UNIVERSITY OF SIDI MOHAMED BEN ABDALLAH

DOCTORAL THESIS

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# Study of some nonlinear elliptic and parabolic systems in different settings

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# Dedication

*"I dedicate this thesis to my parents, my mother Rabha Bouamar (known as Rabha Assou), my father Lahcen Balaadich, my sisters Noura and Siham, and my brothers Younes, Said and Zouhir."*

# Acknowledgment

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# List of Symbols

| | |
|---|---|
| $\forall$ | for all |
| $\exists$ | there exists |
| $\equiv$ | equivalent |
| $\sum$ | summation |
| $\mathbb{N}$ | set of natural numbers |
| $\mathbb{R}$ | set of real numbers |
| $\mathbb{R}^+$ | set of positive real numbers |
| $n$ | positive integer greater than or equals to 1 |
| $m$ | positive integer greater than or equals to 1 |
| $d$ | positive integer greater than or equals to 1 |
| $\mathbb{R}^d$ | Euclidean space of $d$-dimensional vectors |
| $a \otimes b$ | tensor product of two vectors $a, b \in \mathbb{R}^d$ |
| $\mathbb{M}^{m\times n}$ | real space of $m \times n$ matrices |
| $\xi : \eta$ | product of two matrices $\xi, \eta \in \mathbb{M}^{m\times n}$ |
| $\Omega$ | open bounded subset of $\mathbb{R}^n$ |
| $\partial\Omega$ | boudary of $\Omega$ |
| $\overline{\Omega}$ | closure of $\Omega$ (i.e., $\Omega$ plus its boundary) |
| $[t_0, T]$ | closed interval $t_0 \leq t \leq T$ in $\mathbb{R}$ |
| $Q$ | the time-space cylinder $\Omega \times (0, T)$ with $T \in (0, \infty)$ |
| $\partial Q$ | boundary of $Q$ |
| $C(\Omega)$ | continuous functions from $\Omega$ to $\mathbb{R}$ |
| $v : \Omega \to \mathbb{R}^m$ | vector-valued function |

| | |
|---|---|
| $\nabla v$ | gradient of $v$ |
| $Dv$ | symmetric part of $\nabla v$ (i.e., $Dv = 1/2(\nabla v + (\nabla v)^t)$) |
| $C(\Omega;\mathbb{R}^m)$ | continuous functions from $\Omega$ to $\mathbb{R}^m$ |
| $C_0(\Omega;\mathbb{R}^m)$ | continuous functions $g : \Omega \to \mathbb{R}^m$ such that $\lim_{\lvert\lambda\rvert\to\infty} g(\lambda) = 0$ |
| $\mathcal{M}(\Omega;\mathbb{R}^m)$ | space of signed Radon measures (dual space of $C_0(\Omega;\mathbb{R}^m)$) |
| $C^\infty(\Omega;\mathbb{R}^m)$ | infinitely differentiable functions on $\Omega$ |
| $C_0^\infty(\Omega;\mathbb{R}^m)$ | infinitely differentiable functions with compact support on $\Omega$ (denoted also by $D(\Omega;\mathbb{R}^m)$) |
| $p$ | real number such that $1 \leq p \leq \infty$ |
| $p'$ | the Hölder conjugate of $p$ |
| $L^p(\Omega;\mathbb{R}^m)$ | $L^p$-space of equivalent classes of mappings from $\Omega$ to $\mathbb{R}^m$, $\int_\Omega \lvert v(x)\rvert^p dx < \infty$ and $1 \leq p < \infty$ |
| $L^{p'}(\Omega;\mathbb{R}^m)$ | the dual of $L^p(\Omega;\mathbb{R}^m)$ |
| $L^\infty(\Omega;\mathbb{R}^m)$ | essentially bounded $\mathbb{R}^m$-valued measurable functions $v$ on $\Omega$ |
| $W^{1,p}(\Omega;\mathbb{R}^m)$ | usual Sobolev space |
| $W_0^{1,p}(\Omega;\mathbb{R}^m)$ | closure of $C_0^\infty(\Omega;\mathbb{R}^m)$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ (i.e. w.r.t. the norm $\lVert.\rVert_{W^{1,p}}$) |
| $W^{-1,p'}(\Omega;\mathbb{R}^m)$ | dual space of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ |
| $p(.)$ | measurable function (variable exponent) |
| $p^+$ | essential sup of $p(.)$ |
| $p^-$ | essential inf of $p(.)$ |
| $p'(.)$ | Sobolev conjugate of $p(.)$ |
| $L^{p(.)}(\Omega;\mathbb{R}^m)$ | variable exponent Lebesgue space |
| $W^{1,p(.)}(\Omega;\mathbb{R}^m)$ | variable exponent Sobolev space |
| $W_0^{1,p(.)}(\Omega;\mathbb{R}^m)$ | the closure of $C_0^\infty(\Omega;\mathbb{R}^m)$ in $W^{1,p(.)}(\Omega;\mathbb{R}^m)$ |
| $M : \mathbb{R}^+ \to \mathbb{R}^+$ | isotropic N-function |
| $\overline{M} : \mathbb{R}^+ \to \mathbb{R}^+$ | the conjugate function of $M$ |
| $\mathcal{L}_M(\Omega;\mathbb{R}^m)$ | the Orlicz class of measurable functions $v : \Omega \to \mathbb{R}^m$ such that $\int_\Omega M(\lvert v(x)\rvert)dx < \infty$ |

| | |
|---|---|
| $L_M(\Omega;\mathbb{R}^m)$ | the Orlicz space, i.e., the set of measurable functions $v$ such that $\int_\Omega M(\frac{\lvert v(x)\rvert}{\beta})dx < \infty$ for some $\beta > 0$ |
| $L_{\overline{M}}(\Omega;\mathbb{R}^m)$ | the dual of $L_M(\Omega;\mathbb{R}^m)$ |
| $W^1L_M(\Omega;\mathbb{R}^m)$ | the homogeneous Orlicz-Sobolev space |
| $W^1_0L_M(\Omega;\mathbb{R}^m)$ | closure of $C_0^\infty(\Omega;\mathbb{R}^m)$ in $W^1L_M(\Omega;\mathbb{R}^m)$ |
| $W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m)$ | dual space of $W^1_0L_M(\Omega;\mathbb{R}^m)$ |
| $W^{1,x}_0L_M(Q;\mathbb{R}^m)$ | the inhomogeneous Orlicz-Sobolev space |
| $W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$ | dual space of $W^{1,x}_0L_M(Q;\mathbb{R}^m)$ |
| a.e. | almost everywhere |
| $\rightharpoonup$ | weak convergence |
| $\hookrightarrow$ | continuous embedding |
| $\hookrightarrow\hookrightarrow$ | compact embedding |
| $\delta_v$ | the Dirac mass at $v$ |
| $\nu_x : \Omega \to \mathcal{M}$ | Borel probability measure, for $x \in \Omega$ |
| $\langle \nu_x, \varphi \rangle$ | the integral $\int_{\mathbb{R}^m} \varphi(\lambda)d\nu_x(\lambda)$ for $\varphi \in C_0(\mathbb{R}^m)$ |
| $\langle \nu_x, id \rangle$ | the barycenter of $\nu = \{\nu_x\}_{x\in\Omega}$ |
| $\lVert \nu_x \rVert_{\mathcal{M}} = 1$ | $\nu_x$ is a probability measure |

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# Contents

UNIVERSITY OF SIDI MOHAMED BEN ABDALLAH

# *Abstract*

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Doctor of Philosophy

**Study of some nonlinear elliptic and parabolic systems in different settings**

by Farah BALAADICH

In the following monograph we deal with the questions from existence and uniqueness theory to problems of quasilinear elliptic and parabolic systems in divergence form in different settings.

The theory of Young measures and Galerkin method are the basic arguments used through the analysis of this work. Moreover, we allow our functional operators to satisfy different kind of weak (mild) monotonicity assumptions.

Let $n \geq 2$ be an integer and $\Omega$ be a bounded open subset of $\mathbb{R}^n$. By $Q$ we denote the time-space cylinder $\Omega \times (0, T)$ for a given time $T > 0$, and $\mathbb{M}^{m\times n}$ stands for the real space of $m \times n$ matrices equipped with the inner product $\xi : \eta = \sum_{i,j} \xi_{ij}\eta_{ij}$ (for $\xi, \eta \in \mathbb{M}^{m\times n}$). Throughout this text, $u : \Omega \to \mathbb{R}^m$ is a vector-valued function and $Du$ is the symmetric part of the gradient $\nabla u$, i.e., $Du = 1/2(\nabla u + (\nabla u)^t)$.

Consider first the following quasilinear elliptic system given in a generic form:

$$\begin{cases} -\operatorname{div} \sigma(x, u, Du) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{0.0.1}$$

The first part of this thesis is devoted to study the existence of weak solutions to certain elliptic systems similar to (0.0.4) in the setting of Sobolev spaces $W_0^{1,p}(\Omega; \mathbb{R}^m)$ and the main $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ satisfy some conditions of type Leray-Lions. To this end, we will prove the existence of weak solutions to a strongly quasilinear parabolic system given in the form

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} \sigma(x, t, u, Du) + H(x, t, u, Du) &= f \quad \text{in } Q, \\ u(x, t) &= 0 \quad \text{on } \partial Q, \\ u(x, 0) &= u_0(.) \quad \text{in } \Omega, \end{cases} \tag{0.0.2}$$

where $f \in L^{p'}(0, T; W^{-1,p'}(\Omega; \mathbb{R}^m))$ and $H : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$. The needed result is proved under some conditions on the functions $\sigma$ and $H$.

The second part of this thesis concerns the case when the exponent $p$ is not anymore constant, but depends on $x$, i.e., $p \equiv p(x)$. Several types of quasilinear elliptic systems similar to (0.0.4) will be then considered. Some of which are the extension of those earlier treated in the first part to the case of variable exponent Sobolev spaces

$$W^{1,p(x)}(\Omega;\mathbb{R}^m) = \Big\{u \in L^{p(x)}(\Omega;\mathbb{R}^m): \ Du \in L^{p(x)}(\Omega;\mathbb{M}^{m\times n})\Big\}.$$

Tha aim of the third part is to extend the classical growth and coercivity assumptions (i.e., polynomial conditions) stated in [85] to general conditions phrased by N-functions which define the Orlicz-Sobolev space $W_0^1L_M(\Omega;\mathbb{R}^m)$. We will show the existence of weak solutions for the quasilinear elliptic system (0.0.4) for $f \in W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m)$. Furthermore, we will also allow $f$ to depend on the unknown $u: \Omega \to \mathbb{R}^m$ and $Du$, and prove the needed result under further conditions on $f(x,u,Du)$.

The fourth part deals with the existence and uniqueness results for (0.0.5) when $\sigma$ is independent of $u$, $H \equiv 0$ and $f$ belongs to $W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$. Furthermore, we treat the case when $f$ belongs to dual of

$$X(Q) := \Big\{u \in L^2(Q;\mathbb{R}^m) / \ Du \in L_M(Q;\mathbb{M}^{m\times n}); \ u(t) := u(.,t) \in W_0^1L_M(\Omega;\mathbb{R}^m) \\ \text{a.e. } t \in [0,T]\Big\}.$$

Last but not least, the final part of this thesis deals with the existence result for the steady flows of quasi-Newtonian fluids (i.e., Stokes system) associated to (0.0.4) given in the form

$$-\operatorname{div}\sigma(x,u,Du) + u.\nabla u + \nabla\pi = f \quad \text{in } \Omega, \tag{0.0.3}$$

where $\pi : \Omega \to \mathbb{R}$ denotes the pressure and $u.\nabla u$ is the convective term. The presence of this term in the main problem (0.0.6) allows us to define a suitable Orlicz-Sobolev space with free divergence. At the end of this last part, we extend the previous result to the evolutionary case and prove the existence of weak solutions for $f \in W^{-1,x}_{\text{div}}L_{\overline{M}}(Q;\mathbb{R}^m)$ by means of Young measures and weak monotonicity assumptions.

The main tools used in all parts of this thesis are Galerkin method to construct the approximating solutions and the theory of Young measures, which allow the identification of weak limits of functionals and operators, to pass to the limit in the approximating equations.

To the best of our knowledge, the study undertaken in this thesis is in some sense pioneering since both classes elliptic and parabolic partial differential equations have not been the object of previous investigation.

As we know, the research on Young measures generated by sequences in variable exponent Sobolev spaces and Orlicz-Sobolev spaces is still in exploration. The results obtained are original and enrich the theory of existence for such problems by means of the Young measures in different settings.

**Keywords :** Nonlineaire elliptic and parabolic systems; Variable exponents; Orlicz-Sobolev spaces; Galerkin method; Young measures

# Résumé

Dans la monographie suivante, nous abordons les questions de la théorie de l'existence et de l'unicité aux problèmes des systèmes elliptiques et paraboliques quasi-linéaires sous forme de divergence dans différents espaces.

La théorie des mesures de Young et la méthode de Galerkin sont les arguments de base utilisés à travers l'analyse de ce travail. De plus, nous permettons à nos opérateurs fonctionnels de satisfaire différents types d'hypothèses de monotonie faible (légère).

Soit $n \geq 2$ un entier et $\Omega$ un sous-ensemble ouvert borné de $\mathbb{R}^n$. Par $Q$, nous désignons le cylindre d'espace-temps $\Omega \times (0, T)$ pour un temps donné $T > 0$, et $\mathbb{M}^{m\times n}$ représente l'espace réel de $m \times n$-matrices équipées du produit intérieur $\xi : \eta = \sum_{i,j} \xi_{ij}\eta_{ij}$ (pour $\xi, \eta \in \mathbb{M}^{m\times n}$). Dans tout ce texte, $u : \Omega \to \mathbb{R}^m$ est une fonction vectorielle et $Du$ est la partie symétrique du gradient $\nabla u$, c'est-à-dire $Du = 1/2(\nabla u + (\nabla u)^t)$.

Considérons d'abord le système elliptique quasi-linéaire suivant donné sous une forme générique:

$$\begin{cases} -\operatorname{div} \sigma(x, u, Du) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{0.0.4}$$

La première partie de cette thèse est consacrée à l'étude de l'existence de solutions faibles de certains systèmes elliptiques similaires à (0.0.4) dans le cadre d'espaces de Sobolev $W_0^{1,p}(\Omega; \mathbb{R}^m)$ et notre fonction $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ satisfait certaines conditions de type Leray-Lions. À cette fin, nous prouverons l'existence de solutions faibles à un système parabolique fortement non-linéaire donné sous la forme

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} \sigma(x, t, u, Du) + H(x, t, u, Du) &= f \quad \text{in } Q, \\ u(x, t) &= 0 \quad \text{on } \partial Q, \\ u(x, 0) &= u_0(.) \quad \text{in } \Omega, \end{cases} \tag{0.0.5}$$

où $f \in L^{p'}(0, T; W^{-1,p'}(\Omega; \mathbb{R}^m))$ et $H : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$. Le résultat recherché est prouvé sous certaines conditions sur les fonctions $\sigma$ et $H$.

La seconde partie de cette thèse concerne le cas où l'exposant $p$ n'est plus constant, mais dépend de $x$, c'est-à-dire $p \equiv p(x)$. Plusieurs types de systèmes elliptiques quasi-linéaires similaires à (0.0.4) seront alors considérés. Certains d'entre eux sont l'extension de ceux précédemment traités dans la première partie au cas des espaces de Sobolev à exposants variables

$$W^{1,p(x)}(\Omega;\mathbb{R}^m) = \Big\{u \in L^{p(x)}(\Omega;\mathbb{R}^m) : \ Du \in L^{p(x)}(\Omega;\mathbb{M}^{m\times n})\Big\}.$$

L'objectif de la troisième partie est d'étendre les hypothèses classiques de croissance et de coercivité (c'est-à-dire les conditions polynomiales) énoncées dans [85] aux conditions générales formulées par des N-fonctions qui définissent l'espace d'Orlicz-Sobolev $W_0^1 L_M(\Omega;\mathbb{R}^m)$. Nous montrerons l'existence de solutions faibles pour le système elliptique quasi-linéaire (0.0.4) pour $f \in W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m)$. De plus, nous allons également permettre à $f$ de dépendre de l'inconnu $u : \Omega \to \mathbb{R}^m$ et $Du$, et prouver le résultat nécessaire sous d'autres conditions sur $f(x,u,Du)$.

La quatrième partie traite des résultats d'existence et d'unicité pour (0.0.5) lorsque $\sigma$ est indépendante de $u$, $H \equiv 0$ et $f$ appartient à $W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$. De plus, nous traitons le cas où $f$ appartient au dual de

$$X(Q) := \Big\{u \in L^2(Q;\mathbb{R}^m) /\, Du \in L_M(Q;\mathbb{M}^{m\times n});\ u(t) := u(.,t) \in W_0^1 L_M(\Omega;\mathbb{R}^m) \\ \text{a.e. } t \in [0,T]\Big\}.$$

Enfin et surtout, la dernière partie de cette thèse traite du résultat d'existence pour les écoulements réguliers de fluides quasi-newtoniens (c'est-à-dire, les systèmes de Stokes) associés à (0.0.4) donné sous la forme

$$-\operatorname{div}\sigma(x,u,Du) + u.\nabla u + \nabla\pi = f \quad \text{in } \Omega, \tag{0.0.6}$$

où $\pi : \Omega \to \mathbb{R}$ désigne la pression et $u.\nabla u$ est le terme convectif. La présence de ce terme dans le problème principal (0.0.6) nous permet de définir un espace d'Orlicz-Sobolev adapté de divergence nul. A la fin de cette dernière partie, nous étendons le résultat précédent au cas évolutif et prouvons l'existence de solutions faibles pour $f \in W^{-1,x}_{\text{div}}L_{\overline{M}}(Q;\mathbb{R}^m)$ au moyen de mesures de Young et d'hypothèses de monotonie faible.

Les principaux outils utilisés dans toutes les parties de cette thèse sont la méthode de Galerkin pour construire les solutions approximatives et la théorie des mesures de Young, qui permettent d'identifier les limites faibles des fonctionnelles et des opérateurs, pour passer à la limite dans les équations approximatives.

À notre connaissance, l'étude considérée dans cette thèse est en quelque sorte pionnière puisque les deux classes d'équations aux dérivées partielles elliptiques et paraboliques n'ont pas fait l'objet de recherches antérieures.

Comme nous le savons, la recherche sur les mesures de Young générées par des séquences dans les espaces de Sobolev à exposants variables et les espaces d'Orlicz-Sobolev est toujours en cours d'exploration. Les résultats obtenus sont originaux et enrichissent la théorie de l'existence de tels problèmes au moyen des mesures Young dans différents espaces.

**Mots clés :** Systèmes non-linéaires elliptiques et paraboliques; Exposants variables; Espace d'Orlicz-Sobolev; Méthode de Galerkin; Mesures de Young

# Chapter 1

# Introduction

## 1.1 Main objectives

In this thesis, we deal with the existence (and uniqueness) results for some quasilinear elliptic and parabolic of partial differential equations (PDEs) in divergence form. The needed results are obtained by means of the Young measures as technical tools and weak monotonicity assumptions. The growth and coercivity conditions will be phrased in classical Sobolev spaces, variable exponent Sobolev spaces and in terms of N-functions $M : \mathbb{R}^+ \to \mathbb{R}^+$ (a continuous, convex, superlinear, nonnegative function, which will be defined in Chap. 2), which define the Orlicz space $L_M(\Omega; \mathbb{R}^m)$. Here, the solutions in question are vector-valued functions $u : \Omega \to \mathbb{R}^m$, $m \in \mathbb{N}^*$ and $\Omega$ is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. The solvability of the (main) corresponding quasilinear elliptic and parabolic systems follow the Galerkin method, which permits the construction of approximate solutions, and throughout the theory of Young measures.

As we know, weak convergence is a basic tool of modern nonlinear analysis, because it enjoys the same compactness properties that convergence in finite dimensional spaces does. Nonetheless, this notion (i.e., weak convergence) does not behave as one would desire with respect to nonlinear functionals and operators. Young measures are powerful tools to understand and to control these difficulties (c.f. [68]). An example illustrating this situation can be found in [124, Chapter 4].

The aim of this text is to prove the existence of weak solutions for quasilinear elliptic/parabolic and Navier-Stokes problems under standard and nonstandard growth conditions and mild monotonicity assumptions. To explain the motivations of this study, we start with the following part 1.

### 1.1.1 Main results of Part 1

Let us begin by considering the following quasilinear elliptic system of Dirichlet type

$$\begin{cases} -\operatorname{div} A(u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Omega, \end{cases} \tag{1.1.1}$$

where $A : W_0^{1,p}(\Omega;\mathbb{R}^m) \to W^{-1,p'}(\Omega;\mathbb{R}^m)$ is an operator of Leray-Lions type given in the form $A(u) = \sigma(x,u,Du)$, and $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function (i.e., measurable w.r.t. $x \in \Omega$ and continuous w.r.t. other variables) such that

$$|\sigma(x,s,\xi)| \leq d_1(x) + |s|^{p-1} + |\xi|^{p-1}, \quad d_1(x) \in L^{p'}(\Omega), \tag{1.1.2}$$

$$\sigma(x,s,\xi) : \xi \geq \alpha|\xi|^p, \quad \alpha > 0, \tag{1.1.3}$$

$$\big(\sigma(x,s,\xi) - \sigma(x,s,\eta)\big) : (\xi - \eta) > 0, \tag{1.1.4}$$

for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}^m$, $\forall \xi, \eta \in \mathbb{M}^{m\times n}$. Here $\mathbb{M}^{m\times n}$ stands for the real space of $m \times n$ matrices. In (1.1.1), the source term $f$ belongs to $W^{-1,p'}(\Omega;\mathbb{R}^m)$ the dual space of $W_0^{1,p}(\Omega;\mathbb{R}^m)$. The classical monotone operator methods developed by Višik [131], Minty [111], Browder [44], Brézis [42], Lions [105] and others imply that problem (1.1.1) has at least one weak solution $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$. If one consider (1.1.1) with $\sigma$ equal to the derivative with respect to $\xi \in \mathbb{M}^{m\times n}$ of a (potential) function $W : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$, i.e., $\sigma(x,u,\xi) = \big(\partial W/\partial \xi\big)(x,u,\xi) := D_\xi W(x,u,\xi)$, then the above classical methods of monotone operators do not apply if one assumes that $\xi \mapsto W$ is only convex and $C^1$. However, such methods apply if $W$ is strictly convex, which implies the strict monotonicity of $\sigma$. To overcome this situation, Hungerbühler [85] proposed the use of Galerkin method and proceed the proof differently through the theory of Young measures. This notion of Young measures allow to prove the needed compactness of approximating solutions. In that work of Hungerbühler, the author used large monotonicity condition :

$$\big(\sigma(x,s,\xi)-\sigma(x,s,\eta)\big):(\xi-\eta)\geq 0 \tag{1.1.5}$$

instead of (1.1.4). Furthermore, mild monotonicity assmuptions for $\sigma$ were presented (see (H1)-(H3) bellow).

In this part, the first model we will treat is a generalized $p$-Laplacian system associated to (1.1.1) given in the form $A(u)\equiv|Du-\Theta(u)|^{p-2}(Du-\Theta(u))$ with $1<p<\infty$ and $\Theta:\mathbb{R}^m\to\mathbb{M}^{m\times n}$ is a continuous function satisfing $\Theta(0)=0$ and $|\Theta(s)-\Theta(s')|\leq c|s-s'|$ for all $s,s'\in\mathbb{R}^m$ (cf [18]). The needed result follows by applying the Galerkin method and the theory of Young measures.

The second model is when $A(u)=a(x,Du)+\phi(u)$, where $a:\Omega\times\mathbb{M}^{m\times n}\to\mathbb{M}^{m\times n}$ is a function allowed to satisfy some conditions similar to (1.1.2), (1.1.3) and (1.1.5), and $\phi:\mathbb{R}^m\to\mathbb{M}^{m\times n}$ is linear and continuous such that $|\phi(s)|\leq\alpha_0$, where $0<\alpha_0<\alpha$ ($\alpha$ is the constant of the coercivity condition of $a$) (see [27]). Under mild monotonicity assumptions on the function $a$, the needed result follows also by relying on the theory of Young measures.

The aim of the third model is to extend the first one to a general form given by $A(u)\equiv A(x,Du-\Theta(u))$ (cf. [29]). Here $A:\Omega\times\mathbb{M}^{m\times n}\to\mathbb{M}^{m\times n}$ is assumed to satisfy some conditions stated in the sense of (1.1.2), (1.1.3) and (1.1.5), and other mild monotonicity assumptions.

Let $T>0$ be a given final time. Last but not least, the (last) fourth model of Part 1 is a strongly quasilinear parabolic system (cf. [24]) given in the form

$$\begin{cases}\frac{\partial u}{\partial t}-\operatorname{div}A(u)+H(x,t,u,Du) &= f \quad \text{in } Q=\Omega\times(0,T),\\ \qquad\qquad\qquad\qquad u(x,t) &= 0 \quad \text{on } \partial Q=\partial\Omega\times(0,T),\\ \qquad\qquad\qquad\qquad u(x,.) &= u_0(x) \quad \text{in } \Omega,\end{cases} \tag{1.1.6}$$

where $A(u)\equiv\sigma(x,t,u,Du)$ and $f\in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$. The problem (1.1.6) was investigated in [86] with $H\equiv 0$, where the author proved the existence of a weak solution under standard polynomial conditions

$$|\sigma(x,t,s,\xi)| \leq d_1(x,t) + |s|^{p-1} + |\xi|^{p-1}, \quad d_1(x,t) \in L^{p'}(Q),$$

$$\sigma(x,t,s,\xi) : \xi \geq \alpha|\xi|^p - d_2(x,t)|s|^q - d_3(x,t),$$

where $\alpha > 0$, $d_2(x,t) \in L^{(\frac{p}{q})'}(Q)$, $d_3(x,t) \in L^1(Q)$ and $0 < q < p$. By means of the Young measures, we will prove the existence of weak solutions of (1.1.6) under some conditions on $\sigma$ and $H : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$.

### 1.1.2 Main results of Part 2

The main objective of this part is to establish the existence of weak solutions of some quasilinear elliptic systems in variable exponent Sobolev space $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, where $p(x)$ is a measurable function such that $1 < p^- \leq p(x) \leq p^+ < \infty$ (see Chapter 2).

The aim of the first model here is to extend the first model stated in Part 1, with $f \in W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$, to a generalized $p(x)$-Laplacian system in variable exponent Sobolev space by using also the theory of Young measures and Galerkin method (cf. [22]). Note that, in these two models we do not need such Leray-Lions conditions.

The next model is an extension of the previous $p(x)$-Laplacian system to a model arized in physics, given in the form

$$-\text{div}\big(|Du - \Theta(u)|^{p(x,)-2}(Du - \Theta(u))\big) = v(x) + f(x,u) + \text{div}\, g(x,u),$$

which corresponds to a diffusion problem with a source term $v$ in a moving and dissolving substance, the motion is described by $g$ and the dissolution by $f$. Here $v \in W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ (see [30]). Under some conditions on the functions $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ and $g : \Omega \times \mathbb{R}^m \to \mathbb{M}^{m\times n}$ the needed result follows again by Young measure techniques. Moreover, we allow $f$ to depend also on $Du$.

If the datum $f$ in (1.1.1) belongs to Sobolev space of variable exponent $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$, problem (1.1.1) has been treated by Fu and Yang [73], by using the concept of Young measures and mild monotonicity assumptions on $\sigma$, and generalize the result of Hungerbühler [85].

The last model of Part 2 will be concerned to a class of quasilinear elliptic system (1.1.1) with $A(u) = a(|Du|)Du$ and $f \in W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$. Here $a$ is a $C^1$-function defined from $[0,\infty)$ into it self and satisfy some growth/coercivity conditions phrased in variable exponent (cf. [28]) and $-1 \leq \inf_{t>0}\frac{ta'(t)}{a(t)} \leq \sup_{t>0}\frac{ta'(t)}{a(t)} < \infty$. This last condition allows to get the monotonicity of $a$ which will serve us to prove the needed result under some mild monotonicity assumptions.

### 1.1.3 Main results of Part 3

If the growth conditons of $\sigma$ in (1.1.1) are not polynomial, then $W_0^{1,p}(\Omega;\mathbb{R}^m)$ and $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ can not capture the situation. These drawbacks force and motivate us to find a suitable space in order to ensure the existence result. Therefore we consider conditions phrased in terms of N-functions which define the main functional spaces. See [6, 75, 78] for more details. We then formulate growth and coercivity conditions in the following way:

$$|\sigma(x,s,\xi)| \leq d_1(x) + \overline{M}^{-1}P(|s|) + \overline{M}^{-1}M(|\xi|), \quad d_1(x) \in L_{\overline{M}}(\Omega), \tag{1.1.7}$$

$$\sigma(x,s,\xi) : \xi \geq \alpha M(|\xi|) - d_2(x), \quad \alpha > 0,\ d_2(x) \in L^1(\Omega), \tag{1.1.8}$$

where $M$ and $P$ are two N-functions satisfy some conditions. The suitable space to solve (1.1.1) under the conditions (1.1.7), (1.1.8) and (1.1.5) is Orlicz-Sobolev space $W_0^1L_M(\Omega;\mathbb{R}^m)$ for $f \in W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m)$. That is the aim of the first model of this part which generalize the result of [85] and [73]. We define weak solutions for (1.1.1) as follows.

**Definition 1.1.1.** *A weak solution to (1.1.1) is a function $u \in W_0^1L_M(\Omega;\mathbb{R}^m)$ such that*

$$\int_\Omega \sigma(x,u,Du) : D\varphi dx = \langle f, \varphi\rangle$$

*holds for all $\varphi \in W_0^1L_M(\Omega;\mathbb{R}^m)$.*
*Here $\langle .,.\rangle$ is the duality pairing of $\left(W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m), W_0^1L_M(\Omega;\mathbb{R}^m)\right)$*

The main result here can be stated as follows (see [20]):

**Theorem 1.1.1.** *Assume that* (1.1.5)*,* (1.1.7)*,* (1.1.8) *and (H1)-(H3) hold. Then problem* (1.1.1) *has at least one weak solution in the sense of Definition 1.1.1, for every* $f \in W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m)$*.*

The last model of Part 3 is devoted to extend the above result to the case that $f$ depends also on the uknown $u$ and the symmetric part of the gradient $\nabla u$, i.e., $Du$. In this case, we assume further

$$|f(x,s,\xi)| \leq d_3(x) + \overline{M}^{-1}P(|s|) + \overline{M}^{-1}M(|\xi|), \quad d_3(x) \in L_{\overline{M}}(\Omega), \tag{1.1.9}$$

and

$$\sigma(x,s,\xi) : \xi - f(x,s,\xi).s \geq \alpha M(|\xi|) - d_2(x), \quad \alpha > 0,\ d_2(x) \in L^1(\Omega) \tag{1.1.10}$$

instead of (1.1.8) to guarantee the coercivity of an energy functional throughout the Galerkin method (see [21]). In the models considered above, $\sigma$ satisfy one of the following mild monotonicity assumptions:

(H1) $\sigma$ is strictly monotone, i.e., $\sigma$ satisfy (1.1.5) and

$$\big(\sigma(x,s,\xi) - \sigma(x,s,\eta)\big) : (\xi - \eta) = 0 \text{ implies } \xi = \eta.$$

(H2) There exists a (potential) function $W : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,u,\xi) = (\partial W/\partial\xi)(x,u,\xi)$ and $\xi \to W(x,u,\xi)$ is convex and $C^1$.

(H3) $\sigma$ is strictly $M$-quasimonotone on $\mathbb{M}^{m\times n}$, i.e.,

$$\int_{\mathbb{M}^{m\times n}} \big(\sigma(x,u,\lambda) - \sigma(x,u,\overline{\lambda})\big) : (\lambda - \overline{\lambda}) d\nu_x(\lambda) > 0,$$

where $\overline{\lambda} = \langle \nu_x, id\rangle$, $\nu = \{\nu_x\}_{x\in\Omega}$ is any family of Young measures generated by a sequence in $L_M(\Omega)$ and not a Dirac measure for a.e. $x \in \Omega$.

Note that (H3) is weaker than typical strictly monotone conditions. It is phrased in integrand form and in term of Young measure $\nu_x$ generated by a sequence in $L_M(\Omega)$ (see Chap.2 for the definition of $\nu_x$).

It should be noted that, within the above two results, the N-function $M$ and its conjugate $\overline{M}$ are assumed to satisfy the so-called $\Delta_2$-condition (see Chapter 2). This condition, namely $\Delta_2$-condition remain valid through the rest of this thesis.

### 1.1.4 Main results of Part 4

In this part, we investigate the evolutionary case of (1.1.1), where $\sigma$ is independent of $u$. The main problem is given in the following generic form

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} \sigma(x,t,u,Du) &= f \quad \text{in } Q = \Omega \times (0,T), \\ u &= 0 \quad \text{on } \partial Q, \\ u(.,0) &= u_0(.) \quad \text{in } \Omega, \end{cases} \tag{1.1.11}$$

where $\Omega$ is a bounded open domain of $\mathbb{R}^n$ and $T > 0$. Here $f$ is taken in $W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$ the dual space of $W_0^{1,x}L_M(Q;\mathbb{R}^m)$. The first step in the proof of existence of a weak solution is the Galerkin approximation, then Young measures to prove the div-curl inequality which allow the passage to the limit in the approximating equations. We want here to extend (1.1.1) to the parabolic case (1.1.11), but with $\sigma$ independent of $u$ to guarantee the uniqueness result. The result of uniqueness follows by means of the condition of strictly monotone (H1) in the evolutionary case (cf. [19]).

The main objective of this part goes also in the sense to generalize the polynomial growth and coercivity conditions stated in the last model of Part 1 by considering conditions of type (1.1.7) and (1.1.8) (with time dependent). Problems of type (1.1.11) appear in several papers and by different methods, see for example [56, 61, 66, 76, 79, 99].

The last model that will be treated is adressed to the theory of existence and uniqueness of weak solutions for (1.1.11) when $f$ belongs to $X'(Q)$ the dual space of

$$X(Q) = \Big\{ u \in L^2(Q;\mathbb{R}^m) / \; Du \in L_M(\Omega;\mathbb{M}^{m\times n}); \; u(t) := u(.,t) \in W_0^1 L_M(\Omega;\mathbb{R}^m) \quad \text{a.e. } t \in [0,T] \Big\}.$$

The Hahn-Banach theorem implies that for $f \in X'(Q)$ there exist $f_0 \in L^2(Q;\mathbb{R}^m)$ and $F \in L_{\overline{M}}(Q;\mathbb{M}^{m\times n})$ such that $f = f_0 - \operatorname{div} F$ and

$$\langle f,\varphi\rangle_{X',X} = \int_Q f_0\varphi dxdt + \int_Q F : D\varphi dxdt, \quad \forall\varphi \in X(Q).$$

The space $X(Q)$ is reflexive and separable, and $C_0^\infty(Q;\mathbb{R}^m)$ is dense in $X(Q)$. In a similar way to the first model, we prove [23] the existence and uniqueness of weak solutions to (1.1.11) for every $f \in X'(Q)$ under (H1)-(H3) in time dependent. Here, we also used the concept of Young measures to define weak limit of $Du_k$ (constructed by Galerkin method) and to pass to the limit in the approximating equations. On the other side, instead of the conditions (1.1.5) and (H1)-(H3), we will consider (cf. [31]) the following strictly quasimonotone condition

$$\int_Q \big(\sigma(x,t,Du) - \sigma(x,t,Dv)\big) : (Du - Dv)dxdt \geq c\int_Q M(\gamma|Du - Dv|)dxdt,$$

where $c$ and $\gamma$ are positive constants. We prove that this condition implies (H3) under which the existence result to the previous two models is established. The problem (1.1.11) was investigated in [134] with $f = \operatorname{div} g$ and $g \in L^{p'(x)}(Q;\mathbb{M}^{m\times n})$, $p'(x) = p(x)/(p(x)-1)$ the conjugate variable exponent of $p(x)$. The authors used also the techniques of Young measures and mild monotonicity assumptions on $\sigma$. For more results by different methods, see [38, 58, 66, 99, 110].

### 1.1.5 Main results of Part 5

The mathematical analysis of steady flows of an incompressible homogeneous quasi-Newtonian viscous fluid associated to (0.0.4), is the main objective of this part. More precisely, let $u : \Omega \to \mathbb{R}^m$ be the velocity field, $\pi : \Omega \to \mathbb{R}$ the pressure, $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ the Cauchy stress tensor,

$$\begin{cases} -\operatorname{div}\sigma(x,u,Du) + u.\nabla u + \nabla\pi &= f \quad \text{in } \Omega,\\ \operatorname{div} u &= 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \Omega, \end{cases} \tag{1.1.12}$$

where $u.\nabla u$ is the convective term. We prove (cf. [25]), under the conditions (1.1.7), (1.1.8), (1.1.5) and (H1)-(H3), the existence of weak solutions for (1.1.12) in the Orlicz-Sobolev space of divergence free functions defined by

$$W^1_{0,\text{div}}L_M(Q;\mathbb{R}^m) = \Big\{u \in W^1_0 L_M(\Omega;\mathbb{R}^m) : \ \text{div}\, u = 0\Big\}.$$

This result is an extension of that of Arada and Squeira [12] to the non-standard conditions phrased by N-functions. To this end, we will treat the evolutionary case of (1.1.12) (cf. [26]). We show the existence of weak solutions using the ideas developped above, and extend the result of [64].

# Chapter 2

# Preliminaries

In the present chapter we introduce the notations and present some properties about variable exponent Sobolev spaces (see e.g. [69, 92, 122, 141]), Orlicz-Sobolev spaces (see e.g. [4, 93, 94, 61]) and Young measures (see e.g. [33, 84, 68, 106]) and references therein.

## 2.1 Variable exponent Lebesgue and Sobolev spaces

Let $\Omega$ be a bounded open domain of $\mathbb{R}^n$. We denote $C_+(\overline{\Omega})$ the following set

$$C_+(\overline{\Omega}) = \Big\{ p \in C(\overline{\Omega}) : \ p(x) > 1 \quad \text{for all } x \in \overline{\Omega} \Big\}.$$

Throughout this text,

$$p^- := \operatorname*{ess\,inf}_{x \in \Omega} p(x) \quad \text{and} \quad p^+ := \operatorname*{ess\,sup}_{x \in \Omega} p(x)$$

for every $p \in C_+(\overline{\Omega})$. We define the modular of a measurable function $u : \Omega \to \mathbb{R}^m$ by

$$\rho_{p(.)}(u) := \int_\Omega |u(x)|^{p(x)} dx.$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega;\mathbb{R}^m)$ is a Banach space that is the set of all measurable functions $u:\Omega\to\mathbb{R}^m$ such that its modular

$$\rho_{p(.)}(u)<+\infty$$

is finite, equipped with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega;\mathbb{R}^m)}:=\|u\|_{p(x)}=\inf\Big\{\beta>0:\ \rho_{p(.)}\Big(\frac{u}{\beta}\Big)\leq 1\Big\}.$$

Note that the generalized Lebesgue space $L^{p(x)}(\Omega;\mathbb{R}^m)$ is a kind of Musielak-Orlicz space. The space $C_0^\infty(\Omega;\mathbb{R}^m)$ denotes the space of $C^\infty$-functions with compact support in $\Omega$. If

$$1\leq p^-\leq p^+<\infty,$$

$L^{p(x)}(\Omega;\mathbb{R}^m)$ is separable and, in particular, $C_0^\infty(\Omega;\mathbb{R}^m)$ is dense in $L^{p(x)}(\Omega;\mathbb{R}^m)$. If we restrict $p(.)$ to satisfy

$$1<p^-\leq p^+<\infty,$$

then $L^{p(x)}(\Omega;\mathbb{R}^m)$ becomes reflexive, and its dual is given for $p'(x)=p(x)/(p(x)-1)$ by $L^{p'(x)}(\Omega;\mathbb{R}^m)$, where $p'(x)$ is the conjugate of $p(x)$. In these spaces, a version of Hölder's inequality

$$\int_\Omega uv dx\leq\Big(\frac{1}{p^-}+\frac{1}{p^+}\Big)\|u\|_{p(x)}\|v\|_{p'(x)}\leq 2\|u\|_{p(x)}\|v\|_{p'(x)}$$

is valid for $u\in L^{p(x)}(\Omega;\mathbb{R}^m)$ and $v\in L^{p'(x)}(\Omega;\mathbb{R}^m)$. For the relation between the modular $\rho_{p(.)}(.)$ and the norm $\|.\|_{p(x)}$, we recall the following properties: if $u_k,u\in L^{p(x)}(\Omega;\mathbb{R}^m)$ and $1<p^-\leq p^+<\infty$, then:

$$\text{if}\quad \|u\|_{p(x)}>1\Longrightarrow\|u\|_{p(x)}^{p^-}\leq\rho_{p(x)}(u)\leq\|u\|_{p(x)}^{p^+};$$

$$\text{if}\quad \|u\|_{p(x)}<1\Longrightarrow\|u\|_{p(x)}^{p^+}\leq\rho_{p(x)}(u)\leq\|u\|_{p(x)}^{p^-};$$

$$\|u_k\|_{p(x)}\to 0\quad(\text{resp.}+\infty)\Longleftrightarrow\rho_{p(x)}(u_k)\to 0\quad(\text{resp.}+\infty).$$

Let us first review some facts from linear algebra and matrix analysis (c.f. [83]). By

$\mathbb{M}^{m\times n}$ we mean the real space of $m \times n$ matrices. For a $m \times n$-matrix $\xi \in \mathbb{M}^{m\times n}$ we denote by $\xi_{ij}$ the element in the $i^{th}$ row and $j^{th}$ column $(i = 1, ..., m, j = 1, ..., n)$. The matrix space $\mathbb{M}^{m\times n}$ is endowed with the Frobenius matrix (inner) product

$$\xi : \eta = \sum_{i,j} \xi_{ij}\eta_{ij}, \quad \xi, \eta \in \mathbb{M}^{m\times n}.$$

This product induces the norm

$$|\xi| = \sqrt{\sum_{i,j} (\xi_{ij})^2}, \quad \xi \in \mathbb{M}^{m\times n}.$$

For vector-valued functions $u = (u^1, ..., u^m)^t : \Omega \to \mathbb{R}^m$, we define

$$\nabla u := \begin{pmatrix} \partial_1 u^1 & \partial_2 u^1 & . \; . \; . & \partial_m u^1 \\ \partial_1 u^2 & \partial_2 u^2 & . \; . \; . & \partial_m u^2 \\ . & . & . \; . \; . & . \\ . & . & . \; . \; . & . \\ \partial_1 u^m & \partial_2 u^m & . \; . \; . & \partial_m u^m \end{pmatrix},$$

and $Du$ is the symmetric part of $\nabla u$, i.e., $Du = 1/2(\nabla u + (\nabla u)^t)$. Here $Du$ is a matrix-valued function in which all components are distributional partial derivatives of $u$

Now, we define the generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega; \mathbb{R}^m)$ as the set of all $u \in L^{p(x)}(\Omega; \mathbb{R}^m)$ such that $Du \in L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$, which is a Banach space for the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|Du\|_{p(x)}.$$

This space is again a special case of Orlicz-Sobolev spaces and inherits many of the properties of $L^{p(x)}(\Omega; \mathbb{R}^m)$. In particular, $W^{1,p(x)}(\Omega; \mathbb{R}^m)$ is separable if $1 \le p^- \le p^+ < \infty$, and is reflexive if $1 < p^- \le p^+ < \infty$. Further, $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ is the closure of $C_0^\infty(\Omega; \mathbb{R}^m)$ in the norm of $W^{1,p(x)}(\Omega; \mathbb{R}^m)$. If $p(.) \in C_+(\overline{\Omega})$, then an equivalent norm in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ is $\|Du\|_{p(x)}$. The dual space of $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ can be identified with $W^{-1,p'(x)}(\Omega; \mathbb{R}^m)$ for $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. As in [104], the following Poincaré's inequality:

there exists a positive constant $\alpha = \operatorname{diam}(\Omega)$ such that

$$\|u\|_{p(x)} \leq \frac{\alpha}{2}\|Du\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m),$$

together with Hölder's inequality, are central in our analysis.

Let us summarize the above properties in the following proposition:

**Proposition 2.1.1.** *[92] 1) $W^{1,p(x)}(\Omega;\mathbb{R}^m)$ and $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ are Banach spaces which are separable if $p(.) \in L^\infty(\Omega)$ and reflexive if $1 < p^- \leq p^+ < \infty$.*
*2) If $q \in C_+(\overline{\Omega})$ with $q(x) < p^*(x) := \frac{np(x)}{n-p(x)}$ for all $p(x) < n$, then the following compact embedding $W^{1,p(x)} \hookrightarrow L^{q(x)}(\Omega;\mathbb{R}^m)$ holds true. In particular, since $p(x) < p^*(x)$ for all $x \in \Omega$ then*

$$W^{1,p(x)} \hookrightarrow\hookrightarrow L^{p(x)}(\Omega;\mathbb{R}^m).$$

*3) There exists a constant $C > 0$ with $\|u\|_{p(x)} \leq C\|Du\|_{p(x)}$ for all $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, hence $\|Du\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are two equivalent norms on $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.*

## 2.2 Orlicz and Orlicz-Sobolev spaces

### 2.2.1 N-functions

**Definition 2.2.1.** *A function $M : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an N-function if it is a continuous, non-negative, convex function, which satisfy $M(t) = 0$ if and only if $t = 0$, and*

$$M(t)/t \to \begin{cases} 0 & \text{as } t \to 0, \\ \infty & \text{as } t \to \infty. \end{cases}$$

An N-function $M : \mathbb{R}^+ \to \mathbb{R}^+$ can be represented in integral form (see [93])

$$M(t) = \int_0^t m(s)ds, \tag{2.2.1}$$

where $m : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $m(0) = 0$, $m(t) > 0$ and $m(t) \to \infty$ as $t \to \infty$.

**Definition 2.2.2.** *The complementary function $\overline{M}$ of $M$ is also an N-function and defined by*

$$\overline{M}(s) = \sup_{t\in\mathbb{R}^+} \big(ts - M(t)\big) \quad \textit{for } s \in \mathbb{R}^+.$$

The complementary N-function $\overline{M}$ can be also represented in integral form

$$\overline{M}(t) = \int_0^t \overline{m}(s)ds, \tag{2.2.2}$$

where $\overline{m}(t) = \sup\{s : \ m(s) \le t\}$ is the right inverse of $m$. From above definitions, we have $\overline{\overline{M}} = M$ and

$$\overline{m}\big(m(t)\big) \ge t \quad \text{and} \quad m\big(\overline{m}(s)\big) \ge s, \quad \forall s,t \ge 0.$$

The above inequalities become equalities if $m$ is continuous, i.e., $m$ and $\overline{m}$ are mutual inverses.

**Lemma 2.2.1** ([93])**.** *Let $M$ be an N-function given by* (2.2.1)*. Then*

*(1) for all $s,t \ge 0$*

$$ts \le M(t) + \overline{M}(s) \qquad \textit{(Young's inequality)},$$

*(2) $m(t) \le \frac{M(t)}{t} \le \overline{M}^{-1}\big(M(t)\big)$ for all $t > 0$.*

**Remark 2.2.1.** *The Young inequality in Lemma 2.2.1 follows with equality if and only if $t = \overline{m}(s)$ or $s = m(t)$.*

Let $M$ and $P$ be two N-functions. We shall write $M \prec P$ (i.e., $P$ dominate $M$) if there exist positive constants $t_0$ and $k$ such that

$$M(t) \le P(kt) \qquad (t \ge t_0).$$

If $\lim_{t\to\infty}\big(M(t)/P(kt)\big)=0$ for all $k>0$, we say that $M$ grows essentially less rapidly than $M$ and we write $M \ll P$.

**Definition 2.2.3.** *An N-function $M$ is said to satisfy the $\Delta_2$-condition and denote $M\in\Delta_2$, if for some $k>0$*

$$M(2t)\le kM(t),\quad \forall t\ge 0. \tag{2.2.3}$$

*The $\Delta_2$-condition is equivalent to the satisfaction of the inequality*

$$M(lt)\le kM(t)$$

*for large values of $l$, where $l$ can be any number larger than unity.*

### 2.2.2 Orlicz spaces

Let $\Omega$ be an open subset of $\mathbb{R}^n$ ($n\ge 2$) and $M:\mathbb{R}^+\to\mathbb{R}^+$ an N-function. The Orlicz class $\mathcal{L}_M(\Omega;\mathbb{R}^m)$ is the set of measurable vector-valued functions $u:\Omega\to\mathbb{R}^m$ such that

$$\int_\Omega M\big(|u(x)|\big)dx<\infty.$$

The Orlicz space $L_M(\Omega;\mathbb{R}^m)$ is the set of (equivalence classes of) measurable functions $u:\Omega\to\mathbb{R}^m$ which satisfies

$$\int_\Omega M\Big(\frac{|u(x)|}{\beta}\Big)dx<\infty\quad\text{for some }\beta>0.$$

It is a Banach space under the norm of Luxemburg

$$\|u\|_{L_M(\Omega;\mathbb{R}^m)}:=\|u\|_M=\inf\Big\{\beta>0,\ \int_\Omega M\Big(\frac{|u(x)|}{\beta}\Big)dx\le 1\Big\},$$

for $u\in L_M(\Omega;\mathbb{R}^m)$. In general, $L_M(\Omega;\mathbb{R}^m)$ is neither separable nor reflexive. By the superlinear growth of $M$ (i.e., $\lim_{t\to\infty}\big(M(t)/t\big)=\infty$), it result that

$$L_M(\Omega;\mathbb{R}^m)\subset L^1(\Omega;\mathbb{R}^m).$$

We denote by $E_M(\Omega;\mathbb{R}^m)$ the closure of all bounded measurable functions defined on $\Omega$ with respect to $\|.\|_M$.

**Theorem 2.2.1.** *[127, 130] Let $\Omega$ be an open bounded of $\mathbb{R}^n$ and $M$ an N-function. Then*

*(i)* $E_M(\Omega;\mathbb{R}^m) \subset \mathcal{L}_M(\Omega;\mathbb{R}^m) \subset L_M(\Omega;\mathbb{R}^m)$.

*(ii)* $E_M(\Omega;\mathbb{R}^m) = L_M(\Omega;\mathbb{R}^m)$ *if and only if* $M \in \Delta_2$.

*(iii)* $E_M(\Omega;\mathbb{R}^m)$ *is separable and* $C_0^\infty(\Omega;\mathbb{R}^m)$ *is dense in* $E_M(\Omega;\mathbb{R}^m)$.

*(iv)* $L_M(\Omega;\mathbb{R}^m)$ *is reflexive if and only if* $M, \overline{M} \in \Delta_2$.

**Lemma 2.2.2** ([93])**.** *The generalized Hölder inequality*

$$\Big| \int_\Omega u(x)v(x)dx \Big| \leq 2\|u\|_M \|v\|_{\overline{M}}$$

*holds for any pair of functions* $u \in L_M(\Omega;\mathbb{R}^m)$ *and* $v \in L_{\overline{M}}(\Omega;\mathbb{R}^m)$.

In Lemma 2.2.2, $L_{\overline{M}}(\Omega;\mathbb{R}^m)$ is the dual space of $E_M(\Omega;\mathbb{R}^m)$. We also have $L_{\overline{M}} = E_{\overline{M}}$ if and only if $\overline{M} \in \Delta_2$. The functional

$$\rho_M(u) = \int_\Omega M(|u(x)|)dx$$

is a modular of measurable functions $u : \Omega \to \mathbb{R}^m$. A sequence $u_k$ converges modularly to $u$ in $L_M(\Omega;\mathbb{R}^m)$ if there exists $\beta > 0$ such that

$$\rho_M\Big(\frac{u_k - u}{\beta}\Big) = \int_\Omega M\Big(\frac{|u_k - u|}{\beta}\Big)dx \to 0 \quad \text{as } k \to \infty.$$

An important inequality in $L_M(\Omega;\mathbb{R}^m)$ is as follows:

$$\int_\Omega M\Big(\frac{|u(x)|}{\|u\|_{(M)}}\Big)dx \leq 1 \quad \text{for all } u \in L_M(\Omega;\mathbb{R}^m) \setminus \{0\},$$

where $\|.\|_{(M)}$ is the Orlicz norm given by

$$\|u\|_{(M)} = \sup\Big\{ \int_\Omega u(x)v(x)dx : \ v \in E_{\overline{M}} \text{ with } \|v\|_{\overline{M}} \leq 1 \Big\}.$$

The Orlicz norm $\|.\|_{(M)}$ and Luxemburg norm $\|.\|_M$ are equivalent in the following way ([94])

$$\|u\|_M \leq \|u\|_{(M)} \leq 2\|u\|_M, \quad \forall u \in L_M(\Omega;\mathbb{R}^m).$$

### 2.2.3 Homogeneous Orlicz-Sobolev spaces

We now turn to the homogeneous Orlicz-Sobolev spaces. It is sufficient to notice, that the Orlicz space $L_M(\Omega;\mathbb{M}^{m\times n})$ is defined in the similar way as $L_M(\Omega;\mathbb{R}^m)$ (since $\mathbb{M}^{m\times n} \cong \mathbb{R}^{mn}$).

The homogeneous Orlicz-Sobolev space $W^1L_M(\Omega;\mathbb{R}^m)$ is defined to contain all functions $u \in L_M(\Omega;\mathbb{R}^m)$ such that $Du \in L_M(\Omega;\mathbb{M}^{m\times n})$. Similarly,

$$W^1E_M(\Omega;\mathbb{R}^m) = \Big\{u \in E_M(\Omega;\mathbb{R}^m);\ Du \in E_M(\Omega;\mathbb{M}^{m\times n})\Big\}.$$

The space $W^1L_M(\Omega;\mathbb{R}^m)$ (resp. $W^1E_M(\Omega;\mathbb{R}^m)$) is a Banach space under the norm

$$\|u\|_{W^1L_M(\Omega;\mathbb{R}^m)} := \|u\|_{1,M} = \|u\|_{L_M(\Omega;\mathbb{R}^m)} + \|Du\|_{L_M(\Omega;\mathbb{M}^{m\times n})}$$

(resp. $\|u\|_{W^1E_M(\Omega;\mathbb{R}^m)}$, which is defined as $\|u\|_{W^1L_M(\Omega;\mathbb{R}^m)}$).

Recall that a sequence $u_k$ in $W^1L_M(\Omega;\mathbb{R}^m)$ is said to be modular convergent to $u \in W^1L_M(\Omega;\mathbb{R}^m)$, if there exists $\beta > 0$ such that

$$\int_\Omega M\Big(\frac{|Du_k - Du|}{\beta}\Big)dx \to 0 \quad \text{as } k \to \infty.$$

If further $M \in \Delta_2$, then modular convergence coïncides with the norm convergence.

We define $W^1_0E_M(\Omega;\mathbb{R}^m)$ as the (norm) closure of $C^\infty_0(\Omega;\mathbb{R}^m)$ in $W^1E_M(\Omega;\mathbb{R}^m)$. Moreover, $W^1_0E_M(\Omega;\mathbb{R}^m) = W^1_0L_M(\Omega;\mathbb{R}^m)$ if $M$ satisfies the $\Delta_2$-condition. Note that, if $P \ll M$ then $W^1L_M(\Omega;\mathbb{R}^m) \subset E_P(\Omega;\mathbb{R}^m)$ with compact embedding (see [4]). In particular, $W^1L_M(\Omega;\mathbb{R}^m) \subset E_M(\Omega;\mathbb{R}^m)$.

