphi(z_n)sign(u_n) = |\varphi(z_n)|$ on the set $\{|u_n| > k\}$ we have

$$\begin{aligned} C_n + E_n &\le \frac{d}{\alpha}\int_\Omega a(u_n,\nabla u_n)\nabla u_n|\varphi(z_n)|e^{\delta|u_n|}\psi - \delta\int_\Omega a(u_n,\nabla u_n)\nabla u_n\varphi(z_n)e^{\delta|u_n|}sign(u_n)\psi \\ &\le (\frac{d}{\alpha}+\delta)\int_{\{|u_n|\le k\}} a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \\ &+ (\frac{d}{\alpha}-\delta)\int_{\{|u_n|>k\}} a(u_n,\nabla u_n)\nabla u_n\varphi(z_n)e^{\delta|u_n|}\psi \\ &\le (\frac{d}{\alpha}+\delta)\int_{\{|u_n|\le k\}} a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \\ &= (\frac{d}{\alpha}+\delta)\int_{\{|u_n|\le k\}} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))]\nabla z_n|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \\ &+ (\frac{d}{\alpha}+\delta)\int_{\{|u_n|\le k\}} a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \\ &+ (\frac{d}{\alpha}+\delta)\int_{\{|u_n|\le k\}} a(T_k(u_n),\nabla T_k(u))\nabla z_n|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \end{aligned}$$

The last two integrals converge to zero.If we choose λ such that :

$$\lambda \ge 2(\frac{d}{\alpha}+\delta)$$

we have :

$$(\frac{d}{\alpha}+\delta)|\varphi(s)| \leq \frac{\varphi'(s)}{2} \quad \text{for every s in } \mathbb{R}$$

then we can obtain

$$C_n + E_n \leq \frac{1}{2}A_{1,n} + o(1). \tag{3.4.9}$$

Using Remark (3.3.1) we can observe that :

$$D_n \to 0 \tag{3.4.10}$$

For the term F_n we can see that $|\nabla\psi||\varphi z_n|$ converge strongly to zero in $L^{r(x)}(\Omega)$ for every $r(x) > 1$, by (3.2.1) and (3.3.14) the term $a(u_n, \nabla u_n)e^{\delta|u_n|}$ is bounded in $L^{p'(x)}_{loc}(\Omega)$ then we have that :

$$F_n \to 0 \tag{3.4.11}$$

For the term G_n like before we have :

$$\begin{aligned} G_n &= \int_{\{|u|\leq k\}} |g||\nabla z_n|\varphi'(z_n)e^{\delta|u_n|}\psi + \int_{\{|u|>k\}} |g||\nabla z_n|\varphi'(z_n)e^{\delta|u_n|}\psi \\ &= G_{1,n} + G_{2,n} \end{aligned}$$

since

$$|g|\varphi'(z_n)e^{\delta|u_n|}\psi\chi_{\{|u_n|\leq k\}} \to |g|\varphi'(0)e^{\delta|T_k u|}\psi\chi_{\{|u|\leq k\}}$$

almost everywhere in Ω and

$$|g|\varphi'(z_n)e^{\delta|u_n|}\psi\chi_{\{|u_n|\leq k\}} \leq |g|\varphi'(2k)e^{\delta k}\psi$$

Therefore by Lebesgue ?s theorem we have :

$$|g|\varphi'(z_n)e^{\delta|u_n|}\psi\chi_{\{|u_n|\leq k\}} \to |g|\varphi'(0)e^{\delta|T_k u|}\psi\chi_{\{|u|\leq k\}}$$

strongly in $L^{p'(x)}(\Omega)$. Indeed, $\nabla z_n \rightharpoonup 0$ weakly in $L^{p(x)}(\Omega;\mathbb{R}^N)$ then $G_{1,n} \to 0$. Similarly, since $|\nabla z_n|\chi_{\{|u_n|>k\}} = |\nabla T_k(u)|\chi_{\{|u_n|>k\}} \to 0$ strongly in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, while $|g|\varphi'(z_n)e^{\delta|u_n|}\psi$ is bounded in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, by (3.2.7), (3.3.14) and Remark 3.3.1 we obtain : $G_{2,n} \to 0$. Therefore, we have proved that :

$$G_n \to 0 \tag{3.4.12}$$

Moreover

$$|g||\varphi(z_n)|\psi \to 0$$

almost everywhere in Ω and

$$|g||\varphi(z_n)|\psi \leq |g||\varphi(2k)|\psi$$

Therefore by Lebesgue's theorem we have :

$$|g||\varphi(z_n)|\psi \to 0$$

strongly in $L^{p'(x)}(\Omega)$. Indeed, $\nabla u_n e^{\delta|u_n|} \rightharpoonup \nabla u e^{\delta|u|}$ weakly in $L^{p(x)}(\Omega;\mathbb{R}^N)$, then :

$$H_n \to 0 \tag{3.4.13}$$

Finally, $|\nabla\psi||\varphi z_n|$ converge strongly to zero in $L^{r(x)}(\Omega)$ for every $r(x) > 1$, by (3.2.7) and (3.3.14) the term $|g|e^{\delta|u_n|}$ is bounded in $L^{p'(x)}_{loc}(\Omega)$ then we have that :

$$L_n \to 0 \tag{3.4.14}$$

Putting all inequalities (3.4.6), (3.4.7), (3.4.8), (3.4.9), (3.4.10), (3.4.11), (3.4.12), (3.4.13) and (3.4.14) we can conclude :

$$A_n^1 = \int_{\{|u_n|\leq k\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))]\nabla z_n \varphi'(z_n) e^{\delta|T_k(u_n)|}\psi \to 0 \tag{3.4.15}$$

On the other hand we have

$$\begin{aligned}&\int_{\{|u_n|>k\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))]\nabla z_n \varphi'(z_n) e^{\delta|T_k(u_n)|}\psi \\ &= \int_{\{|u_n|>k\}} [a(k, \nabla T_k(u))]\nabla T_k(u)\varphi'(k - T_k(u)) e^{\delta k}\psi \to 0\end{aligned} \tag{3.4.16}$$

From (3.4.15) and (3.4.16) we can conclude that :

$$\int_{\Omega_0} [a(u_n, \nabla T_k(u_n)) - a(u_n, \nabla T_k(u))](\nabla T_k(u_n) - \nabla T_k(u)) \to 0 \tag{3.4.17}$$

Finally, using the Lemma 3.3.1 we have :

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{strongly in } L^{p(x)}(\Omega_0;\mathbb{R}^{\mathbb{N}}) \tag{3.4.18}$$

Step 3 : End of the proof

Observing that :

$$\nabla u_n - \nabla u = \nabla T_k u_n - \nabla T_k u + \nabla G_k u_n - \nabla G_k u$$

Let Ω_0 be an open set compactly contained in Ω, and using (3.4.4) and (3.4.18) we have :

$$\nabla u_n \to \nabla u \quad \text{strongly in } L^{p(x)}(\Omega_0;\mathbb{R}^{\mathbb{N}}) \tag{3.4.19}$$

To obtain (3.4.1) we have to pass to the limit in the distributional formulation of problem (3.3.1) using (3.4.19). Finally, statement (3.4.2) follows easily from Proposition 3.3.1 and (3.4.19), using Fatou's lemma.

3.5 Boundedness of solutions

In this section we will gave some regularity on the solution of the problem 3.1.1 using an adaptation of a classical technique due to Stampacchia. To do this we need the following lemma (see [66]) :

Lemma 3.5.1 *Let ϕ be a non-negative, non-increasing function defined on the half-line $[k_0,\infty)$. Suppose that there exist positive constants A, μ,β, with $\beta > 1$, such that*

$$\phi(h) \leq \frac{A}{(h-k)^{\mu}}\phi(k)^{\beta}$$

for every $h > k \geq k_0$. Then $\phi(k) = 0$ for every $k \geq k_1$, where

$$k_1 = k_0 + A^{1/\mu}2^{\beta/(\beta-1)}\phi(k_0)^{(\beta-1)/\mu}$$

The result that we are going to prove is the following :

Theorem 3.5.1 *Suppose that* (3.2.1)-(3.2.6) *and* (3.2.8) *hold. Then every solution u of* (3.1.1)*, which is specified in 3.4.1 is essentially bounded, and*

$$\|u_n\|_{L^{\infty}(\Omega)} \leq C \tag{3.5.1}$$

The proof relies on the combined use of the well-known technique by Stampacchia (see [66]) and suitable exponential test functions, as in [21].

Proof : Since (3.3.36) we can obtain an estimate for $\int_{\Omega}|u|^{p(x)-1}\varphi(G_k(u))$ then for some constant $k_0 = k(\lambda)$ sufficiently large we have

$$meas(A_{k_0}) < 1 \tag{3.5.2}$$

where

$$A_k = \{x \in \Omega : |u| > k\}$$

as before we can take the test function $\varphi(G_k(u))$ then we have :

$$\begin{aligned}
&\alpha \int_{A_k} |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + \alpha_0 \int_{A_k} |u|^{p(x)-1} |\varphi(G_k(u))|, \\
&\leq d \int_{A_k} |\nabla G_k(u)|^{p(x)} |\varphi(G_k(u))| + \int_{A_k \cap \{|f|>1\}} |f||\varphi(G_k(u))| \\
&+ \int_{A_k \cap \{|f| \leq 1\}} |\varphi(G_k(u))| + \int_{A_k} |g||\nabla G_k(u)|\varphi'(G_k(u))
\end{aligned} \tag{3.5.3}$$

As in the proof of Proposition 3.3.1 one has

$$\begin{aligned}
&\int_{A_k} |g||\nabla G_k(u)|\varphi'(G_k(u)) \\
&\leq \frac{\alpha}{4} \int_{A_k} |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + C_{22} \int_{A_k} |g|^{p'(x)} \varphi'(G_k(u))
\end{aligned}$$

and if $\lambda \geq \frac{4d}{\alpha}$ and $k \geq k_0(\lambda)$(large enough) where

$$\alpha_0 k_0^{p_- - 1} \geq 4 \tag{3.5.4}$$

then

$$\begin{aligned}
&\frac{\alpha}{2} \int_{A_k} |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + \frac{3\alpha_0}{4} \int_{A_k} |u|^{p(x)-1} |\varphi(G_k(u))|, \\
&\leq \int_{(A_k \setminus A_{k+1}) \cap \{|f|>1\}} |f||\varphi(G_k(u))| + \int_{A_{k+1} \cap \{|f|>1\}} |f||\varphi(G_k(u))| \\
&+ C_{22}\varphi'(1) \int_{A_k \setminus A_{k+1}} |g|^{p'(x)} + C_{22} \int_{A_{k+1}} |g|^{p'(x)} \varphi'(G_k(u))
\end{aligned} \tag{3.5.5}$$

using Hölder inequality we have

$$\int_{(A_k \setminus A_{k+1}) \cap \{|f|>1\}} |f||\varphi(G_k(u))| \leq \varphi(1)(\frac{1}{q_-} + \frac{1}{q'_-}) \|f\|_{L^{q(x)}(\{|f|>1\})} (meas(A_k))^{\frac{1}{q'_+}}$$

by Hölder's inequality and interpolation we obtain :

$$\int_{A_{k+1} \cap \{|f|>1\}} |f||\varphi(G_k(u))| \leq \|f\|_{L^{q_-}(A_{k+1} \cap \{|f|>1\})} \|\varphi(G_k(u))\|^{\frac{N}{p_- q_-}}_{L^{p*_-/p_-}(A_{k+1})} \|\varphi(G_k(u))\|^{1-\frac{N}{p_- q_-}}_{L^1(A_{k+1})}$$

while (3.3.22), (3.5.2) and using Young's and sobolev's inequalities we can deduce that :

$$\begin{aligned}
\int_{A_{k+1} \cap \{|f|>1\}} |f||\varphi(G_k(u))| &\leq \frac{\alpha}{8} \|\nabla \psi(G_k(u))\|^{p_-}_{L^{p(x)}(A_k)} + C_{23} \|f\|^{\frac{p_- q_-}{p_- q_- - N}}_{L^{q_-}(\{|f|>1\})} \|\varphi(G_k(u))\|_{L^1(A_k)} \\
&\leq \frac{\alpha}{8} \int_{A_k} |\nabla \psi(G_k(u))|^{p(x)} + 1 dx + C_{23} \|f\|^{\frac{p_- q_-}{p_- q_- - N}}_{L^{q_-}(\{|f|>1\})} \|\varphi(G_k(u))\|_{L^1(A_k)}
\end{aligned}$$

$$\leq \frac{\alpha}{8}\int_{A_k} |\nabla(G_k(u))|^{p(x)}\varphi'(G_k(u))dx + \frac{\alpha}{8} meas(A_k)$$
$$+ C_{23}\|f\|_{L^{q_-}(\{|f|>1\})}^{\frac{p_-q_-}{p_-q_--N}} \|\varphi(G_k(u))\|_{L^1(A_k)}$$

Therfore, choosing k_0 such that :

$$C_{23}\|f\|_{L^{q_-}(\{|f|>1\})}^{\frac{p_-q_-}{p_-q_--N}} \leq \frac{\alpha_0 k_0^{p_--1}}{4} \tag{3.5.6}$$

the second integral in the right-hand side of (3.5.5) can be absorbed by the left-hand side.

In view of Hölder's inequality and (3.5.2) and (3.2.8) we have :

$$C_{22}\varphi'(1)\int_{A_k\setminus A_{k+1}} |g|^{p'(x)} \leq C_{23}\left(\int_{A_k} |g|^{p'(x)}\right)^{\eta} (meas(A_k))^{1-\frac{p'_+}{r_-}}$$
$$\leq C_{24}\left(\|g\|_{L^{r(x)}(A_k)}\right)^{\delta"} (meas(A_k))^{1-\frac{p'_+}{r_-}}$$

where

$$\eta = \begin{cases} \frac{p'_-}{r_+} & \text{if } \int_{A_k} |g|^{p'(x)} \leq 1, \\ \frac{p'_+}{r_-} & \text{if } \int_{A_k} |g|^{p'(x)} > 1. \end{cases}$$
$$\delta" = \begin{cases} \frac{\eta}{r_-} & \text{if } \|g\|_{L^{r(x)}(A_k)} \geq 1, \\ \frac{\eta}{r_+} & \text{if } \|g\|_{L^{r(x)}(A_k)} < 1. \end{cases}$$

Finally, with similar calculations, using (3.3.33) we have :

$$C_{22}\int_{A_{k+1}} |g|^{p'(x)}\varphi'(G_k(u)) \leq C_{24}\int_{A_{k+1}\cap\{|g|>1\}} |g|^{p'(x)}|\varphi(G_k(u))| + C_{24}\int_{A_{k+1}\cap\{|g|\leq 1\}} |\varphi(G_k(u))|$$

If we choose k_0 such that

$$\alpha_0 k_0^{p_--1} > 4C_{24} \tag{3.5.7}$$

then

$$C_{22}\int_{A_{k+1}} |g|^{p'(x)}\varphi'(G_k(u)) \leq C_{24}\int_{A_{k+1}\cap\{|g|>1\}} |g|^{p'_+}|\varphi(G_k(u))| + \frac{\alpha_0}{4}\int_{A_k} |u|^{p(x)-1}|\varphi(G_k(u))|$$

by Hölder's inequality and interpolation we obtain :

$$C_{22}\int_{A_{k+1}} |g|^{p'(x)}\varphi'(G_k(u)) \leq C_{24}\|g\|_{L^{r_-}(\Omega,\mathbb{R}^N)}^{p'_+} \|\varphi(G_k(u))\|_{L^{p^*_-/p_-}(A_k)}^{\frac{p'_+N}{p_-r_-}} \|\varphi(G_k(u))\|_{L^1(A_k)}^{1-\frac{p'_+N}{p_-r_-}}$$
$$+ \frac{\alpha_0}{4}\int_{A_k} |u|^{p(x)-1}|\varphi(G_k(u))|$$

as before while (3.3.22), (3.5.2) and using Young's and sobolev's inequalities we can deduce that :

$$C_{22}\int_{A_{k+1}}|g|^{p'(x)}\varphi'(G_k(u))\leq\frac{\alpha}{8}\int_{A_k}|\nabla(G_k(u))|^{p(x)}\varphi'(G_k(u))dx+\frac{\alpha}{8}meas(A_k)$$
$$+C_{25}\|g\|_{L^{r_-}(\Omega,\mathbb{R}^N)}^{\frac{p'_+p_-r_-}{p_-r_--p'_+N}}\|\varphi(G_k(u))\|_{L^1(A_k)}+\frac{\alpha_0}{4}\int_{A_k}|u|^{p(x)-1}|\varphi(G_k(u))|$$

Therefore, by taking k_0 satisfying (3.5.2), (3.5.4), (3.5.6), (3.5.7) and the further condition :

$$C_{25}\|g\|_{L^{r_-}(\Omega,\mathbb{R}^N)}^{\frac{p'_+p_-r_-}{p_-r_--p'_+N}}\leq\frac{\alpha_0k_0^{p_--1}}{4}\tag{3.5.8}$$

one obtains, for every $k\geq k_0$:

$$\frac{\alpha}{4}\int_{A_k}|\nabla G_k(u)|^{p(x)}\varphi'(G_k(u))\leq\varphi(1)(\frac{1}{q_-}+\frac{1}{q'_-})\|f\|_{L^{q(x)}(\{|f|>1\})}\left(meas(A_k)\right)^{\frac{1}{q'_+}}+\frac{1}{4}meas(A_k)$$
$$+C_{24}\left(\|g\|_{L^{r(x)}(A_k)}\right)^{\delta"}\left(meas(A_k)\right)^{1-\frac{p'_+}{r_-}}$$
$$\leq C_{26}\left(meas(A_k)\right)^{m}$$

where $m=\min(\frac{1}{q'_+},1-\frac{p'_+}{r_-})$. In view of (3.3.20) and Sobolev's inequality we can obtain :

$$\left(\int_{A_k}|\psi(G_k(u))|^{p*(x)}\right)^{\beta}\leq\|\psi(G_k(u))\|_{L^{p*(x)}(A_k)}\leq C_{27}\left(meas(A_k)\right)^{\frac{m}{\alpha}}$$

Where

$$\alpha=\begin{cases}p_+ & \text{if } \|\nabla\psi(G_k(u))\|_{L^{p(x)}(A_k)}\leq 1,\\ p_- & \text{if } \|\nabla\psi(G_k(u))\|_{L^{p(x)}(A_k)}>1.\end{cases}$$

$$\beta=\begin{cases}\frac{N-p_-}{Np_-} & \text{if } \|\psi(G_k(u))\|_{L^{p*(x)}(A_k)}\leq 1,\\ \frac{N-p_+}{Np_+} & \text{if } \|\psi(G_k(u))\|_{L^{p*(x)}(A_k)}>1.\end{cases}$$

We now take $h-k>1$ and recall that there exists $C_{28}(\lambda,p_+,p_-)$ such that
$|\psi(s)|\geq C_{28}|s|$ for every $s\in\mathbb{R}$ so that

$$[C_{28}(h-k)]^{p*_-}meas(A_h)\leq\int_{A_h}|\psi(G_k(u))|^{p*(x)}$$
$$\leq\int_{A_k}|\psi(G_k(u))|^{p*(x)}$$
$$\leq C_{29}(meas(A_k))^{\frac{m}{\alpha\beta}}$$

Then it follows for every h and k (such that $h>k\geq k_0$) that :

$$meas(A_h)\leq\frac{C_{30}}{[h-k]^{p*_-}}(meas(A_k))^{\frac{m}{\alpha\beta}}$$

Since by (3.2.8), $\frac{m}{\alpha\beta} > 1$ Lemma 3.5.1 applied to the function $\phi(h) = meas(A_h)$ gives

$$\|u_n\|_{L^\infty(\Omega)} \leq C$$

Deuxième partie

EXISTENCE RESULTS AND REGULARITY FOR PARABOLIC PROBLEMS IN GENERAL DOMAINS

Chapitre 4

STRONGLY NONLINEAR PARABOLIC EQUATIONS WIYH NATURAL GROWTH IN GENERAL DOMAINS

Abstract

In this chapter, we deal with the existence and boundedness of solutions for nonlinear parabolic problems whose model is :

$$\begin{cases} \partial u_t - \Delta_p u + \mu|u|^{p-2}u = d(x,t)|\nabla u|^p + f(x,t) - div\ g(x,t) & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 & \text{in } \Omega. \end{cases} \tag{4.0.1}$$

Where T is a positive number, $1 < p < N$, $d \in L^\infty(\Omega \times (0,T))$, $\Delta_p u$ is the p-Laplace operator and the lower order terms have a power growth of order p with respect to ∇u. The assumptions on the source terms lead to the existence results though with exponential integrability and the spatial domain can have infinite measure.

4.1 Introduction

The nonlinear parabolic equations in unbounded domains arise naturally in various branches of Mathematical Physics and present specific mathematical difficulties, equations such as (4.0.1) describe diffusion processes accompanied by absorption terms which are proportional to the concentration represented by the solution u. In this chapter we study the existence of bounded weak solutions for the nonlinear Dirichlet problems in unbounded domains, where the principal part of the operator behaves like the p-Laplacian operator, the lower order terms, which depend on the solution u and its gradient ∇u, have a power growth of order $p-1$ and p respectively with respect to these variables, and the source terms belongs to a Lebesgue spaces with a specific conditions.

For the case of bounded domains several studies have obtained, for example we mention the works [[42]-[79]] (and the references therein). In [31] the existence of unbounded solutions was proved under a sign assumption on the first-order term, whilst V. Ferone and al. in [42] proved the existence of unbounded solutions for problem of the type (4.0.1) where $\mu = 0$. Taking in account that there exist various general methods to solve the analogue (4.0.1) when Ω is bounded, these argument break down in the above situation because of losses of compactness.

As to unbounded domains, parabolic problems of the type have been studied under additional regularity assumptions on f and u_0 we recall for instance the results of [30] and [20], authors in [30] proved the existence and some regularity on solution in the case of parabolic equations where $\mu = 0$ and $g \equiv 0$, note that in our work we drop the assumption of smallness of the source term u_0 (cited in [30]) and f in view of the presence of the term $c(x, u)$ with some specific assumptions (see the second section), in [20] Problem (4.0.1) has been studied by assuming that $c(s) = |s|^{\sigma-1}s, \quad d \equiv 0, \quad g \equiv 0$ with $\sigma > p-1$ and $p > 2 - 1/(N + 1)$, we extend the basic result of [20] in different directions actually consider any $N > p > 1$. As far as the corresponding stationary problem is concerned, existence of solutions satisfying Dirichlet boundary conditions was proved in several papers. For the case of constant exponent we can cite for example [[35]-[29]]. In [35] the existence result in weighted Sobolev spaces was proved. Jeff in [69] proved the existence of variational solutions for the equation $Au(x) + c(x, u) = f(x)$ with boundary conditions of either Dirichlet or Neumann type. For the case of variable exponent we can cite [75],[37] and [10](and the references therein). Authors in [44] obtained the existence of solutions for the $p(x)$-Laplacian problem in the super-linear case using Mountain Pass Theorem, while in [10] authors establish an existence result and some regularity for the solution of the strongly nonlinear $p(x)$-elliptic problem without any sign condition on H.

Inspired by the previous works and motivated by techniques used in [[41], [66]], the strategy that we will use consists in approximating the domain Ω by bounded sets Ω_n, and solving a more regular problem in Ω_n, then we pass to the limit in the approximating problems for this reason we can neither use any embedding theorem nor any argument involving the measure of Ω_n. The use of test functions of exponential type allows us to get rid of the term $H(t,x,u,\nabla u)$ and therefore is an essential tool in the proofs. For regularity we will use an adaptation of a classical technique due to Stampacchia [66].

The plan of the chapter is as follows : In the second section we recall some important definitions and results and we introduce the precise assumptions on the data of the problem in order to announce the results of this section. The third section is devoted to prove estimates on u_n, solution of the approximate problems. In the fourth section we establish the main results of the chapter by proving a local strong convergence of ∇u_n and the existence of solutions for a class of equations of the type (4.0.1) by passing to the limit in approximate problem, and again this is done through exponential-type functions, using a local adaptation of a technique by Ferone and Murat (see [41]), finally we prove that, if we assumes a slightly stronger hypothesis on $f(x,t)$, $g(x,t)$ and u_0, then every solution u of (4.0.1) is bounded.

4.2 Notations, assumptions and main result

Fixing a final time $T > 0$ and let Ω be an open subset of $\mathbb{R}^N$, possibly unbounded. We denote by Q_T the cylinder $Q_T := \Omega \times (0,T)$ and Σ_T its lateral boundary $\Sigma_T := \partial\Omega \times (0,T)$ and we consider the following strongly nonlinear parabolic initial-boundary problem :

$$\begin{cases} \frac{\partial u}{\partial t} + Au + c(u) = H(u,\nabla u) + f - div\ g & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(.,0) = u_0 & \text{in } \Omega, \end{cases} \tag{4.2.1}$$

where

$$Au = -\operatorname{div}\ a(x,t,s,\xi).$$

All the functions $a(x,t,s,\xi) : \Omega\times(0,T)\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$, $c(x,t,s) : \Omega\times(0,T)\times\mathbb{R} \to \mathbb{R}$, $H(x,t,s,\xi)$: $\Omega\times(0,T)\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ are always assumed to be Carathéodory functions, i.e., measurable with respect to (x,t) in Q_T for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$ for almost every (x,t) in Q_T.

We assume the following hypotheses on the terms which appear in (4.2.1) :

— There exists a constant $\alpha_0, \beta > 0$ such that :

$$|a(x,t,s,\xi)| \leq \alpha_0(K_1(x,t) + |s|^\beta + |\xi|^{p-1}), \tag{4.2.2}$$

for almost every $(x,t) \in Q_T$ and for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $K_1(x,t)$ is a positive function such that : $K_1^{p'} \in L^{r_1}(0,T; L^{q_1}_{loc}(\Omega))$ with $\frac{1}{r_1'} \geq \max(\frac{N}{pq_1}, \frac{1}{q_1'})$ and $1 < r_1, q_1 < \infty$.

— There exists a constant $\alpha_1 > 0$ such that :

$$a(x,t,s,\xi)\xi \geq \alpha_1 |\xi|^p, \tag{4.2.3}$$

for almost every $(x,t) \in Q_T$ and for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

— For almost every $(x,t) \in Q_T$ and for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, with $\xi \neq \overline{\xi}$ we have :

$$[a(x,t,s,\xi) - a(x,t,s,\overline{\xi})](\xi - \overline{\xi}) > 0. \tag{4.2.4}$$

— there exists $\alpha_2 > 0$ such that, for every (x,s) in $\Omega \times \mathbb{R}$,

$$|c(x,t,s)| \leq \alpha_2 \left(K_2(x,t) + |s|^{p-1}\right), \tag{4.2.5}$$

where $K_2^{p'}$ is a positive function in $L^{r_1}(0,T; L^{q_1}_{loc}(\Omega))$ with $\frac{1}{r_1'} \geq \max(\frac{N}{pq_1}, \frac{1}{q_1'})$ and $1 < r_1, q_1 < \infty$.

Remark 4.2.1 *The assumptions $K_1^{p'}, K_2^{p'} \in L^{r_1}(0,T; L^{q_1}_{loc}(\Omega)$, which appear in* (4.2.2) *or in* (4.2.5) *instead of the more usual hypotheses $k_1(x,t), K_2(x,t) \in L^{p'}(Q_T)$, will be used in the proof of the strong convergence of the gradient of the approximate solutions.*

— there exists μ such that, for every (x,s) in $\Omega \times \mathbb{R}$,

$$c(x,t,s)s \geq \mu |s|^p, \tag{4.2.6}$$

Moreover we assume the further conditions on the remaining terms :

— $f : \Omega \times (0,T) \to \mathbb{R}$ is a measurable function such that :

$$\begin{gathered} f \in L^r(0,T; L^q(\Omega)) \text{ with } \frac{1}{r'} \geq \max(\frac{N}{pq}, \frac{1}{q'}) \\ \text{where } 1 < q \leq p', \quad 1 < r < \infty. \end{gathered} \tag{4.2.7}$$

— $g : \Omega \times (0,T) \to \mathbb{R}^N$ is a measurable function satisfying :

$$g \in (L^{p'}(Q_T))^N \cap (L^{\frac{N+1}{p-1}}(Q_T))^N. \tag{4.2.8}$$

Remark 4.2.2 *Since $p < N$, Assumption* (4.2.8) *is equivalent to*

$$|g| \in L^{p'}(\{|g| \le 1\}), \quad |g| \in L^{\frac{N+1}{p-1}}(\{|g| > 1\}),$$

and this condition holds if and only if it when 1 is replaced by any $H > 0$.

— $u_0 : \Omega \to \mathbb{R}$ is a measurable function satisfying :

$$\int_\Omega e^{\lambda_0 |u_0|} - 1 dx < \infty \qquad \text{for some } \lambda_0 > \frac{dp'}{\alpha_1}. \tag{4.2.9}$$

— The nonlinear term $H(x,t,s,\xi)$ satisfies :

$$|H(x,t,s,\xi)| \le d|\xi|^p, \tag{4.2.10}$$

where $d > 0$, for almost every $(x,t) \in Q_T$ and for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

Remark 4.2.3 *It is obvious from the proofs of the results that the last assumption might be replaced by*

$$|H(x,t,s,\xi)| \le d|\xi|^p + f_1(x,t),$$

with f_1 satisfying the same hypotheses as f.

We now introduce some notation and results which will be useful in the sequel. For $k > 0$, and $s \in \mathbb{R}$ we will denote by $G_k(.)$ and $T_k(.)$ the functions :

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases} \tag{4.2.11}$$

$$G_k(s) = s - T_k(s) = \begin{cases} s - k & \text{if } s > k, \\ 0 & \text{if } |s| \le k, \\ s + k & \text{if } s < -k, \end{cases} \tag{4.2.12}$$

The symbol $\rightharpoonup$ will denote the weak convergence, and the constants C_i, $i = 1, 2, \ldots$ used in each step of proof are independent.

Our main result will be proved by approximating problem (4.2.1) with the following problems on the bounded domains $Q_{n,T} = \Omega_n \times (0,T)$ (where $\Omega_n = \Omega \cap B_n(0)$ $B_n(0)$ is the ball with center 0 and radius n) :

$$\begin{cases} \frac{\partial u_n}{\partial t} + Au_n + c(u_n) = H_n(u_n, \nabla u_n) + f_n - div\ g_n & \text{in } Q_{n,T}, \\ u_n = 0 & \text{on } \partial\Omega_n \times (0,T), \\ u_n(.,0) = u_{0,n} & \text{in } \Omega_n. \end{cases} \tag{4.2.13}$$

Where $H_n(x,t,s,\xi) = T_n(H(x,t,s,\xi))$, $f_n(x,t) = T_n(f(x,t))$, $g_n(x,t) = \frac{g(x,t)}{1+\frac{1}{n}|g(x,t)|}$ and $u_{0,n}$ is a bounded sequence in the same spaces as u_0 ; that is ;

$$\int_\Omega e^{\lambda_0|u_{0,n}|} - 1dx < \infty \text{ with } \lambda_0 > \frac{dp'}{\alpha_1}. \tag{4.2.14}$$

Moreover $u_{0,n}$ is a sequence such that :

$$u_{0,n} \in L^\infty(\Omega_n) \cap W_0^{1,p}(\Omega_n) \qquad u_{0,n} \to u_0 \text{ a.e. in } \Omega, \tag{4.2.15}$$

$$\lim_{n\to\infty} \frac{1}{n}||u_{0,n}||_{W_0^{1,p}(\Omega_n)} = 0. \tag{4.2.16}$$

This condition will be used in the proof of the strong convergence of the gradients ∇u_n. As a consequence, proving existence of at least one weak solution $u_n \in L^p(0,T;W_0^{1,p}(\Omega_n)) \cap L^\infty(Q_{n,T})$ of (4.2.13) is an easy task (see [52]). The main result we are going to prove is the following :

Theorem 4.2.1

Assume that (4.2.2)-(4.2.10) *are satisfied. Then there exists at least one solution* u *of* (4.2.1) *in the sense of distributions, Moreover for every bounded open set* $\Omega^0 \subset \Omega$ *we have :*

$$e^{\frac{\lambda_0}{p}|u|} \in L^p(0,T;W_0^{1,p}(\Omega^0)) \cap L^\infty(0,T;L^p(\Omega^0)), \tag{4.2.17}$$

To prove the boundedness of the solutions we will assume some slightly stronger hypotheses on f and g :

— the assumption (4.2.7) can be replaced by :

$$\begin{aligned} & f \in L^r(0,T;L^q(\Omega)) \text{ with } \frac{1}{r'} > \max(\frac{N}{pq}, \frac{1}{q'}) \\ & \text{where } 1 < q \leq p', \quad 1 < r < \infty. \end{aligned} \tag{4.2.18}$$

— the assumption (4.2.8) can be replaced by :

$$g \in (L^{p'}(Q_T))^N \cap (L^{r_1}(Q_T))^N \text{ with } r_1 > \frac{N+1}{p-1}. \tag{4.2.19}$$

We now state the boundedness theorem :

Theorem 4.2.2 *Suppose that* (4.2.2)-(4.2.6)*,* (4.2.10)*,* (4.2.18)*,* (4.2.19) *and* $||u_0||_{L^\infty(\Omega)} < \infty$ *hold. Then for every solution* u *of* (4.2.1) *there exists a constant C depending on the data such that :*

$$\|u\|_{L^\infty(Q_T)} \leq C. \tag{4.2.20}$$

The proof of Theorem (4.2.2) relies on the combined use of the well-known technique by Stampacchia (see [66]) and suitable exponential test functions.

Our analysis is based on Lebesgue-Sobolev space and theory of operators of monotone type in reflexive Banach spaces, for that let us recall some important known proprieties of these spaces.