**Lemma 2.2.3** ([76]). *If $M \in \Delta_2$, then there exists $\theta > 0$ such that for all $u \in W_0^1 L_M(\Omega; \mathbb{R}^m)$,*

$$\int_\Omega M(|u|)dx \leq \theta \int_\Omega M(|Du|)dx.$$

**Lemma 2.2.4** ([78]). *Let $u_k : \Omega \to \mathbb{R}^m$ be a measurable sequence. Then $u_k$ converges in modular to $u$ in $L_M(\Omega; \mathbb{R}^m)$ if and only if $u_k \to u$ in measure and there exists some $\gamma > 0$ such that $\{M(\gamma|u_k|)\}_k$ is uniformly bounded, i.e.,*

$$\lim_{L\to\infty} \sup_{k\in\mathbb{N}} \int_{\{x\in\Omega:\, |M(\gamma|u_k|)|\geq L\}} M(\gamma|u_k|)dx = 0.$$

If $M, \overline{M} \in \Delta_2$ then $W_0^1 L_M(\Omega; \mathbb{R}^m) (= W_0^1 E_M(\Omega; \mathbb{R}^m))$ is separable and reflexive, and its dual is given by $W^{-1} L_{\overline{M}}(\Omega; \mathbb{R}^m) (= W^{-1} E_{\overline{M}}(\Omega; \mathbb{R}^m))$. Moreover, we denote by $W^1_{0,div} L_M(\Omega; \mathbb{R}^m)$ the Orlicz-Sobolev space with free divergence, i.e.,

$$W^1_{0,div} L_M(\Omega; \mathbb{R}^m) = \Big\{ v \in W_0^1 L_M(\Omega; \mathbb{R}^m) \,:\, \operatorname{div} v = 0 \text{ in } \Omega \Big\}.$$

The dual of this space will be denoted by $W^{-1}_{div} E_{\overline{M}}(\Omega; \mathbb{R}^m) = W^{-1}_{div} L_{\overline{M}}(\Omega; \mathbb{R}^m)$.

### 2.2.4 Inhomogeneous Orlicz-Sobolev spaces

Let us consider the time-space cylinder $Q = \Omega \times (0, T)$, where $\Omega$ is a bounded open subset of $\mathbb{R}^n$ and $T > 0$ is a given time. Let $M$ be an N-function. The definitions and results from above are the same, just replace $\Omega$ by $Q$. The inhomogeneous Orlicz-Sobolev spaces (of order 1) are defined as follows:

$$W^{1,x} L_M(Q; \mathbb{R}^m) = \Big\{ u \in L_M(Q; \mathbb{R}^m) :\ D_x^\alpha u \in L_M(Q; \mathbb{M}^{m\times n}),\ \forall |\alpha| \leq 1 \Big\}$$

and

$$W^{1,x} E_M(Q; \mathbb{R}^m) = \Big\{ u \in E_M(Q; \mathbb{R}^m) :\ D_x^\alpha u \in E_M(Q; \mathbb{M}^{m\times n}),\ \forall |\alpha| \leq 1 \Big\},$$

where $D_x^\alpha$ is the distributional derivative on $Q$ of order $\alpha \in \mathbb{N}^n$ with respect to the variable $x \in \Omega \subset \mathbb{R}^n$. Both are Banach spaces under the norm

$$\|u\|_{W^{1,x}L_M(Q;\mathbb{R}^m)} := \|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D_x^\alpha u\|_M.$$

Note that $W^{1,x}E_M(Q;\mathbb{R}^m)$ is a subspace of $W^{1,x}L_M(Q;\mathbb{R}^m)$. These spaces are equal when $M$ satisfies the $\Delta_2$-condition. In general, without strong assumption on $M$ and $\overline{M}$, we have $L_M(Q;\mathbb{R}^m) \neq L_M(0,T;L_M(\Omega;\mathbb{R}^m))$ (see [61]). If $u \in W^{1,x}L_M(Q;\mathbb{R}^m)$ then the function $t \mapsto u(t) = u(.,t)$ is defined on $(0,T)$ with values in $W^1L_M(\Omega;\mathbb{R}^m)$. When $u \in W^{1,x}E_M(Q;\mathbb{R}^m)$ then $u(.,t)$ is a $W^1E_M(\Omega;\mathbb{R}^m)$-valued and is strongly measurable. The following embedding holds

$$W^{1,x}E_M(Q;\mathbb{R}^m) \subset L^1(0,T;W^1E_M(\Omega;\mathbb{R}^m)).$$

The space $W_0^{1,x}E_M(Q;\mathbb{R}^m)$ is defined as the (norm) closure in $W^{1,x}E_M(Q;\mathbb{R}^m)$ of $C_0^\infty(Q;\mathbb{R}^m)$. Its dual is the space $W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$ which defined as

$$W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m) = \Big\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha;\ f_\alpha \in L_{\overline{M}}(Q;\mathbb{R}^m) \Big\}.$$

Always $W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m) = W^{-1,x}E_{\overline{M}}(Q;\mathbb{R}^m)$ is satisfied if $\overline{M} \in \Delta_2$. Furthermore, $W_0^{1,x}L_M(Q;\mathbb{R}^m)$ is separable and reflexive if $M, \overline{M} \in \Delta_2$.

## 2.3 Young measures

Let $C_0(\mathbb{R}^m)$ denotes the Banach space of continuous functions $\varphi : \mathbb{R}^m \to \mathbb{R}$ satisfying $\lim_{|\lambda|\to\infty} \varphi(\lambda) = 0$ equipped with the $L^\infty$-norm, and $\rightharpoonup^*$ is the weak-star convergence. The dual of $C_0(\mathbb{R}^m)$ can be identified with the space of signed Radon measures with finite mass denoted by $\mathcal{M}(\mathbb{R}^m)$, and the duality pairing between these spaces is given by

$$\langle \nu, \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu(\lambda),$$

for all $\varphi \in C_0(\mathbb{R}^m)$ and $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$. Note that if $\varphi \equiv id$, then $\langle \nu_x, id \rangle = \int_{\mathbb{R}^m} \lambda d\nu_x(\lambda)$ is called the barycenter of $\nu = \{\nu_x\}_{x\in\Omega}$. We also define the support of $\nu \in \mathcal{M}(\mathbb{R}^m)$ by

$$\operatorname{supp}\nu := \Big\{\lambda \in \mathbb{R}^m : \nu\big(B_r(\lambda)\big) > 0 \text{ for all } r > 0\Big\},$$

where $B_r(\lambda) = B(\lambda, r)$ is the ball with center $\lambda$ and radius $r > 0$.

The fundamental theorem of Young measures, due to Ball [33], can be stated as follows:

**Theorem 2.3.1.** *Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable, let $K \subset \mathbb{R}^m$ be closed, and let $u_j : \Omega \to \mathbb{R}^m$, $j \in \mathbb{N}$, be a sequence of Lebesgue measurable functions satisfying $u_j \to K$ in measure as $j \to \infty$, i.e., given any open neighborhood $U$ of $K$ in $\mathbb{R}^m$*

$$\lim_{j\to\infty} \Big|\Big\{x \in \Omega : u_j(x) \notin U\Big\}\Big| = 0.$$

*Then there exist a subsequence $(u_k)$ of $(u_j)$ and a family $(\nu_x)$, $x \in \Omega$, of positive measures on $\mathbb{R}^m$, depending measurably on $x$, such that*

*(i) $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} := \int_{\mathbb{R}^m} d\nu_x(\lambda) \le 1$ for a.e. $x \in \Omega$,*

*(ii) $\operatorname{supp}\nu_x \subset K$ for a.e. $x \in \Omega$, and*

*(iii) $\varphi(u_k) \rightharpoonup^* \langle \nu_x, \varphi\rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu_x(\lambda)$ in $L^\infty(\Omega)$ for $\varphi \in C_0(\mathbb{R}^m)$.*

*Suppose further that $(u_k)$ satisfies the boundedness condition*

$$\forall R > 0 : \lim_{L\to\infty} \sup_{k\in\mathbb{N}} \Big|\Big\{x \in \Omega \cap B_R(0) : |u_k(x)| \ge L\Big\}\Big| = 0. \tag{2.3.1}$$

*Then*

$$\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = 1 \quad \text{for a.e. } x \in \Omega \tag{2.3.2}$$

*(i.e., $\nu_x$ is a probability measure), and there holds*

$$\begin{cases} \text{for any measurable } \Omega' \subset \Omega \text{ and any continuous function } \varphi : \mathbb{R}^m \to \mathbb{R} \\ \text{such that } \{\varphi(u_k)\} \text{ is sequentially weakly relatively compact in} \\ L^1(\Omega) \text{ we have } \varphi(u_k) \rightharpoonup \langle \nu_x, \varphi \rangle \text{ in } L^1(\Omega') \end{cases} \tag{2.3.3}$$

Improved versions of this theorem exist: In [87, Theorem 1.2], it is shown that (2.3.1) is necessary for (2.3.2) and (2.3.3) to hold, and that in fact (2.3.1), (2.3.2) and (2.3.3) are equivalent.

**Lemma 2.3.1** ([68]). *Assume that the sequence $(u_j)$ is bounded in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exist a subsequence $(u_k)$ of $(u_j)$ and a Borel probability measure $\nu_x$ on $\mathbb{R}^m$, for a.e. $x \in \Omega$, such that for almost each $\varphi \in C(\mathbb{R}^m)$ we have*

$$\varphi(u_k) \rightharpoonup^* \overline{\varphi} \quad \text{weakly in } L^\infty(\Omega;\mathbb{R}^m),$$

*where $\overline{\varphi}(x) = \langle \nu_x, \varphi \rangle$ for a.e. $x \in \Omega$.*

Remark that, once the subsequence $(u_k)$ of $(u_j)$ is fixed, $(\nu_x)_{x\in\Omega}$ obtained by this way is unique and is a sub-probability family on $\mathbb{R}^m$ by $(i)$ of Theorem 2.3.1 (means that $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} \leq 1$ for a.e. $x \in \Omega$).

**Definition 2.3.1.** *The $(\nu_x)_{x\in\Omega}$ is called a family of Young measures associated with (generated by) the subsequence $(u_k)$.*

In [33] it is shown, that under hypothesis (2.3.1) for any measurable $\Omega' \subset \Omega$

$$\varphi(.,u_k) \rightharpoonup \langle \nu_x, \varphi(x,.) \rangle = \int_{\mathbb{R}^m} \varphi(x,\lambda) d\nu_x(\lambda) \quad \text{in } L^1(\Omega')$$

for every Carathéodory function $\varphi : \Omega' \times \mathbb{R}^m \to \mathbb{R}$ such that $\{\varphi(.,u_k)\}$ is sequentially weakly relatively compact in $L^1(\Omega')$. Hence, this fact is also equivalent to (2.3.1), (2.3.2) and (2.3.3). Ball has also shows, that if $u_k$ generates the Young measure $\nu_x$, then for $\varphi \in L^1(\Omega; C_0(\mathbb{R}^m))$

$$\lim_{k\to\infty} \int_\Omega \varphi(x,u_k(x))dx = \int_\Omega \langle \nu_x, \varphi(x,.) \rangle dx.$$

Theorem 2.3.1 has useful applications, in particular in non-linear partial differntial equations (PDEs) theory. We recall the following technical statements build the basic tools used through the analysis in this thesis.

**Proposition 2.3.1** ([87]). *(i) If $|\Omega| < \infty$ and $\nu_x$ is the Young measure generated by the (whole) sequence $u_j$, then there holds*

$$u_j \to u \quad \text{in measure if and only if} \quad \nu_x = \delta_{u(x)} \text{ for a.e. } x \in \Omega.$$

*(ii) Let $|\Omega| < \infty$. If the sequences $u_j : \Omega \to \mathbb{R}^m$ and $v_j : \Omega \to \mathbb{R}^d$ generate the Young measure $\nu_x$ and $\delta_{v(x)}$ respectively, then $(u_j, v_j)$ generates the Young measure $\nu_x \otimes \delta_{v(x)}$.*

Note that, the previous properties are still valid for $u_k = Dw_k$, where $w_k : \Omega \to \mathbb{R}^m$ (means that $Dw_k$ is a matrix-valued sequence). Another application of Theorem 2.3.1 is the following Fatou-type inequality, which is of fundamental significance:

**Lemma 2.3.2** ([59]). *Let $\varphi : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and $w_k : \Omega \to \mathbb{R}^m$ a sequence of measurable functions such that $w_k \to w$ in measure and such that $Dw_k$ generates the Young measure $\nu_x$, with $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for almost every $x \in \Omega$. Then*

$$\liminf_{k\to\infty} \int_\Omega \varphi(x, w_k, Dw_k)dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \varphi(x, w, \lambda) d\nu_x(\lambda) dx$$

*provided that the negative part $\varphi^-(x, w_k, Dw_k)$ is equiintegrable.*

# Chapter 3

# Quasilinear elliptic and parabolic problems in Sobolev spaces

This chapter is devoted to study the existence of weak solutions for some quasilinear elliptic/parabolic systems under classical polynomial growth and coercivity conditions.

## 3.1 Generalized $p$-Laplacian system

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$ and $1 < p < \infty$. Consider the following generalized $p$-Laplacian system

$$\begin{cases} -\mathrm{div}\big(\phi(Du - \Theta(u))\big) & = f \quad \text{in } \Omega, \\ & u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.1.1}$$

The source term $f$ is supposed lying in $W^{-1,p'}(\Omega;\mathbb{R}^m)$ the dual space of $W_0^{1,p}(\Omega;\mathbb{R}^m)$, $\phi : \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is given in a simple form $\phi(\xi) = |\xi|^{p-2}\xi$ for $\xi \in \mathbb{M}^{m\times n}$ and $\Theta : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is a continuous function satisfy

$$\Theta(0) = 0 \quad \text{and} \quad |\Theta(a) - \Theta(b)| \leq c|a - b| \tag{3.1.2}$$

for all $a, b \in \mathbb{R}^m$, where $c$ is a positive constant related to the exponent $p$ and the diameter of $\Omega$ (diam$(\Omega)$) by the following:

$$c < \frac{1}{\operatorname{diam}(\Omega)}\left(\frac{1}{2}\right)^{\frac{1}{p}}.$$

For several decades, there have been intensive research activities for equations, or systems, of $p$-Laplacian type. In [128], several examples of degenerate elliptic equations are presented. The author proved the existence of a weak solution by various methods. Dibenedetto and Manfredi [57] considered the following nonlinear elliptic system $\operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F))$, for $F \in L^p_{loc}(\Omega;\mathbb{R}^m)$ and proved the existence of a local weak solution and some estimates of $Du$ in $[BMO_{loc}(\Omega)]^{Nm}$. In [91], the authors proved a regularity result for the quasilinear equation $\operatorname{div}\big((ADu.Du)^{\frac{(p-2)}{2}}ADu\big) = \operatorname{div}(|F|^{p-2}F)$. They studied the regularity of $F$ which reflected to the solutions under minimal assumptions on the coefficient matrix $A$. A collect of some very recent pointwise bounds for the gradient of solutions, and the solutions themselves, to the $p$-Laplace system with right hand side in divergence form were discussed in [41].

In view of [88], our system $-\operatorname{div}\big(|Du-\Theta(u)|^{p-2}(Du-\Theta(u))\big) = f$ is a nonlinear degenerate and singular elliptic system according to the cases $p > 2$ and $1 < p < 2$, respectively.

In the present section, due to the term $\Theta$ in Eq. (3.1.1), we don't have such Leray-Lions conditions and we can't use the main techniques as in [91]. Our aim here is to prove the existence of weak solutions by using the concept of Young measures as technical tools to describe the weak limits of a sequence of approximating solutions.

We say that $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ is a weak solution to (3.1.1) if

$$\begin{aligned}\int_\Omega \phi\big(Du-\Theta(u)\big) : D\varphi dx &:= \int_\Omega |Du-\Theta(u)|^{p-2}(Du-\Theta(u)) : D\varphi dx\\ &= \langle f, \varphi\rangle\end{aligned}$$

holds for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$. Here $\langle .,. \rangle$ denotes the duality pairing between $W^{-1,p'}(\Omega;\mathbb{R}^m)$ and $W_0^{1,p}(\Omega;\mathbb{R}^m)$.

The main result of this part is the following:

**Theorem 3.1.1.** *Suppose that $\Theta$ satisfies (3.1.2), then there exists at least one weak solution of the problem (3.1.1).*

### 3.1.1 Galerkin approximation and *priori* estimates

In what follows, we will use the following Poincaré's inequality (see [104, Lemma 2.2]), there exists a positive constant $\alpha = \mathrm{diam}(\Omega)$ such that

$$\|v\|_p \leq \frac{\alpha}{2}\|Dv\|_p, \quad \forall v \in W_0^{1,p}(\Omega;\mathbb{R}^m). \tag{3.1.3}$$

The relation (3.1.3) and the Hölder inequality are central to establish the required estimates to prove the desired results. We recall the following useful lemma:

**Lemma 3.1.1** ([1]). *Let $a, b \in \mathbb{R}^m$ and let $1 < p < \infty$. We have*

$$\frac{1}{p}|a|^p - \frac{1}{p}|b|^p \leq |a|^{p-2}a.(a-b).$$

Let us define $T : W_0^{1,p}(\Omega;\mathbb{R}^m) \to W^{-1,p'}(\Omega;\mathbb{R}^m)$ in the following way

$$\langle T(u), \varphi\rangle = \int_\Omega \phi\big(Du - \Theta(u)\big) : D\varphi dx - \langle f, \varphi\rangle.$$

Our problem (3.1.1) is then equivalent to find $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ such that $\langle T(u), \varphi\rangle = 0$ for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$.

**Lemma 3.1.2.** *We have the following properties:*

*(i)* *$T : W_0^{1,p}(\Omega;\mathbb{R}^m) \to W^{-1,p'}(\Omega;\mathbb{R}^m)$ is linear, well defined and bounded.*

*(ii)* *The restriction of $T$ to a finite linear subspace of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ is continuous.*

*(iii)* *$T$ is coercive.*

*Proof.* $(i)$ $T$ is trivially linear. For arbitrary $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, by the Hölder inequality, Poincaré's inequality and Eq. (3.1.2), we have

$$\begin{aligned}
\big|\langle T(u),\varphi\rangle\big| &= \Big|\int_\Omega \phi(Du-\Theta(u)) : D\varphi dx - \langle f,\varphi\rangle\Big| \\
&\le \int_\Omega |Du-\Theta(u)|^{p-1}|D\varphi| dx + \|f\|_{-1,p'}\|\varphi\|_{1,p} \\
&\le \Big(\int_\Omega |Du-\Theta(u)|^p dx\Big)^{\frac{1}{p'}}\|D\varphi\|_p + \|f\|_{-1,p'}\|\varphi\|_{1,p} \\
&\le 2^{\frac{(p-1)^2}{p}}\Big(\|Du\|_p^p + \|\Theta(u)\|_p^p\Big)^{\frac{p-1}{p}}\|D\varphi\|_p + \|f\|_{-1,p'}\|\varphi\|_{1,p} \\
&\le c'\|\varphi\|_{1,p}
\end{aligned}$$

for some positive constant $c'$. In the above inequality we have used

$$|a+b|^p \le 2^{p-1}(|a|^p+|b|^p) \qquad (p>1). \tag{3.1.4}$$

It follows that $T$ is well defined and bounded.

$(ii)$ Let $V$ be a subspace of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ with $\dim V = r$ and $(e_i)_{i=1}^r$ a basis of $V$. Let $(u_k = a_k^i e_i)$ be a sequence in $V$ which converges to $u = a^i e_i$ in $V$ (with conventional summation). Then $u_k \to u$ and $Du_k \to Du$ almost everywhere for a subsequence still denoted by $\{u_k\}$. On the other hand, $\|u_k\|_p$ and $\|Du_k\|_p$ are bounded by a constant $C$. Indeed, since $u_k \to u$ strongly in $V$,

$$\int_\Omega |u_k-u|^p dx \to 0 \quad \text{and} \quad \int_\Omega |Du_k-Du|^p dx \to 0,$$

then there exist a subsequence of $\{u_k\}$ still denoted by $\{u_k\}$ and $g_1, g_2 \in L^1(\Omega)$ such that $|u_k-u|^p \le g_1$ and $|Du_k-Du|^p \le g_2$. According to (3.1.4), it follows that

$$\begin{aligned}
|u_k|^p = |u_k-u+u|^p &\le 2^{p-1}\big(|u_k-u|^p+|u|^p\big) \\
&\le 2^{p-1}(g_1+|u|).
\end{aligned}$$

Similarly

$$|Du_k|^p \le 2^{p-1}(g_2+|Du|^p).$$

By the continuity of the function $\Theta$, it follows that

$$\phi(Du_k - \Theta(u_k)) : D\varphi \to \phi(Du - \Theta(u)) : D\varphi \quad \text{almost everywhere.}$$

Let $\Omega' \subset \Omega$ be a measurable subset and $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$. As in the proof of the property $(i)$, we obtain

$$\int_{\Omega'} |\phi(Du_k - \Theta(u_k)) : D\varphi| dx \leq 2^{\frac{(p-1)^2}{p}} \Big( \underbrace{\|Du_k\|_p^p}_{\leq C} + c^p \underbrace{\|u_k\|_p^p}_{\leq C} \Big)^{\frac{p-1}{p}} \Big( \int_{\Omega'} |D\varphi|^p dx \Big)^{\frac{1}{p}}.$$

Since $\int_{\Omega'} |D\varphi|^p dx$ is arbitrary small if the measure of $\Omega'$ is chosen small enough, then $\big(\phi(Du_k - \Theta(u_k)) : D\varphi\big)$ is equiintegrable. Applying the Vitali Theorem, it follows that $T$ is continuous.

$(iii)$ We have

$$\langle T(u), u \rangle = \int_\Omega |Du - \Theta(u)|^{p-2}(Du - \Theta(u)) : Du dx - \langle f, u \rangle. \tag{3.1.5}$$

By Lemma 3.1.1, we have

$$|\xi|^{p-2}\xi : (\xi - \eta) \geq \frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p,$$

then by taking $\xi = Du - \Theta(u)$ and $\eta = -\Theta(u)$, ($\xi - \eta = Du$), we obtain

$$\begin{aligned} &|Du - \Theta(u)|^{p-2}(Du - \Theta(u)) : Du \\ &\qquad = |Du - \Theta(u)|^{p-2}(Du - \Theta(u)) : (Du - \Theta(u) + \Theta(u)) \\ &\qquad \geq \frac{1}{p}|Du - \Theta(u)|^p - \frac{1}{p}|\Theta(u)|^p. \end{aligned}$$

By virtue of (3.1.5), we deduce that

$$\langle T(u), u \rangle \geq \frac{1}{p} \int_\Omega |Du - \Theta(u)|^p dx - \frac{1}{p} \int_\Omega |\Theta(u)|^p dx - \|f\|_{-1,p'} \|u\|_{1,p}.$$

We have

$$\begin{aligned}\frac{1}{2^{p-1}}|Du|^p &= \frac{1}{2^{p-1}}|Du-\Theta(u)+\Theta(u)|^p \\ &\leq \frac{1}{2^{p-1}}\left[2^{p-1}\left(|Du-\Theta(u)|^p+|\Theta(u)|^p\right)\right] \\ &= |Du-\Theta(u)|^p+|\Theta(u)|^p.\end{aligned}$$

This implies that

$$\begin{aligned}\langle T(u),u\rangle &\geq \frac{1}{p}\int_\Omega\left(\frac{1}{2^{p-1}}|Du|^p-|\Theta(u)|^p\right)dx-\frac{1}{p}\int_\Omega|\Theta(u)|^p dx-\|f\|_{-1,p'}\|u\|_{1,p} \\ &\geq \frac{1}{p2^{p-1}}\int_\Omega|Du|^p dx-\frac{2}{p}\int_\Omega|\Theta(u)|^p dx-\|f\|_{-1,p'}\|u\|_{1,p} \\ &\geq \frac{1}{p2^{p-1}}\int_\Omega|Du|^p dx-\frac{1}{p2^p}\int_\Omega|Du|^p dx-\|f\|_{-1,p'}\|u\|_{1,p} \\ &= \frac{1}{p2^p}\int_\Omega|Du|^p dx-\|f\|_{-1,p'}\|u\|_{1,p}.\end{aligned}$$

Consequently, $T$ is coercive. □

To prove Theorem 3.1.1, we will apply a Galerkin schema. Let $V_1\subset V_2\subset \dots \subset W_0^{1,p}(\Omega;\mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\underset{k\geq 1}{\cup} V_k$ is dense in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. Note that the existence of $(V_k)$ is guaranteed by the separability of $W_0^{1,p}(\Omega;\mathbb{R}^m)$.
Now, we can construct the approximating solutions:

**Lemma 3.1.3.** *(i) For all $k\in\mathbb{N}$ there exists $u_k\in V_k$ such that*

$$\langle T(u_k),\varphi\rangle = 0 \quad \textit{for all } \varphi\in V_k. \tag{3.1.6}$$

*(ii) There exists a constant $R>0$ such that*

$$\|u_k\|_{1,p}\leq R \quad \textit{for all } k\in\mathbb{N}. \tag{3.1.7}$$

*Proof.* $(i)$ Let fix $k$ and assume that $\dim V_k = r$. For simplicity, we write $\sum_{1\le i\le r} a^i e_i = a^i e_i$ where $(e_i)_{i=1}^r$ is a basis of $V_k$. Define the map

$$\begin{aligned} S : \mathbb{R}^r &\longrightarrow \mathbb{R}^r \\ (a^1, ..., a^r) &\longrightarrow \Big( \langle T(a^i e_i), e_j \rangle \Big)_{j=1,..,r}. \end{aligned}$$

Remark that $S$ is continuous by Lemma 3.1.2(ii). Let $a \in \mathbb{R}^r$ and $u = a^i e_i \in V_k$, then $\|a\|_{\mathbb{R}^r} \to \infty$ is equivalent to $\|u\|_{1,p} \to \infty$ and

$$S(a).a = \langle T(u), u \rangle.$$

Hence, by Lemma 3.1.2(iii), we have

$$S(a).a \to \infty \quad \text{as} \quad \|a\|_{\mathbb{R}^r} \to \infty.$$

Thus, there exists $R > 0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $S(a).a > 0$. According to the usual topological arguments [139, Proposition 2.8], $S(x) = 0$ has a solution $x \in B_R(0)$. Hence, for all $k$ there exists $u_k \in V_k$ such that

$$\langle T(u_k), \varphi \rangle = 0 \quad \text{for all} \quad \varphi \in V_k.$$

$(ii)$ Since $\langle T(u), u \rangle \to \infty$ as $\|u\|_{1,p} \to \infty$, it follows that there exists $R > 0$ with the property, that $\langle T(u), u \rangle > 1$ whenever $\|u\|_{1,p} > R$. Consequently, for the sequence of Galerkin approximations $u_k \in V_k$ which satisfy $\langle T(u_k), u_k \rangle = 0$ by (3.1.6), we have the uniform bound

$$\|u_k\|_{1,p} \le R \quad \text{for all} \quad k \in \mathbb{N}.$$

□

### 3.1.2 Passage to the limit

This subsection is devoted first to identify weak limits of gradient sequences by means of the Young measures and then we pass to the limit in the approximating equations.

The sequence (or at least a subsequence) of the gradients $Du_k$ generates a Young measure $\nu_x$ (cf. Lemma 2.3.1). Now, we collect some facts about the Young measure $\nu = \{\nu_x\}_x$ in the following lemma:

**Lemma 3.1.4.** *Let $(u_k)$ be the sequence defined in Lemma 3.1.3. Then the Young measure $\nu_x$ generated by $Du_k$ in $L^p(\Omega;\mathbb{M}^{m\times n})$ has the following properties:*

*(i)* *$\nu_x$ is a probability measure, i.e. $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for almost every $x \in \Omega$.*

*(ii)* *The weak $L^1$-limit of $Du_k$ is given by $\langle \nu_x, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)$.*

*(iii)* *$\nu_x$ satisfies $\langle \nu_x, id\rangle = Du(x)$ for almost every $x \in \Omega$.*

*Proof.* $(i)$ Let $\nu_x$ be the Young measure generated by $Du_k$ (see Lemma 2.3.1). Since $(u_k)$ is bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ by (3.1.7), then there exists a constant $C \geq 0$ such that for any $R > 0$,

$$\begin{aligned} C \geq \int_\Omega |Du_k|^p dx &\geq \int_{\{x\in\Omega\cap B_R(0):\,|Du_k|\geq L\}} |Du_k|^p dx \\ &\geq L^p \big|\{x \in \Omega\cap B_R(0) :\, |Du_k| \geq L\}\big|. \end{aligned}$$

Hence

$$\sup_{k\in\mathbb{N}} \big|\{x \in \Omega\cap B_R(0) :\, |Du_k| \geq L\}\big| \leq \frac{C}{L^p} \to 0 \quad \text{as } L \to \infty.$$

According to the Theorem 2.3.1, it follows that $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for almost every $x \in \Omega$.

$(ii)$ Since $L^p(\Omega;M^{m\times n})$ is reflexive ($p > 1$ and $\mathbb{M}^{m\times n} \cong \mathbb{R}^{mn}$) and in view of (3.1.7), we deduce the existence of a subsequence (still denoted by $Du_k$) weakly convergent in $L^p(\Omega;\mathbb{M}^{m\times n})$. Moreover, weakly convergent in $L^1(\Omega;\mathbb{M}^{m\times n})$. By taking $\varphi$ as the identity mapping $Id$ in Theorem 2.3.1(iii), we have

$$Du_k \rightharpoonup \langle \nu_x, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) \text{ weakly in } L^1(\Omega;\mathbb{M}^{m\times n}).$$

$(iii)$ By the equation (3.1.7), a subsequence of $\{u_k\}$ converges weakly in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ to an element denoted by $u$. Thus $u_k \to u$ in $L^p(\Omega;\mathbb{R}^m)$ and $Du_k \rightharpoonup Du$ in $L^p(\Omega;\mathbb{M}^{m\times n})$ (for a subsequence). Owing to $(ii)$, the uniquenesses of the limit implies that

$$\langle \nu_x, id\rangle = Du(x) \quad \text{for a.e. } x \in \Omega.$$

$\square$

Now, we have all ingredients to pass to the limit in the approximating equations and to prove Theorem 3.1.1. Let $(u_k)$ be the sequence constructed in Lemma 3.1.3.

*Proof of Theorem 3.1.1.* Let start by proving that $u_k \to u$ in measure. By (3.1.7), we have (for a subsequence) $u_k \to u$ in $L^p(\Omega;\mathbb{R}^m)$. Let $E_{k,\epsilon} = \{x : |u_k(x) - u(x)| \geq \epsilon\}$, then

$$\int_\Omega |u_k(x) - u(x)|^p dx \geq \int_{E_{k,\epsilon}} |u_k(x) - u(x)|^p dx \geq \epsilon^p |E_{k,\epsilon}|,$$

which implies

$$|E_{k,\epsilon}| \leq \frac{1}{\epsilon^p} \int_\Omega |u_k(x) - u(x)|^p dx \to 0 \quad \text{as } k \to \infty.$$

Therefore, $u_k \to u$ in measure for $k \to \infty$, and we may infer that, after extraction of a suitable subsequence, if necessary,

$$u_k \to u \quad \text{almost everywhere for} \quad k \to \infty.$$

According to a weak limit defined in Lemma 3.1.4 and the continuity of $\Theta$, we can write

$$\begin{aligned} Du_k - \Theta(u_k) \rightharpoonup & \int_{\mathbb{M}^{m\times n}} (\lambda - \Theta(u)) d\nu_x(\lambda) \\ =& \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) - \Theta(u) \underbrace{\int_{\mathbb{M}^{m\times n}} d\nu_x(\lambda)}_{:=1} \\ =& Du - \Theta(u) \end{aligned}$$

weakly in $L^1(\Omega)$, since $(Du_k - \Theta(u_k))$ is equiintegrable by (3.1.3). Therefore

$$|Du_k - \Theta(u_k)|^{p-2}(Du_k - \Theta(u_k)) \rightharpoonup |Du - \Theta(u)|^{p-2}(Du - \Theta(u))$$

weakly in $L^1(\Omega)$. Since $L^p(\Omega)$ is reflexive and $\phi(Du_k - \Theta(u_k))$ is bounded (by (3.1.4)), the sequence $\{\phi(Du_k - \Theta(u_k))\}$ converges in $L^{p'}(\Omega)$. Hence its weak $L^{p'}$-limit is also

$\phi(Du - \Theta(u))$. We may infer that

$$\lim_{k\to\infty} \int_\Omega \phi(Du_k - \Theta(u_k)) : D\varphi(x)dx = \int_\Omega \phi(Du - \Theta(u)) : D\varphi(x)dx \ \forall \varphi \in \bigcup_{k\geq 1} V_k.$$

For any $v \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, since $\bigcup_{k\geq 1} V_k$ is dense in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, there is a sequence $\{v_k\} \subset \bigcup_{k\geq 1} V_k$ such that $v_k \to v$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ as $k \to \infty$. Since

$$\int_\Omega \big(\phi(Du_k - \Theta(u_k)) : Dv - \phi(Du - \Theta(u)) : Dv\big)dx \to 0 \quad \text{as } k \to \infty,$$

we have

$$\begin{aligned}
&\langle T(u_k), v_k\rangle - \langle T(u), v\rangle \\
&= \int_\Omega \big(\phi(Du_k - \Theta(u_k)) : Dv_k - \phi(Du - \Theta(u)) : Dv\big)dx - \langle f, v_k - v\rangle \\
&= \int_\Omega \phi(Du_k - \Theta(u_k)) : (Dv_k - Dv)dx \\
&\qquad + \int_\Omega \big(\phi(Du_k - \Theta(u_k)) - \phi(Du - \Theta(u))\big) : Dvdx \\
&\qquad\qquad - \langle f, v_k - v\rangle.
\end{aligned}$$

The right hand side of the above equation tends to 0 as $k \to \infty$. By virtue of Lemma 3.1.3(i), it follows that $\langle T(u), v\rangle = 0$ for all $v \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ as desired. □

## 3.2 Quasilinear elliptic system in perturbed form

The present section is concerned with the following boundary value system

$$-\mathrm{div}\,\big(\sigma(x, Du) + \phi(u)\big) = f \quad \text{in } \Omega, \tag{3.2.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{3.2.2}$$

where $\Omega$ is a bounded open set of $\mathbb{R}^n$, ($n \geq 2$), $u : \Omega \to \mathbb{R}^m$, $m \in \mathbb{N}$, is a vector-valued function.

In [85] the following quasilinear elliptic system was considered:

$$\begin{aligned} -\operatorname{div} \sigma(x, u, Du) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.2.3}$$

where $f$ belongs to the dual space $W^{-1,p'}(\Omega; \mathbb{R}^m)$ of $W_0^{1,p}(\Omega; \mathbb{R}^m)$. The author proved the existence of weak solutions under weak monotonicity assumptions on the function $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ and by the theory of Young measures. When the right hand side in (3.2.3) is equal to $v(x) + f(x, u) + \operatorname{div} g(x, u)$, the existence of a weak solution under classical regularity, growth and coercivity conditions for $\sigma$, but with only very mild monotonicity assumptions, was proved in [15].

By the same theory (i.e. of Young measures), we have established in [18] (cf. Section 3.1) the existence result for a generalized $p$-Laplacian system of the form

$$-\operatorname{div}(\Phi(Du - \Theta(u)) = f$$

supplemented with Dirichlet condition $u = 0$ on $\partial\Omega$, where $\Phi(\xi) = |\xi|^{p-2}\xi$ for $\xi \in \mathbb{M}^{m\times n}$ and $\Theta$ satisfies some Lipschitz continuity condition. Second-order estimates are established for solutions to the $p$-Laplace system with right hand side in $L^2(\Omega)$ and local estimates for local solutions are provided in [52].

In the scalar case and $f$ belongs to $H^{-1}(\Omega)$, uniqueness in the class of weak solution in $H_0^1(\Omega)$ was proved in [13] if $\phi \equiv 0$ and $\sigma \equiv a(x, u)\nabla u$, and then in [119], where $\phi$ is still assumed to be in $C(\mathbb{R}, \mathbb{R}^n)$ and $f$ belongs to $L^1(\Omega)$. Di Nardo and Perrotta [115] considered the problem (3.2.1) and fixed some structural conditions on $\sigma$ and $\phi$ to prove uniqueness result when $f \in L^1(\Omega)$. For two lower order terms, we refer to [116] where the existence result is obtained as limit of approximations. See also [48, 72].

A large number of papers was devoted to the study of the existence for solutions of elliptic problems of the type (3.2.3) under classical monotone operator methods developed in [44, 105, 111, 131]. These works employ the standard theory of monotone operator on the Sobolev space $W^{1,p}(\Omega)$.

The difficulty that arises in our problem (3.2.1)-(3.2.2) is that we can't use such theory, because we assume only $W$ in **(H3)**(b) (see below) to be convex, but if it is strictly convex, then $\sigma$ becomes strict monotone and the standard method may apply. Moreover, we assume that $\sigma$ is strictly quasimonotone (see **(H3)**(d) below) which allows to proceed the proof differently to [15] and [85]. The presence of the lower term $\phi(u)$ in (3.2.1)-(3.2.2) is an addition difficulty besides previous ones.

In the present problem, a slightly different notions of monotonicity and quasimonotonicity are used. Moreover, we use another condition namely strict quasimontone instead of strict $p$-quasimonotone used in [85] and we proceed the proof differently by using Lemma 3.2.5.

### 3.2.1 Assumptions and main result

Let $\Omega$ be an open bounded set of $\mathbb{R}^n$ ($n \geq 2$). The functions $\sigma$ and $\phi$ are assumed to satisfy the following conditions:
**(H0)** The function $\phi : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is linear and continuous and there exists a constant $\alpha_0$ such that

$$|\phi(u)| \leq \alpha_0.$$

**(H1)** $\sigma : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e. measurable w.r.t. $x \in \Omega$ and continuous w.r.t. $\xi \in \mathbb{M}^{m\times n}$.
**(H2)** There exist $\alpha > \alpha_0 > 0$, $d_1(x) \in L^{p'}(\Omega)$ and $d_2(x) \in L^1(\Omega)$ such that

$$|\sigma(x,\xi)| \leq d_1(x) + |\xi|^{p-1}$$

$$\sigma(x,\xi) : \xi \geq \alpha|\xi|^p - d_2(x), \quad \forall \xi \in \mathbb{M}^{m\times n}.$$

**(H3)** $\sigma$ satisfies one of the following conditions:

(a) For any $x \in \Omega$, $\xi \mapsto \sigma(x,\xi)$ is $C^1$ and monotone, i.e.

$$\big(\sigma(x,\xi) - \sigma(x,\eta)\big) : (\xi - \eta) \geq 0$$

for any $x \in \Omega$ and $\xi, \eta \in \mathbb{M}^{m\times n}$.

(b) There exists a function $W : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,\xi) = \frac{\partial W}{\partial \xi}(x,\xi) := D_\xi W(x,\xi)$ and $\xi \mapsto W(x,\xi)$ is convex and $C^1$.

(c) $\sigma$ is striclty monotone, i.e. $\sigma$ is monotone and

$$\big(\sigma(x,\xi)-\sigma(x,\eta)\big):(\xi-\eta)=0 \Rightarrow \xi=\eta.$$

(d) $\sigma$ is strictly quasimonotone, i.e. there exists a constant $\alpha_1>0$ such that

$$\int_\Omega \big(\sigma(x,Du)-\sigma(x,Dv)\big):(Du-Dv)dx \geq \alpha_1 \int_\Omega |Du-Dv|^p dx.$$

Our main result can be stated as follows:

**Theorem 3.2.1.** *If $\sigma$ and $\phi$ satisfy the conditions **(H0)-(H3)**, then problem (3.2.1)-(3.2.2) has a weak solution for every $f\in W^{-1,p'}(\Omega;\mathbb{R}^m)$.*

**Example 3.2.1.** *As example of problem to which the present result can be applied, we give:*

$$-div\,(|Du|^{p-2}Du+Du)=f,$$

*with $f\in W^{-1,p'}(\Omega;\mathbb{R}^m)$. The conditions **(H3)**(a), (c) and (d) are obvious by direct calculations. For the condition (b), one can take the potential $W=\frac{1}{p}|\xi|^p+\frac{1}{2}|\xi|^2$.*

**Remark 3.2.1.** *The notion of strict quasimonotone in **(H3)**(d) was introduced by Zhang [140] and it implies the strict p-quasimonotone stated in [85] (see [64] for the proof).*

### 3.2.2 Galerkin approximation

Let $V_1\subset V_2\subset ..\subset W_0^{1,p}(\Omega;\mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\underset{i\in\mathbb{N}}{\cup} V_i$ is dense in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. We define the operator

$$\begin{aligned} T:\ & W_0^{1,p}(\Omega;\mathbb{R}^m)\to W^{-1,p'}(\Omega;\mathbb{R}^m)\\ & u\mapsto \Big(w\mapsto \int_\Omega \big(\sigma(x,Du):Dw+\phi(u):Dw\big)dx-\langle f,w\rangle\Big), \end{aligned}$$

where $\langle .,.\rangle$ denotes the dual pairing of $W^{-1,p'}(\Omega;\mathbb{R}^m)$ and $W_0^{1,p}(\Omega;\mathbb{R}^m)$.

**Lemma 3.2.1.** *For arbitrary $u\in W_0^{1,p}(\Omega;\mathbb{R}^m)$, the functional $T(u)$ is linear and bounded.*

*Proof.* $T(u)$ is trivially linear. We have

$$\int_{\Omega} |\sigma(x, Du)|^{p'} dx \leq \int_{\Omega} (|d_1(x)|^{p'} + |Du|^p) dx < \infty,$$

by the growth condition in **(H2)**. It follows from the Hölder inequality that for each $w \in W_0^{1,p}(\Omega; \mathbb{R}^m)$

$$\begin{aligned} |\langle T(u), w\rangle| &= \Big| \int_{\Omega} (\sigma(x, Du) : Dw + \phi(u) : Dw) dx - \langle f, w\rangle \Big| \\ &\leq \big\| |\sigma(x, Du)| \big\|_{p'} \|Dw\|_p + \alpha_0 \|Dw\|_1 + \|f\|_{-1,p'} \|w\|_{1,p} \\ &\leq c \|Dw\|_p, \end{aligned}$$

where we have used Poincaré's inequality and $1 < p$. Thus $T(u)$ is bounded. □

**Lemma 3.2.2.** *The restriction of $T$ to a finite dimensional linear subspace of $W_0^{1,p}(\Omega; \mathbb{R}^m)$ is continuous.*

*Proof.* Let $V$ be a finite subspace of $W_0^{1,p}(\Omega; \mathbb{R}^m)$ such that the dimension of $V$ is equal to $r$ and $(e_i)_{i=1}^r$ a basis of $V$. Let $(u_k = a_k^i e_i)$ be a sequence in $V$ which converges to $u = a^i e_i$ in $V$ (with the standard summation convention). Hence the sequence $(a_k)$ converges to $a$ in $\mathbb{R}^r$. This implies $u_k \to u$ and $Du_k \to Du$ almost everywhere. On the other hand, $\|u_k\|_p$ and $\|Du_k\|_p$ are bounded by a constant $C$. Indeed, we have

$$\int_{\Omega} |u_k - u|^p dx \to 0 \quad \text{and} \quad \int_{\Omega} |Du_k - Du|^p dx \to 0,$$

then there exist a subsequence of $(u_k)$ still denoted by $(u_k)$ and $g_1, g_2 \in L^1(\Omega)$ such that $|u_k - u|^p \leq g_1$ and $|Du_k - Du|^p \leq g_2$. By using the Eq. (3.1.4) we obtain

$$\begin{aligned} |u_k|^p = |u_k - u + u|^p &\leq 2^{p-1} (|u_k - u|^p + |u|^p) \\ &\leq 2^{p-1} (g_1 + |u|^p). \end{aligned}$$

Similarly

$$|Du_k|^p \leq 2^{p-1} (g_2 + |Du|^p).$$

The continuity condition **(H0)** and **(H1)** allow to deduce that $\sigma(x, Du_k) : Dw \to \sigma(x, Du) : Dw$ and $\phi(u_k) : Dw \to \phi(u) : Dw$ almost everywhere. Furthermore, $\big(\sigma(x, Du_k) : Dw\big)$ and $\big(\phi(u_k) : Dw\big)$ are equiintegrable sequences by **(H2)**. Hence, for all $w \in W_0^{1,p}(\Omega; \mathbb{R}^m)$

$$\begin{aligned}
\|T(u_k) - T(u)\|_{-1,p'} &= \sup_{\|w\|_{1,p}=1} \big|\langle T(u_k) - T(u), w\rangle\big| \\
&\le c\big(\|\sigma(x, Du_k) - \sigma(x, Du)\|_{p'} + \|\phi(u_k) - \phi(u)\|_{p'}\big) \\
&\le c.
\end{aligned}$$

□

Now, we fix some $k$ and assume that dim $V_k = r$. Then we define the map

$$\Theta : \mathbb{R}^r \to \mathbb{R}^r, \quad \begin{pmatrix} a^1 \\ a^2 \\ . \\ . \\ a^r \end{pmatrix} \mapsto \begin{pmatrix} \langle T(a^i e_i), e_1\rangle \\ \langle T(a^i e_i), e_2\rangle \\ . \\ . \\ \langle T(a^i e_i), e_r\rangle \end{pmatrix}.$$

**Lemma 3.2.3.** *$\Theta$ is continuous and $\Theta(a).a \to \infty$ as $\|a\|_{\mathbb{R}^r} \to \infty$, where the dote . denotes the inner product of two vectors in $\mathbb{R}^r$.*

*Proof.* The continuity of $\Theta$ can be deduced from that of $T$ restricted to $V_k$. Take $a \in \mathbb{R}^r$ and consider $u = a^i e_i \in V_k$. On the one hand, we have $\Theta(a).a = \langle T(u), u\rangle$ and $\|a\|_{\mathbb{R}^r} \to \infty$ is equivalent to $\|u\|_{1,p} \to \infty$. On the other hand, since $1 < p$ then there exists $\beta = \frac{\alpha}{2\alpha_0} > 0$ such that $\int_\Omega |Du| dx \le \beta \int_\Omega |Du|^p dx$. Therefore

$$\begin{aligned}
\Theta(a).a &= \langle T(a^i e_i), a^i e_i \rangle \\
&= \langle T(u), u \rangle \\
&= \int_\Omega \big(\sigma(x, Du) : Du + \phi(u) : Du\big) dx - \langle f, u \rangle \\
&\ge \int_\Omega \big(\alpha |Du|^p - d_2(x)\big) dx - \alpha_0 \int_\Omega |Du| dx - \|f\|_{-1,p'} \|u\|_{1,p} \\
&\ge \frac{\alpha}{2} \|u\|_{1,p}^p - c - \|f\|_{-1,p'} \|u\|_{1,p} \longrightarrow \infty
\end{aligned}$$

as $\|u\|_{1,p} \to \infty$. □

The properties of $\Theta$ allow the construction of the Galerkin approximations:

**Lemma 3.2.4.** *For all $k \in \mathbb{N}$ there exists $u_k \in V_k$ such that*

$$\langle T(u_k), w \rangle = 0 \quad \textit{for all } w \in V_k. \tag{3.2.4}$$

*Proof.* We have by Lemma 3.2.3, $\Theta(a).a \to \infty$ as $\|a\|_{\mathbb{R}^r} \to \infty$. Then there exists $R > 0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $\Theta(a).a > 0$. The usual topological argument [98] implies that $\Theta(x) = 0$ has a solution $x \in B_R(0)$. Hence, for all $k$ there exists $u_k \in V_k$ such that $\langle T(u_k), w \rangle = 0$ for all $k \in \mathbb{N}$. □

Similar to Lemma 3.1.3, we have $(u_k)$ is uniformly bounded in $W_0^{1,p}(\Omega; \mathbb{R}^m)$. According to Lemma 2.3.1 there exists a Young measure $\nu_x$ generated by $Du_k$ in $L^p(\Omega; \mathbb{M}^{m\times n})$ satisfying the properties of Lemma 3.1.4. Now, before we pass to the limit in the approximating equations, we still need some properties satisfied by $\nu_x$.

Now, the following lemma will serve us to pass to the limit in the approximating equations.

**Lemma 3.2.5.** *If $\sigma$ satisfy **(H1)-(H3)** and $\{Du_k\}$ generates the Young measure $\nu_x$, then the following inequality holds:*

$$\liminf_{k\to\infty} \int_\Omega \big(\sigma(x, Du_k) - \sigma(x, Du)\big) : (Du_k - Du) dx \le 0. \tag{3.2.5}$$

*Proof.* Let us consider the sequence

$$\begin{aligned} I_k &:= \big(\sigma(x, Du_k) - \sigma(x, Du)\big) : (Du_k - Du) \\ &= \sigma(x, Du_k) : (Du_k - Du) - \sigma(x, Du) : (Du_k - Du) \\ &=: I_{k,1} + I_{k,2}. \end{aligned}$$

Remark that since $\sigma(.) \in L^{p'}(\Omega)$ we deduce by Lemma 3.1.4

$$\begin{aligned} \liminf_{k\to\infty} \int_\Omega \sigma(x, Du) : (Du_k - Du)dx &= \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, Du) : (\lambda - Du)d\nu_x(\lambda)dx \\ &= \int_\Omega \sigma(x, Du) : \Big( \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) - Du\Big)dx = 0. \end{aligned}$$

Using Mazur's theorem (see e.g. [136, Theorem 2, p120]) there exists a sequence $(v_k) \subset W_0^{1,p}(\Omega; \mathbb{R}^m)$ such that $v_k \to u$ in $W_0^{1,p}(\Omega; \mathbb{R}^m)$ where each $v_k$ is a convex linear combination of $\{u_1, .., u_k\}$. Thus $v_k \in V_k$. Taking $u_k - v_k$ as a test function in (3.2.4), we get

$$\int_\Omega \sigma(x, Du_k) : (Du_k - Dv_k)dx = \langle f, u_k - v_k\rangle - \int_\Omega \phi(u_k) : (Du_k - Dv_k)dx.$$

Notice that since $\phi$ is linear and continuous and $(u_k)$ is bounded then $\phi(u_k)$ is bounded. By Hölder's inequality we have

$$\begin{aligned} &\Big|\langle f, u_k - v_k\rangle - \int_\Omega \phi(u_k) : (Du_k - Dv_k)dx\Big| \\ &\qquad \le \|f\|_{-1,p'}\|u_k - v_k\|_{1,p} + c_1\|Du_k - Dv_k\|_1 \longrightarrow 0 \end{aligned}$$

by definition of $v_k$, $1 < p$ and

$$\|Du_k - Dv_k\|_p \le \|Du_k - Du\|_p + \|Dv_k - Du\|_p \longrightarrow 0 \quad \text{as } k \to \infty.$$

Thus

$$\int_\Omega \sigma(x, Du_k) : (Du_k - Dv_k)dx \longrightarrow 0 \quad \text{as } k \to \infty.$$

Using this fact and the construction of $v_k$ to deduce that

$$\begin{aligned}
&\liminf_{k\to\infty}\int_\Omega I_k dx\\
&\qquad= \liminf_{k\to\infty}\int_\Omega I_{k,1}dx\\
&\qquad= \liminf_{k\to\infty}\int_\Omega \sigma(x,Du_k):(Du_k-Du)dx\\
&\qquad= \liminf_{k\to\infty}\Big(\int_\Omega \sigma(x,Du_k):(Du_k-Dv_k)dx+\int_\Omega \sigma(x,Du_k):(Dv_k-Du)dx\Big)\\
&\qquad= \liminf_{k\to\infty}\int_\Omega \sigma(x,Du_k):(Dv_k-Du)dx\\
&\qquad\le \liminf_{k\to\infty}\big\||\sigma(x,Du_k)|\big\|_{p'}\|Dv_k-Du\|_p=0.
\end{aligned}$$

Therefore

$$\liminf_{k\to\infty}\int_\Omega \big(\sigma(x,Du_k)-\sigma(x,Du)\big):(Du_k-Du)dx\le 0.$$

□

We have also the following localization of the support of $\nu_x$.