Lemma 4.2.1 *(Stampacchia, see [66])*
Let ϕ *be a non-negative, non-increasing function defined on the half-line* $[k_0, \infty)$*. Suppose that there exist positive constants* A*,* μ*,*β*, with* $\beta > 1$*, such that :*

$$\phi(h) \leq \frac{A}{(h-k)^\mu}\phi(k)^\beta$$

for every $h > k \geq k_0$*. Then* $\phi(k) = 0$ *for every* $k \geq k_1$*, where*

$$k_1 = k_0 + A^{1/\mu}2^{\beta/(\beta-1)}\phi(k_0)^{(\beta-1)/\mu}$$

Characterization of the time mollification of a function u.

To deal with time derivative, we introduce a time mollification of a function u belonging to some Lebesgue space. Thus we define for all $\sigma \geq 0$ and all $(x,t) \in Q_T$

$$u_\sigma = \sigma \int_\infty^t \tilde{u}(x,s)e^{\sigma(s-t)}ds$$

where $\tilde{u}(x,s) = u(x,s)\chi_{(0;T)}(s)$. Note that in this section, we omit the proof of each of the above Proposition and Lemmas, since it is a slight modification of its analogous in [2].

4.3 A Priori Estimate

In this section we look for an estimate on u_n in the case where hypotheses (4.2.7), (4.2.8) and (4.2.9) are assumed, and if we replace (4.2.7), (4.2.8) and(4.2.9) by (4.2.18) (i.e., if the exponents r and q satisfy a strict inequality), (4.2.19) (i.e., if the exponent r_1 satisfies a strict inequality) and $\|u_{0,n}\|_{L^\infty(\Omega_n)}$ respectively we will prove that the solutions u_n are uniformly bounded in $L^\infty Q_{(n,T)}$.

Proposition 4.3.1

Assuming that (4.2.3), (4.2.6)-(4.2.10) *hold, and let* u_n *be any solution of* (4.2.13). *Then there exists a positive constant* C *depending on the data, such that :*

$$\iint_{Q_{n,T}} e^{\lambda_0|u_n|}|\nabla u_n|^p dxdt \leq C. \tag{4.3.1}$$

Remark 4.3.1

The previous estimate yields an estimate for the $|\nabla u_n|$ *in* $L^p((0,T)\times\Omega)$.

Proof : For simplicity of notation we will always omit the index n of the sequence. Let φ a function defined as follows :

$$\varphi(s) = \big(e^{\lambda_0|s|} - 1\big)sign(s), \tag{4.3.2}$$

and its primitive :

$$\psi(s) = \int_0^s \varphi(\sigma)d\sigma. \tag{4.3.3}$$

Let k be a positive number sufficiently large (to be specified later), we take $\varphi(G_k(u))$ as test function in (4.2.13). Integrating on $Q = \Omega\times(0,\tau)$ and in view of hypotheses (4.2.3), (4.2.6) and (4.2.10) we have :

$$\begin{aligned}
&\int_\Omega \psi(G_k(u(\tau)))dx - \int_\Omega \psi(G_k(u(0)))dx + \alpha_1\iint_Q |\nabla G_k(u)|^p\varphi'(G_k(u))\\
&\qquad + \mu\iint_Q |u|^{p-1}|\varphi(G_k(u))|\\
&\leq d\iint_Q |\nabla G_k(u)|^p|\varphi(G_k(u))| + \iint_Q |f||\varphi(G_k(u))| + \iint_Q |g||\nabla G_k(u)|\varphi'(G_k(u))\\
&= d\iint_Q |\nabla G_k(u)|^p|\varphi(G_k(u))| + \iint_{Q\cap\{|f|>1;|G_k(u)|>1\}} |f||\varphi(G_k(u))|\\
&\qquad + \iint_{Q\cap\{|f|>1;|G_k(u)|\leq 1\}} |f||\varphi(G_k(u))| + \iint_{Q\cap\{|f|\leq 1\}} |f||\varphi(G_k(u))|\\
&\qquad + \iint_Q |g||\nabla G_k(u)|\varphi'(G_k(u))\\
&= A + B + C + D + E.
\end{aligned} \tag{4.3.4}$$

Estimation of the integral A :

For every s in $\mathbb{R}$ we have :

$$|\varphi(s)| \leq \frac{1}{\lambda_0}\varphi'(s), \tag{4.3.5}$$

then

$$A \leq \frac{d}{\lambda_0} \iint_Q |\nabla G_k(u)|^p \varphi'(G_k(u)). \tag{4.3.6}$$

Before estimating B, we define :

$$\phi_p(s) = \int_0^{|s|} (\varphi'(\sigma))^{\frac{1}{p}} d\sigma. \tag{4.3.7}$$

It is easy to remark that :

$$\iint_Q |\nabla G_k(u)|^p \varphi'(G_k(u)) = \iint_Q |\nabla \left(\phi_p(G_k(u))\right)|^p. \tag{4.3.8}$$

Moreover, we observe that there exists a positive constant $C_1 = C_1(p, \lambda_0)$ such that :

$$|\varphi(s)| \leq C_1(\phi_p(s))^p \text{ for every s such that } |s| \geq 1, \tag{4.3.9}$$

We can also observe that :

$$|\varphi(s)| \leq C_2 \psi(s) \text{ for every s such that } |s| \geq 1. \tag{4.3.10}$$

Using Hölder and interpolation inequality we have :

$$\begin{aligned} B \leq \int_0^\tau & \left[||f(t)||_{L^q(\{|f|>1\})} ||\varphi(G_k(u))||^{1-\theta}_{L^1(\{|G_k(u)|>1\})} \right] \times \\ & \left[||\varphi(G_k(u))||^{\theta}_{L^{p^*/p}(\{|G_k(u)|>1\})} \right] dt, \end{aligned}$$

where $\theta = \frac{N}{pq}$ and $1 - \theta + \frac{\theta}{p^*/p} = \frac{1}{q'}$.

Let $\epsilon > 0$, by applying Young's inequality, (4.3.9), (4.3.10), Sobolev embedding (see The Sobolev Imbedding Theorem part III and remarks in [1]) and (4.3.8) we have :

$$\begin{aligned} B \leq C_3 \int_0^\tau ||f(t)||^{\frac{1}{1-\theta}}_{L^q(\Omega)} \left[\int_\Omega \psi(G_k(u)) dx \right] dt \\ + \frac{\alpha_1 - \frac{d}{\lambda_0}}{3 + \epsilon} \int_0^\tau \int_\Omega |\nabla G_k(u)|^p \varphi'(G_k(u)) dx dt. \end{aligned} \tag{4.3.11}$$

In fact that (4.2.7) it is easy to check that :

$$\frac{1}{1-\theta} < r. \tag{4.3.12}$$

Estimation of the integral C :

It is easy to check that :

$$C \leq \varphi(1) \iint_{\{|f|>1\}} |f|^q. \tag{4.3.13}$$

Estimation of the integral D :

Now, since we have chosen k large enough such that :

$$\frac{4}{\mu} < k^{p-1}, \tag{4.3.14}$$

then we obtain :

$$D \leq \frac{\mu}{4} \iint_Q |u|^{p-1} |\varphi(G_k(u))|. \tag{4.3.15}$$

Estimation of the integral E :

Thanks to Young's inequality, we have :

$$\begin{aligned} E \leq & \frac{\alpha_1 - \frac{d}{\lambda_0}}{3+\epsilon} \iint_Q |\nabla G_k(u)|^p \varphi'(G_k(u)) + C_4 \iint_Q |g|^{p'} \varphi'(G_k(u)), \\ & = E_1 + E_2. \end{aligned}$$

The integral E_2 can be estimated as follows :

$$\begin{aligned} E_2 \leq & C_4 \lambda e^{\lambda} \iint_{\{|G_k(u)| \leq 1\}} |g|^{p'} + C_4 \iint_{\{|g|>H, |G_k(u)|>1\}} |g|^{p'} \varphi'(G_k(u)) \\ & + C_4 H^{p'} \iint_{\{|g| \leq H, |G_k(u)|>1\}} \varphi'(G_k(u)) \\ & = E_{2,1} + E_{2,2} + E_{2,3}, \end{aligned}$$

where H is a positive number to be chosen hereafter. Since $\varphi'(s) \leq C_5(\phi_p(s))^p$ for every s such that $|s| \geq 1$, one has, by Hölder's inequality,

$$E_{2,2} \leq C_6 \left[\iint_{\{|g|>H\}} |g|^{\frac{N+1}{p-1}} \right]^{\frac{p}{N+1}} \left[\iint_{\{|G_k(u)|>1\}} \phi_p(G_k(u))^{\frac{(N+1)p}{N+1-p}} \right]^{\frac{N+1-p}{N+1}},$$

if we put $p^* = \frac{(N+1)p}{N+1-p}$ in $\mathbb{R}^{N+1}$ and let the mapping $u \to \tilde{u}$ denote zero extension of u :

$$\begin{cases} \tilde{u} = u & \text{in } \Omega \times (0,\tau), \\ \tilde{u} = 0 & \text{elsewhere} \end{cases}$$

the mapping takes $W_0^{1,p}(\Omega \times (0,\tau))$ isometrically into $W^{1,p}(\mathbb{R}^{N+1})$ (see [1]) then

$$E_{2,2} \leq C_6 \left[\iint_{\{|g|>H\}} |g|^{\frac{N+1}{p-1}} \right]^{\frac{p}{N+1}} \left[\int_{\mathbb{R}^{N+1}} \phi_p(G_k(\tilde{u}))^{p^*} \right]^{\frac{p}{p^*}},$$

using Sobolev embedding we obtain that :

$$E_{2,2} \leq C_6 \left[\iint_{\{|g|>H\}} |g|^{\frac{N+1}{p-1}} \right]^{\frac{p}{N+1}} \iint_Q |\nabla G_k(u)|^p \varphi'(G_k(u)).$$

Choosing H large enough, so that

$$C_6 \left[\iint_{\{|g|>H\}} |g|^{\frac{N}{p-1}} \right]^{\frac{p}{N}} \leq \frac{\alpha_1 - \frac{d}{\lambda_0}}{3+\epsilon},$$

then

$$E_{2,2} \leq \frac{\alpha_1 - \frac{d}{\lambda_0}}{3+\epsilon} \iint_Q |\nabla G_k(u)|^p \varphi'(G_k(u)).$$

Finally, using inequality

$$\varphi'(s) \leq C_7 |\varphi(s)|, \quad \text{for every s such that} |s| \geq 1,$$

and choosing k sufficiently large such that :

$$\mu k^{p-1} \geq 4 C_4 C_7 H^{p'}, \tag{4.3.16}$$

we obtain :

$$E_{2,3} \leq \frac{\mu}{4} \int_\Omega |u|^{p-1} |\varphi(G_k(u))|.$$

Therefore :

$$\begin{aligned} E \leq & \frac{2(\alpha_1 - \frac{d}{\lambda_0})}{3+\epsilon} \iint_Q |\nabla G_k(u)|^p \varphi'(G_k(u)) + \frac{\mu}{4} \int_\Omega |u|^{p-1} |\varphi(G_k(u))| \\ & + C_4 \lambda e^{\lambda} \iint_Q |g|^{p'}. \end{aligned} \tag{4.3.17}$$

In conclusion, putting all the inequalities (4.3.4), (4.3.6), (4.3.11), (4.3.13), (4.3.15) and (4.3.17) together, taking into account (4.3.12), and setting

$$h(\tau) = \int_\Omega \psi(G_k(u(\tau)))dx,$$

we get :

$$\begin{aligned} & h(\tau) + \frac{\epsilon(\alpha_1 - \frac{d}{\lambda_0})}{3+\epsilon} \iint_Q |\nabla G_k(u)|^p \varphi'(G_k(u)) + \frac{\mu}{2} \iint_Q |u|^{p-1} |\varphi(G_k(u))| \\ & \leq C_8 + \int_0^\tau g(t) h(t) dt. \end{aligned} \tag{4.3.18}$$

For every k satisfying (4.3.14) and (4.3.16), where the function :

$$g(t) = C_3 ||f(t)||_{L^q(\Omega)}^{\frac{1}{1-\theta}}$$

belongs to $L^1(0,T)$ and

$$C_8 = \varphi(1) \iint_{\{|f|>1\}} |f|^q + \int_\Omega \psi(G_k(u(0)))dx + C_4 \lambda e^{\lambda} \iint_Q |g|^{p'} < \infty$$

(using (4.2.7), (4.3.12), (4.2.9), (4.2.8)). An application of Gronwall-Bellman's Lemma (see (1.2.6)) yields that $h(\tau)$ is a bounded function, and that $\iint_Q |\nabla G_k(u)|^p \varphi'(G_k(u))$ is also bounded.

We fix now k, H such that (4.3.18) and (4.3.16) hold and we take $\varphi(T_k(u)$ as a test function in (4.2.13), then Integrating on $Q_k = (0,\tau) \times \Omega \cap \{|u| \leq k\}$. In view of hypotheses (4.2.3), (4.2.6) and (4.2.10) we have :

$$\begin{aligned}
&\int_{\Omega\cap\{|u|\leq k\}} \psi(u(\tau))dx - \int_{\Omega\cap\{|u|\leq k\}} \psi(u(0))dx + \alpha_1 \iint_{Q_k} |\nabla T_k(u)|^p \varphi'(T_k(u)) \\
&\qquad + \mu \iint_{Q_k} |u|^{p-1} |\varphi(T_k(u))| \\
&\leq d \iint_{Q_k} |\nabla T_k(u)|^p |\varphi(T_k(u))| + \iint_{Q_k} |f||\varphi(T_k(u))| + \iint_{Q_k} |g||\nabla G_k(u)|\varphi'(G_k(u)) \\
&= d \iint_{Q_k} |\nabla T_k(u)|^p |\varphi(T_k(u))| + \iint_{Q_k\cap\{|f|>1\}} |f||\varphi(T_k(u))| \\
&\qquad + \iint_{Q_k\cap\{|f|\leq 1\}} |f||\varphi(T_k(u))| + \iint_{Q_k} |g||\nabla T_k(u)|\varphi'(T_k(u))
\end{aligned} \tag{4.3.19}$$

$= F_1 + F_2 + F_3 + F_4.$

Estimation of the integral F_1 and F_2 :

Using (4.3.5), we have :

$$F_1 \leq \frac{d}{\lambda_0} \iint_{Q_k} |\nabla G_k(u)|^p \varphi'(G_k(u)). \tag{4.3.20}$$

For the second term we have :

$$F_2 \leq \varphi(k) \iint_{Q_k\cap\{|f|>1\}} |f|^q. \tag{4.3.21}$$

Estimation of the integral F_3 :

Remark 4.3.2 *if $meas(\Omega)$ is finite or if $f \in L^1(Q_T)$ it is easy to estimate the integral F_3.*

Let η be a positive constant to be chosen later, we write :

$$F_3 \leq C_9(\eta) \iint_{Q_k\cap\{|f|\leq 1\}} |f|^{p'} + \eta \iint_{Q_k\cap\{|f|\leq 1\}} |\varphi(T_k(u))|^p,$$

Since

$$|\varphi(T_k(u))|^p \leq C_{10}(\lambda_0, p, k)|\varphi(T_k(u))||u|^{p-1},$$

and choosing η such that $\eta C_{10} \leq \frac{\mu}{2}$,we have :

$$F_3 \leq C_9(\eta) \iint_{Q_k \cap \{|f| \leq 1\}} |f|^{p'} + \frac{\mu}{2} \iint_{Q_k \cap \{|f| \leq 1\}} |\varphi(T_k(u))||u|^{p-1}. \tag{4.3.22}$$

Using the fact that $q < p'$ and $r' < q'$ cited in (4.2.7) we have $\iint_{Q_k \cap \{|f| \leq 1\}} |f|^{p'} < \infty$.

Estimation of the integral F_4 :

It easy to check that

$$F_4 \leq \frac{\alpha_1 - \frac{d}{\lambda_0}}{1+\varepsilon} \iint_{Q_k} |\nabla T_k(u)|^p \varphi'(T_k(u)) + C_{11}\varphi'(k) \iint_{Q_k} |g|^{p'}, \tag{4.3.23}$$

where $\varepsilon > 0$.

Putting all the estimations (4.3.19) - (4.3.22) together, we get :

$$\begin{aligned} &\int_{\Omega \cap \{|u| \leq k\}} \psi(u(\tau))dx + \frac{\varepsilon(\alpha_1 - \frac{d}{\lambda_0})}{1+\varepsilon} \iint_{Q_k} |\nabla T_k(u)|^p \varphi'(T_k(u)) \\ &\qquad + \frac{\mu}{2} \iint_{Q_k} |u|^{p-1} |\varphi(T_k(u))| \\ &\leq \varphi(k) \iint_{\{|f|>1\}} |f|^q + C_9(\eta) \iint_{Q_k \cap \{|f| \leq 1\}} |f|^{p'} + C_{11}\varphi'(k) \iint_{Q_k} |g|^{p'} + \int_\Omega \psi(u(0))dx. \end{aligned} \tag{4.3.24}$$

Finally if we combine inequalities (4.3.24), (4.3.18) and assumptions (4.2.7)-(4.2.9) we have (4.3.1).

Remark 4.3.3

— *Using the fact that $h(\tau)$ is bounded (*(4.3.18), (4.3.24)*) we can have :*

$$\sup_{t \in [0,T]} \int_{\Omega^0} e^{\lambda |u_n(x,t)|} dx < \infty, \tag{4.3.25}$$

for every bounded open set $\Omega^0 \subset \Omega$ and $\lambda_0 \geq \lambda \geq 0$.

— *The estimation* (4.3.1) *yields that :*

$$\iint_{Q_{n,T}} e^{\lambda |u_n|} |\nabla u_n|^p dxdt \leq C, \tag{4.3.26}$$

for every λ such that : $\lambda_0 \geq \lambda \geq 0$.

Proposition 4.3.2 *We assume* (4.2.3)*,* (4.2.6)*,* (4.2.10)*,* (4.2.18)*,* (4.2.19) *and $||u_{0,n}||_{L^\infty(\Omega_n)} < \infty$ hold. Then there exists a constant C depending on the data such that :*

$$\|u_n\|_{L^\infty(Q_{n,T})} \leq C. \tag{4.3.27}$$

It will be useful to define the sets :

$$A_{n,k}(t) = \{x : |u_n(x,t)| > k\}, \qquad A_{n,k} = \{(x,t) : |u_n(x,t)| > k\}$$

and the function :

$$\varrho(k) = \sup_{\tau\in[0,T]} mesA_{n,k}(\tau), \qquad \rho(k) = mesA_{n,k}$$

Let us observe that in the a priori estimate we have proved that $h(.)$ and $\iint_{A_{n,k}} |u_n|^{p-1}|\varphi(G_k(u_n)|)$ are bounded where :

$$h(\tau) = \int_\Omega \psi(G_k(u_n(\tau)))dx,$$

these estimates give that $\varrho(.)$ and $\rho(.)$ are respectively bounded for every k such that $k \geq k_1$ (which is large enough).

Proof : We take $\varphi(G_k(u_n))\chi_{(0,\tau)}(t)$ as test function in (4.2.13), where φ is defined in (4.3.2) and for every $H > 0$, choosing k such that :

$$k \geq k_0 = \max\left(||u_{0,n}||_{L^\infty(\Omega)}, \frac{2H}{\mu}, k_1\right).$$

For simplicity of notation we will always omit the index n, as before we have :

$$\begin{aligned}
&\int_\Omega \psi(G_k(u(\tau)))dx + (\alpha_1 - \frac{d}{\lambda_0})\iint_Q |\nabla G_k(u)|^p\varphi'(G_k(u)) + \mu\iint_Q |u|^{p-1}|\varphi(G_k(u))| \\
&\leq \iint_{Q\cap\{|f(t)|\leq H\}} |f||\varphi(G_k(u))| + \iint_{Q\cap\{|f(t)|>H;|G_k(u)|\leq 1\}} |f||\varphi(G_k(u))| \\
&\quad + \iint_{Q\cap\{|f(t)|>H;|G_k(u)|>1\}} |f||\varphi(G_k(u))| + \iint_Q |g||\nabla G_k(u)|\varphi'(G_k(u)),
\end{aligned}$$

where the function $\psi(s)$ is defined by (4.3.3). Since $\mu k_0 \geq 2H$, the first integral in the right-hand side is smaller than the last integral of the left-hand side.

Then :

$$\begin{aligned}
&\int_\Omega \psi(G_k(u(\tau)))dx + (\alpha_1 - \frac{d}{\lambda_0})\iint_Q |\nabla G_k(u)|^p\varphi'(G_k(u)) + \frac{\mu}{2}\iint_Q |u|^{p-1}|\varphi(G_k(u))| \\
&\leq \iint_{Q\cap\{|f(t)|>H;|G_k(u)|>1\}} |f||\varphi(G_k(u))| + \iint_{Q\cap\{|f(t)|>H;|G_k(u)|\leq 1\}} |f||\varphi(G_k(u))| \\
&\quad + \iint_Q |g||\nabla G_k(u)|\varphi'(G_k(u))
\end{aligned}$$

Using the same process used to evaluate the terms B and E in the proof of the Proposition 4.3.1 and choosing H sufficiently large the integrals in the right-hand can be absorbed in the left-hand side and we get :

$$
\begin{aligned}
&C_{12} \sup_{\tau\in[0,T]} \int_\Omega \psi(G_k(u(\tau)))dx + C_{13} \iint_{Q_T} |\nabla G_k(u)|^p \varphi'(G_k(u) \\
\leq & \iint_{Q_T\cap\{|f(t)|>H;|G_k(u)|\leq 1\}} |f||\varphi(G_k(u))| + C_{14} \iint_{A_k} |g|^{p'}.
\end{aligned}
$$

Let us observe that :

$$
|\varphi(s)| \leq C_{15}(\phi_p)^{\frac{p}{r'}} \text{ for every s such that } |s| \leq 1, \tag{4.3.28}
$$

where ϕ_p is defined in (4.3.7).

Thanks to Hölder inequality and to the assumptions (4.2.18), (4.2.19) and using (4.3.28) one has :

$$
\begin{aligned}
&C_{12} \sup_{\tau\in[0,T]} \int_\Omega \psi(G_k(u(\tau)))dx + C_{13} \iint_{Q_T} |\nabla G_k(u)|^p \varphi'(G_k(u)) \\
\leq & C_{16} \left\{ \int_0^T \parallel \phi_p(G_k(u))^p \parallel_{L^{\frac{p^*}{p}}(\Omega)} dt \right\}^{\frac{1}{r'}} \sup_{\tau\in[0,T]} (mesA_k(\tau))^{\frac{1}{q'}-\frac{p}{p^*r'}} \\
&+ C_{14} \|g\|^{p'}_{L^{r_1}(Q_T);\mathbb{R}^N)} (mesA_k)^{1-\frac{p'}{r_1}}.
\end{aligned}
$$

Therefore, using Sobolev's and Young's inequalities we obtain, for every $\epsilon > 0$,

$$
\begin{aligned}
&C_{12} \sup_{\tau\in[0,T]} \int_\Omega \psi(G_k(u(\tau)))dx + C_{13} \iint_{Q_T} |\nabla G_k(u)|^p \varphi'(G_k(u)) \\
\leq & \epsilon \iint_{Q_T} |\nabla G_k(u)|^p \varphi'(G_k(u)) + C_{17}(\epsilon) \sup_{\tau\in[0,T]} (mesA_k(\tau))^{\frac{r}{q'}-\frac{pr}{p^*r'}} + C_{18}(mesA_k)^{1-\frac{p'}{r_1}}.
\end{aligned}
$$

choosing ϵ small enough, we can write :

$$
\begin{aligned}
&C_{12} \sup_{\tau\in[0,T]} \int_\Omega \psi(G_k(u(\tau)))dx + C_{19} \iint_{Q_T} |\nabla G_k(u)|^p \varphi'(G_k(u)) \\
\leq & C_{17}(\epsilon) \sup_{\tau\in[0,T]} (mesA_k(\tau))^{\frac{r}{q'}-\frac{pr}{p^*r'}} + C_{18}(mesA_k)^{1-\frac{p'}{r_1}}.
\end{aligned}
$$

On the other hand, it is easy to confirm that there exist two positive constants ϑ and C_{20} such that $\psi(s) \geq C_{20}|s|^\vartheta$, then for every $h > k \geq k_0$ one has :

$$
\int_{A_k(\tau)} \psi(G_k(u(\tau)))dx \geq C_{20} \int_{A_h(\tau)} |G_k(u(\tau))|^\vartheta dx \geq C_{20}(h-k)^\vartheta mesA_h(\tau).
$$

In view of (4.3.8), Sobolev's inequality and $|\phi_p(s)| \geq C_{21}(\lambda, p)|s|$ for every $s \in \mathbb{R}$ we can obtain :

$$
\iint_{A_k} |\nabla G_k(u)|^p \varphi'(G_k(u)) \geq \iint_{A_h} |\nabla G_k(u)|^p \varphi'(G_k(u)) \geq C_{22}(h-k)^p (mesA_h)^{\frac{p}{p^*}}
$$

for every $h > k \geq k_0$ Finally, we can write :

$$C_{23}(h-k)^{\vartheta} \sup_{\tau\in[0,T]} mesA_h(\tau) + C_{24}(h-k)^p (mesA_h)^{\frac{p}{p*}}$$
$$\leq C_{17}(\epsilon) \sup_{\tau\in[0,T]} (mesA_k(\tau))^{\frac{r}{q'}-\frac{pr}{p^*r'}} + C_{18}(mesA_k)^{1-\frac{p'}{r_1}}.$$

We can treat here two cases :

— If $\max\left\{C_{17}(\epsilon)\sup_{\tau\in[0,T]}(mesA_k(\tau))^{\frac{r}{q'}-\frac{pr}{p^*r'}}; C_{18}(mesA_k)^{1-\frac{p'}{r_1}}\right\}$
$= C_{17}(\epsilon)\sup_{\tau\in[0,T]}(mesA_k(\tau))^{\frac{r}{q'}-\frac{pr}{p^*r'}}$ then :

$$\sup_{\tau\in[0,T]} mesA_h(\tau) \leq \frac{C_{25}}{(h-k)^{\vartheta}} \sup_{\tau\in[0,T]} (mesA_k(\tau))^{\frac{r}{q'}-\frac{pr}{p^*r'}}.$$

— If $\max\left\{C_{17}(\epsilon)\sup_{\tau\in[0,T]}(mesA_k(\tau))^{\frac{r}{q'}-\frac{pr}{p^*r'}}; C_{18}(mesA_k)^{1-\frac{p'}{r_1}}\right\}$
$= C_{18}(mesA_k)^{1-\frac{p'}{r_1}}$ then :

$$mesA_h \leq \frac{C_{26}}{(h-k)^{p*}}(mesA_k)^{\frac{p*}{p}(1-\frac{p'}{r_1})}.$$

It is easy to check that, under hypotheses $\frac{1}{r'} > \frac{N}{pq}$ and $r_1 > \frac{N+1}{p-1}$ the two exponents $\frac{r}{q'} - \frac{pr}{p^*r'}$ and $\frac{p*}{p}(1-\frac{p'}{r_1})$ are greater than 1, therefore one can apply Lemma (4.2.1) to the function $\varrho(h)$ or $\rho(h)$ (following the last two cases) and get the result.

4.4 Proof Of The Result

In this section we will prove the main results of this chapter. Let $\{u_n\}$ be any sequence of solutions of problem (4.2.13), we extend them to zero in $\Omega \setminus \Omega_n$. Let Ω^0 a bounded open subset of Ω and $Q_T^0 = \Omega^0 \times (0,T)$. We limit ourselves to the case where $g \equiv 0$, since the additional term $?div(g)$ can be treated easily (see for instance [10] in the case of variable exponents). Let $(T_k(u))_\nu$ be the mollification with respect to time of $T_k(u)$ and, as before, $\varphi(s) = (e^{\lambda|s|}-1)sign(s)$ and

$$\eta(x) \in C_0^\infty(B_R), \quad 0 \leq \eta \leq 1, \quad \eta \equiv 1 \quad \text{in} \quad \Omega^0, \tag{4.4.1}$$

with B_R is a ball containing Ω^0 (For simplicity we denote $B_R \cap \Omega$ again by B_R).

By Proposition (4.3.1) there exist a subsequence (which we will still denote by u_n) and a function u such that :

$$u_n \rightharpoonup u \text{ weakly in } L^p(0,T;W^{1,p}(\Omega^0)) \text{ and *-weakly in } L^\infty(0,T;L^q(\Omega^0)), \tag{4.4.2}$$

$$e^{\frac{\lambda_0}{p}|u_n|} \rightharpoonup e^{\frac{\lambda_0}{p}|u|} \text{ in } L^p(0,T;W^{1,p}(\Omega^0)), \tag{4.4.3}$$

for every bounded open subset Ω^0 of Ω and for every $q < \infty$. By Remark (4.3.1) and the equation satisfied by u_n, it is easy to check that the sequence $\left(\frac{\partial(\eta u_n)}{\partial t}\right)_{n\in\mathbb{N}}$ is bounded in $L^{p'}(0,T;W^{-1,p'}(B_R) + L^1((0,T)\times B_R)$. Then, by a well-known compactness result (see for instance (1.2.7)), the sequence ηu_n is relatively compact in $L^p(B_R\times(0,T))$. Therefor, by Proposition (4.3.1) and Remark (4.3.3), u_n is bounded in $L^q(Q_T^0)$, for every $q < \infty$. Then still extracting a subsequence,

$$u_n \to u \text{ a.e. } inQ_T^0 \text{ and strongly in } L^q(Q_T^0), \text{ for every } q < \infty. \tag{4.4.4}$$

The proof of the following Lemma is very similar to that in ([30]).

Lemma 4.4.1

Let $\eta \in C_0^\infty(\mathbb{R}^N)$, the following inequality holds for every δ such that $0 < \delta < \lambda_0$:

$$\ll \rho_n, w(x,t) \gg \geq \omega^\nu(n) + \omega(\nu) := \omega(\nu,n)$$

where $\rho_n = div\ a(x,t,u_n,\nabla_n) - c(u_n) + H_n(x,t,u_n,\nabla u_n) + f_n(x,t) - div(g_n)$, $w(x,t) = \eta(x)\varphi(T_k(u_n) - T_k(u)_\nu)e^{\delta|G_k(u_n)|}$ and where we denote by $\omega^\gamma(h)$ a quantity which goes to zero as h goes to infinity, for every γ fixed, and by $\ll .,. \gg$ the duality hook.

We are now up to prove the statement of this part. By Remark (4.3.1), we can say that, up to a subsequence,

$$\nabla u_n \rightharpoonup \nabla u \text{ weakly in } L^p(Q_T^0;\mathbb{R}^N).$$

Step 1 : Strong Convergence Of $\nabla T_k(u_n) in (L^p(Q_T^0))^N$

In this step, we will fix $k > 0$ and prove that $\nabla T_k(u_n) \to \nabla T_k(u)$ strongly in $L^p(Q_T^0;\mathbb{R}^N)$ as $n \to \infty$. In order to prove this result we define :

$$w(x,t) = \eta(x)\varphi(T_k(u_n) - T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|},$$

where $\frac{d}{\alpha_1} < \lambda < \lambda_0$ and n large enough to ensure that $w \in L^p(0,T;W_0^{1,p}(\Omega_n))$, and we take $w(x,t)$ as a test function in (4.2.13), using Lemma (4.4.1) one obtains :

$$\begin{aligned}
A+B=&\iint_{Q_T} a(u_n,\nabla u_n)\nabla(T_k(u_n)-T_k(u)_\nu)\varphi'(T_k(u_n)-T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta\\
&+\iint_{Q_T} c(u_n)\varphi(T_k(u_n)-T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta\\
\le& d\iint_{Q_T}|\nabla u_n|^p\varphi(T_k(u_n)-T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta\\
&+\iint_{Q_T} f_n\varphi(T_k(u_n)-T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta\\
&-\frac{d}{\alpha_1}\iint_{Q_T} a(u_n,\nabla u_n)\nabla(G_k(u_n))sign(u_n)\varphi(T_k(u_n)-T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta\\
&-\iint_{Q_T} a(u_n,\nabla u_n)\nabla\eta\varphi(T_k(u_n)-T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}+\omega(\nu,n)\\
=&C+D+E+F+\omega(\nu,n),
\end{aligned}$$

where the notation $\omega(\nu,n)$ was introduced in the Lemma (4.4.1).

Let us examine the term D :

$$\begin{aligned}
D=&\iint_{Q_T} f_n\varphi(T_k(u_n)-T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta\\
=&\iint_{Q_T} f\varphi(T_k(u)-T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u)|}\eta+\omega^\nu(n)\\
=&\,\omega(\nu,n).
\end{aligned}$$

Indeed, using (4.2.7) we have f_n converges strongly to f in $L^r(0,T;L^q(\Omega))$, $\varphi(T_k(u_n)-T_k(u)_?)$ is bounded and converges almost everywhere. Moreover, by Remark (4.3.3) and (4.2.9) we have $e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta$ is bounded in $L^\infty(0,T;L^p(B_R))\cap L^p(0,T;W_0^{1,p}(B_R))$, therefore by Lemma (1.2.8) (applied with $m=p$)

$$e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta \text{ is bounded in } L^\rho(0,T;L^\sigma(\Omega)) \tag{4.4.5}$$

for every ρ, σ satisfying $\frac{pN}{\sigma}+\frac{p^2}{\rho}=N$, hence $e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta$ converges weakly in $L^{r'}(0,T;L^{q'}(\Omega))$.