**Lemma 3.2.6.** *Suppose (3.2.5) holds, then for almost every $x\in\Omega$*

$$\big(\sigma(x,\lambda)-\sigma(x,Du)\big):(\lambda-Du)=0 \text{ on } \operatorname{supp}\nu_x.$$

*Proof.* From Lemma 3.2.5 we may deduce the following intermediary result, namely div-curl inequality:

$$\int_\Omega\int_{\mathbb{M}^{m\times n}}\big(\sigma(x,\lambda)-\sigma(x,Du)\big):(\lambda-Du)d\nu_x(\lambda)dx\le 0.$$

The naming div-curl inequality is explained by Remark 3.3 in [15]. Indeed, by Lemma 3.2.5 we have

$$\liminf_{k\to\infty}\int_\Omega \sigma(x,Du_k):(Du_k-Du)dx\le 0.$$

Since $\{\sigma(x,Du_k):(Du_k-Du)\}$ is equiintegrable, it follows by Lemma 2.3.2 that

$$\int_\Omega\int_{\mathbb{M}^{m\times n}}\sigma(x,\lambda):(\lambda-Du)d\nu_x(\lambda)dx\le\liminf_{k\to\infty}\int_\Omega \sigma(x,Du_k):(Du_k-Du)dx\le 0.$$

We have that
$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, Du) : (\lambda - Du) d\nu_x(\lambda) dx = 0.$$
Put these results into consideration, we deduce the intermediary result. Now, the monotonicity of $\sigma$ implies that the integral in our intermediary result is non-negative, thus must vanish with respect to the product measure $d\nu_x(\lambda) \otimes dx$. Consequently, for almost every $x \in \Omega$

$$\big(\sigma(x, \lambda) - \sigma(x, Du)\big) : (\lambda - Du) = 0 \text{ on } \operatorname{supp} \nu_x.$$

□

### 3.2.3 Proof of Theorem 3.2.1

Before we present the proof, it is useful to note that, the cases **(H3)**(c) and (d) permit to deduce
$$Du_k \to Du \text{ in measure on } \Omega. \tag{3.2.6}$$
However, this property does not satisfy in the other cases (a) and (b). Let start with the easiest case:
**Case (c)**: By the strict monotonicity of $\sigma$ and Lemma 3.2.6, we deduce that $\operatorname{supp} \nu_x = \{Du(x)\}$ which implies $\nu_x = \delta_{Du(x)}$ for a.e. $x \in \Omega$. We infer from Proposition 2.3.1 that $Du_k \to Du$ in measure on $\Omega$.
**Case (d)**: Remark that for a positive constant $c$

$$\int_\Omega |Du_k - Du|^p dx \le c \int_\Omega \big(\sigma(x, Du_k) - \sigma(x, Du)\big) : (Du_k - Du) dx.$$

Passing to the limit inf and using Lemma 3.2.5, we infer that

$$\lim_{k\to\infty} \int_\Omega |Du_k - Du|^p dx = 0.$$

This implies $Du_k \to Du$ in measure on $\Omega$.
**Case (a)**: We prove that the identity

$$\sigma(x, \lambda) : \mu = \sigma(x, Du) : \mu + \big(\nabla\sigma(x, Du)\mu\big) : (Du - \lambda) \tag{3.2.7}$$

holds on $\operatorname{supp} \nu_x$, for every $\mu \in \mathbb{M}^{m\times n}$. Here $\nabla$ denotes the derivative with respect to the second variable of $\sigma$. On the one hand, the monotonicity of $\sigma$ implies

$$\begin{aligned}0 &\leq \big(\sigma(x,\lambda) - \sigma(x, Du + \tau\mu)\big) : (\lambda - Du - \tau\mu) \\ &= \sigma(x,\lambda) : (\lambda - Du) - \sigma(x,\lambda) : \tau\mu - \sigma(x, Du + \tau\mu) : (\lambda - Du - \tau\mu),\end{aligned}$$

for each $\tau \in \mathbb{R}$. On the other hand, Lemma 3.2.6 permits to write

$$0 \leq \sigma(x, Du) : (\lambda - Du) - \sigma(x,\lambda) : \tau\mu - \sigma(x, Du + \tau\mu) : (\lambda - Du - \tau\mu).$$

Therefore

$$-\sigma(x,\lambda) : \tau\mu \geq -\sigma(x, Du) : (\lambda - Du) + \sigma(x, Du + \tau\mu) : (\lambda - Du - \tau\mu).$$

Note that

$$\begin{aligned}&\sigma(x, Du + \tau\mu) : (\lambda - Du - \tau\mu) \\ &\quad= \sigma(x, Du + \tau\mu) : (\lambda - Du) - \sigma(x, Du + \tau\mu) : \tau\mu \\ &\quad= \sigma(x, Du) : (\lambda - Du) + \nabla\sigma(x, Du)\tau\mu : (\lambda - Du) - \sigma(x, Du) : \tau\mu \\ &\qquad\quad - \nabla\sigma(x, Du)\tau\mu : \tau\mu + o(\tau) \\ &\quad= \sigma(x, Du) : (\lambda - Du) + \tau\big[\big(\nabla\sigma(x, Du)\mu\big) : (\lambda - Du) - \sigma(x, Du) : \mu\big] + o(\tau).\end{aligned}$$

It follows that

$$-\sigma(x,\lambda) : \tau\mu \geq \tau\big[\big(\nabla\sigma(x, Du)\mu\big) : (\lambda - Du) - \sigma(x, Du) : \mu\big] + o(\tau).$$

Since $\tau$ is arbitrary in $\mathbb{R}$, the above inequality implies (3.2.7).
The equation (3.2.7) together with the equiintegrability of the sequence $\sigma(x, Du_k)$ allow

to deduce the weak $L^1$-limit $\overline{\sigma}$ of $\sigma(x, Du_k)$ as follows:

$$\begin{aligned}\overline{\sigma} &= \int_{\operatorname{supp}\nu_x} \sigma(x,\lambda) d\nu_x(\lambda) \\ &= \int_{\operatorname{supp}\nu_x} \sigma(x,Du) d\nu_x(\lambda) + \big(\nabla\sigma(x,Du)\big)^t : \underbrace{\int_{\operatorname{supp}\nu_x} (Du-\lambda) d\nu_x(\lambda)}_{=0} \\ &= \sigma(x,Du).\end{aligned}$$

**Case (b)**: Let show that for almost every $x \in \Omega$, $\operatorname{supp}\nu_x \subset K_x$ where

$$K_x = \Big\{\lambda \in \mathbb{M}^{m\times n} : W(x,\lambda) = W(x,Du) + \sigma(x,Du) : (\lambda - Du)\Big\}.$$

If $\lambda \in \operatorname{supp}\nu_x$, by Lemma 3.2.6 it follows that

$$(1-\tau) : \big(\sigma(x,\lambda) - \sigma(x,Du)\big) : (\lambda - Du) = 0 \text{ for all } \tau \in [0,1]. \tag{3.2.8}$$

Due to the monotonicity of $\sigma$, we have for $\tau \in [0,1]$

$$(1-\tau) : \big(\sigma(x,Du+\tau(\lambda-Du)) - \sigma(x,\lambda)\big) : (Du-\lambda) \geq 0. \tag{3.2.9}$$

Therefore, by subtracting (3.2.8) from (3.2.9), we get

$$(1-\tau) : \big(\sigma(x,Du+\tau(\lambda-Du)) - \sigma(x,Du)\big) : (Du-\lambda) \geq 0. \tag{3.2.10}$$

The monotonicity of $\sigma$ allows again to write

$$\big(\sigma(x,Du+\tau(\lambda-Du)) - \sigma(x,Du)\big) : \tau(\lambda-Du) \geq 0,$$

and since $\tau \in [0,1]$, we have then

$$\big(\sigma(x,Du+\tau(\lambda-Du)) - \sigma(x,Du)\big) : (1-\tau)(\lambda-Du) \geq 0.$$

From the last inequality and Eq. (3.2.10) we deduce

$$\big(\sigma(x, Du + \tau(\lambda - Du)) - \sigma(x, Du)\big) : (\lambda - Du) = 0, \tag{3.2.11}$$

for $\tau \in [0,1]$ and $\lambda \in \operatorname{supp} \nu_x$. Therefore

$$\sigma(x, Du + \tau(\lambda - Du)) : (\lambda - Du) = \sigma(x, Du) : (\lambda - Du).$$

Integrate this equality over $[0,1]$ and using the fact that $\sigma = \frac{\partial W}{\partial \xi}$ to deduce that

$$\begin{aligned} W(x,\lambda) &= W(x, Du) + \int_0^1 \sigma(x, Du + \tau(\lambda - Du)) : (\lambda - Du) d\tau \\ &= W(x, Du) + \sigma(x, Du) : (\lambda - Du). \end{aligned}$$

Consequently $\lambda \in K_x$, i.e. $\operatorname{supp} \nu_x \subset K_x$. By the convexity of $W$ we can write

$$W(x,\lambda) \geq W(x, Du) + \sigma(x, Du) : (\lambda - Du) \quad \forall \lambda \in \mathbb{M}^{m\times n}. \tag{3.2.12}$$

Put $A(\lambda)$ (resp. $B(\lambda)$) the left (resp. the right) hand side of (3.2.12). Since $\lambda \mapsto A(\lambda)$ is $C^1$, it follows for $\tau \in \mathbb{R}$ and $\xi \in \mathbb{M}^{m\times n}$ that

$$\frac{A(\lambda + \tau\xi) - A(\lambda)}{\tau} \geq \frac{B(\lambda + \tau\xi) - B(\lambda)}{\tau} \ \text{ if } \tau > 0,$$

$$\frac{A(\lambda + \tau\xi) - A(\lambda)}{\tau} \leq \frac{B(\lambda + \tau\xi) - B(\lambda)}{\tau} \ \text{ if } \tau < 0.$$

Hence $D_\lambda A = D_\lambda B$. Therefore

$$\sigma(x,\lambda) = \sigma(x, Du) \ \text{ for all } \lambda \in \operatorname{supp} \nu_x \subset K_x.$$

We have then

$$\begin{aligned} \overline{\sigma} = \int_{\mathbb{M}^{m\times n}} \sigma(x,\lambda) d\nu_x(\lambda) &= \int_{\operatorname{supp} \nu_x} \sigma(x, Du) d\nu_x(\lambda) \\ &= \sigma(x, Du) \int_{\operatorname{supp} \nu_x} d\nu_x(\lambda) = \sigma(x, Du). \end{aligned}$$

## Conclusion

For the cases (c) and (d), we have $Du_k \to Du$ in measure on $\Omega$. Now, let $E_{k,\epsilon} = \{x : |u_k(x) - u(x)| \geq \epsilon\}$, then

$$\int_\Omega |u_k(x) - u(x)|^p dx \geq \int_{E_{k,\epsilon}} |u_k(x) - u(x)|^p dx \geq \epsilon^p |E_{k,\epsilon}|,$$

which implies

$$|E_{k,\epsilon}| \leq \frac{1}{\epsilon^p} \int_\Omega |u_k(x) - u(x)|^p dx \to 0 \quad \text{as } k \to 0.$$

Hence $u_k \to u$ in measure for $k \to \infty$. After extracting a suitable subsequence (if necessary), we can infer that $u_k \to u$ and $Du_k \to Du$ for almost every $x \in \Omega$. Then $\sigma(x, Du_k) \to \sigma(x, Du)$ and $\phi(u_k) \to \phi(u)$ almost everywhere, by continuity of $\sigma$ and $\phi$. Furthermore, we have $\sigma(x, Du_k) \to \sigma(x, Du)$ and $\phi(u_k) \to \phi(u)$ in measure. Since $(\sigma(x, Du_k) : Dw)$ and $(\phi(u_k) : Dw)$ are equiintegrable, it follows by Vitali's theorem that

$$\int_\Omega \big(\sigma(x, Du_k) - \sigma(x, Du)\big) : Dwdx \to 0 \quad \text{as } k \to \infty$$

and

$$\int_\Omega \big(\phi(u_k) - \phi(u)\big) : Dwdx \to 0 \quad \text{as } k \to \infty.$$

The proof of Theorem 3.2.1 follows for the cases (c) and (d).

Now, in the cases (a) and (b) we have $\overline{\sigma} = \sigma(x, Du)$. Since $L^p(\Omega; \mathbb{M}^{m\times n})$ is reflexive, the sequences $\{\sigma(x, Du_k)\}$ and $\{\phi(u_k)\}$ converges weakly in $L^{p'}(\Omega; \mathbb{M}^{m\times n})$ and their weak $L^{p'}$-limits are $\sigma(x, Du)$ and $\phi(u)$ (respectively). Hence

$$\int_\Omega \big[\big(\sigma(x, Du_k) - \sigma(x, Du)\big) : Dw + \big(\phi(u_k) - \phi(u)\big) : Dw\big] dx \to 0 \text{ as } k \to \infty.$$

Thus Theorem 3.2.1 follows also for the case (a). For the last case (b), we argue as follows: we consider the Carathéodory function

$$h(x, \lambda) = |\sigma(x, \lambda) - \overline{\sigma}(x)|, \ \lambda \in \mathbb{M}^{m\times n}.$$

Since $h_k(x) := h(x, Du_k)$ is equiintegrable, then

$$h_k \rightharpoonup \overline{h} \quad \text{weakly in } L^1(\Omega),$$

where $\overline{h}$ is given by

$$\begin{aligned}\overline{h}(x) &= \int_{\mathbb{M}^{m\times n}} |\sigma(x,\lambda) - \overline{\sigma}(x)| d\nu_x(\lambda) \\ &= \int_{\operatorname{supp} \nu_x} |\sigma(x,\lambda) - \overline{\sigma}(x)| d\nu_x(\lambda) = 0 \qquad (\text{since } \overline{\sigma} = \sigma(x, Du) = \sigma(x,\lambda)).\end{aligned}$$

Hence

$$\int_\Omega |\sigma(x, Du_k) - \sigma(x, Du)| dx \to 0 \quad (\text{since } h_k \geq 0).$$

Therefore, by Vitali's theorem

$$\int_\Omega \left[\big(\sigma(x, Du_k) - \sigma(x, Du)\big) : Dw + \big(\phi(u_k) - \phi(u)\big) : Dw\right] dx \to 0 \text{ as } k \to \infty.$$

This again accomplishes the proof of Theorem 3.2.1 in the case (b).

## 3.3 Quasilinear elliptic system with perturbed gradient

We consider weak solutions to the Dirichlet problem of quasilinear elliptic system

$$\begin{cases} -\operatorname{div} A\big(x, Du - \Theta(u)\big) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.3.1}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, $\Theta : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is a Lipschitz continuous function, $A : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ satisfies a Carathéodory condition and the source term $f$ belongs to $W^{-1,p'}(\Omega; \mathbb{R}^m)$, $1/p + 1/p' = 1$.

### 3.3.1 Introduction and main result

The motivation in the study of this type of problem (3.3.1) is twofold. On the one hand, consider $A(x, Du - \Theta(u)) = |Du - \Theta(u)|^{p-2}(Du - \Theta(u))$ (i.e., generalized $p$-Laplacian), then (3.3.1) becomes

$$\begin{cases} -\operatorname{div}\left(|Du - \Theta(u)|^{p-2}(Du - \Theta(u))\right) & = f \quad \text{in } \Omega, \\ u & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.3.2}$$

The problem (3.3.2) was considered in [18] (cf. Section 3.1), where we have proved the existence of weak solutions using the concept of Young measures under the condition that $\Theta : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is a continuous function satisfy

$$\Theta(0) = 0 \quad \text{and} \quad \left|\Theta(u) - \Theta(v)\right| \le c|u - v| \quad \forall u, v \in \mathbb{R}^m.$$

On the other hand, for $\Theta \equiv 0$ or $A(x, Du - \Theta(u)) = \sigma(x, u, Du)$ then (3.3.1) becomes

$$\begin{cases} -\operatorname{div} \sigma(x, u, Du) & = f \quad \text{in } \Omega, \\ u & = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.3.3}$$

and this problem has been studied in [85] under classical regularity, growth and coercivity conditions, but with only mild monotonicity assumptions on $\sigma$.

Our interest here, is to extend the result of [18] (i.e., Eq. (3.3.2)) to the case of [85] (i.e., (3.3.3)) by considering the problem (3.3.1) with a perturbation in the symmetric part of the gradient. We will prove the existence of weak solutions under some conditions on the functions $A$ and $\Theta$ based on the Galerkin method to construct the approximating solutions and Young measures to identify weak limits and to pass to the limit in the approximating equations.

For several decades, there have been intensive research activities for equations ($m = 1$)/systems ($m > 1$) of $p$-Laplacian type. DiBenedetto and Manfredi [57] have proved the existence of local weak solutions and some estimates of $Du$ in $[BMO_{\text{loc}}(\Omega)]^{nm}$ for the system $\operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F)$, for $F \in L^p_{\text{loc}}(\Omega; \mathbb{R}^m)$. The quasilinear elliptic equation $\operatorname{div}\left((MDu.Du)^{\frac{p-2}{p}} MDu\right) = \operatorname{div}(|F|^{p-2}F)$ was considered in [104] where the authors have established a regularity result under minimal assumptions on the coefficient matrix $M$. The problem $-\Delta_p u = \pm|\nabla u|^\nu + f(x,u)$, $u \geq 0$ in $\Omega$ and $u = 0$ on $\partial\Omega$, has been investigated in [112]. In this paper the author analysed the interaction between the gradient term and the function $f$ to obtain existence results. For more results, see [41, 52, 123, 128].

In [46], Bulíček and others have established existence, uniqueness and optimal regularity results for very weak solutions to the system

$$\begin{cases} -\operatorname{div} A(x, Du) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.3.4}$$

When the right hand side in (3.3.4) is equal to $\operatorname{div}(|f|^{p-2}f)$, Bulíček and Schwarzacher [47] have studied the corresponding system under mild *a priori* estimate that recovers a duality relation with the right hand side. The existence of three weak solutions ($m = 1$) is proved in [37] by exploiting variational methods to the problem (3.3.4) in the case where $f \equiv \lambda k(x) f(u)$. In [27] (cf. Section 3.2) we have considered the following system

$$\begin{cases} -\operatorname{div}\big(\sigma(x, Du) + \phi(u)\big) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

for $\phi : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ linear, continuous and satisfy $|\phi(u)| \leq c$, and proved the existence of weak solutions using the theory of Young measures. For $m = 1$ and general $p, q$-growth conditions, Cupini et al. [54] have proved that the Dirichlet problem

$$\begin{cases} \sum_{i=1}^n \frac{\partial}{\partial x_i} a^i(x, Du) &= b(x) \quad \text{in } \Omega, \\ u &= u_0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.3.5}$$

has a weak solution $u \in W^{1,q}_{\text{loc}}(\Omega)$ under the assumptions $1 < p \leq q \leq p + 1$ and $q < p\frac{n-1}{n-q}$. As we know, problems of type (3.3.5) comes from derivation of the energy integral $\int_\Omega f(x, Du)dx$ with respect to the gradient. In this sense, the authors Cupini, Marcellini, Mascolo have studied in [55] the local boundedness of minimizers of the above nonuniformly energy integral under $p, q$-growth conditions of the type

$$\lambda(x)|\xi|^p \leq f(x, \xi) \leq \mu(x)(1 + |\xi|^q) \quad \text{for some } q \geq p > 1.$$

See also [65, 102, 109] for more results and [118] for different methods used to solve nonlinear problems.

In order to state the main result of this section, we need the following assumptions:
**(A0)** $\Theta : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is a continuous function satisfy

$$\Theta(0) = 0 \quad \text{and} \quad |\Theta(a) - \Theta(b)| \leq C_\Theta |a-b| \quad \forall a, b \in \mathbb{R}^m,$$

where $C_\Theta$ is a positive constant related to the exponent $p$ and the diameter of $\Omega$ ($\operatorname{diam}(\Omega)$) by the following

$$C_\Theta \leq \frac{1}{2\operatorname{diam}(\Omega)}. \tag{3.3.6}$$

**(A1)** $A : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, that is, $\xi \to A(x,\xi)$ is continuous for a.e. $x \in \Omega$, and $x \to A(x,\xi)$ is measurable for all $\xi \in \mathbb{M}^{m\times n}$.
**(A2)** There exist $d_1 \in L^{p'}(\Omega)$, $d_2 \in L^1(\Omega)$ and $\alpha_0 > 0$ such that

$$|A(x, \xi - \Theta(s))| \leq d_1(x) + |\xi - \Theta(s)|^{p-1},$$

$$A(x, \xi - \Theta(s)) : \xi \geq \alpha_0 |\xi - \Theta(s)|^p - d_2(x), \quad \forall (s,\xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}.$$

**(A3)** The function $A$ satisfies one of the following (monotonicity) conditions:

(1) for all $x \in \Omega$ and all $u \in \mathbb{R}^m$, the map $\xi \mapsto A(x, \xi - \Theta(u))$ is a $C^1$-function and is monotone, that is,

$$\big(A(x, \xi - \Theta(u)) - A(x, \eta - \Theta(u))\big) : (\xi - \eta) \geq 0, \quad \forall \xi, \eta \in \mathbb{M}^{m\times n}.$$

(2) there exists a function (potential) $B : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $A(x, \xi - \Theta(u)) = \big(\partial B / \partial \xi\big)(x, \xi - \Theta(u)) := D_\xi B(x, \xi - \Theta(u))$, and $\xi \mapsto B(x, \xi - \Theta(u))$ is convex and $C^1$-function for all $x \in \Omega$ and $u \in \mathbb{R}^m$.

(3) $A$ is strictly monotone, that is, $A$ is monotone and

$$\big(A(x, \xi - \Theta(u)) - A(x, \eta - \Theta(u))\big) : (\xi - \eta) = 0 \quad \text{implies} \quad \xi = \eta.$$

(4) $A$ is strictly quasimonotone, that is, there exists $\alpha_1 > 0$ such that

$$\int_\Omega \big(A(x, Du - \Theta(u)) - A(x, Dv - \Theta(u))\big) : (Du - Dv)dx \geq \alpha_1 \int_\Omega |Du - Dv|^p dx.$$

**Remark 3.3.1.** *(i) The choice of $C_\Theta$ in (3.3.6) allows to prove that the operator $T$ defined in the Subsection 3.3.3 is coercive.*
*(iii) The result can for example be applied for finding weak solutions to the model given in [18] (cf. Section 3.1).*

We are now in the position to state the main result of this section.

**Theorem 3.3.1.** *Suppose that **(A0)-(A3)** hold. Then problem (3.3.1) possesses at least one weak solution $u \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ for every $f \in W^{-1,p'}(\Omega; \mathbb{R}^m)$.*

### 3.3.2 A convergence result for $A$

This subsection is devoted firstly to present a general convergence result for $A$ which will be proved in the next subsection, secondly to show an elliptic div-curl inequality. This inequality will serve us to pass to the limit in the approximating equations and prove Theorem 3.3.1. The hypotheses in question are the following:

**(H1)** The sequence $(u_k)$ is uniformly bounded in $W_0^{1,p}(\Omega; \mathbb{R}^m)$ for some $p > 1$ and hence a subsequence converges weakly in $W_0^{1,p}(\Omega; \mathbb{R}^m)$ to an element denoted by $u$.

**(H2)** $A : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function.

**(H3)** The sequence $A_k(x) := A(x, Du_k - \Theta(u_k))$ is uniformly bounded in $L^{p'}(\Omega; \mathbb{M}^{m\times n})$ and hence equiintegrable.

**(H4)** The sequence $(A_k(x) : Du_k)^-$ is equiintegrable.

**(H5)** There exists a sequence $(v_k)$ such that $v_k \to u$ in $W_0^{1,p}(\Omega; \mathbb{R}^m)$ and $\int_\Omega A_k(x) : (Du_k - Dv_k)dx \to 0$ as $k \to \infty$.

Remark first that, since $(u_k)$ is bounded in $W_0^{1,p}(\Omega; \mathbb{R}^m)$ by **(H1)**, then according to Lemma 2.3.1 there exists a Young measure $\nu_x$ generated by $Du_k$ in $L^p(\Omega; \mathbb{M}^{m\times n})$. Moreover, this Young measure $\nu_x$ satisfy the properties $(i) - (iii)$ of Lemma 3.1.4.

Now, we can prove the following div-curl inequality under the hypothesis **(H1)-(H5)**.

**Lemma 3.3.1.** *The Young measure $\nu_x$ generated by $Du_k$ in $L^p(\Omega;\mathbb{M}^{m\times n})$ satisfies the following inequality:*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} A(x,\lambda-\Theta(u)) : \lambda d\nu_x(\lambda)dx \leq \int_\Omega \int_{\mathbb{M}^{m\times n}} A(x,\lambda-\Theta(u)) : Du d\nu_x(\lambda)dx.$$

*Proof.* Consider the sequence

$$\begin{aligned} I_k &:= A(x,Du_k-\Theta(u_k)) : (Du_k - Du) \\ &= A_k(x) : Du_k - A_k(x) : Du. \end{aligned}$$

On the one hand, the hypothesis **(H3)** and **(H4)** implies that $I_k^-$ is equiintegrable. On the other hand, the hypothesis **(H1)** gives (up to a subsequence) $u_k \to u$ in measure and almost everywhere in $\Omega$. The continuity of $\Theta$ implies that $\Theta(u_k) \to \Theta(u)$ almost everywhere. Owing to Lemma 2.3.2, one gets

$$\begin{aligned} I &:= \liminf_{k\to\infty} \int_\Omega I_k dx \\ &\geq \int_\Omega \int_{\mathbb{M}^{m\times n}} A(x,\lambda-\Theta(u)) : (\lambda - Du) d\nu_x(\lambda)dx. \end{aligned}$$

To get the needed inequality, we show that $I \leq 0$. To do this, by **(H5)** we have

$$\begin{aligned} I &= \liminf_{k\to\infty} \int_\Omega A_k(x) : (Du_k - Du)dx \\ &= \liminf_{k\to\infty} \Big( \int_\Omega A_k(x) : (Du_k - Dv_k)dx + \int_\Omega A_k(x) : (Dv_k - Du)dx \Big) \\ &= \liminf_{k\to\infty} \int_\Omega A_k(x) : (Dv_k - Du)dx \\ &\leq \liminf_{k\to\infty} \underbrace{\| A_k \| \|_{p'}}_{\leq c} \| Dv_k - Du\|_p = 0, \end{aligned}$$

by Hölder's inequality and **(H3)**. □

As a consequence of Lemma 3.3.1, we have the following localization of the support of $\nu_x$ (following the proof of Lemma 3.2.6):

$$\big(A(x, \lambda - \Theta(u)) - A(x, Du - \Theta(u))\big) : (\lambda - Du) = 0 \quad \text{on } \operatorname{supp} \nu_x. \tag{3.3.7}$$

The convergence result for $A$ is given in the following.

**Proposition 3.3.1.** *Suppose that* **(H1)-(H5)** *are fulfilled and one of the conditions listed in* **(A3)** *holds. Then (after passage to a subsequence) the sequence $A_k$ converges weakly in the space $L^1(\Omega; \mathbb{M}^{m\times n})$ as $k \to \infty$ and the weak limit $\overline{A}$ is given by*

$$\overline{A}(x) = A(x, Du - \Theta(u)).$$

*If* **(A3)***(2), (3) or (4) holds, then $A(x, Du_k - \Theta(u_k)) \to A(x, Du - \Theta(u))$ in $L^1(\Omega; \mathbb{M}^{m\times n})$. In cases* **(A3)***(3) and (4), it follows in addition that (after extraction of a further subsequence, if necessary) $Du_k \to Du$ in measure and almost everywhere in $\Omega$.*

*Proof.* The proof will be divided into four cases listed in **(A3)**.

**Case (A3)(1):** We claim that for almost all $x \in \Omega$ the following identity holds on the support of $\nu_x$

$$A(x, \lambda - \Theta(u)) : \xi = A(x, Du - \Theta(u)) : \xi + \big(\nabla A(x, Du - \Theta(u))\xi\big) : (Du - \lambda), \tag{3.3.8}$$

for all $\xi \in \mathbb{M}^{m\times n}$, where $\nabla$ is the derivative of $A$ with respect to its second variable (i.e., with respect to the gradient).
By the monotonicity of $A$ we have for all $t \in \mathbb{R}$

$$\big(A(x, \lambda - \Theta(u)) - A(x, Du - \Theta(u) + t\xi)\big) : (\lambda - Du - t\xi) \geq 0, \tag{3.3.9}$$

which implies (by Eq. (3.3.7))

$$\begin{aligned}
&-A(x,\lambda-\Theta(u)):t\xi\\
&\geq -A(x,\lambda-\Theta(u)):(\lambda-Du)+A(x,Du-\Theta(u)+t\xi)):(\lambda-Du-t\xi)\\
&=-A(x,Du-\Theta(u)):(\lambda-Du)+A(x,Du-\Theta(u)+t\xi)\big):(\lambda-Du-t\xi).
\end{aligned}$$

We have $A$ is of class $C^1$, thus

$$A(x,Du-\Theta(u)+t\xi)=A(x,Du-\Theta(u))+\nabla A(x,Du-\Theta(u))t\xi+o(t),$$

hence

$$\begin{aligned}
&-A(x,\lambda-\Theta(u)):t\xi\\
&\qquad\geq t\Big(\nabla A(x,Du-\Theta(u))\xi:(\lambda-Du)-A(x,Du-\Theta(u)):\xi\Big)+o(t).
\end{aligned}$$

Since $t$ is arbitrary in $\mathbb{R}$, then our claim (3.3.8) follows. As in the proof of Lemma 3.3.1, $\Theta(u_k)\to\Theta(u)$ almost everywhere for $k\to\infty$. The Vitali Convergence Theorem implies

$$\Theta(u_k)\to\Theta(u)\quad\text{in}\quad L^1(\Omega),$$

by **(A0)**. Using above results and the fact that $A_k$ is equiintegrable by **(H3)**, it follows that its weak $L^1$-limit $\overline{A}$ is given by

$$\begin{aligned}
\overline{A}(x) &:= \int_{\mathbb{M}^{m\times n}} A(x,\lambda-\Theta(u))d\nu_x(\lambda)\\
&= \int_{\operatorname{supp}\nu_x} A(x,\lambda-\Theta(u))d\nu_x(\lambda)\\
&\overset{(3.3.8)}{=} \int_{\operatorname{supp}\nu_x}\Big(A(x,Du-\Theta(u))+\big(\nabla A(x,Du-\Theta(u))\big):(Du-\lambda)\Big)d\nu_x(\lambda)\\
&= A(x,Du-\Theta(u))\underbrace{\int_{\operatorname{supp}\nu_x} d\nu_x(\lambda)}_{=:1}+\big(\nabla A(x,Du-\Theta(u))\big)^t\underbrace{\int_{\operatorname{supp}\nu_x}(Du-\lambda)d\nu_x(\lambda)}_{=:0}\\
&= A(x,Du-\Theta(u)).
\end{aligned}$$

**Case (A3)(2):** In this case, we show that for almost all $x \in \Omega$, the support of $\nu_x$ is in the set where $B$ (the potential) agrees with the supporting hyper-plane

$$E \equiv \Big\{\Big(\lambda, B(x, Du - \Theta(u)) + A(x, Du - \Theta(u)) : (\lambda - Du)\Big)\Big\},$$

i.e., to show that $\operatorname{supp} \nu_x \subset K_x$ where $K_x$ is the set of $\lambda \in \mathbb{M}^{m\times n}$ such that

$$B(x, \lambda - \Theta(u)) = B(x, Du - \Theta(u)) + A(x, Du - \Theta(u)) : (\lambda - Du).$$

Let $\lambda \in \operatorname{supp} \nu_x$, then by (3.3.7) we have for all $t \in [0,1]$

$$(1-t)\big(A(x, Du - \Theta(u)) - A(x, \lambda - \Theta(u))\big) : (Du - \lambda) = 0. \tag{3.3.10}$$

From the monotonicity of $A$ and Eq. (3.3.10) it follows that

$$\begin{aligned} 0 &\le (1-t)\big(A(x, Du - \Theta(u) + t(\lambda - Du)) - A(x, \lambda - \Theta(u))\big) : (Du - \lambda) \\ &= (1-t)\big(A(x, Du - \Theta(u) + t(\lambda - Du)) - A(x, Du - \Theta(u))\big) : (Du - \lambda). \end{aligned} \tag{3.3.11}$$

The monotonicity of $A$ allows again to write

$$\big(A(x, Du - \Theta(u) + t(\lambda - Du)) - A(x, Du - \Theta(u))\big) : t(Du - \lambda) \le 0$$

which gives since $t \in [0,1]$

$$\big(A(x, Du - \Theta(u) + t(\lambda - Du)) - A(x, Du - \Theta(u))\big) : (1-t)(Du - \lambda) \le 0.$$

The above inequality and Eq. (3.3.11) imply that

$$\big(A(x, Du - \Theta(u) + t(\lambda - Du)) - A(x, Du - \Theta(u))\big) : (\lambda - Du) = 0 \quad \forall t \in [0,1]. \tag{3.3.12}$$

By hypothesis, we have

$$A(x, Du - \Theta(u) + t(\lambda - Du)) : (\lambda - Du) = \frac{\partial B}{\partial t}(x, Du - \Theta(u) + t(\lambda - Du)) : (\lambda - Du).$$

Integrate the above equation over $[0,1]$, it results

$$
\begin{aligned}
B(x, &\lambda - \Theta(u)) \\
&= B(x, Du - \Theta(u)) + \int_0^1 A(x, Du - \Theta(u) + t(\lambda - Du)) : (\lambda - Du)dt \\
&\overset{(3.3.12)}{=} B(x, Du - \Theta(u)) + A(x, Du - \Theta(u)) : (\lambda - Du).
\end{aligned}
$$

Therefore $\lambda \in K_x$, i.e., $\operatorname{supp} \nu_x \subset K_x$ for almost all $x \in \Omega$. The convexity of $A$ allows to write

$$
\begin{aligned}
B(x, \lambda - \Theta(u)) &=: F_1(\lambda) \\
&\geq B(x, Du - \Theta(u)) + A(x, Du - \Theta(u)) : (\lambda - Du) =: F_2(\lambda).
\end{aligned}
$$

Since the mapping $\lambda \mapsto F_1(\lambda)$ is, by assumption, continuously differentiable, we get

$$
\frac{F_1(\lambda + t\xi) - F_1(\lambda)}{t} \geq \frac{F_2(\lambda + t\xi) - F_2(\lambda)}{t} \quad \text{for } t > 0
$$

and

$$
\frac{F_1(\lambda + t\xi) - F_1(\lambda)}{t} \leq \frac{F_2(\lambda + t\xi) - F_2(\lambda)}{t} \quad \text{for } t < 0.
$$

Hence $D_\lambda F_1 = D_\lambda F_2$, i.e.,

$$
A(x, \lambda - \Theta(u)) = A(x, Du - \Theta(u)) \quad \text{for all } \lambda \in K_x \supset \operatorname{supp} \nu_x. \tag{3.3.13}
$$

Consequently

$$
\begin{aligned}
\overline{A}(x) &= \int_{\mathbb{M}^{m\times n}} A(x, \lambda - \Theta(u)) d\nu_x(\lambda) \\
&\overset{(3.3.13)}{=} \int_{\operatorname{supp} \nu_x} A(x, Du - \Theta(u)) d\nu_x(\lambda) \\
&= A(x, Du - \Theta(u)).
\end{aligned} \tag{3.3.14}
$$

Now consider the Carathéodory function $\sigma(x, u, \lambda) = |A(x, \lambda - \Theta(u)) - \overline{A}(x)|$. On the one hand, since $\Theta(u_k) \to \Theta(u)$ in measure then $(\Theta(u_k), Du_k)$ generates the Young measure $\delta_{\Theta(u(x))} \otimes \nu_x$, by Proposition 2.3.1. On the other hand, the equiintegrability of

$\sigma_k(x) := \sigma(x, u_k, Du_k)$ implies that $\sigma_k \rightharpoonup \overline{\sigma}$ weakly in $L^1(\Omega)$, where $\overline{\sigma}$ is given by

$$\begin{aligned}\overline{\sigma}(x) &= \int_{\mathbb{R}^m \times \mathbb{M}^{m\times n}} \sigma(x,s,\lambda) d\delta_{\Theta(u(x))}(s) \otimes d\nu_x(\lambda) \\ &= \int_{\mathbb{R}^m \times \mathbb{M}^{m\times n}} \left|A(x,\lambda - \Theta(s)) - \overline{A}(x)\right| d\delta_{\Theta(u(x))}(s) \otimes d\nu_x(\lambda) \\ &= \int_{\operatorname{supp} \nu_x} \left|A(x,\lambda - \Theta(u)) - \overline{A}(x)\right| d\nu_x(\lambda) \overset{(3.3.14)}{=} 0.\end{aligned}$$

Since $\sigma_k \geq 0$ it follows that $\sigma_k \to 0$ strongly in $L^1(\Omega)$.

**Case (A3)(3):** Since $A$ is strict monotone, we have $\nu_x = \delta_{Du(x)}$ by Eq. (3.3.7) for almost all $x \in \Omega$. According to the Proposition 2.3.1 it follows that $Du_k \to Du$ in measure as $k \to \infty$. Therefore $u_k \to u$ and $Du_k \to Du$ almost everywhere. From the continuity of $\Theta$ and $A$ we may infer that

$$A(x, Du_k - \Theta(u_k)) \to A(x, Du - \Theta(u)) \quad \text{almost everywhere in } \Omega.$$

Since, by assumption **(H3)**, $A_k$ is equiintegrable, it follows from the Vitali Convergence Theorem that

$$A(x, Du_k - \Theta(u_k)) \to A(x, Du - \Theta(u)) \quad \text{in } L^1(\Omega; \mathbb{M}^{m\times n}) \text{ for } k \to \infty.$$

**Case (A3)(4):** For a positive constant $c > 0$ we have (by hypothesis)

$$\begin{aligned}\int_\Omega |Du_k - Du|^p dx &\leq c \int_\Omega \big(A(x, Du_k - \Theta(u_k)) - A(x, Du - \Theta(u_k))\big) : (Du_k - Du) dx \\ &= c \int_\Omega \big(A(x, Du_k - \Theta(u_k)) - A(x, Du - \Theta(u))\big) : (Du_k - Du) dx \\ &\quad + c \int_\Omega \big(A(x, Du - \Theta(u)) - A(x, Du - \Theta(u_k))\big) : (Du_k - Du) dx.\end{aligned}$$

On the one hand and, by virtue of Lemma 3.1.4 we get

$$\liminf_{k\to\infty} \int_\Omega \big(A(x, Du - \Theta(u)) - A(x, Du - \Theta(u_k))\big) : (Du_k - Du) dx = 0.$$

On the other hand, by passing to the limes inferior in the rest integral (see the proof of Lemma 3.3.1 if necessary, or Lemma 3.2.5) it follows that

$$\lim_{k\to\infty}\int_\Omega |Du_k - Du|^p dx = 0.$$

Therefore $Du_k \to Du$ in measure for $k \to \infty$. We follow the same arguments as in the Case **(A3)**(3) and the proof of Proposition 3.3.1 is complete. □

### 3.3.3 Existence of a weak solution

The aim of this subsection is to construct the approximating solutions by the Galerkin scheme and to prove Theorem 3.3.1. In *a priori* estimates, the following Poincaré inequality will be needed: there exists a positive constant $\alpha = \operatorname{diam}(\Omega)$ such that

$$\|v\|_p \le \alpha \|Dv\|_p. \tag{3.3.15}$$

Let us define the functional $T(u) : W_0^{1,p}(\Omega;\mathbb{R}^m) \to \mathbb{R}$ (for arbitrary $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$) by

$$\langle T(u), \varphi\rangle = \int_\Omega A(x, Du - \Theta(u)) : D\varphi dx - \langle f, \varphi\rangle,$$

for all $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$. Here $\langle .,.\rangle$ is the duality of $\left(W^{-1,p'}(\Omega;\mathbb{R}^m), W_0^{1,p}(\Omega;\mathbb{R}^m)\right)$.

**Lemma 3.3.2.** *The functional $T(u)$ is well defined, linear and bounded.*

*Proof.* On the one hand, the growth condition in **(A2)** allows to estimate $I_1 \equiv \int_\Omega A(x, Du - \Theta(u)) : D\varphi dx$ for each $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ as follows:

$$\begin{aligned}
|I_1| &\leq \int_\Omega |A(x, Du - \Theta(u))||D\varphi| dx \\
&\leq \int_\Omega d_1(x)|D\varphi| dx + \int_\Omega |Du - \Theta(u)|^{p-1}|D\varphi| dx \\
&\leq \|d_1\|_{p'}\|D\varphi\|_p + \Big(\int_\Omega |Du - \Theta(u)|^p dx\Big)^{\frac{1}{p'}} \|D\varphi\|_p \quad \text{(by Hölder's inequality)} \\
&\leq \|d_1\|_{p'}\|D\varphi\|_p + 2^{\frac{(p-1)^2}{p}} \Big(\|Du\|_p^p + \|\Theta(u)\|_p^p\Big)^{\frac{p-1}{p}} \|D\varphi\|_p \quad \text{(by (3.1.4))} \\
&= \Big(\|d_1\|_{p'} + 2^{\frac{(p-1)^2}{p}} \Big(\|Du\|_p^p + \|\Theta(u)\|_p^p\Big)^{\frac{p-1}{p}}\Big) \|D\varphi\|_p.
\end{aligned}$$

On the other hand, the Hölder inequality implies that

$$|I_2| \equiv |\langle f, \varphi\rangle| \leq \|f\|_{-1,p'}\|\varphi\|_{1,p} \leq c\|f\|_{-1,p}\|D\varphi\| \quad \text{(by Poincaré's inequality)}.$$

Since these two expressions are finite by our assumptions, $T(u)$ is well defined. Furthermore, $T(u)$ is linear and finally we have

$$|\langle T(u), \varphi\rangle| \leq |I_1| + |I_2| \leq c\|D\varphi\|_p,$$

that is to say $T(u)$ is bounded. □

As a consequence of the above lemma, we can define the operator

$$\begin{aligned}
T : W_0^{1,p}(\Omega; \mathbb{R}^m) &\longrightarrow W^{-1,p'}(\Omega; \mathbb{R}^m) \\
u &\longmapsto T(u).
\end{aligned}$$

**Lemma 3.3.3.** *(i) The restriction of $T$ to a finite dimensional linear subspace of $W_0^{1,p}(\Omega; \mathbb{R}^m)$ is continuous.*
*(ii) $T$ is coercive.*

*Proof.* (i) Let $u_k \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ be such that $u_k \to u$ in $W$ a finite dimensional linear subspace of $W_0^{1,p}(\Omega; \mathbb{R}^m)$. The continuity condition in **(A0)** and **(A1)**, and the growth condition in **(A2)** lead to the following estimate

$$\begin{aligned}\big\|T(u_k)-T(u)\big\|_{-1,p'} &= \sup_{\|\varphi\|_{1,p}\equiv 1}\big|\langle T(u_k)-T(u),\varphi\rangle\big| \\ &\leq \big\||A(x,Du_k-\Theta(u_k))-A(x,Du-\Theta(u))|\big\|_{p'} \\ &\leq c,\end{aligned}$$

thanks to the Vitali Theorem. Therefore, the restriction of $T$ to a finite linear subspace of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ is continuous.

(ii) Taking $\varphi = u$ as a test function in the definition of $T$, the coercivity condition in **(A2)** implies (by Hölder's inequality)

$$\begin{aligned}\langle T(u),u\rangle &= \int_\Omega A(x,Du-\Theta(u)):Dudx-\langle f,u\rangle \\ &\geq \alpha_0\int_\Omega |Du-\Theta(u)|^p dx-\int_\Omega d_2(x)dx-\|f\|_{-1,p'}\|u\|_{1,p}.\end{aligned}$$

We have

$$\begin{aligned}\frac{1}{2^{p-1}}|Du|^p &= \frac{1}{2^{p-1}}|Du-\Theta(u)+\Theta(u)|^p \\ &\leq \frac{1}{2^{p-1}}\Big(2^{p-1}\big(|Du-\Theta(u)|^p+|\Theta(u)|^p\big)\Big) \quad \text{(by (3.1.4))} \\ &= |Du-\Theta(u)|^p+|\Theta(u)|^p.\end{aligned} \tag{3.3.16}$$

By virtue of **(A0)**, we deduce that

$$\begin{aligned}\langle T(u),u\rangle &\geq \frac{\alpha_0}{2^{p-1}}\int_\Omega |Du|^p dx-\alpha_0\int_\Omega |\Theta(u)|^p dx-\int_\Omega d_2(x)dx-\|f\|_{-1,p'}\|u\|_{1,p} \\ &\geq \frac{\alpha_0}{2^{p-1}}\int_\Omega |Du|^p dx-\alpha_0 C_\Theta^p\int_\Omega |u|^p dx-\int_\Omega d_2(x)dx-\|f\|_{-1,p'}\|u\|_{1,p} \\ &\geq \frac{\alpha_0}{2^{p}}\int_\Omega |Du|^p dx-\int_\Omega d_2(x)dx-\|f\|_{-1,p'}\|u\|_{1,p} \quad \text{(by (3.3.6) and (3.3.15))}.\end{aligned}$$

Consequently, $\langle T(u),u\rangle \to \infty$ as $\|u\|_{1,p}\to\infty$, that is to say $T$ is coercive. □

Note that, our problem (3.3.1) is equivalent to find $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ such that

$$\langle T(u), \varphi\rangle = 0 \quad \text{for all } \varphi \in W_0^{1,p}(\Omega;\mathbb{R}^m). \tag{3.3.17}$$

As stated above, the aim of this subsection is to find such a solution using a Galerkin scheme. Let $W_1 \subset W_2 \subset ... \subset W_0^{1,p}(\Omega;\mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\cup_{k\in\mathbb{N}} W_k$ is dense in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. The subspaces $(W_k)$ exist since $W_0^{1,p}(\Omega;\mathbb{R}^m)$ is separable. Let us fix some $k$ and assume that dim $W_k = r$ and $e_1, ..., e_r$ is a basis of $W_k$.

Let us define

$$\begin{aligned} F : \mathbb{R}^r &\longrightarrow \mathbb{R}^r \\ (a^i)_{i=1,..,r} &\longmapsto \left(\langle T(a^i e_i), e_j\rangle\right)_{j=1,..,r}. \end{aligned}$$

**Lemma 3.3.4.** *(i) $F$ is continuous and $F(a).a \to \infty$ as $\|a\|_{\mathbb{R}^r} \to \infty$.*
*(ii) For all $k \in \mathbb{N}$ there exists $u_k \in W_k$ such that*

$$\langle T(u_k), \varphi\rangle = 0 \quad \forall \varphi \in W_k. \tag{3.3.18}$$

*(iii) The sequence $(u_k)$ constructed in (ii) above is uniformly bounded, i.e., there exists $R > 0$ such that*

$$\|u_k\|_{1,p} \le R \quad \textit{for all } k \in \mathbb{N}. \tag{3.3.19}$$

*Proof.* The proof is similar to that in Lemma 3.2.3 for ($i$) and Lemma 3.1.3 for ($ii$) and ($iii$). □

Now we have all ingredients to pass to the limit in the approximating equations and to prove Theorem 3.3.1. Before that, let us show that the constructed sequence in Lemma 3.3.4 verify the hypothesis **(H1)-(H5)** listed in Subsection 3.3.2.

- The hypothesis **(H1)** is satisfied by Eq. (3.3.19).
- The hypothesis **(H2)** is exactly the assumption **(A1)**.

- To get **(H3)**, we use the growth condition in **(A2)**

$$\begin{aligned}\int_\Omega |A(x, Du_k - \Theta(u_k))|^{p'} dx &\leq \int_\Omega |d_1(x)|^{p'} dx + \int_\Omega |Du_k - \Theta(u_k)|^p dx \\ &\leq c \qquad \text{(by (3.1.4) and } \textbf{(A0)}\text{)}.\end{aligned}$$

- Next, to verify **(H4)**, let $\Omega'$ be a measurable subset of $\Omega$. Since (by the coercivity condition in **(A2)**)

$$A(x, Du_k - \Theta(u_k)) : Du_k \geq \frac{\alpha_0}{2^{p-1}} |Du_k|^p - \alpha_0 |\Theta(u_k)|^p - d_2(x) \quad \text{(by (3.3.16))},$$

  then

$$\begin{aligned}&\int_{\Omega'} \left| \min\left(A(x,Du_k - \Theta(u_k)) : Du_k, 0\right)\right| dx \\ &\qquad \leq \frac{\alpha_0}{2^{p-1}} \int_{\Omega'} |Du_k|^p dx + \alpha_0 \int_{\Omega'} |\Theta(u_k)|^p dx + \int_{\Omega'} |d_2(x)| dx.\end{aligned}$$

  Since a finite set of integrable functions is equiinterable, the equiintegrability of $(A_k(x) : Du_k))^-$ follows.

- Finally, it remains to verify **(H5)**. According to Mazur's Theorem (cf. [136, Theorem 2, page 120]) there exists a sequence $(v_k) \subset W_0^{1,p}(\Omega; \mathbb{R}^m)$ such that $v_k \to u$ in $W_0^{1,p}(\Omega; \mathbb{R}^m)$. Note that $v_k$ is a convex linear combination of $\{u_1, ..., u_k\}$, means that $v_k$ belongs also to $W_k$. By testing the Eq. (3.3.18) with $u_k - v_k$ as test function, it follows that

$$\int_\Omega A(x, Du_k - \Theta(u_k)) : (Du_k - Dv_k) dx = \langle f, u_k - v_k \rangle. \tag{3.3.20}$$

  Since, by the choice of $v_k$, $u_k - v_k \rightharpoonup 0$ in $W_0^{1,p}(\Omega; \mathbb{R}^m)$, then the right hand side in (3.3.20) goes to zero. Therefore **(H5)** is verified.

From the above results, the sequence $(u_k)$ constructed in Lemma 3.3.4 satisfy the hypothesis **(H1)-(H5)** stated in Subsection 3.3.2. Then we infer from Proposition 3.3.1 that

$$\lim_{k \to \infty} \int_\Omega A(x, Du_k - \Theta(u_k)) : D\varphi dx = \int_\Omega A(x, Du - \Theta(u)) : D\varphi dx \quad \forall \varphi \in \cup_{k \in \mathbb{N}} W_k.$$

We have $\cup_{k\in\mathbb{N}} W_k$ is dense in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, then $u$ is in fact a weak solution of (3.3.1) and the proof of Theorem 3.3.1 is complete.

## 3.4 Strongly quasilinear parabolic system

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and let $Q$ be the cylinder $\Omega \times (0,T)$ with some given $T > 0$. By $\partial Q = \partial\Omega \times (0,T)$ we denote the boundary of $Q$.

### 3.4.1 Introduction

Consider first the quasilinear parabolic initial-boundary value system

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div} \sigma(x,t,u,Du) &= f \text{ in } Q, \\ u(x,t) &= 0 \text{ on } \partial Q, \\ u(x,0) &= u_0(x) \text{ in } \Omega, \end{aligned} \tag{3.4.1}$$

where $u : Q \to \mathbb{R}^m$. In (3.4.1) the right hand side $f$ belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ for some $p \in (1,\infty)$. In [137], Young introduced Young measure as a powerful tool to describe the weak limit of sequences. N. Hungerbühler [86] obtained the existence of a weak solution for (3.4.1) by using the concept of Young measures. The author assumed weak monotonicity assumptions on $\sigma$.

If $A(u) = -\operatorname{div} \sigma(x,t,u,Du)$, $u : Q \to \mathbb{R}$ and $A$ is a classical operator of the Leray-Lions type with respect to the Sobolev space $L^p(0,T;W_0^{1,p}(\Omega))$ for some $1 < p < \infty$, then the existence of solutions for (3.4.1) was proved in [43, 98, 100, 103]. The authors required the strict monotonicity or monotonicity in the variables $(u,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Nevertheless, we will not use the previous type of monotonicity.

In this case, we will be using the Young measures and Galerkin method to prove the existence result for the following strongly quasilinear parabolic system

$$\frac{\partial u}{\partial t} - \operatorname{div} \sigma(x,t,u,Du) + g(x,t,u,Du) = f \text{ in } Q, \tag{3.4.2}$$

$$u(x,t) = 0 \text{ on } \partial Q, \tag{3.4.3}$$

$$u(x,0) = u_0(x) \text{ in } \Omega. \tag{3.4.4}$$

The problem (3.4.2)-(3.4.4) can be seen as a more general form of (3.4.1), with $g : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$. Similar problems to (3.4.2)-(3.4.4) were studied by different methods, we refer the reader to [6, 48, 64].

### 3.4.2 Assumptions and main result

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and set $Q = \Omega \times (0,T)$ for $T > 0$. Throughout this text, we denote $Q_\tau = \Omega \times (0,\tau)$ for every $\tau \in [0,T]$. Consider the problem (3.4.2)–(3.4.4), where $\sigma : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ and $g : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ satisfy the following assumptions:

**(H0)** $\sigma$ and $g$ are Carathéodory's functions (i.e., measurable w.r.t. $(x,t) \in Q$ and continuous w.r.t. other variables).

**(H1)** There exist $c_1 \geq 0$, $\beta > 0$, $d_1 \in L^{p'}(Q)$ and $d_2 \in L^1(Q)$ such that

$$|\sigma(x,t,s,\xi)| \leq d_1(x,t) + c_1\big(|s|^{p-1} + |\xi|^{p-1}\big),$$

$$\sigma(x,t,s,\xi) : \xi + g(x,t,s,\xi).s \geq -d_2(x,t) + \beta|\xi|^p \quad \forall\, (s,\xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times}.$$

**(H2)** $\sigma$ satisfies one of the following conditions:

(a) For all $(x,t) \in Q$, $\xi \mapsto \sigma(x,t,u,\xi)$ is a $C^1$-function and is monotone, that is, for all $(x,t) \in Q$, $u \in \mathbb{R}^m$ and $\xi, \eta \in \mathbb{M}^{m\times n}$, we have

$$\big(\sigma(x,t,u,\xi) - \sigma(x,t,u,\eta)\big) : (\xi - \eta) \geq 0.$$

(b) There exists a function $W : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,t,u,\xi) = \frac{\partial W}{\partial \xi}(x,t,u,\xi) := D_\xi W(x,t,u,\xi)$ and $\xi \to W(x,t,u,\xi)$ is convex and $C^1$ for all $(x,t) \in Q$ and $u \in \mathbb{R}^m$.

(c) $\sigma$ is strictly monotone, that is, $\sigma$ is monotone and

$$\big(\sigma(x,t,u,\xi) - \sigma(x,t,u,\eta)\big) : (\xi - \eta) = 0 \Rightarrow \xi = \eta.$$

(d) $\sigma$ is strictly $p$-quasimonotone on $\mathbb{M}^{m\times n}$, i.e.,

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,\overline{\lambda})\big) : (\lambda - \overline{\lambda}) d\nu(\lambda) dx dt > 0,$$

where $\overline{\lambda} = \langle \nu_{(x,t)}, id \rangle$ and $\nu = \{\nu_{(x,t)}\}_{(x,t)\in Q}$ is any family of Young measures generated by a bounded sequence in $L^p(Q)$ and not a Dirac measure for a.e. $(x,t) \in Q$.

**(H3)** $g$ satisfies one of the following conditions:

(i) There exist $c_2 \geq 0$ and $d_2 \in L^{p'}(Q)$ such that

$$|g(x,t,s,\xi)| \leq d_2(x,t) + c_2\big(|s|^{p-1} + |\xi|^{p-1}\big) \quad \forall\, (s,\xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}.$$

(ii) In addition to $(i)$, the function $g$ is independent of the fourth variable, or, for a.e. $(x,t) \in Q$ and all $u \in \mathbb{R}^m$, the mapping $\xi \to g(x,t,u,\xi)$ is linear.

**Remark 3.4.1.** *Assumptions **(H1)** and **(H3)**(i) state standard growth and coercivity conditions. The assumption **(H1)**(b) allows to take a potential $W(x,t,u,\xi)$ which is only convex but not strictly convex in $\xi \in \mathbb{M}^{m\times n}$ and to consider (3.4.2) with $\sigma(x,t,u,\xi) = \frac{\partial W}{\partial \xi}(x,t,u,\xi)$. Note that if W is assumed to be strictly convex, then $\sigma$ becomes strict monotone. Thus, the standard method may apply. Finally, **(H2)**(d) states the notion of strict p-quasimonotone in terms of gradient Young measures.*

We shall prove the following existence theorem.

**Theorem 3.4.1.** *Suppose that the conditions **(H0)-(H1)** are satisfied. Let $u_0 \in L^2(\Omega;\mathbb{R}^m)$ and $f \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ be given. Then*

(1) *if $\sigma$ satisfies one of the condition* **(H2)***(a) or (b), then for every $g$ satisfying* **(H3)***(ii), the system* (3.4.2)-(3.4.4) *has a weak solution.*

(2) *if $\sigma$ satisfies one of the condition* **(H2)***(c) or (d), then for each $g$ satisfying* **(H3)***(i), the system* (3.4.2)-(3.4.4) *has a weak solution.*

**Example 3.4.1.** *A simple model of our problem is as follows:*

$$\frac{\partial u}{\partial t} - div\left(|Du|^{p-2}Du\right) + |u|^{p-2}u = f \text{ in } Q,$$
$$u(x,t) = 0 \text{ on } \partial Q,$$
$$u(x,0) = u_0(x) \text{ in } \Omega.$$

*For the potential $W$, one can take $W := \frac{1}{p}|\xi|^p$.*

### 3.4.3 Galerkin approximation

We choose an $L^2(\Omega;\mathbb{R}^m)$-orthonormal base $\{w_i\}_{i\geq 1}$ such that

$$\{w_i\}_{i\geq 1} \subset \mathcal{C}_0^\infty(\Omega;\mathbb{R}^m), \quad \mathcal{C}_0^\infty(\Omega;\mathbb{R}^m) \subset \overline{\underset{k\geq 1}{\cup} V_k}^{\mathcal{C}^1(\overline{\Omega};\mathbb{R}^m)},$$

where $V_k = \text{span}\{w_1,\ldots,w_k\}$. Define the following approach for approximating solutions of (3.4.2)-(3.4.4):

$$u_k(x,t) = \sum_{i=1}^{k} \alpha_{ki}(t) w_i(x), \tag{3.4.5}$$

where $\alpha_{ki} : [0,T] \to \mathbb{R}$ are measurable bounded functions. Assume that $u_k \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$. Thus $u_k$ satisfies the boundary condition (3.4.3) by construction. For the initial condition (3.4.4), one can choose the initial coefficients $\alpha_{ki}(0) := (u_0, w_i)_{L^2}$, with $(.,.)$ denotes the inner product of $L^2$, such that

$$u_k(.,0) = \sum_{i=1}^{k} \alpha_{ki}(0) w_i(.) \to u_0 \quad \text{in } L^2(\Omega)$$

as $k \to \infty$. To complete the construction of $u_k$, it remains to determine the coefficients $\alpha_{ki}(t)$. For this, let $k \in \mathbb{N}$ be fixed (for the moment), $0 < \tau < T$ and $I = [0,\tau]$.

Furthermore, we choose $r > 0$ large enough, such that the set $B_r(0) := B(0,r) \subset \mathbb{R}^k$ contains the vectors $(\alpha_{1k}(0), \ldots, \alpha_{kk}(0))$. Consider the function

$$\begin{aligned}
&\Theta : I \times \overline{B_r(0)} \to \mathbb{R}^k \\
&(t, \alpha_1, \ldots, \alpha_k) \mapsto \Big( \langle f(t), w_j \rangle - \int_\Omega \sigma(x, t, \sum_{i=1}^k \alpha_i w_i, \sum_{i=1}^k \alpha_i D w_i) : D w_j dx \\
&\qquad\qquad\qquad - \int_\Omega g(x, t, \sum_{i=1}^k \alpha_i w_i, \sum_{i=1}^k \alpha_i D w_i).w_j dx \Big)_{j=1,\ldots,k},
\end{aligned}$$

where $\langle .,. \rangle$ denotes the dual pairing of $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. The operator $\Theta$ is a Carathéodory function by the condition **(H0)**. Next, we will estimate $\Theta_j$. By using the conditions **(H1)** and **(H3)**(i), one gets together with the Hölder inequality

$$\begin{aligned}
&\Big| \int_\Omega \sigma(x, t, \sum_{i=1}^k \alpha_i w_i, \sum_{i=1}^k \alpha_i D w_i) : D w_j dx \Big| \\
&\leq \Big( \int_\Omega |\sigma(x, t, \sum_{i=1}^k \alpha_i w_i, \sum_{i=1}^k \alpha_i D w_i)|^{p'} dx \Big)^{\frac{1}{p'}} \Big( \int_\Omega |D w_j|^p dx \Big)^{\frac{1}{p}} \\
&\leq c \int_\Omega d_1(x,t) dx + c
\end{aligned} \tag{3.4.6}$$

and

$$\Big| \int_\Omega g(x, t, \sum_{i=1}^k \alpha_i w_i, \sum_{i=1}^k \alpha_i D w_i).w_j dx \Big| \leq c \int_\Omega d_2(x,t) dx + c, \tag{3.4.7}$$

where $c$ depends on $k$ and $r$ but not on $t$.

Note that (3.4.6) and (3.4.7) are obtained by the following arguments: firstly, we have $W_0^{s,2}(\Omega) \subset W_0^{1,p}(\Omega)$ for $s \geq 1 + n(\frac{1}{2} - \frac{1}{p})$, secondly $D w_j \in W^{s-1,2}(\Omega) \subset L^\infty(\Omega)$ for $w_j \in W^{s,2}(\Omega)$. For the first term in the definition of $\Theta$, we have

$$|\langle f(t), w_j \rangle| \leq \|f(t)\|_{-1,p'} \|w_j\|_{1,p}.$$

As a consequence, the $j^{th}$ term of $\Theta$ can be estimated as follows:

$$|\Theta_j(t, \alpha_1, \ldots, \alpha_k)| \leq c(r,k) b(t) \tag{3.4.8}$$

uniformly on $I \times \overline{B_r(0)}$, where $c(r,k)$ is a constant, which depends on $r$ and $k$, and where $b(t) \in L^1(I)$ does not depend on $r$ and $k$. Thus, the Carathéodory existence result on ordinary differential equations (cf. Kamke [90]) applied to the system

$$\begin{cases} \alpha_j'(t) = \Theta_j(t, \alpha_1(t), \ldots, \alpha_k(t)), \\ \alpha_j(0) = \alpha_{kj}(0), \end{cases} \tag{3.4.9}$$

(for $j \in \{1, \ldots, k\}$) ensures the existence of a distributional, continuous solution $\alpha_j$ (depending on $k$) of (3.4.9) on a time interval $[0, \tau')$, where $\tau' > 0$, *a priori*, may depend on $k$. Furthermore, the corresponding integral equation

$$\alpha_j(t) = \alpha_j(0) + \int_0^t \Theta_j(t, \alpha_1(s), \ldots, \alpha_k(s))ds$$

holds on $[0, \tau')$. Hence

$$u_k(x,t) = \sum_{i=1}^{k} \alpha_{ki}(t) w_i(x)$$

is the desired solution to the system of ordinary differential equations

$$(\frac{\partial u_k}{\partial t}, w_j)_{L^2} + \int_\Omega \sigma(x,t,u_k,Du_k) : Dw_j dx + \int_\Omega g(x,t,u_k,Du_k).w_j dx = \langle f(t), w_j \rangle, \tag{3.4.10}$$

with the initial condition $u_k(.,0) = \sum_{i=1}^k \alpha_{ki}(0) w_i(.) \to u_0$ in $L^2(\Omega)$ as $k \to \infty$. Now, we will extend the local solution defined on $[0, \tau')$ to a global one. For this, we multiply each side of (3.4.10) by $\alpha_{kj}(t)$ and we sum. This gives for an arbitrary time $\tau \in [0, T)$

$$\int_{Q_\tau} \frac{\partial u_k}{\partial t} u_k dxdt + \int_{Q_\tau} \big(\sigma(x,t,u_k,Du_k) : Du_k + g(x,t,u_k,Du_k).u_k\big) dxdt$$
$$= \int_0^\tau \langle f(t), u_k \rangle dt,$$

which is denoted as $I_1 + I_2 = I_3$. By integrating and **(H1)**, we have

$$I_1 = \frac{1}{2}\|u_k(.,\tau)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_k(.,0)\|^2_{L^2(\Omega)}$$

and

$$I_2 \geq -\int_{Q_\tau} d_2(x,t)dxdt + \beta \int_{Q_\tau} |Du_k|^p dxdt.$$

By Hölder's inequality

$$|I_3| \leq \|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \|u_k\|_{L^p(0,T;W_0^{1,p}(\Omega))}.$$

From the estimations on $I_\epsilon$, $\epsilon = 1,2,3$, we deduce

$$\|u_k(.,\tau)\|^2_{L^2(\Omega)} = \left|\left(\alpha_{ki}(\tau)\right)_{i=1,\dots,k}\right|^2_{\mathbb{R}^k} \leq \bar{c},$$

where $\bar{c}$ is a constant independent of $\tau$ (and of $k$).
Consider now

$$M := \Big\{ t \in [0,T) : \text{ there exists a weak solution of (3.4.9) on } [0,t) \Big\}.$$

We have $M$ is nonempty, because it contains a local solution. Moreover, thanks to [86], we then have $M$ is an open set, also closed. Thus $M = [0,T)$.

From the estimations on $I_\epsilon$, $\epsilon = 1,2,3$, we conclude that the sequence $(u_k)_k$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$. Therefore, by extracting a suitable subsequence (still denoted by $(u_k)_k$), we may assume

$$u_k \rightharpoonup u \text{ in } L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)), \tag{3.4.11}$$

$$u_k \rightharpoonup^* u \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^m)). \tag{3.4.12}$$

The function $u \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$ is a candidate to be a weak solution for the problem (3.4.2)-(3.4.4). Using the growth condition in **(H1)** and **(H3)**, together with (3.4.11), we can extract a suitable subsequence of $\{-\operatorname{div}\sigma(x,t,u_k,Du_k)\}$ and $\{g(x,t,u_k,Du_k)\}$ such that

$$-\operatorname{div}\sigma(x,t,u_k,Du_k) \rightharpoonup \chi \text{ in } L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)) \tag{3.4.13}$$

and

$$g(x,t,u_k,Du_k) \rightharpoonup \xi \text{ in } L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)), \tag{3.4.14}$$

where $\chi, \xi \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$. Since $(u_k)_k$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, there exists a subsequence, which is again denoted by $(u_k)_k$, such that

$$u_k(.,T) \rightharpoonup u^* \text{ in } L^2(\Omega;\mathbb{R}^m) \text{ as } k \to \infty.$$

We will prove that $u^* = u(.,T)$ and $u(.,0) = u_0(.)$. For simplicity, we denote $u(.,T)$ as $u(T)$ and $u(.,0)$ as $u(0)$. For every $\phi \in \mathcal{C}^\infty([0,T])$ and $v \in V_j$, $j \le k$, we have

$$\begin{aligned}\int_Q \frac{\partial u_k}{\partial t} v\phi dxdt + \int_Q \sigma(x,t,u_k,Du_k) : Dv\phi dxdt + \int_Q g(x,t,u_k,Du_k).v\phi dxdt \\ = \int_Q f.v\phi dxdt.\end{aligned}$$

After integrating, one gets

$$\begin{aligned}\int_\Omega u_k(T)\phi(T)vdx - \int_\Omega u_k(0)\phi(0)vdx = \int_Q f.v\phi dxdt - \int_Q \sigma(x,t,u_k,Du_k) : Dv\phi dxdt \\ - \int_Q g(x,t,u_k,Du_k).v\phi dxdt + \int_Q u_k v\phi' dxdt.\end{aligned}$$

We pass to the limit as $k \to \infty$ in the previous equality

$$\begin{aligned}&\int_\Omega u^*\phi(T)vdx - \int_\Omega u_0\phi(0)vdx \\ &\qquad = \int_Q f.v\phi dxdt - \int_Q \chi.v\phi dxdt - \int_Q \xi.v\phi dxdt + \int_Q uv\phi' dxdt.\end{aligned}$$

Let $\phi(0) = \phi(T) = 0$. Then

$$\begin{aligned}- \int_Q \chi.v\phi dxdt - \int_Q \xi.v\phi dxdt + \int_Q f.v\phi dxdt &= - \int_Q uv\phi' dxdt \\ &= \int_Q \phi vu' dxdt.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\int_\Omega u^*\phi(T)vdx - \int_\Omega u_0\phi(0)vdx &= \int_Q \phi v u' dxdt + \int_Q uv\phi' dxdt\\ &= \int_\Omega u\phi v dx\Big|_0^T\\ &= \int_\Omega u(T)\phi(T)vdx - \int_\Omega u(0)\phi(0)vdx.\end{aligned}$$

If we take $\phi(T) = 0$ and $\phi(0) = 1$, then we have $u(0) = u_0$; if $\phi(T) = 1$ and $\phi(0) = 0$, then $u(T) = u^*$.

The principal difficulty will be to identify $\chi$ with $-\operatorname{div}\sigma(x,t,u,Du)$ and $\xi$ with $g(x,t,u,Du)$.

### 3.4.4 Div-curl inequality

The Young measure (as in previous sections) is a device that comes to overcome the difficulty that may arises when weak convergence does not behave as one desire with respect to nonlinear functionals and operators. The following lemma describes limit points of gradient sequences of approximating solutions.

**Lemma 3.4.1.** *(i) If $(Du_k)_k$ is bounded in $L^p(0,T;L^p(\Omega;\mathbb{R}^m))$, then $(Du_k)$ can generate the Young measure $\nu_{(x,t)}$ which satisfy $\|\nu_{(x,t)}\| = 1$, and there is a subsequence of $(Du_k)$ weakly convergent to $\int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda)$ in $L^1(0,T;L^1(\Omega;\mathbb{R}^m))$.*
*(ii) For almost every $(x,t) \in Q$, $\nu_{(x,t)}$ satisfies $\langle \nu_{(x,t)}, id\rangle = Du(x,t)$.*

The proof of the above lemma is similar to that in Lemma 3.1.4 in the stationary case. However, for completness we present its proof.

*Proof.* (i) To prove the first part of Lemma 3.4.1, it is sufficient to show that $(Du_k)$ satisfies the equation (2.3.1) in Theorem 2.3.1. Since $(Du_k)$ is bounded, it follows that there exists $c \geq 0$ such that

$$\begin{aligned}c \geq \int_Q |Du_k|^p dxdt &\geq \int_{\{(x,t):\,|Du_k(x,t)|\geq L\}} |Du_k|^p dxdt\\ &\geq L^p\big|\{(x,t)\,:\,|Du_k(x,t)| \geq L\}\big|.\end{aligned}$$

Thus
$$\sup_{k\in\mathbb{N}}\left|\{(x,t)\,:\,|Du_k(x,t)|\geq L\}\right|\leq \frac{c}{L^p}\to 0,\text{ as } L\to\infty.$$

According to Theorem 2.3.1(iii), $\|\nu_{(x,t)}\|=1$ for almost every $(x,t)\in Q$.
For the remaining part, the reflexivity of $L^p(0,T;L^p(\Omega))$ implies the existence of a subsequence (still denoted by $(Du_k)$) weakly convergent in $L^p(0,T;L^p(\Omega))$, thus weakly convergent in $L^1(0,T;L^1(\Omega))$. By Theorem 2.3.1(iii) and by taking $\varphi$ as the identity mapping $id$, it result that

$$Du_k\rightharpoonup\langle\nu_{(x,t)},id\rangle=\int_{\mathbb{M}^{m\times n}}\lambda d\nu_{(x,t)}(\lambda)\text{ weakly in } L^1(0,T;L^1(\Omega)).$$

(ii) Since $u_k\rightharpoonup u$ in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ and $u_k\to u$ in $L^p(0,T;L^p(\Omega))$, we have

$$Du_k\rightharpoonup Du\text{ in } L^p(0,T;L^p(\Omega)).$$

Moreover, $Du_k\rightharpoonup Du$ in $L^1(0,T;L^1(\Omega))$ (up to a subsequence). By virtue of Theorem 2.3.1, we can infer that

$$Du(x,t)=\langle\nu_{(x,t)},id\rangle\text{ for a.e. }(x,t)\in Q.$$

□

The following lemma, namely div-curl inequality, is the key ingredient to pass to the limit in the approximating equations and to prove that the weak limit $u$ of the Galerkin approximations $u_k$ is indeed a solution of (3.4.2)-(3.4.4).

**Lemma 3.4.2.** *The Young measure $\nu_{(x,t)}$ generated by the gradient $Du_k$ of the Galerkin approximations $u_k$ has the following property:*

$$\int_Q\int_{\mathbb{M}^{m\times n}}\big(\sigma(x,t,u,\lambda)-\sigma(x,t,u,Du)\big):(\lambda-Du)d\nu_{(x,t)}(\lambda)dxdt\leq 0.$$

*Proof.* Let us consider the sequence

$$\begin{aligned} J_k &:= \big(\sigma(x,t,u_k,Du_k) - \sigma(x,t,u,Du)\big) : (Du_k - Du) \\ &= \sigma(x,t,u_k,Du_k) : (Du_k - Du) - \sigma(x,t,u,Du) : (Du_k - Du) \\ &=: J_{k,1} + J_{k,2}. \end{aligned}$$

We have by the growth condition **(H1)** that

$$\int_Q |\sigma(x,t,u,Du)|^{p'} dxdt \leq c \int_Q \big(|d_1(x,t)|^{p'} + |u|^p + |Du|^p\big) dxdt,$$

and since $u \in L^p(0,T;W_0^{1,p}(\Omega))$ we obtain $\sigma \in L^{p'}(0,T;W^{-1,p'}(\Omega))$. By virtue of Lemma 3.4.1, it follows that

$$\liminf_{k\to\infty} \int_Q J_{k,2} dxdt = \int_Q \sigma(x,t,u,Du)\Big(\int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) - Du\Big) dxdt = 0.$$

Since $(u_k)$ is bounded, then $u_k \rightharpoonup u$ in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ and in measure on $Q$. It follows from the equiintegrability of $\sigma(x,t,u_k,Du_k)$ and Lemma 2.3.2, that

$$\begin{aligned} J := \liminf_{k\to\infty} \int_Q J_k dxdt &= \liminf_{k\to\infty} \int_Q J_{k,1} dxdt \\ &\geq \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : (\lambda - Du) d\nu_{(x,t)}(\lambda) dxdt. \end{aligned} \tag{3.4.15}$$

To get the result, it is sufficient to prove that $J \leq 0$. On the one hand, we have

$$\begin{aligned} \liminf_{k\to\infty} - \int_Q \sigma(x,t,u_k,Du_k) : Dudxdt &= - \int_0^T \langle \chi, u \rangle dt \\ &= \frac{1}{2}\|u(.,T)\|_{L^2}^2 - \frac{1}{2}\|u(.,0)\|_{L^2}^2 - \int_0^T \langle f, u \rangle dt + \int_Q \xi.udxdt, \end{aligned} \tag{3.4.16}$$

where we have used the following energy equality related to $\chi$ and $\xi$:

$$\frac{1}{2}\|u(.,s)\|_{L^2}^2 + \int_0^s \langle \chi, u \rangle dt + \int_0^s \langle \xi, u \rangle dt = \int_0^s \langle f, u \rangle dt + \frac{1}{2}\|u(.,0)\|_{L^2}^2$$

for all $s \in [0, T]$. On the other hand, by the Galerkin equations

$$\int_Q \sigma(x,t,u_k,Du_k) : Du_k dxdt \\ = \int_0^T \langle f, u_k \rangle dt - \int_Q \frac{\partial u_k}{\partial t} u_k dxdt - \int_Q g(x,t,u_k,Du_k).u_k dxdt.$$

We pass to the limit inf in the last equation and using the fact that $u_k(.,0) \to u_0(x) = u(x,0)$ and $u_k(.,T) \rightharpoonup u(.,T)$ in $L^2(\Omega;\mathbb{R}^m)$, we get

$$\liminf_{k\to\infty} \int_Q \sigma(x,t,u_k,Du_k) : Du_k dxdt \\ \leq \int_0^T \langle f, u \rangle dt - \frac{1}{2}\|u(.,T)\|_{L^2}^2 + \frac{1}{2}\|u(.,0)\|_{L^2}^2 - \int_Q \xi.udxdt. \tag{3.4.17}$$

Due to (3.4.16) and (3.4.17)

$$J = \liminf_{k\to\infty} \int_Q \sigma(x,t,u_k,Du_k) : (Du_k - Du)dxdt \leq 0.$$

We have

$$\int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,Du) : (\lambda - Du)d\nu_{(x,t)}(\lambda)dxdt = 0.$$

This together with (3.4.15) imply the needed result. □

It should be noticed that Lemma 3.2.6 holds with time dependent (see its proof).

### 3.4.5 Proof of the main result

In this subsection, we give the proof of Theorem 3.4.1 based on the two cases listed in. We start with the case (2) where we have supposed that $\sigma$ satisfies the condition (c) or (d).

Note that, in these cases, we will prove that we may extract a subsequence with the property

$$Du_k \to Du \quad \text{in measure on} \quad Q. \tag{3.4.18}$$

**Case (c)**: By strict monotonicity, it follows from Lemma 3.2.6 that $\operatorname{supp}\nu_{(x,t)} = \{Du(x,t)\}$, thus $\nu_{(x,t)} = \delta_{Du(x,t)}$ for a.e. $(x,t) \in Q$.
**Case (d)**: Suppose that $\nu_{(x,t)}$ is not a Dirac measure on a set $(x,t) \in Q' \subset Q$ of positive Lebesgue measure $|Q'| > 0$. Since $\|\nu_{(x,t)}\| = 1$ and $Du(x,t) = \langle \nu_{(x,t)}, id\rangle = \overline{\lambda}$, it follows from the strict $p$-quasimonotone that

$$\begin{aligned}0 &< \int_Q \int_{\mathbb{M}^{m\times n}} (\sigma(x,t,u,\lambda) - \sigma(x,t,u,\overline{\lambda})) : (\lambda - \overline{\lambda}) d\nu_{(x,t)}(\lambda) dxdt \\ &= \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : (\lambda - \overline{\lambda}) d\nu_{(x,t)}(\lambda) dxdt.\end{aligned}$$

Hence

$$\int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda) dxdt > \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : Du\, d\nu_{(x,t)}(\lambda) dxdt.$$

From Lemma 3.4.2 and the above inequality, we get

$$\begin{aligned}\int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : Du\, d\nu_{(x,t)}(\lambda) dxdt &\geq \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda) dxdt \\ &> \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : Du\, d\nu_{(x,t)}(\lambda) dxdt\end{aligned}$$

which is a contradiction. Hence $\nu_{(x,t)}$ is a Dirac measure. Assume that $\nu_{(x,t)} = \delta_{h(x,t)}$. Then

$$h(x,t) = \int_{\mathbb{M}^{m\times n}} \lambda d\delta_{h(x,t)}(\lambda) = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) = Du(x,t).$$

Thus $\nu_{(x,t)} = \delta_{Du(x,t)}$.

To complete the proof of this part, we argue as follows: we have $\nu_{(x,t)} = \delta_{Du(x,t)}$ for a.e. $(x,t) \in Q$, then by Proposition 2.3.1 $Du_k \to Du$ in measure on $Q$ as $k \to \infty$, and thus $\sigma(x,t,u_k,Du_k) \to \sigma(x,t,u,Du)$ and $g(x,t,u_k,Du_k) \to g(x,t,u,Du)$ almost everywhere on $Q$ (up to extraction of a further subsequence). Since by **(H1)** and **(H3)**(i) the sequences $\sigma(x,t,u_k,Du_k)$ and $g(x,t,u_k,Du_k)$ are bounded, it follows that $\sigma(x,t,u_k,Du_k) \to \sigma(x,t,u,Du)$ and $g(x,t,u_k,Du_k) \to g(x,t,u,Du)$ in $L^\beta(Q)$, for all $\beta \in [1,p')$ by the Vitali convergence theorem. It then follows that

$$-\operatorname{div}\sigma(x,t,u_k,Du_k) \rightharpoonup \chi = -\operatorname{div}\sigma(x,t,u,Du) \tag{3.4.19}$$

and

$$g(x,t,u_k,Du_k) \rightharpoonup \xi = g(x,t,u,Du). \tag{3.4.20}$$

These properties are sufficient to pass to the limit in the Galerkin equations and to conclude the proof of the part (2) of Theorem 3.4.1.

For the remaining part (i.e., the first part) of Theorem 3.4.1, we note that the property (3.4.18) does not hold (in general), but we will obtain $\sigma(x,t,u_k,Du_k) \rightharpoonup \sigma(x,t,u,Du)$ and $g(x,t,u_k,Du_k) \rightharpoonup g(x,t,u,Du)$ in $L^{p'}(Q)$. To do this, we need the convergence in measure of the sequence $u_k$. Since $(u_k)_k$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$, we have then $u_k \rightharpoonup u$ in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ and in measure on $Q$ as $k \to \infty$.

**Case (a)**: We prove that for a.e. $(x,t) \in Q$ and every $\mu \in \mathbb{M}^{m\times n}$ the following equation holds on supp $\nu_{(x,t)}$

$$\sigma(x,t,u,\lambda) : \mu = \sigma(x,t,u,Du) : \mu + \big(\nabla\sigma(x,t,u,Du)\big) : (\lambda - Du), \tag{3.4.21}$$

where $\nabla$ denotes the derivative with respect to the third variable of $\sigma$. Due to the monotonicity of $\sigma$, we have for all $\tau \in \mathbb{R}$

$$\begin{aligned}
0 &\le \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du+\tau\mu)\big) : (\lambda - Du - \tau\lambda)\\
&= \sigma(x,t,u,\lambda) : (\lambda - Du) - \sigma(x,t,u,\lambda) : \tau\mu - \sigma(x,t,u,Du+\tau\mu) : (\lambda - Du - \tau\mu)\\
&= \sigma(x,t,u,Du) : (\lambda - Du) - \sigma(x,t,u,\lambda) : \tau\mu - \sigma(x,t,u,Du+\tau\mu) : (\lambda - Du - \tau\mu),
\end{aligned}$$

by Lemma 3.2.6. Hence

$$-\sigma(x,t,u,\lambda) : \tau\mu \ge -\sigma(x,t,u,Du) : (\lambda - Du) + \sigma(x,t,u,Du+\tau\mu) : (\lambda - Du - \tau\mu).$$

Using the fact that

$$\sigma(x,t,u,Du+\tau\mu) = \sigma(x,t,u,Du) + \nabla\sigma(x,t,u,Du)\tau\mu + o(\tau)$$

to deduce

$$-\sigma(x,t,u,\lambda) : \tau\mu \ge \tau\Big(\big(\nabla\sigma(x,t,u,Du)\mu\big) : (\lambda - Du) - \sigma(x,t,u,Du) : \mu\Big) + o(\tau).$$

Since the sign of $\tau$ is arbitrary in $\mathbb{R}$, the above inequality implies (3.4.21).
On the other hand, the equiintegrability of $\sigma(x,t,u_k,Du_k)$ implies that its weak $L^1$-limit $\overline{\sigma}$ is given by

$$\begin{aligned}
\overline{\sigma} &:= \int_{\operatorname{supp}\nu_{(x,t)}} \sigma(x,t,u,\lambda)d\nu_{(x,t)}(\lambda) \\
&\overset{(3.4.21)}{=} \int_{\operatorname{supp}\nu_{(x,t)}} \big(\sigma(x,t,u,Du) + \nabla\sigma(x,t,u,Du) : (Du-\lambda)\big)d\nu_{(x,t)}(\lambda) \\
&= \sigma(x,t,u,Du),
\end{aligned}$$

where we have used $\|\nu_{(x,t)}\| = 1$ and $\int_{\operatorname{supp}\nu_{(x,t)}}(Du-\lambda)d\nu_{(x,t)}(\lambda) = 0$. Evidently,

$$\sigma(x,t,u_k,Du_k) \rightharpoonup \sigma(x,t,u,Du) \quad \text{in} \quad L^{p'}(Q).$$

**Case (b)**: We start by proving that for almost all $(x,t)\in Q$, $\operatorname{supp}\nu_{(x,t)} \subset K_{(x,t)}$, where

$$K_{(x,t)} := \Big\{\lambda\in\mathbb{M}^{m\times n} : W(x,t,u,\lambda) = W(x,t,u,Du) + \sigma(x,t,u,Du) : (\lambda - Du)\Big\}.$$

If $\lambda \in \operatorname{supp}\nu_{(x,t)}$, then by Lemma 3.2.6

$$(1-\tau) : \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : (\lambda - Du) = 0 \quad \forall\tau\in[0,1]. \tag{3.4.22}$$

The monotonicity of $\sigma$ together with (3.4.22) imply

$$\begin{aligned}
0 &\le (1-\tau) : \big(\sigma(x,t,u,Du+\tau(\lambda-Du)) - \sigma(x,t,u,\lambda)\big) : (Du-\lambda) \\
&= (1-\tau) : \big(\sigma(x,t,u,Du+\tau(\lambda-Du)) - \sigma(x,t,u,Du)\big) : (Du-\lambda).
\end{aligned} \tag{3.4.23}$$

Again by the monotonicity of $\sigma$ and $\tau\in[0,1]$, it follows that the right hand side of (3.4.23) is nonpositive, because

$$\big(\sigma(x,t,u,Du+\tau(\lambda-Du)) - \sigma(x,t,u,Du)\big) : \tau(\lambda - Du) \ge 0,$$

which implies for all $\tau \in [0,1]$

$$\big(\sigma(x,t,u,Du+\tau(\lambda-Du))-\sigma(x,t,u,Du)\big):(1-\tau)(\lambda-Du)\geq 0.$$

Thus, for all $\tau \in [0,1]$

$$\big(\sigma(x,t,u,Du+\tau(\lambda-Du))-\sigma(x,t,u,Du)\big):(\lambda-Du)=0,$$

whenever $\lambda \in \operatorname{supp}\nu_{(x,t)}$. From the hypothesis of the potential $W$, we get

$$\begin{aligned} W(x,t,u,\lambda) &= W(x,t,u,Du)+\int_0^1 \sigma\big(x,t,u,Du+\tau(\lambda-Du)\big):(\lambda-Du)d\tau \\ &= W(x,t,u,Du)+\sigma(x,t,u,Du):(\lambda-Du). \end{aligned}$$

We conclude that $\lambda \in K_{(x,t)}$, i.e., $\operatorname{supp}\nu_{(x,t)} \subset K_{(x,t)}$. Due to the convexity of $W$, we have for all $\lambda \in \mathbb{M}^{m\times n}$

$$W(x,t,u,\lambda)\geq W(x,t,u,Du)+\sigma(x,t,u,Du):(\lambda-Du).$$

For all $\lambda \in K_{(x,t)}$, put

$$F(\lambda)=W(x,t,u,\lambda) \quad \text{and} \quad G(\lambda)=W(x,t,u,Du)+\sigma(x,t,u,Du):(\lambda-Du).$$

Since $\lambda \mapsto F(\lambda)$ is continuous and differentiable, it follows for $\mu \in \mathbb{M}^{m\times n}$ and $\tau \in \mathbb{R}$

$$\frac{F(\lambda+\tau\mu)-F(\lambda)}{\tau}\geq\frac{G(\lambda+\tau\mu)-G(\lambda)}{\tau} \quad \text{if } \tau>0,$$

$$\frac{F(\lambda+\tau\mu)-F(\lambda)}{\tau}\leq\frac{G(\lambda+\tau\mu)-G(\lambda)}{\tau} \quad \text{if } \tau<0.$$

Consequently, $D_\lambda F = D_\lambda G$, i.e.,

$$\sigma(x,t,u,\lambda)=\sigma(x,t,u,Du) \quad \forall\, \lambda \in K_{(x,t)} \supset \operatorname{supp}\nu_{(x,t)}.$$

Hence

$$
\begin{aligned}
\overline{\sigma} = \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda)d\nu_{(x,t)}(\lambda) &= \int_{\operatorname{supp}\nu_{(x,t)}} \sigma(x,t,u,\lambda)d\nu_{(x,t)}(\lambda) \\
&= \int_{\operatorname{supp}\nu_{(x,t)}} \sigma(x,t,u,Du)d\nu_{(x,t)}(\lambda) \\
&= \sigma(x,t,u,Du).
\end{aligned}
\tag{3.4.24}
$$

This shows that $\sigma(x,t,u_k,Du_k) \rightharpoonup \sigma(x,t,u,Du)$ in $L^1(Q)$, and we will show the strong convergence. Consider the Carathéodory function

$$
h(x,t,s,\lambda) = \big|\sigma(x,t,s,\lambda) - \overline{\sigma}(x,t)\big|, \quad s \in \mathbb{R}^m,\ \lambda \in \mathbb{M}^{m\times n}.
$$

We have $\sigma(x,t,u_k,Du_k)$ is weakly convergent in $L^{p'}(Q)$, hence equiintegrable. This implies the equiintegrability of $h_k(x,t) := h(x,t,u_k,Du_k)$ and

$$
h_k \rightharpoonup \overline{h} \quad \text{in} \quad L^1(Q),
$$

where

$$
\begin{aligned}
\overline{h}(x,t) &= \int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}} h(x,t,s,\lambda)d\delta_{u(x,t)}(s)\otimes d\nu_{(x,t)}(\lambda) \\
&= \int_{\mathbb{M}^{m\times n}} \big|\sigma(x,t,u,\lambda) - \overline{\sigma}(x,t)\big|d\nu_{(x,t)}(\lambda) \\
&= \int_{\operatorname{supp}\nu_{(x,t)}} \big|\sigma(x,t,u,\lambda) - \overline{\sigma}(x,t)\big|d\nu_{(x,t)}(\lambda) = 0,
\end{aligned}
$$

by (3.4.24). Since $h_k \geq 0$, it follows that

$$
h_k \to 0 \quad \text{in} \quad L^1(Q).
$$

Using the fact that $h_k$ is bounded in $L^{p'}(Q)$ together with the Vitali convergence theorem, we conclude that $\sigma(x,t,u_k,Du_k) \rightharpoonup \sigma(x,t,u,Du)$ in $L^{p'}(Q)$.

From cases (a) and (b), we have

$$
\sigma(x,t,u_k,Du_k) \rightharpoonup \sigma(x,t,u,Du) \quad \text{in} \quad L^{p'}(Q).
$$

It remains then to prove that $g(x,t,u_k,Du_k) \rightharpoonup g(x,t,u,Du)$ in $L^{p'}(Q)$. If $g$ does not depend on the third variable, then by the convergence in measure of $u_k$ to $u$ and the continuity of $g$, we get the needed result. On the other hand, if $g$ is linear in $\xi \in \mathbb{M}^{m\times n}$, then

$$\begin{aligned} g(x,t,u_k,Du_k) \rightharpoonup & \int_{\mathbb{M}^{m\times n}} g(x,t,u,\lambda)d\nu_{(x,t)}(\lambda) \\ &= g(x,t,u,.) \circ \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) \\ &= g(x,t,u,.) \circ Du = g(x,t,u,Du), \end{aligned}$$

where we have used $Du(x,t) = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda)$.

In conclusion, we can now pass to the limit in the Galerkin equations. Note that the energy equality

$$\frac{1}{2}\|u(.,T)\|^2_{L^2(\Omega)} + \int_0^T \langle \chi, u\rangle dt + \int_0^T \langle \xi, u\rangle dt = \int_0^T \langle f, u\rangle dt + \frac{1}{2}\|u(.,0)\|^2_{L^2(\Omega)}$$

holds true with $\chi$ replaced by $-\operatorname{div}\sigma(x,t,u,Du)$ and $\xi$ by $g(x,t,u,Du)$.

# Chapter 4

# Quasilinear elliptic systems with variable exponent

In this chapter we are concerned with problems of variable exponent growth and coercivity conditions. This drawback put us to replace the classical Sobolev space with variable exponent Sobolev space $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. We will study the existence of weak solutions for some quasilinear elliptic systems relaying always on the powerful tool of Young measure to achieve the desired results.

## 4.1 Generalized $p(x)$-Laplacian system

### 4.1.1 Introduction and main result

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. In this section we study a Dirichlet problem given by (3.1.1), where $f$ is a given datum in $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$. The term $\Theta : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is a continuous function satisfy $\Theta(0) = 0$ and

$$|\Theta(a) - \Theta(b)| \leq c|a - b| \tag{4.1.1}$$

for all $a, b \in \mathbb{R}^m$, where $c$ is a positive constant related to the exponent variable $p(x)$ and the diameter of $\Omega$ (diam $\Omega$) by the following:

$$c < \frac{1}{\mathrm{diam}(\Omega)} \left(\frac{1}{2}\right)^{\frac{1}{p^+}}.$$

Partial differential equations with nonlinearities involving nonconstant exponents have attached an increasing amont of attention in recent years. The impulse for this perhaps comes from the sound physical applications in play, or perhaps it is just the thrill of developing a mathematical theory where PDEs again meet functional analysis in a truly two-way street.

The operator $-\mathrm{div}\big(|\xi|^{p(x)-2}\xi\big)$ is said to be the $p(x)$-Laplacian, and becomes $p$-Laplacian when $p(x) \equiv p$ (a constant). Problems of type (3.1.1) are nonlinear degenerate and singular elliptic systems according to the cases $p(x) > 2$ and $1 < p(x) < 2$, respectively. The $p(x)$-Laplacian possesses more complicated nonlinearities than the $p$-Laplacian. For example, it is inhomogeneous. Problems with variable exponent growth condition are interesting in applications (see [125]), also appear in the mathematical modeling of stationary thermorheological viscous flows of non-Newtonian fluids and in the mathematical description of the process filtration of an ideal barotropic gas through a porous medium (see [10, 11]). Finally, in image processing (see [49]), the variable nonlinearity is used to outline the borders of the true image and to elliminate possible noise. We refer also to [2, 65, 108] for the case of calculus of variations.

The following quasilinear elliptic equations with data measure on Reifenberg domains

$$\begin{cases} -\mathrm{div}\, a(x, \nabla u) &= \mu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

has been studied in [45], where the authors proved the gradient estimate for renormalized solutions. For elliptic equations containing a convective term and employing the De Giorgi iteration and a localization method, we refer the reader to [82], where the weighted variable Sobolev space were investigated. The reader can also see [3, 17, 36, 70, 122] for the study of the $p(x)$-Laplacian equations and systems, and [16, 32, 89] for degenerate $p(x)$-Laplacian. Note that, in [32], the authors considered the following $p(x)$-curl systems:

$$\begin{cases} \nabla \times (|\nabla \times u|^{p(x)-2}\nabla \times u) & = \lambda g(x,u) - \mu f(x,u), \qquad \nabla . u = 0 \quad \text{in } \Omega, \\ \quad |\nabla \times u|^{p(x)-2}\nabla \times u \times \mathbf{n} & = 0, \qquad u.\mathbf{n} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\nabla \times u$ is the curl of $u = (u_1, u_2, u_3)$. They studied the existence and nonexistence of solutions for this systems which are arising in electromagnetism.

For nonuniformly elliptic energy integrals of the form $\int_\Omega f(x, Dv)dx$ under $p, q$-growth conditions of the type

$$\lambda(x)|\xi|^p \leq f(x,\xi) \leq \mu(x)(1 + |\xi|^q)$$

for some constants $q \geq p > 1$ and with nonnegative functions $\lambda, \mu$, the local boundedness of minimizers of the above nonuniformly energy integral was studied in [55].

In the case where $p(x) \equiv p$ (constant), Azroul and Balaadich proved in [18] (cf. Section 3.1) existence of a weak solution to the main problem given by (3.1.1), by using the concept of Young measure. Marcellini and Papi [109] proved local Lipschitz-continuity and regularity of weak solutions $u$ for a class of nonlinear elliptic differential systems of the form $\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i^\alpha(Du) = 0$ under mild growth conditions. Motivated by the work [18], previous ones and the homogenization in the particular case where $p(.) \equiv 2$ and for a perforated domain with Dirichlet condition on the boundary of the holes in the generalized case, our objective is to prove the existence result when the exponent $p$ is not constant but depends on $x$ and also by using the tool of the Young measure, means to extend (3.1.1) to generalized $p(x)$-Laplace system.

Now, the main result of the present Sect. 4.1 reads as follows:

**Theorem 4.1.1.** *Assume that $\Theta$ satisfies (4.1.1), then there exists at least one weak solution of the problem (3.1.1) for $f \in W^{-1,p'(x)}(\Omega; \mathbb{R}^m)$.*

### 4.1.2 Existence of solutions

In this subsection we will discuss the existence of weak solutions of (3.1.1) with $f \in W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$, by using a Galerkin schema. Let us define first weak solutions for (3.1.1): we say that $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ is a weak solution to the problem (3.1.1) if

$$\int_\Omega \Phi(Du - \Theta(u)) : D\varphi dx = \langle f, \varphi \rangle$$

holds true for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. Here $\langle .,. \rangle$ denotes the duality pairing of $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ and $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

Denote by $C$ the general positive constant (the exact value may change from line to line). Define the mapping $T : W_0^{1,p(x)}(\Omega;\mathbb{R}^m) \to W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ by

$$\langle T(u), \varphi \rangle = \int_\Omega \Phi(Du - \Theta(u)) : D\varphi dx - \langle f, \varphi \rangle.$$

As a matter of fact, our problem (3.1.1) becomes equivalent to find $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ such that $\langle T(u), \varphi \rangle = 0$ for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. The following assertions are important to construct the approximating solutions.

**Assertion 1:** We show that $T(u)$ is linear, well defined and bounded.
$T$ is trivially linear. For arbitrary $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, we have

$$\begin{aligned}
|\langle T(u), \varphi \rangle| &= \Big| \int_\Omega \Phi(Du - \Theta(u)) : D\varphi dx - \langle f, \varphi \rangle \Big| \\
&\leq \int_\Omega |Du - \Theta(u)|^{p(x)-1} |D\varphi| dx + \|f\|_{-1,p'(x)} \|\varphi\|_{1,p(x)} \\
&\leq \Big( \int_\Omega |Du - \Theta(u)|^{p(x)} dx \Big)^{\frac{1}{p'(x)}} \|D\varphi\|_{p(x)} + \|f\|_{-1,p'(x)} \|\varphi\|_{1,p(x)} \\
&\leq \Big( \int_\Omega 2^{p(x)-1} (|Du|^{p(x)} + |\Theta(u)|^{p(x)}) dx \Big)^{\frac{p(x)-1}{p(x)}} \|D\varphi\|_{p(x)} + \|f\|_{-1,p'(x)} \|\varphi\|_{1,p(x)} \\
&\leq 2^{\frac{(p^+-1)^2}{p^-}} \Big( \|Du\|_{p(x)}^{p(x)} + \|\Theta(u)\|_{p(x)}^{p(x)} \Big)^{\frac{p(x)-1}{p(x)}} \|D\varphi\|_{p(x)} + \|f\|_{-1,p'(x)} \|\varphi\|_{1,p(x)} \\
&\leq C \|\varphi\|_{1,p(x)}
\end{aligned}$$

by Hölder's inequality, Poincaré's inequality and the equations (3.1.4) and (4.1.1). Hence

$T$ is well defined and bounded.

**Assertion 2:** We claim that the restriction of $T$ to a finite linear subspace of $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ is continuous.

Let $W$ be a finite subspace of $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ and $\{u_k\} \subset W$ a sequence such that $u_k \to u$ in $W$. For simplicity, we write $T|_W$ as $T$. Then $u_k \to u$ and $Du_k \to Du$ almost everywhere (for a subsequence still denoted by $\{u_k\}$). Since $u_k \to u$ strongly in $W$,

$$\int_\Omega |u_k - u|^{p(x)} dx \to 0 \quad \text{and} \quad \int_\Omega |Du_k - Du|^{p(x)} dx \to 0,$$

then there exist a subsequence still denoted by $\{u_k\}$ and $g_1, g_2 \in L^1(\Omega)$ such that $|u_k - u|^{p(x)} \leq g_1$ and $|Du_k - Du|^{p(x)} \leq g_2$. Owing to (3.1.4), it follows that

$$\begin{aligned} |u_k|^{p(x)} = |u_k - u + u|^{p(x)} &\leq 2^{p^+-1}\big(|u_k - u|^{p(x)} + |u|^{p(x)}\big) \\ &\leq 2^{p^+-1}\big(g_1 + |u|^{p(x)}\big). \end{aligned}$$

Similarly

$$|Du_k|^{p(x)} \leq 2^{p^+-1}\big(g_2 + |Du|^{p(x)}\big).$$

Hence $\|u_k\|_{p(x)}$ and $\|Du_k\|_{p(x)}$ are bounded by a constant $C$. The continuity of $\Theta$ implies that

$$\Phi(Du_k - \Theta(u_k)) : D\varphi \to \Phi(Du - \Theta(u)) : D\varphi \quad \text{almost everywhere.}$$

Now, we prove the equiintegrability of $\big(\Phi(Du_k - \Theta(u_k)) : D\varphi\big)$. To do this, we choose $\Omega' \subset \Omega$ a measurable subset and $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. According to the Assertion 1, we have

$$\int_{\Omega'} |\Phi(Du_k - \Theta(u_k)) : D\varphi| dx \leq 2^{\frac{(p^+-1)^2}{p^-}} \Big(\underbrace{\|Du_k\|_{p(x)}^{p(x)}}_{\leq C} + c^{p^+} \underbrace{\|u_k\|_{p(x)}^{p(x)}}_{\leq C}\Big) \Big(\int_{\Omega'} |D\varphi|^{p(x)} dx\Big)^{\frac{1}{p(x)}}.$$

The equiintegrability of $\big(\Phi(Du_k - \Theta(u_k)) : D\varphi\big)$ follows since $\int_{\Omega'} |D\varphi|^{p(x)} dx$ is arbitrary small if we choose the measure of $\Omega'$ small enough. Applying the Vitali Theorem, we deduce that $T$ is continuous.

**Assertion 3:** We prove that $T$ is coercive.

Thanks to Lemma 3.1.1, we have that

$$\begin{aligned}
&|Du-\Theta(u)|^{p(x)-2}(Du-\Theta(u)) : Du \\
&\qquad\qquad = |Du-\Theta(u)|^{p(x)-2}(Du-\Theta(u)) : (Du-\Theta(u)+\Theta(u)) \\
&\qquad\qquad \geq \frac{1}{p(x)}|Du-\Theta(u)|^{p(x)} - \frac{1}{p(x)}|\Theta(u)|^{p(x)},
\end{aligned}$$

and that (by (3.1.4))

$$\begin{aligned}
\frac{1}{2^{p^+-1}}|Du|^{p(x)} &= \frac{1}{2^{p^+-1}}|Du-\Theta(u)+\Theta(u)|^{p(x)} \\
&\leq \frac{1}{2^{p^+-1}}\Big[2^{p^+-1}\big(|Du-\Theta(u)|^{p(x)}+|\Theta(u)|^{p(x)}\big)\Big] \\
&= |Du-\Theta(u)|^{p(x)}+|\Theta(u)|^{p(x)}.
\end{aligned}$$

Hence (by (3.1.3) and (4.1.1))

$$\begin{aligned}
\langle T(u),u\rangle &= \int_\Omega |Du-\Theta(u)|^{p(x)-2}(Du-\Theta(u)) : Du dx - \langle f,u\rangle \\
&\geq \int_\Omega \frac{1}{p(x)}|Du-\Theta(u)|^{p(x)}dx - \int_\Omega \frac{1}{p(x)}|\Theta(u)|^{p(x)}dx - \|f\|_{-1,p'(x)}\|u\|_{1,p(x)} \\
&\geq \int_\Omega \frac{1}{p(x)}\Big[\frac{1}{2^{p^+-1}}|Du|^{p(x)} - |\Theta(u)|^{p(x)}\Big]dx - \int_\Omega \frac{1}{p(x)}|\Theta(u)|^{p(x)}dx \\
&\qquad\qquad\qquad\qquad - \|f\|_{-1,p'(x)}\|u\|_{1,p(x)} \\
&\geq \int_\Omega \frac{1}{p(x)}\frac{1}{2^{p^+-1}}|Du|^{p(x)}dx - \int_\Omega \frac{2}{p(x)}c^{p(x)}|u|^{p(x)}dx - \|f\|_{-1,p'(x)}\|u\|_{1,p(x)} \\
&\geq \int_\Omega \frac{1}{p(x)}\frac{1}{2^{p^+-1}}|Du|^{p(x)}dx - \int_\Omega \frac{2}{p(x)}\frac{1}{2\alpha^{p^+}}\left(\frac{\alpha}{2}\right)^{p^+}|Du|^{p(x)}dx \\
&\qquad\qquad\qquad\qquad - \|f\|_{-1,p'(x)}\|u\|_{1,p(x)} \\
&= \int_\Omega \frac{1}{p(x)}\frac{1}{2^{p^+}}|Du|^{p(x)}dx - \|f\|_{-1,p'(x)}\|u\|_{1,p(x)} \\
&\geq \frac{1}{p^+}\frac{1}{2^{p^+}}\int_\Omega |Du|^{p(x)}dx - \|f\|_{-1,p'(x)}\|u\|_{1,p(w)}.
\end{aligned}$$

Consequently

$$\lim_{\|u\|_{1,p(x)}\to\infty} \frac{\langle T(u), u\rangle}{\|u\|_{1,p(x)}} = \infty,$$

that is $T$ is coercive.

In order to find a such solution $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ such that $\langle T(u), \varphi\rangle = 0$ for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, we apply a Galerkin scheme to construct the approximating solutions. Let $W_1 \subset W_2 \subset ... \subset W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\bigcup\limits_{k\geq 1} W_k$ is dense in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

**Lemma 4.1.1.** *(1) For all $k \in \mathbb{N}$ there exists $u_k \in W_k$ such that*

$$\langle T(u_k), \varphi\rangle = 0 \quad \textit{for all } \varphi \in W_k. \tag{4.1.2}$$

*(2) There exists a constant $R > 0$ such that*

$$\|u_k\|_{1,p(x)} \leq R \quad \textit{for all } k \in \mathbb{N}. \tag{4.1.3}$$

*Proof.* (1) Let fix $k$ and assume that $\dim W_k = r$. Let $(w_i)_{i=1,..,r}$ be a basis of $W_k$ and we write for simplicity $\sum\limits_{1\leq i\leq r} a^i w_i = a^i w_i$. We define the mapping

$$\begin{aligned} S : \mathbb{R}^r &\longrightarrow \mathbb{R}^r \\ (a^1, ..., a^r) &\longrightarrow \Big(\langle T(a^i w_i), w_j\rangle\Big)_{j=1,..,r}. \end{aligned}$$

Let $a \in \mathbb{R}^r$ and $u = a^i w_i \in W_k$. According to the Assertion 2, $S$ is continuous. Remark that $\|a\|_{\mathbb{R}^r} \to \infty$ is equivalent to $\|u\|_{1,p(x)} \to \infty$ and $S(a).a = \langle T(u), u\rangle$. It follows by the Assertion 3 that

$$S(a).a \to \infty \quad \text{as} \quad \|a\|_{\mathbb{R}^r} \to \infty.$$

Thus, there exists $R > 0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $S(a).a > 0$. Thanks to the usual topological arguments (see e.g. [139, Proposition 2.8]), $S(x) = 0$ has a solution in $B_R(0)$. Consequently, for all $k \in \mathbb{N}$ there exists $u_k \in W_k$ such that $\langle T(u_k), \varphi\rangle = 0$ for all $\varphi \in W_k$.
(2) By the Assertion 3, we have $\langle T(u), u\rangle \to \infty$ as $\|u\|_{1,p(x)} \to \infty$. Hence, it follows that

there exists $R > 0$ with the property that $\langle T(u), u\rangle > 1$ whenever $\|u\|_{1,p(x)} > R$. This is a contradiction with (4.1.2), thus we obtain the uniform bound (4.1.3). $\square$

Now, we collect some facts about the Young measure $\nu = \{\nu_x\}_{x\in\Omega}$ generated by the gradient sequences $Du_k$ in $L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$ (see Eq. (4.1.3) and Lemma 2.3.1).

**Lemma 4.1.2.** *Let $\{u_k\}$ be the sequence defined in Lemma 4.1.1. Then the Young measure $\nu_x$ generated by $Du_k$ in $L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$ satisfy $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for a.e. $x \in \Omega$ and the weak $L^1$-limit of $Du_k$ is given by $\langle \nu_x, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) = Du(x)$ for a.e. $x \in \Omega$.*

We proceed the proof of Lemma 4.1.2 in the similar way as that of Lemma 3.1.4.

*Proof.* Let $\nu_x$ be the Young measure generated by $Du_k$ in $L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$. According to (4.1.3) there exists a constant $C \geq 0$ such that for any $R > 0$

$$\begin{aligned} C \geq \int_\Omega |Du_k|^{p(x)} dx &\geq \int_{\{x\in\Omega\cap B_R(0):\, |Du_k|\geq L\}} |Du_k|^{p(x)} dx \\ &\geq L^{p^-} \big|\{x \in \Omega \cap B_R(0) :\ |Du_k| \geq L\}\big|. \end{aligned}$$

Hence

$$\sup_{k\in\mathbb{N}} \big|\{x \in \Omega \cap B_R(0) :\ |Du_k| \geq L\}\big| \leq \frac{C}{L^{p^-}} \to 0 \quad \text{as } L \to \infty.$$

Thanks to Theorem 2.3.1(iii), we obtain that $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$. On the one hand, since $L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$ is reflexive ($1 < p^- \leq p(x)$ and $\mathbb{M}^{m\times n} \cong \mathbb{R}^{mn}$), we deduce by (4.1.3) the existence of a subsequence (still denoted by $Du_k$) weakly convergent in $L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$, thus weakly convergent in $L^1(\Omega; \mathbb{M}^{m\times n})$. By taking $\varphi$ of Theorem 2.3.1(iii) as the identity mapping $id$, one has

$$Du_k \rightharpoonup \langle \nu_x, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) \quad \text{weakly in } L^1(\Omega; \mathbb{M}^{m\times n}).$$

On the other hand, by (4.1.3), a subsequence of $\{u_k\}$ converges weakly in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ to an element denoted by $u$. Hence $u_k \to u$ in $L^{p(x)}(\Omega; \mathbb{R}^m)$ and $Du_k \rightharpoonup Du$ in $L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$ (for a subsequence). The uniquenesses of limit implies that $\langle \nu_x, id\rangle = Du(x)$ for a.e. $x \in \Omega$. $\square$

Now we have all necessary ingredients to pass to the limit in the approximating equations.

*Proof of Theorem 4.1.1.* . Let $E_{k,\epsilon} = \{x \in \Omega,\ |u_k(x) - u(x)| \geq \epsilon\}$. By the Eq. (4.1.3), $u_k \to u$ in $L^{p(x)}(\Omega; \mathbb{R}^m)$ (for a subsequence). Hence

$$\int_\Omega |u_k(x) - u(x)|^{p(x)} dx \geq \int_{E_{k,\epsilon}} |u_k(x) - u(x)|^{p(x)} dx \geq \epsilon^{p^-} |E_{k,\epsilon}|,$$

thus

$$|E_{k,\epsilon}| \leq \frac{1}{\epsilon^{p^-}} \int_\Omega |u_k(x) - u(x)|^{p(x)} dx \to 0 \quad \text{as } k \to \infty.$$

We deduce that $u_k \to u$ in measure as $k \to \infty$. Therefore, after extraction of a subsequence, if necessary, we infer that

$$u_k \to u \quad \text{almost everywhere for } k \to \infty.$$

Note that $\{Du_k - \Theta(u_k)\}$ is equiintegrable by (4.1.3) and (4.1.1). It follows by the continuity of $\Theta$ and the weak convergence defined in Lemma 4.1.2 that

$$\begin{aligned} Du_k - \Theta(u_k) &\rightharpoonup \int_{\mathbb{M}^{m\times n}} (\lambda - \Theta(u)) d\nu_x(\lambda) \\ &= \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) - \Theta(u) \int_{\mathbb{M}^{m\times n}} d\nu_x(\lambda) \\ &= Du - \Theta(u) \end{aligned}$$

weakly in $L^1(\Omega)$. Therefore

$$\Phi(Du_k - \Theta(u_k)) \rightharpoonup \Phi(Du - \Theta(u)) \quad \text{weakly in } L^1(\Omega).$$

Since $L^{p(x)}(\Omega)$ is reflexive and $\{\Phi(Du_k - \Theta(u_k))\}$ is bounded by the Eq. (3.1.4), the sequence $\{\Phi(Du_k - \Theta(u_k))\}$ converges in $L^{p'(x)}(\Omega)$. Hence its weak $L^{p'(x)}$-limit is also $\Phi(Du - \Theta(u))$. Thus

$$\lim_{k\to\infty} \int_\Omega \Phi(Du_k - \Theta(u_k)) : D\varphi dx = \int_\Omega \Phi(Du - \Theta(u)) : D\varphi dx \quad \forall \varphi \in \bigcup_{k\geq 1} W_k. \tag{4.1.4}$$

Let $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, since $\bigcup\limits_{k\geq 1} W_k$ is dense in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, there exists a sequence $\{\varphi_k\} \subset \bigcup\limits_{k\geq 1} W_k$ such that $\varphi_k \to \varphi$ in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ as $k \to \infty$.

$$\begin{aligned}
&\langle T(u_k), \varphi_k\rangle - \langle T(u), \varphi\rangle \\
&= \int_\Omega \Phi(Du_k - \Theta(u_k)) : D\varphi_k dx - \int_\Omega \Phi(Du - \Theta(u)) : D\varphi dx - \langle f, \varphi_k - \varphi\rangle \\
&= \int_\Omega \Phi(Du_k - \Theta(u_k)) : (D\varphi_k - D\varphi) dx \\
&\qquad + \int_\Omega \big(\Phi(Du_k - \Theta(u_k)) - \Phi(Du - \Theta(u))\big) : D\varphi dx - \langle f, \varphi_k - \varphi\rangle.
\end{aligned}$$

By the Eq. (4.1.4) and the construction of $\varphi_k$, the right hand side of the above equation tends to 0 as $k \to \infty$. Hence

$$\lim_{k\to\infty} \langle T(u_k), \varphi_k\rangle = \langle T(u), \varphi\rangle.$$

According to Lemma 4.1.1, it follows that $\langle T(u), \varphi\rangle = 0$ for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ as desired. □

## 4.2 Generalized $p(x)$-Laplacian with nonlinear physical data

This section aims to extend the previous generalized $p(x)$-Laplacian system to a diffusion problem given by

$$-\text{div}\,\big(\Phi(Du - \Theta(u)\big) = v(x) + f(x,u) + \text{div}\,\big(g(x,u)\big),$$

where the source $v$ is in moving and dissolving substance, the motion is described by $g$ and the dissolution by $f$. By the theory of Young measure we will also prove the existence result in variable exponent Sobolev spaces $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

### 4.2.1 Introduction and main result

Let $\Omega$ be a bounded open domain in $\mathbb{R}^n$, $n \geq 2$. In [18] (cf. first section in Chap. 3), the quasilinear elliptic system (3.1.1) was considered. We have used the theory of Young measure and Galerkin method to prove that (3.1.1) had a weak solution $u \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ under the condition (3.1.2). See also [27] (cf. Section 3.2) for a related topic.

When the exponent $p$ is not constant, but depends on $x$, i.e. $p \equiv p(x)$, Azroul and Balaadich [22] (cf. the previous section) established the existence result for (3.1.1) in the case where $f$ belongs to $W^{-1,p'(x)}(\Omega; \mathbb{R}^m)$ and the constant $c$ in (4.1.1) is assumed to satisfy

$$c < \frac{1}{\operatorname{diam}(\Omega)} \left(\frac{1}{2}\right)^{\frac{1}{p^+}}.$$

They used also the tool of Young measure in establishing their result.

As we know, the $p(x)$-Laplacian is inhomogeneous. This implies that it posesses more complicated nonlinearities than the case of $p$ constant. Problems with variable exponent appear in several domains. For example; in the mathematical modeling of stationary thermorheological viscous of the process filtration of an ideal barotropic gas through a porous medium (cf. [10, 11]). In image processing [92], to outline the borders of a true image and to elliminate possible noise, the variable nonlinearity find its applications. For the case of calculus of variations, the reader can see [2, 65] and references therein.

The authors in [32] considered the following $p(x)$-curl systems

$$\begin{cases} \nabla \times \left(|\nabla \times u|^{p(x)-2} \nabla \times u\right) &= \lambda g(x,u) - \mu f(x,u), \quad \nabla . u = 0 \quad \text{in } \Omega \\ \quad |\nabla \times u|^{p(x)-2} \nabla \times u \times \mathbf{n} &= 0, \quad u.\mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Here $\nabla \times u$ is the curl of $u = (u_1, u_2, u_3)$. They studied the existence and nonexistence of solutions. Note that the above system is arising in electromagnetism.

E. Azroul et al. [16] investigated a class of nonlinear $p(x)$-Laplacian problems, in the scalar case, of the form

$$\begin{cases} -\operatorname{div}\Phi(\nabla u - \Theta(u)) + |u|^{p(x)-2}u + \alpha(u) &= f \quad \text{in } \Omega, \\ \Phi(\nabla u - \Theta(u)).\eta + \gamma(u) &= g \quad \text{on } \partial\Omega, \end{cases}$$

where the source term $f$ was assumed to belong to $L^1(\Omega)$. They used the techniques of entropy solutions to prove the existence of a solution. See also [17, 45].