As far as the term F is concerned, using the assumption (4.2.2) on a and Hölder inequality one has :

$$\begin{aligned}
F\le&\alpha_0\iint_{Q_T}(K_1(x,t)+|u_n|^\beta)|\nabla\eta|e^{\frac{d}{\alpha_1}|G_k(u_n)|}|\varphi(T_k(u_n)-T_k(u)_\nu)|\\
&+\alpha_0\iint_{Q_T}(|\nabla u_n|^{p-1})|\nabla\eta|e^{\frac{d}{\alpha_1}|G_k(u_n)|}|\varphi(T_k(u_n)-T_k(u)_\nu)|
\end{aligned}$$

$$
\begin{aligned}
\leq & \alpha_0 \iint_{Q_T} (K_1(x,t) + |u_n|^{\beta}) |\nabla\eta| e^{\frac{d}{\alpha_1}|G_k(u_n)|} |\varphi(T_k(u_n) - T_k(u)_\nu)| \\
& + \alpha_0 \iint_{Q_T} (|\nabla u_n|^{p-1}) e^{\frac{d}{\alpha_1 p'}|G_k(u_n)|} |\nabla\eta| |\varphi(T_k(u_n) - T_k(u)_\nu)| e^{\frac{d}{\alpha_1 p}|G_k(u_n)|} \\
\leq & \alpha_0 \iint_{Q_T} (K_1(x,t) + |u_n|^{\beta}) |\nabla\eta| e^{\frac{d}{\alpha_1}|G_k(u_n)|} |\varphi(T_k(u_n) - T_k(u)_\nu)| \\
& + C_8 ||(|\nabla u_n|^{p-1}) e^{\frac{d}{\alpha_1 p'}|G_k(u_n)|}||_{L^{p'}(Q_T;\mathbb{R}^N)} \\
& \times ||\nabla\eta\varphi(T_k(u_n) - T_k(u)_\nu) e^{\frac{d}{\alpha_1 p}|G_k(u_n)|}||_{L^{p}(Q_T;\mathbb{R}^N)}.
\end{aligned}
$$

By using the hypothesis on K_1 we observe that $(K_1 + |u_n|^{\beta})$ is bounded in $L^{r_1}(0,T;L^{q_1}(B_R)$ in fact in $L^{r_1 p'}(0,T;L^{q_1 p'}(B_R)$, so using again (4.4.5) with η replaced by $\nabla\eta$ and Proposition (4.3.1) we obtain :

$$F \leq \omega(\nu, n).$$

Similarly, one easily shows that (using (4.2.5)) :

$$
\begin{aligned}
B = & \iint_{Q_T} c(u_n)\varphi(T_k(u_n) - T_k(u)_\nu) e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta \\
& \leq \omega(\nu, n).
\end{aligned}
$$

To deal with the term A we split it as follows :

$$
\begin{aligned}
A = & \iint_{\{u_n \leq k\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] \times \\
& \qquad \nabla(T_k(u_n) - T_k(u))\varphi'(T_k(u_n) - T_k(u)_\nu)\eta \\
& + \iint_{\{u_n \leq k\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] \times \\
& \qquad \nabla(T_k(u) - T_k(u)_\nu)\varphi'(T_k(u_n) - T_k(u)_\nu)\eta \\
& + \iint_{\{u_n \leq k\}} a(T_k(u_n), \nabla T_k(u))\nabla(T_k(u_n) - T_k(u)_\nu)\varphi'(T_k(u_n) - T_k(u)_\nu)\eta \\
& - \iint_{\{u_n > k\}} a(u_n, \nabla u_n)\nabla T_k(u)_\nu \varphi'(T_k(u_n) - T_k(u)_\nu) e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta \\
= & A_1 + A_2 + A_3 + A_4.
\end{aligned}
$$

Using Hölder inequality, the first assumption on a (i.e (4.2.2)) and the Remark (4.3.1) on the term A_2 we can check :

$$|A_2| \leq \varphi'(2k) ||\,[a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))]\,\eta||_{L^{p'}(Q_T;\mathbb{R}^N)} \times$$

$$
\begin{aligned}
&||\nabla(T_k(u)-T_k(u)_\nu)||_{L^p(Q_T;\mathbb{R}^N)}\\
\leq&\omega(\nu,n),
\end{aligned}
$$

while the second norm goes to zero as $\nu\to\infty$. It is easy to check that the same holds for the term A_3.

Now we deal with the term A_4, like before we have :

$$
\begin{aligned}
A_4\leq&C_9\left\|\left(K_1+|u_n|^\beta+|\nabla u_n|^{p-1}\right)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\right\|_{L^{p'}(B_R\times(0,T))}\times\\
&\left\|\nabla T_k(u)_\nu\chi_{\{|u_n|>k\}}\right\|_{L^p(B_R\times(0,T))},
\end{aligned}
$$

the first quantity is bounded by the hypothesis on K_1 and by using again Lemma (1.2.9) (applied with $m=p$, while $\lambda_0>\dfrac{dp'}{\alpha_1}$). In the second quantity the function $|\nabla T_k(u)_\nu|^{p(x)}\chi_{\{|u_n|>k\}}$ converges strongly in L^1 (as n and then ν go to ∞) to $|\nabla T_k(u)|^{p(x)}\chi_{\{|u|>k\}}\equiv 0$. Therefore we have shown that :

$$
A=A_1+\omega(\nu,n).
$$

Let us now examine the two terms C and E together. By employing assumption (4.2.3), and since $sign(u_n)\varphi(T_k(u_n)-T_k(u)_\nu)=|\varphi(T_k(u_n)-T_k(u)_\nu)|$ where $\{u_n>k\}$ one has :

$$
\begin{aligned}
C+E\leq&\frac{d}{\alpha_1}\iint_{Q_T}a(u_n,\nabla u_n)\nabla u_n|\varphi(T_k(u_n)-T_k(u)_\nu)|e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta\\
&-\frac{d}{\alpha_1}\iint_{\{u_n>k\}}a(u_n,\nabla u_n)\nabla u_n|\varphi(T_k(u_n)-T_k(u)_\nu)|e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta\\
\leq&\frac{d}{\alpha_1}\iint_{\{u_n\leq k\}}a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi(T_k(u_n)-T_k(u)_\nu)|\eta\\
=&\frac{d}{\alpha_1}\iint_{\{u_n\leq k\}}\left[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))\right]\\
&\left[\nabla T_k(u_n)-\nabla T_k(u)\right]|\varphi(T_k(u_n)-T_k(u)_\nu)|\eta\\
&+\frac{d}{\alpha_1}\iint_{\{u_n\leq k\}}\left[a(T_k(u_n),\nabla T_k(u))\right]\left[\nabla T_k(u_n)-\nabla T_k(u)\right]|\varphi(T_k(u_n)-T_k(u)_\nu)|\eta\\
&+\frac{d}{\alpha_1}\iint_{\{u_n\leq k\}}a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)|\varphi(T_k(u_n)-T_k(u)_\nu)|\eta\\
\leq&\frac{d}{\alpha_1}\iint_{\{u_n\leq k\}}\left[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))\right]\\
&\left[\nabla T_k(u_n)-\nabla T_k(u)\right]|\varphi(T_k(u_n)-T_k(u)_\nu)|\eta+\omega(\nu,n)\\
\leq&\frac{d}{\alpha_1\lambda}\iint_{\{u_n\leq k\}}\left[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))\right]\\
&\left[\nabla T_k(u_n)-\nabla T_k(u)\right]\varphi'(T_k(u_n)-T_k(u)_\nu)\eta+\omega(\nu,n),
\end{aligned}
$$

the last integral is less than a fraction of the term A_1, and can be canceled. This shows that $A_1 \leq \omega(\nu, n)$, which implies also (since $\varphi'(s) \geq \lambda$)

$$\iint_{\{u_n \leq k\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] \eta \leq \omega(\nu, n).$$

Therefore :

$$\begin{aligned}&\iint_{Q_T} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] \eta \\ &\leq \iint_{\{u_n > k\}} a(T_k(u_n), \nabla T_k(u)) \nabla T_k(u) \eta + \omega(\nu, n) = \omega(\nu, n).\end{aligned}$$

By using the Lemma (5.2.1) it follows that :

$$\nabla T_k(u_n) \to \nabla T_k(u) \text{ strongly in } L^p(Q_T^0; \mathbb{R}^N) \text{ for every } k > 0. \tag{4.4.6}$$

Step 2 : Strong Convergence Of $\nabla e^{\frac{\lambda}{p}|u_n|}$ In $L^p(Q_T^0; \mathbb{R}^N)$ For Every $\lambda < \lambda_0$

Since k is arbitrary, it follows from (4.4.6) that $\nabla u_n \to \nabla u$ a.e. in Q_T^0 (passing to a subsequence). Indeed, using Proposition (4.3.1) and (4.4.3) we deduce that, for every $\lambda < \lambda_0$,

$$\nabla(e^{\frac{\lambda}{p}|u_n|}) \rightharpoonup \nabla(e^{\frac{\lambda}{p}|u_n|}) \text{ weakly in } L^p(Q_T^0; \mathbb{R}^N).$$

Using Proposition (4.3.1) and the fact that $\lambda < \lambda_0$ we have :

$$\begin{aligned}\iint_{Q_T^0} e^{\lambda|u_n|} |\nabla u_n|^p &= \iint_{Q_T^0 \cap \{|u_n| \leq k\}} e^{\lambda|u_n|} |\nabla u_n|^p + \iint_{Q_T^0 \cap \{|u_n| > k\}} e^{\lambda|u_n|} |\nabla u_n|^p \\ &= I_1 + I_2,\end{aligned}$$

from (4.4.6) we have for every $k > 0$:

$$I_1 \to \iint_{Q_T^0 \cap \{|u| \leq k\}} e^{\lambda|u|} |\nabla u|^p \text{ as } n \to \infty,$$

and

$$\begin{aligned}I_2 &\leq e^{(\lambda - \lambda_0)k} \iint_{Q_T^0 \cap \{|u_n| > k\}} e^{\lambda_0|u_n|} |\nabla u_n|^p \\ &\leq e^{(\lambda - \lambda_0)k} C_{10}(\lambda) \to 0 \text{ as } k \to \infty,\end{aligned}$$

which gives that :

$$\nabla e^{\frac{\lambda}{p}|u_n|} \to \nabla e^{\frac{\lambda}{p}|u|} \text{ strongly in } L^p(Q_T^0;\mathbb{R}^N) \text{ for every } \lambda < \lambda_0. \tag{4.4.7}$$

and of course weakly for $\lambda = \lambda_0$, which implies

$$\nabla u_n \to \nabla u \text{ strongly in } L^p(Q_T^0;\mathbb{R}^N). \tag{4.4.8}$$

Step 3 : Compactness Of The Nonlinearities.

In order to pass to the limit in the approximated equation in a very short way, we use the results of the previous steps ((4.4.8), (4.2.2), (4.2.10), (4.4.4)) and like in [8] we can easily show that :

$$H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u) \text{ strongly in } L^1(Q_T^0;\mathbb{R}^N) \text{ by using Vitali's theorem,}$$
$$a(u_n,\nabla u_n) \to a(u,\nabla u) \text{ strongly in } L^{p'}(Q_T^0;\mathbb{R}^N),$$
$$\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } D'(Q_T^0),$$

and therefore, since, by the Assumptions on the sequence $u_{0,n}$, the initial condition is satisfied by u. We can now pass to the limit in (4.2.13), obtaining that u is a distributional solution of (4.2.1). The regularity stated in Theorem (4.2.1) for the solution u follows from Proposition (4.3.1), while, if we use Proposition (4.3.2) we can get the regularity stated in Theorem (4.2.2).

Chapitre 5

EXISTENCES RESULTS AND BOUNDEDNESS OF SOLUTIONS FOR SOME STRONGLY NONLINEAR $p(x)$-PARABOLIC EQUATIONS IN UNBOUNDED DOMAINS

Abstract

This article is devoted to the study of the existence and some regularity to the strongly nonlinear parabolic problem of the form :

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} a(u, \nabla u) + \mu u = H(u, \nabla u) + f & \text{in } Q_T := \Omega \times (0, T), \\ u = 0 & \text{on } \Sigma_T := \partial\Omega \times (0, T), \\ u(., 0) = u_0 & \text{in } \Omega \end{cases} \tag{5.0.1}$$

in unbounded domains. Where T is a positive number, $Au := -div(a(., ., u, \nabla u))$ is a Leray-Lions type operator with variable exponents acting from some functional space V into its topological dual V' and the lower order terms, which depend on the solution u and its gradient ∇u, have a power growth of order $p(x)$ with respect to ∇u. The source term belongs to a Lebesgue space.

5.1 Introduction

The topic of function spaces with variable exponents has undergone an impressive development over the last decades, because these spaces possess a solid background in physics and originate from the study on elastic mechanics and electrorheological fluids. They also have extensive applications in different research branches (see e.g. [[64]-[24]] and the references therein) and raise many difficult mathematical problems.

To study this type of equations we encounter several difficulties arising from the variable exponents and the non boundedness of Ω; there are many differences between Lebesgue and Sobolev spaces with p constant and those with variable exponents, When $p(x) = p$ is a constant, it is well known that $L^p(0, T; W_0^{1,p}(\Omega))$ can be taken as a space of solutions. However, when $p(x)$ is non-constant, then neither $L^{p(.)}(0, T; W_0^{1,p(.)}(\Omega))$ nor $L^{p_-}(0, T; W_0^{1,p(.)}(\Omega))$, where $p_- = ess\min_{x\in\overline{\Omega}} p(x)$, constitute a suitable space of solutions (see [18].) Henceforth, to overcome this difficulty, we shall define below our functional space of solutions V as it was done by Bendahmane and al. in [18]. The variable exponent spaces are not translation invariant, when p is a constant, it is well known that $W_0^{1,p}(\Omega)$ (the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$) is identical to $H_0^{1,p}(\Omega) := \{f \in L^p(\Omega), \nabla f \in L^p(\Omega), f_{|\partial\Omega=0}\}$. However, when p is a function, there exists an interesting Lavrentiev phenomenon [78], which shows that the above two spaces are not equivalent etc, we refer to monograph [33] for details and more references. Therefore, the mathematical results on the variable exponents equations are far from being perfect. Although equations of the above type have been widely studied on bounded domains (see [4], [14], [8] and the references therein), many of the methods which were employed failed for the simplest examples of unbounded domains and lack of compactness in the Sobolev embeddings. To go directly to the theme of the present chapter, we only review some former results in unbounded domains which are closely related to our main results.

We find a lot of work that deals with the case where p is a constant and Ω is an unbounded domain, among others we cite the works (See [28], [48], [29], [63] and [30]) where the authors proved the existence and some regularity on solution in the case of nonlinear elliptic and parabolic equations, in the present setting such an approach cannot be used directly, because of the variability of the function $p(x)$ and the transition from the norm to the modular causes undesirable constant in the proofs. To overcome this obstacle we have partitioned the domain in the approximate problem and we used the fact that p is a bounded continuous function. Based on the above fact and motivated by techniques used in [[41], [66]], the main purpose of this part is to establish the existence result and obtain a global regularity; about exponential summability and boundedness ; of solutions under some higher integrability hypotheses, however by imposing the zero Dirichlet boundary condition.

In the $p(x)$-elliptical frame where Ω is unbounded domain we can cite [75],[37] and [10](and the references therein). Authors in [10] have investigated the existence of at least one weak solution and they have given some regularity for this existence result, they drop the assumption of smallness of the source term f in view of the presence of the term $c(x, u)$ with some specific assumptions which is not appearing in our problem. Similarly in [44] using Mountain Pass Theorem authors obtained the existence of solutions for the $p(x)$-Laplacian problem in the super-linear case. However, to the best of our knowledge, no results on the case of $p(x)$-parabolic equation in general domain have been obtained up to now. Inspired by the previous works, the strategy that we will use consists in approximating the domain Ω by bounded sets Ω_n, and solving a more regular problem in Ω_n, then we pass to the limit in the approximating problems. The use of test functions of exponential type allows us to get rid of the term $H(t, x, u, \nabla u)$ and therefore is an essential tool in the proofs. To this aim, we can neither use any embedding theorem nor any argument involving the measure of Ω_n. For regularity we will use an adaptation of a classical technique due to Stampacchia [66].

The plan of the chapter is as follows : In the second section we will recall some important definitions and results of variable exponent Lebesgue and Sobolev spaces and we will introduce the adequate spaces to discuss the problem (5.0.1). In the third section we give the precise assumptions on the data of the problem. The fourth section is devoted to prove estimates on u_n, solution of the approximate problems. In the fifth section we prove a local strong convergence of ∇u_n and the existence of solutions for a class of equations of the type (5.0.1) by passing to the limit in approximate problem, and again this is done through exponential-type functions, using a local adaptation of a technique by Ferone and Murat (see [41]). Finally in the last section we prove that, if we assumes a slightly stronger hypothesis on $f(x, t)$ and u_0, then every solution u of (5.0.1) is bounded.

Our analysis is built on the variable exponent Lebesgue-Sobolev space and theory of operators of monotone type in reflexive Banach spaces, for that let us recall some important known proprieties of these spaces.