Our purpose here is to prove the existence of weak solutions for the following problem which is motivated by physics or geometry:

$$-\operatorname{div}\big(\Phi(Du - \Theta(u))\big) = v(x) + f(x,u) + \operatorname{div}\big(g(x,u)\big) \quad \text{in } \Omega, \tag{4.2.1}$$

supplemented with the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. Here $v$ belongs to $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$, $\Phi : \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is given in a simple form $\Phi(\xi) = |\xi|^{p(x)-2}\xi$ for all $\xi \in \mathbb{M}^{m\times n}$ and $\Theta : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is a continuous function such that

$$\Theta(0) = 0 \quad \text{and} \quad |\Theta(a) - \Theta(b)| \le c|a-b|, \quad \forall a,b \in \mathbb{R}^m, \tag{4.2.2}$$

where $c$ is a positive constant that satisfies $c < \frac{1}{\operatorname{diam}(\Omega)}\left(\frac{1}{2}\right)^{\frac{1}{p^+}}$. Moreover, $f$ and $g$ satisfy the following continuity and growth conditions:

**(F0)** $f : \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a Carathéodory function, i.e. $x \mapsto f(x,s)$ is measurable for every $s \in \mathbb{R}^m$ and $s \mapsto f(x,s)$ is continuous for a.e. $x \in \Omega$.

**(F1)** There exist $b_1 \in L^{p'(x)}(\Omega)$ and $0 < \gamma(x) < p(x) - 1$ such that

$$|f(x,s)| \le b_1(x) + |s|^{\gamma(x)}.$$

**(G0)** $g : \Omega \times \mathbb{R}^m \longrightarrow \mathbb{M}^{m\times n}$ is a Carathéodory function in the sense of **(F0)**.

**(G1)** There exist $b_2 \in L^{p'(x)}$ and $0 < q(x) < p(x) - 1$ such that

$$|g(x,s)| \le b_2(x) + |s|^{q(x)}.$$

**Remark 4.2.1.** *1) The strict bound $p(x) - 1$ for $\gamma(x)$ and $q(x)$ in the growth conditions **(F1)** and **(G1)** ensures the coercivity of the operator $T$ introduced in the next subsection.*

2) *The function $f$ may depend even on the Jacobien matrix $Du := \frac{1}{2}(\nabla u + (\nabla u)^t)$ and linear with respect to its variable $\xi \in \mathbb{M}^{m\times n}$, see Appendix.*

Let $u : \Omega \to \mathbb{R}^m$ be a vector-valued function.

**Definition 4.2.1.** *A measurable function $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ is called a weak solution to problem (4.2.1) if*

$$\int_\Omega \Phi\big(Du - \Theta(u)\big) : D\varphi dx = \langle v, \varphi\rangle + \int_\Omega f(x,u).\varphi dx - \int_\Omega g(x,u) : D\varphi dx$$

*holds for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. Here $\langle .,.\rangle$ is the duality pairing of $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ and $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.*

We shall prove the following existence theorem:

**Theorem 4.2.1.** *Assume that (4.2.2), **(F0)**, **(F1)**, **(G0)** and **(G1)** hold true. Then there exists at least one weak solution to (4.2.1) in the sense of Definition 4.2.1.*

### 4.2.2 Approximating solutions

To construct the approximating solutions, we will use the Galerkin method. To this purpose, we consider the following map $T : W_0^{1,p(x)}(\Omega;\mathbb{R}^m) \longrightarrow W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ defined for arbitrary $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, by

$$\langle T(u), \varphi\rangle = \int_\Omega \Phi\big(Du - \Theta(u)\big) : D\varphi dx - \langle v, \varphi\rangle - \int_\Omega f(x,u)\varphi dx + \int_\Omega g(x,u) : D\varphi dx$$

for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. As a consequence, our problem (4.2.1) is then equivalent to find $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ such that

$$\langle T(u), \varphi\rangle = 0 \quad \text{for all} \quad \varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m).$$

**Lemma 4.2.1.** *The mapping $T(u)$ is well defined, linear and bounded.*

*Proof.* For arbitrary $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, $T(u)$ is linear. For all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$,

$$\begin{aligned}
&|\langle T(u),\varphi\rangle| \\
&= \Big| \int_\Omega \Phi\big(Du-\Theta(u)\big) : D\varphi dx - \langle v,\varphi\rangle - \int_\Omega f(x,u)\varphi dx + \int_\Omega g(x,u) : D\varphi dx \Big| \\
&\le \int_\Omega \big|Du-\Theta(u)\big|^{p(x)-1}|D\varphi|dx + \big|\langle v,\varphi\rangle\big| \\
&+ \int_\Omega |f(x,u)||\varphi|dx + \int_\Omega |g(x,u)||D\varphi|dx.
\end{aligned}$$

According to the Assertion 1 of Chap. 4,

$$I_1 := \int_\Omega \big|Du-\Theta(u)\big|^{p(x)-1}|D\varphi|dx \le 2^{\frac{(p^+-1)^2}{p^-}} \Big(\|Du\|_{p(x)}^{p(x)} + \|\Theta(u)\|_{p(x)}^{p(x)}\Big)^{\frac{p(x)-1}{p(x)}} \|D\varphi\|_{p(x)}.$$

The generalized Hölder inequality implies that $I_2 := \big|\langle v,\varphi\rangle\big| \le \|v\|_{-1,p'(x)}\|\varphi\|_{1,p(x)}$. On the other hand, it follows from the growth condition **(F1)** (without loss of generality, we can assume that $\gamma(x) = p(x)-1$), that

$$I_3 := \int_\Omega |f(x,u)||\varphi|dx \le \|b_1\|_{p'(x)}\|\varphi\|_{p(x)} + \|u\|_{p(x)}^{p(x)-1}\|\varphi\|_{p(x)}.$$

Finally, the growth condition **(G1)** (without loss of generality, we may assume that $q(x) = p(x)-1$) allows to estimate (by application of the Hölder inequality)

$$I_4 := \int_\Omega |g(x,u)||D\varphi|dx \le \|b_2\|_{p'(x)}\|D\varphi\|_{p(x)} + \|u\|_{p(x)}^{p(x)-1}\|D\varphi\|_{p(x)}.$$

By virtual of the Poincaré inequality (cf. Eq. (3.1.3) or Proposition 2.1.1), the $I_i$ for $i = 1,..,4$ are finite, then $T(u)$ is well defined. Moreover, for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$

$$\big|\langle T(u),\varphi\rangle\big| \le \sum_{i=1}^4 I_i \le C\|D\varphi\|_{p(x)},$$

and this implies that $T(u)$ is bounded. □

**Lemma 4.2.2.** *The restriction of $T$ to a finite dimensional linear subspace $V$ of $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ is continuous.*

*Proof.* Let $r$ be the dimension of $V$ and $(e_i)_{i=1}^r$ a basis of $V$. Let $(u_j = a_j^i e_i)$ be a sequence in $V$ which converges to $u = a^i e_i$ in $V$ (with conventional summation). Then $u_j \to u$ and $Du_j \to Du$ almost everywhere. The continuity of $\Theta$, $f$ and $g$ implies that

$$\Phi(Du_j - \Theta(u_j)) : D\varphi \to \Phi(Du - \Theta(u)) : D\varphi,$$

$$f(x, u_j)\varphi \to f(x, u)\varphi \quad \text{and} \quad g(x, u_j) : D\varphi \to g(x, u) : D\varphi$$

almost everywhere. Since $u_j \to u$ strongly in $V$,

$$\int_\Omega |u_j - u|^{p(x)} dx \longrightarrow 0 \quad \text{and} \quad \int_\Omega |Du_j - Du|^{p(x)} \longrightarrow 0.$$

According to [40] (Chapter IV, Section 3, Theorem 3) there exist a subsequence of $\{u_j\}$ still denoted by $\{u_j\}$ and $h_1, h_2 \in L^1(\Omega)$ such that $|u_j - u|^{p(x)} \leq h_1$, $|Du_j - Du|^{p(x)} \leq h_2$. By virtue of the Eq (3.1.4), we can write

$$\begin{aligned} |u_j|^{p(x)} = |u_j - u + u|^{p(x)} &\leq 2^{p^+-1}\big(|u_j - u|^{p(x)} + |u|^{p(x)}\big) \\ &\leq 2^{p^+-1}\big(h_1 + |u|^{p(x)}\big), \end{aligned}$$

from which (similarly) we get $|Du_j|^{p(x)} \leq 2^{p^+-1}\big(h_2 + |Du|^{p(x)}\big)$. Consequently, the sequences $\|u_j\|_{p(x)}$ and $\|Du_j\|_{p(x)}$ are bounded by a constant denoted $C$. Now, if $\Omega' \subset \Omega$ is a measurable subset and $\varphi \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$, then by Poincaré's inequality

$$\begin{aligned} &\int_{\Omega'} |\Phi(Du_j - \Theta(u_j)) : D\varphi| dx \\ &\qquad \leq 2^{\frac{(p^+-1)^2}{p^-}} \Big(\underbrace{\|Du_j\|_{p(x)}^{p(x)}}_{\leq C} + c^{p^+} \underbrace{\|u_j\|_{p(x)}^{p(x)}}_{\leq C}\Big)^{\frac{p(x)-1}{p(x)}} \Big(\int_{\Omega'} |D\varphi|^{p(x)}\Big)^{\frac{1}{p(x)}}, \end{aligned}$$

where the small $c$ is the constant in (4.2.2), and (without loss of generality, we can assume that $\gamma(x) = p(x) - 1$ and $q(x) = p(x) - 1$)

$$\int_{\Omega'} |f(x, u_j)\varphi| dx \leq C\Big(\|b_1\|_{p'(x)} + \underbrace{\|u_j\|_{p(x)}^{p(x)-1}}_{\leq C}\Big)\Big(\int_{\Omega'} |D\varphi|^{p(x)} dx\Big)^{\frac{1}{p(x)}},$$

and

$$\int_{\Omega'} |g(x,u_j) : D\varphi| dx \le \Big( \|b_2\|_{p'(x)} + \underbrace{\|u_j\|_{p(x)}^{p(x)-1}}_{\le C} \Big) \Big( \int_{\Omega'} |D\varphi|^{p(x)} dx \Big)^{\frac{1}{p(x)}}.$$

Therefore, the sequences $\big(\Phi(Du_j - \Theta(u_j)) : D\varphi\big)$, $\big(f(x,u_j).\varphi\big)$ and $\big(g(x,u_j) : D\varphi\big)$ are equiintegrable, since $\int_{\Omega'} |D\varphi|^{p(x)} dx$ is arbitrary small if the measure of $\Omega'$ is chosen small enough. Applying the Vitali Theorem, it follows for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ that $\lim_{j\to\infty}\langle T(u_j),\varphi\rangle = \langle T(u),\varphi\rangle$ as we desire. □

**Lemma 4.2.3.** *The mapping $T$ is coercive.*

*Proof.* Taking $\varphi = u$ in the definition of $T$, then

$$\langle T(u),u\rangle = \int_\Omega \Phi(Du-\Theta(u)) : Dudx - \langle v,u\rangle - \int_\Omega f(x,u)udx + \int_\Omega g(x,u) : Dudx. \tag{4.2.3}$$

Similar to that in Assertion 3 in Chap.4, we have

$$J_1 = \int_\Omega \Phi(Du-\Theta(u)) : Dudx \ge \frac{1}{p^+}\frac{1}{2^{p^+}} \int_\Omega |Du|^{p(x)} dx.$$

Next, the Hölder inequality implies that

$$|J_2| := |\langle v,u\rangle| \le \|v\|_{-1,p'(x)} \|u\|_{1,p(x)}.$$

Finally, it follows from the growth conditions **(F1)** and **(G1)** that

$$\begin{aligned} J_3 := \int_\Omega f(x,u)udx &\le \|b_1\|_{p'(x)}\|u\|_{p(x)} + \|u\|_{p(x)}^{\gamma(x)+1} \\ &\le C\|b_1\|_{p'(x)}\|Du\|_{p(x)} + C^{\gamma^+ +1}\|Du\|_{p(x)}^{\gamma(x)+1} \end{aligned}$$

and

$$|J_4| := \Big| \int_\Omega g(x,u) : Dudx \Big| \le \|b_2\|_{p'(x)}\|Du\|_{p(x)} + C^{q^+}\|Du\|_{p(x)}^{q(x)+1}.$$

From these estimations it follows that

$$\langle T(u), u\rangle = J_1 - J_2 - J_3 + J_4 \longrightarrow +\infty \quad \text{as} \quad \|u\|_{1,p(x)} \to +\infty$$

since $p^+ > \max\{1, \gamma^+ + 1, q^+ + 1\}$. Hence $T$ is coercive. □

Now, let $V_1 \subset V_2 \subset ... \subset W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\cup_{k\in\mathbb{N}} V_k$ is dense in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$. Notice that such a sequence $(V_k)$ exists since $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ is separable. Let dim $V_k = r$ and $e_1, .., e_r$ be a basis of $V_k$ for a fixed $k$. To construct the approximating solution, we define the map

$$S : \mathbb{R}^r \to \mathbb{R}^r, \qquad \begin{pmatrix} a^1 \\ a^2 \\ . \\ . \\ a^r \end{pmatrix} \mapsto \begin{pmatrix} \langle T(a^i e_i), e_1\rangle \\ \langle T(a^i e_i), e_2\rangle \\ . \\ . \\ \langle T(a^i e_i), e_r\rangle \end{pmatrix}.$$

**Lemma 4.2.4.** *1) The map $S$ is continuous.*
*2) For all $k \in \mathbb{N}$ there exists $u_k \in V_k$ such that*

$$\langle T(u_k), \varphi\rangle = 0 \quad \textit{for all} \quad \varphi \in V_k. \tag{4.2.4}$$

*3) The sequence constructed in 2) is uniformly bounded in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$, i.e. there exists a constant $R > 0$ such that*

$$\|u_k\|_{1,p(x)} \le R \quad \textit{for all} \quad k \in \mathbb{N}. \tag{4.2.5}$$

We omit its proof since it is similar to that of Lemma 4.1.1. Before we pass to the limit in the approximating sequences and so to prove Theorem 4.2.1, notice that since $(u_k)$ is bounded in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ by (4.2.5), it follows by Lemma 2.3.1 the existence of a Young measure $\nu_x$ generated by $Du_k$ in $L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$ which satisfies the properties of Lemma 4.1.2.

### 4.2.3 Proof of Theorem 4.2.1

To apply the convergence described in Lemma 4.1.2 to our approximating problem, we need the convergence in measure of $u_k$ to $u$. To this purpose, consider $E_{k,\epsilon} = \{x \in \Omega;\ |u_k(x) - u(x)| \geq \epsilon\}$. Since $(u_k)$ is bounded in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$, then for a subsequence still denoted $u_k$, $u_k \to u$ in $L^{p(x)}(\Omega; \mathbb{R}^m)$. Therefore

$$\int_\Omega |u_k(x) - u(x)|^{p(x)} \geq \int_{E_{k,\epsilon}} |u_k(x) - u(x)|^{p(x)} \geq \epsilon^{p^-} |E_{k,\epsilon}|,$$

which implies that

$$|E_{k,\epsilon}| \leq \frac{1}{\epsilon^{p^-}} \int_\Omega |u_k(x) - u(x)|^{p(x)} \longrightarrow 0 \quad \text{as } k \to \infty.$$

Hence the sequence $u_k$ converges in measure to $u$ on $\Omega$. On the other hand, since $\{Du_k - \Theta(u_k)\}$ is equiintegrable by the condition (4.2.2) and the boundedness of $(u_k)$, it result that

$$\begin{aligned} Du_k - \Theta(u_k) &\rightharpoonup \int_{\mathbb{M}^{m\times n}} (\lambda - \Theta(u)) d\nu_x(\lambda) \\ &= \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)}_{=:Du(x)} - \Theta(u) \underbrace{\int_{\mathbb{M}^{m\times n}} d\nu_x(\lambda)}_{=:1} \\ &= Du - \Theta(u) \end{aligned}$$

weakly in $L^1(\Omega)$, where we have used Lemma 4.1.2. Further, from the reflexivity of $L^{p'(x)}(\Omega)$ and the boundedness of $\{\Phi(Du_k - \Theta(u_k))\}$, we deduce that $\Phi(Du_k - \Theta(u_k))$ converges in $L^{p'(x)}(\Omega)$ and its weak $L^{p'(x)}$-limit is given by $\Phi(Du - \Theta(u))$. Consequently

$$\lim_{k\to\infty} \int_\Omega \Phi(Du_k - \Theta(u_k)) : D\varphi dx = \int_\Omega \Phi(Du - \Theta(u)) : D\varphi dx \quad \forall \varphi \in \cup_{k\in\mathbb{N}} V_k.$$

Moreover, since $u_k \to u$ in measure for $k \to \infty$, we may infer that, after extraction of a suitable subsequence, if necessary,

$$u_k \longrightarrow u \quad \text{almost everywhere for } k \to \infty.$$

Hence, for arbitrary $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, it follows from the continuity conditions **(F0)** and **(G0)**, that

$$f(x,u_k)\varphi \to f(x,u)\varphi \quad \text{and} \quad g(x,u_k) : D\varphi \to g(x,u) : D\varphi$$

almost everywhere. As in the proof of Lemma 4.2.2, we have $f(x,u_k)\varphi$ and $g(x,u_k) : D\varphi$ are equiintegrable, thus

$$f(x,u_k)\varphi \to f(x,u)\varphi \quad \text{and} \quad g(x,u_k) : D\varphi \to g(x,u) : D\varphi$$

in $L^1(\Omega)$ by the Vitali Convergence Theorem. Consequently

$$\lim_{k\to\infty} \int_\Omega f(x,u_k)\varphi dx = \int_\Omega f(x,u)\varphi dx \quad \forall \varphi \in \cup_{k\geq 1} V_k$$

and

$$\lim_{k\to\infty} \int_\Omega g(x,u_k) : D\varphi dx = \int_\Omega g(x,u) : D\varphi dx \quad \forall \varphi \in \cup_{k\geq 1} V_k.$$

Since $\cup_{k\geq 1} V_k$ is dense in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, $u$ is then a weak solution of (4.2.1).

# Appendix

Consider the function $f$ depends on $\xi \in \mathbb{M}^{m\times n}$, i.e. $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ and satisfies

$$|f(x,s,\xi| \leq b_1(x) + |s|^{\gamma(x)} + |\xi|^{s(x)}, \tag{4.2.6}$$

where $b_1 \in L^{p'(x)}(\Omega)$, $0 < \gamma(x) < p(x) - 1$ and $0 < s(x) < p(x) - 1$. By similar arguments as above (since $p^+ > \max\{1, \gamma^+ + 1, q^+ + 1, s^+ + 1\}$), it follows that

$$\lim_{k\to\infty} \int_\Omega f(x,u_k,Du_k)\varphi dx = \int_\Omega f(x,u,Du)\varphi dx \quad \forall \varphi \in \cup_{k\geq 1} V_k$$

for all $\varphi \in \cup_{k\geq 1} V_k$. Now, assume that $\xi \mapsto f(x,u,\xi)$ is linear. We have $f(x,u_k,Du_k)$ is equiinetgrable (by the growth condition (4.2.6)), this implies

$$\begin{aligned} f(x,u_k,Du_k) &\rightharpoonup \int_{\mathbb{M}^{m\times n}} f(x,u,\lambda)d\nu_x(\lambda) \\ &= f(x,u,.)\underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)}_{=:Du(x)} \\ &= f(x,u,Du) \end{aligned}$$

weakly in $L^1(\Omega)$, by linearity of $f$.

To conclude, let $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, since $\cup_{k\geq 1} V_k$ is dense in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, then there exists a sequence $(\varphi_k) \subset \cup_{k\geq 1} V_k$ such that $\varphi_k \to \varphi$ in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. According to the previous results, we get

$$\lim_{k\to\infty} \langle T(u_k), \varphi_k\rangle = \langle T(u), \varphi\rangle.$$

The equation (4.2.4) implies that $\langle T(u), \varphi\rangle = 0$ as we desire for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

## 4.3 On a class of quasilinear elliptic systems with variable exponent

Our intention here goes into the study of the following quasilinear elliptic system in a Sobolev space with variable exponent:

$$-\mathrm{div}(a(|Du|)Du) = f,$$

where $a$ is a $C^1$-function and $f \in W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$. As usual, the theory of Young measures and weak monotonicity conditions allow to obtain the existence of solutions.

### 4.3.1 Introduction and main result

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with Lipschitz boundary $\partial\Omega$. Consider the following quasilinear elliptic system:

$$\begin{cases} -\operatorname{div}(a(|Du|)Du) & = f \quad \text{in } \Omega, \\ u & = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.3.1}$$

where $a$ is a $C^1$-function defined from $[0, +\infty)$ to $[0, +\infty)$ and $f$ belongs to Sobolev space with variable exponent $W^{-1,p'(x)}(\Omega; \mathbb{R}^m)$. When $a(\xi) = |\xi|^{p-2}$, problem (4.3.1) is the well known $p$-Laplace system. In recent years, there have been a large number of papers on the existence and regularity of solutions of the $p$-Laplace system (see [57, 91, 113] and the references therein). In the case of generalized $p$-Laplacian system where $a(\xi) = |\xi - \Theta(u)|^{p-2}$, $\Theta : \mathbb{R}^m \to \mathbb{M}^{m\times n}$, we have proved in [18] (cf. (3.1.1)) the existence result by using the theory of Young measures and without assuming any conditions of Leray-Lions type. The extension of [18] to the case of variable exponent $p(x)$ can be found in [22] (cf. Section 4.1). For the $p(x)$-Laplace equations, Cianchi and Maz'ya [50] established the Lipschitz continuity of solutions to Dirichlet and Neumann cases. In [3], Acerbi and Mingione proved Caldéron and Zygmund type estimates for a class of $p(x)$-Laplacian system whose right hand side is under the divergence form.

Problems of the form (4.3.1) were studied in [50, 51] under some conditions on the function $a$. Moreover, they treated the corresponding Neumann case. Dirichlet problems of the form (4.3.1) are the main objective of the present section.

Here and after, the function $a : [0, +\infty) \to [0, +\infty)$ is assumed to be of class $C^1([0, +\infty))$, and to fulfill

$$-1 \leq i_a \leq s_a < \infty \tag{4.3.2}$$

where

$$i_a = \inf_{t>0} \frac{ta'(t)}{a(t)} \quad \text{and} \quad s_a = \sup_{t>0} \frac{ta'(t)}{a(t)}. \tag{4.3.3}$$

The function $a$ satisfies the following growth and coercivity conditions: For all $\xi \in \mathbb{M}^{m\times n}$, some constants $c_1, c_2 > 0$ and $l(x) \in L^1(\Omega)$,

$$|a(|\xi|)\xi| \leq c_1 |\xi|^{p(x)-1}, \tag{4.3.4}$$

$$a(|\xi|)\xi : \xi \geq c_2|\xi|^{p(x)} - l(x). \tag{4.3.5}$$

Moreover, we assume that $a$ satisfies one of the following conditions:

**(H0)** There exists a convex and $C^1$-function $b : \mathbb{M}^{m\times n} \to \mathbb{R}$ such that

$$a(|\xi|)\xi = \frac{\partial b(\xi)}{\partial \xi} := D_\xi b(\xi).$$

**(H1)** For $\overline{\lambda} = \langle \nu_x, id\rangle$ where $\nu = \{\nu_x\}_{x\in\Omega}$ is any family of Young measures generated by a sequence in $L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$ and not a Dirac measure for almost every $x \in \Omega$, we have

$$\int_{\mathbb{M}^{m\times n}} \left(a(|\lambda|)\lambda - a(|\overline{\lambda}|)\overline{\lambda}\right) : (\lambda - \overline{\lambda})d\nu(\lambda) > 0.$$

Note that, (4.3.2) and (4.3.3) will serve us to prove that the function $a$ is monotone. The condition **(H0)** allows to take a potential $b$, which is only convex but not strictly convex to avoid the use of the well known classical monotone operator theory, and to consider (4.3.1) with $a(|\xi|)\xi = \partial b(\xi)/\partial\xi$. Assumption **(H1)** may be called strictly $p(x)$-quasimonotone as in the framework $W^{1,p}(\Omega; \mathbb{R}^m)$ (see [85]).

A weak solution for (4.3.1) is a function $u \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ such that

$$\int_\Omega a(|Du|)Du : D\varphi dx = \langle f, \varphi\rangle \quad \text{for all } \varphi \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m).$$

Here $\langle ., .\rangle$ denotes the duality pairing of $W^{-1,p'(x)}(\Omega; \mathbb{R}^m)$ and $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$.

The principal result of this part reads as follows:

**Theorem 4.3.1.** *Under assumptions (4.3.2)-(4.3.5), **(H0)** and **(H1)**, problem (4.3.1) has a weak solution $u \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$.*

### 4.3.2 Approximating solutions

Now, as mentioned in the introduction, we will use the Galerkin method to construct the approximating solutions. To this purpose, we consider the mapping $T$ :

$W_0^{1,p(x)}(\Omega;\mathbb{R}^m) \to W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ defined for $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ as

$$\langle T(u), \varphi\rangle = \int_\Omega a(|Du|)Du : D\varphi dx - \langle f, \varphi\rangle. \tag{4.3.6}$$

As a first remark, the problem (4.3.1) is equivalent to find a such $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ which satisfy $\langle T(u), \varphi\rangle = 0$ for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. In the sequel, we will use a positive constant $c$ which may change values from line to line.

**Lemma 4.3.1.** *The mapping $T$ satisfies the following properties:*

*(i) $T$ is linear, well defined and bounded.*

*(ii) The restriction of $T$ to a finite linear subspace of $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ is continuous.*

*(iii) $T$ is coercive.*

*Proof.* (i) For arbitrary $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, $T(u)$ is trivially linear. We have by (4.3.4)

$$\int_\Omega \left|a(|Du|)Du\right|^{p'(x)} dx \le c\int_\Omega |Du|^{p(x)} dx < \infty$$

where $c$ is a positive constant. It follows by Hölder's inequality that

$$\begin{aligned}\left|\langle T(u), \varphi\rangle\right| &\le c\|Du\|_{p(x)}^{p^+-1}\|D\varphi\|_{p(x)} + c\|f\|_{-1,p'(x)}\|\varphi\|_{1,p(x)}\\ &\le c\|D\varphi\|_{p(x)},\end{aligned}$$

thus $T$ is well defined and bounded.
(ii) Let $W$ be a finite linear subspace of $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ such that $\dim W = r$. For simplicity, we denote $T$ as the restriction $T|_W$ of $T$ to $W$. Let $(u_k = \alpha_{ki}w_i)$ be a sequence in $W$ which converges to $u = \alpha_i w_i$ in $W$ (with conventional summation). Here $\{w_1, .., w_r\}$ is a basis of $W$. We have in one hand, $Du_k \to Du$ almost everywhere and the continuity of the function $a$ gives

$$a(|Du_k|)Du_k \to a(|Du|)Du \quad \text{almost everywhere.}$$

On the other hand, since $Du_k \to Du$ strongly in $W$,

$$\int_\Omega |Du_k - Du|^{p(x)} dx \longrightarrow 0 \quad \text{as } k \to \infty.$$

According to [40, Chap IV, Sec 3, Theorem 3] there exist a subsequence still denoted $(Du_k)$ and $g \in L^1(\Omega)$ such that $|Du_k - Du|^{p(x)} \leq g$. Thanks to (3.1.4), we get

$$|Du_k|^{p(x)} = |Du_k - Du + Du|^{p(x)} \leq 2^{p^+-1}(g + |Du|^{p(x)}).$$

This implies that $\|Du_k\|_{p(x)}$ is bounded by a constant $c$. Now, in order to apply the Vitali Convergence Theorem, we choose $\Omega' \subset \Omega$ to be a measurable subset and by Hölder's inequality

$$\int_{\Omega'} |a(|Du_k|)Du_k : D\varphi| dx \leq c \underbrace{\|Du_k\|_{p(x)}^{p^+-1}}_{\leq c} \Big( \int_{\Omega'} |D\varphi|^{p(x)} dx \Big)^{\frac{1}{p(x)}}.$$

If we choose the measure of $\Omega'$ to be small enough, then $\int_{\Omega'} |D\varphi|^{p(x)} dx$ is arbitrary small, hence $\big(a(|Du_k|)Du_k : D\varphi\big)$ is equiintegrable. By virtue of the Vitali Convergence Theorem, we get $\lim_{k\to\infty} \langle T(u_k), \varphi \rangle = \langle T(u), \varphi \rangle$.
(iii) From Eq. (4.3.5), it follows that

$$\begin{aligned} \langle T(u), u \rangle &= \int_\Omega a(|Du|)Du : Du dx - \langle f, u \rangle \\ &\geq c_2 \int_\Omega |Du|^{p(x)} dx - \int_\Omega l(x) dx - c\|f\|_{-1,p'(x)} \|u\|_{1,p(x)}. \end{aligned}$$

Hence

$$\frac{\langle T(u), u \rangle}{\|u\|_{1,p(x)}} \geq c\|Du\|_{p(x)}^{p(x)-1} - \frac{\|l\|_{L^1}}{\|u\|_{1,p(x)}} - c \longrightarrow \infty \quad \text{as } \|u\|_{1,p(x)} \to \infty.$$

$\square$

Now, in order to find $u \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ such that $\langle T(u), \varphi \rangle = 0$, we consider $W_1 \subset W_2 \subset ... \subset W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ a sequence of finite dimensional subspaces such that $\bigcup_{k\geq 1} W_k$ is dense in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$. The sequence $(W_k)$ exists since $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ is separable. Fix $k$ and assume that $\dim W_k = r$ and $w_1, ..., w_k$ is a basis of $W_k$. We define the map

$$S : \mathbb{R}^r \to \mathbb{R}^r, \qquad \alpha \mapsto \Big( \langle T(\alpha_i w_i), w_j \rangle \Big)_{j=1,..,r}$$

for $\alpha = (\alpha_1, ..., \alpha_r) \in \mathbb{R}^r$.

**Lemma 4.3.2.** *1) The map $S$ is continuous.*
*2) For all $k \in \mathbb{N}$ there exists $u_k \in W_k$ such that*

$$\langle T(u_k), \varphi \rangle = 0 \quad \textit{for all} \quad \varphi \in W_k. \tag{4.3.7}$$

*3) The sequence constructed in 2) is uniformly bounded in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$, i.e. there exists a constant $R > 0$ such that*

$$\|u_k\|_{1,p(x)} \leq R \quad \textit{for all} \quad k \in \mathbb{N}. \tag{4.3.8}$$

The proof is similar to that in Lemma 4.2.4. By (4.3.8) and Lemma 2.3.1, there exists a Young measure $\nu_x$ generated by $Du_k$ in $L^{p(x)}(\Omega; \mathbb{M}^{m\times n})$ satisfying the properties of Lemma 4.1.2. To pass to the limit in the approximating equations, we will use the following usefull lemmas, which can be seen as the key ingredient in the proof of the main result.

**Lemma 4.3.3.** *The Young measure $\nu_x$ generated by $Du_k$ satisfies the following inequality*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big( a(|\lambda|)\lambda - a(|Du|)Du \big) : (\lambda - Du) d\nu_x(\lambda) dx \leq 0.$$

*Proof.* Consider the sequence

$$\begin{aligned} A_k &:= \big( a(|Du_k|)Du_k - a(|Du|)Du \big) : (Du_k - Du) \\ &= a(|Du_k|)Du_k : (Du_k - Du) - a(|Du|)Du : (Du_k - Du) \\ &= A_{k,1} + A_{k,2}. \end{aligned}$$

Since

$$\int_\Omega \left| a(|Du|)Du \right|^{p'(x)} dx \leq c \int_\Omega |Du|^{p(x)} dx < \infty$$

for arbitrary $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, $a(|Du|)Du \in L^{p'(x)}(\Omega;\mathbb{M}^{m\times n})$. Therefore

$$\liminf_{k\to\infty}\int_\Omega A_{k,2}dx = \int_\Omega a(|Du|)Du : \Big(\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) - Du\Big)dx = 0 \tag{4.3.9}$$

by Lemma 4.1.2. We have $\big(a(|Du_k|)Du_k : Du\big)^-$ is equiintegrable (see the proof of Lemma 4.3.1 if necessary). The sequence $\big(a(|Du_k|)Du_k : Du_k\big)$ is easily seen to be equiintegrable. Indeed, by Eq. (4.3.5), we have

$$a(|Du_k|)Du_k : Du_k \geq c_2|Du_k|^{p(x)} - l(x),$$

which implies

$$\int_{\Omega'} \big|\min\big(a(|Du_k|)Du_k : Du_k, 0\big)\big|dx \leq c_2\int_{\Omega'} |Du_k|^{p(x)}dx + \int_{\Omega'} |l(x)|dx < \infty$$

by the boundedness of $(u_k)$. Now, by applying Lemma 2.3.2 to the sequence $\big(a(|Du_k|)Du_k : (Du_k - Du)\big)$, we get

$$\begin{aligned} A := \liminf_{k\to\infty}\int_\Omega A_k dx &= \liminf_{k\to\infty}\int_\Omega A_{k,1}dx \\ &\geq \int_\Omega\int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : (\lambda - Du)d\nu_x(\lambda)dx. \end{aligned}$$

If we arrive at $A \leq 0$, then the needed result follows immediately. Using Mazur's theorem (see [136, Theorem 2, page 120]), it follows the existence of $\varphi_k \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ such that $\varphi_k \to u$ in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, where each $\varphi_k$ is a convex linear combination of $\{u_1, .., u_k\}$, that means $\varphi_k \in W_k$. By taking $u_k - \varphi_k$ as a test function in (4.3.7), we obtain

$$\int_\Omega a(|Du_k|)Du_k : (Du_k - D\varphi_k)dx = \langle f, u_k - \varphi_k\rangle. \tag{4.3.10}$$

From the Hölder inequality, it follows that

$$\big|\langle f, u_k - \varphi_k\rangle\big| \leq c\|f\|_{-1,p'(x)}\|u_k - \varphi_k\|_{1,p(x)}.$$

The right hand side of the above inequality vanishes as $k \to \infty$, since by the construction of $\varphi_k$ we have

$$\|u_k - \varphi_k\|_{1,p(x)} \leq \|u_k - u\|_{1,p(x)} + \|\varphi_k - u\|_{1,p(x)} \longrightarrow 0 \quad \text{as } k \to \infty.$$

Hence, the left hand side in (4.3.10) tends to zero as $k \to \infty$. Using this result and the fact that $\varphi_k \to u$ in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$, we deduce the following

$$\begin{aligned}
A &= \liminf_{k\to\infty} \int_\Omega A_k dx \\
&= \liminf_{k\to\infty} \int_\Omega a(|Du_k|)Du_k : (Du_k - Du)dx \\
&= \liminf_{k\to\infty} \Big( \int_\Omega a(|Du_k|)Du_k : (Du_k - D\varphi_k)dx + \int_\Omega a(|Du_k|)Du_k : (D\varphi_k - Du)dx \Big) \\
&\overset{(4.3.10)}{=} \liminf_{k\to\infty} \Big( \langle f, u_k - \varphi_k \rangle + \int_\Omega a(|Du_k|)Du_k : (D\varphi_k - Du)dx \Big) \\
&= \liminf_{k\to\infty} \int_\Omega a(|Du_k|)Du_k : (D\varphi_k - Du)dx \\
&\leq \liminf_{k\to\infty} c \big\| |a(|Du_k|)Du_k| \big\|_{p'(x)} \|D\varphi_k - Du\|_{p(x)} = 0.
\end{aligned}$$

Consequently, $A \leq 0$ together with (4.3.9) imply the needed result. $\square$

**Lemma 4.3.4.** *If $a$ satisfies (4.3.2) and (4.3.3), then it is monotone, i.e.,*

$$\big(a(|\xi|)\xi - a(|\eta|)\eta\big) : (\xi - \eta) \geq 0 \quad \textit{for all } \xi, \eta \in \mathbb{M}^{m\times n}.$$

*Proof.* For $\xi, \eta \in \mathbb{M}^{m\times n}$ and $t \in [0,1]$ we set $\theta_t = t\xi + (1-t)\eta$, then

$$\begin{aligned}
\big(a(|\xi|)\xi - a(|\eta|)\eta\big) : (\xi - \eta) &= \Big( \int_0^1 \frac{d}{dt}\big(a(|\theta_t|)\theta_t\big)dx \Big) : (\xi - \eta) \\
&= \Big( \int_0^1 \big(a'(|\theta_t|)|\theta_t| + a(|\theta_t|)\big)dx \Big) : (\xi - \eta)^2 \\
&= \Big( \int_0^1 a(|\theta_t|)\big(\frac{a'(|\theta_t|)|\theta_t|}{a(|\theta_t|)} + 1\big)dx \Big) : (\xi - \eta)^2 \geq 0
\end{aligned}$$

by the equations (4.3.2) and (4.3.3). $\square$

We have the following localization of the support of $\nu_x$.

**Lemma 4.3.5.** *The Young measure $\nu_x$ generated by $Du_k$ satisfies*

$$\big(a(|\lambda|)\lambda - a(|Du|)Du\big) : (\lambda - Du) = 0 \quad \textit{on supp}\, \nu_x.$$

*Proof.* According to Lemma 4.3.3, we have

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(a(|\lambda|)\lambda - a(|Du|)Du\big) : (\lambda - Du) d\nu_x(\lambda) dx \le 0,$$

and by virtue of the monotonicity of the function $a$ in Lemma 4.3.4, it follows that the above integral is nonnegative, thus must vanish with respect to the product measure $d\nu_x(\lambda) \otimes dx$. Hence

$$\big(a(|\lambda|)\lambda - a(|Du|)Du\big) : (\lambda - Du) = 0 \quad \text{on supp}\, \nu_x.$$

□

### 4.3.3 Proof of Theorem 4.3.1

Now, we have all ingredients to pass to the limit in the approximating equations by considering both conditions **(H0)** and **(H1)**. Let us tart with the case **(H0)**. We show first that for almost every $x \in \Omega$, supp $\nu_x \subset K_x$, where

$$K_x = \Big\{\lambda \in \mathbb{M}^{m\times n} : \; b(\lambda) = b(Du) + a(|Du|)Du : (\lambda - Du)\Big\}.$$

If $\lambda \in$ supp $\nu_x$, then by Lemma 4.3.5

$$(1-\tau)\big(a(|\lambda|)\lambda - a(|Du|)Du\big) : (\lambda - Du) = 0 \quad \text{for all } \tau \in [0,1]. \tag{4.3.11}$$

It follows by Lemma 4.3.4 that

$$\begin{aligned} 0 &\le (1-\tau)\Big(a(|\lambda|)\lambda - a(|Du + \tau(\lambda - Du)|)(Du + \tau(\lambda - Du))\Big) : (\lambda - Du) \\ &\overset{(4.3.11)}{=} (1-\tau)\Big(a(|Du|)Du - a(|Du + \tau(\lambda - Du)|)(Du + \tau(\lambda - Du))\Big) : (\lambda - Du). \end{aligned} \tag{4.3.12}$$

Remark that, by the monotonicity of the function $a$, we have

$$\Big(a(|Du|)Du - a(|Du+\tau(\lambda-Du)|)(Du+\tau(\lambda-Du))\Big) : \tau(\lambda-Du) \le 0,$$

and since $\tau \in [0,1]$

$$\Big(a(|Du|)Du - a(|Du+\tau(\lambda-Du)|)(Du+\tau(\lambda-Du))\Big) : (1-\tau)(\lambda-Du) \le 0. \tag{4.3.13}$$

From (4.3.12) and (4.3.13), we get for $\tau \in [0,1]$ that

$$\Big(a(|Du|)Du - a(|Du+\tau(\lambda-Du)|)(Du+\tau(\lambda-Du))\Big) : (\lambda-Du) = 0,$$

i.e.,

$$a(|Du|)Du : (\lambda-Du) = a(|Du+\tau(\lambda-Du)|)(Du+\tau(\lambda-Du)) : (\lambda-Du).$$

By integrating the above equality over $[0,1]$ and using the fact that

$$a(|Du+\tau(\lambda-Du)|)(Du+\tau(\lambda-Du)) : (\lambda-Du) = \frac{\partial b}{\partial \tau}(Du+\tau(\lambda-Du)) : (\lambda-Du),$$

we obtain

$$\begin{aligned} b(\lambda) &= b(Du) + \int_0^1 a(|Du|)Du : (\lambda-Du)d\tau \\ &= b(Du) + a(|Du|)Du : (\lambda-Du) \end{aligned}$$

as desired, thus $\lambda \in K_x$, i.e., supp $\nu_x \subset K_x$. Now, the convexity of the potential $b$ implies that

$$b(\lambda) \ge \underbrace{b(Du) + a(|Du|)Du : (\lambda-Du)}_{=:B(\lambda)} \quad \text{for all } \lambda \in \mathbb{M}^{m\times n}.$$

Since the mapping $\lambda \mapsto b(\lambda)$ is of class $C^1$, for every $\xi \in \mathbb{M}^{m\times n}$, $\tau \in \mathbb{R}$

$$\frac{b(\lambda+\tau\xi)-b(\lambda)}{\tau} \ge \frac{B(\lambda+\tau\xi)-B(\lambda)}{\tau} \quad \text{if} \quad \tau > 0$$

$$\frac{b(\lambda+\tau\xi)-b(\lambda)}{\tau} \leq \frac{B(\lambda+\tau\xi)-B(\lambda)}{\tau} \quad \text{if} \quad \tau < 0.$$

Hence $D_\lambda b = D_\lambda B$, i.e.,

$$a(|\lambda|)\lambda = a(|Du|)Du \quad \text{for all } \lambda \in K_x \supset \operatorname{supp} \nu_x. \tag{4.3.14}$$

The equiintegrability of $a(|Du_k|)Du_k$ implies that its weak $L^1$-limit is given by

$$\begin{aligned} \overline{a}(x) &:= \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda d\nu_x(\lambda) \\ &\overset{(4.3.14)}{=} \int_{\operatorname{supp}\nu_x} a(|Du|)Du d\nu_x(\lambda) \\ &= a(|Du|)Du \underbrace{\int_{\operatorname{supp}\nu_x} d\nu_x(\lambda)}_{=:1} = a(|Du|)Du. \end{aligned} \tag{4.3.15}$$

Now, consider the continuous function

$$g(\lambda) = \big|a(|\lambda|)\lambda - \overline{a}(x)\big|, \quad \lambda \in \mathbb{M}^{m\times n}.$$

Since $a(|Du_k|)Du_k$ is equiintegrable, then $g_k(x) := g(Du_k)$ is equiintegrable and its weak $L^1$-limit is given by

$$g_k \rightharpoonup \overline{g} \quad \text{in} \quad L^1(\Omega) \tag{4.3.16}$$

where

$$\begin{aligned} \overline{g}(x) &= \int_{\mathbb{M}^{m\times n}} \big|a(|\lambda|)\lambda - \overline{a}(x)\big| d\nu_x(\lambda) \\ &\overset{(4.3.15)}{=} \int_{\operatorname{supp}\nu_x} \big|a(|Du|)Du - \overline{a}(x)\big| d\nu_x(\lambda) = 0. \end{aligned}$$

As a matter of fact, the convergence in (4.3.16) is strong since $g_k \geq 0$. Therefore

$$\lim_{k\to\infty} \int_\Omega a(|Du_k|)Du_k : D\varphi dx = \int_\Omega a(|Du|)Du : D\varphi dx \quad \forall \varphi \in \bigcup_{k\geq 1} W_k.$$

Now, for the case **(H1)**, we argue by contradiction and suppose that $\nu_x$ is not a Dirac measure on a set $x \in \Omega'$ of positive Lebesgue measure $|\Omega'| > 0$. We have $\overline{\lambda} = \langle \nu_x, id\rangle =$

$Du(x)$ for a.e. $x \in \Omega$, thus

$$\begin{aligned}
\int_{\mathbb{M}^{m\times n}} a(|\overline{\lambda}|)\overline{\lambda} : (\lambda - \overline{\lambda}) d\nu_x(\lambda) &= \int_{\mathbb{M}^{m\times n}} \Big( a(|\overline{\lambda}|)\overline{\lambda} : \lambda - a(|\overline{\lambda}|)\overline{\lambda} : \overline{\lambda} \Big) d\nu_x(\lambda) \\
&= a(|\overline{\lambda}|)\overline{\lambda} : \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)}_{=:\overline{\lambda}} - a(|\overline{\lambda}|)\overline{\lambda} : \overline{\lambda} \underbrace{\int_{\mathbb{M}^{m\times n}} d\nu_x(\lambda)}_{=:1} \\
&= 0.
\end{aligned}$$

By virtue of the strict $p(x)$-quasimonotone in **(H1)**, we obtain then

$$\int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \lambda d\nu_x(\lambda) > \int_{\mathbb{M}^{m\times n}} a(|\lambda|) : \overline{\lambda} d\nu_x(\lambda).$$

Integrating the above inequality over $\Omega$ and using Lemma 4.3.3, we get

$$\begin{aligned}
\int_\Omega \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \lambda d\nu_x(\lambda) dx &> \int_\Omega \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \overline{\lambda} d\nu_x(\lambda) dx \\
&\geq \int_\Omega \int_{\mathbb{M}^{m\times n}} a(|\lambda|)\lambda : \lambda d\nu_x(\lambda) dx,
\end{aligned}$$

which is a contradiction. Therefore $\nu_x$ is a Dirac measure and we can write $\nu_x = \delta_{h(x)}$. Then

$$h(x) = \int_{\mathbb{M}^{m\times n}} \lambda d\delta_{h(x)}(\lambda) = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) = Du(x).$$

Hence $\nu_x = \delta_{Du(x)}$. By virtue of the Proposition 2.3.1, it follows that $Du_k \to Du$ in measure and almost everywhere. The continuity of the function $a$ implies that $a(|Du_k|)Du_k \to a(|Du|)Du$ almost everywhere in $\Omega$. Since $a(|Du_k|)Du_k$ is equiintegrable, the Vitali Theorem gives

$$\int_\Omega \Big( a(|Du_k|)Du_k - a(|Du|)Du \Big) : D\varphi dx \to 0 \quad \text{as} \quad k \to \infty.$$

The density of $\underset{k\geq 1}{\cup} W_k$ in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ implies that $u$ is a weak solution of (4.3.1) and the proof of Theorem 4.3.1 is finish.

# Chapter 5

# Quasilinear elliptic systems in Orlicz-Sobolev spaces

## 5.1 Introduction

In the first part of this chapter, we deal with the existence of solutions for a quasilinear elliptic system in divergence form given by

$$\begin{cases} -\operatorname{div} \sigma(x, u, Du) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.1.1}$$

where $u : \Omega \to \mathbb{R}^m$ is a vector-valued function and $\Omega$ is a bounded open domain of $\mathbb{R}^n$, $n \geq 2$. The data $f$ belongs to $W^{-1}L_{\overline{M}}(\Omega; \mathbb{R}^m)$ the dual space of the Orlicz-Sobolev space $W_0^1 L_M(\Omega; \mathbb{R}^m)$ and $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a function verifying some conditions which will be mentioned later.

The problem (5.1.1) was treated by Hungerbühler [85], where he has proved the existence of a weak solution $u \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, while the data $f$ belongs to $W^{-1,p'}(\Omega; \mathbb{R}^m)$, $p' = p/(p-1)$. In that work, $\sigma$ satisfy the following growth and coercivity conditions:

$$|\sigma(x, s, \xi)| \leq d_1(x) + c_1(|s|^{p-1} + |\xi|^{p-1})$$

and

$$\sigma(x,s,\xi):\xi \geq -d_2(x) - d_3(x)|s|^{\alpha} + c_2|\xi|^p, \quad \text{for } (s,\xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n},$$

where $d_1(x) \in L^{p'}(\Omega)$, $d_2(x) \in L^1(\Omega)$, $d_3(x) \in L^{(p/\alpha)'}(\Omega)$, $0 < \alpha < p$, $0 < q \leq (n(p-1))/(n-p)$ and $c_1, c_2 > 0$. The author used Young measures and only very mild monotonicity assumptions to prove the needed result.

When the exponent $p$ is not anymore constant, but depends on $x$, i.e., $p \equiv p(x)$, Fu and Yang [73] generalized the result of Hungerbühler [85] (always by means of the Young measure technics), where $p(x)$ is a Lipschitz continuous function satisfying $1 < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x)$. The source term $f$ is taken then in $W^{-1,p'(x)}(\Omega;\mathbb{R}^m) = (W_0^{1,p(x)}(\Omega;\mathbb{R}^m))^*$. The growth and coercivity conditions for $\sigma$ are

$$|\sigma(x,s,\xi)| \leq a(x) + c_1(|s|^{q(x)} + |\xi|^{p(x)-1}),$$

$$\sigma(x,s,\xi):\xi \geq -b(x) + c_2|\xi|^{p(x)},$$

where $0 < a(x) \in L^{p'(x)}(\Omega)$, $b(x) \in L^1(\Omega)$, $(p(x)-1)/p(x) < q(x) < (n(p(x)-1))/(n-p(x))$ and $c_1, c_2 \geq 0$. See [6, 27, 28, 75, 78] for related topics and [18, 22] for generalized $p$ and $p(x)$-Laplacian system.

## 5.2 Setting of the problem and formulation of the main result

When we trying to relax the mentioned growth and coercivity conditions (means $\sigma$ satisfies nonpolynomial conditions), we conclude that (5.1.1) can not be formulated respectively neither in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ nor in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. As a consequence, we will use the Orlicz-Sobolev spaces $W_0^1L_M(\Omega;\mathbb{R}^m)$ built from an N-function $M:\mathbb{R}^+ \to \mathbb{R}^+$ (see Chapter 2 for its definition). The existence result will be proved using the concept of Young measure (which was first introduced by Young [137]) and weak monotonicity assumptions on $\sigma$. This concept (i.e., Young measure) has many applications in the calculus of variations, optimal control theory and nonlinear partial differential equations.

In (5.1.1), we suppose that $f$ lies in $W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m)$ the dual of the Orlicz-Sobolev space $W_0^1L_M(\Omega;\mathbb{R}^m)$. From the choice of $f$, it follows then to prove the existence of weak solutions. Let us mention that we treat a class of problems for which the classical monotone operator methods as in [60, 77, 114] do not apply and we will mention the reason behind this (c.f. Remark 5.2.1).

Throughout this chapter (and the rest of this thesis), we assume that the N-function $M$ and its conjugate $\overline{M}$ are satifying the $\Delta_2$-condition (2.2.3). Let $P$ be an N-function such that $P \ll M$. To treat the problem (5.1.1), we state the following assumptions:
**(H0)** $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function (i.e., measurable w.r.t. $x$ and continuous w.r.t. the last variables).
**(H1)** There exist $0 < d_1(x) \in L_{\overline{M}}(\Omega)$, $d_2(x) \in L^1(\Omega)$ and $\alpha, \beta, \gamma > 0$ such that

$$|\sigma(x,s,\xi)| \leq d_1(x) + \overline{M}^{-1}P(\gamma|s|) + \overline{M}^{-1}M(\gamma|\xi|)$$
$$\sigma(x,s,\xi) : \xi \geq -d_2(x) + \alpha M(\frac{|\xi|}{\beta}).$$

**(H2)** $\sigma$ satisfies one of the following conditions:

(a) For any $x \in \Omega$ and $u \in \mathbb{R}^m$, $\xi \mapsto \sigma(x,u,\xi)$ is a $C^1$ and monotone, i.e.,

$$\big(\sigma(x,u,\xi) - \sigma(x,u,\eta)\big) : (\xi - \eta) \geq 0$$

for all $x \in \Omega$, $u \in \mathbb{R}^m$ and $\xi, \eta \in \mathbb{M}^{m\times n}$.

(b) There exists a function $W : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,u,\xi) = \big(\partial W/\partial\xi\big)(x,u,\xi) := D_\xi W(x,u,\xi)$ and $\xi \to W(x,u,\xi)$ is convex and $\mathcal{C}^1$.

(c) $\sigma$ is strictly monotone, i.e., $\sigma$ is monotone and

$$\big(\sigma(x,u,\xi) - \sigma(x,u,\eta)\big) : (\xi - \eta) = 0 \Rightarrow \xi = \eta.$$

(d) $\sigma$ is strictly $M$-quasimonotone on $\mathbb{M}^{m\times n}$, i.e.,

$$\int_{\mathbb{M}^{m\times n}} \big(\sigma(x,u,\lambda) - \sigma(x,u,\overline{\lambda})\big) : (\lambda - \overline{\lambda})d\nu(\lambda) > 0$$

where $\overline{\lambda} = \langle \nu_x, id \rangle$, $\nu = \{\nu_x\}_{x\in\Omega}$ is any family of Young measures generated by a sequence in $L_M(\Omega)$ and not a Dirac measure for a.e. $x \in \Omega$.

**Remark 5.2.1.** *1. **(H1)** states non standard growth and coercivity conditions. **(H2)**(d) is weaker than typical strictly monotone conditions. A feature of this work is that we do not require the classical strict monotonicity. For example, the assumption **(H2)**(b) allows to take a potential $W(x,u,\xi)$, which is only convex but not strictly convex in $\xi \in \mathbb{M}^{m\times n}$, and to consider (5.1.1) with $\sigma(x,u,\xi) = \frac{\partial W}{\partial \xi}(x,u,\xi)$. Notice that if $W$ is assumed to be strictly convex, then $\sigma$ becomes strict monotone and the classical monotone method may apply.*

2. *The naming strict $M$-quasimonotone in **(H2)**(d) comes from its name in the classical case of Sobolev spaces $W_0^{1,p}(\Omega;\mathbb{R}^m)$ (see [85]).*

3. *As in [97], the N-function $P$ is introduced instead of $M$ in **(H1)** only to guarantee the boundedness in $L_{\overline{M}}(\Omega)$ of $\overline{M}^{-1}P(\gamma|u_k|)$ and whenever $u_k$ is bounded in $L_M(\Omega)$, one usually takes $P = M$ in the term $\overline{M}^{-1}P(\gamma|u_k|)$.*

**Example 5.2.1.** *The model examples (by using the second property in Lemma 2.2.1) are the following:*

1. $-div(|Du|^{p-2}Du) = f$, *for some* $p \in (1,\infty)$;
2. $-div(e^{|Du|} - 1) = f$;
3. $-div(log(1 + Du)) = f$

*in $\Omega$, supplemented with a Dirichlet boundary condition.*

The main result of the first part of this chapter can be stated as follows.

**Theorem 5.2.1.** *If $\sigma$ satisfies conditions **(H0)-(H2)**, then the Dirichlet problem (5.1.1) has a weak solution $u \in W_0^1 L_M(\Omega;\mathbb{R}^m)$ for every $f \in W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m)$.*

## 5.3 Proof of the main result

The proof of Theorem 5.2.1 will be divided into 3 steps. In Step 1, we introduce the approximating solution by the Galerkin method and some *a priori* estimates. Step 2 is devoted to prove an inequality of div-curl type which permits to pass to the limit in the approximating equations in Step 3.

In the rest of this chapter, we use $c$ for generic constants which may change values from line to line.

**Step 1.** Let $V_1 \subset V_2 \subset ... \subset W_0^1 L_M(\Omega;\mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\cup_{i\in\mathbb{N}} V_i$ is dense in $W_0^1 L_M(\Omega;\mathbb{R}^m)$. Note that $(V_i)_i$ exist since $W_0^1 L_M(\Omega;\mathbb{R}^m)$ is separable. We define the operator

$$\begin{aligned} T: W_0^1 L_M(\Omega;\mathbb{R}^m) &\to W^{-1} L_{\overline{M}}(\Omega;\mathbb{R}^m) \\ u &\mapsto \Big(\varphi \mapsto \int_\Omega \sigma(x,u,Du) : D\varphi dx - \langle f,\varphi\rangle\Big), \end{aligned}$$

where $\langle .,. \rangle$ is the pairing of $(W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m), W_0^1 L_M(\Omega;\mathbb{R}^m))$

**Lemma 5.3.1.** *For arbitrary $u \in W_0^1 L_M(\Omega;\mathbb{R}^m)$, we have the following properties:*

1. *$T(u)$ is linear, well defined and bounded.*

2. *The restriction of $T$ to a finite linear subspace $V$ of $W_0^1 L_M(\Omega;\mathbb{R}^m)$ is continuous.*

*Proof.* 1. Let $u$ be an arbitrary element of $W_0^1 L_M(\Omega;\mathbb{R}^m)$. Trivially $T(u)$ is linear. By the growth condition in **(H1)**, the continuous embedding $W^1 L_M(\Omega;\mathbb{R}^m) \hookrightarrow L_M(\Omega;\mathbb{R}^m)$ and $P \ll M$, we get

$$\int_\Omega \overline{M}(|\sigma(x,u,Du)|)dx \le c\int_\Omega \big(\overline{M}(d_1(x)) + M(\gamma|u|) + M(\gamma|Du|)\big)dx < \infty.$$

Thus, for each $\varphi \in W_0^1 L_M(\Omega;\mathbb{R}^m)$

$$\begin{aligned} \big|\langle T(u),\varphi\rangle\big| &= \Big|\int_\Omega \sigma(x,u,Du) : D\varphi dx - \langle f,\varphi\rangle\Big| \\ &\le \int_\Omega \big|\sigma(x,u,Du) : D\varphi\big| dx + \big|\langle f,\varphi\rangle\big| \\ &\le 2\big\||\sigma(x,u,Du)|\big\|_{\overline{M}} \|D\varphi\|_M + 2\|f\|_{-1,\overline{M}}\|\varphi\|_{1,M} \\ &\le c\|\varphi\|_{1,M}, \end{aligned}$$

where we have used Hölder's inequality. Hence $T(u)$ is well defined and bounded.
2. Let $u_k \in W_0^1 L_M(\Omega;\mathbb{R}^m)$ be a sequence such that $u_k \to u$ in $V$. The continuity condition in **(H0)** and the growth condition in **(H1)** allow to deduce that

$$\begin{aligned}\|T(u_k)-T(u)\|_{-1,\overline{M}} &= \sup_{\|\varphi\|_{1,M}=1} \left|\langle T(u_k)-T(u),\varphi\rangle\right| \\ &\leq c\big\|\left|\sigma(x,u_k,Du_k)-\sigma(x,u,Du)\right|\big\|_{\overline{M}} \\ &\leq c,\end{aligned}$$

by Vitali's Convergence Theorem. Hence, the restriction of $T$ to a finite linear subspace of $W_0^1 L_M(\Omega;\mathbb{R}^m)$ is continuous. □

Let fix $k$ and assume that $\dim V_k = r$ and $\varphi_1, ..., \varphi_r$ is a basis of $V_k$. Then we define the map

$$\begin{aligned} G : \mathbb{R}^r &\longrightarrow \mathbb{R}^r \\ \begin{pmatrix} a^1 \\ a^2 \\ . \\ . \\ a^r \end{pmatrix} &\mapsto \begin{pmatrix} \langle T(a^i\varphi_i),\varphi_1\rangle \\ \langle T(a^i\varphi_i),\varphi_2\rangle \\ . \\ . \\ \langle T(a^i\varphi_i),\varphi_r\rangle \end{pmatrix}. \end{aligned}$$

**Lemma 5.3.2.** *The map $G$ is continuous and $G(a).a \to \infty$ as $\|a\|_{\mathbb{R}^r} \to \infty$, where $a \in \mathbb{R}^r$ and the dot . is the inner product of two vectors in $\mathbb{R}^r$.*

*Proof.* For the continuity of $G$, it is sufficient to show that $G(a_j) \to G(a_0)$ in $\mathbb{R}^r$ as $a_j \to a_0$ in $\mathbb{R}^r$. Let $u_j = a_j^i\varphi_i \in V_k$ and $u_0 = a_0^i\varphi_i \in V_k$ (with conventional summation). Then $\|a_j\|_{\mathbb{R}^r}$ is equivalent to $\|u_j\|_{1,M}$ and $\|a_0\|_{\mathbb{R}^r}$ is equivalent to $\|u_0\|_{1,M}$. Thus

$$\begin{aligned}\left|\big(G(a_j)-G(a_0)\big)_l\right| &= \left|\langle T(a_j^i\varphi_i - T(a_0^i\varphi_i),\varphi_l\right| \\ &\leq 2\|T(u_j)-T(u_0)\|_{-1,\overline{M}}\|\varphi\|_{1,M}.\end{aligned}$$

Since the restriction of $T$ to $V_k$ is continuous by Lemma 5.3.1, then the continuity of $G$ follows. Moreover, for $u = a^i\varphi_i \in V_k$, we have by the coercivity condition in **(H1)** that

$$\begin{aligned} G(a).a &= \langle T(a^i\varphi_i), a^i\varphi_i\rangle \\ &= \langle T(u), u\rangle \\ &\geq \int_\Omega \Big( - d_2(x) + \alpha M\Big(\frac{|Du|}{\beta}\Big)\Big) dx - \|f\|_{-1,\overline{M}} \|u\|_{1,M}. \end{aligned}$$

We know that there exists $\theta > 0$ such that $\|u\|_M \leq \theta \|Du\|_M$ (see Lemma 2.2.3). Hence

$$\|u\|_{1,M} \leq (1+\theta)\|Du\|_M.$$

Consequently

$$\begin{aligned} \frac{\langle T(u), u\rangle}{\|u\|_{1,M}} &\geq \frac{-\|d_2\|_{L^1}}{\|u\|_{1,M}} + \frac{\alpha \int_\Omega M(\frac{|Du|}{\beta}) dx}{\|u\|_{1,M}} - \|f\|_{-1,\overline{M}} \\ &\geq \frac{-\|d_2\|_{L^1}}{\|u\|_{1,M}} + \frac{\alpha}{1+\theta} \frac{\int_\Omega M(|Du|) dx}{\|Du\|_M} - \|f\|_{-1,\overline{M}}. \end{aligned}$$

Thanks to [63, Remark 2.1], we conclude that

$$\frac{\langle T(u), u\rangle}{\|u\|_{1,M}} \to \infty \quad \text{as } \|u\|_{1,M} \to \infty.$$

Hence the needed result follows. □

Now, we can construct the sequence of approximating solutions in the following way: From Lemma 5.3.2 it follows the existence of a constant $R > 0$ such that for any $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $G(a).a > 0$ and the topological argument [117] gives that $G(x) = 0$ has a solution $x \in B_R(0)$. Then, for each $k \in \mathbb{N}$ there exists $u_k \in V_k$ such that

$$\langle T(u_k), \varphi\rangle = 0 \quad \text{for all } \varphi \in V_k. \tag{5.3.1}$$

**Step 2.** As stated in Chapter 1, the Young measure is a powerful tool to overcome the difficulty that may arises when the weak convergence does not behave as we desire with respect to nonlinear functionals and operators. First, according to Lemma 5.3.2, there exists $R > 0$ with the property, that $\langle T(u), u\rangle > 1$ whenever $\|u\|_{1,M} > R$. Hence, for the sequence of the Galerkin approximations $u_k \in V_k$, constructed above, which satisfy the Eq. (5.3.1), we obtain the uniform bound

$$\|u_k\|_{1,M} \leq R \quad \text{for all } k \in \mathbb{N}. \tag{5.3.2}$$

Consequently, thanks to Lemma 2.3.1, it follows the existence of a Young measure $\nu_x$ generated by $Du_k \in L_M(\Omega; \mathbb{M}^{m\times n})$. Before proving the div-curl inequality, we still need more properties on the Young measure $\nu_x$ associated to $Du_k$, for a.e. $x \in \Omega$.

**Lemma 5.3.3.** *The Young measure $\nu_x$ satisfies the following properties:*

1. $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$, *i.e., $\nu_x$ is a probability measure for a.e. $x \in \Omega$.*
2. *The weak $L^1$-limit of $Du_k$ is given by $\langle \nu_x, id\rangle := \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)$.*
3. *$\nu_x$ satisfies $\langle \nu_x, id\rangle = Du(x)$ for a.e. $x \in \Omega$.*

*Proof.* 1. To prove that $\nu_x$ is a probability measure, it is sufficient to show that $\{Du_k\}$ satisfies Equation (2.3.1) in Theorem 2.3.1. Putting $\epsilon(L) = \min_{|\xi|=L}\left(M(|\xi|)/\xi\right)$, which tends to infinity as $L \to \infty$, by definition of the N-funciton $M$. By (5.3.2), there exists $c \geq 0$ such that for any $r > 0$,

$$\begin{aligned} c \geq \int_\Omega M(|Du_k|)dx &\geq \int_{\{x\in\Omega\cap B_r(0):\,|Du_k(x)|\geq L\}} M(|Du_k|)dx \\ &\geq \epsilon(L) \int_{\{x\in\Omega\cap B_r(0):\,|Du_k(x)|\geq L\}} |Du_k|dx \\ &\geq L\epsilon(L)\big|\{x \in \Omega\cap B_r(0) : |Du_k(x)| \geq L\}\big|. \end{aligned}$$

Thus

$$\sup_k \big|\{x \in \Omega\cap B_r(0) : |Du_k(x)| \geq L\}\big| \leq \frac{c}{L\epsilon(L)} \longrightarrow 0 \text{ as } L \to \infty.$$

By Theorem 2.3.1(iii), it follows that $\|\nu_x\|_{\mathcal{M}} = 1$ for almost every $x \in \Omega$.
2. We have $L_M(\Omega; \mathbb{M}^{m\times n})$ is reflexive $(M, \overline{M} \in \Delta_2)$, and in view of (5.3.2) we deduce the existence of a subsequence (still denoted by $\{Du_k\}$) weakly convergent in $L_M(\Omega; \mathbb{M}^{m\times n}) \subset L^1(\Omega; \mathbb{M}^{m\times n})$, thus weakly convergent in $L^1(\Omega; \mathbb{M}^{m\times n})$. By taking

$\varphi \equiv id$ in Theorem 2.3.1, we get then

$$Du_k \rightharpoonup \langle \nu_x, id \rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) \text{ weakly in } L^1(\Omega;\mathbb{M}^{m\times n}).$$

3. By (5.3.2), we have $u_k \rightharpoonup u$ in $W_0^1 L_M(\Omega;\mathbb{R}^m)$ and $u_k \to u$ in $L_M(\Omega;\mathbb{R}^m)$ (for a subsequence). Hence $Du_k \rightharpoonup Du$ in $L_M(\Omega;\mathbb{M}^{m\times n})$, and this convergence remains true in $L^1(\Omega;\mathbb{M}^{m\times n})$ since $L_M \subset L^1$. Owing to 2., we conclude by the uniqueness of limit that

$$Du(x) = \langle \nu_x, id \rangle \quad \text{for a.e. } x \in \Omega.$$

□

Now, the following lemma, namely div-curl inequality, will serve us to pass to the limit in the approximating equations.

**Lemma 5.3.4.** *The Young measure $\nu_x$ generated by the gradient $Du_k$ is satisfying the following inequality:*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,u,\lambda) - \sigma(x,u,Du)\big) : (\lambda - Du) d\nu_x(\lambda) dx \le 0.$$

*Proof.* Let us consider the sequence

$$\begin{aligned} I_k &:= \big(\sigma(x,u_k,Du_k) - \sigma(x,u,Du)\big) : (Du_k - Du) \\ &= \sigma(x,u_k,Du_k) : (Du_k - Du) - \sigma(x,u,Du) : (Du_k - Du) \\ &= I_{k,1} + I_{k,2}. \end{aligned}$$

By the growth condition in **(H1)** together with the fact that $Du \in L_M(\Omega;\mathbb{M}^{m\times n})$, it follows that $\sigma \in L_{\overline{M}}(\Omega;\mathbb{M}^{m\times n})$. Because of the weak convergence of $\{Du_k\}$ (see Lemma 5.3.3), we obtain

$$I_{k,2} \to 0 \text{ as } k \to \infty.$$

By virtue of Lemma 2.3.2, we have then

$$
\begin{aligned}
I := \liminf_{k\to\infty} \int_\Omega I_k dx = \liminf_{k\to\infty} \int_\Omega I_{k,1} dx \\
&= \liminf_{k\to\infty} \int_\Omega \sigma(x,u_k,Du_k) : (Du_k - Du)dx \\
&\geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : (\lambda - Du)d\nu_x(\lambda)dx.
\end{aligned}
$$

Now we prove that $I \leq 0$. We choose a subsequence $v_k$ which belongs to the same finite dimensional space $V_k$ as $u_k$ such that $v_k \to u$ in $W^1L_M(\Omega;\mathbb{R}^m)$. By using $u_k - v_k$ as a test function in the equation (5.3.1), one obtains

$$
\begin{aligned}
I &= \liminf_{k\to\infty} \int_\Omega \sigma(x,u_k,Du_k) : (Du_k - Du)dx \\
&= \liminf_{k\to\infty} \Big( \int_\Omega \sigma(x,u_k,Du_k) : (Du_k - Dv_k)dx + \underbrace{\int_\Omega \sigma(x,u_k,Du_k) : (Dv_k - Du)dx}_{:=\langle f, v_k - u\rangle} \Big) \\
&\leq \liminf_{k\to\infty} \Big( c\| \, |\sigma(x,u_k,Du_k)| \, \|_{\overline{M}} \|v_k - u_k\|_{1,M} + c\|f\|_{-1,\overline{M}} \|v_k - u\|_{1,M} \Big).
\end{aligned}
$$

The term $\| \, |\sigma(x,u_k,Du_k)| \, \|_{\overline{M}}$ is uniformly bounded in $k$ by the growth condition in **(H1)** together with (5.3.2). Since $\|v_k - u\|_{1,M} \to 0$ and $\|v_k - u_k\|_{1,M} \to 0$ as $k \to \infty$, we conclude that $I \leq 0$. Use the fact that

$$
\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,Du) : (\lambda - Du)d\nu_x(\lambda)dx = 0,
$$

to conclude the desired inequality. □

Note, that we can infer from the monotonicity of $\sigma$ and Lemma 5.3.4 that

$$
\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,u,\lambda) - \sigma(x,u,Du)\big) : (\lambda - Du)d\nu_x(\lambda) \otimes dx = 0.
$$

Therefore, we obtain the following localization of the support of $\nu_x$ as follows:

$$
\big(\sigma(x,u,\lambda) - \sigma(x,u,Du)\big) : (\lambda - Du) = 0 \quad \text{on } \operatorname{supp} \nu_x, \tag{5.3.3}
$$

for a.e. $x \in \Omega$.
**Step3.** We are now in a position to show the existence of solutions for (5.1.1). We consider four cases which correspond to the four cases listed in **(H2)**.
**Case (a):** We claim that for almost $x \in \Omega$ and all $\eta \in \mathbb{M}^{m\times n}$

$$\sigma(x,u,\lambda) : \eta = \sigma(x,u,Du) : \eta + \big(\nabla\sigma(x,u,Du)\eta\big) : (Du - \lambda)$$

holds on $\operatorname{supp} \nu_x$. Here, $\nabla$ denotes the derivative of $\sigma$ with respect to its third variable. The monotonicity of $\sigma$ implies that for all $\tau \in \mathbb{R}$

$$\big(\sigma(x,u,\lambda) - \sigma(x,u,Du+\tau\eta)\big) : (\lambda - Du - \tau\eta) \geq 0,$$

which implies by Equation (5.3.3),

$$\begin{aligned}
&\sigma(x,u,\lambda) : (\lambda - Du) - \sigma(x,u,\lambda) : \tau\eta - \sigma(x,u,Du+\tau\eta) : (\lambda - Du - \tau\eta) \\
&= \sigma(x,u,Du) : (\lambda - Du) - \sigma(x,u,\lambda) : \tau\eta - \sigma(x,u,Du+\tau\eta) : (\lambda - Du - \tau\eta) \geq 0,
\end{aligned}$$

thus

$$-\sigma(x,u,\lambda) : \tau\eta \geq -\sigma(x,u,Du) : (\lambda - Du) + \sigma(x,u,Du+\tau\eta) : (\lambda - Du - \tau\eta).$$

Since $\sigma(x,u,Du+\tau\eta) = \sigma(x,u,Du) + \nabla\sigma(x,u,Du)\tau\eta + o(\tau)$, then

$$\begin{aligned}
&\sigma(x,u,Du+\tau\eta) : (\lambda - Du - \tau\eta) \\
&= \sigma(x,u,Du+\tau\eta) : (\lambda - Du) - \sigma(x,u,Du+\tau\eta) : \tau\eta \\
&= \sigma(x,u,Du) : (\lambda - Du) + \nabla\sigma(x,u,Du)\tau\eta : (\lambda - Du) - \sigma(x,u,Du) : \tau\eta \\
&\quad - \nabla\sigma(x,u,Du)\tau\eta : \tau\eta + o(\tau) \\
&= \sigma(x,u,Du) : (\lambda - Du) + \tau\big[\nabla\sigma(x,u,Du)\eta : (\lambda - Du) - \sigma(x,u,Du) : \eta\big] + o(\tau).
\end{aligned}$$

Hence

$$-\sigma(x,u,\lambda) : \tau\eta \geq \tau\big[\nabla\sigma(x,u,Du)\eta : (\lambda - Du) - \sigma(x,u,Du) : \eta\big] + o(\tau).$$

Since $\tau$ is arbitrary in $\mathbb{R}$, then our claim follows.
We have $\{\sigma(x, u_k, Du_k)\}$ is bounded and equiintegrable, then its weak $L^1$-limit (by definition) is given by

$$\overline{\sigma} = \int_{\operatorname{supp} \nu_x} \sigma(x, u, \lambda) d\nu_x(\lambda).$$

Using our claim to obtain

$$\begin{aligned} \overline{\sigma} &= \int_{\operatorname{supp} \nu_x} \sigma(x, u, Du) d\nu_x(\lambda) + \big(\nabla \sigma(x, u, Du)\big)^t \int_{\operatorname{supp} \nu_x} (Du - \lambda) d\nu_x(\lambda) \\ &= \sigma(x, u, Du), \end{aligned}$$

where we have used the fact that

$$\int_{\operatorname{supp} \nu_x} (Du - \lambda) d\nu_x(\lambda) = Du \int_{\operatorname{supp} \nu_x} d\nu_x(\lambda) - \int_{\operatorname{supp} \nu_x} \lambda d\nu_x(\lambda) = 0.$$

As $L_{\overline{M}}(\Omega; \mathbb{M}^{m\times n})$ is reflexive, it follows then that $\{\sigma(x, u_k, Du_k)\}$ converges weakly in $L_{\overline{M}}(\Omega; \mathbb{M}^{m\times n})$. Hence its weak $L_{\overline{M}}$ -limit is also $\sigma(x, u, Du)$.

**Case (b):** Let show that for almost every $x \in \Omega$

$$\operatorname{supp} \nu_x \subset \Big\{ \lambda \in \mathbb{M}^{m\times n} : \, W(x, u, \lambda) = W(x, u, Du) + \sigma(x, u, Du) : (\lambda - Du) \Big\} =: K_x.$$

Let $\lambda \in \operatorname{supp} \nu_x$. On the one hand, by Eq. (5.3.3)

$$(1 - \tau)\big(\sigma(x, u, \lambda) - \sigma(x, u, Du)\big) : (\lambda - Du) = 0 \text{ for all } \tau \in [0, 1]. \tag{5.3.4}$$

On the other hand, the monotonicity of $\sigma$ implies

$$(1 - \tau)\big(\sigma(x, u, Du + \tau(\lambda - Du)) - \sigma(x, u, \lambda)\big) : (Du - \lambda) \geq 0. \tag{5.3.5}$$

Subtracting (5.3.4) from (5.3.5), we get

$$(1 - \tau)\big(\sigma(x, u, Du + \tau(\lambda - Du)) - \sigma(x, u, Du)\big) : (Du - \lambda) \geq 0 \text{ for all } \tau \in [0, 1],$$

and again by the monotonicity of $\sigma$, it follows that

$$\big(\sigma(x,u,Du+\tau(\lambda-Du))-\sigma(x,u,Du)\big):(\lambda-Du)=0 \text{ for all } \tau\in[0,1].$$

Thus

$$\sigma(x,u,Du):(\lambda-Du)=\sigma(x,u,Du+\tau(\lambda-Du)):(\lambda-Du) \text{ for all } \tau\in[0,1]. \quad (5.3.6)$$

We integrate the equality

$$\sigma(x,u,Du+\tau(\lambda-Du)):(\lambda-Du)=\frac{\partial W}{\partial\tau}(x,u,Du+\tau(\lambda-Du)):(\lambda-Du)$$

over $[0,1]$, this gives together with (5.3.6)

$$W(x,u,\lambda)=W(x,u,Du)+\sigma(x,u,Du):(\lambda-Du).$$

Thus we conclude that $\lambda\in K_x$. By the convexity of W we can write

$$W(x,u,\lambda)\geq W(x,u,Du)+\sigma(x,u,Du):(\lambda-Du) \quad \forall\lambda\in\mathbb{M}^{m\times n}.$$

Putting $A(\lambda)$ and $B(\lambda)$ respectively the left- and the right-hand side of the previous inequality. Using the continuity and differentiability of $\lambda\mapsto W(x,u,\lambda)$, this gives for $\xi\in\mathbb{M}^{m\times n}$ and $\tau\in\mathbb{R}$

$$\frac{A(\lambda+\tau\xi)-A(\lambda)}{\tau}\geq\frac{B(\lambda+\tau\xi)-B(\lambda)}{\tau} \quad \text{if } \tau>0,$$

$$\frac{A(\lambda+\tau\xi)-A(\lambda)}{\tau}\leq\frac{B(\lambda+\tau\xi)-B(\lambda)}{\tau} \quad \text{if } \tau<0.$$

Hence $D_\lambda A=D_\lambda B$ and we obtain

$$\sigma(x,u,\lambda)=\sigma(x,u,Du) \quad \text{for all} \quad \lambda\in K_x\supset\operatorname{supp}\nu_x.$$

Consequently

$$\overline{\sigma} := \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda)d\nu_x(\lambda) = \int_{\operatorname{supp}\nu_x} \sigma(x,u,Du)d\nu_x(\lambda) = \sigma(x,u,Du). \tag{5.3.7}$$

Let consider the Carathéodory function $g(x,s,\xi) = |\sigma(x,s,\xi) - \overline{\sigma}(x)|$. The sequence $g_k(x) := g(x,u_k(x),Du_k(x))$ is equiintegrable and thus

$$g_k \rightharpoonup \overline{g} \quad \text{weakly in } L^1(\Omega),$$

where $\overline{g}$ is given by

$$\begin{aligned}\overline{g}(x) &= \int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}} |\sigma(x,s,\lambda) - \overline{\sigma}(x)| d\delta_{u(x)}(s)\otimes d\nu_x(\lambda)\\ &= \int_{\mathbb{M}^{m\times n}} |\sigma(x,u,\lambda) - \overline{\sigma}(x)| d\nu_x(\lambda) = 0,\end{aligned}$$

because of (5.3.7). Since $g_k \geq 0$ it follows that $g_k \to 0$ strongly in $L^1(\Omega)$.
By Vitali's theorem, we have for $\varphi \in W_0^1L_M(\Omega;\mathbb{R}^m)$

$$\int_\Omega \big(\sigma(x,u_k,Du_k) - \sigma(x,u,Du)\big) : D\varphi dx \to 0 \text{ as } k\to\infty.$$

**Case (c):** By strict monotonicity of $\sigma$, we deduce from Eq. (5.3.3) that $\nu_x = \delta_{Du(x)}$ for almost $x\in\Omega$. Hence, Proposition 2.3.1 implies $Du_k \to Du$ in measure as $k\to\infty$. Thus $\sigma(x,u_k,Du_k) \to \sigma(x,u,Du)$ almost everywhere. We have by the growth condition in **(H1)** that, $\sigma(x,u_k,Du_k)$ is bounded and equiintegrable, thus $\sigma(x,u_k,Du_k) \to \sigma(x,u,Du)$ in $L^1(\Omega)$ (by Vitali's theorem). This implies

$$\langle T(u),\varphi\rangle = 0 \text{ for all } \varphi \in \bigcup_{k\in\mathbb{N}} V_k.$$

Hence $T(u) = 0$.
**Case (d):** Suppose that $\nu_x$ is not a Dirac mass on $\Omega' \subset \Omega$ of positive Lebesgue measure.

Then, by the assumption of $M$-quasimonotonicity of $\sigma$, we have for a.e. $x \in \Omega'$

$$\begin{aligned}\int_{\mathbb{M}^{m\times n}} &\sigma(x,u,\lambda) : \lambda d\nu_x(\lambda)\\ &> \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \overline{\lambda} d\nu_x(\lambda) + \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\overline{\lambda}) : (\lambda - \overline{\lambda}) d\nu_x(\lambda).\end{aligned}$$

Integrating this last inequality over $\Omega$ and using the fact that $\int_{\mathbb{M}^{m\times n}} \sigma(x,u,\overline{\lambda}) : (\lambda - \overline{\lambda}) d\nu_x(\lambda) = 0$ and Lemma 5.3.4, we obtain

$$\begin{aligned}\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) d\nu_x(\lambda) : Du(x)dx &\geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\nu_x(\lambda) dx\\ &> \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) d\nu_x(\lambda) : Du(x) dx\end{aligned}$$

which is a contradiction. Consequently, $\nu_x = \delta_{Du(x)}$ for a.e. $x \in \Omega$. Hence, Proposition 2.3.1 implies $Du_k \to Du$ in measure as $k \to \infty$. We follow then the proof of the case (c).

In conclusion, we have all ingredient to pass to the limit in the approximating equations and prove Theorem 5.2.1. Let $\varphi \in W_0^1 L_M(\Omega;\mathbb{R}^m)$ and since $\cup_{i\in\mathbb{N}} V_i$ is dense in $W_0^1 L_M(\Omega;\mathbb{R}^m)$, there exists a sequence $\varphi_k \in \cup_{i\in\mathbb{N}} V_i$ such that $\varphi_k \to \varphi$ in $W_0^1 L_M(\Omega;\mathbb{R}^m)$ as $k \to \infty$. By all cases in **(H2)** (see Step 3), we obtain

$$\begin{aligned}&\langle T(u_k), \varphi_k\rangle - \langle T(u), \varphi\rangle\\ &= \int_\Omega \sigma(x,u_k,Du_k) : D\varphi_k - \langle f, \varphi_k\rangle - \int_\Omega \sigma(x,u,Du) : D\varphi dx + \langle f, \varphi\rangle\\ &= \int_\Omega \left[\sigma(x,u_k,Du_k) : (D\varphi_k - D\varphi) + \big(\sigma(x,u_k,Du_k) - \sigma(x,u,Du)\big) : D\varphi\right] dx\\ &- \langle f, \varphi_k - \varphi\rangle,\end{aligned}$$

which tends to zero as $k$ tends to infinity. According to (5.3.1), we then deduce that $\langle T(u), \varphi\rangle = 0$ for all $\varphi \in W_0^1 L_M(\Omega;\mathbb{R}^m)$, and the proof of Theorem 5.2.1 is complete.

## 5.4 Extension result

The main objective in this part, is to extend the result of the first part by considering $f$ depends on the unknown $u : \Omega \to \mathbb{R}^m$ and its gradient $Du$. This part is concerned then with the existence of solutions for the Dirichlet quasilinear elliptic system

$$\begin{cases} -\operatorname{div} \sigma(x, u, Du) &= f(x, u, Du) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Omega. \end{cases} \tag{5.4.1}$$

Existence theorems for (5.4.1) in the scalar case (i.e., $m = 1$) were established by different methods in different papers. For example, Akdim et al. [8] have been proved the existence of weak solutions in weighted Sobolev spaces. In [35], the authors proved the existence of solutions in variable Sobolev spaces. Faria and others [71] proved the existence of sub-supsolution, nonlinear regularity theory and strong maximum principle. Pucci and Servadie [121] established a regularity results for weak solutions. In [7], the authors proved the existence of at least one solution by using the topological degree theory when $\sigma(x, u, Du) = |\nabla u|^{p(x)-2}\nabla u$.

It is worthy besides the previous works to mention some other works for the case of systems (i.e., $m > 1$). Fuchs [74] has proved partial regularity theorem under some kind of ellipticity condition. Zhang [140] studied the existence of weak solutions by means of the Young measure under different notions of quasi-monotone mapping and semiconvex functions. Yongqiang and others [135] proved the existence of at least one weak solution. Also the same result has been proved by Dong [62].

Throughout this part, we assume the following hypothesis:
**(H0')** $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ and $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ are Carathéodory functions.
**(H1')** There exist $d_1(x) \in E_{\overline{M}}(\Omega)$, $d_2(x) \in L^1(\Omega)$, $d_3(x) \in E_{\overline{M}}(\Omega)$ and $\alpha, \beta, \gamma > 0$ such that

$$|\sigma(x, s, \xi)| \leq d_1(x) + \overline{M}^{-1}P(\gamma|s|) + \overline{M}^{-1}M(\gamma|\xi|)$$

$$|f(x, s, \xi)| \leq d_3(x) + \overline{M}^{-1}P(\gamma|s|) + \overline{M}^{-1}M(\gamma|\xi|)$$

$$\sigma(x, s, \xi) : \xi - f(x, s, \xi).u \geq \alpha M(\frac{|\xi|}{\beta}) - d_2(x)$$

$$f(x,s,\xi).s \geq 0 \quad \forall (s,\xi) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}.$$

We assume further that **(H2)**, stated in the first part, holds the same. Moreover,
**(H2')**(Linearity) For almost every $x \in \Omega$ and $u \in \mathbb{R}^m$, the mapping $\xi \mapsto f(x,u,\xi)$ is linear.

Now, we can state the main theorem of this part in the following.

**Theorem 5.4.1.** *Under assumptions **(H0')**, **(H1')**, **(H2)** and **(H2')**, problem (5.4.1) has a weak solution $u \in W_0^1 L_M(\Omega;\mathbb{R}^m)$ satisfying*

$$\int_\Omega \sigma(x,u,Du) : D\varphi dx = \int_\Omega f(x,u,Du).\varphi dx,$$

*for all $\varphi \in W_0^1 L_M(\Omega;\mathbb{R}^m)$.*

**Example 5.4.1.** *The model examples for which the result can be applied we give: Going back to Example 5.2.1 and consider $f \equiv |u|^{p-1}u|Du|^{p-2}Du$ for the first example 1., $f \equiv a(u)|Du|^{p-2}Du$ where $a(.)$ is a continuous function satisfying $a(u).u \geq 0$ and $0 < \alpha \leq a(u) \leq \beta$ for the second example 2., finally one can take $f \equiv f(x,u)a(|Du|)Du$ where $f(x,u)$ is any Carathéodory function satisfying $0 < \alpha \leq f(x,u) \leq \beta$ and $f(x,u).u \geq 0$ for the third example 3.*

### 5.4.1 Proof of the main result

Let $V_1 \subset V_2 \subset ... \subset W_0^1 L_M(\Omega;\mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\cup_{i\in\mathbb{N}} V_i$ is dense in $W_0^1 L_M(\Omega;\mathbb{R}^m)$. We define the operator

$$\begin{aligned} T : W_0^1 L_M(\Omega;\mathbb{R}^m) &\to W^{-1} L_{\overline{M}}(\Omega;\mathbb{R}^m) \\ u &\mapsto \Big(\varphi \mapsto \int_\Omega \sigma(x,u,Du) : D\varphi dx - \int_\Omega f(x,u,Du).\varphi dx\Big). \end{aligned}$$

**Lemma 5.4.1.** *For arbitrary $u \in W_0^1 L_M(\Omega;\mathbb{R}^m)$, $T(u)$ is linear, well defined and bounded.*

*Proof.* $T$ is trivially linear. By the growth condition in **(H1')** together with Hölder's inequality, we have

$$\begin{aligned}
&\int_\Omega \sigma(x,u,Du) : D\varphi dx \\
&\quad \leq 2\Big(\int_\Omega \overline{M}(|\sigma(x,u,Du)|)dx\Big)\Big(\int_\Omega M(|D\varphi|)dx\Big) \\
&\quad \leq c\Big(\int_\Omega (\overline{M}(d_1(x)) + P(\gamma|u|) + M(\gamma|Du|))dx\Big)\Big(\int_\Omega M(|D\varphi|)dx\Big),
\end{aligned}$$

and (similarly)

$$\begin{aligned}
&\int_\Omega f(x,u,Du).udx \\
&\quad \leq c\Big(\int_\Omega (\overline{M}(d_3(x)) + P(\gamma|u|) + M(|Du|))\Big)\Big(\int_\Omega M(|D\varphi|)\Big),
\end{aligned}$$

where we have used Lemma 2.2.3, and $c$ is a positive constant. Since $u,\varphi \in W_0^1L_M(\Omega;\mathbb{R}^m)$ and $P \ll M$, we can infer that $T(u)$ is well defined and bounded. $\square$

**Lemma 5.4.2.** *The restriction of $T$ to a finite linear subspace of $W_0^1L_M(\Omega;\mathbb{R}^m)$ is continuous.*

*Proof.* Let $V$ be a subspace of $W_0^1L_M(\Omega;\mathbb{R}^m)$ with $\dim V = r$ and $(\varphi_i)_{i=1}^r$ a basis of $V$. Let $(u_k = a_k^i\varphi_i)$ be a sequence in $V$ which converges to $u = a^i\varphi_i$ in $V$ (with conventional summation). Then on the one hand the sequence $(a_k)$ converges to $a$ in $\mathbb{R}^r$ and so $u_k \to u$ and $Du_k \to Du$ almost everywhere. On the other hand $\|u_k\|_{M,\Omega}$ and $\|Du_k\|_{M,\mathbb{M}^{m\times n}}$ are bounded by a constant $c$. By the continuity condition **(H0')**, it follows that $\sigma(x,u_k,Du_k) : D\varphi \to \sigma(x,u,Du) : D\varphi$ and $f(x,u_k,Du_k).\varphi \to f(x,u,Du).\varphi$ almost everywhere. Let $\Omega' \subset \Omega$ be a measurable subset and $\varphi \in W_0^1L_M(\Omega;\mathbb{R}^m)$. From the inequalities in the proof of Lemma 5.4.1 together with Lemma 2.2.3, we obtain

$$\begin{aligned}
&\int_{\Omega'} |\sigma(x,u_k,Du_k) : D\varphi|dx \\
&\qquad \leq c\Big(\|d_1\|_{\overline{M}} + \theta\underbrace{\|Du_k\|_{M,\mathbb{M}^{m\times n}}}_{\leq c} + \underbrace{\|Du_k\|_{M,\mathbb{M}^{m\times n}}}_{\leq c}\Big)\Big(\int_{\Omega'} M(|D\varphi|)dx\Big)
\end{aligned}$$

and

$$\int_{\Omega'} |f(x,u_k,Du_k).\varphi|dx$$
$$\leq c\theta\Big(\|d_3\|_{\overline{M}} + \theta\underbrace{\|Du_k\|_{M,\mathbb{M}^{m\times n}}}_{\leq c} + \underbrace{\|Du_k\|_{M,\mathbb{M}^{m\times n}}}_{\leq c}\Big)\Big(\int_{\Omega'} M(|D\varphi|)dx\Big)$$

by the Hölder inequality. Note that $\left(\int_{\Omega'} M(|D\varphi|)dx\right)$ is arbitrary small if the measure of $\Omega'$ is chosen small enough. As a consequence, the sequences $\left(\sigma(x,u_k,Du_k) : D\varphi\right)$ and $\left(f(x,u_k,Du_k).\varphi\right)$ are equiintegrable. Applying the Vitali Theorem, it follows that for all $\varphi \in W_0^1 L_M(\Omega;\mathbb{R}^m)$ we have

$$\lim_{k\to\infty} \langle T(u_k), \varphi\rangle = \langle T(u), \varphi\rangle.$$

□

According to the coercivity condition in **(H1')**, we obtain that $T$ is coercive in the sense that $\langle T(u), u\rangle \to \infty$ as $\|u\|_{1,M} \to \infty$. By virtue of Lemma 5.3.2, it follows then, that for all $k \in \mathbb{N}$, there exists $u_k \in V_k$ such that

$$\langle T(u_k), \varphi\rangle = 0 \quad \text{for all } \varphi \in V_k, \tag{5.4.2}$$

where $V_k$ is a finite linear subspace of $W_0^1 L_M(\Omega;\mathbb{R}^m)$. We obtain also, that (5.3.2) is satisfied. Moreover, according to Lemma 2.3.1, there exists a Young measure $\nu_x$ generated by $Du_k$ in $L_M(\Omega;\mathbb{M}^{m\times n})$ satisfying the properties of Lemma 5.3.3.

As in the first part, the following elliptic div-curl inequality is the key ingredient to prove that one can pass to the limit in our quasilinear elliptic system.

**Lemma 5.4.3.** *The Young measure $\nu_x$ generated by $Du_k$ has the property, that*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \left(\sigma(x,u,\lambda) - \sigma(x,u,Du)\right) : (\lambda - Du)d\nu_x(\lambda)dx \leq 0.$$

*Proof.* Let consider the sequence

$$\begin{aligned}I_k &:= \big(\sigma(x,u_k,Du_k)-\sigma(x,u,Du)\big):(Du_k-Du)\\&=\sigma(x,u_k,Du_k):(Du_k-Du)-\sigma(x,u,Du):(Du_k-Du)\\&=: I_{k,1}+I_{k,2}.\end{aligned}$$

Assumption **(H1')** implies

$$\int_\Omega \overline{M}(|\sigma(x,u,Du)|)dx \le c\int_\Omega \big(\overline{M}(d_1(x))+P(\gamma|u|)+M(\gamma|Du|)\big)dx<\infty.$$

Since $u \in W_0^1L_M(\Omega;\mathbb{R}^m)$, $P \ll M$ and by Lemma 2.2.3, it follows that $\sigma \in L_{\overline{M}}(\Omega;\mathbb{M}^{m\times n})$. According to a weak convergence defined in Lemma 5.3.3, we obtain that

$$\liminf_{k\to\infty}\int_\Omega I_{k,2}dx=\int_\Omega \sigma(x,u,Du):\Big(\int_{\mathbb{M}^{m\times n}}\lambda d\nu_x(\lambda)-Du\Big)dx=0.$$

We have $\big(\sigma(x,u_k,Du_k)\ :\ Du\big)^-$ is equiintegrable (see the proof of Lemma 5.4.2 if necessary). The sequence $\big(\sigma(x,u_k,Du_k)\ :\ Du_k\big)^-$ is easily seen to be equiintegrable. Indeed, by the coercivity condition in **(H1')**, we have

$$\begin{aligned}\sigma(x,u_k,Du_k):Du_k &\ge f(x,u_k,Du_k).u_k+\alpha M\Big(\frac{|Du_k|}{\beta}\Big)-d_2(x)\\&\ge \alpha M\Big(\frac{|Du_k|}{\beta}\Big)-d_2(x),\end{aligned}$$

where we have used the sign condition $f(x,u_k,Du_k).u_k\ge 0$, thus

$$\begin{aligned}&\int_{\Omega'}\big|\min\big(\sigma(x,u_k,Du_k):Du_k,0\big)\big|dx\\&\qquad\le \int_{\Omega'}|d_2(x)|dx+\alpha\int_{\Omega'}M\Big(\frac{|Du_k|}{\beta}\Big)dx<\infty \quad \text{(by (5.3.2))}.\end{aligned}$$

We have by (5.3.2), $u_k\to u$ in $L_M(\Omega;\mathbb{R}^m)$ (for a subsequence), and by virtue of Lemma 2.2.4, we get $u_k\to u$ in measure. Hence, we may use lemma 2.3.2 which gives

$$I:=\liminf_{k\to\infty}\int_\Omega I_{k,1}dx\ge\int_\Omega\int_{\mathbb{M}^{m\times n}}\sigma(x,u,\lambda):(\lambda-Du)d\nu_x(\lambda)dx.$$

Is is sufficient to show that $I \leq 0$. According to Mazur's theorem (see, e.g., [136, Theorem 2, page 120]) there exists a sequence $v_k$ in $W_0^1L_M(\Omega;\mathbb{R}^m)$, where each $v_k$ is a convex linear combination of $\{u_1, .., u_k\}$, such that $v_k \to u$ in $W_0^1L_M(\Omega;\mathbb{R}^m)$. This significant that $v_k$ belongs to the same space $V_k$ as $u_k$. By taking $u_k - v_k$ as a test function in (5.4.2), we obtain

$$\int_\Omega \sigma(x,u_k,Du_k) : (Du_k - Dv_k)dx = \int_\Omega f(x,u_k,Du_k).(u_k - v_k)dx. \tag{5.4.3}$$

From the growth condition in **(H1')** and the Hölder inequality, it follows that

$$\begin{aligned} &\Big| \int_\Omega f(x,u_k,Du_k).(u_k - v_k)dx\Big| \\ &\qquad \leq c\Big( \int_\Omega \overline{M}(d_3(x)) + P(\gamma|u_k|) + M(\gamma|Du_k|)dx\Big)\Big( \int_\Omega M(|u_k - v_k|)dx\Big). \end{aligned}$$

The right hand side of this inequality vanishes as $k \to \infty$, since by the construction of $v_k$, we have

$$\|u_k - v_k\|_{M,\Omega} \leq \|u_k - u\|_{M,\Omega} + \|v_k - u\|_{M,\Omega} \to 0 \text{ as } k \to \infty.$$

Hence, the left hand side in (5.4.3) tends to zero as $k \to \infty$. Using this result and the fact that $v_k \to u$ in $W_0^1L_M(\Omega;\mathbb{R}^m)$ to deduce the following

$$\begin{aligned} I &= \liminf_{k\to\infty} \int_\Omega \sigma(x,u_k,Du_k) : (Du_k - Du)dx \\ &= \liminf_{k\to\infty}\Big( \int_\Omega \sigma(x,u_k,Du_k) : (Du_k - Dv_k)dx + \int_\Omega \sigma(x,u_k,Du_k) : (Dv_k - Du)dx\Big) \\ &= \liminf_{k\to\infty} \int_\Omega \sigma(x,u_k,Du_k) : (Dv_k - Du)dx \\ &\leq \liminf_{k\to\infty} c\| \, |\sigma(x,u_k,Du_k)| \, \|_{\overline{M},\mathbb{M}^{m\times n}} \|v_k - u\|_{1,M} = 0. \end{aligned}$$

In view of Lemma 5.3.3, we have

$$\begin{aligned} &\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,Du) : (\lambda - Du)d\nu_x(\lambda)dx \\ &\qquad\qquad = \int_\Omega \sigma(x,u,Du) : \Big( \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) - Du\Big)dx = 0, \end{aligned}$$

and together with $I \leq 0$ we finish the proof of Lemma 5.4.3. $\square$

From Lemma 5.4.3, we can also derive the Eq. (5.3.3). Now, all necessary ingredients are in hand to prove Theorem 5.4.1. Note that the proof of the cases **(H2)**(a)-(d) is similar to that of the first part for $\sigma(x, u_k, Du_k)$, so we omit it. To conclude the proof, it remains to pass to the limit in the term $f(x, u_k, Du_k)$. Since $u_k \to u$ and $Du_k \to Du$ almost everywhere for $k \to \infty$ (see the proof of the first part), it follows from the continuity condition in **(H0')**, that $f(x, u_k, Du_k).\varphi \to f(x, u, Du).\varphi$ almost everywhere, for arbitrary $\varphi \in W_0^1 L_M(\Omega; \mathbb{R}^m)$. According to Vitali's Convergence Theorem, it result that

$$f(x, u_k, Du_k).\varphi \to f(x, u, Du).\varphi \quad \text{in } L^1(\Omega),$$

by the equiintegrability of $(f(x, u_k, Du_k).\varphi)$. This implies

$$\lim_{k\to\infty} \int_\Omega f(x, u_k, Du_k).\varphi dx = \int_\Omega f(x, u, Du).\varphi dx \quad \forall \varphi \in \cup_{k\geq 1} V_k.$$

Now, if $\xi \mapsto f(x, u, \xi)$ is linear by **(H2')**, we argue as follows:

$$\begin{aligned} f(x, u_k, Du_k) &\rightharpoonup \int_{\mathbb{M}^{m\times n}} f(x, u, \lambda) d\nu_x(\lambda) \\ &= f(x, u, .)o \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)}_{:=Du(x)} \\ &= f(x, u, Du) \quad \text{in } L^1(\Omega), \end{aligned}$$

by the equiintegrability of $f(x, u_k, Du_k)$. We come to conclude the proof of this part by saying, that since $\cup_{k\geq 1} V_k$ is dense in $W_0^1 L_M(\Omega; \mathbb{R}^m)$, $u$ is then a weak solution of (5.4.1), and the proof of Theorem 5.4.1 is finish.

# Chapter 6

# Quasilinear parabolic systems in Orlicz-Sobolev spaces

## 6.1 Introductory and background

In this chapter we descuss the existence and uniqueness of a weak solution for an evolutionary problem. The idea is the following: Let $n \geq 2$ be an integer and $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Let $Q$ be $\Omega \times (0, T)$ where $T > 0$ is given. We consider the initial-boundary value problem of the quasilinear parabolic system

$$\frac{\partial u}{\partial t} - \operatorname{div} \sigma(x, t, Du) = f \quad \text{in } Q, \tag{6.1.1}$$

$$u(x, t) = 0 \quad \text{on } \partial Q, \tag{6.1.2}$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \tag{6.1.3}$$

where $\partial Q = \partial\Omega \times (0, T)$ and $u : Q \to \mathbb{R}^m$ is a vector-valued funtion. Here, the source term $f$ belongs to $W^{-1,x}L_{\overline{M}}(Q; \mathbb{R}^m)$ the dual of the inhomogeneous Orlicz-Sobolev space $W_0^{1,x}L_M(Q; \mathbb{R}^m)$ built from an N-function $M$. We will prove the existence and uniqueness of a weak solution to (6.1.1)-(6.1.3) based on the theory of Young measures and weak monotonicity assumptions on the function $\sigma : Q \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$.

Many nonlinear parabolic problems were studied using the classical monotone operator methods developped in [42, 105, 111, 131]. Our propose is to study (6.1.1)-(6.1.3) without using such methods. Norbert Hungerbühler studied the following problem in [86]:

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div} \sigma(x,t,u,Du) = f \quad \text{in } Q \\ u(x,t) = 0 \quad \text{on } \partial Q \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{aligned} \tag{6.1.4}$$

where the source term $f$ is taken in $L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ the dual space of $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ for some $p \in (\frac{2n}{n+2},\infty)$ and $u_0 \in L^2(\Omega;\mathbb{R}^m)$. The author obtained the needed result by means of the Young measures. S. Demoulini [56] used also Young measures to prove an existence result for nonlinear parabolic evolution of forward-backward type $\partial_t u = \nabla.q(\nabla u)$ on $Q_\infty \equiv \Omega \times \mathbb{R}^+$. In [79], the authors considered the problem of existence of weak solutions in Orlicz spaces to the initial boundary problem

$$\begin{cases} \partial u/\partial t &= \operatorname{div} A(x,t,\nabla u) \quad \text{in } Q \\ u(x,t) &= 0 \quad \text{on } \partial Q \\ u(x,0) &= u_0 \quad \text{in } \Omega \end{cases}$$

by using a full anisotropy of the $N$-function and no growth assumption on an $N$-function. We refer to [61, 66, 76, 99, 114] for more results and related topics.

When trying to relax the growth and coercivity conditions of [86] (see also [24]), the problem (6.1.1)-(6.1.3) can not be formulated in the classical Sobolev spaces $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$, and this fact led us to replace $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ with an inhomogeneous Orlicz-Sobolev space $W_0^{1,x}L_M(Q;\mathbb{R}^m)$ built from an Orlicz spaces $L_M$ instead of $L^p$, where the $N$-function $M$ which defines $L_M$ is related to the actual growth and coercivity of $\sigma$ as in Chapter 5.

The main purpose of this chapter is to solve (6.1.1)-(6.1.3) in the setting of Orlicz-Sobolev spaces, and to extend the problem (5.1.1) to evolutionary case; further, we do not require the classical monotonicity claimed in previous works. We will prove the existence and uniqueness of a weak solution $u$ in $W_0^{1,x}L_M(Q;\mathbb{R}^m)$ by using the concept of Young measures.

## 6.2 Assumptions and formulation of the main result

Throughout this chapter, we denote $Q_\tau = \Omega \times (0, \tau)$ for every $\tau \in (0, T)$, where $\Omega$ is an open bounded subset of $\mathbb{R}^n$. Let $M$ be an N-function and $\overline{M}$ its conjugate function, both are satisfying the $\Delta_2$-condition (2.2.3). To study the problem (6.1.1)-(6.1.3), we suppose the following assumptions:

**(H0)** $\sigma : Q \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e., $(x,t) \mapsto \sigma(x,t,\xi)$ is measurable for every $\xi \in \mathbb{M}^{m\times n}$ and $\xi \mapsto \sigma(x,t,\xi)$ is continuous for almost every $(x,t) \in Q$.

**(H1)** There exist $0 \le d_1(x,t) \in E_{\overline{M}}(Q)$, $d_2(x,t) \in L^1(Q)$ and $\alpha, \beta, \gamma > 0$, such that

$$|\sigma(x,t,\xi)| \le d_1(x,t) + \overline{M}^{-1} M(\gamma|\xi|)$$

$$\sigma(x,t,\xi) : \xi \ge -d_2(x,t) + \alpha M\Big(\frac{|\xi|}{\beta}\Big).$$

**(H2)** $\sigma$ satisfies one of the following conditions:

(a) For all $(x,t) \in Q$, $\xi \mapsto \sigma(x,t,\xi)$ is a $\mathcal{C}^1$-function and is monotone, i.e., for all $(x,t) \in Q$, we have

$$\big(\sigma(x,t,\xi) - \sigma(x,t,\eta)\big) : (\xi - \eta) \ge 0.$$

(b) There exists a function $W : Q \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,t,\xi) = \frac{\partial W}{\partial \xi}(x,t,\xi) = D_\xi W(x,t,\xi)$ and $\xi \to W(x,t,\xi)$ is convex and $\mathcal{C}^1$ for all $(x,t) \in Q$.

(c) $\sigma$ is strictly monotone, i.e., $\sigma$ is monotone and

$$\big(\sigma(x,t,\xi) - \sigma(x,t,\eta)\big) : (\xi - \eta) = 0 \Rightarrow \xi = \eta.$$

(d) $\sigma$ is strictly $M$-quasimonotone, i.e.,

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,\lambda) - \sigma(x,t,\overline{\lambda})\big) : (\lambda - \overline{\lambda}) d\nu_{(x,t)}(\lambda) dx dt > 0$$

where $\overline{\lambda} = \langle \nu_{(x,t)}, id\rangle$, $\nu = \{\nu_{(x,t)}\}$ is any family of Young measures generated by a sequence in $L_M(Q)$ and not a Dirac measure for a.e. $(x,t) \in Q$.

Now, the main result of this (part of) Chapter 6 is the following:

**Theorem 6.2.1.** *Assume that* ***(H0)-(H2)*** *hold, then* (6.1.1)-(6.1.3) *has a unique weak solution* $u \in W_0^{1,x}L_M(Q;\mathbb{R}^m) \cap C(0,T;L^2(\Omega;\mathbb{R}^m))$, *for every* $f \in W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$ *and every* $u_0 \in L^2(\Omega;\mathbb{R}^m)$.

## 6.3 Proof of the main result

Our intention is to prove Theorem 6.2.1 using the Galerkin method to construct the approximating solutions, and Young measures together with weak monotonicity assumptions to pass to the limit in the approximate problem.

### 6.3.1 Galerkin method

We choose an $L^2(\Omega;\mathbb{R}^m)$-orthonormal base $\{\varphi_i\}_{i\geq 1}$, such that

$$\{\varphi_i\}_{i\geq 1} \subset \mathcal{C}_0^\infty(\Omega;\mathbb{R}^m), \quad \mathcal{C}_0^\infty(\Omega;\mathbb{R}^m) \subset \overline{\bigcup_{k\geq 1} V_k}^{\mathcal{C}^1(\Omega;\mathbb{R}^m)},$$

where $V_k = span\{\varphi_1, ..., \varphi_k\}$. We consider the following sequence to approach the solutions of the problem (6.1.1)-(6.1.3):

$$u_k(x,t) = \sum_{i=1}^{k} c_{ki}(t)\varphi_i(x), \tag{6.3.1}$$

where $c_{ki} : (0,T) \to \mathbb{R}$ are bounded measurable functions. Notice that each $u_k$ satisfy the condition (6.1.2) by construction in the sense that $u_k \in W_0^{1,x}L_M(Q;\mathbb{R}^m)$. For the condition (6.1.3) we choose the initial coefficients

$$c_{ki}(0) := (u_0, \varphi_i)_{L^2} = \int_\Omega u_0(x)\varphi_i(x)dx,$$

such that

$$u_k(.,0) = \sum_{i=1}^{k} c_{ki}(0)\varphi_i(.) = \sum_{i=1}^{k} (u_0, \varphi_i)_{L^2}\varphi_i(.) \to u_0 \text{ in } L^2(\Omega;\mathbb{R}^m) \text{ as } k \to \infty.$$

**Assertion 1.** We claim that $u_k(x,t)$ in (6.3.1) is the desired solution.

Let's first determine the coefficients $c_{ki}(t)$. Consider the following system of ordinary differential equations

$$(\partial_t u_k, \varphi_j)_{L^2} + \int_\Omega \sigma(x,t,Du_k) : D\varphi_j dx = \langle f(t), \varphi_j \rangle, \tag{6.3.2}$$

(with $j = 1, ..., k$) where $\langle .,. \rangle$ denotes the dual pairing of $W^{-1}E_{\overline{M}}(\Omega;\mathbb{R}^m)$ and $W_0^1 L_M(\Omega;\mathbb{R}^m)$. Equation (6.3.2) is in the sense of distributions. Now, we choose $r > 0$ large enough such that the set $B(0,r) := B_r(0) \subset \mathbb{R}^k$ contains the vectors $(c_{k1}(0), ..., c_{kk}(0))$ for fixed $k \in \mathbb{N}$. Let $\tau > 0$ and consider the functional

$$T : [0,\tau] \times \overline{B_r(0)} \to \mathbb{R}^k$$
$$(t, c_1, .., c_k) \to \Big( \langle f(t), \varphi_j \rangle - \int_\Omega \sigma(x,t,\sum_{i=1}^{k} c_i D\varphi_i) : D\varphi_j dx \Big)_{j=1..k}.$$

By virtue of the assumption **(H0)**, we have $T$ is a Carathéodory function. Moreover, the component $T_j$ may be estimated by

$$\big|T_j(t,c_1,..,c_k)\big| \le c\|f\|_{W^{-1}L_{\overline{M}}}\|\varphi_j\|_{W_0^1 L_M} + c\Big\|\big|\sigma(x,t,\sum_{i=1}^{k} c_i D\varphi_i)\big|\Big\|_{\overline{M},\Omega}\|D\varphi_j\|_{M,\Omega}.$$

where $c$ is a positive constant. Using the growth condition in **(H1)**, the right hand side in the above inequality can be estimated in such a way that

$$\big|T_j(t,c_1,..,c_k)\big| \le C_1(r,k)\psi(t)$$

uniformly on $[0,\tau] \times \overline{B_r(0)}$, where $C_1(r,k)$ is a constant which depends on $r$ and $k$, and $\psi(t) \in L^1([0,\tau])$. Therefore, thanks to the existence result of ordinary differential equation (c.f. [90]), the system

$$\begin{cases} c_j'(t) & = T_j(t, c_1, .., c_k) \\ c_j(0) & = c_{kj}(0) \end{cases} \tag{6.3.3}$$

(for $j = 1, ..., k$) has a continuous solution $c_j$ (depending on $k$) on an interval $[0, \tau')$, where $\tau' > 0$ and may depend on $k$. After integrating (6.3.3), we obtain

$$c_j(t) = c_j(0) + \int_0^t T_j(s, c_1(s), .., c_k(s))ds,$$

which holds on $[0, \tau')$. So $u_k(x,t) := \sum_{i=1}^{k} c_{ki}(t)\varphi_i(x)$ is the desired solution of (6.3.2).

**Assertion 2.** The purpose of this assertion is to extend the local solution constructed in Assertion 1 to the whole interval $[0, T)$.

We multiply each side of (6.3.2) by $c_{ki}(t)$ and we sum. This gives for $\tau \in [0, \tau')$

$$\int_{Q_\tau} \frac{\partial u_k}{\partial t} u_k dxdt + \int_{Q_\tau} \sigma(x, t, Du_k) : Du_k dxdt = \int_0^\tau \langle f(t), u_k \rangle dt,$$

which we denote by $I_1 + I_2 = I_3$. We have

$$I_1 = \frac{1}{2}\|u_k(., \tau)\|_{L^2}^2 - \frac{1}{2}\|u_k(., 0)\|_{L^2}^2.$$

By the coercivity condition in **(H1)**, we can write

$$I_2 = \int_{Q_\tau} \sigma(x, t, Du_k) : Du_k dxdt \geq -\|d_2\|_{\overline{M}} + \alpha\|Du_k\|_M.$$

Finally, by Hölder's inequality, we get

$$I_3 \leq \|f\|_{W^{-1,x}E_{\overline{M}}(Q_\tau;\mathbb{R}^m)}\|u_k\|_{W_0^{1,x}L_M(Q_\tau;\mathbb{R}^m)}.$$

The combination of these three estimates gives

$$\|u_k(., \tau)\|_{L^2(\Omega)}^2 = \left|\left(c_{ki}(\tau)\right)_{i=1}^k\right|_{\mathbb{R}^k}^2 \leq c$$

for a constant $c$ which is independent of $\tau$ and of $k$. Let

$$\Lambda := \Big\{ t \in [0,T) : \text{ there exists a weak solution of (6.3.3) on } [0,t) \Big\}.$$

According to [86], it follows that $\Lambda = [0,T)$.
**Assertion 3.** We claim that $u(.,0) = u_0$ and $u_k(.,T) \rightharpoonup u(.,T)$ in $L^2(\Omega)$.