5.2 Essential Assumption

Let Ω be an open set of $\mathbb{R}^N$ ($N \geq 2$), and $p \in C_+(\bar{\Omega})$. Fixing a final time $T > 0$. Q_T is the cylinder defined by $Q_T := \Omega \times (0,T)$ and Σ_T its lateral boundary $\Sigma_T := \partial\Omega \times (0,T)$. We consider the following strongly nonlinear parabolic initial-boundary problem :

$$\begin{cases} \frac{\partial u}{\partial t} + Au + \mu u = H(u, \nabla u) + f & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(.,0) = u_0 & \text{in } \Omega. \end{cases} \tag{5.2.1}$$

Where

$$Au = -\operatorname{div}\ a(x,t,s,\xi)$$

is an operator which maps from $V(Q_T)$ into its dual $V'(Q_T)$.

Assumptions on $f(x,t)$:

$f : \Omega \times (0,T) \to \mathbb{R}$ is a measurable function such that :

$$\begin{aligned} &i) \text{ If the datum } f \text{ is "large", more precisely on the set } \{|f| > 1\} \text{ we take :} \\ &\quad f \in L^{r_-}(0,T; L^{q(x)}(\Omega)) \text{ with } \max\left(\frac{N}{q_-}, \frac{p_-}{q'_-}\right) \leq \frac{p_-}{r'_+} \leq 1. \\ &ii) \text{ If the datum } f \text{ is "small" we take :} \\ &\quad f \in L^2(\{|f| \leq 1\}), \end{aligned} \tag{5.2.2}$$

where $1 < r(x), q(x) < \infty$.

Remark 5.2.1 *When Ω has finite measure, one can suppose that :*

$$f \in L^{r_-}(0,T; L^{q(x)}(\Omega)) \textit{ with } \frac{N}{q_-} \leq \frac{p_-}{r'_+} \leq 1.$$

Remark 5.2.2 *If we have the presence of the term $|u|^{p(x)-2}u$ in the equation* (5.2.1)*, then we can drop the assumption of smallness of the source term f i.e $f \in L^2(\{|f| \leq 1\})$ even if Ω is unbounded.*

To prove the boundedness of the solutions we will assume some slightly stronger hypothesis on $f(x,t)$; more precisely the assumption (5.2.2) can be replaced by :

$$\begin{aligned} & f \in L^{r_-}(0,T;L^{q(x)}(\Omega)) \text{ with } \max(\frac{N}{q_-},\frac{p_-}{q'_-}) < \frac{p_-}{r'_+} \leq 1 \text{ on the set } \{|f|>1\}, \\ & f \in L^2(\{|f|\leq 1\}). \end{aligned} \tag{5.2.3}$$

Assumptions on u_0 :

$u_0 : \Omega \to \mathbb{R}$ is a measurable function satisfying :

$$\int_\Omega e^{\lambda_0|u_0|}-1dx<\infty \qquad \text{for some } \lambda_0>\max(\frac{dp_+}{\alpha_1},\frac{dp'_+}{\alpha_1}). \tag{5.2.4}$$

Assumptions on $a(x,t,s,\xi)$:

$a : \Omega\times(0,T)\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory vector-valued function (measurable with respect to (x,t) in Q_T for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$ for almost every (x,t) in Q_T) which satisfies the following conditions :

— There exists a constant $\alpha_0,\beta>0$ such that :

$$|a(x,t,s,\xi)|\leq\alpha_0(K_1(x,t)+|s|^\beta+|\xi|^{p(x)-1}), \tag{5.2.5}$$

for almost every $(x,t)\in Q_T$ and for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $K_1(x,t)$ is a positive function such that : $K_1^{p'(x)}\in L^{(r_1)_-}(0,T;L^{q_1(x)}_{loc}(\Omega))$ with
$\max(\frac{N}{(q_1)_-},\frac{p_-}{(q'_1)_-})\leq\frac{p_-}{(r'_1)_+}\leq 1$ and $1<r_1(x),q_1(x)<\infty$.

— There exists a constant $\alpha_1>0$ such that :

$$a(x,t,s,\xi)\xi\geq\alpha_1|\xi|^{p(x)}, \tag{5.2.6}$$

for almost every $(x,t)\in Q_T$ and for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$.

— For almost every $(x,t) \in Q_T$ and for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, with $\xi \neq \overline{\xi}$ we have :

$$[a(x,t,s,\xi) - a(x,t,s,\overline{\xi})](\xi - \overline{\xi}) > 0. \tag{5.2.7}$$

Remark 5.2.3 *The assumption $K_1^{p'(x)} \in L^{(r_1)_-}(0,T;L_{loc}^{q_1(x)}(\Omega)$, with $\max(\frac{N}{(q_1)_-}, \frac{p_-}{(q_1')_-}) \leq \frac{p_-}{(r_1')_+} \leq 1$, which appears in* (5.2.5) *instead of the more usual hypothesis $k_1(x,t) \in L^{p'(x)}(Q_T)$, will be used in the proof of the strong convergence of the gradient of the approximate solutions.*

Assumptions on $H(x,t,s,\xi)$:

The nonlinear term $H(x,t,s,\xi)$ is a Carathéodory function which satisfies :

$$|H(x,t,s,\xi)| \leq d|\xi|^{p(x)}, \tag{5.2.8}$$

where $d > 0$, for almost every $(x,t) \in Q_T$ and for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

Remark 5.2.4 *It is obvious from the proofs of the results that the last assumption might be replaced by*

$$|H(x,t,s,\xi)| \leq d|\xi|^{p(x)} + g(x,t),$$

with g satisfying the same hypotheses as f.

Let us point out that if we consider the function $v(x,t) = e^{\mu t}u(x,t)$; where u is a solution of (5.2.1) ; it is easy to verify that it satisfies :

$$\begin{cases} \frac{\partial v}{\partial t} - div(a_1(x,t,v,\nabla v)) = H_1(v,\nabla v) + f_1 & \text{in } Q_T, \\ v = 0 & \text{on } \Sigma_T, \\ v(.,0) = v_0 & \text{in } \Omega, \end{cases}$$

where $a_1(x,t,s,\xi) = e^{\mu t}a(x,t,e^{-\mu t}s, e^{-\mu t}\xi)$, $H_1(x,t,s,\xi) = e^{\mu t}H(x,t,e^{-\mu t}s,e^{-\mu t}\xi)$ and $f_1(x,t) = e^{\mu t}f(x,t)$ satisfy the same kind of hypotheses as a, H and f, respectively.
We refer to the problem :

$$\begin{cases} \partial v_t - \Delta_{p(x)}v = d(x,t)\frac{|\nabla v|^{p(x)}}{|\nabla v|^{p(x)+1}} + f(x,t) & \text{in } Q_T, \\ v = 0 & \text{on } \Sigma_T, \\ v(.,0) = v_0 & \text{in } \Omega, \end{cases}$$

or

$$\begin{cases} \partial v_t - \Delta_{p(x)} v = d(x,t)|\nabla v|^{p(x)} + f(x,t) & \text{in } Q_T, \\ v = 0 & \text{on } \Sigma_T, \\ v(.,0) = v_0 & \text{in } \Omega, \end{cases}$$

as a model case. Here $\Delta_{p(x)} v$ is the $p(x)$-Laplace operator and d is a function in $L^\infty(Q_T)$.

Notations : The symbol $\rightharpoonup$ will denote the weak convergence, and the constants C_i, $i = 1, 2, \ldots$ used in each step of proof are independent. For $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ is defined by :

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases} \tag{5.2.9}$$

Our main result will be proved by approximating problem (5.2.1) with the following problems on the bounded domains $Q_{n,T} = \Omega_n \times (0,T)$ (where $\Omega_n = \Omega \cap B_n(0)$ $B_n(0)$ is the ball with center 0 and radius n) :

$$\begin{cases} \frac{\partial u_n}{\partial t} + Au_n + \mu u_n = H_n(u_n, \nabla u_n) + f_n & \text{in } Q_{n,T}, \\ u_n = 0 & \text{on } \partial\Omega_n \times (0,T), \\ u_n(.,0) = u_{0,n} & \text{in } \Omega_n. \end{cases} \tag{5.2.10}$$

Where $H_n(x,t,s,\xi) = T_n(H(x,t,s,\xi))$, $f_n(x,t) = T_n(f(x,t))$ and $u_{0,n}$ is a bounded sequence in the same spaces as u_0 ; that is ;

$$\int_\Omega e^{\lambda_0 |u_{0,n}|} - 1 dx < \infty \text{ with } \lambda_0 > \max(\frac{dp_+}{\alpha_1}, \frac{dp'_+}{\alpha_1}). \tag{5.2.11}$$

Moreover $u_{0,n}$ is a sequence such that :

$$u_{0,n} \in L^\infty(\Omega_n) \cap W_0^{1,p(x)}(\Omega_n) \qquad u_{0,n} \to u_0 \text{ a.e. in } \Omega, \tag{5.2.12}$$

$$\lim_{n\to\infty} \frac{1}{n} ||u_{0,n}||_{W_0^{1,p(x)}(\Omega_n)} = 0. \tag{5.2.13}$$

This condition will be used in the proof of the strong convergence of the gradients ∇u_n.

Lemma 5.2.1 ([2])

Let us Assume (5.2.5)-(5.2.7) *hold, and let* $(u_n)_n$ *be a sequence in* $V(Q_T)$ *such that* $u_n \rightharpoonup u$ *in* $V(Q_T)$ *and*

$$\int_{Q_T} [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)] \nabla(u_n - u) dxdt \to 0, \tag{5.2.14}$$

then $u_n \to u \quad in \quad V(Q_T)$.

As a consequence, proving existence of at least one weak solution $u_n \in V \cap L^\infty(Q_{n,T})$ of (5.2.10) is an easy task (see [52]).

5.3 A Priori Estimate

In this Section, we will prove a priori estimates for the solutions u_n.

Proposition 5.3.1

Assuming that (5.2.2)*,* (5.2.4)*,* (5.2.6)*,* (5.2.8) *hold,* $p(.) \in C_+(\overline{\Omega})$ *and let* u_n *be any solution of* (5.2.10)*. Then there exists a positive constant* C *depending on the data, such that :*

$$\iint_{Q_{n,T}} e^{\lambda_0 |u_n|} |\nabla u_n|^{p(x)} dxdt \leq C. \tag{5.3.1}$$

Remark 5.3.1

The previous estimate yields an estimate for the $|\nabla u_n|$ *in* $L^{p(x)}((0,T) \times \Omega)$*.*

Proof : For simplicity of notation we will always omit the index n of the sequence. Let φ a function defined as follows :

$$\varphi(s) = \big(e^{\lambda_0 |s|} - 1\big) sign(s), \tag{5.3.2}$$

and its primitive :

$$\psi(s) = \int_0^s \varphi(\sigma) d\sigma. \tag{5.3.3}$$

Let k be a positive number sufficiently large ($\dfrac{2}{\mu} < k$), we take $\varphi(G_k(u))$ as test function in (5.2.10), where

$$G_k(s) = s - T_k(s) = \begin{cases} s - k & \text{if } s > k, \\ 0 & \text{if } |s| \leq k, \\ s + k & \text{if } s < -k. \end{cases} \tag{5.3.4}$$

Integrating on $Q = \Omega \times (0,\tau)$ and in view of hypotheses (5.2.6),(5.2.8) we have :

$$\begin{aligned}
&\int_\Omega \psi(G_k(u(\tau)))dx - \int_\Omega \psi(G_k(u(0)))dx + \alpha_1 \iint_Q |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)) \\
&\quad + \mu \iint_Q |u||\varphi(G_k(u))| \\
&\le d \iint_Q |\nabla G_k(u)|^{p(x)}|\varphi(G_k(u))| + \iint_Q |f||\varphi(G_k(u))| \\
&= d \iint_Q |\nabla G_k(u)|^{p(x)}|\varphi(G_k(u))| + \iint_{Q\cap\{|f|>1;|G_k(u)|>1\}} |f||\varphi(G_k(u))| \\
&\quad + \iint_{Q\cap\{|f|>1;|G_k(u)|\le 1\}} |f||\varphi(G_k(u))| + \iint_{Q\cap\{|f|\le 1\}} |f||\varphi(G_k(u))| \\
&= A + B + C + D.
\end{aligned} \tag{5.3.5}$$

Estimation of the integral A :

For every s in $\mathbb{R}$ we have :

$$|\varphi(s)| \le \frac{1}{\lambda_0}\varphi'(s), \tag{5.3.6}$$

then

$$A \le \frac{d}{\lambda_0} \iint_Q |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)). \tag{5.3.7}$$

Before estimating B, we define :

$$\phi_{p(x)}(s) = \int_0^{|s|} (\varphi'(\sigma))^{\frac{1}{p(x)}} d\sigma. \tag{5.3.8}$$

It is easy to remark that :

$$\iint_Q |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)) = \iint_Q |\nabla \left(\phi_{p(x)}(G_k(u))\right)|^{p(x)}. \tag{5.3.9}$$

Moreover, we observe that there exists a positive constant $C_1 = C_1(p,\lambda_0)$ such that :

$$|\varphi(s)| \le C_1(\phi_{p(x)}(s))^{p(x)} \text{ for every s such that } |s| \ge 1, \tag{5.3.10}$$

the last inequality remains true for p_+ and p_- with an adequate constant.

We can also observe that :

$$|\varphi(s)| \le C_2\psi(s) \text{ for every s such that } |s| \ge 1. \tag{5.3.11}$$

We can now estimate B by splitting it as follows :

$$B = \iint_{Q\cap\{|f|>1;|G_k(u)|>1\}} |f||\varphi(G_k(u))|$$

$$
\begin{aligned}
&= \iint_{Q\cap\{|f|>1;|G_k(u)|>1;|\nabla G_k(u)|>1\}} |f||\varphi(G_k(u))| \\
&\quad + \iint_{Q\cap\{|f|>1;|G_k(u)|>1;|\nabla G_k(u)|\leq 1\}} |f||\varphi(G_k(u))| \\
&= B_1 + B_2.
\end{aligned}
$$

Using Hölder and interpolation inequality we have :

$$
\begin{aligned}
B_1 \leq \int_0^\tau &\left[||f(t)||_{L^{q_-}(\{|f|>1\})} ||\varphi(G_k(u))||^{1-\theta_+}_{L^1(\{|G_k(u)|>1\})} \right] \times \\
&\left[||\varphi(G_k(u))||^{\theta_+}_{L^{p^*_-/p_-}(\{|G_k(u)|>1;|\nabla G_k(u)|>1\})} \right] dt,
\end{aligned}
$$

where $\theta_+ = \frac{N}{p_- q_-}$ and $1-\theta_+ + \frac{\theta_+}{p^*_-/p_-} = \frac{1}{q'_+}$.

Let $\epsilon > 0$, by applying Young's inequality, 5.3.10, 5.3.11, Sobolev-Poincaré inequality and 5.3.9 we have :

$$
\begin{aligned}
B_1 \leq\, & C_3 \int_0^\tau ||f(t)||^{\frac{1}{1-\theta_+}}_{L^{q_-}(\{|f|>1\})} \left[\int_\Omega \psi(G_k(u)) dx \right] dt \\
&+ \frac{\alpha_1 - \frac{d}{\lambda_0}}{2+\epsilon} \int_0^\tau \int_{\{|\nabla G_k(u)|>1\}} |\nabla G_k(u)|^{p_-} \varphi'(G_k(u)) dx dt \\
\leq\, & C_3 \int_0^\tau ||f(t)||^{\frac{1}{1-\theta_+}}_{L^{q_-}(\Omega)} \left[\int_\Omega \psi(G_k(u)) dx \right] dt \\
&+ \frac{\alpha_1 - \frac{d}{\lambda_0}}{2+\epsilon} \int_0^\tau \int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) dx dt.
\end{aligned}
$$

In the same way we can estimate B_2, using Hölder and interpolation inequality we have :

$$
\begin{aligned}
B_2 \leq \int_0^\tau &\left[||f(t)||_{L^{q_+}(\{|f|>1\})} ||\varphi(G_k(u))||^{1-\theta_-}_{L^1(\{|G_k(u)|>1\})} \right] \times \\
&\left[||\varphi(G_k(u))||^{\theta_-}_{L^{p^*_+/p_+}(\{|G_k(u)|>1;|\nabla G_k(u)|\leq 1\})} \right] dt,
\end{aligned}
$$

where $\theta_- = \frac{N}{p_+ q_+}$ and $1-\theta_- + \frac{\theta_-}{p^*_+/p_+} = \frac{1}{q'_-}$.