From the estimations on $I_\epsilon$, $\epsilon = 1,2,3$ in Assertion 2, it follows that $(u_k)_k$ is bounded in $W_0^{1,x}L_M(Q;\mathbb{R}^m) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$. Therefore, by extracting a suitable subsequence (still denoted by $(u_k)_k$), we may assume

$$u_k \rightharpoonup u \text{ in } W_0^{1,x}L_M(Q;\mathbb{R}^m) \text{ weakly}$$

and

$$u_k \rightharpoonup^* u \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^m)).$$

Since $\sigma$ is monotone, it follows for all $\varphi \in W_0^{1,x}E_M(Q;\mathbb{R}^m)$ that

$$\big(\sigma(x,t,Du_k) - \sigma(x,t,D\varphi)\big) : (Du_k - D\varphi) \geq 0.$$

This gives $\sigma(x,t,Du_k) : D\varphi \leq \sigma(x,t,Du_k) : Du_k - \sigma(x,t,D\varphi) : (Du_k - D\varphi)$, which implies that, since $(u_k)$ is bounded in $W_0^{1,x}L_M(Q;\mathbb{R}^m)$ and $\sigma(x,t,Du_k) : Du_k$ is bounded from the growth condition in **(H1)** and the boundedness of $(u_k)_k$,

$$\int_Q \sigma(x,t,Du_k) : D\varphi dxdt \leq c_\varphi \quad \text{for all } \varphi \in W_0^{1,x}E_M(Q;\mathbb{R}^m),$$

where $c_\varphi$ is a constant depending on $\varphi$ but not on $k$. Therefore, the Banach Steinhauss theorem implies that we can obtain the boundedness of $\big\{ -\operatorname{div}\sigma(x,t,Du_k)\big\}$ in $W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$. Assume that

$$-\operatorname{div}\sigma(x,t,Du_k) \rightharpoonup \chi \quad \text{in } W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m),$$

where $\chi \in W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$. According to [66, Step 2], it follows that $u \in C(0,T;L^2(\Omega;\mathbb{R}^m))$. Note that according to the argument of Aubin-Simon, there is a weak convergence $u_k(.,T) \to u(.,T)$ in $L^2(\Omega)$.

The main principal difficulty will be to identify $\chi$ with $-\operatorname{div}\sigma(x,t,Du)$.

### 6.3.2 Div-curl inequality

Since $(u_k)$ is bounded in $W_0^{1,x}L_M(Q;\mathbb{R}^m)$ (see Assertion 3), it follows according to Lemma 2.3.1, that there exists a Young measure $\nu_{(x,t)}$ generated by $Du_k$ in $L_M(Q;\mathbb{M}^{m\times n})$. Before proving the div-curl inequality, which will be the key ingredient to pass to the limit in the approximating equations of the Galerkin method, we still need some properties of $\nu_{(x,t)}$ associated to $Du_k$.

**Lemma 6.3.1.** *The Young measure $\nu_{(x,t)}$ generated by $Du_k$ in $L_M(Q;\mathbb{M}^{m\times n})$ has the following properties:*

*(i)* $\|\nu_{(x,t)}\|_{\mathcal{M}(\mathbb{M}^{m\times n})}=1$ *and the weak $L^1$-limit of $Du_k$ is* $\langle\nu_{(x,t)},id\rangle=\int_{\mathbb{M}^{m\times n}}\lambda d\nu_{(x,t)}(\lambda)$, *for a.e.* $(x,t)\in Q$.

*(ii) For almost every $(x,t)\in Q$, $\nu_{(x,t)}$ satisfies $\langle\nu_{(x,t)},id\rangle=Du(x,t)$ for a.e. $(x,t)\in Q$.*

The proof of the above lemma is similar to that of Lemma 5.3.3 (for steady case), but for completness we present its proof.

*Proof.* (i) Putting $\epsilon(L)=\min\limits_{|\xi|=L}\frac{M(|\xi|)}{|\xi|}$, which tends to infinity as $L\to\infty$ by definition of the $N$-function $M$. Since $(Du_k)$ is bounded, there exists $c\geq 0$ such that

$$\begin{aligned}
c\geq\int_Q M\big(|Du_k(x,t)|\big)dxdt &\geq \int_{\{(x,t)\in Q:\,|Du_k|\geq L\}} M\big(|Du_k(x,t)|\big)dxdt\\
&\geq \epsilon(L)\int_{\{(x,t)\in Q:\,|Du_k|\geq L\}}|Du_k(x,t)|dxdt\\
&\geq L\epsilon(L)\big|\{(x,t)\in Q:\,|Du_k|\geq L\}\big|,
\end{aligned}$$

where we have used $\epsilon(L)\leq\frac{M(|\xi|)}{|\xi|}$. Therefore

$$\sup_{k\in\mathbb{N}}\big|\{(x,t)\in Q:\,|Du_k|\geq L\}\big|\leq\frac{c}{L\epsilon(L)}\to 0\quad\text{as } L\to\infty.$$

According to Theorem 2.3.1, it follows that $\|\nu_{(x,t)}\|_{\mathcal{M}} = 1$. Since $L_M(Q;\mathbb{M}^{m\times n})$ is reflexive, then there is a subsequence (still denoted by $(Du_k)_k$) weakly convergent in $L_M(Q;\mathbb{M}^{m\times n}) \subset L^1(Q;\mathbb{M}^{m\times n})$, thus weakly convergent in $L^1(Q;\mathbb{M}^{m\times n})$. By virtue again to Theorem 2.3.1, it follows by taking $\varphi \equiv id$, that

$$Du_k \rightharpoonup \langle \nu_{(x,t)}, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) \quad \text{weakly in } L^1(Q;\mathbb{M}^{m\times n}).$$

(ii) Since $(u_k)$ is bounded in $W_0^{1,x}L_M(Q;\mathbb{M}^{m\times n})$, then $Du_k \rightharpoonup Du$ in $L_M(Q;\mathbb{M}^{m\times n})$ (for a subsequence). On the other hand, since $L_M(Q;\mathbb{M}^{m\times n}) \subset L^1(Q;\mathbb{M}^{m\times n})$, it follows then that $Du_k \rightharpoonup Du$ in $L^1(Q;\mathbb{M}^{m\times n})$. Owing to (i) above, we can infer that

$$Du(x,t) = \langle \nu_{(x,t)}, id\rangle \quad \text{for a.e. } (x,t) \in Q.$$

□

Now, we announce the div-curl inequality and its proof.

**Lemma 6.3.2.** *The Young measure $\nu_{(x,t)}$ generated by the gradient $Du_k$ is satisfying the following inequality*

$$\int_Q \int_{\mathbb{M}^{m\times n}} (\sigma(x,t,\lambda) - \sigma(x,t,Du)) : (\lambda - Du) d\nu_{(x,t)}(\lambda) dxdt \leq 0. \tag{6.3.4}$$

*Proof.* Let us consider the sequence

$$\begin{aligned} I_k &:= \big(\sigma(x,t,Du_k) - \sigma(x,t,Du)\big) : (Du_k - Du) \\ &= \sigma(x,t,Du_k) : (Du_k - Du) - \sigma(x,t,Du) : (Du_k - Du) \\ &=: I_{k,1} + I_{k,2}. \end{aligned}$$

By the growth condition in **(H1)** together with the fact that $Du \in L_M(Q;\mathbb{M}^{m\times n})$, it follows that $\sigma \in L_{\overline{M}}(Q;\mathbb{M}^{m\times n})$. According to a weak convergence of $(Du_k)$ (see Lemma 6.3.1), we obtain

$$\liminf_{k\to\infty} \int_Q I_{k,2} dxdt = \int_Q \sigma(x,t,Du) : \Big( \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) - Du \Big) dxdt = 0.$$

By virtue of Lemma 2.3.2, we then have

$$
\begin{aligned}
I := \liminf_{k\to\infty} \int_Q I_k dxdt &= \liminf_{k\to\infty} \int_Q I_{k,1} dxdt \\
&= \liminf_{k\to\infty} \int_Q \sigma(x,t,Du_k) : (Du_k - Du) dxdt \\
&\geq \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,\lambda) : (\lambda - Du) d\nu_{(x,t)}(\lambda) dxdt.
\end{aligned}
$$

Now we prove that $I \leq 0$. Note that the first property of $\chi$ is the following energy equality:

$$
\frac{1}{2}\|u(.,T)\|_{L^2}^2 + \int_0^T \langle \chi, u \rangle dt = \frac{1}{2}\|u(.,0)\|_{L^2}^2 + \int_0^T \langle f, u \rangle dt.
$$

On the one hand, we have

$$
\begin{aligned}
\liminf_{k\to\infty} - \int_Q \sigma(x,t,Du_k) : Du dxdt &= - \int_0^T \langle \chi, u \rangle dt \\
&= \frac{1}{2}\|u(.,T)\|_{L^2}^2 - \frac{1}{2}\|u(.,0)\|_{L^2}^2 - \int_0^T \langle f, u \rangle dt.
\end{aligned}
\tag{6.3.5}
$$

On the other hand, by the Galerkin equations, we obtain

$$
\begin{aligned}
\int_Q \sigma(x,t,Du_k) : Du_k dxdt &= \int_0^T \langle f, u_k \rangle dt - \int_Q u_k \frac{\partial u_k}{\partial t} dxdt \\
&= \int_0^T \langle f, u_k \rangle dt - \frac{1}{2}\|u_k(.,T)\|_{L^2}^2 + \frac{1}{2}\|u_k(.,0)\|_{L^2}^2.
\end{aligned}
$$

By passage to the limit inf in the last expression and using the fact that $u_k(x,0) \to u_0(x) = u(x,0)$ and $u_k(.,T) \rightharpoonup u(.,T)$ in $L^2(\Omega;\mathbb{R}^m)$, we obtain

$$
\liminf_{k\to\infty} \int_Q \sigma(x,t,Du_k) : Du_k dxdt \leq \int_0^T \langle f, u \rangle dt - \frac{1}{2}\|u(.,T)\|_{L^2}^2 + \frac{1}{2}\|u(.,0)\|_{L^2}^2,
$$

which in combination with (6.3.5) gives $I = \liminf_{k\to\infty} \int_Q I_k dxdt \leq 0$. Use the fact that $Du(x,t) = \langle \nu_{(x,t)}, id \rangle$ and

$$
\int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,Du) : (\lambda - Du) d\nu_{(x,t)}(\lambda) dxdt = 0
$$

to conclude (6.3.4). □

**Lemma 6.3.3.** *If $\sigma$ satisfy the equation (6.3.4), then for almost $(x,t) \in Q$,*

$$\big(\sigma(x,t,\lambda) - \sigma(x,t,Du)\big) : (\lambda - Du) = 0 \quad \textit{on supp}\, \nu_{(x,t)}.$$

*Proof.* We have by (6.3.4)

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,\lambda) - \sigma(x,t,Du)\big) : (\lambda - Du) d\nu_{(x,t)}(\lambda) dx dt \le 0.$$

The monotonicity of $\sigma$ permits to deduce that the above integral is nonnegative. Thus, must vanish almost everywhere with respect to the product measure $d\nu_{(x,t)}(\lambda) \otimes dx \otimes dt$. Hence, for almost $(x,t) \in Q$

$$\big(\sigma(x,t,\lambda) - \sigma(x,t,Du)\big) : (\lambda - Du) = 0 \quad \text{on supp}\, \nu_{(x,t)}.$$

□

### 6.3.3 Passage to the limit

Now, we are in a position to prove the main result by considering the 4 cases listed in **(H2)**. Let start with the easiest case.

**Case (c)**: By the strict monotonicity of $\sigma$ and Lemma 6.3.3, it follows that $\text{supp}\, \nu_{(x,t)} = \{Du(x,t)\}$, i.e., $\nu_{(x,t)} = \delta_{Du(x,t)}$ for a.e. $(x,t) \in Q$. According to Proposition 2.3.1, we have $Du_k \to Du$ in measure on $Q$. Thus $\sigma(x,t,Du_k) \to \sigma(x,t,Du)$ a.e. $(x,t) \in Q$. Since by the growth condition in **(H1)**, $\sigma(x,t,Du_k)$ is bounded and equiintegrable, we then have $\sigma(x,t,Du_k) \to \sigma(x,t,Du)$ in $L^1(Q)$ by the Vitali convergence theorem. Now, we take a test function $w \in \bigcup_{i\in\mathbb{N}} V_i$ and $\varphi \in \mathcal{C}_0^\infty([0,T])$ in (6.3.2) and integrate on $(0,T)$ and pass to the limit $k \to \infty$. The resulting equation is

$$\int_\Omega u\varphi w dx\Big|_0^T - \int_Q u\frac{\partial\varphi}{\partial t} w(x) dx dt + \int_Q \sigma(x,t,Du) : Dw(x)\varphi(t) dx dt = \int_Q f.\varphi w dx dt,$$

for arbitrary $w \in \bigcup_{i\in\mathbb{N}} V_i$ and $\varphi \in \mathcal{C}_0^\infty([0,T])$. By the density of the linear span of these functions in $W_0^{1,x}L_M(Q;\mathbb{R}^m)$, it follows that $u$ is in fact a weak solution in this case.

**Case (d)**: Suppose that $\nu_{(x,t)}$ is not a Dirac measure, for a.e. $(x,t) \in Q$. We have by the strict $M$-quasimonotone of $\sigma$

$$\begin{aligned}\int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,\lambda) : \lambda d\nu_{(x,t)}(\lambda) dxdt \\ > \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,\lambda) : \overline{\lambda} d\nu_{(x,t)}(\lambda) dxdt \\ + \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,\overline{\lambda}) : (\lambda - \overline{\lambda}) d\nu_{(x,t)}(\lambda) dxdt,\end{aligned}$$

where $\overline{\lambda} = \langle \nu_{(x,t)}, id \rangle = Du(x,t)$. Since

$$\begin{aligned}\int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,\overline{\lambda}) : (\lambda - \overline{\lambda}) d\nu_{(x,t)}(\lambda) dxdt &= \int_Q \sigma(x,t,\overline{\lambda}) : \Big( \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda)}_{=:\overline{\lambda}} \Big) dxdt \\ &\quad - \int_Q \sigma(x,t,\overline{\lambda}) : \overline{\lambda} \Big( \underbrace{\int_{\mathbb{M}^{m\times n}} d\nu_{(x,t)}(\lambda)}_{=:1} \Big) dxdt \\ &= 0,\end{aligned}$$

it follows from Lemma 6.3.2 that

$$\begin{aligned}\int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,\lambda) : Du d\nu_{(x,t)}(\lambda) dxdt &\geq \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,\lambda) : \lambda d\nu_{(x,t)}(\lambda) dxdt \\ &> \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,\lambda) : Du d\nu_{(x,t)}(\lambda) dxdt,\end{aligned}$$

which is a contradiction. Hence $\nu_{(x,t)}$ is a Dirac measure and we can write $\nu_{(x,t)} = \delta_{g(x,t)}$. On the other hand

$$g(x,t) = \int_{\mathbb{M}^{m\times n}} \lambda d\delta_{g(x,t)}(\lambda) = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) = Du(x,t).$$

Consequently $\nu_{(x,t)} = \delta_{Du(x,t)}$. We follow then the proof of case (c).
**Case (a)**: We claim that for almost $(x,t) \in Q$ and all $\xi \in \mathbb{M}^{m\times n}$ the following equation holds on supp $\nu_{(x,t)}$,

$$\sigma(x,t,\lambda) : \xi = \sigma(x,t,Du) : \xi + \big( \nabla \sigma(x,t,Du)\xi \big) : (Du - \lambda).$$

Here $\nabla$ denotes the derivative with respect to the third variable of $\sigma$. By the monotonicity of $\sigma$, for all $\tau \in \mathbb{R}$, we have

$$\begin{aligned}0 &\leq \big(\sigma(x,t,\lambda) - \sigma(x,t,Du+\tau\xi)\big) : (\lambda - Du - \tau\xi) \\ &= \sigma(x,t,\lambda) : (\lambda - Du) - \sigma(x,t,\lambda) : \tau\xi - \sigma(x,t,Du+\tau\xi) : (\lambda - Du - \tau\xi).\end{aligned}$$

This gives by Lemma 6.3.3

$$\begin{aligned}-\sigma(x,t,\lambda) : \tau\xi &\geq -\sigma(x,t,\lambda) : (\lambda - Du) + \sigma(x,t,Du+\tau\xi) : (\lambda - Du - \tau\xi) \\ &= -\sigma(x,t,Du) : (\lambda - Du) + \sigma(x,t,Du+\tau\xi) : (\lambda - Du - \tau\xi).\end{aligned}$$

Since $\sigma(x,t,Du+\tau\xi) = \sigma(x,t,Du) + \nabla\sigma(x,t,Du)\tau\xi + o(\tau)$, then

$$\begin{aligned}&\sigma(x,t,Du+\tau\xi) : (\lambda - Du - \tau\xi) \\ &= \sigma(x,t,Du+\tau\xi) : (\lambda - Du) - \sigma(x,t,Du+\tau\xi) : \tau\xi \\ &= \sigma(x,t,Du) : (\lambda - Du) + \nabla\sigma(x,t,Du)\tau\xi : (\lambda - Du) \\ &\qquad\qquad - \sigma(x,t,Du) : \tau\xi - \nabla\sigma(x,t,Du) : \tau\xi : \tau\xi + o(\tau) \\ &= \sigma(x,t,Du) : (\lambda - Du) + \tau\big[\big(\nabla\sigma(x,t,Du)\xi\big) : (\lambda - Du) - \sigma(x,t,Du) : \xi\big] + o(\tau).\end{aligned}$$

Therefore

$$-\sigma(x,t,\lambda) : \tau\xi \geq \tau\big[\big(\nabla\sigma(x,t,Du)\xi : (\lambda - Du) - \sigma(x,t,Du) : \xi\big] + o(\tau).$$

Since $\tau$ is arbitrary in $\mathbb{R}$, our claim follows. We have $\{\sigma(x,t,Du_k)\}$ is bounded and equiintegrable, then its weak $L^1$-limit is given by

$$\overline{\sigma} := \int_{\operatorname{supp}\nu_{(x,t)}} \sigma(x,t,\lambda)\,d\nu_{(x,t)}(\lambda).$$

According to our claim, we can write

$$\overline{\sigma} = \int_{\operatorname{supp} \nu_{(x,t)}} \sigma(x,t,Du) d\nu_{(x,t)}(\lambda) + \big(\nabla\sigma(x,t,Du)\big)^t \underbrace{\int_{\operatorname{supp} \nu_{(x,t)}} (Du-\lambda) d\nu_{(x,t)}(\lambda)}_{=0}$$

$$= \sigma(x,t,Du).$$

Since $L_{\overline{M}}(Q;\mathbb{M}^{m\times n})$ is reflexive, thus $\sigma(x,t,Du_k)$ is weakly convergent in $L_{\overline{M}}(\Omega;\mathbb{M}^{m\times n})$ and its weak $L_{\overline{M}}$-limit is also $\sigma(x,t,Du)$.
Now, take $\phi \in \mathcal{C}^1(0,T;V_j)$ for $j \leq k$, then

$$\int_Q \frac{\partial u_k}{\partial t}\phi dxdt + \int_Q \sigma(x,t,Du_k) : D\phi dxdt = \int_Q f.\phi dxdt,$$

which gives after integrating the first term

$$\int_\Omega u_k(.,T)\phi(T)dx - \int_Q u_k(.,0)\phi(0)dx - \int_\Omega u_k \frac{\partial \phi}{\partial t}dxdt$$
$$+ \int_Q \sigma(x,t,Du_k) : D\phi dxdt = \int_Q f.\phi dxdt.$$

Letting $j \to \infty$, then for $\phi \in \mathcal{C}^1(0,T;\mathcal{C}_0^\infty(\Omega))$, we obtain when $k \to \infty$

$$-\int_\Omega u\frac{\partial \phi}{\partial t}dxdt + \int_\Omega u(x,t)\phi(x,t)dx\Big|_0^T + \int_Q \sigma(x,t,Du) : D\phi dxdt = \int_Q f.\phi dxdt.$$

**Case (b)**: Let show that for almost every $x \in \Omega$, $\operatorname{supp} \nu_{(x,t)} \subset K_{(x,t)}$, where

$$K_{(x,t)} := \Big\{\lambda \in \mathbb{M}^{m\times n} : W(x,t,\lambda) = W(x,t,Du) + \sigma(x,t,Du) : (\lambda - Du)\Big\}.$$

If $\lambda \in \operatorname{supp} \nu_{(x,t)}$, then by Lemma 6.3.3

$$(1-\tau)\big(\sigma(x,t,Du) - \sigma(x,t,\lambda)\big) : (Du-\lambda) = 0 \quad \text{for all } \tau \in [0,1].$$

By monotonicity of $\sigma$, we can write for $\tau \in [0,1]$

$$(1-\tau)\big(\sigma(x,t,Du+\tau(\lambda-Du)) - \sigma(x,t,\lambda)\big) : (Du-\lambda) \geq 0.$$

Therefore

$$(1-\tau)\big(\sigma(x,t,Du+\tau(\lambda-Du))-\sigma(x,t,Du)\big):(Du-\lambda)\geq 0. \tag{6.3.6}$$

According to the monotonicity condition, we have

$$\big(\sigma(x,t,Du+\tau(\lambda-Du))-\sigma(x,t,Du)\big):\tau(Du-\lambda)\leq 0,$$

and since $\tau\in[0,1]$, it follows that

$$\big(\sigma(x,t,Du+\tau(\lambda-Du))-\sigma(x,t,Du)\big):(1-\tau)(Du-\lambda)\leq 0. \tag{6.3.7}$$

From (6.3.6) and (6.3.7), we then have

$$\big(\sigma(x,t,Du+\tau(\lambda-Du))-\sigma(x,t,Du)\big):(Du-\lambda)=0 \quad \forall\tau\in[0,1].$$

Hence, for $\tau\in[0,1]$

$$\sigma(x,t,Du+\tau(\lambda-Du)):(\lambda-Du)=\sigma(x,t,Du):(\lambda-Du).$$

We integrate the equality

$$\sigma(x,t,Du+\tau(\lambda-Du)):(\lambda-Du)=\frac{\partial W}{\partial\tau}(x,t,Du+\tau(\lambda-Du)):(\lambda-Du)$$

over $[0,1]$, we obtain

$$\begin{aligned}W(x,t,\lambda)&=W(x,t,Du)+\int_0^1\sigma(x,t,Du+\tau(\lambda-Du)):(\lambda-Du)d\tau\\&=W(x,t,Du)+\sigma(x,t,Du):(\lambda-Du).\end{aligned}$$

Hence, $\lambda\in K_{(x,t)}$, i.e., $\operatorname{supp}\nu_{(x,t)}\subset K_{(x,t)}$. By the convexity of $W$, we can write

$$\underbrace{W(x,t,\lambda)}_{:=A(\lambda)}\geq\underbrace{W(x,t,Du)+\sigma(x,t,Du):(\lambda-Du)}_{:=B(\lambda)}.$$

Since $\lambda \mapsto A(\lambda)$ is continuous and differentiable, then for $\xi \in \mathbb{M}^{m\times n}$ and $\tau \in \mathbb{R}$

$$\frac{A(\lambda+\tau\xi)-A(\lambda)}{\tau} \geq \frac{B(\lambda+\tau\xi)-B(\lambda)}{\tau} \quad \text{if } \tau > 0,$$

$$\frac{A(\lambda+\tau\xi)-A(\lambda)}{\tau} \leq \frac{B(\lambda+\tau\xi)-B(\lambda)}{\tau} \quad \text{if } \tau < 0.$$

Hence $D_\lambda A = D_\lambda B$ and then we have

$$\sigma(x,t,\lambda) = \sigma(x,t,Du) \quad \text{on supp } \nu_{(x,t)} \subset K_{(x,t)}. \tag{6.3.8}$$

Let consider the Carathéodory function $g(x,t,\lambda) := |\sigma(x,t,\lambda) - \overline{\sigma}(x,t)|$. Since $\sigma(x,t,Du_k)$ is weakly convergent in $L_{\overline{M}}(Q;\mathbb{M}^{m\times n})$, then $\sigma(x,t,Du_k)$ is equiintegrable. Therefore, $g_k(x,t) \equiv g(x,t,Du_k)$ is equiintegrable and

$$g_k \rightharpoonup \overline{g} \quad \text{in } L^1(Q),$$

where

$$\begin{aligned}
\overline{g}(x,t) &= \int_{\mathbb{M}^{m\times n}} \big|\sigma(x,t,\lambda) - \overline{\sigma}(x,t)\big| d\nu_{(x,t)}(\lambda) \\
&= \int_{\text{supp } \nu_{(x,t)}} \big|\sigma(x,t,\lambda) - \overline{\sigma}(x,t)\big| d\nu_{(x,t)}(\lambda) \\
&\overset{(6.3.8)}{=} \int_{\text{supp } \nu_{(x,t)}} \big|\sigma(x,t,\lambda) - \sigma(x,t,Du)\big| d\nu_{(x,t)}(\lambda) = 0.
\end{aligned}$$

Since $g_k \geq 0$, we can obtain

$$g_k \to 0 \quad \text{in} \quad L^1(Q).$$

The remainder of the proof is similar to that of case (a).

In order to complete the proof of Theorem 6.2.1, we prove the uniqueness of the weak solution $u \in W_0^{1,x}L_M(Q;\mathbb{R}^m)$. To do this, we consider two solutions $u$ and $v$ of the problem (6.1.1)-(6.1.3) and using $u - v$ as a test function in both equations corresponding to $u$ and $v$, we get

$$\frac{1}{2}\int_\Omega \big(u(t)-v(t)\big)^2 dx\Big|_0^T + \int_Q \big(\sigma(x,t,Du) - \sigma(x,t,Dv)\big) : (Du - Dv)dxdt = 0.$$

By the strict monotonicity of $\sigma$, we obtain $Du = Dv$. Using the fact that $u(0) = v(0) = u_0$, we can deduce that $u(t) = v(t)$ for almost every $t \in (0, T)$. Hence $u = v$ and the proof of Theorem 6.2.1 is complete.

## 6.4 Extension result

In this part, we study the solvability of the initial-boundary value problem (6.1.1)-(6.1.3) when the source term $f$ belongs to dual of $X(Q)$ (see below for the definition). The existence result is proved under nonstandard conditions (**(H0)-(H2)** above) and the theory of Young measures.

Bögelein et al. [38] introduced the concept of variational solutions for the system

$$\partial_t u - \operatorname{div} Df(Du) = 0,$$

under non-standard $p, q$-growth assumption on the convex integarnd $f : \mathbb{M}^{m\times n} \to [0, \infty)$. The same authors [39] considered (6.1.1) with $f = 0$ and $m = 1$, and proved that, when $2 \leq p \leq q < p + \frac{4}{n}$ then any weak solution admits a locally bounded spatial gradient $Du$. Note that the general existence and uniqueness proved in these two articles under general growth and coercivity conditions do not contain the existence and uniqueness result of Theorem 6.4.1 (see bellow), because the nonlinear operator under consideration in Theorem 6.4.1 explicitly depends on $t$ too. We refer the reader to see [110] for general parabolic equations under general $p, q$-growth conditions.

When the vector field $\sigma(x, t, Du)$ has nonstandard $p(x)$-structure, using again the theory of Young measures, the authors in [134] studied the problem (6.1.1) in the case where the source term is in divergence form

$$\frac{\partial u}{\partial t} - \operatorname{div} \sigma(x, t, Du) = -\operatorname{div} f \quad \text{in } Q = \Omega \times (0, T),$$

where $f \in L^{p'(x)}(Q; \mathbb{M}^{m\times n})$ and $u_0 \in L^2(\Omega; \mathbb{R}^m)$. And generalize the previous work of Hüngerbuhler [86].

Inspired by the works mentioned above (especially [86] and [134]), results were extended in our recent paper [19] (i.e., first part of this chapter) to the case that $\sigma$ satisfy nonstandard conditions related to N-functions which define the functional space where we have treated the problem (6.1.1). See also [20, 21] for the steady case in Orlicz-Sobolev spaces and by the same theory of Young measures. For more results and related topics, one can see [56, 66, 79, 114, 138].

Now, the aim of this part, is to study the quasilinear parabolic initial-boudary value problem (6.1.1)-(6.1.3), where $u_0 \in L^2(\Omega; \mathbb{R}^m)$, $f \in X'(Q)$ and $\sigma$ satisfy the conditions **(H0)-(H3)** above. The main difficulty that one can find in dealing with such problems consists in defining the appropriate space in which the problem will be studied. To circumvent this problem we will define a suitable energy space.

### 6.4.1 The energy space and main result

Let $\Omega$ be a bounded open domain in $\mathbb{R}^n$ and $0 < T < +\infty$. For any $\tau \in (0, T]$, denote $Q_\tau = \Omega \times (0, \tau)$ and $Q = Q_T$. Throughout this part, $M$ and its conjugate $\overline{M}$ are satisfuing the $\Delta_2$-condition. We consider the following functional space

$$X(Q) := \Big\{ u \in L^2(Q; \mathbb{R}^m) / \, Du \in L_M(Q; \mathbb{M}^{m\times n}); \\ u(t) := u(.,t) \in W_0^1 L_M(\Omega; \mathbb{R}^m) \text{ a.e. } t \in [0, T] \Big\}.$$

Endowed with the norm

$$\|u\|_{X(Q)} = \|u\|_{L^2(Q;\mathbb{R}^m)} + \|Du\|_{L_M(Q;\mathbb{M}^{m\times n})} := \|u\|_{L^2(Q)} + \|Du\|_M,$$

$X(Q)$ is a Banach space. Moreover, $X(Q)$ is reflexive and separable. Indeed, let define the mapping $\Theta : X(Q) \to L^2(Q; \mathbb{R}^m) \times L_M(Q; \mathbb{M}^{m\times n})$ by $\Theta(u) = (u, Du)$ for every $u \in X(Q)$. It is obvious that $X(Q)$ is isometrically isomorphic to the closed subspace $\Theta(X(Q))$ of $L^2(Q; \mathbb{R}^m) \times L_M(Q; \mathbb{M}^{m\times n})$. As in [58, Theorem 4.6], we have that $C_0^\infty(Q; \mathbb{R}^m)$ is dense in $X(Q)$.

Throughout this part, $\langle ., . \rangle$ denotes the duality pairing between $X(Q)$ and its dual $X'(Q)$, and by virtue of the Hahn-Banach theorem, the elements of $X'(Q)$ can be represented as follows: If $f \in X'(Q)$ then there exist $f_0 \in L^2(Q; \mathbb{R}^m)$ and $F \in$

$L_{\overline{M}}(Q;\mathbb{M}^{m\times n})$ such that $f=f_0-\operatorname{div}F$ and

$$\langle f,\varphi\rangle_{X',X}=\int_Q f_0\varphi dxdt+\int_Q F:D\varphi dxdt$$

for any $\varphi\in X(Q)$.

**Definition 6.4.1.** *A function $u\in X(Q)\cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$ is a weak solution of problem (6.1.1)-(6.1.3) if*

$$-\int_Q u\frac{\partial\varphi}{\partial t}dxdt+\int_\Omega u\varphi dx\Big|_0^T+\int_Q\sigma(x,t,Du):D\varphi dxdt=\langle f,\varphi\rangle$$

*holds for all $\varphi\in C^1(0,T;C_0^\infty(\Omega;\mathbb{R}^m))$.*

Now we state our main result.

**Theorem 6.4.1.** *Let $f\in X'(Q)$ and $u_0\in L^2(\Omega;\mathbb{R}^m)$. Assume that **(H0)-(H2)** hold true. Then there exists a unique weak solution $u\in X(Q)\cap C(0,T;L^2(\Omega;\mathbb{R}^m))$ of the problem (6.1.1)-(6.1.3) in the sense of Definition 6.4.1.*

### 6.4.2 Galerkin approximations

We choose a sequence of functions $\{w_i\}_{i\geq1}\subset C_0^\infty(\Omega;\mathbb{R}^m)$ orthonormal with respect to $L^2(\Omega;\mathbb{R}^m)$ such that $\underset{j\geq1}{\cup}V_j$, where we denote $V_j=\operatorname{span}\{w_1,...,w_j\}$, is dense in $H_0^s(\Omega)$ with $s$ large enough, such as $s>\frac{n}{2}+1$, so that $H_0^s(\Omega)$ is continuously embedded in $C^1(\overline{\Omega})$ (see [5]). Define $W_j=C^1(0,T;V_j)$. Thus, we have $C_0^\infty(Q;\mathbb{R}^m)\subset\overline{\underset{j\geq1}{\cup}W_j}^{C^1(Q;\mathbb{R}^m)}$. Then, for any $f\in X'(Q)$, there exists a sequence $(f_j)\subset\underset{j\geq1}{\cup}W_j$ such that $f_j\to f$ strongly in $X'(Q)$. We also note that there exists a sequence $u_0^j\subset\underset{j\geq1}{\cup}V_j$ such that $u_0^j\to u_0$ in $L^2(\Omega;\mathbb{R}^m)$.

**Definition 6.4.2.** *A function $u_k\in W_k$ is called Galerkin solution of (6.1.1)-(6.1.3) if and only if*

$$\int_\Omega\frac{\partial u_k}{\partial t}vdx+\int_\Omega\sigma(x,t,Du_k):Dvdx=\int_\Omega f_k(t)vdx \tag{6.4.1}$$

*for all $v\in V_k$ and all $t\in[0,T]$ with $u_k(x,0)=u_0^k(x)$.*

Considering $u_k(x,t) = \sum_{i=1}^k d_i(t)w_i(x)$, we then try to look for the coefficients $d_i \in C^1([0,T])$. To this purpose, we define the function $y_k : [0,T] \times \mathbb{R}^k \to \mathbb{R}^k$ for $d = (d_1, ..., d_k)$ by

$$\big(y_k(t,d)\big)_i = \int_\Omega \sigma\big(x,t,\sum_{j=1}^k d_j(t)Dw_j(x)\big) : Dw_i(x)dx,$$

for $i = 1, ..., k$. Since $\sigma$ is a Carathéodory function then $y_k(t,d)$ is continuous. Consequently, we get the following system of ordinary differential equations

$$\begin{cases} d' + y_k(t,d) & = F_k, \\ d(0) & = v_k, \end{cases}$$

where $\big(F_k(t)\big)_i = \int_\Omega f_k(t)w_i dx$ and $(v_k)_i = \int_\Omega u_0^k w_i dx$, $i = 1, ..., k$. We multiply the first equation by $d(t)$, using the fact that by $L_M \subset L^1$ one has $y_k(t,d)d \geq 0$, we apply the Young inequality, it result that

$$\frac{1}{2}\frac{d}{dt}|d(t)|^2 \leq |F_k(t)||d(t)| \leq \frac{1}{2}|F_k(t)|^2 + \frac{1}{2}|d(t)|^2.$$

Using Gronwall's lemma one has

$$|d(t)| \leq C_k(T).$$

Therefore, $|d(t) - d(0)| \leq 2C_k(T)$. Now, let $\theta_k = \max_{t\in[0,T]} |F_k - y_k(t,d(t))|$ and $q = \min\big\{T, \frac{2C_k(T)}{\theta_k}\big\}$. By virtue of the Cauchy-Peano theorem (cf. [9]) we obtain a local solution in $[0,q]$. Starting with the initial value $q$, we obtain a local solution in $[q,2q]$ and so on we get a local solution $d_k$ in $C^1([0,T])$. Consequently, by construction, we know that the function $u_k(x,t) = \sum_{i=1}^k d_{k,i}(t)w_i(x)$, which belongs to $W_k$, is a Galerkin solution of (6.1.1)-(6.1.3) satisfying

$$\int_{Q_\tau} \frac{\partial u_k}{\partial t} v dxdt + \int_{Q_\tau} \sigma(x,t,Du_k) : Dv dxdt = \int_{Q_\tau} f_k v dxdt, \tag{6.4.2}$$

for all $v \in W_k$ and all $\tau \in (0,T]$ with $u_k(x,0) = u_0^k(x)$. In the rest of this part, $c$ will denote a positive constant which may change values from line to line and which depends on the parameters of our problem. Taking $v = u_k$ in (6.4.2), it result after integrating the first term

$$\frac{1}{2}\int_{\Omega}|u_k(\tau)|^2dx + \int_{Q_\tau}\sigma(x,t,Du_k) : Du_k dxdt = \int_{Q_\tau} f_k u_k dxdt + \frac{1}{2}\int_{\Omega}|u_0^k(x)|^2dx, \quad (6.4.3)$$

for every $\tau \in (0,T]$. By the coercivity condition of $\sigma$, one has

$$\frac{1}{2}\|u_k(\tau)\|^2_{L^2(\Omega)} + \int_{Q_\tau} M\Big(\frac{|Du_k|}{\beta}\Big)dxdt \leq \|f_k\|_{X'}\|u_k\|_X + c, \quad (6.4.4)$$

where we have used the fact that $\{u_0^k\}$ is bounded in $L^2(\Omega;\mathbb{R}^m)$ by a constant $c$ which not depend on $k$. By definition of $\|.\|_X$, (6.4.4) implies

$$\frac{1}{2}\|u_k(\tau)\|^2_{L^2(\Omega)} + \int_{Q_\tau} M\Big(\frac{|Du_k|}{\beta}\Big)dxdt \leq \|f_k\|_{X'}\|u_k\|_{L^2(Q_\tau)} + \|f_k\|_{X'}\|Du_k\|_M + c. \quad (6.4.5)$$

If $\|Du_k\|_M$ is unbounded, then $\int_{Q_\tau} M(\frac{|Du_k|}{\beta})dxdt$ is unbounded. This is a contradiction with (6.4.5). Hence

$$\|Du_k\|_M \leq c.$$

Moreover

$$\|u_k(.,\tau)\|^2_{L^2(\Omega)} \leq c.$$

Therefore, there exists a constant $c > 0$, not depending on $k$, such that

$$\|u_k\|_X \leq c. \quad (6.4.6)$$

We also have $\|u\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))} \leq c$. Owing to (6.4.3), we get

$$\int_{Q_\tau}\sigma(x,t,Du_k) : Du_k dxdt \leq c.$$

Since $d_1(x,t) \in E_{\overline{M}}(Q)$ and $\|Du_k\|_M \le c$, it follows by the growth condition in **(H1)** that

$$\begin{aligned}\int_{Q_\tau} \overline{M}\Big(\frac{1}{2\theta_0}|\sigma(x,t,Du_k)|\Big)dxdt &\le \int_{Q_\tau} \overline{M}\Big(\frac{\beta_1}{2\theta_0\beta_1}d_1(x,t) + \frac{1}{2\theta_0}\overline{M}^{-1}M(\gamma|Du_k|)\Big)dx \\ &\le c\int_{Q_\tau} \Big(\overline{M}\big(\beta_1 d_1(x,t)\big) + M(\gamma|Du_k|)\Big)dxdt \\ &\le c,\end{aligned}$$

where $\beta_1$ is such that $\rho_{\overline{M}}(\beta_1 d_1) < \infty$ and $\theta_0 = \max\{1, \frac{1}{\beta_1}\}$. It follows that $\|\sigma(x,t,Du_k)\|_{\overline{M}} \le c$. Therefore, for a subsequence still indexed by $k$ and for some $u \in X$ and $\chi \in L_{\overline{M}}(Q;\mathbb{M}^{m\times n})$

$$\begin{aligned} u_k &\rightharpoonup u \text{ weakly in } X \text{ and weakly in } L^2(Q;\mathbb{R}^m), \\ \sigma(x,t,Du_k) &\rightharpoonup \chi \text{ weakly in } L_{\overline{M}}(Q;\mathbb{M}^{m\times n}). \end{aligned} \tag{6.4.7}$$

Now, we prove that $u_k(.,T) \rightharpoonup u(.,T)$ in $L^2(\Omega)$ and $u(.,0) = u_0(.)$. Take $\tau = T$, we have $\|u_k(.,T)\|^2_{L^2(\Omega)} \le c$, then $\{u_k\}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$. Hence, for a subsequence indexed $k$,

$$u_k(.,T) \rightharpoonup u^* \text{ in } L^2(\Omega;\mathbb{R}^m) \text{ as } k \to \infty.$$

For simplicity, we denote $u(.,T)$ as $u(T)$ and $u(.,0)$ as $u(0)$. Taking $\psi \in C^\infty([0,T])$, $v \in V_j$, $j \le k$, then we have

$$\int_Q \frac{\partial u_k}{\partial t} v\psi dxdt + \int_Q \sigma(x,t,Du_k) : Dv\psi dxdt = \int_Q f_k v\psi dxdt.$$

By integrating the first integral, it result that

$$\begin{aligned}\int_\Omega u_k(T)\psi(T)vdx - \int_\Omega u_k(0)\psi(0)vdx + \int_Q \sigma(x,t,Du_k) : Dv\psi dxdt \\ = \int_Q f_k v\psi dxdt + \int_Q u_k v\psi' dxdt.\end{aligned}$$

We can pass to the limit in the previous equality as $k$ tends to $\infty$ obtaining

$$\int_\Omega u^*\psi(T)vdx - \int_\Omega u_0\psi(0)vdx + \int_Q \chi : Dv\psi dxdt = \int_Q fv\psi dxdt + \int_Q uv\psi' dxdt. \quad (6.4.8)$$

Let $\psi(T) = \psi(0) = 0$, then

$$\int_Q \chi : Dv\psi dxdt - \int_Q fv\psi dxdt = \int_Q uv\psi' dxdt = -\int_Q u'v\psi dxdt.$$

Back to (6.4.8), it becomes then

$$\begin{aligned}\int_\Omega u^*\psi(T)vdx - \int_\Omega u_0\psi(0)vdx &= \int_Q u'v\psi dxdt + \int_Q uv\psi' dxdt \\ &= \int_\Omega u(T)\psi(T)vdx - \int_\Omega u(0)\psi(0)vdx.\end{aligned}$$

Tending $j$ to $\infty$, if we take $\psi(T) = 0$ and $\psi(0) = 1$, then we obtain $u(0) = u_0$, if we take $\psi(T) = 1$ and $\psi(0) = 0$, then $u(T) = u^*$ as desired. Note that, in previous calculations, $f = f_0 - \mathrm{div} F \in X'$.

### 6.4.3 Div-curl inequality

As stated above, $\{Du_k\}_k$ is bounded in $L_M(Q; \mathbb{M}^{m\times n})$. By virtue of Lemma 2.3.1, there exists a Young measure $\nu_{(x,t)}$ generated by $Du_k$ and satisfying the properties (i)-(ii) of Lemma 6.3.1.

The crucial point in the proof of Theorem 6.4.1 is the following lemma which permits the passage to the limit in the approximating equations.

**Lemma 6.4.1.** *If $\sigma$ satisfies* **(H0)-(H2)**, *then the Young measure $\nu_{(x,t)}$ associated to $Du_k$ satisfy*

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,\lambda) - \sigma(x,t,Du)\big) : (\lambda - Du) d\nu_{(x,t)}(\lambda) dxdt \leq 0.$$

*Proof.* Consider $I_k$ as in the proof of Lemma 6.3.2, we then have

$$\begin{aligned}I := \liminf_{k\to\infty}\int_Q I_k dxdt &= \liminf_{k\to\infty}\int_Q I_{k,1} dxdt\\ &\geq \int_Q\int_{\mathbb{M}^{m\times n}} \sigma(x,t,\lambda) : (\lambda - Du)d\nu_{(x,t)}(\lambda)dxdt.\end{aligned}$$

It is sufficient to show that $I \leq 0$. We have

$$\int_Q \frac{\partial u_k}{\partial t}u_k dxdt + \int_Q \sigma(x,t,Du_k) : Du_k dxdt = \int_Q f_k u_k dxdt,$$

then

$$\begin{aligned}I &= \liminf_{k\to\infty}\int_Q \sigma(x,t,Du_k) : (Du_k - Du)dxdt\\ &= \liminf_{k\to\infty}\Big(\int_Q \sigma(x,t,Du_k) : Du_k dxdt - \int_Q \sigma(x,t,Du_k) : Dudxdt\Big) \qquad (6.4.9)\\ &= \liminf_{k\to\infty}\Big(\int_Q f_k u_k dxdt - \int_Q \frac{\partial u_k}{\partial t}u_k dxdt - \int_Q \sigma(x,t,Du_k) : Dudxdt\Big).\end{aligned}$$

Observe that

$$\int_Q f_k u_k dxdt - \int_Q fudxdt = \int_Q (f_k - f)u_k dxdt + \int_Q f(u_k - u)dxdt.$$

Since $\|u_k\|_X \leq c$, $f_k \to f$ in $X'$ and by the the first property in the Eq. (6.4.7), we get

$$\int_Q f_k u_k dxdt - \int_Q fudxdt \to 0 \quad \text{as } k \to \infty.$$

Since $u_k(.,0) \to u_0(.) = u(.,0)$ and $u_k(.,T) \rightharpoonup u(.,T)$ in $L^2(\Omega;\mathbb{R}^m)$, then

$$\|u_k(.,0)\|_{L^2(\Omega)} \to \|u(.,0)\|_{L^2(\Omega)} \quad \text{and} \quad \|u(.,T)\|_{L^2(\Omega)} \leq \liminf_{k\to\infty}\|u_k(.,T)\|_{L^2(\Omega)}.$$

Hence

$$\liminf_{k\to\infty}\Big(-\int_Q u_k\frac{\partial u_k}{\partial t}dxdt\Big)=\liminf_{k\to\infty}\Big(\frac{1}{2}\|u_k(.,0)\|^2_{L^2(\Omega)}-\frac{1}{2}\|u_k(.,T)\|^2_{L^2(\Omega)}\Big)$$
$$\leq\frac{1}{2}\|u(.,0)\|^2_{L^2(\Omega)}-\frac{1}{2}\|u(.,T)\|^2_{L^2(\Omega)}.$$

For the last term on the right hand side of the Eq. (6.4.9), we take $\psi\in C^1(0,T;V_j)$, $j\leq k$,

$$\int_Q\frac{\partial u_k}{\partial t}\psi dxdt+\int_Q\sigma(x,t,Du_k):D\psi dxdt=\int_Q f_k\psi dxdt.$$

Integrating the first integral, we get in passing to the limit as $k$ tends to $\infty$

$$\int_\Omega u(.,T)\psi(T)dx-\int_\Omega u(.,0)\psi(0)dx-\int_Q u\frac{\partial\psi}{\partial t}dxdt$$
$$+\int_Q\chi:D\psi dxdt=\int_Q f\psi dxdt.$$

When $j\to\infty$, the above equality holds true for all $\psi\in C^1(0,T;C^1(\overline{\Omega}))$. Hence

$$-\int_Q u\frac{\partial\psi}{\partial t}dxdt=-\int_Q\chi:D\psi dxdt+\int_Q f\psi dxdt$$

for all $\psi\in C_0^\infty(Q;\mathbb{R}^m)$ and $f\in X'$. Consequently

$$\frac{\partial u}{\partial t}=\operatorname{div}\chi+f.$$

Therefore, for $u\in X$

$$\int_Q u\frac{\partial u}{\partial t}dxdt=-\int_Q\chi:Dudxdt+\int_Q fudxdt.$$

Gathering the above results, we conclude that $I\leq 0$. □

Now, we can pass to the limit in the approximating equations. To conclude the proof of Theorem 6.4.1 in cases **(H2)**(a)-(d), it is sufficient to use similar arguments as in the first part of this chapter. Taking $\psi \in C^1(0,T;V_j)$, $j \leq k$

$$\int_Q \frac{\partial u_k}{\partial t}\psi dxdt + \int_Q \sigma(x,t,Du_k) : D\psi dxdt = \int_Q f_k \psi dxdt.$$

By letting $j \to \infty$ and integrating the first integral, it follows for $\psi \in C^1(0,T;C_0^\infty(\Omega))$ that

$$-\int_Q u\frac{\partial \psi}{\partial t}dxdt + \int_\Omega u\psi dx\Big|_0^T + \int_Q \sigma(x,t,Du) : D\psi dxdt = \int_Q f\psi dxdt$$

as $k \to \infty$. To conclude the proof of the main result, it remains to show the uniqueness of solution. To this purpose, we consider two solutions $u$ and $v$ in $X$ of the problem (6.1.1)-(6.1.3). By using $u - v$ as a test function in both equations corresponding to $u$ and $v$, it result that

$$\frac{1}{2}\int_\Omega \big(u(t)-v(t)\big)^2 dx\Big|_0^T + \int_Q \big(\sigma(x,t,Du) - \sigma(x,t,Dv)\big) : (Du - Dv)dxdt = 0.$$

Owing to the strict monotonicity of $\sigma$, one can get $Du = Dv$. Since $u, v \in X$, we have $u(0) = v(0)$. Therefore $u(t) = v(t)$ for almost every $t \in [0,T]$. Consequently $u = v$ and the proof of Theorem 6.4.1 is complete.

**Remark 6.4.1.** *Instead of the condition* ***(H2)*** *we assume the following (cf. [31]):*
***(H2′)*** *$\sigma$ is strictly quasimonotone, i.e., there exist a constant $\alpha_0 > 0$ and $\gamma > 0$ such that*

$$\int_Q \big(\sigma(x,t,Du) - \sigma(x,t,Dv)\big) : (Du - Dv)dxdt \geq \alpha_0 \int_Q M(\gamma|Du - Dv|)dxdt$$

*for all $u, v \in W_0^{1,x}L_M(Q;\mathbb{R}^m)$.*

Let us make some light on this condition and its relation to the previous condition **(H2)(d)**. Indeed, let $\eta : \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ be a function satisfying the growth condition

$$|\eta(A)| \leq \overline{M}^{-1}M(\gamma|A|) \tag{6.4.10}$$

and the structure condition

$$\int_Q \big(\eta(A + D\varphi) - \eta(A)\big) : D\varphi dxdt \geq \alpha_1 \int_Q M(\gamma|D\varphi|)dxdt$$

for a constant $\alpha_1 > 0$ and for all $\varphi \in C_0^\infty(Q)$ and all $A \in \mathbb{M}^{m\times n}$. Note that the above structure condition was investigated by Zhang [140]. We know that for every $W^{1,x}L_M$ gradient Young measure $\nu$ there exists a sequence $(D\varphi_k)$ generating $\nu$ for which $\{M(\gamma|D\varphi_k|)\}$ is equiintegrable (see Lemma 6.3.1). Hence, there holds

$$\int_Q \big(\eta(A+D\varphi_k)-\eta(A)\big) : D\varphi_k dxdt \geq \alpha_1 \int_Q M(\gamma|D\varphi_k|)dxdt.$$

By the Hölder inequality and Eq. (6.4.10), it follows that $\{\big(\eta(A+D\varphi_k)-\eta(A)\big) : D\varphi_k\}$ is equiintegrable. According to the Fundamental Theorem on Young measures (cf. Theorem 2.3.1), we get

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\eta(A+\lambda)-\eta(\lambda)\big) : \lambda d\nu(\lambda)dxdt \geq \alpha_1 \int_Q \int_{\mathbb{M}^{m\times n}} M(\gamma|\lambda|)d\nu(\lambda)dxdt.$$

We choose first $A+\lambda=\overline{\lambda}$ and then $A=\lambda$, we conclude since $L_M \subset L^1$ that

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\eta(\overline{\lambda})-\eta(\lambda)\big) : (\overline{\lambda}-\lambda)d\nu(\lambda) \geq \alpha_1 \int_Q \int_{\mathbb{M}^{m\times n}} M(\gamma|\overline{\lambda}-\lambda|)d\nu(\lambda) > 0$$

which is exactly the condition **(H2)(d)**. Thus **(H2′)** implies **(H2)(d)**.

If one considers only this condition, the div-curl inequality is not necessary in the proof. In fact, let $\sigma$ satisfy **(H0)**, **(H1)** and **(H2′)**. Consdier the sequence

$$\begin{aligned} I_k &:= \big(\sigma(x,t,Du_k)-\sigma(x,t,Du)\big) : (Du_k - Du) \\ &= \sigma(x,t,Du_k) : (Du_k-Du) - \sigma(x,t,Du) : (Du_k-Du) \\ &=: I_{k,1} + I_{k,2}, \end{aligned}$$

for arbitrary $u \in W_0^{1,x}L_M(Q;\mathbb{R}^m)$. According to the growth condition in **(H1)**,

$$\int_Q \overline{M}(|\sigma(x,t,Du)|)dxdt \leq c\int_Q \big(\overline{M}(d_1(x,t))+M(\gamma|Du|)\big)dxdt < \infty$$

since $d_1 \in E_{\overline{M}}(Q)$. Hence $\sigma(.) \in L_{\overline{M}}(Q;\mathbb{M}^{m\times n})$. Note that, since $(u_k)$ is bounded in $W_0^{1,x}L_M(Q;\mathbb{R}^m) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, it follows by Lemma 2.3.1 the existence of a Young measure $\nu_{(x,t)}$ generated by $Du_k$ in $L_M(Q;\mathbb{M}^{m\times n})$ which satisfies the properties of Lemma 6.3.1. By virtue of the weak limit in Lemma 6.3.1, it follows that

$$\begin{aligned}\liminf_{k\to\infty}\int_Q I_{k,2}dxdt &= \int_Q\int_{\mathbb{M}^{m\times n}}\sigma(x,t,Du):(\lambda-Du)d\nu_{(x,t)}(\lambda)dxdt\\ &= \int_Q\sigma(x,t,Du):\Big(\underbrace{\int_{\mathbb{M}^{m\times n}}\lambda d\nu_{(x,t)}(\lambda)}_{=:Du(x,t)}-Du\Big)dxdt=0.\end{aligned}$$

As above, the sequence $\{-\operatorname{div}\sigma(x,t,Du_k)\}$ is bounded in $W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$. Hence

$$-\operatorname{div}\sigma(x,t,Du_k)\rightharpoonup\chi\quad\text{in } W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$$

for $\chi\in W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$. Hence the first property of $\chi$ is the following energy equality:

$$\frac{1}{2}\|u(.,T)\|^2_{L^2}+\int_Q\chi.udxdt=\frac{1}{2}\|u(.,0)\|^2_{L^2}+\langle f,u\rangle. \tag{6.4.11}$$

It follows that

$$\begin{aligned}\liminf_{k\to\infty}-\int_Q\sigma(x,t,Du_k):Dudxdt &= -\int_Q\chi.udxdt\\ &\overset{(6.4.11)}{=}\frac{1}{2}\|u(.,T)\|^2_{L^2}-\frac{1}{2}\|u(.,0)\|^2_{L^2}-\langle f,u\rangle.\end{aligned} \tag{6.4.12}$$

By virtue of the Galerkin equations, we can write

$$\begin{aligned}\int_Q\sigma(x,t,Du_k):Du_kdxdt &= \langle f,u_k\rangle-\int_Q u_k\frac{\partial u_k}{\partial t}dxdt\\ &= \langle f,u_k\rangle-\frac{1}{2}\|u_k(.,T)\|^2_{L^2}+\frac{1}{2}\|u_k(.,0)\|^2_{L^2}.\end{aligned}$$

Applying the weak limit defined at the end of Subsec. 6.4.2, it follows that

$$\liminf_{k\to\infty}\int_Q\sigma(x,t,Du_k):Du_kdxdt\le\langle f,u\rangle-\frac{1}{2}\|u(.,T)\|^2_{L^2}+\frac{1}{2}\|u(.,0)\|^2_{L^2}. \tag{6.4.13}$$

By combination of (6.4.12) and (6.4.13), we deduce that

$$\liminf_{k\to\infty}\int_Q\sigma(x,t,Du_k):(Du_k-Du)dxdt\le 0.$$

Since
$$\liminf_{k\to\infty}\int_Q \sigma(x,t,Du) : (Du_k - Du)dxdt = 0,$$
we finally get
$$\liminf_{k\to\infty}\int_Q \big(\sigma(x,t,Du_k) - \sigma(x,t,Du)\big) : (Du_k - Du)dxdt \leq 0.$$
Using this inequality and the strict quasimonotone of $\sigma$, it follows that
$$\lim_{k\to\infty}\int_Q M(\gamma|Du_k - Du|)dxdt = 0.$$
Let $E_{k,\epsilon} = \{(x,t) \in Q : |Du_k - Du| \geq \epsilon\}$. Thus
$$\begin{aligned}\int_Q M(\gamma|Du_k - Du|)dxdt &\geq \int_{E_{k,\epsilon}} M(\gamma|Du_k - Du|)dxdt \\ &\geq c\int_{E_{k,\epsilon}} |Du_k - Du|dxdt \\ &\geq c\epsilon|E_{k,\epsilon}|\end{aligned}$$
where $c$ is the constant of the embedding $L_M \subset L^1$. Therefore
$$|E_{k,\epsilon}| \leq \frac{1}{c\epsilon}\int_Q M(\gamma|Du_k - Du|)dxdt \to 0 \quad \text{as } k \to \infty.$$
Hence $Du_k \to Du$ in measure and almost everywhere (up to a subsequence).

# Chapter 7

# Generalized Navier-Stokes system

## 7.1 Introduction On Stokes system

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ ($n \geq 2$). This chapter is devoted first to the existence of weak solutions to the following steady quasi-Newtonien viscous fluid:

$$-\operatorname{div} \sigma(x, u, Du) + u.\nabla u + \nabla \pi = f \quad \text{in} \quad \Omega, \tag{7.1.1}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega, \tag{7.1.2}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{7.1.3}$$

where $u : \Omega \to \mathbb{R}^m$ is the velocity field, $\pi : \Omega \to \mathbb{R}$ the pressure, $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is the Cauchy stress tensor. Second, we will consider the evolutionary case of (7.1.1)-(7.1.3).

Consider first the case when the convective term $u.\nabla u$ is assumed to be small, thus neglected, and $\sigma$ have polynomial growth/coercivity condition with respect to $u$ and $Du$, and with weak monotonicity, it is well known that the problem

$$\begin{cases} -\operatorname{div} \sigma(x, u, Du) &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{7.1.4}$$

was solved by Hungerbühler in [85] for $f \in W^{-1,p'}(\Omega; \mathbb{R}^m)$ ($p' = p/(p-1)$), and

Dolzmann [59] has established the existence result for $f = \mu$ a measure valued function, and the weak derivative $Du$ is replaced by the approximately differentiable $apDu$. We have proved [27] (cf. Section 3.2) existence result for (7.1.4) by using different notions of monotonicity for $\sigma(.) = a(x, Du) + \phi(u)$. In the same setting and with the additional convective term $u.\nabla u$, Arada and Sequeira [12] have proved the existence of weak solutions using polynomial growth and coercivity conditions, they also used weak monotonicity assumptions on the stress tensor $\sigma$.

In [20] (cf. Sect. 6.1), we have investigated the existence of solutions for the quasilinear elliptic system (7.1.4) in the setting of Orlicz-Sobolev spaces. We have used Young measure and mild monotonicity assumptions on $\sigma$. See also [19, 23, 24] for the unsteady case.

The first mathematical investigations on this class of systems (7.1.1)-(7.1.3) were performed by O.A. Ladyzhenskaya [95, 96] and by J.L. Lions [105]. They both considered the unsteady case and showed the existence of a weak solution whenever the coercivity parameter $p$ of the nonlinear elliptic operator related to the stress tensor satisfies $p \geq \frac{3n+2}{n+2}$.

When $\sigma(x, u, Du) = T(x, Du)$ in (7.1.1)-(7.1.3), Gwiazda et al. [78] have shown the existence result in the setting of Musielak-Orlicz when the source term $f$ is equal to div $F$, with $F \in \mathbb{M}^{n\times n}$ and $F \in L_{\overline{M}}(\Omega)$. The authors used the concept of Young measure to define the weak solution and they restricted the N-function to satisfy the following condition: $M(x, F) \geq c|F|^q$ for $F \in \mathbb{M}^{n\times n}$, $c > 0$ and $q \geq \frac{3n}{n+2}$. They also proved that the mapping $T$ belongs to some class of monotone operators, namely the class $(\mathcal{S}_m)$. In [133], the author has established the existence of weak solutions to steady flows of non-Newtonian incompressible fluids with the help of a general $x$-dependent convex function in generalized Orlicz spaces. Further, Gwiazda et.al [81] proved the existence of weak solutions to the generalized Stokes system in anisotropic Orlicz spaces.

It is our purpose in this chapter to extend the result of [12] to a more general space where the growth and coercivity of $\sigma$ are not polynomial. Consequently, the $L^p$-framework will not capture the described situation. For this reason, the homogeneous Orlicz-Sobolev spaces $W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$ is a suitable framework to explore the growth assumptions by means of a convex function, namely an N-function. Further, we extend the result of [20] (i.e., the first part of Chapter 5) to a steady quasi-Newtonian given by (7.1.1)-(7.1.3). We will prove the existence of weak solutions for problem (7.1.1)-(7.1.3) based on the results of [20, 21].

As mentioned above, our aim is to prove the existence result in the setting of Orlicz spaces by using the concept of Young measure as a technical tool to describe weak limits of sequences constructed by the Galerkin approximations due to Landes (c.f [98]). This approach was widely used in the calculus of variations, optimal control theory and non-linear partail differential equations.

Consider two N-functions $M$ and $P$ such that $P$ grows essentially less rapidly than $M$, and $M, \overline{M} \in \Delta_2$. To study the problem (7.1.1)-(7.1.3), we assume that **(H0)-(H3)** stated in Section 5.2 of Chap. 5 hold true.

We define weak solutions for (7.1.1)-(7.1.3) as follows:

**Definition 7.1.1.** *A function $u \in W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$ is said to be a weak solution of problem* (7.1.1)-(7.1.3) *if for all $\varphi \in W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$, it holds*

$$\int_\Omega \big(\sigma(x,u,Du) : D\varphi + u.\nabla u.\varphi\big)dx = \langle f, \varphi\rangle,$$

*where $\langle .,.\rangle$ is the duality pairing of $W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$ and its dual.*

Our main result reads as follows:

**Theorem 7.1.1.** *If $\sigma$ satisfies the conditions **(H0)-(H2)**, then problem* (7.1.1)-(7.1.3) *has a weak solution in the sense of Definition 7.1.1 for every $f \in W^{-1}_{div}L_{\overline{M}}(\Omega;\mathbb{R}^m)$.*

To prove this theorem we will follow the steps of the first part of Chapter 5.

## 7.2 Galerkin approximation

Let $V_1 \subset V_2 \subset ... \subset W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\underset{i\in\mathbb{N}}{\cup} V_i$ is dense in $W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$. Such a sequence $(V_i)$ exists because $W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$ is separable ($M \in \Delta_2$). We define the operator

$$\begin{aligned} T:\ & W^1_{0,div}L_M(\Omega;\mathbb{R}^m) \to W^{-1}_{div}L_{\overline{M}}(\Omega;\mathbb{R}^m) \\ & \qquad u \mapsto \Big(w \mapsto \int_\Omega \sigma(x,u,Du) : Dwdx + \int_\Omega u.\nabla u.wdx - \langle f, w\rangle\Big). \end{aligned}$$

In the sequel, we use a positive constant $c$ which can change values from line to line.

**Lemma 7.2.1.** *For arbitrary $u \in W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$, the functional $T(u)$ is linear and bounded.*

*Proof.* $T(u)$ is trivially linear for arbitrary $u \in W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$. By the growth condition in **(H1)**, $W^1_{0,div}L_M(\Omega;\mathbb{R}^m) \hookrightarrow L_M(\Omega;\mathbb{R}^m)$ and $P \ll M$, we have

$$\int_\Omega \overline{M}(|\sigma(x,u,Du)|)dx \le c\int_\Omega \big(\overline{M}(d_1(x)) + P(\gamma|u|) + M(\gamma|Du|)\big)dx < \infty.$$

Next, assume that

$$|u \otimes u| \le \overline{M}^{-1}P(|u|) + \overline{M}^{-1}M(|u|)$$

which gives

$$\int_\Omega \overline{M}(|u\otimes u|)dx \le c\int_\Omega \big(P(|u|) + M(|u|)\big)dx.$$

Then by Hölder's inequality, it follows that

$$\begin{aligned} \big|\langle T(u), w\rangle\big| &= \Big|\int_\Omega \sigma(x,u,Du) : Dwdx + \int_\Omega u.\nabla u.wdx - \langle f,w\rangle\Big| \\ &\le 2\big\||\sigma(x,u,Du)|\big\|_{\overline{M}}\,\|Dw\|_M + \int_\Omega |u.\nabla u.w|dx + 2\|f\|_{-1,\overline{M}}\,\|w\|_{1,M}. \end{aligned}$$

Since

$$\begin{aligned} \int_\Omega |u.\nabla u.w|dx &= \int_\Omega |(u\otimes u).\nabla w|dx \\ &\le 2\|u\otimes u\|_{\overline{M}}\,\|Dw\|_M, \end{aligned}$$

we get

$$|\langle T(u), w\rangle| \le c\|w\|_{1,M}.$$

This implies that $T(u)$ is well defined and bounded. ∎

**Lemma 7.2.2.** *The restriction of $T$ to a finite linear subspace $V$ of $W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$ is continuous.*

*Proof.* Let $r$ be the dimension of a subspace $V$ of $W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$ and $(e_i)_{i=1}^r$ a basis of $V$. Let $(u_k = a_k^i e_i)$ be a sequence in $V$ which converges to $u = a^i e_i$ in $V$ (with conventional summation). Then $(a_k)$ converges to $a$ in $\mathbb{R}^r$ and $u_k \to u$ and $Du_k \to Du$ almost everywhere. On the other hand $\|u_k\|_M$ and $\|Du_k\|_M$ are bounded by a constant $c$. Thus, the continuity assumption in **(H0)** permits to deduce that $\sigma(x,u_k,Du_k) : Dw \to \sigma(x,u,Du) : Dw$ almost everywhere. Also $(u_k \otimes u_k).\nabla w \to (u \otimes u).\nabla w$ almost everywhere. Hence, by the growth condition in **(H1)**, the Hölder inequality and the Vitali Theorem, it follows for $w \in W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$

$$
\begin{aligned}
&\|T(u_k) - T(u)\|_{-1,\overline{M}} \\
&= \sup_{\|w\|_{1,M}\equiv 1} \left|\langle T(u_k), w\rangle - \langle T(u), w\rangle\right| \\
&= \sup_{\|w\|_{1,M}\equiv 1} \left| \int_\Omega \big(\sigma(x,u_k,Du_k) - \sigma(x,u,Du)\big) : Dw dx + \int_\Omega (u_k \otimes u_k - u \otimes u).\nabla w dx \right| \\
&\le c\Big(\big\||\sigma(x,u_k,Du_k) - \sigma(x,u,Du)|\big\|_{\overline{M},\Omega} + \|u_k \otimes u_k - u \otimes u\|_{\overline{M},\Omega}\Big) \le c.
\end{aligned}
$$

$\square$

We fix some $k$ and assume that the dimension of $V_k$ is $r$ and $e_1, ..., e_r$ is a basis of $V_k$. For somplicity, we write $\sum_{1\le i \le r} a_i e_i = a_i e_i$ and we define the map

$$
G : \mathbb{R}^r \to \mathbb{R}^r, \qquad \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_r \end{pmatrix} \mapsto \begin{pmatrix} \langle T(a_i e_i), e_1\rangle \\ \langle T(a_i e_i), e_2\rangle \\ . \\ . \\ \langle T(a_i e_i), e_r\rangle \end{pmatrix}.
$$

**Lemma 7.2.3.** *$G$ is continuous and*

$$G(a).a \to +\infty \quad as \quad \|a\|_{\mathbb{R}^r} \to +\infty.$$

*Proof.* Let $u_j = a_j^i e_i \in V_k$, $u_0 = a_0^i e_i \in V_k$. Since $T$ is continuous on a finite dimensional subspace and

$$\begin{aligned}|(G(a_j) - G(a))_l| &= |\langle T(a_j^i e_i) - T(a_0^i e_i), e_l\rangle| \\ &\leq \|T(u_j) - T(u_0)\|_{-1,\overline{M}}.\|e_l\|_{1,M},\end{aligned}$$

it follows that $G$ is continuous. Now, take $u = a_i e_i \in V_k$, then $\|a\|_{\mathbb{R}^r} \to +\infty$ is equivalent to $\|u\|_{1,M} \to +\infty$ and

$$G(a).a = \langle T(a_i e_i), a_i e_i\rangle = \langle T(u), u\rangle.$$

The coercivity condition in **(H1)** implies

$$I \equiv \int_\Omega \sigma(x, u, Du) : Dudx \geq \alpha \int_\Omega M(|Du|)dx - c.$$

Next, observe that

$$II \equiv \int_\Omega u.\nabla u.udx = \frac{1}{2}\int_\Omega u^j \frac{\partial}{\partial x_j}|u|^2 dx = -\frac{1}{2}\int_\Omega \operatorname{div} u|u|^2 dx = 0,$$

by the condition (7.1.2). Finally, from Young's inequality and Lemma 2.2.3

$$\begin{aligned}III \equiv \int_\Omega |f||u|dx &= \frac{\alpha}{2\theta}\int_\Omega \frac{2\theta}{\alpha}|f||u|dx \\ &\leq \frac{\alpha}{2\theta}\int_\Omega \overline{M}(\frac{2\theta}{\alpha}|f|)dx + \frac{\alpha}{2\theta}\int_\Omega M(|u|)dx \\ &\leq c + \frac{\alpha}{2}\int_\Omega M(|Du|)dx.\end{aligned}$$

Consequently

$$\begin{aligned}G(a).a = \langle T(u), u\rangle &\geq \alpha \int_\Omega M(|Du|)dx - \frac{\alpha}{2}\int_\Omega M(|Du|)dx - c \\ &= \frac{\alpha}{2}\int_\Omega M(|Du|)dx - c \to +\infty\end{aligned}$$

as $\|u\|_{1,M} \to +\infty$, that is $T$ coercive. □

**Lemma 7.2.4.** *For all $k \in \mathbb{N}$, there exists $u_k \in V_k$ such that*

$$\langle T(u_k), w\rangle = 0 \quad \textit{for all} \quad w \in V_k.$$

*Proof.* We have by Lemma 7.2.3, $G(a).a \to +\infty$ as $\|a\|_{\mathbb{R}^r} \to +\infty$. Then there exists $R > 0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $G(a).a > 0$. The usual topological argument [117] gives that $G(x) = 0$ has a solution $x \in B_R(0)$. Hence, for all $k \in \mathbb{N}$ there exists $u_k \in V_k$ such that $\langle T(u_k), w\rangle = 0$, $\forall w \in V_k$. □

As a consequence of Lemma 7.2.4, the sequence $(u_k)$ is uniformly bounded in $W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$. To see this, suppose that $(u_k)$ is not uniformly bounded. Since $T$ is coercive, then there is $R > 0$ for which $\langle T(u), u\rangle > 1$ whenever $\|u\|_{1,M} > R$. This gives a contradiction with the Galerkin approximation $u_k$ which satisfies Lemma 7.2.4.

According to Lemma 2.3.1, there exists a Young measure $\nu_x$ generated by $Du_k$ in $L_M(\Omega;\mathbb{M}^{m\times n})$ satisfying the properties of Lemma 5.3.3.

## 7.3 Div-curl inequality

The following lemma is the key ingredient to pass to the limit in the approximating equations and to prove the weak limit $u$ of the Galerkin approximations $u_k$ is indeed a solution of (7.1.1)-(7.1.3).

**Lemma 7.3.1** (div-curl inequality)**.** *Assume that $Du_k$ generates the Young measure $\nu_x$. Then the following inequality holds:*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} (\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) d\nu_x(\lambda) dx \le 0. \tag{7.3.1}$$

*Proof.* We consider the sequence

$$\begin{aligned} I_k &:= \big(\sigma(x,u_k,Du_k) - \sigma(x,u,Du)\big) : (Du_k - Du) \\ &= \sigma(x,u_k,Du_k) : (Du_k - Du) - \sigma(x,u,Du) : (Du_k - Du) \\ &=: I_{k,1} + I_{k,2}. \end{aligned}$$

Let start with the sequence $I_{k,2}$. Since

$$\int_\Omega \overline{M}(|\sigma(x,u,Du)|)dx \le c\int_\Omega \left(\overline{M}(d_1(x)) + P(\gamma|u|) + M(\gamma|Du|)\right)dx < \infty$$

by the growth condition in **(H1)** and $P \ll M$, then $\sigma \in L_{\overline{M}}(\Omega;\mathbb{M}^{m\times n})$. It follows according to Lemma 5.3.3 that

$$\liminf_{k\to\infty}\int_\Omega I_{k,2}dx = \int_\Omega\int_{\mathbb{M}^{m\times n}} \sigma(x,u,Du) : (\lambda - Du)d\nu_x(\lambda)dx = 0.$$

For the sequence $I_{k,1}$, take a measurable subset $\Omega' \subset \Omega$ and by the Hölder inequality we have

$$\int_{\Omega'} |\sigma(x,u_k,Du_k) : Du_k|dx \le 2\big\||\sigma(x,u_k,Du_k)|\big\|_{\overline{M},\Omega'}\Big(\int_{\Omega'} M(|Du_k|)\Big)dx.$$

Since $\{u_k\}$ is bounded in $W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$, then by the growth condition in **(H1)** and $W^1_0L_M(\Omega) \hookrightarrow L_M(\Omega)$

$$\int_\Omega \overline{M}(|\sigma(x,u_k,Du_k)|)dx \le c\int_\Omega \overline{M}(d_1(x)) + P(\gamma|u_k|) + M(\gamma|Du_k|)dx \le c.$$

Thus $\big\||\sigma(x,u_k,Du_k)|\big\|_{\overline{M},\Omega'}$ is bounded. Note that the term $\int_{\Omega'} M(|Du|)dx$ is arbitrary small if the measure of $\Omega'$ is chosen small enough. Consequently, the equiintegrability of $I^-_{k,1}$ follows. Since $(u_k)$ is uniformly bounded in $W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$, then $u_k \to u$ in $L_M(\Omega;\mathbb{R}^m)$ (up to a subsequence). Hence

$$\begin{aligned}\int_\Omega M(|u_k - u|)dx &\ge \int_{\{x\in\Omega;|u_k-u|\ge\epsilon\}} M(|u_k-u|)dx\\ &\ge c\int_{\{x\in\Omega;|u_k-u|\ge\epsilon\}} |u_k-u|dx\\ &\ge c\epsilon\big|\{x\in\Omega;|u_k-u|\ge\epsilon\}\big|\end{aligned}$$

for some $\epsilon$ positive and $c$ is the constant of the embedding $L_M \subset L^1$. Therefore, $u_k \to u$ in measure. By virtue of Lemma 2.3.2, one gets

$$\begin{aligned} I := \liminf_{k\to\infty} \int_\Omega I_k dx &= \liminf_{k\to\infty} \int_\Omega I_{k,1} dx \\ &= \liminf_{k\to\infty} \int_\Omega \sigma(x, u_k, Du_k) : (Du_k - Du) dx \\ &\geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, u, \lambda) : (\lambda - Du) d\nu_x(\lambda) dx. \end{aligned}$$

We will see next that $I \leq 0$. Define $dist(u, V_k) = \inf_{v\in V_k} \|u - v\|_{1,M}$ and fix $\epsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ such that $dist(u, V_k) < \epsilon$ for all $k > k_0$, or equivalently

$$dist(u_k - u, V_k) = \inf_{v\in V_k} \|u_k - u - v\|_{1,M} = \inf_{w\in V_k} \|u - w\|_{1,M} = dist(u, V_k) < \epsilon,$$

for any $k > k_0$. Then, for $v_k \in V_k$, we may estimate $I$ as follows

$$\begin{aligned} I &= \liminf_{k\to\infty} \int_\Omega \sigma(x, u_k, Du_k) : (Du_k - Du) dx \\ &= \liminf_{k\to\infty} \int_\Omega \sigma(x, u_k, Du_k) : D(u_k - u - v_k) + \sigma(x, u_k, Du_k) : Dv_k dx \\ &\leq \liminf_{k\to\infty} \Big( 2\| |\sigma(x, u_k, Du_k)| \|_{\overline{M},\Omega} \|D(u_k - u - v_k)\|_{M,\Omega} + \langle f, v_k \rangle - \int_\Omega (u_k \otimes u_k).\nabla v_k dx \Big). \end{aligned}$$

The term $\| |\sigma(x, u_k, Du_k)| \|_{\overline{M},\Omega}$ is uniformly bounded in $k$ by the growth condition in **(H1)**. On the other hand, by choosing $v_k \in V_k$ in such a way that $\|u_k - u - v_k\|_{1,M} < 2\epsilon$ for any $k > k_0$, the term $\|D(u_k - u - v_k)\|_{M,\Omega}$ is bounded by $2\epsilon$. Furthermore, we have

$$\begin{aligned} |\langle f, v_k \rangle| &= |\langle f, v_k - (u_k - u) \rangle + \langle f, u_k - u \rangle| \\ &\leq |\langle f, v_k - (u_k - u) \rangle| + |\langle f, u_k - u \rangle| \\ &\leq 2\epsilon \|f\|_{-1,\overline{M}} + o(k) \end{aligned}$$

and

$$\begin{aligned}|\int_\Omega (u_k \otimes u_k).\nabla v_k dx| &= |\int_\Omega (u_k \otimes u_k).(\nabla(v_k - u_k + u_k))dx| \\ &\leq \underbrace{\int_\Omega |(u_k \otimes u_k).\nabla u_k|dx}_{=0} + \int_\Omega |(u_k \otimes u_k).\nabla(v_k - u_k)|dx \\ &\leq \int_\Omega |(u_k \otimes u_k).\nabla(v_k - u)|dx + \int_\Omega |(u_k \otimes u_k).\nabla(u - u_k)|dx \\ &\leq 2\|u_k \otimes u_k\|_{\overline{M},\Omega}\Big[\|D(v_k - u)\|_{M,\Omega} + \|D(u - u_k)\|_{M,\Omega}\Big].\end{aligned} \tag{7.3.2}$$

Similarly to the proof of Lemma 7.2.1, we have $\|u_k \otimes u_k\|_{\overline{M}}$ is bounded since $(u_k)$ is bounded. Hence, the right hand side in (7.3.2) tends to zero as $k \to +\infty$. Since $\epsilon$ was arbitrary, this proves that $I \leq 0$. Note that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,Du) : (\lambda - Du)d\nu_x(\lambda)dx = 0.$$

This together with $I \leq 0$, the Eq. (7.3.1) follows. □

## 7.4 Proof of Theorem 7.1.1

Now, we have all ingredients to prove Theorem 7.1.1 by considering the conditions (a), (b), (c) and (d) listed in **(H2)**. The proof is similar to that in Step 3 of Chapter 5. It remains to pass to the limit for the convective term. Remark first that, $(u_k, \nabla u_k)$ generates the Young measure $\delta_{u(x)} \otimes \nu_x$ by Proposition 2.3.1. Thus

$$\begin{aligned}u_k.\nabla u_k \rightharpoonup &\int_{\mathbb{R}^m \times \mathbb{R}^{mn}} (s.\lambda)d\delta_{u(x)}(s) \otimes d\nu_x(\lambda) \\ &= \int_{\mathbb{R}^{mn}} (u.\lambda)d\nu_x(\lambda) = u.\nabla u.\end{aligned}$$

In conclusion, let $v \in W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$, since $\underset{i\in\mathbb{N}}{\cup} V_i$ is dense in $W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$, there exists a sequence $v_k \in \underset{i\in\mathbb{N}}{\cup} V_i$ such that $v_k \to v$ in $W^1_{0,div}L_M(\Omega;\mathbb{R}^m)$ for $k \to \infty$. We have

$$\begin{aligned}
&\langle T(u_k), v_k\rangle - \langle T(u), v\rangle \\
&\quad = \int_\Omega \sigma(x, u_k, Du_k) : Dv_k dx + \int_\Omega (u_k.\nabla u_k) v_k dx - \langle f, v_k\rangle \\
&\qquad\qquad - \int_\Omega \sigma(x, u, Du) : Dv dx - \int_\Omega (u.\nabla u) v dx + \langle f, v\rangle \\
&\quad = \int_\Omega \sigma(x, u_k, Du_k) : (Dv_k - Dv) dx + \int_\Omega \big(\sigma(x, u_k, Du_k) - \sigma(x, u, Du)\big) : Dv dx \\
&\qquad\qquad + \int_\Omega (u_k.\nabla u_k) v_k - (u.\nabla u) v) dx - \langle f, v_k - v\rangle.
\end{aligned}$$

According to all cases in **(H2)** and $u_k.\nabla u_k \to u.\nabla u$, the right hand side of the above equality tends to zero as $k$ tends to infinity. By virtue of Lemma 7.2.4, it follows that

$$\langle T(u), v\rangle = 0 \quad \text{for all} \quad v \in W^1_{0,div} L_M(\Omega; \mathbb{R}^m).$$

## 7.5 Navier-Stokes system

Our considerations turn around the existence of weak solutions to a Navier-Stokes system associated to (7.1.1)-(7.1.3), which is motivated by models for electrorheological fluids prescribed by the equations

$$\frac{\partial u}{\partial t} - \operatorname{div} \sigma(x, t, u, Du) + (u.\nabla) u = f - \operatorname{grad}\pi \text{ in } Q, \tag{7.5.1}$$

$$\operatorname{div} u = 0 \text{ in } Q, \tag{7.5.2}$$

$$u = 0 \text{ on } \partial Q, \tag{7.5.3}$$

$$u(., 0) = u_0(.) \text{ in } \Omega, \tag{7.5.4}$$

where $u : Q \to \mathbb{R}^m$ denotes the velocity field, $\pi : Q \to \mathbb{R}$ the pressure, $\sigma$ the stress tensor, $Q = \Omega \times (0, T)$ and $\Omega \subset \mathbb{R}^n$ is a bounded open domain with Lipschitz boundary. Here, $f \in W^{-1,x}_{\text{div}} E_{\overline{M}}(Q; \mathbb{R}^m)$ and $u_0 \in L^2_{\text{div}}(\Omega; \mathbb{R}^m)$, which is the closure of $V = \{\varphi \in C_0^\infty(\Omega; \mathbb{R}^m) : \operatorname{div} \varphi = 0\}$ in the space $L^2(\Omega; \mathbb{R}^m)$. Always $M, \overline{M} \in \Delta_2$.

We want here to extend the theory existence of [64] where the authors proved the existence result in the setting of $L^p(0,T;V)$, where $V$ consists of all functions in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ with vanishing divergence, to a more general class than polynomial growth conditions. Standard growth conditions on the stress tensor $\sigma$, namely polynomial growth are given in the following form:

$$|\sigma(x,t,u,F)| \leq \lambda_1(x,t) + c_1\big(|u|^{p-1} + |F|^{p-1}\big),$$
$$\sigma(x,t,u,F) : F \geq -\lambda_2(x,t) - \lambda_3(x,t)|u|^{\alpha} + c_2|F|^p.$$

When trying to relax these conditions, we find that (7.5.1)-(7.5.4) can not be formulated in the setting of Sobolev spaces $L^p(0,T;V)$. Our purpose is motivated by the process where growth is not polynomial. By this reason, we formulate the growth and coercivity conditions of the stress tensor $\sigma$ by using general convex functions called $N$-functions.

The appropriate spaces to treat such formulated problem (7.5.1)-(7.5.4) are Orlicz-Sobolev spaces. The example of Orlicz space is generalized Lebesgue space while $M(t) = |t|^{p(x)}$. This kind of spaces were applied in [106] to give a description of flow of electrorheological fluid. The classical assumptions considered are: $1 \leq p^- \leq p(x) \leq p^+ < \infty$, where $p \in C^1(\Omega)$ is a function of electric field $E$, i.e., $p = p(|E|^2)$, and $p^+ \geq \frac{3n}{n+2}$ in case of steady flow.

If the flow is assumed to be slow, then the convective term $(u.\nabla)u$ may be assumed to be very small and therefore neglected, hence the whole system reduces to generalized stokes, cf [80]. In this direction, in [19] (cf. Section 6.1) we have considered the following quasilinear parabolic system in an Orlicz-Sobolev space $W_0^{1,x}L_M(Q;\mathbb{R}^m)$:

$$\begin{cases} \partial_t u - \operatorname{div}\sigma(x,t,Du) &= f \quad \text{in } Q \\ u(x,t) &= 0 \quad \text{on } \partial Q \\ u(.,0) &= u_0(.) \quad \text{in } \Omega, \end{cases} \tag{7.5.5}$$

where $f \in W^{-1,x}L_{\overline{M}}(Q;\mathbb{R}^m)$ and $\sigma$ satisfies some conditions and weak monotonicity assumptions. We have proved the existence and uniqueness of solutions by applying the theory of Young measures. When $f = f_0 - \operatorname{div} g$ with $f_0 \in L^2(Q;\mathbb{R}^m)$ and $g \in L_{\overline{M}}(Q;\mathbb{M}^{m\times n})$, the above problem has been treated in [23] where we have proved the existence of a unique weak solution in the space $X(Q) = \{v \in L^2(Q;\mathbb{R}^m)/\ Dv \in$

$L_M(Q;\mathbb{M}^{m\times n})$; $v(t) := v(.,t) \in W_0^1L_M(\Omega;\mathbb{R}^m)$ a.e. $t \in [0,T]\}$. Our purpose in this part, is to extend the previous results to a Navier-Stokes system where the convective term is present and influence the definition of function spaces. Moreover, we allow $\sigma$ to depend on $u$ and introduce an N-function $P$ which grows essentially less rapidly than the N-function $M$.

The case with additional convective term $(u.\nabla)u$ has been studied in [78], where $f$ is assumed belongs to $W^{-1,q'}(Q)$ with $q \geq \frac{3n+2}{n+2}$ and the $N$-function satisfies the condition $M(x,\xi) \geq c|\xi|^q$. Also in [133] the convective term is present, but $f$ is given in the form $f = \mathrm{div}F$ with $F \in \mathbb{M}^{m\times n}$, $F \in L_{\overline{M}}(\Omega,\mathbb{M}^{n\times n})$ and $q \geq \frac{3n}{n+2}$. See also [67, 125] for similar topics.

The difficulty that arises in the corresponding Navier-Stokes system (7.5.1)-(7.5.4) is that we can't use the standard method of type Leray Lions or monotone operators. This fact is due to the following: the tensor $\sigma$ does not need to satisfy the strict monotonicity or monotonicity assumptions in the variable $u$ or in $(u,F)$ as it is usually assumed in [66, 99, 101]. This difficulty leads us to use the tool of Young measures which turns out to be an appropriate and powerful tool to describe weak convergence of sequences and allow to treat such problems under mild monotonicity assumptions for $\sigma$ (see below). For the utilization of the concept of Young measure in partial differential equations, we refer the reader to see [20, 21, 24, 27, 29].

### 7.5.1 Hypothesis and main result

Now, we state our assumptions on $\sigma : Q\times\mathbb{R}^m\times\mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$. Consider two $N$-functions $M$ and $P$ such that $P$ grows essentially less rapidly than $M$ (i.e. $P \ll M$) and $M,\overline{M} \in \Delta_2$.

**(P0)** $\sigma$ is a Carathéodory function (i.e., measurable w.r.t. $(x,t)\in Q$ and continuous w.r.t. the last variables $(s,\xi)\in\mathbb{R}^m\times\mathbb{M}^{m\times n}$).

**(P1)** There exist $0 \leq d_1(x,t) \in L_{\overline{M}}(Q)$, $d_2(x,t)\in L^1(Q)$ and $c_1, c_2, \gamma, \beta > 0$, such that

$$|\sigma(x,t,s,\xi)| \leq d_1(x,t) + c_1\big(\overline{M}^{-1}P(\gamma|s|) + \overline{M}^{-1}M(\gamma|\xi|)\big)$$

$$\sigma(x,t,s,\xi) : \xi \geq c_2 M\Big(\frac{|\xi|}{\beta}\Big) - d_2(x,t).$$

**(P2)** $\sigma$ satisfies one of the following conditions:

(a) The map $F \mapsto \sigma(x,t,u,\xi)$ is a $C^1$-function and is monotone, i.e.,

$$(\sigma(x,t,u,\xi) - \sigma(x,t,u,\eta)) : (\xi - \eta) \geq 0, \quad \forall \xi, \eta \in \mathbb{M}^{m\times n},$$

for all $(x,t) \in Q$, $u \in \mathbb{R}^m$ and $\xi, \eta \in \mathbb{M}^{m\times n}$.

(b) There exists a function $W : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,t,u,\xi) = \frac{\partial W}{\partial \xi}(x,t,u,\xi) = D_\xi W(x,t,u,\xi)$, and $\xi \to W(x,t,u,\xi)$ is convex and $C^1$ for all $(x,t) \in Q$ and all $u \in \mathbb{R}^m$.

(c) $\sigma$ is strictly monotone, i.e., $\sigma$ is monotone and

$$(\sigma(x,t,u,\xi) - \sigma(x,t,u,\xi)) : (\xi - \eta) = 0 \Longrightarrow \xi = \eta.$$

(d) $\sigma$ is strictly M-quasimonotone in $F \in \mathbb{M}^{m\times n}$, i.e.,

$$\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) - \sigma(x,t,u,\overline{\lambda})) : (\lambda - \overline{\lambda}) d\nu_{(x,t)}(\lambda) > 0$$

where $\overline{\lambda} = \langle \nu_{(x,t)}, id \rangle$, $\nu = \{\nu_{(x,t)}\}_{(x,t)\in Q}$ is any family of Young measures generated by a bounded sequence in $L_M(Q)$ and not a Dirac measure for a.e. $(x,t) \in Q$.

Now we define weak solutions of our problem and state the main existence result.

**Definition 7.5.1.** *A function $u \in W^{1,x}_{0,div} L_M(Q;\mathbb{R}^m) \cap L^\infty(0,T;L^2_{div}(\Omega;\mathbb{R}^m))$ is said to be a weak solution of (7.5.1)-(7.5.4) if for all $\varphi \in W^{1,x}_{0,div} L_M(Q;\mathbb{R}^m)$, it holds*

$$\int_Q (-u\varphi_t + \sigma(x,t,u,Du) : D\varphi + u.\nabla u.\varphi) dxdt + \int_\Omega u\varphi dx\Big|_0^T = \langle f, \varphi \rangle,$$

*where $\langle .,. \rangle$ is the dual pairing of $W^{-1,x}_{div} L_{\overline{M}}(Q;\mathbb{R}^m)$ and $W^{1,x}_{0,div} L_M(Q;\mathbb{R}^m)$.*

**Theorem 7.5.1.** *Assume that $\sigma$ satisfies the conditions **(P0)-(P2)**. Given $f \in W^{-1,x}_{div} L_{\overline{M}}(Q;\mathbb{R}^m)$ and $u_0 \in L^2_{div}(\Omega;\mathbb{R}^m)$, then there exists at least one weak solution $u \in W^{1,x}_{0,div} L_M(Q;\mathbb{R}^m) \cap C(0,T;L^2_{div}(\Omega;\mathbb{R}^m))$ of (7.5.1)-(7.5.4) in the sense of Definition 7.5.1.*

### 7.5.2 Proof of Theorem 7.5.1

We construct the approximate Galerkin solutions to (7.5.1)-(7.5.4). We choose an $L^2_{\text{div}}(\Omega;\mathbb{R}^m)$-orthonormal base $\{w_j\}_{j\geq 1}$, such that $\{w_j\}_{j\geq 1} \subset \mathcal{C}^\infty_{0,\text{div}}(\Omega;\mathbb{R}^m) := \{v \in C^\infty_0(\Omega;\mathbb{R}^m),\ \text{div}v = 0\}$ and

$$\mathcal{C}^\infty_{0,\text{div}}(\Omega;\mathbb{R}^m) \subset \overline{\bigcup_{k\geq 1} V_k}^{\mathcal{C}^1_{\text{div}}(\Omega;\mathbb{R}^m)},$$

where $V_k = \text{span}\{w_1, ..., w_k\}$. Let us consider $u_k(x,t) = \sum_{i=1}^{k} c_{ki}(t)w_i(x)$ to approach the solutions of the problem (7.5.1)-(7.5.4), where the coefficients $c_{ki} : (0,T) \to \mathbb{R}$ are bounded measurable functions and solves the following ordinary differential equations

$$\int_\Omega \frac{du_k}{dt} w_i + \int_\Omega \sigma(x,t,u_k,Du_k) : Dw_i + \int_\Omega u_k.\nabla u_k w_i dx = \langle f(t), w_i\rangle. \tag{7.5.6}$$

Each $u_k$ satisfy the conditions (7.5.2) and (7.5.3) by construction in the sense that $u_k \in W^{1,x}_{0,div}L_M(Q;\mathbb{R}^m)$.
For the initial condition (7.5.4) we choose

$$c_{ki}(0) := (u_0, w_i)_{L^2} = \int_\Omega u_0(x)w_i(x)dx,$$

such that

$$u_k(.,0) = \sum_{i=1}^{k} c_{ki}(0)w_i(.) = \sum_{i=1}^{k} (u_0,w_i)_{L^2} w_i(0) \to u_0 \text{ in } L^2(\Omega) \text{ as } k \to \infty.$$

Let us first determine the coefficients $c_{ki}(t)$. Choose $r > 0$ large enough such that the set $B(0,r) = B_r(0) \subset \mathbb{R}^r$ contains the vectors $(c_{k1}(0), .., c_{kk}(0))$ for fixed $k \in \mathbb{N}$. Let $\tau > 0$ and consider the functional $T : [0,\tau] \times \overline{B_r(0)} \to \mathbb{R}^k$ defined by

$$\begin{aligned} T_j(t, c_1, .., c_k) = \langle f(t), w_j\rangle - \int_\Omega \sigma(x,t,\sum_{i=1}^{k} c_i w_i, \sum_{i=1}^{k} c_i Dw_i) : Dw_j dx \\ - B(\sum_{i=1}^{k} c_i w_i, \sum_{i=1}^{k} c_i w_i, w_j), \end{aligned}$$

for $j = 1,..,k$, where we used the short notation $B(u,v,w) = \int_\Omega (u.\nabla)v.wdx$. The assumption **(P0)** implies that $T$ is a Carathéodory function. The three terms in the definition of $T_j$ can be estimated on $[0,\tau] \times \overline{B_r(0)}$ as follows:
For the first term, we have

$$\left|\langle f(t), w_j\rangle\right| \le \|f(t)\|_{W^{-1}_{div}E_{\overline{M}}(\Omega)} \|w_j\|_{W^1_{0,div}L_M(\Omega)}$$

by Hölder's inequality. In the second term, we use the growth condition in **(P1)** and Hölder's inequality to get

$$\Big|\int_\Omega \sigma(x,t,\sum_{i=1}^k c_i w_i, \sum_{i=1}^k c_i Dw_i) : Dw_j dx\Big| \le 2\Big\|\big|\sigma(x,t,\sum_{i=1}^k c_i w_i, \sum_{i=1}^k c_i Dw_i)\big|\Big\|_{\overline{M}} \|Dw_j\|_M.$$

For the third term, we choose $u^2 \le M(u\otimes u)$ which gives in passing to the conjugate form $\overline{M}(u\otimes u) \le u^2$ with *a priori* constant, thus

$$\begin{aligned} |B(u,u,w)| &= \Big|\int_\Omega (u.\nabla)u.wdx\Big| = \Big|\int_\Omega (u\otimes u).\nabla w dx\Big| \\ &\le \|u\otimes u\|_{L_{\overline{M}}} \|\nabla w\|_{L_M} \\ &\le c\|u\|^2_{L^2(\Omega;\mathbb{R}^m)} \|w\|_{W^1_{0,div}L_M(\Omega;\mathbb{R}^m)}, \end{aligned}$$

where $c$ is a positive constant. Note that, instead of $u^2 \le M(u\otimes u)$, one may assume that

$$|u\otimes u| \le \overline{M}^{-1}P(|u|) + \overline{M}^{-1}M(|u|).$$

Using the above estimations, the right hand side in the definition of the component $T_j$ can be estimated in such a way that

$$|T_j(t,c_1,..,c_k)| \le C_1(r,k)\phi(t)$$

uniformly on $[0,\tau] \times \overline{B_r(0)}$, where $C_1(r,k)$ is a constant which depends on $r$ and $k$, and $\phi(t) \in L^1([0,\tau])$. Therefore, thanks to the existence result of ordinary differential equation (see e.g. [90]), the system

$$\begin{cases} c_j'(t) &= T_j(t, c_1, ..., c_k) \\ c_j(0) &= c_{kj}(0) \end{cases} \tag{7.5.7}$$

(for $j = 1, .., k$) has a continuous solution $c_j$ (depending on $k$) on an interval $(0, \tau')$, where $\tau' > 0$ and may depends on $k$. After integrating the Eq. (7.5.7), we obtain

$$c_j(t) = c_j(0) + \int_0^t T_j(t, c_1(s), ..., c_k(s))ds,$$

which holds on $[0, \tau')$. Consequently, $u_k := \sum_{i=1}^{k} c_{ki}(t)w_i(x)$ is the desired solution of the equation (7.5.6).

Now, we extend the local solution constructed above to the whole interval $[0, T)$. To do this, we multiply each side of (7.5.6) by $c_{ki}(t)$ and we sum. This gives for $\tau \in [0, \tau')$

$$\int_{Q_\tau} \frac{\partial u_k}{\partial t} u_k dxdt + \int_{Q_\tau} \sigma(x, t, u_k, Du_k) : Du_k dxdt = \int_0^\tau \langle f(t), u_k \rangle dt,$$

which we denote by $I_1 + I_2 = I_3$. Notice that we have used $\operatorname{div} u_k = 0$ to get

$$\int_{Q_\tau} u_k.\nabla u_k.u_k dxdt = \frac{1}{2} \int_{Q_\tau} u_k^j \frac{\partial}{\partial x_j} |u_k|^2 dxdt = -\frac{1}{2} \int_{Q_\tau} \operatorname{div} u_k |u_k|^2 dxdt = 0.$$

For $I_1$, we have

$$I_1 = \frac{1}{2} \|u_k(., \tau)\|^2_{L^2(\Omega)} - \frac{1}{2} \|u_k(., 0)\|^2_{L^2(\Omega)}.$$

By the coercivity condition in **(P1)**, we can write

$$I_2 = \int_{Q_\tau} \sigma(x, t, u_k, Du_k) : Du_k dxdt \geq -\int_{Q_\tau} d_2 dxdt + c_2 \int_{Q_\tau} M(|Du_k|)dxdt.$$

Finally, by the Hölder inequality, we have

$$|I_3| \leq \|f\|_{W^{-1,x}_{div} L_{\overline{M}}(Q_\tau;\mathbb{R}^m)} \|u_k\|_{W^{1,x}_{0,div} L_M(Q_\tau;\mathbb{R}^m)}.$$

The combination of the three estimates gives

$$\|u_k(.,\tau)\|^2_{L^2(\Omega)} = \left|\left(c_{ki}(\tau)\right)_{i=1}^k\right|^2_{\mathbb{R}^k} \le c$$

for a constant $c$ which is independent of $\tau$ and of $k$. Let

$$\Lambda := \left\{t \in [0,T) : \text{ there exists a weak solution of (7.5.7) on } [0,t)\right\}.$$

We conclude from [86] that $\Lambda = [0,T)$.

In the sequenl, we use $c$ for a generic constant which may change values from line to line.

**Lemma 7.5.1.** *The sequence $\{u_k\}$ is bounded in $W^{1,x}_{0,div}L_M(Q;\mathbb{R}^m) \cap L^\infty(0,T;L^2_{div}(\Omega;\mathbb{R}^m)$ and $\{\sigma(x,t,u_k,Du_k)\}$ is bounded in $L_{\overline{M}}(Q;\mathbb{M}^{m\times n})$.*

*Proof.* From the above estimations on $I_\epsilon$, $\epsilon = 1,2,3$ and the boundedness of $u_{0k}$ in $L^2_{\text{div}}(\Omega;\mathbb{R}^m)$ we have

$$\frac{1}{2}\|u_k(.,\tau)\|^2_{L^2(\Omega)} + c\int_{Q_\tau} M(|Du_k|)dxdt \le \|f\|_{W^{-1,x}_{div}E_{\overline{M}}(Q_\tau;\mathbb{R}^m)}\|u_k\|_{W^{1,x}_{0,div}L_M(Q_\tau;\mathbb{R}^m)} + c$$

If $\|Du_k\|_M$ is unbounded, then $\int_{Q_\tau} M(|Du_k|)dxdt$ is unbounded. This contradict the above inequality. Therefore $\|Du_k\|_M \le c$ and $\|u_k(.,\tau)\|^2_{L^2(\Omega)} \le c$. Hence $(u_k)$ is bounded in $W^{1,x}_{0,div}L_M(Q;\mathbb{R}^m) \cap L^\infty(0,T;L^2_{div}(\Omega;\mathbb{R}^m)$.

Let $\beta_1 > 0$ such that $\rho_{\overline{M}}(\beta_1 d_1) < \infty$ and $\theta_0 = \max\left\{1, \frac{1}{\beta_1}\right\}$. Since $d_1 \in L_{\overline{M}}(Q)$, $\|u_k\|_M \le c$ and $\|Du_k\|_M \le c$, then

$$\begin{aligned}
&\int_{Q_\tau} \overline{M}\Big(\frac{1}{3\theta_0}|\sigma(x,t,u_k,Du_k)|\Big)dxdt \\
&\quad \le \int_{Q_\tau} \overline{M}\Big(\frac{\beta_1}{3\theta_0\beta_1}d_1(x,t) + \frac{c_1}{3\theta_0}\big(\overline{M}^{-1}P(|u_k|) + \overline{M}^{-1}M(|Du_k|)\big)\Big)dxdt \\
&\quad \le c\int_{Q_\tau}\Big(\overline{M}(\beta_1 d_1) + c\big(P(|u_k|) + M(|Du_k|)\big)\Big)dxdt \\
&\quad \le c,
\end{aligned}$$

since $P \ll M$. Therefore $\|\sigma(x,t,u_k,Du_k)\|_{\overline{M}} \le c$. □

According to Lemma 7.5.1, by extracting a suitable subsequence (still denoted by $(u_k)$), we have

$$u_k \rightharpoonup u \text{ in } W^{1,x}_{0,div} L_M(Q; \mathbb{R}^m),$$

$$u_k \rightharpoonup^* u \text{ in } L^\infty(0,T; L^2_{div}(\Omega; \mathbb{R}^m)$$

and

$$\sigma(x,t,u_k,Du_k) \rightharpoonup \chi \quad \text{in } L_{\overline{M}}(Q; \mathbb{M}^{m\times n}),$$

for some $u \in W^{1,x}_{0,\mathrm{div}} L_M(Q; \mathbb{R}^m)$ and $\chi \in L_{\overline{M}}(Q; \mathbb{M}^{m\times n})$.

**Lemma 7.5.2.** *The sequence $\{u_k\}_k$ constructed above satisfy the following properties*

$$u_k(.,T) \rightharpoonup u(.,T) \text{ in } L^2_{div}(\Omega; \mathbb{R}^m),$$

$$u(.,0) = u_0(.).$$

*Proof.* We have $\{u_k\}$ is bounded in $L^\infty(0,T; L^2_{div}(\Omega; \mathbb{R}^m))$. Then there exists a subsequence (still denoted by $\{u_k\}$), such that

$$u_k(.,T) \rightharpoonup u^* \text{ in } L^2_{div}(\Omega; \mathbb{R}^m).$$

For simplicity, we denote $u(.,T)$ as $u(T)$ and $u(.,0)$ as $u(0)$. Let $\phi \in \mathcal{C}^\infty([0,T])$ and $v \in V_j$, $j \leq k$, we have

$$\int_Q \frac{\partial u_k}{\partial t} v\phi dxdt + \int_Q \sigma(x,t,u_k,Du_k) : Dv\phi dxdt + \int_Q (u_k.\nabla)u_k.v\phi dxdt = \langle f, v\phi \rangle.$$

After integrating the first term, we get

$$\begin{aligned}
&\int_\Omega u_k(T)\phi(T)v dx - \int_\Omega u_k(0)\phi(0)v dx \\
&= -\int_Q \sigma(x,t,u_k,Du_k) : Dv\phi dxdt - \int_Q (u_k.\nabla)u_k.v\phi dxdt + \langle f, v\phi\rangle + \int_\Omega u_k v\phi' dxdt.
\end{aligned}$$

We pass to the limit as $k \to \infty$, this implies

$$\begin{aligned}\int_\Omega u^*\phi(T)vdx - &\int_\Omega u_0\phi(0)vdx \\ &= -\int_Q \chi.v\phi dxdt - \int_Q (u.\nabla)u.v\phi dxdt + \langle f, v\phi\rangle + \int_\Omega \phi' uvdxdt.\end{aligned}$$

Choose first $\phi(0) = \phi(T) = 0$, then

$$-\int_Q \chi.v\phi dxdt - \int_Q (u.\nabla)u.v\phi dxdt + \langle f, v\phi\rangle = -\int_\Omega \phi' uvdxdt = \int_\Omega \phi v u' dxdt.$$

Therefore

$$\begin{aligned}\int_\Omega u^*\phi(T)vdx - \int_\Omega u_0\phi(0)vdx &= \int_\Omega \phi v u' dxdt + \int_\Omega \phi' uvdxdt \\ &= \int_\Omega u\phi vdx\Big|_0^T \\ &= \int_\Omega u(T)\phi(T)vdx - \int_\Omega u(0)\phi(0)vdx.\end{aligned}$$

Let $k \to \infty$, if we take $\phi(T) = 0$ and $\phi(T) = 1$, then we have $u(0) = u_0$, and if we take $\phi(T) = 1$ and $\phi(0) = 0$ then we get $u(T) = u^*$ as desired. □

We will prove later that $\chi = \sigma(x, t, u, Du)$ which will be imply that $u$ is a weak solution of problem (7.5.1)-(7.5.4). The following lemma will be the key ingredient to pass to the limit in the approximating equations of the Galerkin method.

**Lemma 7.5.3** (div-curl inequality)**.** *The Young measure $\nu_{(x,t)}$ generated by the gradients $Du_k$ of the Galerkin approximations $u_k$ has the following property*

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : (\lambda - Du) d\nu_{(x,t)}(\lambda) dxdt \le 0. \tag{7.5.8}$$

*Proof.* Consider the sequence

$$
\begin{aligned}
J_k &:= \big(\sigma(x,t,u_k,Du_k) - \sigma(x,t,u,Du)\big) : (Du_k - Du) \\
&= \sigma(x,t,u_k,Du_k) : (Du_k - Du) - \sigma(x,t,u,Du) : (Du_k - Du) \\
&=: J_{k,1} + J_{k,2}.
\end{aligned}
$$

Since $Du \in L_M(Q;\mathbb{M}^{m\times n})$, it follows by the growth condition in **(P1)** that $\sigma \in L_{\overline{M}}(Q;\mathbb{M}^{m\times n})$. From Lemma 6.3.1 we get

$$
\liminf_{k\to\infty} \int_Q J_{k,2} dxdt = \int_Q \sigma(x,t,u,Du) : \Big( \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) - Du \Big) dxdt = 0.
$$

Thanks to [78, Lemma 2.1], we have $u_k \to u$ in measure as $k \to \infty$. By virtue of Lemma 2.3.2, it follows, since $\sigma(x,t,u_k,Du_k)$ is equiintegrable, that

$$
\begin{aligned}
J = \liminf_{k\to\infty} \int_Q J_k dxdt &= \liminf_{k\to\infty} \int_Q J_{k,1} dxdt \\
&= \liminf_{k\to\infty} \int_Q \sigma(x,t,u_k,Du_k) : (Du_k - Du) dxdt \\
&\geq \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : (\lambda - Du) d\nu_{(x,t)}(\lambda) dxdt.
\end{aligned}
$$

Remark first that, since $(u_k, \nabla u_k)$ generates the Young measure $\delta_{u(x,t)} \otimes \nu_{(x,t)}$ by Proposition 2.3.1, then

$$
\begin{aligned}
u_k.\nabla u_k \rightharpoonup & \int_{\mathbb{R}^m \times \mathbb{M}^{m\times n}} s.\lambda d\delta_{u(x,t)}(s) \otimes d\nu_{(x,t)}(\lambda) \\
&= \int_{\mathbb{M}^{m\times n}} u.\lambda d\nu_{(x,t)}(\lambda) \\
&= u.\nabla u.
\end{aligned}
$$

We can conclude the result if we arrive at $J \leq 0$. For this, we use the first property of $\chi$ which is the energy equality:

$$
\frac{1}{2}\|u(.,T)\|_{L^2}^2 + \int_Q \chi : Du \, dxdt = \frac{1}{2}\|u(.,0)\|_{L^2}^2 + \langle f, u \rangle,
$$

where we have used $\int_Q (u.\nabla)u.udxdt = 0$. On the one hand, we have

$$\begin{aligned}\liminf_{k\to\infty} -\int_Q \sigma(x,t,u_k,Du_k) : Dudx &= -\int_Q \chi : Du\, dxdt \\ &=: \frac{1}{2}\|u(.,T)\|^2_{L^2} - \frac{1}{2}\|u(.,0)\|^2_{L^2} - \langle f, u\rangle.\end{aligned} \tag{7.5.9}$$

On the other hand, by the Galerkin equations, we have

$$\begin{aligned}&\int_Q \sigma(x,t,u_k,Du_k) : Du_k dxdt \\ &\quad = \langle f, u_k\rangle - \int_Q (u_k.\nabla)u_k.u_k dxdt - \int_Q u_k \frac{\partial u_k}{\partial t} dxdt \\ &\quad = \langle f, u_k\rangle - \int_Q (u_k.\nabla)u_k.u_k dxd - \frac{1}{2}\|u_k(.,T)\|^2_{L^2} + \frac{1}{2}\|u_k(.,0)\|^2_{L^2}.\end{aligned}$$

By passing to the limit inf in the last expression and using Lemma 7.5.2, it follows that

$$\liminf_{k\to\infty} \int_Q \sigma(x,t,u_k,Du_k) : Du_k dxdt \le \langle f, u\rangle - \frac{1}{2}\|u(.,T)\|^2_{L^2} + \frac{1}{2}\|u(.,0)\|^2_{L^2}.$$

The combination of this inequality with (7.5.9) gives

$$J = \liminf_{k\to\infty} \int_Q J_k dxdt \le 0.$$

□

According to 6.3.3, we have the following localization of the support of $\nu_{(x,t)}$:

$$\big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : (\lambda - Du) = 0 \quad \text{on} \quad supp\, \nu_{(x,t)}. \tag{7.5.10}$$

Now, we can pass to the limit in the approximating equations. To conclude the proof of Theorem 7.5.1 in cases **(P2)**(a)-(d), it is sufficient to use similar arguments as in Subsection 6.3.3 of Chapter 6. Let $\phi \in \mathcal{C}^1(0,T;V_j)$ for $j \le k$, then

$$\int_Q \frac{\partial u_k}{\partial t}\phi dxdt + \int_Q \big(\sigma(x,t,u_k,Du_k) : D\phi + (u_k.\nabla u_k).\phi\big) dxdt = \langle f, \phi\rangle,$$

which gives after integrating the first term

$$\int_\Omega u_k(.,T)\phi(T)dx - \int_Q u_k(.,0)\phi(0)dx - \int_\Omega u_k \frac{\partial\phi}{\partial t} dxdt \\ + \int_Q \big(\sigma(x,t,u_k,Du_k) : D\phi + (u_k.\nabla u_k).\phi\big)dxdt = \langle f, \phi\rangle.$$

Letting $j \to \infty$, then for $\phi \in \mathcal{C}^1(0,T;\mathcal{C}^\infty_{0,\mathrm{div}}(\Omega))$, we obtain when $k \to \infty$

$$-\int_\Omega u\frac{\partial\phi}{\partial t}dxdt + \int_\Omega u(x,t)\phi(x,t)dx\Big|_0^T \\ + \int_Q \big(\sigma(x,t,u,Du) : D\phi + (u.\nabla u).\phi\big)dxdt = \langle f, \phi\rangle.$$

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