Let $\epsilon > 0$, by applying Young's inequality, 5.3.10, 5.3.11, Sobolev-Poincaré inequality and 5.3.9 we have :

$$
\begin{aligned}
B_2 \leq\, & C_4 \int_0^\tau ||f(t)||^{\frac{1}{1-\theta_-}}_{L^{q_+}(\{|f|>1\})} \left[\int_\Omega \psi(G_k(u)) dx \right] dt \\
&+ \frac{\alpha_1 - \frac{d}{\lambda_0}}{2+\epsilon} \int_0^\tau \int_{\{|\nabla G_k(u)|\leq 1\}} |\nabla G_k(u)|^{p_+} \varphi'(G_k(u)) dx dt \\
\leq\, & C_4 \int_0^\tau ||f(t)||^{\frac{1}{1-\theta_-}}_{L^{q_+}(\Omega)} \left[\int_\Omega \psi(G_k(u)) dx \right] dt \\
&+ \frac{\alpha_1 - \frac{d}{\lambda_0}}{2+\epsilon} \int_0^\tau \int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) dx dt.
\end{aligned}
$$

Finally we have :

$$B \leq \int_0^\tau \left(C_4 ||f(t)||_{L^{q_+}(\Omega)}^{\frac{1}{1-\theta_-}} + C_3 ||f(t)||_{L^{q_-}(\Omega)}^{\frac{1}{1-\theta_+}} \right) \left[\int_\Omega \psi(G_k(u))dx \right] dt$$
$$+ \frac{2(\alpha_1 - \frac{d}{\lambda_0})}{2+\epsilon} \int_0^\tau \int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) dx dt. \tag{5.3.12}$$

In fact that (5.2.2) it is easy to check that :

$$\frac{1}{1-\theta_-} < \frac{1}{1-\theta_+} < r_-. \tag{5.3.13}$$

Estimation of the integral C :

It is easy to check that :

$$C \leq \varphi(1) \iint_{\{|f|>1\}} |f|^{q(x)}. \tag{5.3.14}$$

Estimation of the integral D :

Now, since we have chosen k large enough such that :

$$1 \leq \frac{\mu k}{2}, \tag{5.3.15}$$

then we obtain :

$$D \leq \frac{\mu}{2} \iint_Q |u||\varphi(G_k(u))|. \tag{5.3.16}$$

In conclusion, putting all the inequalities (5.3.5), (5.3.7), (5.3.12), (5.3.14) and (5.3.16) together, taking into account (5.3.13), and setting

$$h(\tau) = \int_\Omega \psi(G_k(u(\tau)))dx,$$

we get :

$$h(\tau) + \frac{\epsilon(\alpha_1 - \frac{d}{\lambda_0})}{2+\epsilon} \iint_Q |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + \frac{\mu}{2} \iint_Q |u||\varphi(G_k(u))|$$
$$\leq C_5 + \int_0^\tau g(t)h(t)dt. \tag{5.3.17}$$

For every k satisfying (5.3.15), where the function :

$$g(t) = C_4 ||f(t)||_{L^{q_+}(\Omega)}^{\frac{1}{1-\theta_-}} + C_3 ||f(t)||_{L^{q_-}(\Omega)}^{\frac{1}{1-\theta_+}}$$

belongs to $L^1(0,T)$ and $C_5 = \varphi(1) \iint_{\{|f|>1\}} |f|^{q(x)} + \int_\Omega \psi(G_k(u(0)))dx < \infty$(using (5.2.2), (5.2.4)). An application of Gronwall-Bellman's Lemma (see (1.2.6)) yields that $h(\tau)$ is a bounded function, and that $\iint_Q |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u))$ is also bounded.

We fix now k such that (5.3.17) holds and we take $\varphi(T_k(u)$ as a test function in (5.2.10), then Integrating on $Q_k = (0,\tau)\times\Omega\cap\{|u|\leq k\}$. In view of hypotheses (5.2.6),(5.2.8) we have :

$$\begin{aligned}
&\int_{\Omega\cap\{|u|\leq k\}}\psi(u(\tau))dx - \int_{\Omega\cap\{|u|\leq k\}}\psi(u(0))dx + \alpha_1\iint_{Q_k}|\nabla T_k(u)|^{p(x)}\varphi'(T_k(u))\\
&\qquad + \mu\iint_{Q_k}|u||\varphi(T_k(u))|\\
&\leq d\iint_{Q_k}|\nabla T_k(u)|^{p(x)}|\varphi(T_k(u))| + \iint_{Q_k}|f||\varphi(T_k(u))|\\
&= d\iint_{Q_k}|\nabla T_k(u)|^{p(x)}|\varphi(T_k(u))| + \iint_{Q_k\cap\{|f|>1\}}|f||\varphi(T_k(u))|\\
&\qquad + \iint_{Q\cap\{|f|\leq 1\}}|f||\varphi(T_k(u))|\\
&= E + F + J.
\end{aligned}\tag{5.3.18}$$

Estimation of the integral E and F :

Using (5.3.6), we have :

$$E \leq \frac{d}{\lambda_0}\iint_Q|\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)).\tag{5.3.19}$$

For the second term we have :

$$F \leq \varphi(k)\iint_{Q_k\cap\{|f|>1\}}|f|^{q(x)}.\tag{5.3.20}$$

Estimation of the integral J :

Let ϵ be a positive constant to be chosen later, we write :

$$J \leq C_5(\epsilon)\iint_{Q_k\cap\{|f|\leq 1\}}|f|^2 + \epsilon\iint_{Q_k\cap\{|f|\leq 1\}}|\varphi(T_k(u))|^2,$$

Since

$$|\varphi(T_k(u))|^2 \leq C_6(\lambda_0,k)|\varphi(T_k(u))||u|,$$

and choosing ϵ such that $\epsilon C_6 \leq \frac{\mu}{2}$,we have :

$$J \leq C_5(\epsilon)\iint_{Q_k\cap\{|f|\leq 1\}}|f|^2 + \frac{\mu}{2}\iint_{Q_k\cap\{|f|\leq 1\}}|\varphi(T_k(u))||u|.\tag{5.3.21}$$

Putting all the estimations (5.3.18) - (5.3.21) together, we get :

$$\begin{aligned}
&\int_{\Omega\cap\{|u|\leq k\}}\psi(u(\tau))dx + \left(\alpha_1 - \frac{d}{\lambda_0}\right)\iint_{Q_k}|\nabla T_k(u)|^{p(x)}\varphi'(T_k(u))\\
&\qquad + \frac{\mu}{2}\iint_{Q_k}|u||\varphi(T_k(u))|\\
&\leq \varphi(k)\iint_{\{|f|>1\}}|f|^{q(x)} + C_7\iint_{\{|f|\leq 1\}}|f|^2 + \int_\Omega\psi(u(0))dx.
\end{aligned}\tag{5.3.22}$$

Finally if we combine inequalities (5.3.22), (5.3.17) and assumptions (5.2.4), (5.2.2) we have (5.3.1).

Remark 5.3.2

— *Using the fact that $h(\tau)$ is bounded (*(5.3.17), (5.3.22)*) we can have :*

$$\sup_{t\in[0,T]} \int_{\Omega^0} e^{\lambda|u_n(x,t)|}dx < \infty, \tag{5.3.23}$$

for every bounded open set $\Omega^0 \subset \Omega$ and $\lambda_0 \geq \lambda \geq 0$.

— *The estimation* (5.3.1) *yields that :*

$$\iint_{Q_{n,T}} e^{\lambda|u_n|}|\nabla u_n|^{p(x)}dxdt \leq C, \tag{5.3.24}$$

for every λ such that : $\lambda_0 \geq \lambda \geq 0$.

5.4 Main Result

In this section we will prove the main result of this chapter. Let $\{u_n\}$ be any sequence of solutions of problem (5.2.10), we extend them to zero in $\Omega \setminus \Omega_n$. Let Ω^0 a bounded open subset of Ω and $Q_T^0 = \Omega^0 \times (0,T)$.

Some Technical Results.

Let $(T_k(u))_\nu$ be the mollification with respect to time of $T_k(u)$ and, as before, $\varphi(s) = (e^{\lambda|s|}-1)sign(s)$ and

$$\eta(x) \in C_0^\infty(B_R), \quad 0 \leq \eta \leq 1, \quad \eta \equiv 1 \quad \text{in} \quad \Omega^0, \tag{5.4.1}$$

with B_R is a ball containing Ω^0 (For simplicity we denote $B_R \cap \Omega$ again by B_R).

By Proposition (5.3.1) there exist a subsequence (which we will still denote by u_n) and a function u such that :

$$u_n \rightharpoonup u \text{ weakly in } V_0(Q_T^0) \text{ and *-weakly in } L^\infty(0,T;L^{q(x)}(\Omega^0)), \tag{5.4.2}$$

$$e^{\frac{\lambda_0}{p_+}|u_n|} \rightharpoonup e^{\frac{\lambda_0}{p_+}|u|} \text{ in } V_0(Q_T^0), \tag{5.4.3}$$

for every function $q(x)$ such that $q_+ < \infty$.

Lemma 5.4.1

Still extracting a subsequence,

$$u_n \to u \text{ a.e. in } Q_T^0 \text{ and strongly in } L^{q(x)}(Q_T^0), \quad \text{for every function q(x) such that } q_+ < \infty. \tag{5.4.4}$$

Proof : By Remark (5.3.1) and the equation satisfied by u_n, it is easy to check that the sequence $\left(\frac{\partial(\eta u_n)}{\partial t}\right)_{n\in\mathbb{N}}$ is bounded in $V'(B_R \times (0,T)) + L^1(B_R \times (0,T))$. Then, by a well-known compactness result (see for instance (1.2.7)), the sequence ηu_n is relatively compact in $L^{p(x)}(B_R \times (0,T))$. Therefor, by Proposition (5.3.1) and Remark (5.3.2), u_n is bounded in $L^{q(x)}(Q_T^0)$, for every function $q(x)$ such that $q_+ < \infty$ which completes the proof.

The proof of the following Lemma is very similar to that in ([30]) with constant exponents.

Lemma 5.4.2

Let $\eta \in C_0^\infty(\mathbb{R}^N)$, the following inequality holds for every δ such that $0 < \delta < \lambda_0$:

$$\ll \rho_n, w(x,t) \gg \geq \omega^\nu(n) + \omega(\nu) := \omega(\nu, n)$$

where $\rho_n = div\ a(x,t,u_n,\nabla_n) - \mu u_n + H_n(x,t,u_n,\nabla u_n) + f_n(x,t)$,
$w(x,t) = \eta(x)\varphi(T_k(u_n) - T_k(u)_\nu)e^{\delta|G_k(u_n)|}$ and where we denote by $\omega^\gamma(h)$ a quantity which goes to zero as h goes to infinity, for every γ fixed, and by $\ll .,. \gg$ the duality hook.

We are now up to give and prove the statement of this part.

Theorem 5.4.1

Assume that (5.2.2)*,* (5.2.4)-(5.2.8) *are satisfied. Then there exists at least one solution u of* (5.2.1) *in the sense of distributions such that*

$$e^{\frac{\lambda_0}{p_+}|u|} \in V(Q_T), \tag{5.4.5}$$

Moreover for every bounded open set $\Omega^0 \subset \Omega$ we have :

$$\sup_{t\in[0,T]} \int_{\Omega^0} e^{\lambda_0|u(x,t)|}dx < \infty. \tag{5.4.6}$$

The proof of this theorem is divided into three steps. By Remark (5.3.1), we can say that, up to a subsequence,

$$\nabla u_n \rightharpoonup \nabla u \text{ weakly in } L^{p(x)}(Q_T^0;\mathbb{R}^N).$$

Step 1 : Strong Convergence Of $\nabla T_k(u_n) in (L^{p(x)}(Q_T^0))^N$

In this step, we will fix $k > 0$ and prove that $\nabla T_k(u_n) \to \nabla T_k(u)$ strongly in $L^{p(x)}(Q_T^0; \mathbb{R}^N)$ as $n \to \infty$.

In order to prove this result we define :

$$w(x,t) = \eta(x)\varphi(T_k(u_n) - T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|},$$

where $\frac{d}{\alpha_1} < \lambda < \lambda_0$ and n large enough to ensure that $w \in V_0(Q_{n,T})$, and we take $w(x,t)$ as a test function in (5.2.10), using Lemma (5.4.2) one obtains :

$$\begin{aligned}
A + B = & \iint_{Q_T} a(u_n, \nabla u_n)\nabla(T_k(u_n) - T_k(u)_\nu)\varphi'(T_k(u_n) - T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta \\
& + \mu \iint_{Q_T} u_n\varphi(T_k(u_n) - T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta \\
\leq & d \iint_{Q_T} |\nabla u_n|^{p(x)}\varphi(T_k(u_n) - T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta \\
& + \iint_{Q_T} f_n\varphi(T_k(u_n) - T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta \\
& - \frac{d}{\alpha_1} \iint_{Q_T} a(u_n, \nabla u_n)\nabla(G_k(u_n))sign(u_n)\varphi(T_k(u_n) - T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta \\
& - \iint_{Q_T} a(u_n, \nabla u_n)\nabla\eta\varphi(T_k(u_n) - T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|} + \omega(\nu, n) \\
= & C + D + E + F + \omega(\nu, n),
\end{aligned}$$

where the notation $\omega(\nu, n)$ was introduced in the Lemma (5.4.2).

Let us examine the term D :

$$\begin{aligned}
D = & \iint_{Q_T} f_n\varphi(T_k(u_n) - T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta \\
= & \iint_{Q_T} f\varphi(T_k(u) - T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u)|}\eta + \omega^\nu(n) \\
= & \,\omega(\nu, n).
\end{aligned}$$

Indeed, using (5.2.2) in the set where $\{|f_n| > 1\}$ we have f_n converges strongly to f in $L^{r_-}(0,T; L^{q(x)}(\Omega))$, $\varphi(T_k(u_n) - T_k(u)_?)$ is bounded and converges almost everywhere. Moreover, by Remark (5.3.2) and (5.2.4) we have $e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta$ is bounded in $L^\infty(0,T; L^{p(x)}(B_R)) \cap L^{p_-}(0,T; W_0^{1,p(x)}(B_R))$, therefor by

Corollary (1.2.1) is bounded in $L^{r'_+}(0,T;L^{q'(x)}(\Omega))$, hence $e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta$ converges weakly in $L^{r'_+}(0,T;L^{q'(x)}(\Omega))$. However, in $\{|f_n|\leq 1\}$, by (5.3.23) we have $e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta$ converges weakly.

As far as the term F is concerned, using the assumption (5.2.5) on a and Hölder inequality one has :

$$\begin{aligned}
F \leq& \alpha_0 \iint_{Q_T} (K_1(x,t)+|u_n|^\beta)|\nabla\eta| e^{\frac{d}{\alpha_1}|G_k(u_n)|}|\varphi(T_k(u_n)-T_k(u)_\nu)| \\
&+ \alpha_0 \iint_{Q_T} (|\nabla u_n|^{p(x)-1})|\nabla\eta| e^{\frac{d}{\alpha_1}|G_k(u_n)|}|\varphi(T_k(u_n)-T_k(u)_\nu)| \\
\leq& \alpha_0 \iint_{Q_T} (K_1(x,t)+|u_n|^\beta)|\nabla\eta| e^{\frac{d}{\alpha_1}|G_k(u_n)|}|\varphi(T_k(u_n)-T_k(u)_\nu)| \\
&+ \alpha_0 \iint_{Q_T} (|\nabla u_n|^{p(x)-1}) e^{\frac{d}{\alpha_1 p'(x)}|G_k(u_n)|}|\nabla\eta||\varphi(T_k(u_n)-T_k(u)_\nu)| e^{\frac{d}{\alpha_1 p(x)}|G_k(u_n)|} \\
\leq& \alpha_0 \iint_{Q_T} (K_1(x,t)+|u_n|^\beta)|\nabla\eta| e^{\frac{d}{\alpha_1}|G_k(u_n)|}|\varphi(T_k(u_n)-T_k(u)_\nu)| \\
&+ C_8 ||(|\nabla u_n|^{p(x)-1}) e^{\frac{d}{\alpha_1 p'(x)}|G_k(u_n)|}||_{L^{p'(x)}(Q_T;\mathbb{R}^N)} \\
&\times ||\nabla\eta\varphi(T_k(u_n)-T_k(u)_\nu) e^{\frac{d}{\alpha_1 p(x)}|G_k(u_n)|}||_{L^{p(x)}(Q_T;\mathbb{R}^N)}.
\end{aligned}$$

By using the hypothesis on K_1 we observe that $(K_1+|u_n|^\beta)$ is bounded in $L^{(r_1)_-}(0,T;L^{q_1(x)}(B_R)$, so using again Corollary (1.2.1) as in estimation of term D and the Remark (5.3.2) we obtain :

$$F \leq \omega(\nu,n).$$

Similarly, one easily shows that :

$$\begin{aligned}
B &= \mu \iint_{Q_T} u_n \varphi(T_k(u_n)-T_k(u)_\nu) e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta \\
&\leq \omega(\nu,n).
\end{aligned}$$

To deal with the term A we split it as follows :

$$\begin{aligned}
A =& \iint_{\{u_n\leq k\}} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))] \times \\
&\qquad \nabla(T_k(u_n)-T_k(u))\varphi'(T_k(u_n)-T_k(u)_\nu)\eta \\
&+ \iint_{\{u_n\leq k\}} [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))] \times \\
&\qquad \nabla(T_k(u)-T_k(u)_\nu)\varphi'(T_k(u_n)-T_k(u)_\nu)\eta
\end{aligned}$$

$$
\begin{aligned}
&+\iint_{\{u_n\le k\}} a(T_k(u_n),\nabla T_k(u))\nabla(T_k(u_n)-T_k(u)_\nu)\varphi'(T_k(u_n)-T_k(u)_\nu)\eta \\
&-\iint_{\{u_n>k\}} a(u_n,\nabla u_n)\nabla T_k(u)_\nu\varphi'(T_k(u_n)-T_k(u)_\nu)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta \\
=\ & A_1+A_2+A_3+A_4.
\end{aligned}
$$

Using Hölder inequality, the first assumption on a (i.e (5.2.5)) and the Remark (5.3.1) on the term A_2 we can check :

$$
\begin{aligned}
|A_2| \le&\varphi'(2k)||\,[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))]\,\eta||_{L^{p'(x)}(Q_T;\mathbb{R}^N)}\times \\
&||\nabla(T_k(u)-T_k(u)_\nu)||_{L^{p(x)}(Q_T;\mathbb{R}^N)} \\
\le&\omega(\nu,n).
\end{aligned}
$$

It is easy to check that the same holds for the term A_3.

Now we deal with the term A_4, like before we have :

$$
\begin{aligned}
A_4 \le& C_9\left\|\left(K_1+|u_n|^\beta+|\nabla u_n|^{p(x)-1}\right)e^{\frac{d}{\alpha_1}|G_k(u_n)|}\right\|_{L^{p'(x)}(B_R\times(0,T))}\times \\
&\left\|\nabla T_k(u)_\nu\chi_{\{|u_n|>k\}}\right\|_{L^{p(x)}(B_R\times(0,T))},
\end{aligned}
$$

the first quantity is bounded by the hypothesis on K_1 and by using again Corollary (1.2.1) and Remark (5.3.2) (while $\lambda_0>\dfrac{dp'_+}{\alpha_1}$). In the second quantity the function $|\nabla T_k(u)_\nu|^{p(x)}\chi_{\{|u_n|>k\}}$ converges strongly in L^1 (as n and then ν go to ∞) to $|\nabla T_k(u)|^{p(x)}\chi_{\{|u|>k\}}\equiv 0$. Therefore we have shown that :

$$
A=A_1+\omega(\nu,n).
$$

Let us now examine the two terms C and E together. By employing assumption (5.2.6), and since $sign(u_n)\varphi(T_k(u_n)-T_k(u)_\nu)=|\varphi(T_k(u_n)-T_k(u)_\nu)|$ where $\{u_n>k\}$ one has :

$$
\begin{aligned}
C+E \le&\frac{d}{\alpha_1}\iint_{Q_T} a(u_n,\nabla u_n)\nabla u_n|\varphi(T_k(u_n)-T_k(u)_\nu)|e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta \\
&-\frac{d}{\alpha_1}\iint_{\{u_n>k\}} a(u_n,\nabla u_n)\nabla u_n|\varphi(T_k(u_n)-T_k(u)_\nu)|e^{\frac{d}{\alpha_1}|G_k(u_n)|}\eta \\
\le&\frac{d}{\alpha_1}\iint_{\{u_n\le k\}} a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi(T_k(u_n)-T_k(u)_\nu)|\eta \\
=&\frac{d}{\alpha_1}\iint_{\{u_n\le k\}} [a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))] \\
&[\nabla T_k(u_n)-\nabla T_k(u)]\,|\varphi(T_k(u_n)-T_k(u)_\nu)|\eta
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{d}{\alpha_1}\iint_{\{u_n\le k\}}\left[a(T_k(u_n),\nabla T_k(u))\right]\left[\nabla T_k(u_n)-\nabla T_k(u)\right]|\varphi(T_k(u_n)-T_k(u)_\nu)|\eta\\
&+\frac{d}{\alpha_1}\iint_{\{u_n\le k\}} a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)|\varphi(T_k(u_n)-T_k(u)_\nu)|\eta\\
\le&\frac{d}{\alpha_1}\iint_{\{u_n\le k\}}\left[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))\right]\\
&\left[\nabla T_k(u_n)-\nabla T_k(u)\right]|\varphi(T_k(u_n)-T_k(u)_\nu)|\eta+\omega(\nu,n)\\
\le&\frac{d}{\alpha_1\lambda}\iint_{\{u_n\le k\}}\left[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))\right]\\
&\left[\nabla T_k(u_n)-\nabla T_k(u)\right]\varphi'(T_k(u_n)-T_k(u)_\nu)\eta+\omega(\nu,n),
\end{aligned}
$$

the last integral is less than a fraction of the term A_1, and can be canceled. This shows that $A_1 \le \omega(\nu,n)$, which implies also (since $\varphi'(s)\ge\lambda$)

$$
\iint_{\{u_n\le k\}}\left[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))\right]\left[\nabla T_k(u_n)-\nabla T_k(u)\right]\eta\le\omega(\nu,n).
$$

Therefore :

$$
\begin{aligned}
&\iint_{Q_T}\left[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))\right]\left[\nabla T_k(u_n)-\nabla T_k(u)\right]\eta\\
&\le\iint_{\{u_n>k\}} a(T_k(u_n),\nabla T_k(u))\nabla T_k(u)\eta+\omega(\nu,n)=\omega(\nu,n).
\end{aligned}
$$

By using the Lemma (5.2.1) it follows that :

$$
T_k(u_n)\to T_k(u) \text{ strongly in } V_0(Q_T^0) \text{ for every } k>0. \tag{5.4.7}
$$

Step 2 : Strong Convergence Of $e^{\frac{\lambda}{p_+}|u_n|}$ In $V_0(Q_T^0)$ For Every $\lambda<\lambda_0$

Since k is arbitrary, it follows from (5.4.7) that $\nabla u_n\to\nabla u$ a.e. in Q_T^0 (passing to a subsequence). Indeed, using Proposition (5.3.1), Remark (5.3.1) and Lemma (5.4.1) we deduce that, for every $\lambda<\lambda_0$,

$$
\nabla(e^{\frac{\lambda}{p_+}|u_n|})\rightharpoonup\nabla(e^{\frac{\lambda}{p_+}|u_n|}) \text{ weakly in } L^{p(x)}(Q_T^0;\mathbb{R}^N).
$$

Using Proposition (5.3.1) and the fact that $\lambda<\lambda_0$ we have :

$$
\iint_{Q_T^0} e^{\lambda|u_n|}|\nabla u_n|^{p(x)}=\iint_{Q_T^0\cap\{|u_n|\le k\}} e^{\lambda|u_n|}|\nabla u_n|^{p(x)}+\iint_{Q_T^0\cap\{|u_n|>k\}} e^{\lambda|u_n|}|\nabla u_n|^{p(x)}
$$

$$= I_1 + I_2,$$

from (5.4.7) we have for every $k > 0$:

$$I_1 \to \iint_{Q_T^0 \cap \{|u| \leq k\}} e^{\lambda |u|} |\nabla u|^{p(x)} \text{ as } n \to \infty,$$

and

$$\begin{aligned} I_2 &\leq e^{(\lambda - \lambda_0)k} \iint_{Q_T^0 \cap \{|u_n| > k\}} e^{\lambda_0 |u_n|} |\nabla u_n|^{p(x)} \\ &\leq e^{(\lambda-\lambda_0)k} C_{10}(\lambda) \to 0 \text{ as } k \to \infty, \end{aligned}$$

which gives that :

$$e^{\frac{\lambda}{p_+}|u_n|} \to e^{\frac{\lambda}{p_+}|u|} \text{ strongly in } V_0(Q_T^0). \tag{5.4.8}$$

Step 3 : Compactness Of The Nonlinearities.

In order to pass to the limit in the approximated equation in a very short way, we use the results of the previous steps ((5.4.8), (5.2.5), (5.2.8), (5.4.1)) and like in [8] we can easily show that :

$$\begin{gathered} H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u) \text{ strongly in } L^1(Q_T^0; \mathbb{R}^N) \text{ by using Vitali's theorem,} \\ a(u_n, \nabla u_n) \to a(u, \nabla u) \text{ strongly in } L^{p'(x)}(Q_T^0; \mathbb{R}^N), \\ \frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } D'(Q_T^0), \end{gathered}$$

and therefore, since, by the Assumptions on the sequence $u_{0,n}$, the initial condition is satisfied by u. We can now pass to the limit in (5.2.10), obtaining that u is a distributional solution of (5.2.1). The regularity stated in Theorem (5.4.1) for the solution u follows from Proposition (5.3.1).

5.5 Boundedness Of Solutions

In this section we will give some regularity on the solution of the problem 5.2.1 using an adaptation of a classical technique due to Stampacchia, which is based on the following Lemma (see [66]) :

Lemma 5.5.1 *Let ϕ be a non-negative, non-increasing function defined on the half-line $[k_0, \infty)$. Suppose that there exist positive constants A, μ,β, with $\beta > 1$, such that :*

$$\phi(h) \leq \frac{A}{(h-k)^{\mu}} \phi(k)^{\beta}$$

for every $h > k \geq k_0$. Then $\phi(k) = 0$ for every $k \geq k_1$, where

$$k_1 = k_0 + A^{1/\mu} 2^{\beta/(\beta-1)} \phi(k_0)^{(\beta-1)/\mu}$$

The result that we are going to prove is the following :

Theorem 5.5.1 *Suppose that* (5.2.3), (5.2.5)-(5.2.8) *and $||u_0||_{L^\infty(\Omega)} < \infty$ hold. Then for every solution u of* (5.2.1) *there exists a constant C depending on the data such that :*

$$\|u\|_{L^\infty(Q_T)} \leq C. \tag{5.5.1}$$

It will be useful, moreover, to define the set :

$$A_k(t) = \{x : |u(x,t)| > k\},$$

and the function :

$$\phi(k) = \sup_{\tau \in [0,T]} mes A_k(\tau).$$

Let us observe that in the a priori estimate we have proved that h is bounded function where :

$$h(\tau) = \int_\Omega \psi(G_k(u(\tau)))dx,$$

this estimate gives that $\phi(k)$ is a bounded function with u large enough for k and $k \geq k_0$ to be specified below.

Proof : We take $\varphi(G_k(u))\chi_{(0,\tau)}(t)$ as test function in (5.2.1), where φ is defined in (5.3.2) and for every $H > 0$, choosing k such that :

$$k \geq k_0 = \max\left(||u_0||_{L^\infty(\Omega)}, \frac{H}{\mu}\right),$$

then as before we have :

$$\int_\Omega \psi(G_k(u(\tau)))dx + (\alpha_1 - \frac{d}{\lambda_0}) \iint_Q |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + \mu \iint_Q |u||\varphi(G_k(u))|$$

$$\leq \iint_{Q\cap\{|f(t)|>H;|G_k(u)|>1\}} |f||\varphi(G_k(u))| + \iint_{Q\cap\{|f(t)|>H;|G_k(u)|\leq 1\}} |f||\varphi(G_k(u))|$$
$$+ \iint_{Q\cap\{|f(t)|\leq H\}} |f||\varphi(G_k(u))|,$$

where the function $\psi(s)$ is defined by (5.3.3). Since $\mu k_0 \geq H$, the last integral in the right-hand side is smaller than the last integral of the left-hand side.

Then :

$$\int_\Omega \psi(G_k(u(\tau)))dx + (\alpha_1 - \frac{d}{\lambda_0}) \iint_Q |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u))$$
$$\leq \iint_{Q\cap\{|f(t)|>H;|G_k(u)|>1\}} |f||\varphi(G_k(u))| + \iint_{Q\cap\{|f(t)|>H;|G_k(u)|\leq 1\}} |f||\varphi(G_k(u))|$$

Using the same process used to evaluate the term B in the a priori estimate and choosing H sufficiently large the first integral in the right-hand can be absorbed in the left-hand side and we get :

$$C_{11} \int_\Omega \psi(G_k(u(\tau)))dx + C_{12} \iint_Q |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u))$$
$$\leq \iint_{Q\cap\{|f(t)|>H;|G_k(u)|\leq 1;|\nabla G_k(u)|\leq 1\}} |f||\varphi(G_k(u))|$$
$$+ \iint_{Q\cap\{|f(t)|>H;|G_k(u)|\leq 1;|\nabla G_k(u)|>1\}} |f||\varphi(G_k(u))|.$$

Let us observe that :

$$|\varphi(s)| \leq C_{13} (\phi_{p(x)})^{\frac{p(x)}{r'_+}} \text{ for every s such that } |s| \leq 1, \tag{5.5.2}$$

where $\phi_{p(x)}$ is defined in (5.3.8), the inequality (5.5.2) remains true if we replace $p(x)$ with p_+ or p_- with an adequate constant.

Thanks to Hölder inequality and to the assumption (5.2.3) and using (5.5.2) one has :

$$C_{11} \sup_{\tau\in[0,T]} \int_\Omega \psi(G_k(u(\tau)))dx + C_{12} \int_0^T \int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u))$$
$$\leq C_{14} \left\{ \int_0^T \| \phi_{p_+}(G_k(u))^{p_+} \|_{L^{\frac{p_+^*}{p_+}}(\{|\nabla G_k(u)|\leq 1\})} dt \right\}^{\frac{1}{r'_+}} \sup_{\tau\in[0,T]} (mes A_k(\tau))^{\frac{1}{q'_-} - \frac{p_+}{p_+^* r'_+}}$$
$$+ C_{15} \left\{ \int_0^T \| \phi_{p_-}(G_k(u))^{p_-} \|_{L^{\frac{p_-^*}{p_-}}(\{|\nabla G_k(u)|>1\})} dt \right\}^{\frac{1}{r'_+}} \sup_{\tau\in[0,T]} (mes A_k(\tau))^{\frac{1}{q'_-} - \frac{p_-}{p_-^* r'_+}}.$$

Therefore, using Sobolev's and Young's inequalities we obtain, for every $\epsilon > 0$,

$$C_{11} \sup_{\tau\in[0,T]} \int_\Omega \psi(G_k(u(\tau)))dx + C_{12} \int_0^T \int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u))$$

$$\leq \frac{\epsilon}{2}\int_0^T\int_{\{|\nabla G_k(u)|\leq 1\}}|\nabla G_k(u)|^{p_+}\varphi'(G_k(u)) + C_{16}(\epsilon)\sup_{\tau\in[0,T]}(mesA_k(\tau))^{\frac{r_-}{q'_-}-\frac{p_+r_-}{p^*_+r'_+}}$$

$$+\frac{\epsilon}{2}\int_0^T\int_{\{|\nabla G_k(u)|>1\}}|\nabla G_k(u)|^{p_-}\varphi'(G_k(u)) + C_{17}(\epsilon)\sup_{\tau\in[0,T]}(mesA_k(\tau))^{\frac{r_-}{q'_-}-\frac{p_-r_-}{p^*_-r'_+}},$$

choosing ϵ small enough, we can write :

$$C_{12}\sup_{\tau\in[0,T]}\int_\Omega \psi(G_k(u(\tau)))dx$$
$$\leq C_{18}(\epsilon)\sup_{\tau\in[0,T]}(mesA_k(\tau))^{\frac{r_-}{q'_-}-\frac{p_+r_-}{p^*_+r'_+}} + C_{19}(\epsilon)\sup_{\tau\in[0,T]}(mesA_k(\tau))^{\frac{r_-}{q'_-}-\frac{p_-r_-}{p^*_-r'_+}}.$$

On the other hand, it is easy to confirm that there exist two positive constants ϑ and C_{20} such that $\psi(s)\geq C_{20}|s|^\vartheta$, then for every $h>k\geq k_0$ one has :

$$\int_{A_k(\tau)}\psi(G_k(u(\tau)))dx \geq C_{20}\int_{A_h(\tau)}|G_k(u(\tau))|^\vartheta dx \geq C_{20}(h-k)^\vartheta mesA_h(\tau).$$

Finally, we can write :

$$\sup_{\tau\in[0,T]} mesA_h(\tau)$$
$$\leq \frac{C_{21}}{(h-k)^\vartheta}\max\left\{\sup_{\tau\in[0,T]}(mesA_k(\tau))^{\frac{r_-}{q'_-}-\frac{p_+r_-}{p^*_+r'_+}}, \sup_{\tau\in[0,T]}(mesA_k(\tau))^{\frac{r_-}{q'_-}-\frac{p_-r_-}{p^*_-r'_+}}\right\}$$

It is easy to check that, under hypothesis (5.2.3), the two last exponents are greater than 1, therefore one can apply Lemma (5.5.1) to the function ϕ and get the result.

CONCLUSION AND PERSPECTIVE

In this thesis we have generalized various works of the unbounded domain either in the stationary case or in the case of evolution over time, we established the existence and the main regularity results (about exponential summability and boundedness of solutions), The scheme of the proofs of the main results consists in the following three steps :

— Prove that the truncated problem has a solution u_n.

— Find a priori estimates for the functions $exp(\lambda|u_n|)1$, where u_n is any solution to the approximate problem.

— Extract a weakly converging subsequence and try to pass to the limit in the weak formulation.

Currently, this work raises a number of questions that deserve to be explored further. For example :

— Uuniqueness of solutions.

— The framework of hyperbolic problems.

— Modification of the ordinary assumptions (monotony and coercivity).

— The framework of fractional problems.

— Most of the terms in those problems are controllable indicating that they have good growth, typically $|x|^{p(x)-1}$, It will be interesting to consider some terms having uncontrollable part.

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